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CONSTRUCTIVE MATHEMATICAL ANALYSIS

Editorial Staff

Special issue dedicated to Prof. Dr. Francesco Altomare: on the occasion of his 70th birthday

Prof. Dr. Francesco Altomare
University of Bari Aldo Moro

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Francesco Altomare - the remarkable mathematician and human being

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ABSTRACT. We laconically describe the great contributions of Professor Francesco Altomare to mathematical research and Ph.D education, and his unique status in the mathematical community. In particular, we present and give examples of his innovative and great achievements related to the following areas of mathematics: Functional Analysis, Operator Theory, Potential Theory, Approximation Theory, Probability Theory, Function Spaces, Choquet's Theory, Dirichlet's Problem and Semigroup Theory. Moreover, we report on and give concrete examples of his unique way to work together with Ph.D students, both before and sometimes also after their dissertation. Finally, we shortly describe his remarkable "class travel" from "simple" conditions with no academic traditions in his family in the small town Giovinazzo to finally become the broad, ingenious, and powerful mathematician he is regarded to be today.

Keywords: Francesco Altomare, History of Mathematics, Functional Analysis, Operator Theory, Potential Theory, Approximation Theory, Probability Theory, Function Spaces, Choquet's Theory, Dirichlet's Problem, Semigroup Theory and Evolution Equations, Ph.D education.

2020 Mathematics Subject Classification: 01A99, 31C99, 41A25, 41A63, 46E05, 47D06, 47D07.

1. INTRODUCTION

It is with great enthusiasm that we have accepted the proposal of our friend Professor Tuncer Acar to contribute to the Special Issue in honor of Professor Francesco Altomare's 70th birthday and we have decided to write a paper on his academic life.

First of all, we want to pronounce that in this limited amount of pages it is absolutely impossible to give a fair description of Francesco concerning his work, impact in the mathematical community, and his really unique and strong character. However, we will do our best to give the readers at least some interesting information about this remarkable man and his work. We have the following relations to Francesco: the two first authors are his former students, which still collaborate scientifically with him, while the third author is a typical representative of the mathematical community with broad knowledge about Francesco and his impressive status in the mathematical world.

The paper is organized as follows. In Section 2, we briefly describe the first steps and related personal information about Francesco. In particular, we shortly report on his really remarkable "class travel" from "simple" conditions, in a family with no academic traditions in the small town Giovinazzo, to finally become the broad, innovative and powerful mathematician he is regarded to be today.

In Section 3, we continue by delivering selected information from Francesco's early career from 1975 till around 1990. His strong thesis *Choquet's Theorems and Integral Representation Theory for Convex Compact Subsets* was a perfect basis, which influenced several subsequent research works, sometimes in collaboration with other researchers. Correspondingly, we mention

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some of these collaborations that we know Francesco considers especially important during that period.

Section 4 is used to inform about another activity for which Francesco is especially well known, namely he was the main organizer of six international FAAT (*Functional Analysis and Approximation Theory*) conferences which gathered together more than 900 participants totally. In this section, we also mention some later collaborations that we know have been especially significative for Francesco and his scientific work.

In Section 5, we give some examples of, in our opinion, the most important information from Francesco's very impressive Curriculum Vitae. Besides his obviously strong merits in mathematical research, he is well known for his unique way to supervise Ph.D students (see also Section 7) and as organizer and main speaker at international conferences. Moreover, he is Editor-in-Chief for a very high level international journal as well as Editor of several other journals including the current one. This is well known all over the world but what is less known is that Francesco has great merits and talents also outside the university sphere, e.g., concerning his involvement in inspiring students in basic schools in projects whose aim is to connect mathematics to others fields such as art, music, and literature.

Section 6 is the most important part of this paper. There we make a serious attempt to shortly describe some important parts of Francesco's impressive and innovative scientific contributions related to the following areas of Mathematics: Functional Analysis, Operator Theory, Potential Theory, Approximation Theory, Probability Theory, Function Spaces, Choquet's Theory, Dirichlet's Problem and Semigroup Theory.

Section 7 is reserved for a discussion of the unique and supportive way Francesco uses when supervising and guiding Ph.D students, PostDocs etc.; he sometimes even continues to collaborate with them a long time after they have finished their Ph.D program. We think that this can be a good model and leading star for many of us in the supervision of our own students.

Finally, in Section 8, we do a short summary focused on final words about the remarkable mathematician and human being Francesco, including some especially warm Mirella's, Vita's and Pers(s)onal words related to his anniversary.

2. The first steps and personal information

Francesco Altomare was born on May 18, 1951 in Giovinazzo, a nice town in Puglia (Italy) with a magnificent medieval center, located on the Adriatic Sea, and 18 kilometers away from Bari, the regional capital. Francesco has always lived in this town, to which he is deeply attached.

His father, *Luigi Altomare* (1898–1963), was an esteemed barber, and his mother, *Maria Giordano* (1915–1989), was a pious housewife. Francesco also had three siblings: one sister, *Gesuela* (1949–1999), and two brothers *Salvatore* (1953–1989) and *Giovanni* (1955).

In 1963, his father Luigi died prematurely and so his family suddenly found themselves facing a very difficult and uncertain future.

Despite this, his mother and Gesuela, on the advice of several professors who saw Francesco's potential, taking on themselves great sacrifices and supported by a strong desire for a social redemption, allowed him to continue his high school studies during which he had the opportunity to study and to become keen not only on Mathematics but also on the great Greek and Latin Classics, on art, especially on the Renaissance and the Impressionism, as well as on poetry and Italian literature. Francesco has continued to cultivate the interests in these subjects throughout his life.

In 1970, Francesco enrolled in a degree course in Mathematics at the University of Bari, where he graduated in 1975.

The old town of Giovinazzo.

During his high school and university studies, Francesco contributed to the family purse by doing small jobs, including giving amateur guitar lessons, a musical instrument he loves very much, and which he himself learned to play during the youth protests in 1968.

Between 1968 and 1972, Francesco played in a beat band where he alternated between playing the guitar and the electronic keyboard. Though the band mainly performed beat music, Francesco was also passionate about soul and Neapolitan music.

Raised and educated with a strong sense of religion, Francesco established a deep friendship with some enlightened priests who have been very close to him since his youth. He often participated in religious functions, performing musical pieces on the church organ.

Several sports activities count among his youthful passions, in particular basketball, which he played in provincial championships. He also coordinated a sports association that gathered young people from Giovinazzo, carrying out a noteworthy social action.

April 2019 -The Altomare family on the occasion of the baptism of Nicola Maria. From the left: Francesco, his daughter-in-law Nunzia, Bianca Maria and her husband Vittorio with their son Nicola Maria, Gianluigi and Raffaella.

In 1975, Francesco was invited to partake in the local city politics. He was elected as a member of the city council for a five-year term, the first three of which he was a city councilor as well. To these days, he is still very involved in promoting some cultural activities in Giovinazzo. In particular, he has co-founded a cultural association which, among other things, makes a library for children available to citizens. Moreover, periodically he gives some lectures on Mathematics which are intended to a broad audience and which have the aim to show the importance and usefulness of this Science, together with its harmony and beauty.

In 1968, Francesco met *Raffaella Bavaro* whom he married in 1979. Raffaella accompanied Francesco with love and spirit of self-sacrifice by supporting him throughout the whole scientific and academic activities. They have two children: *Bianca Maria* (1983) and *Gianluigi* (1986). *Bianca Maria*, in turn, gave birth to a son, *Nicola Maria* (2019).

3. The early career (1975-around 1990)

During his degree in Mathematics program, Francesco attended some courses given by Professors *Giuseppe Muni* and *Giovanni Aquaro* (1920–2014) who greatly influenced his future studies in Real and Functional Analysis (Locally Convex Spaces, Integration Theory, Measure Theory, General Topology). They acted as mentors for Francesco throughout his career and they all forged a strong tie of affection and friendship.

Bari, November 1990 – A party on the occasion of the retirement of Professor Giovanni Aquaro.
From the left: Francesco, Giovanni Aquaro, Giuseppe Muni.

Under the supervision of Professor Giuseppe Muni, Francesco wrote his thesis *Choquet's Theorems and Integral Representation Theory for Convex Compact Subsets*. This topic influenced several subsequent research works of his. The thesis contained new results, which were published in 1977 (see [1]). That was the first paper Francesco wrote.

Before completing his degree program in Mathematics, he was awarded a junior C.N.R. (Consiglio Nazionale delle Ricerche) grant. From 1975 to 1978, he was a senior research fellow at the Institute of Mathematical Analysis of the University of Bari. From 1978 until 1985, and from 1985 until 1987, he was an assistant professor and, respectively, an associate professor, at the Faculty of Sciences of the University of Bari. He was appointed full professor at the Faculty of Sciences of the University of Basilicata in Potenza in 1987. In 1990, he moved to the University of Bari, where he is currently employed, on the same chair Professor Aquaro left having reached the retirement age.

During the summer of 1975, Francesco attended a course on locally convex spaces given by Professor *John Horvath* (1924–2015) of the University of Maryland, at a Summer School in Perugia. During the course, Professor Horvath suggested Francesco to discuss the main results of his thesis with Professor *Heinz Bauer* (1928-2002) from the University of Erlangen-Nürnberg. This was the starting point of his long and fruitful cooperation with Professor Bauer, which inspired Francesco's subsequent studies and researches on abstract Choquet boundaries and relevant Dirichlet problems, Korovkin-type Approximation Theory in continuous function spaces, and Probability Theory.

In 1978, Professor Bauer invited Francesco to attend a conference on *Function Spaces* in Oberwolfach. That was Francesco's first conference abroad and, among other things, his first meeting with Professor *Ivan Netuka* (1944-2020) from the Charles University of Praha. The scientific cooperation developed by Professor Netuka and Francesco has been productive and essentially focused on the applications of Choquet Integral Representation Theory to Potential Theory. In 1997, on the invitation of Netuka, Francesco taught a short course on Korovkin-type Approximation Theory in Banach spaces at a Spring School in Paseky, Czech Republic.

Through Professor Bauer, Francesco had the opportunity to meet Professor *George Maltese* (1931–2009) from the University of Münster. The cooperation between them and some Maltese's Ph.D students - *Michael Pannenberg* and *Ferdinand Beckhoff* - focused on the development of Korovin-type Approximation Theory in the framework of Banach algebras. It has also been enriched by several exchanges of visits, one of which, in 1985, was supported by a joint research NATO fellowship.

Thanks to Professor Maltese, in 1980, Francesco came into contact with Professor *Gustave Choquet* (1915-2009), from the University of Paris VI, and his research group - *Richard Becker, Hicham Fakhoury, Gilles Godefroy, Marc Rogalski, Jean Saint-Raymond*. He spent two long research periods in Paris in 1980 and 1981, by also participating in the activities of the *Séminaire Initiation à l'Analyse*.

Erlangen, December 1995 - A conference on the occasion of the retirement of Professor Heinz Bauer. From the left: George Maltese, Gustave Choquet, Heinz Bauer, Francesco.

During the above-mentioned seminar activities, Francesco took up with Professor *Rainer Nagel* from the University of Tübingen. With him Francesco established a fruitful and long cooperation as they were both interested in Positive Operators, Semigroup Theory and Evolution Equations, Markov Semigroups and Stochastic Processes. Several exchanges of visits were realized. Moreover, on the light of the emerging collaboration with Francesco, in 1993, Professor Nagel was appointed the post of full professor at the University of Bari until 1995, when he came back to the University of Tübingen.

In 1987, Francesco was appointed full professor at the University of Basilicata in Potenza where he taught until 1990. This period was one of the most fruitful of his scientific and academic career. The University of Basilicata was a new university, inaugurated in the 1983–84 academic year. It was established after the terrible earthquake of 1980, which devastated the Campania and Basilicata regions. That same year, Francesco was called to manage the newborn Institute of Mathematics and organize it (library, numerical and computer laboratories, administrative offices, teaching and researches activities, and so on).

4. The FAAT conferences and later collaborations

During Francesco's stay in Potenza, many young professors from several Italian and foreign universities took up service at the University of Basilicata. Soon after, an agreeable atmosphere of friendship and strong cohesion was established among them. It was in this atmosphere that Francesco met Professor *Giuseppe Mastroianni*, coming from the University of Naples Federico II. A brotherly friendship built on mutual esteem and collaboration was immediately born between them. Their common scientific interests included Approximation Theory, especially approximation problems by means of positive operators.

Lecce, September 2011: A party on the occasion of Francesco's 60th birthday. From the left: Raffaella, Giuseppe Mastroianni, and Francesco.

They also developed the idea to organize a conference on *Functional Analysis and Approximation Theory* which took place at Acquafredda di Maratea (Potenza) in 1989. The conference was highly appreciated and this encouraged the organization of five further editions (1992, 1996, 2000, 2004, and 2009). Francesco managed the Proceedings volumes of all editions which were published in esteemed mathematical journals.

All six editions received a general appreciation for the organization, level of the topics covered, and the beauty of the locations where they took place. This appreciation is documented, e.g., in the monograph [29, Section 8.2.7.15, p. 626]. For a complete overview of the scientific programs of the Maratea Conferences, we refer the reader to [12].

During the time spent in Potenza, a deep and affectionate friendship with Professor *Michele Campiti* was also consolidated. Professor Campiti graduated in 1985 from the University of Bari and came to the University of Basilicata as a guest lecturer and on an I.N.d.A.M. research fellowship under Francesco's supervision. He is now a full professor at the University of Salento in Lecce, and generously contributed to the organization of all the FAAT conferences. He and Francesco wrote the monograph [14], which has been highly appreciated by the international community since its year of publication in 1994. For many years, it was also listed among the top ten most cited monographs in the Approximation Theory section of Mathematical Reviews.

Other scientific collaborators which characterized the period in Potenza are Biancamaria Della Vecchia, Sapienza University of Rome, who, among other things, enthusiastically participated in the organization of several editions of the Maratea Conferences, as well as Professors *Manuel Valdivia* (1928–2014), University of Valencia, *Vincenzo Moscatelli* (1945–2008), University of Lecce, *Pierluigi Papini*, University of Bologna, and *Carlo Franchetti*, University of Firenze. Those scientific collaborations were mainly concerned with Fréchet spaces, Banach spaces and abstract Approximation Theory.

At the end of 1989, Francesco established the first contacts with Professor *Ioan Raşa* from the Technical University of Cluj-Napoca. This was the start of a long and productive collaboration

Acquafredda di Maratea, September 1992 - Opening Ceremony of the 2nd FAAT Maratea Conference. From the left: Michele Campiti, Delfina Roux (1927–2018), Francesco, Giuseppe Mastroianni, Mario Rosario Occorsio, Biancamaria Della Vecchia.

The Conference poster of the last FAAT Conference.

from 1989 to 2018 characterized by several exchanges of visits, 14 joint published papers and a monograph [20] jointly written with the first and second author of this paper. In 2004, they both spent a research period at the Mathematical Institute of Oberwolfach under the Research in Pair program. Professor Raşa has often attended the FAAT Conferences, being also main speaker in some editions.

We now mention a number of additional collaborations Francesco's research group in Bari developed after 1990 that we know he judges as important.

- Professor *Romulus Cristescu*, University of Bucharest, President of the Section of Mathematical Sciences of the Romanian Academy, and his research group on Riesz spaces and positive operators. Professor Cristescu invited Francesco several times to well-known conferences in Sinaia

on Riesz spaces and positive operators. In 2003, he also invited Francesco as guest speaker of the *5th Congress of Romanian Mathematicians* in Piteşti.
- Professors *Elena Moldovan Popoviciu* (1924–2009), *Dimitrie D. Stancu* (1927-2014), and *Octavian Agratini* from the University Babeş-Bolyai of Cluj-Napoca. Their common scientific interests were concerned with the Approximation Theory by means of positive operators and were developed during several exchanges of visits. Professor Agratini has been one of the main organizers of a series of excellent conferences on *Numerical Analysis and Approximation Theory*, held in Cluj-Napoca in 2006, 2010, 2014, and 2018, to which Francesco was always invited as main speaker.
- Professors *Ioan Gavrea* and *Mircea Ivan* from the Technical University of Cluj-Napoca. They had shared scientific interests in the Approximation Theory by means of positive operators, and soon their interactions extended to their families who are now bound by a warm friendship. Professor Mircea Ivan has been one of the main organizers of the series of *Theodor Angheluţă Seminar Conferences* which were attended by Francesco several times.
- Professors *Pietro Aiena* and *Camillo Trapani* from the University of Palermo and their research group on Operator Theory. Among other things, they developed the national research project "Operator theory, Semigroups and Applications to Evolution Equations and Approximation Problems", sponsored by the Italian Ministry for Education for the years 2003–04 and 2004–05 (PRIN-COFIN 2003).
- Professors *Gianluca Vinti* and *Carlo Bardaro* from the University of Perugia, and their research group on Approximation Theory - *Laura Angeloni, Danilo Costarelli, Ilaria Mantellini, and Luca Zampogni*. Together with their research groups Francesco and Gianluca Vinti developed several national research projects on *Methods of Operator Theory for Approximation Problems and Evolution Equations*.
- Professors *Francisco Javier Muñoz-Delgado* and *Daniel Cárdenas-Morales* from the University of Jaén. They have organized ten editions (2010-2019) of the *Jaén Conferences in Approximation Theory* which took place in the beautiful city of Úbeda, a world heritage site. These conferences and other exchanges of visits consolidated the scientific collaboration in Approximation Theory with Francesco and his research group.
- Professor *Mikio Kato* from Kyushu Institute of Technology, who reignited Francesco's interest in Banach Spaces Theory. Professor Kato visited Italy two times to attend the *6th FAAT Conference* in 2009 and the *Recent Developments in Functional Analysis and Approximation Theory* conference in 2011 in Lecce on the occasion of Francesco's 60th birthday. Francesco visited Japan once in September 2006 to attend the *Second International Symposium on Banach and Function Spaces* 2006, organized by Professor Kato in Kitakyushu. After the conference in Kitakyushu, he was also a guest speaker at the *Autumn Congress of Mathematical Society of Japan*, held in Osaka in the same month.

Other mathematicians with whom in the late 90s Francesco developed fruitful scientific collaborations have been Professor *Elisabetta Mangino* from the University of Salento and Professors *Antonio Attalient*i and *Silvia Romanelli* from the University of Bari.

Very recently, Francesco started new scientific contacts with some Turkish mathematicians and, in particular, with Professor *Tuncer Acar* from the Selcuk University of Konya especially in connection with the development of the new journal *Constructive Mathematical Analysis* founded by him.

The third author of this paper is, like Francesco, Editor of the same journal. Lars-Erik and Francesco met for the first time at the above mentioned conference in Japan in 2006, where both were main speakers. After that they have collaborated and been in contact in several ways both professionally and on a more private level. In particular, Lars-Erik was a main speaker at the above mentioned FAAT conference in 2009.

(A)

(B)

(A) Istanbul, July 2018, Francesco and Tuncer Acar during the International Conference on Mathematics "An Istanbul Meeting for World Mathematicians", Minisymposium on Approximation Theory & Minisymposium on Math Education.
(B) Kitakyushu, September 2006, Francesco and Lars-Erik during the International Conference "Second International Symposium on Banach and Function Spaces".

5. Other selected information from Francesco Altomare's C.V.

During his career, Francesco has been appointed on several official posts. In particular, after being, as already mentioned, Director of the Institute of Mathematics at the University of Basilicata from 1987 to 1990, he has served as
- Director of the Graduate School in Mathematics at the University of Bari from 1993 to 1995.
- Head of the Interuniversity Department of Mathematics of the University and the Polytechnic of Bari from 1997 to 1999.
- Coordinator of the Ph.D. School in Mathematics of the University of Bari from 1999 to 2003.
- Head of the Department of Mathematics of the University of Bari from 2012 to 2015.
- Member of the Academic Senate of the University of Bari from 2012 to 2018.
- Member of the Scientific Committee of Italian Mathematical Union from 2012 to 2015.

During the whole year 2011, Francesco was involved in the groundworks of the Committee for the formulation of the new Statute of the University of Bari. Moreover, in 2014, he was one of the main co-founders of the School of Sciences and Technologies of the University of Bari.

Francesco is member of the Italian Mathematical Union as well as of the European Mathematical Society.

Francesco has carried out various academic and scientific assignments at several Italian universities as well as at the Italian Mathematical Union. In particular, in 2007, he was charged by the Italian Mathematical Union to chair the Organizing Committee of the XVIII Congress of that Society which was held in Bari during the period September 24–29, 2007.

Francesco was invited to deliver lectures and postgraduate short courses as well as developed joint researches at several Italian and foreign universities (Napoli, Lecce, Cosenza, Potenza, Roma, Milano, Bologna, Trieste, Salerno, Palermo, Perugia, Sofia, Annaba, Erlangen, Konya, Passau, Valencia, Praga, Paseky, Siegen, Vienna, Baku). He has also participated in about sixty international meetings as invited speaker (Italy, Germany, U.S.A., Brasil, Romania, Spain, Hungary, Czech Republic, Russia, Taiwan, Austria, Japan, Tunisia, Turkey, Azerbaijan).

Francesco contributes to Mathematical Reviews as a reviewer and acts as a referee for several national and international journals.

Francesco is or has been also a member of the Editorial Board of the following journals: Conferenze del Seminario di Matematica dell'Università di Bari (Italy, 1990–2003), Revue d'Analyse Numérique et de Théorie de l'Approximation (Romania, since 1998), Mathematical Reports (Romania, 2000–2012), Journal of Applied Functional Analysis (U.S.A., 2004–2015), Journal of Interdisciplinary Mathematics (India, 2004–2018), Numerical Functional Analysis and Optimization (U.S.A., since 2008), Bollettino dell'Unione Matematica Italiana (Italy, 2008–2012), Studia Mathematica Universitatis Babeş-Bolyai (Romania, since 2009), Demonstratio Mathematica (Poland, since 2017), Constructive Mathematical Analysis (Turkey, since 2018).

Moreover, since 2004 Francesco is the founding Editor-in-Chief of the international journal *Mediterranean Journal of Mathematics*. It is worth noticing that one of the founding principles and motivations of the journal is to contribute to further integration and collaboration amongst the Mediterranean universities. This aim is reflected in the choice of the title "Mediterranean Journal of Mathematics" and in the composition of the Editorial Board, which includes mathematicians from Mediterranean countries only. Nowadays, the journal has reached an excellent reputation worldwide and publishes about 200 articles per year.

Francesco contributed to the organization of several international meetings and was co-editor of the corresponding proceedings. Among them, we recall the already mentioned six editions of the international conference on *Functional Analysis and Approximation Theory* held in 1989, 1992, 1996, 2000, 2004, and 2009 held in Acquafredda di Maratea (see Section 4).

Francesco has always been passionate about teaching and, during his career, he has taught several undergraduate, graduate and Ph.D courses. He has also developed an interest in communicating Mathematics to the younger generations, pointing out its deep relationship, not only with the other hard sciences, but also with Art, Music, and Literature. For this reason, following a similar project of other Italian universities, he has supported a close collaboration between the Department of Mathematics of the University of Bari and several High Schools in Puglia and has been appointed director of the Professional Training Course *Liceo ad Indirizzo Matematico*, devoted to those high-school teachers who want to approach Mathematics in a more inclusive and comprehensive way.

6. Some important scientific contributions of Francesco Altomare

During his long and fruitful career, Francesco has written more than 90 papers and three monographs (see [9, 14, 20]). An overview of Francesco's scientific activity up to 2011 was presented by Michele Campiti in a talk at the conference *Recent developments in Functional Analysis and Approximation Theory* held in Lecce in 2011 in honor of Francesco's 60th birthday (see [28]). In this section, we focus on some aspects of his scientific career, knowing well enough it is not possible to summarize in a few words the great impact he has had on the mathematical and sciences community.

Francesco's main areas of research as well as his scientific interests may be grouped under the following items:

Real and Functional Analysis (Choquet Integral Representation Theory, Choquet boundaries, continuous function spaces, function algebras and Banach algebras, locally convex vector lattices, positive linear forms and applications to abstract Potential Theory and Harmonic Analysis);

Operator Theory, Probability Theory and Evolution equations (Positive operators, semigroups of operators, positive semigroups and Markov processes, differential operators and applications to evolution equations);

Approximation Theory (Korovkin-type Approximation Theory, positive approximation processes, approximation of semigroups by means of positive operators, asymptotic formulae, iterates of positive linear operators).

One of the characteristics of Francesco's work is that in most of his papers it is not possible to describe the results obtained in one of the above areas without at least one of the others being involved. Francesco's main scientific achievement is in our opinion to have realized how closely connected those areas are.

This fact is especially evident in the two monographs [14] and [20], where several techniques coming from Real and Functional Analysis, Approximation Theory and Operator Theory, Probability and Evolution Equation Theory are combined clearly showing an overview that he has probably always had in his mind.

In fact, some aspects of the research work of Francesco and his collaborators find their roots in the first results he obtained on H-Lion operators (see [1, 2, 3]). Such results are the first contributions of Francesco to the Choquet Integral Representation Theory and to the abstract Dirichlet problem. These investigations have as a consequence, for example, the characterization of a Bauer simplex K of a locally convex space in terms of the existence of a unique continuous positive projection on $C(K)$ such that $T(C(K)) = A(K)$, where $A(K)$ is the space of all affine functions on K, as well as in terms of the existence of a resolvent family $(R_\lambda)_{\lambda>0}$ on K having $A(K)$ as common range and such that $(\lambda R_\lambda)_{\lambda>0}$ is strongly convergent as $\lambda \to +\infty$.

Other contributions to Choquet's Theory can be found in the papers [15, 16, 17], whose authors have considered a particular class of positive projections defined on an adapted subalgebra of continuous real-valued functions that can be characterized by means of representing measures concentrated on the Choquet boundary. We also mention the introduction of new classes of locally convex vector lattices of continuous functions on a locally compact Hausdorff space in whose context some Korovkin-type theorems for continuous positive linear operators with respect to the identity operator, to positive projections, and to finitely defined operators were obtained. Besides, in the above mentioned setting an integral representation theorem for positive linear functionals is presented as well.

Approximation Theory has always been one of Francesco's most beloved topics. He has obtained Korovkin-type theorems not only in continuous function spaces, but also in different contexts, such as abstract Riesz spaces (see, e.g., [4]), L^p spaces, and weighted continuous function spaces or Banach algebras (see, e.g., [5, 6]). Moreover, he has introduced and studied several classes of approximation processes in spaces of continuous and integrable functions.

Moreover, Francesco has deeply studied asymptotic formulae for positive linear operators and their applications, among other things, to the Central Limit Theorem of Probability Theory, to converse theorems of smoothness and saturation theorems ([8, 13, 22]). Besides, much attention has been paid to the study of iterates of positive linear operators and their applications in Ergodic Theory and the constructive approximation of semigroups of operators. For a recent survey on these topics, see [10].

The monograph [14], written by Francesco and Michele Campiti, can be considered as the biggest contribution of these authors to Approximation Theory, since it is highly appreciated by the researchers in that field, which can be judged, for example, by the number of its citations. The monograph presents, for the first time, a comprehensive exposition of the main aspects of the Korovkin-type Approximation Theory. By following an original and systematic approach, the book condenses the main results of the theory obtained during the period 1953–1994, which are documented in more than 700 references. Among other things, the reader can find results concerning Korovkin-type theorems for bounded positive Radon measures, for positive linear operators, and for the identity operator (the classical case), realizing the strong connection between this theory and other areas of functional analysis.

Among Francesco's main achievements pertaining the Korovkin-type Approximation Theory we mention the following:
- A Korovkin-type theorem with respect to limit operators which are positive projections (see [14]).
- An extension of the classical Korovkin theorem in the framework of function spaces on metric spaces (see [9]).
- The equivalence between both the algebraic and trigonometric versions of the Korovkin theorems and the algebraic and trigonometric versions of the Weierstrass theorems (see [9, 14]).
- A characterization of the subalgebras of continuous functions vanishing at infinity on a locally compact Hausdorff space in terms of Korovkin closures. This, in particular, leads to an equivalence between the Bauer extension of the Korovkin theorem in the above mentioned more general setting and the Stone extension of the Weierstrass theorem in the same setting. Such equivalence enlights the deep and beautiful relationship between these milestones of the Approximation Theory which reflects two different points of view: a constructive and a qualitative one, respectively (see [9, 14]).

In [14], there are also several applications to the approximation of continuous functions by positive linear operators, together with applications to the approximation and the representation of the solutions of particular partial differential equations of diffusion type by means of iterates of suitable positive linear operators. The connection between Approximation Theory and Evolution Equations lies in those results.

Subsequently, all aspects treated in [14] were taken up by Francesco.

The monograph [14] already contains the germ of a new theory which would have been developed during the subsequent twenty years by Francesco and his collaborators and which found a rather complete synthesis in the monograph [20], jointly written with the first and second authors of this paper and Ioan Raşa. The main aim of this theory is the study of a large class of (mainly degenerate) initial-boundary evolution problems in connection with the possibility to investigate whether they can be described by positive semigroups and, in such a case, whether it is possible to approximate such semigroups by means of iterates of suitable positive linear operators which also constitute approximation processes in the underlying Banach function space.

By using such kind of approximation, after a careful analysis of the preservation properties of the approximating operators, such as monotonicity, convexity, Hölder continuity and so on, the hope is to infer similar preservation properties for the relevant semigroup and, consequently, some spatial regularity properties of the solutions to the evolution problems as well as their asymptotic behaviour, i.e., their long-term behavior as the "temporal" parameter t tends to infinity.

In what follows, we briefly describe some important differential problems which can be treated by using the ideas developed by Francesco scientific career.

Fix a convex compact subset K of $\mathbb{R}^d$ ($d \geq 1$) with non-empty interior and a positive linear operator on $T : C(K) \to C(K)$ such that $T(\mathbf{1}) = \mathbf{1}$, $\mathbf{1}$ being the constant function of value 1. Then, consider the elliptic second-order differential operator W_T defined as

$$W_T(u) := \frac{1}{2} \sum_{i,j=1}^{d} \alpha_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \quad (u \in C^2(K)),$$

where, for each $i, j = 1, \ldots, d$ and $x \in K$, $\alpha_{ij}(x) := T(pr_i pr_j)(x) - (pr_i pr_j)(x)$, pr_i being the i-th coordinate function (i.e. $pr_i(x) = x_i$ for every $x \in K$).

Operators of the form W_T are of great importance in the study of several diffusion problems arising in population genetics, financial mathematics and other fields.

Because of the degeneracy of W_T on the set $\partial_T K := \{x \in K \,|\, T(f)(x) = f(x) \text{ for every } f \in C(K)\}$ of all interpolation points for T and the fact that the boundary ∂K of K is generally non-smooth, the methods of the theory of partial differential equations might fail in studying such problems. However, there are no such difficulties in the theory developed by Francesco and his co-authors.

Consider the initial boundary value problem associated with $(W_T, C^2(K))$ and the initial value u_0 belonging to a suitable subset of $C^2(K)$:

$$\text{(6.1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} \alpha_{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j}(x,t) & x \in K, \ \ t \geq 0; \\ u(x,0) = u_0(x) & x \in K. \end{cases}$$

One of the first problems deals with showing whether the operator W_T defined on $C^2(K)$ is the (pre)-generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $C(K)$; if this is the case, it is well known that, for any u_0, such a problem has a unique solution given by $u(x,t) = T(t)(u_0)(x)$ $(x \in K, t \geq 0)$, and the solution continuously depends on the initial value u_0.

Therefore, we have at our disposal the solution of the above differential problem in terms of the semigroup (pre)-generated by the elliptic operators underlying it. Unfortunately, usually we do not know an explicit expression for the semigroup, and as a consequence of the solution.

The second aim is to construct a suitable positive approximation process $(L_n)_{n\geq 1}$ on $C(K)$, such that, for every $t \geq 0$ and $f \in C(K)$,

$$T(t)f = \lim_{n\to\infty} L_n^{k_n} f \quad \text{uniformly on } K$$

for a suitable sequence $(k_n)_{n\geq 1}$ of positive integers (independent on f), where each $L_n^{k_n}$ denotes the iterate of L_n of order k_n.

Then, combining the above results, we get an approximation formula for the solutions to (6.1), i.e.,

$$u(x,t) = T(t)(u_0)(x) = \lim_{n\to\infty} L_n^{k_n}(u_0)(x),$$

from which we can try do derive both numerical and qualitative information about u from the study of the operators L_n.

The positive linear operators that are used to prove both the generation and the approximation properties of the semigroup $(T(t))_{\geq 0}$ are the so-called Bernstein-Schnabl operators B_n associated with T and are defined, for every $n \geq 1, f \in C(K), x \in K$, by

$$B_n(f)(x) := \int_K \cdots \int_K f\left(\frac{x_1 + \ldots + x_n}{n}\right) d\tilde{\mu}_x^T(x_1)\cdots d\tilde{\mu}_x^T(x_n)\,,$$

where, for every $x \in K$, $\tilde{\mu}_x^T$ is the unique probability Borel measure on K associated with T via the Riesz representation theorem.

Bernstein-Schnabl operators, first introduced by Schnabl (see [30]) and successively by other authors in different contexts, were intensively studied by Francesco since the late '80s (see [7], [14, Section 6.1] and the references therein). In the above mentioned works, though, it was assumed that the underlying operator T is a positive projection satisfying suitable assumptions. Francesco overcame such a limitation first in the context of the interval [0, 1], in the joint work

[23] with the second author of this paper and Ioan Raşa, and then in the context of compact convex sets of $\mathbb{R}^d$ (see [18, 20]).

Note that similar results hold in the case where K is a convex compact subset of a (not necessarily finite dimensional) locally convex Hausdorff space X. Furthermore, Bernstein-Schnabl operators associated with T are useful not only to constructively approximate suitable Markov semigroups, but they are also an approximation process for continuous functions on K.

Subsequently, Francesco and his collaborators have also tackled similar problems in different function spaces (for instance L^p-spaces) and for differential operators of the complete form $V_T(u) := W_T(u) + \sum_{i=1}^{d} \beta_i \frac{\partial u}{\partial x_i} + \gamma u$ ($u \in C^2(K)$), where $\beta_1, \ldots, \beta_d, \gamma \in C(K)$ satisfy suitable assumptions (see, e.g., [21]).

Other than the construction of suitable approximation processes which would approximate the semigroups, the key properties which guarantee the solution of the above mentioned problems are asymptotic formulae for the involved approximation operators and the assumption that the operator T maps polynomials on K into polynomials of the same degree. This last aspect has led to investigate the delicate problem of the existence of such operators. In [19] (see also [20, Section 4.3]), among other things it is shown that if K is strictly convex, then such operators exist if and only if K is an ellipsoid. In this case, a classification of such operators is given in terms of the Poisson operators associated with suitable strictly elliptic differential operators. Besides, in the two-dimensional case, a complete characterization of those compact subsets K which admit a non-trivial Markov projection on $C(K)$ preserving polynomials of the same degree has been obtained in [27, Section 4] (see also [20, Appendix A.1]). There a complete description of the relevant Markov projections is also given.

In developing the above mentioned theory some new aspects have been disclosed pertaining the classical Approximation Theory. These new aspects have enlightened new perspectives in the study of the positive approximation processes both in the one dimensional and multidimensional settings. It has been shown, indeed, that almost all the most important positive approximation processes studied in Approximation Theory generate positive semigroups whose generators can be explicitly determined.

Accordingly, it turns out that these classical approximation processes are extremely useful, not only to constructively approximate continuous or integrable functions, but also to approximate and to study qualitative properties of the solutions of several classes of initial-boundary value evolution problems which are of interest in applied sciences.

The investigations related to these problems delineated, in fact, a new research field whose contributions have been given by several mathematicians, mainly from Italy, Romania and Germany. For some references to this respect, we refer to [24].

As regards the determination of suitable approximation processes which would approximate the semigroups, we mention that, other than Bernstein-Schnabl operators, several other classes have been constructed, all depending on a given positive linear operator. Among them we cite Lototsky-Schnabl operators (see [14, 20]) and generalized Kantorovich operators (see [21]). The problem of constructing new approximation processes related to a positive linear operator seems to have an independent interest and constitutes a further promising research field.

Moreover Francesco, together with some collaborators, studied semigroups of positive operators also in connection with elliptic boundary value problems and Markov processes, especially in the setting of weighted continuous function spaces. We refer, e.g., to [25] and [26] and the references therein, for more details in this respect.

We end the section by pointing out that very recently Francesco has undertaken the study of convergence criterions for positive linear operators and functionals acting on spaces of bounded functions which are continuous only on suitable subsets of their domains. The first results in this direction are contained in [11].

7. Francesco Altomare as supervisor

Francesco supervised the scientific activities of several undergraduate and postgraduate research fellows as well as Ph.D students. Many of them have later been appointed as professors.

Ph.D students

- *Sabrina Diomede*, Ph.D School at the University of Naples Federico II, 1998-2002. Ph.D thesis: Positive approximation processes on continuous function spaces.
- *Mirella Cappelletti Montano*, Ph.D School at the University of Bari, 2000-2003. Ph.D thesis: Approximation problems by positive operators in adapted spaces.

Rachida Amiar, Ph.D School at the University of Annaba (Algeria), 1998-2005. Ph.D thesis: Méthode des opérateurs positifs pour l'étude des equations de diffusion.

- *Vita Leonessa*, Ph.D School at the University of Bari, 2002-2005. Ph.D thesis: Positive linear operators associated with continuous selections of Borel measures.
- *Sabina Milella*, Ph.D School at the University of Bari, 2002-2005. Ph.D thesis: Evolution equations on real intervals, semigroups and their approximations.
- *Graziana Musceo*, Ph.D School at the University of Bari, 2003-2006. Ph.D thesis: Positive semigroups and evolution equations in weighted continuous function spaces.

Research fellows

- *Daniela Sforza*, Junior C.N.R. research fellowship (1982-1983).
- *Cinzia Lucia Zifarelli*, Senior C.N.R. research fellowship (1987-1988).
- *Michele Campiti*, I.N.d.A.M. research fellowship (1988-1990).
- *Elvira Romita*, Junior C.N.R. research fellowship (1994-1995).
- *Ingrid Carbone*, Senior C.N.R. research fellowship (1995-1996).
- *Marco Romito*, Senior C.N.R. research fellowship (1995-1996).
- *Sabrina Diomede*, Junior C.N.R. research fellowship (1996-1997).
- *Lorenzo D'Ambrosio*, Junior C.N.R. research fellowship (1997-1998).
- *Mirella Cappelletti Montano*, Research Fellowship University of Bari (2005).
- *Vita Leonessa*, I.P.E. Post-doc fellowship (2006).
- *Sabina Milella*, Post-doc fellowship University of Bari (2007-2009).
- *Graziana Musceo*, Post-doc fellowship University of Bari (2009-2011).

Francesco paid a lot of care and attention to his students, not only from a merely scientific point of view, but also from an academic and human one. With most of them he has written joint papers.

Francesco promoted the birth of a Ph.D program in Mathematics at the University of Bari in 1999 and he was its first Coordinator. As a matter of fact, many of his Ph.D students were enrolled in that program. He put many efforts in building a Ph.D School in Bari because he strongly believes in sharing knowledge and experience with students and in creating an environment where they can learn what being a good mathematician means.

Francesco is a very generous supervisor and collaborator. The door of his office is always open and he is very happy to talk with his students and co-authors about (but not only) Mathematics.

Acquafredda di Maratea, 6th FAAT Conference, September 2009 - Francesco and some of his Ph.D students. From the left: Graziana Musceo, Vita Leonessa, Francesco, Sabrina Diomede, Sabina Milella, Mirella Cappelletti Montano.

He thinks that the devil lies in the details, so he approaches every Mathematical problem with rigor and care. He has thought this valuable lesson to all his students as well.

Mirella and Vita, the two first authors of this article, think that they have been lucky enough to be Ph.D students of Francesco and they are still in the lucky situation of continuing to collaborate with him. They are deeply grateful to him for his continuous encouragement over the years and they feel honored for his friendership.

8. Final Words about Francesco Altomare

As seen in our descriptions above the great contributions of Professor Francesco Altomare to mathematical research and Ph.D education, and his unique status in the mathematical community is obvious and worldwide well known. We will finalize our writing by also giving some final words about this remarkable mathematician and human being.
* Francesco is married and he has two children and one grandchild. The family is very important to him.
* Francesco has chosen to live his whole life in his small hometown Giovanazzo, 18 kilometers away from the regional capital Bari.
* Francesco is a really warmhearted and open-minded person, which it is very easy to like and collaborate with.
* Throughout his life Francesco has kept interests concerning subjects other than Mathematics, e.g., Greek and Latin Classics, art (especially on the Renaissance and the Impressionism), poetry and Italian literature.
* During his youth Francesco contributed to the family purse by doing small jobs, including giving amateur guitar lessons. Moreover, he played in a beat band.
* Francesco was also early interested in sports. In particular, he participated in local championships in basketball and coordinated a sports association for young people in his hometown.
* Francesco was invited to partake in the local city politics, and in fact he was city councilor for three years.

* Francesco continues to give contributions to the cultural activities in Giovinazzo. For example, he was the co-founder of a cultural association which, among other things, makes a library for children available to citizens. Moreover, he periodically gives some lectures on the beauty of Mathematics which are intended to a broad audience.
* Francesco was educated with a strong sense of religion. He often participated in religious functions and he has even performed musical pieces on the church organ.

We want to end this paper by thanking Francesco for having inspired many of us in various ways both as an excellent mathematician and as a warmhearted and especially great human being.

Thank you Francesco for everything, in particular,
* for all wonderful new knowledge and innovative ideas you have given to the mathematical sciences;
* for providing us collaborators, including all Ph.D students, with your positive and supportive spirit and ideas to mathematics and life;
* for always being open for and looking forward new adventures in the future;
* for being our dear friend and still a very active main collaborator and ideal for us.

Dear Francesco,
we hereby give you our warmest and most cordial congratulations on your +25 birthday and hope for many years more in collaboration and life with you as our inspirer and friend.

Acknowledgments

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Permission

The pictures present in the paper belong to Prof. Francesco Altomare and we have permission to use them.

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Research Article

Grüss and Grüss-Voronovskaya-type estimates for complex convolution polynomial operators

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ABSTRACT. The aim of this paper is to obtain Grüss and Grüss-Voronovskaya inequalities with exact quantitative estimates (with respect to the degree) for the complex convolution polynomial operators of de la Vallée Poussin, of Zygmund-Riesz and of Jackson, acting on analytic functions.

Keywords: Complex convolution polynomials, de la Vallée-Poussin kernel, Riesz-Zygmund kernel, Jackson kernel, Grüss-type estimate, Grüss-Voronovskaya-type estimate, analytic functions.

2020 Mathematics Subject Classification: 41A10, 41A25, 30E10.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

A classical well-known result in approximation theory is the Grüss inequality for positive linear functionals $L : C[0,1] \to \mathbb{R}$, which gives an upper bound for the Chebyshev-type functional

$$T(f,g) := L(f \cdot g) - L(f) \cdot L(g), \quad f, g \in C[0,1].$$

Starting also from a problem posed in [3], this inequality was investigated in terms of the least concave majorants of the moduli of continuity and for positive linear operators $H : C[0,1] \to C[0,1]$, for the first time in [1] and in the note [5], where the cases of classical Hermite-Fejér and Fejér-Korovkin convolution operators were considered.

Refined versions of the Grüss-type inequality in the spirit of Voronovskaya's theorem were obtained in [4] for Bernstein and Păltaănea operators of real variable and for complex Bernstein, genuine Bernstein-Durrmeyer and Bernstein-Faber operators attached to analytic functions of complex variable.

After the appearance of these results, several papers by other authors have developed these directions of research.

For example, let $C_{2\pi} = \{f : \mathbb{R} \to \mathbb{R}; f$ continuous and 2π periodic on $\mathbb{R}\}$. A classical method to construct trigonometric approximating polynomials for $f \in C_{2\pi}$ is that of convolution of f with various trigonometric even polynomials $K_n(t)$ (called kernels), under the form

$$L_n(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)K_n(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)K_n(x-u)du, \quad x \in \mathbb{R}, n \in \mathbb{N}. \tag{1.1}$$

Upper estimate in the Grüss-type inequality for convolution trigonometric polynomials with respect to general form of the kernel $K_n(t)$, was obtained in [1].

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Now, by analogy, for f analytic in a disk $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ and continuous in the closure of the disk, one can attach the convolution complex (algebraic) polynomials by

$$\mathcal{L}_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it}) \cdot K_n(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{-it}) \cdot K_n(t)dt, z \in \mathbb{D}_R, \quad n \in \mathbb{N}. \tag{1.2}$$

The goal of this paper is to continue the above mentioned directions of research, obtaining Grüss and Grüss-Voronovskaya exact estimates (with respect to the degree) for the de la Vallée-Poussin complex polynomials in Section 2, for Zygmund-Riesz complex polynomials in Section 3 and for Jackson complex polynomials in Section 4.

2. De la Vallée-Poussin complex convolution

In this section, we extend the Grüss and the Grüss-Voronovskaya estimates for the de la Vallée-Poussin complex polynomials given by the general formula (1.2) and based on the convolution with the de la Vallée-Poussin kernel

$$K_n(t) = \frac{1}{2} \cdot \frac{(n!)^2}{(2n)!} \cdot (2\cos(t/2))^{2n},$$

defined by

$$\mathcal{V}_n(f)(z) = \frac{1}{\binom{2n}{n}} \sum_{j=0}^{n} c_j \binom{2n}{n+j} z^j = \sum_{j=0}^{n} c_j \frac{(n!)^2}{(n-j)!(n+j)!} z^j, \tag{2.3}$$

attached to analytic functions in compact disks, $f(z) = \sum_{j=0}^{\infty} c_j z^j$.

Let us denote $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ and $\|f\|_r = \max\{|f(z)|; |z| \le r\}$.

Firstly, we prove a theorem for the general complex convolutions given by (1.2).

Theorem 2.1. *Suppose that $R > 1$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ for all $z \in \mathbb{D}_R$.*

Let $1 \le r < R$ and consider the operators $\mathcal{L}_n$ given by (1.2). For all $n \in \mathbb{N}$, it follows

$$\|\mathcal{L}_n(fg) - \mathcal{L}_n(f)\mathcal{L}_n(g)\|_r \le \sum_{m=0}^{\infty}\left[\sum_{j=0}^{m} |a_j| \cdot |b_{m-j}| \cdot \|A_{n,m,j}\|_r\right],$$

where denoting $A_{n,m,j}(z) = \mathcal{L}_n(e_m)(z) - \mathcal{L}_n(e_j)(z) \cdot \mathcal{L}_n(e_{m-j})(z)$ and $e_m(z) = z^m$, $m \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}\|A_{n,m,j}\|_r \le \|\mathcal{L}_n(e_m) - e_m\|_r + \|e_j\|_r \cdot \|e_{m-j} - \mathcal{L}_n(e_{m-j})\|_r \\ + \|\mathcal{L}_n(e_{m-j})\|_r \cdot \|e_j - \mathcal{L}_n(e_j)\|_r.\end{aligned}$$

Proof. Since $f(z)g(z) = \sum_{m=0}^{\infty} c_m z^m$, where $c_m = \sum_{j=0}^{m} a_j b_{m-j}$, it follows

$$\mathcal{L}_n(fg)(z) = \sum_{m=0}^{\infty}\left[\sum_{j=0}^{m} a_j b_{m-j}\right]\mathcal{L}_n(e_m)(z).$$

Also,

$$\mathcal{L}_n(f)(z) = \sum_{k=0}^{\infty} a_k \mathcal{L}_n(e_k)(z), \;\; \mathcal{L}_n(g)(z) = \sum_{k=0}^{\infty} b_k \mathcal{L}_n(e_k)(z)$$

and

$$\mathcal{L}_n(f)(z)\mathcal{L}_n(g)(z) = \sum_{m=0}^{\infty}\left[\sum_{j=0}^{m} a_j b_{m-j} \mathcal{L}_n(e_j)(z)\mathcal{L}_n(e_{m-j})(z)\right],$$

which immediately implies

$$|\mathcal{L}_n(fg)(z) - \mathcal{L}_n(f)(z)\mathcal{L}_n(g)(z)| = \left|\sum_{m=0}^{\infty}\left[\sum_{j=0}^{m} a_j b_{m-j}\left(\mathcal{L}_n(e_m)(z) - \mathcal{L}_n(e_j)(z)\mathcal{L}_n(e_{m-j})(z)\right)\right]\right|$$
$$\leq \sum_{m=0}^{\infty}\left[\sum_{j=0}^{m} |a_j|\cdot|b_{m-j}|\cdot|\mathcal{L}_n(e_m)(z) - \mathcal{L}_n(e_j)(z)\mathcal{L}_n(e_{m-j})(z)|\right].$$

Then, we get

$$|A_{n,m,j}(z)| = |\mathcal{L}_n(e_m)(z) - \mathcal{L}_n(e_j)(z)\mathcal{L}_n(e_{m-j})(z)|$$
$$\leq |\mathcal{L}_n(e_m)(z) - e_m(z)| + |e_j(z)\cdot e_{m-j}(z) - \mathcal{L}_n(e_j)(z)\mathcal{L}_n(e_{m-j})(z)|$$
$$\leq |\mathcal{L}_n(e_m)(z) - e_m(z)| + |e_j(z)|\cdot|e_{m-j}(z) - \mathcal{L}_n(e_{m-j})(z)|$$
$$+ |\mathcal{L}_n(e_{m-j})(z)|\cdot|e_j(z) - \mathcal{L}_n(e_j)(z)|,$$

which immediately proves the lemma. □

The following Grüss-type estimate holds.

Corollary 2.1. *Suppose that* $1 \leq r < R$ *and* $f, g : \mathbb{D}_R \to \mathbb{C}$ *are analytic in* $\mathbb{D}_R$*, that is* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *and* $g(z) = \sum_{k=0}^{\infty} b_k z^k$ *for all* $z \in \mathbb{D}_R$*.*

For all $n \in \mathbb{N}$*, we have*

$$\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g)\|_r \leq \frac{3}{n}\cdot\sum_{m=1}^{\infty} m^2\left[\sum_{j=0}^{m}|a_j|\cdot|b_{m-j}|\right]r^m,$$

where $\sum_{m=1}^{\infty} m^2\left[\sum_{j=0}^{m}|a_j|\cdot|b_{m-j}|\right]r^m < \infty$*.*

Proof. We estimate $\|A_{n,m,j}\|_r$ in the statement of Theorem 2.1 for $\mathcal{L}_n = \mathcal{V}_n$. For that purpose, by [2, p. 182], we easily get $\|\mathcal{V}_n(e_k)\|_r \leq r^k$, for all $n \in \mathbb{N}$, $k \in \mathbb{N}\bigcup\{0\}$, while from [2, p. 183], we have $\|\mathcal{V}_n(e_k) - e_k\|_r \leq \frac{k^2}{n}r^k$, for all k, n. This implies, for all $n, m, j \in \mathbb{N}$ and $j \leq m$

$$\|A_{n,m,j}\|_r \leq \frac{m^2}{n}r^m + r^j\cdot\frac{(m-j)^2}{n}\cdot r^{m-j} + r^{m-j}\cdot\frac{j^2}{n}\cdot r^j$$
$$\leq \frac{3}{n}\cdot m^2 r^m,$$

which by Theorem 2.1, immediately implies the estimate in the statement of the corollary.

It remains to show that $\sum_{m=1}^{\infty} m^2\left[\sum_{j=0}^{m}|a_j|\cdot|b_{m-j}|\right]r^m < \infty$. Indeed, since f and g are analytic it follows that the series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ converge uniformly for $|z| \leq r$ and all $1 \leq r < R$, that is the series $\sum_{k=0}^{\infty}|a_k|r^k$ and $\sum_{k=0}^{\infty}|b_k|r^k$ converge for all $1 \leq r < R$. Then, by Mertens' theorem (see e.g. [6, Theorem 3.50, p. 74] their (Cauchy) product is a convergent series and therefore

$$\sum_{m=0}^{\infty}\left[\sum_{j=0}^{m}|a_j|\cdot|b_{m-j}|\right]r^m$$

is a convergent series for all $1 \leq r < R$. Denoting $A_m = \sum_{j=0}^{m}|a_j|\cdot|b_{m-j}|$, this means that the power series $F(z) = \sum_{m=0}^{\infty} A_m z^m$ is uniformly convergent for $|z| \leq r$, for all $1 \leq r < R$, which implies that $F''(z) = \sum_{m=2}^{\infty} m(m-1)A_m z^{m-2}$ also is uniformly convergent for $|z| \leq r$, with $1 \leq r < R$ arbitrary, fixed. Indeed, choose an r' with $1 \leq r < r' < R$ and consider the uniformly convergent series $F(z) = \sum_{m=0}^{\infty} A_m z^m$ on $|z| \leq r'$.

Therefore $\sum_{m=2}^{\infty} m(m-1)A_m r^{m-2} < \infty$, which immediately implies that

$$\sum_{m=1}^{\infty} m^2 \left[\sum_{j=0}^{m} |a_j| \cdot |b_{m-j}|\right] r^m < \infty.$$

□

In what follows, it is natural to ask for the limit

$$\lim_{n\to\infty} n[\mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z)].$$

By simple calculation, we have

$$\begin{aligned}
& n[\mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z)] \\
=& n\left\{\mathcal{V}_n(fg)(z) - f(z)g(z) + \frac{z^2}{n}(f(z)g(z))'' + \frac{z}{n}(f(z)g(z))'\right. \\
&- g(z)\left[\mathcal{V}_n(f)(z) - f(z) + \frac{z^2}{n}f''(z) + \frac{z}{n}f'(z)\right] \\
&- \mathcal{V}_n(f)(z)\left[\mathcal{V}_n(g)(z) - g(z) + \frac{z^2}{n}g''(z) + \frac{z}{n}g'(z)\right] \\
&+ \left.\left(\frac{z^2}{n}g''(z) + \frac{z}{n}g'(z)\right)[\mathcal{V}_n(f)(z) - f(z)] - \frac{2z^2}{n}f'(z)g'(z)\right\}.
\end{aligned}$$

Indeed, the above equality easily follows by simple algebraic manipulations, replacing in the right-hand side of the equality $[f(z)g(z)]' = f'(z)g(z) + f(z)g'(z)$, $[f(z)g(z)]'' = f''(z)g(z) + 2f'(z)g'(z) + f(z)g''(z)$ and reducing the corresponding terms.

Taking into account the estimate in [2, Theorem 3.1.2, p. 183] applied successively there for $f \cdot g$, f and g, passing to the limit it easily follows

$$\lim_{n\to\infty} n[\mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z)] = -2z^2 f'(z)g'(z).$$

This suggests us to prove the following Grüss-Voronovskaya-type estimate.

Theorem 2.2. *Suppose that $1 \leq r < R$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$. Then, for all $|z| \leq r$, there exists a constant $C(r, f, g) > 0$ depending on r, f, g, such that*

$$\left|\mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z) + \frac{2z^2}{n}f'(z)g'(z)\right| \leq \frac{C(r,f,g)}{n^2},\ n \in \mathbb{N}.$$

Proof. Firstly, note that we have the decomposition formula

$$\begin{aligned}
& \mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z) + \frac{2z^2}{n}f'(z)g'(z) \\
=& \left[\mathcal{V}_n(fg)(z) - (fg)(z) + \frac{z^2}{n}(f(z)g(z))'' + \frac{z}{n}(f(z)g(z))'\right] \\
& -f(z)\left[\mathcal{V}_n(g)(z) - g(z) + \frac{z^2}{n}g''(z) + \frac{z}{n}g'(z)\right] \\
& -g(z)\left[\mathcal{V}_n(f)(z) - f(z) + \frac{z^2}{n}f''(z) + \frac{z}{n}f'(z)\right] \\
& +[g(z) - \mathcal{V}_n(g)(z)] \cdot [\mathcal{V}_n(f)(z) - f(z)].
\end{aligned}$$

Passing to modulus with $|z| \le r$ and taking into account the estimates in [2, Theorem 3.1.1, (i), p. 182] and [2, Theorem 3.1.2, p. 183], we get

$$\begin{aligned}
&|\mathcal{V}_n(fg)(z) \ -\mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z) + \frac{2z^2}{n}f'(z)g'(z)\Big| \\
\le &\left|\mathcal{V}_n(fg)(z) - (fg)(z) + \frac{z^2}{n}(f(z)g(z))'' + \frac{z}{n}(f(z)g(z))'\right| \\
&+|f(z)|\left|\mathcal{V}_n(g)(z) - g(z) + \frac{z^2}{n}g''(z) + \frac{z}{n}g'(z)\right| \\
&+|g(z)|\left|\mathcal{V}_n(f)(z) - f(z) + \frac{z^2}{n}f''(z) + \frac{z}{n}f'(z)\right| \\
&+|g(z) - \mathcal{V}_n(g)(z)| \cdot |\mathcal{V}_n(f)(z) - f(z)|. \\
\le &\frac{C_1(r,f,g)}{n^2} + \|f\|_r \cdot \frac{C_2(r,g)}{n^2} + \|g\|_r \cdot \frac{C_3(r,f)}{n^2} + \frac{C_4(r,g)}{n} \cdot \frac{C_5(r,f)}{n} \\
\le &\frac{C(r,f,g)}{n^2},
\end{aligned}$$

for all $n \in \mathbb{N}$ and $|z| \le r$, with $C(r,f,g) > 0$ independent of n and depending on r, f, g. □

In what follows, the above theorem is used to obtain a lower estimate in the Grüss-type inequality.

Corollary 2.2. *Suppose that $1 \le r < R$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$. Then there exists an $n_0 \in \mathbb{N}$, depending only on r, f and g, such that*

$$\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g)\|_r \ge \frac{1}{n} \cdot \|e_2 f'g'\|_r, \ n \ge n_0.$$

Proof. We can write

$$\begin{aligned}
&\mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z) \\
=&\frac{1}{n}\left\{-2z^2f'(z)g'(z) + \frac{1}{n}\left[n^2\left(\mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z) + \frac{2z^2}{n}f'(z)g'(z)\right)\right]\right\}.
\end{aligned}$$

Applying to the above identity, the obvious inequality

$$\|F+G\|_r \ge |\|F\|_r - \|G\|_r| \ge \|F\|_r - \|G\|_r,$$

and denoting $e_2(z) = z^2$, we obtain

$$\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g)\|_r \ge \frac{1}{n}\left\{\|2e_2f'g'\|_r - \frac{1}{n}\left[n^2\left\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g) + \frac{2e_2}{n}f'g'\right\|_r\right]\right\}.$$

Since f and g are not constant functions, we easily get $\|2e_2f'g'\|_r > 0$.

Taking into account that by Theorem 2.2, we get

$$n^2\left\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g) + \frac{2e_2}{n}f'g'\right\|_r \le C(r,f,g)$$

and that $\frac{1}{n} \to 0$, there exists an index n_0 (depending only on r, f, g), such that for all $n \geq n_0$, we have

$$\begin{aligned} \|2e_2 f'g'\|_r - \frac{1}{n}\left[n^2 \left\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g) + \frac{2e_2}{n} f'g'\right\|_r\right] &\geq \frac{\|2e_2 f'g'\|_r}{2} \\ &= \|e_2 f'g'\|_r \\ &> 0, \end{aligned}$$

which for all $n \geq n_0$ implies

$$\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g)\|_r \geq \frac{1}{n} \cdot \|e_2 f'g'\|_r.$$

□

As an immediate consequence of Corollary 2.1 and Corollary 2.2, we obtain the following exact estimate.

Corollary 2.3. *Suppose that $1 \leq r < R$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$. If f and g are not constant functions, then there exists $n_0 \in \mathbb{N}$ depending only on r, f and g, such that we have*

$$\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g)\|_r \sim \frac{1}{n},\ n \in \mathbb{N}, n \geq n_0,$$

where the constants in the equivalence are independent of n but depend on r, f, g.

3. Zygmund-Riesz complex convolution

This section deals with the Grüss and the Grüss-Voronovskaya estimates for the Zygmund-Riesz complex polynomials based on the convolution with the Zygmund-Riesz kernel

$$K_{n,s}(t) = \frac{1}{2} + \sum_{j=1}^{n-1}\left(1 - \frac{j^s}{n^s}\right)\cos(jt),\ s \in \mathbb{N} \text{ fixed},$$

defined by

$$\mathcal{R}_{n,s}(f)(z) = \sum_{j=0}^{n-1} c_j \left[1 - \left(\frac{j}{n}\right)^s\right] z^j, \tag{3.4}$$

attached to analytic functions in compact disks, $f(z) = \sum_{j=0}^{\infty} c_j z^j$.

Firstly, as a consequence of Theorem 2.1, the following Grüss-type estimate holds for Zygmund-Riesz complex polynomial convolution.

Corollary 3.4. *Suppose that $1 \leq r < R$, $s \in \mathbb{N}$ are fixed arbitrary and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ for all $z \in \mathbb{D}_R$.*

For all $n \in \mathbb{N}$, we have

$$\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g)\|_r \leq \frac{3}{n^s}\sum_{m=1}^{\infty} m^s \left[\sum_{j=0}^{m} |a_j| \cdot |b_{m-j}|\right] r^m,$$

where $\sum_{m=1}^{\infty} m^s \left[\sum_{j=0}^{m} |a_j| \cdot |b_{m-j}|\right] r^m < +\infty$.

Proof. Denote $e_m(z) = z^m$. We will estimate $\|A_{n,m,j}\|_r$ in the case when in Theorem 2.1, we take $\mathcal{L}_n = \mathcal{R}_{n,s}$.

From the formula (3.4), we immediately get that $\mathcal{R}_{n,s}(e_k)(z) = 0$ if $k \geq n$ and that $\mathcal{R}_{n,s}(e_k)(z) = \left[1 - \frac{k^s}{n^s}\right] e_k(z)$ if $k \leq n-1$. This immediately implies $\|\mathcal{R}_{n,s}(e_k)\|_r \leq r^k$ for all n, k. Also,

$\|\mathcal{R}_{n,s}(e_k) - e_k\|_r = r^k \leq \frac{k^s}{n^s} \cdot r^k$ if $k \geq n$ and $\|\mathcal{R}_{n,s}(e_k) - e_k\|_r \leq \frac{k^s}{n^s} r^k$ if $k \leq n-1$, which easily implies

$$\|A_{m,n,j}\|_r \leq \frac{m^s}{n^s} r^m + r^j \cdot \frac{(m-j)^s}{n^s} \cdot r^{m-j} + r^{m-j} \cdot \frac{j^s}{n^s} \cdot r^j \leq \frac{3}{n^s} m^s r^m.$$

It remains to show that $\sum_{m=1}^{\infty} m^s \left[\sum_{j=0}^{m} |a_j| \cdot |b_{m-j}|\right] r^m < +\infty$. This follows immediately by reasoning exactly as in the proof of Corollary 2.1. Indeed, keeping the notation there for the series $F(z) = \sum_{m=0}^{\infty} A_m z^m$, we analogously get that for any $1 \leq r < R$, all the series $F'(z)$, ..., $F^{(s)}(z)$ are uniformly convergent for $|z| \leq r$.

In conclusion, we obtain the conclusions in the statement. □

In what follows, it is natural to ask for the limit

$$\lim_{n\to\infty} n^s[\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z)].$$

For this purpose, for arbitrary $k, s \in \mathbb{N}$, let us denote the coefficients $\alpha_{j,s} \in \mathbb{N}$, independent of k which satisfy (see, e.g., [2, Lemma 3.1.7, p. 190])

$$k^s = \sum_{j=1}^{s} \alpha_{j,s} k(k-1) \cdot \ldots \cdot (k-(j-1)), \tag{3.5}$$

and the recurrence formula

$$\alpha_{j,s+1} = \alpha_{j-1,s} + j\alpha_{j,s}, j = 2, ..., s, s \geq 2, \text{ with } \alpha_{1,s} = \alpha_{s,s} = 1, \text{ for all } s \geq 1. \tag{3.6}$$

By simple calculation (see the indications for the relation after the proof of Corollary 2.1), we have

$$\begin{aligned}
&n^s[\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z)] \\
=&n^s \Bigg\{ \mathcal{R}_{n,s}(fg)(z) - f(z)g(z) + \frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} z^j (f(z)g(z))^{(j)} \\
&- g(z) \left[\mathcal{R}_{n,s}(f)(z) - f(z) + \frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} z^j f^{(j)}(z) \right] \\
&- \mathcal{R}_{n,s}(f)(z) \left[\mathcal{R}_{n,s}(g)(z) - g(z) + \frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} z^j g^{(j)}(z) \right] \\
&+ \left(\frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} z^j g^{(j)}(z) \right) [\mathcal{R}_{n,s}(f)(z) - f(z)] + E_{n,s}(f,g)(z) \Bigg\},
\end{aligned}$$

where $E_{n,s}(f,g)(z) = \frac{1}{n^s} G_s(f,g)(z)$ with

$$G_s(f,g)(z) = \sum_{j=1}^{s} \alpha_{j,s} z^j [f(z)g^{(j)}(z) + g(z)f^{(j)}(z) - (f(z)g(z))^{(j)}]. \tag{3.7}$$

Taking into account the estimate in [2, Theorem 3.1.8, p. 190], applied successively there for $f \cdot g$, f and g, passing to the limit it easily follows

$$\lim_{n\to\infty} n^s[\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z)] = G_s(f,g)(z).$$

This suggests us to prove the following Grüss-Voronovskaya-type estimate.

Theorem 3.3. *Suppose that $1 \le r < R$, $s \in \mathbb{N}$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$. Then, for all $|z| \le r$, there exists a constant $C(r, s, f, g) > 0$ depending on r, s, f, g, such that*

$$\left|\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z) - \frac{1}{n^s}G_s(f,g)(z)\right| \le \frac{C(r,s,f,g)}{n^{s+1}},\ n \in \mathbb{N}.$$

Proof. Firstly, note that we have the decomposition formula

$$\begin{aligned}
&\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z) - \frac{1}{n^s}G_s(f,g)(z)\\
&= \left[\mathcal{R}_{n,s}(fg)(z) - (fg)(z) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}z^j(f(z)g(z))^{(j)}\right]\\
&-f(z)\left[\mathcal{R}_{n,s}(g)(z) - g(z) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}z^j g^{(j)}(z)\right]\\
&-g(z)\left[\mathcal{R}_{n,s}(f)(z) - f(z) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}z^j f^{(j)}(z)\right]\\
&+[g(z) - \mathcal{R}_{n,s}(g)(z)]\cdot[\mathcal{R}_{n,s}(f)(z) - f(z)].
\end{aligned}$$

Passing to modulus with $|z| \le r$ and taking into account the estimates in the second line of the proof of [2, Theorem 3.1.6, p. 189] and [2, Theorem 3.1.8, p. 190], we get

$$\begin{aligned}
&\left|\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z) - \frac{1}{n^s}G_s(f,g)(z)\right|\\
&\le \left|\mathcal{R}_{n,s}(fg)(z) - (fg)(z) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}z^j(f(z)g(z))^{(j)}\right|\\
&+|f(z)|\left|\mathcal{R}_{n,s}(g)(z) - g(z) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}z^j g^{(j)}(z)\right|\\
&+|g(z)|\left|\mathcal{R}_{n,s}(f)(z) - f(z) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}z^j f^{(j)}(z)\right|\\
&+|g(z) - \mathcal{R}_{n,s}(g)(z)|\cdot|\mathcal{R}_{n,s}(f)(z) - f(z)|\\
&\le \frac{C_1(r,s,f,g)}{n^{s+1}} + \|f\|_r\cdot\frac{C_2(r,s,g)}{n^{s+1}} + \|g\|_r\cdot\frac{C_3(r,s,f)}{n^{s+1}} + \frac{C_4(r,s,g)}{n^s}\cdot\frac{C_5(r,s,f)}{n^s}\\
&\le \frac{C(r,s,f,g)}{n^{s+1}},
\end{aligned}$$

for all $n \in \mathbb{N}$ and $|z| \le r$, with $C(r, s, f, g) > 0$ independent of n and depending on r, s, f, g. □

Remark 3.1. *Taking $s = 1$ in Theorem 3.3 and using that $G_1(f, g)(z) = 0$ for all $z \in \mathbb{D}_R$, in this case we get a better estimate in the Grüss-type inequality than that in Corollary 3.4, namely*

$$\|\mathcal{R}_{n,1}(fg) - \mathcal{R}_{n,1}(f)\mathcal{R}_{n,1}(g)\|_r \le \frac{C(f,g)}{n^2}.$$

In what follows, the above theorem is used to obtain lower estimate in the Grüss-type inequality.

Corollary 3.5. *Suppose that* $1 \leq r < R$, $s \in \mathbb{N}$, $s \geq 2$ *and* $f, g : \mathbb{D}_R \to \mathbb{C}$ *are analytic in* $\mathbb{D}_R$. *Then there exists* $n_0 \in \mathbb{N}$ *depending on* r, s, f *and* g, *such that*

$$\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g)\|_r \geq \frac{1}{n^s} \cdot \frac{\|G_s(f,g)\|_r}{2}, \; n \in \mathbb{N}, n \geq n_0,$$

where $G_s(f,g)(z)$ *is given by relation (3.7).*

Proof. We can write

$$\begin{aligned}&\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z)\\ =&\frac{1}{n^s}\left\{G_s(f,g)(z) + \frac{1}{n^s}\left[n^{2s}\left(\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z) - \frac{1}{n^s}G_s(f,g)(z)\right)\right]\right\}.\end{aligned}$$

Applying to the above identity, the obvious inequality

$$\|F + G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r,$$

we obtain

$$\begin{aligned}&\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g)\|_r\\ \geq&\frac{1}{n^s}\left\{\|G_s(f,g)\|_r - \frac{1}{n^s}\left[n^{2s}\left\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g) - \frac{1}{n^s}G_s(f,g)\right\|_r\right]\right\}.\end{aligned}$$

By hypothesis, we easily get $\|G_s(f,g)\|_r > 0$.
Taking into account that by Theorem 3.3, we get

$$n^{s+1}\left\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g) + \frac{1}{n^s}G_s(f,g)\right\|_r \leq C(r,s,f,g)$$

and that $\frac{1}{n} \to 0$, there exists an index n_0 (depending only on r, f, g), such that for all $n \geq n_0$, we have

$$\begin{aligned}&\|G_s(f,g)\|_r - \frac{1}{n^s}\left[n^{2s}\left\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g) + \frac{1}{n^s}G_s(f,g)\right\|_r\right]\\ \geq&\|G_s(f,g)\|_r - \frac{K(r,s,f,g)}{n}\\ \geq&\frac{\|G_s(f,g)\|_r}{2}\\ >&0,\end{aligned}$$

which for all $n \geq n_0$ implies

$$\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g)\|_r \geq \frac{1}{n^s} \cdot \frac{\|G_s(f,g)\|_r}{2}.$$

The corollary is proved. □

As an immediate consequence of Corollary 3.4 and Corollary 3.5, we obtain the following exact estimate.

Corollary 3.6. *Suppose that* $1 \leq r < R$, $s \in \mathbb{N}$, $s \geq 2$ *and* $f, g : \mathbb{D}_R \to \mathbb{C}$ *are analytic in* $\mathbb{D}_R$. *If* f *and* g *are such that* $G_s(f,g)(z)$ *is not identical zero in* $\mathbb{D}_r$, *then there exists* $n_0 \in \mathbb{N}$ *depending only on* r, s, f, g, *such that we have*

$$\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g)\|_r \sim \frac{1}{n^s}, \; n \in \mathbb{N}, n \geq n_0,$$

where the constants in the equivalence are independent of n *but depend on* r, s, f, g.

Remark 3.2. *The statements of Corollaries 3.5 and 3.6 suggest to be of interest to examine the pair of functions* f, g, *for which* $G_s(f,g)(z) \equiv 0$. *For example, in the particular case* $s = 2$, *taking into account the formula for* $G_s(f,g)(z)$ *in (3.7), we easily obtain that*

$$f(z)g''(z) + f''(z)g(z) - [f(z)g(z)]'' \equiv 0.$$

This easily one reduces to $f'(z)g'(z) \equiv 0$, which means that f is a constant function and g is an arbitrary analytic function, or f is an arbitrary analytic function and g is a constant function.

The cases $s \geq 3$ are more complicated and remain as open questions.

4. Jackson complex convolution

In this section, we study the Jackson complex polynomials based on the convolution with the Jackson kernel

$$K_n(t) = \frac{3}{2n(2n^2+1)} \cdot \left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^4,$$

defined by

$$\mathcal{J}_n(f)(z) = c_0 + \sum_{j=1}^{2n-2} c_j \cdot \lambda_{j,n} \cdot z^j, \tag{4.8}$$

attached to analytic functions on compact disks, $f(z) = \sum_{j=0}^{\infty} c_j z^j$, where $\lambda_{j,n} = \frac{4n^3-6j^2n+3j^3-3j+2n}{2n(2n^2+1)}$ if $1 \leq j \leq n$, $\lambda_{j,n} = \frac{j-2n-(j-2n)^3}{2n(2n^2+1)}$ if $n \leq j \leq 2n-2$.

As a consequence of Theorem 2.1, the following Grüss-type estimate holds for Jackson complex convolution.

Corollary 4.7. *Suppose that* $1 \leq r < R$ *and* $f, g : \mathbb{D}_R \to \mathbb{C}$ *are analytic in* $\mathbb{D}_R$, *that is* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *and* $g(z) = \sum_{k=0}^{\infty} b_k z^k$ *for all* $z \in \mathbb{D}_R$.

For all $n \in \mathbb{N}$, *we have*

$$\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g)\|_r \leq \frac{3C_r}{n^2} \sum_{m=1}^{\infty} m^2 \left[\sum_{j=0}^{m} |a_j| \cdot |b_{m-j}|\right] r^m.$$

Here, $C_r > 0$ *is a constant depending only on* r.

Proof. Denote $e_m(z) = z^m$. We will estimate $\|A_{n,m,j}\|_r$ in the case when in Theorem 2.1, we take $\mathcal{L}_n = \mathcal{J}_n$.

From the formula for $\mathcal{J}_n$ in (4.8), we get $\mathcal{J}_n(e_k)(z) = 0$, if $k > 2n-2$ and $\mathcal{J}_n(e_k) = \lambda_{k,n} e_k(z)$ if $0 \leq k \leq 2n-2$, which implies that $\|\mathcal{J}_n(e_k)\|_r \leq r^k$, for all k, n (here we take into account that by e.g. [2, Remark 3, p. 195], we have $0 \leq \lambda_{k,n} \leq 1$ for all k, n).

Also, from [2, Theorem 3.1.10, (iv), p. 195], combined with the mean value theorem applied to the divided difference of the complex valued function $g(t) = f(re^{it})$, we immediately get

$$
\begin{aligned}
|\mathcal{J}_n(f)(z) - f(z)| &\leq C_r \omega_2(f;1/n)_{\partial\mathbb{D}_r} \\
&\leq \frac{C_r}{n^2}\|g''\|_{[0,2\pi]} \\
&\leq \frac{C_r}{n^2}\left[\|f'\|_r + \|f''\|_r\right] \\
&\leq \frac{C_r}{n^2}\left[\sum_{k=1}^{\infty}|c_k|kr^{k-1} + \sum_{k=2}^{\infty}|c_k|(k-1)kr^{k-2}\right] \\
&\leq \frac{C_r}{n^2}\sum_{k=1}^{\infty}|c_k|\cdot k^2\cdot r^k.
\end{aligned}
$$

Note that here, the constant C_r depends only on r and is different at each occurrence.

It is worth noting here that the above estimate corrects a little the constant in the estimate in [2, Corollary 3.1.11, (i)] (where instead of $\sum_{k=1}^{\infty}|c_k|\cdot k^2\cdot r^k$ we got the incorrect constant $\sum_{k=1}^{\infty}|c_k|\cdot k(k-1)\cdot r^{k-2}$, which appears because in [2, p. 196] we used the incorrect estimate $\|g''\|_{[0,2\pi]} \leq \|f''\|_r$).

Now, if we put above e_k instead of f, we easily arrive at

$$
\|\mathcal{J}_n(e_k) - e_k\|_r \leq \frac{C_r}{n^2}\cdot k^2 r^k
$$

for all k, n.

Therefore, for all $j \leq m$, it follows

$$
\begin{aligned}
\|A_{m,n,j}\|_r &\leq \frac{C_r}{n^2}m^2r^m + r^j\cdot\frac{C_r}{n^2}(m-j)^2r^{m-j} + r^{m-j}\cdot\frac{C_r}{n^2}j^2r^j \\
&\leq \frac{3C_r}{n^2}\cdot m^2r^m,
\end{aligned}
$$

which combined with Theorem 2.1 proves the corollary. □

In what follows, it is natural to ask for the limit

$$
\lim_{n\to\infty} n^2[\mathcal{J}_n(fg)(z) - \mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z)].
$$

By simple calculation, we have (see the indications for the relation after the proof of Corollary 2.1)

$$
\begin{aligned}
& n^2[\mathcal{J}_n(fg)(z) - \mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z)] \\
=& n^2\left\{\mathcal{J}_n(fg)(z) - f(z)g(z) + \frac{3z^2}{2n^2}(f(z)g(z))'' + \frac{3z}{2n^2}(f(z)g(z))'\right. \\
&- g(z)\left[\mathcal{J}_n(f)(z) - f(z) + \frac{3z^2}{2n^2}f''(z) + \frac{3z}{2n^2}f'(z)\right] \\
&- \mathcal{J}_n(f)(z)\left[\mathcal{J}_n(g)(z) - g(z) + \frac{3z^2}{2n^2}g''(z) + \frac{3z}{2n^2}g'(z)\right] \\
&\left. + \left(\frac{3z^2}{2n^2}g''(z) + \frac{3z}{2n^2}g'(z)\right)[\mathcal{J}_n(f)(z) - f(z)] - \frac{3z^2}{n^2}f'(z)g'(z)\right\}.
\end{aligned}
$$

Taking into account the estimate in [2, Theorem 3.1.12, p. 196], applied successively there for $f \cdot g$, f and g, passing to the limit it easily follows

$$\lim_{n\to\infty} n^2[\mathcal{J}_n(fg)(z) - \mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z)] = -3z^2 f'(z)g'(z).$$

This suggests us to prove the following Grüss-Voronovskaya-type estimate.

Theorem 4.4. *Suppose that $1 \le r < R$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$. Then, for all $|z| \le r$, there exists a constant $C(r, f, g) > 0$ depending on r, f, g, such that*

$$\left|\mathcal{J}_n(fg)(z) - \mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z) + \frac{3z^2}{n^2} f'(z)g'(z)\right| \le \frac{C(r, f, g)}{n^3},\ n \in \mathbb{N}.$$

Proof. Firstly, note that we have the decomposition formula

$$\begin{aligned}
&\mathcal{J}_n(fg)(z) - \mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z) + \frac{3z^2}{n^2} f'(z)g'(z)\\
&= \left[\mathcal{J}_n(fg)(z) - (fg)(z) + \frac{3z^2}{2n^2}(f(z)g(z))'' + \frac{3z}{2n^2}(f(z)g(z))'\right]\\
&-f(z)\left[\mathcal{J}_n(g)(z) - g(z) + \frac{3z^2}{2n^2} g''(z) + \frac{3z}{2n^2} g'(z)\right]\\
&-g(z)\left[\mathcal{J}_n(f)(z) - f(z) + \frac{3z^2}{2n^2} f''(z) + \frac{3z}{2n^2} f'(z)\right]\\
&+[g(z) - \mathcal{J}_n(g)(z)] \cdot [\mathcal{J}_n(f)(z) - f(z)].
\end{aligned}$$

Passing to modulus with $|z| \le r$ and taking into account the estimates in [2, Theorem 3.1.12, p. 196] and the estimate in the proof of Corollary 4.7, we get

$$\begin{aligned}
&\left|\mathcal{J}_n(fg)(z) - \mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z) + \frac{3z^2}{n^2} f'(z)g'(z)\right|\\
&\le \left|\mathcal{J}_n(fg)(z) - (fg)(z) + \frac{3z^2}{2n^2}(f(z)g(z))'' + \frac{3z}{2n^2}(f(z)g(z))'\right|\\
&+|f(z)|\left|\mathcal{J}_n(g)(z) - g(z) + \frac{3z^2}{2n^2} g''(z) + \frac{3z}{2n^2} g'(z)\right|\\
&+|g(z)|\left|\mathcal{J}_n(f)(z) - f(z) + \frac{3z^2}{2n^2} f''(z) + \frac{3z}{2n^2} f'(z)\right|\\
&+|g(z) - \mathcal{J}_n(g)(z)| \cdot |\mathcal{J}_n(f)(z) - f(z)|\\
&\le \frac{C_1(r, f, g)}{n^3} + \|f\|_r \cdot \frac{C_2(r, g)}{n^3} + \|g\|_r \cdot \frac{C_3(r, f)}{n^3} + \frac{C_4(r, g)}{n^2} \cdot \frac{C_5(r, f)}{n^2}\\
&\le \frac{C(r, f, g)}{n^3}
\end{aligned}$$

for all $n \in \mathbb{N}$ and $|z| \le r$, with $C(r, f, g) > 0$ independent of n and depending on r, f, g. □

In what follows, the above theorem is used to obtain a lower estimate in the Grüss-type inequality.

Corollary 4.8. *Suppose that $1 \le r < R$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$. Then there exists an $n_0 \in \mathbb{N}$, depending only on r, f and g, such that*

$$\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g)\|_r \ge \frac{1}{n^2} \cdot \frac{\|3e_2 f' \cdot g'\|_r}{2}, \quad n \in \mathbb{N}, n \ge n_0.$$

Proof. We can write

$$\mathcal{J}_n(fg)(z) - \mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z)$$
$$= \frac{1}{n^2}\left\{-3z^2 f'(z)g'(z) + \frac{1}{n^2}\left[n^4\left(\mathcal{J}_n(fg)(z) - \mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z) + \frac{3z^2}{n^2}f'(z)g'(z)\right)\right]\right\}.$$

Applying to the above identity, the obvious inequality

$$\|F+G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r,$$

and denoting $e_2(z) = z^2$, we obtain

$$\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g)\|_r \geq \frac{1}{n^2}\left\{\|3e_2 f'g'\|_r - \frac{1}{n^2}\left[n^4\left\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g) + \frac{3e_2}{n^2}f'g'\right\|_r\right]\right\}.$$

Since f and g are not constant functions, we easily get $\|3e_2 f'g'\|_r > 0$.
Taking into account that by Theorem 4.4, we get

$$n^3\left\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g) + \frac{3e_2}{n^2}f'g'\right\|_r \leq C(r,f,g)$$

and that $\frac{1}{n} \to 0$, there exists an index n_0 (depending only on r, f, g), such that for all $n \geq n_0$, we have

$$\|3e_2 f'g'\|_r - \frac{1}{n}\left[n^3\left\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g) + \frac{3e_2}{n^2}f'g'\right\|_r\right] \geq \frac{\|3e_2 f'g'\|_r}{2} > 0,$$

which for all $n \geq n_0$ implies

$$\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g)\|_r \geq \frac{1}{n^2} \cdot \frac{\|3e_2 f'g'\|_r}{2}.$$

The corollary is proved. □

As an immediate consequence of Corollary 4.7 and Corollary 4.8, we obtain the following exact estimate.

Corollary 4.9. *Suppose that $1 \leq r < R$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$. If f and g are not constant functions, then there exists $n_0 \in \mathbb{N}$ depending only on r, f and g, such that we have*

$$\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g)\|_r \sim \frac{1}{n^2}, \quad n \in \mathbb{N}, n \geq n_0,$$

where the constants in the equivalence are independent of n but depend on r, f, g.

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Research Article

Abstract generalized fractional Landau inequalities over $\mathbb{R}$

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ABSTRACT. We present uniform and L_p mixed Caputo-Bochner abstract generalized fractional Landau inequalities over $\mathbb{R}$ of fractional orders $2 < \alpha \leq 3$. These estimate the size of first and second derivatives of a composition with a Banach space valued function over $\mathbb{R}$. We give applications when $\alpha = 2.5$.

Keywords: Abstract generalized fractional Landau inequality, right and left Caputo abstract generalized fractional derivatives.

2020 Mathematics Subject Classification: 26A33, 26D10, 26D15.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of f, such that

(1) $$\|f'\|_{p,I} \leq C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \|f''\|_{p,I}^{\frac{1}{2}},$$

where $\|\cdot\|_{p,I}$ is the p-norm on the interval I, see [1], [5]. The research on these inequalities started by E. Landau [10] in 1913. For the case of $p = \infty$, he proved that

(2) $$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2}$$

are the best constants in (1). In 1932, G. H. Hardy and J. E. Littlewood [7] proved (1) for $p = 2$, with the best constants

(3) $$C_2(\mathbb{R}_+) = \sqrt{2} \text{ and } C_2(\mathbb{R}) = 1.$$

In 1935, G. H. Hardy, E. Landau and J. E. Littlewood [8] showed that the best constants $C_p(\mathbb{R}_+)$ in (1) satisfies the estimate

(4) $$C_p(\mathbb{R}_+) \leq 2, \text{ for } p \in [1, \infty),$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$.

In fact, in [6] and [9] was shown that $C_p(\mathbb{R}) \leq \sqrt{2}$. We need the following concepts from abstract generalized fractional calculus. Our integrals next are of Bochner type [11]. We need the following definition.

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Definition 1.1. *([4], p. 104) Let $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ a Banach space, $g \in C^1([a,b])$ and increasing, $f \in C([a,b], X)$, $\nu > 0$. We define the left Riemann-Liouville generalized fractional Bochner integral operator*

(5)
$$\left(J_{a;g}^{\nu} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_a^x (g(x) - g(z))^{\nu-1} g'(z) f(z)\, dz,$$

$\forall\, x \in [a,b]$, where Γ is the gamma function. The last integral is of Bochner type. Since $f \in C([a,b], X)$, then $f \in L_\infty([a,b], X)$. By Theorem 4.10, p. 98, [4], we get that $\left(J_{a;g}^{\nu} f\right) \in C([a,b], X)$. Above we set $J_{a;g}^{0} f := f$ and see that $\left(J_{a;g}^{\nu} f\right)(a) = 0$.

We need the following definition.

Definition 1.2. *([4], p. 105) Let $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ a Banach space, $g \in C^1([a,b])$ and increasing, $f \in C([a,b], X)$, $\nu > 0$. We define the right Riemann-Liouville generalized fractional Bochner integral operator*

(6)
$$\left(J_{b-;g}^{\nu} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_x^b (g(z) - g(x))^{\nu-1} g'(z) f(z)\, dz,$$

$\forall\, x \in [a,b]$, where Γ is the gamma function. The last integral is of Bochner type. Since $f \in C([a,b], X)$, then $f \in L_\infty([a,b], X)$. By Theorem 4.11, p. 101, [4], we get that $\left(J_{b-;g}^{\nu} f\right) \in C([a,b], X)$. Above we set $J_{b-;g}^{0} f := f$ and see that $\left(J_{b-;g}^{\nu} f\right)(b) = 0$.

We also need the following definition.

Definition 1.3. *([4], p. 106) Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. We define the left generalized g-fractional derivative X-valued of f of order α as follows:*

(7)
$$\left(D_{a+;g}^{\alpha} f\right)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x) - g(t))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)}(g(t))\, dt,$$

$\forall\, x \in [a,b]$. The last integral is of Bochner type. Ordinary vector valued derivative is as in [12], similar to numerical one. If $\alpha \notin \mathbb{N}$, by Theorem 4.10, p. 98, [4], we have that $\left(D_{a+;g}^{\alpha} f\right) \in C([a,b], X)$. We see that

(8)
$$\left(J_{a;g}^{n-\alpha}\left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)(x) = \left(D_{a+;g}^{\alpha} f\right)(x), \quad \forall\, x \in [a,b].$$

We set

(9)
$$D_{a+;g}^{n} f(x) := \left(\left(f \circ g^{-1}\right)^{n} \circ g\right)(x) \in C([a,b], X), \quad n \in \mathbb{N},$$

$$D_{a+;g}^{0} f(x) = f(x), \quad \forall\, x \in [a,b].$$

When $g = id$, then

(10)
$$D_{a+;g}^{\alpha} f = D_{a+;id}^{\alpha} f = D_{*a}^{\alpha} f,$$

the usual left X-valued Caputo fractional derivative, see [4, Chapter 1].

We mention the following definition.

Definition 1.4. *([4], p. 107) Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. We define the right generalized g-fractional derivative X-valued of f of order α as follows:*

$$(11) \qquad \left(D^{\alpha}_{b-;g} f\right)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (g(t) - g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt,$$

$\forall\, x \in [a,b]$. The last integral is of Bochner type. If $\alpha \notin \mathbb{N}$, by Theorem 4.11, p. 101, [4], we have that $\left(D^{\alpha}_{b-;g} f\right) \in C([a,b], X)$. We see that

$$(12) \qquad J^{n-\alpha}_{b-;g} \left((-1)^n \left(f \circ g^{-1}\right)^{(n)} \circ g \right)(x) = \left(D^{\alpha}_{b-;g} f\right)(x), \;\; a \leq x \leq b.$$

We set

$$(13) \qquad D^{n}_{b-;g} f(x) := (-1)^n \left(\left(f \circ g^{-1}\right)^{n} \circ g \right)(x) \in C([a,b], X), \;\; n \in \mathbb{N},$$

$$D^{0}_{b-;g} f(x) := f(x), \;\; \forall\, x \in [a,b].$$

When $g = id$, then

$$(14) \qquad D^{\alpha}_{b-;g} f(x) = D^{\alpha}_{b-;id} f(x) = D^{\alpha}_{b-} f,$$

the usual right X-valued Caputo fractional derivative, see [4, Chapter 2].

We mention the generalized left fractional Taylor formula:

Theorem 1.1. *([4], p. 107) Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$, $a \leq x \leq b$. Then,*

$$\begin{aligned} f(x) &= f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D^{\alpha}_{a+;g} f\right)(t)\, dt \\ &= f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) \\ (15) \qquad &+ \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} \left(\left(D^{\alpha}_{a+;g} f\right) \circ g^{-1} \right)(z)\, dz. \end{aligned}$$

We also mention the generalized right fractional Taylor formula:

Theorem 1.2. *([4], p. 108) Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$,*

$a \leq x \leq b$. Then,

$$f(x) = f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b))$$
$$+ \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \left(D_{b-;g}^{\alpha} f\right)(t)\, dt$$
$$= f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b))$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{\alpha-1} \left(\left(D_{b-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz. \tag{16}$$

By convention, we suppose that

$$\left(D_{x_0+;g}^{\alpha} f\right)(x) = 0, \text{ for } x < x_0,$$
$$\left(D_{x_0-;g}^{\alpha} f\right)(x) = 0, \text{ for } x > x_0, \tag{17}$$

for any $x, x_0 \in [a, b]$.

The author has already done an extensive amount of work on fractional Landau inequalities, see [3], and on abstract fractional Landau inequalities, see [4]. However, there the proving methods came out of applications of fractional Ostrowski inequalities ([2], [4]) and the derived inequalities were for small fractional orders, i.e. $\alpha \in (0, 1)$. Usually there the domains where $[A, +\infty)$ or $(-\infty, B]$, with $A, B \in \mathbb{R}$ and in one mixed case the domain was all of $\mathbb{R}$.

In this work with less assumptions, we establish uniform and L_p type mixed Caputo-Bochner abstract generalized fractional Landau inequalities over $\mathbb{R}$ for fractional orders $2 < \alpha \leq 3$. The method of proving is based on left and right Caputo-Bochner generalized fractional Taylor's formulae with integral remainder, see Theorems 1.1,1.2. We give also applications for $\alpha = 2.5$. Certainly, we are also inspired by [3], [4].

2. Main Results

We give the following abstract mixed generalized fractional Landau inequalities over $\mathbb{R}$.

Theorem 2.3. *Let $2 < \alpha \leq 3$ and $f \in C^3(\mathbb{R}, X)$, where $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1(\mathbb{R})$, strictly increasing, such that $g^{-1} \in C^3(g(\mathbb{R}))$. We assume that $\|\|f\|\|_{\infty,\mathbb{R}} < \infty$ and that*

$$K := \max\left\{ \left\|\left\|\left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\|\right\|_{\infty,\mathbb{R}\times g(\mathbb{R})}, \left\|\left\|\left(\left(D_{a-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\|\right\|_{\infty,\mathbb{R}\times g(\mathbb{R})} \right\} < \infty, \tag{18}$$

where $(a, z) \in \mathbb{R} \times g(\mathbb{R})$. Then,

$$\left\|\left\|\left(f \circ g^{-1}\right)' \circ g\right\|\right\|_{\infty,\mathbb{R}} \leq \alpha \left(\frac{K}{\Gamma(\alpha+1)}\right)^{\frac{1}{\alpha}} \left(\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{\alpha - 1}\right)^{\frac{\alpha-1}{\alpha}} \tag{19}$$

and

$$\left\|\left\|\left(f \circ g^{-1}\right)'' \circ g\right\|\right\|_{\infty,\mathbb{R}} \leq \alpha \left(\frac{K}{\Gamma(\alpha+1)}\right)^{\frac{2}{\alpha}} \left(\frac{4\|\|f\|\|_{\infty,\mathbb{R}}}{\alpha - 2}\right)^{\frac{\alpha-2}{\alpha}}. \tag{20}$$

That is,

$$\left\|\left\|\left(f \circ g^{-1}\right)' \circ g\right\|\right\|_{\infty,\mathbb{R}}, \left\|\left\|\left(f \circ g^{-1}\right)'' \circ g\right\|\right\|_{\infty,\mathbb{R}} < \infty.$$

Proof. Here $2 < \alpha \leq 3$, i.e. $\lceil \alpha \rceil = 3$. Let $f \in C^3(\mathbb{R}, X)$, where $(X, \|\cdot\|)$ is a Banach space, $a \in \mathbb{R}$ is fixed momentarily. We need the following abstract generalized fractional Taylor formulae for $n = 3$. By Theorem 1.1, we get

$$f(x) - f(a) = (g(x) - g(a)) \left(f \circ g^{-1}\right)' (g(a)) + \frac{(g(x) - g(a))^2}{2} \left(f \circ g^{-1}\right)'' (g(a)) + \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} \left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz, \quad \forall\, x \geq a. \tag{21}$$

And by Theorem 1.2, we get

$$f(x) - f(a) = (g(x) - g(a)) \left(f \circ g^{-1}\right)' (g(a)) + \frac{(g(x) - g(a))^2}{2} \left(f \circ g^{-1}\right)'' (g(a)) + \frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(a)} (z - g(x))^{\alpha-1} \left(\left(D_{a-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz, \quad \forall\, x \leq a. \tag{22}$$

Let $x_1 > a$, then

$$(g(x_1) - g(a)) \left(f \circ g^{-1}\right)' (g(a)) + \frac{(g(x_1) - g(a))^2}{2} \left(f \circ g^{-1}\right)'' (g(a)) = (f(x_1) - f(a)) - \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x_1)} (g(x_1) - z)^{\alpha-1} \left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz =: A, \tag{23}$$

and let $x_2 < a$, then

$$(g(x_2) - g(a)) \left(f \circ g^{-1}\right)' (g(a)) + \frac{(g(x_2) - g(a))^2}{2} \left(f \circ g^{-1}\right)'' (g(a)) = (f(x_2) - f(a)) - \frac{1}{\Gamma(\alpha)} \int_{g(x_2)}^{g(a)} (z - g(x_2))^{\alpha-1} \left(\left(D_{a-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz =: B. \tag{24}$$

Let $h > 0$, we can choose x_1 such that $g(x_1) - g(a) = h$ and we can choose x_2 such that $g(a) - g(x_2) = h$. That is $g(x_1) = g(a) + h$ and $g(x_2) = g(a) - h$, and $g(x_2) - g(a) = -h$. Furthermore, it holds $g(x_2) - g(x_1) = -2h$. We can rewrite (23) as

$$h \left(f \circ g^{-1}\right)' (g(a)) + \frac{h^2}{2} \left(f \circ g^{-1}\right)'' (g(a)) = A, \tag{25}$$

and we can rewrite (24) as

$$-h \left(f \circ g^{-1}\right)' (g(a)) + \frac{h^2}{2} \left(f \circ g^{-1}\right)'' (g(a)) = B. \tag{26}$$

Solving the system of (25) and (26), we find

$$\left(f \circ g^{-1}\right)' (g(a)) = \frac{A - B}{2h} \tag{27}$$

and

$$\left(f \circ g^{-1}\right)'' (g(a)) = \frac{A + B}{h^2}.$$

We assumed that

$$\left\|\left\|\left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\|\right\|_{\infty, \mathbb{R} \times g(\mathbb{R})}, \left\|\left\|\left(\left(D_{a-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\|\right\|_{\infty, \mathbb{R} \times g(\mathbb{R})} < \infty.$$

We obtain,

$$\left\|\left(f \circ g^{-1}\right)' (g(a))\right\| = \frac{1}{2h} \|A - B\|$$

and

$$\left\| \left(f \circ g^{-1}\right)'' (g(a)) \right\| \leq \frac{1}{h^2} (\|A\| + \|B\|). \tag{28}$$

We get

$$\begin{aligned}
\|A\| &= \left\| (f(x_1) - f(a)) - \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x_1)} (g(x_1) - z)^{\alpha-1} \left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1} \right)(z)\, dz \right\| \\
&\leq 2 \||f\||_{\infty,\mathbb{R}} + \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x_1)} (g(x_1) - z)^{\alpha-1} \left\| \left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1} \right)(z) \right\| dz \\
&\leq 2 \||f\||_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha)} \left(\int_{g(a)}^{g(x_1)} (g(x_1) - z)^{\alpha-1} dz \right) \\
&= 2 \||f\||_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha+1)} (g(x_1) - g(a))^{\alpha} = 2 \||f\||_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha+1)} h^{\alpha}. \qquad (29)
\end{aligned}$$

That is,

$$\|A\| \leq 2 \||f\||_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha+1)} h^{\alpha}, \quad h > 0. \tag{30}$$

Similarly, it holds

$$\begin{aligned}
\|B\| &= \left\| (f(x_2) - f(a)) - \frac{1}{\Gamma(\alpha)} \int_{g(x_2)}^{g(a)} (z - g(x_2))^{\alpha-1} \left(\left(D_{a-;g}^{\alpha} f\right) \circ g^{-1} \right)(z)\, dz \right\| \\
&\leq 2 \||f\||_{\infty,\mathbb{R}} + \frac{1}{\Gamma(\alpha)} \int_{g(x_2)}^{g(a)} (z - g(x_2))^{\alpha-1} \left\| \left(\left(D_{a-;g}^{\alpha} f\right) \circ g^{-1} \right)(z) \right\| dz \\
&\leq 2 \||f\||_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha)} \left(\int_{g(x_2)}^{g(a)} (z - g(x_2))^{\alpha-1} dz \right) \\
&= 2 \||f\||_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha+1)} (g(a) - g(x_2))^{\alpha} = 2 \||f\||_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha+1)} h^{\alpha}. \qquad (31)
\end{aligned}$$

That is,

$$\|B\| \leq 2 \||f\||_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha+1)} h^{\alpha}, \quad h > 0. \tag{32}$$

Furthermore, we have

$$\|A\| + \|B\| \leq 4 \||f\||_{\infty,\mathbb{R}} + \frac{2K}{\Gamma(\alpha+1)} h^{\alpha}, \quad h > 0. \tag{33}$$

We also notice that

$$\|A-B\| = \left\| f(x_1) - f(a) - \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x_1)} (g(x_1)-z)^{\alpha-1} \left(\left(D_{a+;g}^{\alpha} f \right) \circ g^{-1} \right)(z)\, dz \right.$$

$$\left. -f(x_2) + f(a) + \frac{1}{\Gamma(\alpha)} \int_{g(x_2)}^{g(a)} (z-g(x_2))^{\alpha-1} \left(\left(D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z)\, dz \right\|$$

$$\leq \|f(x_1)-f(x_2)\| + \frac{1}{\Gamma(\alpha)} \left[\int_{g(a)}^{g(x_1)} (g(x_1)-z)^{\alpha-1} \left\| \left(\left(D_{a+;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\| dz \right. \tag{34}$$

$$\left. + \int_{g(x_2)}^{g(a)} (z-g(x_2))^{\alpha-1} \left\| \left(\left(D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\| dz \right]$$

$$\leq 2 \left\|\|f\|\right\|_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha)} \left[\int_{g(a)}^{g(x_1)} (g(x_1)-z)^{\alpha-1} dz + \int_{g(x_2)}^{g(a)} (z-g(x_2))^{\alpha-1} dz \right]$$

$$= 2 \left\|\|f\|\right\|_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha+1)} \left[(g(x_1)-g(a))^{\alpha} + (g(a)-g(x_2))^{\alpha} \right]$$

$$= 2 \left\|\|f\|\right\|_{\infty,\mathbb{R}} + \frac{2Kh^{\alpha}}{\Gamma(\alpha+1)}.$$

That is,

$$\frac{\|A-B\|}{2} \leq \left\|\|f\|\right\|_{\infty,\mathbb{R}} + \frac{Kh^{\alpha}}{\Gamma(\alpha+1)}, \quad h>0. \tag{35}$$

Consequently, we obtain

$$\left\| \left(f \circ g^{-1} \right)' (g(a)) \right\| \overset{((28),(35))}{\leq} \frac{\left\|\|f\|\right\|_{\infty,\mathbb{R}}}{h} + \frac{Kh^{\alpha-1}}{\Gamma(\alpha+1)}$$

and

$$\left\| \left(f \circ g^{-1} \right)'' (g(a)) \right\| \overset{((28),(33))}{\leq} \frac{4 \left\|\|f\|\right\|_{\infty,\mathbb{R}}}{h^2} + \frac{2Kh^{\alpha-2}}{\Gamma(\alpha+1)}, \tag{36}$$

$h>0$, for any $a \in \mathbb{R}$. Hence,

$$\left\| \left\| \left(f \circ g^{-1} \right)' \circ g \right\| \right\|_{\infty,\mathbb{R}} \leq \frac{\left\|\|f\|\right\|_{\infty,\mathbb{R}}}{h} + \frac{Kh^{\alpha-1}}{\Gamma(\alpha+1)}$$

and

$$\left\| \left\| \left(f \circ g^{-1} \right)'' \circ g \right\| \right\|_{\infty,\mathbb{R}} \leq \frac{4 \left\|\|f\|\right\|_{\infty,\mathbb{R}}}{h^2} + \frac{2Kh^{\alpha-2}}{\Gamma(\alpha+1)}, \tag{37}$$

true $\forall\, h>0$, $2<\alpha\leq 3$. Call

$$\mu := \left\|\|f\|\right\|_{\infty,\mathbb{R}}, \quad \theta = \frac{K}{\Gamma(\alpha+1)}, \tag{38}$$

both are greater than zero. Set also $\rho := \alpha - 1 > 1$. We consider the function

$$y(h) := \frac{\mu}{h} + \theta h^{\rho}, \quad \forall\, h>0. \tag{39}$$

We have

$$y'(h) = -\frac{\mu}{h^2} + \rho\theta h^{\rho-1} = 0,$$

then

$$\rho\theta h^{\rho+1} = \mu,$$

with a unique solution

(40) $$h_0 := h_{crit.no} = \left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}}.$$

We have that

(41) $$y''(h) = 2\mu h^{-3} + \rho(\rho-1)\theta h^{\rho-2}.$$

We observe that

$$y''(h_0) = 2\mu\left(\frac{\mu}{\rho\theta}\right)^{-\frac{3}{\rho+1}} + \rho(\rho-1)\theta\left(\frac{\mu}{\rho\theta}\right)^{\frac{((\rho+1)-3)}{\rho+1}}$$

(42) $$= \left(\frac{\mu}{\rho\theta}\right)^{-\frac{3}{\rho+1}}[2\mu + \mu(\rho-1)] = \mu\left(\frac{\mu}{\rho\theta}\right)^{-\frac{3}{\rho+1}}(\rho+1) > 0.$$

Therefore, y has a global minimum at $h_0 = \left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}}$, which is

$$y(h_0) = \frac{\mu}{\left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}}} + \theta\left(\frac{\mu}{\rho\theta}\right)^{\frac{\rho}{\rho+1}}$$

$$= (\rho\theta)^{\frac{1}{\rho+1}}\frac{\mu}{\mu^{\frac{1}{\rho+1}}} + \frac{\theta\mu^{\frac{\rho}{\rho+1}}}{\rho^{\frac{\rho}{\rho+1}}\theta^{\frac{\rho}{\rho+1}}}$$

(43) $$= (\theta\mu^{\rho})^{\frac{1}{\rho+1}}\left(\rho^{\frac{1}{\rho+1}} + \frac{1}{\rho^{\frac{\rho}{\rho+1}}}\right) = (\theta\mu^{\rho})^{\frac{1}{\rho+1}}\left(\frac{\rho+1}{\rho^{\frac{\rho}{\rho+1}}}\right)$$

$$= (\theta\mu^{\rho})^{\frac{1}{\rho+1}}(\rho+1)\rho^{-\frac{\rho}{\rho+1}}.$$

That is,

(44) $$y(h_0) = (\theta\mu^{\rho})^{\frac{1}{\rho+1}}(\rho+1)\rho^{-\frac{\rho}{\rho+1}}.$$

Consequently,

(45) $$y(h_0) = \left(\frac{K}{\Gamma(\alpha+1)}\|\|f\|\|_{\infty,\mathbb{R}}^{\alpha-1}\right)^{\frac{1}{\alpha}}\alpha(\alpha-1)^{-\left(\frac{\alpha-1}{\alpha}\right)}.$$

We have proved that

(46) $$\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,\mathbb{R}} \leq \left(\frac{K}{\Gamma(\alpha+1)}\|\|f\|\|_{\infty,\mathbb{R}}^{\alpha-1}\right)^{\frac{1}{\alpha}}\alpha(\alpha-1)^{-\left(\frac{\alpha-1}{\alpha}\right)}.$$

Next call

(47) $$\xi := 4\|\|f\|\|_{\infty,\mathbb{R}}, \quad \psi = \frac{2K}{\Gamma(\alpha+1)},$$

both are greater than zero. Set also $\varphi := \alpha - 2 > 0$. We consider the function

(48) $$\gamma(h) := \frac{\xi}{h^2} + \psi h^{\varphi} = \xi h^{-2} + \psi h^{\varphi}, \ \forall\, h > 0.$$

We have

$$\gamma'(h) = -2\xi h^{-3} + \varphi\psi h^{\varphi-1} = 0,$$

then

$$\varphi\psi h^{\varphi+2}=2\xi,$$

with a unique solution

$$h_0:=h_{crit.no}=\left(\frac{2\xi}{\varphi\psi}\right)^{\frac{1}{\varphi+2}}. \tag{49}$$

We have that

$$\gamma''(h)=6\xi h^{-4}+\varphi(\varphi-1)\psi h^{\varphi-2}. \tag{50}$$

We see that

$$\begin{aligned}\gamma''(h_0)&=6\xi\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{4}{\varphi+2}}+\varphi(\varphi-1)\psi\left(\frac{2\xi}{\varphi\psi}\right)^{\frac{(\varphi+2)-4}{\varphi+2}}\\&=\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{4}{\varphi+2}}[6\xi+(\varphi-1)2\xi]=2\xi\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{4}{\varphi+2}}(\varphi+2)>0.\end{aligned} \tag{51}$$

Therefore, γ has a global minimum at $h_0=\left(\frac{2\xi}{\varphi\psi}\right)^{\frac{1}{\varphi+2}}$, which is

$$\begin{aligned}\gamma(h_0)&=\xi\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{2}{\varphi+2}}+\psi\left(\frac{2\xi}{\varphi\psi}\right)^{\frac{\varphi+2-2}{\varphi+2}}\\&=\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{2}{\varphi+2}}\left[\xi+\psi\frac{2\xi}{\varphi\psi}\right]=\frac{\xi}{\varphi}\left(\frac{\xi}{\varphi}\right)^{-\frac{2}{\varphi+2}}\left(\frac{2}{\psi}\right)^{-\frac{2}{\varphi+2}}(\varphi+2)\\&=\left(\frac{\xi}{\varphi}\right)^{\frac{\varphi}{\varphi+2}}\left(\frac{\psi}{2}\right)^{\frac{2}{\varphi+2}}(\varphi+2).\end{aligned} \tag{52}$$

That is,

$$\gamma(h_0)=\left(\frac{\xi}{\varphi}\right)^{\frac{\varphi}{\varphi+2}}\left(\frac{\psi}{2}\right)^{\frac{2}{\varphi+2}}(\varphi+2). \tag{53}$$

Consequently,

$$\gamma(h_0)=\left(\frac{4\left\|\|f\|\right\|_{\infty,\mathbb{R}}}{\alpha-2}\right)^{\frac{\alpha-2}{\alpha}}\left(\frac{K}{\Gamma(\alpha+1)}\right)^{\frac{2}{\alpha}}\alpha. \tag{54}$$

We have proved that

$$\left\|\left\|\left(f\circ g^{-1}\right)''\circ g\right\|\right\|_{\infty,\mathbb{R}}\leq\left(\frac{4\left\|\|f\|\right\|_{\infty,\mathbb{R}}}{\alpha-2}\right)^{\frac{\alpha-2}{\alpha}}\left(\frac{K}{\Gamma(\alpha+1)}\right)^{\frac{2}{\alpha}}\alpha. \tag{55}$$

The theorem is established. □

We also give an L_p analog of a generalized fractional Landau inequality

Theorem 2.4. *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $2<\alpha\leq3$ and $f\in C^3(\mathbb{R},X)$, where $(X,\|\cdot\|)$ is a Banach space. Let $g\in C^1(\mathbb{R})$, strictly increasing, such that $g^{-1}\in C^3(g(\mathbb{R}))$. We assume that*

$\||f\||_{\infty,\mathbb{R}} < \infty$, *and that*

$$M := \max\left\{\sup_{a\in\mathbb{R}} \left\|\left\|\left(\left(D^{\alpha}_{a+;g}f\right)\circ g^{-1}\right)(z)\right\|\right\|_{p,g(\mathbb{R})},\right.$$

$$\left.\sup_{a\in\mathbb{R}} \left\|\left\|\left(\left(D^{\alpha}_{a-;g}f\right)\circ g^{-1}\right)(z)\right\|\right\|_{p,g(\mathbb{R})}\right\} < \infty. \tag{56}$$

Then,
1)

$$\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,\mathbb{R}}$$

$$\leq \left(\alpha-\frac{1}{p}\right)\left(\frac{M}{\Gamma(\alpha)\left(q(\alpha-1)+1\right)^{\frac{1}{q}}}\right)^{\left(\frac{1}{\alpha-\frac{1}{p}}\right)}\left(\frac{\||f\||_{\infty,\mathbb{R}}}{\alpha-1-\frac{1}{p}}\right)^{\left(\frac{\alpha-1-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)} \tag{57}$$

2) under the additional assumption $2+\frac{1}{p}<\alpha\leq 3$, *we have*

$$\left\|\left\|\left(f\circ g^{-1}\right)''\circ g\right\|\right\|_{\infty,\mathbb{R}}$$

$$\leq \left(\alpha-\frac{1}{p}\right)\left(\frac{M}{\Gamma(\alpha)\left(q(\alpha-1)+1\right)^{\frac{1}{q}}}\right)^{\left(\frac{2}{\alpha-\frac{1}{p}}\right)}\left(\frac{4\||f\||_{\infty,\mathbb{R}}}{\alpha-2-\frac{1}{p}}\right)^{\left(\frac{\alpha-2-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)}. \tag{58}$$

That is,

$$\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,\mathbb{R}},\ \left\|\left\|\left(f\circ g^{-1}\right)''\circ g\right\|\right\|_{\infty,\mathbb{R}} < \infty.$$

Proof. We continue with the proof of Theorem 2.3. By (23), we have

$$\|A\| = \left\|\left(f(x_1)-f(a)\right)-\frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(x_1)}\left(g(x_1)-z\right)^{\alpha-1}\left(\left(D^{\alpha}_{a+;g}f\right)\circ g^{-1}\right)(z)\,dz\right\|$$

$$\leq 2\||f\||_{\infty,\mathbb{R}}+\frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(x_1)}\left(g(x_1)-z\right)^{\alpha-1}\left\|\left(\left(D^{\alpha}_{a+;g}f\right)\circ g^{-1}\right)(z)\right\|dz$$

$$\leq 2\||f\||_{\infty,\mathbb{R}}+\frac{1}{\Gamma(\alpha)}\left(\int_{g(a)}^{g(x_1)}\left(g(x_1)-z\right)^{q(\alpha-1)}dz\right)^{\frac{1}{q}}$$

$$\times\left(\int_{g(a)}^{g(x_1)}\left\|\left(\left(D^{\alpha}_{a+;g}f\right)\circ g^{-1}\right)(z)\right\|^{p}dz\right)^{\frac{1}{p}} \tag{59}$$

$$\leq 2\||f\||_{\infty,\mathbb{R}}+\frac{1}{\Gamma(\alpha)}\frac{\left(g(x_1)-g(a)\right)^{\frac{(q(\alpha-1)+1)}{q}}}{\left(q(\alpha-1)+1\right)^{\frac{1}{q}}}\left\|\left\|\left(\left(D^{\alpha}_{a+;g}f\right)\circ g^{-1}\right)(z)\right\|\right\|_{p,g(\mathbb{R})}$$

$$\leq 2\||f\||_{\infty,\mathbb{R}}+\frac{1}{\Gamma(\alpha)}\frac{h^{\alpha-\frac{1}{p}}}{\left(q(\alpha-1)+1\right)^{\frac{1}{q}}}\left(\sup_{a\in\mathbb{R}}\left\|\left\|\left(\left(D^{\alpha}_{a+;g}f\right)\circ g^{-1}\right)(z)\right\|\right\|_{p,g(\mathbb{R})}\right)$$

$$\leq 2\||f\||_{\infty,\mathbb{R}}+\frac{h^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left(q(\alpha-1)+1\right)^{\frac{1}{q}}}M.$$

That is,

$$\|A\| \leq 2 \||f\||_{\infty,\mathbb{R}} + \frac{M}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{1}{q}}} h^{\alpha-\frac{1}{p}}, \quad h>0. \tag{60}$$

Similarly, from (24), we get

$$\|B\| = \left\| (f(x_2) - f(a)) - \frac{1}{\Gamma(\alpha)} \int_{g(x_2)}^{g(a)} (z - g(x_2))^{\alpha-1} \left(\left(D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z)\, dz \right\|$$

$$\leq 2 \||f\||_{\infty,\mathbb{R}} + \frac{1}{\Gamma(\alpha)} \int_{g(x_2)}^{g(a)} (z - g(x_2))^{\alpha-1} \left\| \left(\left(D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\| dz$$

$$\leq 2 \||f\||_{\infty,\mathbb{R}} + \frac{1}{\Gamma(\alpha)} \left(\int_{g(x_2)}^{g(a)} (z - g(x_2))^{q(\alpha-1)} dz \right)^{\frac{1}{q}}$$

$$\times \left(\int_{g(x_2)}^{g(a)} \left\| \left(\left(D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\|^p dz \right)^{\frac{1}{p}}$$

$$\leq 2 \||f\||_{\infty,\mathbb{R}} + \frac{1}{\Gamma(\alpha)} \frac{(g(a) - g(x_2))^{\alpha-\frac{1}{p}}}{(q(\alpha-1)+1)^{\frac{1}{q}}} \left(\sup_{a\in\mathbb{R}} \left\|\left\| \left(\left(D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\|\right\|_{p,g(\mathbb{R})} \right) \tag{61}$$

$$\leq 2 \||f\||_{\infty,\mathbb{R}} + \frac{h^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{1}{q}}} M.$$

That is,

$$\|B\| \leq 2 \||f\||_{\infty,\mathbb{R}} + \frac{M}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{1}{q}}} h^{\alpha-\frac{1}{p}}, \quad h>0. \tag{62}$$

Hence, it holds

$$\|A+B\| \leq \|A\| + \|B\|$$

$$\overset{\text{(by (60), (62))}}{\leq} 4 \||f\||_{\infty,\mathbb{R}} + \frac{2M}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{1}{q}}} h^{\alpha-\frac{1}{p}}, \quad h>0. \tag{63}$$

Furthermore, we have

$$\|A-B\| \overset{(34)}{\leq} 2 \||f\||_{\infty,\mathbb{R}}$$

$$+ \frac{1}{\Gamma(\alpha)} \left[\left(\int_{g(a)}^{g(x_1)} (g(x_1) - z)^{q(\alpha-1)} dz \right)^{\frac{1}{q}} \left\|\left\| \left(\left(D_{a+;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\|\right\|_{p,g(\mathbb{R})} \right.$$

$$\left. + \left(\int_{g(x_2)}^{g(a)} (z - g(x_2))^{q(\alpha-1)} dz \right)^{\frac{1}{q}} \left\|\left\| \left(\left(D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\|\right\|_{p,g(\mathbb{R})} \right]$$

$$\leq 2 \||f\||_{\infty,\mathbb{R}} + \frac{M}{\Gamma(\alpha)} \left[\frac{2h^{\alpha-\frac{1}{p}}}{(q(\alpha-1)+1)^{\frac{1}{q}}} \right]. \tag{64}$$

We have proved that

$$\frac{\|A-B\|}{2} \leq \||f\||_{\infty,\mathbb{R}} + \frac{M}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{1}{q}}} h^{\alpha-\frac{1}{p}}, \quad h>0. \tag{65}$$

From (27), (65), we have

$$\left\| \left(f \circ g^{-1}\right)' (g(a)) \right\| \leq \frac{\|\|f\|\|_{\infty,\mathbb{R}}}{h} + \frac{M}{\Gamma(\alpha)\left(q(\alpha-1)+1\right)^{\frac{1}{q}}} h^{\alpha-\frac{1}{p}-1}, \tag{66}$$

$h > 0$, any $a \in \mathbb{R}$. And from (27), (63), we get that

$$\left\| \left(f \circ g^{-1}\right)'' (g(a)) \right\| \leq \frac{4\|\|f\|\|_{\infty,\mathbb{R}}}{h^2} + \frac{2M}{\Gamma(\alpha)\left(q(\alpha-1)+1\right)^{\frac{1}{q}}} h^{\alpha-\frac{1}{p}-2}, \tag{67}$$

$h > 0$, any $a \in \mathbb{R}$. Hence,

$$\left\|\left\| \left(f \circ g^{-1}\right)' \circ g \right\|\right\|_{\infty,\mathbb{R}} \leq \frac{\|\|f\|\|_{\infty,\mathbb{R}}}{h} + \left(\frac{M}{\Gamma(\alpha)\left(q(\alpha-1)+1\right)^{\frac{1}{q}}} \right) h^{\left(\alpha-\frac{1}{p}-1\right)} \tag{68}$$

and

$$\left\|\left\| \left(f \circ g^{-1}\right)'' \circ g \right\|\right\|_{\infty,\mathbb{R}} \leq \frac{4\|\|f\|\|_{\infty,\mathbb{R}}}{h^2} + \left(\frac{2M}{\Gamma(\alpha)\left(q(\alpha-1)+1\right)^{\frac{1}{q}}} \right) h^{\left(\alpha-\frac{1}{p}-2\right)}, \tag{69}$$

true $\forall\, h > 0$, $2 < \alpha \leq 3$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. In (69), we restrict ourselves to $2 + \frac{1}{p} < \alpha \leq 3$. Call

$$\mu := \|\|f\|\|_{\infty,\mathbb{R}}\,, \quad \theta = \frac{M}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{1}{q}}}, \tag{70}$$

both are greater than zero. Set also $\rho := \alpha - 1 - \frac{1}{p} > \frac{1}{q} > 0$. We consider the function

$$y(h) := \frac{\mu}{h} + \theta h^{\rho}, \quad \forall\, h > 0. \tag{71}$$

As in the proof of Theorem 2.3, it has only one critical number

$$h_0 := h_{crit.no} = \left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}} \tag{72}$$

and a global minimum

$$y(h_0) = \theta^{\frac{1}{\rho+1}} \mu^{\frac{\rho}{\rho+1}} (\rho+1)\, \rho^{-\frac{\rho}{\rho+1}}. \tag{73}$$

Consequently,

$$y(h_0) = \left(\frac{M}{\Gamma(\alpha)\left(q(\alpha-1)+1\right)^{\frac{1}{q}}} \right)^{\frac{1}{\left(\alpha-\frac{1}{p}\right)}} \left(\|\|f\|\|_{\infty,\mathbb{R}}\right)^{\left(\frac{\alpha-1-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)}$$

$$\left(\alpha - \frac{1}{p}\right)\left(\alpha - 1 - \frac{1}{p}\right)^{-\left(\frac{\alpha-1-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)}. \tag{74}$$

We have proved that (see (68))

$$\left\|\left\| \left(f \circ g^{-1}\right)' \circ g \right\|\right\|_{\infty,\mathbb{R}}$$

$$\leq \left(\frac{M}{\Gamma(\alpha)\left(q(\alpha-1)+1\right)^{\frac{1}{q}}} \right)^{\frac{1}{\left(\alpha-\frac{1}{p}\right)}} \left(\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{\alpha-1-\frac{1}{p}} \right)^{\left(\frac{\alpha-1-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)} \left(\alpha - \frac{1}{p}\right). \tag{75}$$

We also call

$$\xi := 4\|\|f\|\|_{\infty,\mathbb{R}}\,, \quad \psi = \frac{2M}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{1}{q}}}, \tag{76}$$

both are greater than zero. Set also $\varphi := \alpha - 2 - \frac{1}{p} > 0$. We consider the function

$$\gamma(h) := \frac{\xi}{h^2} + \psi h^{\varphi}, \quad \forall\, h > 0. \tag{77}$$

As in the proof of Theorem 2.3, γ has a global minimum at

$$h_0 = \left(\frac{2\xi}{\varphi\psi}\right)^{\frac{1}{\varphi+2}}, \tag{78}$$

which is

$$\gamma(h_0) = \left(\frac{\xi}{\varphi}\right)^{\frac{\varphi}{\varphi+2}} \left(\frac{\psi}{2}\right)^{\frac{2}{\varphi+2}} (\varphi + 2). \tag{79}$$

Consequently,

$$\gamma(h_0) = \left(\frac{4\,\|\|f\|\|_{\infty,\mathbb{R}}}{\alpha - 2 - \frac{1}{p}}\right)^{\left(\frac{\alpha-2-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)} \left(\frac{M}{\Gamma(\alpha)\,(q(\alpha-1)+1)^{\frac{1}{q}}}\right)^{\left(\frac{2}{\alpha-\frac{1}{p}}\right)} \left(\alpha - \frac{1}{p}\right). \tag{80}$$

We have proved that (see (69))

$$\begin{aligned} &\left\|\left\|\left(f \circ g^{-1}\right)'' \circ g\right\|\right\|_{\infty,\mathbb{R}} \\ &\leq \left(\frac{4\,\|\|f\|\|_{\infty,\mathbb{R}}}{\alpha - 2 - \frac{1}{p}}\right)^{\left(\frac{\alpha-2-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)} \left(\frac{M}{\Gamma(\alpha)\,(q(\alpha-1)+1)^{\frac{1}{q}}}\right)^{\left(\frac{2}{\alpha-\frac{1}{p}}\right)} \left(\alpha - \frac{1}{p}\right). \end{aligned} \tag{81}$$

The theorem is established. □

Next, we apply Theorems 2.3, 2.4 for $g(t) = e^t$, $t \in \mathbb{R}$ and $\alpha = 2.5$.

Corollary 2.1. *Let $f \in C^3(\mathbb{R}, X)$, where $(X, \|\cdot\|)$ is a Banach space. We assume that $\|\|f\|\|_{\infty,\mathbb{R}} < \infty$, and that*

$$\begin{aligned} K_{2.5} := \max\Big\{ &\left\|\left\|\left(\left(D^{2.5}_{a+;e^t} f\right) \circ \ln\right)(z)\right\|\right\|_{\infty,\mathbb{R}\times(0,\infty)}, \\ &\left\|\left\|\left(\left(D^{2.5}_{a-;e^t} f\right) \circ \ln\right)(z)\right\|\right\|_{\infty,\mathbb{R}\times(0,\infty)} \Big\} < \infty, \end{aligned} \tag{82}$$

where $(a, z) \in \mathbb{R} \times (0, \infty)$. Then,

$$\left\|\left\|(f \circ \ln)' \circ e^t\right\|\right\|_{\infty,\mathbb{R}} \leq 1.21136\,(K_{2.5})^{0.4} \left(\|\|f\|\|_{\infty,\mathbb{R}}\right)^{0.6} \tag{83}$$

and

$$\left\|\left\|(f \circ \ln)'' \circ e^t\right\|\right\|_{\infty,\mathbb{R}} \leq 1.44713\,(K_{2.5})^{0.8} \left(\|\|f\|\|_{\infty,\mathbb{R}}\right)^{0.2}. \tag{84}$$

That is,

$$\left\|\left\|(f \circ \ln)' \circ e^t\right\|\right\|_{\infty,\mathbb{R}}, \left\|\left\|(f \circ \ln)'' \circ e^t\right\|\right\|_{\infty,\mathbb{R}} < \infty.$$

Proof. By Theorem 2.3. □

We finish with the following result.

Corollary 2.2. *(case of $g(t) = e^t$, $\alpha = 2.5$, $p = q = 2$) Let $f \in C^3(\mathbb{R}, X)$, where $(X, \|\cdot\|)$ is a Banach space. We assume that $\||f\||_{\infty,\mathbb{R}} < \infty$, and that*

$$M_{2.5} := \max \left\{ \sup_{a\in\mathbb{R}} \left\| \left\| \left(\left(D^{2.5}_{a+;e^t} f \right) \circ \ln \right)(z) \right\| \right\|_{2,(0,\infty)}, \right.$$

$$\left. \sup_{a\in\mathbb{R}} \left\| \left\| \left(\left(D^{2.5}_{a-;e^t} f \right) \circ \ln \right)(z) \right\| \right\|_{2,(0,\infty)} \right\} < \infty. \tag{85}$$

Then,

$$\left\| \left\| (f \circ \ln)' \circ e^t \right\| \right\|_{\infty,\mathbb{R}} \leq 1.226583057 \sqrt{M_{2.5} \||f\||_{\infty,\mathbb{R}}} < \infty. \tag{86}$$

Proof. By Theorem 2.4, (57). □

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Survey Article

A survey on recent results in Korovkin's approximation theory in modular spaces

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ABSTRACT. In this paper, we give a survey about recent versions of Korovkin-type theorems for modular function spaces, a class which includes L^p, Orlicz, Musielak-Orlicz spaces and many others. We consider various kinds of modular convergence, using certain summability processes, like triangular matrix statistical convergence, and filter convergence (which are generalizations of the statistical convergence). Finally, we consider an abstract axiomatic convergence which includes the previous ones and even almost convergence, which is not generated by any filter, as we show by an example.

Keywords: Korovkin's theorem, modular space, filter convergence, abstract convergence.

2020 Mathematics Subject Classification: 41A35, 47G10, 46E30, 40A35.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

Korovkin's approximation theory is one of the main research topics of Professor Francesco Altomare. His famous monography ([3]), written in collaboration with Professor Michele Campiti, represents a fundamental reference point for any mathematicians who wish to study approximation theory by positive linear operators. Later on (see [2]), Altomare introduced a different approach to the theory developed in [3], obtaining general and unifying results.

The origin of this theory is the classical Bernstein proof of the Weierstrass theorem, where the author shows the uniform convergence of the Bernstein polynomials of a continuous function f over the compact interval $[0, 1]$ by stating it only for the test functions $\{1, x, x^2\}$, see [13]. This method was first generalized by H. Bohman ([18], who applied it to certain interpolating discrete operators acting on continuous functions over $[0, 1]$) and then by P.P. Korovkin, for general positive linear operators [33] (see also the monograph [34]). In the meantime, also trigonometric versions of this basic result were obtained, by using the test functions $\{1, \cos x, \sin x\}$, (see e.g. [20]), and more generally functions which form a so-called Chebyshev system (that is a set of functions such that a linear combination of them has no more than two zeros on an interval, whenever its coefficients are not simultaneously null).

Several extensions in multivariate frame are also available in literature (see e.g. [12], [25], [43]).

The Korovkin theorem was successfully applied for stating uniform convergence for a very large class of integral or discrete positive operators, acting on continuous functions defined over compact intervals of the real line or also on compact sets in Euclidean spaces. Later,

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several extensions are obtained when the functions are defined on not necessarily compact intervals, using certain suitable weight functions, so obtaining weighted uniform convergence (see e.g. [29], [30]).

However, a unifying and general approach to Korovkin-type approximation was given in the quoted book [3] and in the extensive article [2], in which the authors expose the theory in the spaces $C(X)$, where X is a locally compact topological space, together with its numerous applications.

Recently, versions of the Korovkin theorem were obtained in different functional spaces, namely L^p-spaces, or more general Lebesgue spaces (see [21], [26], [32], [42]), Orlicz-type spaces (see [37]). In this respect, many authors gave contributions to this theory and we refer to the exhaustive list of references of [2].

Our aim is to report a group of Korovkin-type theorems obtained by us in the abstract setting of the so-called modular spaces, a class of function spaces which includes L^p spaces, Orlicz and Musielak–Orlicz spaces, weighted spaces, certain interpolation spaces, and many others. The theory of modular spaces was started by H. Nakano ([41]), but extensively studied by J. Musielak ([40]) and later on by W.M. Kozlowski ([36]). An abstract approach to the theory of approximation in modular spaces was given in [10]. The theory of Korovkin-type approximation in modular spaces enables one to obtain a unifying approach, which includes by a unique method, many previous results on the subject. Also, it is possible to obtain general results using a different notion of convergence, namely, filter-type convergence, including the case of the statistical convergence, relative uniform convergence with respect to scale functions, or an axiomatic abstract convergence, which includes filter convergence, triangular matrix statistical convergence and even almost convergence, which is not generated by any filter (see also [1], [4], [11], [15], [16], [17], [23], [24], [27], [28], [31], [39], [44]).

In Section 2, we introduce the modular function spaces for functions defined over nonempty open sets of an Hausdorff locally compact topological space. Then in Section 3, we report a Korovkin-type theorem for a family of positive linear operators acting on suitable subspaces of the modular spaces established in [8], [9], and we discuss some interesting examples. Then in Section 4, we report an extension of the previous results employing filter convergence, showing by examples that these new results are authentic extensions. Section 5 contains extensions of the Korovkin theorem with respect to generalized versions of statistical modular convergence, defined through certain summability processes involving regular matrices, and we give a comparison between this kind of convergence and filter convergence. In Section 6, we report some extensions of the Korovkin theorem to an abstract convergence introduced axiomatically, which includes the previous ones and other kinds of convergences not generated by any filter (see [5], [6], [17] and the references therein).

2. Modular function spaces

Let G be a Hausdorff locally compact topological space endowed with a regular measure μ defined on the Borel sets of G.

We will denote by $L^0(G)$ the space of all real-valued Borel measurable functions $f : G \to \mathbb{R}$ provided with equality μ-a.e. A functional $\varrho : L^0(G) \to \widetilde{\mathbb{R}_0^+}$ is said to be a modular on $L^0(G)$ if

i) $\varrho[f] = 0 \Leftrightarrow f = 0$, a.e. in G,
ii) $\varrho[-f] = \varrho[f]$, for every $f \in L^0(G)$,
iii) $\varrho[\alpha f + \beta g] \leq \varrho[f] + \varrho[g]$, for every $f, g \in L^0(G), \alpha, \beta \geq 0, \alpha + \beta = 1$.

We will say that a modular ϱ is Q-quasi convex if there is a constant $Q \geq 1$ such that

$$\varrho[\alpha f + \beta g] \leq Q\alpha\varrho[Qf] + Q\beta\varrho[Qg],$$

for every $f, g \in L^0(G), \alpha, \beta \geq 0, \alpha + \beta = 1$. If $Q = 1$, we will say that ϱ is convex.
By means of the functional ϱ, we introduce the vector subspace of $L^0(G)$, denoted by $L^\varrho(G)$, defined by

$$L^\varrho(G) = \{f \in L^0(G) : \lim_{\lambda \to 0^+} \varrho[\lambda f] = 0\}.$$

The subspace $L^\varrho(G)$ is called the modular space generated by ϱ. It is easy to see that when ϱ is Q-quasi-convex, we have the following characterization of the modular space $L^\varrho(G)$:

$$L^\varrho(G) = \{f \in L^0(G) : \varrho[\lambda f] < +\infty \text{ for some } \lambda > 0\},$$

see for example [40] and [10]. The subspace of $L^\varrho(G)$defined by

$$E^\varrho(G) = \{f \in L^\varrho(G) : \varrho[\lambda f] < +\infty \text{ for all } \lambda > 0\}$$

is called the space of the finite elements of $L^\varrho(G)$, see [40].
The following assumptions on modulars will be used:

a) ϱ is monotone, i.e. for $f, g \in L^0(G)$ and $|f| \leq |g|$, then $\varrho[f] \leq \varrho[g]$.
b) ϱ is finite, i.e. denoting by e_0 the function $e_0(t) = 1$ for every $t \in G$, $e_0 \in L^\varrho(G)$.
c) ϱ is absolutely finite, i.e. ϱ is finite and for every $\varepsilon > 0, \lambda > 0$ there is $\delta > 0$ such that $\varrho[\lambda\chi_B] < \varepsilon$ for any measurable subset $B \subset G$ with $\mu(B) < \delta$. Here, χ_B denotes the characteristic function of the set B.
d) ϱ is strongly finite, i.e. $e_0 \in E^\varrho(G)$.
e) ϱ is absolutely continuous, i.e. there exists $\alpha > 0$ such that for every $f \in L^0(G)$, with $\varrho[f] < +\infty$, the following condition is satisfied: for every $\varepsilon > 0$ there is $\delta > 0$ such that $\varrho[\alpha f \chi_B] < \varepsilon$, for every measurable subset $B \subset G$ with $\mu(B) < \delta$.

For the above notions, see [10], [40]. Note that, since $\mu(G) < +\infty$, if ϱ is strongly finite and absolutely continuous, then it is also absolutely finite.
Classical examples of modular spaces are given by the Orlicz spaces generated by a φ-function φ or, more generally, by any Musielak-Orlicz space generated by a φ-function φ depending on a parameter, satisfying some growth condition with respect to the parameter (see [10], [36], [40] in some special cases). The modular functionals generating the above spaces satisfy all the previous assumptions.

We say that a sequence of functions $(f_n)_{n\in\mathbb{N}} \subset L^\varrho(G)$ is modularly convergent to a function $f \in L^\varrho(G)$, if there exists $\lambda > 0$ such that

$$\lim_{n\to+\infty} \varrho[\lambda(f_n - f)] = 0.$$

This notion extends the norm-convergence in L^p-spaces. Moreover, it is weaker than the F-norm-convergence induced by the Luxemburg F-norm generated by ϱ and defined by

$$\|f\|_\rho \equiv \inf\{u > 0 : \varrho[f/u] \leq u\}.$$

We recall that a sequence of functions $(f_n)_{n\in\mathbb{N}}$ is F-norm-convergent (or strongly convergent) to f if

$$\lim_{n\to+\infty} \varrho[\lambda(f_n - f)] = 0$$

for every $\lambda > 0$. The two notions of convergence are equivalent if and only if the modular satisfies the Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\varrho[2f] \leq M\varrho[f]$, for every $f \in L^0(G)$, see [40]. For example, this happens for all L^p-spaces and Orlicz spaces generated by φ-functions with the Δ_2-regularity condition (see [10], [40]). The modular convergence induces a topology on $L^\varrho(G)$, called modular topology. Given a subset $\mathcal{B} \subset L^\varrho(G)$, we will denote by $\overline{\mathcal{B}}$ the closure of $\mathcal{B}$ with respect to the modular topology. Then $f \in \overline{\mathcal{B}}$ if there is a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ such that f_n is modularly convergent to f.

Let $A \subset G$ be an open set such that $\overline{A}$ is compact. We will denote by $L^0(A)$, $L^\varrho(A)$, $E^\varrho(A)$, the corresponding spaces. We denote by $C(A)$ the space of all continuous and bounded real functions defined on A and by $C_u(A)$ the subset of $C(A)$ whose elements have a continuous extension to $\overline{A}$. It is easy to show that $C(A) \subset L^\varrho(A)$,whenever ϱ is monotone and finite. Indeed, for $\lambda > 0$ we have $\varrho[\lambda f] \leq \varrho[\lambda \|f\|_\infty e_0]$, and so, since $e_0 \in L^\varrho(A)$,we have $\lim_{\lambda \to 0^+} \varrho[\lambda f] = 0$, that is $f \in L^\varrho(A)$. Analogously, if ϱ is monotone and strongly finite, then $C(A) \subset E^\varrho(A)$.

We will denote by $C_c(G)$ the subspace of $C(G)$ comprising all the functions with compact support on G. If ϱ is monotone and finite, then $C_c(G) \subset L^\varrho(G)$, and if moreover ϱ is strongly finite, $C_c(G) \subset E^\varrho(G)$.

We have the following (see [38]).

Proposition 1. *Let ϱ be a monotone, strongly finite and absolutely continuous modular on $L^0(G)$. Then $\overline{C_c(G)} = L^\varrho(G)$. Moreover, if $A \subset G$ is an open set such that $\overline{A}$ is compact, we have $\overline{C_u(A)} = L^\varrho(A)$.*

3. A Korovkin theorem for the modular convergence

Let $A \subset G$ be an open set with compact closure and let $e_1, \dots e_m$ be m functions in $C_u(A)$ such that the following property (P) holds: there exist continuous functions $a_i \in C_u(A)$, $i = 1, \dots m$ such that the function

$$P_s(t) = \sum_{i=1}^{m} a_i(s) e_i(t), s, t \in \overline{A} \tag{3.1}$$

is positive and is equal to zero if and only if $s = t$.

Let $\mathbf{T} = (T_n)_{n \in \mathbb{N}}$ be a family of positive linear operators $T_n : \mathcal{D} \to L^0(A)$, where $C_u(A) \subset \mathcal{D} \subset L^0(A)$. Here, $\mathcal{D}$ is the domain of the operators T_n.

We will assume that the family $(T_n)_{n \in \mathbb{N}}$ satisfies the following property $(*)$: there exist a subset $X_{\mathbf{T}} \subset \mathcal{D} \cap L^\varrho(A)$ with $C_u(A) \subset X_{\mathbf{T}}$ and a constant $R > 0$ such that for every function $f \in X_{\mathbf{T}}$, we have $T_n f \in L^\varrho(A)$ and

$$\limsup_{n \to +\infty} \varrho[\lambda(T_n f)] \leq R \varrho[\lambda f]$$

for every $\lambda > 0$.

Note that if $T_n : \mathcal{D} \to L^0(A)$ are equi-continuous operators in $L^\varrho(A)$, i.e.

$$\varrho[\lambda T_n f] \leq R \varrho[\lambda f]$$

for an absolute constant $R > 0$, for every $\lambda > 0$ and for every $f \in \mathcal{D} \cap L^\varrho(A)$, then clearly we can take $X_{\mathbf{T}} = L^\varrho(A) \cap \mathcal{D}$.

In what follows, we will assume that

$$\lim_{n \to +\infty} T_n e_i = e_i, i = 1, \dots m \text{ modularly in } L^\varrho(A). \tag{3.2}$$

A first result concerns the space $C_u(A)$ (see [8, Lemma 4]).

Theorem 1. *Let ϱ be a finite, monotone and Q-quasi-convex modular. Let the assumptions (P) and (3.2) be satisfied. Then for every $f \in C_u(A)$, we have*

$$\lim_{n \to +\infty} T_n f = f \textit{ modularly in } L^\varrho(A).$$

Then, using Theorem 1 and the density result expressed by Proposition 1, the following result holds (see [8, Theorem 1]).

Theorem 2. *Let ϱ be a monotone, strongly finite, absolutely continuous and Q-quasi-convex modular on $L^0(A)$. Let $\mathbf{T} = (T_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators satisfying property $(*)$. Let the assumption (P) be satisfied. If*

$$\lim_{n\to+\infty} T_n e_i = e_i i = 1, \ldots, m, \text{ strongly in } L^\varrho(A),$$

then $\lim_{n\to+\infty} T_n f = f$, *modularly in* $L^\varrho(A)$ *for each* $f \in L^\varrho(A) \cap \mathcal{D}$ *such that* $f - C_u(A) \subset X_{\mathbf{T}}$.

Theorem 2 can be applied to several positive linear operators in functional spaces. As example, it can be applied to positive operators of the form

$$(S_n f)(x) := \sum_{k=0}^{r(n)} K_n(x, \nu_{n,k}) f(\nu_{n,k}) \quad (n \in \mathbb{N},\, x \in A) \quad (f \in L^\varrho(A)).$$

Here, $A \subset \mathbb{R}^N$ is a bounded open set, $r(n)$ is an increasing sequence of natural numbers, and $\Gamma_n = (\nu_{n,k})_{k=0,1,\ldots,r(n)} \subset A$, $\nu_{n,k} = (\nu^1_{n,k}, \ldots, \nu^N_{n,k})$ is a sequence of points such that their union is dense in A.

In this and in the next example, we set $e_0(t) = 1,\ e_i(t) = t_i, i = 1, 2, \ldots, N$ and $e_{N+1}(t) = |t|^2$, for every $t = (t_1, \ldots, t_N) \in A$. For details, see [8, Section 4].

Another example is given by linear integral operators with positive kernel of Mellin-type. Let us consider $A = [0, 1]^N$ and for any vectors $t = (t_1, \ldots, t_N), x = (x_1, \ldots, x_N) \in A$, we put $tx = (t_1x_1, \ldots, t_Nx_N)$. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of kernel functions $K_n : A \to \mathbb{R}^+_0$ such that

$$\int_A K_n(t)dt = 1 \text{ and } \int_A \frac{K_n(t)}{t_1 \cdots t_N} dt \le W$$

for every $n \in \mathbb{N}$, where W is an absolute constant.
For any function $f \in L^\varrho(A)$, we define the positive linear operator

$$(T_n f)(x) = \int_A K_n(t) f(tx) dt, x \in A.$$

In this instance, we can show that $L^\varrho(A) \subset \mathcal{D} = Dom\mathbf{T} = \bigcap_{n\in\mathbb{N}} DomT_n$, where $DomT_n$ is the subset of $L^0(A)$ on which $T_n f$ is well defined as a measurable function of $x \in A$. For details, see [8, Section 5].

4. Korovkin's theorem for filter modular convergence

Here, we report extensions of Korovkin's theorem in modular spaces when the convergence is taken in a generalized form, namely involving filters. We begin with some properties of the filters of $\mathbb{N}$.

A nonempty family $\mathcal{F}$ of subsets on $\mathbb{N}$ is called a *filter* of $\mathbb{N}$ iff $\emptyset \notin \mathcal{F},\ A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and $A \subset B$ we have $B \in \mathcal{F}$. A sequence $(x_n)_n \subset \mathbb{R}$ is said to be $\mathcal{F}$-convergent to $x \in \mathbb{R}$ iff for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - x| \le \varepsilon\}$ is an element of $\mathcal{F}$. We denote this by writing $x = (\mathcal{F}) \lim_n x_n$. We now introduce the filter versions of the limsup and liminf operations. Let $\underline{x} = (x_n)_n$ be a sequence in $\mathbb{R}$ and set

$$A_{\underline{x}} = \{a \in \mathbb{R} : \{n \in \mathbb{N} : x_n \ge a\} \notin \mathcal{F}\},\ B_{\underline{x}} = \{b \in \mathbb{R} : \{n \in \mathbb{N} : x_n \le b\} \notin \mathcal{F}\}.$$

The $\mathcal{F}$-limit superior and the $\mathcal{F}$-limit inferior of $(x_n)_n$ are defined by

(4.3) $$(\mathcal{F}) \limsup_n x_n = \begin{cases} \sup B_{\underline{x}} & \text{if } B_{\underline{x}} \ne \emptyset \\ -\infty & \text{if } B_{\underline{x}} = \emptyset \end{cases},\ (\mathcal{F}) \liminf_n x_n = \begin{cases} \inf A_{\underline{x}} & \text{if } A_{\underline{x}} \ne \emptyset \\ +\infty & \text{if } A_{\underline{x}} = \emptyset \end{cases},$$

respectively. For a related concept, see [22].

Among the examples of filters, we mention here the following two: the filter $\mathcal{F}_{\text{cofin}}$ of all subsets on $\mathbb{N}$ whose complement is finite, and the filter $\mathcal{F}_d$ associated with the statistical convergence, that is the set of all subsets of $\mathbb{N}$ whose asymptotic density is 1 (see e.g. [35]). In the first example, the notions of limsup and liminf coincide with the usual ones. The filter $\mathcal{F}_{\text{cofin}}$ is known as the *Fréchet filter*.

A filter $\mathcal{F}$ is said to be *free* if it contains the Fréchet filter. In the following, we assume that all the involved filters are free. We will use a uniformity $\mathcal{U} \subset 2^{G\times G}$ which generates the topology of the locally compact Hausdorff space G. Let $\mathcal{B}$ be the σ-algebra of all Borel subsets of G and let μ be a σ-finite and regular measure on $\mathcal{B}$. Here, we denote by $C_b(G)$ the subspace of $C(G)$ comprising all the bounded functions on G. As before, $C_c(G)$ will denote the subspace of $C_b(G)$ containing all the functions with compact support on G. We define now the notion of modular convergence in a modular space $L^\varrho(G)$ in the context of filter convergence. A sequence $(f_n)_n \subset L^\varrho(G)$ is said to be $\mathcal{F}$-modularly convergent to $f \in L^\varrho(G)$ if there exists $\lambda > 0$ such that

$$(\mathcal{F}) \lim_n \varrho(\lambda(f_n - f)) = 0.$$

Note that in case of the Fréchet filter, the filter modular convergence coincides with usual modular convergence. Also, with the same method, it is possible to introduce a notion of strong modular convergence in the context of filters.

In order to obtain an extension of the results of Section 3, let e_i, $i = 0, 1, \dots m$, be functions in $L^\varrho(G)$ such that (3.1) holds, in which we assume that it holds for every $s, t \in G$ and the following further conditions hold:

(P.1) $P_s(s) = 0$ for all $s \in G$,
(P.2) for every $U \in \mathcal{U}$ there is $\delta > 0$ such that for $s, t \in G$ such that $(s, t) \notin \mathcal{U}$ one has $P_s(t) \geq \delta$.

In [7, Example 4.1], there are described several examples of functions $P_s(t)$ satisfying the above assumptions. Moreover, we will modify slightly assumption $(*)$ simply replacing the space $C_u(A)$ by $C_b(G)$.

We have the following extension.

Theorem 3. *Let ϱ be a strongly finite, monotone and Q-quasi convex modular. Assume that the functions e_i and a_i, $i = 0, 1, \dots, m$, satisfy assumptions $(P.1)$ and $(P.2)$. Let $(T_n)_n$ be a sequence of positive linear operators satisfying the modified property $(*)$. If $T_n e_i$ is $\mathcal{F}$-modularly convergent to e_i for $i = 0, \dots, m$ in $L^\varrho(G)$, then $T_n f$ converges $\mathcal{F}$-modularly to f in $L^\varrho(G)$ for every $f \in C_c(G)$. If $T_n e_i$ is $\mathcal{F}$-strongly convergent to e_i, $i = 0, 1, \dots, m$ in $L^\varrho(G)$, then $T_n f$ is $\mathcal{F}$-strongly convergent to $f \in L^\varrho(G)$ for every $f \in C_c(G)$.*

The next theorem, which uses as a Lemma the previous theorem, extends also some previous Korovkin-type theorems in the setting of the statistical convergence, since the filter modular convergence is a generalization of the statistical convergence.

Theorem 4. *Let ϱ be a strongly finite, monotone, absolutely continuous and Q-quasi convex modular on $L^0(G)$. Let $(T_n)_n$ be a sequence of positive linear operators satisfying the modified property $(*)$. If $T_n e_i$ is $\mathcal{F}$-strongly convergent to e_i, $i = 0, 1, \dots, m$, in $L^\varrho(G)$, then $T_n f$ is $\mathcal{F}$-modularly convergent to f in $L^\varrho(G)$ for all $f \in L^\varrho(G) \cap \mathcal{D}$ such that $f - C_b(G) \subset X_{\mathbf{T}}$.*

Further extensions of Theorem 4 can be also obtained for not necessarily positive linear operators, in case of functions defined over bounded intervals $I \subset \mathbb{R}$ with certain regularity assumptions (see [4], [7]).

5. Double sequences of positive linear operators and Korovkin's theorem with respect to a generalized version of statistical convergence

Here, we report a generalized version of the Korovkin theorem in case of double sequences of positive linear operators, employing a generalized concept of statistical convergence which involves regular matrices and their submatrices. This theory started in [5] in which a kind of "triangular matrix statistical convergence" was introduced for double sequences of positive linear operators in the space of continuous functions over compact subsets of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. This notion represents an extension of the known A-statistical convergence, defined through a regular matrix A, extensively studied in several previous papers (see e.g. [4], [24], [27], [39]).

For what concerns modular function spaces, in [6] we introduce a generalization of the triangular A-statistical convergence. In the present section, we describe this new method.

Let $A = (a_{i,j})_{i,j}$ be a two-dimensional infinite matrix. Given a double sequence $x = (x_{i,j})_{i,j}$ of real numbers, set

$$(Ax)_i := \sum_{j=1}^{\infty} a_{i,j} x_{i,j},$$

provided that the series is convergent. We say that A is *regular* if it maps every convergent sequence into a convergent sequence with the same limit. We now recall the *Silverman-Toeplitz conditions*, which are a characterization of regular two-dimensional matrix transformations (see e.g. [4]). Here, $\Psi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is a fixed function.

(i) $\|A\| = \sup_{i \in \mathbb{N}} \sum_{j=1}^{\infty} |a_{i,j}| < \infty$,

(ii) $\lim_i a_{i,j} = 0$ for each $j \in \mathbb{N}$,

(iii) $\lim_i \sum_{j=1}^{\infty} a_{i,j} = 1$.

We say that A is a *summability matrix* iff it satisfies the following conditions:

$(A1)$ $\sum_{j \in \mathbb{N}, \Psi(i,j) \geq 0} a_{i,j} \leq 1$ for each $i \in \mathbb{N}$,

$(A2)$ $\lim_i \sum_{j \in \mathbb{N}, \Psi(i,j) \geq 0} a_{i,j} > 0$,

$(A3)$ $\lim_i a_{i,j} = 0$ for every $j \in \mathbb{N}$

(see also [17]).

Let $A = (a_{i,j})_{i,j}$ be a nonnegative regular matrix, $K \subset \mathbb{N}^2$, $K_i := \{j \in \mathbb{N}\colon (i,j) \in K, \Psi(i,j) \geq 0\}$ and let $|K_i|$ denote the cardinality of K_i, for each $i \in \mathbb{N}$. The Ψ-A-density of K is defined by

$$\delta_A^{\Psi}(K) := \lim_i \sum_{j \in K_i} a_{i,j},$$

provided that the limit on the right-hand side exists in $\mathbb{R}$.

It is not difficult to check that the Ψ-A-density satisfies the following properties:

j) $\delta_A^{\Psi}(\mathbb{N}^2) = 1$,

jj) if $H_1 \subset H_2$, then $\delta_A^{\Psi}(H_1) \leq \delta_A^{\Psi}(H_2)$,

jjj) $\delta_A^{\Psi}(\mathbb{N}^2 \setminus K) = 1 - \delta_A^{\Psi}(K)$, provided that K has a Ψ-A density.

Note that from j), jj) and jjj), it follows that the family

$$\mathcal{F}_A^{\Psi} := \{K \subset \mathbb{N}^2 : \delta_A^{\Psi}(\mathbb{N}^2 \setminus K) = 0\} \tag{5.4}$$

is a filter of $\mathbb{N}^2$ (Observe that the notion of (free) filter of $\mathbb{N}^2$ and the corresponding concept of filter convergence can be given analogously as those of filter of $\mathbb{N}$ and filter convergence presented in Section 4). In order to prove that $\mathcal{F}_A^\Psi$ is free, it is enough to see that, if $p, q \in \mathbb{N}$ and $K := \{(i,j) \in \mathbb{N}^2 \colon i \geq p, j \geq q\}$, then $\delta_A^\Psi(\mathbb{N}^2 \setminus K) = 0$. Indeed, we have $\mathbb{N}^2 \setminus K \subset K^{(1)} \cup K^{(2)}$, where

$$K^{(1)} = \bigcup_{j=1}^{q-1} (\mathbb{N} \times \{j\}), \quad K^{(2)} = \bigcup_{i=1}^{p-1} (\{i\} \times \mathbb{N}).$$

From $(A3)$, it follows that

$$\delta_A^\Psi(K^{(1)}) = \lim_i \sum_{j \in [1, q-1], \Psi(i,j)=0} a_{i,j} \leq \lim_i \sum_{j=1}^{q-1} a_{i,j} = 0.$$

Furthermore, note that $\delta_A^\Psi(K^{(2)}) = 0$, since $K_i^{(2)} = \emptyset$ for all $i \geq p$. Hence, $\delta_A^\Psi(\mathbb{N}^2 \setminus K) = 0$, that is the claim.

Let $A = (a_{i,j})_{i,j}$ be a nonnegative regular summability matrix. The double sequence $x = (x_{i,j})_{i,j}$ is said to be Ψ-*A-statistically convergent* to L if for any $\varepsilon > 0$ we get

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where $K_i(\varepsilon) = \{j \in \mathbb{N} \colon \Psi(i,j) \geq 0, |x_{i,j} - L| \geq \varepsilon\}$, and we write $st_A^\Psi\text{-}\lim_i x_{i,j} = L$.

By $st_A^\Psi\text{-}\limsup_i x_{i,j}$ and $st_A^\Psi\text{-}\liminf_i x_{i,j}$, we denote the quantities

$$\limsup_i \sum_{j \in K_i(\varepsilon)} a_{i,j} \quad \text{and} \quad \liminf_i \sum_{j \in K_i(\varepsilon)} a_{i,j},$$

respectively.

If we take $\Psi(i,j) = i - j$, we obtain the notion of triangular A-statistical convergence (see also [5]). Observe that, if $A = C_1$ is the Cesàro matrix, defined by setting

$$a_{i,j} := \begin{cases} \dfrac{1}{i} & \text{if } j \leq i, \\ 0 & \text{otherwise,} \end{cases}$$

then the Ψ-A-statistical convergence can be considered as a generalized concept of the classical statistical convergence.

The Ψ-density $\delta^\Psi(K)$ is defined by

$$\delta^\Psi(K) = \lim_i \frac{1}{i} |K_i|.$$

The double sequence $x = (x_{i,j})_{i,j}$ is said to be Ψ-statistically convergent to L if for each $\varepsilon > 0$ the set $K(\varepsilon) := \{(i,j) \in \mathbb{N}^2 \colon \Psi(i,j) \geq 0, |x_{i,j} - L| \geq \varepsilon\}$ has triangular density zero, and we write $st^\Psi\text{-}\lim_i x_{i,j} = L$.

We denote by st_A^Ψ the set of all Ψ-A-statistically convergent sequences.

Observe that, thanks to (5.4), filter convergence in $\mathbb{N}^2$ is weaker than Ψ-A-statistical convergence, but in general these two convergences are not equal (see also [17, Section 4]). Indeed, we claim that there is some filter $\mathcal{F}$ of $\mathbb{N}^2$ such that, for each summability matrix A, there exists a set $K \in \mathcal{F} \setminus \mathcal{F}_A^\Psi$. Pick arbitrarily a summability matrix $A = (a_{i,j})_{i,j}$. Thanks to $(A1)$ and $(A2)$, it is possible to find a real number $B_0 \in (0,1]$ and an infinite subset $S \subset \mathbb{N}$ with

$\sum_{j\in\mathbb{N},\Psi(i,j)\geq 0} a_{i,j} \geq B_0$ for all $i \in S$ (see [17]). Let S_1 and S_2 be two disjoint infinite subsets of S, with $S = S_1 \cup S_2$. Let $K := \{(i,1)\colon i \in \mathbb{N}\} \cup \{(i,j) : i \in S_1,\ \Psi(i,j) \geq 0\}$. Taking into account $(A3)$, we get

$$
\begin{aligned}
0 &\leq \liminf_{i\in\mathbb{N}\setminus S_1} \sum_{j\in\mathbb{N}:(i,j)\in K,\Psi(i,j)\geq 0} a_{i,j} \leq \limsup_{i\in\mathbb{N}\setminus S_1} \sum_{j\in\mathbb{N}:(i,j)\in K,\Psi(i,j)\geq 0} a_{i,j} \\
&\leq \limsup_i a_{i,1} = \lim_i a_{i,1} = 0. \qquad (5.5)
\end{aligned}
$$

Thus, all inequalities in (5.5) are equalities, and in particular it is

$$\lim_{i\in\mathbb{N}\setminus S_1} \sum_{j\in\mathbb{N}:(i,j)\in K,\Psi(i,j)\geq 0} a_{i,j} = 0. \qquad (5.6)$$

Since $\limsup_{i\in S_1} \sum_{j\in\mathbb{N}:(i,j)\in K,\Psi(i,j)\geq 0} a_{i,j} \geq B_0$, from this and (5.6), it follows that $\mathbb{N}^2 \setminus K \notin \mathcal{F}_A^{\Psi}$. From this and $(A2)$, it follows that $K \notin \mathcal{F}_A^{\Psi}$. So, any ultrafilter $\mathcal{F}$ of $\mathbb{N}$ contains at least a set not belonging to $\mathcal{F}_A^{\Psi}$, since it contains either K or $\mathbb{N}^2 \setminus K$. This gets the claim.

We now define the modular and strong convergences in the context of the Ψ-A-statistical convergence. In the following, G is a locally compact Hausdorff space satisfying the assumptions of Section 4, with the uniform structure $\mathcal{U} \subset 2^{G\times G}$.

We say that a double sequence $(f_{i,j})_{i,j}$ of functions in $L^{\varrho}(G)$ is st_A^{Ψ}-modularly convergent to $f \in L^{\varrho}(G)$ if there is a $\lambda > 0$ with

$$st_A^{\Psi}\text{-}\lim_i \varrho[\lambda(f_{i,j} - f)] = 0.$$

A double sequence $(f_{i,j})_{i,j}$ in $L^{\varrho}(G)$ is st_A^{Ψ}-strongly convergent to $f \in L^{\varrho}(G)$ if

$$st_A^{\Psi}\text{-}\lim_i \varrho[\lambda(f_{i,j} - f)] = 0$$

for every $\lambda > 0$.

For $\Psi(i,j) = i - j$, we obtain the corresponding notions for the "triangular" modular and strong convergences.

Let T be a double sequence of linear operators $T_{i,j} : \mathcal{D} \to L^0(G)$, $i,j \in \mathbb{N}$, with $C_b(G) \subset \mathcal{D} \subset L^0(G)$. Here, the set $\mathcal{D}$ is the domain of the operators $T_{i,j}$.

We introduce now the following extension of the property $(*)$ for double sequences of positive linear operators. We say that the double sequence T, together with the modular ϱ, satisfies the *property* $(*)$ if there exist a subset $X_T \subset \mathcal{D}\cap L^{\varrho}(G)$ with $C_b(G) \subset X_T$ and a positive real constant N with $T_{i,j}f \in L^{\varrho}(G)$ for any $f \in X_T$ and $i,j \in \mathbb{N}$, and $st_A^T\text{-}\limsup_i \varrho[\tau(T_{i,j}f)] \leq N\varrho[\tau f]$ for every $f \in X_T$ and $\tau > 0$.

The first Korovkin-type theorem in the present setting is the following (see [6]).

Theorem 5. *Let ϱ be a strongly finite, monotone and Q-quasi convex modular. Assume that e_r and a_r, $r = 0,\dots,m$, satisfy (P1) and (P2). Let $T_{i,j}$, $i,j \in \mathbb{N}$, be a double sequence of positive linear operators with property $(*)$. If $(T_{i,j}e_r)_{i,j}$ is st_A^{Ψ}-modularly convergent to e_r in $L^{\varrho}(G)$ for each $r = 0,\dots,m$, then $(T_{i,j}f)_{i,j}$ is st_A^{Ψ}-modularly convergent to f in $L^{\varrho}(G)$ for every $f \in C_c(G)$.*

If $(T_{i,j}e_r)_{i,j}$ is st_A^{Ψ}-strongly convergent to e_r, $r = 0,\dots,m$ in $L^{\varrho}(G)$, then $(T_{i,j}f)_{i,j}$ is st_A^{Ψ}-strongly convergent to f in $L^{\varrho}(G)$ for every $f \in C_c(G)$.

Employing Theorem 5, we can prove the following general Korovkin-type theorem in modular spaces (see [6]).

Theorem 6. *Let ϱ be a monotone, strongly finite, absolutely continuous and Q-quasi convex modular on $L^0(G)$, and $T_{i,j}$, $i,j \in \mathbb{N}$ be a double sequence of positive linear operators fulfilling $(*)$. If $(T_{i,j}e_r)_{i,j}$ is st^{Ψ}_A-strongly convergent to e_r, $r = 0, \ldots, m$ in $L^{\varrho}(G)$, then $(T_{i,j}e_r)_{i,j}$ is st^{Ψ}_A-modularly convergent to f in $L^{\varrho}(G)$ for every $f \in L^{\varrho}(G) \cap \mathcal{D}$ with $f - C_b(G) \subset X_T$, where $\mathcal{D}$ and X_T are as before.*

The theory developed in this section can be applied to two-dimensional Mellin-type integral operators for functions defined in compact intervals of $\mathbb{R}^2$. As example, let $L^{\varphi}(G)$ be an Orlicz space generated by the convex φ-function φ and $G = [0,1]^2$.

For every $(x_1, x_2) \in [0,1]^2$, let $e_0(x_1,x_2) = a_3(x_1,x_2) = 1$, $e_1(x_1,x_2) = x_1$, $e_2(x_1,x_2) = x_2$, $e_3(x_1,x_2) = a_0(x_1,x_2) = x_1^2 + x_2^2$, $a_1(x_1,x_2) = -2\,x_1$, $a_2(x_1,x_2) = -2\,x_2$. For each $i, j \in \mathbb{N}$, set $A_{i,j} = \left[\frac{1}{i}, 1\right] \times \left[\frac{1}{j}, 1\right]$,

$$c_{i,j} = \int_0^1 \int_0^1 t_1\, t_2 (t_1^2 + t_2^2)^{i+j} \chi_{A_{i,j}}(t_1,t_2)\, dt_1\, dt_2, \quad d_{i,j} = \frac{1}{c_{i,j}}. \tag{5.7}$$

For every $i, j \in \mathbb{N}$, t_1, $t_2 \in [0,1]$, define $K_{i,j}(t_1,t_2) = d_{i,j} t_1\, t_2 (t_1^2 + t_2^2)^{i+j} \chi_{A_{i,j}}(t_1,t_2)$. Moreover, let C_1 be the Cesàro matrix. Finally, let us consider the double sequences of operators defined by

$$(M_{i,j}f)(x_1,x_2) = \int_0^1 \int_0^1 K_{i,j}(t_1,t_2) f(t_1 x_1, t_2 x_2)\, dt_1\, dt_2, \quad f \in C([0,1]^2), \quad i,j \in \mathbb{N}.$$

It is proved that the operators $M_{i,j}$ satisfy all the assumptions of Theorem 5.

6. Korovkin theorems with respect to abstract convergence

In this section, we extend the results of the previous sections to abstract convergences, including filter and triangular matrix statistical convergences. In [17], Theorems 3, 4, 5 and 6 were generalized by considering an axiomatic abstract convergence, defined as follows (see also [14]).

Let $\mathcal{T}$ be the set of all real-valued sequences $(x_n)_n$. A *convergence* is a pair $(\mathcal{S}, \ell)$, where $\mathcal{S}$ is a linear subspace of $\mathcal{T}$ and $\ell : \mathcal{S} \to \mathbb{R}$ is a function, satisfying the following axioms.

- (a) $\ell((a_1\, x_n + a_2\, y_n)_n) = a_1\, \ell((x_n)_n) + a_2\, \ell((y_n)_n)$ for every pair of sequences $(x_n)_n$, $(y_n)_n \in \mathcal{S}$ and for each a_1, $a_2 \in \mathbb{R}$ (linearity).
- (b) If $(x_n)_n$, $(y_n)_n \in \mathcal{S}$ and $x_n \le y_n$ definitely, then $\ell((x_n)_n) \le \ell((y_n)_n)$ (monotonicity).
- (c) If $(x_n)_n$ satisfies $x_n = l$ definitely, then $(x_n)_n \in \mathcal{S}$ and $\ell((x_n)_n) = l$.
- (d) If $(x_n)_n \in \mathcal{S}$, then $(|x_n|)_n \in \mathcal{S}$ and $\ell((|x_n|)_n) = |\ell((x_n)_n)|$.
- (e) Given three sequences $(x_n)_n$, $(y_n)_n$, $(z_n)_n$, satisfying $(x_n)_n$, $(z_n)_n \in \mathcal{S}$, $\ell((x_n)_n) = \ell((z_n)_n)$ and $x_n \le y_n \le z_n$ definitely, then $(y_n)_n \in \mathcal{S}$.

Note that $\mathcal{S}$ is the space of all convergent sequences, and ℓ is the "limit" according to this approach.

We now give the axiomatic definition of the operators "limit superior" and "limit inferior" associated with a convergence $(\mathcal{S}, \ell)$.

Let $\mathcal{T}$, $\mathcal{S}$ be as above. We define two functions $\overline{\ell}$, $\underline{\ell} : \mathcal{T} \to \widetilde{\mathbb{R}}$, satisfying the following axioms:

- (f) If $(x_n)_n$, $(y_n)_n \in \mathcal{T}$, then $\underline{\ell}((x_n)_n) \le \overline{\ell}((x_n)_n)$ and $\overline{\ell}((x_n)_n) = -\underline{\ell}((-x_n)_n)$.
- (g) If $(x_n)_n \in \mathcal{T}$, then
  - (i) $\overline{\ell}((x_n + y_n)_n) \le \overline{\ell}((x_n)_n) + \overline{\ell}((y_n)_n)$ (subadditivity);
  - (ii) $\underline{\ell}((x_n + y_n)_n) \ge \underline{\ell}((x_n)_n) + \underline{\ell}((y_n)_n)$ (superadditivity).
- (h) If $(x_n)_n$, $(y_n)_n \in \mathcal{T}$ and $x_n \le y_n$ definitely, then $\overline{\ell}((x_n)_n) \le \overline{\ell}((y_n)_n)$ and $\underline{\ell}((x_n)_n) \le \underline{\ell}((y_n)_n)$ (monotonicity).

(j) A sequence $(x_n)_n \in \mathcal{T}$ belongs to $\mathcal{S}$ if and only if $\overline{\ell}((x_n)_n) = \underline{\ell}((x_n)_n)$.

It is not difficult to check that the $\mathcal{F}$-limit superior and the $\mathcal{F}$-limit inferior defined in (4.3) fulfill the above axioms (f)-(j) (see also [22, Theorems 3 and 4]).

Now, we define the modular and strong convergences in this setting.

A sequence $(f_n)_n$ of functions in $L^\rho(G)$ is (ℓ)-*modularly convergent* to $f \in L^\rho(G)$ if there is a positive real number λ such that

$$(\ell) \lim_n \rho[\lambda(f_n - f)] = 0.$$

A sequence $(f_n)_n$ in $L^\rho(G)$ is (ℓ)-*strongly convergent* to $f \in L^\rho(G)$ if

$$(\ell) \lim_n \rho[\lambda(f_n - f)] = 0 \text{ for all } \lambda > 0.$$

The Korovkin-type theorems obtained in the context of this abstract axiomatic convergence are the following.

Theorem 7. ([17, Theorem 3.2]) *Let ρ be a strongly finite, monotone and Q-quasi convex modular. Assume that e_r and a_r, $r = 0, \ldots, m$, satisfy* (P1) *and* (P2). *Let $(T_n)_n$ be a sequence of positive linear operators satisfying property $(*)$. If $(T_n e_r)_n$ is (ℓ)-modularly convergent to e_r in $L^\rho(G)$ for each $r = 0, \ldots, m$, then $(T_n f)_n$ is (ℓ)-modularly convergent to f in $L^\rho(G)$ for all $f \in C_c(G)$.*

If $(T_n e_r)_n$ is (ℓ)-strongly convergent to e_r, $r = 0, \ldots, m$ in $L^\rho(G)$, then $(T_n f)_n$ is (ℓ)-strongly convergent to f in $L^\rho(G)$ for every $f \in C_c(G)$.

Theorem 8. ([17, Theorem 3.3]) *Let ρ be a monotone, strongly finite, absolutely continuous and Q-quasi convex modular on $L^0(G)$, and $(T_n)_n$ be a sequence of positive linear operators satisfying $(*)$. If $(T_n e_r)_n$ is (ℓ)-strongly convergent to e_r, $r = 0, \ldots, m$ in $L^\rho(G)$, then $(T_n e_r)_n$ is (ℓ)-modularly convergent to f in $L^\rho(G)$ for every $f \in L^\rho(G) \cap \mathcal{D}$ with $f - C_b(G) \subset X_T$, where $\mathcal{D}$ and X_T are as above.*

Now, we give an example of convergence, which satisfies axioms (a)-(j) introduced in this section, but is not generated by any (free) filter (see also [7, Section 6]).

A sequence $(x_n)_n$ in $\mathbb{R}$ is *almost convergent* to $x_0 \in \mathbb{R}$ (shortly, $(A) \lim_n x_n = x_0$) if

$$\lim_n \frac{x_{m+1} + x_{m+2} + \ldots + x_{m+n}}{n} = x_0$$

uniformly with respect to $m \in \mathbb{N}$. It is not difficult to check that almost convergence satisfies axioms (a)-(j).

Let $\mathcal{F}$ be any fixed free filter of $\mathbb{N}$. A function $f : \mathbb{R} \to \mathbb{R}$ is *$\mathcal{F}$-continuous at* $x_0 \in \mathbb{R}$ if $(\mathcal{F}) \lim_n f(x_n) = f(x_0)$ whenever $(\mathcal{F}) \lim_n x_n = x_0$, and is *A-continuous at* $x_0 \in \mathbb{R}$ if $(A) \lim_n f(x_n) = f(x_0)$ whenever $(A) \lim_n x_n = x_0$. In [35, Proposition 3.3] it is shown that $\mathcal{F}$-continuity is equivalent to usual continuity, while in [19, Theorem 1] it is proved that every (A)-continuous function at any fixed point x_0 is linear. Thus, the concepts of (A)- and $\mathcal{F}$-continuity do not coincide, and therefore almost convergence is not generated by any free filter.

As a final remark, note that in [6, Section 4] and [17, Theorems 3.4 and 3.5] some results about the rate of modular convergence were also stated, using suitable moduli of continuity, when G is a metric space (G, d) satisfying a suitable assumption, which is naturally verified in every Euclidean space. Following an approach already used in [7], it is also possible to obtain a version of the previous theorems for not necessarily positive linear operators, in certain particular situations (see [6, Theorem 12] and [17, Theorem 4.3]).

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Research Article

Congruence and metaplectic covariance: rational biquadratic reciprocity and quantum entanglement

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ABSTRACT. The purpose of the paper is to elucidate the cyclotomographic applications of the coadjoint orbit methodology to the Legendre-Hilbert-Artin symbolic tower of class field theory in the sense of the number field theories of Chevalley, Hasse, Weil and Witt. The Witt arithmetics concludes with the law of rational biquadratic reciprocity and quantum entanglement.

Keywords: Third order principle of spinor triality, spaces of even and odd half-spinors, Hopf principal circle bundle, metaplectic Lie group $\mathrm{Mp}(2,\mathbb{R})$, the metaplectic coadjoint orbit model and spherical contact geometry, the first maxim of the geometric quantization principle, half-spinor Maslov index, Witt quartic groups, cyclotomography, Legendre-Hilbert-Artin symbolic tower of class field theory, valuation and module function, adéles and idéles, differential idéles, rational quantum entanglement, pure spinor invariants of octonion geometry, magnetic resonance tomography and angiography, symplectic spectroscopy at molecular level.

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Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

The law of quadratic reciprocity is one of the cornerstones of the classical theory of numbers. The theorem on primes in arithmetic progressions was a key ingredient in Adrien-Marie Legendre's (1752-1833) attempted proof of the law of quadratic reciprocity. He coined the term "reciprocity". The most penetrating *classical* approach of quadratic reciprocity which is suggestive of substantial generalizations is due to Carl Friedrich Gauß (1777-1855). His very first proof of the law of quadratic reciprocity by means of a remarkable induction argument over the primes was published in the treatise *Disquisitiones Arithmeticæ* of 1801. The concept of Gaussian sum for quadratic forms has included the laws of quadratic and biquadratic reciprocity into the field of constructive mathematical analysis.

In a letter to the astronomer Heinrich Wilhelm Matthias Olbers, Gauß wrote in 1805:

"This lack of sign has overcast everything else I have found, and since four years hardly a week has passed in which I did not make an attempt to no avail of resolving this knot. But all the brooding, all the searching was in vain, and each time I was forced to put down the pen in sorrow. Finally, a few days ago, I was successful - but not by my arduous search but only by the grace of God, as I would say. Like lightening strikes the riddle was solved; I myself would be unable to tell you the connection between what I knew before, in my last attempts - and the idea by which I finally succeeded. Curiously the solution of the problem now appears to be easier than many other results which have not cost me as

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many days as the present problem cost me years, and certainly no one will get any idea about the tight squeeze of feeling which besieged me for so a long time when I eventually present this matter."

The original form of the law of quadratic reciprocity that Gauß proved reads

$$\left(\frac{p}{q}\right) = \left(\frac{(-1)^{\frac{q-1}{2}}q}{p}\right)$$

for any distinct odd primes $\{p, q\}$. This equation makes already the quantum entanglement phenomenon apparent. However Gauß never explicitly employed the concept of group, but many group theoretical results are found in the treatise *Démonstration de quelques théorèmes concernants les périodes des classes de formes binaires du second degré* of 1876. The concept of symplectic Lie group $\mathrm{Sp}(2,\mathbb{R}) \cong \mathrm{SL}(2,\mathbb{R})$, its two-fold covering group which is formed by the metaplectic Lie group

$$\mathrm{Mp}(2,\mathbb{R}) \hookrightarrow \mathrm{U}\big(L^2(\mathbb{R}),\mathbb{C}\big)$$

and its strong operator action as a *deus ex machina* of the analytic theory of quadratic forms remained outside his way of mathematical reasoning. This has been done later on by the investigations of Carl Ludwig Siegel (1896-1981) ([39], [21]). The present paper offers a Galois cohomology approach to the metaplectic Schaar-Landsberg construction. It can be considered as an outgrowth of local class field theory and the third order principle of spinor triality which emphasizes impressively the extraordinary role of the finite place 2 of the prime field $\mathbb{Q}$ of rational numbers corresponding to the embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_2$ into the quasifactor $\mathbb{Q}_v$ of the topological ring of adéles $\mathbb{Q}_{\mathbb{A}}$ belonging to the associated 2-adic valuation v ([8], [38]).

The motivation of Gauß in seeking new proofs of the law of quadratic reciprocity was to develop methods for treating higher reciprocity laws. This can be read off the treatise *Theorematis fundamentalis in doctrina de residuis quadraticis demonstrationes et amplicationes novæ* of 1817 ([7]). The metaplectic group realization

$$\mathrm{Mp}(4,\mathbb{R}) \hookrightarrow \mathrm{U}\big(L^2(\mathbb{R}\oplus\mathbb{R}),\mathbb{C}\big)$$

suggests an extension to the law of biquadratic reciprocity on $\mathbb{C}\oplus\mathbb{C} \hookrightarrow \mathbb{O}$. In his second memoir on biquadratic residues, Gauß stated, without proof, the law of biquadratic reciprocity in 1832. Subsequently, Ferdinand Gotthold Max Eisenstein (1823-1852) published several proofs of the *mysteries of the higher arithmetic* in 1844.

The merit of the approach of Gauß and Eisenstein is that they pointed out the way of the laws of reciprocity of classical number theory to Artin's law of reciprocity and the duality determined by Hilbert's symbol. Hasse's law of reciprocity is in close connection with Artin's law. Actually, Helmut Hasse (1898-1979) opened the road jointly with Hensel to the work of Chevalley and Artin.

2. The Legendre-Hilbert-Artin symbolic tower

The ability to detect ultra-precisely the interaction of light and matter at the single-particle level represents a particular important application of the formalism of local class field theory to quantum optics and quantum information processing. In terms of class field theory over the local fields $\mathbb{R}$ and $\mathbb{C}$ of characteristic zero, ultra-precise ion clocks represent a type of factor-sets which are attached to the cyclic Galois extension of degree 2 of the groundfield

$$\mathbb{R} \cong \mathbb{Q}_\infty.$$

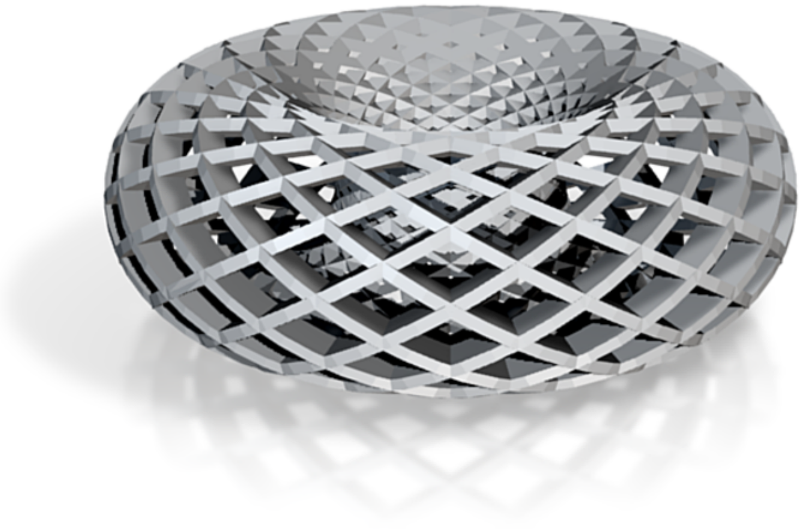

FIGURE 1. Cyclotomographic visualization of a two-fold covering of the two-dimensional torus $\mathbb{T}^2$. The leaves of the foliation define pairs of gradient controllable Villarceau circles. The fundamental group is $\pi_1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathrm{WQ}(\mathbb{Q}_2)$ to implement the law of rational biquadratic reciprocity by the spinor triality and the action of the Witt quartic group $\mathrm{WQ}(\mathbb{Q}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_2$.

According to Galois cohomology, it is sufficient to use a quantum entangled state to amplify the momentum an ion receives upon scattering a photon. In this vein, the following result of Galois cohomological quantum metrology arises:

Theorem 2.1. *Every equivalence class of simple central associative algebras over the groundfield $\mathbb{Q}_\infty \cong \mathbb{R}$ contains a cyclic algebra $\left[\mathbb{C} \,|\, \mathbb{R}; \{\chi, \theta\}\right]$ of non-trivial character χ of the Galois group $\mathcal{G}$ of $\mathbb{C}$ over $\mathbb{R}$ given by $\chi(\varsigma) = e^{\pi i} = -1$, and the coboundary factor $\theta \in \mathbb{R}^\times$ of quantum entanglement so that the mapping $\theta \rightsquigarrow \{\chi, \theta\}$ is a morphism of $\mathbb{R}^\times$ into the group $\mathrm{H}(\mathbb{R})$ of cyclic factor-classes of $\mathbb{R}$ with kernel formed by the group of norms of $C^\times$ of the cyclic Galois extension $C = \mathbb{C} \,|\, \mathbb{R}$ of ramification index 2 over $\mathbb{R}$ attached to the character $\chi \in \widehat{\mathcal{G}}$ of second order, and the coboundary factor θ which corresponds to the four-fold half-spinor Maslov index.*

The only non-trivial cyclic algebra over the groundfield $\mathbb{R} \cong \mathbb{Q}_\infty$ of this kind is the four-dimensional real division algebra of classical quaternions

$$\mathbb{H} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \left[\mathbb{C} \,|\, \mathbb{R}; \{\chi, -1\}\right]$$

with non-trivial character $\chi \in \widehat{\mathcal{G}}$ of second order, and non-trivial coboundary factor $\theta = -1 \in \mathbb{R}^\times_-$ induced by reflection of the norm image $N_{\mathbb{C}\,|\,\mathbb{R}}\left(\mathbb{C}^\times\right) = \mathbb{R}^\times_+$ which is a subgroup of $\mathbb{R}^\times$ of index 2. Corresponding to the Pauli spin matrices in $\mathrm{SU}(2, \mathbb{C})$, the vectors of the canonical basis of $\mathbb{H}$ satisfy the same relations as the matrices

$$\left\{\begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}, \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\right\}.$$

With exception of the matrix of the identity transformation with trace 2, the other matrices are traceless. The last matrix of the quadruple is the symplectic matrix

$$-J = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

with $J^{-1} = {}^tJ = -J \in \mathrm{Sp}(2, \mathbb{R}) \cong \mathrm{SL}(2, \mathbb{R})$. Referring to the nomenclature of the Steinberg symbolic calculus ([29], [30]), the real division algebra $\mathbb{H}$ takes the form

$$(+1, +1)_{\mathbb{R}} \cong (+1, -1)_{\mathbb{R}} \cong (-1, -1)_{\mathbb{R}} \cong (-1, +1)_{\mathbb{R}}$$

provided the symbols on right hand side represent a skew field, or equivalently, their multiplicative norm form is non-isotropic and therefore hyperbolic. In particular, the symbolic calculus involves the isomorphies

$$(a, b)_{\mathbb{R}} \cong \left(aa'^2, bb'^2\right)_{\mathbb{R}}$$

for $\{a, a', b, b'\} \subset \mathbb{R}^{\times}$.

In view of the third order principle of spinor triality, it is a remarkable observation that the non-zero *pure* quaternions in $\mathbb{H} \hookrightarrow \mathbb{O}$ are characterized by the fact that they are not belonging to the groundfield $\mathbb{R} \cong \mathbb{Q}_{\infty}$, but their squares are. Thus, the three-dimensional vector subspace of $\mathbb{H}$ consisting of the pure quaternions takes the form

$$\left\{q \in (a, b)_{\mathbb{R}} \mid q^2 \in \mathbb{R},\ q \notin \mathbb{R}^{\times}\right\}.$$

In accordance with the theorem of Alexander Merkurjev concerning the quaternion symbol ([23], [32]), the central simple algebra $\mathbb{H}$ gives rise to the:

Corollary 2.1. *The coboundary factor $\theta \in \mathbb{R}^{\times}$ of quantum entanglement attached to the quaternion quantization procedure is given by the quaternion symbol*

$$(\bullet, \bullet')_{\mathbb{R}} : \mathbb{R}^{\times} \times \mathbb{R}^{\times} \longrightarrow \mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}_2 \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}.$$

It is universal in the sense that every other quaternion symbol can be factored over it in the third Galois cohomology $\mathrm{H}^3(\mathbb{R}, \mathbb{Z}_2)$ of the third order principle of spinor triality.

A standard functorial argument establishes the fact that a universal symbol is uniquely determined up to a unique isomorphism. Since the half-spinor norm form of the metaplectic coadjoint orbit model determines completely its quaternion algebra, the following result arises in the rational case:

Theorem 2.2. *Up to an isometry the quaternion symbol $(-1. -1)_{\mathbb{Q}_2}$ defines the only non-split quaternion algebra over the commutative field of 2-adic rational numbers $\mathbb{Q}_2$.*

Notice that the quasifactor $\mathbb{Q}_2$ of the adélic ring $\mathbb{Q}_{\mathbb{A}}$ is either the *completion* $\mathbb{Q}_v$ of the prime field $\mathbb{Q}$ of rational numbers with respect to the ultrametric distance associated to the 2-adic valuation

$$v = |\bullet|_2$$

or $\mathbb{Q}_v$ denotes for $v = \infty$ the completion $\mathbb{Q}_{\infty} = \mathbb{R}$. Of course, the topology associated to the ultrametric distance is locally compact and not discrete. The closure $\mathbb{Z}_2$ of the rational integers $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in $\mathbb{Q}_2$ defines a compact neighborhood of 0 in $\mathbb{Q}_2$. The algebraic theory of quadratic forms originated in a seminal paper of 1937 by Ernst Witt (1911-1991) ([30], [32]). The Witt quartic group $\mathrm{WQ}(\mathbb{Q}_2)$ will help to detect the pure half-spinors acting by rational quantum entanglement according to the third order principle of spinor triality.

3. The symplectic spinor pair of groups $\big(\mathrm{Mp}(2,\mathbb{R}), \mathrm{Br}(\mathbb{R})\big)$

Let $\widehat{\mathcal{N}}$ denote the unitary dual of the (2+1)-dimensional real unipotent Heisenberg Lie group $\mathcal{N}$ consisting of the equivalence classes of irreducible unitary linear representations of $\mathcal{N}$ in complex Hilbert spaces ([15]). The geometric model of $\widehat{\mathcal{N}}$ derives in spherical contact geometry from the first maxim of the geometric quantization principle:

"Never look at the orbits of the adjoint action - rather always look at the orbits of the coadjoint *action".*

Actually, it is the quantization maxim which permits the transition from number theory to symplectic spectroscopy at molecular level in terms of the third order principle of spinor triality.

Let the subgroup $G \hookrightarrow \mathrm{U}\big(L^2(\mathbb{R}), \mathbb{C}\big)$ be the covariance group associated with $\widehat{\mathcal{N}}$. Then, the exact sequence

$$\{1\} \longrightarrow \mathbb{T} \longrightarrow G \xrightarrow{\nu} \mathrm{Sp}(2,\mathbb{R}) \longrightarrow \{1\}$$

arises, where $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong \mathrm{U}(1,\mathbb{C})$ denotes the one-dimensional compact torus group. The mapping $\nu : G \longrightarrow \mathrm{Sp}(2,\mathbb{R})$ is a Lie group homomorphism and the differential ν_* a real Lie algebra isomorphism. Concerning the inverse Fourier transforms $\mathcal{F}_\mathbb{R}$ and $\bar{\mathcal{F}}_\mathbb{R}$, it is important to note the spin echo projections

$$\nu\big(\mathcal{F}_\mathbb{R}\big) = J, \quad \nu\big(\bar{\mathcal{F}}_\mathbb{R}\big) = J^{-1}.$$

The restriction $\mu = \nu|\mathrm{Mp}(2,\mathbb{R})$ defines the exact sequence

$$\{1\} \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Mp}(2,\mathbb{R}) \xrightarrow{\mu} \mathrm{Sp}(2,\mathbb{R}) \longrightarrow \{1\}.$$

Taking into account the half-spinor Maslov index $-\frac{1}{2}$ of $\mathrm{Mp}(2,\mathbb{R})$, the 2-cocycle $e^{-\frac{\pi i}{4}}$ of $\mathrm{Sp}(2,\mathbb{R})$, the bijective differential μ_* of $\mu : \mathrm{Mp}(2,\mathbb{R}) \longrightarrow \mathrm{Sp}(2,\mathbb{R})$ and the symplectic spinor configuration inside the real dual vector space $\mathfrak{Lie}(\mathcal{N})^*$ of the real nilpotent Heisenberg Lie algebra $\mathfrak{Lie}(\mathcal{N})$, the universal quaternion symbol $(\bullet, \bullet')_\mathbb{R}$ affords by character composition of the entangled ingredients the metaplectic Schaar-Landsberg construction of the third Galois cohomology $\mathrm{H}^3(\mathbb{R}, \mathbb{Z}_2)$ of the third order principle of spinor triality ([5], [8]).

Theorem 3.3. *For any integral numbers p and $q \geq 1$, the metaplectically entangled Schaar-Landsberg identity*

$$\frac{e^{-\frac{\pi i}{8}}}{\sqrt{p}} \sum_{0 \leq m \leq p-1} e^{2\pi i m^2 \frac{q}{p}} = \frac{e^{\frac{\pi i}{8}}}{\sqrt{2q}} \sum_{0 \leq n \leq 2q-1} e^{-\frac{\pi i n^2}{2} \frac{p}{q}}$$

holds. The opposite 22.5° phase factors on both sides of the equality derive from the Galois cohomological meaning of the 2-cocycle as an octonionic half-spinor root of unity. The action of the symplectic spinor can be visualized by the Picard tori of the Hopf principal circle bundle.

The metaplectically entangled Schaar-Landsberg identity was first discovered in 1850 by Mathieu Schaar (1817-1867) who proved it using the Poisson summation formula, and proceeded to derive from it the law of quadratic reciprocity. In 1893, Georg Landsberg (1865-1912),

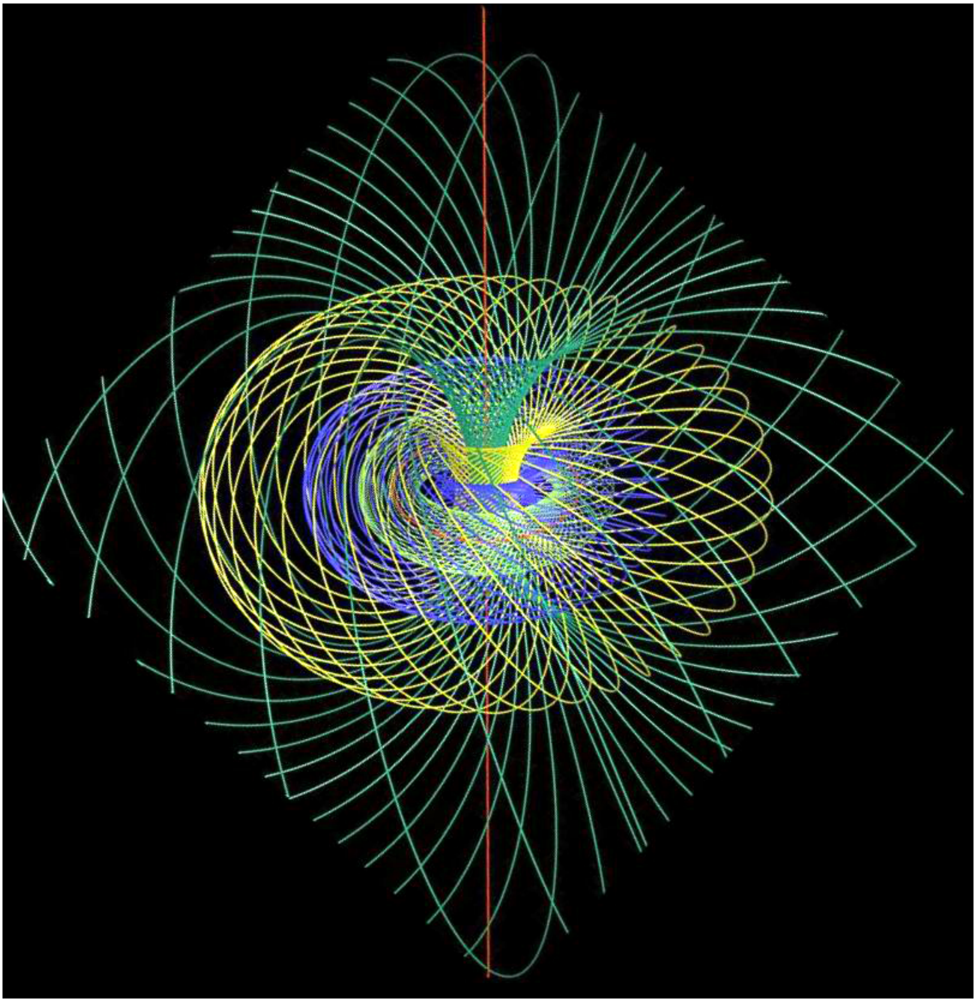

FIGURE 2. Cyclotomography of the Riemann-Roch structure on the torus $\mathbb{T}^2$: The Hopf principal circle bundle visualizing the cyclic factor-sets of the first Galois cohomology $\mathrm{H}^1\big(\mathbb{R}, \mathrm{Spin}(3, \mathbb{R})\big)$ in the context of the third cohomology $\mathrm{H}^3\big(\mathbb{R}, \mathbb{Z}_2\big)$ which represents the third order principle of spinor triality and implements the law of quadratic reciprocity. In terms of the linear theory of theta functions with $\mathbb{T}$-valued quadratic characters, the Hopf fibration receives its Riemann-Roch structure on tori from the cyclic extension of the Poisson summation formula to the adélic ring of the field $\mathbb{R}$. For the period sublattice $\Lambda_1 \hookrightarrow \mathbb{C}$, the coordinatization by the affine $\mathbb{C}$-basis $\big\{1, \wp_{\Lambda_1}, \wp'_{\Lambda_1}\big\}$ defines a projective embedding of the two-dimensional Picard torus $(\mathrm{mod}\,\Lambda_1)$ by a non-singular elliptic curve $E(\mathbb{C})$ in the complex projective plane $\mathbb{P}_2(\mathbb{C}) \hookrightarrow \mathbb{P}_{\mathbb{C}}\big(\mathfrak{Lie}(\mathcal{N})^*\big)$ inside the projectivized dual of the real nilpotent Heisenberg Lie algebra $\mathfrak{Lie}(\mathcal{N})$.

who was the coauthor of Hensel's treatise on algebraic functions of one variable and was unaware of Schaar's work, rediscovered a slightly more general version of the relation. The arithmetician Kurt Hensel (1861-1941) created the p-adic methodology which places the theory of quadratic reciprocity in the natural frame of modern number theory ([40]). Then, its adélic language permits to unify the various techniques of significantly stepping forward. In Section 6 *infra*, an interpretation of the metaplectically entangled Schaar-Landsberg construction will be given in terms of theta differential idéles of module 1.

The evaluation of the basic Gaussian quadratic sum is performed by putting $q = 1$:

$$\sum_{0\leq n\leq p-1} e^{\frac{2\pi i n^2}{p}} = \sqrt{\frac{p}{2}}\, e^{\frac{\pi i}{4}}\left(1+e^{-\frac{\pi i}{2}}\right) = \sqrt{p}\, i^{(\frac{p-1}{2})^2}.$$

Due to the congruence modulo 4

$$\left(\frac{pq-1}{2}\right)^2 - \left(\frac{p-1}{2}\right)^2 - \left(\frac{q-1}{2}\right)^2 = \frac{(p-1)(q-1)}{2},$$

the law of quadratic reciprocity for any distinct odd primes $\{p, q\}$ takes the standard form in terms of the Legendre quadratic symbol

$$\left(\frac{p}{q}\right) = \left(\frac{p}{q}\right)_2 = \begin{cases} \;\;1 & : \; p \text{ is a quadratic residue mod } q \\ -1 & : \; p \text{ is a quadratic nonresidue mod } q \end{cases}$$

over the field $\mathbb{Q}$ of rational numbers

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}$$

as the product of multiplicative characters of $\mathbb{Z}_p^\times$ and $\mathbb{Z}_q^\times$, respectively. The Legendre quadratic symbol $\left(\frac{\bullet}{\bullet'}\right) = \left(\frac{\bullet}{\bullet'}\right)_2$, invariably so written by most classical authors is defined whenever the lower variable is a prime and the upper variable an integer prime to the lower variable, which admits the value + 1 when the upper variable is a quadratic residue modulo the lower variable, and - 1 otherwise ([2]). The quantum entangled Schaar-Landsberg identity can also used to establish the so-called supplementary theorem which includes the first supplement

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$$

and the second supplement

$$\left(\frac{+2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$$

on the quadratic residue behaviour of -1 and + 2, respectively. The multiplicativity of the Legendre quadratic symbol implies the equation

$$\left(\frac{-1}{q}\right)^{\frac{p-1}{2}}\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$$

so that the first supplement affords the law of quadratic reciprocity. Tensoring rational quaternion algebras over the field of p-adic numbers $\mathbb{Q}_p$ including $\mathbb{Q}_\infty \cong \mathbb{R}$ provides by means of the quaternion symbol $(\bullet, \bullet')_{\mathbb{Z}}$ the Hilbert reciprocity law which generalizes the quadratic reciprocity law. The global reciprocity law which relies on normic aspects of class field theory forms a monumental generalization of the classical law of quadratic reciprocity.

In terms of class field theory, the classical quadratic reciprocity law is equivalent to the assertion that an extension of the field of rational numbers $\mathbb{Q}$ admits a Hecke L-function. Over the years, however, the L-functions have been expunged from class field theory and have been replaced by an algebraic edifice based on Galois cohomology. The avoidance of the concepts of cohomology theory gives rise to an artificial flair. In the context of harmonic analysis on Abelian groups, the universal quaternion symbol $(\bullet, \bullet')_{\mathbb{R}}$ allows to circumvent the standard reference of the famous functional equation for Jacobi's theta function as a special case of the Poisson summation formula and its cyclic Eisenstein extension ([38]).

Remark 3.1. *The induction arguments of Gauß to prove the* Theorema Aureum *are closely related to the calculations of the Witt quartic group isomorphisms by splitting off metabolic subspaces* ([32]). *Using the conventions of basic number theory* ([38], [32])*, the Witt geometric method yields*

$$\mathrm{WQ}(\mathbb{F}_2) \cong \mathbb{Z}_2\,, \quad \mathrm{WQ}(2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_8\,, \quad \mathrm{WQ}(\mathbb{Q}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_2$$

in terms of the half-spinor Maslov index associated as an octonionic root of unity

$$e^{\frac{\pi i}{4}} = \frac{1 + e^{\frac{\pi i}{2}}}{\sqrt{2}} = \frac{1+i}{\sqrt{2}}$$

with the generic coadjoint orbit $\mathcal{O}_1 \in \mathfrak{Lie}(\mathcal{N})^*/\mathrm{CoAd}(\mathcal{N})$. *Thus, the Witt quartic groups* $\mathrm{WQ}(2) \hookrightarrow \mathrm{WQ}(\mathbb{Q}_2)$ *are adapted to the third order principle of spinor triality by the projection onto the direct factor relative to the direct product representation*

$$\mathrm{WQ}(\mathbb{Q}) \longrightarrow \mathrm{WQ}(\mathbb{Q}_2) \longrightarrow \mathbb{Z}_8$$

which actually is a Gaussian sum. The invariant of the order $2^3 = \dim_{\mathbb{R}} \mathbb{O}$ *of the alternative and biassociative division* $\mathbb{R}$*-algebra of octonions* $\mathbb{O}$ *on the Cayley-Dickson scale*

$$\mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{O}$$

with continuous module function $\mathrm{mod}_{\mathbb{R}}$ *of respective Haar measures*

$$|\bullet|,\ |\bullet'|^2,\ |\bullet''|^4,\ |\bullet'''|^8$$

is due to Johann Peter Gustav Lejeune Dirichlet (1805-1859) ([31], [35])*. Moreover, the module function of the 2-adic case affords the identity*

$$\mathrm{mod}_{\mathbb{Q}_2}(2) = \frac{1}{2}$$

which defines a $\mathrm{WQ}(\mathbb{Q}_2)$*-invariant model of the plane coadjoint orbit* $\mathcal{O}_{\frac{1}{2}} \in \mathfrak{Lie}(\mathcal{N})^*/\mathrm{CoAd}(\mathcal{N})$ *in concordance with Theorem 1* supra*. Due to the third order principle of spinor triality, the coadjoint orbit model* $\mathfrak{Lie}(\mathcal{N})^*/\mathrm{CoAd}(\mathcal{N})$ *of the unitary dual* $\widehat{\mathcal{N}}$ *of the (2+1)-dimensional real unipotent Heisenberg Lie group* $\mathcal{N}$ *implements the non-invasive imaging modality of gradient controlled clinical magnetic resonance tomography and angiography of radiological diagnostics in current clinical use* ([1], [4], [33])*. To include dynamical phenomena into the spherical contact geometry of* $\mathfrak{Lie}(\mathcal{N})^*/\mathrm{CoAd}(\mathcal{N})$*, functional magnetic resonance permits the modalities of diffusion and perfusion magnetic resonance imaging* ([18], [19], [25], [36], [37])*.*

Just as in the earlier studies of Legendre, Gauß distinguishes eight separate cases according to the different nature of the individual primes in order to present a natural crystallization of the special cases that had been discovered earlier by Leonhard Euler (1707-1783). It was Dirichlet's modification of the Gaussian proof which completed in 1854 the induction procedure over the upper and lower sequences of prime variables $\{P, Q\}$ that are enumerated by the

FIGURE 3. Gradient controlled clinical magnetic resonance tomography: Non-invasive image of the musculoskeletal anatomy of a normal right shoulder joint (*Articulatio glenohumeralis*). The supraspinatus, infraspinatus and teres minor muscles and tendons are shown. They all attach to the greater tuberosity. The tendons and rotator cuff muscles act to stabilize the shoulder joint during movements.

Witt quartic group $\mathrm{WQ}(\mathbb{Z}) \cong \mathrm{WQ}(\mathbb{R}) \cong \mathbb{Z}$ in the idélic product

$$\left(\frac{P}{Q}\right)\left(\frac{Q}{P}\right) = \prod_{\{p,q\}} \left(\frac{p}{q}\right)\left(\frac{q}{p}\right).$$

Finally, the inductive conclusion over the primes reads

$$\left(\frac{P}{Q}\right)\left(\frac{Q}{P}\right) = (-1)^{\frac{(P-1)(Q-1)}{4}}$$

as desired ([2], [11]). The seventh proof of Gauß is based on cyclotomy which anticipates the idélic procedure of pasting together local cyclic data into a global object. Originally these global

FIGURE 4. Gradient controlled clinical magnetic resonance angiography: Intercranial time-of-flight non-invasively performed on a healthy male adult acquired at a 3.0 Tesla magnetic field density. Maximum intensity projection in the axial direction demonstrates excellent blood to background contrast and depiction of small peripheral intercranial vessels. The high resolution of the angiogram can be enhanced by electrocardiogram gating and the administratio of a contrast agent. Higher static magnetic field density permits to increase the spatial resolution in time-of-flight and the temporal resolution in dynamic magnetic resonance angiography.

objects were known as ideal elements or *idéles*. In modern terms partly due to the seminal work of Chevalley and Weil, the cyclotomic extension to cyclotomography combines the factor-sets of quadratic periods of the cylotomographic equation

$$\frac{X^{p-1}-1}{X-1}=0$$

for positive odd primes p with cyclic Galois extensions. Gauß's seventh proof, which has been admired by Dirichlet, introduces theta type data of a Frobenius decomposition of quadratic periods by means of the inverse affine linear tomographic gradient mapping

$$X \rightsquigarrow 2\,X+1=Y$$

of metaplectically invariant quadratic periods within the generic coadjoint orbit $\mathcal{O}_1 \in \mathfrak{Lie}(\mathcal{N})^*/\mathrm{CoAd}(\mathcal{N}$ ([21]). The discriminant-congruence $\bmod\ q$ of the cylotomographic equation transforms into the solvability condition in original form of Gauß

$$\left(\frac{(-1)^{\frac{p-1}{2}}p}{q}\right)=+1$$

for positive odd primes p and q, or into the condition

$$(-1)^{\frac{p-1}{2}\frac{q-1}{2}}p^{\frac{q-1}{2}} \equiv +1 \pmod q.$$

However, if the solvability condition $\bmod\ q$ does not admit integral roots, then

$$(-1)^{\frac{p-1}{2}\frac{q-1}{2}}p^{\frac{q-1}{2}} \equiv -1 \pmod q$$

holds. By comparison with David Hilbert's Theorem 90 of group cohomology theory ([24], [26], [29]), applied to the quaternionic Galois extension

$$\mathbb{H}=(1,1)_{\mathbb{R}}=\left(\frac{-1,-1}{\mathbb{R}}\right)$$

of degree 2, the law of quadratic reciprocity with its supplements follows. In the same vein, the law of quadratic reciprocity is a consequence of the statement that all quadratic fields are contained in cyclotomic fields. More general, the celebrated Kronecker-Weber theorem states that every Abelian field extension of the rational numbers $\mathbb{Q}$ extends to a subfield of the cyclotomic field, so expressible in terms of roots of unity. In other words, a maximal Abelian extension of the field $\mathbb{Q}$ is generated by the torsion points of the action of the ring of integers

$$\mathbb{Z} \ni m \rightsquigarrow w^m \in \mathbb{C}^{\times}$$

on $w \in \mathbb{C}^{\times}$ ([10], [27]).

The exponents of the primitive roots modulo p separated within the primitive root power difference of quadratic residues and quadratic non-residues

$$Y_1-Y_2=i^{(\frac{p-1}{2})^2}\sqrt{p}$$

provides the Gaussian theta series

$$\left(\frac{p}{q}\right)(Y_1-Y_2)=\sum_{0\le n\le q-1}\left(\frac{n}{q}\right)e^{\frac{2\pi i p}{q}\,n^2}$$

and the $\mathrm{Mp}(2,\mathbb{R})$ invariant square

$$(Y_1-Y_2)^2=(-1)^{\frac{p-1}{2}}\,p$$

so that the quadratic pairing $\left((Y_1 - Y_2)^2, Y_1 - Y_2\right)$ permits to derive the law of quadratic reciprocity by virtue of the identity

$$i^{(\frac{pq-1}{2})^2}\sqrt{pq} = \left(\frac{p}{q}\right)\left(\frac{q}{p}\right) i^{(\frac{p-1}{2})^2+(\frac{q-1}{2})^2}\sqrt{p}\,\sqrt{q}$$

and in accordance with the third order principle of spinor triality. In accordance with the half-spinor Maslov index, the law of quadratic reciprocity implies for the Legendre quadratic symbols

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$$

if either $p \equiv 1 \pmod 4$ or $q \equiv 1 \pmod 4$, and

$$\left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right)$$

if both $p \equiv 3 \pmod 4$ and $q \equiv 3 \pmod 4$.

The fourth and seventh proofs of Gauß are able to open a route to the cohomological aspect of the law of quadratic reciprocity as indicated earlier in connection with the half-spinor Maslov index which appeared in the context of the two-fold Brauer group $\mathrm{Br}_2(\mathbb{R}) \cong \mathrm{Br}(\mathbb{R})$ of order 2. Nevertheless, the half-spinor Maslov index of the metaplectically entangled Schaar-Landsberg demonstrates that the central extension $\mathrm{Mp}(2,\mathbb{R})$ of the symplectic Lie group $\mathrm{Sp}(2,\mathbb{R})$ has nothing lost of its "mysterious" character as a *deus ex machina* of quantum field theory.

The $\mathbb{R}$-linear independence of the generators of the Brauer group $\mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}_2$ gives rise to:

Theorem 3.4. *The cyclotomographic procedure of the proof of the law of quadratic reciprocity implements the trivial first cohomologies*

$$\mathrm{H}^1(\mathbb{C}, \mathbb{Z}_2) = \{0\},$$

and correspondingly

$$\mathrm{H}^1(\mathbb{C}^\times, \mathbb{Z}_2) = \{1\}.$$

It is known that the Hilbert symbol over the local field of p-adic numbers $\mathbb{Q}_p$ and denoted by

$$\left(\frac{a,b}{p}\right)$$

is completely determined once its values are known, first for all rational integers $\{a, b\}$ that are prime to p, and secondly for all rational integers a that are prime to p with $b = p$. For an odd prime number p and rational integers $\{a, b\}$ prime to p, one obtains the equations over $\mathbb{Q}_p$

$$\left(\frac{a,b}{p}\right) = 1, \quad \left(\frac{a,p}{p}\right) = \left(\frac{a}{p}\right).$$

In the context of global class field theory, the Hilbert symbol determines a duality between the quotient group of the quadratic reciprocity law and itself by means of which it can be identified with its own dual. The Hilbert symbol over 2-adic numbers $\mathbb{Z}_2 \cong \mathcal{G}$ of rational integers $\{a, b\}$ prime to $p = 2$ yields

$$\left(\frac{a,b}{2}\right) = (-1)^{\frac{(a-1)(b-1)}{4}}, \quad \left(\frac{a,2}{2}\right) = (-1)^{\frac{a^2-1}{8}}.$$

The Hilbert reciprocity law adopts the product form

$$\prod_{n} \left(\frac{p, q}{n} \right) = 1, \tag{3.1}$$

where n runs through all prime places including ∞ with $\left(\frac{\bullet,\bullet'}{\infty}\right) = 1$. For any prime number n distinct from $\{p, q, 2\}$, it follows $\left(\frac{p,q}{n}\right) = 1$. Hence,

$$\left(\frac{p, q}{p} \right) \left(\frac{p, q}{q} \right) = \left(\frac{p, q}{2} \right).$$

The equation implies the classical law of quadratic reciprocity including the two supplement theorems that are obtained from the product equations

$$\prod_{n} \left(\frac{-1, p}{n} \right) = 1, \quad \prod_{n} \left(\frac{+2, p}{n} \right) = 1.$$

The duality between the quotient group of the power reciprocity law and itself gives rise to the equivalence of the power reciprocity law and the classical quadratic reciprocity law based on the Abelian field extension by the formalism of local class field theory over the field of rational numbers

$$\mathbb{Q}\left(\sqrt{(-1)^{\frac{p-1}{2}} p} \right) | \mathbb{Q}$$

and Galois group $\mathcal{G} \cong \mathbb{Z}_2$, where p is a positive odd prime ([6], [10], [14], [38]); the field extension of $\mathbb{Q}$ is only ramified at $p > 2$. The generalization of the power residue symbol to the Artin symbol and the closely related Hasse's law of reciprocity is regarded as the central result in global class field theory. In the Abelian case, the Artin symbol coincides with the Frobenius symbol.

Theorem 3.5. *The classical law of quadratic reciprocity*

$$1 = (-1)^{\frac{(p-1)(q-1)}{4}} \left(\frac{p}{q} \right) \left(\frac{q}{p} \right)$$

follows for any distinct odd primes p and $q \neq 2$ from purely local data.

In his fundamental paper [39], Weil wrote in 1964: *Contrary to its appearances, the proof of the law of quadratic reciprocity exposed above does not differ in substance of the classical proof in terms of theta functions and Gaussian sums.*

The Artin symbol gives rise to the idélic group epimorphism $J_{\mathbb{Q}} \longrightarrow \mathcal{G}$ with normic kernel in the associated multiplicative idélic vector group $J_{\mathbb{Q}}$ and image of the vector $(1, \ldots, 1, q, 1, \ldots) \in J_{\mathbb{Q}}$, where the qth place is in the component at pth valuation. The image of the vector $(-1, 1, 1, \ldots) \in J_{\mathbb{Q}}$, where - 1 is in the component at ∞, is given by the Legendre symbol

$$(-1)^{\frac{p-1}{2}} = \left(\frac{-1}{p} \right).$$

For $q = 2$, the supplementary identity

$$1 = \left(\frac{2}{p} \right) (-1)^{\frac{p^2-1}{8}}$$

is a consequence of the fact that $(-1)^{\frac{p-1}{2}} p \equiv 1 \pmod 8$ if and only if $p \equiv \pm 1 \pmod 8$.

It is known from the formalism of global class field theory that the locally compact topological group $J_{\mathbb{Q}}$ contains the discrete subgroup $\mathbb{Q}^{\times}$. Algebraically, the idéles of the field of rational numbers $\mathbb{Q}$ can be regarded as a subset of the adéles under the natural injection. As pastings together of local cyclic data, the idéles form the units of the topological ring of adéles $\mathbb{Q}_{\mathbb{A}}$, which is the locally compact adélic ring equipped with the restricted direct product topology and the discrete subring $\mathbb{Q}$ under the diagonal embedding. The duality theory of idéles and adéles endowed with *their own* restricted direct product topologies represent a natural *global* extension of the harmonic analysis and Schwartz-Bruhat distribution theory on the adélic ring $\mathbb{Q}_{\mathbb{A}}$ ([38], [11]).

As has been emphasized by Weil in his studies of Siegel's work on quadratic forms, a complicated mathematical theory is not simplified by its specialization, but by neither superficial nor artificial generalizations ([40]). A generalization of this kind is given by the concept of idélization. The idéles are defined in terms of all of the places of a number field K as a finite extension of the field $\mathbb{Q}$. Because the idéles of K carry global information about K in terms of local information about K at each of its places, they are a successful implementation of the local-global principle, which is a recurring theme in quantum field theory and modern number theory, whereby harmonic analysis on the unipotent Heisenberg Lie group $\mathcal{N}$ and its companions have been seen to play an increasingly important role.

4. Cyclotomographic extensions

In his two-part treatise on the theory of biquadratic residues, entitled *Theoria residuorum biquadraticorum* of 1828, Gauß claims that the theory of quadratic residues had been brought to such a state of perfection that nothing more could be wished. On the other hands, *"the theory of cubic and biquadratic residues is by far more difficult ... the previously accepted principles of arithmetic are in no way sufficient for the foundations of a general theory, that rather such a theory necessarily demands that to a certain extent the domain of higher arithmetic needs to be endlessly enlarged ..."* In modern language, Gauß is calling for the algebraic theory of spinors and Clifford algebras which culminates in the principle of spinor triality which is due to Élie Cartan (1865-1951). An elegant metaplectically invariant spinor version is due to Claude Chevalley (1909-1984) ([5], [8]).

To derive the law of biquadratic reciprocity in the Euclidean ring $\mathbb{Z}[i] \hookrightarrow \mathbb{C}$ of Gaussian integers with units $\{-1, -i, 1, i\}$ by use of the symplectic machinery, the (4 + 1)-dimensional real unipotent Heisenberg Lie group $\mathcal{N}_2$ and real nilpotent Heisenberg Lie algebra $\mathfrak{Lie}(\mathcal{N}_2)$ yields the identity

$$\nu(\mathcal{F}_{\mathbb{R}\oplus\mathbb{R}}) = J_4,$$

where the inclusion

$$J_4 = \begin{pmatrix} 0 & -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{R})$$

holds. If

$$N : w \rightsquigarrow w\bar{w} = |w|^2$$

denotes the lattice norm in $\mathbb{Z}[i]$, the half-spinor Maslov index -1 of $\mathrm{Mp}(4, \mathbb{R})$ and relatively prime primary elements $\{\rho, \sigma\}$ of $\mathbb{Z}[i]$ give rise to the general law of biquadratic reciprocity, which adopts in terms of the quartic Legendre symbols $\left(\frac{\bullet}{\bullet'}\right)_4$ the following form

$$\left(\frac{\rho}{\sigma}\right)_4 \left(\frac{\sigma}{\rho}\right)_4 = (-1)^{\frac{N\rho-1}{4}\frac{N\sigma-1}{4}}.$$

In the context of biquadratic reciprocity, the equation $\left(\frac{\rho}{\sigma}\right)_4 = 1$ holds if and only if the congruence $X^4 \equiv \rho$ modulo σ is solvable in the ring $\mathbb{Z}[i]$ of Gaussian integers, and $\left(\frac{\rho}{\sigma}\right)_4 = -1, -i$ or i otherwise. Thus

Theorem 4.6. *The metaplectic coadjoint orbit model* $\mathfrak{Lie}(\mathcal{N}_2)^*/\mathrm{CoAd}(\mathcal{N}_2)$ *of nilpotent harmonic analysis implements the law of biquadratic reciprocity.*

A closely related approach to the law of cubic reciprocity by way of the third order principle of spinor triality proceeds by the ring $\mathbb{Z}[\omega] \hookrightarrow \mathbb{C}$, where

$$\omega = e^{\frac{2\pi i}{3}} = \frac{1}{2}(-1+\sqrt{-3}), \quad \bar{\omega} = e^{-\frac{2\pi i}{3}} = \omega^2 = \frac{1}{2}(-1-\sqrt{-3})$$

and $1+\omega+\omega^2 = 0$. The units $\{1, -1, \omega, -\omega, \omega^2, -\omega^2\}$ of $\mathbb{Z}[\omega]$ form a cyclic group of order 3. If the rational integer $p \in \mathbb{Z}$ satisfies the congruence $p \equiv 1 \pmod 3$, then $p = \rho\bar{\rho}$, where $\rho \in \mathbb{Z}[\omega]$ is prime in $\mathbb{Z}[\omega]$. If $p \equiv 2 \pmod 3$, then p is prime in $\mathbb{Z}[\omega]$. The half-spinor isomorphism of the triality principle can be applied to the residue characters of order three $\left(\frac{\bullet}{\bullet'}\right)_3$ or cubic residue characters which play the same role in the theory of cubic residues as the Legendre symbols play in the theory of quadratic residues. The behaviour of cubic residue characters under complex conjugation reads

$$\overline{\left(\frac{\bullet}{\bullet'}\right)_3} = \left(\frac{\bar{\bullet}}{\bar{\bullet}'}\right)_3 = \left(\frac{\bullet}{\bullet'}\right)_3^2 = \left(\frac{\bullet^2}{\bullet'}\right)_3.$$

A decomposition of the third power Gaussian sums yields:

Theorem 4.7. *For primary prime elements* $\{\rho_1, \rho_2\} \hookrightarrow \mathbb{Z}[\omega]$ *with* $N\rho_1 \neq 3, N\rho_2 \neq 3$, *and* $N\rho_1 \neq N\rho_2$ *the order three cubic residue character equation*

$$\left(\frac{\rho_1}{\rho_2}\right)_3 = \left(\frac{\rho_2}{\rho_1}\right)_3.$$

In terms of quantum optics, the cubic residue character equation embodies the quantum entanglement concept of spherical contact geometry.

Remark 4.2. *Galois cohomology on the real dual* $\mathfrak{Lie}(\mathcal{N})^*$ *of the (2 + 1)-dimensional real nilpotent Heisenberg Lie algebra* $\mathfrak{Lie}(\mathcal{N})$ *gives rise to an imaginative background of the multiplicative group* $\mathrm{H}(\mathbb{R})$ *of cyclic factor-classes by the quotient of cyclic factor-sets of degree 2*

$$\frac{\text{covariant factor} - \text{sets}}{\text{coboundary factor} - \text{sets}}$$

of the groundfield $\mathbb{R}$ *([38], [14]). On smooth differentiable manifolds, the de Rham cohomology is based on the quotient*

$$\frac{\text{closed differential forms}}{\text{exact differential forms}}$$

to define the de Rham cohomology groups in terms of the coboundaries of closed differential forms. It opens the way to the Kronecker diophantine approximation in number theory via symplectic spinors as symbols of the Fourier integral operators associated with metaplectic mappings ([20], [12]).

The quaternions represent a quantum field theoretic analogue of the Foucault spherical pendulum device with an open-book foliation of spin echo-stabilized, symplectic swing-planes ([34]). Using a single molecular ion confined in a laser cooled cavity of a linear trap, interferometry experiments permit to ultrasensitively detect the frequency and phase displacement of the pendulum's quantum-enhanced dynamical states which are excited by a sequence of laser-beam driven bichromatic sideband pulses. A calibrated displacement is implemented by exposing the ion to an electric field oscillating at the trapping frequency. The displacement amplitudes of the two-level qubits are in correspondence to the reduced norm of the quaternions.

5. Rational biquadratic reciprocity

It is known that the field of rational numbers $\mathbb{Q}$ has one infinite place corresponding to the embedding

$$\mathbb{Q} \hookrightarrow \mathbb{Q}_\infty = \mathbb{R};$$

as mentioned *supra* this place is denoted by ∞. The finite places of $\mathbb{Q}$ are in bijective correspondence with the rational primes, with which they will be identified, the place p corresponding to the embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p \hookrightarrow \mathbb{Q}_\infty$. Due to these embeddings, the metaplectic coadjoint orbit model $\mathfrak{Lie}(\mathcal{N}_2)^*/\mathrm{CoAd}(\mathcal{N}_2)$ of the unitary dual $\widehat{\mathcal{N}_2}$ of the (4 + 1)-dimensional real unipotent Heisenberg Lie group $\mathcal{N}_2 \hookrightarrow \mathrm{SL}(5, \mathbb{R})$ is able to derive by means of quantum entanglement the law of rational biquadratic reciprocity. This law answers the following problem: If distinct primes $p \equiv 1 \pmod 4$ and $q \equiv 1 \pmod 4$ are given so that p is a fourth power modulo q, establish necessary and sufficient conditions that q is a fourth power modulo p ([3]). Actually, the spinor triality principle leads to a number theoretic interpretation of the second Kepplerian law of planetary dynamics through its spherical contact geometry.

The multiplicative group $\mathbb{Z}_p^\times$ admits a unique subgroup of order $\frac{p-1}{4}$ consisting of the residues of fourth powers of integers. Let $\left(\frac{\rho}{p}\right)_4$ denote the biquadratic residue character defined by an irreducible $\rho \in \mathbb{Z}[i]$ dividing p. Then, $\left(\frac{\rho}{q}\right)_4 = 1$ if and only if the quartic congruence

$$X^4 \equiv q \pmod p$$

admits a solution $X \in \mathbb{Z} \hookrightarrow \mathbb{Z}[i]$.

Theorem 5.8. *For the quadratic residue character assume $\left(\frac{p}{q}\right) = 1$ the law of rational biquadratic reciprocity*

$$\left(\frac{\rho}{q}\right)_4 \left(\frac{\sigma}{p}\right)_4 = (-1)^{\frac{q-1}{4}} \left(\frac{p \wedge q}{q}\right)$$

holds in terms of the Witt invariant -1 of the symbol of norm $\bullet\bar{\bullet}$ of the four-dimensional involutive central simple $\mathbb{R}$-algebra of quaternions $\mathbb{H} \cong \mathbb{C} \oplus \mathbb{C}$ under quantum entangled cyclotomographic spinor coordinates.

In the context of the spherical contact geometry of the real dual $\mathfrak{Lie}(\mathcal{N})^*$, the group isomorphisms

$$\mathrm{Spin}(4, \mathbb{R}) \cong \mathrm{Spin}(3, \mathbb{R}) \times \mathrm{Spin}(3, \mathbb{R})$$

and

$$\mathrm{Spin}(3, \mathbb{R}) \cong \mathrm{SU}(2, \mathbb{C}) \cong \mathrm{SL}(1, \mathbb{H}) \cong \mathrm{Aut}(\mathbb{H})$$

emphasize the spinor character of the law of rational biquadratic reciprocity. In view of the fact that there is no algebra of quaternions over the field $\mathbb{C}$, the quantum optical phenomenon of entanglement is attached to the local field $\mathbb{R}$ and the locally compact connected $\mathbb{R}$-field $\mathbb{H}$ of classical quaternions. The quaternion group Q_8 with cyclic center $\mathbb{Z}_8$ represents a non-Abelian 2-group.

6. Coherent theta differential idéles

Similarly to the theory of zeta-functions of number theory, the metaplectic Schaar-Landsberg construction depends essentially on the concept of Fourier transforms of the metaplectic coadjoint orbit model $\mathfrak{Lie}(\mathcal{N})^*/\mathrm{CoAd}(\mathcal{N})$ of the unitary dual $\widehat{\mathcal{N}}$ of the (2 + 1)-dimensional real unipotent Heisenberg Lie group $\mathcal{N} \hookrightarrow \mathrm{SL}(3, \mathbb{R})$.

Let E denote a vector space of finite dimension over an $\mathbb{A}$-field k and topological ring $k_{\mathbb{A}}$ of adéles of k in the sense of number theory ([38]). A Haar measure on the tensor product $E_{\mathbb{A}} = E \otimes_k k_{\mathbb{A}}$ can be defined by choosing a Haar measure α_v on the tensor product $E_v = E \otimes_k k_v$ with symplectic basis E_v° for each place v so that $\alpha_v\left(E_v^\circ\right) = 1$ for almost all v. Then, the product measure

$$\alpha = \prod_v \alpha_v$$

and its dual are coherent measures. The measure α on $E_{\mathbb{A}}$ for which $\alpha(E_{\mathbb{A}}/E) = 1$ is known as the Tamagawa measure on $E_{\mathbb{A}}$. Its dual is the Tamagawa measure on $E_{\mathbb{A}}^*$.

The field of rational numbers $k = \mathbb{Q}$ admits maximal order $\mathbb{Z}$ and discriminant 1, $k_v = \mathbb{R} = \mathbb{Q}_\infty$ for each place v, $E_{\mathbb{R}} = E \otimes_{\mathbb{Q}} \mathbb{R}$, and the generic flat coadjoint orbit

$$\mathcal{O}_1 = E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = E_{\mathbb{C}}$$

inside the foliation $\mathfrak{Lie}(\mathcal{N})^*/\mathrm{CoAd}(\mathcal{N})$ of the unitary dual $\widehat{\mathcal{N}}$; the one-dimensional center of the real nilpotent Heisenberg Lie algebra $\mathfrak{Lie}(\mathcal{N})$ is able to parametrize $\widehat{\mathcal{N}}$.

Theorem 6.9. *The embedding of the metaplectic coadjoint orbit model $\mathfrak{Lie}(\mathcal{N})^*/\mathrm{CoAd}(\mathcal{N})$ into the octonionic triality principle of spinor algebra in terms of the coherent dual Tamagawa measure on the dual group $\mathbb{Z} \times \mathbb{Z}$ of the compact two-dimensional torus group $\mathbb{T} \times \mathbb{T} = \mathbb{T}^2 \hookrightarrow \mathbb{H}$ of the Hopf principal circle bundle with unitary theta characters attached to the rectangular lattice $\Lambda_1 = \mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathrm{WQ}(\mathbb{Q}_2)$ as collection $J_{\mathbb{Q}} \in \mathbb{Q}_{\mathbb{A}}$ of second order differential idéles of module $|J_{\mathbb{Q}}|_{\mathbb{A}} = 1$ on $E_{\mathbb{C}} = \mathcal{O}_1$ affords the metaplectic Schaar-Landsberg identity.*

The concepts of coherent measure and metaplectic entanglement underline the connection of number theory to the laser realization of spherical contact geometry in the field of quantum optics ([28]).

7. Entanglement of the rational half-spinor norm group

Every rational number is a p-adic number, whereby p is a rational prime. A direct calculation of the half-spinor norm with formal power series yields:

Theorem 7.10. *The half-spinor norm group $\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2}$ which is realized by the law of rational biquadratic reciprocity consists of eight elements represented by {1,3,5,7,2,6,10,14}.*

The ensuing multiplication table for the square classes of $\mathbb{Q}_2^\times$ reads ([32]):

	1	3	5	7	2	6	10	14
1	1	3	5	7	2	6	10	14
3		1	7	5	6	2	14	10
5			1	3	10	14	2	6
7				1	14	10	6	2
2					1	3	5	7
6						1	7	5
10							1	3
14								1

Theorem 7.11. *In terms of the Witt quartic group* $\mathrm{WQ}(\mathbb{Q}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_2$*, the symmetries of the spinor principle of triality are represented by the generators*

$$< -1, 2 >, \quad < 1 >, \quad < -1, 5 > .$$

Since the Witt quartic group $\mathrm{WQ}(\mathbb{Q}_2)$ contains 32 elements, the square class $< 1, 7 >$ of two-dimensional quadratic forms over $\mathbb{Q}_2$ endows number theoretically the generic flat coadjoint orbit $\mathcal{O}_1$ inside the tomographic foliation $\mathfrak{Lie}(\mathcal{N})^*/\mathrm{CoAd}(\mathcal{N})$ of quantum field theory with the structure of a hyperbolic plane ([9]).

Corollary 7.2. *The action of* $\mathbb{Z}_2 \times \mathbb{Z}_2 \hookrightarrow \mathrm{WQ}(\mathbb{Q}_2)$ *generates the phenomenon of rational quantum entanglement on the two-dimensional hyperbolic space over* $\mathcal{O}_1 \in \mathfrak{Lie}(\mathcal{N})^*/\mathrm{CoAd}(\mathcal{N})$ *in terms of the Klein four-group* $\mathfrak{V}_4$.

Thus, the Witt arithmetics admits unexpected applications to the area of quantum field theory.

8. By way of conclusion

When Hermann Weyl (1885-1955), one of the pioneers in introducing non-commutative harmonic analysis of the unipotent Heisenberg Lie group $\mathcal{N}$ into quantum mechanics, wrote his book "The classical groups" in 1939, he overlooked the natural occurrence of the Heisenberg group and the compact, triality conformally triangulated, homogeneous Heisenberg nilmanifold $\Lambda\backslash\mathcal{N}$, exploitation of which opens the gateway to results which one feels Weyl and his former assistant Brauer, the authors of the paper 1935 paper on "Spinors in n Dimensions", would have liked very much. The symbolic calculus of the spherical contact geometry of symplectic spectroscopy at molecular level can be considered as a heritage of Weyl and Brauer ([16], [17]).

Brauer was a former doctoral student of Issai Schur (1875-1911) who supervised his dissertation devoted to the representations of the rotation group by groups of linear transformations. Brauer and Weyl gave a much simpler presentation over Cartan's theory of spinors, based on the close ties with the structures of associative Clifford/Graßmann algebra models. It prepared the portal of the algebraic theory of spinors to the fields of de Rham cohomology theory and supersymmetry ([5], [13]). Brauer's earlier results have opened the way for his definition of the Abelian group of classes of central simple algebras over a commutative groundfield k, today called the Brauer group, and denoted by $\mathrm{Br}(k)$ ([38], [32]). Two central simple algebras are said to belong to the same class if the division algebras associated with them by Wedderburn's fundamental theorem are isomorphic. The isomorphy of $\mathrm{Br}(k)$ with the equivalence classes of factor-sets over k, along with other results of importance for the study of cyclic field extensions such as David Hilbert's Theorem 90, led to the highly elaborate machinery known today as Galois cohomology of local class field theory ([14]). Together with the Stone-von Neumann-Segal theorem, the Brauer group $\mathrm{Br}(\mathbb{R})$ and the twofold Brauer group $\mathrm{Br}_2(\mathbb{R})$ give rise to an implementation of the basic concepts of quantum entanglement and spin echo, respectively, as outlined in Sections 3 and 7 *supra*.

A summary of the paper can be given by a quotation of Weyl whose formulation is in his lucid style that only to him was available:
"The problems of Mathematics are not problems in a vacuum. There pulses in them the life of ideas which realize themselves in concreto *through our human endeavours in our historical existence, but forming an indissoluble whole transcending any particular science."*

The geometric quantization principle mentioned above fits to this philosophy.

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Research Article

Heun equations and combinatorial identities

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ABSTRACT. Heun functions are important for many applications in mathematics, physics and in thus in interdisciplinary phenomena modelling. They satisfy second order differential equations and are usually represented by power series. Closed forms and simpler polynomial representations are useful. Therefore, we study and derive closed forms for several families of Heun functions related to classical entropies. By comparing two expressions of the same Heun function, we get several combinatorial identities generalizing some classical ones.

Keywords: Heun functions, entropies, combinatorial identities.

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Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

Consider the general Heun equation (see, e.g., [15], [8], [9] and the references therein)

$$u''(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right)u'(x) + \frac{\alpha\beta x - q}{x(x-1)(x-a)}u(x) = 0, \tag{1.1}$$

where $a \notin \{0,1\}$, $\gamma \notin \{0,-1,-2,\dots\}$ and $\alpha+\beta+1 = \gamma+\delta+\epsilon$. Its solution $u(x)$ normalized by the condition $u(0) = 1$ is called the *(local) Heun function* and is denoted by $Hl(a,q;\alpha,\beta;\gamma,\delta;x)$.

The confluent Heun equation is

$$u''(x) + \left(4p + \frac{\gamma}{x} + \frac{\delta}{x-1}\right)u'(x) + \frac{4p\alpha x - \sigma}{x(x-1)}u(x) = 0, \tag{1.2}$$

where $p \neq 0$. The solution $u(x)$ normalized by $u(0) = 1$ is called the *confluent Heun function* and is denoted by $HC(p,\gamma,\delta,\alpha,\sigma;x)$.

It was proved in [14] that

$$Hl\left(\frac{1}{2}, -n; -2n, 1; 1, 1; x\right) = \sum_{k=0}^{n}\left(\binom{n}{k}x^k(1-x)^{n-k}\right)^2, \tag{1.3}$$

$$Hl\left(\frac{1}{2}, n; 2n, 1; 1, 1; -x\right) = \sum_{k=0}^{\infty}\left(\binom{n+k-1}{k}x^k(1+x)^{-n-k}\right)^2, \tag{1.4}$$

$$HC\left(n, 1, 0, \frac{1}{2}, 2n; x\right) = \sum_{k=0}^{\infty}\left(e^{-nx}\frac{(nx)^k}{k!}\right)^2. \tag{1.5}$$

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More general results, providing closed forms of the functions $Hl\left(\frac{1}{2}, -2n\theta; -2n, 2\theta; \gamma, \gamma; x\right)$ and $Hl\left(\frac{1}{2}, 2n\theta; 2n, 2\theta; \gamma, \gamma; x\right)$, and explicit expressions for some confluent Heun functions can be found in [4].

In this paper, we give closed forms for several families of Heun functions and confluent Heun functions, extending (1.3), (1.4) and (1.5). Basic tools will be the results of [7] and [16] concerning the derivatives of Heun functions, respectively confluent Heun functions; see also [13] and [4].

By comparing two expressions of the same Heun function, we get several combinatorial identities; very particular forms of them can be traced in the classical book [5]. Recently, Ulrich Abel and Georg Arends gave in [1] purely combinatorial proofs of some similar identities presented in [2].

It is well known that the Heun functions and the Heun equations have important applications in Physics; see, e.g., [6]. Let us mention that the families of (confluent) Heun functions investigated in this paper are naturally related to some classical entropies: see [13], [14], [4], [3], [12].

Throughout the paper, we shall use the notation

$$(x)_0 := 1, \quad (x)_k := x(x+1)\dots(x+k-1), \quad k \geq 1,$$

$$a_{nj} := 4^{-n}\binom{2j}{j}\binom{2n-2j}{n-j}, \tag{1.6}$$

$$r_{nj} := \binom{n}{j}^{-1} a_{nj}. \tag{1.7}$$

2. Heun functions

Let $\alpha\beta \neq 0$. As a consequence of the results of [7], we have (see [4, Prop. 1] and [4, (14)]):

$$Hl\left(\frac{1}{2}, \frac{1}{2}(\alpha+2)(\beta+2); \alpha+2, \beta+2; \gamma+1, \gamma+1; x\right) = \frac{\gamma}{\alpha\beta}(1-2x)^{-1}\frac{d}{dx}Hl\left(\frac{1}{2}, \frac{1}{2}\alpha\beta; \alpha, \beta; \gamma, \gamma; x\right).$$

From [4, (6)], [4, (22)] and (1.6), we obtain

$$Hl\left(\frac{1}{2}, -n; -2n, 1; 1, 1; x\right) = \sum_{j=0}^{n} a_{nj}(1-2x)^{2j}. \tag{2.8}$$

Theorem 2.1. *Let $0 \leq m \leq n$. Then*

$$\begin{aligned} &Hl\left(\frac{1}{2}, (2m+1)(m-n); 2(m-n), 2m+1; m+1, m+1; x\right) \\ =&4^m\binom{n}{m}^{-1}\binom{2m}{m}^{-1}\sum_{j=0}^{n-m}\binom{m+j}{m}a_{n,m+j}(1-2x)^{2j} \\ =&\sum_{j=0}^{n-m}4^j\binom{n-m}{j}\frac{(m+1/2)_j}{(m+1)_j}(x^2-x)^j. \end{aligned} \tag{2.9}$$

Proof. We shall prove the first equality by induction with respect to m. For $m = 0$, it follows from (2.8). Suppose that it is valid for a certain $m < n$. Then, (2) implies

$$\begin{aligned}
&Hl\left(\frac{1}{2}, \frac{1}{2}(2m+3)(m+1-n); 2(m+1-n), 2m+3; m+2, m+2; x\right)\\
=&\frac{(m+1)(1-2x)^{-1}}{2(m-n)(2m+1)}\frac{d}{dx}Hl\left(\frac{1}{2}, (2m+1)(m-n); 2(m-n), 2m+1; m+1, m+1; x\right)\\
=&\frac{(m+1)(1-2x)^{-1}}{2(m-n)(2m+1)}4^m\binom{n}{m}^{-1}\binom{2m}{m}^{-1}\sum_{i=1}^{n-m}\binom{m+i}{m}a_{n,m+i}(-4i)(1-2x)^{2i-1}\\
=&4^{m+1}\binom{n}{m+1}^{-1}\binom{2m+2}{m+1}^{-1}\sum_{j=0}^{n-m-1}\binom{m+1+j}{m+1}a_{n,m+1+j}(1-2x)^{2j},
\end{aligned}$$

and so the desired equality is true for $m+1$; this finishes the proof by induction.
In order to prove that the first member and the last member of (2.9) are equal, it suffices to use [4, Th. 1] with $\gamma = m+1$, $\theta = m + \frac{1}{2}$, and n replaced by $n - m$. □

Corollary 2.1. *Let $0 \le i \le n-m$, $0 \le j \le n-m$. Then*

$$\sum_{j=i}^{n-m}(-1)^{j-i}\binom{n-m}{j}\frac{(m+1/2)_j}{(m+1)_j}\binom{j}{i} = 4^m\binom{n}{m}^{-1}\binom{2m}{m}^{-1}\binom{m+i}{m}a_{n,m+i} \tag{2.10}$$

and

$$\sum_{i=j}^{n-m}\binom{m+i}{m}\binom{i}{j}a_{n,m+i} = 4^{-m}\binom{n}{m}\binom{2m}{m}\frac{(m+1/2)_j}{(m+1)_j}\binom{n-m}{j}. \tag{2.11}$$

Proof. It suffices to combine the last equality in (2.9) with

$$(x^2-x)^j = 4^{-j}\left((1-2x)^2-1\right)^j,$$

respectively with

$$(1-2x)^{2j} = \left(1+4(x^2-x)\right)^j.$$

□

Example 2.1. *For $i = m = 0$, (2.10) reduces to*

$$\sum_{j=0}^{n}\left(-\frac{1}{4}\right)^j\binom{n}{j}\binom{2j}{j} = 4^{-n}\binom{2n}{n}, \tag{2.12}$$

which is (3.85) in [5].
For $j = m = 0$, (2.11) becomes

$$\sum_{i=0}^{n}\binom{2i}{i}\binom{2n-2i}{n-i} = 4^n, \tag{2.13}$$

which is (3.90) in [5].
From [4, (7)], [4, (23)] and (1.6), we know that

$$Hl\left(\frac{1}{2}, n+1; 2n+2, 1; 1, 1; x\right) = \sum_{j=0}^{n}a_{nj}(1-2x)^{2j-2n-1}. \tag{2.14}$$

Theorem 2.2. *For $m \geq 0$, we have*

$$Hl\left(\frac{1}{2}, (2m+1)(m+n+1); 2(m+n+1), 2m+1; m+1, m+1; x\right)$$
$$=\binom{n+m}{n}^{-1}\sum_{j=0}^{n}\binom{2n+2m-2j}{2m}\binom{n+m-j}{m}^{-1}a_{nj}(1-2x)^{2j-2n-2m-1}$$
$$=(1-2x)^{-2n-2m-1}\sum_{j=0}^{n}4^j\binom{n}{j}\frac{(1/2)_j}{(m+1)_j}(x^2-x)^j.$$

Proof. As in the proof of Theorem 2.1, the first equality can be proved by induction with respect to m, if we use (2.14) and (2). The equality of the first member and the last member follows from [4, Cor. 2] by choosing $\gamma = m+1$, $\theta = m+1/2$, and replacing n by $n+m+1$. □

Corollary 2.2. *Let $0 \leq i \leq n$, $0 \leq j \leq n$. Then*

$$\sum_{j=i}^{n}(-1)^{j-i}\binom{n}{j}\frac{(1/2)_j}{(m+1)_j}\binom{j}{i} = \binom{2n+2m-2i}{2m}\binom{n+m}{n}^{-1}\binom{n+m-i}{m}^{-1}a_{ni} \tag{2.15}$$

and

$$\sum_{i=j}^{n}\binom{2n+2m-2i}{2m}\binom{n+m-i}{m}^{-1}\binom{i}{j}a_{ni} = \binom{n+m}{n}\binom{n}{j}\frac{(1/2)_j}{(m+1)_j}. \tag{2.16}$$

The proof is similar to the proof of Corollary 2.1.
For $i = m = 0$, (2.15) reduces to (2.12), i.e., (3.85) in [5].
For $j = m = 0$, (2.16) reduces to (2.13), i.e., (3.90) in [5].

Let again $\alpha\beta \neq 0$. According to the results of [7] (see [4, Prop. 1] and [4, (15)]), we have

$$Hl\left(\frac{1}{2}, \frac{1}{2}(2\gamma-\alpha)(2\gamma-\beta); 2\gamma-\alpha, 2\gamma-\beta; \gamma+1, \gamma+1; x\right)$$
$$=\frac{\gamma}{\alpha\beta}(1-2x)^{\alpha+\beta+1-2\gamma}\frac{d}{dx}Hl\left(\frac{1}{2}, \frac{1}{2}\alpha\beta; \alpha, \beta; \gamma, \gamma; x\right). \tag{2.17}$$

Using (2.8), (2.17) and the above methods of proof, we obtain the following identities:

$$Hl\left(\frac{1}{2}, (2k+1)(k-n); 2(k-n), 2k+1; 2k+1, 2k+1; x\right)$$
$$=4^k\binom{n+k}{n}^{-1}\binom{n}{k}^{-1}\sum_{i=0}^{n-k}\binom{2n-2i}{2k}\binom{n}{i}r_{n,k+i}(1-2x)^{2i}$$
$$=\sum_{j=0}^{n-k}4^j\binom{n-k}{j}\frac{(k+1/2)_j}{(2k+1)_j}(x^2-x)^j, \quad 0 \leq k \leq n. \tag{2.18}$$

As a consequence of (2.18), one gets

$$\sum_{j=i}^{n-k}(-1)^{j-i}\binom{n-k}{j}\frac{(k+1/2)_j}{(2k+1)_j}\binom{j}{i} \tag{2.19}$$
$$=4^k\binom{n+k}{n}^{-1}\binom{n}{k}^{-1}\binom{2n-2i}{2k}\binom{n}{i}r_{n,k+i}, \quad 0 \leq i \leq n-k$$

and

$$\sum_{i=j}^{n-k}\binom{2n-2i}{2k}\binom{n}{i}\binom{i}{j}r_{n,k+i} =4^{-k}\binom{n+k}{n}\binom{n}{k}\binom{n-k}{j}\frac{(k+1/2)_j}{(2k+1)_j},\quad 0\le j\le n-k. \tag{2.20}$$

For $i=k=0$, (2.19) reduces to (2.12); for $j=k=0$, (2.20) becomes (2.13). Moreover,

$$\begin{aligned} &Hl\left(\frac{1}{2},(2k-1)(k+n);2(k+n),2k-1;2k,2k;x\right)\\ =&2^{2k-1}\binom{n+k-1}{k-1}^{-1}\binom{n-1}{k-1}^{-1}\sum_{i=0}^{n-k}\binom{2n-2i-2}{2k-2}\binom{n-1}{i}r_{n,k+i}(1-2x)^{1-2n+2i}\\ =&(1-2x)^{1-2n}\sum_{j=0}^{n-k}4^j\binom{n-k}{j}\frac{(k+1/2)_j}{(2k)_j}(x^2-x)^j,\quad 1\le k\le n. \end{aligned} \tag{2.21}$$

From (2.21), we derive

$$\sum_{j=i}^{n-k}(-1)^{j-i}\binom{n-k}{j}\frac{(k+1/2)_j}{(2k)_j}\binom{j}{i} =2^{2k-1}\binom{n+k-1}{k-1}^{-1}\binom{n-1}{k-1}^{-1}\binom{2n-2i-2}{2k-2}\binom{n-1}{i}r_{n,k+i},\quad 0\le i\le n-k, \tag{2.22}$$

$$\sum_{i=j}^{n-k}\binom{2n-2i-2}{2k-2}\binom{n-1}{i}\binom{i}{j}r_{n,k+i} =2^{1-2k}\binom{n+k-1}{k-1}\binom{n-1}{k-1}\binom{n-k}{j}\frac{(k+1/2)_j}{(2k)_j},\quad 0\le j\le n-k. \tag{2.23}$$

For $i=0$, $k=1$ and replacing n by $n+1$, from (2.22), we obtain

$$\sum_{j=0}^{n}\left(-\frac{1}{4}\right)^j\binom{n}{j}\binom{2j+1}{j}=\frac{1}{(n+1)4^n}\binom{2n}{n}. \tag{2.24}$$

With $j=0$, $k=1$ and replacing n by $n+1$, (2.23) yields

$$\sum_{i=0}^{n}(i+1)\binom{2i+2}{i+1}\binom{2n-2i}{n-i}=\frac{n+1}{2}4^{n+1}. \tag{2.25}$$

It is a pleasant calculation to prove (2.24) and (2.25) directly.

Using (2.14) and (2.17), we get

$$
\begin{aligned}
&Hl\left(\frac{1}{2},(2k+1)(k+n+1);2(k+n+1),2k+1;2k+1,2k+1;x\right)\\
=&4^k\binom{n+k}{n}^{-1}\binom{n}{k}^{-1}\sum_{j=0}^{n-k}\binom{2k+2j}{2j}\binom{n}{k+j}r_{nj}(1-2x)^{-2k-1-2j}\\
=&(1-2x)^{-2n-1}\sum_{j=0}^{n-k}4^j\binom{n-k}{j}\frac{(k+1/2)_j}{(2k+1)_j}(x^2-x)^j,\quad 0\le k\le n. \qquad (2.26)
\end{aligned}
$$

Taking into account that $r_{n,n-j}=r_{nj}$, from (2.26), we derive (2.19) and (2.20). Moreover,

$$
\begin{aligned}
&Hl\left(\frac{1}{2},(2k+1)(k-n);2(k-n),2k+1;2k+2,2k+2;x\right)\\
=&4^k\frac{2k+1}{n+1}\binom{n+k+1}{n}^{-1}\binom{n}{k}^{-1}\sum_{j=0}^{n-k}\binom{2k+2j+2}{2j}\binom{n+1}{k+j+1}r_{nj}(1-2x)^{2n-2k-2j}\\
=&\sum_{j=0}^{n-k}4^j\binom{n-k}{j}\frac{(k+1/2)_j}{(2k+2)_j}(x^2-x)^j,\quad 0\le k\le n. \qquad (2.27)
\end{aligned}
$$

From (2.27), we derive

$$
\begin{aligned}
&\sum_{j=i}^{n-k}(-1)^{j-i}\binom{n-k}{j}\frac{(k+1/2)_j}{(2k+2)_j}\binom{j}{i},\quad 0\le i\le n-k \qquad (2.28)\\
=&4^k\frac{2k+1}{n+1}\binom{n+k+1}{n}^{-1}\binom{n}{k}^{-1}\binom{2n-2i+2}{2k+2}\binom{n+1}{i}r_{n,k+i}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=j}^{n-k}\binom{2n-2i+2}{2k+2}\binom{n+1}{i}\binom{i}{j}r_{n,k+i},\quad 0\le j\le n-k \qquad (2.29)\\
=&4^{-k}\frac{n+1}{2k+1}\binom{n+k+1}{n}\binom{n}{k}\binom{n-k}{j}\frac{(k+1/2)_j}{(2k+2)_j}.
\end{aligned}
$$

For $i=k=0$, (2.28) becomes

$$
\sum_{j=0}^{n}\left(-\frac{1}{4}\right)^j\binom{n+1}{j+1}\binom{2j}{j}=\frac{2n+1}{4^n}\binom{2n}{n}. \qquad (2.30)
$$

Let us recall the formula (7.6) in [5]:

$$
\sum_{j=0}^{n}\left(-\frac{1}{4}\right)^j\binom{n}{j}\binom{2j}{j}\binom{j+h}{h}^{-1}=\frac{1}{4^n}\binom{2n+2h}{n+h}\binom{2h}{h}^{-1}. \qquad (2.31)
$$

For $h=1$, (2.31) reduces to (2.30). For $j=k=0$, (2.29) becomes

$$
\sum_{i=0}^{n}(2n-2i+1)\binom{2i}{i}\binom{2n-2i}{n-i}=(n+1)4^n, \qquad (2.32)
$$

which can be proved also directly.

3. Confluent Heun functions

The hypergeometric function $v(t) = {}_1F_1(\alpha;\gamma;t)$ satisfies (see [10, p. 336], [11, 13.2.1]) $v(0) = 1$ and

$$tv''(t) + (\gamma - t)v'(t) - \alpha v(t) = 0. \tag{3.33}$$

Moreover (see [10, p. 338, 5.6], [11, 13.3.15]),

$${}_1F_1(\alpha+1;\gamma+1;t) = \frac{\gamma}{\alpha}\frac{d}{dt}{}_1F_1(\alpha;\gamma;t). \tag{3.34}$$

With the above notation, we have:

Theorem 3.3. *For $\alpha p \neq 0$, the confluent Heun function $HC(p,\gamma,0,\alpha,4p\alpha;x)$ satisfies*

$$HC(p,\gamma,0,\alpha,4p\alpha;x) = {}_1F_1(\alpha;\gamma;-4px), \tag{3.35}$$

$$HC(p,\gamma+1,0,\alpha+1,4p(\alpha+1);x) = -\frac{\gamma}{4p\alpha}\frac{d}{dx}HC(p,\gamma,0,\alpha,4p\alpha;x), \tag{3.36}$$

$$HC(p,\gamma+j,0,\alpha+j,4p(\alpha+j);x) = \frac{(-1)^j(\gamma)_j}{(4p)^j(\alpha)_j}\frac{d^j}{dx^j}HC(p,\gamma,0,\alpha,4p\alpha;x), \tag{3.37}$$

for all integers $j \geq 0$ with $(\alpha)_j \neq 0$.

Proof. According to (1.2), the function $u(x) = HC(p,\gamma,0,\alpha,4p\alpha;x)$ satisfies $u(0) = 1$ and

$$xu''(x) + (4px+\gamma)u'(x) + 4p\alpha u(x) = 0. \tag{3.38}$$

From (3.33) and (3.38), it is easy to deduce that $u(x) = v(-4px)$, and this entails (3.35). Now, (3.36) is a consequence of (3.35) and (3.34); (3.37) can be proved by induction with respect to j. Let us remark that (3.36) coincides with (30) in [4]. □

Corollary 3.3. *Let $K_n(x) := HC\left(n,1,0,\frac{1}{2},2n;x\right)$ be the function given by (1.5). Then*

$$K_n(x) = {}_1F_1\left(\frac{1}{2};1;-4nx\right) \tag{3.39}$$

and

$$K_n(x) = \frac{1}{\pi}\int_{-1}^{1} e^{-2nx(1+t)}\frac{dt}{\sqrt{1-t^2}}. \tag{3.40}$$

Proof. (3.39) follows from (3.35) with $\alpha = 1/2$, $\gamma = 1$ and $p = n$. By using (3.39) and [10, p. 338, 5.9], [11, 13.4(i)], we get (3.40). Let us remark that (3.40) coincides with (69) in [14]. □

Using (3.37) with $p = n$, $\gamma = 1$, $\alpha = 1/2$, we get

$$HC\left(n,j+1,0,j+\frac{1}{2},2n(2j+1);x\right) = \frac{(-1)^j}{n^j}\binom{2j}{j}^{-1}K_n^{(j)}(x), \quad j \geq 0. \tag{3.41}$$

From (3.41) and [4, (34)], we obtain

$$K_n^{(j)}(0) = (-n)^j\binom{2j}{j}, \tag{3.42}$$

which is (35) in [4].
On the other hand, (3.40) implies (with $t = \sin\varphi$)

$$K_n^{(j)}(0) = \frac{(-2n)^j}{\pi} \sum_{k=0}^{j} \binom{j}{k} \int_{-\pi/2}^{\pi/2} \sin^k \varphi d\varphi$$
$$= (-2n)^j \sum_{i=0}^{[j/2]} \binom{j}{2i} \binom{2i}{i} 4^{-i}.$$

Combined with (3.42), this produces

$$\sum_{i=0}^{[j/2]} \binom{j}{2i} \binom{2i}{i} 4^{-i} = 2^{-j} \binom{2j}{j},$$

which is (3.99) in [5].
Finally, we give closed forms for some families of confluent Heun functions.

Theorem 3.4. *(i) For $0 \le j \le n$, we have*

(3.43) $$HC\left(p, j + \frac{1}{2}, 0, j - n, 4p(j-n); x\right) = \frac{(2j)!}{j!} \sum_{k=0}^{n-j} \binom{n-j}{k} \frac{(n-k)!}{(2n-2k)!} (16px)^{n-j-k}.$$

(ii) More generally, for $0 \le j \le n$ and $\lambda > -1$,

(3.44) $$HC(p, j + 1 + \lambda, 0, j - n, 4p(j-n); x)$$
$$= \frac{(\lambda+1)_j \Gamma(\lambda+1)}{\Gamma(n+\lambda+1)} \sum_{k=0}^{n-j} (\lambda + n + 1 - k)_k \binom{n-j}{k} (4px)^{n-j-k}.$$

Proof. By using the relation between the function ${}_1F_1$ and the Hermite polynomials (see [10, p. 340, 5.16], [10, p. 235, (4.51)], [11, 13.6.16]), we have

(3.45) $${}_1F_1\left(-n; \frac{1}{2}; x\right) = n! \sum_{k=0}^{n} \frac{1}{k!(2n-2k)!} (-4x)^{n-k}.$$

From (3.35) and (3.45), it follows that

(3.46) $$HC\left(p, \frac{1}{2}, 0, -n, -4pn; x\right) = n! \sum_{k=0}^{n} \frac{(16px)^{n-k}}{k!(2n-2k)!}.$$

Now, (3.43) is a consequence of (3.46) and (3.37).
In order to prove (3.44), we need the relation between ${}_1F_1$ and the Laguerre polynomials (see [10, p. 340, 5.14], [11, 13.6.19]):

(3.47) $${}_1F_1(-n; \lambda + 1; x) = \frac{n!\Gamma(\lambda+1)}{\Gamma(n+\lambda+1)} L_n^\lambda(x), \quad \lambda > -1,$$

where (see [10, p. 245, (4.61)], [11, 18.5.12])

(3.48) $$L_n^\lambda(x) = \sum_{k=0}^{n} (-1)^k \frac{(\lambda+k+1)_{n-k}}{k!(n-k)!} x^k.$$

From (3.35), (3.47) and (3.48), we get

(3.49) $$HC(p, \lambda + 1, 0, -n, -4pn; x) = \frac{n!\Gamma(\lambda+1)}{\Gamma(n+\lambda+1)} L_n^\lambda(-4px).$$

Combined with (3.37), (3.49) produces(3.44), and this concludes the proof. □

4. OTHER COMBINATORIAL IDENTITIES

Let us return to (2.10). Since

$$\frac{(m+1/2)_j}{(m+1)_j} = 4^{-j}\binom{2m+2j}{m+j}\binom{2m}{m}^{-1}, \tag{4.50}$$

it becomes

$$\sum_{j=i}^{n-m}(-1)^{j-i}4^{-j}\binom{n-m}{j}\binom{2m+2j}{m+j}\binom{j}{i}$$
$$=4^{m-n}\binom{n}{m}^{-1}\binom{m+i}{m}\binom{2m+2i}{m+i}\binom{2n-2m-2i}{n-m-i}.$$

Set $i+m=r$, $j=r-m+k$, and replace n by $n+r$; we get

$$\sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k\binom{n+r-m}{n-k}\binom{2r+2k}{r+k}\binom{r-m+k}{k} \tag{4.51}$$
$$=4^{-n}\binom{n+r}{m}^{-1}\binom{r}{m}\binom{2r}{r}\binom{2n}{n},\quad n\geq 0, r\geq m\geq 0.$$

Here are some particular cases of (4.51).

$$r=m=n:\quad \sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k\binom{n}{k}\binom{2n+2k}{n+k}=4^{-n}\binom{2n}{n}.$$

$$r=m:\quad \sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k\binom{n}{k}\binom{2r+2k}{r+k}=4^{-n}\binom{n+r}{r}^{-1}\binom{2r}{r}\binom{2n}{n}.$$

$$m=0:\quad \sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k\binom{n+r}{n-k}\binom{2r+2k}{r+k}\binom{r+k}{k}=4^{-n}\binom{2r}{r}\binom{2n}{n}.$$

$$r=n:\quad \sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k\binom{2n-m}{n-k}\binom{2n+2k}{n+k}\binom{n-m+k}{k}=4^{-n}\binom{2n}{m}^{-1}\binom{n}{m}\binom{2n}{n}^2.$$

$$m=n:\quad \sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k\binom{r}{n-k}\binom{2r+2k}{r+k}\binom{r-n+k}{k}=4^{-n}\binom{n+r}{r}^{-1}\binom{r}{n}\binom{2r}{r}\binom{2n}{n}.$$

Now, let us return to (2.11); use (4.50), set $j+m=r$, $i=r-m+k$, and replace n by $n+r$. We get

$$\sum_{k=0}^{n}\binom{r+k}{m}\binom{r+k-m}{k}\binom{2r+2k}{r+k}\binom{2n-2k}{n-k} \tag{4.52}$$
$$=4^n\binom{n+r}{m}\binom{2r}{r}\binom{n+r-m}{n},\quad n\geq 0, r\geq m\geq 0.$$

For $r=m=n$, (4.52) reduces to

$$\sum_{k=0}^{n}\binom{n+k}{n}\binom{2n+2k}{n+k}\binom{2n-2k}{n-k}=4^n\binom{2n}{n}^2.$$

Clearly, there are many other particular cases of (4.52).
Several other particular combinatorial identities can be obtained starting with other general formulas from the preceding sections, but we omit the details.

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Research Article

Infimal generators and monotone sublinear operators

SEMËN S. KUTATELADZE*

ABSTRACT. We provide a version of Korovkin-type theorems for monotone sublinear operators in vector lattices and discuss the possibilities of further extensions and generalizations.

Keywords: Supremal generator, Korovkin-type theorem, sublinear operator.

2020 Mathematics Subject Classification: 41A36, 46N99.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

S. G. Gal and C. P. Niculescu recently proved a Korovkin-type theorem for monotone sublinear operators in spaces of continuous functions in N-dimensions [1, Theorem 2]. As the test functions they use the coordinate projections and the sum of their squares whose span is a classical supremal generator in $C(Q)$ with Q a compact subset of the N-dimensional Euclidean space $\mathbb{R}^N$. In this short note, we extend this theorem to order convergence.

2. THE MAIN RESULT

We will proceed with a vector sublattice X of a Dedekind complete vector lattice Y. Recall that a subset H of X is a supremal generator of X_+ with respect to Y provided that $x = \sup_Y\{h \in H : h \leq x\}$ for all $x \in X_+$, where as usual $X_+ = \{x \in X : x \geq 0\}$. Using the reverse order, we define an infimal generator of X_+ with respect to Y for convenience. We let $L_+(X, Y)$ stand for the set of positive linear operators from X to Y. Assume that Id_Y is the identity operator on Y. Then, the order interval $[0, \mathrm{Id}_Y]$ is the set of multipliers of Y.

Theorem 2.1. *Let H be a convex cone in X. Then, the following are equivalent:*

(1) H is an infimal generator of X_+ with respect to Y;
(2) there is a multiplier $\lambda \in [0, \mathrm{Id}_Y]$ such that $o{-}\limsup_{\alpha\in\mathrm{A}} P_\alpha(x) = \lambda x$ for all $x \in X_+$ and every net $(P_\alpha)_{\alpha\in\mathrm{A}}$ of monotone sublinear operators from X to Y such that $o{-}\limsup_\alpha P_{\alpha\in\mathrm{A}}(h) \leq h$ for all $h \in H$;
(3) H is cofinal and if $T \in L_+(X, Y)$ such that $Th \leq h$ for all $h \in H$; then T is a multiplier in Y.

Proof. (1) $\to$ (2): Clearly, $P(x) := o - \limsup_{\alpha\in\mathrm{A}} P_\alpha(x)$ exists for every $x \in X$. So, $P : X \to Y$ is a monotone sublinear operator and $P(x) = \sup\{Tx : T \in \partial(P)\}$, where $\partial(P)$ is the subdifferential of P.

Take $T \in \partial(P)$, $x \in X_+$, and $h \geq x, h \in H$. Since H is an infimal generator of X_+ with respect to Y and $Th \leq P(h) \leq h$; therefore, $Tx \leq x$ by the operator principle of preservation of

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inequalities [2, Theorem 2.1]. Consequently, $T \in [0, \mathrm{Id}_Y]$ and so $P(x) = \lambda x$ for all $x \in X_+$ since the pointwise supremum of multipliers is a multiplier itself.

(2)→ (3): This is obvious.

(3) → (1): Let $q_H(x) := \inf\{h \in H : h \geq x\}$. Then, q_H is a monotone sublinear operator on X and $\partial(q_H) = \{T \in L_+(X, Y) : (\forall h \in H\,)Th \leq h\}$. By (3), q_H is a multiplier itself.

□

3. Comments and Extensions

In this section, we will discuss applications and extensions of Theorem 2.1 using the terminology and techniques of vector lattices, subdifferential calculus, and Boolean-valued analysis as presented in [3] and [4].

The ideas behind the main theorem make it possible to abstract the effects of monotone sublinear approximations along all lines thoroughly presented in [5]. For instance, if H is a finite-dimensional subspace of X, then X is a vector lattice of bounded elements. Moreover, H is coinitial to X (in fact, a supremal generator as well) and we have an analog of Theorem 2.1 for relatively uniform convergence to the embedding of X to Y. This yields uniform approximation by monotone sublinear operators in spaces of bounded continuous functions on compact subsets of finite-dimensional Euclidean space.

Many opportunities are open by Boolean valued analysis for Korovkin-type results in the realm of ordered modules over the ring of orthomorphisms of a Dedekind complete vector lattice Y. We can consider approximation of the embedding by monotone module-sublinear operators along the lines of Theorem 2.1. The rest of the matter is that the modular situation can be lifted to the Boolean valued universe $\mathbb{V}^{\mathbb{B}}$ over the base $\mathbb{B}$ of Y (i.e., the set of band projections in Y). Since by transfer Theorem 2.1 and its analogs are valid inside $\mathbb{V}^{\mathbb{B}}$, we can use their Boolean valued interpretations which yield, for instance, some Korovkin-type theorems for approximation of extensional continuous vector-valued functions on the so-called procompact spaces which are usually noncompact (for instance, the order intervals between two Lebesgue measurable functions).

We will not dwell on all these new model-theoretic opportunities since the possibilities of their application for the needs of the working mathematician seem dim these days.

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Research Article

On Hölder continuity and equivalent formulation of intrinsic Harnack estimates for an anisotropic parabolic degenerate prototype equation

SIMONE CIANI* AND VINCENZO VESPRI

ABSTRACT. We give a proof of Hölder continuity for bounded local weak solutions to the equation

$$(*) \qquad u_t = \sum_{i=1}^{N} (|u_{x_i}|^{p_i-2} u_{x_i})_{x_i}, \quad \text{in} \quad \Omega_T = \Omega \times (0,T], \quad \text{with} \quad \Omega \subset\subset \mathbb{R}^N,$$

under the condition $2 < p_i < \bar{p}(1+2/N)$ for each $i = 1,..,N$, being $\bar{p}$ the harmonic mean of the p_is, via recently discovered intrinsic Harnack estimates. Moreover, we establish an equivalent formulation of these Harnack estimates within the proper intrinsic geometry.

Keywords: Anisotropic p-Laplacian, Hölder continuity, Harnack estimates, intrinsic scaling.

2020 Mathematics Subject Classification: 35K65, 35K92, 35B65.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION AND MAIN RESULT

Equations of the kind of (*) fall into the wide class of degenerate equations, because their coordinate modulus of ellipticity $|u_{x_i}|^{p_i-2}$ vanishes as soon as the partial derivative u_{x_i} approaches zero. This behaviour is classically studied in equations evolving as the degenerate p-Laplacian

$$(1.1) \qquad u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0, \quad \text{in} \quad \Omega_T = \Omega \times (0,T], \quad \Omega \subset\subset \mathbb{R}^N,$$

whose modulus of ellipticity $|\nabla u|^{p-2}\nabla u$ goes to zero when the whole gradient of the solution vanishes. Within a rich variety of other different techniques, the method of intrinsic scaling is one of the keys to access the theory of regularity for degenerate parabolic equations (see for instance [7], [10]): it provides a correct interpretation of the evolution of the equation, interpreted in a particular geometry dictated by the solution itself, hence the name. This method has proven to be powerful and flexible enough to be adapted to a wide class of equations; a sketchy example for the doubly nonlinear case can be found in [4]. Nevertheless, the application of this method to the anisotropic case is not straightforward, because the degenerative behavior of the equation is purely directional, i.e., some partial derivatives can vanish while some other ones may direct the diffusion. When $p_i \equiv p$, the equation (*) is a different equation from (1.1) and bears the name of *orthotropic p-Laplacian*. The prototype equation (*) reflects the

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modeling of many materials that reveal different diffusion rates along different directions, such as liquid crystals, wood or earth's crust (see for an example the book [26]). The regularity of bounded local weak solutions to equations as $(*)$ with measurable and bounded coefficients is still an open problem. The main difference with standard nonlinear regularity theory is the directional growth of the operator, usually referred to as *nonstandard growth* (see for instance [1]). This requires the definition of a new class of function spaces, called anisotropic Sobolev spaces (see Section 2), and whose study is open and challenging.

1.1. **The elliptic problem.** Even in the elliptic case, a standard statement of regularity for such equations requires a bound on the sparseness of the powers p_i. Indeed, in general, the weak solution can be unbounded, as proved in [15], [19]. Lipschitz bounds were obtained by Marcellini for the p, q nonstandard growth in his work [21], supposing the coefficients are regular enough. This work opened one way to the regularity theory of the so-called nonstandard growth conditions. See also the work by Uralt'seva and Urdaletova in [28], and [17], [18] for more general equations. In [2], the authors proved the boundedness of solutions under the assumption

$$\bar{p} < N, \quad \max\{p_1, .., p_N\} < \bar{p}^*, \tag{1.2}$$

where $\bar{p}$ and $\bar{p}^*$ are defined respectively in (2.11) and (2.12). Regularity properties are proved assuming strong conditions on the regularity of the coefficients (see [20], [21]). In this context, we quote also the contribution [12]. A recent result of Lipschitz regularity has been proven by Brasco and Bousquet in [3] for the elliptic counterpart of $(*)$, by assuming that solutions are just bounded. On the other hand, structure conditions are left in big generalisation while a tighter condition on the spareness of p_i is used in the paper [14] to obtain Lipschitz bounds. This list is far from being complete, we refer to the survey [22] for an exhaustive bibliography. Nevertheless even in the elliptic case, when the coefficients are rough, Hölder continuity remains still nowadays an open problem. Indeed, continuity conditioned to boundedness has been proved in [11] through the intrinsic scaling method, but with a condition of stability on the exponents p_i which is only qualitative. Removability of singularities has been considered in [27], where the idea of working with fundamental solutions in the anisotropic framework had yet been taken into consideration.

1.2. **The parabolic problem.** To the best of our knowledge, the boundedness of local weak solutions to equations behaving as $(*)$ has been proved in [13], [23]. More precisely, they prove that local weak solutions are bounded if

$$p_i < \bar{p}\left(1 + \frac{2}{N}\right), \quad i = 1, .., N. \tag{1.3}$$

Again in [13], the authors find some useful L^∞ estimates, together with finite speed of propagation and lower semicontinuity of solutions. These have been the starting point for the study of fundamental solutions to $(*)$ (see for instance [5]), and the behaviour of their support in [6]. Recently, an approach based on an expansion of positivity relying on the behaviour of fundamental solutions has brought the authors to prove in [6] the following Harnack inequality, properly structured in an intrinsic anisotropic geometry that we are about to describe. Fix numbers $\theta, \rho > 0$ to be defined later, and define the anisotropic cubes

$$\mathcal{K}_\rho(\theta) := \prod_{i=1}^N \left\{ |x_i| < \theta^{\frac{p_i - \bar{p}}{p_i}} \rho^{\frac{\bar{p}}{p_i}} \right\}. \tag{1.4}$$

Next, define the following centered, forward and backward anisotropic cylinders, for a generic point (x_0, t_0):

$$
(1.5) \quad \begin{cases} \text{centered cylinders: } (x_0, y_0) + \mathcal{Q}_\rho(\theta) = \{x_0 + \mathcal{K}_\rho(\theta)\} \times (t_0 - \theta^{2-\bar{p}}\rho^{\bar{p}}, t_0 + \theta^{2-\bar{p}}\rho^{\bar{p}}); \\ \text{forward cylinders: } (x_0, y_0) + \mathcal{Q}_\rho^+(\theta) = \{x_0 + \mathcal{K}_\rho(\theta)\} \times [t_0, t_0 + \theta^{2-\bar{p}}\rho^{\bar{p}}); \\ \text{backward cylinders: } (x_0, y_0) + Q_\rho^-(\theta) = \{x_0 + \mathcal{K}_\rho(\theta)\} \times (t_0 - \theta^{2-\bar{p}}\rho^{\bar{p}}, t_0]. \end{cases}
$$

We use the following Theorem 1.1 as an essential tool to prove the Hölder continuity of solutions, in a similar fashion to the approach firstly used by J. Moser in [24] and successively developed in [8] for degenerate parabolic equations of p-Laplacian type with measurable coefficients.

Theorem 1.1. *Let u be a non-negative local weak solution to ($*$) such that for some point $(x_0, t_0) \in \Omega_T$, we have $u(x_0, t_0) > 0$. There exist constants c, γ depending only upon p_i, N such that the following inequality holds for all intrinsic cylinders $(x_0, t_0) + \mathcal{Q}_{4\rho}^+(\theta)$ contained in Ω_T*

$$
(1.6) \qquad u(x_0, t_0) \le \gamma \inf_{x_0 + \mathcal{K}_\rho(\theta)} u(x, t_0 + \theta^{2-\bar{p}}\rho^{\bar{p}}), \qquad \theta = \left(\frac{c}{u(x_0, t_0)}\right).
$$

It is remarkable that estimate (1.6) is prescribed on a *space* configuration dependent on the solution. This property differs substantially from the isotropic case because it reveals a typical anisotropic intrinsic geometry. In this setting, an expansion of positivity can be performed by means of the comparison principle (see [6]). In the present work, we show that Theorem 1.1 implies local Hölder continuity of local weak solutions to ($*$).

Theorem 1.2. *Let u be a local weak solution to ($*$). Then u is locally Hölder continuous in Ω_T, i.e., there exist constants $\gamma > 1$, $\alpha \in (0, 1)$ depending only upon p_i, N, such that for each compact set $K \subset\subset \Omega_T$ we have*

$$
(1.7) \qquad |u(x, t) - u(y, s)| \le \gamma ||u||_\infty \left(\frac{\sum_{i=1}^N |x_i - y_i|^{\frac{p_i}{\bar{p}}} ||u||_\infty^{\frac{\bar{p}-p_i}{p_i}} + |t - s|^{\frac{1}{\bar{p}}} ||u||_\infty^{\frac{\bar{p}-2}{\bar{p}}}}{\pi\text{-dist}(K, \partial\Omega_T)} \right)^\alpha,
$$

for every pair of points (x, t), $(y, s) \in K$, with

(1.8)

$$
\pi\text{-dist}(K, \partial\Omega_T) := \inf \left\{ \left(|x_i - y_i|^{\frac{p_i}{\bar{p}}} ||u||_\infty^{\frac{\bar{p}-p_i}{p_i}} \wedge |t - s|^{\frac{1}{\bar{p}}} ||u||_\infty^{\frac{\bar{p}-2}{\bar{p}}} \right) : (x, t) \in K, (y, s) \in \partial\Omega_T, \ i = 1..N \right\}.
$$

Moreover, through a similar approach to the isotropic case in [9], we show that the classical Pini-Hadamard estimate can be recovered (see [25] for the complete reference)

$$
(1.9) \qquad \gamma^{-1} \sup_{K_\rho(x_0)} u(\cdot, t_0 - \rho^2) \le u(x_0, t_0)n \le \gamma \inf_{K_\rho(x_0)} u(\cdot, t_0 + \rho^2), \quad \gamma > 0,
$$

when $p_i \equiv 2$ for all $i = 1, .., N$ and provided the parabolic cylinders $(x_0, t_0) + \mathcal{Q}_{4\rho}^\pm$ are contained in Ω_T. Indeed, the following theorem can be shown to be sole consequence of Theorem 1.1.

Theorem 1.3. *Let u be a non-negative local weak solution to ($*$) such that for some point $(x_0, t_0) \in \Omega_T$ we have $u(x_0, t_0) > 0$. There exist constants c, γ depending only upon p_i, N such that for all intrinsic cylinders $(x_0, t_0) + \mathcal{Q}_{4\rho}(\theta)$ contained in Ω_T as in (1.5) we have*

(1.10)

$$
\gamma^{-1} \sup_{x_0 + \mathcal{K}_\rho(\theta)} u(x, t_0 - \theta^{2-\bar{p}}\rho^{\bar{p}}) \le u(x_0, t_0) \le \gamma \inf_{x_0 + \mathcal{K}_\rho(\theta)} u(x, t_0 + \theta^{2-\bar{p}}\rho^{\bar{p}}), \qquad \theta = \left(\frac{c}{u(x_0, t_0)}\right).
$$

Remark 1.1. *When the elliptic counterpart is considered, we deal with stationary solutions to (*$*$*), so that the behavior at each time is always the same. In this context, easily deductible by (*1.10*), we get the usual sup-inf estimate with no need of a waiting time. What is essential (at least for the proof of (*1.6*) in [*6*]) and deeply different from the isotropic case where the elliptic estimate holds in classic cubes, is the intrinsic space geometry (*1.4*) of* $\mathcal{K}_\rho(\theta)$*.*

2. Preliminaries

Given $\mathbf{p} := (p_1, .., p_N)$, $\mathbf{p} > 1$ with the usual meaning, we assume that the harmonic mean is smaller than the dimension of the space variables

$$\bar{p} := \left(\frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \right)^{-1} < N, \tag{2.11}$$

and we define the Sobolev exponent of the harmonic mean $\bar{p}$,

$$\bar{p}^* := \frac{N\bar{p}}{N - \bar{p}}. \tag{2.12}$$

We will suppose all along this note that the p_is are ordered increasingly, without loss of generality. Next, we introduce the natural parabolic anisotropic spaces. Given $T > 0$ and a bounded open set $\Omega \subset \mathbb{R}$, we let $\Omega_T = \Omega \times (0, T]$ and we define

$$W_o^{1,\mathbf{p}}(\Omega) := \{ u \in W_o^{1,1}(\Omega) |\, D_i u \in L^{p_i}(\Omega) \},$$

$$L^{\mathbf{p}}_{loc}(0, T; W_o^{1,\mathbf{p}}(\Omega)) := \{ u \in L^1_{loc}(0, T; W_o^{1,1}(\Omega)) |\, D_i u \in L^{p_i}_{loc}(0, T; L^{p_i}_{loc}(\Omega)) \}.$$

A function

$$u \in C^0_{loc}(0, T; L^2_{loc}(\Omega)) \cap L^{\mathbf{p}}_{loc}(0, T; W_o^{1,\mathbf{p}}(\Omega))$$

is a *local weak solution* of ($*$) if for all $0 < t_1 < t_2 < T$ and any test function $\varphi \in C^\infty_{loc}(0, T; C_o^\infty(\Omega))$ it satisfies

$$\int_\Omega u\varphi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega (-u\, \varphi_t + \sum_{i=1}^{N} |u_{x_i}|^{p_i - 2} u_{x_i} \varphi_{x_i}) \, dxdt = 0. \tag{2.13}$$

By a density and approximation argument, this actually holds for any test function of the kind

$$\varphi \in W^{1,2}_{loc}(0, T; L^2_{loc}(\Omega)) \cap L^{\mathbf{p}}_{loc}(0, T; W_o^{1,\mathbf{p}}(\Omega))$$

for any p_i*-semirectangular domain* $\Omega \subset\subset \mathbb{R}^N$, where traces can be properly defined (see Theorem 3 in [16]).

Definition 2.1. *([*16*]) If the set of N elements of the vector $(p_1, .., p_N)$ consists of L distinct values, let us denote the multiplicity of each of the values by n_i, $i = 1, ..., L$ such that $n_1 + ... + n_L = N$. We say that a bounded domain $\Omega \subset \mathbb{R}^n$ satisfies the p_i-semirectangular restriction related to the vector $(p_1, .., p_N)$, if there exist bounded Lipschitz domains $\Omega_i \subset \mathbb{R}^{n_i}$, $i = 1, .., L$, such that $\Omega = \Omega_1 \times ... \times \Omega_L$.*

We will suppose all along the work that $\Omega \subset\subset \mathbb{R}^N$ is a p_i*-semirectangular domain*, being considerations and estimates of local nature.

3. Proof of Theorem 1.2

We employ the intrinsic Harnack inequality (1.6) to establish locally quantitative Hölder estimates for local, weak solutions u of ($*$), conditioned to the boundedness condition $2 < p_i < \bar{p}(1+2/N)$ for each $i = 1,..,N$. Fix a point $(x_0,t_0) \in \Omega_T$ which, up to translations, we will consider to be the origin in $\mathbb{R}^{N+1}$, and for an initial radius $\rho_0 > 0$, consider the cylinder $Q' = K_{\rho_0} \times (-\rho_0^2, 0]$, with vertex at $(0,0)$, and set

$$M_0 = \sup_{Q'} u, \quad m_0 = \inf_{Q'} u, \quad \text{and} \quad \omega_0 = M_0 - m_0 = \operatorname*{osc}_{Q'} u.$$

With ω_0 at hand, we can construct the initial cylinder for our purpose, intrinsically scaled:

$$\mathcal{Q}_0 = \mathcal{K}_{\rho_0} \times (-\theta_0^{\bar{p}-2}\rho_0^{\bar{p}}, 0] = \prod_{i=1}^{N} \left\{ |x_i| < \theta_0^{\frac{\bar{p}-p_i}{p_i}} \rho_0^{\frac{\bar{p}}{p_i}} \right\} \times \Big(-\theta_0^{\bar{p}-2}\rho_0^{\bar{p}}, 0 \Big], \text{ where} \quad \theta_0 = \left(\frac{c}{\omega_0} \right),$$

and c is a constant to be determined later only in terms of the data and independent of u, ρ_0. The accommodation of degeneracy deals with the following fact: if $\omega_0 > c\rho_0$, then $\mathcal{Q}_0 \subset Q'$. Converse inequality would lead directly to continuity in $\mathcal{Q}_0$.

Proposition 3.1. *Either $\omega_0 \leq c\rho_0$, or there exist numbers $\gamma > 1$, $\delta, \varepsilon \in (0,1)$, that can be quantitatively determined only in terms of the data p_i, N and independent of u, ρ_0, such that if we set*

$$\omega_n = \delta\omega_{n-1}, \quad \theta_n = \left(\frac{c}{\omega_n} \right), \quad \rho_n = \varepsilon\rho_{n-1}, \tag{3.14}$$

and

$$\mathcal{Q}_n = \mathcal{Q}_{\rho_n}(\theta_n) = \prod_{i=1}^{N} \left\{ |x_i| < \theta_n^{\frac{\bar{p}-p_i}{p_i}} \rho_n^{\frac{\bar{p}}{p_i}} \right\} \times (-\theta_n^{\bar{p}-2}\rho_n^{\bar{p}}, 0], \quad \textit{for} \quad n \in \mathbb{N}, \quad \textit{holding} \quad \mathcal{Q}_{n+1} \subset \mathcal{Q}_n,$$

then the oscillation in $\mathcal{Q}_n$ can be controlled with the same constant by the oscillation in $\mathcal{Q}_{n-1}$, i.e.,

$$\operatorname*{osc}_{\mathcal{Q}_n} u \leq \omega_n. \tag{3.15}$$

Proof. The proof is by induction: we show constants δ, ε, c depending only upon the data p_i, N, such that if the statement holds for the n-th step, then it holds the the $(n+1)$-th. Coherently with the accommodation of degeneracy, the first inductive step has already been stated. Assume now that $\mathcal{Q}_n$ has been constructed and that the statement holds up to n. Set

$$M_n = \sup_{\mathcal{Q}_n} u, \quad m_n = \inf_{\mathcal{Q}_n} u, \quad \text{and} \quad P_n = \left(0, -\frac{1}{2}\theta_n \rho_n^{\bar{p}} \right).$$

The point P_n is roughly speaking the point whose coordinates are the mid-point of each coordinate of $\mathcal{Q}_n$. On a first glance, we observe that we can assume

$$\omega_n \leq M_n - m_n = \operatorname*{osc}_{\mathcal{Q}_n} u,$$

because otherwise there is nothing to prove. At least one of the two inequalities

$$M_n - u(P_n) > \frac{1}{4}\omega_n, \quad \text{or} \quad u(P_n) - m_n > \frac{1}{4}\omega_n$$

must hold. Indeed, if it is not so, we arrive at the contradiction

$$\omega_n \leq M_n - m_n \leq \frac{1}{2}\omega_n.$$

We assume that the first inequality holds true, the proof for the second case is similar. The function $(M_n - u)$ is a nonnegative weak solution of ($*$) in $\mathcal{Q}_n$, and satisfies the intrinsic Harnack inequality (1.6) with respect to P_n, if its waiting time and space levels

$$\left(\frac{c_1}{M_n - u(P_n)}\right)^{\bar{p}-2}, \left(\frac{c_1}{M_n - u(P_n)}\right)^{\frac{\bar{p}-p_i}{p_i}},$$

are inside the respective time and space ranges of $\mathcal{Q}_n$. To this aim, we define c to be greater than c_1 so that

$$\left(\frac{c_1}{M_n - u(P_n)}\right) \leq \left(\frac{c}{\omega_n}\right), \quad \text{as} \quad \omega_n \leq 4\Big(M_n - u(P_n)\Big).$$

Finally, we have the estimate

$$\inf_{\mathcal{Q}_{\rho_n/4}} (M_n - u) \geq \frac{1}{\gamma}(M_n - u(P_n)) > \frac{1}{4\gamma}\omega_n.$$

This means, as $\inf(-u) = -\sup u$, that

$$M_n \geq \sup_{\mathcal{Q}_{\rho_n/4}} u + \frac{1}{4\gamma}\omega_n \geq \sup_{\mathcal{Q}_{n+1}} + \frac{1}{4\gamma}\omega_n, \quad \text{if} \quad \mathcal{Q}_{n+1} \subset Q_{\rho_n/4} \subset \mathcal{Q}_{\rho_n},$$

leading us, by subtracting $\inf_{\mathcal{Q}_{n+1}} u$ from both sides, to

$$M_n - \inf_{\mathcal{Q}_n} u \geq M_n - \inf_{\mathcal{Q}_{n+1}} u \geq \operatorname*{osc}_{\mathcal{Q}_{n+1}} u + \frac{1}{4\gamma}, \quad \text{if} \qquad \mathcal{Q}_{n+1} \subset Q_{\rho_n/4} \subset \mathcal{Q}_{\rho_n},$$

and thus to

$$\operatorname*{osc}_{\mathcal{Q}_{n+1}} u \leq \delta\omega_n = \omega_{n+1}, \quad \text{if} \qquad \mathcal{Q}_{n+1} \subset Q_{\rho_n/4} \subset \mathcal{Q}_{\rho_n}.$$

By choosing

$$\delta = 1 - \frac{1}{4\gamma} \quad \text{and} \quad \varepsilon = \frac{1}{4}\delta^{\frac{\bar{p}-2}{\bar{p}}}, \tag{3.16}$$

we manage to have both the inclusion $\mathcal{Q}_{n+1} \subset Q_{\rho_n/4} \subset \mathcal{Q}_{\rho_n}$ and the $(n+1)$-th conclusion of the iterative step (3.15). Indeed, by direct computation,

$$\begin{aligned}
\theta_{n+1}^{\bar{p}-2}\rho_{n+1}^{\bar{p}} &= \left(\frac{c}{\omega_{n+1}}\right)^{\bar{p}-2}\left(\frac{\rho_n^{\bar{p}}}{4^{\bar{p}}(\frac{4\gamma-1}{4\gamma})^{\bar{p}-2}}\right) \\
&= \left(\frac{4\gamma c}{(4\gamma-1)\omega_{n+1}}\right)^{\bar{p}-2}\left(\frac{\rho_n}{4}\right)^{\bar{p}} \\
&= \left(\frac{c}{\omega_n}\right)^{\bar{p}-2}\left(\frac{\rho_n}{4}\right)^{\bar{p}} \\
&= \theta_n^{\bar{p}-2}(\rho_n/4)^{\bar{p}},
\end{aligned}$$

precisely, and for each $i \in \{1, .., N\}$ as $p_i > 2$, it holds

$$\begin{aligned}
\theta_{n+1}^{\frac{\bar{p}-p_i}{p_i}} \rho_{n+1}^{\frac{\bar{p}}{p_i}} &= \left(\frac{c}{\omega_{n+1}}\right)^{\frac{\bar{p}-p_i}{p_i}} \left(\frac{\rho_n}{4}\left(1-\frac{1}{4\gamma}\right)^{\frac{\bar{p}-2}{\bar{p}}}\right)^{\frac{\bar{p}}{p_i}} \\
&= \left(\frac{c}{(1-\frac{1}{4\gamma})\omega_n}\right)^{\frac{\bar{p}-p_i}{p_i}} \left(1-\frac{1}{4\gamma}\right)^{\bar{p}-2} p_i \left(\frac{\rho_n}{4}\right)^{\frac{\bar{p}}{p_i}} \\
&= \left(\frac{c}{\omega_n}\right)^{\frac{\bar{p}-p_i}{p_i}} \left(\frac{\rho_n}{4}\right)^{\frac{\bar{p}}{p_i}} \left(1-\frac{1}{4\gamma}\right)^{\frac{p_i-\bar{p}}{p_i}+\frac{\bar{p}-2}{p_i}} \\
&= \theta_n \left(\frac{\rho_n}{4}\right)^{\frac{\bar{p}}{p_i}} \left(1-\frac{1}{4\gamma}\right)^{\frac{p_i-2}{p_i}} \le \theta_n^{\frac{\bar{p}-p_i}{p_i}} \left(\frac{\rho_n}{4}\right)^{\frac{\bar{p}}{p_i}}.
\end{aligned}$$

□

3.1. **Conclusion of the proof of Theorem 1.2.** Owing to the previous Proposition, we conclude that on each such intrinsically scaled cylinder $\mathcal{Q}_n$ it holds $\operatorname{osc}_{\mathcal{Q}_n} u \le \omega_n$, so that by induction and the definition of ω_n, we have

$$\operatorname*{osc}_{\mathcal{Q}_n} u \le \delta^n \omega_0.$$

Let now $0 < \rho < r$ be fixed, and observe that there exists a $n \in \mathbb{Z}$ such that, by use of (3.16), we have

$$\epsilon^{n+1} r \le \rho \le \epsilon^n r.$$

This implies

$$(n+1) \ge \ln\left(\frac{\rho}{r}\right)^{\frac{1}{\ln(\epsilon)}} \Rightarrow \delta^n \le \frac{1}{\delta}\left(\frac{\rho}{r}\right)^{\alpha}, \quad \text{with} \quad \alpha = \frac{|\ln(\delta)|}{|\ln(\epsilon)|}, \tag{3.17}$$

by an easy change of basis on the logarithm. Thus, by (3.16) and (3.17), we get

$$\operatorname*{osc}_{\mathcal{Q}_0} u \le \operatorname*{osc}_{\mathcal{Q}_n} u \le \frac{\omega_0}{\delta}\left(\frac{\rho}{r}\right)^{\alpha}. \tag{3.18}$$

Now finally, we give Hölder conditions to each variable, irrespective to the others. Fix $(x,t), (y,s) \in K$, $s > t$, let $R > 0$ to be determined later, and construct the intrinsic cylinder $(y,s) + \mathcal{Q}_R(M)$, where $M = ||u||_{L^\infty(\Omega_T)}$. This cylinder is contained in Ω_T if the variables satisfy for each $i = 1, .., N$,

$$M^{\frac{p_i-\bar{p}}{p_i}} R^{\frac{\bar{p}}{p_i}} \le \inf\left\{|x_i - y_i|, \quad \text{for} \quad x \in K,\, y \in \partial\Omega\right\} \quad \text{and} \quad M^{\frac{2-\bar{p}}{\bar{p}}} R \le \inf_{t \in K} t^{\frac{1}{\bar{p}}}.$$

This is easily achieved if we set, for instance,

$$2R = \pi\text{-dist}(K; \partial\Omega).$$

To prove the Hölder continuity in the variable t, we first assume that $(s-t) \le M^{2-\bar{p}} R^{\bar{p}}$. Then $\exists \rho_0 \in (0, R)$ such that $(s-t)^{\frac{1}{\bar{p}}} M^{\frac{\bar{p}-2}{\bar{p}}} = \rho_0$, and the oscillation (3.18) gives

$$\operatorname*{osc}_{\mathcal{Q}_{\rho_0}} u \le \gamma\omega_0 \left(\frac{\rho_0}{R}\right)^{\alpha},$$

implying

$$|u(x,s) - u(x,t)| \le \gamma M \left(\frac{M^{\frac{\bar{p}-2}{\bar{p}}} |s-t|^{\frac{1}{\bar{p}}}}{\pi\text{-dist}(K; \partial\Omega_T)}\right)^{\alpha},$$

as claimed. If otherwise $s-t \geq M^{2-\bar{p}}R^{\bar{p}}$ then, exploiting the fact that $\rho_0^\alpha \leq 4R$, we have

$$|u(x,s)-u(x,t)| \leq |u(x,s)|+|u(x,t)| \leq 2M \leq 4M\left(\frac{M^{\frac{\bar{p}-2}{\bar{p}}}|s-t|^{\frac{1}{\bar{p}}}}{\pi\text{-dist}(K;\partial\Omega)}\right)^\alpha.$$

About the space variables, we have for each i-th one the following alternative:

- If $|y_i-x_i| < M^{\frac{p_i-\bar{p}}{p_i}}R^{\frac{\bar{p}}{p_i}}$, and then analogously $\exists \rho_0 \in (0,R)$ such that $\rho_0 = |y_i-x_i|^{\frac{p_i}{\bar{p}}}M^{\frac{\bar{p}-p_i}{p_i}}$ and the oscillation reduction (3.18) gives

$$\operatorname*{osc}_{\mathcal{Q}_0} u \leq \operatorname*{osc}_{\mathcal{Q}_{\rho_0}(\theta_0)} u \leq \gamma\omega_0\left(\frac{\rho_0}{R}\right)^\alpha \quad \Rightarrow \quad |u(y_i,t)-u(x_i,t)| \leq \gamma M\left(\frac{|y_i-x_i|^{\frac{p_i}{\bar{p}}}M^{\frac{\bar{p}-p_i}{p_i}}}{\pi\text{-dist}(k;\partial\Omega_T)}\right)^\alpha.$$

- If otherwise $|y_i-x_i| \geq M^{\frac{p_i-\bar{p}}{p_i}}R^{\frac{\bar{p}}{p_i}}$, then similarly

$$|u(y_i,t)-u(x_i,t)| \leq 2M \leq 4M\left(\frac{|y_i-x_i|^{\frac{p_i}{\bar{p}}}M^{\frac{\bar{p}-p_i}{p_i}}}{\pi\text{-dist}(k;\partial\Omega_T)}\right)^\alpha.$$

The proof is completed.

4. Proof of Theorem 1.3

We take as hypothesis that for each radius $r>0$ such that the intrinsic cylinder $\mathcal{Q}_{4r}(\theta)$ is contained in Ω_T, the right-hand Harnack estimate (1.6) holds and we show that the full Harnack estimate (1.10) comes as a consequence.

4.1. **Step 1.** Let us suppose that there exists a time $t_1 < t_0$ such that

(4.19) $$u(x_0,t_1) = 2\gamma u(x_0,t_0),$$

where $\gamma, c > 0$ are the constants in (1.6). For such a time, it must hold

(4.20) $$t_0-t_1 > \theta_{t_1}^{\bar{p}-2}r^{\bar{p}} := c\,u(x_0,t_1)^{2-\bar{p}}r^{\bar{p}} = c\,\frac{u(x_0,t_0)^{2-\bar{p}}}{(2\gamma)^{\bar{p}-2}}r^{\bar{p}},$$

owing last equality to (4.19). Indeed, if (4.20) were violated then $t_0 \in [t_1, t_1+\theta_{t_1}^{\bar{p}-2}r^{\bar{p}}]$, and by applying (1.6) evaluated in (x_0,t_1) for a radius $r>0$ small enough, we would incur a contradiction

$$u(x_0,t_1) \leq \gamma u(x_0,t_0) \quad \Longleftrightarrow \quad 2\gamma u(x_0,t_0) \leq u(x_0,t_0).$$

So (4.20) holds, and we set t_2 to be the time

(4.21) $$t_2 = t_0 - \theta_{t_1}^{\bar{p}-2}r^{\bar{p}}.$$

By (4.20), we deduce that $t_1 < t_2 < t_0$ and again by the right-hand Harnack estimate (1.6), we have that

(4.22) $$u(x_0,t_0) = \frac{u(x_0,t_1)}{2\gamma} \leq u(x_0,t_2) < 2\gamma u(x_0,t_0),$$

where the last inequality comes from t_1 being the first time before t_0 respecting (4.19). The contradiction of (4.19) is, in our context $u(x_0,t_2) < 2\gamma u(x_0,t_0)$, because the converse inequality conflicts with our hypothesis (1.6). Now, let $r>0$ be fixed, as in (1.6), and consider the vector $\xi \in \mathbb{R}^N$ whose components are

(4.23) $$\xi_i := \theta^{\frac{p_i-\bar{p}}{p_i}}r^{\frac{\bar{p}}{p_i}}, \quad \theta = \left(\frac{c}{u(x_0,t_0)}\right).$$

Now for each vector of parameters $s \in [0,1]^N$ define $\xi_s = (s_1\xi_1, ..., s_N\xi_N)$. As s varies in $[0,1]^N$, the configuration $x_0 + \xi_s$ describes all points of $x_0 + \mathcal{K}_r(\theta)$. Consider $\bar{s} \in [0,1]^N$ such that the vector $\xi_{\bar{s}}$ satisfies $u(x_0 + \xi_{\bar{s}}, t_2) = 2\gamma u(x_0, t_0)$. We claim that such an $\bar{s}$ does not exist or that $\bar{s} \geq 1$: in either case the conclusion is that

$$\sup_{x_0+\mathcal{K}_r(\theta)} u(\cdot, t_2) \leq 2\gamma\, u(x_0, t_0). \tag{4.24}$$

Thus to establish the claim, assume that such vector $\bar{s}$ exists and that $\bar{s} < 1$. Apply the estimate (1.6) in the point (x_2, t_2) with $x_2 = x_0 + \xi_{\bar{s}}$ to get

$$u(x_2, t_2) \leq \gamma \inf_{x_2+\mathcal{K}_r(\theta_{t_2})} u(\cdot, t_2 + \theta_{t_2}^{\bar{p}-2} r^{\bar{p}}) = \inf_{x_2+\mathcal{K}_r(\theta_{t_2})} u(\cdot, t_0), \quad \text{being} \quad \theta_{t_2} = \frac{c}{u(x_2, t_2)},$$

where last equality holds because of

$$\begin{aligned} t_2 + \theta_{t_2}^{\bar{p}-2} r^{\bar{p}} &= t_0 - \left(\frac{c}{2\gamma u(x_0, t_0)}\right)^{\bar{p}-2} r^{\bar{p}} + \left(\frac{c}{u(x_2, t_2)}\right)^{\bar{p}-2} \\ &= t_0 - \left(\frac{c}{2\gamma u(x_0, t_0)}\right)^{\bar{p}-2} r^{\bar{p}} + \left(\frac{c}{2\gamma u(x_0, t_0)}\right)^{\bar{p}-2} r^{\bar{p}} = t_0, \end{aligned} \tag{4.25}$$

being x_2 the point for which holds $u(x_2, t_2) = u(x_0 + \xi_{\bar{s}}, t_2) = 2\gamma\, u(x_0, t_0)$ by assumption. But since $\bar{s} < 1$, then $x_0 \in \{x_2 + \mathcal{K}_r(\theta_{t_2})\}$ and we arrive to the contradiction

$$2\gamma\, u(x_0, t_0) = u(x_2, t_2) \leq \gamma \inf_{x_2+\mathcal{K}_r(\theta_{t_2})} u(\cdot, t_0) \leq \gamma\, u(x_0, t_0).$$

Finally, the contradiction implies that (4.24) holds, which means that for each $r > 0$ such that $\mathcal{Q}_{4r}(\theta) \subseteq \Omega_T$ it holds

$$\sup_{x_0+\mathcal{K}_r(\theta)} u\left(\cdot\,, \left(\frac{c}{2\gamma\, u(x_0, t_0)}\right)^{\bar{p}-2} r^{\bar{p}}\right) \leq 2\gamma\, u(x_0, t_0).$$

Let $\rho > 0$ be such that the right hand side of (1.10) holds, then by choosing $r = \rho(2\gamma)^{\frac{\bar{p}-2}{\bar{p}}}$ we obtain, by suitably redefining the constants, the full estimate (1.10).

4.2. **Step 2.** Suppose on the contrary that such a time $t < t_0$ for which holds true (4.19) does not exist. In this case, we have

$$u(x_0, t) < 2\gamma u(x_0, t_0), \quad \text{for all} \quad t \in [t_0 - \theta(4r)^{\bar{p}}, t_0], \tag{4.26}$$

because the converse inequality would be in conflict with the holding Harnack estimate. We establish by contradiction that this in turn implies

$$\sup_{x_0+\mathcal{K}_r(\theta)} u(\cdot, t_0 - \theta^{\bar{p}-2} r^{\bar{p}}) \leq 2\gamma^2 u(x_0, t_0). \tag{4.27}$$

If not, it simultaneously holds (4.26) and a fortiori

$$\sup_{x_0+\mathcal{K}_\rho(\theta)} u(\cdot, \bar{t}) > 2\gamma^2 u(x_0, t_0) > u(x_0, \bar{t}), \qquad \text{for} \quad \bar{t} = t_0 - \theta^{\bar{p}-2} r^{\bar{p}}. \tag{4.28}$$

Thus, by the proven continuity in space, there must exist by the intermediate value theorem a point $\bar{x} \in x_0 + \mathcal{K}_r(\theta)$ such that

$$u(\bar{x}, \bar{t}) = 2\gamma u(x_0, t_0). \tag{4.29}$$

We apply the Harnack estimate (1.6) centered in $(\bar{x}, \bar{t})$ to get

$$u(\bar{x}, \bar{t}) \leq \gamma \inf_{\bar{x}+\mathcal{K}_r(\theta_{\bar{t}})} u(\cdot, \bar{t} + \theta_{\bar{t}}^{\bar{p}-2} r^{\bar{p}}), \quad \text{where} \quad \theta_{\bar{t}} = \frac{c}{u(\bar{x}, \bar{t})}.$$

Now, as $\gamma > 1$ and $p_i > 2$ for each $i \in \{1, .., N\}$, we have

$$\begin{cases} \theta_{\bar{t}}^{\frac{p_i-\bar{p}}{p_i}} r^{\frac{\bar{p}}{p_i}} &= \left(\frac{2}{2\gamma u(x_0,t_0)}\right)^{\frac{p_i-\bar{p}}{p_i}} r^{\frac{\bar{p}}{p_i}} \geq \left(\frac{2}{2u(x_0,t_0)}\right)^{\frac{p_i-\bar{p}}{p_i}} r^{\frac{\bar{p}}{p_i}} = \theta^{\frac{p_i-\bar{p}}{p_i}} r^{\frac{\bar{p}}{p_i}} \quad \Rightarrow \quad x_0 \in \{\bar{x} + \mathcal{K}_r(\theta_{\bar{t}})\}, \\ \bar{t} + \theta_{\bar{t}}^{\bar{p}-2} r^{\bar{p}} &= t_0 - \left(\frac{c}{u(x_0,t_0)}\right)^{\bar{p}-2} r^{\bar{p}} + \left(\frac{c}{2\gamma u(x_0,t_0)}\right)^{\bar{p}-2} < t_0, \end{cases}$$

and thus, finally,

$$2\gamma^2 u(x_0, t_0) = u(\bar{x}, \bar{t}) \leq \gamma u(x_0, \bar{t} + \theta_{\bar{t}}^{\bar{p}-2} r^{\bar{p}}) < 2\gamma^2 u(x_0, t_0),$$

owing last inequality to (4.28) and establishing (4.27) by contradiction. Finally, the estimate (4.27), by possibly redefining the constants, is the desired left-hand estimate of (1.10).

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Research Article

Weak A-frames and weak A-semi-frames

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ABSTRACT. After reviewing the interplay between frames and lower semi-frames, we introduce the notion of lower semi-frame controlled by a densely defined operator A or, for short, a *weak lower A-semi-frame* and we study its properties. In particular, we compare it with that of lower atomic systems, introduced by one of us (GB). We discuss duality properties and we suggest several possible definitions for weak A-upper semi-frames. Concrete examples are presented.

Keywords: *A-frames, weak (upper and lower) A-semi-frames, lower atomic systems, G-duality.*

2020 Mathematics Subject Classification: 41A99, 42C15.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION AND BASIC FACTS

We consider an infinite dimensional Hilbert space $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle$, linear in the first entry, and norm $\|\cdot\|$. $GL(\mathcal{H})$ denotes the set of all invertible bounded operators on $\mathcal{H}$ with bounded inverse. Given a linear operator A, we denote its domain by $\mathcal{D}(A)$, its range by $\mathcal{R}(A)$ and its adjoint by A^*, if A is densely defined. Given a locally compact, σ-compact space (X,μ) with a (Radon) measure μ, a function $\psi : X \mapsto \mathcal{H}, x \mapsto \psi_x$ is said to be *weakly measurable* if for every $f \in \mathcal{H}$ the function $x \mapsto \langle f|\psi_x\rangle$ is measurable. As a particular case, we obtain a discrete situation if $X = \mathbb{N}$ and μ is the counting measure. Given a weakly measurable function ψ, the operator $C_\psi : \mathcal{D}(C_\psi) \subseteq \mathcal{H} \to L^2(X,\, \mathrm{d}\mu)$ with domain

$$\mathcal{D}(C_\psi) := \left\{ f \in \mathcal{H} : \int_X |\langle f|\psi_x\rangle|^2 \, \mathrm{d}\mu(x) < \infty \right\}$$

and $(C_\psi f)(x) = \langle f|\psi_x\rangle$, $f \in \mathcal{D}(C_\psi)$, C_ψ is called the *analysis* operator of ψ.

Remark 1.1. *In general, the domain of C_ψ is not dense, hence C_ψ^* is not well-defined. An example of function whose analysis operator is densely defined can be found in* [10, Example 2.8], *where $\mathcal{D}(C_\psi)$ coincides with the domain of a densely defined sesquilinear form associated to ψ. Moreover, a sufficient condition for $\mathcal{D}(C_\psi)$ to be dense in $\mathcal{H}$ is that $\psi_x \in \mathcal{D}(C_\psi)$ for every $x \in X$, see* [3, Lemma 2.3].

Proposition 1.1. [3, Lemma 2.1] *Let (X,μ) be a locally compact, σ-compact space, with a Radon measure μ and $\psi : x \in X \mapsto \psi_x \in \mathcal{H}$ a weakly measurable function. Then, the analysis operator C_ψ is closed.*

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Consider the set $\mathcal{D}(\Omega_\psi) = \mathcal{D}(C_\psi)$ and the mapping $\Omega_\psi : \mathcal{D}(C_\psi) \times \mathcal{D}(C_\psi) \to \mathbb{C}$ defined by

$$\Omega_\psi(f,g) := \int_X \langle f|\psi_x\rangle\langle\psi_x|g\rangle \, \mathrm{d}\mu(x). \tag{1.1}$$

Ω_ψ is clearly a nonnegative symmetric sesquilinear form which is well defined for every $f, g \in \mathcal{D}(C_\psi)$ because of the Cauchy-Schwarz inequality. It is unbounded in general. Moreover, since $\mathcal{D}(C_\psi)$ is the largest domain such that Ω_ψ is defined on $\mathcal{D}(C_\psi) \times \mathcal{D}(C_\psi)$, it follows that

$$\Omega_\psi(f,g) = \langle C_\psi f | C_\psi g\rangle, \quad \forall f, g \in \mathcal{D}(C_\psi), \tag{1.2}$$

where C_ψ is the analysis operator defined above. Since C_ψ is a closed operator, the form Ω_ψ is closed, see e.g. [16, Example VI.1.13]. If $\mathcal{D}(C_\psi)$ is dense in $\mathcal{H}$, then by Kato's first representation theorem [16, Theorem VI.2.1], there exists a positive self-adjoint operator T_ψ *associated to the sesquilinear form* Ω_ψ on

$$\mathcal{D}(\mathsf{T}_\psi) = \left\{ f \in \mathcal{D}(\Omega_\psi) : h \mapsto \int_X \langle f|\psi_x\rangle\langle\psi_x|h\rangle \, \mathrm{d}\mu(x) \text{ is bounded in } \mathcal{D}(C_\psi) \right\} \tag{1.3}$$

defined by

$$\mathsf{T}_\psi f := h \tag{1.4}$$

with h as in (1.3). The density of $\mathcal{D}(\Omega_\psi)$ ensures the uniqueness of the vector h. The operator T_ψ is the greatest one whose domain is contained in $\mathcal{D}(\Omega_\psi)$ and such that

$$\Omega_\psi(f,g) = \langle \mathsf{T}_\psi f | g\rangle, \quad f \in \mathcal{D}(\mathsf{T}_\psi),\ g \in \mathcal{D}(\Omega_\psi).$$

The set $\mathcal{D}(\mathsf{T}_\psi)$ is dense in $\mathcal{D}(\Omega_\psi)$, see [16, p. 279]. In addition, by Kato's second representation theorem [16, Theorem VI.2.23], we have $\mathcal{D}(\Omega_\psi) = \mathcal{D}(\mathsf{T}_\psi^{1/2})$ and

$$\Omega_\psi(f,g) = \langle \mathsf{T}_\psi^{1/2} f | \mathsf{T}_\psi^{1/2} g\rangle, \quad \forall f, g \in \mathcal{D}(\Omega_\psi),$$

hence, comparing with (1.2), we deduce $\mathsf{T}_\psi = C_\psi^* C_\psi = |C_\psi|^2$ on $\mathcal{D}(\mathsf{T}_\psi)$.

Definition 1.1. *The operator* $\mathsf{T}_\psi : \mathcal{D}(\mathsf{T}_\psi) \subset \mathcal{H} \to \mathcal{H}$ *defined by* (1.4) *will be called the* generalized frame operator *of the function* $\psi : x \in X \to \psi_x \in \mathcal{H}$.

Now, we recall a series of notions well-known in the literature, see e.g. [1, 3, 15]. A weakly measurable function ψ is said to be

- μ-total if $\langle f|\psi_x\rangle = 0$ for a.e. $x \in X$ implies that $f = 0$;
- a *continuous frame* of $\mathcal{H}$ if there exist constants $0 < \mathsf{m} \leq \mathsf{M} < \infty$ (the frame bounds) such that
$$\mathsf{m}\|f\|^2 \leq \int_X |\langle f|\psi_x\rangle|^2 \, \mathrm{d}\mu(x) \leq \mathsf{M}\,\|f\|^2, \qquad \forall f \in \mathcal{H};$$
- a *Bessel mapping* of $\mathcal{H}$ if there exists $\mathsf{M} > 0$ such that
$$\int_X |\langle f|\psi_x\rangle|^2 \, \mathrm{d}\mu(x) \leq \mathsf{M}\,\|f\|^2, \qquad \forall f \in \mathcal{H};$$
- an *upper semi-frame* of $\mathcal{H}$ if there exists $\mathsf{M} < \infty$ such that
$$0 < \int_X |\langle f|\psi_x\rangle|^2 \, \mathrm{d}\mu(x) \leq \mathsf{M}\,\|f\|^2, \qquad \forall f \in \mathcal{H},\ f \neq 0,$$
i.e., if it is a μ-total Bessel mapping;
- a *lower semi-frame* of $\mathcal{H}$ if there exists a constant $\mathsf{m} > 0$ such that
$$\mathsf{m}\,\|f\|^2 \leq \int_X |\langle f|\psi_x\rangle|^2 \, \mathrm{d}\mu(x), \qquad \forall f \in \mathcal{H}. \tag{1.5}$$

Note that the integral on the right hand side in (1.5) may diverge for some $f \in \mathcal{H}$, namely, for $f \notin \mathcal{D}(C_\psi)$. Moreover, if ψ satisfies (1.5), then it is automatically μ-total.

2. From semi-frames to frames and back

Starting from a lower semi-frame, one can easily obtain a genuine frame, albeit in a smaller space. Indeed, we have proved a theorem [8, Prop.3.5], which implies the following:

Proposition 2.2. *A weakly measurable function ϕ on $\mathcal{H}$ is a lower semi-frame of $\mathcal{H}$ whenever $\mathcal{D}(C_\phi)$ is complete for the norm $\|f\|^2_{C_\phi} = \int_X |\langle f|\phi_x\rangle|^2 \, \mathrm{d}\mu(x) = \|C_\phi f\|^2$, continuously embedded into $\mathcal{H}$ and for some $\alpha, \mathsf{m}, \mathsf{M} > 0$, one has*

$$\alpha \, \|f\| \leq \|f\|_{C_\phi} \text{ and} \tag{2.1}$$

$$\mathsf{m} \, \|f\|^2_{C_\phi} \leq \int_X |\langle f|\phi_x\rangle|^2 \, \mathrm{d}\mu(x) \leq \mathsf{M} \, \|f\|^2_{C_\phi}, \ \forall f \in \mathcal{D}(C_\phi). \tag{2.2}$$

Note that (2.2) is trivial here. Following the notation of our previous papers, denote by $\mathcal{H}(\mathsf{T}_\phi^{1/2})$ the Hilbert space $\mathcal{D}(\mathsf{T}_\phi^{1/2})$ with the norm $\|f\|^2_{1/2} = \left\|\mathsf{T}_\phi^{1/2} f\right\|^2$, where T_ϕ is the generalized frame operator defined in (1.4). In the same way, denote by $\mathcal{H}(C_\phi)$ the Hilbert space $\mathcal{D}(C_\phi)$ with the inner product $\langle \cdot|\cdot\rangle_{C_\phi} = \langle C_\phi \cdot | C_\phi \cdot\rangle$, and the corresponding norm $\|f\|^2_{C_\phi} = \|C_\phi f\|^2$. Then, clearly $\mathcal{H}(\mathsf{T}_\phi^{1/2}) = \mathcal{H}(C_\phi)$. What we have obtained in Proposition 2.2 is a frame in $\mathcal{H}(C_\phi) = \mathcal{H}(\mathsf{T}_\phi^{1/2})$. Indeed, assume that $\mathcal{D}(C_\phi)$ is dense. Then, for every $x \in X$, the map $f \mapsto \langle f|\phi_x\rangle$ is a bounded linear functional on the Hilbert space $\mathcal{H}(C_\phi)$. By the Riesz Lemma, there exists an element $\chi^\phi_x \in \mathcal{D}(C_\phi)$ such that

$$\langle f|\phi_x\rangle = \langle f|\chi^\phi_x\rangle_{C_\phi} \quad \forall f \in \mathcal{D}(C_\phi).$$

By Proposition 2.2, χ^ϕ is a frame. Actually, one can say more [8]. The norm $\|f\|^2_{1/2} = \left\|\mathsf{T}_\phi^{1/2} f\right\|^2$ is equivalent to the the graph norm of $\mathsf{T}_\phi^{1/2}$. Hence, $\langle f|\phi_x\rangle = \langle f|\chi^\phi_x\rangle_{C_\phi} = \langle f|\mathsf{T}_\phi \chi^\phi_x\rangle$ for all $f \in \mathcal{D}(C_\phi)$. Thus, $\chi^\phi_x = \mathsf{T}_\phi^{-1}\phi_x$ for all $x \in X$, i.e., χ^ϕ is the canonical dual Bessel mapping of ϕ (we recall that ϕ may have several duals).

Proposition 2.3. *Let ϕ be a lower semi-frame of $\mathcal{H}$ with $\mathcal{D}(C_\phi)$ dense. Then, the canonical dual Bessel mapping of ϕ is a tight frame for the Hilbert space $\mathcal{H}(C_\phi)$.*

Conversely, starting with a frame $\chi \in \mathcal{D}(C_\phi)$, does there exists a lower semi-frame η of $\mathcal{H}$ such that χ is the frame χ^η constructed from η in the way described above. The answer is formulated in the following [13, Prop. 6].

Proposition 2.4. *Let χ be a frame of $\mathcal{H}(C_\phi) = \mathcal{H}(\mathsf{T}_\phi^{1/2})$. Then,*

(i) there exists a lower semi-frame η of $\mathcal{H}$ such that $\chi = \chi^\eta$ if and only if $\chi \in \mathcal{D}(\mathsf{T}_\phi)$;
(ii) if $\chi = \chi^\eta$ for some lower semi-frame η of $\mathcal{H}$, then $\eta = \mathsf{T}_\phi \chi$.

So far, we have discussed the interplay between frames and lower semi-frames. But, one question remains: how does one obtain semi-frames? A standard construction is to start from an unbounded operator A and build a lattice of Hilbert spaces out of it, as described in [4] and in [8]. As we will see in Section 6 (1) and (2) below, this approach indeed generates a weak lower A-semi-frame. Before that, we need a new ingredient, namely the notion of metric operator. Given a closed unbounded operator S with dense domain $\mathcal{D}(S)$, define the operator $G = I + S^*S$, which is unbounded, with $G > 1$ and bounded inverse. This is a *metric operator*, that is, a strictly positive self-adjoint operator G, that is, $G > 0$ or $\langle Gf|f\rangle \geq 0$ for every $f \in \mathcal{D}(G)$ and $\langle Gf|f\rangle = 0$ if and only if $f = 0$. Then, the norm $\|f\|_{G^{1/2}} = \|G^{1/2}f\|$ is equivalent to the graph norm of $G^{1/2}$ on $\mathcal{D}(G^{1/2}) = \mathcal{D}(S)$ and makes the latter into a Hilbert space continuously embedded into $\mathcal{H}$, denoted by $\mathcal{H}(G)$. Then $\mathcal{H}(G^{-1})$, built in the same way from G^{-1}, coincides, as a vector space, with the conjugate dual of $\mathcal{H}(G)$. On the other hand, G^{-1} is bounded. Hence, we get the triplet

$$\mathcal{H}(G) \ \subset \ \mathcal{H} \ \subset \ \mathcal{H}(G^{-1}) = \mathcal{H}(G)^{\times}. \tag{2.3}$$

Two developments arise from these relations. First, the triplet (2.3) is the central part of the discrete scale of Hilbert spaces $V_{\mathcal{G}}$ built on the powers of $G^{1/2}$. This means that $V_{\mathcal{G}} := \{\mathcal{H}_n, n \in \mathbb{Z}\}$, where $\mathcal{H}_n = \mathcal{D}(G^{n/2}), n \in \mathbb{N}$, with a norm equivalent to the graph norm, and $\mathcal{H}_{-n} = \mathcal{H}_n^{\times}$:

$$\ldots \subset \ \mathcal{H}_2 \ \subset \ \mathcal{H}_1 \ \subset \ \mathcal{H} \ \subset \ \mathcal{H}_{-1} \subset \ \mathcal{H}_{-2} \subset \ \ldots .$$

Thus $\mathcal{H}_1 = \mathcal{H}(G^{1/2}) = \mathcal{D}(S)$, $\mathcal{H}_2 = \mathcal{H}(G) = \mathcal{D}(S^*S)$, and $\mathcal{H}_{-2} = \mathcal{H}(G^{-1})$, and so on. What we have obtained in this way is a Lattice of Hilbert Spaces (LHS), the simplest example of a Partial Inner Product Spaces (PIP-space). See our monograph [2] about this structure.

One may also add the end spaces of the scale, namely,

$$\mathcal{H}_{\infty}(G) := \cap_{n\in\mathbb{Z}}\mathcal{H}_n, \qquad \mathcal{H}_{-\infty}(G) := \bigcup_{n\in\mathbb{Z}} \mathcal{H}_n. \tag{2.4}$$

In this way, we get a genuine Rigged Hilbert Space:

$$\mathcal{H}_{\infty}(G) \subset \mathcal{H} \subset \mathcal{H}_{-\infty}(G).$$

In fact, one can go one more step farther. Namely, following [2, Sec. 5.1.2], we can use quadratic interpolation theory [12] and build a continuous scale of Hilbert spaces $\mathcal{H}_\alpha, \alpha \geq 0$, where $\mathcal{H}_\alpha = \mathcal{D}(G^{\alpha/2})$, with the graph norm $\|\xi\|_\alpha^2 = \|\xi\|^2 + \|G^{\alpha/2}\xi\|^2$ or, equivalently, the norm $\left\|(I+G)^{\alpha/2}\xi\right\|^2$. Indeed, every $G^\alpha, \alpha \geq 0$, is an unbounded metric operator. Next, we define $\mathcal{H}_{-\alpha} = \mathcal{H}_\alpha^{\times}$ and thus obtain the full continuous scale $V_{\widetilde{\mathcal{G}}} := \{\mathcal{H}_\alpha, \alpha \in \mathbb{R}\}$. Of course, one can replace $\mathbb{Z}$ by $\mathbb{R}$ in the definition (2.4) of the end spaces of the scale. A second development of the previous analysis is that we have made a link to the formalism based on metric operators that we have developed for the theory of pseudo-Hermitian operators, in particular non-self-adjoint Hamiltonians, as encountered in the so-called pseudo-Hermitian or $\mathcal{PT}$-symmetric quantum mechanics. This is not the place, however, to go into details, instead we refer the reader to [4, 5] for a complete mathematical treatment.

3. Weak lower A-semi-frames

The following concept was introduced and studied in [10].

Definition 3.2. *Let A be a densely defined operator on $\mathcal{H}$. A* (continuous) weak A-frame *is a function $\phi : x \in X \mapsto \phi_x$ such that, for all $u \in \mathcal{D}(A^*)$, the map $x \mapsto \langle u|\phi_x\rangle$ is a measurable function on X and, for some $\alpha > 0$,*

$$\alpha \left\|A^* u\right\|^2 \leq \int_X |\langle u|\phi_x\rangle|^2 \,\mathrm{d}\mu(x) < \infty, \qquad \forall\, u \in \mathcal{D}(A^*). \tag{3.1}$$

If $X = \mathbb{N}$ and μ is the counting measure, we recover the discrete situation (so that the word "continuous" is superfluous in the definition above). We get a simpler situation when A is bounded and ϕ is Bessel. This is in fact the construction of Găvruţa [14]. Now, we introduce a structure that generalizes both concepts of lower semi-frame and weak A-frame. We follow mostly the terminology of [10] and keep the term "weak" because the notion leads to a weak decomposition of the range of the operator A (see Theorem 4.2). We begin with giving the following definitions.

Definition 3.3. *Let A be a densely defined operator on $\mathcal{H}$, $\phi : x \in X \mapsto \phi_x$ a function such that, for all $u \in \mathcal{D}(A^*)$, the map $x \mapsto \langle u|\phi_x\rangle$ is measurable on X. We say that a closed operator B is a ϕ-*extension *of A if*

$$A \subset B \;\text{ and }\; \mathcal{D}(B^*) \subset \mathcal{D}(C_\phi).$$

We denote by $\mathcal{E}_\phi(A)$ the set of ϕ-extensions of A.

Remark 3.2. *It is worth noting that, if A has a ϕ-extension, then A is automatically closable.*

Definition 3.4. *Let A and ϕ be as in Definition 3.3. Then ϕ is called a* weak lower A-semi-frame *if A admits a ϕ-extension B such that ϕ is a weak B-frame.*

Let us put $\mathcal{D}(A, \phi) := \mathcal{D}(A^*) \cap \mathcal{D}(C_\phi)$. If ϕ and A are as in Definition 3.3, and $B := (A^* \upharpoonright \mathcal{D}(A, \phi))^*$ is a ϕ-extension of A, it would be the smallest possible extension for which ϕ is a weak B-frame, but in general, we could have a larger extension enjoying the same property. Indeed, if B is a closed extension of A such that ϕ is a weak B-frame, we have

$$A \subset A^{**} \subset (A^* \upharpoonright \mathcal{D}(A, \phi))^* \subset B.$$

Remark 3.3.

(1) *If A is bounded, $\mathcal{D}(A^*) = \mathcal{H}$ and we recover the notion of lower semi-frame, under some minor restrictions on A, hence the name (see Proposition 5.5).*

(2) *If A is a densely defined operator on $\mathcal{H}$ such that the integral on the right hand side of (3.1) is finite for every $f \in \mathcal{D}(A^*)$, then $\mathcal{D}(A^*) \subset \mathcal{D}(C_\phi)$ and the weak lower A-semi-frame ϕ is, in fact, a weak A-frame, in the sense of Definition 3.2.*

(3) *Let us assume that ϕ is both a lower semi-frame and a weak A-frame, then we have simultaneously*

$$\mathsf{m}\,\|f\|^2 \le \int_X |\langle f|\phi_x\rangle|^2 \, d\mu(x), \quad \forall\, f \in \mathcal{H},$$
$$\alpha\,\|A^* f\|^2 \le \int_X |\langle f|\phi_x\rangle|^2 \, d\mu(x) < \infty, \quad \forall\, f \in \mathcal{D}(A, \phi) = \mathcal{D}(A^*) \cap \mathcal{D}(C_\phi).$$

It follows that

$$\alpha'(\|f\|^2 + \|A^* f\|^2) \le \int_X |\langle f|\phi_x\rangle|^2 \, d\mu(x) < \infty, \quad \forall\, f \in \mathcal{D}(A^*) \cap \mathcal{D}(C_\phi) \tag{3.2}$$

with $\alpha' \le \frac{1}{2}\min\{\mathsf{m}, \alpha\}$. If we consider the domain $\mathcal{D}(A^*)$ with its graph norm $(\|f\|_{A^*} = (\|f\|^2 + \|A^* f\|^2)^{1/2}$, $f \in \mathcal{D}(A^*))$, we are led to the triplet of Hilbert spaces

$$\mathcal{H}(A^*) \subset \mathcal{H} \subset \mathcal{H}(A^*)^\times,$$

as discussed in Section 2. Let us consider the sesquilinear form Ω_ϕ defined in (1.1) and suppose in particular that $\mathcal{D}(A^*) = \mathcal{D}(\Omega_\phi) = \mathcal{D}(C_\phi)$. Then, using Proposition 1.1, it is not difficult to prove that Ω_ϕ is closed in $\mathcal{H}(A^*)$ and then bounded. Thus, there exists $\gamma > 0$ such that, for every $f \in \mathcal{D}(A^*)$,

$$\alpha'(\|f\|^2 + \|A^* f\|^2) \le \int_X |\langle f|\phi_x\rangle|^2 \, d\mu(x) \le \gamma(\|f\|^2 + \|A^* f\|^2).$$

One could notice that (3.2) is similar to a frame condition. The inequality (3.2) says that the sesquilinear form Ω_ϕ defined in (1.1) is coercive on $\mathcal{H}(A^*)$ and thus the Lax-Milgram theorem *applies* [16, VI §2, 2] *or* [18, Lemma 11.2]. *This means that for every* $F \in \mathcal{H}(A^*)^\times$ *there exists* $w \in \mathcal{H}(A^*)$ *such that*

$$\langle F|f\rangle = \Omega_\phi(w, f) = \int_X \langle w|\phi_x\rangle\langle\phi_x|f\rangle \,\mathrm{d}\mu(x), \quad \forall f \in \mathcal{H}(A^*).$$

Therefore, in the case under consideration, we get expansions in terms of ϕ *of elements that do not belong to the domain of* A^**; in particular, those of* $\mathcal{H}$. *The price to pay is that the form of this expansion is necessarily weak since vectors of* $\mathcal{H}$ *do not belong to the domain of the analysis operator* C_ϕ.

In the sequel, we will need the following:

Lemma 3.1. [11, Lemma 3.8] *Let* $(\mathcal{H}, \|\cdot\|), (\mathcal{H}_1, \|\cdot\|_1)$ *and* $(\mathcal{H}_2, \|\cdot\|_2)$ *be Hilbert spaces and* $T_1 : \mathcal{D}(T_1) \subseteq \mathcal{H}_1 \to \mathcal{H}$, $T_2 : \mathcal{D}(T_2) \subseteq \mathcal{H} \to \mathcal{H}_2$ *densely defined operators. Assume that* T_1 *is closed and* $\mathcal{D}(T_1^*) = \mathcal{D}(T_2)$. *If* $\|T_1^* f\|_1 \leq \lambda\|T_2 f\|_2$ *for all* $f \in \mathcal{D}(T_1^*)$ *and some* $\lambda > 0$, *then there exists a bounded operator* $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ *such that* $T_1 = T_2^* U$.

Remark 3.4. *Lemma 3.1 is still valid if we replace closedness of* T_1 *by its closability, and in this hypothesis,* $\overline{T_1} = T_2^* U$.

In the literature [19], two measurable functions ψ and ϕ are said to be *dual* to each other if one has

$$\langle f|g\rangle = \int_X \langle f|\phi_x\rangle\langle\psi_x|g\rangle \,\mathrm{d}\mu(x), \qquad \forall f, g \in \mathcal{H}. \tag{3.3}$$

If ϕ is a lower semi-frame of $\mathcal{H}$, then its dual ψ is a Bessel mapping of $\mathcal{H}$ [8]. In addition, if $\mathcal{D}(C_\phi)$ is dense, its dual ψ is an upper semi-frame. However, this definition is too general, in the sense that the right hand side may diverge for arbitrary $f, g \in \mathcal{H}$. A more useful definition will be given below, namely (5.3). A notion of duality related to a given operator G can be formulated as follows.

Definition 3.5. *Let* G *be a densely defined operator and* $\phi : x \in X \mapsto \phi_x$ *a function such that, for all* $u \in \mathcal{D}(G^*)$ *the map* $x \mapsto \langle u|\phi_x\rangle$ *is a measurable function on* X. *Then a function* $\psi : x \in X \mapsto \psi_x \in \mathcal{H}$ *such that, for all* $f \in \mathcal{D}(G)$ *the map* $x \mapsto \langle f|\psi_x\rangle$ *is a measurable function on* X *is called a weak G-dual of* ϕ *if*

$$\langle Gf|u\rangle = \int_X \langle f|\psi_x\rangle\langle\phi_x|u\rangle \,\mathrm{d}\mu(x), \quad \forall f \in \mathcal{D}(G) \cap \mathcal{D}(C_\psi), \forall u \in \mathcal{D}(G^*) \cap \mathcal{D}(C_\phi). \tag{3.4}$$

This is a generalization of the notion of weak G-dual in [10].

Remark 3.5.

(i) *The weak G-dual* ψ *of* ϕ *is not unique, in general. On the other hand, Definition 3.5 could be meaningless. For instance, if either* $\mathcal{D}(G) \cap \mathcal{D}(C_\psi) = \{0\}$ *or* $\mathcal{D}(G^*) \cap \mathcal{D}(C_\phi) = \{0\}$, *then everything is "dual".*

(ii) *Note that, if* ϕ *is a weak G-frame, then there exists a weak G-dual* ψ *of* ϕ *such that relation* (3.4) *must hold only for* $\forall f \in \mathcal{D}(G), \forall u \in \mathcal{D}(G^*)$ *indeed* $\mathcal{D}(G^*) \subset \mathcal{D}(C_\phi)$ *and by Theorem 3.20 in* [10], *there exists a Bessel weak G-dual* ψ *of* ϕ, *hence* $\mathcal{D}(G) \subset \mathcal{D}(C_\psi) = \mathcal{H}$.

Example 3.1. *Given a densely defined operator G on a separable Hilbert space* $\mathcal{H}$, *we show two examples of G-duality (see* [10, Ex. 3.10]*).*

(i) Let (X, μ) *be a locally compact,* σ*-compact measure space and let* $\{X_n\}_{n\in\mathbb{N}}$ *be a covering of* X *made up of countably many measurable disjoint sets of finite measure. Without loss of generality, we suppose that* $\mu(X_n) > 0$ *for every* $n \in \mathbb{N}$. *Let* $\{e_n\} \subset \mathcal{D}(G)$ *be an orthonormal basis*

of $\mathcal{H}$ and consider ϕ, with $\phi_x = \frac{Ge_n}{\sqrt{\mu(X_n)}}$, $x \in X_n, \forall n \in \mathbb{N}$, then ϕ is a weak G-frame, see [10, Example 3.10]. *One can take ψ with $\psi_x = \frac{e_n}{\sqrt{\mu(X_n)}}$, $x \in X_n, \forall n \in \mathbb{N}$.*

(ii) If $\phi := G\zeta$, where $\zeta : x \in X \mapsto \zeta_x \in \mathcal{D}(G) \subset \mathcal{H}$ is a continuous frame for $\mathcal{H}$, then one can take as ψ any dual frame of ζ.

4. Lower atomic systems

Theorem 4.1. *Let (X, μ) be a locally compact, σ-compact measure space, A a densely defined operator and $\phi : x \in X \mapsto \phi_x \in \mathcal{H}$ a map such that, for every $u \in \mathcal{D}(A^*)$, the function $x \mapsto \langle u|\phi_x\rangle$ is measurable on X. Then, the following statements are equivalent:*

(i) ϕ is a weak lower A-semi-frame for $\mathcal{H}$;

(ii) $\mathcal{E}_\phi(A) \neq \emptyset$ and for every $B \in \mathcal{E}_\phi(A)$, there exists a closed densely defined extension R of C_ϕ^*, with $\mathcal{D}(R^*) = \mathcal{D}(B^*)$, such that B can be decomposed as $B = RM$ for some $M \in \mathcal{B}(\mathcal{H}, L^2(X, \mu))$.

Proof. We proceed as in [10, Theor. 3.16].

(i)⇒(ii): If ϕ is a weak lower A-semi-frame for $\mathcal{H}$, by definition, there exists $B \in \mathcal{E}_\phi(A)$. Consider $E : \mathcal{D}(B^*) \to L^2(X, \mu)$ given by $(Eu)(x) = \langle u|\phi_x\rangle$, $\forall u \in \mathcal{D}(B^*)$, $x \in X$ which is a restriction of the analysis operator C_ϕ. E is closable and densely defined.

Apply Lemma 3.1 to $T_1 := B$, and $T_2 := E$, noting that $\|Eu\|_2^2 = \int_X |\langle u|\phi_x\rangle|^2 \,\mathrm{d}\mu(x)$, $u \in \mathcal{D}(B^*)$. Thus, there exists $M \in \mathcal{B}(\mathcal{H}, L^2(X, \mu))$ such that $B = E^*M$. Then the statement is proved by taking $R = E^*$, indeed $R = E^* \supseteq C_\phi^*$ and $\mathcal{D}(R) \supset \mathcal{D}(C_\phi^*)$ is dense because C_ϕ is closed and densely defined. Note that, we have $\mathcal{D}(B^*) = \mathcal{D}(R^*)$; indeed $\mathcal{D}(R^*) = \mathcal{D}(\overline{E})$,

$$\mathcal{D}(B^*) \subset \mathcal{D}(\overline{E}) = \mathcal{D}(M^*\overline{E}) \subset \mathcal{D}((E^*M)^*) = \mathcal{D}(B^*),$$

hence, in particular, E is closed, recalling that $\mathcal{D}(E) = \mathcal{D}(B^*)$.

(ii)⇒(i): Let $B \in \mathcal{E}_\phi(A)$. For every $u \in \mathcal{D}(B^*) = \mathcal{D}(R^*)$,

$$\|B^*u\|^2 = \|M^*R^*u\|^2 \le \|M^*\|^2\|R^*u\|^2 = \|M^*\|^2 \int_X |\langle u|\phi_x\rangle|^2 \,\mathrm{d}\mu(x) < \infty$$

since $R^* \subset C_\phi$. This proves that ϕ is a weak lower A-semi-frame. □

Generalizing the notion of continuous weak atomic system for A [10], we consider the following:

Definition 4.6. *Let A be a densely defined operator on $\mathcal{H}$. A* lower atomic system *for A is a function $\phi : x \in X \mapsto \phi_x \in \mathcal{H}$ such that*

(i) *for all $u \in \mathcal{D}(A^*)$, the map $x \mapsto \langle u|\phi_x\rangle$ is a measurable function on X;*

(ii) *the operator A has a closed extension B such that $\mathcal{D}(B^*) \subset \mathcal{D}(C_\phi)$; i.e., $\mathcal{E}_\phi(A) \neq \emptyset$;*

(iii) *there exists $\gamma > 0$ such that, for every $f \in \mathcal{D}(A)$, there exists $a_f \in L^2(X, \mu)$, with $\|a_f\|_2 = \left(\int_X |a_f(x)|^2 \,\mathrm{d}\mu(x)\right)^{1/2} \le \gamma\|f\|$ and*

$$\langle Af|u\rangle = \int_X a_f(x)\langle\phi_x|u\rangle \,\mathrm{d}\mu(x), \qquad \forall u \in \mathcal{D}(B^*).$$

We have chosen not to call ϕ a *weak* lower atomic system for A for brevity, even if it leads to a weak decomposition of the range of the operator A. Theorem 3.20 of [10], gives a characterization of weak atomic systems for A and weak A-frames. The next theorem yields the corresponding result for weak lower A-semi-frames.

Theorem 4.2. *Let (X, μ) be a locally compact, σ-compact measure space, A a densely defined operator in $\mathcal{H}$ and $\phi : x \in X \mapsto \phi_x \in \mathcal{H}$ a function such that, for all $u \in \mathcal{D}(A^*)$, the map $x \mapsto \langle u|\phi_x\rangle$ is measurable on X. Then, the following statements are equivalent:*

(i) *ϕ is a lower atomic system for A;*
(ii) *ϕ is a weak lower A-semi-frame for $\mathcal{H}$;*
(iii) *$\mathcal{E}_\phi(A) \neq \emptyset$ and for every $B \in \mathcal{E}_\phi(A)$, ϕ has a Bessel weak B-dual ψ.*

Proof.
(i)⇒(ii): Consider a ϕ-extension B of A. By the density of $\mathcal{D}(A)$, we have, for every $u \in \mathcal{D}(B^*)$

$$\begin{aligned}\|B^*u\| &= \sup_{f\in\mathcal{H},\|f\|=1} |\langle B^*u|f\rangle| = \sup_{f\in\mathcal{D}(A),\|f\|=1} |\langle B^*u|f\rangle| \\ &= \sup_{f\in\mathcal{D}(A),\|f\|=1} |\langle u|Bf\rangle| = \sup_{f\in\mathcal{D}(A),\|f\|=1} |\langle u|Af\rangle| \\ &= \sup_{f\in\mathcal{D}(A),\|f\|=1} \left|\int_X \overline{a_f(x)}\langle u|\phi_x\rangle \,\mathrm{d}\mu(x)\right| \\ &\le \sup_{f\in\mathcal{D}(A),\|f\|=1} \left(\int_X |a_f(x)|^2 \,\mathrm{d}\mu(x)\right)^{1/2} \left(\int_X |\langle u|\phi_x\rangle|^2 \,\mathrm{d}\mu(x)\right)^{1/2} \\ &\le \gamma \left(\int_X |\langle u|\phi_x\rangle|^2 \,\mathrm{d}\mu(x)\right)^{1/2} < \infty\end{aligned}$$

for some $\gamma > 0$, the last but one inequality is due to the fact that ϕ is a lower atomic system for A and the last one to the inclusion $\mathcal{D}(B^*) \subset \mathcal{D}(C_\phi)$. Then, ϕ is a weak lower A-semi-frame.
(ii)⇒(iii): Following the proof of Theorem 4.1, for every ϕ-extension B of A, there exists a closed densely defined extension R of C_ϕ^*, with $\mathcal{D}(R^*) = \mathcal{D}(B^*)$, such that $B = RM$ for some $M \in \mathcal{B}(\mathcal{H}, L^2(X,\mu))$. By the Riesz representation theorem, for every $x \in X$ there exists a unique vector $\psi_x \in \mathcal{H}$ such that $(Mh)(x) = \langle h|\psi_x\rangle$, for every $h \in \mathcal{H}$. The function $\psi : x \in X \mapsto \psi_x \in \mathcal{H}$ is Bessel. Indeed,

$$\begin{aligned}\int_X |\langle h|\psi_x\rangle|^2 \,\mathrm{d}\mu(x) &= \int_X |(Mh)(x)|^2 \,\mathrm{d}\mu(x) \\ &= \|Mh\|_2^2 \le \|M\|^2\|h\|^2, \qquad \forall h \in \mathcal{H}.\end{aligned}$$

Hence $\mathcal{D}(C_\psi) = \mathcal{H}$. Moreover, for $f \in \mathcal{D}(B) \cap \mathcal{D}(C_\psi) = \mathcal{D}(B)$, $u \in \mathcal{D}(B^*) = \mathcal{D}(R^*) \subset \mathcal{D}(C_\phi)$

$$\begin{aligned}\langle Bf|u\rangle &= \langle RMf|u\rangle = \langle Mf|R^*u\rangle_2 \\ &= \int_X \langle f|\psi_x\rangle\langle\phi_x|u\rangle \,\mathrm{d}\mu(x).\end{aligned}$$

(iii)⇒(i): It suffices to take, for every fixed ϕ-extension B of A, $a_f : x \in X \mapsto a_x(f) = \langle f|\psi_x\rangle \in \mathbb{C}$ for all $f \in \mathcal{D}(B)$. Indeed, $a_f \in L^2(X,\mu)$ and, for some $\gamma > 0$, we have $\int_X |a_x(f)|^2 \,\mathrm{d}\mu(x) = \int_X |\langle f|\psi_x\rangle|^2 \,\mathrm{d}\mu(x) \le \gamma\|f\|^2$, since ψ is a Bessel function. Moreover, by definition of weak B-dual, we have $\langle Bf|u\rangle = \int_X a_f(x)\langle\phi_x|u\rangle \,\mathrm{d}\mu(x)$, for $f \in \mathcal{D}(C_\psi) \bigcap \mathcal{D}(B) = \mathcal{D}(B), u \in \mathcal{D}(B^*) \subset \mathcal{D}(C_\phi)$. Indeed, we note that $\mathcal{D}(C_\psi) = \mathcal{H}$ since ψ is a Bessel function. □

Remark 4.6. *We don't know if ψ is a weak upper A-semi-frame, in the sense of Definition 5.7, indeed ψ need not be μ-total, that is, $\int_X |\langle f|\psi_x\rangle|^2 \neq 0$ for every $f \in \mathcal{H}$, $f \neq 0$.*

5. Duality and Weak upper A-semi-frames

If $C \in GL(\mathcal{H})$, a *frame controlled by the operator* C or C*- controlled frame* [9] is a family of vectors $\phi = (\phi_n \in \mathcal{H} : n \in \Gamma)$, such that there exist two constants $\mathsf{m}_A > 0$ and $\mathsf{M}_A < \infty$ satisfying

$$\mathsf{m_A} \|f\|^2 \leq \sum_n \langle f|\phi_n\rangle\langle C\phi_n|f\rangle \leq \mathsf{M_A} \|f\|^2, \forall f \in \mathcal{H} \tag{5.1}$$

or, to put it in a continuous form:

$$\mathsf{m_A} \|f\|^2 \leq \int_X \langle f|\phi_x\rangle \langle C\phi_x|f\rangle \, \mathrm{d}\mu(x) \leq \mathsf{M_A} \|f\|^2, \ \forall f \in \mathcal{H}. \tag{5.2}$$

According to Proposition 3.2 of [9], an A-controlled frame is in fact a classical frame when the controlling operator belongs to $GL(\mathcal{H})$. A similar result holds true for a weak lower A-semi-frame if A is bounded as we show in Proposition 5.5. From there it follows that, if A is bounded, a weak lower A-semi-frame has an upper semi-frame dual to it.

Remark 5.7. *We recall that a bounded operator A is surjective if and only if A^* is injective and $\mathcal{R}(A^*)$ is norm closed (if and only if A^* is injective and $\mathcal{R}(A)$ is closed)* [17, Theor. 4.14 and 4.15].

Proposition 5.5. *Let $A \in \mathcal{B}(\mathcal{H})$ and ϕ be a weak lower A-semi-frame. Assume that anyone of the following assumptions is satisfied:*
- (i) *A^* injective, with $\mathcal{R}(A^*)$ norm closed or*
- (ii) *A^* injective, with $\mathcal{R}(A)$ closed or*
- (iii) *A surjective.*

Then,
- (a) *ϕ is a lower semi-frame of $\mathcal{H}$ in the sense of* (1.5),
- (b) *there exists an upper semi-frame ψ dual to ϕ.*

Proof. (a) By Remark 5.7, it suffices to prove (iii). By Theorem 4.15 in [17], A is surjective if and only if there exists $\gamma > 0$ such that $\|A^* f\| \geq \gamma\|f\|$, for every $f \in \mathcal{H}$, then

$$\gamma^2\alpha\|f\|^2 \leq \alpha\|A^*f\|^2 \leq \int_X |\langle f|\phi_x\rangle|^2 \, \mathrm{d}\mu(x), \qquad \forall f \in \mathcal{H}.$$

(b) The thesis follows from (a) and Proposition 2.1 (ii) in [7] (with $\{e_n\}$ an ONB of $\mathcal{H}$).

As explained above, the notion of duality given in (3.3) is too general. Therefore, in what follows ψ will be said to be *dual to* ϕ if one has

$$\langle f|g\rangle = \int_X \langle f|\phi_x\rangle\langle\psi_x|g\rangle \, \mathrm{d}\mu(x), \qquad \forall f \in \mathcal{D}(C_\phi),\, g \in \mathcal{D}(C_\psi). \tag{5.3}$$

□

An interesting question is to identify a weak A-dual of a weak lower A-semi-frame. We expect one should generalize to the present situation the notion of upper semi-frame. We first consider the next definition and examine its consequences.

Definition 5.7. *Let A be a densely defined operator on $\mathcal{H}$. A* weak upper A-semi-frame *for $\mathcal{H}$ is a function $\psi : x \in X \mapsto \psi_x \in \mathcal{H}$ such that, for all $f \in \mathcal{D}(A)$, the map $x \mapsto \langle f|\psi_x\rangle$ is measurable on X and there exists a closed extension F of A and a constant $\alpha > 0$ such that*

$$\int_X |\langle u|\psi_x\rangle|^2 \, \mathrm{d}\mu(x) \leq \alpha\|F^*u\|^2, \qquad \forall u \in \mathcal{D}(F^*). \tag{5.4}$$

Remark 5.8.
- (i) *From Definition 5.7, it is clear that $\mathcal{D}(F^*) \subset \mathcal{D}(C_\psi)$.*

(ii) *If $A \in \mathcal{B}(\mathcal{H})$, then ψ it is clearly a Bessel family.*

Corollary 5.1. *Let ψ be a Bessel mapping of $\mathcal{H}$, and $A \in \mathcal{B}(\mathcal{H})$. Assume that anyone of the following statements is satisfied:*

(i) *A^* injective, with $\mathcal{R}(A^*)$ norm closed or*
(ii) *A^* injective, with $\mathcal{R}(A)$ closed or*
(iii) *A surjective.*

Then, ψ is a weak upper A-semi-frame.

Proof. By Remark 5.7, it suffices to prove (iii). By Theorem 4.15 in [17], we have just to note that

$$\int_X |\langle f|\psi_x\rangle|^2 \, d\mu(x) \leq \gamma\|f\|^2 \leq \alpha^2\gamma\|A^*f\|^2, \qquad \forall\, f \in \mathcal{H}.$$

□

Remark 5.9. *The previous result is true* a fortiori *if ψ is an upper semi-frame of $\mathcal{H}$.*

Summarizing Proposition 5.5, Corollary 5.1 together with the preceding results we have that:

Corollary 5.2. *Let $A \in \mathcal{B}(\mathcal{H})$. Assume that anyone of the following assumptions is satisfied:*

(i) *A^* injective, with $\mathcal{R}(A^*)$ norm closed or*
(ii) *A^* injective, with $\mathcal{R}(A)$ closed or*
(iii) *A surjective*

and let ϕ be a weak lower A-semi-frame. Then, there exists a weak upper A-semi-frame ψ dual to ϕ.

Theorem 5.3. *Let (X, μ) be a locally compact, σ-compact measure space, A a densely defined operator and $\psi : x \in X \mapsto \psi_x \in \mathcal{H}$ a map such that, for every $f \in \mathcal{D}(A)$, the function $x \mapsto \langle f|\psi_x\rangle$ is measurable on X. Then, the following statements are equivalent:*

(i) ψ is a weak upper A-semi-frame for $\mathcal{H}$;
(ii) For every closed, densely defined extension F of A such that (5.4) holds true, there exists a closed, densely defined extension Q of C_ψ^* such that $Q = FN$ for some $N \in \mathcal{B}(L^2(X, \mu), \mathcal{H})$.

Proof.
(i)⇒(ii): Let ψ be a weak upper A-semi-frame, then for every closed extension F of A for which (5.4) holds true, consider the operator $E = C_\psi \upharpoonright \mathcal{D}(F^*)$. It is densely defined, closable since C_ψ is closed. Define an operator O on $R(F^*) \subseteq \mathcal{H}$ as $OF^*f = Ef \in L^2(X, \mu)$. Then, O is a well-defined bounded operator by (5.4). Now, we extend O to the closure of $R(F^*)$ by continuity and define it to be zero on $R(F^*)^\perp$. Therefore $O \in \mathcal{B}(\mathcal{H}, L^2(X, \mu))$ and $OF^* = E$, i.e., $E^* = FO^*$ and the statement is proved by taking $Q = E^*$ and $N = O^*$.
(ii)⇒(i): From $Q = FN$, with Q a densely defined closed extension of C_ψ^*, we have that $Q^* = N^*F^* \subset C_\psi$. For every $u \in \mathcal{D}(F^*) = \mathcal{D}(N^*F^*) = \mathcal{D}((FN)^*) \subset \mathcal{D}(C_\psi)$,

$$\|C_\psi u\|_2^2 = \int_X |\langle u|\psi_x\rangle|^2 \, d\mu(x) = \|N^*F^*u\|_2^2 \leq \alpha\|F^*u\|^2$$

for some $\alpha > 0$. □

We can now prove the following duality result, which suggests that Definition 5.7 is convenient in this context.

Proposition 5.6. *Let A be a densely defined operator and ψ a weak upper A-semi-frame. Let F be a closed extension of A satisfying (5.4) for some $\alpha > 0$. Assume that $\phi \subset \mathcal{D}(A)$ is a weak F-dual of ψ such that*

(a) *$F^*\mathcal{D}(F^*) \subset \mathcal{D}(C_\phi)$,*

(b) *the function* $x \to \|A\phi_x\|$ *is in* $L^2(X,\mu)$.

Then, $F \in \mathcal{E}_{A\phi}(A)$ *and* $A\phi$ *is a weak lower A-semi-frame with* F *as* $(A\phi)$*-extension and lower bound* α^{-1}*, i.e.,*

$$\alpha^{-1}\,\|F^*u\|^2 \le \int_X |\langle u|A\phi_x\rangle|^2\,\mathrm{d}\mu(x), \qquad \forall\, u \in \mathcal{D}(F^*) \cap \mathcal{D}(C_\phi). \tag{5.5}$$

Proof. For every $u \in \mathcal{D}(FF^*)$,

$$\begin{aligned}
\|F^*u\|^2 &= \langle F^*u|F^*u\rangle = \langle FF^*u|u\rangle \\
&= \int_X \langle F^*u|\phi_x\rangle\langle\psi_x|u\rangle\,\mathrm{d}\mu(x), \;\; \text{by weak } F\text{-duality} \\
&\le \left(\int_X |\langle u|\psi_x\rangle|^2\,\mathrm{d}\mu(x)\right)^{1/2} \left(\int_X |\langle F^*u|\phi_x\rangle|^2\,\mathrm{d}\mu(x)\right)^{1/2} \\
&\le \alpha^{1/2}\,\|F^*u\| \left(\int_X |\langle F^*u|\phi_x\rangle|^2\,\mathrm{d}\mu(x)\right)^{1/2}.
\end{aligned}$$

The right hand side of the previous inequality is finite because of (a). Hence,

$$\|F^*u\| \le \alpha^{1/2}\left(\int_X |\langle u|A\phi_x\rangle|^2\,\mathrm{d}\mu(x)\right)^{1/2}, \quad \forall u \in \mathcal{D}(FF^*).$$

Now, we take into account that $\mathcal{D}(FF^*)$ is a core for F^* by von Neumann's theorem [16, Theorem 3.24]. Therefore, for every $u \in \mathcal{D}(F^*)$, there exists a sequence $\{u_n\} \subset \mathcal{D}(FF^*)$ such that $\|u_n - u\| \to 0$ and $\|F^*u_n - F^*u\| \to 0$. This implies, of course, that $\langle F^*u_n|\phi_x\rangle \to \langle F^*u|\phi_x\rangle$, for every $x \in X$. Moreover, since $\{u_n\}$ is bounded, we have

$$|\langle F^*u_n|\phi_x\rangle| = |\langle u_n|F\phi_x\rangle| \le M\|F\phi_x\|$$

for some $M > 0$ and for every $x \in X$. The assumption that $x \to \|A\phi_x\|$ is in $L^2(X,\mu)$ allows us to apply the dominated convergence theorem and conclude that

$$\|F^*u\| \le \alpha^{1/2}\left(\int_X |\langle u|A\phi_x\rangle|^2\,\mathrm{d}\mu(x)\right)^{1/2}, \quad \forall u \in \mathcal{D}(F^*).$$

The right hand side of the latter inequality is finite again by (a), hence $\mathcal{D}(F^*) \subset \mathcal{D}(C_{A\phi})$. This fact also implies that $F \in \mathcal{E}_{A\phi}(A)$ since, if $u \in \mathcal{D}(F^*)$, we get

$$\int_X |\langle u|A\phi_x\rangle|^2\,\mathrm{d}\mu(x) = \int_X |\langle F^*u|\phi_x\rangle|^2\,\mathrm{d}\mu(x) = \|C_{A\phi}u\|^2 < \infty.$$

□

Remark 5.10.

(1) Note (5.5) can obviously be also written

$$\alpha^{-1}\,\|h\|^2 \le \int_X |\langle h|\phi_x\rangle|^2\,\mathrm{d}\mu(x), \qquad \forall\, h \in \mathcal{R}(F^*).$$

(2) For every $f \in \mathcal{D}(F^*)$*, with our new definition, by*

$$\alpha^{-1}\int_X |\langle f|\psi_x\rangle|^2\,\mathrm{d}\mu(x) \le \|A^*f\|^2 \le \alpha\int_X |\langle f|A\phi_x\rangle|^2\,\mathrm{d}\mu(x),$$

it follows that $\|C_\psi f\| \le \alpha\|C_{A\phi}f\|$*, for every* $f \in \mathcal{D}(F^*)$*. Since* C_ψ *is closed, then* $\mathcal{D}(C_\psi^*)$ *is dense and (5.4) and (5.5) imply that* $\mathcal{D}(C_{A\phi}) \subseteq \mathcal{D}(F^*) \subseteq \mathcal{D}(C_\psi)$*, hence the latter is dense too.*

Another possibility is to mimic the notion of controlled frame (5.1) or (5.2), introduced in [9, Definition 3.1]. Because the operator A is supposed to belong to $GL(\mathcal{H})$, we end up with a generalized frame (and actually a genuine frame). It would be interesting to extend the definition to an unbounded operator or at least to operators less regular than elements of $GL(\mathcal{H})$. A different strategy is to investigate the following generalization. Let B be a linear operator with domain $\mathcal{D}(B)$. Suppose that $\psi_n \in \mathcal{D}(B)$ for all n. Put

$$\Omega_B(f,g) = \sum_n \langle f|\psi_n\rangle\langle B\psi_n|g\rangle, \quad \forall\, f,g \in \mathcal{D}(\Omega_B),$$

where $\mathcal{D}(\Omega_B)$ is some domain of the sesquilinear form defined formally on the rhs. Following [13, Sec. 4], we may consider the form Ω_B as the form generated by two sequences, $\{\psi_n\}$ and $\{B\psi_n\}$. Then, the operator associated to the form Ω_B is precisely B, since one has $\langle Bf|g\rangle = \Omega_B(f,g)$. A continuous version of (5.1) would be

$$\mathsf{m}_A \|f\|^2 \le \int_X \langle f|\psi_x\rangle\langle A\psi_x|f\rangle \,\mathrm{d}\mu(x) \le \mathsf{M}_A \|f\|^2, \quad \text{for all } f \in \mathcal{H},$$

and the sesquilinear form becomes

$$\Omega_A(f,g) = \int_X \langle f|\psi_x\rangle\langle A\psi_x|g\rangle \,\mathrm{d}\mu(x) \le \mathsf{M}_A \|f\|^2, \quad \text{for all } f,g \in \mathcal{D}(\Omega_A).$$

From the last relation, we might infer two alternative possible definitions of an *upper A-semi-frame*, namely:

$$\int_X |\langle Af|\psi_x\rangle|^2 \,\mathrm{d}\mu(x) \le \mathsf{M}\|f\|^2, \quad \forall\, f \in \mathcal{D}(A),$$

$$\int_X \langle f|\psi_x\rangle\, \langle \psi_x|Af\rangle \,\mathrm{d}\mu(x) \le \mathsf{M}\|f\|^2, \quad \forall\, f \in \mathcal{D}(A). \tag{5.6}$$

Actually the definition (5.6) leads to that of an *A-Bessel map*, provided that $\psi_x \in \mathcal{D}(A^*)$, for all $x \in X$:

$$\int_X \langle f|\psi_x\rangle\, \langle A^*\psi_x|f\rangle \,\mathrm{d}\mu(x) \le \mathsf{M}\|f\|^2, \quad \forall\, f \in \mathcal{D}(A).$$

Further, study will hopefully reveal which of the three definitions of an upper A-semi-frame is the most natural one.

6. Examples

(1) A reproducing kernel Hilbert space. We start from the example of a lower semi-frame in a reproducing kernel Hilbert space described in increasing generality in [6, 8]. Let $\mathcal{H}_K$ be a reproducing kernel Hilbert space of (nice) functions on a measure space (X,μ), with kernel function $k_x, x \in X$, that is, $f(x) = \langle f|k_x\rangle_K, \forall f \in \mathcal{H}_K$. Choose a (real valued, measurable) weight function $m(x) > 1$ and consider the unbounded self-adjoint multiplication operator $(Mf)(x) = m(x)f(x), \forall x \in X$, with dense domain $\mathcal{D}(M)$. For each $n \in \mathbb{N}$, define $\mathcal{H}_n = \mathcal{D}(M^n)$, equipped with its graph norm, and $\mathcal{H}_{\overline{n}} := \mathcal{H}_{-n} = \mathcal{H}_n^\times$ (conjugate dual). Then, we have the Hilbert scale $\{\mathcal{H}_n,\, n \in \mathbb{Z}\}$:

$$\ldots \mathcal{H}_n \subset \ldots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}_0 = \mathcal{H}_K \subset \mathcal{H}_{\overline{1}} \subset \mathcal{H}_{\overline{2}} \ldots \subset \mathcal{H}_{\overline{n}} \ldots.$$

As an operator on the scale, which is a partial inner product space [2], the operator M has continuous representatives $M_{n+1} \to M_n, n \in \mathbb{Z}$. Fix some $n > 1$ and define the measurable functions $\phi_x = k_x m^n(x), \psi_x = k_x m^{-n}(x)$, for every $x \in X$. Then $\psi_x \in \mathcal{H}_n$, for every $x \in X$, and ψ is an upper semi-frame, whereas $\phi_x \in \mathcal{H}_{\overline{n}}$, for every $x \in X$, and ϕ is a lower semi-frame. Also, $C_\psi : \mathcal{H}_K \to \mathcal{H}_n$, $C_\phi : \mathcal{H}_K \to \mathcal{H}_{\overline{n}}$ continuously. One has indeed, for every

$g \in \mathcal{H}_K$, $\langle\psi_x|g\rangle_K = \overline{g(x)}\, m^{-n}(x) \in \mathcal{H}_n$ and $\langle\phi_x|g\rangle_K = \overline{g(x)}\, m^n(x) \in \mathcal{H}_{\overline{n}}$. Next, choose a real valued, measurable function $x \mapsto a(x)$ such that $a(x) \leq m^n(x), \forall x \in X$, and define $A = A^*$ as the multiplication operator by $a : (Af)(x) = a(x)f(x), \forall\, x \in X$. Let $\mathcal{D}(A) = \mathcal{H}_n$. Then $A \in \mathcal{B}(\mathcal{H}_n)$ since $\|af\| \leq \|m^n f\| < \infty$, for every $f \in \mathcal{H}_n$ and since $a(x)m^{-n}(x) < 1$ for every $x \in X$ and for every $f \in \mathcal{H}_n$, then $\mathcal{R}(A) \subset \mathcal{D}(M^n) = \mathcal{H}_n$. As an operator on the scale, A has continuous representatives $A_{p,n+p} : \mathcal{H}_{n+p} \to \mathcal{H}_p$. Then, we have, $\forall f \in \mathcal{D}(A) = \mathcal{H}_n \subset \mathcal{D}(C_\phi)$,

$$\|Af\|^2 = \int_X |f(x)|^2\, a(x)^2\, \mathrm{d}\mu(x) \leq \int_X |f(x)|^2\, m^{2n}(x)\, \mathrm{d}\mu(x) = \int_X |\langle f|\phi_x\rangle_K|^2\, \mathrm{d}\mu(x) < \infty,$$

that is, ϕ is a weak A-frame for $\mathcal{H}_K$. The same holds for every self-adjoint operator A' which is the multiplication operator by the measurable function $x \mapsto a'(x)$ such that $a'(x) \leq m^n(x), \forall x \in X$, and $\mathcal{D}(A') = \mathcal{H}_n$.

Let now the closed operator B be a ϕ-extension of A, that is,

$$A \subset B \text{ and } \mathcal{H}_n = \mathcal{D}(A) \subset \mathcal{D}(B^*) \subset \mathcal{D}(C_\phi)$$

and

$$\|B^* f\|^2 \leq \int_X |\langle f|\phi_x\rangle_K|^2\, \mathrm{d}\mu(x) < \infty,\ \forall\, f \in \mathcal{D}(B^*).$$

Then, ϕ is a weak lower A-semi-frame for $\mathcal{H}_K$.

(2) A discrete example. A more general situation may be derived from the discrete example of Section 5.2 of [6]. Take a weight sequence $m := \{|m_n|\}_{n\in\mathbb{N}}, m_n \neq 0$, where $m \in \ell^\infty$ has a subsequence converging to zero (or $m \in c_0$). Then consider the space ℓ^2_m with norm $\|\xi\|_{\ell^2_m} := \sum_{n\in\mathbb{N}} |m_n \xi_n|^2$. Thus, we have the following triplet

$$\ell^2_{1/m} \subset \ell^2 \subset \ell^2_m.$$

Next, for each $n \in \mathbb{N}$, define $\psi_n = m_n e_n$, where $e := \{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in ℓ^2. Then ψ is an upper semi-frame and $C_\psi : \mathcal{H} \to \ell^2_{1/m}$, continuously. On the other hand, $\phi := \{(1/\overline{m_n})e_n\}_{n\in\mathbb{N}}\}$ is a lower semi-frame and $C_\phi : \mathcal{H} \to \ell^2_m$, continuously. In other words, $\psi = Me$ and $\phi = M^{-1}e$, where M is the diagonal operator $M_n = m_n, n \in \mathbb{N}$. In order to define a weak lower A-semi-frame for ℓ^2, we take another diagonal operator $A = \{a_n\}$ such that, for each $n \in \mathbb{N}$ one has $|a_n| \leq |m_n|^{-1}$. Then, $\forall f \in \mathcal{D}(A)$,

$$\begin{aligned}\|Af\|^2 &= \sum_{n\in\mathbb{N}} |a_n|^2 |f_n|^2 = \sum_{n\in\mathbb{N}} |a_n|^2 |\langle f|e_n\rangle|^2 \leq \sum_{n\in\mathbb{N}} |m_n|^{-2} |\langle f|e_n\rangle|^2 \\ &= \sum_{n\in\mathbb{N}} |\langle f|\phi_n\rangle|^2.\end{aligned}$$

Thus, ϕ is a weak A-frame for ℓ^2. As in Example (1), we get a weak lower A-semi-frame for ℓ^2 if we have a ϕ-extension B of A. The same result holds true if one replaces the ONB $\{e_n\}$ by a frame $\{\theta_n\}_{n\in\mathbb{N}}$:

$$\alpha \left\|f\right\|^2 \leq \sum_n |\langle f|\theta_n\rangle|^2 \leq \beta \left\|f\right\|^2, \forall f \in \mathcal{H},$$

for some $\alpha, \beta > 0$. Since $m \in \ell^\infty$, we can as well assume that $|m_n| < \delta$, $\forall n \in \mathbb{N}$ for some $\delta > 0$. Thus $|1/\overline{m_n}| > 1/\delta$, $\forall n$. Then, for every $f, g \in \mathcal{H}$, we have

$$\sum_n |\langle f|\psi_n\rangle|^2 = \sum_n |m_n|^2 |\langle f|\theta_n\rangle|^2 \le \delta^2 \sum_n |\langle f|\theta_n\rangle|^2 \le \delta^2 \beta \|f\|^2 ,$$

$$\sum_n |\langle g|\phi_n\rangle|^2 = \sum_n \left|\frac{1}{m_n}\right|^2 |\langle g|\theta_n\rangle|^2 \ge \frac{1}{\delta^2} \sum_n |\langle g|\theta_n\rangle|^2 \ge \frac{1}{\delta^2} \alpha \|g\|^2 .$$

Thus, indeed, ψ is an upper semi-frame and ϕ is a lower semi-frame. The rest of the construction follows.

(3) A standard construction. As explained in Section 2, a standard construction of lower semi-frames stems from the consideration of a metric operator induced by an unbounded operator. Given a closed, densely defined, unbounded operator S with dense domain $\mathcal{D}(S)$, define the metric operator $G = I + S^*S$, which is unbounded with bounded inverse. Then, if we take an ONB $\{e_n\}$ of $\mathcal{D}(G^{1/2}) = \mathcal{D}(S)$, contained in $\mathcal{D}(S^*S)$, then $\{\phi_n\} = \{Ge_n\} = \{(I + S^*S)e_n\}$ is a lower semi-frame of $\mathcal{H}$ on $\mathcal{D}(S)$. Now, if A is a densely defined operator that satisfies the equation

$$\alpha \|A^* f\| \le \|f\|_{C_\phi} , \ \forall f \in \mathcal{D}(A^*)$$

instead of (2.1), then ϕ is a weak A-frame for $\mathcal{H}$. As for the equivalent of (2.2), it is of course trivial.

7. Conclusion

In the search of expansions of certain functions into simpler ones, the usual strategy is to pass from orthonormal bases (e.g. Fourier series or integral) to frames, and then to semi-frames, lower or upper. In each case, one obtains more flexibility. The aim of this paper is to apply the same philosophy to more recent structures.

Given a densely defined linear operator A on a Hilbert space $\mathcal{H}$, the notion of weak A-frame was introduced in [10], as explained in Def. 3.2. Following the strategy described above, we obtain the notion of *weak lower A-semi-frame* given in Def. 3.4 and discussed in Section 3. A parallel notion is that of weak atomic system for A, also introduced in [10], from which we obtain that of *lower atomic system for A*. As explained in Section 4, the two original structures go hand in hand and the equivalence extends to the new ones as well. In the same way, one defines a *weak upper A-semi-frame* (Def. 5.7), although this definition is only tentative. Finally, there is a recurrent property of duality, namely, a weak upper A-semi-frame generates by duality a lower one.

The conclusion is that, in each case, one can obtain more flexibility for expansions by passing from frames to semi-frames, as illustrated by the examples provided. Of course, more work is needed. The results presented here are in fact a first step toward a generalization of the notions of weak A-frames and weak atomic system to lower semi-frames.

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Research Article

On the Korovkin-type approximation of set-valued continuous functions

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ABSTRACT. This paper is devoted to some Korovkin approximation results in cones of Hausdorff continuous set-valued functions and in spaces of vector-valued functions. Some classical results are exposed in order to give a more complete treatment of the subject. New contributions are concerned both with the general theory than in particular with the so-called convexity monotone operators, which are considered in cones of set-valued function and also in spaces of vector-valued functions.

Keywords: Korovkin approximation, approximation of vector-valued functions, approximation of set-valued functions.

2020 Mathematics Subject Classification: 41A65, 41A36, 41A25, 41A63.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION AND NOTATION

This paper is devoted to some aspects of the Korovkin approximation theory which has been deeply studied in the last decades and is concerned with the approximation of operators between cones of Hausdorff continuous set-valued functions. It is partially a survey of preceding results but contains also some new results which completes the analysis carried out in the following sections. My attention to this topic was pointed out and strongly encouraged by Professor Francesco Altomare throughout many discussions on the pioneristic work by Keimel and Roth [13, 14]. Some important consequences are also concerned with Korovkin approximation theory in spaces of continuous real functions and also in this case we shall give some new contributions.

The paper is organized as follows. Section 2 is devoted to classical results together with some new results strictly connected to the classical theory. In Section 3, we consider a particular class of linear operators, the so-called convexity monotone operators, which allow to point out how the analysis of set-valued functions is strictly connected with that one of vector-valued functions. We shall state some consequences of the results in Section 4 and in the same section we shall state some new results concerning the Korovkin approximation of vector-valued functions which generalize some recent results obtained in the space $C([0,1])$. We can also state further developments concerning Korovkin approximation of set-valued functions.

In the following sections, we shall denote by **B** the closed unit ball with center 0 in $\mathbb{R}^d$ ($d \geq 1$) and by $e_1, \dots, e_d$ the canonical base of $\mathbb{R}^d$. Moreover, $\mathcal{K}(\mathbb{R}^d)$ will denote the cone of all non-empty compact convex subsets of $\mathbb{R}^d$ endowed with the natural addition and multiplication by

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positive scalars. In $\mathcal{K}(\mathbb{R}^d)$, we consider the Hausdorff distance

$$d_H(A,B) := \inf\{\varepsilon > 0 \mid A \subset B + \varepsilon \cdot \mathbf{B}, B \subset A + \varepsilon \cdot \mathbf{B}\}.$$

If X is an Hausdorff topological space, we shall deal with the cone $C(X, \mathcal{K}(\mathbb{R}^d))$ of all Hausdorff continuous set-valued functions; hence, $f \in C(X, \mathcal{K}(\mathbb{R}^d))$ if and only if for every $x_0 \in X$ and $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $f(x) \subset f(x_0) + \varepsilon \cdot \mathbf{B}$ and $f(x_0) \subset f(x) + \varepsilon \cdot \mathbf{B}$ whenever $x \in U$. The space $C(X, \mathcal{K}(\mathbb{R}^d))$ is naturally ordered by inclusion, that is

$$f \leq g \;\Leftrightarrow\; \forall x \in X: \; f(x) \subset g(x)$$

whenever $f, g \in C(X, \mathcal{K}(\mathbb{R}^d))$. An operator $T : C(X, \mathcal{K}(\mathbb{R}^d)) \to C(X, \mathcal{K}(\mathbb{R}^d))$ is called *linear* if it preserves addition and multiplication by positive scalars and is called *monotone* if it preserves inclusions. If $\varphi \in C(X, \mathbb{R}^d)$, it will be useful to denote by $\{\varphi\}$ the set-valued function (in $C(X, \mathcal{K}(\mathbb{R}^d))$) defined by setting $\{\varphi\}(x) = \{\varphi(x)\}$ for every $x \in X$. Moreover, if $\varphi_1, \ldots, \varphi_n \in C(X, \mathbb{R}^d)$ are set-valued functions on X, we can consider the set-valued function $\mathrm{co}(\varphi_1, \ldots, \varphi_n)$ defined by

$$\mathrm{co}(\varphi_1, \ldots, \varphi_n)(x) = \mathrm{co}(\varphi_1(x), \ldots, \varphi_n(x))$$

for every $x \in X$, where $\mathrm{co}(\varphi_1(x), \ldots, \varphi_n(x))$ denotes the convex hull of $\varphi_1(x), \ldots, \varphi_n(x)$. Moreover, if $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, we shall denote by $\mathrm{Sel}(f)$ the convex subset of $C(X, \mathcal{K}(\mathbb{R}^d))$ consisting of all continuous selections of f. It is well-known that (see e.g. [13, Lemma 4.1. and Theorem 3.2] or [4, Proposition 1.1])

$$f = \bigcup_{\varphi \in \mathrm{Sel}(f)} \{\varphi\}$$

for every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, that is $f(x) = \bigcup_{\varphi \in \mathrm{Sel}(f)} \{\varphi(x)\}$ for every $x \in X$.

2. Some classical results

The starting point is the extension of the classical Korovkin theorem to the setting of Hausdorff continuous set-valued functions. We recall that a subset M of $C(X, \mathbb{R})$ is called a *Korovkin system* in $C(X, \mathbb{R})$ if whenever $(T_i)^{\leq}_{i \in I}$ is an equicontinuous net of positive linear operators from $C(X, \mathbb{R})$ into itself satisfying the condition $\lim^{\leq}_{i \in I} T_i(\varphi) = \varphi$ for every $\varphi \in M$, we also have $\lim^{\leq}_{i \in I} T_i(\psi) = \psi$ for every $\psi \in C(X, \mathbb{R})$. If the limit operator is not the identity operator but a linear positive operator $T : C(X, \mathbb{R}) \to C(X, \mathbb{R})$, we shall say that M is called a T-Korovkin system (or a Korovkin system for T) in $C(X, \mathbb{R})$. In the space $C(X, \mathcal{K}(\mathbb{R}^d))$, we have a similar definition (see [13, 4]). Namely, let $\mathcal{C}$ be a subcone of $C(X, \mathcal{K}(\mathbb{R}^d))$; a subset H of $\mathcal{C}$ is called a *Korovkin system* in $\mathcal{C}$ if for every equicontinuous net $(L_i)^{\leq}_{i \in I}$ of linear monotone operators from $\mathcal{C}$ into itself satisfying $\lim^{\leq}_{i \in I} L_i(h) = h$ for every $h \in H$, we also have $\lim^{\leq}_{i \in I} L_i(f) = f$ for every $f \in \mathcal{C}$. Also in this case, we shall call H a L-Korovkin system (or a Korovkin system for L) in $\mathcal{C}$ in the case where the limit operator is not the identity operator but a continuous linear monotone operator $L : \mathcal{C} \to \mathcal{C}$. The existence of Korovkin systems in $C(X, \mathbb{R}^d)$ has been widely studied and a complete treatment can be found in [1, 2]. In despite of this, we do not have a complete analysis of Korovkin systems in $C(X, \mathcal{K}(\mathbb{R}^d))$ and our aim is to add some contribution to this topic. First of all, an extension of Korovkin systems to the case of set-valued functions was obtained by Keimel and Roth in [13] and can be stated as follows.

Theorem 2.1. **([13, Theorem 3.1])** *Let X be a compact Hausdorff topological space and let M be a Korovkin system of positive functions in $C(X, \mathbb{R})$. Then, the subset consisting of all the functions $f \cdot \mathbf{B}$ (for every $x \in X$: $f \cdot \mathbf{B}(x) := f(x) \cdot \mathbf{B}$) and the constant functions $\mathbf{B}$, $e_1, \ldots, e_d$, is a Korovkin system in $C(X, \mathcal{K}(\mathbb{R}^d))$.*

It follows in particular that the functions

$$x \mapsto \mathbf{B}, \qquad x \mapsto x \cdot \mathbf{B}, \qquad x \mapsto x^2 \cdot \mathbf{B}$$

form a Korovkin system in $C(X, \mathcal{K}(\mathbb{R}))$. A refinement of this result has been obtained in [4] in terms of upper and lower envelopes.

Theorem 2.2. **([4, Theorem 2.2])** *Let X be a compact Hausdorff topological space and let H be a subset of $C(X, \mathcal{K}(\mathbb{R}^d))$. Assume that there exists a Korovkin system of positive functions M in $C(X, \mathbb{R})$ such that, for every $\varphi \in M$, $x_0 \in X$ and $\varepsilon > 0$,*

$$(\varphi(x_0) + \varepsilon) \cdot \boldsymbol{B} = \overline{\bigcup_{f \in H, f \leq (\varphi+\varepsilon)\cdot \boldsymbol{B}} f(x_0)} = \bigcap_{f \in H, (\varphi+\varepsilon)\cdot \boldsymbol{B} \leq f} f(x_0).$$

Then, H is a Korovkin system in $C(X, \mathcal{K}(\mathbb{R}^d))$.

The assumptions in Theorem 2.1 ensure that for every $\varphi \in M$ and $\varepsilon > 0$, the function $(\varphi + \varepsilon) \cdot \mathbf{B} \in H$ and hence the assumptions in Theorem 2.2 are trivially satisfied. Therefore, Theorem 2.2 generalizes Theorem 2.1. A more meaningful extension of the preceding results can be found in [5] in a more general setting and even in the case where the limit operator is not necessarily the identity operator. In this regard, we recall the following result obtained in [5, Theorem 2.4 and Corollary 2.5].

Theorem 2.3. **([5, Theorem 2.4])** *Let X be a compact Hausdorff topological space, $\mathcal{C}$ a subcone of $C(X, \mathcal{K}(\mathbb{R}^d))$ containing the single-valued functions and $L : \mathcal{C} \to \mathcal{C}$ a continuous monotone linear operator satisfying the following conditions:*

a) *For every $\varphi \in C(X, \mathbb{R}^d)$, $L(\{\varphi\})$ is single-valued;*
b) *For every $f \in \mathcal{C}$ and $x \in X$:*

$$L(f)(x) = \bigcup_{\varphi \in \mathrm{Sel}(f)} L(\{\varphi\})(x).$$

Let H be a subset of $\mathcal{C}$ such that, for every $f \in \mathcal{C}$, $x_0 \in X$ and $\varepsilon > 0$, there exists $h \in H$ satisfying the following conditions:

$$f \leq h, \qquad L(h)(x_0) \subset L(f)(x_0) + \varepsilon \cdot \boldsymbol{B}. \tag{2.1}$$

Then, H is a Korovkin system for L in $\mathcal{C}$.

As observed in [5, Remark 2.6], if H contains the constant set-valued functions, then condition (2.1) can be weakened with the following condition

$$f \leq h + \varepsilon \cdot \mathbf{B}, \qquad L(h)(x_0) \subset L(f)(x_0) + \varepsilon \cdot \mathbf{B}. \tag{2.2}$$

Obviously, the identity operator $I : \mathcal{C} \to \mathcal{C}$ satisfies conditions a) and b) of Theorem 2.3. Theorems 2.2 and 2.3 hold also if we replace $\mathcal{K}(\mathbb{R}^d)$ with $\mathcal{K}(E)$, where E is a Fréchet space. However, in the setting of finite dimensional spaces, we can add the following new criteria for Korovkin systems for the identity operator.

Theorem 2.4. *Let (X, σ) be a compact metric space, $\mathcal{C}$ a subcone of $C(X, \mathcal{K}(\mathbb{R}^d))$ and H a subset of $\mathcal{C}$ which contains the functions*

$$x \mapsto K + \lambda\sigma(x, x_0) \cdot \boldsymbol{B}$$

whenever $K \in \mathcal{K}(\mathbb{R}^d)$, $x_0 \in X$ and $\lambda \geq 0$ (in particular, H contains the constant functions). Then, H is a Korovkin system in $\mathcal{C}$.

Proof. Let $f \in \mathcal{C}$ and $x_0 \in X$. Since f is Hausdorff continuous and X is compact, we can find $M > 0$ such that, for every $x \in X$, $f(x) \subset f(x_0) + M \cdot \mathbf{B}$. Now, let $\varepsilon > 0$; from the Hausdorff continuity of f, we can find $\delta > 0$ such that, for every $x \in X$ satisfying $\sigma(x, x_0) \leq \delta$, we have $f(x) \subset f(x_0) + \frac{\varepsilon}{2} \cdot \mathbf{B}$. Now, consider the function $h : X \to \mathcal{K}(\mathbb{R}^d)$ defined by setting, for every $x \in X$,

$$h(x) := f(x_0) + \frac{\varepsilon}{2} \cdot \mathbf{B} + \frac{\sigma(x, x_0)}{\delta} \cdot \mathbf{B}.$$

The function h is obviously Hausdorff continuous and our assumptions ensure that $h \in H \subset \mathcal{C}$. Now, we show that $f \leq h$. Indeed, if $\sigma(x, x_0) \leq \delta$, we have

$$f(x) \subset f(x_0) + \frac{\varepsilon}{2} \cdot \mathbf{B} \subset h(x)$$

and similarly, if $\sigma(x, x_0) > \delta$, we have

$$f(x) \subset f(x_0) + M \cdot \mathbf{B} \subset f(x_0) + \frac{\varepsilon}{2} \cdot \mathbf{B} + \frac{\sigma(x, x_0)}{\delta} \cdot \mathbf{B} = h(x).$$

Finally, we obviously have

$$h(x_0) = f(x_0) + \frac{\varepsilon}{2} \cdot \mathbf{B}.$$

Hence, the subset H satisfies the assumptions in Theorem 2.3 and therefore is a Korovkin system in $\mathcal{C}$. □

The proof of the following result is similar by considering the function $k : X \to \mathcal{K}(\mathbb{R}^d)$ defined by setting, for every $x \in X$,

$$k(x) := f(x_0) + \frac{\varepsilon}{2} \cdot \mathbf{B} + \frac{\sigma(x, x_0)^2}{\delta^2} \cdot \mathbf{B}$$

in place of h. We omit the details for the sake of brevity.

Theorem 2.5. *Let (X, σ) be a compact metric space, $\mathcal{C}$ a subcone of $C(X, \mathcal{K}(\mathbb{R}^d))$ and H a subset of $\mathcal{C}$ which contains the functions*

$$x \mapsto K + \lambda\sigma(x, x_0)^2 \cdot \boldsymbol{B}$$

whenever $K \in \mathcal{K}(\mathbb{R}^d)$, $x_0 \in X$ and $\lambda \geq 0$. Then, H is a Korovkin system in $\mathcal{C}$.

As a particular case, we can state the preceding Theorems 2.4 and 2.5 in the case where X is a compact subset of $\mathbb{R}$.

Corollary 2.1. *Let X be a compact subset of $\mathbb{R}$, $\mathcal{C}$ a subcone of $C(X, \mathcal{K}(\mathbb{R}^d))$ and H a subset of $\mathcal{C}$ which contains the functions*

$$x \mapsto K + \lambda\,|x - x_0| \cdot \boldsymbol{B}$$

(or alternatively, contains the functions

$$x \mapsto K + \lambda\,(x - x_0)^2 \cdot \boldsymbol{B}\ )$$

whenever $K \in \mathcal{K}(\mathbb{R}^d)$, $x_0 \in X$ and $\lambda \geq 0$ (in particular, H contains the constant functions). Then, H is a Korovkin system in $\mathcal{C}$.

In particular, if $X = [0, 1]$ and $d = 1$, we obtain that the subcone H of $C([0, 1], \mathcal{K}(\mathbb{R}))$ containing the functions

$$x \mapsto (\lambda + \mu|x - x_0|) \cdot [-1, 1]$$

for every $x_0 \in [0, 1]$, is a Korovkin system in $C([0, 1], \mathcal{K}(\mathbb{R}))$. In the alternative formulation, we have to require that the functions

$$x \mapsto (\lambda + \mu(x - x_0)^2) \cdot [-1, 1], \qquad x_0 \in [0, 1],$$

belong to H, and hence we find the classical Korovkin system in $C([0,1],\mathcal{K}(\mathbb{R}))$ consisting of the functions

$$x \mapsto \mathbf{B}, \qquad x \mapsto x \cdot \mathbf{B}, \qquad x \mapsto x^2 \cdot \mathbf{B}.$$

We conclude this section with the following result which generalizes Theorem 2.3. It has been established in [8, Theorem 1.3] only in the particular case of the identity operator. Here, we give a proof in the general case.

Theorem 2.6. *Let X be a compact Hausdorff topological space, $\mathcal{C}$ a subcone of $C(X,\mathcal{K}(\mathbb{R}^d))$ containing the set*

$$C(X,\mathbb{R}) \otimes \mathcal{K}(\mathbb{R}^d) := \{\varphi \cdot A \mid \varphi \in C(X,\mathbb{R}^d),\ A \in \mathcal{K}(\mathbb{R}^d)\}$$

and $L : \mathcal{C} \to \mathcal{C}$ a continuous monotone linear operator. If H is a subset of $\mathcal{C}$ such that, for every $f \in \mathcal{C}$, $x_0 \in X$ and $\varepsilon > 0$, there exist $h_1,\ldots,h_m \in H$ such that

$$f \le h_j,\ j = 1,\ldots,m\,, \qquad \bigcap_{j=1}^{m} L(h_j)(x_0) \subset L(f)(x_0) + \varepsilon \cdot \mathbf{B},$$

then H is a Korovkin set for L in $\mathcal{C}$.

Proof. Let $(L_i)_{i\in I}$ be an equicontinuous net of convexity monotone linear operators such that $\lim^{\le}_{i\in I} L_i(h) = L(h)$ for every $h \in H$. Let $f \in \mathcal{C}$.
First step. Assume that $f = \{\varphi\}$ with $\varphi \in C(X,\mathbb{R}^d)$. Let $\varepsilon > 0$ and $x_0 \in X$. The assumptions on H ensure the existence of $h_1,\ldots,h_m \in H$ such that

$$f \le h_j,\ j = 1,\ldots,m\,, \qquad \bigcap_{j=1}^{m} L(h_j)(x_0) \subset L(f)(x_0) + \frac{\varepsilon}{4} \cdot \mathbf{B}.$$

Since $L(f)$ and each $L(h_j)$, $j = 1,\ldots,m$ are Hausdorff continuous at x_0, there exists a neighborhood U of x_0 such that, for every $x \in U$,

$$L(f)(x_0) \subset L(f)(x) + \frac{\varepsilon}{4} \cdot \mathbf{B}\,, \qquad L(h_j)(x) \subset L(h_j)(x_0) + \frac{\varepsilon}{4} \cdot \mathbf{B}\,,\ j = 1,\ldots,m.$$

Hence, for every $x \in U$, we have

$$\bigcap_{j=1}^{m} L(h_j)(x) \subset \bigcap_{j=1}^{m} L(h_j)(x_0) + \frac{\varepsilon}{4} \cdot \mathbf{B} \subset L(f)(x_0) + \frac{\varepsilon}{2} \cdot \mathbf{B} \subset L(f)(x) + \frac{3\varepsilon}{4} \cdot \mathbf{B}.$$

Since $\lim^{\le}_{i\in I} L_i(h_j) = L(h_j)$ for every $j = 1,\ldots,m$, there exists $\alpha \in I$ such that, for every $i \in I$, $i \ge \alpha$, $j = 1\ldots,m$ and $x \in X$,

$$L_i(h_j)(x) \subset L(h_j)(x) + \frac{\varepsilon}{4} \cdot \mathbf{B}\,, \qquad L(h_j)(x) \subset L_i(h_j)(x) + \frac{\varepsilon}{4} \cdot \mathbf{B}.$$

It follows, for every $i \in I$, $i \ge \alpha$, $j = 1\ldots,m$ and $x \in U$,

$$L_i(f)(x) \subset L_i(h_j)(x) \subset L(h_j)(x) + \frac{\varepsilon}{4} \cdot \mathbf{B}$$

and hence

$$L_i(f)(x) \subset \bigcap_{j=1}^{m} L(h_j)(x) + \frac{\varepsilon}{4} \cdot \mathbf{B} \subset L(f)(x) + \varepsilon \cdot \mathbf{B}.$$

Since X is compact, we can deduce the existence of $\beta \in I$ such that $L_i(f)(x) \subset L(f)(x) + \varepsilon \cdot \mathbf{B}$ for every $i \in I$, $i \ge \beta$. Since f is single-valued, we have also $L(f)(x) \subset L_i(f)(x) + \varepsilon \cdot \mathbf{B}$ for every $i \in I$, $i \ge \beta$ and the proof is complete in this case.

Second step. Assume that $f = \varphi \cdot A$ with $\varphi \in C(X, \mathbb{R})$ and $A \in \mathcal{K}(\mathbb{R}^d)$. Let $\varepsilon > 0$ and $x_0 \in X$. Since A is compact, there exist $y_1, \ldots, y_p \in A$ such that

$$f(x_0) = \varphi(x_0) \cdot A \subset \bigcup_{s=1}^{p} \varphi(x_0) \cdot \left(\{y_s\} + \frac{\varepsilon}{4} \cdot \mathbf{B}\right)$$

for every $s = 1, \ldots, p$, we consider the set-valued function $g_s = \{\varphi \cdot y_s\}$ which satisfies $g_s \leq f$. From the assumptions on H, there exist of $h_1, \ldots, h_m \in H$ such that

$$f \leq h_j,\ j = 1, \ldots, m, \qquad \bigcap_{j=1}^{m} L(h_j)(x_0) \subset L(f)(x_0) + \frac{\varepsilon}{4} \cdot \mathbf{B}.$$

Since $L(f)$ and each $L(g_s)$, $s = 1, \ldots, p$ and $L(h_j)$, $j = 1, \ldots, m$ are Hausdorff continuous at x_0, we can apply the same argument of the first step and obtain a neighborhood U of x_0 such that, for every $x \in U$,

$$L(f)(x) \subset \bigcup_{s=1}^{p} L(g_s)(x) + \frac{3\varepsilon}{4} \cdot \mathbf{B}, \qquad \bigcap_{j=1}^{m} L(h_j)(x) \subset L(f)(x) + \frac{3\varepsilon}{4} \cdot \mathbf{B}.$$

For every $s = 1, \ldots, p$, the function g_s is single-valued and therefore the net $(L_i(g_s))_{i \in I}$ converges to $L(g_s)$. Moreover, $\lim^{\leq}_{i \in I} L_i(h_j) = L(h_j)$ for every $j = 1, \ldots, m$ and therefore we can find $\alpha \in I$ such that, for every $i \in I$, $i \geq \alpha$, $s = 1, \ldots, p$, $j = 1, \ldots, m$ and $x \in X$,

$$L_i(h_j)(x) \subset L(h_j)(x) + \frac{\varepsilon}{4} \cdot \mathbf{B}, \qquad L(g_s)(x) \subset L_i(g_s)(x) + \frac{\varepsilon}{4} \cdot \mathbf{B} \subset L_i(f)(x) + \frac{\varepsilon}{4} \cdot \mathbf{B}.$$

It follows, for every $i \in I$, $i \geq \alpha$, $j = 1, \ldots, m$ and $x \in U$,

$$L(f)(x) \subset \bigcup_{s=1}^{p} L(g_s)(x) + \frac{3\varepsilon}{4} \cdot \mathbf{B} \subset L_i(f)(x) + \varepsilon \cdot \mathbf{B}.$$

Similarly, since $L_i(f)(x) \subset L_i(h_j)(x) \subset L(h_j) + \frac{3\varepsilon}{4} \cdot \mathbf{B}$, we have

$$L_i(f)(x) \subset \bigcap_{j=1}^{m} L(h_j)(x) + \frac{\varepsilon}{4} \cdot \mathbf{B} \subset L(f)(x) + \varepsilon \cdot \mathbf{B}.$$

Since X is compact, we can conclude the proof as in the first step.

Third step. Let $f \in \mathcal{C}$. From [8, Lemma 1.2], for every $\varepsilon > 0$, we can find $\varphi_1, \ldots, \varphi_p \in C(X, \mathbb{R})$ and $A_1, \ldots, A_p \in \mathcal{K}(\mathbb{R}^d)$ such that

$$f \leq \sum_{s=1}^{p} \varphi_s \cdot A_s + \varepsilon \cdot \mathbf{B}, \qquad \sum_{s=1}^{p} \varphi_s \cdot A_s + \varepsilon \cdot \mathbf{B} \leq f,$$

and from the second step we easily obtain the convergence of $(L_i(f))_{i \in I}$ to $L(f)$ also in this case. □

3. Convexity monotone operators and Korovkin approximation of vector-valued functions

Korovkin approximation of set-valued functions is strictly related to Korovkin approximation of single vector-valued functions. In this section, we consider a particular class of set-valued and vector-valued operators which will allow us to state some applications of the results in Section 2 concerning vector-valued functions and conversely to deepen the analysis of set-valued function in the next section. First of all, let X be a Hausdorff compact topological space and let $T : C(X, \mathbb{R}^d) \to C(X, \mathbb{R}^d)$ a linear operator. We recall that T is called a *convexity monotone operator* if it satisfies the following property, for every $\varphi_1, \varphi_2 \in C(X, \mathbb{R}^d)$ having disjoint graphs

$$\varphi \in \mathrm{co}(\varphi_1, \varphi_2) \;\Rightarrow\; T(\varphi) \in \mathrm{co}(T(\varphi_1), T(\varphi_2)). \tag{3.3}$$

A simple equivalent formulation of the preceding definition states that T is convexity monotone if and only if for every $\varphi, \varphi_1 \in C(X, \mathbb{R}^d)$ such that $\varphi_1(x) \neq 0$ for every $x \in X$, we have (see [9, Proposition 2.4])

$$\varphi \in \mathrm{co}(0, \varphi_1) \;\Rightarrow\; T(\varphi) \in \mathrm{co}(0, T(\varphi_1)).$$

The convexity monotone property generalizes that of positive operator in the case $d = 1$. In this regard, different notions have been introduced with the same aim. In [16], Nishishiraho introduced the quasi-positive operators; namely a linear operator $T : C(X, \mathbb{R}^d) \to C(X, \mathbb{R}^d)$ is said to be *quasi positive* if, for every $\varphi, \psi \in C(X, \mathbb{R})$ such that $|\varphi| \leq \psi$, we have

$$\|T(\varphi \cdot y)(x)\| \leq \|T(\psi \cdot y)(x)\|$$

for every $x \in X$ and $y \in \mathbb{R}^d$. The connections between convexity monotone and quasi positive operators have been investigated in [7, 9]. In general, convexity monotone operators are quasi-positive operators ([9, Proposition 2.6]); in the case $d = 1$, we have the following characterization [7, Proposition 2.5].

Proposition 3.1. *Let X a connected compact Hausdorff topological space and $T : C(X, \mathbb{R}) \to C(X, \mathbb{R})$ a continuous linear operator. Then, the following statements are equivalent:*

a) *T is convexity monotone.*
b) *T is quasi-positive.*
c) *There exist two closed subsets X_T^+ and X_T^- of X such that $X = X_T^+ \cup X_T^-$ and*

$$\varphi \in C(X, \mathbb{R}),\ \varphi \geq 0 \;\Rightarrow\; T(\varphi) \geq 0 \text{ on } X_T^+,\ T(\varphi) \leq 0 \text{ on } X_T^-.$$

d) *There exist two closed subsets X_T^+ and X_T^- of X such that $X = X_T^+ \cup X_T^-$ and*

$$\varphi, \psi \in C(X, \mathbb{R}),\ \varphi \leq \psi \;\Rightarrow\; T(\varphi) \leq T(\psi) \text{ on } X_T^+,\ T(\varphi) \geq T(\psi) \text{ on } X_T^-.$$

If $T : C(X, \mathbb{R}) \to C(X, \mathbb{R})$ is convexity monotone, the two subsets X_T^+ and X_T^- provided in c) and d) of the preceding proposition are given by

$$X_T^+ := \{x \in X \mid L(\mathbf{1})(x) \geq 0\}, \qquad X_T^- := \{x \in X \mid L(\mathbf{1})(x) \leq 0\}.$$

We observe that if $x \in X_T^+ \cap X_T^-$, we have necessarily $T(\xi)(x) = 0$ for every $\xi \in C([0, 1], \mathbb{R})$. Condition c) in Proposition 3.1 has been used in [10] in order to extend the results in [12, 15] to sequences of convexity monotone operators (not necessarily positive) in the space $C([0, 1])$. Our aim is a more general extension in the case where X is not necessarily the interval $[0, 1]$. Classical examples of convexity monotone operators $T : C([a, b]) \to C([a, b])$ which are not positive are integral operators $T(\xi)(x) := \int_{x_0}^{x} \xi$ with $x_0 \in]a, b[$ (in this case $X_T^+ = [a, x_0]$ and $X_T^- = [x_0, b]$). The notion of convexity monotone operator introduced in [5, 6] for operators in spaces of vector-valued functions can be also considered for operators in the setting of set-valued functions. Namely, a continuous monotone linear operator $L : C(X, \mathcal{K}(\mathbb{R}^d)) \to C(X, \mathcal{K}(\mathbb{R}^d))$ is said to be *convexity monotone* if it satisfies the condition

$$\varphi \in \mathrm{co}(\varphi_1, \varphi_2) \;\Rightarrow\; L(\{\varphi\}) \subset \mathrm{co}(L(\{\varphi_1\}), L(\{\varphi_2\})) \tag{3.4}$$

for every $\varphi_1, \varphi_2 \in C(X, \mathbb{R}^d)$ having disjoint graphs. The preceding condition is equivalent to the following, for every $\varphi \in C(X, \mathbb{R}^d)$,

$$L(\mathrm{co}(0, \varphi)) \subset \mathrm{co}(0, L(\{\varphi\})).$$

We begin by stating some results obtained in [6, 7, 8]. We recall that $\varphi_1, \dots, \varphi_n$ are *affinely independent* if, for every $x \in X$, we have

$$\lambda_1, \dots, \lambda_n \in [0,1], \ \sum_{i=1}^{n} \lambda_i = 0, \ \sum_{i=1}^{n} \lambda_i \varphi_i(x) = 0 \ \Rightarrow \ \lambda_1 = 0, \dots, \lambda_n = 0.$$

If $\mathcal{C}$ is a subcone of $C(X, \mathcal{K}(\mathbb{R}^d))$ and $L : \mathcal{C} \to \mathcal{C}$ is a convexity monotone linear continuous operator, a subset H of $\mathcal{C}$ is called a *Korovkin system in $\mathcal{C}$ with respect to convexity monotone operators* for L if for every equicontinuous net $(L_i)_{i\in I}^{\leq}$ of convexity monotone linear operators from $\mathcal{C}$ into itself satisfying $\lim_{i\in I}^{\leq} L_i(h) = L(h)$ for every $h \in H$, we also have $\lim_{i\in I}^{\leq} L_i(f) = L(f)$ for every $f \in \mathcal{C}$. Similarly, a subset M of $C(X, \mathbb{R}^d)$ is called a *Korovkin system in $C(X, \mathbb{R}^d)$ with respect to convexity monotone operators* for a convexity monotone linear continuous operator $T : C(X, \mathbb{R}^d) \to C(X, \mathbb{R}^d)$ if for every equicontinuous net $(T_i)_{i\in I}^{\leq}$ of convexity monotone linear operators from $C(X, \mathbb{R}^d)$ into itself satisfying $\lim_{i\in I}^{\leq} T_i(\varphi) = T(\varphi)$ for every $\varphi \in M$, we also have $\lim_{i\in I}^{\leq} T_i(\varphi) = T(\varphi)$ for every $\varphi \in C(X, \mathbb{R}^d)$.

Theorem 3.7. **([7, Theorem 2.6])** *Let X be a connected compact Hausdorff topological space, $T : C(X, \mathbb{R}^d) \to C(X, \mathbb{R}^d)$ a convexity monotone linear operator and M a subset of $C(X, \mathbb{R}^d)$ such that, for every $\varphi \in C(X, \mathbb{R}^d)$, $x_0 \in X$ and $\varepsilon > 0$, there exist affinely independent functions $\varphi_1, \dots, \varphi_m \in C(X, \mathbb{R}^d)$ satisfying the following conditions*

$$\varphi \in \mathrm{co}(\varphi_1, \dots, \varphi_m),$$

$$\mathrm{co}(T(\varphi_1)(x_0), \dots, T(\varphi_m)(x_0)) \subset T(\varphi)(x_0) + \varepsilon \cdot \boldsymbol{B}.$$

Then, M is a Korovkin system in $C(X, \mathbb{R}^d)$ for T with respect to convexity monotone operators.

We state some consequences of the preceding result. In the case $d = 1$, the following result generalizes a classical property of Korovkin systems obtained by Berens and Lorentz [3] in the case of the identity operator and by Ferguson and Rusk [11] in the case of positive operators (see also [1, Chapter 3]).

Corollary 3.2. **([7, Corollaries 2.8])** *Let X be a connected compact Hausdorff topological space, $T : C(X, \mathbb{R}) \to C(X, \mathbb{R})$ a convexity monotone linear operator and M a subset of $C(X, \mathbb{R})$ such that, for every $\varphi \in C(X, \mathbb{R})$, $x_0 \in X$ and $\varepsilon > 0$, there exist $\psi, \chi \in C(X, \mathbb{R})$ satisfying the following conditions*

$$\psi \leq \varphi \leq \chi, \qquad |T(\chi)(x_0) - T(\psi)(x_0)| \leq \varepsilon.$$

Then, M is a Korovkin system in $C(X, \mathbb{R})$ for T with respect to convexity monotone operators.

In the particular case where T is the identity operator, we have the following further consequences.

Corollary 3.3. **([7, Corollaries 2.7, 2.9] and [8, Corollary 3.3])** *Let X be a connected compact Hausdorff topological space and Γ a Korovkin set in $C(X, \mathbb{R})$. Then, the following subsets of $C(X, \mathbb{R}^d)$:*

$$M_1 := \{\varphi \cdot y \mid \varphi \in \Gamma, y \in \mathbb{R}^d\},$$

$$M_2 := \{(\delta_{i,1}\varphi, \dots, \delta_{i,d}\varphi) \mid i = 1, \dots, d, \varphi \in \Gamma\}$$

($\delta_{i,j}$ denotes as usual the Kronecker symbol) are Korovkin systems in $C(X, \mathbb{R}^d)$ for the identity operator with respect to convexity monotone operators.

The next result is a set-valued version of Theorem 3.7.

Theorem 3.8. *Let X be a connected compact Hausdorff topological space, $\mathcal{C}$ a subcone of $C(X, \mathcal{K}(\mathbb{R}^d))$ containing the single-valued continuous functions and the constant function $1 \cdot \boldsymbol{B}$ and $L : \mathcal{C} \to \mathcal{C}$ a convexity monotone continuous linear operator satisfying conditions a) and b) of Theorem 2.3. Let H be a subset of $\mathcal{C}$ containing the function $\boldsymbol{B}$ and such that, for every $f \in \mathcal{C}$, $x_0 \in X$ and $\varepsilon > 0$, there exist affinely independent functions $\varphi_1, \ldots, \varphi_m \in C(X, \mathbb{R}^d)$ satisfying the following conditions*

$$f \leq \mathrm{co}(\varphi_1, \ldots, \varphi_m) \quad (\textit{i.e.}, \forall\, x \in X: \ f(x) \subset \mathrm{co}(\varphi_1(x), \ldots, \varphi_m(x)))$$

and

$$\mathrm{co}(L(\{\varphi_1\})(x_0), \ldots, L(\{\varphi_m\})(x_0)) \subset L(\{\varphi\})(x_0) + \varepsilon \cdot \boldsymbol{B}.$$

Then, H is a Korovkin system in $\mathcal{C}$ for L with respect to convexity monotone operators.

Proof. Let $(L_i)_{i \in I}$ be an equicontinuous net of convexity monotone linear operators converging to $L(h)$ for every $h \in H$. First, we show that if $\varphi_1, \ldots, \varphi_m$ are affinely independent functions in $C(X, \mathbb{R}^d)$, then

$$\lim_{i \in I}{}_{\leq} L_i(\mathrm{co}(\varphi_1, \ldots, \varphi_m)) = L(\mathrm{co}(\varphi_1, \ldots, \varphi_m)).$$

Let $\varepsilon > 0$; since $(L_i(\mathbf{B}))_{i \in I}$ converges to $L(\mathbf{B})$, we can find $M \geq 1$ such that

$$L_i(1 \cdot \mathbf{B}) \subset M \cdot \mathbf{B}\,, \qquad L(1 \cdot \mathbf{B}) \subset M \cdot \mathbf{B}.$$

For every $x \in X$, the set $\mathrm{co}(\varphi_1(x), \ldots, \varphi_m(x))$ is compact and therefore there exist $\lambda(s) := (\lambda_1(s), \ldots, \lambda_m(s))$, $s = 1, \ldots, p$, such that $\lambda_j(s) \geq 0$ for every $j = 1, \ldots, m$ and $\sum_{j=1}^{p} \lambda_j(s) = 1$ for every $s = 1, \ldots, p$ and further

$$\mathrm{co}(\varphi_1(x), \ldots, \varphi_m(x)) \subset \bigcup_{s=1}^{p} \left(\sum_{j=1}^{m} \lambda_j(s)\{\varphi_j(x)\} + \frac{\varepsilon}{2M} \cdot \mathbf{B} \right).$$

Since X is compact, the preceding formula can be extended to every $x \in X$. From Theorem 3.7, we have the convergence of $(L_i(\{\varphi\}))_{i \in I}$ to $L(\{\varphi\})$ for every $\varphi \in C(X, \mathbb{R}^d)$ and consequently the nets $\left(\sum_{j=1}^{m} \lambda_j(s) L_i(\{\varphi_j\})\right)_{i \in I}$ converge to $\sum_{j=1}^{m} \lambda_j(s) L(\{\varphi_j\})$ for every $s = 1, \ldots, p$. Hence, we can find $\alpha \in I$ such that, for every $i \in I$, $i \geq \alpha$ and $s = 1, \ldots, p$,

$$\sum_{j=1}^{m} \lambda_j(s) L(\{\varphi_j\}) \subset \sum_{j=1}^{m} \lambda_j(s) L_i(\{\varphi_j\}) + \frac{\varepsilon}{2} \cdot \mathbf{B},$$
$$\sum_{j=1}^{m} \lambda_j(s) L_i(\{\varphi_j\}) \subset \sum_{j=1}^{m} \lambda_j(s) L(\{\varphi_j\}) + \frac{\varepsilon}{2} \cdot \mathbf{B}.$$

It follows, for every $i \in I$, $i \geq \alpha$,

$$\begin{aligned} L(\mathrm{co}(\varphi_1, \ldots, \varphi_m)) &\subset \bigcup_{s=1}^{p} \left(\sum_{j=1}^{m} \lambda_j(s) L(\{\varphi_j\}) + L\left(\frac{\varepsilon}{2M} \cdot \mathbf{B}\right) \right) \\ &\subset \bigcup_{s=1}^{p} \left(\sum_{j=1}^{m} \lambda_j(s) L(\{\varphi_j\}) + \frac{\varepsilon}{2} \cdot \mathbf{B} \right) \\ &\subset \bigcup_{s=1}^{p} \left(\sum_{j=1}^{m} \lambda_j(s) L_i(\{\varphi_j\}) + \varepsilon \cdot \mathbf{B} \right) \\ &\subset L_i(\mathrm{co}(\varphi_1, \ldots, \varphi_m)) + \varepsilon \cdot \mathbf{B}, \end{aligned}$$

and conversely

$$\begin{aligned} L_i(\mathrm{co}(\varphi_1,\dots,\varphi_m)) &\subset \bigcup_{s=1}^{p}\left(\sum_{j=1}^{m}\lambda_j(s)L_i(\{\varphi_j\}) + L_i\left(\frac{\varepsilon}{2M}\cdot\mathbf{B}\right)\right) \\ &\subset \bigcup_{s=1}^{p}\left(\sum_{j=1}^{m}\lambda_j(s)L_i(\{\varphi_j\}) + \frac{\varepsilon}{2}\cdot\mathbf{B}\right) \\ &\subset \bigcup_{s=1}^{p}\left(\sum_{j=1}^{m}\lambda_j(s)L(\{\varphi_j\}) + \varepsilon\cdot\mathbf{B}\right) \\ &\subset L(\mathrm{co}(\varphi_1,\dots,\varphi_m)) + \varepsilon\cdot\mathbf{B}. \end{aligned}$$

Hence, the net $(L_i(\mathrm{co}(\varphi_1,\dots,\varphi_m)))_{i\in I}$ converges to $L(\mathrm{co}(\varphi_1,\dots,\varphi_m))$. At this point, we observe that the set

$$\{\mathrm{co}(\varphi_1,\dots,\varphi_m) \mid \varphi_1,\dots,\varphi_m \text{ affinely independent in } C(X,\mathbb{R}^n)\}$$

satisfies condition (2.1) in Theorem 2.3; hence we can proceed just as in the proof of Theorem 2.3 in order to show the convergence of $(L_i(f))_{i\in I}$ to $L(f)$ for every $f\in\mathcal{C}$ and this completes the proof. □

4. Further developments on the approximation of set-valued functions

In this section, we consider some recent developments of Korovkin approximation theory in the case where the existence of the limit operator is not assigned. The following results have been established in [10] in the case $X=[0,1]$. Here, we consider the more general setting of metric spaces. Let (X,σ) be a compact metric space. Functions in $C(X,\mathcal{K}(\mathbb{R}^d))$ can be approximated by functions in $C(X,\mathbb{R}^d)$, as shown in the following lemma.

Lemma 4.1. *Let $f\in C(X,\mathcal{K}(\mathbb{R}^d))$. Then, for every $\varepsilon>0$, there exist $\varphi_1,\dots,\varphi_m\in\mathrm{Sel}(f)$ such that, for every $x\in X$,*

$$f(x)\subset\bigcup_{i=1}^{m}\left(\{\varphi_i(x)\}+\varepsilon\cdot\mathbf{B}\right). \tag{4.5}$$

Proof. The proof follows the same line of that of [10, Lemma 2.1], but for the sake of completeness we give some details. Let $\varepsilon>0$ and $x_0\in X$. From the compactness of $f(x_0)$, we can find $y_1,\dots,y_m\in f(x_0)$ such that

$$f(x_0)\subset\bigcup_{i=1}^{m}\left(\{y_i\}+\frac{\varepsilon}{3}\cdot\mathbf{B}\right).$$

Moreover, from [5, Proposition 1.1], we can find $\varphi_1,\dots,\varphi_m\in\mathrm{Sel}(f)$ such that $\varphi_i(x_0)=y_i$ for every $i=1,\dots,m$. Since f and $\varphi_1,\dots,\varphi_m$ are continuous, there exists $\delta>0$ such that

$$f(x)\subset f(x_0)+\frac{\varepsilon}{3}\cdot\mathbf{B}\,,\qquad \|\varphi_i(x)-\varphi_i(x_0)\|\le\frac{\varepsilon}{3},$$

for every $x\in X$ such that $\sigma(x,x_0)<\delta$ and $i=1,\dots,m$. Hence, for every $x\in X$ such that $\sigma(x,x_0)<\delta$,

$$f(x)\subset f(x_0)+\frac{\varepsilon}{3}\cdot\mathbf{B}\subset\bigcup_{i=1}^{m}\left(\{\varphi_i(x_0)\}+\frac{2\varepsilon}{3}\cdot\mathbf{B}\right)\subset\bigcup_{i=1}^{m}\left(\{\varphi_i(x)\}+\varepsilon\cdot\mathbf{B}\right).$$

At this point, from a compactness argument on X, we obtain the proof. □

We observe that if $\varphi_1, \ldots, \varphi_m \in C(X, \mathbb{R}^d)$, then the set-valued function $\mathrm{co}(\varphi_1, \ldots, \varphi_m) \in C(X, \mathcal{K}(\mathbb{R}^d))$ and since f takes convex values, (4.5) can be written as

$$\mathrm{co}(\varphi_1, \ldots, \varphi_m) \leq f \leq \mathrm{co}(\varphi_1, \ldots, \varphi_m) + \varepsilon \cdot \mathbf{B}. \tag{4.6}$$

In the following proposition, we state some general properties of continuous monotone linear operators on $C(X, \mathcal{K}(\mathbb{R}^d))$. We only sketch the proof since it is similar to that of [10, Proposition 2.2].

Proposition 4.2. *Let $L : C(X, \mathcal{K}(\mathbb{R}^d)) \to C(X, \mathcal{K}(\mathbb{R}^d))$ be a continuous monotone linear operator on $C(X, \mathcal{K}(\mathbb{R}^d))$. Then,*

(i) *$L(\{0\}) = \{0\}$ and consequently L maps single-valued continuous functions into single-valued continuous functions, i.e., for every $\varphi \in C(X, \mathbb{R}^d)$ there exists $\psi \in C(X, \mathbb{R}^d)$ such that $L(\{\varphi\}) = \{\psi\}$.*

(ii) *If $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, then*

$$L(f) = \overline{\bigcup_{\varphi \in \mathrm{Sel}(f)} L(\{\varphi\})} = \overline{\bigcup_{\varphi_1, \ldots, \varphi_m \in \mathrm{Sel}(f)} L(\mathrm{co}(\varphi_1, \ldots, \varphi_m))}, \tag{4.7}$$

i.e.,

$$L(f)(x) = \overline{\bigcup_{\varphi \in \mathrm{Sel}(f)} L(\{\varphi\})(x)} = \overline{\bigcup_{\varphi_1, \ldots, \varphi_m \in \mathrm{Sel}(f)} L(\mathrm{co}(\varphi_1, \ldots, \varphi_m))(x)}$$

for every $x \in X$.

Proof. Obviously $\{0\} \subset L(\{0\})$ and moreover $L(\{0\}) = L(\{0\}) + L(\{0\})$; hence $L(\{0\}) = \{0\}$. Now, let $\varphi \in C(X, \mathbb{R}^d)$, $x \in X$ and $y, z \in \{\varphi\}(x)$. From [4, Proposition 1.1], we can find $\psi_1, \psi_2 \in \mathrm{Sel}(L(\{\varphi\}))$ such that $\psi_1(x) = y$ and $\psi_2(x) = z$. The linearity of L yields $z - y = \psi_2(x) - \psi_1(x) \in L(\{\varphi\}) + L(\{-\varphi\}) = L(\{0\}) = \{0\}$ and therefore $y = z$. Thus, $L(\{\varphi\})$ must be single-valued. As regards to the proof of the first equality in (4.7), the monotonicity property of L yields the inequality $\bigcup_{\varphi \in \mathrm{Sel}(f)} L(\{\varphi\}) \leq L(f)$ and since $L(f)(x)$ is closed for every $x \in [0,1]$, we get $\overline{\bigcup_{\varphi \in \mathrm{Sel}(f)} L(\{\varphi\})(x)} \leq L(f)(x)$. In order to show the converse inequality, first observe that since L is continuous and takes compact values on the compact space X, there exists $\alpha > 0$ such that $L(e_0 \cdot \mathbf{B}) \leq \alpha \cdot \mathbf{B}$. If $\varepsilon > 0$, from Lemma 4.1, there exist $\varphi_1, \ldots, \varphi_m \in \mathrm{Sel}(f)$ such that, for every $x \in [0,1]$,

$$f(x) \subset \bigcup_{i=1}^{m} (\{\varphi_i(x)\} + \varepsilon \cdot \mathbf{B}) = \left(\bigcup_{i=1}^{m} \{\varphi_i(x)\} \right) + \varepsilon \cdot \mathbf{B} \subset \mathrm{co}(\varphi_1, \ldots, \varphi_m)(x) + \varepsilon \cdot \mathbf{B}.$$

From the monotonicity property of f, we obtain

$$L(\mathrm{co}(\varphi_1, \ldots, \varphi_m)) \leq L(f) \leq L(\mathrm{co}(\varphi_1, \ldots, \varphi_m)) + \varepsilon\alpha \cdot \mathbf{B}.$$

Since $\mathrm{co}(\varphi_1, \ldots, \varphi_m) \subset \bigcup_{\varphi \in \mathrm{Sel}(f)} L(\{\varphi\})$, we have that the uniform distance between $L(f)$ and $\overline{\bigcup_{\varphi \in \mathrm{Sel}(f)} L(\{\varphi\})}$ is less or equal to $\alpha\varepsilon$. From the arbitrarily of $\varepsilon > 0$, this distance must be 0. Hence, the first equality in (4.7) is valid. Finally, the second equality is a consequence of the following inclusions

$$\overline{\bigcup_{\varphi \in \mathrm{Sel}(f)} L(\{\varphi\})(x)} \subset \overline{\bigcup_{\varphi_1, \ldots, \varphi_m \in \mathrm{Sel}(f)} L(\mathrm{co}(\varphi_1, \ldots, \varphi_m))(x)} \subset L(f)(x)$$

which are obviously true for every $x \in X$. $\square$

The preceding Proposition 4.2 allows us to associate a single-valued bounded linear operator to every set-valued continuous monotone linear operator. This result has been established in [10, Lemma 2.3] in the particular case $X = [0,1]$, while here we consider the general case.

Lemma 4.2. *Let $L : C(X,\mathcal{K}(\mathbb{R}^d)) \to C(X,\mathcal{K}(\mathbb{R}^d))$ be a continuous monotone linear operator. Then, there exists a bounded linear operator $T_L : C(X,\mathbb{R}^d) \to C(X,\mathbb{R}^d)$ such that, for every $\varphi \in C(X,\mathbb{R}^d)$ and $x \in X$,*

$$\{T_L(\varphi)(x)\} = L(\{\varphi\})(x). \tag{4.8}$$

Moreover, if $L : C(X,\mathcal{K}(\mathbb{R}^d))$ is convexity monotone, then the operator T_L is itself convexity monotone.

The proof is at all similar to that of [10, Lemma 2.3] and therefore we omit the details. We only point out that the operator $T_L : C(X,\mathbb{R}^d) \to C(X,\mathbb{R}^d)$ is defined by setting, for every $\varphi \in C(X,\mathbb{R}^d)$ and $x \in X$, $T_L(\varphi)(x)$ as the unique element of $L(\{\varphi\})(x)$. Conversely, if $T : C(X,\mathbb{R}^d) \to C(X,\mathbb{R}^d)$ is a bounded linear operator satisfying (3.3), we can define the operator $L_T : C(X,\mathcal{K}(\mathbb{R}^d)) \to C(X,\mathcal{K}(\mathbb{R}^d))$ by setting, for every $f \in C(X,\mathcal{K}(\mathbb{R}^d))$ and $x \in X$,

$$L_T(f)(x) = \bigcup_{\varphi\in \mathrm{Sel}(f)} \{T(\varphi)(x)\}. \tag{4.9}$$

In [5, Proposition 1.5], it has been shown that $L_T(f)(x)$ is a convex compact subset of $\mathbb{R}^d$ and $L_T(f)$ is continuous and hence L_T is well-defined; moreover it has also been shown that L_T is a continuous monotone linear operator and satisfies the condition of convexity monotonicity. Now, we are in a position to state the main results of this section which generalizes the Korovkin-type approximation of vector-valued and set-valued functions to the case where the limit operator is not assigned. Some results in this setting have been obtained in [12, 15] and more recently in [10, Theorems 1.1, 1.2 and 2.4]. All the preceding results have been established in the case $X = [0,1]$, while we consider here a non trivial extension to the multivariable case. For the sake of simplicity, we consider the case where X is a compact convex subset of $\mathbb{R}^2$. We begin with the case of vector-valued continuous functions by considering first the case of real valued functions and then the general case.

Theorem 4.9. *Let X be a convex compact subset of $\mathbb{R}^2$ and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of continuous linear operators on $C(X,\mathbb{R})$ satisfying condition (3.3) and assume that*

(i) *The sequence $(T_n(\mathrm{pr}_i^2))_{n\in\mathbb{N}}$, $i = 1,2$, converges to a real continuous function $h_i \in C(X,\mathbb{R})$ (pr_i denotes the canonical projection $(x_1,x_2) \mapsto x_i$ of $\mathbb{R}^2$ onto $\mathbb{R}$).*

(ii) *The sequence $(|T_n(\xi)(x,y)|)_{n\in\mathbb{N}}$ is nonincreasing with respect to n for any convex function $\xi \in C(X,\mathbb{R})$ and $(x,y) \in X$.*

Then, there exists a continuous linear operator $T : C(X,\mathbb{R}) \to C(X,\mathbb{R})$ satisfying condition (3.3) and such that $\lim_{n\to+\infty} T_n(\xi)(x,y) = T(\xi)(x,y)$ for every $\xi \in C(X,\mathbb{R})$ and uniformly with respect to $(x,y) \in X$.

Proof. We observe that the sequence $(T_n(\xi))_{n\in\mathbb{N}}$ is constant in n for every linear function ξ and therefore converges to a function $T(\xi) = T_0(\xi)$; in particular $(T_n(\mathbf{1}))_{n\in\mathbb{N}}$ converges to $T(\mathbf{1})$ and $(T_n(\mathrm{pr}_i))_{n\in\mathbb{N}}$ converges to $T(\mathrm{pr}_i)$, $i = 1,2$. Now, let $X^+ := \{(x,y) \in X \mid T(\mathbf{1})(x,y) \geq 0\}$ and $X^- := \{(x,y) \in X \mid T(\mathbf{1})(x,y) \leq 0\}$ and, for every $n \in \mathbb{N}$, $X_n^+ := \{(x,y) \in X \mid T_n(\mathbf{1})(x,y) \geq 0\}$ and $X_n^- := \{(x,y) \in X \mid T_n(\mathbf{1})(x,y) \leq 0\}$. First, we observe that the sequence $(T_n)_{n\in\mathbb{N}}$ is equibounded. Indeed, if $\xi \in C(X,\mathbb{R})$ is positive and $\|\xi\| \leq 1$, we have $\xi \in \mathrm{co}(0,\mathbf{1})$ and from (3.3) it follows $T_n(\xi) \in \mathrm{co}(0,T_n(\mathbf{1}))$ for every $n \in \mathbb{N}$. This yields $0 \leq T_n(\xi)(x,y) \leq T_n(\mathbf{1})(x,y)$ for every $(x,y) \in X_n^+$ and $T_n(\mathbf{1})(x,y) \leq T_n(\xi)(x,y) \leq 0$ for every $(x,y) \in X_n^-$ and therefore

$\|T_n(\xi)\| \leq \|T_n(\mathbf{1})\|$. If ξ is not positive, we apply the above argument to ξ_+ and ξ_- and we easily obtain $\|T_n(\xi)\| = \|T_n(\xi_+) - T_n(\xi_-)\| \leq \|T_n(\xi_+)\| + \|T_n(\xi_-)\| \leq 2\|T_n(\mathbf{1})\|$. Thus

$$\sup_{n\in\mathbb{N}} \|T_n\| \leq 2 \sup_{n\in\mathbb{N}} \|T_n(T_\infty(\mathbf{1}))\| < +\infty.$$

Hence, the sequence $(T_n)_{n\in\mathbb{N}}$ is equibounded and therefore it is enough to show the convergence of the sequence $(T_n(\xi))_{n\in\mathbb{N}}$ for every $\xi \in C^2(X,\mathbb{R})$. Let $\xi \in C^2(X,\mathbb{R})$ and consider the two functions

$$\gamma_\pm(x,y) := a(x^2+y^2) \pm \xi(x,y)\,, \qquad (x,y) \in X,$$

where

$$a := \max\left\{ \left\|\frac{\partial^2\xi}{\partial x^2}\right\|, \left\|\frac{\partial^2\xi}{\partial y^2}\right\|, \sqrt{\frac{1}{2}\left(\left\|\left(\frac{\partial^2\xi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\xi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\xi}{\partial x\partial y}\right)^2 - \frac{\partial^2\xi}{\partial x^2}\,\frac{\partial^2\xi}{\partial y^2}\right\|\right)} \right\}.$$

We have, for every $(x,y) \in X$,

$$\begin{aligned}
\frac{\partial^2\gamma_\pm}{\partial x^2}(x,y) &= 2a \pm \frac{\partial^2\xi}{\partial x^2}(x,y),\\
\frac{\partial^2\gamma_\pm}{\partial y^2}(x,y) &= 2a \pm \frac{\partial^2\xi}{\partial y^2}(x,y),\\
\frac{\partial^2\gamma_\pm}{\partial x\partial y}(x,y) &= \pm\frac{\partial^2\xi}{\partial x\partial y}(x,y),
\end{aligned}$$

and therefore $\frac{\partial^2\gamma_\pm}{\partial x^2} \geq 0$ and

$$\begin{aligned}
\frac{\partial^2\gamma_\pm}{\partial x^2}\,\frac{\partial^2\gamma_\pm}{\partial y^2} - \left(\frac{\partial^2\gamma_\pm}{\partial x\partial y}\right)^2 &= \left(2a \pm \frac{\partial^2\xi}{\partial x^2}\right)\left(2a \pm \frac{\partial^2\xi}{\partial y^2}\right) - \left(\frac{\partial^2\xi}{\partial x\partial y}\right)^2\\
&= 4a^2 \pm 2a\left(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2}\right) + \frac{\partial^2\xi}{\partial x^2}\,\frac{\partial^2\xi}{\partial y^2} - \left(\frac{\partial^2\xi}{\partial x\partial y}\right)^2\\
&= a^2 \pm 2a\frac{\partial^2\xi}{\partial x^2} + \left(\frac{\partial^2\xi}{\partial x^2}\right)^2\\
&\quad + a^2 \pm 2a\frac{\partial^2\xi}{\partial y^2} + \left(\frac{\partial^2\xi}{\partial y^2}\right)^2\\
&\quad + 2a^2 + \frac{\partial^2\xi}{\partial x^2}\,\frac{\partial^2\xi}{\partial y^2} - \left(\frac{\partial^2\xi}{\partial x\partial y}\right)^2 - \left(\frac{\partial^2\xi}{\partial x^2}\right)^2 - \left(\frac{\partial^2\xi}{\partial y^2}\right)^2\\
&= \left(a \pm \frac{\partial^2\xi}{\partial x^2}\right)^2 + \left(a \pm \frac{\partial^2\xi}{\partial y^2}\right)^2\\
&\quad + 2a^2 + \frac{\partial^2\xi}{\partial x^2}\,\frac{\partial^2\xi}{\partial y^2} - \left(\frac{\partial^2\xi}{\partial x\partial y}\right)^2 - \left(\frac{\partial^2\xi}{\partial x^2}\right)^2 - \left(\frac{\partial^2\xi}{\partial y^2}\right)^2\\
&\geq 0.
\end{aligned}$$

Hence, the functions $\gamma_\pm$ are convex and from assumption (ii), it follows that the sequences $(|T_n(\gamma_\pm)(x,y)|)_{n\in\mathbb{N}}$ are nonincreasing in n for every $(x,y) \in X$. Let $(x,y) \in X$; if $T(\mathbf{1})(x,y) > 0$, we have $(x,y) \in X^+$ and moreover, taking into account that the sequence $(T_n(\mathbf{1}))_{n\in\mathbb{N}}$ is constant, we also have $(x,y) \in X_n^+$ for every $n \in \mathbb{N}$. Hence, for every $n,p \in \mathbb{N}$, we have

$T_n(\gamma_\pm)(x,y) - T_{n+p}(\gamma_\pm)(x,y) \geq 0$, $i = 1,2$ and consequently

$$\begin{aligned}|(T_n - T_{n+p})(\xi)(x,y)| \leq & \|\xi\| \, (T_n - T_{n+p})(\mathbf{1})(x,y) + a\,(T_n - T_{n+p})(\mathrm{pr}_1^2)(x,y) \\ & + a\,(T_n - T_{n+p})(\mathrm{pr}_2^2)(x,y) \\ = & a\,\left((T_n - T_{n+p})(\mathrm{pr}_1^2)(x,y) + (T_n - T_{n+p})(\mathrm{pr}_2^2)(x,y)\right),\end{aligned}$$

which ensures that the sequence $(T_n(\xi)(x,y))_{n\in\mathbb{N}}$ is a Cauchy sequence. We get the same result if $T(\mathbf{1})(x,y) < 0$ using the fact that $(x,y) \in X^-$ and $(x,y) \in X_n^-$ for every $n \in \mathbb{N}$. Finally, assume that $T(\mathbf{1})(x,y) = 0$, that is $\lim_{n\to+\infty} T_n(\mathbf{1})(x,y) = 0$; if ξ is positive, we have $\xi \in \mathrm{co}(0, \|\xi\|\,\mathbf{1})$ and from Proposition 3.1, it follows $0 \leq T_n(\xi)(x,y) \leq \|\xi\| T_n(\mathbf{1})(x,y)$ for every $n \in \mathbb{N}$; hence we have $\lim_{n\to+\infty} T_n(\xi)(x,y) = 0$. If ξ is not positive, it is enough to apply the previous argument to the positive and negative part of ξ. Thus, we have shown that $(T_n(\xi)(x,y))_{n\in\mathbb{N}}$ is a Cauchy sequence for every $(x,y) \in X$. A straightforward compact argument on X yields that $(T_n(\xi))_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(X,\mathbb{R})$ and therefore it converges. Hence, we have obtained the existence of the limit operator $T : C(X,\mathbb{R}) \to C(X,\mathbb{R})$. If $(x,y) \in X^+ \setminus X^-$, we have $T(\mathbf{1})(x,y) > 0$ and the above argument shows that $T(\xi)(x,y) \geq 0$ for every positive $\xi \in C(X,\mathbb{R})$; analogously if $(x,y) \in X^- \setminus X^+$, we have $T(\mathbf{1})(x,y) < 0$ and $T(\xi)(x,y) \leq 0$ for every positive $\xi \in C(X,\mathbb{R})$. If $(x,y) \in X^+ \cap X^-$, we have $T(\mathbf{1})(x,y) = 0$ and we have shown that in this case $T(\xi)(x,y) = 0$ for every positive $\xi \in C(X,\mathbb{R})$. Hence, T satisfies condition c) in Proposition 3.1. □

A quantitative estimate of the convergence can be easily obtained by applying the same arguments in [10, (3)] (see also the proof of [15, Theorem 1.2]) to the cases $(x,y) \in X^+$ and $(x,y) \in X^-$. We omit the details for the sake of brevity. As an immediate consequence of Theorem 4.9, we can state the following result concerning the convergence of sequences of vector-valued functions. As usual, we shall denote by $\mathbf{e}_j = (\delta_{ij})_{i=1,2}$, $j = 1,2$, the canonical basis of $\mathbb{R}^2$ and by $\langle\cdot,\cdot\rangle$ the scalar product in $\mathbb{R}^2$.

Theorem 4.10. *Let X be a convex compact subset of $\mathbb{R}^2$ and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of continuous linear operators on $C(X,\mathbb{R}^2)$ satisfying condition (3.3) and assume that*

(i) *The sequence $(T_n(\mathrm{pr}_i^2 \cdot \mathbf{e}_j))_{n\in\mathbb{N}}$, $i = 1,2$, converges to a vector-valued continuous function $h_i \in C(X,\mathbb{R}^2)$.*

(ii) *The sequence $(\|T_n(\varphi)(x,y)\|)_{n\in\mathbb{N}}$ is nonincreasing with respect to n for any function $\varphi \in C(X,\mathbb{R}^2)$ having convex components and $(x,y) \in X$.*

Then, there exists a continuous linear operator $T : C(X,\mathbb{R}^2) \to C(X,\mathbb{R}^2)$ satisfying (3.3) and such that $\lim_{n\to+\infty} T_n(\varphi)(x,y) = T(\varphi)(x,y)$ for every $\varphi \in C(X,\mathbb{R}^2)$ uniformly with respect to $(x,y) \in X$.

Proof. The proof follows the same line of [10, Theorem 1.2]. For every $j,k = 1,2$ and $n \in \mathbb{N}$, consider the operator $T_{j,k,n} : C(X,\mathbb{R}) \to C(X,\mathbb{R})$ defined by setting

$$T_{j,k,n}(\xi)(x,y) := \langle T_n(\xi \cdot \mathbf{e}_j)(x,y), \mathbf{e}_k\rangle \tag{4.10}$$

whenever $\xi \in C(X,\mathbb{R})$ and $(x,y) \in X$. Then, the sequence $(T_{j,k,n})_{n\in\mathbb{N}}$ satisfies all assumptions of Theorem 4.9 and consequently if strongly converges to an operator $T_{j,k} : C(X,\mathbb{R}) \to C(X,\mathbb{R})$ satisfying (3.3). Now, consider the operator $T : C(X,\mathbb{R}^2) \to C(X,\mathbb{R}^2)$ defined by setting, for every $\varphi \in C(X,\mathbb{R}^2)$ and $(x,y) \in X$,

$$T(\varphi)(x,y) := (T_{1,1}(\varphi_1)(x,y) + T_{2,1}(\varphi_2)(x,y), T_{1,2}(\varphi_1)(x,y) + T_{2,2}(\varphi_2)(x,y)),$$

where $\varphi_j := \langle\varphi, \mathbf{e}_j\rangle$, $j = 1,2$, denotes the function $\varphi_j(x,y) = \langle\varphi(x,y), \mathbf{e}_j\rangle$, $(x,y) \in X$. Then, from (4.10), it follows that the sequence $(T_n)_{n\in\mathbb{N}}$ converges strongly to T and this also implies that T is convexity monotone; indeed (3.3) is satisfied by each T_n and passing to the limit at every point $(x,y) \in X$ it is also satisfied by T. □

Finally, we state the corresponding result for set-valued continuous functions.

Theorem 4.11. *Let* $(L_n)_{n\in\mathbb{N}}$ *be a sequence of continuous monotone linear operators on* $C(X,\mathcal{K}(\mathbb{R}^2))$ *satisfying condition* (3.4) *and assume that*

(i) *The sequence* $(L_n(\{\mathrm{pr}_i^2 \cdot \boldsymbol{e}_j\})_{n\in\mathbb{N}}$, $i,j = 1,2$, *converges to a set-valued continuous function* $h_{i,j} \in C(X,\mathbb{R}^2)$.

(ii) *For every* $j,k = 1,2$, *the sequence* $(|\langle \eta_{n,j}(x,y), \boldsymbol{e}_k\rangle|)_{n\in\mathbb{N}}$ *is nonincreasing whenever* $\xi \in C(X,\mathbb{R})$ *is a convex function,* $(x,y) \in X$ *and* $\eta_{n,j} \in C(X,\mathbb{R}^2)$ *is such that* $L_n(\{\xi \cdot \boldsymbol{e}_j\})(x) = \{\eta_{n,j}(x)\}$.

(ii) *The sequence* $(|T_n(\varphi)(x,y)|)_{n\in\mathbb{N}}$ *is nonincreasing with respect to* n *for any function* $\varphi \in C(X,\mathbb{R}^2)$ *having convex components and* $(x,y) \in X$.

Then, there exists a continuous monotone linear operator $L : C(X,\mathcal{K}(\mathbb{R}^2)) \to C(X,\mathcal{K}(\mathbb{R}^2))$ *such that* $\lim_{n\to+\infty} L_n(f)(x,y) = L(f)(x,y)$ *for every* $f \in C(X,\mathcal{K}(\mathbb{R}^2))$ *uniformly with respect to* $(x,y) \in X$.

Proof. Also in this case the proof follows the same line of [10, Theorem 2.4]. For every $n \in \mathbb{N}$, consider the operator $T_n := T_{L_n}$ defined as in (4.8). From Lemma 4.2, the sequence $(T_n)_{n\in\mathbb{N}}$ satisfies the assumptions in Theorem 4.10 and therefore there exists a continuous linear operator $T : C(X,\mathbb{R}^2) \to C(X,\mathbb{R}^2)$ satisfying condition (3.3) and such that $\lim_{n\to+\infty} T_n(\varphi) = T(\varphi)$ for every $\varphi \in C(X,\mathbb{R}^2)$. From (4.9), we can define the continuous monotone linear operator $L := L_T : C(X,\mathcal{K}(\mathbb{R}^2)) \to C(X,\mathcal{K}(\mathbb{R}^2))$. First, we observe that the sequence $(L_n(\mathbf{1}\cdot\mathbf{B}))_{n\in\mathbb{N}}$ is equibounded. Indeed, $\mathbf{1}\cdot\mathbf{B} \subset \mathrm{co}(\delta_{kj}\mathbf{1}\cdot\mathbf{e}_j; k,j = 1,2)$ and since every L_n is convexity monotone, we have $L_n(\mathbf{1}\cdot\mathbf{B}) \subset \mathrm{co}(\delta_{kj}L_n(\{\mathbf{1}\cdot\mathbf{e}_j\})$, $k,j = 1,2$; from assumption (i), the sequence $(L_n(\{\mathbf{1}\cdot\mathbf{e}_j\}))_{n\in\mathbb{N}}$ converges to $\{h_{0,j}\}$ for every $j = 1,2$ and therefore there exists $M > 0$ such that $L_n(\mathbf{1}\cdot\mathbf{B}) \subset M\cdot\mathbf{B}$. Now, we show that for every $\varphi_1,\ldots,\varphi_m \in C([0,1],\mathbb{R}^2)$,

$$\lim_{n\to+\infty} L_n(\mathrm{co}(\varphi_1,\ldots,\varphi_m)) = \mathrm{co}(L(\{\varphi_1\}),\ldots,L(\{\varphi_m\})). \tag{4.11}$$

Indeed, for every $j = 1,\ldots,m$ we know that $\lim_{n\to+\infty} L_n(\{\varphi_j\}) = L(\{\varphi_j\})$ (i.e., for the associated operators, $\lim_{n\to+\infty} T_n(\varphi_j) = T(\varphi_j)$) and therefore, if $\varepsilon > 0$, there exists $\nu \in \mathbb{N}$ such that $\|T_n(\varphi_j) - T(\varphi_j)\| \le \varepsilon$ whenever $n \ge \nu$. It follows, for every $n \ge \nu$,

$$\begin{aligned}
&\mathrm{co}(L_n(\{\varphi_1\}),\ldots,L_n(\{\varphi_m\})) \subset \mathrm{co}(L(\{\varphi_1\}),\ldots,L(\{\varphi_m\})) + \varepsilon\cdot\mathbf{B},\\
&\mathrm{co}(L(\{\varphi_1\}),\ldots,L(\{\varphi_m\})) \subset \mathrm{co}(L_n(\{\varphi_1\}),\ldots,L_n(\{\varphi_m\})) + \varepsilon\cdot\mathbf{B}.
\end{aligned} \tag{4.12}$$

Indeed, if $\varphi(x,y) := \sum_{j=1}^m \lambda_j(x,y)T_n(\varphi_j)(x,y)$ with $\lambda_j(x,y) \ge 0$ and $\sum_{j=1}^m \lambda_j(x,y) = 1$, we can write

$$\varphi(x,y) := \sum_{j=1}^m \lambda_j(x,y)T(\varphi_j)(x,y) + \sum_{j=1}^m \lambda_j(x,y)(T_n(\varphi_j)(x,y) - T(\varphi_j)(x,y))$$

and since $\|T_n(\varphi_j)(x,y) - T(\varphi_j)(x,y)\| \le \varepsilon$, we obtain $\varphi \in \mathrm{co}(L(\{\varphi_1\}),\ldots,L(\{\varphi_m\})) + \varepsilon\cdot\mathbf{B}$; the second inclusion in (4.12) is similar. From (4.12) and taking into account that every L_n and L are convexity monotone, (4.11) follows. Finally, let $f \in C(X,\mathcal{K}(\mathbb{R}^2))$ and fix $\varepsilon > 0$. From Lemma 4.1 and (4.6), there exist $\varphi_1,\ldots,\varphi_m \in \mathrm{Sel}(f)$ such that

$$\mathrm{co}(\varphi_1,\ldots,\varphi_m) \le f \le \mathrm{co}(\varphi_1,\ldots,\varphi_m) + \varepsilon\cdot\mathbf{B}.$$

Every L_n is convexity monotone and therefore

$$\mathrm{co}(L_n(\{\varphi_1\}),\ldots,L_n(\{\varphi_m\})) \le L_n(f) \le \mathrm{co}(L_n(\{\varphi_1\}),\ldots,L_n(\{\varphi_m\})) + \varepsilon M\cdot\mathbf{B}.$$

Since ε is arbitrarily chosen, from (4.11), we can conclude that $\lim_{n\to+\infty} L_n(f) = L(f)$. □

Many sequences of set-valued operators are obtained from their analogs in the single-valued setting. In this sense, the assumptions in Theorem 4.11 are quite natural since allow to obtain the convergence in the set-valued setting from the same assumptions in the single-valued setting.

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