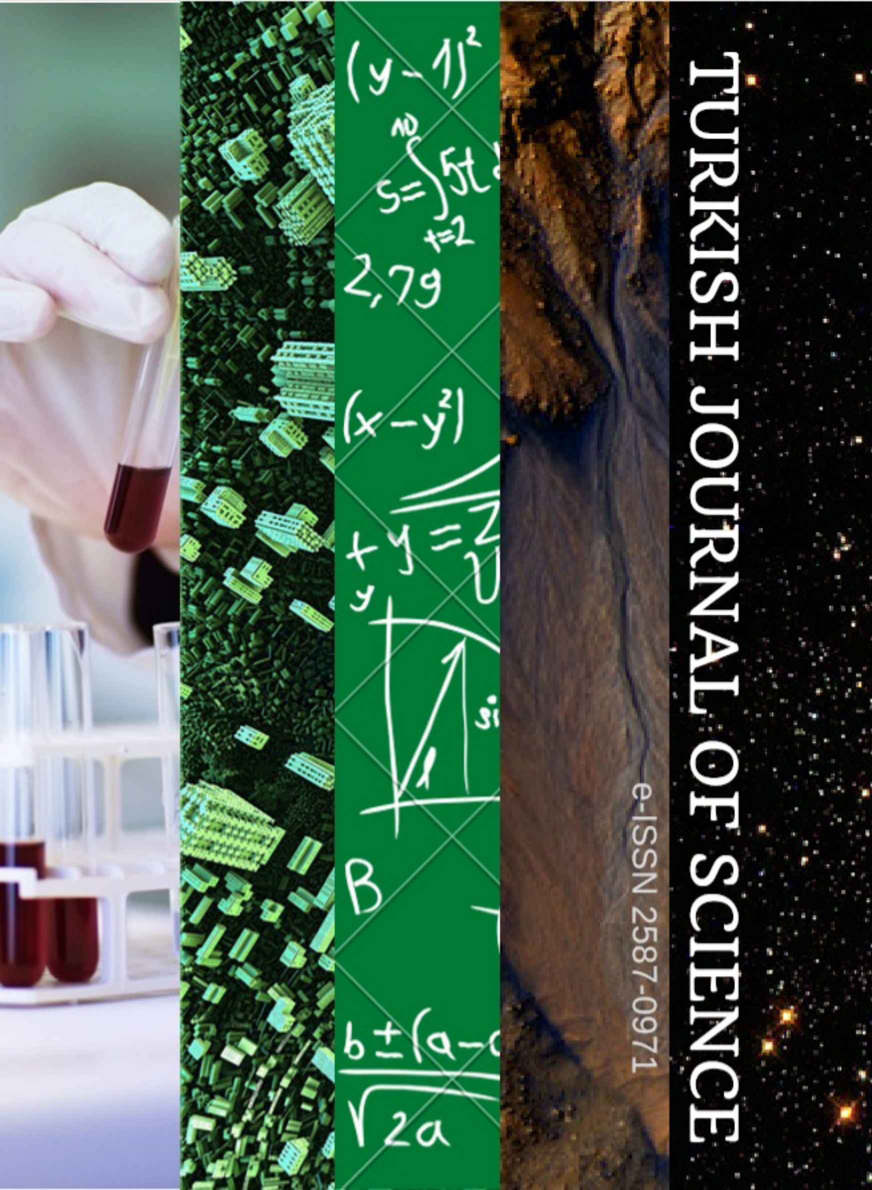

TURKISH JOURNAL OF SCIENCE
e-ISSN 2587-0971

TURKISH JOURNAL OF SCIENCE

(An International Peer-Reviewed Journal)
ISSN: 2587-0971

Volume: VII, Issue: I, 2022

Turkish Journal of Science (TJOS) is published electronically yearly. It publishes, in English or Turkish, full-length original research papers and solicited review articles. TJOS provides a forum to scientists, researchers, engineers and academicians to share their ideas and new research in the field of mathematical and related sciences as well as theirs applications. TJOS is a high-quality double-blind refereed journal. TJOS is also an international research journal that serves as a forum for individuals in the field to publish their research efforts as well as for interested readers to acquire latest development information in the field. TJOS facilitate communication and networking among researchers and scientists in a period where considerable changes are taking place in scientific innovation. It provides a medium for exchanging scientific research and technological achievements accomplished by the international community.

Abstracting & Indexing in:
1- CiteFactor
https://www.citefactor.org
2- ResearchBib
http://journalseeker.researchbib.com/view/issn/2587-0971
3- Scientific Indexing Services
https://www.sindexs.org/JournalList.aspx?ID=4519
4- COSMOS
http://www.cosmosimpactfactor.com/page/journals_details/5755.html
5- ASOS
https://asosindex.com.tr/index.jsp?modul=journal-page&journal-id=1655
6- J-Gate
https://jgateplus.com/home/j-gate/

Correspondence Address
Turkish Journal of Science (TJOS)
http://dergipark.gov.tr/tjos

CONTENTS

A Logarithmic Finite Difference Method for Numerical Solutions of the Generalized Huxley Equation	*Gonca ÇELİKTEN*	1-6
Independent Transversal Domination Number of Corona and Join Operation in Path Graphs	*Betül ATAY ATAKUL*	7-13
Invariant and Lacunary Invariant Statistical Convergence of Order η for Double Set Sequences	*Uğur ULUSU and Erdinç DÜNDAR*	14-20
Lightlike Hypersurfaces of Poly-Norden Semi-Riemannian Manifolds	*Erol KILIÇ, Tuba ACET and Selcen YÜKSEL PERKTAŞ*	21-30
Some special Smarandache ruled surfaces by Frenet Frame in E^3 -I	*Süleyman ŞENYURT, Davut CANLI and Elif ÇAN*	31-42
Analysis of the spread of Hookworm infection with Caputo-Fabrizio fractional derivative	*Esin İLHAN*	43-52

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 1, 1-6
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

A Logarithmic Finite Difference Method for Numerical Solutions of the Generalized Huxley Equation

Gonca Çelikten[a]

[a]*Kafkas University, Faculty of Science and Letters, Department of Mathematics, 36100, Kars, Turkey.*

Abstract. In this paper, numerical solutions of generalized Huxley equation are obtained by using a new scheme: Implicit logarithmic finite difference method (I-LFDM). The efficiency of the presented method is illustrated by a numerical example for different cases of parameters which confirm that obtained results are in good agreement with the exact solutions and numerical solutions obtained by some other methods in literature. The method is analyzed by von-Neumann stability analysis method and it is displayed that the method is unconditionally stable.

1. INTRODUCTION

Nonlinear partial differential equations are often used to model most of the problems in various fields such as physics, chemistry, biology, mathematics and engineering. One of these nonlinear partial differential equations is generalized Huxley equation.
The generalized Huxley equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \beta u\left(1 - u^{\delta}\right)\left(u^{\delta} - \gamma\right), \qquad a < x < b, \quad t > 0 \tag{1}$$

with initial condition

$$u(x, 0) = f(x), \qquad a < x < b$$

and boundary conditions

$$u(a, t) = g_1(t), \quad u(b, t) = g_2(t), \quad t > 0$$

describes the propagation of a nerve impulse in nerve fibers and the movement of the wall in liquid crystals. Where $f(x)$, $g_1(t)$ and $g_2(t)$ are known functions, $\delta, \beta \geq 0$and $\gamma \in (0.1)$are given parameters.
Various numerical methods have been used to solve the equation (1) numerically by many researchers. Hashim et. al. [9] applied the Adomian decomposition method to solve the equation numerically. Variational iteration method (VIM) has been used to obtain the numerical solutions of the equation by Batiha et. al. [2]. Hashemi et. al. [8] used the homotopy perturbation method (HPM) and then Hemida and Mohamed [10] used the homotopy analysis method (HAM) for obtaining the numerical solutions of the

Corresponding author: GÇ mail address: gnc.cnbk@gmail.com ORCID:https://orcid.org/0000-0002-2639-2490
Received: 8 November 2021; Accepted: 2 March 2022; Published: 20 April 2022
Keywords. (Generalized Huxley equation; implicit logarithmic finite difference method; von Neumann Satbility analysis.)
2010 *Mathematics Subject Classification*. 65M06
Cited this article as: Çelikten G. A Logarithmic Finite Difference Method for Numerical Solutions of the Generalized Huxley Equation, Turkish Journal of Science, 2022, 7(1), 1-6.

equation. Inan [12, 13] used the explicit exponential finite difference method and implicit exponential finite difference method (I-EFDM) to solve the equation.
In this study, we present the implicit logarithmic finite difference method to obtain the numerical solutions of the generalized Huxley equation. Logarithmic finite difference methods have been used to solve various equations in literature. İsmail and Al-Basyoni [14] used the closed logarithmic finite difference method to solve the Troesch problem numerically. Srivastava et al. [16] used the closed logarithmic finite difference method to solve two-dimensional Burgers equation systems. The one-dimensional coupled Burgers equation was solved by Srivastava et al. [15] using the closed logarithmic finite difference method. Aljaboori [1] used the Crank-Nicolson logarithmic finite difference method to solve the combined Burgers equation numerically. El-Azab et al. [7] obtained numerical solutions of the Korteweg de Vries Burger (KdVB) equation using the open logarithmic finite difference method. Celikten et. al. [3] used the explicit logarithmic finite difference schemes to solve the Burgers equation. Modified Burgers equation as solved by Celikten [4] using the explicit logarithmic finite difference schemes. Celikten [5] obtained the numerical solutions of Burgers equation by using implicit and fully implicit logarithmic finite difference methods. Celikten and Sürek [6] used the explicit logarithmic finite difference method to solve the generalized Burgers-Fisher equation numerically.

2. MATERIALS AND METHODS

2.1. IMPLICIT LOGARITHMIC FINITE DIFFERENCE METHOD

We demonstrate the finite difference approximation of $u(x,t)$at the node point (x_i, t_n) by u_i^n in which $x_i = ih(i = 0,1,\ldots,N)$, $t_n = t_0 + nk(n = 0,1,2,\ldots)$, $h = \frac{b-a}{N}$ is the node size in x direction and $k$is the time step.
We reorganize Equation (1) to acquire

$$\frac{\partial u}{\partial t} = \beta u \left(1 - u^{\delta}\right)\left(u^{\delta} - \gamma\right) + \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

Multiplying equation (2) by e^u, we acquire the following equation:

$$\frac{\partial e^u}{\partial t} = e^u \left(\beta u \left(1 - u^{\delta}\right)\left(u^{\delta} - \gamma\right) + \frac{\partial^2 u}{\partial x^2}\right) \tag{3}$$

using the finite difference approximations for derivatives in Equation (3) the following implicit logarithmic finite difference scheme is acquired
I-EFDM

$$u_i^{n+1} = u_i^n + \ln\left\{1 + k\left[\beta u_i^n \left(1 - \left(u_i^n\right)^{\delta}\right)\left(\left(u_i^n\right)^{\delta} - \gamma\right) + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2}\right]\right\} \tag{4}$$

where $1 \leq \mathrm{i} \leq \mathrm{N} - 1$.
Equation (4) is a system of nonlinear difference equations. We assume this nonlinear system of equations in the form

$$G(W) = 0 \tag{5}$$

where $G = \left[g_1, g_2, \ldots, g_{N-1}\right]^T$ and $W = \left[u_1^{n+1}, u_2^{n+1}, \ldots, u_{N-1}^{n+1}\right]^T$. The nonlinear Equation (5) is linearized using Newton's iterative approach, which yields the following iteration:
1) Determine $W^{(0)}$, a first guess.
2) For $m = 0,1,2,3\ldots$up to convergency do:
Resolve $J\left(W^{(m)}\right)\delta^{(m)} = -G\left(W^{(m)}\right)$;
Adjust $W^{(m+1)} = W^{(m)} + \delta^{(m)}$ where $J(W^{(m)})$ the Jacobian matrix which is appraised analytically. The initial estimate is based on the solution from the previous time step. The Newton iteration is halted at every time step when $\left\|G(W^{(m)})\right\| \leq 10^{-5}$.

2.2. LOCAL TRUNCATION ERROR AND CONSISTENCY

In order to analyze the local truncation errors of the numerical scheme (4), the nonlinear term of the scheme has been linearized by replacing the quantity $\left(u_i^n\right)^\delta$ by local constant $\tilde{U}$. Hence the numerical scheme (4), convert into

$$u_i^{n+1} = u_i^n + \ln\left\{1 + k\left[\beta u_i^n\left(1-\tilde{U}\right)\left(\tilde{U}-\gamma\right) + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2}\right]\right\} \tag{6}$$

Since the scheme (6) is logarithmic, the examination will be improved by expanding the logarithmic term of the scheme into a Taylor's series. Hilal et al. [11] applied the same procedure to calculate the local truncation error of exponential finite difference schemes and examine their stability. If the scheme's logarithmic term is expanded to a Taylor series and the first term is used, the scheme can be expressed as:

$$u_i^{n+1} = u_i^n + k\beta u_i^n\left(1-\tilde{U}\right)\left(\tilde{U}-\gamma\right) + k\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2}\right] \tag{7}$$

Expansion of the terms u_i^{n+1}, u_{i+1}^{n+1} and u_{i-1}^{n+1}about the point $(x_i, t_n)$by Taylor' s series and substitution into

$$T_i^n = u_i^{n+1} - u_i^n - k\beta u_i^n\left(1-\tilde{U}\right)\left(\tilde{U}-\gamma\right) - k\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2}\right]$$

leads to

$$T_i^n = \left[\frac{\partial u}{\partial t} - \beta u\left(1-\tilde{U}\right)\left(\tilde{U}-\gamma\right) - \frac{\partial^2 u}{\partial x^2}\right]_i^n + \frac{k}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n - \frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n + \ldots$$

Therefore the principal part of the local truncation error is as follows:

$$\frac{k}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n - \frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n$$

Hence the local truncation error is $T_i^n = O(k) + O\left(h^2\right)$
Since $\lim\limits_{h,k\to 0}\left[O(k) + O\left(h^2\right)\right] = 0$ presented scheme is consistent. And the scheme is first order in time and second order in space.

2.3. STABILITY ANALYSIS

We will utilize the von Neumann stability analysis to analyze the scheme's stability, where the growth factor of a characteristic Fourier mode is specified as follows:

$$u_i^n = \varepsilon^n e^{I\phi ih},\ I = \sqrt{-1}. \tag{8}$$

von Neumann stability analysis is used to analyze the stability of finite difference schemes applied to linear partial differential equations. So we will investigate the stability of linear form of the scheme. By substituting the (8) equality into the (7) linear form of the scheme, we get the growth factors as follows:

$$\varepsilon = \frac{1 + k\beta\left(1-\tilde{U}\right)\left(\tilde{U}-\gamma\right)}{1 + \frac{2k}{h^2}\sin^2\frac{\phi h}{2}}.$$

Stability condition in von-Neumann method is $|\varepsilon| \leq 1$
$|\varepsilon| \leq 1$since $\beta \geq 0$ and $\gamma \in (0.1)$.Therefore I-LFDM generalized Huxley equation is unconditionally stable.

3. NUMERICAL RESULTS AND DISCUSSION

Implicit logarithmic finite difference method is used to acquire the numerical solutions of the generalized Huxley equation. To demonstrate the correctness of results L_2 and L_∞error norms:

$$L_2 = \|U - u_N\|_2 = \sqrt{h \sum_{j=0}^{N} \left|U_j - (u_N)_j\right|^2},$$

$$L_\infty = \|U - u_N\|_\infty = \max_j \left|U_j - (u_N)_j\right|$$

are used, in which U and u indicate the exact and computed numerical solutions, respectively. In all numerical computations we took as $h = 0.01$ and $k = 0.0001$.

3.1. *NUMERICAL EXAMPLE OF GENERALIZED HUXLEY EQUATION*

Consider the generalized Huxley equation of the form Equation (1) in domain $0 \le x \le 1,\ t > 0$ with initial condition

$$u(x, 0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}$$

and boundary conditions

$$u(0,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left\{\sigma\gamma\left\{\frac{(1+\delta-\gamma)\rho}{2(1+\delta)}\right\}t\right\}\right]^{\frac{1}{\delta}}, u(1,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left\{\sigma\gamma\left(1 + \left\{\frac{(1+\delta-\gamma)\rho}{2(1+\delta)}\right\}t\right)\right\}\right]^{\frac{1}{\delta}}.$$

The exact solution of this problem is [17]:

$$u(x,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left\{\sigma\gamma\left(x + \left\{\frac{(1+\delta-\gamma)\rho}{2(1+\delta)}\right\}t\right)\right\}\right]^{\frac{1}{\delta}}$$

where $\rho = \sqrt{4\beta(1+\delta)}$ and $\sigma = \delta\rho/4(1+\delta)$.

The numerical solutions of Generalized Huxley Equation obtained by I-LFDM are compared with the exact solutions and numerical solutions obtained by some other methods [2,8-10,12] in literature in Table 1-3. The comparisons for the case $\delta = 1$, $\beta = 1$and $\gamma = 0.001$ are shown in Table 1 while the comparisons for the case $\delta = 2$, $\beta = 1$and $\gamma = 0.001$ are shown in Table 2 and for the case $\delta = 3$, $\beta = 1$and $\gamma = 0.001$ are shown in Table 3. As can be seen from the tables, numerical solutions obtained by the presented method are quite compatible with exact solutions and numerical solutions obtained by some other methods in the literature. In addition, the numerical solutions obtained by the method presented at time $t = 1$ are better than the numerical solutions obtained by some other methods in the literature. L_2 and L_∞ error norms for the case $\delta = 1$, $\gamma = 0.01$ and different values of β are given in Table 4. L_2 and L_∞ error norms for the case $\delta = 1$,$\beta = 1$ and different values of γ are given in Table 5. Table 6 presents L_2 and L_∞ error norms for the case $\beta = 1$, $\gamma = 0.001$ and different values of δ. As it can be seen from the tables, the L_2 and L_∞ error norms acquired by the I-LFDM are quite small in all cases.

4. CONCLUSION

In this study, implicit logarithmic finite difference method is used to obtain the numerical solutions of the generalized Huxley equation. The comparison of the numerical solutions obtained by presented method with the exact solutions and the numerical solutions obtained by previous studies in the literature is given by tables. It is clear from the tables that the numerical solutions obtained by I-LFDM are in good agreement with the exact solutions and better than numerical solutions obtained by some other methods in literature. The presented method is an efficient technique for finding numerical solutions for various kinds of nonlinear problems.

Table 1: Exact and numerical solutions for the case $\delta = 1$, $\beta = 1$and $\gamma = 0.001$.

x	t	Exact	I-LFDM	VIM [2], HPM [8], ADM [9]	HAM [10]	I-EFDM [12]
0.1	0.05	5.000302E-4	5.000199E-4	5.000052E-4	5.000100E-4	5.000125 E-4
	0.1	5.000427E-4	5.000276E-4	4.999927E-4	5.000030E-4	5.000102 E-4
	1	5.002676E-4	5.002451E-4	4.997678E-4	4.998680E-4	5.000064 E-4
0.5	0.05	5.001009E-4	5.000778E-4	5.000759E-4	5.000810E-4	5.000768 E-4
	0.1	5.001134E-4	5.000750E-4	5.000634E-4	5.000730E-4	5.000692 E-4
	1	5.003383E-4	5.002758E-4	4.998385E-4	4.999380E-4	5.000572 E-4
0.9	0.05	5.001716E-4	5.001613E-4	5.001466E-4	5.001520E-4	5.001540 E-4
	0.1	5.001841E-4	5.001691E-4	5.001341E-4	5.001440E-4	5.001516 E-4
	1	5.004090E-4	5.003865E-4	4.999092E-4	5.000090E-4	5.001479 E-4

Table 2: Exact and numerical solutions for the case $\delta = 2$, $\beta = 1$and $\gamma = 0.001$.

x	t	Exact	I-LFDM	VIM [2]	HPM [8], ADM [9]	HAM [10]	I-EFDM [12]
0.1	0.05	2.236188E-2	2.236142E-2	2.236077E-2	2.236077E-2	2.236100E-2	2.236110E-2
	0.1	2.236244E-2	2.236177E-2	2.236021E-2	2.236021E-2	2.236070E-2	2.236099E-2
	1	2.237250E-2	2.237149E-2	2.235015E-2	2.235015E-2	2.223546E-2	2.236082E-2
0.5	0.05	2.236447E-2	2.236343E-2	2.236335E-2	2.236335E-2	2.236360E-2	2.236339E-2
	0.1	2.236503E-2	2.236331E-2	2.236279E-2	2.236279E-2	2.236320E-2	2.236305E-2
	1	2.237508E-2	2.237229E-2	2.235273E-2	2.235273E-2	2.235720E-2	2.236251E-2
0.9	0.05	2.236705E-2	2.236659E-2	2.236593E-2	2.236593E-2	2.236620E-2	2.236114E-2
	0.1	2.236761E-2	2.236693E-2	2.236537E-2	2.236537E-2	2.236580E-2	2.236615E-2
	1	2.237766E-2	2.237665E-2	2.235532E-2	2.235531E-2	2.235980E-2	2.236599E-2

Table 3: Exact and numerical solutions for the case $\delta = 3$, $\beta = 1$and $\gamma = 0.001$.

x	t	Exact	I-LFDM	VIM [2]	HPM [8], ADM [9]	HAM [10]	I-EFDM [12]
0.1	0.05	7.937402E-2	7.937239E-2	7.937005E-2	7.937005E-2	7.937080E-2	7.937122E-2
	0.1	7.937601E-2	7.937361E-2	7.936807E-2	7.936807E-2	7.936970E-2	7.937084E-2
	1	7.941169E-2	7.940812E-2	7.933236E-2	7.933234E-2	7.934820E-2	7.937025E-2
0.5	0.05	7.938196E-2	7.937829E-2	7.937799E-2	7.937799E-2	7.937880E-2	7.937814E-2
	0.1	7.938394E-2	7.937784E-2	7.937601E-2	7.937601E-2	7.937760E-2	7.937692E-2
	1	7.941962E-2	7.940971E-2	7.934031E-2	7.934029E-2	7.935620E-2	7.937501E-2
0.9	0.05	7.938989E-2	7.938826E-2	7.938592E-2	7.938592E-2	7.938670E-2	7.938709E-2
	0.1	7.939187E-2	7.938948E-2	7.938394E-2	7.938394E-2	7.938550E-2	7.938671E-2
	1	7.942755E-2	7.942398E-2	7.934825E-2	7.934823E-2	7.936410E-2	7.938612E-2

Table 4: L_2 and L_∞ error norms for the case $\delta = 1$ and $\gamma = 0.01$.

t	L_2			L_∞		
	$\beta = 1$	$\beta = 10$	$\beta = 100$	$\beta = 1$	$\beta = 10$	$\beta = 100$
0.01	4.390519E-7	4.390371E-6	4.388844E-5	4.974336E-7	4.974196E-6	4.972792E-5
0.1	2.851070E-6	2.850957E-5	2.847622E-4	3.825356E-6	3.825217E-5	3.821034E-4
1	4.541154E-6	4.531847E-5	3.835394E-4	6.218224E-6	6.205651E-5	5.263925E-4
10	4.529978E-6	3.594694E-5	1.047479E-7	6.202945E-6	4.923570E-5	1.442949E-7

Table 5: L_2 and L_∞ error norms for the case $\delta = 1$ and $\beta = 1$.

t	L_2			L_∞		
	$\gamma = 0.01$	$\gamma = 0.001$	$\gamma = 0.0001$	$\gamma = 0.01$	$\gamma = 0.001$	$\gamma = 0.0001$
0.01	4.390519E-7	4.410392E-9	4.412417E-11	4.974336E-7	4.996848E-9	7.049136E-12
0.1	2.851070E-6	2.863973E-8	2.865266E-10	3.825356E-6	3.842667E-8	5.420848E-11
1	4.541154E-6	4.561835E-8	4.563772E-10	6.218224E-6	6.246538E-8	8.811778E-11
10	4.529978E-6	4.561961E-8	4.563810E-10	6.202945E-6	6.246721E-8	8.811849E-11

Table 6: L_2 and L_∞ error norms for the case $\beta = 1$ and $\gamma = 0.001$.

t	L_2			L_∞		
	$\delta = 1$	$\delta = 2$	$\delta = 4$	$\delta = 1$	$\delta = 2$	$\delta = 4$
0.01	4.410392E-9	1.972431E-7	1.318968E-6	4.996848E-9	2.234709E-7	1.494356E-6
0.1	2.863973E-8	1.280826E-6	8.564763E-6	3.842667E-8	1.718520E-6	1.149162E-5
1	4.561835E-8	2.039346E-6	1.362625E-5	6.246538E-8	2.792493E-6	1.865864E-5
10	4.561961E-8	2.030148E-6	1.344094E-5	6.246721E-8	2.779902E-6	1.840493E-5

References

[1] Aljaboori, M. A. H. (2018). Exponenetial And Logarithmic Crank-Nicolson Methods For Solving Coupled Burger's Equations. Misan Journal for Academic Studies, 33, 50-61.

[2] Batiha, B., Noorani, M.S.M., Hashim, I. (2007). Numerical simulation of the generalized Huxley equation by He's variational iteration method, Applied Mathematics and Computations, 186: 1322-1325.

[3] Celikten, G., Göksu, A. and Yagub, G. (2017). Explicit Logarithmic Finite Difference Schemes For Numerical Solution of Burgers Equation. European International Journal of Science and Technology, 6(5), 57-67.

[4] Celikten, G. (2021). Numerical Solutions of the Modified Burgers Equation by Explicit Logarithmic Finite Difference Schemes. Sohag Journal of Mathematics, 8(3), 73-79.

[5] Celikten, G. (2020). Logarithmic Finite Difference Methods for Numerical Solutions of Burgers Equation. Erzincan University Journal of Science and Technology, 13(3), 984-994.

[6] Celikten, G. (2020). Numerical Solution of the Generalized Burgers – Fisher Equation with Explicit Logarithmic Finite Difference Method. Gümüshane University Journal of Science and Technology, 10(3), 752-761.

[7] El-Azab, M. S., El-Kalla, I. L. and El-Morsy, S. A. (2014). Composite Finite Difference Scheme Applied to a Couple of Nonlinear Evolution Equations. Electronic Journal Of Mathematical Analysis And Applications, 2(2), 254-263.

[8] Hashemi, S. H., Daniali, H. R. M., Ganji, D. D. (2007). Numerical simulation of the generalized Huxley equation by He's homotopy perturbation method, Applied Mathematics and Computations, 192: 157-161.

[9] Hashim, I., Noorani, M. S. M., Batiha, B. (2006). A note on the Adomian decomposition method for the generalized Huxley Equation, Applied Mathematics and Computations, 181: 1439-1445.

[10] Hemida, K., Mohamed, M. S. (2012). Application of homotopy analysisi method to fractional order generalized Huxley equation, Journal of Applied Functional Analysis, 7(4): 367-372.

[11] Hilal, N. Injrou, S., Karroum, R. (2020). Exponential finite difference methods for solving Newell–Whitehead–Segel equation, Arabian Journal of Mathematics, 9, 367–379.

[12] Inan, B.(2016). A New Numerical Scheme for the Generalized Huxley Equation, Bulletin of Mathematical Sciences and Applications, 16: 105-111.

[13] Inan, B.(2017). Finite difference methods for the generalized Huxley and Burgers-Huxley equations, Kuwait Journal of Science, 44(3): 20-27.

[14] Ismail, M.S. and Al-Basyoni, K.S. (2018). A Logarithmic Finite Difference Method for Troesch's Problem. Applied Mathematics, 9, 550-559.

[15] Srivastava, V. K., Singh, S. and Awasthi M. K. (2013). Numerical solutions of coupled Burgers' equations by an implicit finite-difference scheme. AIP ADVANCES, 3, 082131.

[16] Srivastava, V. K., Awasthi, M. K. And Singh, S. (2013). An implicit logarithm finite difference technique for two dimensional coupled viscous Burgers' equation. AIP Advances, 3, 122105.

[17] Wang, X.Y., Zhu, Z.S., Lu, Y. K. (1990). Solitary wave solutions of the generalized Burgers-Huxley equation, Journal of Physics A: Mathematical and General, 23: 271-274.

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 1, 7–13
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

Independent Transversal Domination Number of Corona and Join Operation in Path Graphs

Betül ATAY ATAKUL[a]

[a]*Dept. of Computer Education and Instructional Technology*
Ağri İbrahim Çeçen University
04100, Ağri, Turkey.

Abstract. A dominating set of a graph G which intersects every independent set of a maximum cardinality in G is called an independent transversal dominating set. The minimum cardinality of an independent transversal dominating set is called the independent transversal domination number of G and is denoted by $\gamma_{it}(G)$. In this paper we investigate the independent transversal domination number of the path graph P_n with the star graph $S_{1,m}$, the wheel graph $W_{1,m}$ and the complete graph K_n under neihgbourhood corona, edge corona and join operation providing $\beta(P_n) > \beta(G)$.

1. Introduction

In this paper, we consider simple finite undirected graphs without loops and multiple edges. Let $G = (V(G), E(G))$ be a graph. For a vertex x of G, $N(x)$ denotes the set of all neighbours of x in G. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them. The *diameter* of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$. The number of the neighbor vertices of the vertex v is called degree of v and denoted by $deg_G(v)$. The minimum and maximum degrees of a vertex of G are denoted by $\delta(G)$ and $\Delta(G)$. A vertex v is said to be pendant vertex if $deg_G(v) = 1$. A vertex u is called support if u is adjacent to a pendant vertex [7]. The eccentricity $e(u)$ of a vertex u in G is the distance from u to a vertex farthest from u. The minimum eccentricity of the vertices of the graph G is the *radius* of G denoted by $rad(G)$, while the *diameter* of G is the greatest eccentricity[4].
Let G be a graph and $S \subseteq V(G)$. We denote by $< S >$ the subgraph of G induced by S. A set S is said to be an *independent set* of G, if no pair of vertices of S are adjacent in G. The *independence number* of G, denoted by $\beta(G)$, is the cardinality of a maximum independent set of G. We denote by $\Omega(G)$ the set of all maximum independent sets of G. A vertex and an edge are said to *cover* each other if they are incident. A set of vertices which cover all the edges of a graph G is called a *vertex cover* for G, while a set of edges which covers all the vertices is an *edge cover*. The smallest number of vertices in any vertex cover for G is called its *vertex covering number* and is denoted by $\alpha(G)$ [7]. For any graph G of order n, $\alpha(G) + \beta(G) = n$.
A *dominating set* S in a graph G is a set of vertices of G such that every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a

Corresponding author: BAA mail address: batay@agri.edu.tr ORCID:0000-0003-1964-3287
Received: 30 November 2021; Accepted: 20 January 2022; Published: 30 April 2022
Keywords. Domination number, Independent transversal domination number, Independence number.
2010 *Mathematics Subject Classification.* 05C69, 05C40, 68M10, 68R10
Cited this article as: Atay Atakul B. Independent Transversal Domination Number of Corona and Join Operation in Path Graphs, Turkish Journal of Science. 2022, 7(1), 7-13.

dominating set of G [8, 9]. It is clear that for a graph G, $\gamma(G) \leq \beta(G)$, and If G has no isolated vertices, $\gamma(G) \leq \alpha(G)$.

Given a graph G and a collection of subsets of its vertices, a subset of $V(G)$ is called a *transversal* of G if it intersects each subset of the collection. If we think of the graph as modeling a communication network, many graph theoretical parameters have been used to describe the stability of communication networks including connectivity, toughness, integrity, binding nımber, domination, exponential domination, independent transversal domination etc. The independent transversal domination number is one of the measures of the graph vulnerability. A transversal of a collection of sets is a set of distinct representatives of the elements in the collection. It is possible to find transversals regarding several types of vertex sets in graphs such that the domination number, the chromatic number and the independence number of a graph. In [2], the concept "partition domination number" was defined as the largest integer k such that given any partition of the vertex set of the graph having at most k elements in every set of the partition, there is transversal of the partition being a dominating set. Some complexity results regarding the associated desicion problems and some bounds or exact values for some specific families of graphs were presented in [2]. A recent work in new style of transversal-type concepts has been presented in [6]: the independent transversal domination number[1].

A dominating set of G which intersects every independent set of maximum cardinality in G is called an *independent transversal dominating set*. The minimum cardinality of an independent transversal dominating set is called the *independent transversal domination number* of G and is denoted by $\gamma_{it}(G)$. An independent transversal dominating set of cardinality $\gamma_{it}(G)$ is called a $\gamma_{it}(G) - set$. Thus, if D is an ITD-set of G, then D is a dominating set of G and $\beta(G) > \beta(G - D)$. The notion of independent transversal domination was first introduced by Hamid [3, 6].

In this paper, firstly known results are given. Then, we investigate the independent transversal domination number for the neighbourhood corona, the edge corona of the path graph with the star graph $S_{1,m}$, the wheel graph $W_{1,m}$, the complete graph K_m and join operation of the path graph with some graphs G providing $\beta(P_n) > \beta(G)$. Lastly, the conclusion section is presented.

2. Known Results

Theorem 2.1. [6] *If G is a complete multipartite graph having r maximum independent sets, then*

$$\gamma_{it}(G) = \begin{cases} 2 & , r = 1 \\ r & , otherwise \end{cases}$$

Theorem 2.2. [6] *For complete graph with order n and complete bipartite graph with order $m + n$, $\gamma_{it}(K_n) = n$ and $\gamma_{it}(K_{m,n}) = 2$, respectively.*

Theorem 2.3. [6] *For any path P_n of order n, we have*

$$\gamma_{it}(P_n) = \begin{cases} 2 & , n = 2, 3 \\ 3 & , n = 6 \\ \lceil \frac{n}{3} \rceil & , otherwise \end{cases}$$

Theorem 2.4. [6] *For any cycle C_n of order n, we have*

$$\gamma_{it}(C_n) = \begin{cases} 3 & , n = 3, 5 \\ \lceil \frac{n}{3} \rceil & , otherwise \end{cases}$$

Theorem 2.5. [6] *If W_n is a wheel on n vertices, then*

$$\gamma_{it}(W_n) = \begin{cases} 2, & if\ n = 5 \\ 3, & if\ n \geq 7 \text{ and is odd or } n = 6 \\ 4, & otherwise \end{cases}$$

Theorem 2.6. [6] *If G is a disconnected graph with compenents $G_1, G_2, ..., G_r$, then $\gamma_{it}(G) = min_{1\leq i\leq r}\{\gamma_{it}(G_i) + \sum_{j=1,j\neq i}^{r} \gamma(G_j)\}$.*

Theorem 2.7. [6] *If G has an isolated vertex, then $\gamma_{it}(G) = \gamma(G)$.*

Theorem 2.8. [6] *For any graph G, we have $1 \leq \gamma_{it}(G) \leq n$. Further $\gamma_{it}(G) = n$ if and only if $G = K_n$.*

Theorem 2.9. [6] *Let G be a graph on n vertices. Then $\gamma_{it}(G) = n - 1$ if and only if $G = P_3$.*

Theorem 2.10. [6] *Let G be a non-complete connected graph with $\beta(G) \geq \frac{n}{2}$. Then $\gamma_{it}(G) \leq \frac{n}{2}$.*

Theorem 2.11. [6] *If G is bipartite, then $\gamma_{it}(G) \leq \frac{n}{2}$.*

Theorem 2.12. [6] *Let a and b be two positive integers with $b \geq 2a - 1$. Then there exists a graph G on b vertices such that $\gamma_{it}(G) = a$.*

Theorem 2.13. [6] *If G is a non-complete connected graph on n vertices, then $\gamma_{it}(G) \leq \lceil \frac{n}{2} \rceil$.*

Theorem 2.14. [6] *For any graph G, we have $\gamma(G) \leq \gamma_{it}(G) \leq \gamma(G) + \delta(G)$.*

Corollary 2.15. [6] *If T is a tree, then $\gamma_{it}(T)$ is either $\gamma(T)$ or $\gamma(T) + 1$.*

Theorem 2.16. [6] *If G is a graph with $diam(G) = 2$, then $\gamma_{it}(G) \leq \delta(G) + 1$.*

Theorem 2.17. [3] *If G is a connected graph and u is a vertex of minimum degree in G, then*

$$\gamma_{it}(G) \leq \begin{cases} \delta(G) + 1 & if\ ecc_G(u) \leq 2 \\ \frac{n(G)}{2} + 1, & if\ ecc_G(u) \geq 3 \end{cases}$$

and these bounds are tight.

Theorem 2.18. [3] *If G is a graph with $\beta(G) \geq \frac{n(G)}{2}$, then $\gamma_{it}(G) \leq \gamma(G) + 1$, and this bound is tight.*

3. Independent Transversal Domination Number for the Neighbourhood Corona of the Path Graph

Definition 3.1. [6] *A dominating set $S \subseteq V$ of a graph G is said to be an independent transversal dominating set if S intersects every maximum independent set of G. The minimum cardinality of an independent transversal dominating set of G is called the independent transversal domination number of G and is denoted by $\gamma_{it}(G)$. An independent transversal dominating set S of G with $|S| = \gamma_{it}(G)$ is called a $\gamma_{it} - set$.*

The following figure shows the independent transversal domination number of a graph G.

where, $\beta(G) = 4, \gamma(G) = 4$. The maximum independent set of the graph consists of four pendant vertices or two two support vertices on cycle having $deg(v) = 2$. The dominating set of the graph consists of four pendant vertices or four support vertices on cycle having $deg(v) = 2$. Let S be an independent transversal dominating set. If we pick the support vertices on cycle for S, then all vertices of the graph G are dominated. But the independence number of the graph doesn't decrease. $V - S$ contains at least one $\beta - set$. So, we must also add any pendant vertex to S. Hence, $\beta(G) > \beta(G - S)$ and $\gamma_{it}(G) = 5$.

Definition 3.2. [5]
The graph $G_1 * G$ which is obtained by neighbourhood corona operation of a connected graph G_1 and graph G is formed as follows: Every vertex u_i of graph G_1 correspond to a graph G and every vertex v_{ij} of G is adjacent to every neighbour vertex of the corresponding vertex u_i of G_1, where $i = \overline{1, |G_1|}$ and $j = \overline{1, |G|}$.

The neighbourhood corona of the graph $P_6 * P_2$ can be depicted as in the following figure:

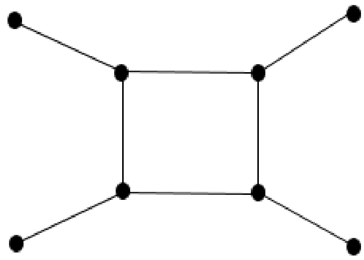


Figure 1: The graph G

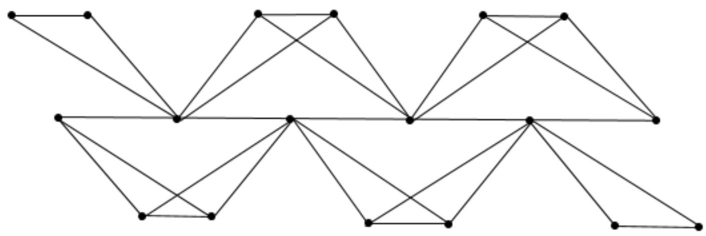


Figure 2: The graph $P_6 * P_2$

Theorem 3.3. *Let P_n and P_m be any path graphs with order n and m, respectively. Then,*

$$\gamma_{it}(P_n * P_m) = \begin{cases} \frac{n}{2}+1, & if\ n \equiv 0(mod\ 4)\ and\ m\ is\ odd, \\ \frac{n}{2}+2, & if\ n \equiv 0(mod\ 4)\ and\ m\ is\ even \\ \lfloor\frac{n}{2}\rfloor+2, & if\ n \equiv 1,2,3(mod\ 4)\ and\ m\ is\ odd, \\ \lfloor\frac{n}{2}\rfloor+3, & if\ n \equiv 1,2,3(mod\ 4)\ and\ m\ is\ even. \end{cases}$$

Proof. We denote the vertices of P_n with u_i, $i = \overline{1,n}$ and the corresponding vertices of P_m with v_j, $j = \overline{1,m}$. Let D be a $\gamma - set$ of the graph $P_n * P_m$. So, for $k \in \{0, 1, ..., \lfloor\frac{n}{4}\rfloor - 1\}$,
$D = \{u_{4k+2}, u_{4k+3}\}$ and $|D| = \frac{n}{2}, if\ n \equiv 0(\text{mod } 4)$;
$D = \{u_{4k+2}, u_{4k+3}, u_{n-1}\}$ and $|D| = \lfloor\frac{n}{2}\rfloor + 1, if\ n \equiv 1(\text{mod } 4)$;
$D = \{u_{4k+2}, u_{4k+3}, u_{n-1}, u_n\}$ and $|D| = \lfloor\frac{n}{2}\rfloor + 1, if\ n \equiv 2, 3(\text{mod } 4)$.
The vertices of the maximum independent set of $P_n * P_m$ consist of the maximum independent sets of every P_m. Then, $\beta(P_n * P_m) = n\lceil\frac{m}{2}\rceil$. Independence number of $< V(P_n * P_m) - D >$ is the same as the independence number of $V(P_n * P_m)$. Let S be the independent transversal dominating set of the graph $P_n * P_m$. $S = D \cup \{v_{11}\}$ if m is odd and $S = D \cup \{v_{11}, v_{12}\}$ if m is even. So, we have $\beta(V(P_n * P_m) - S) < \beta(P_n * P_m)$ and this means that $< V(P_n * P_m) - S >$ doesn't contain any $\beta - set$ of $P_n * P_m$. So,

$$\gamma_{it}(P_n * P_m) = \begin{cases} \frac{n}{2}+1, & if\ n \equiv 0(\text{mod } 4)\ and\ m\ is\ odd, \\ \frac{n}{2}+2, & if\ n \equiv 0(\text{mod } 4)\ and\ m\ is\ even \\ \lfloor\frac{n}{2}\rfloor+2, & if\ n \equiv 1,2,3(\text{mod } 4)\ and\ m\ is\ odd, \\ \lfloor\frac{n}{2}\rfloor+3, & if\ n \equiv 1,2,3(\text{mod } 4)\ and\ m\ is\ even. \end{cases}$$

The proof is completed. □

Theorem 3.4. *Let P_n be any path graph with order n and $S_{1,m}$ be a star graph with order $m + 1$. Then,*

$$\gamma_{it}(P_n * S_{1,m}) = \begin{cases} \frac{n}{2}+1, & if\ n \equiv 0(mod\ 4) \\ \lfloor\frac{n}{2}\rfloor+2, & if\ n \equiv 1,2,3(mod\ 4) \end{cases}$$

Proof. The proof is similar to the proof of Theorem 3.1. □

Theorem 3.5. *Let P_n be any path graph with order n and $W_{1,m}$ be a wheel graph with order $m+1$ for $m>3$ and $m \neq 9$. Then,*

$$\gamma_{it}(P_n * W_{1,m}) = \begin{cases} \frac{n}{2}+2, & if\ n \equiv 0(mod\ 4)\ and\ m\ is\ even, \\ \frac{n}{2}+3, & if\ n \equiv 0(mod\ 4)\ and\ m\ is\ odd, \\ \lfloor\frac{n}{2}\rfloor+3, & if\ n \equiv 1,2,3(mod\ 4)\ and\ m\ is\ even, \\ \lfloor\frac{n}{2}\rfloor+4, & if\ n \equiv 1,2,3(mod\ 4)\ and\ m\ is\ odd. \end{cases}$$

Proof. The proof is similar to the proof of Theorem 3.1. □

Theorem 3.6. *Let P_n be any path graph with order n and K_m be any complete graph with order m. Then,*

$$\gamma_{it}(P_n * K_m) = \begin{cases} \frac{n}{2}+m, & if\ n \equiv 0(mod\ 4), \\ \frac{n}{2}+1+m, & if\ n \equiv 2(mod\ 4), \\ \lceil\frac{n}{2}\rceil, & if\ n \equiv 1,3(mod\ 4), \end{cases}$$

Proof. Let D be a $\gamma-set$ of the graph $P_n * K_m$. $|D| = \frac{n}{2}$ if $n \equiv 0(\text{mod } 4)$, $|D| = \lfloor\frac{n}{2}\rfloor+1$ otherwise. $< V(P_n * K_m) - D >$ contains a maximum independent set so, $\gamma_{it}(P_n * K_m) > \gamma(P_n * K_m)$. We denote the vertices of P_n with u_i, $i = \overline{1,n}$ and the corresponding vertices of K_m with v_{ij}, $j = \overline{1,m}$. We have three cases:
Case 1. $n \equiv 0(\text{mod } 4)$
In this case $D = \{u_{4k+2}, u_{4k+3}\}$, $k = \overline{0, \lfloor\frac{n}{4}\rfloor - 1}$ is a dominating set and the maximum independent sets are $\beta_1 = \{u_1, v_{1j}, u_3, v_{3j}, u_5, v_{5j}, ..., u_{n-1}, v_{n-1j}\}$, $\beta_2 = \{u_2, v_{2j}, u_4, v_{4j}, u_6, v_{6j}, ..., u_n, v_{nj}\}$ and $\beta_3 = \{v_{1j}, v_{2j}, v_{3j}, ..., v_{nj}\}$, where each j is related to exactly one member of $\{1,2,...,m\}$. So, $\beta(P_n * K_m) = n$. Let S be an independent transversal dominating set of $P_n * K_m$. We must add $\frac{n}{2}$ vertices from the graph P_n and m vertices from any graph K_m to S so that $< V(P_n * K_m) - S >$ doesn't contain any $\beta - set$. So, we have $\gamma_{it}(P_n * K_m) = \frac{n}{2} + m$.
Case 2. $n \equiv 2(\text{mod } 4)$
In this case $D = \{u_{4k+2}, u_{4k+3}, u_{n-1,u_n}\}$, $k = \overline{0, \lfloor\frac{n}{4}\rfloor - 1}$ is a dominating set and the maximum independent sets are $\beta_1 = \{u_1, v_{1j}, u_3, v_{3j}, u_5, v_{5j}, ..., u_{n-1}, v_{n-1j}\}$, $\beta_2 = \{u_2, v_{2j}, u_4, v_{4j}, u_6, v_{6j}, ..., u_n, v_{nj}\}$ and $\beta_3 = \{v_{1j}, v_{2j}, v_{3j}, ..., v_{nj}\}$, where each j is related to exactly one member of $\{1,2,...,m\}$. So $\beta(P_n * K_m) = n$. For the independent transversal dominating set selected as $S = D \cup K_m$, $< V(P_n * K_m) - S >$ doesn't contain any $\beta - set$ and all vertices of the graph $P_n * K_m$ are dominated. So, we have $\gamma_{it}(P_n * K_m) = \frac{n}{2} + 1 + m$.
Case 3. $n \equiv 1,3(\text{mod } 4)$
In this case the maximum independent set of $P_n * K_m$ is $\{u_1, v_{1j}, u_3, v_{3j}, ..., u_n, v_{nj}\}$, where each j is related to exactly one member of $\{1,2,...,m\}$. The vertex set that occurs one vertex from every graph K_m isn't a maximum independent set. For $k = \overline{0, \lfloor\frac{n}{4}\rfloor - 1}$, $D = \{u_{4k+2}, u_{4k+3}, ..., u_{n-1}\}$ if $n \equiv 1(\text{mod } 4)$ and $D = \{u_{4k+2}, u_{4k+3}, ..., u_{n-1}, v_{nj}\}$ if $n \equiv 3(\text{mod } 4)$ is a dominating set. So, $< V(P_n * K_m) - D >$ doesn't contain any $\beta - set$ and $\gamma_{it}(P_n * K_m) = \gamma(P_n * K_m) = \lceil\frac{n}{2}\rceil$.
The proof is completed. □

4. Independent Transversal Domination Number for the Edge Corona of the Path Graph

Definition 4.1. [10]
The graph $G_1 \diamond G$ which is obtained by edge corona operation of a connected graph G_1 and graph G is formed as follows: Every edge e_i of graph G_1 correspond to a graph G and every vertex vij of G is adjacent to two end vertices of the corresponding edge e_i of G_1, $i = \overline{1, |E(G_1)|}$ and $j = \overline{1, |V(G)|}$.

The edge corona of the graph $P_6 \diamond P_2$ can be depicted as in the following figure:

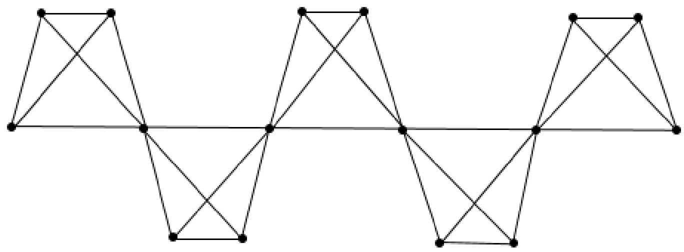


Figure 3: The graph $P_6 \diamond P_2$

Theorem 4.2. *Let P_n and P_m be any path graphs with order n and m, respectively. Then,*

$$\gamma_{it}(P_n \diamond P_m) = \begin{cases} \lfloor \frac{n}{2} \rfloor + 1, & if\ m \equiv 1(mod\ 2) \\ \lfloor \frac{n}{2} \rfloor + 2, & if\ m \equiv 0(mod\ 2) \end{cases}$$

Proof. The domination set of $(P_n \diamond P_m)$ is $D = \{u_{2k}\}, k = \overline{1, 2, ..., \lfloor \frac{n}{2} \rfloor}$. Also, the maximum independence number $\beta((P_n \diamond P_m) = (n-1)\beta(P_m)$. Let S be the independent transversal dominating set of $P_n \diamond P_m$. $< V(P_n \diamond P_m) - D >$ contains $\beta - set$. So, $\gamma_{it}((P_n \diamond P_m) > \gamma((P_n \diamond P_m)$. If we also add the vertex v_{11} in case $m \equiv 1(\text{mod } 2)$ and the vertices v_{11}, v_{12} in case $m \equiv 0(\text{mod } 2)$ to the S with $V(D)$, then $< V(P_n \diamond P_m) - S >$ doesn't contain any $\beta - set$. Hence, we have

$$\gamma_{it}(P_n \diamond P_m) = \begin{cases} \lfloor \frac{n}{2} \rfloor + 1, & if\ m \equiv 1(\text{mod } 2) \\ \lfloor \frac{n}{2} \rfloor + 2, & if\ m \equiv 0(\text{mod } 2) \end{cases}$$

The proof is completed. □

Theorem 4.3. *Let P_n be any path graph with order n and $S_{1,m}$ be a star graph with order $m + 1$. Then,*

$$\gamma_{it}(P_n \diamond S_{1,m}) = \lfloor \frac{n}{2} \rfloor + 1.$$

Proof. The proof is similar to the proof of Theorem 4.1. □

Theorem 4.4. *Let P_n be any path graph with order n and $W_{1,m}$ be a wheel graph with order $m + 1$. Then,*

$$\gamma_{it}(P_n \diamond W_{1,m}) = \begin{cases} \lfloor \frac{n}{2} \rfloor + 2, & if\ m\ is\ even, \\ \lfloor \frac{n}{2} \rfloor + 3, & if\ m\ is\ odd, \end{cases}$$

Proof. The proof is similar to the proof of Theorem 4.1. □

Theorem 4.5. *Let P_n be any path graph with order n and K_m be any complete graph with order m. Then,*

$$\gamma_{it}(P_n \diamond K_m) = \lfloor \frac{n}{2} \rfloor + m.$$

Proof. The proof is similar to the proof of Theorem 4.1. □

5. Independent Transversal Domination Number for the Join Operation of the Path Graph

Definition 5.1. [7] *Graphs G_1 and G_2 have disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Their union $G = G_1 \cup G_2$ has, as expected, $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Their join is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining V_1 with V_2.*

The join operation of the graph $P_2 + P_3$ can be depicted as in the following figure:

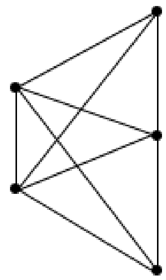


Figure 4: The graph $P_2 + P_3$

Theorem 5.2. *Let G be any graph with order m and P_n be any path graph with order n. If $\beta(P_n) > \beta(G)$ then,*

$$\gamma_{it}(G + P_n) = \begin{cases} 2, & if\ n \equiv 1(mod\ 2), \\ 3, & if\ n \equiv 0(mod\ 2). \end{cases}$$

Proof. We label the vertices as $u_i \in G$ and $v_j \in P_n$ of the graph $G + P_n$, for $i = \overline{1,m}$ and for $j = \overline{1,n}$. We can dominate all vertices of P_n with any vertex u_i and all vertices of G with any vertex v_j since $d(u_i, v_j) = 1$ $\forall u_i, v_j$. So, $\gamma(G + P_n) = 2$. Let S be any independent transversal domination set. If $n \equiv 1(\text{mod } 2)$ then, $\beta(P_n) = \beta(P_{n-1})$. In this case $S = \{u_i, v_1\}$ doesn't contain any $\beta - set$. If $n \equiv 0(\text{mod } 2)$ then, $\beta(P_n) = \beta(P_{n-1}) + 1$. In this case $S = \{u_i, v_1, v_2\}$ doesn't contain any $\beta - set$, where each i is related to exactly one member of $\{1, 2, ..., m\}$ So, we have

$$\gamma_{it}(G + P_n) = \begin{cases} 2, & if\ n \equiv 1(\text{mod } 2), \\ 3, & if\ n \equiv 0(\text{mod } 2). \end{cases}$$

The proof is completed. □

References

[1] Ahangar H.A., Samodivkin V. and Yero I.G., Independent Transversal Dominating Sets in Graphs. Complexity and Structural Properties. Filomat. 30(2), 2016, 293–303.
[2] Alon N., Fellows M.R. and Hare D.R., Vertex Transversals that Dominate. Journal of Graph Theory. 21(1), 1996, 21–31.
[3] Brause C., Henning M., Ozeki K., Schiermeyer I. and Vumar E., On Upper Bounds For The Independent Transversal Domination Number. Discrete Applied Mathematics. Vol. 236, 2018, 66–72.
[4] Chartrand G. and Lesniak L., Graphs and Digraphs, Fourth Edition, 2005.
[5] Gopalapillai I., The spectrum of Neigbourhood Corona of Graphs. Kragujevac Journal of Mathematics. 35(3), 2011, 493–500.
[6] Hamid I.S., Independent Transversal Domination ın Graphs. Discussiones Mathematicae Graph Theory. Vol.32, 2012, 5–7.
[7] Harary F., Graph Theory, Addition-Wesley Publishing Co., Reading, MA/Menlo Park, CA/London, 1969.
[8] Haynes T. W., Hedeniemi S. T. and Slater P. J., Fundamentals of Domination in Graphs, Marcel Dekker, Inc, New York, 1998.
[9] Henning M.A., Domination in Graphs: a survey. Cong. Number., In G. Chartrand and M. Jacobson, editors, Surveys in Graph Theory. Vol. 116, 1996, 139–172.
[10] Hou Y. and Shiu W., The spectrum of edge corona of graphs. Electronic Journal of Linear Algebra. Vol. 20, 2010, 586–594.

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 1, 14-20
ISSN: 2587–0971 http://dergipark.gov.tr/tjos

Invariant and Lacunary Invariant Statistical Convergence of Order η for Double Set Sequences

Uğur Ulusu[a], Erdinç Dündar[b]

[a]*Sivas Cumhuriyet University, 58140 Sivas, Turkey*
[b]*Afyon Kocatepe University, 03200 Afyonkarahisar, Turkey*

Abstract. In this study, for double set sequences, we introduced the notions of invariant and lacunary invariant statistical convergence of order η $(0 < \eta \leq 1)$ in the Wijsman sense. Also, we investigated the inclusion relations between them.

1. Introduction

Long after the notion of convergence for double sequences was introduced by Pringsheim [12], this notion was respectively extended to the notions of statistical convergence, lacunary statistical convergence and double σ-convergent lacunary statistical sequence by Mursaleen and Edely [5], Patterson and Savaş [11] and Savaş and Patterson [13]. Recently, for double sequences, on two new convergence concepts called double almost statistical and double almost lacunary statistical convergence of order α were studied by Savaş [14, 15].

Over the years, on the various convergence notions for set sequences have been studied by many authors. One of them, discussed in this study, is the notion of convergence in the Wijsman sense [1, 2, 6]. Using the notions of statistical convergence, double lacunary sequence and invariant mean, this notion was extended to new convergence notions for double set sequences by some authors [7–9]. In [8], Nuray and Ulusu studied on the notions of invariant and lacunary invariant statistical convergence in the Wijsman sense for double set sequences.

In this paper, using order η, we studied on new convergence notions in the Wijsman sense for double set sequences.

More information on the notions of convergence for real or set sequences can be found in [3, 4, 6, 10, 16–20].

2. Definitions and Notations

Firstly, let us remind the basic notions that need for a better understanding of our study (see, [7–9, 11]).

Corresponding author: UU ugurulusu@cumhuriyet.edu.tr ORCID:0000-0001-7658-6114, ED ORCID:0000-0002-0545-7486
Received: 13 December 2021; Accepted: 27 January 2022; Published: 30 April 2022
Keywords. Double lacunary sequence, invariant statistical convergence, order η, double set sequences, convergence in the Wijsman sense
2010 *Mathematics Subject Classification*. 40G15, 40D25
Cited this article as: Ulusu U. and Dündar E. Invariant and Lacunary Invariant Statistical Convergence of Order η for Double Set Sequences, Turkish Journal of Science. 2022, 7(1), 14-20.

For a metric space (Y,d), $\mu(y,C)$ denote the distance from y to C where

$$\mu(y,C) := \mu_y(C) = \inf_{c\in C} d(y,c)$$

for any $y \in Y$ and any non-empty set $C \subseteq Y$.

For a non-empty set Y, let a function $g : \mathbb{N} \to P_Y$ (the power set of Y) is defined by $g(m) = C_m \in P_Y$ for each $m \in \mathbb{N}$. Then, the sequence $\{C_m\} = \{C_1, C_2, \ldots\}$, which is the codomain elements of g, is called set sequences.

Throughout this study, (Y,d) will be considered as a metric space and C, C_{mn} will be considered as any non empty closed subsets of Y.

A double set sequence $\{C_{mn}\}$ is called convergent to the set C in the Wijsman sense if each $y \in Y$,

$$\lim_{m,n\to\infty} \mu_y(C_{mn}) = \mu_y(C).$$

A double set sequence $\{C_{mn}\}$ is called statistically convergent to the set C in the Wijsman sense if every $\xi > 0$ and each $y \in Y$,

$$\lim_{p,q\to\infty} \frac{1}{pq}\left|\left\{(m,n) : m \le p, n \le q, \left|\mu_y(C_{mn}) - \mu_y(C)\right| \ge \xi\right\}\right| = 0.$$

A double sequence $\theta_2 = \{(j_u, k_v)\}$ is called a double lacunary sequence if there exist increasing sequences (j_u) and (k_v) of the integers such that

$$j_0 = 0,\ h_u = j_u - j_{u-1} \to \infty \ \text{ and }\ k_0 = 0,\ \bar{h}_v = k_v - k_{v-1} \to \infty \ \text{ as }\ u, v \to \infty.$$

In general, the following notations is used for any double lacunary sequence:

$$\ell_{uv} = j_u k_v,\ \ h_{uv} = h_u \bar{h}_v,\ \ I_{uv} = \left\{(m,n) : j_{u-1} < m \le j_u \ \text{ and }\ k_{v-1} < n \le k_v\right\},$$

$$q_u = \frac{j_u}{j_{u-1}} \ \text{ and }\ q_v = \frac{k_v}{k_{v-1}}.$$

Throughout this study, $\theta_2 = \{(j_u, k_v)\}$ will be considered as a double lacunary sequence.

A double set sequence $\{C_{mn}\}$ is called lacunary statistically convergent to the set C in the Wijsman sense if every $\xi > 0$ and each $y \in Y$,

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}}\left|\left\{(m,n) \in I_{uv} : \left|\mu_y(C_{mn}) - \mu_y(C)\right| \ge \xi\right\}\right| = 0.$$

Let σ be a mapping such that $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$ (the set of positive integers). A continuous linear functional ψ on ℓ_∞ is called an invariant mean (or a σ-mean) if it satisfies the following conditions:

1. $\psi(x_s) \ge 0$, when the sequence (x_s) has $x_s \ge 0$ for all s,
2. $\psi(e) = 1$, where $e = (1, 1, 1, \ldots)$ and
3. $\psi(x_{\sigma(s)}) = \psi(x_s)$ for all $(x_s) \in \ell_\infty$.

The mappings σ are assumed to be one to one and such that $\sigma^m(s) \neq s$ for all $m, s \in \mathbb{N}^+$, where $\sigma^m(s)$ denotes the m th iterate of the mapping σ at s. Thus ψ extends the limit functional on c, in the sense that $\psi(x_s) = \lim x_s$ for all $(x_s) \in c$.

A double set sequence $\{C_{mn}\}$ is called invariant statistically convergent to the set C in the Wijsman sense if every $\xi > 0$ and each $y \in Y$,

$$\lim_{p,q\to\infty} \frac{1}{pq}\left|\left\{(m,n) : m \le p, n \le q, \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \ge \xi\right\}\right| = 0$$

uniformly in s, t.

The set of all invariant statistically convergent double set sequences in the Wijsman sense is denoted by $\left\{W_2 S_\sigma\right\}$.

A double set sequence $\{C_{mn}\}$ is called lacunary invariant statistically convergent to the set C in the Wijsman sense if every $\xi > 0$ and each $y \in Y$,

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}}\left|\left\{(m,n) \in I_{uv} : \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \ge \xi\right\}\right| = 0$$

uniformly in s, t.

3. Main Results

In this section, for double set sequences, we introduced the notions of invariant and lacunary invariant statistical convergence of order η $(0 < \eta \le 1)$ in the Wijsman sense. Also, we investigated the inclusion relations between them.

Definition 3.1. *The double set sequence $\{C_{mn}\}$ is invariant statistically convergent of order η to the set C in the Wijsman sense if every $\xi > 0$ and each $y \in Y$,*

$$\lim_{p,q\to\infty} \frac{1}{(pq)^\eta}\left|\left\{(m,n) : m \le p, n \le q, \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \ge \xi\right\}\right| = 0$$

uniformly in s, t where $0 < \eta \le 1$ and we denote this in $C_{mn} \xrightarrow{W_2 S_\sigma^\eta} C$ format.

Example 3.2. *Let $Y = \mathbb{R}^2$ and a double set sequence $\{C_{mn}\}$ be defined as following:*

$$C_{mn} := \begin{cases} \left\{(a,b) \in \mathbb{R}^2 : a^2 + (b+1)^2 = \frac{1}{mn}\right\} & ; \text{ if } m \text{ and } n \text{ are square integers} \\ \{(-1,0)\} & ; \text{ otherwise.} \end{cases}$$

In this case, the double set sequence $\{C_{mn}\}$ is invariant statistically convergent of order η $(0 < \eta \le 1)$ to the set $C = \{(-1,0)\}$ in the Wijsman sense.

Remark 3.3. *For $\eta = 1$, the notion of invariant statistical convergence of order η in the Wijsman sense coincides with the notion of invariant statistical convergence in the Wijsman sense for double set sequences in [8].*

Definition 3.4. *The double set sequence $\{C_{mn}\}$ is lacunary invariant statistically convergent of order η to the set C in the Wijsman sense if every $\xi > 0$ and each $y \in Y$,*

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}^\eta}\left|\left\{(m,n) \in I_{uv} : \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \ge \xi\right\}\right| = 0$$

uniformly in s, t where $0 < \eta \le 1$ and we denote this in $C_{mn} \xrightarrow{W_2 S_{\sigma\theta}^\eta} C$ format.

The set of all lacunary invariant statistically convergent double set sequences of order η in the Wijsman sense is denoted by $\left\{W_2 S_{\sigma\theta}^\eta\right\}$.

Example 3.5. *Let $Y = \mathbb{R}^2$ and a double set sequence $\{C_{mn}\}$ be defined as following:*

$$C_{mn} := \begin{cases} \left\{(a,b) \in \mathbb{R}^2 : (a+m)^2 + (b-n)^2 = 1\right\} & ; \ \textit{if } (m,n) \in I_{uv}, \ m \textit{ and } n \textit{ are square integers} \\ \{(1,1)\} & ; \ \textit{otherwise.} \end{cases}$$

In this case, the double set sequence $\{C_{mn}\}$ is lacunary invariant statistically convergent of order η $(0 < \eta \le 1)$ to the set $C = \{(1,1)\}$ in the Wijsman sense.

Remark 3.6. *For $\eta = 1$, the notion of lacunary invariant statistical convergence of order η in the Wijsman sense coincides with the notion of lacunary invariant statistical convergence in the Wijsman sense for double set sequences in [8].*

Theorem 3.7. *If*

$$\liminf_u q_u^\eta > 1 \ \textit{ and } \ \liminf_v q_v^\eta > 1$$

where $0 < \eta \le 1$, then

$$C_{mn} \overset{W_2 S_\sigma^\eta}{\longrightarrow} C \Rightarrow C_{mn} \overset{W_2 S_{\sigma\theta}^\eta}{\longrightarrow} C.$$

Proof. Let $0 < \eta \le 1$ and suppose that $\liminf_u q_u^\eta > 1$ and $\liminf_v q_v^\eta > 1$. Then, there exist $\alpha, \beta > 0$ such that $q_u^\eta \ge 1 + \alpha$ and $q_v^\eta \ge 1 + \beta$ for all u, v, which implies that

$$\frac{h_{uv}}{\ell_{uv}} \ge \frac{\alpha\beta}{(1+\alpha)(1+\beta)} \Rightarrow \frac{h_{uv}^\eta}{\ell_{uv}^\eta} \ge \frac{\alpha^\eta \beta^\eta}{(1+\alpha)^\eta (1+\beta)^\eta}.$$

For every $\xi > 0$ and each $y \in Y$, we have

$$\begin{aligned} &\frac{1}{\ell_{uv}^\eta}\left|\left\{(m,n) : m \le j_u, n \le k_v, \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \ge \xi\right\}\right| \\ &\ge \frac{1}{\ell_{uv}^\eta}\left|\left\{(m,n) \in I_{uv} : \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \ge \xi\right\}\right| \\ &= \frac{h_{uv}^\eta}{\ell_{uv}^\eta}\frac{1}{h_{uv}^\eta}\left|\left\{(m,n) \in I_{uv} : \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \ge \xi\right\}\right| \\ &\ge \frac{\alpha^\eta \beta^\eta}{(1+\alpha)^\eta (1+\beta)^\eta}\frac{1}{h_{uv}^\eta}\left|\left\{(m,n) \in I_{uv} : \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \ge \xi\right\}\right| \end{aligned}$$

for all s, t. If $C_{mn} \overset{W_2 S_\sigma^\eta}{\longrightarrow} C$, then for each $y \in Y$ the term on the left side of the above inequality convergent to 0 and this implies that

$$\frac{1}{h_{uv}^\eta}\left|\left\{(m,n) \in I_{uv} : \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \ge \xi\right\}\right| \to 0$$

uniformly in s, t. Thus, we get $C_{mn} \overset{W_2 S_{\sigma\theta}^\eta}{\longrightarrow} C$. □

Theorem 3.8. *If*

$$\limsup_u q_u < \infty \ \textit{ and } \ \limsup_v q_v < \infty,$$

then

$$C_{mn} \overset{W_2 S_{\sigma\theta}^\eta}{\longrightarrow} C \Rightarrow C_{mn} \overset{W_2 S_\sigma^\eta}{\longrightarrow} C$$

where $0 < \eta \le 1$.

Proof. Let $\limsup_u q_u < \infty$ and $\limsup_v q_v < \infty$. Then, there exist $M, N > 0$ such that $q_u < M$ and $q_v < N$ for all u, v. Also, we suppose that $C_{mn} \overset{W_2 S^{\eta}_{\sigma\theta}}{\longrightarrow} C$ (where $0 < \eta \leq 1$) and $\xi > 0$, and let

$$\kappa_{uv} := \left|\left\{(m,n) \in I_{uv} : \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \xi\right\}\right|.$$

Then, there exist $u_0, v_0 \in \mathbb{N}$ such that for every $\xi > 0$, each $y \in Y$ and all $u \geq u_0, v \geq v_0$

$$\frac{\kappa_{uv}}{h^{\eta}_{uv}} < \xi$$

for all s, t. Now, let

$$\gamma := \max\{\kappa_{uv} : 1 \leq u \leq u_0,\ 1 \leq v \leq v_0\},$$

and let p and q be any integers satisfying $j_{u-1} < p \leq j_u$ and $k_{v-1} < q \leq k_v$. Then, for each $y \in Y$ we have

$$\begin{aligned}
&\frac{1}{(pq)^{\eta}}\left|\left\{(m,n) : m \leq p, n \leq q, \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \xi\right\}\right| \\
&\leq \frac{1}{\ell^{\eta}_{(u-1)(v-1)}}\left|\left\{(m,n) : m \leq j_u, n \leq k_v, \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \xi\right\}\right| \\
&= \frac{1}{\ell^{\eta}_{(u-1)(v-1)}}\left\{\kappa_{11} + \kappa_{12} + \kappa_{21} + \kappa_{22} + \cdots + \kappa_{u_0 v_0} + \cdots + \kappa_{uv}\right\} \\
&\leq \frac{u_0 v_0}{\ell^{\eta}_{(u-1)(v-1)}}\left(\max_{\substack{1 \leq m \leq u_0 \\ 1 \leq n \leq v_0}}\{\kappa_{mn}\}\right) \\
&\quad + \frac{1}{\ell^{\eta}_{(u-1)(v-1)}}\left\{h^{\eta}_{u_0(v_0+1)}\frac{\kappa_{u_0(v_0+1)}}{h^{\eta}_{u_0(v_0+1)}} + h^{\eta}_{(u_0+1)v_0}\frac{\kappa_{(u_0+1)v_0}}{h^{\eta}_{(u_0+1)v_0}}\right. \\
&\qquad\qquad \left. + h^{\eta}_{(u_0+1)(v_0+1)}\frac{\kappa_{(u_0+1)(v_0+1)}}{h^{\eta}_{(u_0+1)(v_0+1)}} + \cdots + h^{\eta}_{uv}\frac{\kappa_{uv}}{h^{\eta}_{uv}}\right\} \\
&\leq \frac{u_0 v_0 \gamma}{\ell^{\eta}_{(u-1)(v-1)}} + \frac{1}{\ell^{\eta}_{(u-1)(v-1)}}\left(\sup_{\substack{u > u_0 \\ v > v_0}}\frac{\kappa_{uv}}{h^{\eta}_{uv}}\right)\left(\sum_{m,n \geq u_0, v_0}^{u,v} h^{\eta}_{mn}\right) \\
&\leq \frac{u_0 v_0 \gamma}{\ell^{\eta}_{(u-1)(v-1)}} + \frac{1}{\ell_{(u-1)(v-1)}}\left(\sup_{\substack{u > u_0 \\ v > v_0}}\frac{\kappa_{uv}}{h^{\eta}_{uv}}\right)\left(\sum_{m,n \geq u_0, v_0}^{u,v} h_{mn}\right) \\
&\leq \frac{u_0 v_0 \gamma}{\ell^{\eta}_{(u-1)(v-1)}} + \xi\frac{(j_u - j_{u_0})(k_v - k_{v_0})}{\ell_{(u-1)(v-1)}} \\
&\leq \frac{u_0 v_0 \gamma}{\ell^{\eta}_{(u-1)(v-1)}} + \xi\, q_u\, q_v \\
&\leq \frac{u_0 v_0 \gamma}{\ell^{\eta}_{(u-1)(v-1)}} + \xi\, M N
\end{aligned}$$

for all s,t. Since $j_{u-1}, k_{v-1} \to \infty$ as $p,q \to \infty$, it follows that for each $y \in Y$

$$\frac{1}{(pq)^{\eta}}\left|\left\{(m,n):\ m \leq p, n \leq q, \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \xi\right\}\right| \to 0$$

uniformly in s,t. Thus, we get $C_{mn} \overset{W_2S_{\sigma}^{\eta}}{\longrightarrow} C$. $\square$

Theorem 3.9. *If*

$$1 < \liminf_u q_u^{\eta} \leq \limsup_u q_u < \infty \ \ and \ \ 1 < \liminf_v q_v^{\eta} \leq \limsup_v q_v < \infty$$

where $0 < \eta \leq 1$, *then*

$$C_{mn} \overset{W_2S_{\sigma\theta}^{\eta}}{\longrightarrow} C \Leftrightarrow C_{mn} \overset{W_2S_{\sigma}^{\eta}}{\longrightarrow} C.$$

Proof. This can be obtained from Theorem 3.7 and Theorem 3.8, immediately. $\square$

Theorem 3.10. *If*

$$\liminf_{u,v\to\infty} \frac{h_{uv}^{\eta}}{\ell_{uv}} > 0$$

where $0 < \eta \leq 1$, *then*

$$\left\{W_2S_{\sigma}\right\} \subseteq \left\{W_2S_{\sigma\theta}^{\eta}\right\}.$$

Proof. For every $\xi > 0$ and each $y \in Y$, it is obvious that

$$\left\{(m,n): m \leq j_u, n \leq k_v, \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \xi\right\} \supset \left\{(m,n) \in I_{uv}: \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \xi\right\}.$$

Thus, we have

$$\begin{aligned}
&\frac{1}{\ell_{uv}}\left|\left\{(m,n):\ m \leq j_u, n \leq k_v, \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \xi\right\}\right| \\
&\qquad \geq \frac{1}{\ell_{uv}}\left|\left\{(m,n) \in I_{uv}: \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \xi\right\}\right| \\
&\qquad = \frac{h_{uv}^{\eta}}{\ell_{uv}}\frac{1}{h_{uv}^{\eta}}\left|\left\{(m,n) \in I_{uv}: \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \xi\right\}\right|
\end{aligned}$$

for all s,t. If $C_{mn} \overset{W_2S_{\sigma}}{\longrightarrow} C$, then for each $y \in Y$ the term on the left side of the above inequality convergent to 0 and this implies that

$$\frac{1}{h_{uv}^{\eta}}\left|\left\{(m,n) \in I_{uv}: \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \xi\right\}\right| \to 0$$

uniformly in s,t. Thus, we get $C_{mn} \overset{W_2S_{\sigma\theta}^{\eta}}{\longrightarrow} C$. Consequently,

$$\{W_2S_{\sigma}\} \subseteq \{W_2S_{\sigma\theta}^{\eta}\}.$$

$\square$

References

[1] Baronti, M. and Papini, P. Convergence of sequences of sets. In: Methods of Functional Analysis in Approximation Theory (pp.133–155). Birkhäuser, Basel, (1986).
[2] Beer, G. Wijsman convergence: A survey. *Set-Valued Anal.*, **2**(1) (1994), 77–94.
[3] Çolak, R. Statistical convergence of order α. In: Modern Methods in Analysis and Its Applications (pp.121–129). Anamaya Publishers, New Delhi, (2010).
[4] Gülle, E. and Ulusu, U. Double Wijsman lacunary statistical convergence of order α. *J. Appl. Math. Inform.*, **39**(3-4) (2021), 303–319.
[5] Mursaleen, M. and Edely, O.H.H. Statistical convergence of double sequences. *J. Math. Anal. Appl.*, **288**(1) (2003), 223–231.
[6] Nuray, F. and Rhoades, B.E. Statistical convergence of sequences of sets. *Fasc. Math.*, **49** (2012), 87–99.
[7] Nuray, F., Ulusu, U. and Dündar, E. Lacunary statistical convergence of double sequences of sets. *Soft Comput.*, **20**(7) (2016), 2883–2888.
[8] Nuray, F. and Ulusu, U. Lacunary invariant statistical convergence of double sequences of sets. *Creat. Math. Inform.*, **28**(2) (2019), 143–150.
[9] Nuray, F., Dündar, E. and Ulusu, U. Wijsman statistical convergence of double sequences of sets. *Iran. J. Math. Sci. Inform.*, **16**(1) (2021), 55–64.
[10] Pancaroğlu, N. and Nuray, F. On invariant statistically convergence and lacunary invariant statistical convergence of sequences of sets. *Progress Appl. Math.*, **5**(2) (2013), 23–29.
[11] Patterson, R.F. and Savaş, E. Lacunary statistical convergence of double sequences. *Math. Commun.*, **10**(1) (2005), 55–61.
[12] Pringsheim, A. Zur theorie der zweifach unendlichen Zahlenfolgen. *Math. Ann.*, **53**(3) (1900), 289–321.
[13] Savaş, E. and Patterson, R.F. Double σ-convergence lacunary statistical sequences. *J. Comput. Anal. Appl.*, **11**(4) (2009), 610–615.
[14] Savaş, E. Double almost statistical convergence of order α. *Adv. Difference Equ.*, **2013**(62) (2013), 9 pages.
[15] Savaş, E. Double almost lacunary statistical convergence of order α. *Adv. Difference Equ.*, **2013**(254) (2013), 10 pages.
[16] Savaş, E. On $\mathcal{I}$-lacunary statistical convergence of order α for sequences of sets. *Filomat*, **29**(6) (2015), 1223–1229.
[17] Şengül, H. and Et, M. On lacunary statistical convergence of order α. *Acta Math. Sci. Ser. B*, **34**(2) (2014), 473–482.
[18] Şengül, H. and Et, M. On $\mathcal{I}$-lacunary statistical convergence of order α of sequences of sets. *Filomat*, **31**(8) (2017), 2403–2412.
[19] Ulusu, U. and Nuray, F. Lacunary statistical convergence of sequences of sets. *Progress Appl. Math.*, **4**(2) (2012), 99–109.
[20] Ulusu, U. and Gülle, E. Some statistical convergence types of order α for double set sequences. *Facta Univ. Ser. Math. Inform.*, **35**(3) (2020), 595–603.

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 1, 21-30
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

Lightlike Hypersurfaces of Poly-Norden Semi-Riemannian Manifolds

Erol KILIÇ[a], Tuba ACET[b], Selcen YÜKSEL PERKTAŞ[c]

[a]*Faculty of Arts and Sciences, Department of Mathematics, İnönü University, Malatya, TURKEY*
[b]*Faculty of Arts and Sciences, Department of Mathematics, İnönü University, Malatya, TURKEY*
[c]*Faculty of Arts and Sciences, Department of Mathematics, Adıyaman University, Adıyaman, TURKEY*

Abstract. In this article, we initiate the study of lightlike hypersurfaces in a poly-Norden semi-Riemannian manifold. We introduce invariant and screen semi-invariant lightlike hypersurfaces of a poly-Norden semi-Riemannian manifold. Also, we give some examples of such hypersurfaces.

1. Introduction

In differential geometry, submanifolds equipped with different geometric structure have been studied widely. A submanifold of a semi-Riemann manifold is known a lightlike submanifold if the induced metric is degenerate. The general theory of lightlike submanifold has been examined in [1] (see also [4]). On this subject, some applications of the theory mathematical physics is inspired, especially electromagnetisms [1], black hole theory [4] and general relativity [5]. Many studies on lightlike submanifolds have been reported by many geometers (see [2], [3], [6], [7], [8]).

The golden proportion and the golden rectangle have been found in the harmonious proportion of temples, fractals, paintings etc. Golden structure was revealed by the golden proportion which was charecterized by J. Kepler. The number φ, which is the real positive root of

$$x^2 - x - 1 = 0,$$

(hence $\varphi = \frac{1+\sqrt{5}}{2}$) is the golden proportion. In [9], inspired by golden ratio, golden Riemannian manifolds were introduced. Then many geometers have studied golden (semi) Riemannian manifolds on different manifolds ([10], [11], [12], [13]).

As a generalization of the golden mean, metallic mean family was studied by V. W. de Spinadel [14]. The positive solution of the equation

$$x^2 - px - q = 0,$$

is called member of the metallic means family, where p and q are fixed two positive integers. These number denoted by;

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2},$$

Corresponding author: TA mail address: tubaact@gmail.com ORCID:0000-0002-5096-3388, EK ORCID:0000-0001-7536-0404 SYP ORCID:0000-0002-8848-0621

Received: 25 April 2021; Accepted: 25 February 2022; Published: 30 April 2022
Keywords. Almost poly-Norden structure, poly-Norden manifold, Lightlike hypersurface
2010 *Mathematics Subject Classification*. 53C15; 53C40
Cited this article as: Kılıç E, Acet T, Yüksel Perktaş S. Lightlike Hypersurfaces of Poly-Norden Semi-Riemannian Manifolds, Turkish Journal of Science. 2022, 7(1), 21-30.

are also known $(p,q)-$metallic numbers. Recently many paper about metallic mean have been published ([15], [16], [17], [18]).

On the other hand in [19], the authors has defined Bronze mean which is different from Bronze mean given in [20]. Also note that there is no inclusion relation between the Bronze mean defined in [19] and metallic mean.

In [21], B. Şahin introduce as a new type of manifold which is called almost poly-Norden manifolds and study the geometry of such manifolds. Recently S. Yüksel Perktaş defined and studied submanifolds of almost poly-Norden Riemannian manifolds in [22].

In this article, by inspring from [21] and [15], we study lightlike hypersurfaces of almost poly-Norden manifolds.

2. Preliminaries

The bronze mean [19] which is the positive solution of the equation $x^2 - mx + 1 = 0$, is defined by

$$B_m = \frac{m + \sqrt{m^2-4}}{2}. \tag{1}$$

The Bronze Fibonacci numbers $(f_{m,n})$ (resp., the Bronze Lucas numbers $(l_{m,n})$) are the family of sequences defined by recurrence

$$f_{m,n+2} = mf_{m,n+1} - f_{m,n}, \quad (\text{resp., } l_{m,n+2} = ml_{m,n+1} - l_{m,n}),$$

where $f_{m,0} = 0$ and $f_{m,1} = 1$ (resp., $l_{m,0} = 2$ and $l_{m,1} = m$). The Bronze Fibonacci numbers and Bronze Lucas numbers are related by

$$B_m^n = \frac{l_{m,n} + f_{m,n}\sqrt{m^2-4}}{2}.$$

Also note that the recurrence relation $B_m^{n+2} = mB_m^{n+1} - B_m^n$ is satisfied and the covergents of B_m^a are $\frac{f_{m,a(n+1)}}{f_{m,an}}$ [19].

By being inspired of the Bronze mean (1) defined by S. Kalia [19], a new structure on a differentiable manifold which is called a poly-Norden structure was introduced by B. Şahin [21].

Definition 2.1. *[21] On a manifold $\breve{M}$, a poly-Norden structure is defined by a $(1,1)$-tensor field Φ which satisfies*

$$\Phi^2 = m\Phi - I, \tag{2}$$

where I is the identity operator on $\breve{M}$. So, $(\breve{M},\Phi)$ is called an almost poly-Norden manifold.

Example 2.2. *[21] Let Φ be a map defined by*

$$\begin{array}{rcl} \Phi & : & \mathbb{R}^4 \to \mathbb{R}^4 \\ (u_1,u_2,u_3,u_4) & \to & (B_m u_1, B_m u_2, \bar{B}_m u_3, \bar{B}_m u_4), \end{array}$$

where $B_m = \frac{m+\sqrt{m^2-4}}{2}$, $\bar{B} = m - B_m$ and (u_1,u_2,u_3,u_4) is the standard coordinate system on $\mathbb{R}^4$. One can easily see that Φ satisfies (2). Thus $(\mathbb{R}^4,\Phi)$ is a poly-Norden manifold.

A semi-Riemannian metric $\breve{g}$ is called Φ-compatible, if it satisfies

$$\breve{g}(\Phi X, \Phi Y) = m\breve{g}(\Phi X, Y) - \breve{g}(X,Y), \tag{3}$$

which yields

$$\breve{g}(\Phi X, Y) = \breve{g}(X, \Phi Y). \tag{4}$$

Definition 2.3. *[21] Let $(\breve{M},\breve{g})$ be a semi-Riemannian manifold endowed with a poly-Norden structure Φ. If the semi-Riemannian metric $\breve{g}$ is Φ-compatible, then the manifold is named an almost poly-Norden semi-Riemannain manifold and $(\breve{g},\Phi)$ is called an almost poly-Norden semi-Riemannian structure on $\breve{M}$.*

From now on, we shall consider that $m \neq 0$.

We note that the eigenvalues of Φ are $\frac{m+\sqrt{m^2-4}}{2}$ and $\frac{m-\sqrt{m^2-4}}{2}$. The inverse of Φ is not an almost poly-Norden structure. Additionally, each complex structure on a semi-Riemannian manifold allows defining two poly-Norden structures in the forms [21]

$$\Phi_1 = \frac{m}{2}I + \left(\frac{\sqrt{4-m^2}}{2}\right)J, \quad \Phi_1 = \frac{m}{2}I - \left(\frac{\sqrt{4-m^2}}{2}\right)J, \quad -2 < m < 2.$$

Conversely, each poly-Norden structure Φ on the manifold gives rise to following two almost complex structures on this manifold,

$$J_1 = \frac{-m}{\sqrt{4-m^2}}I + \frac{2}{\sqrt{4-m^2}}\Phi, \quad J_2 = \frac{m}{\sqrt{4-m^2}}I - \frac{2}{\sqrt{4-m^2}}\Phi, \quad -2 < m < 2.$$

Definition 2.4. *[21] Let $(\breve{M}, \breve{g}, \Phi)$ be an almost poly-Norden semi-Riemannian manifold. If the almost poly-Norden structure is parallel with respect to the Levi-Civita connection $\breve{\nabla}$ then $(\breve{M}, \breve{g}, \Phi)$ is called a poly-Norden semi-Riemannian manifold.*

Let $\breve{M}$ be a semi-Riemannian manifold equipped with a semi-Riemannian metric $\breve{g}$ of index q, $0 < q < 2n+1$, and M is a hypersurface of $\breve{M}$ with the induced metric $g = \breve{g}\mid_M$. If the induced metric g is degenerate and the orthogonal complement $TM^{\perp}$ of tangent space TM, given as

$$TM^{\perp} =_{p\in M} \{V_p \in T_p\breve{M} : g_p(U_p, V_p) = 0, \forall U \in \Gamma(T_pM)\}$$

is a distribution of rank 1 on M, then M is called a lightlike hypersurface of $\breve{M}$ [1]. In this case, $TM^{\perp} \subset TM$ and then it coincides with the radical distribution $Rad\,TM = TM \cap TM^{\perp}$.

The complementary bundle of $TM^{\perp}$ in TM, namely screen distribution, is a non-degenerate distribution of constant rank $2n-1$ over M and denoted by $S(TM)$.

Theorem 2.5. *[1] Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $\breve{M}$. Then there exists a unique rank 1 vector subbundle $ltr(TM)$ of $T\breve{M}$, with base space M, such that for any non-zero section E of $Rad\,TM$ on a coordinate neighbourhood $\mathfrak{I} \subset M$, there exists a unique section N of $ltr(TM)$ on $\mathfrak{I}$ satisfying:*

$$\breve{g}(N,N) = 0, \quad \breve{g}(N,W) = 0, \quad \breve{g}(N,E) = 1, \ \text{for } W \in \Gamma(S(TM)\mid_{\mathfrak{I}}.$$

$ltr(TM)$ is called the lightlike transversal vector bundle of M with respect to $S(TM)$.

Therefore we get

$$TM = S(TM) \perp Rad\,TM, \tag{5}$$

$$\begin{aligned} T\breve{M} &= TM \oplus ltr(TM) \\ &= S(TM) \perp \{Rad\,TM \oplus ltr(TM)\}. \end{aligned} \tag{6}$$

Let $\omega : \Gamma(TM) \to \Gamma(S(TM))$ be the projection morphism of TM. So we have

$$\breve{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{7}$$

$$\breve{\nabla}_X N = -A_N X + \tau(X)N, \tag{8}$$

$$\nabla_X \omega Y = \nabla^*_X \omega Y + C(X, \omega Y)E, \tag{9}$$

$$\nabla_X E = -A^*_E X - \tau(X)E, \tag{10}$$

where ∇ (resp., ∇^*) is a linear connection on M (resp., $S(TM)$) and B, A_N and τ are called the local second fundamental form, the local shape operator, the transversal differential $1-$form, respectively.

The induced linear connection ∇ is not a metric connection in general and we have

$$(\nabla_X g)(Y,Z) = B(X,Z)\theta(Y) + B(X,Y)\theta(Z), \tag{11}$$

where θ is a differential 1-form such that

$$\theta(X) = \breve{g}(N,X). \tag{12}$$

3. LIGHTLIKE HYPERSURFACES OF ALMOST POLY-NORDEN SEMI-RIEMANNIAN MANIFOLDS

Let M be a lightlike hypersurface of an almost poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then, for every $X \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$, we write

$$\Phi X = \phi X + u(X)N, \tag{13}$$

$$\Phi N = \zeta + v(E)N, \tag{14}$$

where ϕX, $\zeta \in \Gamma(TM)$ and u, v are 1-forms given by

$$u(X) = g(X, \Phi E), \qquad v(X) = g(X, \Phi N). \tag{15}$$

Lemma 3.1. *Let M be a lightlike hypersurface of an almost poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then we have*

$$\phi^2 X = m\phi X - X - u(X)\zeta, \tag{16}$$

$$u(\phi X) = -mu(X) - u(X)v(E), \tag{17}$$

$$\phi\zeta = m\zeta - v(E)\zeta, \tag{18}$$

$$v(E)^2 = mv(E) - 1 - u(\zeta), \tag{19}$$

$$g(\phi X, Y) = g(X, \phi Y) - u(X)\theta(Y) + u(Y)\theta(X), \tag{20}$$

$$\begin{aligned} g(\phi X, \phi Y) &= mg(X, \phi Y) - g(X, Y) + mu(Y)\theta(X) \\ &\quad -u(Y)g(\phi X, N) - u(X)g(\phi Y, N). \end{aligned} \tag{21}$$

In case of $\breve{M}$ is being a poly-Norden semi-Riemannian manifold, we give the following:

Lemma 3.2. *Let M be a lightlike hypersurface of a poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then we have*

$$(\nabla_X \phi)Y = u(Y)\,(A_N X) + B(X, Y)\zeta, \tag{22}$$

$$(\nabla_X u)Y = v(E)\,(B(X, Y)) - B(X, \phi Y) - u(Y)\tau(X), \tag{23}$$

$$\nabla_X \zeta = -\phi A_N X + \tau(X)\zeta + v(E)\,(A_N X)\,, \tag{24}$$

$$X(v(E)) = -B(X, \zeta) - u(A_N X). \tag{25}$$

4. INVARIANT LIGHTLIKE HYPERSURFACES OF A POLY-NORDEN SEMI-RIEMANNIAN MANIFOLD

Definition 4.1. *Let M be a lightlike hypersurface of an almost poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then M is called an invariant lightlike hypersurface of $\breve{M}$ if*

$$\begin{aligned} \Phi(Rad\,TM) &= Rad\,TM, \\ \Phi(ltr(TM)) &= ltr(TM). \end{aligned} \tag{26}$$

Example 4.2. *Let $\breve{M} = \mathbb{R}^7_3$ be a semi-Euclidean space with coordinate system $(x_1, x_2, ..., x_7)$ and signature $(-, +, -, +, -, +, +)$. Taking*

$$\Phi(x_1, x_2, ..., x_7) = (B_m x_1, B_m x_2, B_m x_3, B_m x_4, B_m x_5, B_m x_6, B_m x_7),$$

then Φ is an almost poly-Norden structure on $\breve{M}$.

Now, we consider a hypersurface M of $\breve{M}$ with

$$x_5 = x_7.$$

Then TM of M is spanned by

$$\Pi_1 = \frac{\partial}{\partial x_1}, \quad \Pi_2 = \frac{\partial}{\partial x_2},$$
$$\Pi_3 = \frac{\partial}{\partial x_3}, \quad \Pi_4 = \frac{\partial}{\partial x_4}, \quad \Pi_5 = \frac{\partial}{\partial x_6},$$
$$\Pi_6 = \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7}.$$

In this case, $Rad\,TM$ *and* $ltr(TM)$ *are given by*

$$Rad\,TM = Sp\{E = \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7}\},$$

and

$$ltr(TM) = Sp\{N = -\frac{1}{2}(\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_7})\},$$

respectively. Thus, we find

$$\Phi E = B_m E \qquad and \qquad \Phi N = B_m N,$$

which implies that M is an invariant lightlike hypersurface of $\breve{M}$.

Theorem 4.3. *Let M be a lightlike hypersurface of an almost poly-Norden semi-Riemannian manifold* $(\breve{M}, \breve{g}, \Phi)$. *Then* ϕ *is an almost poly-Norden structure on M.*

Proof. It is well known that, M is an invariant lightlike hypersurface if and only if

$$\Phi X = \phi X,$$

that is

$$u(X) = 0.$$

Then, from (16) and (20), we get

$$\phi^2 X = m\phi X - X,$$

and

$$g(\phi X, Y) = g(X, \phi Y).$$

So, we get our assertion. □

Theorem 4.4. *Let M be an invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold* $(\breve{M}, \breve{g}, \Phi)$. *Then we have*

$$\begin{aligned} B(X, \Phi Y) &= B(\Phi X, Y) = \Phi B(X, Y), \\ B(\Phi X, \Phi Y) &= mB(X, \Phi Y) + B(X, Y). \end{aligned}$$

Proof. It is obvious from (7). □

5. SCREEN SEMI-INVARIANT LIGHTLIKE HYPERSURFACES OF A POLY-NORDEN SEMI-RIEMANNIAN MANIFOLD

Definition 5.1. *Let M be a lightlike hypersurface of an almost poly-Norden semi-Riemannian manifold* $(\breve{M}, \breve{g}, \Phi)$. *If*

$$\begin{aligned} \Phi(Rad\,TM) &\subset S(TM), \\ \Phi(ltr(TM)) &\subset S(TM), \end{aligned} \tag{27}$$

then M is called a screen semi-invariant lightlike hypersurface of $\breve{M}$.

Example 5.2. *Let $\breve{M} = \mathbb{R}_2^5$ be semi-Euclidean space with coordinate system $(x^1, x^2, x^3, x^4, x^5)$ and signature $(-, +, -, +, +)$. Taking*

$$\Phi(x^1, x^2, x^3, x^4, x^5) = ((m - B_m)x^1, (m - B_m)x^2, B_m x^3, B_m x^4, B_m x^5),$$

then we can say that Φ is a poly-Norden structure on $\breve{M}$.

Consider a hypersurface M of $\breve{M}$ with

$$x^5 = B_m x^1 + B_m x^2 + x^3.$$

Then TM of M is spanned by

$$\Omega_1 = \frac{\partial}{\partial x^1} + B_m \frac{\partial}{\partial x^5}, \qquad \Omega_2 = \frac{\partial}{\partial x^2} + B_m \frac{\partial}{\partial x^5}$$

$$\Omega_3 = \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^5}, \qquad \Omega_4 = \frac{\partial}{\partial x^4}$$

So, $Rad\,TM$ and $ltr(TM)$ are given by

$$Rad\,TM = Sp\{E = B_m\Omega_1 - B_m\Omega_2 + \Omega_3\},$$

$$ltr(TM) = Sp\left\{N = \frac{1}{2}\left(-B_m \frac{\partial}{\partial x^1} + B_m \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^5}\right)\right\}.$$

Also $S(TM)$ is spanned by $\{\Pi_1, \Pi_2, \Pi_3\}$, where

$$\Pi_1 = -\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + B_m \frac{\partial}{\partial x^3} + B_m \frac{\partial}{\partial x^5},$$

$$\Pi_2 = \frac{1}{2}\left\{ \begin{array}{c} -\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} \\ -B_m \frac{\partial}{\partial x^3} + B_m \frac{\partial}{\partial x^5} \end{array} \right\},$$

$$\Pi_3 = \frac{\partial}{\partial x_4}.$$

Thus we arrive at

$$\Pi_1 = \Phi E \qquad \text{and} \qquad \Pi_2 = \Phi N,$$

which imply that M is a screen semi-invariant lightlike hypersurface of $\breve{M}$.

We know that $S(TM)$ is non-degenerate, so we can define a distribution ϑ such that

$$S(TM) = \{\Phi(Rad\,TM) \oplus \Phi(ltr(TM))\} \perp \vartheta, \tag{28}$$

from which

$$TM = \{\Phi(Rad\,TM) \oplus \Phi(ltr(TM))\} \perp \vartheta \perp Rad\,TM, \tag{29}$$

$$T\breve{M} = \{\Phi(Rad\,TM) \oplus \Phi(ltr(TM))\} \perp \vartheta \perp \{Rad\,TM \oplus ltr(TM)\}. \tag{30}$$

Taking $\hat{D} = Rad\,TM \perp \Phi(Rad\,TM) \perp \vartheta$ and $\mathring{D} = \Phi(ltr(TM))$ on M. So, we get

$$TM = \hat{D} \oplus \mathring{D}. \tag{31}$$

Let $\xi = \Phi N$ and $\Psi = \Phi E$ be local lightlike vector fields. For $X \in \Gamma(TM)$, we can write

$$X = RX + QX, \tag{32}$$

where R and Q are projections of TM into $\hat{D}$ and $\mathring{D}$, respectively.

Also, for $X, Y \in \Gamma(TM)$, $\xi \in \mathring{D}$ and $\Psi \in \hat{D}$,

$$\phi^2 X = m\phi X - X - u(X)\xi, \tag{33}$$

$$u(\phi X) = mu(X), \quad u(\xi) = -1, \tag{34}$$

$$g(X, \phi Y) = g(\phi X, Y) + u(X)\theta(Y) - u(Y)\theta(X), \tag{35}$$

$$\begin{aligned} g(\phi X, \phi Y) &= mg(X, \phi Y) - g(X, Y) - mu(Y)\theta(X) \\ &\quad - u(Y)g(\phi X, N) - u(X)g(\phi Y, N), \end{aligned} \tag{36}$$

$$(\nabla_X \phi)Y = g(A^*_E X, Y)\xi + u(Y)A_N X, \tag{37}$$

$$\nabla_X \xi = -\phi A_N X + \tau(X)\xi, \tag{38}$$

$$\nabla_X \Psi = -\phi A^*_E X - \tau(X)\Psi \tag{39}$$

$$B(X, \xi) = -C(X, \Psi). \tag{40}$$

Theorem 5.3. *Assume that M is a screen semi-invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $\breve{M}$. Then the lightlike vector field Ψ is parallel on M if and only if*
i) M is totally geodesic on $\breve{M}$,
ii) $\tau = 0$.

Proof. Assume that Ψ is a parallel vector fields. From (13) and (39), for any $X \in \Gamma(TM)$, we have

$$\begin{aligned} 0 &= -\phi A^*_E X - \tau(X)\Psi \\ &= -\Phi A^*_E X + u(A^*_E X)N - \tau(X)\Psi. \end{aligned} \tag{41}$$

Applying Φ to (41) and in view of (13) with (2), we get

$$-m\phi(A^*_E X) - mu(A^*_E X)N + A^*_E X - m\tau(X)\Psi + \tau(X)E + u(A^*_E X)\xi = 0. \tag{42}$$

Taking tangential and transversal part of equation (42), we arrive at

$$A^*_E X = -\tau(X)E - u(A^*_E X)\xi, \quad mu(A^*_E X) = 0.$$

So, we get the proof of our assertion. □

Theorem 5.4. *Assume that M is a screen semi-invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $\breve{M}$. Then the lightlike vector field ξ is parallel on M if and only if M and $S(TM)$ is totally geodesic on $\breve{M}$.*

Proof. Since ξ is parallel vector fields on M, in view of (13) and (38), for any $X \in \Gamma(TM)$, we have

$$\begin{aligned} 0 &= -\phi A_N X + \tau(X)\xi \\ &= -\Phi A_N X + u(A_N X)N + \tau(X)\xi. \end{aligned} \tag{43}$$

Applying Φ to (43) and by use of (13) with (2), we get

$$-m\phi(A_N X) - mu(A_N X)N + A_N X + m\tau(X)\xi - \tau(X)N + u(A_N X)\xi = 0. \tag{44}$$

Taking tangential and transversal part of equation (44), we find

$$A_N X = -u(A_N X)\xi, \quad mu(A_N X) = \tau(X).$$

This completes the proof. □

Definition 5.5. *Let M be a screen semi-invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. If the second fundamental form*

$$B(X, Z) = 0,$$

for any $X \in \Gamma(\hat{D})$ and $Z \in \Gamma(\mathring{D})$, then M is called a mixed geodesic lightlike hypersurface.

Theorem 5.6. *Let M be a screen semi-invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then M is a mixed geodesic lightlike hypersurface if and only if*
i) There is no component of A_N, $\hat{D}-$valuable.
ii) There is no component of A^*_E, $\mathring{D}-$valuable.

Proof. Assume that M is mixed geodesic, i.e.

$$B(X, \xi) = 0. \tag{45}$$

In view of (4) and (8) in (45), we have

$$\begin{aligned} 0 &= B(X, \xi) = B(X, \Phi N) \\ &= \breve{g}(\breve{\nabla}_X \Phi N, E) \\ &= \breve{g}((\breve{\nabla}_X \Phi)N + \Phi \breve{\nabla}_X N, E) \\ &= \breve{g}(\breve{\nabla}_X N, \Phi E) \\ &= -\breve{g}(A_N X, \Phi E). \end{aligned}$$

Therefore we arrive at (*i*).
On the other hand, since

$$-\breve{g}(A_N X, \Phi E) = \breve{g}(A^*_E X, \Phi N),$$

we obtain (*ii*). □

Now, we consider the distribution ϑ. From (29) and taking

$$\beta = \{\Phi(Rad(TM)) \oplus \Phi(ltr(TM))\} \perp Rad(TM),$$

for any $X \in \Gamma(TM)$, $Y \in \Gamma(\vartheta)$ and $Z \in \Gamma(\beta)$, we can write

$$\nabla_X Y = \overset{\vartheta}{\nabla}_X Y + \overset{\vartheta}{h}(X, Y), \tag{46}$$

$$\nabla_X Z = -\overset{\vartheta}{A}_Z X + \nabla^{\perp}_X Z, \tag{47}$$

where $\overset{\vartheta}{h} : \Gamma(TM) \times \Gamma(\vartheta) \to \Gamma(\beta)$ is an $\Im(M)$ bilinear, $\overset{\vartheta}{A}$ is an $\Im(M)$ linear operator on $\Gamma(\vartheta)$, $\overset{\vartheta}{\nabla}$ and $\nabla^{\perp}$ is a linear connection on ϑ and β, respectively.
Let $\Im \subset M$ be a coordinate neighborhood. If we consider the decomposition (29), we take

$$\begin{aligned} \rho_1(U, Y) &= -g(\overset{\vartheta}{h}(U, Y), \Phi N), \\ \rho_2(U, Y) &= -g(\overset{\vartheta}{h}(U, Y), \Phi E), \\ \rho_3(U, Y) &= g(\overset{\vartheta}{h}(U, Y), N), \end{aligned} \tag{48}$$

for any $U, Y \in \Gamma(\vartheta|_{\Im})$.
Therefore, from (3), we get

$$\breve{\nabla}_U Y = \overset{\vartheta}{\breve{\nabla}}_U Y - \rho_1(U, Y)\Phi E - \rho_2(U, Y)\Phi N + \rho_3(U, Y)E. \tag{49}$$

If we compute ρ_1, ρ_2 and ρ_3 in terms of B and C we arrive at

$$\begin{aligned} g(\breve{\nabla}_U Y, \Phi N) = \rho_1(U, Y) &= -C(U, \Phi Y), \\ g(\breve{\nabla}_U Y, \Phi E) = \rho_2(U, Y) &= -B(U, \Phi Y), \\ g(\breve{\nabla}_U Y, N) = \rho_3(U, Y) &= -C(U, Y). \end{aligned} \tag{50}$$

Thus, we can rewrite equation (49) with

$$\nabla_U Y = \overset{\vartheta}{\breve{\nabla}}_U Y - C(U, \Phi Y)\Phi E - B(U, \Phi Y)\Phi N - C(U, Y)E. \tag{51}$$

Theorem 5.7. *Let M be a screen semi-invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then the distribution ϑ is integrable if and only if*

$$C(U, \Phi Y) = C(\Phi U, Y), \quad B(U, \Phi Y) = B(\Phi U, Y), \quad C(U, Y) = C(Y, U), \tag{52}$$

for every $U, Y \in \Gamma(\vartheta)$.

Proof. We know that $\breve{\nabla}$ is a linear connection. Therefore, in view of (51), we have

$$\begin{aligned} [U, Y] &= \overset{\vartheta}{\nabla}_U Y - \overset{\vartheta}{\nabla}_Y U \\ &\quad +(C(U, \Phi Y) - C(\Phi U, Y))\Phi E \\ &\quad +(B(U, \Phi Y) - B(\Phi U, Y))\Phi N \\ &\quad +(C(U, Y) - C(Y, U))E. \end{aligned}$$

If ϑ is integrable then the components of $[U, Y]$ with respect to $\Phi E, \Phi N$ and E vanish. Thus, we get (52).
Contrary to, if (52) is satisfied we arrive at

$$[U, Y] \in \Gamma(\vartheta).$$

This completes the proof. □

Theorem 5.8. *Let M be a screen semi-invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then the distribution $\hat{D}$ is integrable if and only if*

$$B(\Phi U, \Phi Y) = mB(\Phi U, Y) - B(U, Y), \tag{53}$$

for every $U, Y \in \Gamma(\hat{D})$.

Proof. If we take $Y \in \Gamma(\hat{D})$, we get $\Phi Y \in \Gamma(\hat{D})$. Then $\hat{D}$ is integrable if and only if

$$\begin{aligned} \breve{g}([\Phi U, Y], \Phi E) &= \breve{g}(\breve{\nabla}_{\Phi U} Y, \Phi E) - \breve{g}(\breve{\nabla}_Y \Phi U, \Phi E) \\ &= \breve{g}(\Phi \breve{\nabla}_{\Phi U} Y, E) - \breve{g}(\Phi \breve{\nabla}_Y U, \Phi E) \\ &= \breve{g}(\breve{\nabla}_{\Phi U} \Phi Y, E) - m\breve{g}(\breve{\nabla}_Y U, \Phi E) + \breve{g}(\breve{\nabla}_Y U, E) \\ &= B(\Phi U, \Phi Y) - mB(\Phi U, Y) + B(U, Y), \end{aligned}$$

which yields (53). □

Theorem 5.9. *Let M be a screen semi-invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then the distribution $\hat{D}$ is parallel if and only if $\hat{D}$ is totally geodesic on M.*

Proof. From the definition of the distribution $\mathring{D}$ we know that $\mathring{D}$ is parallel if and only if

$$g(\nabla_U Y, \Psi) = 0.$$

From this equation, we get

$$\begin{aligned} 0 &= g(\breve{\nabla}_U Y, \Psi) \\ &= \breve{g}(\breve{\nabla}_U Y, \Psi) \\ &= \breve{g}(\breve{\nabla}_U Y, \Phi E) \\ &= \breve{g}(\Phi \nabla_U Y, E) \\ &= \breve{g}(-((\breve{\nabla}_U \Phi)Y + \breve{\nabla}_U \Phi Y, E) \\ &= \breve{g}(\breve{\nabla}_U \Phi Y, E) = B(U, \tilde{J} Y), \end{aligned}$$

which gives the proof of our assertion. □

References

[1] Duggal KL, Bejancu A. Lightlike submanifolds of semi-Riemannian manifolds and applications, Mathematics and Its Applications. Kluwer Publisher, 1996.
[2] Duggal KL, Şahin B. Generalized Cauchy-Riemann lightlike submanifolds of Kaehler manifolds. Acta Math Hungarica. 112, 2006, 107 -– 130.
[3] Duggal KL, Şahin B. Lightlike submanifolds of indefinite Sasakian manifolds. Int J Math and Math Sci. Article ID 57585, 2007.
[4] Duggal KL, Şahin B. Differential geometry of lightlike submanifolds. Frontiers in Mathematics, 2010.
[5] Galloway GJ. Lecture notes on spacetime geometry. Beijing Int. Math. Research Center. 1-55, 2007.
[6] Acet BE, Yüksel Perktaş S, Kılıç, E. Symmetric lightlike hypersurfaces of a para-Sasakian space form. Analele Stiintifice Ale Universitatii Al.I. Cuza Iasi (N.S.) 2, 2016, 915 -– 926.
[7] Acet BE, Yüksel Perktaş S, Kılıç, E. On lightlike geometry of para-Sasakian manifolds. Scientific Work J. Article ID 696231, 2014.
[8] Yüksel Perktaş S, Erdoğan FE. On generalized CR-lightlike submanifolds. Palestine J Math. 8(1), 2019, 200 -– 208.
[9] Crasmareanu MC, Hretcanu CE. Golden differential geometry. Chaos, Solitons and Fractals. 38(5), 2008, 1229 -– 1238.
[10] Hretcanu CE, Crasmareanu MC. On some invariant submanifolds in a Riemannian manifold with golden structure. Analele Stiintifice Ale Universitatii Al.I. Cuza Iasi (N.S.) 53(1), 2007, 199 -– 211.
[11] Gezer A, Cengiz N, Salimov A. On integrability of golden Riemannian structure. Turk J Math. 37, 2013, 693 -– 703.
[12] Erdoğan FE, Yıldırım C. On a study of the totally umbilical semi-invariant submanifolds of golden Riemannian manifolds. J Polytech. 21(4), 2018, 967 -– 970.
[13] Önen Poyraz N, Yaşar E. Lightlike hypersurfaces of a golden semi-Riemannian manifold. Mediter J Math. 14(204), 2017, 1 -– 20.
[14] Spinadel VW. The metallic means family and forbidden symmetries. Int Math J. 2(3), 2002, 279 -– 288.
[15] Acet BE. Lightlike hypersurfaces of metallic semi-Riemannian manifolds. Int J Geo Meth in Modern Phys. 15(12), 2018, 185 -– 201.
[16] Erdoğan FE. Transversal lightlike submanifolds of metallic semi-Riemannian manifolds. Turk J Math. 42, 2018, 3133 -– 3148.
[17] Hretcanu CE, Blaga AM. Submanifolds in metallic semi-Riemannian manifolds. Differ Geom Dynm Syst. 20, 2018, 83 -– 97.
[18] Erdoğan FE,Yüksel Perktaş S, Acet BE. Blaga AM. Screen transversal lightlike submanifolds of metallic semi-Riemannian manifolds. J Geom Phys. 142, 2019, 111 -– 120.
[19] Kalia S. The generalizations of the golden ratio, their powers, continued fractions and convergents, http://math.mit.edu/research/highschool/primes/papers.php
[20] Hretcanu CE, Crasmareanu MC. Metallic structure on Riemannian manifolds. Rev Un Mat Argentina. 54(2), 2013, 15 -– 27.
[21] Şahin B. Almost poly-Norden manifolds. Int J Maps in Math. 1(1), 2018, 68 -– 79.
[22] Yüksel Perktaş S. Submanifolds of almost poly-Norden Riemannian manifolds. Turk J Math. 44(1), 2020, 31 -– 49.

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 1, 31-42
ISSN: 2587–0971

http:/dergipark.gov.tr/tjos

Some special Smarandache ruled surfaces by Frenet Frame in E^3 -I

Süleyman Şenyurt[a], Davut Canlı[a], Elif Çan[a]

[a]*Department of Mathematics, Ordu University, Ordu, Turkey*

Abstract. The paper introduces some new special ruled surfaces with the base TNB- Smarandache curve where the unit vector of the generator is taken as one of other Frenet vectors and their linear combinations. The geometric properties with reference to fundamental forms such as minimality and developability of each generated surface are examined by Gauss and mean curvatures. An example is also given by considering the famous Viviani's curve.

1. Introduction

The theory of surfaces is an important branch of differential geometry. A typical surface is defined as an image of a function with two real valued variables (domain) by a mapping to 2- or 3-dimensional space. As a special type of surfaces, the ruled surfaces are defined to be one parameter family of lines. The simplest formulation makes these surfaces popular to refer for purposes on geometric modeling. Therefore, they are subjected in many areas such as engineering, architectural designs, computer graphics, automobile industry, etc [1, 2]. Since they are mostly referred in geometric designs sometimes to deal with real world problems and more frequently to model the real objects, introducing new ruled surfaces generated by different methods will lead new potentials to the related fields. Providing their characteristics may also enable easy adaptations for interested researchers. The basic theory related to ruled surfaces can be found in many differential geometry textbooks such as [3–7]. Recently, Ouarab, (2021a) put forth a method to generate new ruled surfaces in by taking the advantage of the idea of Smarandache geometry. By assigning the base curve as one of the Smarandache curves and taking the generator as the another vector element of Frenet frame, she introduced these ruled surfaces as Smarandache ruled surfaces according to Frenet frame in [8]. The same method of generating such ruled surfaces is applied to the Darboux frame by Ouarab, (2021b) in [9] and according to the alternative frame by Ouarab, (2021c) in [10]. Motivated by this, in this study, we address new ruled surfaces by considering some linear combinations of Frenet vectors as a Smarandache curve. Then, we study some characteristics of these ruled surfaces and present an example regarding to Viviani's curve to illustrate each surface.

Corresponding author: SŞ mail address: senyurtsuleyman52@gmail.com ORCID:0000-0003-1097-5541, DC ORCID:0000-0003-0405-9969, EÇ ORCID:0000-0001-5870-114X

Received: 25 January 2022; Accepted: 27 February 2022; Published: 30 April 2022
Keywords. Frenet frame; ruled surfaces; Smarandache geometry; fundamental forms
2010 *Mathematics Subject Classification*. 53A04, 53A05
Cited this article as: Şenyurt S, Canlı D, Çan E, Some special Smarandache ruled surfaces by Frenet Frame in E^3 -I, Turkish Journal of Science. 2022, 7(1), 31-42.

2. Preliminaries

We comprise the basic concepts which will be used throughout the paper in this section. Let $\alpha : I \to E^3$ be a regular unit speed curve. The very well-known Frenet apparatus is given by following identities:

$$T = \alpha', \quad N = \frac{\alpha''}{\|\alpha''\|}, \quad B = T \wedge N, \quad \kappa = \|\alpha''\|, \quad \tau = \langle N', B \rangle,$$
$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N, \ [12]$$

On the other hand, a Smarandache curve of is a regular curve generated by the position vector of the following form

$$\gamma = \frac{fT + gN + hB}{\sqrt{f^2 + g^2 + h^2}} \tag{1}$$

where f, g and h are real functions. For $\forall s \in I$ the vector γ corresponds to a differentiable curve. If each f, g and h is considered to be a constant function then the curves drawn by the γ vector are known as Smarandache curves [11]. There are many studies in the literature with the context of Smarandache curves by applying different frames and considering different spaces. For more detail see [11–13].
A ruled surface, on the other hand is a one parameter family of lines and it has the following parameterization

$$X(s, v) = \alpha(s) + vr(s). \tag{2}$$

The normal vector field of the ruled surface , is given as

$$N_X = \frac{X_s \wedge X_v}{\|X_s \wedge X_v\|}, \tag{3}$$

while the Gauss and mean curvatures are defined by

$$K = \frac{eg - f^2}{EG - F^2}, \quad H = \frac{Eg - 2fF + eG}{2(EG - F^2)}, \tag{4}$$

respectively [1–5]. The coefficients appeared at (4) are known to be the coefficients of first and second fundamental forms and calculated by followings:

$$E = \langle X_s, X_s \rangle, \quad F = \langle X_s, X_v \rangle, \quad G = \langle X_v, X_v \rangle, \tag{5}$$
$$e = \langle X_{ss}, N_X \rangle, \quad f = \langle X_{sv}, N_X \rangle, \quad g = \langle X_{vv}, N_X \rangle. \tag{6}$$

3. Some special Smarandache Ruled Surfaces according to Frenet Frame in E^3

Let us recall the relation (1) . If $f = g = h = 1$, then the corresponding curve whose position vector is $\vec{\gamma} = \dfrac{\vec{T} + \vec{N} + \vec{B}}{\sqrt{3}}$ is called as the $TNB-$ Smarandache curve. Next, let us consider the ruled surfaces whose base is $TNB-$ Smarandache curve and the genarator is the one of following unit vectors

$$\vec{T}, \quad \vec{N}, \quad \vec{B}, \quad \vec{r_1} = \frac{\vec{T} + \vec{N}}{\sqrt{2}}, \quad \vec{r_2} = \frac{\vec{T} + \vec{B}}{\sqrt{2}}, \quad \vec{r_3} = \frac{\vec{N} + \vec{B}}{\sqrt{2}}, \quad \vec{r_4} = \frac{\vec{T} + \vec{N} + \vec{B}}{\sqrt{3}}.$$

We examine the properties of these seven ruled surfaces by means of Gaussian and mean curvatures.

Definition 3.1. *Let $\alpha : I \subset R \to R$ be a regular unit speed curve and denote $\{T,\ N,\ B\}$ as its Frenet frame. We define and consider the ruled surface where the unit vector $\vec{T}$ moves along on the $TNB-$ Smarandache curve of α. The parametric form of this is given as*

$$\mathcal{F}(s, v) = \frac{1}{\sqrt{3}}\left(\left(1 + \sqrt{3}v\right)T + N + B\right).$$

The first and second partial derivatives of the surface $\mathcal{F}(s,v)$ are given in respective order as follows:

$$\mathcal{F}_s = \frac{1}{\sqrt{3}}\left(-\kappa T + \left(\kappa - \tau + \sqrt{3}v\kappa\right)N + \tau B\right), \quad \mathcal{F}_v = T, \quad \mathcal{F}_{sv} = \kappa N, \quad \mathcal{F}_{vv} = 0,$$

$$\mathcal{F}_{ss} = \frac{1}{\sqrt{3}}\left(\left(-\kappa' - \kappa^2 + \tau\kappa - \sqrt{3}v\kappa^2\right)T - \left(\kappa^2 + \tau^2 - \kappa' + \tau' - \sqrt{3}v\kappa'\right)N + \left(\tau' + \tau\kappa - \tau^2 + \sqrt{3}v\tau\kappa\right)B\right).$$

To formulate the normal vector field of $\mathcal{F}(s,v)$ denoted by $N_{\mathcal{F}}$, we first compute

$$\mathcal{F}_s \wedge \mathcal{F}_v = \frac{1}{\sqrt{3}}\left(\tau N - \left(\kappa - \tau + \sqrt{3}v\kappa\right)B\right).$$

When the norm is taken, we have

$$\|\mathcal{F}_s \wedge \mathcal{F}_v\| = \frac{1}{\sqrt{3}}\sqrt{\tau^2 + \left(\kappa - \tau + \sqrt{3}v\kappa\right)^2} = \frac{1}{\sqrt{3}}\sqrt{\kappa^2 + 2\tau^2 - 2\kappa\tau + 2\sqrt{3}v\left(\kappa^2 - \kappa\tau\right) + 3v^2\kappa^2}.$$

Hence, we obtain

$$N_{\mathcal{F}} = \frac{\tau N - \left(\kappa - \tau + \sqrt{3}v\kappa\right)B}{\sqrt{\kappa^2 + 2\tau^2 - 2\kappa\tau + 2\sqrt{3}v\left(\kappa^2 - \kappa\tau\right) + 3v^2\kappa^2}}.$$

Moreover, from the relations (5) and (6) the coefficients of first and second fundamental forms are calculated as

$$E_{\mathcal{F}} = \frac{1}{3}\left(\kappa^2 + \tau^2 + \left(\kappa - \tau + \sqrt{3}v\kappa\right)^2\right), \qquad F_{\mathcal{F}} = \frac{\kappa}{\sqrt{3}}, \qquad G_{\mathcal{F}} = 1,$$

$$e_{\mathcal{F}} = \frac{-\tau\left(\kappa^2 + \tau^2 - \kappa' + \tau' - \sqrt{3}v\kappa'\right) - \left(\kappa - \tau + \sqrt{3}v\kappa\right)\left(\tau' + \tau\kappa - \tau^2 + \sqrt{3}v\tau\kappa\right)}{\sqrt{3}\sqrt{\kappa^2 + 2\tau^2 - 2\kappa\tau + 2\sqrt{3}v\left(\kappa^2 - \kappa\tau\right) + 3v^2\kappa^2}},$$

$$f_{\mathcal{F}} = \frac{\kappa\tau}{\sqrt{3}\sqrt{\kappa^2 + 2\tau^2 - 2\kappa\tau + 2\sqrt{3}v\left(\kappa^2 - \kappa\tau\right) + 3v^2\kappa^2}}, \qquad g_{\mathcal{F}} = 0.$$

Finally, by referring the relation (4) the Gaussian and mean curvatures are obtained as

$$K_{\mathcal{F}} = -\frac{\kappa^2\tau^2}{\left(\kappa^2 + 2\tau^2 - 2\kappa\tau + 2\sqrt{3}v\left(\kappa^2 - \kappa\tau\right) + 3v^2\kappa^2\right)^2},$$

$$H_{\mathcal{F}} = -\frac{\sqrt{3}\tau\left(\kappa^2 + \tau^2 - \kappa' + \tau' - \sqrt{3}v\kappa'\right) + \sqrt{3}\left(\kappa - \tau + \sqrt{3}v\kappa\right)\left(\tau' + \tau\kappa - \tau^2 + \sqrt{3}v\tau\kappa\right) + 2\kappa^2\tau}{2\left(\kappa^2 + 2\tau^2 - 2\kappa\tau + 2\sqrt{3}v\left(\kappa^2 - \kappa\tau\right) + 3v^2\kappa^2\right)^{\frac{3}{2}}}.$$

Corollary 3.2. *If α is a planar curve then the ruled surface $\mathcal{F}(s,v)$ is both developable and minimal.*

Definition 3.3. *Let $\alpha : I \subset R \to R$ be a regular unit speed curve and denote $\{T,\ N,\ B\}$ as its Frenet frame. We define and consider the ruled surface where the unit vector $\vec{N}$ moves along on the $TNB-$ Smarandache curve of α. The parametric form of this is given as*

$$\mathcal{U}(s,v) = \frac{1}{\sqrt{3}}\left(T + \left(1 + \sqrt{3}v\right)N + B\right).$$

The first and second partial derivatives of the surface $\mathcal{U}(s,v)$ are given in respective order as follows:

$$\mathcal{U}_s = \frac{1}{\sqrt{3}}\left(-\kappa\left(1+\sqrt{3}v\right)T + (\kappa-\tau)N + \tau\left(1+\sqrt{3}v\right)\right), \qquad \mathcal{U}_v = N, \qquad \mathcal{U}_{sv} = -\kappa T + \tau B, \qquad \mathcal{U}_{vv} = 0,$$
$$\mathcal{U}_{ss} = \frac{1}{\sqrt{3}}\left(-\left(\kappa'\left(1+\sqrt{3}v\right) + \kappa(\kappa-\tau)\right)T + \left(\kappa' - \tau' - \left(1+\sqrt{3}v\right)\left(\kappa^2+\tau^2\right)\right)N + \left(\tau(\kappa-\tau) + \tau'\left(1+\sqrt{3}v\right)\right)B\right).$$

To formulate the normal vector field of $\mathcal{U}(s,v)$ denoted by $N_{\mathcal{U}}$, we first compute

$$\mathcal{U}_s \wedge \mathcal{U}_v = \frac{1}{\sqrt{3}}\left(\tau\left(1+\sqrt{3}v\right)T - \kappa\left(1+\sqrt{3}v\right)B\right).$$

When the norm is taken, we have

$$\|\mathcal{U}_s \wedge \mathcal{U}_v\| = \frac{1}{\sqrt{3}}\left(1+\sqrt{3}v\right)\sqrt{\kappa^2+\tau^2}.$$

Hence, we obtain

$$N_{\mathcal{U}} = \frac{\tau T - \kappa B}{\sqrt{\kappa^2+\tau^2}}.$$

Moreover, from the relations (5) and (6) the coefficients of first and second fundamental forms are calculated as

$$E_{\mathcal{U}} = \frac{1}{3}\left(\left(\kappa^2+\tau^2\right)\left(1+\sqrt{3}v\right)^2 + (\kappa-\tau)^2\right), \qquad F_{\mathcal{U}} = \frac{(\kappa-\tau)}{\sqrt{3}}, \qquad G_{\mathcal{U}} = 1,$$
$$e_{\mathcal{U}} = -\frac{(\tau\kappa' + \kappa\tau')\left(1+\sqrt{3}v\right) + 2\tau\kappa(\kappa-\tau)}{\sqrt{3}\sqrt{\kappa^2+\tau^2}}, \qquad f_{\mathcal{U}} = -\frac{2\kappa\tau}{\sqrt{\kappa^2+\tau^2}}, \qquad g_{\mathcal{U}} = 0.$$

Finally, by referring the relation (4) the Gaussian and mean curvatures are obtained as

$$K_{\mathcal{U}} = -\frac{12\kappa^2\tau^2}{\left(1+\sqrt{3}v\right)^2(\kappa^2+\tau^2)^2},$$
$$H_{\mathcal{U}} = -\frac{3(\tau\kappa' + \kappa\tau')\left(1+\sqrt{3}v\right) + 18\kappa\tau(\kappa-\tau)}{2\sqrt{3}\left(1+\sqrt{3}v\right)(\kappa^2+\tau^2)^{\frac{3}{2}}}.$$

Corollary 3.4.

- *If α is a planar curve, then the ruled surface $\mathcal{F}(s,v)$ is both developable and minimal.*
- *If α is a circular helix with equal curvatures, then the ruled surface $\mathcal{F}(s,v)$ is minimal.*

Definition 3.5. *Let $\alpha : I \subset R \to R$ be a regular unit speed curve and denote $\{T,\ N,\ B\}$ as its Frenet frame. We define and consider the ruled surface where the unit vector $\vec{B}$ moves along on the TNB$-$ Smarandache curve of α. The parametric form of this is given as*

$$\mathcal{Z}(s,v) = \frac{1}{\sqrt{3}}\left(T + N + \left(1+\sqrt{3}v\right)B\right).$$

The first and second partial derivatives of the surface $\mathcal{Z}(s,v)$ are given in respective order as follows:

$$\mathcal{Z}_s = \frac{1}{\sqrt{3}}\left(-\kappa T + \left(\kappa - \tau - \sqrt{3}v\tau\right)N + \tau B\right), \mathcal{Z}_v = B, \qquad \mathcal{Z}_{sv} = -\tau N, \qquad \mathcal{Z}_{vv} = 0,$$
$$\mathcal{Z}_{ss} = \frac{1}{\sqrt{3}}\left(-\left(\kappa' T + \kappa^2 - \tau\kappa - \sqrt{3}v\kappa\tau\right)T + \left(\kappa' - \tau' - \sqrt{3}v\tau' - \kappa^2 - \tau^2\right)N + \left(\tau' + \tau\kappa - \tau^2 - \sqrt{3}v\tau^2\right)B\right).$$

To formulate the normal vector field of $\mathcal{Z}(s, v)$ denoted by $N_{\mathcal{Z}}$, we first compute

$$\mathcal{Z}_s \wedge \mathcal{Z}_v = \frac{1}{\sqrt{3}}\left(\left(\kappa - \tau - \sqrt{3}v\tau\right)T - \kappa N\right).$$

When the norm is taken, we have

$$\|\mathcal{Z}_s \wedge \mathcal{Z}_v\| = \frac{1}{\sqrt{3}}\sqrt{\left(\kappa - \tau - \sqrt{3}v\tau\right)^2 + \kappa^2} = \frac{1}{\sqrt{3}}\sqrt{2\kappa^2 + \tau^2 - 2\kappa\tau - 2\sqrt{3}v\left(\kappa\tau - \tau^2\right) + 3v^2\tau^2}.$$

Hence, we obtain

$$N_{\mathcal{Z}} = \frac{\left(\kappa - \tau - \sqrt{3}v\tau\right)T - \kappa N}{\sqrt{2\kappa^2 + \tau^2 - 2\kappa\tau - 2\sqrt{3}v\left(\kappa\tau - \tau^2\right) + 3v^2\tau^2}}.$$

Moreover, from the relations (5) and (6) the coefficients of first and second fundamental forms are calculated as

$$\begin{aligned}
E_{\mathcal{Z}} &= \frac{1}{3}\left(\left(\kappa^2 + \tau^2\right)\left(\kappa - \tau - \sqrt{3}v\tau\right)^2\right), \qquad F_{\mathcal{Z}} = \frac{\tau}{\sqrt{3}}, \qquad G_{\mathcal{Z}} = 1,\\
e_{\mathcal{Z}} &= \frac{-\left(\kappa' T + \kappa^2 - \tau\kappa - \sqrt{3}v\kappa\tau\right)\left(\kappa - \tau - \sqrt{3}v\tau\right) - \kappa\left(\kappa' - \tau' - \sqrt{3}v\tau' - \kappa^2 - \tau^2\right)}{\sqrt{3}\sqrt{2\kappa^2 + \tau^2 - 2\kappa\tau - 2\sqrt{3}v\left(\kappa\tau - \tau^2\right) + 3v^2\tau^2}},\\
f_{\mathcal{Z}} &= \frac{\kappa\tau}{\sqrt{2\kappa^2 + \tau^2 - 2\kappa\tau - 2\sqrt{3}v\left(\kappa\tau - \tau^2\right) + 3v^2\tau^2}}, \qquad g_{\mathcal{Z}} = 0.
\end{aligned}$$

Finally, by referring the relation (4) the Gaussian and mean curvatures are obtained as

$$\begin{aligned}
K_{\mathcal{Z}} &= \frac{-3\kappa^2\tau^2}{\left(2\kappa^2 + \tau^2 - 2\kappa\tau - 2\sqrt{3}v\left(\kappa\tau - \tau^2\right) + 3v^2\tau^2\right)^2},\\
H_{\mathcal{Z}} &= \frac{-\sqrt{3}\left(\kappa' T + \kappa^2 - \tau\kappa - \sqrt{3}v\kappa\tau\right)\left(\kappa - \tau - \sqrt{3}v\tau\right) - \sqrt{3}\left(\kappa\kappa' - \kappa\tau' - \sqrt{3}v\kappa\tau' - \kappa^3 + \kappa\tau^2\right)}{2\left(\kappa^2 + \tau^2 - 2\kappa\tau - 2\sqrt{3}v\left(\kappa\tau - \tau^2\right) + 3v^2\tau^2.\right)^{\frac{3}{2}}}.
\end{aligned}$$

Corollary 3.6. *If α is a planar curve, then the ruled surface $\mathcal{F}(s, v)$ is developable.*

Definition 3.7. *Let $\alpha : I \subset R \to R$ be a regular unit speed curve and denote $\{T, N, B\}$ as its Frenet frame. We define and consider the ruled surface where the unit vector $\vec{r_1}$ moves along on the TNB− Smarandache curve of α. The parametric form of this is given as*

$$\mathcal{S}(s, v) = \frac{1}{\sqrt{3}}(T + N + B) + \frac{v}{\sqrt{2}}(T + N).$$

The first and second partial derivatives of the surface $\mathcal{S}(s, v)$ are given in respective order as follows:

$$\begin{aligned}
\mathcal{S}_s &= \frac{1}{\sqrt{6}}\left(-\kappa\left(\sqrt{2} + v\sqrt{3}\right)T + \left(\kappa\left(\sqrt{2} + v\sqrt{3}\right) - \sqrt{2}\tau\right)N + \tau\left(\sqrt{2} + v\sqrt{3}\right)B\right),\\
\mathcal{S}_v &= \frac{1}{\sqrt{2}}(T + N), \quad \mathcal{S}_{sv} = \frac{1}{\sqrt{2}}(-\kappa T + \kappa N + \tau B), \quad \mathcal{S}_{vv} = 0,\\
\mathcal{S}_{ss} &= \frac{1}{\sqrt{6}}\left\{\begin{array}{c}
\left(-\sqrt{2}\left(\kappa' + \kappa^2 - \kappa\tau\right) - v\sqrt{3}\left(\kappa' + \kappa^2\right)\right)T\\
+\left(\sqrt{2}\left(\kappa' - \tau' - \kappa^2 - \tau^2\right) + v\sqrt{3}\left(\kappa' - \kappa^2 - \tau^2\right)\right)N\\
+\left(\sqrt{2}\left(\tau' - \tau^2 + \kappa\tau\right) + v\sqrt{3}\left(\tau' + \kappa\tau\right)\right)B
\end{array}\right\}
\end{aligned}$$

To formulate the normal vector field of $\mathcal{S}(s,v)$ denoted by $N_{\mathcal{S}}$, we first compute

$$\mathcal{S}_s \wedge \mathcal{S}_v = \frac{1}{2\sqrt{6}}\left(-\tau\left(2+\sqrt{6}v\right)T + \tau\left(2+\sqrt{6}v\right)N + \left(2\tau - 2\kappa\left(2+\sqrt{6}v\right)\right)B\right).$$

When the norm is taken, we have

$$\|\mathcal{S}_s \wedge \mathcal{S}_v\| = \frac{1}{\sqrt{6}}\sqrt{(4\kappa^2+3\tau^2-4\kappa\tau)+2\sqrt{6}v\,(2\kappa^2+\tau^2-\kappa\tau)+3v^2\,(2\kappa^2+\tau^2)}.$$

Hence, we obtain

$$N_{\mathcal{S}} = \frac{-\tau\left(2+\sqrt{6}v\right)T + \tau\left(2+\sqrt{6}v\right)N + \left(2\tau - 2\kappa\left(2+\sqrt{6}v\right)\right)B}{2\sqrt{(4\kappa^2+3\tau^2-4\kappa\tau)+2\sqrt{6}v\,(2\kappa^2+\tau^2-\kappa\tau)+3v^2\,(2\kappa^2+\tau^2)}}.$$

Moreover, from the relations (5) and (6) the coefficients of first and second fundamental forms are calculated as

$$E_{\mathcal{S}} = \frac{1}{6}\left(4\left(\kappa^2-\kappa\tau+\tau^2\right)+2\sqrt{6}v\left(2\kappa^2-\kappa\tau+\tau^2\right)+3v^2\left(2\kappa^2+\tau^2\right)\right),$$
$$F_{\mathcal{S}} = -\frac{\tau}{\sqrt{6}}, \qquad G_{\mathcal{S}} = 1,$$

$$e_{\mathcal{S}} = \frac{\begin{array}{l}4\kappa'\tau - 4\kappa\tau' - 4\tau^3 + \sqrt{6}\tau\tau' + \sqrt{6}\kappa\tau^2 + 6\kappa\tau^2 - 6\kappa^2\tau \\ +\sqrt{6}v\left(4\kappa'\tau - 2\tau\tau' - 4\kappa\tau' - 3\kappa^2\tau + 2\kappa\tau^2 - 2\kappa^2\tau - 2\tau^3\right) + 3v^2\left(2\tau\kappa' - 2\kappa\tau' - 2\kappa^2\tau - \tau^3\right)\end{array}}{2\sqrt{3}\left(4\kappa^2+3\tau^2-4\kappa\tau+2\sqrt{6}v\,(2\kappa^2+\tau^2-\kappa\tau)+3v^2\,(2\kappa^2+\tau^2)\right)^{\frac{1}{2}}},$$

$$f_{\mathcal{S}} = \frac{\tau^2 - 2\kappa\tau - \sqrt{6}\kappa\tau v}{\sqrt{2}\sqrt{(4\kappa^2+3\tau^2-4\kappa\tau)+2\sqrt{6}v\,(2\kappa^2+\tau^2-\kappa\tau)+3v^2\,(2\kappa^2+\tau^2)}}, \qquad g_{\mathcal{S}} = 0.$$

Finally, by referring the relation (4) the Gaussian and mean curvatures are obtained as

$$K_{\mathcal{S}} = \frac{-3\left(\tau^2 - 2\kappa\tau - \sqrt{6}\kappa\tau v\right)^2}{\left(4\kappa^2+3\tau^2-4\kappa\tau+2\sqrt{6}v\,(2\kappa^2+\tau^2-\kappa\tau)+3v^2\,(2\kappa^2+\tau^2)\right)^2},$$

$$H_{\mathcal{S}} = \frac{\begin{array}{l}4\kappa'\tau - 4\kappa\tau' - 6\tau^3 + \sqrt{6}\tau\tau' + \left(\sqrt{6}+10\right)\kappa\tau^2 - 6\kappa^2\tau \\ +\sqrt{6}v\left(4\kappa'\tau - 2\tau\tau' - 4\kappa\tau' - 3\kappa^2\tau + 2\kappa\tau^2 - 2\kappa^2\tau - 2\tau^3\right) + 3v^2\left(2\tau\kappa' - 2\kappa\tau' - 2\kappa^2\tau - \tau^3\right)\end{array}}{\left(2/\sqrt{3}\right)\left(4\kappa^2+3\tau^2-4\kappa\tau+2\sqrt{6}v\,(2\kappa^2+\tau^2-\kappa\tau)+3v^2\,(2\kappa^2+\tau^2)\right)^{\frac{3}{2}}}.$$

Corollary 3.8. *If α is a planar curve then the ruled surface $\mathcal{S}(s,v)$ is developable.*

Definition 3.9. *Let $\alpha : I \subset R \to R$ be a regular unit speed curve and denote $\{T,\ N,\ B\}$ as its Frenet frame. We define and consider the ruled surface where the unit vector $\vec{r_2}$ moves along on the TNB− Smarandache curve of α. The parametric form of this is given as*

$$\mathcal{Q}(s,v) = \frac{1}{\sqrt{3}}(T+N+B) + \frac{v}{\sqrt{2}}(T+B).$$

The first and second partial derivatives of the surface $Q(s,v)$ are given in respective order as follows:

$$Q_s = \frac{1}{\sqrt{6}}\left(-\sqrt{2}\kappa T + (\kappa-\tau)\left(\sqrt{2}+v\sqrt{3}\right)N + \sqrt{2}\tau B\right),$$
$$Q_v = \frac{1}{\sqrt{2}}(T+B), \quad Q_{vv}=0, \quad Q_{sv} = \frac{1}{\sqrt{2}}(\kappa-\tau)N,$$
$$Q_{ss} = \frac{1}{\sqrt{6}}\left\{\begin{array}{c} \left(\sqrt{2}\left(-\kappa'-\kappa^2+\tau\kappa\right)+\sqrt{3}v\left(-\kappa^2+\tau\kappa\right)\right)T \\ +\left(-\sqrt{2}\left(\kappa^2+\tau^2+\tau'-\kappa'\right)-\sqrt{3}v\left(\tau'-\kappa'\right)\right)N \\ +\left(\sqrt{2}\left(\tau'+\kappa\tau-\tau^2\right)+\sqrt{3}v\left(\tau\kappa-\tau^2\right)\right)B \end{array}\right\}$$

To formulate the normal vector field of $Q(s,v)$ denoted by N_Q, we first compute

$$Q_s \wedge Q_v = \frac{1}{2\sqrt{6}}\left\{(\kappa-\tau)\left(2+\sqrt{6}v\right)T + 2(\kappa+\tau)N - (\kappa-\tau)\left(2+\sqrt{6}v\right)B\right\}.$$

When the norm is taken, we have

$$\|Q_s \wedge Q_v\| = \frac{1}{\sqrt{6}}\sqrt{3\kappa^2+3\tau^2-2\kappa\tau+v2\sqrt{6}(\kappa-\tau)^2+3v^2(\kappa-\tau)^2}.$$

Hence, we obtain

$$N_Q = \frac{(\kappa-\tau)\left(2+\sqrt{6}v\right)T + 2(\kappa+\tau)N - (\kappa-\tau)\left(2+\sqrt{6}v\right)B}{2\sqrt{3\kappa^2+3\tau^2-2\kappa\tau+v2\sqrt{6}(\kappa-\tau)^2+3v^2(\kappa-\tau)^2}}.$$

Moreover, from the relations (5) and (6) the coefficients of first and second fundamental forms are calculated as

$$E_Q = \frac{1}{6}\left(4\kappa^2+4\tau^2-4\tau\kappa+2\sqrt{6}v(\kappa-\tau)^2+3v^2(\kappa-\tau)^2\right),$$
$$F_Q = \frac{\tau-\kappa}{\sqrt{6}}, \qquad G_Q = 1,$$

$$e_Q = \frac{4\sqrt{3}\left(\tau\kappa'-\kappa\tau'-\kappa^3-\tau^3\right)+4\sqrt{3}v\left(-\kappa\tau'+\tau\kappa'-\kappa^3-\tau^3+\kappa\tau^2+\kappa\tau^2\right)-3\sqrt{3}v^2\left(\kappa^2-\tau^2\right)(\kappa-\tau)}{6\sqrt{3\kappa^2+3\tau^2-2\kappa\tau+v2\sqrt{6}(\kappa-\tau)^2+3v^2(\kappa-\tau)^2}},$$
$$f_Q = \frac{\sqrt{2}\left(\kappa^2-\tau^2\right)}{2\sqrt{3\kappa^2+3\tau^2-2\kappa\tau+v2\sqrt{6}(\kappa-\tau)^2+3v^2(\kappa-\tau)^2}}, \qquad g_Q = 0.$$

Finally, by referring the relation (4) the Gaussian and mean curvatures are obtained as

$$K_Q = \frac{-3\left(\kappa^2-\tau^2\right)^2}{\left(3\kappa^2+3\tau^2-2\kappa\tau+v2\sqrt{6}(\kappa-\tau)^2+3v^2(\kappa-\tau)^2\right)^2},,$$

$$H_Q = \frac{\begin{array}{c} 4\sqrt{3}\left(\tau\kappa'-\kappa\tau'-\kappa^3-\tau^3\right)+\left(\kappa^2-\tau^2\right)(\kappa-\tau) \\ +4\sqrt{3}v\left(-\kappa\tau'+\tau\kappa'-\kappa^3-\tau^3+\kappa\tau^2+\kappa\tau^2\right)-3\sqrt{3}v^2\left(\kappa^2-\tau^2\right)(\kappa-\tau) \end{array}}{2\left(3\kappa^2+3\tau^2-2\kappa\tau+v2\sqrt{6}(\kappa-\tau)^2+3v^2(\kappa-\tau)^2\right)^{\frac{3}{2}}}.$$

Corollary 3.10. *If α is a circular helix with equal curvatures then the ruled surface $\boldsymbol{Q}(s,v)$ is developable.*

Definition 3.11. *Let $\alpha : I \subset R \to R$ be a regular unit speed curve and denote $\{T,\ N,\ B\}$ as its Frenet frame. We define and consider the ruled surface where the unit vector $\vec{r_3}$ moves along on the TNB− Smarandache curve of α. The parametric form of this is given as*

$$\mathcal{M}(s,v) = \frac{1}{\sqrt{3}}(T+N+B) + \frac{v}{\sqrt{2}}(N+B).$$

The first and second partial derivatives of the surface $\mathcal{M}(s,v)$ are given in respective order as follows:

$$\begin{aligned}
\mathcal{M}_s &= \frac{1}{\sqrt{6}}\left(-\left(\sqrt{2}\kappa + v\sqrt{3}\kappa\right)T + \left(\sqrt{2}(\kappa-\tau) - v\sqrt{3}\tau\right)N + \left(\sqrt{2}\tau + v\sqrt{3}\tau\right)B\right)\\
\mathcal{M}_v &= \frac{1}{\sqrt{2}}(N+B), \qquad \mathcal{M}_{vv} = 0, \qquad \mathcal{M}_{sv} = \frac{1}{\sqrt{2}}(-\kappa T - \tau N + \tau B),\\
\mathcal{M}_{ss} &= \frac{1}{\sqrt{6}}\begin{pmatrix} -\left(\sqrt{2}\left(\kappa' + \kappa^2 - \kappa\tau\right) + \sqrt{3}v(\kappa' - \kappa\tau)\right)T \\ +\left(\sqrt{2}\left(\kappa' - \tau' - \kappa^2 - \tau^2\right) + \sqrt{3}v\left(\tau' - \kappa^2 - \tau^2\right)\right)N \\ +\left(\sqrt{2}\left(\tau' + \kappa\tau - \tau^2\right) + \sqrt{3}v\left(\tau' - \tau^2\right)\right)B \end{pmatrix}
\end{aligned}$$

To formulate the normal vector field of $\mathcal{M}(s,v)$ denoted by $N_{\mathcal{M}}$, we first compute

$$\mathcal{M}_s \wedge \mathcal{M}_v = \frac{1}{2\sqrt{6}}\left\{\left(2\kappa - 2\tau\left(2+\sqrt{6}v\right)\right)T + \kappa\left(2+\sqrt{6}v\right)N - \kappa\left(2+\sqrt{6}v\right)B\right\}.$$

When the norm is taken, we have

$$\|\mathcal{M}_s \wedge \mathcal{M}_v\| = \frac{1}{2\sqrt{6}}\sqrt{3\kappa^2 + 4\tau^2 - 4\kappa\tau + 2\sqrt{6}v(\kappa^2 + 2\tau^2 - \kappa\tau) + 3v^2(\kappa^2 + 2\tau^2)}.$$

Hence, we obtain

$$N_{\mathcal{M}} = \frac{\left(2\kappa - 4\tau - 2\sqrt{6}\tau v\right)T + \kappa\left(2+\sqrt{6}v\right)N - \kappa\left(2+\sqrt{6}v\right)B}{\sqrt{3\kappa^2 + 4\tau^2 - 4\kappa\tau + 2\sqrt{6}v(\kappa^2 + 2\tau^2 - \kappa\tau) + 3v^2(\kappa^2 + 2\tau^2)}}.$$

Moreover, from the relations (5) and (6) the coefficients of first and second fundamental forms are calculated as

$$\begin{aligned}
E_{\mathcal{M}} &= \frac{1}{6}\left(4\kappa^2 + 4\tau^2 - 4\kappa\tau + 2\sqrt{6}v\left(\kappa^2 + 2\tau^2 - \kappa\tau\right) + 3v^2\left(\kappa^2 + 2\tau^2\right)\right),\\
F_{\mathcal{M}} &= \frac{\kappa}{\sqrt{6}}, \qquad G_{\mathcal{M}} = 1,\\
e_{\mathcal{M}} &= \frac{-4\sqrt{2}\left(\kappa'\tau - \kappa\tau' - \kappa\tau^2 + \tau\kappa^2 + \kappa^3\right) + 2\sqrt{3}v\left(4\kappa'\tau - 4\kappa\tau^2 + 3\tau\kappa^2 - \kappa\kappa'\right) + v^2 6\sqrt{2}\left(\kappa'\tau - \kappa\tau^2\right)}{\sqrt{3\kappa^2 + 4\tau^2 - 4\kappa\tau + 2\sqrt{6}v(\kappa^2 + 2\tau^2 - \kappa\tau) + 3v^2(\kappa^2 + 2\tau^2)}},\\
f_{\mathcal{M}} &= \frac{-\sqrt{2}\kappa^2}{\sqrt{3\kappa^2 + 4\tau^2 - 4\kappa\tau + 2\sqrt{6}v(\kappa^2 + 2\tau^2 - \kappa\tau) + 3v^2(\kappa^2 + 2\tau^2)}}, \qquad g_{\mathcal{M}} = 0.
\end{aligned}$$

Finally, by referring the relation (4) the Gaussian and mean curvatures are obtained as

$$K_{\mathcal{M}} = -\frac{12\kappa^4}{\left(3\kappa^2+4\tau^2-4\kappa\tau+2\sqrt{6}v\left(\kappa^2+2\tau^2-\kappa\tau\right)+3v^2\left(\kappa^2+2\tau^2\right)\right)^2},$$

$$H_{\mathcal{M}} = \frac{\begin{array}{l}-12\sqrt{2}\left(\kappa'\tau-\kappa\tau'-\kappa\tau^2+\tau\kappa^2+\kappa^3\right)+2\sqrt{3}\kappa^3\\ +6\sqrt{3}v\left(4\kappa'\tau-4\kappa\tau^2+3\tau\kappa^2-\kappa\kappa'\right)+v^2 18\sqrt{2}\left(\kappa'\tau-\kappa\tau^2\right)\end{array}}{\left(3\kappa^2+4\tau^2-4\kappa\tau+2\sqrt{6}v\left(\kappa^2+2\tau^2-\kappa\tau\right)+3v^2\left(\kappa^2+2\tau^2\right)\right)^{\frac{3}{2}}}$$

Corollary 3.12. *The ruled surface $\mathcal{M}(s,v)$ cannot be a developable surface.*

Definition 3.13. *Let $\alpha : I \subset R \to R$ be a regular unit speed curve and denote $\{T,\ N,\ B\}$ as its Frenet frame. We define and consider the ruled surface where the unit vector $\vec{r_4}$ moves along on the TNB$-$ Smarandache curve of α. The parametric form of this is given as*

$$\Gamma(s,v) = \frac{1}{\sqrt{3}}(T+N+B) + \frac{v}{\sqrt{3}}(T+N+B).$$

The first and second partial derivatives of the surface $\Gamma(s,v)$ are given in respective order as follows:

$$\begin{aligned}
\Gamma_s &= \frac{1}{\sqrt{3}}(1+v)\left(-\kappa T+(\kappa-\tau)N+\tau B\right),\\
\Gamma_v &= \frac{1}{\sqrt{3}}(T+N+B),\\
\Gamma_{ss} &= \frac{1}{\sqrt{3}}(1+v)\left\{\left(-\kappa'-\kappa^2+\kappa\tau\right)T+\left(\kappa'-\tau'-\kappa^2-\tau^2\right)N+\left(\tau'+\kappa\tau-\tau^2\right)B\right\},\\
\Gamma_{sv} &= \frac{1}{\sqrt{3}}\left(-\kappa T+(\kappa-\tau)N+\tau B\right), \qquad \Gamma_{vv}=0.
\end{aligned}$$

To formulate the normal vector field of $\Gamma(s,v)$ denoted by N_Γ, we first compute

$$\Gamma_s \wedge \Gamma_v = \frac{1}{3}(1+v)\left((\kappa-2\tau)T+(\kappa+\tau)N+(\tau-2\kappa)B\right).$$

When the norm is taken, we have

$$\|\Gamma_s \wedge \Gamma_v\| = \frac{\sqrt{6}}{3}(1+v)\sqrt{\kappa^2-\kappa\tau+\tau^2}.$$

Hence, we obtain

$$N_\Gamma = \frac{(\kappa-2\tau)T+(\kappa+\tau)N+(\tau-2\kappa)B}{\sqrt{6}\sqrt{\kappa^2-\kappa\tau+\tau^2}}.$$

Moreover, from the relations (5) and (6) the coefficients of first and second fundamental forms are calculated as

$$E_\Gamma = \frac{2}{3}(1+v)^2\left(\kappa^2-\kappa\tau+\tau^2\right), \qquad F_\Gamma = 0, \qquad G_\Gamma = 1,$$

$$e_\Gamma = \frac{(1+v)\left\{-2\left(\kappa^3+\tau^3\right)-2\kappa^2\tau+3\left(\kappa'\tau-\kappa\tau'\right)\right\}}{3\sqrt{2}\sqrt{\kappa^2-\kappa\tau+\tau^2}}, \qquad f_\Gamma = 0, \qquad g_\Gamma = 0.$$

Finally, by referring the relation (4) the Gaussian and mean curvatures are obtained as

$$K_\Gamma = 0, \qquad H_\Gamma = \frac{-2\left(\kappa^3+\tau^3\right)-2\kappa^2\tau+3\left(\kappa'\tau-\kappa\tau'\right)}{2\sqrt{2}(1+v)\left(\kappa^2-\kappa\tau+\tau^2\right)^{\frac{3}{2}}}.$$

Remark 3.14. *Note that since the directrix of this ruled surface can be collapsed to a point, it clearly corresponds to a cone and can be parameterized as in the following form:*

$$\Gamma(s,v) = \frac{1}{\sqrt{3}}(1+v)(T+N+B).$$

As known from the literature that for any conical surface, the coefficients F_Γ and f_Γ of first and second fundamental forms in respective order vanish, which corresponds to the relation $K_\Gamma = 0$. Therefore, the predefined ruled surface forms always a developable cone. However, we find it worth to do the calculations for validation purposes and providing the relation for mean curvature.

Example 3.15. *Let us consider the well known Viviani's curve parameterized as*

$$\gamma(t) = \left(a(1+cost), asint, 2asin\frac{1}{2}t\right), \quad t \in [-2\pi, 2\pi], \quad \text{[Gray, 1997 p.201]}.$$

For $a = 0.5$ and by changing the parameter as $t = 2s$, we easily represent the given Viviani's curve as in the following way

$$\alpha(s) = \left(\cos^2(s), \sin(s)\cos(s), \sin(s)\right), \quad s \in [-\pi, \pi].$$

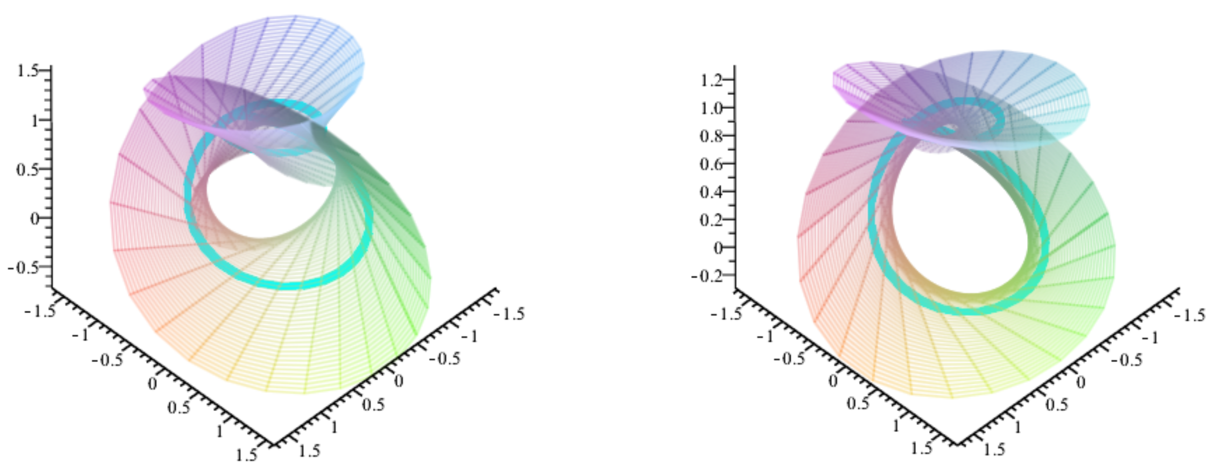
Then, the Frenet apparatus of $\alpha = \alpha(s)$ are given as

$$T(s) = \frac{2}{\sqrt{2\cos(2s)+6}}\left(-\sin(2s), \cos(2s), \cos(s)\right),$$

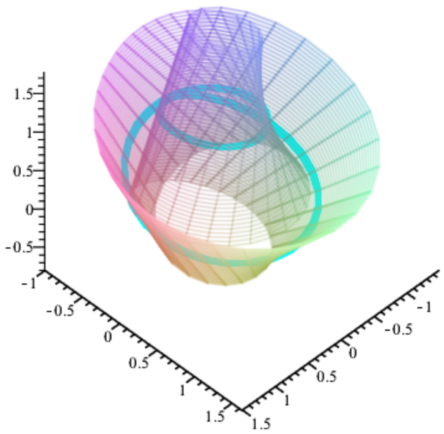
$$N(s) = \frac{-1}{\sqrt{2\cos(2s)+6}\sqrt{6\cos(2s)+26}}\begin{pmatrix} \cos(4s)+12\cos(2s)+3, \\ \sin(4s)+12\sin(2s), \\ 4\sin(s) \end{pmatrix},$$

$$B(s) = \frac{1}{\sqrt{6\cos(2s)+26}}\left(\sin(3s)+3\sin(s), -\cos(3s)-3\cos(s), 4\right).$$

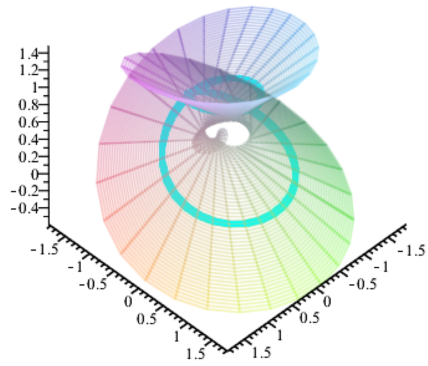
For $s \in [-\pi, \pi]$ and $v \in [-1, 1]$, the ruled surfaces $\mathcal{F}(s,v)$, $\mathcal{U}(s,v)$, $\mathcal{Z}(s,v)$, $\mathcal{S}(s,v)$, $\mathcal{Q}(s,v)$, $\mathcal{M}(s,v)$ and $\Gamma(s,v)$ are sketched in the following figures from (a) to (g).



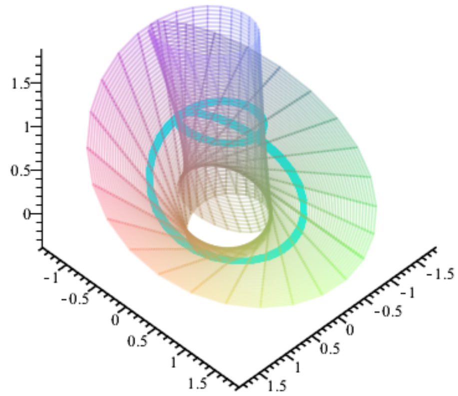
(a) generated by the unit vector $\vec{T}$ (b) generated by the unit vector $\vec{N}$



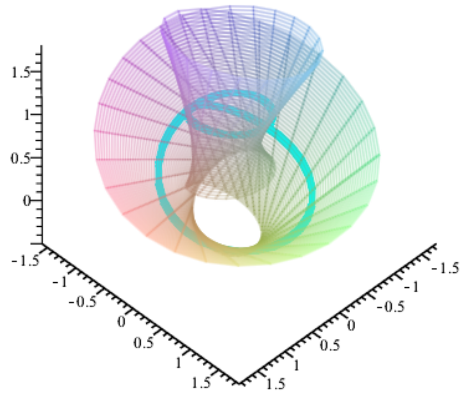
(c) generated by the unit vector $\vec{B}$



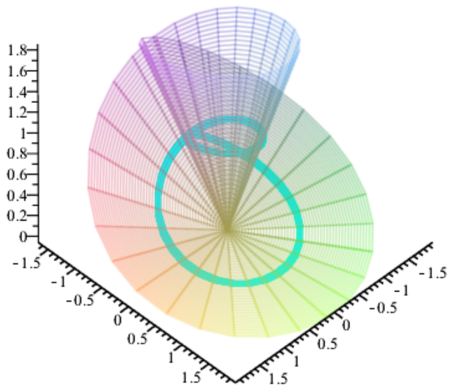
(d) generated by the unit vector $\vec{r_1}$



(e) generated by the unit vector $\vec{r_2}$



(f) generated by the unit vector $\vec{r_3}$



(g) generated by the unit vector $\vec{r_4}$

Figure 1: The ruled surfaces $\mathcal{F}(s,v)$, $\mathcal{U}(s,v)$, $\mathcal{Z}(s,v)$, $\mathcal{S}(s,v)$, $\mathcal{Q}(s,v)$, $\mathcal{M}(s,v)$ and $\Gamma(s,v)$ with the base TNB-Smarandache curve

References

[1] Pottmann, H., Asperl, A., Hofer, M. ve Killian, A., Architectural Geometry, Bentley, D. (ed.), First Edition, Bentley Institute Press, 2007

[2] Berk, A., A Structural Basis for Surface Discretization of Free Form Structures: Integration of Geometry, Materials and Fabrication, Ph. D Thesis, Michigan University, ABD, 2012

[3] Do-Carmo, P.M., Differential geometry of curvesand surfaces. IMPA, 1976

[4] Fenchel,W., On the Differential Geometry of Closed Space Curves, Bulletin of American Mathematical Society, 57(44-54), 1951

[5] Gray A. Modern differential geometry of curves and surfaces with Mathematica, 1997

[6] Pressley A.,Elementary Differential Geometry, Second Edition, Springer London, 2010

[7] Struik D. J., Lectures on classical differential geometry, Addison-Wesley Publishing Company, 1961

[8] Ouarab, S., Smarandache Ruled Surfaces according to Frenet-Serret Frame of a Regular Curve in E3, Hindawi Abstract and Applied Analysis, Vol. 2021, Article ID 5526536, 8 pages

[9] Ouarab, S., Smarandache ruled Surfaces according to Darboux Frame in E^3, Hindawi Journal of Mathematics, Vol. 2021, Article ID 9912624, 10 pages

[10] Ouarab,S., NC-Smarandache ruled surface and NW-Smarandache ruled surface according to alternative moving frame in E^3, Hindawi Journal of Mathematics ,Vol. 2021, Article ID 9951434, 6 pages

[11] Turgut, N., S. Yılmaz, S., "Smarandache curves in Minkowski space-time," International Journal of Mathematical Combinatorics, vol. 3, pp. 51–55, 2008.

[12] Ali, A.T., Special Smarandache curves in the Euclidean space, International Journal of Mathematical Combinatorics, 2(2010),30-36, 2010.

[13] Şenyurt, S., Sivas, S., An Application of Smarandache Curve, Ordu Univ. J. Sci. Tech., 3(1),46-6, 2013

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 1, 43-52
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

Analysis of the spread of Hookworm infection with Caputo-Fabrizio fractional derivative

Esin İlhan[a]

[a] *Faculty of Engineering and Architecture, Kirsehir Ahi Evran University, Kirsehir, Türkiye*

Abstract. This research study provides a mathematical analysis for the spread of Hookworm infection model. Firstly, the proposed disease model is extended by means of the Caputo-Fabrizio fractional derivative. Then, existence and uniqueness of the solution is presented for the fractional-type Hookworm infection model with the help of the fixed-point theorem. Theoretical results of the model under consideration show the advantages of the fractional differential operators.

1. Introduction

In comparison to traditional mathematical models, fractional-order models are more advantageous since they generally produce better outcomes than classical order models [1]. Many researchers have concentrated on studying non-linear dynamical systems based on different types of fractional differential operators, inspired by the growth of fractional calculus, by creating a number of analytical or numerical techniques in order to obtain solutions [2, 3]. In order to analyze and investigate these systems, Riemann-Liouville (RL), Caputo, Caputo-Fabrizio (CF), Atangana-Baleanu (AB), as well as other non-local fractional derivatives, are employed to reach more detailed results. Recently, a new-type fractional derivative including a non-singular kernel has been presented as can be seen in [4]. The kernel of this non-local non-singular fractional operator has the form of the exponential function. Some type of fractional operators, on the other hand, have a power-law kernel and are limited in their ability to describe physical situations. Therefore, Caputo and Fabrizio proposed an additional fractional differential operator with an exponential decay kernel to overcome this challenge in [1]. The CF fractional derivative operator, which has a non-singular kernel, is a new approach to the fractional calculus that has captivated the interest of many researchers. Additionally, the CF operator is one of the best suited for simulating real-world problems that follow the exponential decay law. Developing a mathematical model employing the CF fractional-order derivative became a well-known subject of study over time [10–12].

Inspired by the above information, the Hookworm infection model [5] is investigated in this study utilizing CF fractional-type derivative and integral operator. First, the model is updated to use CF fractional operator. The existence and uniqueness of solutions are then determined under initial conditions utilizing the fixed point theorem.

Corresponding author: Eİ mail address: eilhan@ahievran.edu.tr ORCID:0000-0002-0839-0942

Received: 18 January 2022; Accepted: 22 February 2022; Published: 30 April 2022
Keywords. Modeling, Fractional derivative
2010 *Mathematics Subject Classification*. 26A33, 34K28
Cited this article as: İlhan E. Analysis of the spread of Hookworm infection with Caputo-Fabrizio fractional derivative, Turkish Journal of Science. 2022, 7(1), 43-52.

2. Preliminaries

In the current portion, some fundamental definitions of fractional derivative and integral are presented. For more information on fractional calculus, we refer the readers to [6–9].

Definition 2.1. *Let $n \in \mathbb{N}$ and $n-1 < \nu < n$, then Caputo fractional derivative is defined by [7]:*

$$ {}_{a}^{C}D_{t}^{\nu} f(t)) = \frac{1}{\Gamma(n-\nu)} \int_{a}^{t} \frac{f^{(n)}(r)}{(t-r)^{\nu+1-n}} dr. \tag{1}$$

Definition 2.2. *For $f \in H^1(a,b), b > a, \nu \in (0,1)$, the CF fractional derivative is presented as [4]:*

$$ {}_{a}^{CF}\mathfrak{D}_{t}^{\nu}(f(t)) = \frac{\nu M(\nu)}{1-\nu} \int_{a}^{t} \frac{df(x)}{dx} exp\Big[-\nu \frac{t-x}{1-\nu}\Big] dx. \tag{2}$$

Here $M(\nu)$ is a normalization constants given by $M(0) = M(1) = 1$. Also, the definition of CF operator can be given as below:

$$ {}_{a}^{CF}\mathfrak{D}_{t}^{\nu}(f(t)) = \frac{\nu M(\nu)}{1-\nu} \int_{a}^{t} \Big(f(t) - f(x)\Big) exp\Big[-\nu \frac{t-x}{1-\nu}\Big] dx.$$

Remark 2.3. *If $\eta = \frac{1-\nu}{\nu} \in (0,\infty), \nu = \frac{1}{1+\eta} =\in [0,1]$, then the above equation supposes the following expression*

$$ \mathfrak{D}_{t}^{\eta}(f(t)) = \frac{N(\eta)}{\eta} \int_{a}^{t} \frac{df(x)}{dx} exp\Big[-\frac{t-x}{\eta}\Big] dx, \quad N(0) = N(\infty) = 1. \tag{3}$$

Furthermore,

$$ \lim_{\nu \to 0} \frac{1}{\nu} exp\Big[-\frac{t-x}{\nu}\Big] = \delta(x-t).$$

It should be noted that according to the definition, the fractional integral of Caputo type function with order ν is an average between function f and its integral of order one. Hence, this means that

$$ M(\nu) = \frac{2}{2-\nu}, \quad 0 \le \nu \le 1. \tag{4}$$

Owing to the above expression, Nieto and Losada presented the new Caputo type derivative of order ν can be rewritten as follows:

Definition 2.4. *The fractional derivative of order ν is [6],*

$$ {}^{CF}\mathfrak{D}_{\star}^{\nu}(f(t)) = \frac{1}{1-\nu} \int_{0}^{t} f'(x) exp\Big[-\nu \frac{t-x}{1-\nu}\Big] dx. \tag{5}$$

At this instant subsequent to the preface of the novel derivative, the connected anti-derivative turns out to be imperative; the connected integral of the derivative was proposed by Nieto and Losada [6],

Definition 2.5. *Let $0 < \nu < 1.$, then the fractional integral with order ν of a function f is given by*

$$ {}^{CF}I^{\nu} f(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)} u(t) + \frac{2\nu}{(2-\nu)M(\nu)} \int_{0}^{t} u(s) ds, \qquad t \ge 0. \tag{6}$$

3. Fractional Model

In this section, we expand the spread of Hookworm infection model [5] to the fractional CF derivative. Classic integer order model is reformulated in the nonlinear system of differential in equations (7):

$$\begin{cases} \frac{S(t)}{dt} = \Lambda - \mu S(t)L_2(t) - \rho S(t) + \beta R(t) \\ \frac{E(t)}{dt} = \mu S(t)L_2(t) - \rho E(t) - \alpha\gamma E(t) - (1-\alpha)\gamma E(t) \\ \frac{I_1(t)}{dt} = (1-\alpha)\gamma E(t) - (\eta + \mu + \psi_1)I_1(t) \\ \frac{I_2(t)}{dt} = \alpha\gamma E(t) + \eta I_1(t) - (\mu + \rho + \psi_2)I_2(t) \\ \frac{R(t)}{dt} = \psi_1 I_1(t) + \psi_2 I_2(t) - (\rho + \beta)R(t) \\ \frac{F(t)}{dt} = \phi I_1(t) + \phi I_2(t) - (w + \chi)F(t) \\ \frac{L_1(t)}{dt} = \chi F(t) - (\delta + \zeta)L_1(t) \\ \frac{L_2(t)}{dt} = \zeta L_1(t) - kL_2(t). \end{cases} \tag{7}$$

In the above system (3.1), $S(t), E(t), I_1(t), I_2(t), R(t), F(t), L_1(t)$ and $L_2(t)$ represent the the dynamics of hookworm and human populations, susceptible humans,exposed humans, infective humans with moderate infection, infective humans with heavy infection, recovered humans and, worm eggs, non infective rhabditiform larvae, infective filariaform larvae respectively. All the parameters are positive constants and Λ is the recruited at the rate of the population, μ is the individuals from the recovery class at the rate, η is the moderate infectious individual progresses at the rate of the population, ψ_1 is the rate of recovery from moderate infection , ψ_2 is the rate of heavy infection, the natural death rate of human and the disease induced related mortality rate are denoted by ρ and μ while w, δ and k are respective death rates for eggs.

The spread of Hookworm infection model is integrated via Caputo–Fabrizio fractional derivative with the model and can be written as follows:

$$\begin{cases} {}^{CF}_{0}\mathfrak{D}^{\nu}_{t} S(t) = \Lambda - \mu S(t)L_2(t) - \rho S(t) + \beta R(t), \\ {}^{CF}_{0}\mathfrak{D}^{\nu}_{t} E(t) = \mu S(t)L_2(t) - \rho E(t) - \alpha\gamma E(t) - (1-\alpha)\gamma E(t), \\ {}^{CF}_{0}\mathfrak{D}^{\nu}_{t} I_1(t) = (1-\alpha)\gamma E(t) - (\eta + \mu + \psi_1)I_1(t), \\ {}^{CF}_{0}\mathfrak{D}^{\nu}_{t} I_2(t) = \alpha\gamma E(t) + \eta I_1(t) - (\mu + \rho + \psi_2)I_2(t), \\ {}^{CF}_{0}\mathfrak{D}^{\nu}_{t} R(t) = \psi_1 I_1(t) + \psi_2 I_2(t) - (\rho + \beta)R(t), \\ {}^{CF}_{0}\mathfrak{D}^{\nu}_{t} F(t) = \phi I_1(t) + \phi I_2(t) - (w + \chi)F(t), \\ {}^{CF}_{0}\mathfrak{D}^{\nu}_{t} L_1(t) = \chi F(t) - (\delta + \zeta)L_1(t), \\ {}^{CF}_{0}\mathfrak{D}^{\nu}_{t} L_2(t) = \zeta L_1(t) - kL_2(t). \end{cases} \tag{8}$$

where $\nu \in (0, 1)$ is the order of the fractional derivative operator. Then the initial values are as follows:

$$\begin{cases} S_{(0)}(t) = S(0), \quad E_{(0)}(t) = E(0), \quad I_{1_{(0)}}(t) = I_1(0), \\ I_{2_{(0)}}(t) = I_2(0), \quad R_{(0)}(t) = R(0), F_{(0)}(t) = F(0), \\ L_{1_{(0)}}(t) = L_1(0), \quad L_{2_{(0)}}(t) = L_2(0). \end{cases}$$

4. Existence and Uniqueness of Hookworm infection Model

Utilizing fixed point theorem, we show the existence of the model under investigation in this section. We utilize the CF integral operator on (9) in order to get

$$\begin{cases} S(t)-S(0)={}_0^{CF}I_t^{\nu}\{\Lambda-\mu S(t)L_2(t)-\rho S(t)+\beta R(t)\},\\ E(t)-E(0)={}_0^{CF}I_t^{\nu}\{\mu S(t)L_2(t)-\rho E(t)-\alpha\gamma E(t)-(1-\alpha)\gamma E(t)\},\\ I_1(t)-I_1(0)={}_0^{CF}I_t^{\nu}\{(1-\alpha)\gamma E(t)-(\eta+\mu+\psi_1)I_1(t)\},\\ I_2(t)-I_2(0)={}_0^{CF}I_t^{\nu}\{\alpha\gamma E(t)+\eta I_1(t)-(\mu+\rho+\psi_2)I_2(t)\},\\ R(t)-R(0)={}_0^{CF}I_t^{\nu}\{\psi_1 I_1(t)+\psi_2 I_2(t)-(\rho+\beta)R(t)\}\\ F(t)-F(0)={}_0^{CF}I_t^{\nu}\{\phi I_1(t)+\phi I_2(t)-(w+\chi)F(t)\},\\ L_1(t)-L_1(0)={}_0^{CF}I_t^{\nu}\{\chi F(t)-(\delta+\zeta)L_1(t)\},\\ L_2(t)-L_2(0)={}_0^{CF}I_t^{\nu}\{\zeta L_1(t)-kL_2(t)\}. \end{cases} \tag{9}$$

By using the approach in [6], we have

$$\begin{cases} S(t)-S(0)=\frac{2(1-\nu)}{(2-\nu)M(\nu)}\{\Lambda-\mu S(t)L_2(t)-\rho S(t)+\beta R(t)\}\\ \qquad +\frac{2\nu}{(2-\nu)M(\nu)}\int_0^t\{\Lambda-\mu S(r)L_2(r)-\rho S(r)+\beta R(r)\}dr,\\ E(t)-E(0)=\frac{2(1-\nu)}{(2-\nu)M(\nu)}\{\mu S(t)L_2(t)-\rho E(t)-\alpha\gamma E(t)-(1-\alpha)\gamma E(t)\}\\ \qquad +\frac{2\nu}{(2-\nu)M(\nu)}\int_0^t\{\mu S(r)L_2(r)-\rho E(r)-\alpha\gamma E(r)-(1-\alpha)\gamma E(r)\}dr,\\ I_1(t)-I_1(0)=\frac{2(1-\nu)}{(2-\nu)M(\nu)}\{(1-\alpha)\gamma E(t)-(\eta+\mu+\psi_1)I_1(t)\}\\ \qquad +\frac{2\nu}{(2-\nu)M(\nu)}\int_0^t\{(1-\alpha)\gamma E(r)-(\eta+\mu+\psi_1)I_1(r)\}dr,\\ I_2(t)-I_2(0)=\frac{2(1-\nu)}{(2-\nu)M(\nu)}\{\alpha\gamma E(t)+\eta I_1(t)-(\mu+\rho+\psi_2)I_2(t)\}\\ \qquad +\frac{2\nu}{(2-\nu)M(\nu)}\int_0^t\{\alpha\gamma E(r)+\eta I_1(r)-(\mu+\rho+\psi_2)I_2(r)\}dr,\\ R(t)-R(0)=\frac{2(1+\nu)}{(2-\nu)M(\nu)}\{\psi_1 I_1(t)+\psi_2 I_2(t)-(\rho+\beta)R(t)\}\\ \qquad +\frac{2\nu}{(2-\nu)M(\nu)}\int_0^t\{\psi_1 I_1(r)+\psi_2 I_2(r)-(\rho+\beta)R(r)\}dr,\\ F(t)-F(0)=\frac{2(1-\nu)}{(2-\nu)M(\nu)}\{\phi I_1(t)+\phi I_2(t)-(w+\chi)F(t)\}\\ \qquad +\frac{2\nu}{(2-\nu)M(\nu)}\int_0^t\{\phi I_1(r)+\phi I_2(r)-(w+\chi)F(r)\}dr,\\ L_1(t)-L_1(0)=\frac{2(1-\nu)}{(2-\nu)M(\nu)}\{\chi F(t)-(\delta+\zeta)L_1(t)\}\\ \qquad +\frac{2\nu}{(2-\nu)M(\nu)}\int_0^t\{\chi F(r)-(\delta+\zeta)L_1(r)\}dr,\\ L_2(t)-L_2(0)=\frac{2(1-\nu)}{(2-\nu)M(\nu)}\{\zeta L_1(t)-kL_2(t)\}\\ \qquad +\frac{2\nu}{(2-\nu)M(\nu)}\int_0^t\{\zeta L_1(r)-kL_2(r)\}dr. \end{cases} \tag{10}$$

For simplicity, we replace as follows:

$$\begin{cases} G_1(t,S)=\Lambda-\mu S(t)L_2(t)-\rho S(t)+\beta R(t),\\ G_2(t,E)=\mu S(t)L_2(t)-\rho E(t)-\alpha\gamma E(t)-(1-\alpha)\gamma E(t),\\ G_3(t,I_1)=(1-\alpha)\gamma E(t)-(\eta+\mu+\psi_1)I_1(t),\\ G_4(t,I_2)=\alpha\gamma E(t)+\eta I_1(t)-(\mu+\rho+\psi_2)I_2(t),\\ G_5(t,R)T=\psi_1 I_1(t)+\psi_2 I_2(t)-(\rho+\beta)R(t),\\ G_6(t,F)=\phi I_1(t)+\phi I_2(t)-(w+\chi)F(t),\\ G_7(t,L_1)T=\chi F(t)-(\delta+\zeta)L_1(t),\\ G_8(t,L_2)=\zeta L_1(t)-kL_2(t). \end{cases}$$

For proving our results, we assume the following assumption (H). For the following continuous functions $S(t), E(t), I_1(t), I_2(t), R(t), F(t), L_1(t), L_2(t) \in L[0,1]$, such that $\|S(t)\| \le c_1, \|E(t)\| \le c_2, \|I_1(t)\| \le c_3, \|I_2(t)\| \le c_4, \|R(t)\| \le c_5, \|F(t)\| \le c_6, \|L_1(t)\| \le c_7, \|L_2(t)\| \le c_8$.

Theorem 4.1. *The kernels $G_1, G_2, G_3, G_4, G_5, G_6, G_7$ and G_8 satisfy the Lipschitz condition if the assumption H is true and they are contractions provied that $\Phi_i < 1$ for $\forall \in i = 1, \dots, 8$.*

Proof. We start with G_1. Suppose that S and S_1 are two functions, then we obtain,

$$\begin{aligned}\|G_1(t,S) - G_1(t,S_1)\| &= (\Lambda - \mu S(t)L_2(t) - \rho S(t) + \beta R(t)) - (\Lambda - \mu S_1(t)L_2(t) - \rho S_1(t) + \beta R(t))\|.\\ &\leq \{\mu L_2(t) + \rho\}\|(S(t) - S_1(t))\|\\ &\leq \{\mu c_8 + \rho\}\|(S(t) - S_1(t))\|\\ &\leq \Phi_1\|(S(t) - S_1(t))\|.\end{aligned}$$

Next, we prove for G_2. Suppose that E and E_1 are two functions, then we calculate in below,

$$\begin{aligned}\|G_2(t,E) - G_2(t,E_1)\| &= (\mu S(t)L_2(t) - \rho E(t) - \alpha\gamma E(t) - (1-\alpha)\gamma E(t))\\ &\quad - (\mu S(t)L_2(t) - \rho E_1(t) - \alpha\gamma E_1(t) - (1-\alpha)\gamma E_1(t))\|.\\ &\leq \{\rho + \alpha\gamma + (1-\alpha)\gamma\}\|(E(t) - E_1(t))\|\\ &\leq \{\rho + 1\}\|(E(t) - E_1(t))\|\\ &\leq \Phi_2\|(E(t) - E_1(t))\|.\end{aligned}$$

Then we show for G_3. Suppose that I_1 and I_{1_1} are two functions, then one can reach

$$\begin{aligned}\|G_3(t,I_1) - G_3(t,I_{1_1})\| &= ((1-\alpha)\gamma E(t) - (\eta + \mu + \psi_1)I_1(t))\\ &\quad - ((1-\alpha)\gamma E(t) - (\eta + \mu + \psi_1)I_{1_1}(t)))\|.\\ &\leq \{(\eta + \mu + \psi_1)\}\|I_1(t) - I_{1_1}(t))\|\\ &\leq \Phi_3\|I_1(t) - I_{1_1}(t))\|.\end{aligned}$$

Similarly, we prove for G_4. Suppose that I_2 and I_{2_1} are two functions, then

$$\begin{aligned}\|G_4(t,I_2) - G_4(t,I_{2_1})\| &= (\alpha\gamma E(t) + \eta I_1(t) - (\mu + \rho + \psi_2)I_2(t))\\ &\quad - (\alpha\gamma E(t) + \eta I_1(t) - (\mu + \rho + \psi_2)I_{2_1}(t))\|.\\ &\leq \{(\mu + \rho + \psi_2)\}\|I_2(t) - I_{2_1}(t))\|\\ &\leq \Phi_4\|I_2(t) - I_{2_1}(t))\|.\end{aligned}$$

For G_5, we suppose that R and R_1 are two functions, then we have

$$\begin{aligned}\|G_5(t,R) - G_5(t,R_1)\| &= (\psi_1 I_1(t) + \psi_2 I_2(t) - (\rho + \beta)R(t))\\ &\quad - (\psi_1 I_1(t) + \psi_2 I_2(t) - (\rho + \beta)R_1(t))\|.\\ &\leq \{(\rho + \beta)\}\|(R(t) - R_1(t))\|\\ &\leq \Phi_5\|(R(t) - R_1(t))\|.\end{aligned}$$

Now suppose that F and F_1 are two functions, then for G_6 one can readily get

$$\begin{aligned}\|G_6(t,F) - G_6(t,F_1)\| &= (\phi I_1(t) + \phi I_2(t) - (w + \chi)F(t))\\ &\quad - (\phi I_1(t) + \phi I_2(t) - (w + \chi)F_1(t))\|.\\ &\leq \{(w + \chi)\}\|(F(t) - F_1(t))\|\\ &\leq \Phi_6\|(F(t) - F_1(t))\|.\end{aligned}$$

For G_7, supposing that L_1 and L_{1_1} are two functions, we can obtain

$$
\begin{aligned}
\|G_7(t,L_1)-G_3(t,L_{1_1})\| &= (\chi F(t)-(\delta+\zeta)L_1(t)) \\
&\quad -(\chi F(t)-(\delta+\zeta)L_1(t))\|. \\
&\leq \{(\delta+\zeta)\}\|L_1(t)-L_{1_1}(t))\| \\
&\leq \Phi_7\|L_1(t)-L_{1_1}(t))\|,
\end{aligned}
$$

and for G_8, suppose that L_2 and L_{2_1} are two functions, then we reach

$$
\begin{aligned}
\|G_8(t,L_2)-G_3(t,L_{2_1})\| &= (\zeta L_1(t)-kL_2(t)) \\
&\quad -(\zeta L_1(t)-kL_2(t))\|. \\
&\leq \{(k)\}\|L_2(t)-L_{2_1}(t))\| \\
&\leq \Phi_8\|L_2(t)-L_{2_1}(t))\|.
\end{aligned}
$$

All kernels which $G_i, i=1,\ldots,8$ satisfy the conditions, so that they are contractions with $\Phi_i, i=1,\ldots,8$. Therefore, this completes the proof.

Using notations for kernels, with all the initial values zero equation (9) becomes

$$
\begin{cases}
S(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}G_1(t,S) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_1(r,S))dr, \\
E(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}G_2(t,E) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_2(r,E))dr, \\
I_1(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}G_3(t,I_1) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_3(r,I_1))dr, \\
I_2(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}G_4(t,I_2) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_4(r,I_2))dr, \\
R(t) = \frac{2(1+\nu)}{(2-\nu)M(\nu)}G_5(t,R) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_5(r,R))dr, \\
F(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}G_6(t,F) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_6(r,F))dr, \\
L_1(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}G_7(t,L_1) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_7(r,L_1))dr, \\
L_2(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}G_8(t,L_2) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_8(r,L_2))dr,
\end{cases}
$$

The following recursive formula is presented:

$$
\begin{cases}
S_n(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}G_1(t,S_{n-1}) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_1(r,S_{n-1}))dr, \\
E_n(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}G_2(t,E_{n-1}) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_2(r,E_{n-1}))dr, \\
I_{1(n)}(t) = \frac{2(1+\nu)}{(2-\nu)M(\nu)}G_3(t,I_{1(n-1)}) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_3(r,I_{1(n-1)}))dr, \\
I_{2(n)}(t) = \frac{2(1+\nu)}{(2-\nu)M(\nu)}G_4(t,I_{2(n-1)}) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_4(r,I_{2(n-1)}))dr, \\
R_n(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}G_5(t,R_{n-1}) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_5(r,R_{n-1}))dr, \\
F_n(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}G_6(t,F_{n-1}) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_6(r,F_{n-1}))dr, \\
L_{1(n)}(t) = \frac{2(1+\nu)}{(2-\nu)M(\nu)}G_7(t,L_{1(n-1)}) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_7(r,L_{1(n-1)}))dr, \\
L_{2(n)}(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}G_8(t,L_{2(n-1)}) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t (G_8(r,L_{2(n-1)}))dr.
\end{cases}
\tag{11}
$$

and

$$
\begin{cases}
S_{(0)}(t) = S(0), \\
E_{(0)}(t) = E(0), \\
I_{1(0)}(t) = I_1(0), \\
I_{2(0)}(t) = I_2(0), \\
R_{(0)}(t) = R(0), \\
F_{(0)}(t) = F(0), \\
L_{1(0)}(t) = L_1(0), \\
L_{2(0)}(t) = L_2(0).
\end{cases}
$$

where $S_{(0)}(t), E_{(0)}(t), M_{(0)}(t), I_{1(0)}(t), I_{2(0)}(t), R_{(0)}(t), F_{(0)}(t), L_{1(0)}(t)$ and $L_{2(0)}(t)$ are the initial conditions. The difference of the succeeding terms is obtained as

$$\begin{cases} \Psi_{1n}(t) & = S_n(t) - S_{n-1}(t) \\ & = \frac{2(1-\nu)}{(2-\nu)M(\nu)}(G_1(t, S_{n-1}) - G_1(t, S_{n-2}) \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \int_0^t (G_1(r, S_{n-1}) - G_1(r, S_{n-2})dr, \\ \Psi_{2n}(t) & = E_n(t) - E_{n-1}(t) \\ & = \frac{2(1-\nu)}{(2-\nu)M(\nu)}(G_2(t, E_{n-1}) - G_2(t, E_{n-2}) \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \int_0^t (G_2(r, E_{n-1}) - G_2(r, E_{n-2})dr, \\ \Psi_{3n}(t) & = I_{1(n)}(t) - I_{1(n-1)}(t) \\ & = \frac{2(1-\nu)}{(2-\nu)M(\nu)}(G_3(t, I_{1(n-1)}) - G_3(t, I_{1(n-2)}) \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \int_0^t (G_3(r, I_{1(n-1)}) - G_3(r, I_{1(n-2)})dr, \\ \Psi_{4n}(t) & = I_{2(n)}(t) - I_{2(n-1)}(t) \\ & = \frac{2(1-\nu)}{(2-\nu)M(\nu)}(G_4(t, I_{2(n-1)}) - G_4(t, I_{2(n-2)}) \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \int_0^t (G_4(r, I_{2(n-1)}) - G_4(r, I_{2(n-2)})dr, \\ \Psi_{5n}(t) & = R_n(t) - R_{n-1}(t) \\ & = \frac{2(1-\nu)}{(2-\nu)M(\nu)}(G_5(t, R_{n-1}) - G_5(t, R_{n-2}) \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \int_0^t (G_5(r, R_{n-1}) - G_5(r, R_{n-2})dr, \\ \Psi_{6n}(t) & = F_n(t) - F_{n-1}(t) \\ & = \frac{2(1-\nu)}{(2-\nu)M(\nu)}(G_6(t, F_{n-1}) - G_6(t, F_{n-2}) \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \int_0^t (G_6(r, F_{n-1}) - G_6(r, F_{n-2})dr, \\ \Psi_{7n}(t) & = L_{1(n)}(t) - L_{1(n-1)}(t) \\ & = \frac{2(1-\nu)}{(2-\nu)M(\nu)}(G_7(t, L_{1(n-1)}) - G_7(t, L_{1(n-2)}) \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \int_0^t (G_7(r, L_{1(n-1)}) - G_7(r, L_{1(n-2)})dr, \\ \Psi_{8n}(t) & = L_{2(n)}(t) - L_{2(n-1)}(t) \\ & = \frac{2(1-\nu)}{(2-\nu)M(\nu)}(G_8(t, L_{2(n-1)}) - G_8(t, L_{2(n-2)}) \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \int_0^t (G_8(r, L_{2(n-1)}) - G_8(r, L_{2(n-2)})dr. \end{cases}$$

Notive that

$$\begin{cases} S_n(t) = \sum_{i=1}^n \Psi_{1i}(t), \\ E_n(t) = \sum_{i=1}^n \Psi_{2i}(t), \\ I_{1(n)}(t) = \sum_{i=1}^n \Psi_{3i}(t), \\ I_{2(n)}(t) = \sum_{i=1}^n \Psi_{4i}(t), \\ R_n(t) = \sum_{i=1}^n \Psi_{5i}(t), \\ F_n(t) = \sum_{i=1}^n \Psi_{6i}(t), \\ L_{1(n)}(t) = \sum_{i=1}^n \Psi_{7i}(t), \\ L_{2(n)}(t) = \sum_{i=1}^n \Psi_{8i}(t). \end{cases}$$

Now we continue the same process and we have the following form,

$$\begin{cases} \|\Psi_{1n}(t)\| & = \|S_n(t) - S_{n-1}(t)\| \\ & = \|\frac{2(1-\nu)}{(2-\nu)M(\nu)}(G_1(t, S_{n-1}) - G_1(t, S_{n-2}) \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \int_0^t (G_1(r, S_{n-1}) - G_1(r, S_{n-2})dr\|. \end{cases}$$

Using the triangular inequality, equation (11) is simplified to

$$\begin{cases} \|S_n(t) - S_{n-1}(t)\| & \leq \frac{2(1-\nu)}{(2-\nu)M(\nu)} \|(G_1(t, S_{n-1}) - G_1(t, S_{n-2})\| \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \| \int_0^t (G_1(r, S_{n-1}) - G_1(r, S_{n-2}) dr\|. \end{cases}$$

Because of the fact that the kernel satisfyies the Lipschitz condition, then we can get

$$\begin{cases} \|S_n(t) - S_{n-1}(t)\| & \leq \frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_1 \|S_{n-1} - S_{n-2}\| \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \Phi_1 \| \int_0^t \|S_{n-1} - S_{n-2}\| dr. \end{cases} \tag{12}$$

Then we have

$$\|\Psi_{1n}(t)\| \leq \frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_1 \|\Psi_{1(n-1)}(t)\| + \frac{2\nu}{(2-\nu)M(\nu)} \Phi_1 \int_0^t \|\Psi_{1(n-1)}(r)\| dr.$$

Accordingly, we attain the results as below:

$$\begin{cases} \|\Psi_{2n}(t)\| \leq \frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_2 \|\Psi_{2(n-1)}(t)\| + \frac{2\nu}{(2-\nu)M(\nu)} \Phi_2 \int_0^t \|\Psi_{2(n-1)}(r)\| dr, \\ \|\Psi_{3n}(t)\| \leq \frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_3 \|\Psi_{3(n-1)}(t)\| + \frac{2\nu}{(2-\nu)M(\nu)} \Phi_3 \int_0^t \|\Psi_{3(n-1)}(r)\| dr, \\ \|\Psi_{4n}(t)\| \leq \frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_4 \|\Psi_{4(n-1)}(t)\| + \frac{2\nu}{(2-\nu)M(\nu)} \Phi_4 \int_0^t \|\Psi_{4(n-1)}(r)\| dr, \\ \|\Psi_{5n}(t)\| \leq \frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_5 \|\Psi_{5(n-1)}(t)\| + \frac{2\nu}{(2-\nu)M(\nu)} \Phi_5 \int_0^t \|\Psi_{5(n-1)}(r)\| dr, \\ \|\Psi_{6n}(t)\| \leq \frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_6 \|\Psi_{6(n-1)}(t)\| + \frac{2\nu}{(2-\nu)M(\nu)} \Phi_6 \int_0^t \|\Psi_{6(n-1)}(r)\| dr, \\ \|\Psi_{7n}(t)\| \leq \frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_7 \|\Psi_{7(n-1)}(t)\| + \frac{2\nu}{(2-\nu)M(\nu)} \Phi_7 \int_0^t \|\Psi_{7(n-1)}(r)\| dr, \\ \|\Psi_{8n}(t)\| \leq \frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_8 \|\Psi_{8(n-1)}(t)\| + \frac{2\nu}{(2-\nu)M(\nu)} \Phi_8 \int_0^t \|\Psi_{8(n-1)}(r)\| dr. \end{cases}$$

We shall then state the following theorem. □

Theorem 4.2. *The Hookworm infection model (9) has unique solution if the conditions below hold.*

$$\frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_1 - \frac{2\nu}{(2-\nu)M(\nu)} \Phi_1 t < 1.$$

Proof. Since all the functions $S(t), E(t), I_1(t), I_2(t), R(t), F(t), L_1(t)$ and $L_2(t)$ are bounded, we can say that the kernels satisfy the Lipschitz condition, so by using the recursive method, we get the succeeding relation as

$$\begin{cases} \|\Psi_{1n}(t)\| \leq \|S_n(0)\| \Big[\Big(\frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_1 \Big) + \Big(\frac{2\nu}{(2-\nu)M(\nu)} \Phi_1 t \Big) \Big]^n, \\ \|\Psi_{2n}(t)\| \leq \|E_n(0)\| \Big[\Big(\frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_2 \Big) + \Big(\frac{2\nu}{(2-\nu)M(\nu)} \Phi_2 t \Big) \Big]^n, \\ \|\Psi_{3n}(t)\| \leq \|I_{1(n)}(0)\| \Big[\Big(\frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_3 \Big) + \Big(\frac{2\nu}{(2-\nu)M(\nu)} \Phi_3 t \Big) \Big]^n, \\ \|\Psi_{4n}(t)\| \leq \|I_{2(n)}(0)\| \Big[\Big(\frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_4 \Big) + \Big(\frac{2\nu}{(2-\nu)M(\nu)} \Phi_4 t \Big) \Big]^n, \\ \|\Psi_{5n}(t)\| \leq \|R_n(0)\| \Big[\Big(\frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_5 \Big) + \Big(\frac{2\nu}{(2-\nu)M(\nu)} \Phi_5 t \Big) \Big]^n, \\ \|\Psi_{6n}(t)\| \leq \|F_n(0)\| \Big[\Big(\frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_6 \Big) + \Big(\frac{2\nu}{(2-\nu)M(\nu)} \Phi_6 t \Big) \Big]^n, \\ \|\Psi_{7n}(t)\| \leq \|L_{1(n)}(0)\| \Big[\Big(\frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_7 \Big) + \Big(\frac{2\nu}{(2-\nu)M(\nu)} \Phi_7 t \Big) \Big]^n, \\ \|\Psi_{8n}(t)\| \leq \|L_{2(n)}(0)\| \Big[\Big(\frac{2(1-\nu)}{(2-\nu)M(\nu)} \Phi_8 \Big) + \Big(\frac{2\nu}{(2-\nu)M(\nu)} \Phi_8 t \Big) \Big]^n. \end{cases} \tag{13}$$

Thus, the existence and continuity of the solutions is proved. Moreover, in order to ensure that the above function is a solution of equation (9), we continue as below:

$$\begin{cases} S(t) - S(0) = S_n(t) - A_n(t), \\ E(t) - E(0) = E_n(t) - B_n(t), \\ I_1(t) - I_1(0) = I_{1n}(t) - C_n(t), \\ I_2(t) - I_2(0) = T_{2n}(t) - D_n(t). \\ R(t) - R(0) = R_n(t) - G_n(t), \\ F(t) - F(0) = F_n(t) - H_n(t), \\ L_1(t) - L_1(0) = L_{1n}(t) - M_n(t), \\ L_2(t) - L_2(0) = L_{2n}(t) - N_n(t). \end{cases} \tag{14}$$

Therefore, we have

$$\begin{cases} \|A_n(t)\| & = \|\frac{2(1-\nu)}{(2-\nu)M(\nu)}(G_1(t, S_n) - G_1(t, S_{n-1}) \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \int_0^t (G_1(r, S_n) - G_1(r, S_{n-1}))dr\| \\ & \leq \frac{2(1-\nu)}{(2-\nu)M(\nu)}\|(G_1(t, S_n) - G_1(t, S_{n-1}))\| \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \int_0^t \|(G_1(r, S_n) - G_1(r, S_{n-1}))\|dr \\ & \leq \frac{2(1-\nu)}{(2-\nu)M(\nu)}\Phi_1\|K - K_{n-1}\| + \frac{2\nu}{(2-\nu)M(\nu)}\Phi_1\|S - S_{n-1}\|t. \end{cases}$$

Using the process in a recursive manner gives

$$\|A_n(t)\| \leq \left(\frac{2(1-\nu)}{(2-\nu)M(\nu)} + \frac{2\nu}{(2-\nu)M(\nu)}t\right)^{n-1} \Phi_1^{n+1} a. \tag{15}$$

By applying the limit on equation (4.8) as n tends to infinity, we get

$$\|A_n(t)\| \to 0.$$

Similarly,

$$\|B_n(t)\| \to 0, \quad \|C_n(t)\| \to 0, \quad \|D_n(t)\| \to 0,$$

$$\|G_n(t)\| \to 0, \quad \|H_n(t)\| \to 0, \|M_n(t)\| \to 0, \|N_n(t)\| \to 0$$

For the uniqueness system (9) solution, we take on contrary that there exists another solution of (9) given by $S_1(t), E_1(t), I_{11}(t), I_{12}(t), R_1(t), F_1(t), L_{11}(t)$ and $L_{12}(t)$. Then

$$\begin{cases} S(t) - S_1(t) & = \frac{2(1-\nu)}{(2-\nu)M(\nu)}(G_1(t, S_n) - G_1(t, S_{n-1}) \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \int_0^t (G_1(r, S_n) - G_1(r, S_{n-1}))dr. \end{cases} \tag{16}$$

Taking norm on equation (16), we get

$$\begin{cases} \|S(t) - S_1(t)\| & \leq \frac{2(1-\nu)}{(2-\nu)M(\nu)}\|(G_1(t, S_n) - G_1(t, S_{n-1})\| \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \int_0^t \|(G_1(r, S_n) - G_1(r, S_{n-1}))\|dr. \end{cases}$$

If we apply the Lipschitz condition of kernel, we have

$$\begin{cases} \|S(t) - S_1(t)\| & \leq \frac{2(1-\nu)}{(2-\nu)M(\nu)}\Phi_1\|S(t) - S_1(t)\| \\ & \quad + \frac{2\nu}{(2-\nu)M(\nu)} \int_0^t \Phi_1 t\|S(t) - S_1(t)\|dr. \end{cases}$$

It gives

$$\|S(t) - S_1(t)\|\left(1 - \frac{2(1-\nu)}{(2-\nu)M(\nu)}\Phi_1 - \frac{2\nu}{(2-\nu)M(\nu)}\Phi_1 t\right) \leq 0. \tag{17}$$

□

Theorem 4.3. *The model (9) solution will be unique if*

$$\left(1 - \frac{2(1-\nu)}{(2-\nu)M(\nu)}\Phi_1 - \frac{2\nu}{(2-\nu)M(\nu)}\Phi_1 t\right) > 0. \tag{18}$$

Proof. If condition (18) holds, then (17) implies that

$$\|S(t) - S_1(t)\| = 0.$$

Hence, we can attain

$$S(t) = S_1(t).$$

On employing the same procedure, we get

$$\begin{cases} E(t) = E_1(t), \\ I_1(t) = I_{11}(t), \\ I_2(t) = I_{21}(t). \\ R(t) = R_1(t), \\ F(t) = F_1(t), \\ L_1(t) = L_{11}(t), \\ L_2(t) = L_{21}(t). \end{cases}$$

□

5. Conclusion

The Hookworm infection model is analyzed employing the fractional derivative and integral operator presented by Caputo and Fabrizio. First, the model revised to the fractional derivative of Caputo–Fabrizio. Then, using the fixed point theorem, existence and uniqueness solutions were performed under initial conditions.

References

[1] Ismael HF, Atas SS, Bulut H, Osman MS. Analytical solutions to the M-derivative resonant Davey–Stewartson equations. Modern Physics Letters B. 2021, 35(30), 2150455.
[2] Ismael HF, Bulut H, Baskonus HM, Gao W. Dynamical behaviors to the coupled Schrödinger-Boussinesq system with the beta derivative. AIMS Mathematics. 2021, 6(7), 7909-28.
[3] Gao W, Ismael HF, Mohammed SA, Baskonus HM, Bulut H. Complex and real optical soliton properties of the paraxial non-linear Schrödinger equation in Kerr media with M-fractional. Frontiers in Physics. 2019, 7, 197.
[4] Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. Progr Fract Differ Appl 1 (2), 73–85.
[5] Mustapha UT, Qureshi S, Yusuf A, Hincal E. Fractional modeling for the spread of Hookworm infection under Caputo operator. Chaos, Solitons & Fractals. 2020, 137, 109878.
[6] Losada J, Nieto JJ. Properties of a new fractional derivative without singular kernel. Progr. Fract. Differ. Appl. 2015, 1(2), 87-92.
[7] Caputo M. Linear models of dissipation whose Q is almost frequency independent—II. Geophysical Journal International. 1967, 13(5), 529-39.
[8] Podlubny I. Fractional differential equations, mathematics in science and engineering. San Diego: Academic Press, 1999.
[9] Bulut H, Ilhan E. Fractional vector-borne disease model with lifelong immunity under Caputo operator. Physica Scripta. 2021, 96(8), 084006.
[10] Dokuyucu, M.A. and Celik, E., Analyzing a novel coronavirus model (COVID-19) in the sense of caputo-fabrizio fractional operator. Applied and Computational Mathematics, 2021, 20(1), 49-69.
[11] Dokuyucu MA. Analysis of the Nutrient Phytoplankton Zooplankton system with Non Local and Non Singular Kernel. Turkish J. Inequalities. 2020, 4, 58-69.
[12] Dokuyucu MA. Analysis of a novel finance chaotic model via ABC fractional derivative. Numerical Methods for Partial Differential Equations. 2021,37(2),1583-90.