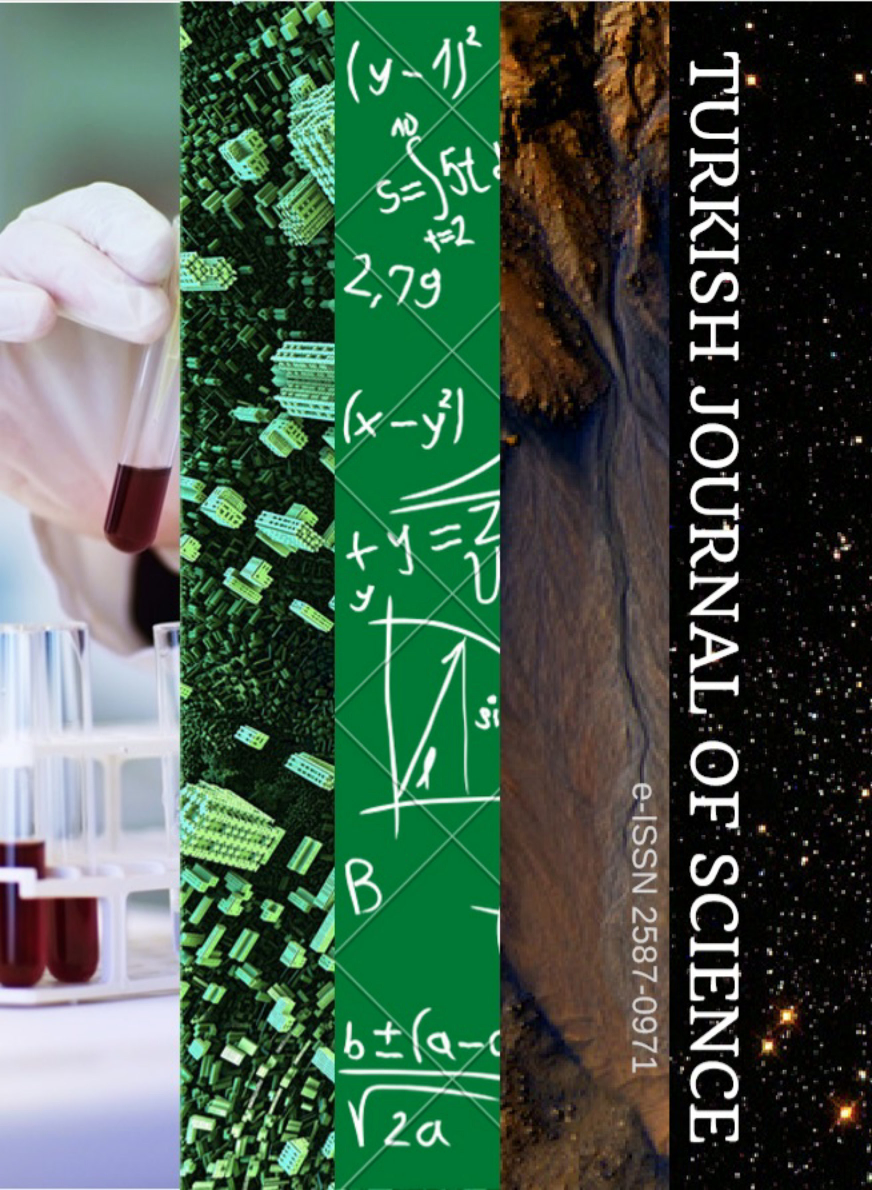
TURKISH JOURNAL OF SCIENCE
e-ISSN 2587-0971

TURKISH JOURNAL OF SCIENCE

(An International Peer-Reviewed Journal)
ISSN: 2587-0971

Volume: VII, Issue: III, 2022

Turkish Journal of Science (TJOS) is a scientific journal devoted to the principle of single-blinded peer-reviewed. TJOS focuses on the new ideas of researchers, original research articles with the potential to bring innovation to science, and innovative studies that will contribute to applied sciences, covering all branches of mathematics. The journal has determined its open access policy and publishes issues regularly in April-September and December. TJOS, whose publication language is English, attaches importance to the fact that the original research articles to be published have the potential to attract the attention of researchers and readers working in the field of mathematical sciences, and that they contain motivating ideas that will contribute to science at an intellectual level. With its distinguished and expert editors, TJOS conducts article evaluation and publication processes in line with transparent principles. As a result of these processes, quality and original articles are selected.

TJOS expects contributions from researchers for the submission of original research articles related to all branches of mathematics and some of the current topics that stand out among them.

Approximation theory, fractional calculus, applied functional analysis, linear algebra, discrete mathematics, combinatorics, control theory, financial mathematics, fuzzy sets and logic, game theory, graph theory, inverse problems, numerical analysis, ordinary and partial differential equations.

Correspondence Address
Turkish Journal of Science (TJOS)
http://dergipark.gov.tr/tjos

CONTENTS

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 3, 146–156
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

Analysis of Inverse Euler-Bernoulli Equation with periodic boundary conditions

Irem Baglan[a], Timur Canel[b]

[a] *Department of Mathematics, Kocaeli University, 41380, Kocaeli, Turkey*
[b] *Department of Physics, Kocaeli University, 41380, Kocaeli, Turkey*

Abstract. In this study, which aims to solve the inverse problem of a linear Euler-Bernoulli equation, the boundary condition has been periodically defined and integral overdetermination conditions. The conditions of the data used in the generalized Fourier method used to solve the problem have regularity and consistency.

1. Introduction

$T(t,x)$ is the displacement at time t and at position x , $o(x)$ is the bending stiffness, and $k(x) > 0$ is the linear mass on the Euler-Bernoulli problem . The behavior of an unloaded thin beam moving transversely can be described using the fourth-order partial differential equation:

$$k(x)(\partial^2 T)/(\partial t^2) + o(x)(\partial^2 T)/(\partial x^4) = 0, t > 0, 0 < x < L. \tag{1}$$

[1] studied isospectral properties and inhomogeneous variants of this equation. [2] used the Lie symmetry approach. [3] tried to solve it with Cartan's equivalence method. [4] obtained exact equivalence transformations by dealing with this problem initially [3] with some ambiguous functions. [5] investigated the transverse vibrations of a beam moving with time using the symmetry method and obtained approximate solutions for the problem.

If elastic modulus, area of inertia, mass per unit length, transverse displacement position x at time t and applied load are described as $E, l, \alpha, T(x,t)$ and f respectively, the PDE which is fourth-order can be given as below [6];

$$(EIT_{xx})_{xx} + \alpha T_{tt} = f(x,t), t > 0, 0 < x < L. \tag{2}$$

$$T_{xxxx} + T_{tt} = f(x,t), t > 0, 0 < x < L. \tag{3}$$

where E, I, α as constants [7].

Corresponding author: IB, mail address: isakinc@kocaeli.edu.tr ORCID:0000-0002-2877-9791, TC ORCID:0000-0002-4282-1806
Received: 10 October 2022; Accepted: 15 December 2022; Published: 30 December 2022
Keywords. inverse problem,euler bernoulli equation,periodic boundary condition
2010 *Mathematics Subject Classification*. 35K55, 35K70.
Cited this article as: Baglan I. and Canel T. Analysis of Inverse Euler-Bernoulli Equation with periodic boundary conditions, Turkish Journal of Science, 2022, 7(3), 146-156.

Vibration, buckling and dynamic behavior, which are frequently encountered in many fields from engineering to medicine, can be defined in a much broader way with the Euler-Bernoulli equations [8–16]. Many studies have been conducted on linear and quasi-linear equations and their applications in different fields [8–13].

Since periodic boundary conditions are encountered in many events, especially heat transfer, it has many application areas [14–16]. The existence and uniqueness of the solution of the problem are proved in section 2 using the Fourier and iteration methods. The stability of the method used to solve the problem is shown in section 3. Finally, a numerical procedure for solving the problem is presented in section 4.

Let $T > 0$ be fixed number and denote by $\Omega := \{0 < x < \pi, 0 < t < T\}$.

In case it is desired to obtain the function pairs $\{q(t), T(x,t)\}$ that will provide the equation given by Equation 4:

$$\frac{\partial^2 T}{\partial t^2} + \frac{\partial^4 T}{\partial x^4} = q(t) f(x,t),\ (x,t) \in \Omega \tag{4}$$

$$\begin{aligned} T(0,t) &= T(\pi,t) \\ T_x(0,t) &= T_x(\pi,t) \\ T_{xx}(0,t) &= T_{xx}(\pi,t) \\ T_{xxx}(0,t) &= T_{xxx}(\pi,t),\ t \in [0,T] \end{aligned} \tag{5}$$

$$T(x,0) = \varphi(x), T_t(x,0) = \psi(x),\ x \in [0,\pi] \tag{6}$$

$$H(t) = \int_0^\pi x T(x,t) dx,\ t \in [0,T] \tag{7}$$

The known functions $f(x,t)$,$\varphi(x)$, $\psi(x)$ and $H(t)$ expressed in equations (4)-/7) are known functions and are always continuous and have positive values. gets. The functions $u(x,t)$ and $r(t)$ are unknown. In the heat dissipation in a thin rod, studies have been made to obtain the total amount of heat dissipated [**?**].

Definition 1.1. *$\{q(t), T(x,t)\}$ is called the inverse problem .*

Definition 1.2. *$v(x,t) \in C(\overline{\Omega})$ is a test function and satisfies these conditions;*

$$v(x,T) = v_t(x,T) = 0, v(0,t) = v(\pi,t), v_x(0,t) = v_x(\pi,t), v_{xx}(0,t) = v_{xx}(\pi,t), v_{xxx}(0,t,) = v_{xxx}(\pi,t), t\epsilon\,[0,T].$$

Definition 1.3. *$u(x,t) \in C(\overline{\Omega})$ can be called as generalized equation. Following equation can be obtained with the generalized equation.:*

$$\int_0^T \int_0^\pi \left(\left\{ \frac{\partial^2 v}{\partial t^2} + \frac{\partial^4 v}{\partial x^4} \right\} u - r(t) f v \right) dx dt - \int_0^\pi v(x,0) \psi(x) dx + \int_0^\pi v_t(x,0) \varphi(x) dx = 0.$$

Nomenclature
$\varphi(x), \psi(x)$ Initial condition
$q(t)$ Unknown coefficient
$H(t)$ Energy
$T(x,t)$ Temperature distribution
$f(x,t)$ Source function
$T_0(t), T_{ck}(t), T_{sk}(t)$ Fourier coefficients
M_1, M_2, M_3 constants
$F(t)$ Continous function
$K(t,\tau)$ Kernel function
$\Omega := \{0 < x < \pi,\ 0 < t < T\}$ Domain of x, t

2. Solution of this problem

Let us look for solution of (1)-(4) in the form:

$$T(x,t) = \frac{u_0(t)}{2} + \sum_{k=1}^{\infty} \left(T_{ck}(t)\cos 2kx + T_{sk}(t)\sin 2kx\right)$$

The Fourier coefficients in Equation 8 can be obtained by applying the standard procedure of the Fourier method:

$$\begin{aligned} T(x,t) \;=\; & \frac{1}{2}\left[\varphi_0 + \psi_0 t + \frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)q(\tau)f(\xi,\tau)d\xi d\tau\right] \\ & + \sum_{k=1}^{\infty}\left[\varphi_{ck}\cos(2k)^2 t + \frac{\psi_{ck}}{\pi(2k)^2}\sin(2k)^2 t\right]\cos 2kx \\ & + \sum_{k=1}^{\infty}\left[\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi f(\xi,\tau,T)q(\tau)\sin(2k)^2(t-\tau)\cos 2k\xi d\xi d\tau\right]\cos 2kx \\ & + \sum_{k=1}^{\infty}\left[\varphi_{sk}\cos(2k)^2 t + \frac{\psi_{sk}}{\pi(2k)^2}\sin(2k)^2 t\right]\sin 2kx \\ & + \sum_{k=1}^{\infty}\left[\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi f(\xi,\tau,T)q(\tau)\sin(2k)^2(t-\tau)\sin 2k\xi d\xi d\tau\right]\sin 2kx \end{aligned} \tag{8}$$

Definition 2.1. *The pair* $\{q(t), T(x,t)\} \in C\left(\overline{\Omega}\right)$ *is called the classical solution of the problems (1)-(4) .*

Theorem 2.2. *Suppose that the following conditions hold:*

(A1) $H(t) \in C^2[0,T]$,
(A2) $\varphi(x) \epsilon C^3[0,\pi]$, $\psi(x)\epsilon C^1[0,\pi]$,
$\varphi(0) = \varphi(\pi), \varphi'(0) = \varphi'(\pi), \varphi''(0) = \varphi''(\pi), \psi(0) = \psi(\pi), \psi'(0) = \psi'(\pi)$,
(A3) $f(x,t)\epsilon C\left(\overline{\Omega}\right), f(0,t) = f(\pi,t), f_x(0,t) = f_x(\pi,t)$,
(A4) $\int_0^\pi x f(x,t)dx \neq 0, \forall x \epsilon [0,\pi]$
then the solution of system (1)-(4) has unique solutions.

Proof. The assumptions $\varphi(0)=\varphi(\pi)$, $\varphi^{'}(0)=\varphi^{'}(\pi), \psi(0)=\psi(\pi)$, $f(0,t)=f(\pi,t)$, are verify for the representation (7) of the solution $T(x,t)$. Further, under $\varphi(x)\epsilon C^3[0,\pi]$, $\psi(x)\epsilon C[0,\pi]$, $f(x,t)\,\epsilon C\left(\overline{\Omega}\right)$, the series (7) converge uniformly in $\overline{\Omega}$ since their majorizing sums are absolutely convergent. Under the conditions, since the majorizing sum of the t-partial derivative series are convergent, $T_t(x,t), T_{tt}(x,t)$ i is continuous in $\overline{\Omega}$. because the majorizing sum of t-partial derivative series is absolutely convergent under the conditions $\varphi(0)=\varphi(\pi)$, $\varphi^{'}(0)=\varphi^{'}(\pi), \varphi^{''}(0)=\varphi^{''}(\pi), \psi(0)=\psi(\pi), \psi^{'}(0)=\psi^{'}(\pi)$, $f(0,t)=f(\pi,t), f_x(0,t)=f_x(\pi,t)$ in $\overline{\Omega}$.

From the (5) and under the condition $(A1)$ to obtain:

$$H^{''}(t)=\int_0^\pi xT_{tt}(x,t)dx \tag{9}$$

The formulas (5)-(6) yield the following equation:

$$q(t)=\frac{H^{''}(t)-\pi\sum\limits_{k=1}^{\infty}(2k)^3\left\{\varphi_{sk}\cos(2k)^2t+\frac{\psi_{sk}}{(2k)^2}\sin(2k)^2t+\frac{1}{(2k)^2}\int\limits_0^t f_{sk}(\tau)q(\tau)\sin(2k)^2(t-\tau)d\tau\right\}}{\int\limits_0^\pi xf(x,t)dx}$$

From The second kind Volterra integral equation:

$$q(t)=F(t)+\int_0^t K(t,\tau)q(\tau)d\tau, t\epsilon\,[0,T] \tag{10}$$

$$F(t)=\frac{H^{''}(t)-\pi\sum\limits_{k=1}^{\infty}(2k)^3\,\varphi_{sk}\cos(2k)^2t-\pi\sum\limits_{k=1}^{\infty}(2k)\,\psi_{sk}\sin(2k)^2t}{\int\limits_0^\pi xf(x,t)dx}, \tag{11}$$

$$K(t,\tau)=\frac{-\pi\sum\limits_{k=1}^{\infty}(2k)\int\limits_0^t f_{sk}(\tau)q(\tau)\sin(2k)^2(t-\tau)d\tau}{\int\limits_0^\pi xf(x,t)dx}. \tag{12}$$

Let $F(t)$ and the kernel $K(t,\tau)$ are continuous functions:

$$F(t)=\frac{H^{''}(t)-\pi\sum\limits_{k=1}^{\infty}(2k)^3\left(\int\limits_0^\pi\varphi(\xi)\sin 2k\xi d\xi\right)\cos(2k)^2t-\pi\sum\limits_{k=1}^{\infty}(2k)\left(\int\limits_0^\pi\psi(\xi)\sin 2k\xi d\xi\right)\sin(2k)^2t}{\int\limits_0^\pi xf(x,t)dx},$$

we applying partial integration method for convergence ,

$\varphi_{sk}=\frac{2}{\pi}\int\limits_0^\pi\varphi(\xi)\sin 2k\xi d\xi=-\frac{1}{2k}\varphi_{ck}^{'}=\frac{1}{(2k)^2}\varphi_{sk}^{''}=\frac{-1}{(2k)^3}\varphi_{ck}^{'''}$

$\psi_{sk}=\frac{2}{\pi}\int\limits_0^\pi\psi(\xi)\sin 2k\xi d\xi=-\frac{1}{2k}\psi_{ck}^{'}$

$$F(t)=\frac{H^{''}(t)+\pi\sum\limits_{k=1}^{\infty}(2k)^3\,\frac{1}{(2k)^3}\varphi_{ck}^{'''}\cos(2k)^2t+\pi\sum\limits_{k=1}^{\infty}(2k)\,\frac{1}{2k}\psi_{ck}^{'}\sin(2k)^2t}{\int\limits_0^\pi xf(x,t)dx},$$

$$F(t) = \frac{H''(t) + \pi \sum_{k=1}^{\infty} \varphi_{ck}''' \cos(2k)^2 t + \pi \sum_{k=1}^{\infty} \psi_{ck}' \sin(2k)^2 t}{\int_0^\pi x f(x,t)dx},$$

$$|F(t)| \leq \frac{\left|H''(t)\right| + \pi \sum_{k=1}^{\infty} \left|\varphi_{ck}'''\right| + \pi \sum_{k=1}^{\infty} \left|\psi_{ck}'\right|}{\left|\int_0^\pi x f(x,t)dx\right|}$$

$$|F(t)| \leq \frac{2\left(\left|H''(t)\right| + \pi \sum_{k=1}^{\infty} \left|\varphi_{ck}'''\right| + \pi \sum_{k=1}^{\infty} \left|\psi_{ck}'\right|\right)}{M\pi^2}$$

Taking maximum both of sides

$$\|F(t)\| \leq \frac{2\left(\left\|H''(t)\right\| + \pi \sum_{k=1}^{\infty} \left\|\varphi_{ck}'''\right\| + \pi \sum_{k=1}^{\infty} \left\|\psi_{ck}'\right\|\right)}{M\pi^2}.$$

$$K(t,\tau) = \frac{-\pi \sum_{k=1}^{\infty} (2k) \int_0^t f_{sk}(\tau) q(\tau) \sin(2k)^2 (t-\tau) d\tau}{\int_0^\pi x f(x,t)dx}$$

$f_{sk} = \frac{2}{\pi} \int_0^\pi f(\xi,\tau) \sin 2k\xi d\xi = -\frac{1}{2k} \left(f_{ck}\right)_x$

$$K(t,\tau) = \frac{-\pi \sum_{k=1}^{\infty} (2k) \int_0^t \frac{2}{\pi} \int_0^\pi f(\xi,\tau) q(\tau) \sin(2k)^2 (t-\tau) \sin 2k\xi \, d\xi d\tau}{\int_0^\pi x f(x,t)dx}$$

$$K(t,\tau) = \frac{\pi \sum_{k=1}^{\infty} \int_0^t \frac{(2k)}{(2k)} \left(f_{ck}\right)_x \sin(2k)^2 (t-\tau) d\tau}{\int_0^\pi x f(x,t)dx}$$

$$|K(t,\tau)| \leq \frac{\pi \sum_{k=1}^{\infty} \left|\left(f_{ck}\right)_x\right| \left|\int_0^t \sin(2k)^2 (t-\tau) d\tau\right|}{\left|\int_0^\pi x f(x,t)dx\right|}$$

Taking maximum both of sides

$$\|K(t,\tau)\| \leq \frac{2 \sum_{k=1}^{\infty} \left\|\left(f_{ck}\right)_x\right\| \|T\|}{M\pi}$$

Under the assumption (A1)-(A2) and according to Weierstrass M test the function $F(t)$ and the kernel $K(t,\tau)$ are continuous functions The unique solution of the inverse problem (1)-(4) according to Volterra Theorem. □

3. Stability of Problem

The result in the theorem given below is valid for solving problems from equality (1) to (4).

Theorem 3.1. $\Phi = \{\varphi, \psi,\ H,\ f\}$ *satisfy the assumptions (A1)-(A4) of theorem 1 then the solution* (u, r) *of the problem (1)-(4) depends continuously upon the data* f, φ, ψ, H.

Proof. Suppose that there exist positive constants M_i, $i = 1, 2, 3$.

Let us denote $\|\Phi\| = (\|H\|_{C^1[0,T]} + \|\varphi\|_{C^1[0,\pi]} + \|\psi\|_{C[0,\pi]} + \|f\|_{C(\overline{\Omega})})$. Let (u, r) and $(\overline{u}, \overline{r})$ be solutions of inverse problems (1)-(4) corresponding to the data $\Phi = \{\varphi, \psi,\ H,\ f\}$ and $\overline{\Phi} = \left\{\overline{\varphi}, \overline{\psi}, \overline{H}, \overline{E}, \overline{f}\right\}$ respectively.

$$F(t) - \overline{F(t)} = \frac{H''(t) - \overline{H''(t)} + \pi \sum\limits_{k=1}^{\infty} \left(\varphi'''_{ck} - \overline{\varphi'''_{ck}}\right) \cos(2k)^2 t + \pi \sum\limits_{k=1}^{\infty} \left(\psi'_{ck} - \overline{\psi'_{ck}}\right) \sin(2k)^2 t}{\int\limits_0^\pi x f(x, t) dx}$$

Equation (13) can be obtained with the maximum of both sides of this equation:

$$\left\|F - \overline{F}\right\| \leq \frac{2}{\pi^2 M} \left\{ \left\|H'(t) - \overline{H'(t)}\right\| + \pi \sum_{k=1}^{\infty} \left\|\varphi'''_{ck} - \overline{\varphi'''_{ck}}\right\| + \left\|\psi'_{ck} - \overline{\psi'_{ck}}\right\| \right\}. \tag{13}$$

$$K(t, \tau) = \frac{\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t (f_{ck})_x \sin(2k)^2 (t - \tau) d\tau}{\int\limits_0^\pi x f(x, t) dx}$$

$$K - \overline{K} = \frac{\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t (f_{ck})_x \sin(2k)^2 (t - \tau) d\tau}{\int\limits_0^\pi x f(x, t) dx} - \frac{\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t \overline{(f_{ck})_x} \sin(2k)^2 (t - \tau) d\tau}{\int\limits_0^\pi x \overline{f(x, t)} dx}$$

$$K - \overline{K} = \frac{\left(\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t (f_{ck})_x \sin(2k)^2 (t - \tau) d\tau\right)\left(\int\limits_0^\pi x \overline{f(x, t)} dx\right) - \left(\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t \overline{(f_{ck})_x} \sin(2k)^2 (t - \tau) d\tau\right)\left(\int\limits_0^\pi x f(x, t) dx\right)}{\left(\int\limits_0^\pi x f(x, t) dx\right)\left(\int\limits_0^\pi x \overline{f(x, t)} dx\right)}$$

$$K-\overline{K} = \frac{\left(\pi\sum_{k=1}^{\infty}\int_0^t (f_{ck})_x \sin(2k)^2(t-\tau)d\tau\right)\left(\int_0^\pi x\overline{f(x,t)}dx\right)}{\left(\int_0^\pi xf(x,t)dx\right)\left(\int_0^\pi x\overline{f(x,t)}dx\right)} - \frac{\left(\pi\sum_{k=1}^{\infty}\int_0^t \overline{(f_{ck})_x} \sin(2k)^2(t-\tau)d\tau\right)\left(\int_0^\pi xf(x,t)dx\right)}{\left(\int_0^\pi xf(x,t)dx\right)\left(\int_0^\pi x\overline{f(x,t)}dx\right)}$$
$$+\frac{\left(\pi\sum_{k=1}^{\infty}\int_0^t \overline{(f_{ck})_x} \sin(2k)^2(t-\tau)d\tau\right)\left(\int_0^\pi x\overline{f(x,t)}dx\right)}{\left(\int_0^\pi xf(x,t)dx\right)\left(\int_0^\pi x\overline{f(x,t)}dx\right)} - \frac{\left(\pi\sum_{k=1}^{\infty}\int_0^t \overline{(f_{ck})_x} \sin(2k)^2(t-\tau)d\tau\right)\left(\int_0^\pi x\overline{f(x,t)}dx\right)}{\left(\int_0^\pi xf(x,t)dx\right)\left(\int_0^\pi x\overline{f(x,t)}dx\right)}$$

$$K-\overline{K} = \frac{\left(\pi\sum_{k=1}^{\infty}\int_0^t \left((f_{ck})_x-\overline{(f_{ck})_x}\right) \sin(2k)^2(t-\tau)d\tau\right)\left(\int_0^\pi x\overline{f(x,t)}dx\right)}{\left(\int_0^\pi xf(x,t)dx\right)\left(\int_0^\pi x\overline{f(x,t)}dx\right)}$$
$$\frac{\left(\int_0^\pi x\left(\overline{f(x,t)}-f(x,t)\right)dx\right)\left(\pi\sum_{k=1}^{\infty}\int_0^t \overline{(f_{ck})_x} \sin(2k)^2(t-\tau)d\tau\right)}{\left(\int_0^\pi xf(x,t)dx\right)\left(\int_0^\pi x\overline{f(x,t)}dx\right)}$$

$$\left|K-\overline{K}\right| \le \frac{\frac{\pi^3}{2}M\sum_{k=1}^{\infty}\left|(f_{ck})_x-\overline{(f_{ck})_x}\right|}{\frac{\pi^4}{2}M^2} + \frac{\frac{\pi^3}{2}M\left|f-\overline{f}\right|}{\frac{\pi^4}{2}M^2}$$

Equation (14) can be obtained with the maximum of both sides of this equation:

$$\left\|K-\overline{K}\right\| \le \frac{2}{\pi M}\|T\|\left\|f-\overline{f}\right\| + \frac{2}{\pi M}\|T\|\sum_{k=1}^{\infty}\left\|(f_{ck})_x-\overline{(f_{ck})_x}\right\| \tag{14}$$

Using same estimations, we obtain

$$\left\|q-\overline{q}\right\| \le \left\|F-\overline{F}\right\| + |T|\,\|K\|\left\|r-\overline{r}\right\| + \left\|\overline{r}\right\|\left\|K-\overline{K}\right\|,$$

From (10)-(11) we also obtain that

$$
\begin{aligned}
\left\|q-\overline{q}\right\| &\leq \frac{2}{\pi^2 M(1-|T|\,|K|)}\left\{\left\|H^{'}(t)-\overline{H^{'}(t)}\right\|+\pi\sum_{k=1}^{\infty}\left\|\varphi_{ck}^{'''}-\overline{\varphi_{ck}^{'''}}\right\|+\left\|\psi_{ck}^{'}-\overline{\psi_{ck}^{'}}\right\|\right\} \\
& \frac{2\left\|\overline{r}\right\|\|T\|}{\pi M(1-|T|\,|K|)}\left\|f-\overline{f}\right\|+\frac{2\left\|\overline{r}\right\|\|T\|^2}{\pi M(1-|T|\,|K|)}\sum_{k=1}^{\infty}\left\|(f_{ck})_x-\overline{(f_{ck})_x}\right\|
\end{aligned}
$$

We obtain the difference u and $\overline{u}$ from (5):

$$
\begin{aligned}
T-\overline{T} &= \frac{1}{2}\left[(\varphi_0-\overline{\varphi_0})+\left(\psi_0-\overline{\psi_0}\right)t+\int_0^t (t-\tau)q(\tau)\left(f_0-\overline{f_0}\right)d\tau\right] \\
&+\sum_{k=1}^{\infty}\left[(\varphi_{ck}-\overline{\varphi_{ck}})\cos(2k)^2t+\frac{\left(\psi_{ck}-\overline{\psi_{ck}}\right)}{(2k)^2}\sin(2k)^2t\right]\cos 2kx \\
&+\sum_{k=1}^{\infty}\left[\frac{1}{(2k)^2}\int_0^t\left(f_{ck}-\overline{f_{ck}}\right)q(\tau)\sin(2k)^2(t-\tau)d\tau\right]\cos 2kx \\
&+\sum_{k=1}^{\infty}\left[(\varphi_{sk}-\overline{\varphi_{sk}})\cos(2k)^2t+\frac{\left(\psi_{sk}-\overline{\psi_{sk}}\right)}{(2k)^2}\sin(2k)^2t\right]\sin 2kx \\
&+\sum_{k=1}^{\infty}\left[\frac{1}{(2k)^2}\int_0^t\left(f_{sk}-\overline{f_{sk}}\right)q(\tau)\sin(2k)^2(t-\tau)d\tau\right]\sin 2kx
\end{aligned}
\tag{15}
$$

Taking maximum both of sides

$$
\begin{aligned}
\left\|T-\overline{T}\right\| &\leq \frac{1}{2}\left\|\varphi_0-\overline{\varphi_0}\right\|+\frac{1}{2}|T|\left\|\psi_0-\overline{\psi_0}\right\|+\frac{1}{2}|T|\left\|f_0-\overline{f_0}\right\| \\
&+\sum_{k=1}^{\infty}\left(\left\|\varphi_{ck}-\overline{\varphi_{ck}}\right\|+\left\|\varphi_{sk}-\overline{\varphi_{sk}}\right\|\right) \\
&+\sum_{k=1}^{\infty}\frac{1}{(2k)^2}\left(\left\|\psi_{ck}-\overline{\psi_{ck}}\right\|+\left\|\psi_{sk}-\overline{\psi_{sk}}\right\|\right) \\
&+\sum_{k=1}^{\infty}|T|\left\|f_{ck}-\overline{f_{ck}}\right\|\left\|q\right\|+\sum_{k=1}^{\infty}|T|\left\|f_{sk}-\overline{f_{sk}}\right\|\|r\| \\
&+\sum_{k=1}^{\infty}|T|\left\|\overline{f_{ck}}\right\|\left\|q-\overline{q}\right\|+\sum_{k=1}^{\infty}|T|\left\|\overline{f_{sk}}\right\|\left\|q-\overline{q}\right\|
\end{aligned}
$$

Applying Hölder inequality,

$$\begin{aligned}
\left\|u-\overline{u}\right\| &\leq \frac{1}{2}\left\|\varphi_0-\overline{\varphi_0}\right\|+\frac{1}{2}\left|T\right|\left\|\psi_0-\overline{\psi_0}\right\|+\frac{1}{2}\left|T\right|\left\|f_0-\overline{f_0}\right\| \\
&+\sum_{k=1}^{\infty}\left(\left\|\varphi_{ck}-\overline{\varphi_{ck}}\right\|+\left\|\varphi_{sk}-\overline{\varphi_{sk}}\right\|\right) \\
&+\frac{1}{4}\left(\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\left[\left\|\psi_{ck}-\overline{\psi_{ck}}\right\|+\left\|\psi_{sk}-\overline{\psi_{sk}}\right\|\right]^2\right)^{\frac{1}{2}} \\
&+\frac{1}{4}\left(\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^2\right)^{\frac{1}{2}}\sum_{k=1}^{\infty}\left(\left|T\right|\left\|f_{ck}-\overline{f_{ck}}\right\|\left\|q\right\|\right)^{\frac{1}{2}} \\
&+\frac{1}{4}\left(\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^2\right)^{\frac{1}{2}}\sum_{k=1}^{\infty}\left(\left|T\right|\left\|f_{sk}-\overline{f_{sk}}\right\|\left\|q\right\|\right)^{\frac{1}{2}} \\
&+\frac{1}{4}\left(\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^2\right)^{\frac{1}{2}}\sum_{k=1}^{\infty}\left(\left|T\right|\left\|\overline{f_{ck}}\right\|\left\|q-\overline{q}\right\|\right)^{\frac{1}{2}} \\
&+\frac{1}{4}\left(\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^2\right)^{\frac{1}{2}}\sum_{k=1}^{\infty}\left(\left|T\right|\left\|\overline{f_{sk}}\right\|\left\|q-\overline{q}\right\|\right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
\left\|T-\overline{T}\right\| &\leq \frac{1}{2}\left\|\varphi_0-\overline{\varphi_0}\right\|+\frac{1}{2}\left|T\right|\left\|\psi_0-\overline{\psi_0}\right\|+\frac{1}{2}\left|T\right|\left\|f_0-\overline{f_0}\right\| \\
&+\sum_{k=1}^{\infty}\left(\left\|\varphi_{ck}-\overline{\varphi_{ck}}\right\|+\left\|\varphi_{sk}-\overline{\varphi_{sk}}\right\|\right) \\
&+\frac{\pi^2}{24}\sum_{k=1}^{\infty}\left\|\psi_{ck}-\overline{\psi_{ck}}\right\|+\left\|\psi_{sk}-\overline{\psi_{sk}}\right\| \\
&+\frac{\pi^2}{24}\sum_{k=1}^{\infty}\left|T\right|\left\|f_{ck}-\overline{f_{ck}}\right\|\left\|q\right\| \\
&+\frac{\pi^2}{24}\sum_{k=1}^{\infty}\left|T\right|\left\|f_{sk}-\overline{f_{sk}}\right\|\left\|q\right\| \\
&+\frac{\pi^2}{24}\sum_{k=1}^{\infty}\left|T\right|\left\|\overline{f_{ck}}\right\|\left\|q-\overline{q}\right\| \\
&+\frac{\pi^2}{24}\sum_{k=1}^{\infty}\left|T\right|\left\|\overline{f_{sk}}\right\|\left\|q-\overline{q}\right\|
\end{aligned}$$

$$\left\|T-\overline{T}\right\|\leq M_1\left\|\varphi-\overline{\varphi}\right\|+M_2\left\|\psi-\overline{\psi}\right\|+M_3\left\|f-\overline{f}\right\|+M_4\left\|H''-\overline{H''}\right\|$$

where

$$\begin{aligned} M_1 &= \max\left\{\frac{1}{2}, 1, \frac{|T|\,\pi}{6(1-|T|\,|K|)}\right\}, \\ M_2 &= \max\left\{\frac{\pi^2}{24}, \frac{|T|\,\pi}{6(1-|T|\,|K|)}\right\}, \\ M_3 &= \max\left\{\frac{|T|}{2}, |T|\left\|q\right\|, \frac{|T|\,\pi\left\|\overline{q}\right\|}{6(1-|T|\,|K|)}, \frac{|T|^3\,\pi\left\|\overline{q}\right\|}{6(1-|T|\,|K|)}\right\}, \\ M_4 &= \max\left\{\frac{|T|}{6(1-|T|\,|K|)}\right\} \end{aligned}$$

we also obtain that

$$\left\|T-\overline{T}\right\| \le M_5\left\|\Phi-\overline{\Phi}\right\|,$$

where

$$M_5 = \max\{M_1, M_2, M_3, M_4\}.$$

For $\Phi \to \overline{\Phi}$ then $u \to \overline{u}$. □

4. Numerical Method

We use finite-difference approximation for discretizing problem (1)-(3):

$$\frac{1}{\tau^2}\left(T_i^{j+1}-2T_i^j+T_i^{j-1}\right)+\frac{1}{h^4}\left(T_{i+2}^j-4T_{i+1}^j+6T_i^j-4T_{i-1}^j+T_{i-2}^j\right)=q^j\widetilde{f}_i^j$$

$$T_i^0=\phi_i, \frac{1}{\tau}\left(T_i^1-T_i^0\right)=\psi_i \tag{16}$$

$$T_0^j=T_{N_x+1}^j, \tag{17}$$

$$T_1^j=T_{N_x+2}^j, \tag{18}$$

$$T_{-1}^j=T_{N_x}^j, \tag{19}$$

$$T_2^j-T_{-2}^j=T_{N_x+3}^j-T_{N_x-1}^j, \tag{20}$$

The domain $[0,\pi]\times[0,T]$ is divided into an $N_x \times N_t$ mesh with the spatial step size $h=\pi/N_x$ in x direction and the time step size $\tau=T/N_t$, respectively.

x_i, t_j are defined by
$x_i = ih;\ i=0;1;2;...;N_x;$
$t_j = j\tau;\ j=0;1;2;...;N_t;$
$T_i^j = T(x_i,t_j),\ \widetilde{f}_i^j=\widetilde{f}(x_i,t_j),\ q^j=q(t_j).$

Let us integrate the equation (1) respect to x from 0 to π , we obtain

$$q(t)=\frac{H''(t)}{\int_0^\pi \widetilde{f}(x,t)dx}. \tag{21}$$

The finite difference approximation of (18) is

$$q^j = \frac{\left(\left(H^{j+1} - 2H^j + H^{j-1}\right)/\tau^2\right)}{\left(\int_0^\pi \widetilde{f_i^j} dx\right)}.$$

where $H^j = H(t_j)$, $q^j = q(t_j)$, $j = 0, 1, ..., N_t$. We mention that the integral is numerically calculated using Simpson's rule of integration.The system of equations (13)-(17) is solved and u_i^j, q^j
is determined.The condition for stopping the iteration depends on the value difference between the two iterations. Iteration should be stopped when this difference is equal to the tolerance predicted previously.

References

[1] H. P. W. Gottlieb, "Isospectral Euler-Bernoulli beams with continuous density and rigidity functions," *Proceedings of the Royal Society of London Series A: Mathematical, Physical and Engineering Sciences*, vol. 413, no. 1844, pp. 235–250, 1987.

[2] C. W. Soh, "Euler-Bernoulli beams from a symmetry standpoint—characterization of equivalent equations," *JournalofMathematicalAnalysisandApplications*, vol. 345, no. 1, pp. 387–395, 2008.

[3] O. I. Morozov and C. W. Soh, "The equivalence problem for the Euler-Bernoulli beam equation via Cartan's method," *Journal of Physics A: Mathematical and Theoretical*, vol. 41, no. 13, 135206, pp. 135–206, 2008.

[4] J. C. Ndogmo, "Equivalence transformations of the Euler-Bernoulli equation," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 5, pp. 2172–2177, 2012.

[5] E. Ozkaya and M. Pakdemirli, "Group-theoretic approach to ¨ axially accelerating beam problem," *Acta Mechanica*, vol. 155, no. 1-2, pp. 111–123, 2002.

[6] A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, Dover Publications, New York, NY, USA, 4th edition,1944.

[7] A. H. Bokhari, F. M. Mahomed, and F. D. Zaman, "Invariantboundary value problems for a fourth-order dynamic EulerBernoulli beam equation," *Journal of Mathematical Physics*, vol.53, no. 4, 2012.

[8] He X.Q., Kitipornchai S., Liew K.M.,"Buckling analysis of multi-walled carbon nanotubes a continuum model accounting for van der Waals interaction", *Journal of the Mechanics and Physics of Solids*, 53, 303-326, 2005.

[9] Natsuki T., Ni Q.Q., Endo M.,"Wave propagation in single-and double-walled carbon nano tubes filled with fluids", *Journal of Applied Physics*, 101, 034319, 2007.

[10] Yana Y., Heb X.Q., Zhanga L.X., Wang C.M., "Dynamic behavior of triple-walled carbon nano-tubes conveying fluid", *Journal of Sound and Vibration* ,319, 1003-1018, 2010.

[11] T.S. Jang, "A new solution procedure for a nonlinear infinite beam equation of motion", *Commun. Nonlinear Sci. Numer. Simul.*, 39 , 321–331,2016.

[12] T.S.Jang, " A general method for analyzing moderately large deflections of a non-uniform beam: an infinite Bernoulli–Euler–von Karman beam on a non-linear elastic foundation", *Acta Mech* , 225 , pp. 1967-1984,2014.

[13] Mohebbi A. & Abbasi M. , "A fourth-order compact difference scheme for the parabolic inverse problem with an overspecification at a point", *Inverse Problems in Science and Engineering*, 23:3, 457-478, DOI:10.1080/17415977.2014.922075,2014.

[14] Pourgholia, R, Rostamiana, M. and Emamjome, M., "A numerical method for solving a nonlinear inverse parabolic problem", *Inverse Problems in Science and Engineering*, 18;8 ,1151-1164,2010.

[15] I. Baglan and F.Kanca, "An inverse coefficient problem for a quasilinear parabolic equation with periodic boundary and integral overdetermination condition",*Math. Meth. Appl. Sci.* , DOI: 10.1002/mma.3112,2015.

[16] Hill, G.W., On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon, Acta Mathematica, 8, 1-36,1986.

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 3, 157–167
ISSN: 2587–0971

http:/dergipark.gov.tr/tjos

Bi-Periodic Generalized Fibonacci Polynomials

Yasemin Taşyurdu[a]

[a] *Erzincan Binali Yıldırım University, Faculty of Arts and Sciences, Department of Mathematics, Erzincan, Turkey*

Abstract. In this paper, we define bi-periodic generalized Fibonacci polynomials, which generalize Fibonacci, Pell, Jacobsthal, Fermat, Chebyshev polynomials and the other well-known polynomials. We obtain generating functions, Binet formulas and some properties of these polynomials. Also, we prove some fundamental identities conform to the known results of Fibonacci polynomials.

1. Introduction

Polynomials in many fields of mathematics and science are emerged as the generalizations of numbers. Fibonacci polynomials, one of the special polynomials in the literature, are a generalization of well-known Fibonacci numbers defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ with initial terms $f_0 = 0$, $f_1 = 1$ [1]. The nth Fibonacci polynomial $f_n(x)$, is defined by the recurrence relation

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x), \quad n \geq 2$$

with initial terms $f_0(x) = 0$, $f_1(x) = 1$, and terms of the sequence $\left\{0, 1, x, x^2+1, x^3+2x, x^4+3x^2+1, \ldots\right\}$ are Fibonacci polynomials. Many polynomials related to numbers defined by the recurrence relations have been presented in different ways as generalizations of the Fibonacci polynomials called generalized Fibonacci and generalized Fibonaci type polynomials. One of the ways of generalization is to add integers or variables to the recurrence relation of the Fibonacci polynomials. For instance, Pell polynomials are defined by the recurrence relation $p_n(x) = 2xp_{n-1}(x) + p_{n-2}(x)$ with initial terms $p_0(x) = 0$, $p_1(x) = 1$ for $n \geq 2$. Then Jacobsthal polynomials are defined by the recurrence relation $j_n(x) = j_{n-1}(x) + 2xj_{n-2}(x)$ with initial terms $J_0(x) = 0$, $J_1(x) = 1$ for $n \geq 2$ [2, 3]. For the parameter variables x and y in the recurrence relation, bivariate Fibonacci polynomials are introduced by the recurrence relation

$$f_n(x, y) = xf_{n-1}(x, y) + yf_{n-2}(x, y), \quad f_n(x, y) = 0,\ f_1(x, y) = 1, \quad n \geq 2$$

where $x, y \neq 0$, $x^2 + 4y \neq 0$ and generalized identities of these polynomials are obtained [4, 5]. Then, $h(x)$-Fibonacci polynomials as another generalization of Fibonacci polynomials are defined by the recurrence relation

$$f_{h,n}(x) = h(x) f_{h,n-1}(x) + f_{h,n-2}(x), \quad f_{h,n}(x) = 0,\ f_{h,1}(x) = 1, \quad n \geq 2$$

Corresponding author: YT mail address: ytasyurdu@erzincan.edu.tr ORCID: 0000-0002-9011-8269
Received: 4 October 2022; Accepted: 25 December 2022; Published: 30 December 2022
Keywords. Bi-periodic Fibonacci polynomials; Fibonacci polynomial, generalized Fibonacci polynomials
2010 *Mathematics Subject Classification*. 11B39
Cited this article as: Taşyurdu Y. Bi-Periodic Generalized Fibonacci Polynomials, Turkish Journal of Science, 2022, 7(3), 157-167.

where $h(x)$ be a polynomial with real coefficients [6]. Further generalizations of Fibonacci polynomials have been presented by many authors as Fermat, Chebyshev, Morgan-Voyce, Vieta polynomials. The generating functions, exponential generating functions, the Binet-like formulas, sums formulas, matrix representations and periods according to the m modulo of Fibonacci polynomial sequences are presented [7–10].

Motivated by of the above-cited studies, it is introduced a new generalization of the Fibonacci numbers and polynomials called generalized Fibonacci polynomials. For $n \geq 2$, the generalized Fibonacci polynomial sequences, $\{\mathcal{F}_n(x)\}_{n\geq 0}$ are defined by the recurrence relation

$$\mathcal{F}_n(x) = d(x)\mathcal{F}_{n-1}(x) + g(x)\mathcal{F}_{n-2}(x) \tag{1}$$

with initial terms $\mathcal{F}_0(x) = 0$ and $\mathcal{F}_1(x) = 1$ where $d(x)$ and $g(x)$ are fixed nonzero polynomials in $\mathbb{Q}[x]$ [11]. Obviously, for $d(x) = x$ and $g(x) = 1$ we obtain classical Fibonacci polynomial and $\mathcal{F}_n(1) = f_n$ where f_n is the nth classical Fibonacci number. Binet formulas for the generalized Fibonacci polynomial sequences are given by

$$\mathcal{F}_n(x) = \frac{\sigma^n(x) - \rho^n(x)}{\sigma(x) - \rho(x)}$$

where $\sigma(x)$ and $\rho(x)$ are the roots of the quadratic equation $t^2 - d(x)t - g(x) = 0$ of equation (1). The readers can find more detailed information about the generalized Fibonacci polynomial in [12, 13].

In other generalizations of Fibonacci numbers and polynomials, nonzero real numbers are taken into account, bi-periodic Fibonacci number sequences, $\{q_n\}$ are defined by

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases} \qquad n \geq 2$$

with initial terms $q_0 = 0$, $q_1 = 1$ [14] and bi-periodic Fibonacci polynomial sequences, $\{q_n(x)\}$ are defined by

$$q_n(x) = \begin{cases} aq_{n-1}(x) + q_{n-2}(x), & \text{if } n \text{ is even} \\ bq_{n-1}(x) + q_{n-2}(x), & \text{if } n \text{ is odd} \end{cases} \qquad n \geq 2$$

with initial terms $q_0(x) = 0$, $q_1(x) = 1$ where a and b are any two nonzero real numbers. Also, some identities related to these bi-periodic sequences are given, respectively [15].

The aim of this study is to define new generalizations of the Fibonaci and the Fibonacci type polynomials, the bi-periodic Fibonacci and the bi-periodic Fibonacci type polynomials, which we shall call bi-periodic generalized Fibonacci polynomials. It is to present generating functions, general formulas and well-known identities for these polynomials. It is also to give special cases of the bi-periodic generalized Fibonacci polynomials and generalize all the results.

2. Bi-Periodic Generalized Fibonacci Polynomials

In this section we define a new kind of generalized Fibonacci polynomials, called bi-periodic generalized Fibonacci polynomials, which are Fibonacci polynomials, $h(x)$-Fibonacci polynomials, Fibonacci polynomials with two variables, Pell polynomials, Jacobsthal polynomials, Fermat polynomials, Chebyshev second kind polynomials, Morgan-Voyce first kind polynomials and Vieta polynomials. Generating functions, Binet formulas, some basic properties as well as the Catalan's identity, Cassini's identity, d'Ocagne's identity for these polynomials are obtained.

Definition 2.1. *For any two nonzero real numbers a and b, the nth bi-periodic generalized Fibonacci polynomial is defined by the recurrence relation*

$$\mathbb{F}_n(x) = \begin{cases} ad(x)\,\mathbb{F}_{n-1}(x) + g(x)\,\mathbb{F}_{n-2}(x), & \text{if } n \text{ is even} \\ bd(x)\,\mathbb{F}_{n-1}(x) + g(x)\,\mathbb{F}_{n-2}(x), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2 \tag{2}$$

with initial terms $\mathbb{F}_0(x) = 0$, $\mathbb{F}_1(x) = 1$ *for* $n \geq 2$, *where* $d(x)$ *and* $g(x)$ *are fixed nonzero polynomials in* $\mathbb{Q}[x]$. *The bi-periodic generalized Fibonacci polynomial sequences are denoted by* $\{\mathbb{F}_n(x)\}_{n\in\mathbb{N}}$.

The bi-periodic generalized Fibonacci polynomial sequences are as follows

$$\{\mathbb{F}_n(x)\}_{n\in\mathbb{N}} = \big\{0, 1, ad(x), abd^2(x) + g(x), a^2bd^3(x) + 2ad(x)g(x),\ a^2b^2d^4(x) + 3abd^2(x)g(x) + g^2(x),$$
$$a^3b^2d^5(x) + 4a^2bd^3(x)g(x) + 3ad(x)g^2(x), a^3b^3d^6(x) + 5a^2b^2d^4(x)g(x) + 6abd^2(x)g^2(x) + g^3(x), \ldots\}$$

Note that $d(x) = x$ and $g(x) = 1$, we get the bi-periodic Fibonacci polynomial $\mathbb{F}_n(x) = F_n(x)$. Similar special cases of the bi-periodic generalized Fibonacci polynomials are given in the Table 1

Table 1: Special cases of the polynomials $\mathbb{F}_n(x)$

Bi-Periodic Generalized Fibonacci Polynomials	$\mathbb{F}_n$	$d(x)$	$g(x)$
Bi-periodic Fibonacci polynomials	$F_n(x)$	x	1
Bi-periodic $h(x)$-Fibonacci polynomials	$F_{h,n}(x)$	$h(x)$	1
Bi-periodic Fibonacci polynomials with two variables	$F_n(x,y)$	x	y
Bi-periodic Pell polynomials	$P_n(x)$	$2x$	1
Bi-periodic Jacobsthal polynomials	$J_n(x)$	1	$2x$
Bi-periodic Fermat polynomials	$\Phi_n(x)$	$3x$	-2
Bi-periodic Chebyshev second kind polynomials	$U_n(x)$	$2x$	-1
Bi-periodic Morgan-Voyce first kind polynomials	$B_n(x)$	$x+2$	-1
Bi-periodic Vieta polynomials	$V_n(x)$	x	-1

Since the all results given throughout the study are provided for all the bi-periodic generalized Fibonacci polynomials, the values given in Table 1 can be used in the relevant theorem or corollary for any bi-periodic polynomials.

From Definition 2.1, alternative recurrence relations can be given for the bi-periodic generalized Fibonacci polynomials where $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function, i.e.,

$$\xi(n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}.$$

Let a and b be any two nonzero real numbers, nth bi-periodic generalized Fibonacci polynomial is given by

$$\mathbb{F}_n(x) = a^{1-\xi(n)}b^{\xi(n)}d(x)\,\mathbb{F}_{n-1}(x) + g(x)\,\mathbb{F}_{n-2}(x), \quad n \geq 2 \tag{3}$$

with initial terms $\mathbb{F}_0(x) = 0$, $\mathbb{F}_1(x) = 1$ where $d(x)$ and $g(x)$ are fixed nonzero polynomials in $\mathbb{Q}[x]$.

The quadratic equation of the bi-periodic generalized Fibonacci polynomials is

$$t^2 - d(x)abt - g(x)ab = 0$$

and their roots are $\gamma(x) = \frac{d(x)ab + \sqrt{d^2(x)a^2b^2 + 4g(x)ab}}{2}$ and $\delta(x) = \frac{d(x)ab - \sqrt{d^2(x)a^2b^2 + 4g(x)ab}}{2}$. In this case, the following relations are obtained between the roots $\gamma(x)$ and $\delta(x)$

$$\gamma(x) + \delta(x) = d(x)\,ab$$

$$\gamma(x) - \delta(x) = \sqrt{d^2(x)\,a^2b^2 + 4g(x)\,ab}$$

$$\gamma(x)\,\delta(x) = -g(x)\,ab$$

$$d(x)\,\gamma(x) + g(x) = \frac{\gamma^2(x)}{ab}$$

$$d(x)\,\delta(x) + g(x) = \frac{\delta^2(x)}{ab}.$$

2.1. Generating Functions and Binet Formulas of Polynomials $\mathbb{F}_n(x)$

In this section, we construct the generating functions of the bi-periodic generalized Fibonacci polynomial the sequences, $\{\mathbb{F}_n(x)\}_{n\in\mathbb{N}}$. Let the generating functions of these sequences be $G_n(x,t)$ such that

$$G_n(x,t) = \sum_{n=0}^{\infty} \mathbb{F}_n(x)\,t^n \tag{4}$$

where $\mathbb{F}_n(x)$ is the nth bi-periodic generalized Fibonacci polynomial and $d(x)$, $g(x)$ are fixed nonzero polynomials in $\mathbb{Q}[x]$. First, the identities for the odd and even subscript terms of the bi-periodic generalized Fibonacci polynomials are given in the following lemma used to derive these functions.

Lemma 2.2. *The bi-periodic generalized Fibonacci polynomial sequences,* $\{\mathbb{F}_n(x)\}_{n\in\mathbb{N}}$ *satisfy the following identities*

i. $\mathbb{F}_{2n}(x) = \left(abd^2(x) + 2g(x)\right)\mathbb{F}_{2n-2}(x) - g^2(x)\,\mathbb{F}_{2n-4}(x)$

ii. $\mathbb{F}_{2n+1}(x) = \left(abd^2(x) + 2g(x)\right)\mathbb{F}_{2n-1}(x) - g^2(x)\,\mathbb{F}_{2n-3}(x)$

Proof. Using the equation (2)
i.

$$\begin{aligned}
\mathbb{F}_{2n}(x) &= ad(x)\,\mathbb{F}_{2n-1}(x) + g(x)\,\mathbb{F}_{2n-2}(x)\\
&= ad(x)\left(bd(x)\,\mathbb{F}_{2n-2}(x) + g(x)\,\mathbb{F}_{2n-3}(x)\right) + g(x)\,\mathbb{F}_{2n-2}(x)\\
&= \left(abd^2(x) + g(x)\right)\mathbb{F}_{2n-2}(x) + ad(x)\,g(x)\,\mathbb{F}_{2n-3}(x)\\
&= \left(abd^2(x) + g(x)\right)\mathbb{F}_{2n-2}(x) + g(x)\,\mathbb{F}_{2n-2}(x) - g^2(x)\,\mathbb{F}_{2n-4}(x)\\
&= \left(abd^2(x) + 2g(x)\right)\mathbb{F}_{2n-2}(x) - g^2(x)\,\mathbb{F}_{2n-4}(x)
\end{aligned}$$

ii.

$$\begin{aligned}
\mathbb{F}_{2n+1}(x) &= bd(x)\,\mathbb{F}_{2n}(x) + g(x)\,\mathbb{F}_{2n-1}(x)\\
&= bd(x)\left(ad(x)\,\mathbb{F}_{2n-1}(x) + g(x)\,\mathbb{F}_{2n-2}(x)\right) + g(x)\,\mathbb{F}_{2n-1}(x)\\
&= \left(abd^2(x) + g(x)\right)\mathbb{F}_{2n-1}(x) + bd(x)\,g(x)\,\mathbb{F}_{2n-2}(x)\\
&= \left(abd^2(x) + g(x)\right)\mathbb{F}_{2n-1}(x) + g(x)\,\mathbb{F}_{2n-1}(x) - g^2(x)\,\mathbb{F}_{2n-3}(x)\\
&= \left(abd^2(x) + 2g(x)\right)\mathbb{F}_{2n-1}(x) - g^2(x)\,\mathbb{F}_{2n-3}(x)
\end{aligned}$$

Thus, the proof is completed.

Using the Lemma 2.2, the generating functions of the sequences $\{\mathbb{F}_n(x)\}_{n\in\mathbb{N}}$ are given in the following Theorem.

Theorem 2.3. *The generating functions for the bi-periodic generalized Fibonacci polynomial sequences are*

$$G_n(x,t) = \frac{t + ad(x)t^2 - g(x)t^3}{1 - (abd^2(x) + 2g(x))t^2 + g^2(x)t^4}.$$

Proof. Using equation 4, we get

$$G_n(x,t) = \sum_{n=0}^{\infty} \mathbb{F}_n(x)t^n = \mathbb{F}_0(x) + \mathbb{F}_1(x)t + \mathbb{F}_2(x)t^2 + \ldots + \mathbb{F}_n(x)t^n + \ldots$$

Let generating functions $G_n(x,t)$ be the sum of the odd subscript and even subscript terms separately. Then

$$G_n(x,t) = G_n^{Ç}(x,t) + G_n^{T}(x,t) \tag{5}$$

where $G_n^{Ç}(x,t)$ is the sum of the even subscript terms and $G_n^{T}(x,t)$ is the sum of the odd subscript terms. Therefore,

$$G_n^{Ç}(x,t) = \sum_{i=0}^{\infty} \mathbb{F}_{2i}(x)t^{2i} = \mathbb{F}_0(x) + \mathbb{F}_2(x)t^2 + \mathbb{F}_4(x)t^4 + \ldots \tag{6}$$

If both sides of equation (6) are multiplied by $-\left(abd^2(x) + 2g(x)\right)t^2$ and $g^2(x)t^4$, then we get

$$-\left(abd^2(x) + 2g(x)\right)t^2 G_n^{Ç}(x,t) = -abd^2(x) + 2g(x)\sum_{i=0}^{\infty} \mathbb{F}_{2i}(x)t^{2i+2} \tag{7}$$

and

$$g^2(x)t^4 G_n^{Ç}(x,t) = g^2(x)\sum_{i=0}^{\infty} \mathbb{F}_{2i}(x)t^{2i+4} \tag{8}$$

If we add the equations (6), (7) and (8) side by side, we obtain

$$\begin{aligned}
\left(1 - \left(abd^2(x) + 2g(x)\right)t^2 + g^2(x)t^4\right)G_n^{Ç}(x,t) &= \mathbb{F}_0(x) + \mathbb{F}_2(x)t^2 + \sum_{i=2}^{\infty} \mathbb{F}_{2i}(x)t^{2i} \\
&\quad - \left(abd^2(x) + 2g(x)\right)\sum_{i=0}^{\infty} \mathbb{F}_{2i}(x)t^{2i+2} + g^2(x)\sum_{i=0}^{\infty} \mathbb{F}_{2i}(x)t^{2i+4} \\
&= ad(x)t^2 + \sum_{i=2}^{\infty} \mathbb{F}_{2i}(x)t^{2i} - \left(abd^2(x) + 2g(x)\right)\sum_{i=2}^{\infty} \mathbb{F}_{2i-2}(x)t^{2i} \\
&\quad + g^2(x)\sum_{i=2}^{\infty} \mathbb{F}_{2i-4}(x)t^{2i} \\
&= ad(x)t^2 + \sum_{i=2}^{\infty} \left(\mathbb{F}_{2i}(x) - \left(abd^2(x) + 2g(x)\right)\mathbb{F}_{2i-2}(x) + g^2(x)\mathbb{F}_{2i-4}(x)\right)t^{2i}.
\end{aligned}$$

Using Lemma 2.2, i., generating functions for even subscript terms in the bi-periodic generalized Fibonacci polynomial sequences are obtained as

$$G_n^{Ç}(x,t) = \frac{ad(x)t^2}{1 - (abd^2(x) + 2g(x))t^2 + g^2(x)t^4}.$$

Now let consider the sum of the odd subscript terms in the generating function. Therefore,

$$G_n^T(x,t) = \sum_{i=0}^{\infty} \mathbb{F}_{2i+1}(x)\, t^{2i+1} = \mathbb{F}_1(x)\, t + \mathbb{F}_3(x)\, t^3 + \mathbb{F}_5(x)\, t^5 + ... \tag{9}$$

If both sides of equation (9) are multiplied by $-\left(abd^2(x) + 2g(x)\right)t^2$ and $g^2(x)\, t^4$, , then we get

$$-\left(abd^2(x) + 2g(x)\right)t^2 G_n^T(x,t) = -\left(abd^2(x) + 2g(x)\right)\sum_{i=0}^{\infty} \mathbb{F}_{2i+1}(x)\, t^{2i+3} \tag{10}$$

and

$$g^2(x)\, t^4 G_n^T(x,t) = g^2(x) \sum_{i=0}^{\infty} \mathbb{F}_{2i+1}(x)\, t^{2i+5} \tag{11}$$

If we add the equations (9), (10) and (11) side by side, we obtain

$$\begin{aligned}
\left(1-\left(abd^2(x)+2g(x)\right)t^2+g^2(x)\,t^4\right)G_n^T(x,t) &= \mathbb{F}_1(x)\,t+\mathbb{F}_3(x)\,t^3+\sum_{i=2}^{\infty}\mathbb{F}_{2i+1}(x)\,t^{2i+1}\\
&\quad -\left(abd^2(x)+2g(x)\right)\mathbb{F}_1(x)\,t^3-\left(abd^2(x)+2g(x)\right)\sum_{i=1}^{\infty}\mathbb{F}_{2i+1}(x)\,t^{2i+3}\\
&\quad + g^2(x)\sum_{i=0}^{\infty}\mathbb{F}_{2i+1}(x)\,t^{2i+5}\\
&= t+\left(abd^2(x)+g(x)\right)t^3+\sum_{i=2}^{\infty}\mathbb{F}_{2i+1}(x)\,t^{2i+1}-\left(abd^2(x)+2g(x)\right)t^3\\
&\quad -\left(abd^2(x)+2g(x)\right)\sum_{i=2}^{\infty}\mathbb{F}_{2i-1}(x)\,t^{2i+1}+g^2(x)\sum_{i=2}^{\infty}\mathbb{F}_{2i-3}(x)\,t^{2i+1}\\
&= t+\left(abd^2(x)+g(x)\right)t^3-\left(abd^2(x)+2g(x)\right)t^3\\
&\quad +\sum_{i=2}^{\infty}\left(\mathbb{F}_{2i+1}(x)-\left(abd^2(x)+2g(x)\right)\mathbb{F}_{2i-1}(x)+g^2(x)\,\mathbb{F}_{2i-3}(x)\right)t^{2i+1}.
\end{aligned}$$

Using Lemma 2.2, ii., generating functions for even subscript terms in the bi-periodic generalized Fibonacci polynomial sequences are obtained as

$$G_n^T(x,t) = \frac{t - g(x)\, t^3}{1 - \left(abd^2(x) + 2g(x)\right)t^2 + g^2(x)\, t^4}$$

From equation (5), generating functions for the bi-periodic generalized Fibonacci polynomial sequences are

$$G_n(x,t) = \frac{t + ad(x)\, t^2 - g(x)\, t^3}{1 - \left(abd^2(x) + 2g(x)\right)t^2 + g^2(x)\, t^4}.$$

Thus, the proof is completed.

Now we give Binet formulas that allow us to calculate the nth terms of sequences $\{\mathbb{F}_n(x)\}_{n\epsilon\mathbb{N}}$ in the following theorem.

Theorem 2.4. *The Binet formulas for the bi-periodic generalized Fibonacci polynomial sequences are given by*

$$\mathbb{F}_n(x) = \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\right) \frac{\gamma^n(x) - \delta^n(x)}{\gamma(x) - \delta(x)}$$

where $\gamma(x) = \frac{d(x)ab + \sqrt{d^2(x)a^2b^2 + 4g(x)ab}}{2}$, $\delta(x) = \frac{d(x)ab - \sqrt{d^2(x)a^2b^2 + 4g(x)ab}}{2}$ *and* $\xi(n) = n - 2\left[\frac{n}{2}\right]$.

Proof. By induction method on n. The result is obviously valid for $n = 0, 1$. Suppose that result is true for $n \in \mathbb{N}$, we shall show that it is true for $n + 1$. Using equation (3) and the hypothesis of induction, we have

$$\begin{aligned}
\mathbb{F}_{n+1}(x) &= a^{1-\xi(n+1)} b^{\xi(n+1)} d(x) \mathbb{F}_n(x) + g(x) \mathbb{F}_{n-1}(x) \\
&= a^{1-\xi(n+1)} b^{\xi(n+1)} d(x) \left(\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \right) \frac{\gamma^n(x) - \delta^n(x)}{\gamma(x) - \delta(x)} \right) + g(x) \left(\left(\frac{a^{1-\xi(n-1)}}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}} \right) \frac{\gamma^{n-1}(x) - \delta^{n-1}(x)}{\gamma(x) - \delta(x)} \right) \\
&= \frac{a^{1-\xi(n+1)} \gamma^{n-1}(x)}{\gamma(x) - \delta(x)} \left(\frac{a^{1-\xi(n)} b^{\xi(n+1)} d(x) \gamma(x)}{(ab)^{\lfloor \frac{n}{2} \rfloor}} + \frac{a^{1-\xi(n-1)} g(x)}{a^{1-\xi(n+1)} (ab)^{\lfloor \frac{n-1}{2} \rfloor}} \right) \\
&\quad - \frac{a^{1-\xi(n+1)} \delta^{n-1}(x)}{\gamma(x) - \delta(x)} \left(\frac{a^{1-\xi(n)} b^{\xi(n+1)} d(x) \delta(x)}{(ab)^{\lfloor \frac{n}{2} \rfloor}} + \frac{a^{1-\xi(n-1)} g(x)}{a^{1-\xi(n+1)} (ab)^{\lfloor \frac{n-1}{2} \rfloor}} \right) \\
&= \frac{a^{1-\xi(n+1)} \gamma^{n-1}(x)}{\gamma(x) - \delta(x)} \left(\frac{abd(x) \gamma(x)}{a^{\xi(n)} b^{1-\xi(n+1)} (ab)^{\lfloor \frac{n}{2} \rfloor}} + \frac{abg(x)}{(ab)^{\lfloor \frac{n-1}{2} \rfloor + 1}} \right) \\
&\quad - \frac{a^{1-\xi(n+1)} \delta^{n-1}(x)}{\gamma(x) - \delta(x)} \left(\frac{abd(x) \delta(x)}{a^{\xi(n)} b^{1-\xi(n+1)} (ab)^{\lfloor \frac{n}{2} \rfloor}} + \frac{abg(x)}{(ab)^{\lfloor \frac{n-1}{2} \rfloor + 1}} \right) \\
&= \frac{a^{1-\xi(n+1)} \gamma^{n-1}(x)}{\gamma(x) - \delta(x)} \left(\frac{ab(d(x) \gamma(x) + g(x))}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \right) - \frac{a^{1-\xi(n+1)} \delta^{n-1}(x)}{\gamma(x) - \delta(x)} \left(\frac{ab(d(x) \delta(x) + g(x))}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \right) \\
&= \frac{a^{1-\xi(n+1)} \gamma^{n-1}(x)}{\gamma(x) - \delta(x)} \left(\frac{\gamma^2(x)}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \right) - \frac{a^{1-\xi(n+1)} \delta^{n-1}(x)}{\gamma(x) - \delta(x)} \left(\frac{\delta^2(x)}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \right) \\
&= \left(\frac{a^{1-\xi(n+1)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \right) \frac{\gamma^{n+1}(x) - \delta^{n+1}(x)}{\gamma(x) - \delta(x)}
\end{aligned}$$

where $d(x)\gamma(x) + g(x) = \frac{\gamma^2(x)}{ab}$, $d(x)\delta(x) + g(x) = \frac{\delta^2(x)}{ab}$ and $\xi(n) + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor$, $1 - \xi(n+1) + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor$. This completes the proof.

2.2. *Identities for Polynomials* $\mathbb{F}_n(x)$

In this section, we give various identities for consecutive terms and negative subscript terms of the bi-periodic generalized Fibonacci polynomial sequences and present the Catalan's identity, Cassini's identity, d'Ocagne's identity for these polynomials.

Theorem 2.5. *The limit of the ratio of consecutive terms of the bi-periodic generalized Fibonacci polynomial sequences is*

i. $lim_{n\to\infty} \frac{\mathbb{F}_{2n+1}(x)}{\mathbb{F}_{2n}(x)} = \frac{\gamma(x)}{a}$

ii. $lim_{n\to\infty} \frac{\mathbb{F}_{2n}(x)}{\mathbb{F}_{2n-1}(x)} = \frac{\gamma(x)}{b}$

where $\mathbb{F}_n(x)$ *is the nth bi-periodic generalized Fibonacci polynomial.*

Proof. Using Binet formula for nth bi-periodic generalized Fibonacci polynomial given in Theorem 2.4, we have

i.

$$\begin{aligned}
lim_{n\to\infty}\frac{\mathbb{F}_{2n+1}(x)}{\mathbb{F}_{2n}(x)} &= lim_{n\to\infty}\frac{\left(\frac{a^{1-\xi(2n+1)}}{(ab)^{\lfloor\frac{2n+1}{2}\rfloor}}\right)\left(\frac{\gamma^{2n+1}(x)-\delta^{2n+1}(x)}{\gamma(x)-\delta(x)}\right)}{\left(\frac{a^{1-\xi(2n)}}{(ab)^{\lfloor\frac{2n}{2}\rfloor}}\right)\left(\frac{\gamma^{2n}(x)-\delta^{2n}(x)}{\gamma(x)-\delta(x)}\right)}\\
&= lim_{n\to\infty}\frac{\frac{1}{(ab)^n}\left(\frac{\gamma^{2n+1}(x)-\delta^{2n+1}(x)}{\gamma(x)-\delta(x)}\right)}{\frac{a}{(ab)^n}\left(\frac{\gamma^{2n}(x)-\delta^{2n}(x)}{\gamma(x)-\delta(x)}\right)}\\
&= lim_{n\to\infty}\frac{1}{a}\frac{\gamma^{2n+1}(x)\left(1-\left(\frac{\delta(x)}{\gamma(x)}\right)^{2n+1}\right)}{\gamma^{2n}(x)\left(1-\left(\frac{\delta(x)}{\gamma(x)}\right)^{2n}\right)}\\
&= \frac{\gamma(x)}{a}
\end{aligned}$$

ii.

$$\begin{aligned}
lim_{n\to\infty}\frac{\mathbb{F}_{2n}(x)}{\mathbb{F}_{2n-1}(x)} &= lim_{n\to\infty}\frac{\left(\frac{a^{1-\xi(2n)}}{(ab)^{\lfloor\frac{2n}{2}\rfloor}}\right)\left(\frac{\gamma^{2n}(x)-\delta^{2n}(x)}{\gamma(x)-\delta(x)}\right)}{\left(\frac{a^{1-\xi(2n-1)}}{(ab)^{\lfloor\frac{2n-1}{2}\rfloor}}\right)\left(\frac{\gamma^{2n-1}(x)-\delta^{2n-1}(x)}{\gamma(x)-\delta(x)}\right)}\\
&= lim_{n\to\infty}\frac{\frac{a}{(ab)^n}\left(\frac{\gamma^{2n}(x)-\delta^{2n}(x)}{\gamma(x)-\delta(x)}\right)}{\frac{1}{(ab)^{n-1}}\left(\frac{\gamma^{2n-1}(x)-\delta^{2n-1}(x)}{\gamma(x)-\delta(x)}\right)}\\
&= lim_{n\to\infty}\frac{a}{ab}\frac{\gamma^{2n}(x)\left(1-\left(\frac{\delta(x)}{\gamma(x)}\right)^{2n}\right)}{\gamma^{2n-1}(x)\left(1-\left(\frac{\delta(x)}{\gamma(x)}\right)^{2n-1}\right)}\\
&= \frac{\gamma(x)}{b}
\end{aligned}$$

where $|\delta(x)| < \gamma(x)$ and $lim_{n\to\infty}\left(\frac{\delta(x)}{\gamma(x)}\right)^n = 0$. This completes the proof.

Theorem 2.6. *Negative subscript terms of the bi-periodic generalized Fibonacci polynomial sequences are obtained as*

$$\mathbb{F}_{-n}(x) = (-1)^{n+1}(g(x))^{-n}\,\mathbb{F}_n(x).$$

Proof. Using Binet formula for nth bi-periodic generalized Fibonacci polynomial given in Theorem 2.4, we have

$$\begin{aligned}
\mathbb{F}_{-n}(x) &= \left(\frac{a^{1-\xi(-n)}}{(ab)^{\lfloor\frac{-n}{2}\rfloor}}\right)\frac{\gamma^{-n}(x)-\delta^{-n}(x)}{\gamma(x)-\delta(x)}\\
&= (-1)\left(\frac{a^{1-\xi(-n)}}{(ab)^{\lfloor\frac{-n}{2}\rfloor}}\right)\frac{\gamma^{n}(x)-\delta^{n}(x)}{(-g(x)\,ab)^n\,(\gamma(x)-\delta(x))}\\
&= (-1)(-g(x))^{-n}\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\right)\frac{\gamma^{n}(x)-\delta^{n}(x)}{\gamma(x)-\delta(x)}\\
&= (-1)^{n+1}(g(x))^{-n}\,\mathbb{F}_n(x)
\end{aligned}$$

where $\gamma(x)\delta(x) = -g(x)ab$. Thus, the proof is completed.

Now we present some basic identities for the bi-periodic generalized Fibonacci polynomials, such as Catalan's identity, Cassini's identity and d'Ocagne's identity.

Theorem 2.7. *(Catalan's Identity) Let n and r be nonnegative integers. For $n \geq r$, we have*

$$a^{\xi(n-r)}b^{1-\xi(n-r)}\mathbb{F}_{n-r}(x)\,\mathbb{F}_{n+r}(x) - a^{\xi(n)}b^{1-\xi(n)}\mathbb{F}_n^{\,2}(x) = -(-g(x))^{n-r}a^{\xi(r)}b^{1-\xi(r)}\mathbb{F}_r^{\,2}(x)$$

where $\mathbb{F}_n(x)$ is the nth bi-periodic generalized Fibonacci polynomial.

Proof. Using Binet formula for nth bi-periodic generalized Fibonacci polynomial given in Theorem 2.4, we have

$$\begin{aligned}
&a^{\xi(n-r)}b^{1-\xi(n-r)}\mathbb{F}_{n-r}(x)\,\mathbb{F}_{n+r}(x) - a^{\xi(n)}b^{1-\xi(n)}\mathbb{F}_n^{\,2}(x)\\
&= a^{\xi(n-r)}b^{1-\xi(n-r)}\left(\frac{a^{1-\xi(n-r)}}{(ab)^{\lfloor\frac{n-r}{2}\rfloor}}\right)\left(\frac{a^{1-\xi(n+r)}}{(ab)^{\lfloor\frac{n+r}{2}\rfloor}}\right)\left(\frac{\gamma^{n-r}(x)-\delta^{n-r}(x)}{\gamma(x)-\delta(x)}\right)\left(\frac{\gamma^{n+r}(x)-\delta^{n+r}(x)}{\gamma(x)-\delta(x)}\right)\\
&- a^{\xi(n)}b^{1-\xi(n)}\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\right)\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\right)\left(\frac{\gamma^{n}(x)-\delta^{n}(x)}{\gamma(x)-\delta(x)}\right)\left(\frac{\gamma^{n}(x)-\delta^{n}(x)}{\gamma(x)-\delta(x)}\right)\\
&= \frac{a^{2-\xi(n-r)}b^{1-\xi(n-r)}}{(ab)^{\lfloor\frac{n-r}{2}\rfloor+\lfloor\frac{n+r}{2}\rfloor}}\left(\frac{\gamma^{2n}(x)-\gamma^{n-r}(x)\delta^{n+r}(x)-\delta^{n-r}(x)\gamma^{n+r}(x)+\delta^{2n}(x)}{(\gamma(x)-\delta(x))^2}\right)\\
&- \frac{a^{2-\xi(n)}b^{1-\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}\left(\frac{\gamma^{2n}(x)-2\gamma^{n}(x)\delta^{n}(x)+\delta^{2n}(x)}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{a^{2-\xi(n-r)}b^{1-\xi(n-r)}}{(ab)^{n-\xi(n-r)}}\left(\frac{\gamma^{2n}(x)-(\gamma(x)\delta(x))^{n-r}\left(\gamma^{2r}(x)+\delta^{2r}(x)\right)+\delta^{2n}(x)}{(\gamma(x)-\delta(x))^2}\right)\\
&- \frac{a^{2-\xi(n)}b^{1-\xi(n)}}{(ab)^{n-\xi(n)}}\left(\frac{\gamma^{2n}(x)-2(\gamma(x)\delta(x))^{n}+\delta^{2n}(x)}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{a}{(ab)^{n-1}}\left(\frac{-(\gamma(x)\delta(x))^{n-r}\left(\gamma^{2r}(x)+\delta^{2r}(x)\right)+2(\gamma(x)\delta(x))^{n}}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{-a(\gamma(x)\delta(x))^{n-r}}{(ab)^{n-1}}\left(\frac{\gamma^{r}(x)-\delta^{r}(x)}{\gamma(x)-\delta(x)}\right)^2\\
&= \frac{-a(-g(x)ab)^{n-r}}{(ab)^{n-1}}\frac{(ab)^{2\lfloor\frac{r}{2}\rfloor}}{a^{2-2\xi(r)}}\mathbb{F}_r^{\,2}(x)\\
&= -(-g(x))^{n-r}\frac{a(ab)^{2\lfloor\frac{r}{2}\rfloor}}{(ab)^{\xi(r)+2\lfloor\frac{r}{2}\rfloor-1}a^{2-2\xi(r)}}\mathbb{F}_r^{\,2}(x)\\
&= -(-g(x))^{n-r}a^{\xi(r)}b^{1-\xi(r)}\mathbb{F}_r^{\,2}(x)
\end{aligned}$$

where $\xi(n) = n - 2\left\lfloor\frac{n}{2}\right\rfloor$ and $\left\lfloor\frac{n-r}{2}\right\rfloor + \left\lfloor\frac{n+r}{2}\right\rfloor = n - \xi(n-r)$. This completes the proof.

Theorem 2.8. *(Cassini's Identity) Let n be nonnegative integer. Then, we have*

$$\left(\frac{a}{b}\right)^{\xi(n-1)}\mathbb{F}_{n-1}(x)\,\mathbb{F}_{n+1}(x) - \left(\frac{a}{b}\right)^{\xi(n)}\mathbb{F}_n^{\,2}(x) = -(-g(x))^{n-1}\frac{a}{b}$$

Proof. The proof can be seen in an obvious way by taking $r = 1$ in the Catalan's identity.

Theorem 2.9. *(d'Ocagne's Identity) Let n and r be nonnegative integers. For $n \geq r$, we have*

$$a^{\xi(nr+n)}b^{\xi(nr+r)}\mathbb{F}_n(x)\,\mathbb{F}_{r+1}(x) - a^{\xi(nr+r)}b^{\xi(nr+n)}\mathbb{F}_{n+1}(x)\,\mathbb{F}_r(x) = (-g(x))^r\, a^{\xi(n-r)}\mathbb{F}_{n-r}(x)$$

where $\mathbb{F}_n(x)$ is the nth bi-periodic generalized Fibonacci polynomial.

Proof. Using Binet formula for nth bi-periodic generalized Fibonacci polynomial given in Theorem 2.4, we have

$$a^{\xi(nr+n)}b^{\xi(nr+r)}\mathbb{F}_n(x)\,\mathbb{F}_{r+1}(x) - a^{\xi(nr+r)}b^{\xi(nr+n)}\mathbb{F}_{n+1}(x)\,\mathbb{F}_r(x)$$

$$\begin{aligned}
&= a^{\xi(nr+n)}b^{\xi(nr+r)}\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\right)\left(\frac{a^{1-\xi(r+1)}}{(ab)^{\lfloor\frac{r+1}{2}\rfloor}}\right)\left(\frac{\gamma^n(x)-\delta^n(x)}{\gamma(x)-\delta(x)}\right)\left(\frac{\gamma^{r+1}(x)-\delta^{r+1}(x)}{\gamma(x)-\delta(x)}\right)\\
&- a^{\xi(nr+r)}b^{\xi(nr+n)}\left(\frac{a^{1-\xi(n+1)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}\right)\left(\frac{a^{1-\xi(r)}}{(ab)^{\lfloor\frac{r}{2}\rfloor}}\right)\left(\frac{\gamma^{n+1}(x)-\delta^{n+1}(x)}{\gamma(x)-\delta(x)}\right)\left(\frac{\gamma^{r}(x)-\delta^{r}(x)}{\gamma(x)-\delta(x)}\right)\\
&= \frac{ab^{\xi(nr+r)}a^{1-\xi(n)-\xi(r+1)+\xi(nr+n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor+\lfloor\frac{r+1}{2}\rfloor}}\left(\frac{\gamma^{n+r+1}(x)-\gamma^n(x)\,\delta^{r+1}(x)-\delta^n(x)\,\gamma^{r+1}(x)+\delta^{n+r+1}(x)}{(\gamma(x)-\delta(x))^2}\right)\\
&- \frac{ab^{\xi(nr+n)}a^{1-\xi(n+1)-\xi(r)+\xi(nr+r)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor+\lfloor\frac{r}{2}\rfloor}}\left(\frac{\gamma^{n+r+1}(x)-\gamma^{n+1}(x)\,\delta^{r}(x)-\delta^{n+1}(x)\,\gamma^{r}(x)+\delta^{n+r+1}(x)}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{ab^{\xi(nr+r)}a^{\xi(n-r)-\xi(nr+n)}}{(ab)^{\frac{n-r-\xi(n-r)}{2}+\xi(nr+r)+r}}\left(\frac{\gamma^{n+r+1}(x)+\delta^{n+r+1}(x)-(\gamma(x)\,\delta(x))^r\left(\gamma(x)\,\delta^{n-r}(x)+\delta(x)\,\gamma^{n-r}(x)\right)}{(\gamma(x)-\delta(x))^2}\right)\\
&- \frac{ab^{\xi(nr+n)}a^{\xi(n-r)-\xi(nr+r)}}{(ab)^{\frac{n-r-\xi(n-r)}{2}+\xi(nr+n)+r}}\left(\frac{\gamma^{n+r+1}(x)+\delta^{n+r+1}(x)-(\gamma(x)\,\delta(x))^r\left(\gamma^{n-r+1}(x)+\delta^{n-r+1}(x)\right)}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{ab^{\xi(nr+r)}a^{\xi(nr+r)}}{(ab)^{\frac{n-r-\xi(n-r)}{2}+\xi(nr+r)+r}}\left(\frac{\gamma^{n+r+1}(x)+\delta^{n+r+1}(x)-(\gamma(x)\,\delta(x))^r\left(\gamma(x)\,\delta^{n-r}(x)+\delta(x)\,\gamma^{n-r}(x)\right)}{(\gamma(x)-\delta(x))^2}\right)\\
&- \frac{ab^{\xi(nr+n)}a^{\xi(nr+n)}}{(ab)^{\frac{n-r-\xi(n-r)}{2}+\xi(nr+n)+r}}\left(\frac{\gamma^{n+r+1}(x)+\delta^{n+r+1}(x)-(\gamma(x)\,\delta(x))^r\left(\gamma^{n-r+1}(x)+\delta^{n-r+1}(x)\right)}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{a\,(ab)^{-r}}{(ab)^{\frac{n-r-\xi(n-r)}{2}}}\left(\frac{(\gamma(x)\,\delta(x))^r\left(-\gamma(x)\,\delta^{n-r}(x)-\delta(x)\,\gamma^{n-r}(x)+\gamma^{n-r+1}(x)+\delta^{n-r+1}(x)\right)}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{a\,(ab)^{-r}}{(ab)^{\lfloor\frac{n-r}{2}\rfloor}}\left(\frac{(-g(x)\,ab)^r\,(\gamma(x)-\delta(x))\,(\gamma^{n-r}(x)-\delta^{n-r}(x))}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{a\,(-g(x))^r}{(ab)^{\lfloor\frac{n-r}{2}\rfloor}}\left(\frac{\gamma^{n-r}(x)-\delta^{n-r}(x)}{\gamma(x)-\delta(x)}\right)\\
&= (-g(x))^r\,a^{\xi(n-r)}\mathbb{F}_{n-r}(x)
\end{aligned}$$

where

$$\xi(n)+\xi(r+1)-2\xi(nr+n) = \xi(n+1)+\xi(r)-2\xi(nr+r) = 1-\xi(n-r)$$

$$\xi(n-r) = \xi(nr+n)+\xi(nr+r)$$

$$\frac{n-r-\xi(n-r)}{2}+\xi(nr+r)+r=\left\lfloor\frac{n}{2}\right\rfloor+\left\lfloor\frac{r+1}{2}\right\rfloor$$

$$\frac{n-r-\xi(n-r)}{2}+\xi(nr+n)+r=\left\lfloor\frac{n+1}{2}\right\rfloor+\left\lfloor\frac{r}{2}\right\rfloor$$

$$\frac{n-r-\xi(n-r)}{2}=\left\lfloor\frac{n-r}{2}\right\rfloor.$$

This completes the proof.

3. Conclusion and Suggestion

The most interesting applications of the Fibonacci numbers have been on its generalizations, also called families of Fibonacci numbers. Large classes of polynomials are emerged as the well-known generalization of Fibonacci numbers. In this paper, the bi-periodic generalized Fibonacci polynomials, which generalize well-known Fibonacci polynomials, the $h(x)$-Fibonacci polynomials, the Fibonacci polynomials with two variable, the Pell polynomials, the Jacobsthal polynomials, the Fermat polynomials, the Chebyshev second kind polynomials, the Morgan-Voyce first kind polynomials, the Vieta polynomials, are defined. Also the bi-periodic Fibonacci polynomials, the bi-periodic $h(x)$-Fibonacci polynomials, the bi-periodic Fibonacci polynomials with two variable, the bi-periodic Pell polynomials, the bi-periodic Jacobsthal polynomials, the bi-periodic Fermat polynomials, the bi-periodic Chebyshev second kind polynomials, the bi-periodic Morgan-Voyce first kind polynomials, the bi-periodic Vieta polynomials are presented. Binet formulas that allow us to calculate the nth term of these polynomial sequences and some properties of their consecutive terms are given. Also generating functions, Catalan's identity, Cassini's identity, and d'Ocagne's identity are obtained.

It would be interesting to study these polynomials in matrix theory. More general formulas that allow us to calculate the nth terms of these polynomial sequences and sums formulas can be explored.

References

[1] Horadam, AF. A Generalized Fibonacci sequence. The American Mathematical Monthly. 68(5), 1961, 455 – 459.
[2] Hoggatt Jr. VE, Bicknell M. Generalized Fibonacci polynomials and Zeckendorf 's theorem. The The Fibonacci Quarterly. 11(4), 1973, 399 – 419.
[3] Horadam, AF. Jacobsthal and Pell Curves. The Fibonacci Quarterly. 26, 1988, 79 – 83.
[4] Catalani M. Some formulae for bivariate Fibonacci and Lucas polynomials. 2009, 1 – 9. Site: https://doi.org/10.48550/arXiv.math/0406323
[5] Panwar YK, Gupta VK, and Bhandari J. Generalized identities of bivariate Fibonacci and bivariate Lucas polynomials. Journal of Amasya University the Institute of Sciences and Technology. 1(2), 2020, 146 – 154.
[6] Nalli A, Haukkanen P. On generalized Fibonacci and Lucas polynomials. Chaos, Solitons and Fractals. 45(5), 2009, 3179 – 3186.
[7] Koshy T. Fibonacci and Lucas Numbers with Applications. Wiley-Interscience Publications. 2nd Edition, vol. 1, 2017, 704p.
[8] Çağman A. Repdigits as product of Fibonacci and Pell numbers. Turkish Journal of Science. 6(1), 2021, 31 – 35.
[9] Gültekin İ, Taşyurdu Y. On period of the sequence of Fibonacci Polynomials modulo m. Dynamics in Nature and Society. 2013, 3p.
[10] Taşyurdu Y, Deveci Ö.The Fibonacci Polynomials in Rings. Ars Combinatoria. 133, 2017, 355 – 366.
[11] Florez R, McAnally N, Mukherjee A. Identities for the generalized Fibonacci polynomial, Integers. 18B, Article number A12, 2018, 28p.
[12] Florez R, Higuita RA, Mukherjee A. Characterization of the strong divisibility property for generalized Fibonacci polynomials, Integers. 18B, Article number A14, 2018, 28p.
[13] Florez R,Higuita RA, Mukherjee A. Alternating sums in the Hosoya polynomial triangle, Journal of Integer Sequences. 17, Article number 14.9.5, 2014, 17p.
[14] Edson M, Yayenie O. A New Generalization of Fibonacci Sequence and Extended Binet's Formula. Integers. 9. 2009, 639 – 654.
[15] Yılmaz N, Çoşkun A, Taşkara N. On properties of bi-periodic Fibonacci and Lucas polynomials. AIP Conference Proceedings 1863, 310002, 2017.

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 3, 168–176
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

Some Curves on 3-Dimensional Normal almost Contact Pseudo-metric Manifolds

Müslüm Aykut Akgün[a]

[a] *Adıyaman University, Technical Sciences Vocational High School*

Abstract. In this study, we characterize Frenet curves in 3-dimensional normal almost contact pseudo-metric manifolds. We give Frenet equations and the Frenet elements of such curves. Also, we obtain the curvatures of non-geodesic Frenet curves on 3-dimensional almost contact pseudo-metric manifolds. Finally we present some corollaries about these curves.

1. Introduction

The differential geometry of curves on manifolds is an attractive topic in differential geometry. Especially the curves in contact and para-contact manifolds drew attention and studied by many authors. Olszak [10], gave the conditions for an a.c.m structure on a manifold to be normal and gave examples for this structure.

Welyczko [14], gave some of the results for Legendre curves to the case of 3-dimensional normal a.c.m. manifolds, especially, quasi-Sasakian manifolds. Acet and Perktaş [1] obtained curvature and torsion of Legendre curves in 3-dimensional (ε, δ) trans-Sasakian manifolds.

Yıldırım [15] obtained the curvatures of non-geodesic Frenet curves on three dimensional normal almost contact manifolds and gave some results for these characterizations. De and Mondal [6] studied ξ-projectively flat and φ-projectively flat 3-dimensional normal almost contact metric manifolds and gave an illustrative example.

Calvaruso and Perrone [3] introduced a systematic study of contact structures with pseudo-Riemannian associated metrics, emphasizing analogies and differences with respect to the Riemannian case. In particular, they classified contact pseudo-metric manifolds of constant sectional curvature, three dimensional locally symmetric contact pseudo-metric manifolds and three-dimensional homogeneous contact Lorentzian manifolds.

Takahashi [11] defined Sasakian manifold with pseudo-Riemannian metric and discussed the classification of Sasakian manifolds. Venkatesha V. [13] examined 3-dimensional normal almost contact pseudo-metric manifold and gave the conditions for these manifolds to be normal. studied the almost contact pseudo-metric manifolds of dimension three which are normal and derived certain necessary and sufficient conditions for an almost contact pseudo-metric manifold to be normal.

This paper is organized as: Section 2 with three subsections, we give basic definitions and propositions of an almost contact pseudo-metric manifold. In the second subsection we give the properties of 3-dimensional

Corresponding author: MAA muslumakgun@adiyaman.edu.tr ORCID:0000-0002-8414-5228
Received: 9 June 2022; Accepted: 16 September 2022; Published: 30 December 2022
Keywords. (Contact pseudo-metric manifolds,Frenet curves, Frenet elements.)
2010 *Mathematics Subject Classification*. 53B30, 53C25
Cited this article as: AKGÜN MA. Some curves on 3-dimensional normal almost contact pseudo-metric manifolds, Turkish Journal of Science, 2022, 7(3), 168-176.

almost contact pseudo-metric manifolds. We give Frenet equations of a curve in 3-dimensional almost contact pseudo-metric manifolds in the last subsection of this section.

We finally give the Frenet elements of a Frenet curve in such manifolds and give corollaries for the Frenet curves in the third section.

2. Preliminaries

2.1. *Normal Almost Contact Pseudo-metric Manifolds*

A $(2n+1)$-dimensional smooth connected manifold M is said to be an almost contact manifold if there exists on M a (1,1) tensor field φ, a vector field ξ and a 1-form η such that [2]

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \qquad (1)$$
$$\varphi(\xi) = 0, \quad \eta \circ \varphi = 0.$$

If an almost contact manifold is endowed with a pseudo-Riemannian metric g such that

$$\bar{g}(\varphi X, \varphi Y) = \bar{g}(X, Y) - \varepsilon \eta(X)\eta(Y), \qquad (2)$$

where $\varepsilon = \mp 1$, for all $X, Y \in TM$, then $(\bar{N}, \varphi, \xi, \eta, \bar{g})$ is called an almost contact pseudo-metric manifold[13]. From (2) we have

$$\eta(X) = \varepsilon \bar{g}(X, \xi) \quad \textit{and} \quad \bar{g}(\varphi X, Y) = -\bar{g}(X, \varphi Y). \qquad (3)$$

In particular, for an almost contact pseudo-metric manifold $\bar{g}(\xi, \xi) = \varepsilon$. Thus, the characteristic vector field ξ is a unit vector field, which is either spacelike or timelike, but cannot be ligtlike. The fundamental 2-form of an almost contact pseudo-metric manifold $(\bar{N}, \varphi, \xi, \eta, \bar{g})$ is defined by

$$\Phi(X, Y) = \bar{g}(X, \varphi(Y)), \qquad (4)$$

where $\eta \wedge \Phi^n \neq 0$[13]. An almost contact pseudo-metric manifold is said to be contact pseudo-metric manifold if $d\eta = \Phi$, where

$$d\eta(X, Y) = \frac{1}{2}\left(X\eta(Y) - Y\eta(X) - \eta([X, Y])\right). \qquad (5)$$

[3]In an almost contact pseudo-metric manifold $(\bar{N}, \varphi, \xi, \eta, \bar{g})$ there always exists a special kind of local pseudo-orthonormal basis $\{e_i, \varphi e_i, \xi\}_{i=1}^n$, called a local φ-basis.

Let $\bar{N}$ be a (2n+1)-dimensional almost contact pseudo-metric manifold with structure (φ, ξ, η) and consider the manifold $\bar{N} \times R$. We denote a vector field on $\bar{N} \times R$ by $X, f\frac{d}{dt}$, where $X \in T\bar{N}$, t is the coordinate on $\mathfrak{R}$ and f is a C^∞ function on $\bar{N} \times R$. Then the structure J on $\bar{N} \times R$ defined by

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}, \qquad (6)$$

is an almost complex structure. If the almost complex structure J is integrable, then we say that the almost contact pseudo-metric structure (φ, ξ, η) is normal. Necessary and sufficient condition for integrability of J is

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0, \qquad (7)$$

where $[\varphi, \varphi]$ is the Nijenhius torsion of φ.[3]

Proposition 2.1. *[12] An almost contact pseudo-metric manifold is normal if and only if*

$$(\nabla_{\varphi X}\varphi)Y - \varphi(\nabla_X \varphi)Y + (\nabla_X \eta)(Y)\xi = 0, \qquad (8)$$

where ∇ is the Levi-Civita connection.

2.2. *Three dimensional normal almost contact pseudo-metric(n.a.c.p-m) manifold*

Lemma 2.2. *[13] A three dimensional n.a.c.p-m manifold $\bar{N}$ is normal if and only if*

$$\nabla_{\varphi X}\xi = \varphi\nabla_X\xi. \tag{9}$$

Theorem 2.3. *[13] For a three dimensional n.a.c.p-m manifold $\bar{N}$, the following three conditions are mutually equivalent:*
(1) *$\bar{N}$ is normal*
(2) *there exist smooth functions α, β on $\bar{N}$ such that*

$$\nabla_X\xi = \alpha\{X - \eta(X)\xi\} - \beta\varphi X, \tag{10}$$

(3) *there exist smooth functions α, β on $\bar{N}$ such that*

$$(\nabla_{X\varphi})Y = \alpha\{\varepsilon\bar{g}(\varphi X, Y)\xi - \eta(Y)\varphi X\} + \beta\{\varepsilon\bar{g}(X, Y)\xi - \eta(Y)X\}. \tag{11}$$

In particular, the functions appearing above are given by

$$2\alpha = div(\xi), \quad 2\beta = tr(\varphi\nabla_X). \tag{12}$$

Corollary 2.4. *[13] For a three dimensional n.a.c.p-m manifold, the vector field ξ is geodesic, i.e., $\nabla_\xi\xi = 0$ and $d\eta = \varepsilon\beta\Phi$.*

From (11) we can give the following definition.

Definition 2.5. *[13] A three dimensional n.a.c.p-m manifold is called*
(i) *cosymplectic if $\alpha = \beta = 0$,*
(ii) *quasi-Sasakian if $\alpha = 0$ and $\beta \neq 0$, and β-Sasakian pseudo-metric manifold if $\alpha = 0$ and β is non-zero constant. If $\beta = \varepsilon$ it is the Sasakian pseudo-metric manifold,*
(iii) *an almost α-Kenmotsu pseudo-metric manifold if $\beta = 0$ and $\alpha \neq 0$, and α-Kenmotsu pseudo-metric manifold if $\beta = 0$ and α is a non-zero constant. If $\alpha = 1$ it is the Kenmotsu pseudo-metric manifold.*

Lemma 2.6. *[13] For a three dimensional n.a.c.p-m manifold $\xi(\beta) + 2\alpha\beta = 0$ holds.*

2.3. *Frenet Curves*

Let $\bar{N}$ be a three dimensional n.a.c.p-m manifold with Levi-Civita connection ∇ and $\vartheta : I \to \bar{N}$ be a unit speed curve parametrized by arc length s in $\bar{N}$ where I is an open interval. A unit speed curve ϑ is called timelike or spacelike if its casual character is -1 or 1, respectively. Also, ϑ is called a Frenet curve if $\bar{g}(\vartheta', \vartheta') \neq 0$. A Frenet curve ϑ admits an orthonormal frame field $\{t = \vartheta', n, b\}$ along ϑ. Then the following Frenet equations holds:

$$\begin{aligned}
\nabla_{\vartheta'} t &= \kappa n, \\
\nabla_{\vartheta'} n &= -\kappa t + \varepsilon\tau b, \\
\nabla_{\vartheta'} b &= -\varepsilon\tau n,
\end{aligned}$$

where $\kappa = |\nabla_{\vartheta'}\vartheta'|$ is the geodesic curvature of ϑ and τ is geodesic torsion. The vector fields t, n and b are called the tangent vector field, the principal normal vector field and the binormal vector field of ϑ, respectively.
A Frenet curve ϑ is a geodesic if and only if $\kappa = 0$. A Frenet curve ϑ with constant geodesic curvature and zero geodesic torsion is called a pseudo-circle. A pseudo-helix is a Frenet curve ϑ whose geodesic curvature and torsion are constant.

A curve in a 3-dimensional n.a.c.p-m manifold is said to be slant if its tangent vector field has constant angle with the Reeb vector field,i.e.$\eta(\vartheta') = \varepsilon\bar{g}(\vartheta', \xi) = cos\theta = constant$. If the condition $\eta(\vartheta') = \varepsilon\bar{g}(\vartheta', \xi) = 0$ holds then ϑ is a Legendre curve[14].

3. Main Results

Let us consider a 3-dimensional normal almost contact pseudo-metric manifold $\bar{N}$. Let $\vartheta : I \to \bar{N}$ be a non-geodesic ($\kappa \neq 0$) Frenet curve given with the arc-parameter s and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{N}$. From the basis $(\vartheta^{'}, \varphi\vartheta^{'}, \xi)$ we obtain an orthonormal basis $\{z_1, z_2, z_3\}$ defined by

$$\begin{aligned} z_1 &= \vartheta^{'}, \\ z_2 &= \frac{\varphi\vartheta^{'}}{\sqrt{1-\varepsilon\varrho^2}}, \\ z_3 &= \frac{\xi - \varepsilon\varrho\vartheta^{'}}{\sqrt{1-\varepsilon\varrho^2}}, \end{aligned} \tag{13}$$

where

$$\eta(\vartheta^{'}) = \varepsilon\bar{g}(\vartheta^{'}, \xi) = \varepsilon\varrho. \tag{14}$$

Moreover we have

$$\bar{\nabla}_{\vartheta'} z_1 = \nu z_2 + \mu z_3 \tag{15}$$

such that

$$\nu = \bar{g}(\bar{\nabla}_{\vartheta'} z_1, z_2) \tag{16}$$

is a function. Then we obtain μ by

$$\mu = \bar{g}(\bar{\nabla}_{\vartheta'} z_1, z_3) = \frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}} - \alpha\sqrt{1-\varepsilon\varrho^2}. \tag{17}$$

So, we have

$$\bar{\nabla}_{\vartheta'} z_2 = -\nu z_1 + \left(\beta - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}}\right) z_3 \tag{18}$$

and

$$\bar{\nabla}_{\vartheta'} z_3 = -\mu z_1 - \left(\beta - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}}\right) z_2. \tag{19}$$

The fundamental forms of the tangent vector ϑ' on the basis of the equation (13) is

$$[\omega_{ij}(\vartheta')] = \begin{bmatrix} 0 & \nu & \mu \\ -\nu & 0 & -\beta + \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} \\ -\mu & \beta - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} & 0 \end{bmatrix} \tag{20}$$

and the Darboux vector connected to the vector ϑ' is

$$\omega(\vartheta') = \left(-\beta + \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}}\right) z_1 - \mu z_2 + \nu z_3. \tag{21}$$

Then, we have

$$\bar{\nabla}_{\vartheta'} z_i = \omega(\vartheta') \wedge \varepsilon z_i \quad (1 \leq i \leq 3). \tag{22}$$

Furthermore, for any vector field $Z = \sum_{i=1}^{3} \theta^i z_i \in \chi(\bar{N})$ is strictly dependent on the curve ϑ on $\bar{N}$, there exists the following equation

$$\bar{\nabla}_{\vartheta'} Z = \omega(\vartheta') \wedge Z + \varepsilon\sum_{i=1}^{3} z_i[\theta^i] z_i. \tag{23}$$

3.1. Frenet Elements of the curve ϑ

Let $\vartheta : I \to \bar{N}$ be a non-geodesic ($\kappa \neq 0$) Frenet curve given with the arc parameter s and the elements $\{t, n, b, \kappa, \tau\}$.

From (15) we have

$$\kappa n = \bar{\nabla}_{\vartheta'} z_1 = \nu z_2 + \mu z_3. \tag{24}$$

From the equations (17) and (23) we find

$$\kappa = \sqrt{\nu^2 + \left(\frac{\varepsilon \varrho'}{\sqrt{1-\varepsilon\varrho^2}} - \alpha \sqrt{1-\varepsilon\varrho^2} \right)^2}. \tag{25}$$

On the other hand

$$\begin{aligned} \bar{\nabla}_{\vartheta'} n =& \left(\frac{\nu}{\varepsilon\kappa}\right)' z_2 + \frac{\nu}{\varepsilon\kappa} \nabla_{\vartheta'} z_2 + \left(\frac{\mu}{\varepsilon\kappa}\right)' z_3 + \frac{\mu}{\varepsilon\kappa} \nabla_{\vartheta'} z_3 \\ =& -\kappa t + \varepsilon \tau B. \end{aligned} \tag{26}$$

By using the equations (18) and (19) we find

$$\begin{aligned} \tau b =& \left[\left(\frac{\nu}{\varepsilon\kappa}\right)' + \frac{\mu}{\varepsilon\kappa} \left(\beta - \varepsilon \frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} \right) \right] z_2 \\ &+ \left[\left(\frac{\mu}{\varepsilon\kappa}\right)' + \frac{\nu}{\varepsilon\kappa} \left(\beta - \varepsilon \frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} \right) \right] z_3. \end{aligned} \tag{27}$$

By a direct computation we find following equation

$$\left[\left(\frac{\nu}{\varepsilon\kappa}\right)' \right]^2 + \left[\left(\frac{\mu}{\varepsilon\kappa}\right)' \right]^2 = \left[-\left(\frac{\nu}{\varepsilon\kappa}\right)' \frac{\mu}{\varepsilon\kappa} + \frac{\nu}{\varepsilon\kappa} \left(\frac{\mu}{\varepsilon\kappa}\right)' \right]^2. \tag{28}$$

If we take the norm of the this equation and use the equations (17) and (28) in (27) we get

$$\tau = \left| \left(\beta - \varepsilon \frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} \right) - \sqrt{ \left[\left(\frac{\nu}{\varepsilon\kappa}\right)' \right]^2 + \left[\left(\frac{ \frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}} - \alpha\sqrt{1-\varepsilon\varrho^2} }{\varepsilon\kappa} \right)' \right]^2 } \right|. \tag{29}$$

Theorem 3.1. *Let $\bar{N}$ be a three dimensional n.a.c.p-m manifold and ϑ be a Frenet curve on $\bar{N}$. Then t, n and b can be given as*

$$\begin{aligned} t =& \vartheta' = z_1, \\ n =& \frac{\nu}{\varepsilon\kappa} z_2 + \frac{\mu}{\varepsilon\kappa} z_3, \\ b =& \frac{1}{\varepsilon\tau} \left[\left(\frac{\nu}{\kappa}\right)' - \frac{\mu}{\kappa} \left(\beta - \varepsilon \frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} \right) \right] z_2 \\ &+ \frac{1}{\varepsilon\tau} \left[\left(\frac{\mu}{\kappa}\right)' + \frac{\nu}{\kappa} \left(\beta - \varepsilon \frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} \right) \right] z_3. \end{aligned} \tag{30}$$

Moreover we can write

$$\xi = \varepsilon\varrho t + \frac{\mu\sqrt{1-\varepsilon\varrho^2}}{\kappa} n - \varepsilon \frac{\sqrt{1-\varepsilon\varrho^2}}{\tau} \left[\left(\frac{\mu}{\kappa}\right)' + \frac{\nu}{\kappa} \left(\beta - \varepsilon \frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} \right) \right] b. \tag{31}$$

Theorem 3.2. *Let $\bar{N}$ be a three dimensional n.a.c.p-m manifold and ϑ be a Frenet curve on $\bar{N}$. ϑ is a slant curve on $\bar{N}$ if and only if the Frenet elements $\{t, n, b, \kappa, \tau\}$ of this curve ϑ are as follows*

$$\begin{aligned} t =& z_1 = \vartheta', \\ n =& z_2 = \frac{\varphi\vartheta'}{\sqrt{1-\varepsilon cos^2\theta}}, \\ b =& z_3 = \frac{\xi - \varepsilon cos\theta\vartheta'}{\sqrt{1-\varepsilon cos^2\theta}}, \\ \kappa =& \sqrt{\alpha^2(1-\varepsilon cos^2\theta)+\nu^2}, \\ \tau =& \left| \left(\beta - \varepsilon \frac{cos\theta\nu}{\sqrt{1-\varepsilon\cos^2\theta}}\right) - \sqrt{\left[\left(\frac{\nu}{\varepsilon\kappa}\right)'\right]^2 + \left[\left(\frac{\alpha\sqrt{1-\varepsilon\cos^2\theta}}{\varepsilon\kappa}\right)'\right]^2} \right|. \end{aligned} \tag{32}$$

Proof. Let the curve ϑ be a slant curve on $\bar{N}$. By considering the condition $\varrho = \eta(\vartheta') = cos\theta = constant$ in the equations (13), (25) and (29) we arrive at (32). If (32) holds, it is obvious from the definition of slant curves, ϑ is slant. □

From Theorem 3.2, we easily give the above corollaries.

Corollary 3.3. *Let $\bar{N}$ be a three dimensional n.a.c.p-m manifold and ϑ be a slant curve on $\bar{N}$. If κ is a non-zero constant, then $\tau = \left|\left(\beta - \varepsilon\frac{cos\theta\nu}{\sqrt{1-\varepsilon\cos^2\theta}}\right)\right|$ and ϑ is a pseudo-helix on $\bar{N}$.*

Corollary 3.4. *Let $\bar{N}$ be a three dimensional n.a.c.p-m and ϑ be a slant curve on this manifold $\bar{N}$. If κ is not constant and $\tau = 0$ then ϑ is a plane curve on $\bar{N}$ and the following equation satisfies*

$$\bar{g}(\nabla_{\vartheta'} z_2, z_3) = \frac{\nu^2(\frac{\alpha}{\nu})'\sqrt{1-\varepsilon\cos^2\theta}}{\nu^2+\alpha^2(1-\varepsilon\cos^2\theta)}. \tag{33}$$

Theorem 3.5. *Let $\bar{N}$ be a three dimensional n.a.c.p-m manifold and ϑ be a Frenet curve on $\bar{N}$. ϑ is a Legendre curve($\varrho = \eta(\vartheta') = 0$) on this manifold if and only if the Frenet elements $\{t, n, b, \kappa, \tau\}$ of this curve ϑ are as follows*

$$\begin{aligned} t =& z_1 = \vartheta', \\ n =& z_2 = \varphi\vartheta', \\ b =& z_3 = \xi, \\ \kappa =& \sqrt{\nu^2+\alpha^2}, \\ \tau =& \left|\beta - \sqrt{\left[\left(\frac{\nu}{\varepsilon\kappa}\right)'\right]^2 + \left[\left(\frac{\alpha}{\varepsilon\kappa}\right)'\right]^2}\right|. \end{aligned} \tag{34}$$

Proof. Let the curve ϑ be a Legendre curve on $\bar{N}$. By considering $\varrho = \eta(\vartheta') = 0$ in the equations (13), (25) and (29) we arrive at(34). If the equations in (34) hold, from the definition of Legendre curves it is obvious that the curve ϑ is a Legendre curve on $\bar{N}$. □

Corollary 3.6. *Let the curve ϑ is a Legendre curve in three dimensional n.a.c.p-m manifold $\bar{N}$. If κ is non-zero constant and $\tau = 0$ then ϑ is a plane curve on $\bar{N}$ and $\beta = 0$.*

Moreover we can give the following corollaries.

Corollary 3.7. *Let $\bar{N}$ be a three dimensional n.a.c.p-m manifold and ϑ be a Frenet curve on this manifold. If $\bar{N}$ is cosymplectic, then from the equations* (25) *and* (29) *the curvatures of ϑ are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}}\right)^2} \tag{35}$$

and

$$\tau = \left|\varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} + \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\varrho'}{\kappa\sqrt{1-\varepsilon\varrho^2}}\right)'\right]^2}\right|. \tag{36}$$

i) *If ϑ is a slant, then we get*

$$\kappa = \nu \quad and \quad \tau = \left|\varepsilon\frac{cos\theta}{\sqrt{1-\varepsilon\cos^2\theta}}\right|\kappa. \tag{37}$$

ii) *If ϑ is a Legendere curve, then we get*

$$\kappa = \nu \quad and \quad \tau = 0. \tag{38}$$

Corollary 3.8. *Let ϑ be a curve on three dimensional quasi Sasakian pseudo-metric manifold $\bar{N}$. Then, the curvatures of ϑ are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}}\right)^2} \tag{39}$$

and

$$\tau = \left|\beta - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} + \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\varrho'}{\kappa\sqrt{1-\varepsilon\varrho^2}}\right)'\right]^2}\right|. \tag{40}$$

If the curve ϑ is a slant curve on $\bar{N}$, then we get

$$\kappa = \nu \quad and \quad \tau = \left|\beta - \varepsilon\frac{cos\theta}{\sqrt{1-\varepsilon\cos^2\theta}}\right|\kappa. \tag{41}$$

If the curve ϑ is a Legendre curve on $\bar{N}$, then we obtain

$$\kappa = \nu \quad and \quad \tau = |\beta|. \tag{42}$$

Corollary 3.9. *Let ϑ be a curve on three dimensional β-Sasakian pseudo-metric manifold $\bar{N}$. Then, the curvatures of ϑ are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}}\right)^2} \tag{43}$$

and

$$\tau = \left|\beta - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} + \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\varrho'}{\kappa\sqrt{1-\varepsilon\varrho^2}}\right)'\right]^2}\right|. \tag{44}$$

The curvatures of ϑ are

$$\kappa = \nu \quad and \quad \tau = \left|\beta - \varepsilon\frac{cos\theta}{\sqrt{1-\varepsilon\cos^2\theta}}\right|\kappa \tag{45}$$

where ϑ is a slant curve in three dimensional β-Sasakian pseudo-metric manifold $\bar{N}$ and

$$\kappa = \nu \quad and \quad \tau = |\beta| \tag{46}$$

where ϑ is a Legendre curve in three dimensional β-Sasakian pseudo-metric manifold $\bar{N}$.

Corollary 3.10. *From* (25) *and* (29) *the curvatures of ϑ on tree dimensional Sasakian pseudo-metric manifold $\bar{N}$ are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}}\right)^2} \tag{47}$$

and

$$\tau = \left|\varepsilon\left(1 - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}}\right) - \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\varrho'}{\kappa\sqrt{1-\varepsilon\varrho^2}}\right)'\right]^2}\right|. \tag{48}$$

i) *If ϑ is a slant curve, then we have*

$$\kappa = \nu \quad and \quad \tau = \left|\varepsilon\left(1 - \varepsilon\frac{cos\theta}{\sqrt{1-\varepsilon\cos^2\theta}}\right)\right|\kappa. \tag{49}$$

ii) *If ϑ is a Legendere curve, then we get*

$$\kappa = \nu \quad and \quad \tau = 1. \tag{50}$$

Corollary 3.11. *Let ϑ be a curve on three dimensional α-Kenmotsu pseudo-metric manifold $\bar{N}$. Then the curvatures of ϑ are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}} - \alpha\sqrt{1-\varepsilon\varrho^2}\right)^2} \tag{51}$$

and

$$\tau = \left|\varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} + \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}} - \alpha\sqrt{1-\varepsilon\varrho^2}}{\varepsilon_2\kappa}\right)'\right]^2}\right|. \tag{52}$$

If ϑ is a slant curve on $\bar{N}$, then we obtain

$$\kappa = \sqrt{\nu^2 + \alpha^2(1-\varepsilon\cos^2\theta)}, \tag{53}$$

$$\tau = \left|\varepsilon\frac{\nu cos\theta}{\sqrt{1-\varepsilon\cos^2\theta}} + \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\alpha\sqrt{1-\varepsilon\cos^2\theta}}{\kappa}\right)'\right]^2}\right|. \tag{54}$$

If ϑ is a Legendre curve on $\bar{N}$, then we get

$$\kappa = \sqrt{\nu^2+\alpha^2} \quad and \quad \tau = \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\alpha}{\kappa}\right)'\right]^2}. \tag{55}$$

Corollary 3.12. *Let ϑ be a curve on three dimensional Kenmotsu pseudo-metric manifold $\bar{N}$. Then, the curvatures of ϑ are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}} - \sqrt{1-\varepsilon\varrho^2}\right)^2} \tag{56}$$

and

$$\tau = \left|\varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} + \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}} - \sqrt{1-\varepsilon\varrho^2}}{\varepsilon_2\kappa}\right)'\right]^2}\right|. \tag{57}$$

The curvatures of ϑ are

$$\kappa = \sqrt{\nu^2 + (1 - \varepsilon \cos^2 \theta)}, \tag{58}$$

$$\tau = \left| \varepsilon \frac{\nu cos\theta}{\sqrt{1 - \varepsilon \cos^2 \theta}} + \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\sqrt{1 - \varepsilon \cos^2 \theta}}{\kappa}\right)'\right]^2} \right|. \tag{59}$$

where ϑ is a slant curve in three dimensional Kenmotsu pseudo-metric manifold $\bar{N}$ and

$$\kappa = \sqrt{\nu^2 + 1} \quad and \quad \tau = \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{1}{\kappa}\right)'\right]^2} \tag{60}$$

where ϑ is a Legendre curve in three dimensional Kenmotsu pseudo-metric manifold $\bar{N}$.

4. Conclusion

In this paper we constructed the Frenet apparatus of a non-geodesic Frenet curve on three dimensional normal almost contact pseudo-metric manifold. We gave some theorems about these curves and find their Frenet elements $\{t, n, b, \kappa, \tau\}$. Moreover we gave corollaries for these curves to be slant curve and Legendre curve. So, we characterized some curves on three dimensional normal almost contact pseudo-metric manifolds by using their Frenet elements.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

References

[1] B. E. Acet and S. Y. Perktaş, Curvature and torsion of a legendre curve in (ε, δ) Trans-Sasakian manifolds, Malaya Journal of Matematik, Volume 6, No:1 (2018), 140–144.
[2] D. E. Blair, Progr. Math., Birkhäuser, Boston, 2002.
[3] G. Calvaruso and D. Perrone , Contact pseudo-metric manifolds, Differential Geometry and its Applications, Volume 28 (2010), 615–634.
[4] G. Calvaruso,Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds, Geom. Dedicata, Volume 127 (2007), 99–119.
[5] J. T. Cho, J. Ioguchi and Ji-Eun Lee, On Slant Curves in Sasakian 3-Manifolds, Bulletin of Australian Mahematical Society, Volume 74 (2006), 359–367.
[6] U. C. De and A. K. Mondal, The structure of some classes of 3-dimensional normal almost contact metric manifolds, Bull. Malays. Math. Sci. Soc., Volume 33(2) (2013), 501–509.
[7] J. I. Inoguchi and J. E. Lee, On slant curves in normal almost contact metric 3-manifolds, Beitr Algebra Geom., Volume 55 (2014), 603–620.
[8] Ji-Eun Lee, Slant Curves and Contact Magnetic Curves in Sasakian Lorentzian 3-Manifolds, Symmetry, Volume 11(6) (2019), 784.
[9] B. O'Neill, Semi-Riemannian Geometry, Academic Press, New York, 1983.
[10] B. Olszak,Normal Almost Contact Metric Manifolds of Dimension Three, Annales Polonici Mathematici, Volume XLVII (1986).
[11] T. Takahashi, Sasakian manifold with pseudo-Riemannian metrics, Tôhoku Math. J., Volume 21 (1969), 271–290.
[12] S. Tanno, Almost complex structures in bundle spaces over almost contact manifolds, J. Math. Soc. Japan, Volume 17(2) (1965), 167–186.
[13] V. Venkatesha and D. M. Naik, On 3-dimensional normal almost contact pseudo-metric manifolds, Afrika Matematika, Volume 32 (2021), 139–150.
[14] J. Welyczko, On Legendre Curves in 3-Dimensional Normal Almost Contact Metric Manifolds, Soochow Journal of Mathematics, Volume 33(4) (2007), 929–937.
[15] A. Yıldırım, On curves in 3-dimensional normal almost contact metric manifolds, International Journal of Geometric Methods in Modern Physics, (2020), 1–18.
[16] M. A. Akgün, Frenet curves in 3-dimensional δ-Lorentzian trans Sasakian manifolds,AIMS Mathematics, (2022), 7(1), 199-211

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 3, 177–184
ISSN: 2587–0971

http:/dergipark.gov.tr/tjos

Spherical Curves with Modified Orthogonal Frame with Torsion

Nural Yuksel[a], Murat Kemal Karacan[b], Tuğba Demirkıran[c]

[a]*Erciyes University, Faculty of Sciences, Department of Mathematics, 38030- Melikgazi / KAYSERİ*
[b]*Usak University, Faculty of Sciences and Arts, Department of Mathematics,1 Eylul Campus, 64200,Usak-TURKEY*
[c]*Yalova University, Çınarcık Vocational High School Yalova-TURKEY*

Abstract. In this paper, we studied the spherical curves according to modified orthogonal frame with torsion in 3 dimensional Euclidean space. We obtained the center, the radius and spherical condition of spherical curves according to the 3 dimensional Euclidean space.

1. Introduction

The theory of curves is one of the most important areas of study in differential geometry.The concept of the curve that Euler defined in plane moved to three-dimensional Euclidean space by Fujiwara (1914)[3]. It is well known from the literature that; in order to examine the geometry of a given curve, Frenet equations belonging to this curve must be known. These equations are also known as the Serret-Frenet equations, and it can be understood whether a curve is planar or a line. Studies on this subject were first made for space curves [1, 2]. Considering that the given curve can also be found on a surface, the geometry of these types of curves has been investigated by many mathematicians on the subject [7, 8]. These investigations have been made especially for curves on a sphere, which are called spherical curves [4, 9, 10]. Wong (1963) stated that a global formulation of the condition for a curve to lie in a sphere [4]. This formülation has taken its place as a necessary and sufficient condition for a curve to lie in a sphere in books written on differential geometry. Wong (1972) reached an explicit characterization of spherical curves [5].Considering the definition of the sphere, it is clear that the sphere is actually related to the given dot product. When the subject is considered from this point of view, it can be thought that spherical curves can have very different characterizations in Euclidean and semi-Euclidean spaces. In this study, the spherical curve studies, which were done according to the Serret-Frenet frame previously defined in Euclidean space, which were done according to modified orthogonal frame previously defined in Euclidean space, were made according to the orthogonal frame modified with torsion, also defined in the Euclidean space [6].

Corresponding author: NY mail address: yukseln@erciyes.edu.tr ORCID:0000-0003-3360-5148, MKK ORCID:0000-0002-2832-9444, TD ORCID:0000-0003-0099-6170 .

Received: 24 June 2022; Accepted: 15 September 2022; Published: 30 December 2022
Keywords. (Spherical curves, Modified orthogonal frame with torsion, Frenet-Serret frame.)
2010 *Mathematics Subject Classification*. 53A04, 53A05.
Cited this article as: Yüksel N. Karacan MK. Demirkıran T. Spherical Curves with Modified Orthogonal Frame with Torsion, Turkish Journal of Science, 2022, 7(3), 177–184.

2. Preliminaries

We initially give the classical basic theorem of space curves in 3 dimensional Euclidean space. We assume that the curve $\beta(u)$ in $\mathbb{C}^3$ is parametrized by arc-length. In addition we suppose that its curvature $\kappa(s)$ never vanish. Then orthonormal frame $\{t, n, b\}$ which satisfies the Frenet-Serret equation is as follows:

$$\begin{pmatrix} t'(s) \\ n'(s) \\ b'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} \tag{1}$$

in which t, n, b are the unit tangent, principal normal and binormal vectors, respectively, and $\tau(s)$ is the torsion. Then an orthonormal frame $\{t, n, b\}$ exists satisfying the equation (2.1). Now we assume that the curvature $\kappa(s)$ of β is not identically zero. We define an orthogonal frame $\{T, N, B\}$ by

$$T = \frac{d\beta}{ds}, N = \frac{dT}{ds}, B = T \wedge N$$

in which $\wedge$ denotes the vector product. Then we can give the relation between $\{T, N, B\}$ and $\{t, n, b\}$ as follows:

$$\begin{aligned} T &= t \\ N &= \tau n \\ B &= \tau b \end{aligned} \tag{2}$$

From the definition of $\{T, N, B\}$ or equation (2.2), we can write matrix form as:

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \frac{\kappa(s)}{\tau(s)} & 0 \\ -\kappa(s)\tau(s) & \frac{\tau'(s)}{\tau(s)} & \tau(s) \\ 0 & -\tau(s) & \frac{\tau'(s)}{\tau(s)} \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \tag{3}$$

In addition to, $\{T, N, B\}$ satisfies:

$$\langle T, T\rangle = 1, \langle N, N\rangle = \langle B, B\rangle = \tau^2$$
$$\langle T, N\rangle = \langle T, B\rangle = \langle N, B\rangle = 0$$

in which $\langle , \rangle$ represents the Euclidean inner product [6].

3. Spherical Curves With Modified Orthogonal Frame With Torsion

Definition 3.1. *Let β be in $\mathbb{E}^3$ given by coordinate neighborhood (I, β). If $\beta \subset \mathbb{E}^3$ then β is defined by a spherical curve of $\mathbb{E}^3$ [6].*

Definition 3.2. *The sphere having sufficiently close common four points at $m \in \beta$ the curve $\beta \subset \mathbb{E}^3$ is called the osculating sphere or curvature sphere of the curve β at the point $m \in \beta$ [6].*

Theorem 3.3. *Assume that β is in $\mathbb{E}^3$ given with coordinate neighborhood (I, β). The geometric locus of the centers of the spherical curves with 3-contact points with the curve β providing the modified orthogonal frame with torsion vectors $\{T, N, B\}$ at the point $\beta(s)$, $s \in I$ is*

$$a(s) = \beta(s) + m_2(s)N(s) + m_3(s)B(s)$$

in which

$$m_2 : I \to R, m_2(s) = \frac{1}{\kappa\tau}, \quad and \quad m_3(s) = \frac{\pm 1}{\kappa\tau}\sqrt{r^2k^2 - 1}.$$

Proof. Let (I, β) be a coordinate neighborhood, s is arc length parameter. Let also a be the center and, r be the radius of the sphere with 3-contact points with β. From this, let us consider

$$f : I \to R$$

$$s \to f(s) = \langle a - \beta(s), a - \beta(s) \rangle - r^2 \tag{4}$$

Since

$$f(s) = f^{'}(s) = f^{''}(s) = 0 \tag{5}$$

at the point $\beta(s)$, then the sphere

$$S^2 = \left\{ x \in \mathbb{E}^3 : \langle x - a, x - a \rangle = r^2 \right\}, \text{(x generic point of the sphere)},$$

with the curve β at this point passes sufficiently close three points. Therefore, considering equations (3.1) and (3.2)

$$f(s) = \langle a - \beta(s), a - \beta(s) \rangle - r^2 = 0$$

$$f^{'}(s) = 0 \implies \langle T, a - \beta(s) \rangle = 0$$

is obtained. From this, since $f^{''}(s) = 0$, we get

$$\left\langle T^{'}, a - \beta(s) \right\rangle + \langle T, -\beta'(s) \rangle = 0$$

is obtained. Considering equation (2.3) with this, we have

$$\langle N, a - \beta(s) \rangle = \frac{\tau}{\kappa}$$

On the other words, for the base $\{T, N, B\}$,

$$a - \beta(s) = m_1(s)T(s) + m_2(s)N(s) + m_3(s)B(s) \tag{6}$$

is obtained. But, from using using equation (3.2), we have

$$m_1(s) = \langle a - \beta(s), T(s) \rangle = 0 \tag{7}$$

and

$$m_2(s)\tau^2 = \langle a - \beta(s), N(s) \rangle \implies m_2 = \frac{1}{\kappa\tau} \tag{8}$$

With the assistance of $f(s) = 0$, we have

$$\langle a - \beta(s), a - \beta(s) \rangle = r^2 \to m_1^2(s) + m_2^2(s)\tau^2 + m_3^2(s)\tau^2 = r^2 \tag{9}$$

Considering equation (3.4) and equation (3.5),we have

$$m_3(s) = \frac{\pm 1}{\kappa\tau} \sqrt{r^2 k^2 - 1} = \lambda \in R \tag{10}$$

Therefore, subtituting equations (3.4), (3.5) and (3.7) into equation (3.3)

$$a(s) = \beta(s) + \frac{1}{\kappa\tau} N(s) \pm \frac{1}{\kappa\tau} \sqrt{r^2 k^2 - 1} B(s)$$

Thus, the proof of the theorem is completed. □

Corollary 3.4. *Assume that β is in $\mathbb{E}^3$ given by coordinate neighborhood (I, β). Then the centers of the spheres with 3-contact points with the β at the points $\beta(s) \in \beta$ lie on a straight line.*

Proof. From Theorem 1, we have

$$a(s) = \beta(s) + \frac{1}{\kappa\tau}N(s) + \lambda B(s)$$

The equation with λ parameter denotes a line which pass through the point $C(s) = \beta(s) + \frac{1}{\kappa\tau}N(s)$ and is parallel to the B. □

Definition 3.5. *The line $a(s) = \beta(s) + \frac{1}{\kappa\tau}N(s) + \lambda B(s)$ is the geometric locus of the centers of the spheres with 3-contact points with the curve at $\beta \subset \mathbb{E}^3$ at the point $m \in \beta$ is called curvature the axis at the point $m \in \beta$ of curve $\beta \subset \mathbb{E}^3$. The point*

$$C(s_0) = \beta(s_0) + \frac{1}{\kappa\tau}N(s_0)$$

on curvature the axis is called curvature the center at the point $m = \beta(s_0)$ of curve $\beta \subset \mathbb{E}^3$.

Theorem 3.6. *Assume that β is in $\mathbb{E}^3$ given with coordinate neighborhood (I, β). If*

$$a(s) = \beta(s) + m_2(s)N(s) + m_3(s)B(s)$$

is the center of the osculating sphere at the point $\beta(s) \in \beta$ then

$$m_2(s) = \frac{1}{\kappa\tau} \quad \text{and} \quad m_3(s) = \frac{-\kappa'}{\kappa^2\tau^2}$$

Proof. The proof of the theorem is similar to the proof of Theorem 1. The osculating sphere with the curve β have sufficiently close common four points. So, from $f''(s) = 0$ in equation (3.2) thus $f'''(s) = 0$. Then we get

$$\frac{\kappa'}{\kappa} - \frac{\tau'}{\tau} + \frac{\kappa}{\tau}(-\kappa\tau\langle T, a - \beta(s)\rangle + \frac{\tau'}{\tau}\langle N, a - \beta(s)\rangle + \tau\langle B, a - \beta(s)\rangle) = 0$$

Considering equations (3.4) and (3.5) in the last equality, we get

$$\langle B, a - \beta(s)\rangle = \frac{-\kappa'}{\kappa^2}$$

or

$$m_3(s) = \frac{-\kappa'}{\kappa^2\tau^2}$$

□

Corollary 3.7. *Suppose that β is in $\mathbb{E}^3$ given with coordinate neighborhood (I, β). The radius of the osculating sphere is:*

$$r = \sqrt{(m_2^2(s) + m_3^2(s))\tau^2} = \sqrt{\frac{1}{\kappa^2} + \left(\frac{\kappa'}{\tau\kappa^2}\right)^2}$$

Proof. From Theorem 1,

$$a(s) = \beta(s) + m_2(s)N(s) + m_3(s)B(s)$$

Therefore, we have

$$r = \left\|a - \beta(s)\right\| = \sqrt{(m_2^2\langle N, N\rangle + m_3^2\langle B, B\rangle} = \sqrt{(m_2^2(s) + m_3^2(s))\tau^2} = \sqrt{\frac{1}{\kappa^2} + \left(\frac{\kappa'}{\tau\kappa^2}\right)^2}$$

□

Theorem 3.8. *Let S_0^2 be a sphere centered at zero and also $\beta \subset S_0^2$ be a spherical curve. In this case,*

$$-m_1(s) = \langle \beta(s), T \rangle, \quad -m_2(s) = \frac{\langle \beta(s), N \rangle}{\tau^2} \quad \textit{and} \quad -m_3(s) = \frac{\langle \beta(s), B \rangle}{\tau^2}$$

Proof. Since $\beta \subset S_0^2$ for all $s \in I$, and r is radius, then we have

$$\overrightarrow{0} = \beta(s) + m_1 T + m_2 N + m_3 B$$

and

$$\langle \beta(s), \beta(s) \rangle = r^2$$

Thus, by differentiation of the above equations with respect to s we have

$$-m_1 = \langle \beta(s), T \rangle = 0$$

by differentiation of the above equations with respect to s we have

$$\langle \beta(s), N \rangle = \frac{-\tau}{\kappa}$$

and

$$-m_2(s) = \frac{\langle \beta(s), N \rangle}{\tau^2}$$

and

$$\langle \beta(s), B \rangle = \frac{\kappa'}{\kappa^2}$$

Thus, since $-m_3(s) = \frac{\kappa'}{\kappa^2 \tau^2}$, we can write the last equality as

$$-m_3(s) = \frac{\langle \beta(s), B \rangle}{\tau^2}$$

□

Theorem 3.9. *$S_0^2 \subset \mathbb{E}^3$ be a sphere whose center is at the origin. If β is a curve on S_0^2, then the osculating sphere of the curve β at every point is S_0^2.*

Proof. suppose thet the curve β with (I, β) neighbouring coordinate such that $s \in I$ is arclength parameter. By Theorem 2

$$a(s) = \beta(s) + m_2(s)N(s) + m_3(s)B(s)$$

By Theorem 3, this expression can be written as

$$a(s) = \beta(s) - \frac{\langle \beta(s), N \rangle}{\tau^2} N(s) - \frac{\langle \beta(s), B \rangle}{\tau^2} B(s)$$

Since $\langle \beta(s), T \rangle = 0$, we get

$$a(s) = \beta(s) - \frac{\langle \beta(s), N \rangle}{\tau^2} N(s) - \frac{\langle \beta(s), B \rangle}{\tau^2} B(s)$$

Thus we get

$$a = \beta(s) - \beta(s) = 0$$

This completes the proof of the theorem. □

Theorem 3.10. *Let $\beta : I \to E^3$ be a given curve whose $\tau \neq 0$ for all $s \in I$ and let $m_3(s) \neq 0$. The radius of the osculating sphere at the point $\beta(s)$ is constant for all $s \in I$ if and only if the centers of the osculating sphere are the same point.*

Proof. $\Longrightarrow$: By Corollary 2, we can write as follows

$$r^2 = (m_2^2(s) + m_3^2(s))\tau^2$$

Since r=constant, by differentiation of with this equation respect to s, we have

$$(2m_2m_2' + 2m_3m_3')\tau^2 + 2\tau\tau'(m_2^2 + m_3^2) = 0$$

or

$$m_3' + \frac{\tau'}{\tau}m_3 = \frac{-\tau'}{\tau}\frac{m_2}{m_3}m_2 - \frac{m_2}{m_3}m_2'$$

Inverting values $m_2 = \frac{1}{\kappa\tau}$,$m_3 = \frac{-\kappa'}{\kappa^2\tau^2}$ and $m_2' = \frac{-\kappa'\tau+\tau'\kappa}{\kappa^2\tau^2}$ in right side of the last equality, we obtain

$$m_3' + \frac{\tau'}{\tau}m_3 = \frac{-1}{\kappa} = -\tau m_2$$

Finally, since $m_2 = \frac{1}{\kappa\tau}$, we get

$$m_3' + \frac{\tau'}{\tau}m_3 + \tau m_2 = 0 \tag{11}$$

Otherwise for base $\{T, N, B\}$ we get

$$a(s) = \beta(s) + m_1(s)T(s) + m_2(s)N(s) + m_3(s)B(s)$$

From derivative with respect to s of the last equality, we get

$$a'(s) = (1 - \kappa\tau m_2)T + (m_2' + m_2\frac{\tau'}{\tau} - \tau m_3)N + (\tau m_2 + m_3' + \frac{\tau'}{\tau}m_3)B \tag{12}$$

Inverting values $m_2 = \frac{1}{\kappa\tau}$,$m_3 = \frac{-\kappa'}{\kappa^2\tau^2}$ and $m_2' = \frac{-\kappa'\tau+\tau'\kappa}{\kappa^2\tau^2}$ in right side of the last equality, we obtain

$$a'(s) = (\tau m_2 + m_3' + \frac{\tau'}{\tau}m_3)B$$

So by Equation (3.8) $m_3' + \frac{\tau'}{\tau}m_3 + \tau m_2 = 0$ we find $a'(s) = 0$ and so $a = constant$ for all $s \in I$.
Conversely, suppose that $a(s) = constant$ for all $s \in I$. According to the equation

$$\langle a(s) - \beta(s), a(s) - \beta(s)\rangle = r^2,$$

taking differentiation of this equation with respect to s,we made

$$r(s)r'(s) = 0$$

Here, either $r(s) = 0$ or $r'(s) = 0$. If $r(s) = 0$, then by Corollary 2, we have

$$(m_2^2(s) + m_3^2(s))\tau^2 = 0, \tau \neq 0$$

or

$$m_2^2(s) = -m_3^2(s) = 0$$

But this contradicts the theorem. So $r'(s) = 0$. Thus, $r(s)$ is constant for all $s \in I$. □

Theorem 3.11. *Let $\beta : I \to E^3$ be a curve such that $m_3(s) \neq 0$, for all $s \in I$ and $\tau \neq 0$. Then, the curve β lies on a sphere if and only if the centers of the osculating spheres of the curve β are all the same point.*

Proof. Let β be a curve on S_b^2 which have the radius r and centered at any point b. By Theorem 4, the proof is clear. Conversely, let the centers of the osculating curve be the point b in $\beta(s) \in \beta$ all $s \in I$. Then by Theorem 5 all osculating spheres have the same radius r. Therefore

$$d(\beta(s), b) = r$$

for all $s \in I$. This completes the proof of the theorem. □

Theorem 3.12. *Let the curve β in $\mathbb{E}^3$ be given with coordinate neighborhood (I, β) and $m_3(s) \neq 0$, $\tau \neq 0$ such that s is a arclength parameter, then, β is a spherical curve if and only if*

$$(\frac{-\kappa^{'}}{\kappa^2\tau^2})^{'} - \frac{\tau^{'}\kappa^{'}}{\kappa^2\tau^3} + \frac{1}{\kappa} = 0$$

Proof. Let β be a spherical curve. By Theorem 6 , for all $s \in I$, the center $a(s)$ of the osculating spheres are constant. Moreover, the equation (3.8) yields

$$m_3^{'} + \frac{\tau^{'}}{\tau} m_3 + \tau m_2 = 0$$

or

$$(\frac{-\kappa^{'}}{\kappa^2\tau^2})^{'} - \frac{\tau^{'}\kappa^{'}}{\kappa^2\tau^3} + \frac{1}{\kappa} = 0$$

Conversely, let $m_3^{'} + \frac{\tau^{'}}{\tau} m_3 + \tau m_2 = 0$ By Theorem 5 and $a^{'}(s) = 0$. Therefore $a(s) = constant$. So by Theorem 6, β is a spherical curve. □

Example 3.13. *Let the curve β such that $c = 2\sqrt{ab}$ and $r = a + b$.*

$$\beta(t) = (acost + bcos3t, asint - bsin3t, csin2t)$$

We find radius and center of the osculating sphere at the point $t = 0$. Since

$$\beta'(t) = (-asint - 3bsin3t, acost - 3bcos3t, 2ccos2t)$$

$$\left\|\beta'(t)\right\| = \sqrt{(a + 3b)^2 + (ccos2t)^2}.$$

For $t = 0$, $a = 1$, $b = 1$, $c = 2$, $r = 2$, Since $\left\|\beta'(0)\right\| = 2\sqrt{5}$ t is a arbitrary parameter. From this, we can find $\{t, n, b\}$ Frenet vectors.

$$t = \frac{\beta^{'}}{\left\|\beta'\right\|} = (0, \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$$

$$b = \frac{\beta^{'}(0) \wedge \beta^{''}(0)}{\left\|\beta^{'}(0) \wedge \beta^{''}(0)\right\|} = (0, \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}})$$

$$n = b \wedge t = (-1, 0, 0)$$

The curvature and torsion of the curve β are as follows:

$$\kappa = \frac{\left\|\beta^{'}(0) \wedge \beta^{''}(0)\right\|}{\left\|\beta'(0)\right\|^3} = \frac{1}{2}$$

$$\tau = \frac{det(\beta', \beta'', \beta''')}{\left\|\beta^{'}(0) \wedge \beta^{''}(0)\right\|^2} = \frac{-9}{25}$$

From this, we can find $\{T, N, B\}$ *Frenet vectors of the modified orthogonal frame with torsion.*

$$T = t = (0, \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$$

$$N = \tau n = (\frac{9}{25}, 0, 0)$$

$$B = \tau b = (0, \frac{18}{25\sqrt{5}}, \frac{9}{25\sqrt{5}})$$

We can find coordinates of the center of the osculating sphere at the point $\beta(0)$. *From* $a(t) = \beta(t) + m_2(t)N(t)$, $a(0) = (4, 0, 0)$. *We can find the radius of the osculating sphere at the point* $\beta(0)$ *as seen in Figure 1 .*

$$r = \sqrt{(m_2^2(t) + m_3^2(t))\tau^2} = 2$$

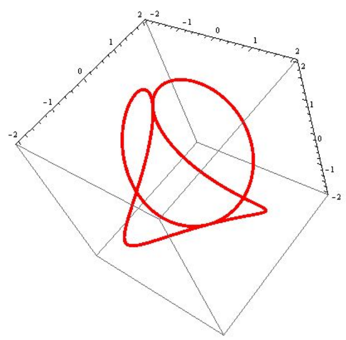

Figure.1. β spherical curve .

4. Conclusion

In this study , we obtained the center, the radius and spherical condition of spherical curves according to modified orthogonal frame with torsion. An example of a spherical curve is given according to this frame. The results obtained can also be found according to other frame, for future works.

References

[1] Carmo, M.D. Differential Geometry of Curves and Surfaces, Prentice-Hall, New Jersey, (1976).

[2] Milman, R.S., Parker, G.D. Elements of Differential Geometry, Prentice-Hall Inc., Englewood Clifs, New Jersey, (1977).

[3] Fujivara, M. On Space Curves Of Constant Breadth, Thoku Math. J., 5, 179-184, (1914).

[4] Wong, Y.C. A Global Formulation of the Condition for a curve to lie in a sphere, Monatsh.Math., 67, 363-365, (1963).

[5] Wong, Y.C. On an Explicit Characterization of Spherical Curves, Proc. Amer. Math. Soc., 34, 239-242, (1972).

[6] Bukcu, B., Karacan, M.K. Spherical Curves with Modified Orthogonal Frame, Journal of New Results in Science, (2016).

[7] Cakmak, A., Sahin, V. Characterizations of adjoint curves according to alternative moving frame, Fundamental Journal of Mathematics and Applications, 5 (1) (2022) 42-50.

[8] Yuksel, N.,Vanlı, A., Damar, E. A New Approach For Geometric Properties of DNA Structure in E3, Life Science Journal,(2015) 12-2.

[9] Yuksel, N., Saltık, B., Damar, E. Spherical Images of Salkowski Curve in 3- Dimensional Minkowski Space, International Marmara Scientific Research and Innovation Congress,(2022).

[10] Şenyurt, S., Uzun, M. Smarandache Curves of Anti-Salkowski Curve According to The Spherical İndicatrix Curve of The Unit Darboux Vector, GÜFBED/GUSTIJ (2021) 11 (4): 1304-1314.

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 3, 185-201
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

New Variants of Hermite-Hadamard Type Inequalities via Generalized Fractional Operator for Differentiable Functions

Jamshed Nasir[a], Saad Ihsan Butt[a], Mustafa Ali Dokuyucu[b], Ahmet Ocak Akdemir[b]

[a]*COMSATS University of Islamabad, Lahore Campus, Lahore, Pakistan*
[b]*Department of Mathematics, Faculty of Science and Letters, Ağri Ibrahim Cecen University, Ağri, Turkey*

Abstract. The main motivation of this study is to present new Hermite-Hadamard (HH) type inequalities via a certain fractional operators. We establish two new identities and give new estimations of HH- type inequalities for differentiable and convex mapping via Katugampola-fractional operators. Here, we gave new Lemmas having identities for differentiable functions and construct related inequalities. Main findings of this study would provide elegant connections and general variants of well known results established recently. These results can be extended to different kinds of convex functions as well as pre-invex functions.

1. Introduction

Convexity is a very functional concept in programming, statistics and numerical analysis as in many different branches of mathematics. In theory of inequality, the concept of convexity exists in the proof of many classical inequalities, but has been a source of inspiration for many new and useful inequalities.

Definition 1.1. *[22]. The function $f : [c_1, c_2] \to \Re$, is said to be convex, if we have*

$$f(t\kappa + (1-t)\tau) \leq t f(\kappa) + (1-t) f(\tau)$$

for all $\kappa, \tau \in [c_1, c_2]$ and $t \in [0, 1]$.

In addition to the use of convex functions in many fields, inequality has increased its reputation in theory with the Hermite-Hadamard inequality (See [22]). This celebrated inequality can be stated as: If a mapping $\Upsilon : J \subseteq \Re \to \Re$ is a convex function on J and $r, s \in J$, $r < s$, then

$$\Upsilon\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r} \int_r^s \Upsilon(\lambda) d\lambda \leq \frac{\Upsilon(r) + \Upsilon(s)}{2}.$$

Fractional calculus is a good expansion of the concept of derivative operator from integer order n to arbitrary order a. Fractional derivative operators are accepted as the inverse of fractional integral operators. Recently, the multiplicity of applications in many fields of engineering, physics, statistics and mathematics

Corresponding author: MAD: mail address: mustafaalidokuyucu@gmail.com ORCID:0000-0001-9331-8592, JN ORCID:0000-0002-7141-4089, SIB ORCID:0000-0001-7192-8269, AOA ORCID:0000-0003-2466-0508
Received: 24 November 2022; Accepted: 15 December 2022; Published: 30 December 2022
Keywords. Convex functions; Hermite-Hadamard inequalities; Katugampola-fractional integrals.
2010 *Mathematics Subject Classification*. 26A33, 26A51, 26D10, 26D07.
Cited this article as: Nasir J. Butt SI. Dokuyucu MA. and Akdemir AO. New Variants of Hermite-Hadamard type inequalities Via Generalized fractional operator for Differentiable Functions, Turkish Journal of Science, 2022, 7(3), 185-201.

has led to the study of fractional integrals by many researchers. The fact that they are a more effective tool than the results in classical analysis has resulted in more use of these operators on real world problems. Since the definition of the convex functions has been given as an inequality, this concept has established a powerful link between convexity and inequalities. It is now become a trending aspect of mathematical research to generalize classical known results via fractional integral operator. Although fractional analysis is basically a generalization of classical analysis, it has developed rapidly with the concepts of fractional order operators. Fractional analysis has recently become a popular topic with its applications in many fields such as modeling, physics, approximation theory, engineering, control theory and mathematical biology, based on applied mathematics problems (see [1], [3], [8], [9]-[11], [17], [18]-[21] and [23]-[26]).

Recently in [14], the author introduced a new concept to unify Riemann-Liouville and Hadamard fractional integral operators which a certain general form for fractional integral operators. Also the conditions are given so that the operator is bounded in an extended Lebesgue measurable space. The corresponding fractional derivative approach to this new generalized operator can be seen in [15]. Moreover, Katugampola worked for the Mellin transforms of the fractional integrals and derivatives (see [16]).

Definition 1.2. *([14]) Let $[\kappa, \tau] \subset \Re$ be a finite interval. Then the left-sided and right-sided Katugampola fractional integrals of order $\xi > 0$ of $\Upsilon \in X_c^{\nu}(\kappa^{\nu}, \tau^{\nu})$ are defined as follows:*

$$({}^{\nu}I^{\xi}_{\kappa+}\Upsilon)(x) = \frac{\nu^{1-\xi}}{\Gamma(\xi)}\int_{\kappa}^{x}\frac{\Upsilon(\lambda)}{(x-\lambda)^{1-\xi}}\lambda^{\nu-1}d\lambda,\ x > \kappa$$

and

$$({}^{\nu}I^{\xi}_{\tau-}\Upsilon)(x) = \frac{\nu^{1-\xi}}{\Gamma(\xi)}\int_{x}^{\tau}\frac{\Upsilon(\lambda)}{(\lambda-x)^{1-\xi}}\lambda^{\nu-1}d\lambda,\ x < \tau,$$

with $\kappa < x < \tau$ and $\nu > 0$, if the integrals exist.

Theorem 1.3. *([14]) If $\xi > 0$ and $\nu > 0$, then for $x > \kappa$*

$$1) \lim \nu \to 1({}^{\nu}I^{\xi}_{\kappa^+}\Upsilon)(x) = (J^{\xi}_{\kappa^+}\Upsilon)(x)$$

$$2) \lim \nu \to (0^+)({}^{\nu}I^{\xi}_{\kappa^+}\Upsilon)(x) = (H^{\xi}_{\kappa^+}\Upsilon)(x).$$

The main motivation point of the study is to prove the HH type inequalities with specific and general forms for the functions whose absolute values of derivatives are convex and concave functions with the help of the fractional integral operator, which has a general kernel structure. The main results are reduced to the results available in the literature in some special cases, as well as giving new approximations and estimates for differentiable and convex functions. To obtain our results, we used some known proof methods alongside classical inequalities such as the Hölder inequality, Power mean inequality, and Weighted Hölder inequality.

2. Hermite-Hadamard Type inequalities for Katugampola-Fractional Integrals

We will start with the following identities that will be useful to prove our main findings via Katugampola fractional integrals:

Lemma 2.1. *Let $\xi \in (0,1)$ and $\nu > 0$ and $f : [\kappa^\nu, \tau^\nu] \to \Re$ be a twice differentiable mapping on (κ^ν, τ^ν) with $0 < \kappa^\nu < \tau^\nu$. Then the following equality holds for Katugampola fractional integral operators:*

$$\begin{aligned}|A| =& \frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\left(\tau^\nu - \kappa^\nu\right)^\xi}\Big(\big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\big)f\big(\tau^\nu\big) + \big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\big)f\big(\kappa^\nu\big)\Big) - f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)\\
=& \frac{(\tau^\nu - \kappa^\nu)}{4}\Big[\int_0^1 t^{\nu\xi}t^{\nu-1}f'\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)dt + \int_0^1 t^{\nu\xi}t^{\nu-1}f'\Big(\frac{t^\nu}{2}\tau^\nu + \frac{2-t^\nu}{2}\kappa^\nu\Big)dt\Big]\\
=& \frac{(\tau^\nu - \kappa^\nu)^2}{8(\xi+1)}\Big[\int_0^1 t^{\nu(\xi+1)}t^{\nu-1}f''\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)dt\\
&+ \int_0^1 t^{\nu(\xi+1)}t^{\nu-1}f''\Big(\frac{t^\nu}{2}\tau^\nu + \frac{2-t^\nu}{2}\kappa^\nu\Big)dt\Big].\end{aligned}$$

Proof. By applying integration by parts to the right hand side of the equality, we have

$$\begin{aligned}& k_1\\
=& \int_0^1 t^{\nu\xi}t^{\nu-1}f'\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)dt\\
=& f'\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)\frac{t^{\nu(\xi+1)}}{\nu(\xi+1)}\Big|_0^1\\
&- \int_0^1 \frac{t^{\nu(\xi+1)}}{\nu(\xi+1)}f''\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)\nu t^{\nu-1}\Big(\frac{\kappa^\nu-\tau^\nu}{2}\Big)dt\\
=& \frac{1}{\nu(\xi+1)}f'\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)\\
&- \frac{\left(\frac{\kappa^\nu-\tau^\nu}{2}\right)}{(\xi+1)}\int_0^1 t^{\nu(\xi+1)}t^{\nu-1}f''\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)dt.\end{aligned}$$

Similarly, we can write

$$\begin{aligned}k_2 =& \int_0^1 t^{\nu\xi}t^{\nu-1}f'\Big(\frac{t^\nu}{2}\tau^\nu + \frac{2-t^\nu}{2}\kappa^\nu\Big)dt\\
=& \frac{1}{\nu(\xi+1)}f'\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)\\
&- \frac{\left(\frac{\tau^\nu-\kappa^\nu}{2}\right)}{(\xi+1)}\int_0^1 t^{\nu(\xi+1)}t^{\nu-1}f''\Big(\frac{t^\nu}{2}\tau^\nu + \frac{2-t^\nu}{2}\kappa^\nu\Big)dt.\end{aligned}$$

Now, by taking into account $(k_1 - k_2)$, we obtain

$$\begin{aligned}& (k_1 - k_2) \qquad (1)\\
=& \frac{(\tau^\nu - \kappa^\nu)}{2(\xi+1)}\Big[\int_0^1 t^{\nu(\xi+1)}t^{\nu-1}f''\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)dt\\
&+ \int_0^1 t^{\nu(\xi+1)}t^{\nu-1}f''\Big(\frac{t^\nu}{2}\tau^\nu + \frac{2-t^\nu}{2}\kappa^\nu\Big)dt\Big].\end{aligned}$$

On the other hand, we have

$$I_1 = \int_0^1 (t^\nu)^\xi t^{\nu-1}f'\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)dt$$

$$= (t^\nu)^\xi t^{\nu-1}\frac{f\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)}{\nu t^{\nu-1}\left(\frac{\kappa^\nu-\tau^\nu}{2}\right)}\Big|_0^1-\int_0^1 \xi(t^\nu)^{\xi-1}\nu t^{\nu-1}\frac{t^{\nu-1}f\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)}{\nu t^{\nu-1}\left(\frac{\kappa^\nu-\tau^\nu}{2}\right)}dt$$

$$= \frac{-2f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)}{\nu(\kappa^\nu-\tau^\nu)}+\frac{2^{\xi+1}\xi\Gamma(\xi)}{(\tau^\nu-\kappa^\nu)^{\xi+1}\nu^{1-\xi}}\left(\left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{+}}\right)f\left(\tau^\nu\right)\right).$$

By a similar way, it is easy to see that

$$\begin{aligned} I_2 &= \int_0^1 (t^\nu)^\xi t^{\nu-1}f'\left(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\right)dt \\ &= \frac{2f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)}{\nu(\tau^\nu-\kappa^\nu)} \\ &\quad -\frac{2^{\xi+1}\xi\Gamma(\xi)}{(\tau^\nu-\kappa^\nu)^{\xi+1}\nu^{1-\xi}}\left(\left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{-}}\right)f\left(\kappa^\nu\right)\right). \end{aligned}$$

Thus,

$$\begin{aligned} &(I_1-I_2) \\ &= \frac{-4f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)}{\nu(\tau^\nu-\kappa^\nu)}+\frac{2^{\xi+1}\Gamma(\xi+1)}{\nu^{1-\xi}(\tau^\nu-\kappa^\nu)^{\xi+1}} \\ &\quad \times\left[\left(\left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{+}}\right)f\left(\tau\right)\right)+{}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{-}}f(\kappa^\nu)\right]. \end{aligned} \tag{2}$$

Multiplying (1) and (2) by $\frac{\nu(\tau^\nu-\kappa^\nu)}{4}$, we get the desired result. □

Lemma 2.2. *Let $\xi \in (0,1)$ and $\nu > 0$ and $f : [\kappa^\nu, \tau^\nu] \to \Re$ be a twice differentiable mapping on (κ^ν, τ^ν) with $0 < \kappa < \tau$. Then, the following equality holds for Katugampola fractional integral operators:*

$$\begin{aligned} |B| = \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}&\left[\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}f''\left(\frac{1-t^\nu}{2}\kappa^\nu+\frac{1+t^\nu}{2}\tau^\nu\right)dt\right. \\ &\left.+\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}f''\left(\frac{1-t^\nu}{2}\tau^\nu+\frac{1+t^\nu}{2}\kappa^\nu\right)dt\right] \\ = \frac{2^{\xi-1}\Gamma(\xi+1)\nu^\xi}{(\tau^\nu-\kappa^\nu)^\xi}&\left[\left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{+}}\right)f\left(\tau^\nu\right)+\left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{-}}\right)f\left(\kappa^\nu\right)-f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)\right]. \end{aligned}$$

Proof. By integration by parts for the right hand side of the equality, we have

$$\begin{aligned} I_1 &= \int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}f''\left(\frac{1-t^\nu}{2}\kappa^\nu+\frac{1+t^\nu}{2}\tau^\nu\right)dt \\ &= \frac{-2f'(\frac{\kappa^\nu+\tau^\nu}{2})}{\nu(\tau^\nu-\kappa^\nu)}+\frac{2(\xi+1)}{\tau^\nu-\kappa^\nu} \\ &\quad -\left[\frac{2f(\frac{\kappa^\nu+\tau^\nu}{2})}{\tau^\nu-\kappa^\nu}+\frac{2\xi}{\tau^\nu-\kappa^\nu}\int_0^1 (1-t^\nu)^{\xi-1}t^{\nu-1}f(\frac{1-t^\nu}{2}\kappa^\nu+\frac{1+t^\nu}{2}\tau^\nu)\right]. \end{aligned}$$

By changing of the variables, we get

$$\begin{aligned} I_1 &= \int_0^1 (1-t^\nu)^{\xi+1} t^{\nu-1} f''\Big(\frac{1-t^\nu}{2}\kappa^\nu + \frac{1+t^\nu}{2}\tau^\nu\Big)dt \\ &= \frac{-2f'(\frac{\kappa^\nu+\tau^\nu}{2})}{\nu(\tau^\nu-\kappa^\nu)} + \frac{2(\xi+1)}{\tau^\nu-\kappa^\nu} \\ &\quad \times\Bigg[\frac{-2f(\frac{\kappa^\nu+\tau^\nu}{2})}{\tau^\nu-\kappa^\nu} + \frac{2^{\xi+1}\xi}{(\tau^\nu-\kappa^\nu)^{\xi+1}} \int_{(\frac{\kappa^\nu+\tau^\nu}{2})^{\frac{1}{\nu}}}^{\tau} (\tau^\nu-u^\nu)^{\xi-1} u^{\nu-1} f(u^\nu)du\Bigg] \end{aligned}$$

Multiplying the resulting equality by $\frac{\Gamma(\xi)\nu^{1-\xi}}{\Gamma(\xi)\nu^{1-\xi}}$, we obtain

$$\begin{aligned} I_1 &= \int_0^1 (1-t^\nu)^{\xi+1} t^{\nu-1} f''\Big(\frac{1-t^\nu}{2}\kappa^\nu + \frac{1+t^\nu}{2}\tau^\nu\Big)dt \\ &= \frac{-2f'(\frac{\kappa^\nu+\tau^\nu}{2})}{\nu(\tau^\nu-\kappa^\nu)} + \frac{2(\xi+1)}{\tau^\nu-\kappa^\nu} \\ &\quad -\Bigg[\frac{2f(\frac{\kappa^\nu+\tau^\nu}{2})}{\tau^\nu-\kappa^\nu} + \frac{2^{\xi+1}\xi\Gamma(\xi)}{(\tau^\nu-\kappa^\nu)^{\xi+1}\nu^{1-\xi}}\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(\tau^\nu\big)\Bigg]. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= \int_0^1 (1-t^\nu)^{\xi+1} t^{\nu-1} f''\Big(\frac{1-t^\nu}{2}\tau^\nu + \frac{1+t^\nu}{2}\kappa^\nu\Big)dt \\ &= \frac{2f'(\frac{\kappa^\nu+\tau^\nu}{2})}{\nu(\tau^\nu-\kappa^\nu)} + \frac{2(\xi+1)}{\tau^\nu-\kappa^\nu} \\ &\quad -\Bigg[\frac{2f(\frac{\kappa^\nu+\tau^\nu}{2})}{\tau^\nu-\kappa^\nu} + \frac{2^{\xi+1}\xi\Gamma(\xi)}{(\tau^\nu-\kappa^\nu)^{\xi+1}\nu^{1-\xi}}\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^\nu\big)\Bigg]. \end{aligned}$$

Namely,

$$\begin{aligned} & I_1 + I_2 \\ = \; & -\frac{8(\xi+1)f(\frac{\kappa^\nu+\tau^\nu}{2})}{\nu(\tau^\nu-\kappa^\nu)^2} + \frac{2^{\xi+2}\Gamma(\xi+2)}{(v^\nu-\kappa^\nu)^{\xi+2}\nu^{1-\xi}} \\ & \times\Bigg[\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(\tau^\nu\big) + \Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^\nu\big)\Bigg] \end{aligned}$$

Now, multiplying both sides by $\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}$, we provide

$$\begin{aligned} & \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big(I_1 + I_2\Big) \\ = \; & -f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big) \\ & +\frac{2^{\xi-1}\Gamma(\xi+1)\nu^\xi}{(\tau^\nu-\kappa^\nu)^\xi}\Big[\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(\tau^\nu\big) + \Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^\nu\big)\Big]. \end{aligned}$$

Which completes the proof. □

Theorem 2.3. *Suppose that* $f : [\kappa^\nu, \tau^\nu] \to \Re$ *be a twice differentiable function on* (κ^ν, τ^ν) *with* $0 \le \kappa < \tau$. *If* $|f''|$ *is convex function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\left(\tau^\nu-\kappa^\nu\right)^{\xi}}\left(\left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{+}}\right)f\left(\tau^\nu\right)+\left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{-}}\right)f\left(\kappa^\nu\right)\right)-f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)$$

$$\le \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\left(\frac{1}{\nu(\xi+2)}\right)\Big[|f''(\kappa^\nu)|+|f''(\tau^\nu)|\Big].$$

Proof. By using right hand side of the Lemma (2.1), we can write

$$|A| \le \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\left[\int_0^1 t^{\xi\nu+\nu}t^{\nu-1}\left[\frac{t^\nu}{2}\left|f''(\kappa^\nu)\right|+\frac{2-t^\nu}{2}\left|f''(\tau^\nu)\right|\right]dt + \int_0^1 t^{\xi\nu+\nu}\left[\frac{t^\nu}{2}\left|f''(\tau^\nu)\right|+\frac{2-t^\nu}{2}\left|f''(\kappa^\nu)\right|\right]dt\right].$$

By making use of the necessary calculations, we get

$$|A| \le \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\left(\frac{1}{\nu(\xi+2)}\right)\Big[|f''(\kappa^\nu)|+|f''(\tau^\nu)|\Big].$$

Which completes the proof. □

Theorem 2.4. *Suppose that* $f : [\kappa^\nu, \tau^\nu] \to \Re$ *be a twice differentiable function on* (κ^ν, τ^ν) *with* $0 \le \kappa < \tau$. *If* $|f''|$ *is convex function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\left(\tau^\nu-\kappa^\nu\right)^{\xi}}\left(\left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{+}}\right)f\left(\tau^\nu\right)+\left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{-}}\right)f\left(\kappa^\nu\right)\right)-f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)$$

$$\le \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\left(\frac{1}{\nu s(\xi+2)-s+1}\right)^{\frac{1}{q}}$$

$$\left[\left(\frac{1}{2(\nu+1)}|f''(\kappa^\nu)|^r+\frac{2\nu+1}{2(\nu+1)}|f''(\tau^\nu)|^r\right)^{\frac{1}{p}}+\left(\frac{1}{2(\nu+1)}|f''(\tau^\nu)|^r+\frac{2\nu+1}{2(\nu+1)}|f''(\kappa^\nu)|^r\right)^{\frac{1}{p}}\right].$$

for $p > 1$ *and* $q > 1$.

Proof. From the right hand side of Lemma (2.1), we have

$$|A| \le \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\left[\int_0^1 t^{\nu\xi+\nu}t^{\nu-1}\left|f''\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)+f''\left(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\right)\right|dt\right]$$

By using the Hölder inequality, we get

$$|A| \le \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\left[\left(\int_0^1 (t^{\nu(\xi+2)-1})^q dt\right)^{\frac{1}{q}}\left(\int_0^1\left|f''\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)\right|^p dt\right)^{\frac{1}{p}} + \left(\int_0^1 \left(t^{\nu(\xi+2)-1}\right)^q dt\right)^{\frac{1}{q}}\left(\int_0^1\left|f''\left(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\right)\right|^p dt\right)^{\frac{1}{p}}\right]$$

$$\leq \frac{(\tau^{\nu}-\kappa^{\nu})^{2}}{8(\xi+1)}\Big(\int_{0}^{1}(t^{\nu(\xi+2)-1})^{q}dt\Big)^{\frac{1}{q}}$$

$$\Big[\int_{0}^{1}\Big(\frac{t^{\nu}}{2}|f''(\kappa^{\nu})|^{p}+\frac{2-t^{\nu}}{2}|f''(\tau^{\nu})|^{p}\Big)dt+\int_{0}^{1}\Big(\frac{t^{\nu}}{2}|f''(\tau^{\nu})|^{p}+\frac{2-t^{\nu}}{2}|f''(\kappa^{\nu})|^{p}\Big)dt\Big]^{\frac{1}{p}}$$

Thus, we provide

$$|A|\leq \frac{(\tau^{\nu}-\kappa^{\nu})^{2}}{8(\xi+1)}\left(\frac{1}{\nu q(\xi+2)-q+1}\right)^{\frac{1}{q}}$$

$$\left[\left(\frac{1}{2(\nu+1)}|f''(\kappa^{\nu})|^{r}+\frac{2\nu+1}{2(\nu+1)}|f''(\tau^{\nu})|^{r}\right)^{\frac{1}{r}}+\left(\frac{1}{2(\nu+1)}|f''(\tau^{\nu})|^{r}+\frac{2\nu+1}{2(\nu+1)}|f''(\kappa^{\nu})|^{r}\right)^{\frac{1}{r}}\right].$$

This completes the proof. □

Theorem 2.5. *If $f:[\kappa^{\nu},\tau^{\nu}]\rightarrow\Re$ be differentiable function on $(\kappa^{\nu},\tau^{\nu})$ with $\kappa^{\nu}<\tau^{\nu}$ and $f''\in L_{1}[\kappa^{\nu},\tau^{\nu}]$. If $|f''|$ is a concave function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\left(\tau^{\nu}-\kappa^{\nu}\right)^{\xi}}\Big(\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(\tau^{\nu}\big)+\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^{\nu}\big)\Big)-f\Big(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\Big)$$

$$\leq\frac{(\tau^{\nu}-\kappa^{\nu})^{2}}{8(\xi+1)}\left[\Big(\frac{1}{\nu(\xi+2)}\Big)\left[\left|f''\left(\frac{\kappa^{\nu}}{2\nu(\xi+3)}+\left(\frac{2}{\nu(\xi+2)}-\frac{1}{\nu(\xi+3)}\right)\frac{\tau^{\nu}}{2}\right)\right|\right.\right.$$

$$\left.\left.+\left|f''\left(\frac{\tau^{\nu}}{2\nu(\xi+3)}+\left(\frac{2}{\nu(\xi+2)}-\frac{1}{\nu(\xi+3)}\right)\frac{\kappa^{\nu}}{2}\right)\right|\right]\right].$$

Proof. From Lemma 2.1, we have

$$\left|\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\left(\tau^{\nu}-\kappa^{\nu}\right)^{\xi}}\Big(\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(b^{\nu}\big)+\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^{\nu}\big)\Big)-f\Big(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\Big)\right|$$

$$\leq\frac{(\tau^{\nu}-\kappa^{\nu})^{2}}{8(\xi+1)}\left[\int_{0}^{1}t^{\nu(\xi+1)}t^{\nu-1}\left|f''\Big(\frac{t^{\nu}}{2}\kappa^{\nu}+\frac{2-t^{\nu}}{2}\tau^{\nu}\Big)\right|dt\right.$$

$$\left.+\int_{0}^{1}t^{\nu(\xi+1)}t^{\nu-1}\left|f''\Big(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu}\Big)\right|dt\right]$$

By applying Jensen Integral inequality, we get

$$|A|\leq\frac{(\tau^{\nu}-\kappa^{\nu})^{2}}{8(\xi+1)}\left[\Big(\int_{0}^{1}t^{\nu(\xi+2)-1}dt\Big)\left|f''\left(\frac{\int_{0}^{1}t^{\nu(\xi+2)-1}\Big(\frac{t^{\nu}}{2}\kappa^{\nu}+\frac{2-t^{\nu}}{2}\tau^{\nu}\Big)dt}{\int_{0}^{1}t^{\nu(\xi+2)-1}dt}\right)\right|\right.$$

$$\left.+\Big(\int_{0}^{1}t^{\nu(\xi+2)-1}dt\Big)\left|f''\left(\frac{\int_{0}^{1}t^{\nu(\xi+2)-1}\Big(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu}\Big)dt}{\int_{0}^{1}t^{\nu(\xi+2)-1}dt}\right)\right|\right]$$

$$=\frac{(\tau^{\nu}-\kappa^{\nu})^{2}}{8(\xi+1)}\left[\Big(\frac{1}{\nu(\xi+2)}\Big)\left[\left|f''\left(\frac{\kappa^{\nu}}{2\nu(\xi+3)}+\left(\frac{2}{\nu(\xi+2)}-\frac{1}{\nu(\xi+3)}\right)\frac{\tau^{\nu}}{2}\right)\right|\right.\right.$$

$$\left.\left.+\left|f''\left(\frac{\tau^{\nu}}{2\nu(\xi+3)}+\left(\frac{2}{\nu(\xi+2)}-\frac{1}{\nu(\xi+3)}\right)\frac{\kappa^{\nu}}{2}\right)\right|\right]\right].$$

Which completes the proof. □

Theorem 2.6. *If $f:[\kappa^\nu,\tau^\nu]\to\Re$ be differentiable function on (κ^ν,τ^ν) with $\kappa^\nu<\tau^\nu$ and $f''\in L_1[\kappa^\nu,\tau^\nu]$. If $|f''|^q$ is a convex function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\left(\tau^\nu-\kappa^\nu\right)^{\xi}}\Big(\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(\tau^\nu\big)+\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^\nu\big)\Big)-f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)$$

$$\le\frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{\nu(\xi+2)}\Big)^{1-\frac{1}{q}}\Bigg[\frac{|f''(\kappa^\nu)|^q+|f''(\tau^\nu)|^q}{\nu(\xi+2)}\Bigg]^{\frac{1}{q}}\Bigg].$$

Proof. From Lemma 2.1, we have

$$\left|\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\left(\tau^\nu-\kappa^\nu\right)^{\xi}}\Big(\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(\tau^\nu\big)+\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^\nu\big)\Big)-f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)\right|$$

$$\le\frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\int_0^1 t^{\nu(\xi+1)}t^{\nu-1}\Big|f''\Big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\Big)\Big|dt$$

$$+\int_0^1 t^{\nu(\xi+1)}t^{\nu-1}\Big|f''\Big(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\Big)\Big|dt\Bigg]$$

By applying Power-mean inequality, we get

$$|A|\le\frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\int_0^1 t^{\nu(\xi+2)-1}\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^{\nu(\xi+2)-1}\Big|f''\Big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 t^{\nu(\xi+2)-1}\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^{\nu(\xi+2)-1}\Big|f''\Big(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}\Bigg]$$

By using convexity of $|f''|^q$, we get

$$|A|\le\frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}$$

$$\Bigg[\Big(\int_0^1 t^{\nu(\xi+2)-1}\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^{\nu(\xi+2)-1}\frac{t^\nu}{2}|f''(\kappa^\nu)|^q dt+\int_0^1\frac{2-t^\nu}{2}|f''(\tau^\nu)|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 t^{\nu(\xi+2)-1}\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^{\nu(\xi+2)-1}\frac{t^\nu}{2}|f''(\tau^\nu)|^q dt+\int_0^1\frac{2-t^\nu}{2}|f''(\kappa^\nu)|^q dt\Big)^{\frac{1}{q}}\Bigg]$$

$$=\frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{\nu(\xi+2)}\Big)^{1-\frac{1}{q}}\Bigg[\frac{|f''(\kappa^\nu)|^q}{2}\Big(\frac{2}{\nu\xi+2\nu}\Big)+\frac{|f''(\tau^\nu)|^q}{2}\Big(\frac{2}{\nu\xi+2\nu}\Big)\Bigg]^{\frac{1}{q}}\Bigg].$$

By simplfying the above inequality, we obtain

$$|A\le\frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{\nu(\xi+2)}\Big)^{1-\frac{1}{q}}\Bigg[\frac{|f''(\kappa^\nu)|^q+|f''(\tau^\nu)|^q}{\nu(\xi+2)}\Bigg]^{\frac{1}{q}}\Bigg].$$

Which completes the proof. □

Theorem 2.7. *Let $f : I^\circ \subset \Re \to \Re$ be a differentiable mapping on I° and $\kappa^\nu, \tau^\nu \in I^\circ$ with $\kappa^\nu < \tau^\nu$ and $q \geq 1$. If the mapping $|f''|^q$ is convex on the interval (κ^ν, τ^ν), then the following inequality holds for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\left(\tau^\nu-\kappa^\nu\right)^{\xi}}\Big(\big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\big)f\big(\tau^\nu\big)+\big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\big)f\big(\kappa^\nu\big)\Big)-f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)$$

$$\leq \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{(-p+2p\nu+p\xi\nu+1)(2p\nu+p\xi\nu-p+2)}\Big)^{\frac{1}{p}}$$

$$\Bigg[\Big(\frac{|f''(\kappa^\nu)|^q}{2(\nu+1)(\nu+2)}+\frac{\nu^2+3\nu+1}{2(\nu+1)(\nu+2)}|f''(\tau^\nu)|^q\Big)^{\frac{1}{q}}$$

$$+\Big(\frac{|f''(\tau^\nu)|^q}{2(\nu+1)(\nu+2)}+\frac{\nu^2+3\nu+1}{2(\nu+1)(\nu+2)}|f''(\kappa^\nu)|^q\Big)^{\frac{1}{q}}\Bigg]+\Big(\frac{1}{2p\nu+p\nu\xi-p+2}\Big)^{\frac{1}{p}}$$

$$\Bigg[\Big(\frac{|f''(\kappa^\nu)|^q}{2(\nu+2)}+\frac{(\nu+1)|f''(\tau^\nu)|^q}{2(\nu+2)}\Big)^{\frac{1}{q}}+\Big(\frac{|f''(\tau^\nu)|^q}{2(\nu+2)}+\frac{(\nu+1)|f''(\kappa^\nu)|^q}{2(\nu+2)}\Big)^{\frac{1}{q}}\Bigg]\Bigg].$$

Proof. From Lemma 2.1, we can write

$$|A| \leq \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\int_0^1 t^{\nu\xi+\nu}t^{\nu-1}f''\Big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\Big)dt$$

$$+\int_0^1 t^{\nu\xi+\nu}t^{\nu-1}f''\Big(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\Big)dt\Bigg].$$

Let us denote

$$k_1=\int_0^1 t^{\nu\xi+\nu}t^{\nu-1}f''\Big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\Big)dt.$$

By using Hölder-Işcan inequality, we have

$$|A| \leq \Big(\int_0^1(1-t)|t^{\nu(\xi+2)-1}|^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1(1-t)\Big|f''\big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 t|t^{\nu(\xi+2)-1}|^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t\Big|f''\big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$\leq \Big(\int_0^1(1-t)(t^{p\nu(\xi+2)-p})dt\Big)^{\frac{1}{p}}\Big(\int_0^1(1-t)\frac{t^\nu}{2}\Big|f''(\kappa^\nu)\Big|^q dt+\int_0^1(1-t)\frac{2-t^\nu}{2}\Big|f''(\tau^\nu)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 t(t^{p\nu(\xi+2)-p})dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t\frac{t^\nu}{2}\Big|f''(\kappa^\nu)\Big|^q dt+\int_0^1 t\frac{2-t^\nu}{2}\Big|f''(\tau^\nu)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$\leq \Big(\frac{1}{(-p+2p\nu+p\xi\nu+1)(-p+2p\nu+p\xi\nu+2)}\Big)^{\frac{1}{p}}$$

$$\Big(\frac{1}{2(\nu+1)(\nu+2)}|f''(\kappa^\nu)|^q+\frac{\nu^2+3\nu+1}{2(\nu+1)(\nu+2)}|f''(\tau^\nu)|^q\Big)^{\frac{1}{q}}$$

$$+\Big(\frac{1}{2p\nu+p\xi\nu+2}\Big)^{\frac{1}{p}}\Big(\frac{1}{2(\nu+2)}|f''(\kappa^{\nu})|^{q}+\frac{\nu+1}{2(\nu+2)}|f''(\tau^{\nu})|^{q}\Big)^{\frac{1}{q}}.$$

Similarly,

$$k_2=\int_0^1 t^{\nu\xi+\nu}t^{\nu-1}f''\Big(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu}\Big)dt.$$

By using Hölder-Işcan inequality, we have

$$|A|\leq\Big(\int_0^1(1-t)|t^{\nu(\xi+2)-1}|^{p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1(1-t)\Big|f''(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu})\Big|^{q}dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 t|t^{\nu(\xi+2)-1}|^{p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t\Big|f''(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu})\Big|^{q}dt\Big)^{\frac{1}{q}}$$

$$\leq\Big(\int_0^1(1-t)(t^{p\nu(\xi+2)-p})dt\Big)^{\frac{1}{p}}\Big(\int_0^1(1-t)\frac{t^{\nu}}{2}\Big|f''(\tau^{\nu})\Big|^{q}dt+\int_0^1(1-t)\frac{2-t^{\nu}}{2}\Big|f''(\kappa^{\nu})\Big|^{q}dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 t(t^{p\nu(\xi+2)-p})dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t\frac{t^{\nu}}{2}\Big|f''(\tau^{\nu})\Big|^{q}dt+\int_0^1 t\frac{2-t^{\nu}}{2}\Big|f''(\kappa^{\nu})\Big|^{q}dt\Big)^{\frac{1}{q}}$$

$$\leq\Big(\frac{1}{(-p+2p\nu+p\xi\nu+1)(-p+2p\nu+p\xi\nu+2)}\Big)^{\frac{1}{p}}$$

$$\Big(\frac{1}{2(\nu+1)(\nu+2)}|f''(\tau^{\nu})|^{q}+\frac{\nu^{2}+3\nu+1}{2(\nu+1)(\nu+2)}|f''(\kappa^{\nu})|^{q}\Big)^{\frac{1}{q}}$$

$$+\Big(\frac{1}{2p\nu+p\xi\nu+2}\Big)^{\frac{1}{p}}\Big(\frac{1}{2(\nu+2)}|f''(\tau^{\nu})|^{q}+\frac{\nu+1}{2(\nu+2)}|f''(\kappa^{\nu})|^{q}\Big)^{\frac{1}{q}}.$$

Now, k_1+k_2

$$|A|\leq\frac{(\tau^{\nu}-\kappa^{\nu})^{2}}{8(\xi+1)}\Big[\Big(\frac{1}{(-p+2p\nu+p\xi\nu+1)(2p\nu+p\xi\nu-p+2)}\Big)^{\frac{1}{p}}$$

$$\Big[\Big(\frac{|f''(\kappa^{\nu})|^{q}}{2(\nu+1)(\nu+2)}+\frac{\nu^{2}+3\nu+1}{2(\nu+1)(\nu+2)}|f''(\tau^{\nu})|^{q}\Big)^{\frac{1}{q}}$$

$$+\Big(\frac{|f''(\tau^{\nu})|^{q}}{2(\nu+1)(\nu+2)}+\frac{\nu^{2}+3\nu+1}{2(\nu+1)(\nu+2)}|f''(\kappa^{\nu})|^{q}\Big)^{\frac{1}{q}}\Big]$$

$$+\Big(\frac{1}{2p\nu+p\nu\xi-p+2}\Big)^{\frac{1}{p}}$$

$$\Big[\Big(\frac{|f''(\kappa^{\nu})|^{q}}{2(\nu+2)}+\frac{(\nu+1)|f''(\tau^{\nu})|^{q}}{2(\nu+2)}\Big)^{\frac{1}{q}}+\Big(\frac{|f''(\tau^{\nu})|^{q}}{2(\nu+2)}+\frac{(\nu+1)|f''(\kappa^{\nu})|^{q}}{2(\nu+2)}\Big)^{\frac{1}{q}}\Big]\Big].$$

Which is the desired result. □

Theorem 2.8. *Let $f: I^{\circ}\subset\Re\to\Re$ be a differentiable mapping on I° and $\kappa^{\nu},\tau^{\nu}\in I^{\circ}$ with $\tau^{\nu}<\kappa^{\nu}$ and $q\geq 1$. If the mapping $|f''|^{q}$ is convex on the interval $(\kappa^{\nu},\tau^{\nu})$, then the following inequality holds for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\big(\tau^{\nu}-\kappa^{\nu}\big)^{\xi}}\Big(\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(b^{\nu}\big)+\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^{\nu}\big)\Big)-f\Big(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\Big)$$

$$\leq \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}\Big)^{1-\frac{1}{q}}\Bigg[\Bigg(\frac{|f''(\kappa^\nu)|^q}{2\nu(\xi+3)(\xi\nu+3\nu+1)}$$

$$+\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}-\frac{1}{2\nu(\xi+3)(\xi\nu+3\nu+1)}\Big)|f''(\tau^\nu)|^q\Bigg)^{\frac{1}{q}}$$

$$+\Big(\frac{|f''(\tau^\nu)|^q}{2\nu(\xi+3)(\xi\nu+3\nu+1)}$$

$$+\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}-\frac{1}{2\nu(\xi+3)(\xi\nu+3\nu+1)}\Big)|f''(\kappa^\nu)|^q\Big)^{\frac{1}{q}}\Bigg]$$

$$+\Big(\frac{1}{\nu(\xi+2)+1}\Big)^{1-\frac{1}{q}}\Bigg[\Bigg(\frac{|f''(\kappa^\nu)|^q}{2(3\nu+\xi\nu+1)}+\frac{\xi\nu+4\nu+1}{2(2\nu+\xi\nu+1)(3\nu+\xi\nu+1)}|f''(\tau^\nu)|^q\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\frac{|f''(\tau^\nu)|^q}{2(3\nu+\xi\nu+1)}+\frac{\xi\nu+4\nu+1}{2(2\nu+\xi\nu+1)(3\nu+\xi\nu+1)}|f''(\kappa^\nu)|^q\Bigg)^{\frac{1}{q}}\Bigg]\Bigg].$$

Proof. Let us denote

$$k_1=\int_0^1 t^{\nu\xi+\nu}t^{\nu-1}f''\Big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\Big)dt$$

By using Improved Power mean inequality, we get

$$|A|\leq\Big(\int_0^1(1-t)t^{\nu(\xi+2)-1}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1(1-t)t^{\nu(\xi+2)-1}\Big|f''\Big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 tt^{\nu(\xi+2)-1}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 tt^{\nu(\xi+2)-1}\Big|f''\Big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$\leq\Big(\frac{1}{2\nu+\xi\nu}-\frac{1}{2\nu+\xi\nu+1}\Big)^{1-\frac{1}{q}}\Bigg(\int_0^1(1-t)t^{\nu(\xi+2)-1}\frac{t^\nu}{2}|f''(\kappa^\nu)|^q dt$$

$$+\int_0^1(1-t)t^{\nu(\xi+2)-1}\frac{2-t^\nu}{2}|f''(b^\nu)|^q dt\Bigg)^{\frac{1}{q}}+\Big(\frac{1}{\nu(\xi+2)+1}\Big)^{1-\frac{1}{q}}$$

$$\Bigg(\int_0^1 t^{\nu(\xi+2)}\frac{t^\nu}{2}|f''(\kappa^\nu)|^q dt+\int_0^1 t^{\nu(\xi+2)}\frac{2-t^\nu}{2}|f''(\tau^\nu)|^q dt\Bigg)^{\frac{1}{q}}.$$

By taking into account the facts that

$$\int_0^1(1-t)t^{\nu(\xi+2)-1}(1-\frac{t^\nu}{2})dt=\int_0^1(1-t)t^{\nu(\xi+2)-1}dt-\int_0^1(1-t)t^{\nu(\xi+2)-1}\frac{t^\nu}{2}dt$$

$$=\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}-\frac{1}{2\nu(\xi+3)(\xi\nu+3\nu+1)}.$$

It is clear to see that

$$|A|\leq\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}\Big)^{1-\frac{1}{q}}$$

$$\Bigg(\frac{|f''(\kappa^\nu)|^q}{2\nu(\xi+3)(\xi\nu+3\nu+1)}+\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}-\frac{1}{2\nu(\xi+3)(\xi\nu+3\nu)+1}\Big)|f''(\tau^\nu)|^q\Bigg)^{\frac{1}{q}}$$

$$+\Big(\frac{1}{\nu(\xi+2)+1}\Big)^{1-\frac{1}{q}}\Big(\frac{|f''(\kappa^{\nu})|^q}{2(3\nu+\xi\nu+1)}+\frac{\xi\nu+4\nu+1}{2(2\nu+\xi\nu+1)(3\nu+\xi\nu+1)}|f''(\tau^{\nu})|^q\Big)^{\frac{1}{q}}.$$

Let

$$k_2=\int_0^1 t^{\nu\xi+\nu}t^{\nu-1}f''\Big(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu}\Big)dt$$

By using Improved Power mean inequality

$$|A|\le\Big(\int_0^1(1-t)t^{\nu(\xi+2)-1}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1(1-t)t^{\nu(\xi+2)-1}\Big|f''\Big(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu}\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 tt^{\nu(\xi+2)-1}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 tt^{\nu(\xi+2)-1}\Big|f''\Big(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu}\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$\le\Big(\frac{1}{2\nu+\xi\nu}-\frac{1}{2\nu+\xi\nu+1}\Big)^{1-\frac{1}{q}}\Bigg(\int_0^1(1-t)t^{\nu(\xi+2)-1}\frac{t^{\nu}}{2}|f''(\tau^{\nu})|^q dt$$

$$+\int_0^1(1-t)t^{\nu(\xi+2)-1}\frac{2-t^{\nu}}{2}|f''(\kappa^{\nu})|^q dt\Bigg)^{\frac{1}{q}}+\Big(\frac{1}{\nu(\xi+2)+1}\Big)^{1-\frac{1}{q}}$$

$$\Bigg(\int_0^1 t^{\nu(\xi+2)}\frac{t^{\nu}}{2}|f''(\tau^{\nu})|^q dt+\int_0^1 t^{\nu(\xi+2)}\frac{2-t^{\nu}}{2}|f''(\kappa^{\nu})|^q dt\Bigg)^{\frac{1}{q}}.$$

By computing the above integrals, we have

$$|A|\le\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}\Big)^{1-\frac{1}{q}}$$

$$\Bigg(\frac{|f''(\tau^{\nu})|^q}{2\nu(\xi+3)(\xi\nu+3\nu+1)}+\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}-\frac{1}{2\nu(\xi+3)(\xi\nu+3\nu)+1}\Big)|f''(\kappa^{\nu})|^q\Bigg)^{\frac{1}{q}}$$

$$+\Big(\frac{1}{\nu(\xi+2)+1}\Big)^{1-\frac{1}{q}}\Big(\frac{|f''(\tau^{\nu})|^q}{2(3\nu+\xi\nu+1)}+\frac{\xi\nu+4\nu+1}{2(2\nu+\xi\nu+1)(3\nu+\xi\nu+1)}|f''(\kappa^{\nu})|^q\Big)^{\frac{1}{q}}.$$

Now, k_1+k_2

$$|A|\le\frac{(\tau^{\nu}-\kappa^{\nu})^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}\Big)^{1-\frac{1}{q}}\Bigg[\Bigg(\frac{|f''(\kappa^{\nu})|^q}{2\nu(\xi+3)(\xi\nu+3\nu+1)}$$

$$+\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}-\frac{1}{2\nu(\xi+3)(\xi\nu+3\nu+1)}\Big)|f''(\tau^{\nu})|^q\Bigg)^{\frac{1}{q}}$$

$$+\Big(\frac{|f''(\tau^{\nu})|^q}{2\nu(\xi+3)(\xi\nu+3\nu+1)}$$

$$+\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}-\frac{1}{2\nu(\xi+3)(\xi\nu+3\nu+1)}\Big)|f''(\kappa^{\nu})|^q\Big)^{\frac{1}{q}}\Bigg]$$

$$+\Big(\frac{1}{\nu(\xi+2)+1}\Big)^{1-\frac{1}{q}}\Bigg[\Bigg(\frac{|f''(\kappa^{\nu})|^q}{2(3\nu+\xi\nu+1)}+\frac{\xi\nu+4\nu+1}{2(2\nu+\xi\nu+1)(3\nu+\xi\nu+1)}|f''(\tau^{\nu})|^q\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\frac{|f''(\tau^{\nu})|^q}{2(3\nu+\xi\nu+1)}+\frac{\xi\nu+4\nu+1}{2(2\nu+\xi\nu+1)(3\nu+\xi\nu+1)}|f''(\kappa^{\nu})|^q\Bigg)^{\frac{1}{q}}\Bigg]\Bigg].$$

Which is the desired result. □

Theorem 2.9. *Suppose that $f:[\kappa^\nu,\tau^\nu]\to\Re$ be a twice differentiable function on (κ^ν,τ^ν) with $0\le\kappa<\tau$. If $|f''|$ is convex function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^\xi}{(\tau^\nu-\kappa^\nu)^\xi}\Big[\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\Big)f\big(\tau^\nu\big)+\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\Big)f\big(\kappa^\nu\big)\Big]-f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)$$

$$\le\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\frac{1}{\nu(\xi+2)}\Big]\Big[|f''(\kappa^\nu)|+|f''(\tau^\nu)|\Big].$$

Proof. By using the property of modulus on R.H.S of lemma (2.2), we can write

$$|B|\le\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}\Big|f''\Big(\frac{1-t^\nu}{2}\kappa^\nu$$

$$+\frac{1+t^\nu}{2}\tau^\nu\Big)\Big|dt+\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}\Big|f''\Big(\frac{1-t^\nu}{2}\tau^\nu+\frac{1+t^\nu}{2}\kappa^\nu\Big)\Big|dt\Big]$$

$$\le\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}\Big[\frac{1-t^\nu}{2}|f''(\kappa^\nu)|+\frac{1+t^\nu}{2}|f''(\tau^\nu)|\Big]dt$$
$$+\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}\Big[\frac{1-t^\nu}{2}|f''(\tau^\nu)|+\frac{1+t^\nu}{2}|f''(\kappa^\nu)|\Big]dt\Big]$$

$$=\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}\Big[|f''(\kappa^\nu)|+|f''(\tau^\nu)|\Big]dt\Big]$$
$$=\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\frac{1}{\nu(\xi+2)}\Big]\Big[|f''(\kappa^\nu)|+|f''(\tau^\nu)|\Big]$$
$$|B|\le\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\frac{1}{\nu(\xi+2)}\Big]\Big[|f''(\kappa^\nu)|+|f''(\tau^\nu)|\Big].$$

This completes the proof. □

Theorem 2.10. *Suppose that $f:[\kappa^\nu,\tau^\nu]\to\Re$ be a twice differentiable function on (κ^ν,τ^ν) with $0\le\kappa<\tau$. If $|f''|$ is convex function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^\xi}{(\tau^\nu-\kappa^\nu)^\xi}\Big[\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\Big)f\big(\tau^\nu\big)+\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\Big)f\big(\kappa^\nu\big)\Big]-f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)$$

$$\le\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\Big(\frac{1}{s(\xi+1)+1}\Big)^{\frac{1}{s}}\Big[\Big(\frac{|f''(\kappa^\nu)|^q+3|f''(\tau^\nu)|^q}{4\nu}\Big)^{\frac{1}{q}}+\Big(\frac{3|f''(\kappa^\nu)|^q+|f''(\tau^\nu)|^q}{4\nu}\Big)^{\frac{1}{q}}\Big]\Big]$$

for $r>1$ and $s>1$.

Proof. Using Hölder Inequality in lemma (2.2), we get

$$|B|\le\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\Big(\int_0^1|(1-t^\nu)^{\xi+1}|^p t^{\nu-1}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t^{\nu-1}\Big|f''\Big(\frac{1-t^\nu}{2}a^\nu+\frac{1+t^\nu}{2}\tau^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 |(1-t^\nu)^{\xi+1}|^p t^{\nu-1}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t^{\nu-1}\Big|f''\Big(\frac{1-t^\nu}{2}\tau^\nu+\frac{1+t^\nu}{2}\kappa^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}\Bigg].$$

By using the convexity of $|f''|$, we have

$$|B|\le\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\int_0^1(1-t^\nu)^{p(\xi+1)}t^{\nu-1}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t^{\nu-1}(\frac{1-t^\nu}{2})|f''(a^\nu)|^q dt$$

$$+\int_0^1 t^{\nu-1}(\frac{1+t^\nu}{2})|f''(\tau^\nu)|^q dt\Big)^{\frac{1}{q}}+\Big(\int_0^1(1-t^\nu)^{p(\xi+1)}t^{\nu-1}dt\Big)^{\frac{1}{p}}$$

$$\Big(\int_0^1 t^{\nu-1}(\frac{1-t^\nu}{2})|f''(\tau^\nu)|^q dt+\int_0^1 t^{\nu-1}(\frac{1+t^\nu}{2})|f''(\kappa^\nu)|^q dt\Big)^{\frac{1}{q}}\Bigg].$$

$$|B|\le\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{p(\xi+1)+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}\Big[(\frac{1}{2\nu})|f''(\kappa^\nu)|^q+(\frac{3}{2\nu})|f''(\tau^\nu)|^q\Big]\Big)^{\frac{1}{q}}$$

$$\Big(\frac{1}{p(\xi+1)+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}\Big[(\frac{1}{2\nu})|f''(\tau^\nu)|^q+(\frac{3}{2\nu})|f''(\kappa^\nu)|^q\Big]\Big)^{\frac{1}{q}}\Bigg].$$

Namely,

$$|B|\le\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{p(\xi+1)+1}\Big)^{\frac{1}{p}}\Bigg[\Big(\frac{|f''(\kappa^\nu)|^q+3|f''(\tau^\nu)|^q}{4\nu}\Big)^{\frac{1}{q}}+\Big(\frac{3|f''(\kappa^\nu)|^q+|f''(\tau^\nu)|^q}{4\nu}\Big)^{\frac{1}{q}}\Bigg]\Bigg].$$

This completes the proof. □

Theorem 2.11. *If $f:[\kappa^\nu,\tau^\nu]\to\Re$ be differentiable function on (κ^ν,τ^ν) with $\kappa^\nu<\tau^\nu$ and $f''\in L_1[\kappa^\nu,\tau^\nu]$. If $|f''|$ is a concave function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^\xi}{(\tau^\nu-\kappa^\nu)^\xi}\Bigg[\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\Big)f\big(\tau^\nu\big)+\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\Big)f\big(\kappa^\nu\big)\Bigg]-f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)$$

$$\le\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\frac{1}{\nu(\xi+2)}\Bigg[f''\Bigg(\frac{\Big(\frac{1}{\nu(\xi+3)}\Big)\frac{\kappa^\nu}{2}+\Big(\frac{\xi+4}{\nu(\xi+2)(\xi+3)}\Big)\frac{\tau^\nu}{2}}{\frac{1}{\nu(\xi+2)}}\Bigg)$$

$$+f''\Bigg(\frac{\Big(\frac{\xi+4}{\nu(\xi+2)(\xi+3)}\Big)\frac{\kappa^\nu}{2}+\Big(\frac{1}{\nu(\xi+3)}\Big)\frac{\tau^\nu}{2}}{\frac{1}{\nu(\xi+2)}}\Bigg)\Bigg]\Bigg].$$

Proof. By applying Jensen inequality on R.H.S of lemma (2.2), we can write

$$|B|\le\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}\Big|f''\Big(\frac{1-t^\nu}{2}\kappa^\nu+\frac{1+t^\nu}{2}\tau^\nu\Big)\Big|dt$$

$$+\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}\Big|f''\Big(\frac{1+t^\nu}{2}\kappa^\nu+\frac{1-t^\nu}{2}\tau^\nu\Big)\Big|dt\Bigg]$$

$$\le\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}dt\Big)\Bigg|f''\Bigg(\frac{\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}\Big(\frac{1-t^\nu}{2}\kappa^\nu+\frac{1+t^\nu}{2}\tau^\nu\Big)dt}{\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}dt}\Bigg)\Bigg|$$

$$+\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}dt\Big)\Big|f''\Big(\frac{\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}\Big(\frac{1+t^\nu}{2}\kappa^\nu+\frac{1-t^\nu}{2}\tau^\nu\Big)dt}{\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}dt}\Big)\Big|.$$

By a simple computation, one has

$$|B| \leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\frac{1}{\nu(\xi+2)}\Big[f''\Big(\frac{\Big(\frac{1}{\nu(\xi+3)}\Big)\frac{\kappa^\nu}{2}+\Big(\frac{\xi+4}{\nu(\xi+2)(\xi+3)}\Big)\frac{\tau^\nu}{2}}{\frac{1}{\nu(\xi+2)}}\Big)$$

$$+f''\Big(\frac{\Big(\frac{\xi+4}{\nu(\xi+2)(\xi+3)}\Big)\frac{\kappa^\nu}{2}+\Big(\frac{1}{\nu(\xi+3)}\Big)\frac{\tau^\nu}{2}}{\frac{1}{\nu(\xi+2)}}\Big)\Big]\Big].$$

This is the desired result. □

Theorem 2.12. *If $f:[\kappa^\nu,\tau^\nu]\to\Re$ be differentiable function on (κ^ν,τ^ν) with $\kappa^\nu<\tau^\nu$ and $f''\in L_1[\kappa^\nu,\tau^\nu]$. If $|f''|^q$ is a convex function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^\xi}{(\tau^\nu-\kappa^\nu)^\xi}\Big[\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\Big)f\big(\tau^\nu\big)+\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\Big)f\big(\kappa^\nu\big)\Big]-f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)$$

$$\leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\Big(\frac{1}{\nu(\xi+2)}\Big)^{1-\frac{1}{q}}\Big[\frac{|f''(a^\nu)|^q}{2}\Big(\frac{2}{\nu(\xi+2)}\Big)+\frac{|f''(\tau^\nu)|^q}{2}\Big(\frac{2}{\nu(\xi+2)}\Big)\Big]^{\frac{1}{q}}\Big].$$

Proof. By applying Power mean inequality on R.H.S of lemma (2.2), we have

$$|B| \leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}\Big|f''\Big(\frac{1-t^\nu}{2}\kappa^\nu+\frac{1+t^\nu}{2}\tau^\nu\Big)\Big|dt$$

$$+\int_0^1 \Big((1-t^\nu)^{\xi+1}t^{\nu-1}\Big)\Big|f''\Big(\frac{1+t^\nu}{2}\kappa^\nu+\frac{1-t^\nu}{2}\tau^\nu\Big)\Big|dt\Big]$$

$$\leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}dt\Big)^{1-\frac{1}{q}}$$

$$\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}\Big|f''\Big(\frac{1-t^\nu}{2}\kappa^\nu+\frac{1+t^\nu}{2}\tau^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}\Big|f''\Big(\frac{1+t^\nu}{2}\kappa^\nu+\frac{1-t^\nu}{2}\tau^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}\Big]$$

By using convexity of $|f''|^q$, we get

$$|B| \leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}\Big(\frac{1-t^\nu}{2}\Big)|f''(\kappa^\nu)|^q dt$$

$$+\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}\Big(\frac{1+t^\nu}{2}\Big)|f''(\tau^\nu)|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}\Big(\frac{1+t^\nu}{2}\Big)|f''(\kappa^\nu)|^q dt$$

$$+\int_0^1 (1-t^\nu)^{\xi+1} t^{\nu-1} (\frac{1-t^\nu}{2})|f''(\tau^\nu)|^q dt\Big)^{\frac{1}{q}}\Big].$$

By computing the above integrals, we obtain

$$|B| \leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\Big(\frac{1}{\nu(\xi+2)}\Big)^{1-\frac{1}{q}}\Big[\frac{|f''(\kappa^\nu)|^q}{2}\Big(\frac{1}{\nu(\xi+3)}\Big)+\frac{|f''(\tau^\nu)|^q}{2}\Big(\frac{\xi+4}{\nu(\xi+2)(\xi+3)}\Big)$$

$$+\frac{|f''(\tau^\nu)|^q}{2}\Big(\frac{1}{\nu(\xi+3)}\Big)+\frac{|f''(\kappa^\nu)|^q}{2}\Big(\frac{\xi+4}{\nu(\xi+2)(\xi+3)}\Big)\Big]^{\frac{1}{q}}\Big].$$

This is the desired result. □

3. Conclusion

In the literature, there are many studies of different researchers that include Katugampola integral operators for functions whose absolute values of first derivatives are convex. The main motivation point of the study is to obtain the inequalities with the help of Katugampola integral operators for the functions whose absolute value of the second derivatives are convex and concave functions. In this sense, the findings contribute to the improvement in convex analysis and take the discussion one step further. In addition, Hölder's inequality is used to prove the main results and new approaches are obtained.

Recently, researchers working in the field of inequalities frequently use fractional integral operators and thus obtain new generalizations associated with the certain types of inequalities. Katugampola integral operators structurally combine Riemann-Liouville and Hadamard fractional integral operators and contribute to the effectiveness of the results with its generalized kernel structure.

The results can be performed for different kinds of convexity and operators.These results can be applied in convex analysis, optimization and different areas of pure and applied sciences. The authors hope that these results will serve as a motivation for future work in this fascinating area.

Acknowledgement

The research of the first author has been fully supported by H.E.C. Pakistan under NRPU project 7906.

References

[1] A. Kilbas, H.M. Srivastava and J.J. Trujillo,*Theory and applications of fractional differential equations*, Elsevier B.V., Amsterdam, Netherlands, (2006).

[2] J. E. Pečarić, D. S. Mitrinovic and A. M. Fink, *Classical and New Inequalities in Analysis*, (1993): 1-2.

[3] K. S. Miller and B. Ross, *An introduction to fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, John Wiley and Sons, Inc., New York, (1993).

[4] H. Chen, U.N. Katugampola, *Hermite-Hadamard and Hermite-Hadamar-Fejer type inequalityies for generalized fractional integrals* , J.Math. Anal. Appl. , 26(2013), 742-753.

[5] H. Yaldiz, A.O. Akdemir, *Katugampola Fractional Integrals within the Class of s-convex function*, Turkish Journal of Science, 3 (1), 2018, 40-50.

[6] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, *Hermite-Hadamards inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, 57 (2013) 2403-2407, doi:10.1016/j.mcm.2011.12.048.

[7] M.Z. Sarikaya and H. Yildirim, *On Hermite-Hadamard type inequalities for Riemann Liouville fractional integrals*, Miskolc Mathematical Notes, 17 (2016), No. 2, 1049-1059.

[8] R. Gorenflo, *Fractional calculus: Some numerical methods. In: A. Carpinteri, F. Mainardi (Eds), Fractals and Fractional Calculus in Continuum Mechanics*, Springer-Verlag, Wien- New York, (1997), 277-290.

[9] F.A. Aliev, N.A. Aliev and N.A. Safarova, Transformation of the Mittag-Leffler function to an exponential function and some of its applications to problems with a fractional derivative, Applied and Computational Mathematics, V.18, N.3, 2019, pp.316-325.

[10] M. E. Özdemir, A. Ekinci, A.O. Akdemir, Some new integral inequalities for functions whose derivatives of absolute values are convex and concave, TWMS Journal of Pure and Applied Mathematics, vol. 2, no. 10, pp. 212-224, Oct. 2019.

[11] E. Set, A.O. Akdemir and F. Özata, Grüss Type Inequalities for Fractional Integral Operator Involving the Extended Generalized Mittag Leffler Function, Applied and Computational Mathematics, vol. 19, no. 3, pp. 402-414, Oct. 2020.

[12] S.S. Dragomir and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula* Appl. Math. lett. 11(5),(1998), 91-95.

[13] S. Kermausuor, E.R. Nwaeze and A.M. Tameru, *New Integral Inequalities via the Katugampola Fractional Integrals for Functions Whose Second Derivatives Are Strongly η-Convex* Mathematics 2019, 7, 183; doi:10.3390/math7020183.

[14] U.N. Katugampola, *New approach to a Generalized Fractional Integral*, Appl. Math. Comput. (2011), 218(3), 860-865.

[15] U.N. Katugampola, *A New approach to a Generalized Fractional Derivatives*, Bull. Math. Anal. Appl. (2014), 6(4), 1-15.

[16] U.N. Katugampola, *Mellin transforms of the Generalized Fractional Integrals and Derivatives* , Appl. Math. Comput. (2015), 257, 566-580.

[17] M.Z. Sarikaya, N. Alp, On Hermite-Hadamard-Fejer type integral inequalities for generalized convex functions via local fractional integrals, Open Journal of Mathematical Sciences, Vol. 3 (2019), Issue 1, pp. 273-284.

[18] S.I. Butt, A.O. Akdemir, J. Nasir, and F. Jarad, Some Hermite-Jensen-Mercer like inequalities for convex functions through a certain generalized fractional integrals and related results. Miskolc Mathematical Notes, 21(2), 689-715, (2020).

[19] A.O. Akdemir, S.I. Butt, M. Nadeem, and M.A. Ragusa, New general variants of Chebyshev type inequalities via generalized fractional integral operators. Mathematics, 9(2), 122, (2021).

[20] S.I. Butt, M. Nadeem, S. Qaisar, A.O. Akdemir, and T. Abdeljawad, Hermite–Jensen–Mercer type inequalities for conformable integrals and related results. Advances in Difference Equations, 2020 (1), 1-24, (2020).

[21] A. Ekinci and M. E. Ozdemir, Some New Integral Inequalities via Riemann Liouville Integral Operators, Applied and Computational Mathematics, 3, 288–295, (2019).

[22] S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[23] S.I. Butt, S. Yousaf, A.O. Akdemir, M.A. Dokuyucu, New Hadamard-type integral inequalities via a general form of fractional integral operators. Chaos, Solitons and Fractals, 148, 111025, 2021.

[24] S.I. Butt, J. Nasir, S. Qaisar, K.M. Abualnaja, $k-$Fractional Variants of Hermite-Mercer-Type Inequalities via-Convexity with Applications, Journal of Function Spaces, 2021.

[25] S.I. Butt, M. Tariq, A. Aslam, H. Ahmad, T.A. Nofal, Hermite-Hadamard type inequalities via generalized harmonic exponential convexity and applications. Journal of Function Spaces, 2021.

[26] M. Vivas-Cortez, A. Kashuri, S.I. Butt, M. Tariq, J. Nasir, Exponential Type p-Convex Function with Some Related Inequalities and their Applications. Appl. Math, 15(3), 253-261, 2021.

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 3, 202–210
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

The Complex-type Cyclic-Pell Sequence and its Applications

Özgür Erdağ[a], Ömür Deveci[a], Erdal Karaduman[b]

[a]*Department of Mathematics, Faculty of Science and Letters, Kafkas University, 36100, Turkey*
[b]*Department of Mathematics, Faculty of Science, Ataturk University, 25240 Erzurum, Turkey*

Abstract. In this paper, we define the complex-type cyclic-Pell sequence and then, we give miscellaneous properties of this sequence by using matrix method. Also, we study the complex-type cyclic-Pell sequence modulo m. In addition, we describe the complex-type cyclic-Pell sequence in a 2-generator group and we investigate that in finite groups in detail. Finally, we obtain the lengths of the periods of the complex-type cyclic-Pell sequences in dihedral groups D_2, D_3, D_4, D_6, D_8, D_{16} and D_{32} with respect to the generating pair (x, y).

1. Introduction

The well-known the Pell sequence $\{P_n\}$ is defined by the following recurrence relation:

$$P_n = 2P_{n-1} + P_{n-2}$$

for $n \geq 2$ and with initial conditions $P_0 = 0$ and $P_1 = 1$.

The complex Fibonacci sequence $\{F_n^*\}$ is defined [21] by the following equation: for $n \geq 0$

$$F_n^* = F_n + iF_{n+1}$$

where $i = \sqrt{-1}$ is the imaginary unit and F_n is the n^{th} Fibonacci number (cf. [5, 22]).

Suppose that $\left\{c_j\right\}_{j=0}^{k-1}$, $(k \geq 2)$ is a sequence of real numbers such that $c_{k-1} \neq 0$. The k-generalized Fibonacci sequence $\{a_n\}_{n=0}^{+\infty}$ is defined as

$$a_{n+k} = c_{k-1}a_{n+k-1} + c_{k-2}a_{n+k-2} + \cdots + c_0 a_n$$

for $n \geq 0$ and where $a_0, a_1, \ldots, a_{k-1}$ are specified by the initial conditions.

Corresponding author: ÖE mail address: ozgur_erdag@hotmail.com ORCID:0000-0001-8071-6794, ÖD ORCID:0000-0001-5870-5298, EK ORCID:0000-0001-7915-1036

Received: 24 September 2022; Accepted: 27 December 2022; Published: 30 December 2022
Keywords. The complex-type cyclic-Pell sequence, Matrix, Group, Period.
2010 *Mathematics Subject Classification*. 11K31, 39B32, 11B50, 11C20, 20F05.
Cited this article as: Erdağ Ö. Deveci Ö. Karaduman E. The Complex-type Cyclic-Pell Sequence and its Applications, Turkish Journal of Science, 2022, 7(3), 202–210.

In [23], Kalman gave a number of closed-form formulas for the generalized sequence using the companion matrix as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Also, he proved that

$$(A_k)^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

In the literature, many interesting properties and applications of the recurrence sequences relevant to this paper have been studied by many authors; see for example, [3, 7–9, 14, 15, 28, 29]. Especially, in [18] and [17], the authors defined the new sequences using the quaternions and complex numbers and then they gave miscellaneous properties and many applications of the sequences defined. In the first part of this paper, we define the complex-type cyclic-Pell sequence and then, we give miscellaneous properties of this sequence by the aid of the matrix method.

We recall that when a sequence is composed only of repetitions of a fixed subsequence A sequence is periodic if after a certain points it consists only of repetitions of a fixed subsequence. We refer to the number of members in the shortest repeating subsequence as the period of the sequence. For instance, when a sequence with the terms $x, y, z, t, y, z, t, y, z, t, \ldots$ is considered, one would say it is periodic after the initial term k and it has period 3. Also, the first r terms in a sequence form a repeating subsequence, then it is said to be simply periodic with period r. For instance, when a sequence with the terms $x, y, z, t, x, y, z, t, x, y, z, t, \ldots$ is considered, one would say it is simply periodic with period 4.

The study of the linear recurrence sequences modulo m began with the earlier work of Wall [30] where the periods of the ordinary Fibonacci sequences modulo m were investigated. Recently, the theory extended to some special linear recurrence sequences by several authors; see for example, [20, 26].

For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_n\}$, the sequence $x_u = a_{u+1}$, $0 \le u \le n-1$, $x_{n+u} = \prod_{v=1}^{n} x_{u+v-1}$, $u \ge 0$ is called the Fibonacci orbit of G with respect to the generating set A, denoted as $F_A(G)$ in [11].

A k-nacci (k-step Fibonacci) sequence in a finite group is a sequence of group elements $x_0, x_1, x_2, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, x_1, x_2, \ldots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \cdots x_{n-1} & \text{for } j \le n < k, \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \ge k. \end{cases}$$

We also require that the initial elements of the sequence $x_0, x_1, x_2, \ldots, x_{j-1}$ generate the group, thus forcing the k-nacci sequence to reflect the structure of the group. The k-nacci sequence of a group G generated by $x_0, x_1, x_2, \ldots, x_{j-1}$ is denoted by $F_k\left(G; x_0, x_1, x_2, \ldots, x_{j-1}\right)$ in [25].

Note also that the orbit of a k-generated group is a k-nacci sequence.

From [17], we use the following definition as our preliminary information.

Definition 1.1. *Let G be a k-generated group. For a generating k-tuple* $(x_1, x_2, \ldots, x_k)$, *the complex-type k-Finonacci orbit is defined by* $a_i = x_{i+1}$, $(0 \le i \le k-1)$,

$$a_{n+k} = (a_n)^{i^k} (a_{n+1})^{i^{k-1}} \ldots (a_{n+k-1})^{i}, \ n \ge 0$$

where the following conditions are achieved for any $x, y \in G$ and any integer u:

(i). *Let e be the identity of G and consider $z = a + ib$, where a, b are integers, then*

$* \; x^z \equiv x^{a(\text{mod } |x|)+ib(\text{mod } |x|)} = x^{a(\text{mod } |x|)}x^{ib(\text{mod } |x|)} = x^{ib(\text{mod } |x|)}x^{a(\text{mod } |x|)} = x^{ib(\text{mod } |x|)+a(\text{mod } |x|)}$,

$* \; x^{ia} = \left(x^i\right)^a = (x^a)^i$,

$* \; e^u = e$,

$* \; x^{0+i0} = e$.

(ii). *Given $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, where a_1, b_1, a_2 and b_2 are integers, $y^{-z_2}x^{-z_1} = (x^{z_1}y^{z_2})^{-1}$.*

(iii). *If $yx \neq xy$, then $y^ix^i \neq x^iy^i$.*

(iv). *$y^ix^i = (xy)^i$ and $x^{-1}y^{-1} = \left(x^iy^i\right)^i$,*

(v). *$y^ix = xy^i$ and so $x^iy^{-1} = \left(xy^i\right)^i$ and $x^{-1}y^i = \left(x^iy\right)^i$.*

The study of the recurrence sequences in groups began with the earlier work of Wall [30]. In the mid-eighties, Wilcox studied the Fibonacci sequences in abelian groups in [31]. In [12], the theory was expanded to some finite simple groups by Campbell et al.. There, they defined the Fibonacci length of the Fibonacci orbit and the basic Fibonacci length of the basic Fibonacci orbit in a 2-generator group. The concept of Fibonacci length for more than two generators has also been considered; see, for example, [10, 11]. In [25], Knox signified that a k-nacci (k-step Fibonacci) sequence in a finite group is periodic. Recently, the theory has been extended to some special linear recurrence sequences by several authors; see for example, [1, 2, 4, 13, 16, 19, 24, 27]. Deveci and Shannon [17] defined the complex-type k-Fibonacci orbit of a k-generator group. They proved that the complex-type k-Fibonacci orbit of a k-generator group is periodic if the group is finite. In the second part of this paper, we redefine the complex-type cyclic-Pell sequence by means of the elements of 2-generator groups which is called the complex-type cyclic-Pell orbit. Then we examine the sequence in finite groups in detail. Finally, we obtain the lengths of the periods of the complex-type cyclic-Pell orbits of the dihedral group D_n for some $n \geq 2$ as applications of the results obtained.

2. The Complex-type Cyclic-Pell Sequence

Now we define the complex-type cyclic-Pell sequence by the following homogeneous linear recurrence relation for $n \geq 1$

$$p_{n+2}^{(c,i)} = \begin{cases} 2p_{n+1}^{(c,i)} + p_n^{(c,i)} & n \equiv 0 \,(\text{mod } 4) \\ i\left(2p_{n+1}^{(c,i)} + p_n^{(c,i)}\right) & n \equiv 1 \,(\text{mod } 4) \\ -2p_{n+1}^{(c,i)} - p_n^{(c,i)} & n \equiv 2 \,(\text{mod } 4) \\ -i\left(2p_{n+1}^{(c,i)} + p_n^{(c,i)}\right) & n \equiv 3 \,(\text{mod } 4) \end{cases}$$

where $p_1^{(c,i)} = 0$, $p_2^{(c,i)} = 1$ and $i = \sqrt{-1}$.

Letting

$$M = \begin{bmatrix} -13 & -6-2i \\ -6+2i & -3 \end{bmatrix}. \tag{1}$$

and by using an induction method on n, we find the relationship between the elements of the sequence $\left\{p_n^{(c,i)}\right\}$ and the matrix M as follows:

$$(M)^n = \begin{bmatrix} p_{4n+2}^{(c,i)} & \overline{p_{4n+1}^{(c,i)}} \\ p_{4n+1}^{(c,i)} & \text{Re}\left(p_{4n}^{(c,i)}\right) - \text{Im}\left(p_{4n+1}^{(c,i)}\right) \end{bmatrix}.$$

In [6], Bicknell defined the generating matrix of the Pell numbers, P-matrix as follows:

$$N = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Using the matrices M and N, we have the following useful result.

Proposition 2.1. *For $n \geq 0$*

$$\det(M)^n = (-1)^n \cdot \det(N)^{4n}.$$

Proof. It is well-known that the *nth* powers of the matrix N is as follows:

$$(N)^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} \tag{2}$$

for $n \geq 0$. Since $\det M = \det(N)^4$ and from the (1) and (2), we have conclusion. □

We use the above definitions and define the matrices:

$$B_1 = \begin{bmatrix} 2i & i \\ 1 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} -2i & -i \\ 1 & 0 \end{bmatrix}$$

and

$$B_4 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let $M = B_4 B_3 B_2 B_1$. Using the above identities, we define the following matrix:

$$E^n = B_u B_{u-1} \dots B_1 M^k$$

where $n = 4k + u$ such that $u, k \in \mathbb{N}$. So we get

$$E^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{n+1}^{(c,i)} \\ p_n^{(c,i)} \end{bmatrix} \tag{3}$$

for $n = 4k + u$ such that $u, k \in \mathbb{N}$.

Now we investigate the Simpson formulas of the complex-type cyclic-Pell sequence.

If $n = 4k + 1$ ($k \in \mathbb{N}$), then

$$E^n = B_1 M^k = \begin{bmatrix} p_{n+2}^{(c,i)} & -2\mathrm{Re}\left(p_{n+2}^{(c,i)}\right) + i \cdot \left[\mathrm{Re}\left(p_{n+3}^{(c,i)}\right) + \mathrm{Im} p_{n+2}^{(c,i)}\right] \\ p_{n+1}^{(c,i)} & \overline{p_n^{(c,i)}} \end{bmatrix}.$$

So we get

$$\left(p_{n+2}^{(c,i)}\right)\left(\overline{p_n^{(c,i)}}\right) - \left(p_{n+1}^{(c,i)}\right)\left(-2\mathrm{Re}\left(p_{n+2}^{(c,i)}\right) + i \cdot \left[\mathrm{Re}\left(p_{n+3}^{(c,i)}\right) + \mathrm{Im} p_{n+2}^{(c,i)}\right]\right) = (-1)^{k+1} \cdot i.$$

If $n = 4k + 2$ ($k \in \mathbb{N}$), then

$$E^n = B_2 B_1 M^k = \begin{bmatrix} p_{n+2}^{(c,i)} & \overline{p_{n+1}^{(c,i)}} \\ p_{n+1}^{(c,i)} & -2\mathrm{Re}\left(p_{n+1}^{(c,i)}\right) + i \cdot \left[\mathrm{Re}\left(p_{n+2}^{(c,i)}\right) + \mathrm{Im}\left(p_{n+1}^{(c,i)}\right)\right] \end{bmatrix}.$$

So we get

$$\left(p_{n+2}^{(c,i)}\right)\left(-2\mathrm{Re}\left(p_{n+1}^{(c,i)}\right) + i \cdot \left[\mathrm{Re}\left(p_{n+2}^{(c,i)}\right) + \mathrm{Im}\left(p_{n+1}^{(c,i)}\right)\right]\right) - \left(p_{n+1}^{(c,i)}\right)\left(\overline{p_{n+1}^{(c,i)}}\right) = (-1)^{k+1} \cdot i.$$

If $n = 4k+3$ ($k \in \mathbb{N}$), then

$$E^n = B_3B_2B_1M^k = \begin{bmatrix} p_{n+2}^{(c,i)} & \mathrm{Re}\left(p_{n+1}^{(c,i)}\right) - \mathrm{Im}\left(p_n^{(c,i)}\right) \\ p_{n+1}^{(c,i)} & \overline{p_n^{(c,i)}} \end{bmatrix}.$$

So we get

$$\left(p_{n+2}^{(c,i)}\right)\left(\overline{p_n^{(c,i)}}\right) - \left(p_{n+1}^{(c,i)}\right)\left[\mathrm{Re}\left(p_{n+1}^{(c,i)}\right) - \mathrm{Im}\left(p_n^{(c,i)}\right)\right] = (-1)^k .$$

If $n = 4k+4$ ($k \in \mathbb{N}$), then

$$E^n = M^{k+1} = \begin{bmatrix} p_{n+2}^{(c,i)} & \overline{p_{n+1}^{(c,i)}} \\ p_{n+1}^{(c,i)} & \mathrm{Re}\left(p_n^{(c,i)}\right) - \mathrm{Im}\left(p_{n-1}^{(c,i)}\right) \end{bmatrix}.$$

So we get

$$\left(p_{n+2}^{(c,i)}\right)\left[\mathrm{Re}\left(p_n^{(c,i)}\right) - \mathrm{Im}\left(p_{n-1}^{(c,i)}\right)\right] - \left(p_{n+1}^{(c,i)}\right)\left(\overline{p_{n+1}^{(c,i)}}\right) = (-1)^{k+1} .$$

3. The Complex-type Cyclic-Pell Sequence in Groups

If we reduce the sequence $\left\{p_n^{(c,i)}\right\}$ modulo m, taking least nonnegative residues, then we get the following recurrence sequence:

$$\left\{p_n^{(c,i)}(m)\right\} = \left\{p_1^{(c,i)}(m),\ p_2^{(c,i)}(m),\ \ldots,\ p_j^{(c,i)}(m),\ \ldots\right\}$$

where $p_j^{(c,i)}(m)$ is used to mean the nth element of the complex-type cyclic-Pell sequence when read modulo m. We note here that the recurrence relations in the sequences $\left\{p_n^{(c,i)}(m)\right\}$ and $\left\{p_n^{(c,i)}\right\}$ are the same.

Theorem 3.1. *The sequence $\left\{p_n^{(c,i)}(m)\right\}$ is periodic and the length of its period is divisible by* 4.

Proof. Consider the set

$$\begin{aligned} R \;=\; & \{(z_1, z_2) \mid z_k\text{'s are complex numbers } a_k + ib_k \text{ where} \\ & a_k \text{ and } b_k \text{ are integers such that } 0 \le a_k, b_k \le m-1 \text{ and } k \in \{1,2\}\}. \end{aligned}$$

Let $|R|$ be the cardinality of the set R. Since the set R is finite, there are $|R|$ distinct 2-tuples of the complex-type cyclic-Pell sequence modulo m. Thus, it is clear that at least one of these 2-tuples appears twice in the sequence $\left\{p_n^{(c,i)}(m)\right\}$. Let $p_u^{(c,i)}(m) \equiv p_v^{(c,i)}(m)$ and $p_{u+1}^{(c,i)}(m) \equiv p_{v+1}^{(c,i)}(m)$. If $v - u \equiv 0 \,(\mathrm{mod}\ 4)$, then we get $p_{u+2}^{(c,i)}(m) \equiv p_{v+2}^{(c,i)}(m)$, $p_{u+3}^{(c,i)}(m) \equiv p_{v+3}^{(c,i)}(m)$, ... So, it is easy to see that the subsequence following this 2-tuple repeats; that is, $\left\{p_n^{(c,i)}(m)\right\}$ is a periodic sequence and the length of its period must be divided by 4. □

We denote the lengths of periods of the sequence $\left\{p_n^{(c,i)}(m)\right\}$ by $h_{p_n^{(c,i)}}(m)$. It is easy to see from the equation (3), $h_{p_n^{(c,i)}}(m)$ is the smallest positive integer α such that $E^\alpha \equiv I \,(\mathrm{mod}\ m)$.

Given an integer matrix $A = \left[a_{ij}\right]$, $A\,(\mathrm{mod}\ m)$ means that all entries of A are modulo m, that is, $A\,(\mathrm{mod}\ m) = \left(a_{ij}\,(\mathrm{mod}\ m)\right)$. Let us consider the set $\langle A\rangle_m = \{(A)^n\,(\mathrm{mod}\ m) \mid n \ge 0\}$. If $(\det A, m) = 1$, then the set $\langle A\rangle_m$ is a cyclic group; if $(\det A, m) \ne 1$, then the set $\langle A\rangle_m$ is a semigroup. Since $\det M = -1$, the set $\langle M\rangle_m$ is a cyclic group for every positive integer $m \ge 2$. From (3), it is easy to see that $h_{p_n^{(c,i)}}(m) = 2\,|\langle M\rangle_m|$.

Theorem 3.2. *Let ε be a prime. If s is the smallest positive integer such that $|\langle M\rangle_{\varepsilon^{s+1}}| \ne |\langle M\rangle_{\varepsilon^s}|$, then $|\langle M\rangle_{\varepsilon^{s+1}}| = \varepsilon\,|\langle M\rangle_{\varepsilon^s}|$.*

Proof. Suppose that α is a positive integer and $|\langle M\rangle_m|$ is denoted by $l_{p_n^{(c,i)}}(m)$. Let I be 2×2 identity matrix and $(M)^{l_{p_n^{(c,i)}}(\varepsilon^{\alpha+1})} \equiv I\left(\text{mod}\varepsilon^{\alpha+1}\right)$. Then we can derive $(M)^{l_{p_n^{(c,i)}}(\varepsilon^{\alpha+1})} \equiv I\left(\text{mod}\varepsilon^{\alpha}\right)$, which means that $l_{p_n^{(c,i)}}(\varepsilon^{\alpha})$ divides $l_{p_n^{(c,i)}}\left(\varepsilon^{\alpha+1}\right)$. Moreover, we may write $(M)^{l_{p_n^{(c,i)}}(\varepsilon^{s})} = I + \left(m_{i,j}^{(\alpha)} \cdot \varepsilon^{s}\right)$, by the binomial theorem. Hence, we obtain:

$$(M)^{l_{f_n^{(c,i)}}(\varepsilon^{\alpha})\cdot\varepsilon} = \left(I + \left(m_{i,j}^{(\alpha)} \cdot \varepsilon^{\alpha}\right)\right)^{\varepsilon} = \sum_{n=0}^{\varepsilon} \binom{\varepsilon}{i} \left(m_{i,j}^{(\alpha)} \cdot \varepsilon^{\alpha}\right)^{n} \equiv I\left(\text{mod}\varepsilon^{\alpha+1}\right).$$

Then we have $(M)^{l_{p_n^{(c,i)}}(\varepsilon^{\alpha})\cdot\varepsilon} \equiv I\left(\text{mod}\varepsilon^{\alpha+1}\right)$, which implies that $l_{p_n^{(c,i)}}\left(\varepsilon^{\alpha+1}\right)$ divides $l_{p_n^{(c,i)}}(\varepsilon^{s}) \cdot \varepsilon$. According to these results, it is seen that $l_{p_n^{(c,i)}}\left(\varepsilon^{\alpha+1}\right) = l_{p_n^{(c,i)}}(\varepsilon^{\alpha})$ or $l_{p_n^{(c,i)}}\left(\varepsilon^{\alpha+1}\right) = l_{p_n^{(c,i)}}(\varepsilon^{\alpha}) \cdot \varepsilon$, and the latter holds if and only if there is a $m_{i,j}^{(\alpha)}$ which is not divisible by ε. Due to fact that we assume s is the smallest positive integer such that $l_{p_n^{(c,i)}}\left(\varepsilon^{s+1}\right) \neq l_{p_n^{(c,i)}}(\varepsilon^{s})$, there is an $m_{i,j}^{(t)}$ which is not divisible by ε. This shows that $l_{p_n^{(c,i)}}\left(\varepsilon^{s+1}\right) = l_{p_n^{(c,i)}}(\varepsilon^{s}) \cdot \varepsilon$. So we have the conclusion. □

Theorem 3.3. *Let m_1 and m_2 be positive integers with $m_1, m_2 \geq 2$, then $\left|\langle M\rangle_{\text{lcm}[m_1,m_2]}\right| = \text{lcm}\left[\left|\langle M\rangle_{m_1}\right|, \left|\langle M\rangle_{m_2}\right|\right]$.*

Proof. Let $|\langle M\rangle_m|$ is denoted by $l_{p_n^{(c,i)}}(m)$ and let $\text{lcm}[m_1, m_2] = m$. Clearly, $(M)^{l_{p_n^{(c,i)}}(m_1)} \equiv I(\text{mod}m_1)$ and $(M)^{l_{p_n^{(c,i)}}(m_2)} \equiv I(\text{mod}m_2)$. Using the least common multiple operation this implies that$(M)^{l_{p_n^{(c,i)}}(m)} \equiv I(\text{mod}m_1)$ and $(M)^{l_{p_n^{(c,i)}}(m)} \equiv I(\text{mod}m_2)$. So we get $\left|\langle M\rangle_{m_1}\right| \mid |\langle M\rangle_m|$ and $\left|\langle M\rangle_{m_2}\right| \mid |\langle M\rangle_m|$, which means that $\text{lcm}\left[\left|\langle M\rangle_{m_1}\right|, \left|\langle M\rangle_{m_2}\right|\right]$ divides $\left|\langle M\rangle_{\text{lcm}[m_1,m_2]}\right|$. Now we consider as $\text{lcm}\left[\left|\langle M\rangle_{m_1}\right|, \left|\langle M\rangle_{m_2}\right|\right] = \rho$. Then we can write $M^{\rho} \equiv I(\text{mod}m_1)$ and $M^{\rho} \equiv I(\text{mod}m_2)$, which yields that $M^{\rho} \equiv I(\text{mod}m)$. Thus, it is seen that $\text{lcm}\left[\left|\langle M\rangle_{m_1}\right|, \left|\langle M\rangle_{m_2}\right|\right]$ is divisible by $\left|\langle M\rangle_{\text{lcm}[m_1,m_2]}\right|$. So we have the conclusion. □

Let G be a finite j-generator group and let X be the subset of $\underbrace{G \times G \times \cdots \times G}_{j \text{ times}}$ such that $\left(x_1, x_2, \ldots, x_j\right) \in X$ if and only if G is generated by $x_1, x_2, \ldots, x_j$. $\left(x_1, x_2, \ldots, x_j\right)$ is said to be a generating j-tuple for G.

Definition 3.4. *Let G be a 2-generator group and let (x_1, x_2) be a generating 2-tuple of G. Then, we define the complex-type cyclic-Pell orbit by*

$$c_1 = x_1,\ c_2 = x_2, c_n = \begin{cases} (c_{n-2})(c_{n-1})^{2} & \textit{for } n \equiv 0\,(\text{mod}4) \\ (c_{n-2})^{i}(c_{n-1})^{2i} & \textit{for } n \equiv 1\,(\text{mod}4) \\ (c_{n-2})^{-1}(c_{n-1})^{-2} & \textit{for } n \equiv 2\,(\text{mod}4) \\ (c_{n-2})^{-i}(c_{n-1})^{-2i} & \textit{for } n \equiv 3\,(\text{mod}4) \end{cases},\ \ (n > 2).$$

Let the notation $P_{(x_1,x_2)}^{(i,c)}(G)$ denote the complex-type cyclic-Pell orbit of G for generating 2-tuple (x_1, x_2).

Theorem 3.5. *If G is finite, then the complex-type cyclic-Pell orbit of G is a periodic sequence and the length of its period is divisible by 4.*

Proof. Consider the set

$$\begin{aligned} W \;=\; & \left\{\left((w_1)^{a_1(\text{mod}|w_1|)+ib_1(\text{mod}|w_1|)}, (w_2)^{a_2(\text{mod}|w_2|)+ib_2(\text{mod}|w_2|)}\right) :\right. \\ & \left. i = \sqrt{-1}, w_1, w_2 \in G \text{ and } a_1, a_2, b_1, b_2 \in Z\right\}. \end{aligned}$$

Since the group G is finite, W is a finite set. Then for any $u \geq 0$, there exists $v > u$ such that $c_u = c_v$ and $c_{u+1} = c_{v+1}$. If $v - u \equiv 0\,(\text{mod}4)$, then we get $c_{u+2} = c_{v+2}$, $c_{u+3} = c_{v+3}$, ... Because of the repeating, for all generating pairs, the sequence $P_{(x_1,x_2)}^{(i,c)}(G)$ is periodic and the length of its period must be divided by 4. □

We denote the length of the period of the orbit $P^{(i,c)}_{(x_1,x_2)}(G)$ by $LP^{(i,c)}_{(x_1,x_2)}(G)$. From the definition of the orbit $P^{(i,c)}_{(x_1,x_2)}$ it is clear that the length of the period of this sequence in a finite group depends on the chosen generating set and the order in which the assignments of x_1, x_2 are made.

We will now address the lengths of the periods of the orbits $P^{(i,c)}_{(x,y)}(D_2)$, $P^{(i,c)}_{(x,y)}(D_3)$, $P^{(i,c)}_{(x,y)}(D_4)$, $P^{(i,c)}_{(x,y)}(D_6)$, $P^{(i,c)}_{(x,y)}(D_8)$, $P^{(i,c)}_{(x,y)}(D_{16})$ and $P^{(i,c)}_{(x,y)}(D_{32})$. The dihedral group D_n of order $2n$ is defined as follows:

$$D_n = \langle x, y \mid x^n = y^2 = (xy)^2 = e \rangle$$

for every $n \geq 2$. Note that $|x| = n$, $|y| = 2$, $xy = yx^{-1}$ and $yx = x^{-1}y$. By direct calculation, we obtain the orbit $P^{(i,c)}_{(x,y)}(D_n)$ as follows:

$$\begin{aligned}
c_1 &= x, c_2 = y, c_3 = x^{i},\\
c_4 &= x^{-2i}y, c_5 = x^{-3}, c_6 = x^{6-2i}y,\\
c_7 &= x^{4-3i}, c_8 = x^{14+8i}y, c_9 = x^{13-4i},\\
c_{10} &= x^{-12}y, c_{11} = x^{4+13i}, c_{12} = x^{-4-26i}y,\\
c_{13} &= x^{-39-4i}, c_{14} = x^{74-34i}y, c_{15} = x^{72-39i},\\
c_{16} &= x^{218+112i}y, c_{17} = x^{185-72i}, c_{18} = x^{-152-32i}y,\\
c_{19} &= x^{136+185i}, c_{20} = x^{120-338i}y, c_{21} = x^{-491-136i},\\
c_{22} &= x^{1102-610i}y, c_{23} = x^{1356-491i}, c_{24} = x^{3814+1592i}y,\\
c_{25} &= x^{2693-1356i}, c_{26} = x^{-1572-1120i}y, c_{27} = x^{3596+2693i},\\
c_{28} &= x^{5620-4266i}y, c_{29} = x^{-5839-3596i}, c_{30} = x^{17298-11458i}y,\\
c_{31} &= x^{26512-5839i}, c_{32} = x^{70322+23136i}y, c_{33} = x^{40433-26512i},\\
c_{34} &= x^{-10544-29888i}y, c_{35} = x^{86288+40433i}, c_{36} = x^{162032-50978i}y,\\
c_{37} &= x^{-61523-86288i}, c_{38} = x^{285078-223554i}y, c_{39} = x^{533396-61523i},\\
c_{40} &= x^{1351870+346600i}y, c_{41} = x^{631677-533396i}, c_{42} = x^{88516-720192i}y,\\
c_{43} &= x^{1973780+631677i}, c_{44} = x^{4036076-543162i}y, c_{45} = x^{-454647-1973780i},\\
c_{46} &= x^{4945370-4490722i}y, c_{47} = x^{10955224-454647i}, c_{48} = x^{26855818+5400016i}y,\\
c_{49} &= x^{10345385-10955224i}, c_{50} = x^{6165048-16510432i}y, c_{51} = x^{43976088+10345385i},\\
c_{52} &= x^{94117224-4180338i}y, c_{53} = x^{1984709-43976088i}, c_{54} = x^{90147806-92132514i}y,\\
c_{55} &= x^{228241116+1984709i}, c_{56} = x^{546630038+88163096i}y, c_{57} = x^{178310901-228241116i},\\
c_{58} &= x^{190008236-368319136i}y, c_{59} = x^{964879388+178310901i}, c_{60} = x^{2119767012+11697334i}y,\\
c_{61} &= x^{201705569-964879388i}, c_{62} = x^{1716355874-1918061442i}y, c_{63} = x^{4801002272+201705569i},\\
c_{64} &= x^{11318360418+1514650304i}y, c_{65} = x^{3231006177-4801002272i}, c_{66} = x^{4856348064-8087354240i}y,\\
c_{67} &= x^{20975710752+3231006177i}, c_{68} = x^{46807769568+1625341886i}y, c_{69} = x^{6481689949-20975710752i},\\
c_{70} &= x^{33844389670-40326079618i}y, c_{71} = x^{101627869988+6481689949i}, c_{72} = x^{237100129646+27362699720i}y,\\
c_{73} &= x^{61207089389-101627869988i}, c_{74} = x^{114685950868-175893040256i}y, c_{75} = x^{453413950500+61207089389i},\\
c_{76} &= x^{1021513851868+53478861478i}y, c_{77} = x^{168164812345-453413950500i}, c_{78} = x^{685184227178-853349039522i}y,\\
c_{79} &= x^{2160112029544+168164812345i}, c_{80} = x^{5005408286266+517019414832i}y, c_{81} = x^{1202203642009-2160112029544i},\\
c_{82} &= x^{2601001002248-3803204644256i}y, c_{83} = x^{9766521318056+1202203642009i}, c_{84} = x^{22134043638360+1398797360238i}y,\\
c_{85} &= x^{3999798362485-9766521318056i}, c_{86} = x^{14134446913390-18134245275874i}y, c_{87} = x^{46035011869804+3999798362485},\\
c_{88} &= x^{106204470652998+101346485550904i}y, c_{89} = x^{24269095464293-46035011869804i}, c_{90} = x^{57666279724412-81935375188704i}y,\\
c_{91} &= x^{209905762247212+24269095464293i}, c_{92} = x^{477477804218836+33397184260118i}y, c_{93} = x^{91063463984529-209905762247212i},
\end{aligned}$$

$$\begin{aligned}
c_{94} &= x^{295350876249778-386414340234306i}y, c_{95} = x^{982734442715824+91063463984529i}, c_{96} = x^{2260819761681426+204287412265248i}y,\\
c_{97} &= x^{499638288515025-982734442715824i}, c_{98} = x^{1261543184651376-1761181473166400i}y, c_{99} = x^{4505097389048624+499638288515025i},\\
c_{100} &= x^{1027173796274862 4+761904896136350i}y, c_{101} = x^{2023448080787725-4505097389048624i}, c_{102} = x^{6224841801173174-8248289881960898i}y,\\
c_{103} &= x^{21001677152970420+2023448080787725i}, c_{104} = x^{48228196107114014+4201393720385448i}y, c_{105} = x^{10426235521558621-21001677152970420i},\\
c_{106} &= x^{27375725063996772-37801960585555392i}y, c_{107} = x^{96605598324081204+10426235521558621i}, c_{108} = x^{220\,586\,921\,712\,159\,180+16\,949\,489\,542\,438\,150i}y.
\end{aligned}$$

Using the above information, the orbits $P^{(i,c)}_{(x,y)}(D_2)$, $P^{(i,c)}_{(x,y)}(D_3)$, $P^{(i,c)}_{(x,y)}(D_4)$, $P^{(i,c)}_{(x,y)}(D_6)$, $P^{(i,c)}_{(x,y)}(D_8)$, $P^{(i,c)}_{(x,y)}(D_{16})$ and $P^{(i,c)}_{(x,y)}(D_{32})$ become, respectively:

$$\begin{aligned}
c_5 &= x^{-3} = x = c_1, c_6 = x^{6-2i}y = y = c_2,\\
c_7 &= x^{4-3i} = x^i = c_3, c_8 = x^{14+8i}y = y = c_4, \ldots,
\end{aligned}$$

$$\begin{aligned}
c_{105} &= x^{10426235521558621-21001677152970420i} = x = c_1, c_{106} = x^{27375725063996772-37801960585555392i}y = y = c_2,\\
c_{107} &= x^{96605598324081204+10426235521558621i} = x^i = c_3, c_{108} = x^{220\,586\,921\,712\,159\,180+16\,949\,489\,542\,438\,150i}y = y = c_4, \ldots,
\end{aligned}$$

$$\begin{aligned}
c_9 &= x^{13-4i} = x = c_1, c_{10} = x^{-12}y = y = c_2,\\
c_{11} &= x^{4+13i} = x^i = c_3, c_{12} = x^{-4-26i}y = y = c_4, \ldots,
\end{aligned}$$

$$\begin{aligned}
c_{105} &= x^{10426235521558621-21001677152970420i} = x = c_1, c_{106} = x^{27375725063996772-37801960585555392i}y = y = c_2,\\
c_{107} &= x^{96605598324081204+10426235521558621i} = x^i = c_3, c_{108} = x^{220\,586\,921\,712\,159\,180+16\,949\,489\,542\,438\,150i}y = y = c_4, \ldots,
\end{aligned}$$

$$\begin{aligned}
c_{17} &= x^{185-72i} = x = c_1, c_{18} = x^{-152-32i}y = y = c_2,\\
c_{19} &= x^{136+185i} = x^i = c_3, c_{20} = x^{120-338i}y = y = c_4, \ldots,
\end{aligned}$$

$$\begin{aligned}
c_{33} &= x^{40433-26512i} = x = c_1, c_{34} = x^{-10544-29888i}y = y = c_2,\\
c_{35} &= x^{86288+40433i} = x^i = c_3, c_{36} = x^{162032-50978i}y = y = c_4, \ldots,
\end{aligned}$$

and

$$\begin{aligned}
c_{65} &= x^{3231006177-4801002272i} = x = c_1, c_{66} = x^{4856348064-8087354240i}y = y = c_2,\\
c_{67} &= x^{20975710752+3231006177i} = x^i = c_3, c_{68} = x^{46807769568+1625341886i}y = y = c_4, \ldots.
\end{aligned}$$

So we get $LP^{(i,c)}_{(x,y)}(D_2) = 4$, $LP^{(i,c)}_{(x,y)}(D_3) = 104$, $LP^{(i,c)}_{(x,y)}(D_4) = 8$, $LP^{(i,c)}_{(x,y)}(D_6) = 104$, $LP^{(i,c)}_{(x,y)}(D_8) = 16$, $LP^{(i,c)}_{(x,y)}(D_{16}) = 32$ and $LP^{(i,c)}_{(x,y)}(D_{32}) = 64$.

Corollary 3.6. *For $n = 2^k$ such that $k \geq 2$, the length of the period of the complex-type cyclic-Pell orbit $LP^{(i,c)}_{(x,y)}(D_n)$ is $2n$.*

Proof. From the orbit $P^{(i,c)}_{(x,y)}(D_n)$, we can deduce the following:

$$\begin{aligned}
c_1 &= x, c_2 = y, \ldots,\\
c_9 &= x^{13-4i}, c_{10} = x^{-12}y, \ldots,\\
c_{17} &= x^{185-72i}, c_{18} = x^{-152-32i}y, \ldots,\\
c_{8u+1} &= x^{4u\lambda_1+1-4u\lambda_2 i}, c_{8u+2} = x^{-4u\lambda_3-4u\lambda_4 i}y, \ldots,
\end{aligned}$$

where $\gcd(\beta_1, \beta_2) = 1$. So we need an $u \in \mathbb{N}$ such that $4u = \tau n$ for $\tau \in \mathbb{N}$. If $n = 2^k$ such that $k \geq 2$, then $u = \frac{n}{4}$, and we obtain $LP^{(i,c)}_{(x,y)}(D_n) = 8\frac{n}{4} = 2n$. □

4. Conclusion

In Section 2, we defined the complex-type cyclic-Pell sequence and then, we obtained the relationships among the elements of the sequence and the generating matrix of the sequence. Also, we gave the Simpson formula of the complex-type cyclic-Pell sequence. In Section 3, we studied the complex-type cyclic-Pell sequence modulo m. Furthermore, we got the cyclic groups generated by reducing the multiplicative orders of the generating matrices and the auxiliary equations of these sequences modulo m and then, we investigated the orders of these cyclic groups. Moreover, using the terms of 2-generator groups which is called the complex-type cyclic-Pell orbit, we redefined the complex-type cyclic-Pell sequence. Also, the sequence in finite groups was examined in detail. Finally, for some $n \geq 2$ as applications of the results obtained, we got the lengths of the periods of the complex-type cyclic-Pell orbits of the dihedral group D_n and we reached the length of the period of the complex-type cyclic-Pell orbit $LP^{(i,c)}_{(x,y)}(D_n)$ for $n = 2^k$ when $k \geq 2$.

References

[1] Akuzum Y, Deveci O. The Hadamard-type k-step Fibonacci sequences in groups. Communications in Algebra. 48(7), 2020, 2844−2856.

[2] Akuzum Y, Deveci O, Rashedi ME. The Hadamard-type k-step Pell sequences in groups. Caspian Journal of Mathematical Sciences. 11(1), 2022, 304−312.

[3] Akuzum Y, Deveci O, Shannon AG. On the Pell p-circulant sequences. Notes on Number Theory and Discrete Mathematics. 23(2), 2017, 91-103.

[4] Aydin H, Dikici R. General Fibonacci sequences in finite groups. Fibonacci Quarterly. 36(3), 1998, 216−221.

[5] Berzsenyi G. Sums of products of generalized Fibonacci numbers. Fibonacci Quarterly. 13(4), 1975,343−344.

[6] Bicknell M. A Primer on the Pell Sequences and Related Sequences. Fibonacci Quarterly. 14(4), 1975, 345−349.

[7] Cagman A. Explicit Solutions of Powers of Three as Sums of Three Pell Numbers Based on Baker's Type Inequalities. Turkish Journal of Inequalities. 5(1), 2021, 93−103.

[8] Cagman A. An approach to Pillai's problem with the Pell sequence and the powers of 3. Miskolc Mathematical Notes. 22(2), 2021, 599−610.

[9] Cagman A, Polat K. On a Diophantine equation related to the difference of two Pell numbers. Contributions to Mathematics. 3, 2021, 37−42.

[10] Campbell CM, Campbell PP, Doostie H, Robertson EF. On the Fibonacci length of powers of dihedral groups. In Applications of Fibonacci numbers. F. T. Howard, Ed., vol. 9, 2004, pp. 69−85, Kluwer Academic Publisher, Dordrecht, The Netherlands.

[11] Campbell CM, Campbell PP. The Fibonacci lengths of binary polyhedral groups and related groups. Congressus Numerantium. 194, 2009, 95−102.

[12] Campbell CM, Doostie H, Robertson EF. Fibonacci length of generating pairs in groups. In: Bergum, G. E., ed. Applications of Fibonacci Numbers. Vol. 3, 1990, pp. 27−35, Springer, Dordrecht: Kluwer Academic Publishers.

[13] Deveci O, Akdeniz M, Akuzum Y. The Periods of The Pell p-Orbits of Polyhedral and Centro-Polyhedral Groups. Jordanian Journal of Mathematics and Statistics. 10(1), 2017, 1−9.

[14] Deveci O, Akuzum Y, Karaduman E. The Pell-Padovan p-sequences and its applications. Utilitas Mathematica. 98, 2015, 327−347.

[15] Deveci O, Hulku S, Shannon AG. On the co-complex-type k-Fibonacci numbers. Chaos, Solitons and Fractals. 153, 2021, 111522.

[16] Deveci O, Karaduman E, Campbell CM. On the k-nacci sequences in finite binary polyhedral groups. Algebra Colloquium. 18(1), 2011, 945−954.

[17] Deveci O, Shannon AG. The complex-type k-Fibonacci sequences and their applications. Communications in Algebra. 49(3), 2021, 1352−1367.

[18] Deveci O, Shannon AG. The quaternion-Pell sequence. Communications in Algebra. 46(12), 2018, 5403−5409.

[19] Doostie H, Hashemi M. Fibonacci lengths involving the Wall number $K(n)$. Journal of Applied Mathematics and Computing. 20(1), 2006, 171−180.

[20] Falcon S, Plaza A. k-Fibonacci sequences modulo m. Chaos Solitons & Fractals 41(1), 2009, 497−504.

[21] Horadam AF. A generalized Fibonacci sequence. American Mathematical Monthly. 68(5), 1961, 455−459.

[22] Horadam AF. Complex Fibonacci numbers and Fibonacci quaternions. American Mathematical Monthly. 70(3), 1963, 289−291.

[23] Kalman D. Generalized Fibonacci numbers by matrix methods. Fibonacci Quarterly. 20(1), 1982, 73−76.

[24] Karaduman E, Aydın H. k-nacci sequences in some special groups of finite order. Mathematical and Computer Modelling of Dynamical Systems. 50(1-2), 2009, 53−58.

[25] Knox SW. Fibonacci sequences in finite groups. Fibonacci Quarterly. 30(2), 1992, 116−120.

[26] Lu K, Wang J. k-step Fibonacci sequence modulo m. Utilitas Mathematica. 71, 2006, 169−177.

[27] Ozkan, E. Truncated Lucas sequences and its period. Applied Mathematics and Computation. 232, 2014, 285−291.

[28] Ozkan E, Alp T. Bigaussian Pell and Pell-Lucas Polynomials. Mathematica Montisnigri. 53(3), 2022, 17−25.

[29] Tastan M, Ozkan E. On the Gauss k-Fibonacci Polynomials. Electronic Journal of Mathematical Analysis and Applications. 9(1), 2021, 124−130.

[30] Wall DD. Fibonacci series modulo m, American Mathematical Monthly. 67(6), 1960, 525−532.

[31] Wilcox HJ. Fibonacci sequences of period n in Groups. Fibonacci Quarterly. 24(4), 1986, 356−361.

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 3, 211–218
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

Coefficient Bound Estimates and Fekete-Szegö Problem for a Certain Subclass of Analytic and Bi-univalent Functions

Nizami Mustafa[a], Semra Korkmaz[a]

[a] *Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars, Turkey*

Abstract. In this study, we introduce and examine a certain subclass of analytic and bi-univalent functions in the open unit disk in the complex plane. Here, we give coefficient bound estimates and examine the Fekete-Szegö problem for this class. Some interesting special cases of the results obtained here are also discussed.

1. Introduction and preliminaries

Let A denote the class of all complex valued functions $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = z + a_2z^2 + a_3z^3 + ... + a_nz^n + ... = z + \sum_{n=2}^{\infty} a_nz^n, \ z \in \mathbb{C}, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. By S, we will denote the class of all univalent functions in the set A. For $\alpha \in [0, 1)$, some of the important and well-investigated subclasses of S include the classes $S^* (\alpha)$ and $C (\alpha)$, respectively, starlike and convex function classes of order α in U.

It is well-known that (see [3]) every function $f \in S$ has an inverse f^{-1} defined by

$$f^{-1} (f(z)) = z, z \in U, \ f^{-1} (f(w)) = w, w \in U_0 = \{w \in \mathbb{C} : |w| < r_0 (f)\}, r_0 (f) \geq \frac{1}{4}$$

and

$$f^{-1}(w) = w + b_2w^2 + b_3w^3 + ... + b_nw^n + ... = w + \sum_{n=2}^{\infty} b_nw^n, \ w \in U_0,$$

where

$$b_2 = -a_2, \ b_3 = 2a_2^2 - a_3, \ b_4 = -5a_2^3 + 5a_2a_3 - a_4.$$

.

Corresponding author: NM mail address: nizamimustafa@gmail.com ORCID:0000-0002-2758-0274, SK ORCID:0000-0002-7846-9779

Received: 5 July 2022; Accepted: 12 December 2022; Published: 30 December 2022
Keywords. Coefficient bound estimate, Fekete-Szegö problem, bi-univalent function, q-derivative
2010 *Mathematics Subject Classification*. 30C45, 30C50, 30C55, 30C80
Cited this article as: Mustafa N. Korkmaz S. Coefficient Bound Estimates and Fekete-Szegö Problem for a Certain Subclass of Analytic and Bi-univalent Functions, Turkish Journal of Science, 2022, 7(3), 211–218.

A function $f \in A$ is called bi-univalent in U if both f and f^{-1} are univalent in U and $f(U)$ respectively. Let Σ denote the class of bi-univalent functions in the set S.

For the functions f and g which are analytic in U, f is said to be subordinate to g and denoted as $f(z) \prec g(z)$ if there exists an analytic function ω such that

$$\omega(0) = 0, |\omega(z)| < 1 \text{ and } f(z) = g(w(z)).$$

As is known that the coefficient problem is one of the important subjects of the theory of geometric functions. Firstly, by Lewin was introduced [7] a subclass of bi-univalent functions and obtained the estimate $|a_2| \leq 1.51$ for the function belonging to this class. Subsequently, Brannan and Clunie [1] developed the result of Lewin to $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Later, Netanyahu [11] showed that $|a_2| \leq \frac{4}{3}$ for this class functions. By Brannan and Taha [2] were introduced certain subclasses of bi-univalent function class Σ, namely bi-starlike function of order α denoted $S^*_\Sigma(\alpha)$ and bi-convex function of order α denoted $C_\Sigma(\alpha)$, respectively. For each of the function classes $S^*_\Sigma(\alpha)$ and $C_\Sigma(\alpha)$, non-sharp estimates on the first two coefficients for the functions belonging to these classes were found by Brannan and Taha (see [2]). Many researchers have introduced and investigated several interesting subclasses of bi-univalent function class Σ and they have found non-sharp estimates on the first two coefficients for the functions belonging to these classes (see [13, 15]).

It is also well known that the important tools in the theory of analytic functions is the functional $H_2(1) = a_3 - a_2^2$, which is known as the Fekete-Szegö functional and one usually considers the further generalized functional $H_2(1) = a_3 - \mu a_2^2$, where μ is a complex or real number (see [5]). Estimating the upper bound of $\left|a_3 - \mu a_2^2\right|$ is known as the Fekete-Szegö problem in the theory of analytic functions. The Fekete-Szegö problem has been investigated by many mathematicians for several subclasses of analytic functions (see [8, 9, 14]). Very soon, Mustafa and Mrugusundaramoorthy [10] examine the Fekete-Szegö problem for the subclass of bi-univalent functions related to shell shaped region.

Now, let's give some concepts that we will use throughout our study.

For $q \in (0,1)$, in his fundamental paper by Jackson [6] introduced $q-$derivative operator D_q of an analytic function f as follows:

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & if \quad z \neq 0, \\ f' & if \quad z = 0. \end{cases} \tag{2}$$

It follows from that $D_q z^n = [n]_q z^{n-1}, n \in \mathbb{N}$, where $[n]_q = 1 + q + q^2 + ... + q^{n-1} = \sum\limits_{k=1}^{n} q^{k-1}$ is $q-$analogue of the natural numbers n. Also, it can be easily shown that $\lim\limits_{q\to 1^{-1}} [n]_q = n$, $[n]_q \frac{1-q^n}{1-q}$, $[0]_q = 0, [1]_q = 1$.

Using definition (2) for the first and second $q-$ derivative of the function $f \in A$, we write

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q z^{n-1} \text{ and } D_q^2 f(z) = D_q\left(D_q f(z)\right) = \sum_{n=2}^{\infty} [n]_q [n-1]_q z^{n-2}.$$

Also, it is clear that $\lim\limits_{q\to 1^-} D_q f(z) = f'(z)$ for an analytic function f.

For the function $f \in A$, Salagean (see [12]) introduced the following differential operator, which is called the Salagean operator

$$S^0 f(z) = f(z), S^1 f(z) = zSf(z) = zf'(z),$$
$$S^2 f(z) = zS(Sf(z)) = zf''(z), ..., S^n f(z) = zS\left(S^{n-1} f(z)\right), n = 1, 2, ...$$

It follows from that

$$S^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, z \in U, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Now, let we define the following subclass of analytic and bi-univalent functions.

Definition 1.1. *For $q \in (0,1)$, a function $f \in \Sigma$ is said to be in the class $C_{q,\Sigma}(n,\varphi)$ if the following conditions are satisfied*

$$1+\frac{zD_q^2\left(S^n f(z)\right)}{D_q\left(S^n f(z)\right)} \prec \varphi(z), z \in U \text{ and } 1+\frac{zD_q^2\left(S^n f^{-1}(w)\right)}{D_q\left(S^n f^{-1}(w)\right)} \prec \varphi(w), w \in U_0.$$

In this definition $\varphi(z) = z + \sqrt{1+z^2}$ and the branch of the square root is chosen to be principal one, that $\varphi(0) = 1$. It can be easily seen that the function $\varphi(z) = z + \sqrt{1+z^2}$ maps the unit disc U onto a shell shaped region on the right half plane and it is analytic and univalent in U. The range $\varphi(U)$ is symmetric respect to real axis and φ is a function with positive real part in U, with $\varphi(0) = \varphi'(0) = 1$ Moreover, it is a starlike domain with respect to point $\varphi(0) = 1$.

In the case $n = 0$, from the Definition 1.1 we have the subclass $C_{q,\Sigma}(\varphi) = C_{q,\Sigma}(0,\varphi)$. Also, we have the subclass $C_{\Sigma}(n,\varphi)$, when $q \to 1^-$.

Let, P be the set of the functions $p(z)$ analytic in U and satisfying $\Re(p(z)) > 0$, $z \in U$ and $p(0) = 1$ with power series

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... + p_n z^n + ... = 1 + \sum_{n=1}^{\infty} p_n z^n, \ z \in U,$$

In order to prove our main results in this paper, we shall need the following lemmas (see [3, 4]).

Lemma 1.2. *Let $p \in P$, then $|p_n| \le 2$, $n = 1,2,3,....$ These inequalities are sharp. In particular, equality holds for the function $p(z) = (1+z)/(1-z)$ for all $n = 1,2,3,....$*

Lemma 1.3. *Let $p \in P$, then $|p_n| \le 2$, $n = 1,2,3,...$ and*

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)x,$$

$$4p_3 = p_1^3 + 2\left(4-p_1^2\right)p_1 x - 2\left(4-p_1^2\right)p_1 x^2 + 2\left(4-p_1^2\right)\left(1-|x|^2\right)z$$

for some x and z with $|x| < 1$ and $|z| < 1$.

Remark 1.4. *As can be seen from the serial expansion of the function φ given in Definition 1.1, this function belong to the class P.*

In this paper, we give coefficient bound estimates and examine the Fekete-Szegö problem for the class $C_{q,\Sigma}(n,\varphi)$.

2. Main Results

In this section, firstly we give the following theorem on the coefficient bound estimates for the class $C_{q,\Sigma}(n,\varphi)$.

Theorem 2.1. *Let the function f given by (1) be in the class $C_{q,\Sigma}(n,\varphi)$. Then*

$$|a_2| \le \frac{1}{[2]_q 2^n}, |a_3| \le \begin{cases} \frac{1}{[2]_q[3]_q 3^n}, & \frac{[3]_q}{[2]_q} \le \left(\frac{4}{3}\right)^n, \\ \frac{1}{[2]_q^2 4^n} & \frac{[3]_q}{[2]_q} > \left(\frac{4}{3}\right)^n. \end{cases}$$

Moreover,

$$|a_4| \leq \begin{cases} \max\left\{\lambda(q,n), \nu(q,n)\right\}, & \theta_1(q,n) \geq 0, \\ \max\left\{\lambda(q,n), \nu(q,n), \sqrt{\frac{-4\theta_2^3(q,n)}{27\theta_1(q,n)}}\right\}, & \theta_1(q,n) < 0, \end{cases}$$

where

$$\lambda(q,n) = \frac{\left([2]_q + 1\right)[3]_q 3^n - [2]_q^2 4^n}{[2]_q^2 [3]_q [4]_q 16^n}, \nu(q,n) = \frac{1}{[3]_q [4]_q 4^n},$$

$$\theta_1(q,n) = \frac{\left([2]_q + 1\right)[3]_q 3^n - [2]_q^2 4^n}{8[2]_q^2 [3]_q [4]_q 16^n} - \frac{5}{16[2]_q^2 [3]_q 6^n} - \frac{1}{[3]_q [4]_q 4^n},$$

$$\theta_2(q,n) = \frac{5}{4[2]_q^2 [3]_q 6^n} + \frac{1}{[3]_q [4]_q 4^{n-1}}.$$

Proof. Let $f \in C_{q,\Sigma}(n,\varphi)$. Then, according to Definition 1.1 there are analytic functions $\omega : U \to U$ and $\varpi : U_0 \to U_0$ with $\omega(0) = 0 = \varpi(0), |\omega(z)| < 1$ and $|\varpi(z)| < 1$ satisfying the following conditions

$$\begin{aligned} 1 + \frac{zD_q^2(S^n f(z))}{D_q(S^n f(z))} &= \varphi(\omega(z)) = \omega(z) + \sqrt{1 + \omega^2(z)}, \ z \in U, \quad (3) \\ 1 + \frac{zD_q^2\left(S^n f^{-1}(w)\right)}{D_q(S^n f^{-1}(w))} &= \varphi(\varpi(w)) = \varpi(w) + \sqrt{1 + \varpi^2(w)}, \ w \in U_0. \end{aligned}$$

Now, we define the functions $p, \phi \in P$ as follows:

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... + p_n z^n + ... = 1 + \sum_{n=1}^{\infty} p_n z^n, \ z \in U$$

$$\phi(w) = \frac{1 + \varpi(z)}{1 - \varpi(z)} = 1 + \phi_1 w + \phi_2 w^2 + \phi_3 w^3 + ... + \phi_n w^n + ... = 1 + \sum_{n=1}^{\infty} \phi_n w^n, \ w \in U_0.$$

From here, we find the following equalities for the functions ω and ϖ

$$\begin{aligned} \omega(z) &= \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 - p_1 p_2 + \frac{p_1^2}{4}\right)z^3 + ...\right], \ z \in U, \quad (4) \\ \varpi(w) &= \frac{\phi(w) - 1}{\phi(w) + 1} = \frac{1}{2}\left[\phi_1 w + \left(\phi_2 - \frac{\phi_1^2}{2}\right)w^2 + \left(\phi_3 - \phi_1 \phi_2 + \frac{\phi_1^2}{4}\right)w^3 + ...\right], \ w \in U_0, \end{aligned}$$

Changing the expression of the functions $\omega(z)$ and $\varpi(z)$ in (3) with expressions in (4), we can write the following equalities

$$\begin{aligned} & 1 + \frac{zD_q^2(S^n f(z))}{D_q(S^n f(z))} \quad (5) \\ =& 1 + \frac{p_1}{2}z + \left(\frac{p_2}{2} - \frac{p_1^2}{8}\right)z^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{4}\right)z^3 + ..., z \in U, \\ & 1 + \frac{zD_q^2\left(S^n f^{-1}(w)\right)}{D_q(S^n f^{-1}(w))} \\ =& 1 + \frac{\phi_1}{2}w + \left(\frac{\phi_2}{2} - \frac{\phi_1^2}{8}\right)w^2 + \left(\frac{\phi_3}{2} - \frac{\phi_1 \phi_2}{4}\right)w^3 + ..., w \in U_0. \end{aligned}$$

If the operations and simplifications on the left side of (5) are made and then the coefficients of the terms of the same degree are equalized, are obtained the following equalities for the coefficients a_2, a_3 and a_4

$$[2]_q\, 2^n a_2 = \frac{p_1}{2},\ [2]_q\, [3]_q\, 3^n a_3 - [2]_q^2\, 4^n a_2^2 = \frac{p_2}{2} - \frac{p_1^2}{8}$$

$$[3]_q\, [4]_q\, 4^n a_4 - [2]_q\, [3]_q \left([2]_q + 1\right) 6^n a_2 a_3 + [2]_q^2\, 8^n a_2^3 = \frac{p_3}{2} - \frac{p_1 p_2}{4}$$

and

$$-[2]_q\, 2^n a_2 = \frac{\phi_1}{2}, -[2]_q\, [3]_q\, 3^n a_3 + \left\{2\,[2]_q\, [3]_q\, 3^n - [2]_q^2\, 4^n\right\} a_2^2 = \frac{\phi_2}{2} - \frac{\phi_1^2}{8}$$

$$\begin{aligned} & -[3]_q\, [4]_q\, 4^n a_4 + \left[5\,[3]_q\, [4]_q\, 4^n - [2]_q\, [3]_q \left([2]_q + 1\right) 6^n\right] a_2 a_3 \\ & -\left[5\,[3]_q\, [4]_q\, 4^n - 2\,[2]_q\, [3]_q \left([2]_q + 1\right) 6^n + [2]_q^2\, 8^n\right] a_2^3 \\ = & \ \frac{\phi_3}{2} - \frac{\phi_1 \phi_2}{4}. \end{aligned}$$

From these equalities, we write

$$\frac{p_1}{[2]_q\, 2^{n+1}} = a_2 = -\frac{\phi_1}{[2]_q\, 2^{n+1}},\ p_1 = -\phi_1, \tag{6}$$

$$a_3 = a_2^2 + \frac{p_2 - \phi_2}{4\,[2]_q\, [3]_q\, 3^n}, \tag{7}$$

$$a_4 = \frac{5\left(p_2 - \phi_2\right)}{16\,[2]_q^2\, [3]_q\, 6^n} + \frac{[3]_q \left([2]_q + 1\right) 3^n - [2]_q^2\, 4^n}{[2]_q^2\, [3]_q\, [4]_q\, 2^{4n+3}} p_1^3 + \frac{p_3 - \phi_3}{[3]_q\, [4]_q\, 4^{n+1}} - \frac{\left(p_2 + \phi_2\right) p_3}{2\,[3]_q\, [4]_q\, 4^{n+1}}. \tag{8}$$

By applying the Lemma 1.2 to equality (6), obtained immediately first result of theorem.

Now, firstly using the Lemma 1.3 and then applying triangle inequality and Lemma 1.2 to the equality (7), we get

$$|a_3| = \frac{t^2}{[2]_q\, 4^{n+1}} + \frac{4 - t^2}{8\,[2]_q\, [3]_q\, 3^n} \left(\xi + \eta\right).$$

with $\left|p_1\right| = t$, $|x| = \xi$ and $\left|y\right| = \eta$ for some x and y with $|x| < 1$ and $\left|y\right| < 1$. Then, maximizing the right-hand side of the last inequality according to the parameters $\xi \in (0, 1)$ and $\eta \in (0, 1)$, we obtain the following inequality

$$|a_3| \leq c\,(q, n)\, t^2 + \frac{1}{[2]_q\, [3]_q\, 3^n}, t \in [0, 2]\,, c\,(q, n) = \frac{[3]_q\, 3^n - [2]_q\, 4^n}{4\,[2]_q^2\, [3]_q\, 12^n}.$$

From the last inequality obtained the second result of theorem.

Finally, let's find an upper bound estimate for $|a_4|$. By applying Lemma 1.3 and then triangle inequality and Lemma 1.2 to expression of a_4 in the equality (8), we obtain

$$8\,|a_4| \leq c_1\,(t) + c_2\,(t)\,(\xi + \eta) + c_3\,(t) \left(\xi^2 + \eta^2\right), \tag{9}$$

with $|x| = \xi \in (0, 1)$ and $\left|y\right| = \eta \in (0, 1)$, where

$$c_1(t) = \frac{[3]_q\left([2]_q + 1\right)3^n - [2]_q^2 4^n}{[2]_q^2 [3]_q [4]_q 2^{4n+3}} t^3 + \frac{1}{[3]_q [4]_q 4^{n+1}}\left(4 - t^2\right),$$

$$c_2(t) = \left[\frac{5}{32[2]_q^2 [3]_q 6^n} + \frac{1}{[3]_q [4]_q 4^{n+1}}\right]\left(4 - t^2\right)t,\ c_3(t) = \frac{\left(4 - t^2\right)(t-2)}{[3]_q [4]_q 4^{n+2}}$$

If we maximize the right-hand side of the inequality (9) firstly according to the parametres ξ and η and then to the parameter, we get the desired estmate for $|a_4|$.

Thus, the proof of Theorem 2.1 is completed. □

From the Teorem 2.1, we obtain the following results.

Corollary 2.2. *Let $f \in C_\Sigma(n, \varphi)$. Then,*

$$|a_2| \le \frac{1}{2^{n+1}}, n = 0, 1, 2, ..., |a_3| \le \begin{cases} \frac{1}{4^{n+1}}, & if\ \ n = 0, 1, \\ \frac{1}{2.3^{n+1}} & otherwise \end{cases}$$

and

$$|a_4| \le \max\left\{\left(\frac{3}{16}\right)^{n+1} - \frac{1}{3.4^{n+1}}, \frac{1}{3.4^{n+1}}, \sqrt{\frac{-4\theta_2^3(n)}{27\theta_1(n)}}\right\}$$

where

$$\theta_1(q, n) = \frac{3^{n+1}}{8.16^{n+1}} - \frac{5}{32.6^n} - \frac{3}{8.4^{n+1}},$$

$$\theta_2(q, n) = \frac{5}{8.6^{n+1}} + \frac{1}{3.4^n}.$$

Corollary 2.3. *Let $C_{q,\Sigma}(\varphi)$. Then*

$$|a_2| \le \frac{1}{q+1}, |a_3| \le \frac{1}{(q+1)^2}$$

Moreover,

$$|a_4| \le \max\left\{\lambda(q), \nu(q), \sqrt{\frac{-4\theta_2^3(q)}{27\theta_1(q)}}\right\},$$

where

$$\lambda(q) = \frac{q^3 + 2q^2 + q + 1}{(q+1)^2(q^2+q+1)(q^3+q^2+q+1)}, \nu(q) = \frac{1}{(q^2+q+1)(q^3+q^2+q+1)},$$

$$\theta_1(q) = \frac{-3q^3 - 17q^2 - 35q - 19}{16(q+1)^2(q^2+q+1)(q^3+q^2+q+1)},$$

$$\theta_2(q, n) = \frac{5q^3 + 21q^2 + 37q + 16}{4(q+1)^2(q^2+q+1)(q^3+q^2+q+1)}.$$

Now, we give the following theorem on the Fekete-Szegö problem for the class $C_{q,\Sigma}(n, \varphi)$.

Theorem 2.4. *Let the function f given by (1) be in the class $C_{q,\Sigma}(n,\varphi)$. Then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{1}{[2]_q[3]_q 3^n} & if \quad \left|1-\mu\right| \le \frac{[2]_q}{[3]_q}\left(\frac{4}{3}\right)^n, \\ \frac{\left|1-\mu\right|}{[2]_q^2 4^n} & if \quad \left|1-\mu\right| > \frac{[2]_q}{[3]_q}\left(\frac{4}{3}\right)^n. \end{cases}$$

Proof. Let $f \in C_{q,\Sigma}(n,\varphi)$ and $\mu \in \mathbb{C}$. Then, from the expressions for a_2 and a_3 in the equalities (6) and (7), we can write the following equality for $a_3-\mu a_2^2$

$$a_3-\mu a_2^2 = (1-\mu)\,a_2^2 + \frac{p_2-\phi_2}{4\,[2]_q\,[3]_q\,3^n}.$$

According to Lemma 1.3, from the last equality we can write

$$a_3-\mu a_2^2 = (1-\mu)\,a_2^2 + \frac{4-p_1^2}{8\,[2]_q\,[3]_q\,3^n}\,(x-y)$$

for some x and y with $|x|<1$ and $\left|y\right|<1$.

Then, using triangle inequality and considering that $\left|p_1\right| = t \le 2$ from the last equality, we get

$$\left|a_3-\mu a_2^2\right| \le \frac{(1-\mu)}{[2]_q^2\,4^{n+1}}t^2 + \frac{4-t^2}{8\,[2]_q\,[3]_q\,3^n}\,(\xi+\eta), \tag{10}$$

with $|x| = \xi \in (0,1)$ and $\left|y\right| = \eta \in (0,1)$. If we maximize the right-hand side of the inequality (10) according to the parametres ξ and η we get the following inequality

$$\left|a_3-\mu a_2^2\right| \le \frac{1}{[2]_q^2\,4^{n+1}}\left\{\left|1-\mu\right| - \frac{[2]_q}{[3]_q}\left(\frac{4}{3}\right)^n\right\}t^2 + \frac{1}{[2]_q\,[3]_q\,3^n}, t \in [0,2].$$

From here, by maximizing the right hand side of the last inequality according to the parameter t, obtained the result of theorem. Thus, the proof of Theorem 2.4 is completed. □

From the Theorem 2.4 obtained the following results.

Corollary 2.5. *Let $f \in C_{\Sigma}(n,\varphi)$. Then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{1}{2.3^{n+1}} & if \quad \left|1-\mu\right| \le \frac{1}{2}\left(\frac{4}{3}\right)^{n+1}, \\ \frac{\left|1-\mu\right|}{4^{n+1}} & if \quad \left|1-\mu\right| > \frac{1}{2}\left(\frac{4}{3}\right)^{n+1}. \end{cases}$$

Corollary 2.6. *Let $f \in C_{q,\Sigma}(\varphi)$. Then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{1}{[2]_q[3]_q} & if \quad \left|1-\mu\right| \le \frac{[2]_q}{[3]_q}, \\ \frac{\left|1-\mu\right|}{[2]_q^2} & if \quad \left|1-\mu\right| > \frac{[2]_q}{[3]_q}. \end{cases}$$

Corollary 2.7. *Let $f \in C_{q,\Sigma}(n,\varphi)$. Then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{1}{[2]_q[3]_q 3^n} & if \quad \frac{[3]_q}{[2]_q} \le \left(\frac{4}{3}\right)^n, \\ \frac{\left|1-\mu\right|}{[2]_q^2 4^n} & if \quad \frac{[3]_q}{[2]_q} > \left(\frac{4}{3}\right)^n. \end{cases}$$

Corollary 2.8. *Let $f \in C_{\Sigma}(n,\varphi)$. Then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{1}{4^{n+1}} & if \;\; n = 0,1, \\ \frac{1}{2.3^{n+1}} & otherwise. \end{cases}$$

Corollary 2.9. *Let* $f \in C_{q,\Sigma}(\varphi)$*. Then*

$$|a_3| \leq \frac{1}{[2]_q^2}.$$

Remark 2.10. *The Corollary 2.7 confirm the second result obtained in the Theorem 2.1.*

References

[1] Brannan DA, Clunie J. Aspects of contemporary complex analysis. Academic Press, London and New York, USA. 1980.
[2] Brannan DA, Taha TS. On some classes of bi-univalent functions. Studia Univ. Babes-Bolyai Mathematics. 31, 1986, 70–77.
[3] Duren PL. Univalent Functions. In: Grundlehren der Mathematischen Wissenschaften, Band 259, New-York, Berlin, Heidelberg and Tokyo, Springer- Verlag. 1983.
[4] Grenander U, Szegö G. Toeplitz form and their applications. California Monographs in Mathematical Sciences, University California Press, Berkeley. 1958.
[5] Fekete M, Szegö G. Eine Bemerkung Über ungerade schlichte Funktionen. Journal of the London Mathematical Society. 8, 1993, 85–89.
[6] Jackson FH. On definite integrals. The Quarterly Journal of Pure and Applied Mathematics. 41, 1910, 193–203.
[7] Lewi M. On a coefficient problem for bi-univalent functions. Proceedings of the American Mathematical Society. 18, 1967, 63–68.
[8] Mustafa N. Fekete-Szegö Problem for certain subclass of analytic and bi-univalent functions. Journal of Scientific and Engineering Research. 4(8), 2017, 30–400.
[9] Mustafa N, Gündüz MC. The Fekete-Szegö problem for certain class of Analytic and univalent functions. Journal of Scientific and Engineering Research. 6(5), 2019, 232–239.
[10] Mustafa N, Mrugusundaramoorthy G. Second Hankel for Mocanu type bi-starlike functions related to shell shaped region. Turkish Journal of Mathematics. 45, 2021, 1270–1286.
[11] Netanyahu E. The minimal distance of the image boundary from the origin and the second coefficient of a univalent function. Archive for Rational Mechanics and Analysis. 32, 1969, 100–112.
[12] Salagean GS. Subclasses of univalent functions. Complex Analysis. 103, 1983, 362–372.
[13] Srivastava HM, Mishra AK, Gochhayat P. Certain subclasses of analytic and bi-univalent functions. Applied Mathematics Letters. 23, 2010, 1188–1192.
[14] Zaprawa P. On the Fekete-Szegö problem for the classes of bi-univalent functions. Bulletin of the Belgian Mathematical Society. 21, 2014, 169–178.
[15] Xu QH, Xiao G, Srivastava HM. A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems. Applied Mathematics and Computation. 218, 2012, 11461–11465.

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 3, 219–230
ISSN: 2587–0971 http://dergipark.gov.tr/tjos

Generalized Inequalities for Quasi-Convex Functions via Generalized Riemann-Liouville Fractional Integrals

Recep Türker[a], Havva Kavurmacı-Önalan[a]

[a]*Department of Mathematics, Faculty of Science and Arts, Van Yüzüncü Yıl University, Van-TURKEY*
[b]*Department of Mathematics Education, Faculty of Education, Van Yüzüncü Yıl University, Van-TURKEY*

Abstract. We establish some new Generalized Hermite-Hadamard-type inequalities involving generalized fractional integrals for quasi-convex functions. Our results are consistent with previous findings in the literature. The analysis used in the proofs is fairly elementary and based on the use of Hölder inequality and the power inequality.

1. Introduction

The H-H inequality shows that the mean value of a continuous convex function is greater than the value of the function at the midpoint of this range and less than the arithmetic mean of its endpoints and it has many applications for real analysis. So, it has been studied by many researchers.

Let us give this unique inequality which is named as H-H inequality in the literature: Let $g : I \longrightarrow \mathbb{R}$ be a convex mapping defined on the interval $I \subseteq \mathbb{R}$ and $\varepsilon, \delta \in I$ with $\varepsilon < \delta$,then

$$g\left(\frac{\varepsilon+\delta}{2}\right) \leq \frac{1}{\delta-\varepsilon}\int_{\varepsilon}^{\delta} g(x)\,dx \leq \frac{g(\varepsilon)+g(\delta)}{2}. \tag{1}$$

In the case where g is concave, the above inequality is reversed.

Later, many researchers used different classes of convex functions to generalize, improve, and extend this inequality. (See [3], [7], [8]-[11], [14]-[19], [21], [23]-[44]).

Some researchers have been proven that studies for the inequality of H-H can be generalized with the help of fractional integrals. So new studies have been carried out in the field of convex functions and inequalities using the concepts of fractional derivatives and fractional integrals. (For interested researchers [1], [3]-[6], [11]-[22], [26], [28] and [32]-[44]).

Let's remind some definitions and inequalities as following:

Corresponding author: HKÖ mail address: havvaonalan@yyu.edu.tr ORCID:0000-0002-0034-778X, RT ORCID:0000-0001-6047-9007
Received: 27 October 2022; Accepted: 18 December 2022; Published: 30 December 2022
Keywords. Quasi-convex functions; Generalized inequalities; Hölder inequality.
2010 *Mathematics Subject Classification*. 26D15, 26A51, 32F99, 41A17.
Cited this article as: Türker R. Kavurmacı Önalan H. Generalized Inequalities for Quasi-Convex Functions via Generalized Riemann-Liouville Fractional Integrals, Turkish Journal of Science, 2022, 7(3), 219–230.

$\vartheta : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

$$\begin{aligned} &\int_0^1 \frac{\vartheta(l)}{l} dl < \infty, \\ &\frac{1}{A_1} \leq \frac{\vartheta(u)}{\vartheta(t)} \leq A_1 \text{ for } \frac{1}{2} \leq \frac{u}{t} \leq 2 \\ &\frac{\vartheta(t)}{t^2} \leq A_2 \frac{\vartheta(u)}{u^2} \text{ for } u \leq t \\ &\left| \frac{\vartheta(u)}{u^2} - \frac{\vartheta(t)}{t^2} \right| \leq A_3 |t-u| \frac{\vartheta(t)}{t^2} \text{ for } \frac{1}{2} \leq \frac{u}{t} \leq 2 \end{aligned} \tag{2}$$

where $A_1, A_2, A_3 > 0$ are independent of $t, u > 0$. If $\vartheta(t) t^{\alpha}$ is increasing for some $\alpha \geq 0$ and $\frac{\vartheta(t)}{t^{\beta}}$ is decreasing for some $\beta \geq 0$, then ϑ satisfies (2).

In [32], Sarıkaya and Ertuğral defined new left-sided and right-sided generalized fractional integral operators which are useful in the proofs of our main results, respectively, as following:

Definition 1.1. *Let $g \in L[\varepsilon, \delta]$. The generalized fractional integrals ${}_{\varepsilon^+}I_{\vartheta}g$ and ${}_{\delta^-}I_{\vartheta}g$ with $\varepsilon \geq 0$ are defined by*

$$ {}_{\varepsilon^+}I_{\vartheta}g(x) = \int_{\varepsilon}^{x} \frac{\vartheta(x-l)}{x-l} g(l)\, dl, x > \varepsilon \tag{3}$$

$$ {}_{\delta^-}I_{\vartheta}g(x) = \int_{x}^{\delta} \frac{\vartheta(x-l)}{x-l} g(l)\, dl, x < \delta \tag{4}$$

where $\vartheta : [0,\infty) \longrightarrow [0,\infty)$ a function which satisfies $\int_0^1 \frac{\vartheta(l)}{l} dl < \infty$.

The above generalized fractional integrals produce different kinds of fractional integrals as R-L, k−R-L, Katugampola, conformable, Hadamard, etc... You can find the different cases of the above integral operators (3) and (4) in the study [32]. (For interested researchers [5], [11], [18]-[21], [26], [34]-[40].)

In [32], Ertuğral and Sarıkaya achieved the basic H-H inequality with the help of generalized fractional integrals in (3) and (4) as follows:

Theorem 1.2. *Let $g : [\varepsilon, \delta] \longrightarrow \mathbb{R}$ be a convex function on (ε, δ) with $\varepsilon < \delta$,then the following inequalities for generalized fractional integral hold:*

$$ g\left(\frac{\varepsilon+\delta}{2}\right) \leq \frac{1}{2\Lambda(1)} \left[{}_{\varepsilon^+}I_{\vartheta}g(\delta) + {}_{\delta^-}I_{\vartheta}g(\varepsilon) \right] \leq \frac{g(\varepsilon)+g(\delta)}{2}, \tag{5}$$

where $\Lambda(1) = \int_0^1 \frac{\vartheta((\delta-\varepsilon)l)}{l} dl$ and $\Lambda(1) \neq 0$.

The following lemma is used to obtain some inequalities that is trapezoid inequalities for generalized fractional integrals as in [32]:

Lemma 1.3. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be a differentiable mapping on (ε, δ) with $\varepsilon < \delta$. If $g' \in L[\varepsilon, \delta]$, then the following equality for generalized fractional integrals holds:*

$$\begin{aligned} &\frac{g(\varepsilon)+g(\delta)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{\varepsilon^+}I_{\vartheta}g(\delta) + {}_{\delta^-}I_{\vartheta}g(\varepsilon) \right] \\ = &\frac{\delta-\varepsilon}{2\Lambda(1)} \int_0^1 \left[\Lambda(1-l) - \Lambda(l)\right] g'(l\varepsilon + (1-l)\delta)\, dl, \end{aligned} \tag{6}$$

where $\Lambda(1) = \int_0^1 \frac{\vartheta((\delta-\varepsilon)l)}{l} dl$ and $\Lambda(1) \neq 0$.

The following theorem is an inequality for generalized fractional integrals via the right side of the H-H inequality obtained by using Lemma 1.3:

Theorem 1.4. *Let $g:[\varepsilon,\delta]\to\mathbb{R}$ be a differentiable mapping on (ε,δ) with $\varepsilon<\delta$. If $\left|g'\right|$ is convex on $[\varepsilon,\delta]$, then the following inequality for generalized fractional integrals hold:*

$$\begin{aligned}&\left|\frac{g(\varepsilon)+g(\delta)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{\varepsilon^+}I_{\vartheta}g(\delta)+{}_{\delta^-}I_{\vartheta}g(\varepsilon)\right]\right| \\ \leq\ &\frac{(\delta-\varepsilon)}{\Lambda(1)}\int_0^1 l\left|\left[\Lambda(1-l)-\Lambda(l)\right]\right|dl\frac{[g(\varepsilon)+g(\delta)]}{2}.\end{aligned} \tag{7}$$

You can find some results for this and other generalized fractional integrals in [3], [12] and [42]-[44].

Now, let's remind some inequalities that we encountered in the results obtained in our study. Firstly, we give the basic H-H inequality via fractional integrals which is proved by Sarıkaya et al. in [34]:

Theorem 1.5. *Let $g:[\varepsilon,\delta]\longrightarrow\mathbb{R}$ be a positive function with $0\leq\varepsilon<\delta$ and $g\in L_1[\varepsilon,\delta]$. If g is a convex function on $[\varepsilon,\delta]$, then the following inequalities fractional integrals hold:*

$$g\left(\frac{\varepsilon+\delta}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(\delta-\varepsilon)^{\alpha}}\left[J_{\varepsilon^+}^{\alpha}g(\delta)+J_{\delta^-}^{\alpha}g(\varepsilon)\right]\leq\frac{g(\varepsilon)+g(\delta)}{2} \tag{8}$$

with $\alpha>0$.

Since the results which are obtained in this study by using quasi-convex functions, let us remind the definition of quasi-convex functions [30]:

Definition 1.6. *The function $g:I\subset\mathbb{R}\longrightarrow\mathbb{R}$ is said to be quasi-convex if for every $x,y\in I$ and $\omega\in[0,1]$ we have*

$$g(\omega x+(1-\omega)y)\leq\max\{g(x),g(y)\}. \tag{9}$$

Quasi-convexity is a weaker condition than classical convexity. Cause of this situation, you can say every convex function is quasi-convex but there are quasi-convex functions that are not convex (See [16]).

The classical H-H inequality for quasi-convex functions was obtained by Dragomir and Pearce in [8] as follows:

Theorem 1.7. *Let $g:I\to\mathbb{R}$ be a quasi-convex map on I and nonnegative, and suppose $\varepsilon,\delta\in I\subseteq\mathbb{R}$ with $\varepsilon<\delta$ and $g\in L_1[\varepsilon,\delta]$. Then we have the inequality*

$$\frac{1}{\delta-\varepsilon}\int_{\varepsilon}^{\delta}g(x)\,dx\leq\max\{g(\varepsilon),g(\delta)\}. \tag{10}$$

The following theorems which are H-H type inequalities for via quasi-convex function was obatained by Ion in [16] as follows:

Theorem 1.8. *Assume $\varepsilon,\delta\in\mathbb{R}$ with $\varepsilon<\delta$ and $g:[\varepsilon,\delta]\to\mathbb{R}$ is a differentiable function on (ε,δ). If $\left|g'\right|$ is quasi-convex on $[\varepsilon,\delta]$, then the following inequality holds true*

$$\left|\frac{g(\varepsilon)+g(\delta)}{2}-\frac{1}{\delta-\varepsilon}\int_{\varepsilon}^{\delta}g(x)\,dx\right|\leq\frac{(\delta-\varepsilon)\sup\left\{\left|g'(\varepsilon)\right|,\left|g'(\delta)\right|\right\}}{4}. \tag{11}$$

Theorem 1.9. *Assume $\varepsilon,\delta\in\mathbb{R}$ with $\varepsilon<\delta$ and $g:[\varepsilon,\delta]\to\mathbb{R}$ is a differentiable function on (ε,δ). Assume $p\in\mathbb{R}$ with $p>1$. If $\left|g'\right|^{p\backslash(p-1)}$ is quasi-convex on $[\varepsilon,\delta]$ then the following inequality holds true*

$$\begin{aligned}&\left|\frac{g(\varepsilon)+g(\delta)}{2}-\frac{1}{\delta-\varepsilon}\int_{\varepsilon}^{\delta}g(x)\,dx\right| \\ \leq\ &\frac{(\delta-\varepsilon)}{2(p+1)^{1\backslash p}}\left[\sup\left\{\left|g'(\varepsilon)\right|^{p\backslash(p-1)},\left|g'(\delta)\right|^{p\backslash(p-1)}\right\}\right]^{(p-1)\backslash p}.\end{aligned} \tag{12}$$

The following theorems that are H-H type inequalities for via quasi-convex function was obatained Alomari et al. in [2] as follows:

Theorem 1.10. *Let $g : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\varepsilon, \delta \in I^\circ$ with $\varepsilon < \delta$. If $\left|g'\right|^q$ is quasi-convex on $[\varepsilon, \delta]$, $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(\varepsilon) + f(\delta)}{2} - \frac{1}{\delta - \varepsilon} \int_{\varepsilon}^{\delta} g(x)\, dx\right| \leq \frac{(\delta - \varepsilon)}{4} \left(\sup\left\{\left|g'(\varepsilon)\right|^q, \left|g'(\delta)\right|^q\right\}\right)^{\frac{1}{q}}. \tag{13}$$

Theorem 1.11. *Let $g : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\varepsilon, \delta \in I^\circ$ with $\varepsilon < \delta$. If $\left|g'\right|$ is quasi-convex on $[\varepsilon, \delta]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{\delta - \varepsilon} \int_{\varepsilon}^{\delta} g(x)\, dx - g\left(\frac{\varepsilon + \delta}{2}\right)\right| \qquad (14) \\ \leq\ & \frac{\delta - \varepsilon}{8} \left[\max\left\{\left|g'\left(\frac{\varepsilon + \delta}{2}\right)\right|, \left|g'(\delta)\right|\right\} + \max\left\{\left|g'\left(\frac{\varepsilon + \delta}{2}\right)\right|, \left|g'(\varepsilon)\right|\right\}\right]. \end{aligned}$$

Theorem 1.12. *Let $g : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $\varepsilon < \delta$, . If $\left|g'\right|^{p\backslash(p-1)}$ is quasi-convex on $[\varepsilon, \delta]$, $p > 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{\delta - \varepsilon} \int_{\varepsilon}^{\delta} g(x)\, dx - g\left(\frac{\varepsilon + \delta}{2}\right)\right| \qquad (15) \\ \leq\ & \frac{(\delta - \varepsilon)}{4(p+1)^{1\backslash p}} \left[\left(\max\left\{\left|g'\left(\frac{\varepsilon + \delta}{2}\right)\right|^{p\backslash(p-1)}, \left|g'(\delta)\right|^{p\backslash(p-1)}\right\}\right)^{(p-1)\backslash p} \right. \\ & \left. + \left(\max\left\{\left|g'\left(\frac{\varepsilon + \delta}{2}\right)\right|^{p\backslash(p-1)}, \left|g'(\varepsilon)\right|^{p\backslash(p-1)}\right\}\right)^{(p-1)\backslash p}\right]. \end{aligned}$$

Theorem 1.13. *Let $g : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $\varepsilon, \delta \in I^\circ$ with $\varepsilon < \delta$. If $\left|g'\right|^q$ is quasi-convex on $[\varepsilon, \delta]$, $q \geq 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{\delta - \varepsilon} \int_{\varepsilon}^{\delta} g(x)\, dx - g\left(\frac{\varepsilon + \delta}{2}\right)\right| \qquad (16) \\ \leq\ & \frac{\delta - \varepsilon}{8} \left[\left(\max\left\{\left|g'\left(\frac{\varepsilon + \delta}{2}\right)\right|^q, \left|g'(\delta)\right|^q\right\}\right)^{\frac{1}{q}} \right. \\ & \left. + \left(\max\left\{\left|g'\left(\frac{\varepsilon + \delta}{2}\right)\right|^q, \left|g'(\varepsilon)\right|^q\right\}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

In [26], Özdemir and Çetin established some fractional inequalities for differentiable quasi-convex mappings which are connected with H-H inequality as following:

Theorem 1.14. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$, be a positive function with $0 \leq \varepsilon < \delta$ and $g \in L_1[\varepsilon, \delta]$. If g is a quasi-convex function on $[\varepsilon, \delta]$, then the following inequality for fractional integrals holds:*

$$\frac{\Gamma(\alpha + 1)}{2(\delta - \varepsilon)^{\alpha}} \left[J_{\varepsilon+}^{\alpha} g(\delta) + J_{\delta-}^{\alpha} g(\varepsilon)\right] \leq \max\{g(\varepsilon), g(\delta)\} \tag{17}$$

with $\alpha > 0$.

Theorem 1.15. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$, be a differentiable mapping on (ε, δ) with $\varepsilon < \delta$. If $|g'|$ is quasi-convex on $[\varepsilon, \delta]$ and $\alpha > 0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left| \frac{g(\varepsilon) + g(\delta)}{2} - \frac{\Gamma(\alpha+1)}{2(\delta-\varepsilon)^{\alpha}} \left[J^{\alpha}_{\varepsilon+} g(\delta) + J^{\alpha}_{\delta-} g(\varepsilon) \right] \right| \\ \leq & \frac{\delta - \varepsilon}{\alpha + 1} \left(1 - \frac{1}{2^{\alpha}} \right) \max \left\{ |g'(\varepsilon)|, |g'(\delta)| \right\}. \end{aligned} \tag{18}$$

Theorem 1.16. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$, be a differentiable mapping on (ε, δ) with $\varepsilon < \delta$such that $g' \in L_1 [\varepsilon, \delta]$. If $|g'|^q$ is quasi-convex on $[\varepsilon, \delta]$, and $p > 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left| \frac{g(\varepsilon) + g(\delta)}{2} - \frac{\Gamma(\alpha+1)}{2(\delta-\varepsilon)^{\alpha}} \left[J^{\alpha}_{\varepsilon+} g(\delta) + J^{\alpha}_{\delta-} g(\varepsilon) \right] \right| \\ \leq & \frac{\delta - \varepsilon}{2(\alpha p + 1)^{\frac{1}{p}}} \left(\max \left\{ |g'(\varepsilon)|^q, |g'(\delta)|^q \right\} \right)^{\frac{1}{q}} \end{aligned} \tag{19}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0, 1]$.

Here we remind a previous basic inequality for generalized fractional integral inequality and a lemma that produces left sided H-H type inequalities related this basic inequality [3].

Theorem 1.17. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be a function with $\varepsilon < \delta$ and $g \in L_1 [\varepsilon, \delta]$. If g is a convex function on $[\varepsilon, \delta]$, then we have the following inequalities for generalized fractional integral operators:*

$$g\left(\frac{\varepsilon+\delta}{2}\right) \leq \frac{1}{2\Psi(1)} \left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^+} I_{\vartheta} g(\delta) + {}_{\left(\frac{\varepsilon+\delta}{2}\right)^-} I_{\vartheta} g(\varepsilon) \right] \leq \frac{g(\varepsilon) + g(\delta)}{2} \tag{20}$$

where the mapping $\Psi : [0, 1] \to \mathbb{R}$ is defined by

$$\Psi(x) = \int_0^x \frac{\vartheta\left(\left(\frac{\delta-\varepsilon}{2}\right) l\right)}{l} dl. \tag{21}$$

Lemma 1.18. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be differentiable function on (ε, δ) with $\varepsilon < \delta$. If $g' \in L[\varepsilon, \delta]$, then we have the following identity for generalized fractional integral operators:*

$$\begin{aligned} & \frac{1}{2\Psi(1)} \left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^+} I_{\vartheta} g(\delta) + {}_{\left(\frac{\varepsilon+\delta}{2}\right)^-} I_{\vartheta} g(\varepsilon) \right] - g\left(\frac{\varepsilon+\delta}{2}\right) \\ = & \frac{\delta - \varepsilon}{4\Psi(1)} \left[\int_0^1 \Psi(l) g'\left(\frac{l\varepsilon}{2} + \frac{(2-l)\delta}{2}\right) dl - \int_0^1 \Psi(l) g'\left(\frac{(2-l)\varepsilon}{2} + \frac{l\delta}{2}\right) dl \right] \end{aligned} \tag{22}$$

where the mapping $\Psi(l)$ is defined as in Theorem 1.16.

The following results for quasi-convex functions with the help of $k-$Riemann-Liouville fractioal integral operators obtained by Hussain et al. in [15].

Theorem 1.19. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be positive function and $g \in L_1 [\varepsilon, \delta]$. If g is quasi-convex on $[\varepsilon, \delta]$, the subsequent inequality for $k-$fractional integrals is valid:*

$$\frac{\Gamma_k(\alpha + k)}{2(\delta - \varepsilon)^{\frac{\alpha}{k}}} \left[{}_k J^{\alpha}_{\varepsilon^+} g(\delta) + {}_k J^{\alpha}_{\delta^-} g(\varepsilon) \right] \leq \max\{g(\varepsilon), g(\delta)\} \tag{23}$$

with $\frac{\alpha}{k} > 0$.

Theorem 1.20. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be a differentiable function on (ε, δ) such that $g' \in L_1 [\varepsilon, \delta]$. If $|g'|^q$ is quasi-convex on $[\varepsilon, \delta]$ and $q > 1$, the subsequent inequality for k–fractional integrals is valid:*

$$\begin{aligned} &\left| \frac{g(\varepsilon) + g(\delta)}{2} - \frac{\Gamma_k(\alpha + k)}{2(\delta - \varepsilon)^{\frac{\alpha}{k}}} \left[{}_k J^{\alpha}_{\varepsilon^+} g(\delta) + {}_k J^{\alpha}_{\delta^-} g(\varepsilon) \right] \right| \\ \leq\ & \frac{\delta - \varepsilon}{2\left(\frac{\alpha}{k} p + 1\right)^{\frac{1}{p}}} \left(\max\left\{ |g'(\varepsilon)|^q, |g'(\delta)|^q \right\} \right)^{\frac{1}{q}} \end{aligned} \tag{24}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{\alpha}{k} \in [0, 1]$.

Theorem 1.21. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be a differentiable function on (ε, δ) such that $g' \in L_1 [\varepsilon, \delta]$. If $|g'|^q$ is quasi-convex on $[\varepsilon, \delta]$ and $q \geq 1$, the subsequent inequality for k–fractional integrals is valid:*

$$\begin{aligned} &\left| \frac{g(\varepsilon) + g(\delta)}{2} - \frac{\Gamma_k(\alpha + k)}{2(\delta - \varepsilon)^{\frac{\alpha}{k}}} \left[{}_k J^{\alpha}_{\varepsilon^+} g(\delta) + {}_k J^{\alpha}_{\delta^-} g(\varepsilon) \right] \right| \\ \leq\ & \frac{\delta - \varepsilon}{\left(\frac{\alpha}{k} + 1\right)} \left(1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) \left(\max\left\{ |g'(\varepsilon)|^q, |g'(\delta)|^q \right\} \right)^{\frac{1}{q}} \end{aligned} \tag{25}$$

with $\frac{\alpha}{k} \in [0, 1]$.

Corollary 1.22. *In Theorem 1.5 of [1], if we take $g(x) = 1$, we get the inequality:*

$$\begin{aligned} &\left| \frac{g(\varepsilon) + g(\delta)}{2} - \frac{\Gamma_k(\alpha + k)}{2(\delta - \varepsilon)^{\frac{\alpha}{k}}} \left[{}_k J^{\alpha}_{\varepsilon^+} g(\delta) + {}_k J^{\alpha}_{\delta^-} g(\varepsilon) \right] \right| \\ \leq\ & \frac{\delta - \varepsilon}{\left(\frac{\alpha}{k} + 1\right)} \left(1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) \left(\max\left\{ |g'(\varepsilon)|, |g'(\delta)| \right\} \right). \end{aligned} \tag{26}$$

with $\frac{\alpha}{k} \in [0, 1]$.

By using the above results we build new inequalities related to left-sided and right-sided H-H-type generalized fractional integral inequalities via quasi-convex functions by using elementary analysis such as Hölder inequality, properties of modulus, power mean inequality.

2. Main Results

The point of this study is to generalize the inequalities for quasi-convex functions found in the literature with the help of a new fractional integral operator. Throughout this study, for brevity, we use

$$\Lambda(\mu) = \int_0^{\mu} \frac{\vartheta((\delta - \varepsilon) l)}{l} dl \text{ and } \Lambda(1) \neq 0. \tag{27}$$

Firstly, let us obtain H-H inequality for the quasi-convex functions by using this new fractional integral operator given in (3) and (4).

Theorem 2.1. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be a positive function with $0 \leq \varepsilon < \delta$ and $g \in L_1 [\varepsilon, \delta]$. If g is a quasi-convex function on $[\varepsilon, \delta]$, then we have the following inequality for generalized fractional integral operators:*

$$\frac{1}{2\Lambda(1)} \left[{}_{\varepsilon^+} I_{\vartheta} g(\delta) + {}_{\delta^-} I_{\vartheta} g(\varepsilon) \right] \leq \max\{g(\varepsilon), g(\delta)\}. \tag{28}$$

Proof. Since g is quasi-convex on $[\varepsilon, \delta]$, we have

$$g(\varepsilon l + (1-l)\delta) \leq \max\{g(\varepsilon), g(\delta)\} \tag{29}$$

and

$$g((1-l)\varepsilon + l\delta) \leq \max\{g(\varepsilon), g(\delta)\}. \tag{30}$$

By adding the inequalities (29) and (30), we obtain

$$\frac{1}{2}\left[g(l\varepsilon + (1-l)\delta) + g((1-l)\varepsilon + l\delta)\right] \leq \max\{g(\varepsilon), g(\delta)\}. \tag{31}$$

Multiplying both sides of (31) by $\frac{\vartheta((\delta-\varepsilon)l)}{l}$, then integrating the resulting inequality with respect to l over $(0,1]$, we get

$$\begin{aligned} &\frac{1}{2}\left[\int_0^1 \frac{\vartheta((\delta-\varepsilon)l)}{l} g(l\varepsilon + (1-l)\delta)\,dl + \int_0^1 \frac{\vartheta((\delta-\varepsilon)l)}{l} g((1-l)\varepsilon + l\delta)\,dl\right] \\ \leq\ & \max\{g(\varepsilon), g(\delta)\}\int_0^1 \frac{\vartheta((\delta-\varepsilon)l)}{l}dl. \end{aligned}$$

Then by using the definition of generalized fractional integral operators, we get the inequality in (28). So the proof is completed. □

Corollary 2.2. *If we choose $\vartheta(l) = l$ in Theorem 2.1, the inequality (28) reduces to the inequality (10).*

Corollary 2.3. *If we choose $\vartheta(l) = \frac{l^\alpha}{\Gamma(\alpha)}$ in Theorem 2.1, the inequality (28) reduces to the inequality (17).*

Corollary 2.4. *If we choose $\vartheta(l) = \frac{l^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.1, the inequality (28) reduces to the inequality (23).*

Remark 2.5. *Other results for different fractional integral operators as Katugampola, conformable, Hadamard, etc... can also be found by changing the operator $\vartheta(l)$ in Theorem 2.1.*

Now, by using a lemma in the literature we present new generalized inequalities for quasi-convex functions via generalized fractional integral operators.

Theorem 2.6. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$, be a differentiable mapping on (ε, δ) with $\varepsilon < \delta$. If $|g'|$ is quasi-convex on $[\varepsilon, \delta]$ and $g \in L_1[\varepsilon, \delta]$, $\alpha > 0$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned} &\left|\frac{g(\varepsilon) + g(\delta)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{\varepsilon^+}I_\vartheta g(\delta) + {}_{\delta^-}I_\vartheta g(\varepsilon)\right]\right| \\ \leq\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\max\left\{|g'(\varepsilon)|, |g'(\delta)|\right\}\int_0^1 |\Lambda(1-l) - \Lambda(l)|\,dl \end{aligned} \tag{32}$$

$\Lambda(\mu)$ is as in (27).

Proof. Using Lemma 1.3, the properties of modulus and the quasi-convexity of $|g'|$, we get

$$\begin{aligned} &\left|\frac{g(\varepsilon) + g(\delta)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{\varepsilon^+}I_\vartheta g(\delta) + {}_{\delta^-}I_\vartheta g(\varepsilon)\right]\right| \\ \leq\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\int_0^1 |\Lambda(1-l) - \Lambda(l)|\left|g'(l\varepsilon + (1-l)\delta)\right|dl \\ \leq\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\int_0^1 |\Lambda(1-l) - \Lambda(l)|\max\left\{|g'(\varepsilon)|, |g'(\delta)|\right\}dl \end{aligned}$$

The proof of inequality (32) is completed. □

Corollary 2.7. *If we choose $\vartheta(l) = l$ in Theorem 2.6, the inequality (32) reduces to the inequality (11).*

Corollary 2.8. *If we choose $\vartheta(l) = \frac{l^{\alpha}}{\Gamma(\alpha)}$ in Theorem 2.6, the inequality (32) reduces to the inequality (18).*

Corollary 2.9. *If we choose $\vartheta(l) = \frac{l^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.6, the inequality (32) reduces to the inequality (26).*

Corollary 2.10. *Other results for different fractional integral operators as Katugampola, conformable, Hadamard, etc... can also be found by changing the operator $\vartheta(l)$ in Theorem 2.6.*

Theorem 2.11. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$, be a differentiable mapping on (ε, δ) with $\varepsilon < \delta$. If $|g'|^q$ is quasi-convex on $[\varepsilon, \delta]$ and $g' \in L[\varepsilon, \delta]$, $p > 1$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned} &\left| \frac{g(\varepsilon) + g(\delta)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{\varepsilon^+}I_{\vartheta} g(\delta) + {}_{\delta^-} I_{\vartheta} g(\varepsilon) \right] \right| \\ \leq\ & \frac{\delta - \varepsilon}{2\Lambda(1)} \left[\max\left\{ |g'(\varepsilon)|^q, |g'(\delta)|^q \right\} \right]^{\frac{1}{q}} \left(\int_0^1 |\Lambda(1-l) - \Lambda(l)|^p \, dl \right)^{\frac{1}{p}} \end{aligned} \tag{33}$$

$\Lambda(\mu)$ is as in (27).

Proof. Using Lemma 1.3, properties of modulus and Hölder inequality, we have

$$\begin{aligned} &\left| \frac{g(\varepsilon) + g(\delta)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{\varepsilon^+}I_{\vartheta} g(\delta) + {}_{\delta^-} I_{\vartheta} g(\varepsilon) \right] \right| \\ \leq\ & \frac{\delta - \varepsilon}{2\Lambda(1)} \int_0^1 |\Lambda(1-l) - \Lambda(l)| \left| g'(l\varepsilon + (1-l)\delta) \right| dl \\ \leq\ & \frac{\delta - \varepsilon}{2\Lambda(1)} \left(\int_0^1 |\Lambda(1-l) - \Lambda(l)|^p \, dl \right)^{\frac{1}{p}} \left(\int_0^1 \left| g'(l\varepsilon + (1-l)\delta) \right|^q dl \right)^{\frac{1}{q}}. \end{aligned}$$

Since the quasi-convexity of $|g'|^q$ on $[\varepsilon, \delta]$, we get

$$\begin{aligned} &\left| \frac{g(\varepsilon) + g(\delta)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{\varepsilon^+}I_{\vartheta} g(\delta) + {}_{\delta^-} I_{\vartheta} g(\varepsilon) \right] \right| \\ \leq\ & \frac{\delta - \varepsilon}{2\Lambda(1)} \left(\int_0^1 |\Lambda(1-l) - \Lambda(l)|^p \, dl \right)^{\frac{1}{p}} \left[\max\left\{ |g'(\varepsilon)|^q, |g'(\delta)|^q \right\} \right]^{\frac{1}{q}}. \end{aligned}$$

So the proof is completed. □

Corollary 2.12. *If we choose $\vartheta(l) = l$ in Theorem 2.11, the inequality (32) reduces to the inequality (12).*

Corollary 2.13. *If we choose $\vartheta(l) = \frac{l^{\alpha}}{\Gamma(\alpha)}$ in Theorem 2.11, the inequality (32) reduces to the inequality (19).*

Corollary 2.14. *If we choose $\vartheta(l) = \frac{l^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.11, the inequality (32) reduces to the inequality (24).*

Corollary 2.15. *Other results for different fractional integral operators as Katugampola, conformable, Hadamard, etc... can also be found by changing the operator $\vartheta(l)$ in Theorem 2.11.*

Theorem 2.16. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$, be a differentiable mapping on (ε, δ) with $\varepsilon < \delta$. If $|g'|^q$ is quasi-convex on $[\varepsilon, \delta]$ and $g' \in L[\varepsilon, \delta]$, $q \geq 1$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned} &\left| \frac{g(\varepsilon) + g(\delta)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{\varepsilon^+}I_{\vartheta} g(\delta) + {}_{\delta^-} I_{\vartheta} g(\varepsilon) \right] \right| \\ \leq\ & \frac{\delta - \varepsilon}{2\Lambda(1)} \left(\int_0^1 |\Lambda(1-l) - \Lambda(l)| \, dl \right) \left[\max\left\{ |g'(\varepsilon)|^q, |g'(\delta)|^q \right\} \right]^{\frac{1}{q}} \end{aligned} \tag{34}$$

$\Lambda(\mu)$ is as in (27).

Proof. Using Lemma 1.3 and power-mean integral inequality, we have

$$\begin{aligned}&\left|\frac{g(\varepsilon)+g(\delta)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{\varepsilon^+}I_\vartheta g(\delta)+{}_{\delta^-}I_\vartheta g(\varepsilon)\right]\right|\\ \leq\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\int_0^1|\Lambda(1-l)-\Lambda(l)|\left|g'(l\varepsilon+(1-l)\delta)\right|dl\\ \leq\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\left(\int_0^1|\Lambda(1-l)-\Lambda(l)|\,dl\right)^{1-\frac{1}{q}}\left(\int_0^1|\Lambda(1-l)-\Lambda(l)|\left|g'(l\varepsilon+(1-l)\delta)\right|^q dl\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|g'|^q$ is quasi-convex on $[\varepsilon,\delta]$, we have desired result. So, the proof is completed. □

Corollary 2.17. *If we choose $\vartheta(l)=l$ in Theorem 2.16, the inequality (34) reduces to the inequality 13.*

Corollary 2.18. *If we choose $\vartheta(l)=\frac{l^\alpha}{\Gamma(\alpha)}$ in Theorem 2.16, the inequality (34) reduces to the inequality (18).*

Corollary 2.19. *If we choose $\vartheta(l)=\frac{l^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.16, the inequality (34) reduces to the inequality (25).*

Corollary 2.20. *Other results for different fractional integral operators as Katugampola, conformable, Hadamard, etc... can also be found by changing the operator $\vartheta(l)$ in Theorem 2.16.*

Now we give some new inequalities for generalized fractional integral operators with Lemma 1.18 obtained by Budak et al. in [3].

Theorem 2.21. *Let $g:[\varepsilon,\delta]\to\mathbb{R}$ be differentiable function on (ε,δ) with $\varepsilon<\delta$. If $|g'|$ is quasi-convex on $[\varepsilon,\delta]$ and $g'\in L[\varepsilon,\delta]$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned}&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^+}I_\vartheta g(\delta)+{}_{\left(\frac{\varepsilon+\delta}{2}\right)^-}I_\vartheta g(\varepsilon)\right]-g\left(\frac{\varepsilon+\delta}{2}\right)\right| \qquad (35)\\ \leq\ & \frac{\delta-\varepsilon}{4\Psi(1)}\left(\int_0^1|\Psi(l)|\,dl\right)\left[\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|,\left|g'(\delta)\right|\right\}\right.\\ &\left.+\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|,\left|g'(\varepsilon)\right|\right\}\right]\end{aligned}$$

where $\Psi(1)$is as in (21).

Proof. Using Lemma 1.18 and the quasi-convexity of $|g'|$ on $[\varepsilon,\delta]$,we get

$$\begin{aligned}&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^+}I_\vartheta g(\delta)+{}_{\left(\frac{\varepsilon+\delta}{2}\right)^-}I_\vartheta g(\varepsilon)\right]-g\left(\frac{\varepsilon+\delta}{2}\right)\right|\\ \leq\ & \frac{\delta-\varepsilon}{4\Psi(1)}\left[\int_0^1|\Psi(l)|\left|g'\left(\frac{l}{2}\varepsilon+\frac{2-l}{2}\delta\right)\right|dl+\int_0^1|\Psi(l)|\left|g'\left(\frac{2-l}{2}\varepsilon+\frac{l}{2}\delta\right)\right|dl\right]\\ \leq\ & \frac{\delta-\varepsilon}{4\Psi(1)}\int_0^1|\Psi(l)|\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|,\left|g'(\delta)\right|\right\}dl\\ &+\int_0^1|\Psi(l)|\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|,\left|g'(\varepsilon)\right|\right\}dl\end{aligned}$$

By making the necessary arrangements the desired result is achieved. □

Corollary 2.22. *If we choose $\vartheta(l)=l$ in Theorem 2.16, we get the inequality (14).*

Theorem 2.23. *Let $g:[\varepsilon,\delta]\to\mathbb{R}$ be differentiable function on (ε,δ) with $\varepsilon<\delta$. If $\left|g'\right|^{q}$ is quasi-convex on $[\varepsilon,\delta]$ and $g'\in L[\varepsilon,\delta]$, $p>1$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned}&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^{+}}I_{\vartheta}g(\delta)+{}_{\left(\frac{\varepsilon+\delta}{2}\right)^{-}}I_{\vartheta}g(\varepsilon)\right]-g\left(\frac{\varepsilon+\delta}{2}\right)\right|\\ \leq\;&\frac{\delta-\varepsilon}{4\Psi(1)}\left(\int_{0}^{1}|\Psi(l)|^{p}dl\right)^{\frac{1}{p}}\left[\left(\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|^{\frac{p}{p-1}},\left|g'(\delta)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right.\\ &\left.+\left(\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|^{\frac{p}{p-1}},\left|g'(\varepsilon)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right]\end{aligned}$$

where $\Psi(1)$is as in (21).

Proof. Using Lemma 1.18 and Hölder inequality, we get

$$\begin{aligned}&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^{+}}I_{\vartheta}g(\delta)+{}_{\left(\frac{\varepsilon+\delta}{2}\right)^{-}}I_{\vartheta}g(\varepsilon)\right]-g\left(\frac{\varepsilon+\delta}{2}\right)\right|\\ \leq\;&\frac{\delta-\varepsilon}{4\Psi(1)}\left(\int_{0}^{1}|\Psi(l)|^{p}dl\right)^{\frac{1}{p}}\left[\left(\int_{0}^{1}\left|g'\left(\frac{l}{2}\varepsilon+\frac{2-l}{2}\delta\right)\right|^{\frac{p}{p-1}}dl\right)^{\frac{p-1}{p}}\right.\;.\end{aligned}$$

If we use the quasi-convexity of $\left|g'\right|$ on $[\varepsilon,\delta]$ last inequality, the desired result is achieved. □

Corollary 2.24. *If we choose $\vartheta(l)=l$ in Theorem 2.23, we get the inequality in (15).*

Theorem 2.25. *Let $g:[\varepsilon,\delta]\to\mathbb{R}$ be differentiable function on (ε,δ) with $\varepsilon<\delta$. If $\left|g'\right|^{q}$ is quasi-convex on $[\varepsilon,\delta]$ and $g'\in L[\varepsilon,\delta]$, $q\geq1$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned}&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^{+}}I_{\vartheta}g(\delta)+{}_{\left(\frac{\varepsilon+\delta}{2}\right)^{-}}I_{\vartheta}g(\varepsilon)\right]-g\left(\frac{\varepsilon+\delta}{2}\right)\right|\\ \leq\;&\frac{\delta-\varepsilon}{4\Psi(1)}\left(\int_{0}^{1}|\Psi(l)|dl\right)\left[\left(\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|^{q},\left|g'(\delta)\right|^{q}\right\}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|^{q},\left|g'(\varepsilon)\right|^{q}\right\}\right)^{\frac{1}{q}}\right]\end{aligned}$$

where $\Psi(1)$is as in (21).

Proof. Using Lemma 1.18 and power-mean inequality, we get

$$\begin{aligned}&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^{+}}I_{\vartheta}g(\delta)+{}_{\left(\frac{\varepsilon+\delta}{2}\right)^{-}}I_{\vartheta}g(\varepsilon)\right]-g\left(\frac{\varepsilon+\delta}{2}\right)\right| \qquad (36)\\ \leq\;&\frac{\delta-\varepsilon}{4\Psi(1)}\left[\int_{0}^{1}|\Psi(l)|\left|g'\left(\frac{l}{2}\varepsilon+\frac{2-l}{2}\delta\right)\right|dl\right.\\ &\left.+\int_{0}^{1}|\Psi(l)|\left|g'\left(\frac{2-l}{2}\varepsilon+\frac{l}{2}\delta\right)\right|dl\right]\\ \leq\;&\frac{\delta-\varepsilon}{4\Psi(1)}\left(\int_{0}^{1}|\Psi(l)|dl\right)^{1-\frac{1}{q}}\left[\left(\int_{0}^{1}|\Psi(l)|\left|g'\left(\frac{l}{2}\varepsilon+\frac{2-l}{2}\delta\right)\right|^{q}dl\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{0}^{1}|\Psi(l)|\left|g'\left(\frac{2-l}{2}\varepsilon+\frac{l}{2}\delta\right)\right|^{q}dl\right)^{\frac{1}{q}}\right].\end{aligned}$$

If we use the quasi-convexity of $|g'|$ in (36), the desired result is achieved. □

Corollary 2.26. *If we choose $\vartheta(l) = l$ in Theorem 2.25, we get the inequality in (16).*

Acknowledgement

This paper has been produced from the Master thesis of the first author.

References

[1] Ali, A., Gulshan, G., Hussain, R., Latif, A. Muddassar, M. and Park, J., Generalized Inequalities of the type of Hermite-Hadamard-Fejer with Quasi-Convex Functions by way of k-Fractional Derivatives, J. Computational Analysis and applications, 22(7) (2017), 1208-1219.

[2] Alomari, M., Darus, M., Dragomir, S.S., Inequalities of Hermite-Hadamard's type for functions whose derivatives absolute values are quasi-convex. RGMIA Res. Rep. Coll., 12: Supplement, Article 14 2009, 1-11.

[3] Budak, H., Ertuğral, F., Sarıkaya, M.Z., New Generalization of Hermite-Hadamard Type Inequalities via Generalized Fractional Integrals. ResearchGate Article (2017).
https://www.researchgate.net/publication/321760465.

[4] Belarbi S., Dahmani Z., On some new fractional integral inequalities, J. Ineq. Pure Appl. Math., 10(3) (2009), Art. 86

[5] Carter, M., Brunt, B.V.: The Lebesgue-Stieljies Integral: A Practical Introduction. New York, Springer (2000).

[6] Dahmani Z., New inequalities in fractional integrals, Int. J. Nonlinear Sci., 9(4) (2010), 493-497.

[7] Dragomir, S.S, Agarwal, R.P.: Two Inequalities for Di erentiable Mappings and Applications To Special Meansof Real Numbers and to Trapezoidal Formula. Appl. Math. Lett. 11 (1999), 91-95.

[8] Dragomir, S.S., Pearce, C.E.M., Quasi-Convex Functions and Hadamard's Inequality. B. Aust. Math. Soc. 57 (1998), 377-385.

[9] Dragomir, S.S., Pearce, C.E.M., Selected Topics on Hermite-Hadamard Inequalities and Applications. RGMIA, Monographs. Victoria University, 2000.

[10] Dragomir, S.S., Pecaric, J., and Persson, L.E., Some Inequalities of Hadamard Type. Soochow J. of Math. 21 (1995), 335-341.

[11] Ertuğral, F. Sarıkaya, M.Z., Budak, H., On Hermite-Hadamard Type Inequalities Associated With The Generalized Fractional Integrals. ResearchGate Article (2019).
https://www.researchgate.net/publication/334634529.

[12] Gidergelmez, H.F., Akkurt, A., Yıldırım, H., Hermite-Hadamard Type Inequalities for Generalized Fractional Integrals via Strongly Convex Functions. KJM. 7 (2019), 268-273.

[13] Gorenflo R., Mainardi F., Fractional Calculus: Integral and Differential Equations of Fractional Order, Springer Verlag, Wien., (1997), 223-276.

[14] Hadamard, J.: ėtude sur les propri ėtės des fonctions entières en particulier dűne fonction considėrėe par Riemann. Journal de mathėmatiques pures et appliquėes, 4e sėrie. 9 (1893), 171-216.

[15] Hussain, R., Ali, A., Latif, A. and Gulshan, G., Some $k-$Fractional associates of Hermite-Hadamard's Inequality for Quasi-Convex Functions and Applications to Special Means, Fractional Differential Calculus, 7(2) (2017),301-309.

[16] Ion, D. A., Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, Annals of University of Craiova, Math. Sci. Ser., 34 (2007), 82–87.

[17] Iscan I., New general integral inequalities for quasi-geometrically convex functions via fractional integrals, J. Inequal. Appl., (491) (2013), 1-15.

[18] I scan, I.: Hermite-Hadamard-Fejer type inequalities for convex functions via fractional integrals. Stud. Univ. Babes -Bolyai Math. 60 (2015), 355-366.

[19] Khan, M.A., Khurshid, Y., Ali, T.: Hermite{Hadamard Inequality for Fractional Integrals via -Convex Functions. Acta Math. Univ. Comen. 86 (2017), 153-164.

[20] K lbas, A.A., Srivastava, H.M., Trujillo J.J.: Theory and Applications of Fractional Di erential Equations. Elsevier, Amsterdam (2006).

[21] Kunt, M., Karap nar, D., Turhan, S., İscan, İ.: The Left Riemann-Liouville Fractional Hermite-Hadamard Type Inequalities for Convex Functions. Math. Slovaca. 69 (2019), 773-784.

[22] Miller S., Ross B., An introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley. Soons. USA., 1993.

[23] Mitrinovi c, D.S.: Analytic Inequalities. Springer, Berlin (1970).

[24] Mitrinovi c, D.S., Pe cari c, J.E., Fink, A.M.: Classical and New Inequalities in Analysis. Kluwer, Dordrecht (1993).

[25] Niculescu, C.P., Persson, L.E.: Convex Functions and Their Applications: A Contemporary Approach. Springer, New York (2006).

[26] Özdemir, M. E., Yıldız, Ç., Annals of the University of Craiova, Mathematics and Computer Science., 40 (2), (2013),167-173.

[27] Pearce C. E. M., Quasi-convexity, fractional programming and extrenal traffic congestion, in "Frontiers in Global Optimization", Kluwer, Dordrecht, "Nonlinear Optimization and its Applications", 74 (2004), 403-409.

[28] Pearce C. E. M. and Rubinov A. M., P- functions, quasi-convex functions and Hadamard type inequalities, J. Math. Anal. Applic., 240 (1999), 92-104.

[29] Pe cari c, J.E., Proschan, F.,Tong, Y.L.: Convex Functions, Partial Orderings and Statistical Applications. Boston, Academic Press (1992).

[30] Ponstein, J., Seven types of convexity, SIAM Review 9 (1967), 115-119.

[31] Roberts, A.W., Varberg, D.E.: Convex Functions. Academic Press, New York (1973).
[32] Sar kaya, M.Z., Ertuğral, F.: On The Generalized Hermite-Hadamard Inequalities. Annals of University of Craiova, Math. Comp. Sci. Ser.,47 (1) (2020),193-213.
[33] Sarıkaya M.Z., Ogunmez H., On new inequalities via Riemann-Liouville fractional integration, Abst. Appl. Anal., Art.ID 428983, (2012), 10 pages. http://dx.doi.org./10.1155/2012/428983.
[34] Sar kaya, M.Z., Set, E., Yaldiz, H., Ba sak, N.: Hermite-Hadamard's Inequalities for Fractional Integrals and Related Fractional Inequalities. Math. Comput. Modell. 57 (2013), 2403-2407.
[35] Sarikaya, M.Z. and Yildirim, H., On generalization of the Riesz potential, Indian Jour. of Math. and Mathematical Sci. 3 (2007), no. 2, 231-235.
[36] Set, E. and Çelik, B, Fractional Hermite-Hadamard Type Inequalities for Quasi-convex functions, Ordu Univ. J. Sci. Tech. 6, 1 (2016), 137–149.
[37] Set, E., Karata s S.S., Khan, M.A.: Hermite{Hadamard Type Inequalities Obtained via Fractional Integrals for Di erentiable m-Convex and (;m)-Convex Functions. Int. J. Anal (2016). https://doi.org/10.1155/2016/4765691.
[38] Set, E., Sar kaya, M.Z., Ozdemir, M.E., Yıldırım, H.: The Hermite-Hadamard's Inequality for Some Convex Functions via Fractional Integrals and Related Results. JAMSI. 10 (2014), 69-83.
[39] Shi, D.-P., Xi, B.-Y., Qi, F.: Hermite-Hadamard Type Inequalities for Reimann-Liouville Fractional Integrals of (alpha;m)-Convex Functions. Fractional Di er. Calc. 4 (2014), 31-43.
[40] Tun c, M.: On New Inequalities for h-Convex Functions via Riemann-Liouville Fractional Integration. Filomat, 27(2013), 559-565.
[41] Varosanec, S.: On h-convexity. J. Math. Anal. Appl. 326 (2007), 303-311.
[42] Yaldiz, H., Set, E.: Re nements Hermite-Hadamard-Fejer Type Inequalities For Generalized Fractional Integrals.ResearchGate Article (2018). https://www.researchgate.net/publication/323357856
[43] Yıldırım, M.E., Sarıkaya, M.Z., Yldırım, H.: The Generalized Hermite-Hadamard-Fejer Type Inequalities For Generalized Fractional Integrals, ResearchGate Article (2018) https://www.researchgate.net/publication/322592667.
[44] Zhao, D.M., Ali, A., Kashuri, A., Budak, H.: Generalized Fractional Integral Inequalities of Hermite-Hadamard Type for Harmonically Convex Functions. Adv. Di er. Equ. 1 (2020),1-14.

TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 3, 231–244
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

Exponentially $m-$ and $(\alpha, m)-$Convex Functions on the Coordinates and Related Inequalities

Sinan Aslan[a], Ahmet Ocak Akdemir[b], Mustafa Ali Dokuyucu[b]

[a]*Agri Ibrahim Cecen University, Institute of Graduate Studies, Department of Mathematics, Agri-Türkiye*
[b]*Agri Ibrahim Cecen University, Faculty of Science and Letters, Department of Mathematics, Agri-Türkiye*

Abstract. In the present paper, new classes of convexity, namely, exponentially $m-$ and$(\alpha, m)-$convex functions on the co-ordinates are defined. Then, some new integral inequalities are proved by using some classical inequalities and properties of exponentially $m-$ ad $(\alpha, m)-$convex functions on the co-ordinates.

1. Introduction

In [8], Toader defined $m-$convex functions as following:

Definition 1.1. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be $m-$convex, where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m(b)$ the class of all $m-$convex functions on $[0, b]$ for which $f(0) \le 0$. Obviously, if we choose $m = 1$, we have ordinary convex functions on $[0, b]$.

In [7], Miheşan introduced $(\alpha, m)-$convexity as following:

Definition 1.2. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be $(\alpha, m)-$convex, where $(\alpha, m) \in [0, 1]^2$, if we have*

$$f(tx + m(1-t)y) \le t^{\alpha} f(x) + m(1-t^{\alpha})f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m^{\alpha}(b)$ the class of all $(\alpha, m)-$convex functions on $[0, b]$ for which $f(0) \le 0$. If we choose $(\alpha, m) = (1, m)$, it can be easily seen that $(\alpha, m)-$convexity reduces to $m-$convexity and for $(\alpha, m) = (1, 1)$, we have ordinary convex functions on $[0, b]$.

For several results related to above definitions we refer interest of readers to [4], [5], [6], [7], [8], [9] and [11].

Corresponding author: SA mail address: sinanaslan0407@gmail.com ORCID:0000-0001-5970-1926, AOA ORCID:0000-0003-2466-0508, MAD ORCID:0000-0001-9331-8592

Received: 14 October 2022; Accepted: 10 December 2022; Published: 30 December 2022

Keywords. coordinates, exponentially convex functions. Hölder inequality.

2010 *Mathematics Subject Classification*. 26D15, 26A51

Cited this article as: Aslan S. Akdemir A.O. Dokuyucu M.A. Exponentially $m-$ and $(\alpha, m)-$Convex Functions on the Coordinates and Related Inequalities, Turkish Journal of Science, 2022, 7(3), 231–244.

We will start by expressing an important inequality proved for convex functions. This inequality is presented on the basis of averages and give bounds for the mean value of a convex function.
Assume that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the interval I of $\mathbb{R}$ where $a < b$. The following statement;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

holds and known as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave.
In [1], Dragomir mentions an expansion of the concept of convex function, which is used in many inequalities in the field of inequality theory and has applications in different fields of mathematics, especially convex programming.

Definition 1.3. *Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$. Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex on Δ if the following inequality holds,*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w)$$

for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

Expressing convex functions in coordinates brought up the question that it is possible for Hermite-Hadamard inequality to expand into coordinates. The answer to this motivating question has been found in Dragomir's paper (see [1]) and has taken its place in the literature as the expansion of Hermite-Hadamard inequality to a rectangle from the plane $\mathbb{R}^2$ stated below.

Theorem 1.4. *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities;*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \qquad (1)\\ \leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ \leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\ \leq\ & \frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x,c)dx + \frac{1}{(b-a)}\int_a^b f(x,d)dx \right. \\ & \left. + \frac{1}{(d-c)}\int_c^d f(a,y)dy + \frac{1}{(d-c)}\int_c^d f(b,y)dy\right] \\ \leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

The above inequalities are sharp.

The concept of exponentially convex function on the coordinates and the associated results are presented as the followings:

Definition 1.5. *(See [12]) Let us consider the interval such as $\Delta = [\epsilon_1, \epsilon_2] \times [\epsilon_3, \epsilon_4]$ in $\mathbb{R}^2$ with $\epsilon_1 < \epsilon_2$, $\epsilon_3 < \epsilon_4$. The function $\Psi : \Delta \to \mathbb{R}$ is exponentially convex on Δ if*

$$\Psi((1-\zeta)\,u_1 + \zeta u_3, (1-\zeta)\,u_2 + \zeta u_4) \leq (1-\zeta)\,\frac{\Psi(u_1,u_2)}{e^{\alpha(u_1+u_2)}} + \zeta\frac{\Psi(u_3,u_4)}{e^{\alpha(u_3+u_4)}}$$

for all $(u_1,u_2),(u_3,u_4) \in \Delta$, $\alpha \in \mathbb{R}$ and $\zeta \in [0,1]$.

An equivalent definition of the exponentially convex function definition in coordinates can be done as follows:

Definition 1.6. *(See [12]) The mapping $\Psi : \Delta \longrightarrow \mathbb{R}$ is exponentially convex function on the co-ordinates on Δ, if*

$$\begin{aligned} & \Psi\left(\zeta\epsilon_1 + (1-\zeta)\epsilon_2, \xi\epsilon_3 + (1-\xi)\epsilon_4\right) \\ \leq \;& \zeta\xi\frac{\Psi(\epsilon_1,\epsilon_3)}{e^{\alpha(\epsilon_1+\epsilon_3)}} + \zeta(1-\xi)\frac{\Psi(\epsilon_1,\epsilon_4)}{e^{\alpha(\epsilon_1+\epsilon_4)}} + (1-\zeta)\xi\frac{\Psi(\epsilon_2,\epsilon_3)}{e^{\alpha(\epsilon_2+\epsilon_3)}} + (1-\zeta)(1-\xi)\frac{\Psi(\epsilon_2,\epsilon_4)}{e^{\alpha(\epsilon_2+\epsilon_4)}} \end{aligned}$$

for all $(\epsilon_1,\epsilon_3), (\epsilon_1,\epsilon_4), (\epsilon_2,\epsilon_3), (\epsilon_2,\epsilon_4) \in \Delta, \alpha \in \mathbb{R}$ and $\zeta, \xi \in [0,1]$.

The main motivation of this paper is to define exponentially $m-$ and $(\alpha, m)-$convex functions on the co-ordinates. We have proved several integral inequalities for these classes of functions.

2. Exponentially $m-$convex functions on the co-ordinates

Definition 2.1. *Let us consider the bidimensional interval $\Delta = [0,b] \times [0,d]$ in R^2 with $0 < a < b < \infty$ and $c < d$. The mapping $f : \Delta \longrightarrow R$ is exponentially $m-$convex function on the co-ordinates on Δ, if the following inequality holds,*

$$f\left(tx + (1-t)z, ty + m(1-t)w\right) \leq t\frac{f(x,y)}{e^{\alpha(x+y)}} + m(1-t)\frac{f(z,w)}{e^{\alpha(z+w)}}.$$

for all $(x,y),(z,w) \in \Delta$,$\alpha \in R, m \in (0,1]$ and $t \in [0,1]$.

An equivalent definition of the exponentially $m-$convex function definition in coordinates can be done as follows:

Definition 2.2. *The mapping $f : \Delta \longrightarrow R$ is exponential convex on the co-ordinates on Δ, if the following inequality holds,*

$$\begin{aligned} & f\left(ta + (1-t)b, sc + m(1-s)d\right) \\ \leq \;& ts\frac{f(a,c)}{e^{\alpha(a+c)}} + t(1-s)m\frac{f(a,d)}{e^{\alpha(a+d)}} + (1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}} + (1-t)(1-s)m\frac{f(b,d)}{e^{\alpha(b+d)}} \end{aligned}$$

for all $(a,c), (a,d), (b,c), (b,d) \in \Delta, \alpha \in R$ and $m, t, s \in [0,1]$

Lemma 2.3. *A function $f : \Delta \longrightarrow R$ will be called exponential $m-$convex function on the co-ordinates on Δ, if the partial mappings $f_y : [a,b] \longrightarrow R$, $f_y(u) = e^{\alpha y} f(u,y)$ and $f_x : [c,d] \longrightarrow R$, $f_x(v) = e^{\alpha x} f(x,v)$ are exponentially $m-$convex function on the co-ordinates on Δ, where defined for all $y \in [c,d]$ and $x \in [a,b]$.*

Proof. From the definition of partial mapping f_x, we can write

$$\begin{aligned} f_x(tv_1 + m(1-t)v_2) &= e^{\alpha x} f(x, tv_1 + m(1-t)v_2) \\ &= e^{\alpha x} f(tx + (1-t)x, tv_1 + m(1-t)v_2) \\ &\leq e^{\alpha x}\left[t\frac{f(x,v_1)}{e^{\alpha(x+v_1)}} + m(1-t)\frac{f(x,v_2)}{e^{\alpha(x+v_2)}}\right] \\ &= t\frac{f(x,v_1)}{e^{\alpha v_1}} + m(1-t)\frac{f(x,v_2)}{e^{\alpha v_2}} \\ &= t\frac{f_x(v_1)}{e^{\alpha v_1}} + m(1-t)\frac{f_x(v_2)}{e^{\alpha v_2}} \end{aligned}$$

Similarly,

$$\begin{aligned} f_y(tu_1 + m(1-t)u_2) &= e^{\alpha y} f(tu_1 + m(1-t)u_2, y) \\ &= e^{\alpha y} f(tu_1 + m(1-t)u_2, ty + (1-t)y) \\ &\leq e^{\alpha y}\left[t\frac{f(u_1,y)}{e^{\alpha(u_1+y)}} + m(1-t)\frac{f(u_2,y)}{e^{\alpha(u_2+y)}}\right] \\ &= t\frac{f(u_1,y)}{e^{\alpha u_1}} + m(1-t)\frac{f(u_2,y)}{e^{\alpha u_2}} \\ &= t\frac{f_y(u_1)}{e^{\alpha u_1}} + m(1-t)\frac{f_y(u_2)}{e^{\alpha u_2}}. \end{aligned}$$

The proof is completed. □

Theorem 2.4. *Let $f : \Delta = [0,b] \times [0,d] \to R$ be partial differentiable mapping on $\Delta = [0,b] \times [0,d]$ in R^2 with $0 < a < b < \infty$ and $0 < c < md < \infty$, $f \in L(\Delta)$, $\alpha \in R$. If f is exponentially $m-$convex function on the co-ordinates on Δ, then the following inequality holds;*

$$\begin{aligned} &\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy \\ \leq\ &\frac{1}{4}\left[\frac{f(a,c)}{e^{\alpha(a+c)}} + \frac{f(b,c)}{e^{\alpha(b+c)}} + m\left(\frac{f(b,d)}{e^{\alpha(b+d)}} + \frac{f(a,d)}{e^{\alpha(a+d)}}\right)\right]. \end{aligned}$$

Proof. By the definition of the exponentially $m-$convex functions on the co-ordinates on Δ, we can write

$$\begin{aligned} &f(ta + (1-t)b, sc + m(1-s)d) \\ \leq\ &ts\frac{f(a,c)}{e^{\alpha(a+c)}} + mt(1-s)\frac{f(a,d)}{e^{\alpha(a+d)}} + (1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}} + m(1-t)(1-s)\frac{f(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

By integrating both sides of the above inequality with respect to t, s on $[0,1]^2$, we have

$$\begin{aligned} &\int_0^1\int_0^1 f(ta + (1-t)b, sc + m(1-s)d)\,dtds \\ \leq\ &\int_0^1\int_0^1 ts\frac{f(a,c)}{e^{\alpha(a+c)}}dtds + \int_0^1\int_0^1 t(1-s)m\frac{f(a,d)}{e^{\alpha(a+d)}}dtds \\ &+ \int_0^1\int_0^1 (1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}}dtds + \int_0^1\int_0^1 (1-t)(1-s)m\frac{f(b,d)}{e^{\alpha(b+d)}}dtds. \end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Theorem 2.5. *Let $f : \Delta = [0,b] \times [0,d] \to R$ be partial differentiable mapping on $\Delta = [0,b] \times [0,d]$ in R^2 with $0 < a < b < \infty$ and $0 < c < md < \infty$, $f \in L(\Delta)$, $\alpha \in R$. If $\left|f\right|$ is exponentially $m-$convex function on the co-ordinates on Δ, $p > 1$ and $m \in (0,1]$, then the following inequality holds;*

$$\begin{aligned} &\left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\ \leq\ &\left(\frac{1}{(p+1)^2}\right)^{\frac{1}{p}}\left(\frac{\left|f(a,c)\right|}{e^{\alpha(a+c)}} + \frac{\left|mf(a,d)\right|}{e^{\alpha(a+d)}} + \frac{\left|f(b,c)\right|}{e^{\alpha(b+c)}} + \frac{\left|mf(b,d)\right|}{e^{\alpha(b+d)}}\right). \end{aligned}$$

Proof. By the definition of the exponentially $m-$convex functions on the co-ordinates on Δ, we can write

$$\begin{aligned}
& f\left(ta+(1-t)b, sc+m(1-s)d\right) \\
\leq\; & ts\frac{f(a,c)}{e^{\alpha(a+c)}}+t(1-s)m\frac{f(a,d)}{e^{\alpha(a+d)}}+ \\
& (1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t)(1-s)m\frac{f(b,d)}{e^{\alpha(b+d)}}
\end{aligned}$$

The absolute value property is used in integral and by integrating both sides of the above inequality with respect to t,s on $[0,1]^2$, we can write

$$\begin{aligned}
& \left|\int_0^1\int_0^1 f\left(ta+(1-t)b, sc+m(1-s)d\right)dtds\right| \\
\leq\; & \int_0^1\int_0^1\left|ts\frac{f(a,c)}{e^{\alpha(a+c)}}\right|dtds+\int_0^1\int_0^1\left|t(1-s)m\frac{f(a,d)}{e^{\alpha(a+d)}}\right|dtds \\
& +\int_0^1\int_0^1\left|(1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}}\right|dtds+\int_0^1\int_0^1\left|(1-t)(1-s)m\frac{f(b,d)}{e^{\alpha(b+d)}}\right|dtds
\end{aligned}$$

If we apply the Hölder's inequality to the right-hand side of the inequality, we get

$$\begin{aligned}
& \left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\
\leq\; & \left(\int_0^1\int_0^1 t^p s^p dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 t^p(1-s)^p dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{mf(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 (1-t)^p s^p dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 (1-t)^p(1-s)^p dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{mf(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right)^{\frac{1}{q}}
\end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Theorem 2.6. *Let $f:\Delta=[0,b]\times[0,d]\to R$ be partial differentiable mapping on $\Delta=[0,b]\times[0,d]$ in R^2 with $0<a<b<\infty$ and $0<c<md<\infty$, $f\in L(\Delta)$, $\alpha\in R$. If $\left|f\right|$ is exponentially $m-$convex function on the co-ordinates on Δ, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds;*

$$\begin{aligned}
& \left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\
\leq\; & \left(\frac{4}{p(p+1)^2}\right) \\
& +\frac{1}{q}\left(\frac{\left|f(a,c)\right|^q}{e^{\alpha q(a+c)}}+\frac{\left|mf(a,d)\right|^q}{e^{\alpha q(a+d)}}+\frac{\left|f(b,c)\right|^q}{e^{\alpha q(b+c)}}+\frac{\left|mf(b,d)\right|^q}{e^{\alpha q(b+d)}}\right).
\end{aligned}$$

Proof. By the definition of the exponentially $m-$convex functions on the co-ordinates on Δ, we can write

$$\begin{aligned} & f\left(ta+(1-t)b, sc+m(1-s)d\right) \\ \leq\ & ts\frac{f(a,c)}{e^{\alpha(a+c)}}+t(1-s)m\frac{f(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t)(1-s)m\frac{f(b,d)}{e^{\alpha(b+d)}} \end{aligned}$$

By the absolute value property and by integrating both sides of the above inequality with respect to t,s on $[0,1]^2$, we can write

$$\begin{aligned} & \left|\int_0^1\int_0^1 f\left(ta+(1-t)b, sc+m(1-s)d\right)dtds\right| \\ \leq\ & \int_0^1\int_0^1\left|ts\frac{f(a,c)}{e^{\alpha(a+c)}}\right|dtds+\int_0^1\int_0^1\left|t(1-s)m\frac{f(a,d)}{e^{\alpha(a+d)}}\right|dtds \\ & +\int_0^1\int_0^1\left|(1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}}\right|dtds+\int_0^1\int_0^1\left|(1-t)(1-s)m\frac{f(b,d)}{e^{\alpha(b+d)}}\right|dtds \end{aligned}$$

If we apply the Young's inequality to the right-hand side of the inequality, we get

$$\begin{aligned} & \left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\ \leq\ & \frac{1}{p}\left(\int_0^1\int_0^1 t^p s^p dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right) \\ & +\frac{1}{p}\left(\int_0^1\int_0^1 t^p(1-s)^p dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{mf(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right) \\ & +\frac{1}{p}\left(\int_0^1\int_0^1 (1-t)^p s^p dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right) \\ & +\frac{1}{p}\left(\int_0^1\int_0^1 (1-t)^p(1-s)^p dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{mf(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right). \end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Proposition 2.7. *If $f,g:\Delta\to R$ are two exponentially $m-$convex functions on the co-ordinates on Δ, then $f+g$ is exponentially $m-$convex function on the co-ordinates on Δ.*

Proof. By the definition of the exponentially $m-$convex functions on the co-ordinates on Δ, we can write

$$\begin{aligned} & f\left(ta+(1-t)b, sc+m(1-s)d\right)+g\left(ta+(1-t)b, sc+m(1-s)d\right) \\ \leq\ & ts\left(\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{g(a,c)}{e^{\alpha(a+c)}}\right)+t(1-s)m\left(\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{g(a,d)}{e^{\alpha(a+d)}}\right) \\ & +(1-t)s\left(\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{g(b,c)}{e^{\alpha(b+c)}}\right)+(1-t)(1-s)m\left(\frac{f(b,d)}{e^{\alpha(b+d)}}+\frac{g(b,d)}{e^{\alpha(b+d)}}\right). \end{aligned}$$

Namely,

$$\begin{aligned} & (f+g)\left(ta+(1-t)b, sc+m(1-s)d\right) \\ \leq\ & ts\frac{(f+g)(a,c)}{e^{\alpha(a+c)}}+t(1-s)m\frac{(f+g)(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t)s\frac{(f+g)(b,c)}{e^{\alpha(b+c)}}+(1-t)(1-s)m\frac{(f+g)(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

Therefore, $(f+g)$ is exponentially $m-$convex functions on the co-ordinates on Δ. □

Proposition 2.8. *If $f : \Delta \to R$ is exponential $m-$convex functions on the co-ordinates on Δ and $k \geq 0$ then kf is exponential $m-$convex functions on the co-ordinates on Δ.*

Proof. By the definition of the exponentially $m-$convex functions on the co-ordinates on Δ, we can write

$$\begin{aligned} & f(ta+(1-t)b, sc+m(1-s)d) \\ \leq\ & ts\frac{f(a,c)}{e^{\alpha(a+c)}} + t(1-s)m\frac{f(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}} + (1-t)(1-s)m\frac{f(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

If both sides are multiplied by k, we have

$$\begin{aligned} & (kf)(ta+(1-t)b, sc+m(1-s)d) \\ \leq\ & ts\frac{(kf)(a,c)}{e^{\alpha(a+c)}} + t(1-s)m\frac{(kf)(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t)s\frac{(kf)(b,c)}{e^{\alpha(b+c)}} + (1-t)(1-s)m\frac{(kf)(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

Therefore (kf) is exponentially $m-$convex functions on the co-ordinates on Δ. □

3. Exponentially $(\alpha, m)-$convex functions on the co-ordinates

Definition 3.1. *Let us consider the bidimensional interval $\Delta = [0,b] \times [0,d]$ in R^2 with $0 < a < b < \infty$ and $c < d$. The mapping $f : \Delta \longrightarrow R$ is exponentially (α_1, m)-convex on the co-ordinates on Δ, if the following inequality holds,*

$$f(tx+(1-t)z, ty+m(1-t)w) \leq t^{\alpha_1}\frac{f(x,y)}{e^{\alpha(x+y)}} + m(1-t^{\alpha_1})\frac{f(z,w)}{e^{\alpha(z+w)}}$$

for all $(x,y), (z,w) \in \Delta, \alpha \in R, (\alpha_1, m) \in [0,1]^2$ and $t \in [0,1]$.

An equivalent definition of the exponentially (α_1, m)-convex function definition in coordinates can be done as follows:

Definition 3.2. *The mapping $f : \Delta \longrightarrow R$ is exponentially (α_1, m)-convex on the co-ordinates on Δ, if the following inequality holds,*

$$\begin{aligned} & f(ta+(1-t)b, sc+m(1-s)d) \\ \leq\ & t^{\alpha_1}s^{\alpha_1}\frac{f(a,c)}{e^{\alpha(a+c)}} + t^{\alpha_1}(1-s^{\alpha_1})m\frac{f(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t^{\alpha_1})s^{\alpha_1}\frac{f(b,c)}{e^{\alpha(b+c)}} + (1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{f(b,d)}{e^{\alpha(b+d)}} \end{aligned}$$

for all $(a,c), (a,d), (b,c), (b,d) \in \Delta, \alpha \in R, (\alpha_1, m) \in [0,1]^2$ and $t, s \in [0,1]$

Lemma 3.3. *A function $f : \Delta \longrightarrow R$ will be called exponentially (α_1, m)-convex on the co-ordinates on Δ, if the partial mappings $f_y : [a,b] \longrightarrow R$, $f_y(u) = e^{\alpha y}f(u,y)$ and $f_x : [c,d] \longrightarrow R$, $f_x(v) = e^{\alpha x}f(x,v)$ are exponentially (α_1, m)-convex on the co-ordinates on Δ, where defined for all $y \in [c,d]$ and $x \in [a,b]$.*

Proof. From the definition of partial mapping f_x, we can write

$$\begin{aligned} f_x(tv_1 + m(1-t)v_2) &= e^{\alpha x} f(x, tv_1 + m(1-t)v_2) \\ &= e^{\alpha x} f(tx + (1-t)x, tv_1 + m(1-t)v_2) \\ &\leq e^{\alpha x}\left[t^{\alpha_1}\frac{f(x,v_1)}{e^{\alpha(x+v_1)}} + m(1-t^{\alpha_1})\frac{f(x,v_2)}{e^{\alpha(x+v_2)}}\right] \\ &= t^{\alpha_1}\frac{f(x,v_1)}{e^{\alpha v_1}} + m(1-t^{\alpha_1})\frac{f(x,v_2)}{e^{\alpha v_2}} \\ &= t^{\alpha_1}\frac{f_x(v_1)}{e^{\alpha v_1}} + m(1-t^{\alpha_1})\frac{f_x(v_2)}{e^{\alpha v_2}}. \end{aligned}$$

Similarly,

$$\begin{aligned} f_y(tu_1 + m(1-t)u_2) &= e^{\alpha y} f(tu_1 + m(1-t)u_2, y) \\ &= e^{\alpha y} f(tu_1 + m(1-t)u_2, ty + (1-t)y) \\ &\leq e^{\alpha y}\left[t^{\alpha_1}\frac{f(u_1,y)}{e^{\alpha(u_1+y)}} + m(1-t^{\alpha_1})\frac{f(u_2,y)}{e^{\alpha(u_2+y)}}\right] \\ &= t^{\alpha_1}\frac{f(u_1,y)}{e^{\alpha u_1}} + m(1-t^{\alpha_1})\frac{f(u_2,y)}{e^{\alpha u_2}} \\ &= t^{\alpha_1}\frac{f_y(u_1)}{e^{\alpha u_1}} + m(1-t^{\alpha_1})\frac{f_y(u_2)}{e^{\alpha u_2}}. \end{aligned}$$

The proof is completed. □

Theorem 3.4. *Let $f : \Delta = [0,b] \times [0,d] \to R$ be partial differentiable mapping on $\Delta = [0,b] \times [0,d]$ in R^2 with $0 < a < b < \infty, 0 < c < md < \infty, f \in L(\Delta), (\alpha_1, m) \in [0,1]^2$ and $\alpha \in R$. If f is exponentially $(\alpha_1, m)-$convex function on the co-ordinates on Δ, then the following inequality holds;*

$$\begin{aligned} &\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy \\ \leq\; &\frac{1}{(\alpha_1+1)^2}\frac{f(a,c)}{e^{\alpha(a+c)}} + \frac{\alpha_1}{(\alpha_1+1)^2}\left(\frac{mf(a,d)}{e^{\alpha(a+d)}} + \frac{f(b,c)}{e^{\alpha(b+c)}}\right) \\ &+\frac{\alpha_1^2}{(\alpha_1+1)^2}\frac{mf(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

Proof. By the definition of the exponentially (α_1, m)-convex on the co-ordinates on Δ, we can write

$$\begin{aligned} &f(ta + (1-t)b, sc + m(1-s)d) \\ \leq\; &t^{\alpha_1}s^{\alpha_1}\frac{f(a,c)}{e^{\alpha(a+c)}} + t^{\alpha_1}(1-s^{\alpha_1})m\frac{f(a,d)}{e^{\alpha(a+d)}} \\ &+(1-t^{\alpha_1})s^{\alpha_1}\frac{f(b,c)}{e^{\alpha(b+c)}} + (1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{f(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

By integrating both sides of the above inequality with respect to t, s on $[0,1]^2$, we have

$$\begin{aligned} &\int_0^1\int_0^1 f(ta + (1-t)b, sc + m(1-s)d)\,dtds \\ \leq\; &\int_0^1\int_0^1 t^{\alpha_1}s^{\alpha_1}\frac{f(a,c)}{e^{\alpha(a+c)}}dtds + \int_0^1\int_0^1 t^{\alpha_1}(1-s^{\alpha_1})m\frac{f(a,d)}{e^{\alpha(a+d)}}dtds \\ &+\int_0^1\int_0^1 (1-t^{\alpha_1})s^{\alpha_1}\frac{f(b,c)}{e^{\alpha(b+c)}}dtds + \int_0^1\int_0^1 (1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{f(b,d)}{e^{\alpha(b+d)}}dtds. \end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Theorem 3.5. *Let $f : \Delta = [0,b] \times [0,d] \to R$ be partial differentiable mapping on $\Delta = [0,b] \times [0,d]$ in R^2 with $0 < a < b < \infty, 0 < c < md < \infty, f \in L(\Delta), (\alpha_1, m) \in [0,1]^2$ and $\alpha \in R$. If $|f|$ is exponentially (α_1, m)-convex on the co-ordinates on Δ, $p > 1$, then the following inequality holds;*

$$\begin{aligned} & \left| \frac{1}{(b-a)(md-c)} \int_a^b \int_c^{md} f(x,y)dxdy \right| \\ \leq & \left(\frac{1}{(p\alpha_1+1)^2} \right)^{\frac{1}{p}} \frac{|f(a,c)|}{e^{\alpha(a+c)}} \\ & + \left(\frac{p\alpha_1}{(p\alpha_1+1)^2} \right)^{\frac{1}{p}} \left(\frac{m|f(a,d)|}{e^{\alpha(a+d)}} + \frac{|f(b,c)|}{e^{\alpha(b+c)}} \right) \\ & + \left(\frac{p^2\alpha_1^2}{(p\alpha_1+1)^2} \right)^{\frac{1}{p}} \frac{m|f(b,d)|}{e^{\alpha(b+d)}}. \end{aligned}$$

Proof. By the definition of the exponentially (α_1, m)-convex on the co-ordinates on Δ, we can write

$$\begin{aligned} & f(ta + (1-t)b, sc + m(1-s)d) \\ \leq & t^{\alpha_1} s^{\alpha_1} \frac{f(a,c)}{e^{\alpha(a+c)}} + t^{\alpha_1}(1-s^{\alpha_1}) m \frac{f(a,d)}{e^{\alpha(a+d)}} \\ & + (1-t^{\alpha_1}) s^{\alpha_1} \frac{f(b,c)}{e^{\alpha(b+c)}} + (1-t^{\alpha_1})(1-s^{\alpha_1}) m \frac{f(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

The absolute value property is used in integral and by integrating both sides of the above inequality with respect to t, s on $[0,1]^2$, we can write

$$\begin{aligned} & \left| \int_0^1 \int_0^1 f(ta + (1-t)b, sc + m(1-s)d)\, dtds \right| \\ \leq & \int_0^1 \int_0^1 \left| t^{\alpha_1} s^{\alpha_1} \frac{f(a,c)}{e^{\alpha(a+c)}} \right| dtds + \int_0^1 \int_0^1 \left| t^{\alpha_1}(1-s^{\alpha_1}) m \frac{f(a,d)}{e^{\alpha(a+d)}} \right| dtds \\ & + \int_0^1 \int_0^1 \left| (1-t^{\alpha_1}) s^{\alpha_1} \frac{f(b,c)}{e^{\alpha(b+c)}} \right| dtds + \int_0^1 \int_0^1 \left| (1-t^{\alpha_1})(1-s^{\alpha_1}) m \frac{f(b,d)}{e^{\alpha(b+d)}} \right| dtds \end{aligned}$$

If we apply the Hölder's inequality to the right-hand side of the inequality, we get

$$\begin{aligned} & \left| \frac{1}{(b-a)(md-c)} \int_a^b \int_c^{md} f(x,y)dxdy \right| \\ \leq & \left(\int_0^1 \int_0^1 t^{p\alpha_1} s^{p\alpha_1} dtds \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{f(a,c)}{e^{\alpha(a+c)}} \right|^q dtds \right)^{\frac{1}{q}} \\ & + \left(\int_0^1 \int_0^1 t^{p\alpha_1}(1-s^{\alpha_1})^p dtds \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{mf(a,d)}{e^{\alpha(a+d)}} \right|^q dtds \right)^{\frac{1}{q}} \\ & + \left(\int_0^1 \int_0^1 (1-t^{\alpha_1})^p s^{p\alpha_1} dtds \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{f(b,c)}{e^{\alpha(b+c)}} \right|^q dtds \right)^{\frac{1}{q}} \\ & + \left(\int_0^1 \int_0^1 (1-t^{\alpha_1})^p (1-s^{\alpha_1})^p dtds \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{mf(b,d)}{e^{\alpha(b+d)}} \right|^q dtds \right)^{\frac{1}{q}} \end{aligned}$$

By using the fact that $\left|1-(1-t)^{\theta}\right|^{\beta} \leq 1-(1-t)^{\theta\beta}$ for $\theta > 0, \beta > 0$, we can write

$$\begin{aligned}
&\left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\
\leq\;& \left(\int_0^1\int_0^1 t^{p\alpha_1}s^{p\alpha_1}dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1 t^{p\alpha_1}(1-s^{p\alpha_1})dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{mf(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1 (1-t^{p\alpha_1})s^{p\alpha_1}dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1 (1-t^{p\alpha_1})(1-s^{p\alpha_1})dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right)^{\frac{1}{q}}.
\end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Theorem 3.6. *Let $f : \Delta = [0,b]\times[0,d] \to R$ be partial differentiable mapping on $\Delta = [0,b]\times[0,d]$ in R^2 with $0 < a < b < \infty, 0 < c < md < \infty, f \in L(\Delta), (\alpha_1, m) \in [0,1]^2$ and $\alpha \in R$. If $\left|f\right|$ is exponentially (α_1, m)-convex on the co-ordinates on Δ, $p, q > 1, \frac{1}{p}+\frac{1}{q} = 1$, then the following inequality holds;*

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\
\leq\;& \left(\frac{1}{p\,(p\alpha_1+1)^2}\right)+\frac{\left|f(a,c)\right|^q}{qe^{\alpha q(a+c)}} \\
&+\left(\frac{\alpha_1}{(p\alpha_1+1)^2}\right)+\left(\frac{\left|mf(a,d)\right|^q}{qe^{\alpha q(a+d)}}+\frac{\left|f(b,c)\right|^q}{qe^{\alpha q(b+c)}}\right) \\
&+\left(\frac{p\alpha_1^2}{(p\alpha_1+1)^2}\right)+\frac{\left|mf(b,d)\right|^q}{qe^{\alpha q(b+d)}}.
\end{aligned}$$

Proof. By the definition of the exponentially (α_1, m)-convex on the co-ordinates on Δ, we can write

$$\begin{aligned}
&f\left(ta+(1-t)b, sc+m(1-s)d\right) \\
\leq\;& t^{\alpha_1}s^{\alpha_1}\frac{f(a,c)}{e^{\alpha(a+c)}}+t^{\alpha_1}(1-s^{\alpha_1})m\frac{f(a,d)}{e^{\alpha(a+d)}} \\
&+(1-t^{\alpha_1})s^{\alpha_1}\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{f(b,d)}{e^{\alpha(b+d)}}.
\end{aligned}$$

The absolute value property is used in integral and by integrating both sides of the above inequality with respect to t, s on $[0,1]^2$, we can write

$$\begin{aligned}
&\left|\int_0^1\int_0^1 f\left(ta+(1-t)b, sc+m(1-s)d\right)dtds\right| \\
\leq\;& \int_0^1\int_0^1\left|t^{\alpha_1}s^{\alpha_1}\frac{f(a,c)}{e^{\alpha(a+c)}}\right|dtds+\int_0^1\int_0^1\left|t^{\alpha_1}(1-s^{\alpha_1})m\frac{f(a,d)}{e^{\alpha(a+d)}}\right|dtds \\
&+\int_0^1\int_0^1\left|(1-t^{\alpha_1})s^{\alpha_1}\frac{f(b,c)}{e^{\alpha(b+c)}}\right|dtds+\int_0^1\int_0^1\left|(1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{f(b,d)}{e^{\alpha(b+d)}}\right|dtds
\end{aligned}$$

If we apply the Young's inequality to the right-hand side of the inequality, we get

$$\begin{aligned}
&\left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\
\leq\ & \frac{1}{p}\left(\int_0^1\int_0^1 t^{p\alpha_1}s^{p\alpha_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 t^{p\alpha_1}(1-s^{\alpha_1})^p dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{mf(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 (1-t^{\alpha_1})^p s^{p\alpha_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 (1-t^{\alpha_1})^p(1-s^{\alpha_1})^p dtds\right) \\
&+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{mf(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right)
\end{aligned}$$

By using the fact that $\left|1-(1-t)^{\theta}\right|^{\beta}\leq 1-(1-t)^{\theta\beta}$ for $\theta>0,\beta>0$, we can write

$$\begin{aligned}
&\left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\
\leq\ & \frac{1}{p}\left(\int_0^1\int_0^1 t^{p\alpha_1}s^{p\alpha_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 t^{p\alpha_1}(1-s^{p\alpha_1})dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{mf(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 (1-t^{p\alpha_1})s^{p\alpha_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 (1-t^{p\alpha_1})(1-s^{p\alpha_1})dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{mf(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right).
\end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Proposition 3.7. *If $f,g:\Delta\to R$ are two exponentially (α_1,m)-convex on the co-ordinates on Δ, then $f+g$ is exponentially convex functions on the co-ordinates on Δ.*

Proof. By the definition of the exponentially (α_1,m)-convex on the co-ordinates on Δ, we can write

$$\begin{aligned}
&f(ta+(1-t)b,sc+m(1-s)d) \\
&+g(ta+(1-t)b,sc+m(1-s)d) \\
\leq\ & t^{\alpha_1}s^{\alpha_1}\left(\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{g(a,c)}{e^{\alpha(a+c)}}\right) \\
&+t^{\alpha_1}(1-s^{\alpha_1})m\left(\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{g(a,d)}{e^{\alpha(a+d)}}\right) \\
&+(1-t^{\alpha_1})s^{\alpha_1}\left(\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{g(b,c)}{e^{\alpha(b+c)}}\right) \\
&+(1-t^{\alpha_1})(1-s^{\alpha_1})m\left(\frac{f(b,d)}{e^{\alpha(b+d)}}+\frac{g(b,d)}{e^{\alpha(b+d)}}\right).
\end{aligned}$$

Namely,

$$\begin{aligned} & (f+g)(ta+(1-t)b, sc+m(1-s)d) \\ \leq \quad & t^{\alpha_1}s^{\alpha_1}\frac{(f+g)(a,c)}{e^{\alpha(a+c)}}+t^{\alpha_1}(1-s^{\alpha_1})m\frac{(f+g)(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t^{\alpha_1})s^{\alpha_1}\frac{(f+g)(b,c)}{e^{\alpha(b+c)}} \\ & +(1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{(f+g)(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

Therefore $(f+g)$ is exponentially (α_1, m)-convex on the co-ordinates on Δ. □

Proposition 3.8. *If $f:\Delta\to R$ is exponentially (α_1, m)-convex on the co-ordinates on Δ and $k\geq 0$ then kf is exponentially (α_1, m)-convex on the co-ordinates on Δ.*

Proof. By the definition of the exponentially (α_1, m)-convex functions on the co-ordinates on Δ, we can write

$$\begin{aligned} & f(ta+(1-t)b, sc+m(1-s)d) \\ \leq \quad & t^{\alpha_1}s^{\alpha_1}\frac{f(a,c)}{e^{\alpha(a+c)}}+t^{\alpha_1}(1-s^{\alpha_1})m\frac{f(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t^{\alpha_1})s^{\alpha_1}\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t^{\alpha_1})(1-s^{\alpha_1})m)\frac{f(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

If both sides are multiplied by k, we have,

$$\begin{aligned} & (kf)(ta+(1-t)b, sc+m(1-s)d) \\ \leq \quad & t^{\alpha_1}s^{\alpha_1}\frac{(kf)(a,c)}{e^{\alpha(a+c)}}+t^{\alpha_1}(1-s^{\alpha_1})m\frac{(kf)(a,d)}{e^{\alpha(a+d)}}+ \\ & (1-t^{\alpha_1})s^{\alpha_1}\frac{(kf)(b,c)}{e^{\alpha(b+c)}}+(1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{(kf)(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

Therefore (kf) is exponentially (α_1, m)-convex functions on the co-ordinates on Δ. □

References

[1] Dragomir, S.S. (2001). On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese Journal of Math.*, 5, 775-788.

[2] Bakula M.K. and Pečarić, J. (2006). On the Jensen's inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese Journal of Math.*, 5, 1271-1292.

[3] Özdemir, M.E. Set, E., Sarıkaya, M.Z. (2011). Some new Hadamard's type inequalities for co-ordinated $m-$convex and $(\alpha, m)-$convex functions, Hacettepe Journal of Mathematics and Statistics , 40(2), 219-229.

[4] Bakula M.K., Özdemir, M.E. and Pečarić, J. (2007). Hadamard-type inequalities for $m-$convex and $(\alpha, m)-$convex functions, *J. Inequal. Pure and Appl. Math.*, 9, (4), Article 96.

[5] Bakula M.K., Pečarić, J.and Ribibić, M. (2006). Companion inequalities to Jensen's inequality for $m-$convex and $(\alpha, m)-$convex functions, *J. Inequal. Pure and Appl. Math.*, 7 (5), Article 194.

[6] Dragomir, S.S. and Toader, G. (1993). Some inequalities for $m-$convex functions, Studia University Babes Bolyai, *Mathematica*, 38 (1), 21-28.

[7] Miheşan, V.G. (1993). A generalization of the convexity, *Seminar of Functional Equations, Approx. and Convex*, Cluj-Napoca (Romania) (1993).

[8] Toader, G. (1984). Some generalization of the convexity, *Proc. Colloq. Approx. Opt.*, Cluj-Napoca, 329-338.

[9] Set, E., Sardari, M., Ozdemir, M.E. and Rooin, J. (2012). On generalizations of the Hadamard inequality for $(\alpha, m)-$convex functions, Kyungpook Mathematical Journal , 52, 307-317.

[10] Sarıkaya, M.Z., Set, E., Özdemir, M.E. and Dragomir, S.S. (2012). New some Hadamard's type inequalities for co-ordinated convex functions, Tamsui Oxford Journal of Information and Mathematical Sciences, 28(2), 137-152.

[11] Özdemir, M.E., Avcı M. and Set, E. (2010). On some inequalities of Hermite-Hadamard type via $m-$convexity, *Applied Mathematics Letters*, 23, 1065-1070.

[12] Aslan, S. and Akdemir, A.O. (2022). Exponentially convex functions on the co-ordinates and related integral inequalities, Proceedings of the 8th International Conference on Control and Optimization with Industrial Applications, Volume II, pp. 120-122, 24-26 August 2022.