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WEIGHTED OSTROWSKI AND ČEBYŠEV TYPE INEQUALITIES WITH APPLICATIONS

S. HUSSAIN[1] AND M.W. ALOMARI[2]

Abstract. Weighted Ostrowski and Čebyšev type inequalities on time scales for single and double integrals have been derived which unify the corresponding continuous and discrete versions and some applications for quantum calculus are also given.

1. Introduction

In 1937, Ostrowski gave a useful formula to estimate the absolute value of derivation of a differentiable function by its integral mean, the so called Ostrowski's inequality [11]

$$\left| f(t) - \frac{1}{b-a}\int_a^b f(s)\ ds \right| \le \sup_{a\le t\le b} |f'(t)|(b-a)\left[\frac{1}{4} + \frac{\left(t-\frac{a+b}{2}\right)^2}{(b-a)^2}\right], \tag{1.1}$$

by means of the Montgomery's identity [9, p.565].

In 1980, Pečarić [13] gave the weighted generalization of the Montgomery identity as:

$$f(x) = \int_a^b w(t)\ f(t)\ dt + \int_a^b Q_w(x,t)\ f'(t)\ dt, \tag{1.2}$$

where the weighted Peano kernel Q_w is defined by:

$$Q_w(x,t) = \begin{cases} W(t), & t\in[a,x] \\ W(t)-1, & t\in(x,b], \end{cases}$$

with $W(t) = \int_a^t w(x)\ dx$ for $t\in[a,b]$.

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K. Boukerrioua et. al [2], further generalized (1.2), while on the other hand in 1882, P. L. Čebyšev [5] proved the following inequality, the so called Čebyšev inequality, for two absolutely continuous functions $f, g : [a, b] \to \mathbb{R}$

$$|T(f, g)| \leq \frac{(b-a)^2}{12} \ \|f'\|_{\infty;[a,b]} \ \ \|g'\|_{\infty;[a,b]}, \tag{1.3}$$

where

$$T(f, g) = \frac{1}{b-a}\int_a^b f(x)\ g(x)\ dx - \left(\frac{1}{b-a}\int_a^b f(x)\ dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\ dx\right).$$

During the past few years, many researchers have given considerable attention to the inequalities (1.1) and (1.3) and various generalizations, extensions, variants of these have appeared in literature [3, 4, 6, 7, 12, 14, 15, 16, 17, 18], and the references therein. The main aim of this paper is to prove some new results in general time scales, generalizing some results in literature, which unify discrete, continuous and many other cases. As a consequence some new Ostrowski and Čebyšev type inequalities have been proved with some new applications. In the whole paper $\mathbb{T}_i$, $1 \leq i \leq 2$, is considered as a time scale.

2. Ostrowski type inequalities for single integral

We may begin with the following lemma:

Lemma 2.1. *Let $a, b \in \mathbb{T}_1$; $c, d \in \mathbb{R}_+ \cup \{0\}$ and let $f : [a, b] \to \mathbb{R}$ be a differentiable function. If f^Δ is an integrable on $[a, b]$ and $w : [a, b] \to \mathbb{R}$ is a weight function such that $\int_a^b w(t)\ \Delta t = d$. Let $\phi : [c, d] \to \mathbb{R}$ be a continuously differentiable function such that $\phi(0) = 0$ and $\phi(d) \neq 0$, then*

$$f(x) = \frac{1}{\phi(d)}\int_a^b [(\phi\circ W)(t)]^\Delta f^\sigma(t)\Delta t + \frac{1}{\phi(d)}\int_a^b K_{w,\phi}(x, t) f^\Delta(t)\Delta t, \tag{2.1}$$

where, $K_{w,\phi}$ is the generalized weighted Peano kernel defined by:

$$K_{w,\phi}(x, t) = \begin{cases} \phi(W(t)), & t \in [a, x] \\ \\ \phi(W(t)) - \phi(d), & t \in (x, b], \end{cases}$$

with $W : [a, b] \to [c, d]$ defined by $W(t) = \int_a^t w(x)\ \Delta x$ for $t \in [a, b]$.

Proof. By integration

$$\begin{aligned}
\int_a^b K_{w,\phi}(x,t) f^\Delta(t)\Delta t &= \int_a^x K_{w,\phi}(x,t) f^\Delta(t)\Delta t + \int_x^b K_{w,\phi}(x,t) f^\Delta(t)\Delta t. \\
&= \int_a^x \phi(W(t)) f^\Delta(t)\Delta t + \int_x^b [\phi(W(t)) - \phi(d)] f^\Delta(t)\Delta t. \\
&= \int_a^b \phi(W(t)) f^\Delta(t)\Delta t - \phi(d)[f(b) - f(x)]. \\
&= \phi(W(b)) f(b) - \phi(W(a)) f(a) - \int_a^b [(\phi\circ W)(t)]^\Delta f^\sigma(t)\Delta t \\
&\quad + f(x)\phi(d) - \phi(d) f(b),
\end{aligned}$$

which is equivalent to (2.1) □

Remark 2.1. For $\mathbb{T}_1 = \mathbb{R}$ and $d = 1$, Lemma 2.1 coincides [2, Theorem 2.1]

Corollary 2.1. *(Discrete case) Let* $\mathbb{T}_1 = \mathbb{Z}$, $a = 0, b = n, c = 0, d = m, x = i, t = k, s = l$ *and* $f(x) = x_i$, *then* (2.1) *reduces to*

$$x_i = \frac{1}{\phi(m)} \sum_{k=1}^{n} f(k) \Delta\phi \left(\sum_{p=1}^{k-1} w(p) \right) + \frac{1}{\phi(m)} \sum_{k=1}^{n} \Delta f(k-1) K_{w,\phi}(i, k-1),$$

where

$$K_{w,\phi}(i,k) = \begin{cases} \phi\left(\sum_{p=0}^{k-1} w(p)\right), & 1 \le k \le i \\ \phi\left(\sum_{p=0}^{k-1} w(p)\right) - \phi(m), & i+1 \le k \le n. \end{cases} \tag{2.2}$$

Corollary 2.2. (*Quantum calculus case*) *Let* $\mathbb{T}_1 = q_1^{\mathbb{N}_0}$ *with* $q_1 > 1$. *Suppose* $a = q_1^i, b = q_1^j, d = q_2^l$ *for some* $i < j$, $i < r$; $l, r \in \mathbb{N}_0$ *and* $q_2 > 1$, *then* (2.1) *reduces to*

$$f\left(q_1^m\right) = \frac{1}{\phi\left(q_2^l\right)} \sum_{r=i}^{j-1} \frac{f(q_1^{r+1}) \ \Delta\phi\left(\sum_{u=i}^{r-1} w\left(q_1^u\right)\right)}{q_1^r(q_1-1)} - \frac{1}{\phi\left(q_2^l\right)} \sum_{r=i}^{j-1} K_{w,\phi}\left(q_1^m, q_1^r\right) \frac{\Delta f\left(q_1^r\right)}{(q_1-1)q_1^r},$$

where Δ *is the forward difference operator defined by:*

$$\Delta\phi\left(\sum_{u=i}^{r-1} w\left(q_1^u\right)\right) = \phi\left(\sum_{u=i}^{r} w\left(q_1^u\right)\right) - \phi\left(\sum_{u=i}^{r-1} w\left(q_1^u\right)\right).$$

Remark 2.2. By setting $\phi(x) = x$; $d = 1$ and $W(t) = \frac{t-a}{b-a}$, Lemma 2.1 reduces to [8, Lemma A].

A generalization of Ostrowski's inequality on time scale may be considered as follows:

Theorem 2.1. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable function on* (a,b) *such that* f^{Δ} *is bounded on* (a,b), *that is,* $\left\|f^{\Delta}\right\|_{\infty;[a,b]} := \sup_{t\in(a,b)} \left|f^{\Delta}(t)\right| < \infty$ *and* $\|\phi \circ W\| = \int_a^b |\phi(W(t))| \Delta t$. *If* f^{Δ} *is integrable on* $[a,b]$, *then for* $x \in [a,b]$

$$\left| f(x) - \frac{1}{\phi(d)} \int_a^b \left[(\phi \circ W)(t)\right]^{\Delta} f^{\sigma}(t) \Delta t \right| \le \frac{\left\|f^{\Delta}\right\|_{\infty;[a,b]}}{|\phi(d)|} \left[\|\phi \circ W\| + |\phi(d)|(b-x)\right], \tag{2.3}$$

Proof. By properties of modulus and Lemma 2.1

$$\left|f(x)-\frac{1}{\phi(d)}\int_a^b[(\phi\circ W)(t)]^{\Delta}f^{\sigma}(t)\,\Delta t\right|=\left|\frac{1}{\phi(d)}\int_a^b K_{W,\phi}(x,t)\,f^{\Delta}(t)\,\Delta t\right|$$
$$\leq\frac{1}{|\phi(d)|}\int_a^b|K_{W,\phi}(x,t)|\left|f^{\Delta}(t)\right|\Delta t$$
$$\leq\frac{1}{|\phi(d)|}\left\{\sup_{t\in(a,b)}\left|f^{\Delta}(t)\right|\right\}\left[\int_a^x|\phi(W(t))|\,\Delta t+\int_x^b|\phi(W(t))-\phi(d)|\,\Delta t\right]$$
$$\leq\frac{\left\|f^{\Delta}\right\|_{\infty;[a,b]}}{|\phi(d)|}\left[\int_a^x|\phi(W(t))|\,\Delta t+\int_x^b|\phi(W(t))|\,\Delta t+\int_x^b|\phi(d)|\,\Delta t\right]$$
$$=\frac{\left\|f^{\Delta}\right\|_{\infty;[a,b]}}{|\phi(d)|}\left[\int_a^b|\phi(W(t))|\,\Delta t+|\phi(d)|(b-x)\right]$$
$$=\frac{\left\|f^{\Delta}\right\|_{\infty;[a,b]}}{|\phi(d)|}\left[\|\phi\circ W\|+|\phi(d)|(b-x)\right],$$

which is as required. □

Corollary 2.3. *(Continuous case) Let $\mathbb{T}_1=\mathbb{R}$, then* (2.3) *becomes*

$$\left|f(x)-\frac{1}{\phi\left(\int_a^b w(t)\,dt\right)}\int_a^b w(t)\,\phi'\left(\int_a^t w(x)\,dx\right)f(t)\,dt\right|$$
$$\leq\frac{\|f'\|_{\infty;[a,b]}}{\left|\phi\left(\int_a^b w(t)\,dt\right)\right|}\left[\int_a^b\left|\phi\left(\int_a^t w(x)\,dx\right)\right|dt+\left|\phi\left(\int_a^b w(t)\,dt\right)\right|(b-x)\right].\tag{2.4}$$

For instance, let $w(t)=\frac{1}{b-a}$ and $\phi(t)=t$, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\|f'\|_{\infty;[a,b]}\left[\frac{b-a}{2}+(b-x)\right].\tag{2.5}$$

Corollary 2.4. *(Discrete case) Let $\mathbb{T}_1=\mathbb{Z}$, $a=0,b=n,c=0,d=m,x=i,t=k,s=l$ and $f(x)=x_i$, then* (2.3) *reduces to*

$$\left|x_i-\frac{1}{\phi(m)}\sum_{k=1}^{n}f(k)\Delta\phi\left(\sum_{p=1}^{k-1}w(p)\right)\right|$$
$$\leq\frac{1}{|\phi(m)|}\left[\sum_{k=1}^{n}\phi\left(\sum_{p=1}^{k-1}w(p)\right)+|\phi(m)|(n-i)\right]\max_{1\leq i\leq n-1}|\Delta x_i|.\tag{2.6}$$

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that f^{Δ} is integrable on $[a,b]$. If $q=\frac{p}{p-1}$ for $p>1$ and $\left\|f^{\Delta}\right\|_{q;[a,b]}=\left(\int_a^b\left|f^{\Delta}(t)\right|^q\Delta t\right)^{1/q}$,* *then for* $x \in [a,b]$

$$\left|f(x)-\frac{1}{\phi(d)}\int_a^b [(\phi\circ W)(t)]^{\Delta} f^{\sigma}(t)\,\Delta t\right| \le \frac{\left\|f^{\Delta}\right\|_{q;[a,b]}}{|\phi(d)|} \times \left[\|\phi\circ W\|_{p;[a,x]}+\|\phi\circ W\|_{p;[x,b]}+|\phi(d)|(b-x)^{1/p}\right]. \tag{2.7}$$

Proof. By Hölder's inequality and Lemma 2.1

$$\begin{aligned}
&\left|f(x)-\frac{1}{\phi(d)}\int_a^b [(\phi\circ W)(t)]^{\Delta} f^{\sigma}(t)\,\Delta t\right| \\
&=\left|\frac{1}{\phi(d)}\int_a^b K_{W,\phi}(x,t)\, f^{\Delta}(t)\,\Delta t\right| \\
&\le \frac{1}{|\phi(d)|}\int_a^b |K_{W,\phi}(x,t)|\left|f^{\Delta}(t)\right|\Delta t \\
&\le \frac{1}{|\phi(d)|}\left(\int_a^b |K_{W,\phi}(x,t)|^p\,\Delta t\right)^{1/p}\left(\int_a^b \left|f^{\Delta}(t)\right|^q\Delta t\right)^{1/q}
\end{aligned}$$

But,

$$\begin{aligned}
&\left(\int_a^b |K_{W,\phi}(x,t)|^p\,\Delta t\right)^{1/p} \\
&=\left(\int_a^x |\phi(W(t))|^p\,\Delta t+\int_x^b |\phi(W(t))-\phi(d)|^p\,\Delta t\right)^{1/p} \\
&\le \left(\int_a^x |\phi(W(t))|^p\,\Delta t\right)^{1/p}+\left(\int_x^b |\phi(W(t))-\phi(d)|^p\,\Delta t\right)^{1/p} \\
&\le \left(\int_a^x |\phi(W(t))|^p\,\Delta t\right)^{1/p}+\left(\int_x^b |\phi(W(t))|^p\,\Delta t+\int_x^b |\phi(d)|^p\,\Delta t\right)^{1/p} \\
&\le \left(\int_a^x |\phi(W(t))|^p\,\Delta t\right)^{1/p}+\left(\int_x^b |\phi(W(t))|^p\,\Delta t\right)^{1/p}+\left(\int_x^b |\phi(d)|^p\,\Delta t\right)^{1/p} \\
&=\|\phi\circ W\|_{p;[a,x]}+\|\phi\circ W\|_{p;[x,b]}+|\phi(d)|(b-x)^{1/p}
\end{aligned}$$

Combining the above obtained inequalities we get the required result. □

Theorem 2.3. *Let* $f,g:[a,b]\to\mathbb{R}$ *be differentiable functions such that* f^{Δ}, g^{Δ} *are integrable functions on* $[a,b]$*. If* ϕ, w *are as in Lemma 2.1, then*

$$|S(f,g;\phi,W)| \le \frac{H^2(x)}{\phi^2(d)}\parallel f^{\Delta}\parallel_{\infty;[a,b]}\parallel g^{\Delta}\parallel_{\infty;[a,b]}\parallel \phi'\parallel^2_{\infty;[a,b]},$$

where

$$S(f,g;\phi,W)=f(x)g(x)-\frac{1}{\phi(d)}\left\{f(x)\int_a^b\left[(\phi\circ W)(t)\right]^{\Delta}g^{\sigma}(t)\Delta t\right. \tag{2.8}$$

$$\left.+\ g(x)\int_a^b\left[(\phi\circ W)(t)\right]^{\Delta}f^{\sigma}(t)\Delta t\right\}+\frac{1}{\phi^2(d)}\left[\int_a^b\left[(\phi\circ W)(t)\right]^{\Delta}f^{\sigma}(t)\Delta t\right]$$

$$\times\left[\int_a^b\left[(\phi\circ W)(t)\right]^{\Delta}g^{\sigma}(t)\Delta t\right],$$

and $H(x)=\parallel w\parallel_{\infty;[a,b]}(b-a)(1-a)+|d|(b-x)$.

Proof. By Lemma 2.1 for functions f and g

$$f(x)-\frac{1}{\phi(d)}\int_a^b\left[(\phi\circ W)(t)\right]^{\Delta}f^{\sigma}(t)\Delta t=\frac{1}{\phi(d)}\int_a^b K_{w,\phi}(x,t)f^{\Delta}(t)\Delta t. \tag{2.9}$$

$$g(x)-\frac{1}{\phi(d)}\int_a^b\left[(\phi\circ W)(t)\right]^{\Delta}g^{\sigma}(t)\Delta t=\frac{1}{\phi(d)}\int_a^b K_{w,\phi}(x,t)g^{\Delta}(t)\Delta t. \tag{2.10}$$

Multiplying equations (2.9) and (2.10) simultaneously and using properties of modulus:

$$\begin{aligned}|S(f,g;\phi,W)|&=\frac{1}{\phi^2(d)}\left|\left\{\int_a^b K_{w,\phi}(x,t)f^{\Delta}(t)\Delta t\right\}\left\{\int_a^b K_{w,\phi}(x,t)g^{\Delta}(t)\Delta t\right\}\right|\\&\leq\frac{1}{\phi^2(d)}\max_{t\in[a,b]}\left|f^{\Delta}(t)\right|\max_{t\in[a,b]}\left|g^{\Delta}(t)\right|\left[\int_a^b\left|K_{w,\phi}(x,t)\right|\Delta t\right]^2\\&=\frac{\|f^{\Delta}\|_{\infty;[a,b]}\|g^{\Delta}\|_{\infty;[a,b]}}{\phi^2(d)}\left[\int_a^x\left|\phi(W(t))\right|\Delta t+\int_x^b\left|\phi(W(t))-\phi(d)\right|\Delta t\right]^2\end{aligned}$$

By Mean Value Theorem, there exist $\eta_t,\xi_t\in[c,d]$ such that

$$\phi(W(t))=\phi'(\xi_t)W(t)\quad\text{and}\quad\phi(W(t))-\phi(d)=\phi'(\eta_t)(W(t)-d).$$

$$\begin{aligned}&|S(f,g;\phi,W)|\\&\leq\frac{\parallel\phi'\parallel^2_{\infty;[a,b]}\parallel f^{\Delta}\parallel_{\infty;[a,b]}\parallel g^{\Delta}\parallel_{\infty;[a,b]}}{\phi^2(d)}\left\{\int_a^x|W(t)|\Delta t+\int_x^b|W(t)|\Delta t+|d|\int_x^b\Delta t\right\}^2\\&\leq\frac{\parallel\phi'\parallel^2_{\infty;[a,b]}\parallel f^{\Delta}\parallel_{\infty;[a,b]}\parallel g^{\Delta}\parallel_{\infty;[a,b]}}{\phi^2(d)}\left\{\int_a^b\left(\int_a^t|w(x)|\Delta x\right)\Delta t+|d|(b-x)\right\}^2\\&\leq\frac{\parallel\phi'\parallel^2_{\infty;[a,b]}\parallel f^{\Delta}\parallel_{\infty;[a,b]}\parallel g^{\Delta}\parallel_{\infty;[a,b]}}{\phi^2(d)}\left[\parallel w\parallel_{\infty;[a,b]}(b-a)(1-a)+|d|(b-x)\right]^2.\end{aligned}$$

This completes the proof of the theorem. □

Applications of theorem 2.3 to discrete and continuous cases give the following results.

Corollary 2.5. (*continuous case*) *Let $\mathbb{T}_1 = \mathbb{R}$. In this case delta integral is the usual Riemann integral from calculus, then*

$$\begin{aligned}\Bigg| f(x)g(x) &- \frac{1}{\phi(d)}\left\{ f(x)\int_a^b g(t)\ w(t)\ \phi'\left(W_1(t)\right)\ dt + g(x)\int_a^b f(t)\ w(t)\phi'\left(W_1(t)\right) dt \right\} \\ &+ \frac{1}{\phi^2(d)}\left[\int_a^b f(t)\ w(t)\ \phi'\left(W_1(t)\right)\ dt\right]\left[\int_a^b g(t)\ w(t)\ \phi'\left(W_1(t)\right)\ dt\right]\Bigg| \\ &\leq \frac{\| f' \|_{\infty;[a,b]} \| g' \|_{\infty;[a,b]} \| \phi' \|^2_{\infty;[a,b]}}{\phi^2(d)}\ H^2(x),\end{aligned}$$

where

$$W_1(u) = \int_a^u w(l)\ dl. \tag{2.11}$$

Corollary 2.6. (*Discrete case*) *Let $\mathbb{T}_1 = \mathbb{Z}$, $a = 0, b = n, d = m, x = i$, $f(x) = x_i$ and $g(x) = y_i$, then*

$$\begin{aligned}\Bigg| x_i y_i &- \frac{1}{\phi(m)}\left\{ x_i \sum_{k=1}^{n} y_k\ \Delta\phi\left(\sum_{p=1}^{k-1} w(p)\right) + y_i \sum_{k=1}^{n} x_k\ \Delta\phi\left(\sum_{p=1}^{k-1} w(p)\right)\right\} \\ &+ \frac{1}{\phi^2(m)} \times \left[\sum_{k=1}^{n} y_k\ \Delta\phi\left(\sum_{p=1}^{k-1} w(p)\right)\right]\left[\sum_{k=1}^{n} x_k\ \Delta\phi\left(\sum_{p=1}^{k-1} w(p)\right)\right]\Bigg| \\ &\leq \frac{1}{\phi^2(m)} \max_{1\leq i\leq n-1} |\Delta x_i| \max_{1\leq i\leq n-1} |\Delta y_i| \left[\max_{1\leq i\leq n-1} |\phi'(i)|\right]^2 \\ &\quad \times \left[n \max_{1\leq i\leq n-1} |w(i)| + |m|(n-i)\right]^2.\end{aligned}$$

Corollary 2.7. (*Quantum calculus case*) *Let $\mathbb{T}_1 = q_1^{\mathbb{N}_0}$ with $q_1 > 1$. Suppose $a = q_1^i, b = q_1^j, c = q_2^k, d = q_2^l$ for some $i < j$; $i < r$; $k < l$; $l, r \in \mathbb{N}_0$ and $q_2 > 1$, then*

$$\begin{aligned}\Bigg| f\left(q_1^m\right) g\left(q_1^m\right) &- \frac{1}{q_1^r(q_1-1)\phi\left(q_2^l\right)} \sum_{r=i}^{j-1} \left(f\left(q_1^m\right) g\left(q_1^{r+1}\right) + g\left(q_1^m\right) f\left(q_1^{r+1}\right)\right) \\ \times \Delta\phi\left(\sum_{u=i}^{r-1} w\left(q_1^u\right)\right) &- \frac{1}{q_1^{2r}(q_1-1)^2\phi^2\left(q_2^l\right)}\left[\sum_{r=i}^{j-1} g\left(q_1^{r+1}\right)\Delta\phi\left(\sum_{u=i}^{r-1} w\left(q_1^u\right)\right)\right] \\ &\times \left[\sum_{r=i}^{j-1} f\left(q_1^{r+1}\right)\Delta\phi\left(\sum_{u=i}^{r-1} w\left(q_1^u\right)\right)\right]\Bigg| \\ \leq \frac{1}{\phi^2\left(q_2^l\right)(q_1-1)^2} &\sup_{i<r<j-1}\left|\frac{\Delta f\left(q_1^r\right)}{q_1^r}\right| \sup_{i<r<j-1}\left|\frac{\Delta g\left(q_1^r\right)}{q_1^r}\right| \left[\sup_{k<s<l-1}\left|\phi'\left(q_2^s\right)\right|\right]^2 \\ &\times \left[\max_{i<r<j-1} \left|w\left(q_1^r\right)\right| \left(\sum_{r=i}^{j-1} q_1^r\right)\left(1-q_1^i\right) + q_2^l \sum_{p=m}^{j-1} q_1^p\right]^2.\end{aligned}$$

3. Ostrowski type inequality for double integrals

Lemma 3.1. *Let $a, b \in \mathbb{T}_1$; $c, d \in \mathbb{T}_2$ and let $f : [a, b] \times [c, d] \to \mathbb{R}$ be such that the partial derivatives $\frac{\partial f(t,s)}{\Delta_1 t}$, $\frac{\partial f(t,s)}{\Delta_2 s}$ and $\frac{\partial^2 f(t,s)}{\Delta_2 s\ \Delta_1 t}$ exist and are continuous on $[a, b] \times [c, d]$, then*

(3.1)
$$f(x,y) = \frac{1}{\phi^2(d)} \left[\int_a^b \int_c^d K_{w,\phi}(x,t) K_{w,\phi}(y,s) \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t}\ \Delta_2 s \Delta_1 t + \int_a^b \int_c^d K_{w,\phi}(x,t) \right.$$
$$\times \left[(\phi \circ W)(s)\right]^{\Delta} \frac{\partial f(t, \sigma_2(s))}{\Delta_1 t} \Delta_2 s \Delta_1 t + \int_a^b \int_c^d \left[(\phi \circ W)(t)\right]^{\Delta} K_{w,\phi}(y,s) \frac{\partial f(\sigma_1(t), s)}{\Delta_2 s}$$
$$\left. \times\ \Delta_2 s \Delta_1 t + \int_a^b \int_c^d \left[(\phi \circ W)(t)\right]^{\Delta} \left[(\phi \circ W)(s)\right]^{\Delta} f(\sigma_1(t), \sigma_2(s)) \Delta_2 s \Delta_1 t \right]$$

Proof. By Lemma 2.1 for partial delta map $f(., y)$ we have

$$f(x,y) = \frac{1}{\phi(d)} \int_a^b K_{w,\phi}(x,t) \frac{\partial f(t,y)}{\Delta_1 t} \Delta_1 t + \frac{1}{\phi(d)} \int_a^b \left[(\phi \circ W)(t)\right]^{\Delta} f(\sigma_1(t), y) \Delta_1 t, \tag{3.2}$$

for all $(x, y) \in [a, b] \times [c, d]$. Also application of the same lemma for the partial delta map $f(\sigma_1(t), .)$ yields:

$$f(\sigma_1(t), y) = \frac{1}{\phi(d)} \int_c^d K_{w,\phi}(y,s) \frac{\partial f(\sigma_1(t), s)}{\Delta_2 s} \Delta_2 s + \frac{1}{\phi(d)} \int_c^d \left[(\phi \circ W)(s)\right]^{\Delta} f(\sigma_1(t), \sigma_2(s)) \Delta_2 s. \tag{3.3}$$

Similarly for the partial delta map $\frac{\partial f(t,.)}{\Delta_1 t}$, Lemma 2.1 provides:

$$\frac{\partial f(t,y)}{\Delta_1 t} = \frac{1}{\phi(d)} \int_c^d K_{w,\phi}(y,s) \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t} \Delta_2 s + \frac{1}{\phi(d)} \int_c^d \left[(\phi \circ W)(s)\right]^{\Delta} \frac{\partial f(t, \sigma_2(s))}{\Delta_1 t} \Delta_2 s. \tag{3.4}$$

From equations (3.2)-(3.4) we obtain (3.1). □

Application of Lemma 3.1 to different time scales gives some new results.

Corollary 3.1. (*Continuous case*) *Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$. In this case delta integral is the Riemann integral from calculus, then*

$$f(x,y) = \frac{1}{\phi^2(d)} \left[\int_a^b \int_c^d K_{w,\phi}(x,t) K_{w,\phi}(y,s) \frac{\partial^2 f(t,s)}{\partial s\ \partial t}\ ds\ dt + \int_a^b \int_c^d K_{w,\phi}(x,t) \right.$$
$$\times\ w(s) \phi' (W_1(s)) \frac{\partial f(t,s)}{\partial t}\ ds\ dt + \int_a^b \int_c^d K_{w,\phi}(y,s)\ w(t) \phi' (W_1(t)) \frac{\partial f(t,s)}{\partial s}\ ds\ dt$$
$$\left. + \int_a^b \int_c^d w(t)\ w(s) \phi' (W_1(t))\ \ \phi' (W_1(s)) f(t,s)\ ds\ dt \right]$$

Corollary 3.2. (*Discrete case*) *Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, $a = 0, b = n, d = m, x = i, y = j, t = k, s = l$ and $f(p,q) = x_p y_q$, then*

$$f(i,j) = \frac{1}{\phi^2(m)} \left[\sum_{k=1}^{n} \sum_{l=1}^{m} K_{w,\phi}(i,k)\ K_{w,\phi}(j,l)\ \Delta x_k\ \Delta y_l + \sum_{k=1}^{n} \sum_{l=1}^{m} K_{w,\phi}(i,k)\ w(l) \right.$$
$$\times \Delta\phi \left(\sum_{q=1}^{l-1} w(q) \right) \Delta x_k + \sum_{k=1}^{n} \sum_{l=1}^{m} K_{w,\phi}(j,l) \Delta\phi \left(\sum_{p=1}^{k-1} w(p) \right) \Delta y_l$$
$$\left. + \sum_{k=1}^{n} \sum_{l=1}^{m} w(k)\ w(l)\ \Delta\phi \left(\sum_{q=1}^{l-1} w(q) \right) \Delta\phi \left(\sum_{p=1}^{k-1} w(p) \right) \right],$$

where, $K_{w,\phi}$ is given by (2.2).

Corollary 3.3. (*Quantum calculus case*) *Let $\mathbb{T}_1 = q_1^{\mathbb{N}_0}$ and $\mathbb{T}_2 = q_2^{\mathbb{N}_0}$ with $q_1 > 1$ and $q_2 > 1$. Suppose $a = q_1^i, b = q_1^j, c = q_2^k, d = q_2^l$ for some $i < j$; $k < l$, then*

$$f\left(q_1^m, q_2^n\right) = \frac{1}{\phi^2\left(q_2^l\right)} \left[\sum_{r=i}^{j-1} \sum_{s=k}^{l-1} K_{w,\phi}\left(q_1^m, q_1^r\right) K_{w,\phi}\left(q_2^n, q_2^s\right) \left[\Delta_1 f\left(q_1^r, q_2^{s+1}\right) - \Delta_1 f\left(q_1^r, q_2^s\right)\right] \right.$$
$$+ \sum_{r=i}^{j-1} \sum_{s=k}^{l-1} K_{w,\phi}\left(q_1^m, q_1^r\right) \Delta\phi \left(\sum_{v=j}^{s-1} w\left(q_2^v\right) \right) \frac{\Delta_1 f\left(q_1^r, q_2^{s+1}\right)}{q_1^r q_2^s (q_1-1)(q_2-1)} + \sum_{r=i}^{j-1} \sum_{s=k}^{l-1} K_{w,\phi}\left(q_2^n, q_2^s\right)$$
$$\times \Delta\phi \left(\sum_{u=i}^{r-1} w\left(q_1^u\right) \right) \frac{\Delta_2 f\left(q_1^{r+1}, q_2^s\right)}{q_1^r q_2^s (q_1-1)(q_2-1)} + \sum_{r=i}^{j-1} \sum_{s=k}^{l-1} \Delta\phi \left(\sum_{v=j}^{s-1} w\left(q_2^v\right) \right) \Delta\phi \left(\sum_{u=i}^{r-1} w\left(q_1^u\right) \right)$$
$$\left. \times \frac{f\left(q_1^{r+1}, q_2^{s+1}\right)}{q_1^r q_2^s (q_1-1)(q_2-1)} \right],$$

where, Δ_1 is the forward difference operator with respect to first component and Δ_2 is with respect to second component.

Theorem 3.1. *Let the conditions of Lemma 3.1 be satisfied, then*

$$\left| f(x,y) - \frac{1}{\phi^2(d)} \int_a^b \int_c^d \left[(\phi \circ W)(t)\right]^\Delta \left[(\phi \circ W)(s)\right]^\Delta f(\sigma_1(t), \sigma_2(s)) \Delta_2 s \Delta_1 t \right| \tag{3.5}$$
$$\leq (M_1 + M_2 + M_3) \|w\|_{\infty;[a,b]} \|w\|_{\infty;[c,d]} \left[\int_a^b \Psi(t) \Delta_1 t \right] \left[\int_c^d \Psi(s) \Delta_2 s \right]$$
$$+ (M_1 + M_3)|\phi(d)|(d-y)\|w\|_{\infty;[a,b]} \left[\int_a^b \Psi(t) \Delta_1 t \right] + (M_2 + M_3)|\phi(d)|(b-x)$$
$$\times \|w\|_{\infty;[c,d]} \left[\int_c^d \Psi(s) \Delta_2 s \right] + M_3 \phi^2(d)(b-x)(d-y),$$

for all $(x,y) \in [a,b] \times [c,d]$, where

$$M_1 = \sup_{s \in [c,d]} \left| \frac{\partial f(t,s)}{\Delta_2 s} \right|, \quad M_2 = \sup_{t \in [a,b]} \left| \frac{\partial f(t,s)}{\Delta_1 t} \right| \quad \text{and } M_3 = \sup_{(t,s) \in [a,b] \times [c,d]} \left| \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t} \right|$$

and

$$\Psi(u)=\int_0^1 \phi'(W(u)+h\mu(u)\ w(u))\ dh.$$

Proof. By properties of modulus and Lemma 3.1

$$\left|f(x,y)-\frac{1}{\phi^2(d)}\int_a^b\int_c^d \left[(\phi\circ W)(t)\right]^{\Delta}\left[(\phi\circ W)(s)\right]^{\Delta} f(\sigma_1(t),\sigma_2(s))\Delta_2 s\Delta_1 t\right|$$

$$
\begin{aligned}
&= \frac{1}{\phi^2(d)} \left| \int_a^b \int_c^d K_{w,\phi}(x,t) K_{w,\phi}(y,s) \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t}\ \Delta_2 s \Delta_1 t + \int_a^b \int_c^d K_{w,\phi}(x,t) \left[(\phi \circ W)(s)\right]^{\Delta} \right. \\
&\quad \left. \times \frac{\partial f(t,\sigma_2(s))}{\Delta_1 t} \Delta_2 s \Delta_1 t + \int_a^b \int_c^d \left[(\phi \circ W)(t)\right]^{\Delta} K_{w,\phi}(y,s) \frac{\partial f(\sigma_1(t),s)}{\Delta_2 s} \Delta_2 s \Delta_1 t \right| \\
&\le \frac{1}{\phi^2(d)} \left\{ \int_a^b \int_c^d |K_{w,\phi}(x,t)|\ |K_{w,\phi}(y,s)|\ \left| \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t} \right| \Delta_2 s \Delta_1 t + \int_a^b \int_c^d |K_{w,\phi}(x,t)| \right. \\
&\quad \times \left| \left[(\phi \circ W)(s)\right]^{\Delta} \right|\ \left| \frac{\partial f(t,\sigma_2(s))}{\Delta_1 t} \right| \Delta_2 s \Delta_1 t + \int_a^b \int_c^d \left| \left[(\phi \circ W)(t)\right]^{\Delta} \right| \\
&\quad \left. \times |K_{w,\phi}(y,s)|\ \left| \frac{\partial f(\sigma_1(t),s)}{\Delta_2 s} \right| \Delta_2 s \Delta_1 t \right\} \\
&\le \frac{1}{\phi^2(d)} \left\{ \sup_{(t,s)\in[a,b]\times[c,d]} \left| \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t} \right| \int_a^b \int_c^d |K_{w,\phi}(x,t)|\ |K_{w,\phi}(y,s)| \Delta_2 s \Delta_1 t \right. \\
&\quad + \sup_{t\in[a,b]} \left| \frac{\partial f(t,s)}{\Delta_1 t} \right| \int_a^b \int_c^d |K_{w,\phi}(x,t)| \left| \left[(\phi \circ W)(s)\right]^{\Delta} \right| \Delta_2 s \Delta_1 t \\
&\quad \left. + \sup_{s\in[c,d]} \left| \frac{\partial f(t,s)}{\Delta_2 s} \right| \int_a^b \int_c^d \left| \left[(\phi \circ W)(t)\right]^{\Delta} \right| |K_{w,\phi}(y,s)| \Delta_2 s \Delta_1 t \right\} \\
&= \frac{1}{\phi^2(d)} \left[M_3 \left\{ \int_a^b |K_{w,\phi}(x,t)|\ \Delta_1 t \right\} \left\{ \int_c^d\ |K_{w,\phi}(y,s)| \Delta_2 s \right\} + M_2 \left\{ \int_a^b |K_{w,\phi}(x,t)| \Delta_1 t \right\} \right. \\
&\quad \left. \times \left\{ \int_c^d \left| \left[(\phi \circ W)(s)\right]^{\Delta} \right| \Delta_2 s \right\} + M_1 \left\{ \int_a^b \left| \left[(\phi \circ W)(t)\right]^{\Delta} \right| \Delta_1 t \right\} \left\{ \int_c^d |K_{w,\phi}(y,s)| \Delta_2 s \right\} \right] \\
&\le \frac{1}{\phi^2(d)} \left[M_3 \left\{ \int_a^b |w(t)|\ \left| \int_0^1 \phi'(W(t) + h\ \mu(t)\ w(t)) dh \right| \Delta_1 t + |\phi(d)|(b-x) \right\} \right. \\
&\quad \times \left\{ \int_c^d |w(s)|\ \left| \int_0^1 \phi'(W(s) + h\ \mu(s)\ w(s)) dh \right| \Delta_2 s + |\phi(d)|(d-y) \right\} \\
&\quad + M_2 \left\{ \int_a^b |w(t)|\ \left| \int_0^1 \phi'(W(t) + h\ \mu(t)\ w(t)) dh \right| \Delta_1 t + |\phi(d)|(b-x) \right\} \\
&\quad \times \left\{ \int_c^d |w(s)|\ \left| \int_0^1 \phi'(W(s) + h\ \mu(s)\ w(s)) dh \right| \Delta_2 s \right\} \\
&\quad + M_1 \left\{ \int_a^b |w(t)|\ \left| \int_0^1 \phi'(W(t) + h\ \mu(t)\ w(t)) dh \right| \Delta_1 t \right\} \\
&\quad \left. \times \left\{ \int_c^d |w(s)|\ \left| \int_0^1 \phi'(W(s) + h\ \mu(s)\ w(s)) dh \right| \Delta_2 s + |\phi(d)|(d-y) \right\} \right] \\
&\le \frac{1}{\phi^2(d)} \left[M_3 \left\{ \sup_{t\in[a,b]} |w(t)| \int_a^b \left| \int_0^1 \phi'(W(t) + h\ \mu(t)\ w(t)) dh \right| \Delta_1 t + |\phi(d)|(b-x) \right\} \right. \\
&\quad \times \left\{ \sup_{s\in[c,d]} |w(s)| \int_c^d \left| \int_0^1 \phi'(W(s) + h\ \mu(s)\ w(s)) dh \right| \Delta_2 s + |\phi(d)|(d-y) \right\}
\end{aligned}
$$

$$+ M_2 \left\{ \sup_{t\in[a,b]} |w(t)| \int_a^b \left| \int_0^1 \phi'(W(t) + h\ \mu(t)\ w(t))dh \right| \Delta_1 t + |\phi(d)|(b-x) \right\}$$
$$\times \left\{ \int_c^d |w(s)| \ \left| \int_0^1 \phi'(W(s) + h\ \mu(s)\ w(s))dh \right| \Delta_2 s \right\}$$
$$+ M_1 \left\{ \sup_{t\in[a,b]} |w(t)| \int_a^b \left| \int_0^1 \phi'(W(t) + h\ \mu(t)\ w(t))dh \right| \Delta_1 t \right\}$$
$$\times \left\{ \sup_{s\in[c,d]} |w(s)| \int_c^d \left| \int_0^1 \phi'(W(s) + h\ \mu(s)\ w(s))dh \right| \Delta_2 s + |\phi(d)|(d-y) \right\} \Bigg]$$
$$= \frac{1}{\phi^2(d)} \Bigg[M_3 \left\{ \|w\|_{\infty;[a,b]} \int_a^b |\Psi(t)|\Delta_1 t + |\phi(d)|(b-x) \right\}$$
$$\times \left\{ \|w\|_{\infty;[c,d]} \int_c^d |\Psi(s)|\Delta_2 s + |\phi(d)|(d-y) \right\}$$
$$+ M_2 \left\{ \|w\|_{\infty;[a,b]} \int_a^b |\Psi(t)|\Delta_1 t + |\phi(d)|(b-x) \right\} \left\{ \|w\|_{\infty;[c,d]} \int_c^d |\Psi(s)|\Delta_2 s \right\}$$
$$+ M_1 \left\{ \|w\|_{\infty;[a,b]} \int_a^b |\Psi(t)|\Delta_1 t \right\} \left\{ \|w\|_{\infty;[c,d]} \int_c^d |\Psi(s)|\Delta_2 s + |\phi(d)|(d-y) \right\} \Bigg],$$

which is equivalent to (3.5). □

The followings are the discrete and continuous cases of the Theorem 3.1.

Corollary 3.4. (*continuous case*) *Let* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$. *In this case delta integral is the Riemann integral from calculus, then*

$$\left| f(x,y) - \frac{1}{\phi^2(d)} \int_a^b \int_c^d w(t)\ w(s)\phi'(W_1(t))\phi'(W_1(s))\ f(t,s)\ ds\ dt \right|$$
$$\leq (M_1 + M_2 + M_3)\|w\|_{\infty;[a,b]}\|w\|_{\infty;[c,d]} \left[\int_a^b \phi'(W_1(t))\ dt \right] \left[\int_c^d \phi'(W_1(s))\ ds \right]$$
$$+ (M_1 + M_3)|\phi(d)|(d-y)\|w\|_{\infty;[a,b]} \left[\int_a^b \phi'(W_1(t))\ dt \right] + (M_2 + M_3)|\phi(d)|(b-x)$$
$$\times \|w\|_{\infty;[c,d]} \left[\int_c^d \phi'(W_1(s))\ ds \right] + M_3\ \phi^2(d)(b-x)(d-y),$$

where, $W_1(u)$ *is given by* (2.11).

Corollary 3.5. (*Discrete case*) *Let* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, $a = 0 = c, b = n, d = m, x = i, y = j, t = k, s = l$ *and* $f(p,q) = x_p\ y_q$, *then*

$$\left| x_i\ y_j - \frac{1}{\phi^2(m)} \sum_{k=1}^{n} \sum_{l=1}^{m} \Delta\phi \left(\sum_{q=1}^{l-1}\ w(q) \right)\ \Delta\phi \left(\sum_{p=1}^{k-1}\ w(p) \right) \times x_k\ y_l \right|$$
$$\leq (M_1 + M_2 + M_3)$$
$$\times \max_{1 \leq i \leq n-1} |w(i)| \times \max_{1 \leq j \leq m-1} |w(j)| \sum_{k=1}^{n} \sum_{l=1}^{m} \frac{1}{w(k)\ w(l)} \Delta\phi \left(\sum_{p=1}^{k-1}\ w(p) \right) \Delta\phi \left(\sum_{q=1}^{l-1}\ w(q) \right)$$
$$+ (M_1 + M_3)|\phi(m)|(m-j) \max_{1 \leq i \leq n-1} |w(i)| \sum_{k=1}^{n} \frac{1}{w(k)} \Delta\phi \left(\sum_{p=1}^{k-1}\ w(p) \right) + (M_2 + M_3)|\phi(m)|$$
$$\times (n-i) \max_{1 \leq j \leq m-1} |w(j)| \sum_{l=1}^{m} \frac{1}{w(l)} \Delta\phi \left(\sum_{q=1}^{l-1}\ w(q) \right) + M_3 \phi^2(m)(n-i)(m-j),$$

where

$$M_1 = \max_{1 \leq l \leq m-1} |\Delta y_l|\,,\ M_2 = \max_{1 \leq k \leq n-1} |\Delta x_k| \ \text{ and } M_3 = \max_{1 \leq l \leq m-1; 1 \leq k \leq n-1} |\Delta x_k \Delta y_l|$$

4. Čebyšev type inequalities

Theorem 4.1. *Let the conditions of Theorem 2.3 be satisfied, then*

$$|T(f,g;\phi,W)| \leq \frac{\| f^{\Delta} \|_{\infty} \| g^{\Delta} \|_{\infty} \| w \|_{\infty}}{\phi^2(d)} \int_a^b |\Psi(x)|\ G^2(x)\ \Delta x, \tag{4.1}$$

where

$$T(f,g;\phi,W) = \int_a^b S(f,g;\phi,W)\ [(\phi \circ W)(x)]^{\Delta}\ \Delta x,$$

$$\text{and}\ \ G(x) = \|w\|_{\infty;[a,b]} \int_a^b |\Psi(t)| \Delta_1 t + |\phi(d)|(b-x).$$

Proof. By Lemma 2.1, the following identities hold for all $x \in [a,b]$

$$f(x) - \frac{1}{\phi(d)} \int_a^b [(\phi \circ W)(t)]^{\Delta}\ f^{\sigma}(t) \Delta t = \frac{1}{\phi(d)} \int_a^b K_{w,\phi}(x,t) f^{\Delta}(t) \Delta t. \tag{4.2}$$

$$g(x) - \frac{1}{\phi(d)} \int_a^b [(\phi \circ W)(t)]^{\Delta}\ g^{\sigma}(t) \Delta t = \frac{1}{\phi(d)} \int_a^b K_{w,\phi}(x,t) g^{\Delta}(t) \Delta t. \tag{4.3}$$

Multiplying both sides of equations (4.2) and (4.3) we get (2.8), then multiplying by $[(\phi\circ W)(x)]^{\Delta}$ and integrating over $x\in[a,b]$

$$\begin{aligned}
&\int_a^b f(x)g(x)\,[(\phi\circ W)(x)]^{\Delta}\,\Delta x\\
&\quad-\frac{1}{\phi(d)}\left\{\int_a^b f(x)[(\phi\circ W)(x)]^{\Delta}\Delta x\right\}\left\{\int_a^b[(\phi\circ W)(t)]^{\Delta}\ \times g^{\sigma}(t)\Delta t\right\}\\
&\quad-\frac{1}{\phi(d)}\left\{\int_a^b g(x)[(\phi\circ W)(x)]^{\Delta}\Delta x\right\}\left\{\int_a^b[(\phi\circ W)(t)]^{\Delta}f^{\sigma}(t)\Delta t\right\}\\
&\quad+\frac{1}{\phi^2(d)}\left[\int_a^b[(\phi\circ W)(t)]^{\Delta}f^{\sigma}(t)\Delta t\right]\left[\int_a^b[(\phi\circ W)(t)]^{\Delta}g^{\sigma}(t)\Delta t\right]\left[\int_a^b[(\phi\circ W)(x)]^{\Delta}\Delta x\right]\\
&=\frac{1}{\phi^2(d)}\int_a^b[(\phi\circ W)(x)]^{\Delta}\left[\int_a^b K_{w,\phi}(x,t)g^{\Delta}(t)\Delta t\right]\left[\int_a^b K_{w,\phi}(x,t)f^{\Delta}(t)\Delta t\right]\Delta x,
\end{aligned}$$

that is,

$$\begin{aligned}
T(f,g;\phi,W)&=\int_a^b S(f,g;\phi,W)[(\phi\circ W)(x)]^{\Delta}\Delta x\\
&=\frac{1}{\phi^2(d)}\int_a^b[(\phi\circ W)(x)]^{\Delta}\left[\int_a^b K_{w,\phi}(x,t)g^{\Delta}(t)\Delta t\right]\left[\int_a^b K_{w,\phi}(x,t)f^{\Delta}(t)\Delta t\right]\Delta x.
\end{aligned}$$

By using properties of modulus

$$\begin{aligned}
&|T(f,g;\phi,W)|\\
&\le\frac{\|f^{\Delta}\|_{\infty;[a,b]}\|g^{\Delta}\|_{\infty;[a,b]}}{\phi^2(d)}\int_a^b\left|[(\phi\circ W)(x)]^{\Delta}\right|\left[\int_a^b|K_{w,\phi}(x,t)|\,\Delta t\right]^2\Delta x\\
&\le\frac{\|f^{\Delta}\|_{\infty;[a,b]}\|g^{\Delta}\|_{\infty;[a,b]}}{\phi^2(d)}\\
&\qquad\times\int_a^b|\Psi(x)|\;|w(x)|\left[\|w\|_{\infty;[a,b]}\int_a^b|\Psi(t)|\;\Delta t+|\phi(d)|(b-x)\right]^2\Delta x\\
&\le\frac{\|f^{\Delta}\|_{\infty;[a,b]}\|g^{\Delta}\|_{\infty;[a,b]}\|w\|_{\infty;[a,b]}}{\phi^2(d)}\\
&\qquad\times\int_a^b|\Psi(x)|\left[\|w\|_{\infty;[a,b]}\int_a^b|\Psi(t)|\;\Delta t+|\phi(d)|(b-x)\right]^2\Delta x
\end{aligned}$$

□

Corollary 4.1. (*continuous case*) *Let $\mathbb{T}_1 = \mathbb{R}$. In this case delta integral is the Riemann integral from calculus, then*

$$\begin{aligned}
&\Bigg| \int_a^b f(x)\ g(x)\ w(x)\ \phi'(W(x))\ dx - \frac{1}{\phi(d)} \left(\int_a^b f(x)\ w(x)\ \phi'(W(x))\ dx \right) \\
&\quad \times \left(\int_a^b\ w(t)\ \phi'(W(t))\ g(t)\ dt \right) - \frac{1}{\phi(d)} \left(\int_a^b g(x)\ w(x)\ \phi'(W(x))\ dx \right) \\
&\quad \times \left(\int_a^b\ w(t)\ \phi'(W(t))\ f(t)\ dt \right) + \frac{1}{\phi^2(d)} \left(\int_a^b g(t)\ w(t)\ \phi'(W(t))\ dt \right) \\
&\quad \times \left(\int_a^b\ w(t)\ \phi'(W(t))\ f(t)\ dt \right) \left(\int_a^b w(x)\ \phi'(W(x))\ dx \right) \Bigg| \\
&\leq \frac{\parallel f' \parallel_{\infty;[a,b]} \parallel g' \parallel_{\infty;[a,b]} \parallel w \parallel_{\infty;[a,b]}}{\phi^2(d)} \int_a^b |\phi'(W(x))|\ G^2(x)\ \Delta x,
\end{aligned}$$

where

$$G(x) = \|w\|_{\infty;[a,b]} \int_a^b |\phi'(W(t))|\ dt + |\phi(d)|(b-x).$$

Remark 4.1. For Theorems 3.1 and 4.1 the applications for quantum calculus can also be given.

References

[1] Bohner, M. and Lutz, D. A., Asymptotic behaviour of dynamic equations on time scales J. Differ. Equations Appl., 7(1) (2001) 21-50.

[2] Boukerrioua, K. and Guezane-Lakoud, A., On Generalization of Čebyšev type inequalities, J. Inequal. Pure and Appl. Math., 8 (2) 2007.

[3] Barnet, N. S., Ceron, P., Dragomir, S. S., Pinheiro, M. R. and Sofo, A., Ostrowski type inequalities for functions whose modulus of derivatives are convex and applications, RGMIA Res. Collec., 5 (2) (2002) 219-231.

[4] Cerone, P. and Dragomir, S.S., Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions, Demonstratio Math., 37 (2) (2004) 299-308.

[5] Čebyšev, P. L., Sur les expressions approximative des integrals par les auters prises entre les mêmes limites, Proc. Math. Soc. Charkov, 2 (1882), 93-98.

[6] Dragomir, S.S. and Rassias, Th. M., (Eds.), Ostrowski type inequalities and applications in Numerical integration, Kluwer Academic Publishers, Dordrect, 2002.

[7] Dragomir, S.S. and Sofo, A., Ostrowski type inequalities for functions whose derivatives are convex, Proceedind of the 4th International Conference on Modelling and Simulation, November 2002. Victoria University, Melbourne Australia, RGMIA Res. Rep. Collec., 5 (2002) Supp., Art. 30.

[8] Basak Karpuz and Umut Mutlu Özkan, Generalized ostrowski's inequality on time scales, J. Inequal. Pure and Appl. Math., 9 (4) 2008.

[9] Mitrinović, D. S., Pečarić, J. E. and Fink, A. M., Inequalities involving functions and their integrals and derivatives, Mathematics and its applications, Dordrecht, Kluwer Academic Publishers, Vol. 53, 1991.

[10] Mitrinović, D. S., Pečarić, J. E. and Fink, A. M., Classical and new inequalities in analysis, Kluwer Academic Publishers, Dordrecht, 1993.

[11] Ostrowski, A., Über die absolutabweichung einer differentierbaren funktion von ihren integralmittelwert, Comment. Math. Helv., 10 (1938), 226-227.

[12] Ozkan, U. M., Sarikaya M. Z., and Yildirim, H., Extensions of certain integral inequalities on time scales, *Applied Mathematics Letters*, 21 (10) (2008), 993–1000.

[13] Pečarić, J. E., On the Čebysev inequality, Bul. Sti. Tehn. Inst. Politehn. Tralan Vuia Timisora (Romania), 25 (39) (1) (1980), 5-9.

[14] Pachpatte, B. G., On Ostrowski-Grüss-Čebyšev type inequalities for functions whose modulus of derivatives are convex, JIPAM 6(4) (2005) 1-14.
[15] Pečarić, J. E., Proschan F. and Tang Y. L., Convex functions, Partial orderings and statistical Applications, Academic Press, New York, 1991.
[16] Sarikaya, M. Z., A Note on Grüss type inequalities on time scales, *Dynamic Systems and Applications*, 17 (2008), 663-666.
[17] Sarikaya, M. Z., On weighted Iyengar type inequalities on time scales, *Applied Mathematics Letters*, 22 (2009), 1340–1344.
[18] Sarikaya, M. Z., Aktan, N., and Yildirim, H., On Cebyšev–Grüss type inequalities on time scales, *Journal of Mathematical Inequalities*, Volume 2, Number 2 (2008), 185–195.

[1]Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan.
E-mail address: sabirhus@gmail.com

[2]Department of Mathematics, Faculty of Science, Jerash University, 26150 Jerash, Jordan.
E-mail address: mwomath@gmail.com

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SOME INEQUALITIES FOR DIFFERENTIABLE PREQUASIINVEX FUNCTIONS WITH APPLICATIONS

MUHAMMAD AMER LATIF

Abstract. In this paper, we present several inequalities of Hermite-Hadamard type for differentiable prequasiinvex functions.Our results generalize those results proved in [2] and hence generalize those given in [7], [11] and [23]. Applications of the obtained results are given as well.

1. Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated as (see [25]):

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I on real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave.

For several results which generalize, improve and extend the inequalities (1.1), we refer the interested reader to [7, 8, 9], [11]-[14], [23, 24], [27]-[32].

In [7], Dragomir and Agarwal obtained the following inequalities for differentiable functions which estimate the difference between the middle and the rightmost terms in (1.1):

Theorem 1.1. [7] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$, and $f' \in L(a,b)$. If $\left|f'\right|$ is convex function on $[a,b]$, then the*

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following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{8}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{1.2}$$

Theorem 1.2. [7] *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, where a, $b\in I$ with $a<b$,and $f'\in L(a,b)$. If $\left|f'\right|^{\frac{p}{p-1}}$ is convex function on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\left|f'(a)\right|^{\frac{p}{p-1}}+\left|f'(b)\right|^{\frac{p}{p-1}}\right], \tag{1.3}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

In [23], Pearce and J. Pečarić gave an improvement and simplification of the constant in Theorem 1.2 and consolidated this results with Theorem 1.1. The following is the main result from [23]:

Theorem 1.3. [23] *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, where a, $b\in I$ with $a<b$, and $f'\in L(a,b)$. If $\left|f'\right|^{q}$ is convex function on $[a,b]$, for some $q\geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left[\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right]^{\frac{1}{q}}. \tag{1.4}$$

If $\left|f'\right|^{q}$ is concave on $[a,b]$, for some $q\geq 1$. Then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{1.5}$$

Now, we recall that the notion of quasi-convex functions generalizes the notion of convex functions. More exactly, a function $f:[a,b]\to\mathbb{R}$ is said quasi-convex on $[a,b]$ if

$$f(tx+(1-t)y)\leq\max\{f(x),f(y)\}$$

for all $x;y\in[a;b]$ and $t\in[0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex

functions which are not convex, (see [11]).

Recently, Ion [11] introduced two inequalities of the right hand side of Hadamard's type for quasi-convex functions, as follows:

Theorem 1.4. [11] *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, where a, $b\in I$ with $a<b$. If $\left|f'\right|$ is quasi-convex function on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\} \tag{1.6}$$

Theorem 1.5. [11] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$. If $\left|f'\right|^p$ is quasi-convex function on $[a, b]$, for some $p > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx - \frac{1}{b-a}\int_a^b f(x)g\left(x\right)dx\right| \leq \frac{b-a}{2\left(p+1\right)^{\frac{1}{p}}}\left(\sup\left\{\left|f'\left(a\right)\right|^{\frac{p}{p-1}}, \left|f'\left(b\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}, \tag{1.7}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In [2], Alomari, Darus and Kirmaci established Hermite-Hadamard-type inequalities for quasi-convex functions which give refiments of those given above in Theorem 1.4 and Theorem 1.5.

Theorem 1.6. [2] *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$. If the mapping $\left|f'\right|$ is quasi-convex function on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[\sup\left\{\left|f'\left(a\right)\right|, \left|f'\left(\frac{a+b}{2}\right)\right|\right\} + \sup\left\{\left|f'\left(b\right)\right|, \left|f'\left(\frac{a+b}{2}\right)\right|\right\}\right]. \tag{1.8}$$

Theorem 1.7. [2] *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$. If $\left|f'\right|^p$ is convex function on $[a, b]$, for $p > 1$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{4\left(p+1\right)^{\frac{1}{p}}}\left[\left(\sup\left\{\left|f'\left(a\right)\right|^{\frac{p}{p-1}}, \left|f'\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right. \\ &\quad \left.+\left(\sup\left\{\left|f'\left(b\right)\right|^{\frac{p}{p-1}}, \left|f'\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right].\end{aligned} \tag{1.9}$$

Theorem 1.8. [2] *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$. If $\left|f'\right|^q$ is convex function on $[a, b]$, for $p > 1$, then the* *following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{1.10}$$
$$\leq \frac{b-a}{8}\left[\left(\sup\left\{\left|f^{'}(a)\right|^q,\left|f^{'}\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\sup\left\{\left|f^{'}(b)\right|^q,\left|f^{'}\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right].$$

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. These studies include among others the work of Hanson in [10], Ben-Israel and Mond [5], Pini [22], M.A.Noor [19, 20], Yang and Li [34] and Weir [33]. Mond [5], Weir [32] and Noor [18, 19], have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems. Hanson in [10], introduced invex functions as a significant generalization of convex functions. Ben-Israel and Mond [4], gave the concept of preinvex function which is special case of invexity. Pini [22], introduced the concept of prequasiinvex functions as a generalization of invex functions.

Let us recall some known results concerning preinvexity and prequasiinvexity.

Let K be a closed set in $\mathbb{R}^n$ and let $f:K\to\mathbb{R}$ and $\eta:K\times K\to\mathbb{R}$ be continuous functions. Let $x\in K$, then the set K is said to be invex at x with respect to $\eta(\cdot,\cdot)$, if

$$x+t\eta(y,x)\in K, \forall x,y\in K, t\in[0,1].$$

K is said to be an invex set with respect to η if K is invex at each $x\in K$. The invex set K is also called a η-connected set.

Definition 1.1. [33] The function f on the invex set K is said to be preinvex with respect to η, if

$$f(u+t\eta(v,u))\leq(1-t)f(u)+tf(v), \forall u,v\in K, t\in[0,1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(x,y)=x-y$ but the converse is not true see for instance [32].

Definition 1.2. [21] The function f on the invex set K is said to be prequasiinvex with respect to η, if

$$f(u+t\eta(v,u))\leq\max\{f(u),f(v)\}, \forall u,v\in K, t\in[0,1].$$

Also Every quasi-convex function is a prequasiinvex with respect to the map $\eta(v,u)$ but the converse does not hold, see for example [35].

In the recent paper, Noor [17] has obtained the following Hermite-Hadamard inequalities for the preinvex functions:

Theorem 1.9. [17]*Let* $f:[a,a+\eta(b,a)]\to(0,\infty)$ *be a preinvex function on the interval of the real numbers* K° *(the interior of* K*) and* a, $b\in K^{\circ}$ *with* $a<a+\eta(b,a)$. *Then the following inequality holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right)\leq\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\leq\frac{f(a)+f(b)}{2}.$$

Barani, Ghazanfari and Dragomir in [4], presented the following estimates of the right-side of a Hermite- Hadamard type inequality in which some preinvex functions are involved.

Theorem 1.10. [4] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $\left|f'\right|$ is preinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b,a)|}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{1.11}$$

Theorem 1.11. [4] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $\left|f'\right|^{\frac{p}{p-1}}$ is preinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b,a)|}{2(1+p)^{\frac{1}{p}}}\left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}}+\left|f'(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{1.12}$$

In [3], Barani, Ghazanfari and Dragomir gave similar results for quasi-preinvex functions as follows:

Theorem 1.12. [3] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $\left|f'\right|$ is prequasiinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b,a)|}{8}\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}. \tag{1.13}$$

Theorem 1.13. [3] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $\left|f'\right|^{\frac{p}{p-1}}$ is preinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b,a)|}{2(1+p)^{\frac{1}{p}}}\left(\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}. \tag{1.14}$$

For several new results on inequalities for preinvex functions we refer the interested reader to [4, 21, 26] and the references therein.

In the present paper we give new inequalities of Hermite-Hadamard for functions whose derivatives in absolute value are preinvex and prequasiinvex. Our results generalize those results presented in a very recent paper of Alomari, Darus and Kirmaci [2].

2. Main Results

The following Lemma is essential in establishing our main results in this section:

Lemma 2.1. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a,\, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f^{'} \in L([a, a+\eta(b,a)])$. Then the following equality holds:*

$$\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx = \frac{\eta(b,a)}{4}$$
$$\times \left[\int_0^1 (-t) f^{'}\left(a + \left(\frac{1-t}{2}\right)\eta(b,a)\right) dt + \int_0^1 t f^{'}\left(a + \left(\frac{1+t}{2}\right)\eta(b,a)\right) dt \right],$$

Proof. It suffices to note that

$$\begin{aligned}
I_1 &= \int_0^1 (-t) f^{'}\left(a + \left(\frac{1-t}{2}\right)\eta(b,a)\right) dt \\
&= \left. \frac{2(-t) f\left(a + \left(\frac{1-t}{2}\right)\eta(b,a)\right)}{-\eta(b,a)} \right|_0^1 - \frac{2}{\eta(b,a)} \int_0^1 f\left(a + \left(\frac{1-t}{2}\right)\eta(b,a)\right) dt \\
&= \frac{2f(a)}{\eta(b,a)} - \frac{2}{\eta(b,a)} \int_0^1 f\left(a + \left(\frac{1-t}{2}\right)\eta(b,a)\right) dt.
\end{aligned}$$

Setting $x = a + \left(\frac{1-t}{2}\right)\eta(b,a)$ and $dx = -\frac{\eta(b,a)}{2} dt$, which gives

$$I_1 = \frac{2f(a)}{\eta(b,a)} - \frac{4}{(\eta(b,a))^2} \int_a^{a+\frac{1}{2}\eta(b,a)} f(x)\,dx.$$

Similarly, we also have

$$I_2 = \frac{2f(a+\eta(b,a))}{\eta(b,a)} - \frac{4}{(\eta(b,a))^2} \int_{a+\frac{1}{2}\eta(b,a)}^{a+\eta(b,a)} f(x)\,dx.$$

Thus

$$\frac{\eta(b,a)}{4}[I_1 + I_2] = \frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx.$$

which is the required result. □

Remark 2.1. If we take $\eta(b,a) = b - a$, then Lemma 2.1 reduces to Lemma 2.1 from [2].

Now using Lemma 2.1, we shall propose some new upper bound for the right-hand side of Hadamard's inequality for prequasiinvex mappings, which is better than the inequality had done in [3]. our results generalize those results proved in [2] as well.

Theorem 2.1. *Let $K \subseteq [0,\infty)$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a,\, b \in K$ with $a < a+\eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $\left|f'\right|$ is prequasiinvex on K, then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \tag{2.1}$$
$$\leq \frac{\eta(b,a)}{8}\left[\sup\left\{\left|f'(a)\right|, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\} + \sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|, \left|f'(a+\eta(b,a))\right|\right\}\right].$$

Proof. From Lemma 2.1 and by using the prequasiinvex of $\left|f'\right|$ on K, for any $t \in [0,1]$ we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{\eta(b,a)}{4}\left[\int_0^1 t\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt + \int_0^1 t\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right]$$
$$\leq \frac{\eta(b,a)}{4}\left[\sup\left\{\left|f'(a)\right|, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}\int_0^1 tdt + \sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|, \left|f'(a+\eta(b,a))\right|\right\}\int_0^1 tdt\right]$$
$$= \frac{\eta(b,a)}{8}\left[\sup\left\{\left|f'(a)\right|, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\} + \sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|, \left|f'(a+\eta(b,a))\right|\right\}\right].$$

This completes the proof of the theorem. ☐

Corollary 2.1. *Let f be as in Theorem 2.1, if in addition*

(1) $\left|f'\right|$ *is increasing, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \tag{2.2}$$
$$\leq \frac{\eta(b,a)}{8}\left[\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right| + \left|f'(a+\eta(b,a))\right|\right]$$

(2) $\left|f'\right|$ *is decreasing, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \tag{2.3}$$
$$\leq \frac{\eta(b,a)}{8}\left[\left|f'(a)\right| + \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right]$$

Proof. The proof follows directly from Theorem 2.1. □

Remark 2.2. We note that the inequalities (2.2) and (2.3) are two new refinements of the trapezoid inequality for prequasiinvex functions, and thus for preinvex functions.

Remark 2.3. If we take $\eta(b,a) = b - a$ in Theorem 2.1, then the inequality reduces to the inequality (1.8). If we take $\eta(b,a) = b - a$ in corollary 2.1, then (2.2) and (2.3) reduce to the related corollary of Theorem 1.6 from [2].

Another similar result may be extended in the following theorem.

Theorem 2.2. *Let $K \subseteq [0,\infty)$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a,\, b \in K$ with $a < a+\eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f^{'} \in L([a, a+\eta(b,a)])$. If $\left|f^{'}\right|^{p}$ is prequasiinvex on K from some $p > 1$, then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \tag{2.4}$$
$$\leq \frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left(\sup\left\{\left|f^{'}(a)\right|^{\frac{p}{p-1}}, \left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right.$$
$$\left. + \sup\left\{\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}, \left|f^{'}(a+\eta(b,a))\right|^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\right].$$

Proof. From Lemma 2.1 and using the well known Hölder's inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \tag{2.5}$$
$$\leq \frac{\eta(b,a)}{4}\left[\int_0^1 t\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt + \int_0^1 t\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right]$$
$$\leq \frac{\eta(b,a)}{4}\left[\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

By the prequasiinvexity of $\left|f^{'}\right|^{p}$ on K from some $p > 1$, we have for every $a, b \in K$ with $\eta(b,a) > 0$ and $t \in [0,1]$ that

$$\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q \leq \sup\left\{\left|f^{'}(a)\right|^q, \left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}$$

and

$$\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q \leq \sup\left\{\left|f^{'}(a+\eta(b,a))\right|^q, \left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\},$$

where $\frac{1}{p}+\frac{1}{q} = 1$. Using the above inequalities in (2.5), we get the required result. This completes the proof of the theorem as well. □

Corollary 2.2. *Let f be as in Theorem 2.2, if in addition*

(1) $\left|f^{'}\right|^{\frac{p}{p-1}}$ *is increasing, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \leq \frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|+\left|f^{'}(a+\eta(b,a))\right|\right] \tag{2.6}$$

(2) $\left|f^{'}\right|^{\frac{p}{p-1}}$ *is decreasing, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \leq \frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left|f^{'}(a)\right|+\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right] \tag{2.7}$$

Proof. It is a direct consequence of Theorem 2.2. □

Remark 2.4. If we take $\eta(b,a)=b-a$ in Theorem 2.2, then the inequality reduces to the inequality (1.9). If we take $\eta(b,a)=b-a$ in corollary 2.2, then (2.6) and (2.7) reduce to the related corollary of Theorem 1.7 from [2].

An improvement of the constants in Theorem 2.2 and a consolidation of this result with Theorem 2.1 are given in the following theorem.

Theorem 2.3. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$ and $a,\, b\in K$ with $a<a+\eta(b,a)$ Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K such that $f^{'}\in L([a,a+\eta(b,a)])$. If $\left|f^{'}\right|^{q}$ for $q\geq 1$, is prequasiinvex on K, then for every $a,b\in K$ with $\eta(b,a)>0$ we have the following inequality:*

$$\begin{aligned}\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| &\\ \leq \frac{\eta(b,a)}{8}\left[\left(\sup\left\{\left|f^{'}(a)\right|^{q},\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right.&\\ \left.+\sup\left\{\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q},\left|f^{'}(a+\eta(b,a))\right|^{q}\right\}^{\frac{1}{q}}\right].&\end{aligned} \tag{2.8}$$

Proof. From Lemma 2.1, using the power-mean integral inequality and using the prequasiinvexity of $\left|f^{'}\right|^{q}$ on K for $q \geq 1$, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)\,dx\right| \qquad (2.9)\\
&\leq \frac{\eta(b,a)}{4}\left[\int_0^1 t\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt+\int_0^1 t\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right]\\
&\leq \frac{\eta(b,a)}{4}\left[\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]\\
&\leq \frac{\eta(b,a)}{8}\left[\left(\sup\left\{\left|f^{'}(a)\right|^{q},\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right.\\
&\quad\left.++\sup\left\{\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q},\left|f^{'}(a+\eta(b,a))\right|^{q}\right\}^{\frac{1}{q}}\right].
\end{aligned}
$$

which completes the proof □

Corollary 2.3. *Let f be as in Theorem 2.3, if in addition*

(1) $\left|f^{'}\right|^{\frac{1}{q}}$ *is increasing, then we have the inequality (2.2).*

(2) $\left|f^{'}\right|^{\frac{1}{q}}$ *is decreasing, then we have the inequality (2.3).*

Remark 2.5. If we take $\eta(b,a)=b-a$ in Theorem 2.3, then the inequality reduces to the inequality (1.10). If we take $\eta(b,a)=b-a$ in corollary 2.3, then we get the results of the related corollary of Theorem 1.8 from [2].

Remark 2.6. For $q=1$, (2.8) reduces to Theorem 2.1. For $q=\frac{p}{p-1}$ $(p>1)$ we have an improvement of the constants in Theorem 2.2, since $4^{p}>p+1$ if $p>1$ and accordingly

$$\frac{1}{8}<\frac{1}{(p+1)^{\frac{1}{p}}}.$$

3. Applications to Special Means

In what follows we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 3.1. [6]A function $M:\mathbb{R}_{+}^{2}\rightarrow\mathbb{R}_{+}$, is called a Mean function if it has the following properties:

(1) Homogeneity: $M(ax,ay)=aM(x,y)$, for all $a>0$,
(2) Symmetry : $M(x,y)=M(y,x)$,
(3) Reflexivity : $M(x,x)=x$,
(4) Monotonicity: If $x\leq x^{'}$ and $y\leq y^{'}$, then $M(x,y)\leq M(x^{'},y^{'})$,
(5) Internality: $\min\{x,y\}\leq M(x,y)\leq\max\{x,y\}$.

We consider some means for arbitrary positive real numbers α, β (see for instance [6]).

(1) The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}$$

(2) The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}$$

(3) The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}$$

(4) The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \ r \geq 1$$

(5) The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}}, & \alpha \neq \beta \\ \alpha, & \alpha = \beta \end{cases}$$

(6) The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\alpha - \beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|$$

(7) The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}, \ \alpha \neq \beta, \ p \in \mathbb{R}\backslash\{-1, 0\}.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$.

Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(a, b) : [a, a + \eta(b, a)] \times [a, a + \eta(b, a)] \to \mathbb{R}^+$, which is one of the above mentioned means, therefore one can obtain variant inequalities for these means as follows:

Setting $\eta(b, a) = M(b, a)$ in (2.1), (2.4) and (2.8), one can obtain the following interesting inequalities involving means:

$$\begin{aligned} &\left|\frac{f(a) + f(a + M(b,a))}{2} - \frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx\right| \\ &\leq \frac{M(b,a)}{8}\left[\sup\left\{\left|f'(a)\right|, \left|f'\left(a + \frac{1}{2}M(b,a)\right)\right|\right\}\right. \\ &\qquad \left. + \sup\left\{\left|f'\left(a + \frac{1}{2}M(b,a)\right)\right|, \left|f'(a + M(b,a))\right|\right\}\right]. \end{aligned} \tag{3.1}$$

$$\left|\frac{f(a)+f(a+M(b,a))}{2}-\frac{1}{M(b,a)}\int_a^{a+M(b,a)}f(x)\,dx\right| \tag{3.2}$$
$$\leq \frac{M(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left(\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'\left(a+\frac{1}{2}M(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right.$$
$$\left.+\sup\left\{\left|f'\left(a+\frac{1}{2}M(b,a)\right)\right|^{\frac{p}{p-1}},\left|f'(a+M(b,a))\right|^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\right],$$

for $p>1$, and

$$\left|\frac{f(a)+f(a+M(b,a))}{2}-\frac{1}{M(b,a)}\int_a^{a+M(b,a)}f(x)\,dx\right| \tag{3.3}$$
$$\leq \frac{M(b,a)}{8}\left[\left(\sup\left\{\left|f'(a)\right|^{q},\left|f'\left(a+\frac{1}{2}M(b,a)\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\sup\left\{\left|f'\left(a+\frac{1}{2}M(b,a)\right)\right|^{q},\left|f'(a+M(b,a))\right|^{q}\right\}^{\frac{1}{q}}\right],$$

for $q\geq 1$. Letting $M=A$, G, H, P_r, I, L, L_p in (3.1), (3.2) and (3.3), we can get the required inequalities, and the details are left to the interested reader.

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References

[1] T. Antczak, Mean value in invexity analysis, Nonl. Anal., 60 (2005), 1473-1484.
[2] M. Alomari , M. Darus, U.S. Kirmaci, Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, Computers and Mathematics with Applications 59 (2010) 225 232.
[3] A. Barani, A.G. Ghazanfari, S.S. Dragomir, Hermite-Hadamard inequality through prequsi-invex functions, RGMIA Research Report Collection, 14(2011), Article 48, 7 pp.
[4] A. Barani, A.G. Ghazanfari, S.S. Dragomir, Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex, RGMIA Research Report Collection, 14(2011), Article 64, 11 pp.
[5] A. Ben-Israel and B. Mond, What is invexity?, J. Austral. Math. Soc., Ser. B, 28(1986), No. 1, 1-9.
[6] P.S. Bullen, Handbook of Means and Their Inequalities, Kluwer Academic Publishers, Dordrecht, 2003.
[7] S. S. Dragomir, and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula, Appl. Math. Lett., 11(5)(1998), 91–95.
[8] S. S. Dragomir, Two mappings in connection to Hadamard's inequalities, J. Math. Anal. Appl., 167(1992), 42–56.
[9] D. -Y. Hwang, Some inequalities for differentiable convex mapping with application to weighted trapezoidal formula and higher moments of random variables, Appl. Math. Comp., 217(23)(2011), 9598-9605.
[10] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl. 80 (1981) 545-550.

[11] D. A. Ion, Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, Annals of University of Craiova, Math. Comp. Sci. Ser. 34 (2007) 82 87.
[12] U. S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 147(1)(2004), 137-146.
[13] U. S. Kırmacı and M. E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 153(2)(2004), 361-368.
[14] K. C. Lee and K. L. Tseng, On a weighted generalization of Hadamard's inequality for G-convex functions, Tamsui-Oxford J. Math. Sci., 16(1)(2000), 91–104.
[15] A. Lupas, A generalization of Hadamard's inequality for convex functions, Univ. Beograd. Publ. Elek. Fak. Ser. Mat. Fiz., 544-576(1976), 115–121.
[16] S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl. 189 (1995), 901–908.
[17] M. Aslam Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, Preprint.
[18] M. A. Noor, Variational-like inequalities, Optimization, 30 (1994), 323–330.
[19] M. A. Noor, Invex equilibrium problems, J. Math. Anal. Appl., 302 (2005), 463–475.
[20] M. A. Noor, Some new classes of nonconvex functions, Nonl. Funct. Anal. Appl.,11(2006),165-171
[21] M. A. Noor, On Hadamard integral inequalities involving two log-preinvex functions, J. Inequal. Pure Appl. Math., 8(2007), No. 3, 1-14.
[22] R. Pini, Invexity and generalized convexity, Optimization 22 (1991) 513-525.
[23] C. E. M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13(2)(2000), 51–55.
[24] F. Qi, Z. -L.Wei and Q. Yang, Generalizations and refinements of Hermite-Hadamard's inequality, Rocky Mountain J. Math., 35(2005), 235–251.
[25] J. Pečarić, F. Proschan and Y. L. Tong, Convex functions, partial ordering and statistical applications, Academic Press, New York, 1991.
[26] M. Z. Sarikaya, H. Bozkurt and N. Alp, On Hermite-Hadamard type integral inequalities for preinvex and log-preinvex functions, arXiv:1203.4759v1.
[27] M. Z. Sarıkaya and N. Aktan, On the generalization some integral inequalities and their applications Mathematical and Computer Modelling, 54(9-10)(2011), 2175-2182.
[28] M. Z. Sarikaya, M. Avci and H. Kavurmaci, On some inequalities of Hermite-Hadamard type for convex functions, ICMS International Conference on Mathematical Science, AIP Conference Proceedings 1309, 852(2010).
[29] M. Z. Sarikaya, O new Hermite-Hadamard Fejér type integral inequalities, Stud. Univ. Babeş-Bolyai Math. 57(2012), No. 3, 377-386.
[30] A. Saglam, M. Z. Sarikaya and H. Yıldırım and, Some new inequalities of Hermite-Hadamard's type, Kyungpook Mathematical Journal, 50(2010), 399-410.
[31] C. -L. Wang and X. -H. Wang, On an extension of Hadamard inequality for convex functions, Chin. Ann. Math., 3(1982), 567–570.
[32] S. -H. Wu , On the weighted generalization of the Hermite-Hadamard inequality and its applications, The Rocky Mountain J. of Math., 39(2009), no. 5, 1741–1749.
[33] T. Weir, and B. Mond, Preinvex functions in multiple bjective optimization, Journal of Mathematical Analysis and Applications, 136 (1998) 29-38.
[34] X. M. Yang and D. Li, On properties of preinvex functions, J. Math. Anal. Appl. 256 (2001), 229-241.
[35] X.M. Yang, X.Q. Yang and K.L. Teo, Characterizations and applications of prequasiinvex functions, properties of preinvex functions, J. Optim. Theo. Appl. 110 (2001) 645-668.
[36] X. M. Yang, X. Q. Yang, K.L. Teo, Generalized invexity and generalized invariant monotonocity, Journal of Optimization Theory and Applications 117 (2003) 607-625.

College of Science, Department of Mathematics,, University of Hail, Hail 2440, Saudi Arabia
E-mail address: m_amer_latif@hotmail.com

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ON AN OPEN PROBLEM BY B. SROYSANG

JULIUS FERGY T. RABAGO

Abstract. In this short note, we answer an open problem posed by B. Sroysang [1]. More precisely, we find all solutions of the Diophantine equation $8^x + 17^y = z^2$ where x, y and z are non-negative integers.

1. Introduction

In a recent paper [1], B. Sroysang showed that (1, 0, 3) is a unique solution (x, y, z) to the Diophantine equation $8^x + 19^y = z^2$ where x, y and z are non-negative integers. His findings contradicts the result suggested by Peker and Cenberci in [2]: the Diophantine equation $8^x + 19^y = z^2$ has no non-negative integer solution. Also, in the end of his paper, Sroysang [1] posed the question "What is the set of all solutions (x, y, z) for the Diophantine equation $8^x + 17^y = z^2$ where x, y and z are non-negative integers?". In this short note, we answer this question of Sroysang.

2. Main Results

We begin this section by stating Catalan's conjecture and proving a helpul Lemma.

Proposition 2.1. [2] *The solution to the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $min\{a, b, x, y\} > 1$ is unique and is given by $(a, b, x, y) = (3, 2, 2, 3)$.*

Lemma 2.1. *Let x and z be non-negative integers. Then, the solutions (x, z) to the Diophantine equation $8^x + 17 = z^2$ are $(1, 5), (2, 9)$ and $(3, 23)$.*

Proof. The case $x = 0$ and $z = 0$ are obvious. So we only consider the case when $x, z > 0$. We note that $1 \equiv 8^x + 17 \equiv z^2 \pmod 4$. So, z is either of the form $4k + 1$ or $4k + 3$, $k = 0$ or a natural number. Hence, we have the following cases.

Case 1. $z = 4k + 1$. If $z = 4k + 1$ then we have $8^x + 17 = (4k + 1)^2 = 16k^2 + 8k + 1$. So, $8^x + 16 = 16k^2 + 8k$ and this implies that $8^{x-1} + 2 = 2k^2 + k$.

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Thus, $(2^{x-1})^3 + 1 = 2k^2 + k - 1$. Expressing both sides as product of their prime factors, we have $(2^{x-1} + 1)((2^{x-1})^2 - 2^{x-1} + 1) = (2k - 1)(k + 1)$. Therefore, we have two possibilities.

$$\begin{cases} 2^{x-1} + 1 &= k + 1, \\ (2^{x-1})^2 - 2^{x-1} + 1 &= 2k - 1 \end{cases}$$

or

$$\begin{cases} 2^{x-1} + 1 &= 2k - 1, \\ (2^{x-1})^2 - 2^{x-1} + 1 &= k + 1. \end{cases}$$

For the first set of equalities, we have $2^{x-1} + 1 = k + 1$ implies that $k = 2^{x-1}$. So, $(2^{x-1})^2 - 2^{x-1} + 1 = 2k - 1 = 2(2^{x-1}) - 1$. Hence, $(2^{x-1})^2 - 3(2^{x-1}) + 2 = 0$, which is a quadratic equation and is factorable. In particular, $(2^{x-1} - 1)(2^{x-1} - 2) = 0$. Here we'll obtain, $2^{x-1} = 1$ and $2^{x-1} = 2$. This gives us the values $x = 1$ and $x = 2$, respectively. This follows that $k = 1$ and $k = 2$. For $k = 1$, we have $(x, z) = (1, 5)$ and, for $k = 2$, we have $(x, z) = (2, 9)$. On the otherhand, it could be verified easily that the second set of equalities will give us the solution $(x, z) = (2, 9)$.

Case 2. $z = 4k + 3$. If $z = 4k + 3$ then $8^x + 17 = (4k + 3)^2 = 16k^2 + 24k + 9$. Hence, $8^x + 8 = 16k^2 + 24k$ and this implies that $8^{x-1} + 1 = 2k^2 + 3k$. Therefore, $(2^{x-1})^3 + 1 = (2^{x-1} + 1)((2^{x-1})^2 - 2^{x-1} + 1) = k(2k + 3)$. So, we have the following equalities

$$\begin{cases} 2^{x-1} + 1 &= k, \\ (2^{x-1})^2 - 2^{x-1} + 1 &= 2k + 3. \end{cases}$$

Eliminating k we have, $(2^{x-1})^2 - 2^{x-1} + 1 = 2(2^{x-1} + 1) + 3 = 2(2^{x-1}) + 5$. Here we obtain the quadratic equation $(2^{x-1})^2 - 3(2^{x-1}) - 4 = 0$ which is equivalent to $(2^{x-1} + 1)(2^{x-1} - 4) = 0$. Solving for zeros, we have $2^{x-1} = -1$, which is impossible and $2^{x-1} = 4$, which is true for $x = 3$. This gives us the value $k = 5$. Thus, we have the solution $(x, y) = (3, 23)$. This completes the proof of the theorem. □

Theorem 2.1. *The only solutions to the Diophantine equation* $8^x + 17^y = z^2$ *in non-negative integers are given by* $(x, y, z) \in \{(1, 0, 3), (1, 1, 5), (2, 1, 9), (3, 1, 23)\}$.

Proof. The case when $z = 0$ is obvious so we only consider the following cases.

Case 1. $x = 0$. Suppose $8^x + 17^y = z^2$ is possible in non-negative integers for $x = 0$ then $z^2 - 1 = (z + 1)(z - 1) = 17^y$. So, $2 = (z + 1) - (z - 1) = 17^\beta - 17^\alpha$, where $\alpha + \beta = y$ and $\alpha < \beta$. It follows that $7^\alpha(7^{\beta-\alpha} - 1) = 2$. Hence, $7^\alpha = 1$ which is true for $\alpha = 0$. Thus, $7^\beta = 3$, a contradiction. Therefore, $17^y + 1 = z^2$ is not possible in non-negative integers.

Case 2. $y = 0$. If $y = 0$ we have $z^2 - 1 = (z + 1)(z - 1) = 2^{3x}$. Then, $2 = (z+1) - (z-1) = 2^\beta - 2^\alpha$, where $\alpha + \beta = 3x$ and $\alpha < \beta$. So, $2^{\alpha-1}(2^{\beta-\alpha} - 1) = 1$. Here we obtain $\alpha = 1$ and $2^{\beta-1} = 2$, which is true for $\beta = 2$. Thus, $x = 1$ and $z = 3$. Therefore, we have the solution $(x, y, z) = (1, 0, 3)$ to the Diophantine equation $8^x + 17^y = z^2$.

Case 3. $x, y, z > 0$. First note that for $y = 1$, Lemma 2.2 implies the following solutions $(x, y, z) = (1, 1, 5), (2, 1, 9), (3, 1, 23)$. Now suppose that $8^x + 17^y = z^2$ is possible in non-negative integers x, y, z for $y > 1$. We consider two sub-cases.

Subcase 3.1 y is even. If we let y be even, *i.e.* $y = 2n$ for some natural number n, then $z^2 - (17^n)^2 = 2^{3x}$. So, $(z + 17^n) - (z - 17^n) = 2^\beta - 2^\alpha$, again, $\alpha + \beta = 3x$ and $\alpha < \beta$. Hence, $2^{\alpha-1}(2^{\beta-\alpha} - 1) = 17^n$. This gives us a value $\alpha = 1$. It follows that, $2^{\beta-1} - 17^n = 1$, a contradiction to Catalan's conjecture. Thus, $8^x + 17^y = z^2$, for y even is impossible in positive integers.

Subcase 3.2 y is odd. If $y = 2n + 1$ then we have $8^x + 17^{2k+1} = z^2$. Since $1 \equiv 8^x + 17^{2k+1} \equiv z^2 \pmod 4$ then, either $z = 4k + 1$ or $z = 4k + 3$, where $k = 0$ or a natural number. Hence, we have the following

$$\begin{cases} 8^{x-1} + 1 = 2k^2 + k - \left(\dfrac{17^{2n+1} - 9}{8}\right), & \text{for } z = 4k + 1 \quad (1), \\ \\ 8^{x-1} + 1 = 2k^2 + 3k - \left(\dfrac{17^{2n+1} - 17}{8}\right), & \text{for } z = 4k + 3 \quad (2). \end{cases}$$

Take note that k is an integer. Hence, the RHS of (1) must be factorable. That is,

$$(1)^2 - (4)(2)\left(-\frac{17^{2n+1} - 9}{8}\right) = m^2,$$

for some non-negative integer m. Then, $17^{2n+1} - 8 = m^2$. Adding both sides by -9 we obtain $17(17^{2n} - 1) = (z + 3)(z - 3) = z^2 - 9$. This gives us a value $z = 20$. Thus, $17^{2n} = 24$ which is a contradiction. On the otherhand, the RHS of (2) must also be an integer, more precisely

$$(3)^2 - (4)(2)\left(-\frac{17^{2n+1} - 17}{8}\right) = m^2,$$

where m is a non-negative integer. Hence, $17^{2n+1} - 8 = m^2$, which is again a contradiction. Thus, $8^x + 17^y = z^2$, for y odd is not solvable in positive integers. This proves the theorem. $\square$

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References

[1] Sroysang, B., More on the Diophantine Equation $8^x + 19^y = z^2$, Int. J. of Pure App. Math. Vol:81, No.4 (2012), 601-604.

[2] Peker, B., Cenberci, S., On the Diophantine equations of $(2^n)^x + p^y = z^2$ type, Amer. J. Math. Sci. Vol:1, (2012), 195-199.

[3] Mihailescu, P., Primary cycolotomic units and a proof of Catalan's conjecture, J. Reine Angew. Math. Vol:27, (2004), 167-195.

Central Luzon State University, College of Arts and Sciences, Department of Mathematics and Physics, Science City of Muñoz 3120, Nueva Ecija, Philippines

E-mail address: `julius_fergy.rabago@up.edu.ph`

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ON THE HADAMARD'S TYPE INEQUALITIES FOR L-LIPSCHITZIAN MAPPING

MEHMET ZEKI SARIKAYA AND HATICE YALDIZ

Abstract. In this paper, we establish some new inequalities of Hadamard's type for L-Lipschitzian mapping in two variables.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. the following double inequality is well known in the literature as the Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Let us now consider a bidemensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x,y) + (1-t)f(z,w)$$

holds, for all $(x,y), (z,w) \in \Delta$ and $t \in [0,1]$ .A function $f : \Delta \to \mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b]$ and $y \in [c,d]$ (see [3]).

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1.1. A function $f : \Delta \to \mathbb{R}$ will be called co-ordinated canvex on Δ, for all $t, s \in [0,1]$ and $(x,y), (u,w) \in \Delta$,if the following inequality holds:

$$\begin{aligned} & f(tx + (1-t)y, su + (1-s)w) \\ \leq\ & tsf(x,u) + s(1-t)f(y,u) + t(1-s)f(x,w) + (1-t)(1-s)f(y,w). \end{aligned} \tag{1.1}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex, (see, [3]). For several recent results concerning Hermite-Hadamard's inequality for some convex function on the

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co-ordinates on a rectangle from the plane $\mathbb{R}^2$, we refer the reader to ([1]-[3], [5], [6], [8], [9] and [11]).

In [3], Dragomir establish the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1.1. *Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
\leq\ & \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right. \\
& \left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

The above inequalities are sharp.

Definition 1.2. Consider a function $f:V\to\mathbb{R}$ defined on a subset V of $\mathbb{R}^n$, $n\in\mathbb{N}$. Let $L=(L_1,L_2,...,L_n)$ where $L_i\geq 0,\ i=1,2,...,n$. We say that f is L-Lipschitzian function if

$$|f(x)-f(y)|\leq\sum_{i=1}^n L\,|x_i-y_i|$$

for all $x,y\in V$.

For several recent results concerning Hadamard's type inequality for some L-Lipschitzian function, we refer the reader to ([4], [7], [10]).

The main purpose of this paper is to establish some Hadamard's type ineqaulities for L-Lipschitzian mapping in two variables.

2. Hadamard's Type Inequalities

Firstly, we will start the proof of the Theorem 1.1 by using the definition of the co-ordinated convex functions as follows:

Theorem 2.1. *Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}\tag{2.1}$$

Proof. According to (1.1) with $x = t_1a + (1 - t_1)b,\ y = (1 - t_1)a + t_1b,\ u = s_1c + (1 - s_1)d,\ w = (1 - s_1)c + s_1d$ and $t = s = \frac{1}{2}$, we find that

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq\ & \frac{1}{4}\left[f(t_1a + (1-t_1)b, s_1c + (1-s_1)d) + f((1-t_1)a + t_1b, s_1c + (1-s_1)d)\right. \\ & \left. + f(t_1a + (1-t_1)b, (1-s_1)c + s_1d) + f((1-t_1)a + t_1b, (1-s_1)c + s_1d)\right]. \end{aligned}$$

Thus, by integrating with respect to t_1, s_1 on $[0,1] \times [0,1]$, we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq\ & \frac{1}{4}\left[\int_0^1\int_0^1 \left[f(t_1a + (1-t_1)b, s_1c + (1-s_1)d) + f((1-t_1)a + t_1b, s_1c + (1-s_1)d)\right] ds_1dt_1\right. \\ & \left. + \int_0^1\int_0^1 \left[f(t_1a + (1-t_1)b, (1-s_1)c + s_1d) + f((1-t_1)a + t_1b, (1-s_1)c + s_1d)\right] ds_1dt_1\right]. \end{aligned}$$

Using the change of the variable, we get

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx, \tag{2.2}$$

which the first inequality is proved. The proof of the second inequality follows by using (1.1) with $x = a,\ y = b,\ u = c$ and $w = d$, and integrating with respect to t, s over $[0,1] \times [0,1]$,

$$\begin{aligned} & \int_0^1\int_0^1 f(ta + (1-t)b, sc + (1-s)d)\,dsdt \\ \leq\ & \int_0^1\int_0^1 \left[tsf(a,c) + s(1-t)f(b,c) + t(1-s)f(a,d) + (1-t)(1-s)f(b,d)\right] dsdt \\ =\ & \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \end{aligned}$$

Here, using the change of the variable $x = ta + (1-t)b$ and $y = sc + (1-s)d$ for $s, t \in [0,1]$, we have

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \tag{2.3}$$

We get the inequality (2.1) from (2.2) and (2.3). The proof is complete. □

Theorem 2.2. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ satisfy L-Lipschitzian conditions. That is, for (t_1, s_1) and (t_2, s_2) belong to $\Delta := [a,b] \times [c,d]$, then we have*

$$|f(t_1,s_1) - f(t_2,s_2)| \leq L_1|t_1 - t_2| + L_2|s_1 - s_2|$$

where L_1 and L_2 are positive constants. Then, we have the following inequalities:

$$\left|f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right| \leq \frac{1}{16}\left(M_1|b-a| + M_2|d-c|\right) \tag{2.4}$$

$$\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dydx \right| \tag{2.5}$$
$$\leq \frac{1}{12}\left(M_1 |b-a| + M_2 |d-c|\right)$$

where $M_1 = [L_1 + L_3 + L_5 + L_7]$ *and* $M_2 = [L_2 + L_4 + L_6 + L_8]$.

Proof. Let $t, s \in [0,1]$. Since $ts + s(1-t) + t(1-s) + (1-t)(1-s) = 1$, then we have

$$\begin{aligned}
& |tsf(a,c) + s(1-t)f(b,c) + t(1-s)f(a,d) + (1-t)(1-s)f(b,d) \\
& -f(ta+(1-t)b, sc+(1-s)d)| \\
= & |ts[f(a,c) - f(ta+(1-t)b, sc+(1-s)d)] \\
& +s(1-t)[f(b,c) - f(ta+(1-t)b, sc+(1-s)d)] \\
& +t(1-s)[f(a,d) - f(ta+(1-t)b, sc+(1-s)d)] \\
& +(1-t)(1-s)[f(b,d) - f(ta+(1-t)b, sc+(1-s)d)]| \\
\leq & ts[(1-t)L_1|b-a| + (1-s)L_2|d-c|] + s(1-t)[tL_3|b-a| + (1-s)L_4|d-c|] \\
& +t(1-s)[(1-t)L_5|b-a| + sL_6|d-c|] + (1-t)(1-s)[tL_7|b-a| + sL_8|d-c|] \\
= & (ts(1-t)[L_1+L_3] + t(1-s)(1-t)[L_5+L_7])|b-a| \\
& + (ts(1-s)[L_2+L_6] + s(1-s)(1-t)[L_4+L_8])|d-c|.
\end{aligned} \tag{2.6}$$

If we choose $t = s = \frac{1}{2}$ in (2.6), we get

$$\begin{aligned}
& \left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right| \\
\leq & \frac{1}{8}\left([L_1+L_3+L_5+L_7]|b-a| + [L_2+L_6+L_4+L_8]|d-c|\right).
\end{aligned} \tag{2.7}$$

Thus, if we put $ta+(1-t)b$ instead of a, $(1-t)a+tb$ instead of b, $sc+(1-s)d$ instead of c and $(1-s)c+sd$ instead of d in (2.7), respectively, then it follows that

$$\begin{aligned}
&\left|\frac{f(ta+(1-t)b,sc+(1-s)d)+f(ta+(1-t)b,(1-s)c+sd)}{4}\right. \\
&+\frac{f((1-t)a+tb,sc+(1-s)d)+f((1-t)a+tb,(1-s)c+sd)}{4} \\
&\left.-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right| \\
\leq\ &\frac{1}{8}\left([L_1+L_3+L_5+L_7]\,|1-2t|\,|b-a|+[L_2+L_6+L_4+L_8]\,|1-2s|\,|d-c|\right)
\end{aligned} \tag{2.8}$$

for all $t,s\in[0,1]$. If we integrate the inequality (2.8) with respect to s,t on $[0,1]\times[0,1]$

$$\begin{aligned}
&\left|\frac{1}{4}\int_0^1\int_0^1\left[f(ta+(1-t)b,sc+(1-s)d)+f(ta+(1-t)b,(1-s)c+sd)\right]dsdt\right. \\
&+\frac{1}{4}\int_0^1\int_0^1\left[f((1-t)a+tb,sc+(1-s)d)+f((1-t)a+tb,(1-s)c+sd)\right]dsdt \\
&\left.-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right| \\
\leq\ &\frac{1}{8}\left\{[L_1+L_3+L_5+L_7]\,|b-a|\int_0^1\int_0^1|1-2t|\,dsdt\right. \\
&\left.+[L_2+L_6+L_4+L_8]\,|d-c|\int_0^1\int_0^1|1-2s|\,dsdt\right\}.
\end{aligned}$$

Thus, using the change of the variable $x=ta+(1-t)b$, $y=(1-t)a+tb$, $u=sc+(1-s)d$ and $w=(1-s)c+sd$ for $t,s\in[0,1]$, and

$$\int_0^1\int_0^1|1-2t|\,dsdt=\int_0^1\int_0^1|1-2s|\,dsdt=\frac{1}{2}$$

we obtain the inequality (2.4).

Note that, by the inequality (2.6), we write

$$\begin{aligned}
&|tsf(a,c)+s(1-t)f(b,c)+t(1-s)f(a,d)+(1-t)(1-s)f(b,d) \\
&-f(ta+(1-t)b,sc+(1-s)d)| \\
\leq\ &(ts(1-t)\,[L_1+L_3]+t(1-s)(1-t)\,[L_5+L_7])\,|b-a| \\
&+(ts(1-s)\,[L_2+L_6]+s(1-s)(1-t)\,[L_4+L_8])\,|d-c|\,.
\end{aligned} \tag{2.9}$$

for all $t, s \in [0,1]$. If we integrate the inequality (2.9) with respect to s, t on $[0,1] \times [0,1]$, we have

$$\begin{aligned}
&\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dydx \right| \\
\leq\ & \frac{1}{12}\left([L_1+L_3+L_5+L_7]\,|b-a| + [L_2+L_6+L_4+L_8]\,|d-c| \right)
\end{aligned}$$

and so we have the inequality (2.5), where we use the fact that

$$\int_0^1 \int_0^1 st(1-t)dsdt = \int_0^1 \int_0^1 s(1-s)(1-t)dsdt = \frac{1}{12}.$$

This completes the proof. $\square$

3. The Mapping H

For a L-Lipschitzian function $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$, we can define a mapping $H : [0,1] \times [0,1] \to \mathbb{R}$ by

$$H(t,s) := \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2} \right) dydx.$$

Now, we give some properties of this mapping as follows:

Theorem 3.1. *Suppose that $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be L-Lipschitzian on $\Delta := [a,b] \times [c,d]$. Then:*

(i) The mapping H is L-Lipschitzian on $[0,1] \times [0,1]$.
(ii) We have the following inequalities

$$\left| H(t,s) - f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right| \leq \frac{L_1 t}{4}(b-a) + \frac{L_2 s}{4}(d-c) \tag{3.1}$$

$$\left| H(t,s) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dydx \right| \leq \frac{L_1(1-t)}{4}(b-a) + \frac{L_2(1-s)}{4}(d-c). \tag{3.2}$$

Proof. *(i)* Let $t_1, t_2, s_1, s_2 \in [0,1]$. Then, we have

$$\begin{aligned}
& |H(t_2,s_2) - H(t_1,s_1)| \\
= & \frac{1}{(b-a)(d-c)} \left| \int_a^b \int_c^d f\left(t_2 x + (1-t_2)\frac{a+b}{2}, s_2 y + (1-s_2)\frac{c+d}{2}\right) dydx \right. \\
& \left. - \int_a^b \int_c^d f\left(t_1 x + (1-t_1)\frac{a+b}{2}, s_1 y + (1-s_1)\frac{c+d}{2}\right) dydx \right| \\
\leq & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left| f\left(t_2 x + (1-t_2)\frac{a+b}{2}, s_2 y + (1-s_2)\frac{c+d}{2}\right) \right. \\
& \left. - f\left(t_1 x + (1-t_1)\frac{a+b}{2}, s_1 y + (1-s_1)\frac{c+d}{2}\right) dydx \right| \\
= & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left[L_1 |t_2 - t_1| \left| x - \frac{a+b}{2} \right| + L_2 |s_2 - s_1| \left| y - \frac{c+d}{2} \right| \right] dydx \\
= & \frac{L_1(b-a)}{4} |t_2 - t_1| + \frac{L_2(d-c)}{4} |s_2 - s_1|,
\end{aligned}$$

i.e., for all $t_1, t_2, s_1, s_2 \in [0,1]$,

$$|H(t_2,s_2) - H(t_1,s_1)| \leq \frac{L_1(b-a)}{4} |t_2 - t_1| + \frac{L_2(d-c)}{4} |s_2 - s_1|, \tag{3.3}$$

which yields that the mapping H is L-Lipschitzian on $[0,1] \times [0,1]$.

(ii) The inequalities (3.1) and (3.2) follow from (3.3) by choosing $t_1 = 0,\ t_2 = t,\ s_1 = 0,\ s_2 = s$ and $t_1 = 1,\ t_2 = t,\ s_1 = 1,\ s_2 = s$, respectively. □

Another result which is connected in a sense with the inequality (2.5) is also given in the following:

Theorem 3.2. *Under the assumptions Theorem 3.1, then we get the following inequality*

$$\begin{aligned}
& \left| \frac{f\left(at + (1-t)\frac{a+b}{2}, cs + (1-s)\frac{c+d}{2}\right) + f\left(at + (1-t)\frac{a+b}{2}, ds + (1-s)\frac{c+d}{2}\right)}{4} \right. \\
& + \frac{f\left(bt + (1-t)\frac{a+b}{2}, cs + (1-s)\frac{c+d}{2}\right) + f\left(bt + (1-t)\frac{a+b}{2}, ds + (1-s)\frac{c+d}{2}\right)}{4} \\
& \left. - \frac{1}{(n_2-n_1)(m_2-m_1)} \int_{n_1}^{n_2} \int_{m_1}^{m_2} f(u,w) dwdu \right| \\
\leq & \frac{1}{12} \left(M_1 |n_2 - n_1| t + M_2 |m_2 - m_1| s \right)
\end{aligned} \tag{3.4}$$

where $M_1 = [L_1 + L_3 + L_5 + L_7]$ *and* $M_2 = [L_2 + L_4 + L_6 + L_8]$.

Proof. If we denote $n_1 = at + (1-t)\frac{a+b}{2}$, $n_2 = bt + (1-t)\frac{a+b}{2}$, $m_1 = cs + (1-s)\frac{c+d}{2}$ and $m_2 = ds + (1-s)\frac{c+d}{2}$, then, we have

$$H(t,s) = \frac{1}{(n_2-n_1)(m_2-m_1)} \int_{n_1}^{n_2} \int_{m_1}^{m_2} f(u,w)dwdu.$$

Now, using the inequality (2.5) applied for n_1, n_2, m_1 and m_2, we have

$$\begin{aligned} & \left| \frac{f(n_1,m_1) + f(n_1,m_2) + f(n_2,m_1) + f(n_2,m_2)}{4} \right. \\ & \left. - \frac{1}{(n_2-n_1)(m_2-m_1)} \int_{n_1}^{n_2} \int_{m_1}^{m_2} f(u,w)dwdu \right| \\ \leq & \ \frac{1}{12} \left(M_1 |n_2 - n_1| + M_2 |m_2 - m_1| \right) \end{aligned}$$

from which we have the inequality (3.4). This completes the proof. □

References

[1] M. Alomari and M. Darus, *Co-ordinated s-convex function in the first sense with some Hadamard-type inequalities*, Int. J. Contemp. Math. Sciences, 3 (32) (2008), 1557-1567.

[2] M. Alomari and M. Darus, *On the Hadamard's inequality for log-convex functions on the coordinates*, J. of Inequal. and Appl, Article ID 283147, (2009), 13 pages.

[3] S.S. Dragomir, *On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwanese Journal of Mathematics, 4 (2001), 775-788.

[4] S.S. Dragomir, Y. J. Cho and S. S. Kim, *Inequalities of Hadamard's type for Lipschitzian mappings and their applications*, J. Math. Anal. Appl., 245 (2000), 489-501.

[5] M. A. Latif and M. Alomari, *Hadamard-type inequalities for product two convex functions on the co-ordinetes*, Int. Math. Forum, 4(47), 2009, 2327-2338.

[6] M. A. Latif and M. Alomari, *On the Hadamard-type inequalities for h-convex functions on the co-ordinetes*, Int. J. of Math. Analysis, 3(33), 2009, 1645-1656.

[7] M. Matic and J. Pecaric, *On inequalities of Hadamard's type for Lipschizian mappings*, Tamkang J. Math. 32, 2 (2001), 127-130.

[8] M.E. Özdemir, E. Set and M.Z. Sarikaya, *New some Hadamard's type inequalities for co-ordinated m-convex and (α, m)-convex functions*, Hacettepe Journal of Mathematics and Statistics , 40(2), 2011, 219-229.

[9] F. Qi, Z.-L. Wei and Q. Yang, *Generalizations and refinements of Hermite–Hadamard's inequality*, Rocky Mountain J. Math. 35 (2005), pp. 235–251.

[10] G. S. Yang and K. L. Tseng, *Inequalities of Hadamard's Type for Lipschizian Mappings*, J. Math. Anal. Appl., 260 (2001), 230-238.

[11] S.-H. Wu, *On the weighted generalization of the Hermite-Hadamard inequality and its applications*, The Rocky Mountain J. of Math., vol. 39, no. 5, pp. 1741–1749, 2009.

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

E-mail address: sarikayamz@gmail.com

E-mail address: yaldizhatice@gmail.com

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THE RELATION BETWEEN ADDING MACHINE AND $p-$ADIC INTEGERS

MUSTAFA SALTAN

Abstract. In this paper, we equip $Aut(X^*)$ with a natural metric and give an elementary proof that the closure of the adding machine group, a subgroup of the automorphism group, is both isometric and isomorphic to the group of $p-$adic integers. This also shows that the group of $p-$adic integers can be isometrically embedded into the metric space $Aut(X^*)$.

1. Introduction

The adding machine group is one of the most important examples of self-similar automorphism groups of the rooted tree X^* ([2], [5], [7]). In this paper, we denote this group by A. A is a cyclic group generated by

$$a = (\underbrace{1, 1, \ldots, 1}_{p-1 \text{ times}}, a)\sigma$$

where a is an automorphism of the $p-$ary rooted tree and $\sigma = (012\ldots(p-1))$ is a permutation in S_p on $X = \{0, 1, 2, \ldots, (p-1)\}$. Since A is a infinite cyclic group, it is isomorphic to $\mathbb{Z}$. On the other hand, one can consider the automorphism a as adding one to a $p-$adic integer. This is a reason of the term adding machine introduced in [3]. In [6], a $p-$adic integer is pictured on a tree. This picture shows that any ultrametric space can be drawn on a tree. Moreover, in [3], the properties of $p-$adic adding machine are given in detail.

It is well-known that the closure of the group generated by the adding machine automorphism of a regular rooted tree is topologically isomorphic to the group of $p-$adic integers. In this paper, more clearly, by using a different way, we present a proof. So, we firstly equip $Aut(X^*)$ with a natural metric and prove that the group of $p-$adic integers is both isometric and isomorphic to the closure of the adding machine group which is denoted by $\overline{A}$, a subgroup of the automorphism group of the $p-$ary rooted tree. Consequently, we identify any $p-$adic integers with an element of $\overline{A}$.

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2. Preliminaries

The following definitions and notions are given in [4], [8] and [9].
$p-$adic integers: A $p-$adic integer is a formal series

$$\sum_{i\geq 0} x_i p^i$$

for each $x_i \in \{0,1,2,\ldots,(p-1)\}$ and the set of all $p-$adic integers is denoted by $\mathbb{Z}_p$ ([8]).

Suppose that $x = \sum_{i\geq 0} x_i p^i$ and $y = \sum_{i\geq 0} y_i p^i$ be elements of $\mathbb{Z}_p$. Then, the addition $z = \sum_{i\geq 0} z_i p^i$ of x and y is defined by

$$\sum_{i=0}^{m} z_i p^i \equiv \sum_{i=0}^{m} (x_i + y_i) p^i \pmod{p^{m+1}} \tag{2.1}$$

for each $m \in \{0,1,2,\ldots\}$ where $z_i \in \{0,1,\ldots,(p-1)\}$. If $x = \sum_{i\geq 0} x_i p^i$ is an element of $\mathbb{Z}_p$, then $-x = \sigma(x) + 1$ is the inverse of x where

$$\sigma(x) = \sum_{i\geq 0} (p-1-x_i) p^i.$$

$\mathbb{Z}_p$ is a group with this operation and is called the group of $p-$adic integers.

Let $x = \sum_{i\geq 0} x_i p^i$ be an element of $\mathbb{Z}_p$ and let $x \neq 0$. Thus, there is a first index $v(x) \geq 0$ such that $x_v \neq 0$. This index is called the order of x and is denoted by $ord_p(x)$. If $ord_p(x) = \infty$, then $x_i = 0$ for $i = 0,1,2,\ldots$. On the other hand, the $p-$adic value of x is denoted by

$$|x|_p = \begin{cases} 0 & \text{if } x_i = 0 \text{ for } i = 0,1,2,\ldots, \\ p^{-ord_p(x)} & \text{otherwise} \end{cases}$$

and induces the metric $d_p(x,y) = |x-y|_p$ for $x, y \in \mathbb{Z}_p$ ([8]).

A $p-$adic number is a formal series

$$\sum_{i=-\infty}^{\infty} a_i p^i$$

where $a_i \in \{0,1,2,\ldots,(p-1)\}$ for each $i \in \mathbb{Z}$ and $a_{-i} = 0$ for large i. The set of all $p-$adic numbers is denoted by $\mathbb{Q}_p$. Addition in $\mathbb{Z}_p$ which is defined by equation (2.1) can be naturally extended to $\mathbb{Q}_p$. Hence, $\mathbb{Q}_p$ is a group. Moreover, $\mathbb{Q}_p$ is the metric completion of $\mathbb{Q}$ with respect to the $p-$adic metric. It is easily seen that the group of $p-$adic numbers is a topological group. Moreover, the group of $p-$adic integers is expressed as

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p |\ |x|_p \leq 1\}$$

and is an important subgroup of $\mathbb{Q}_p$.

The following definitions and notions are given in [2], [3], [5] and [7].
The automorphism group of the rooted tree: Let X be a finite set (alphabet) and let

$$X^* = \{x_1 x_2 \ldots x_n \mid x_i \in X, n \geqslant 0\ \}$$

be the set of all finite words over the alphabet X, including the empty word $\emptyset$. In other terms, X^* is the free monoid generated by X ([7]). The length of a word $v = x_1 x_2 \ldots x_n \in X^*$ is the number of its letters and is denoted by $|v|$. The product of $v_1, v_2 \in X^*$ is naturally defined by concatenation $v_1 v_2$. One can think of X^* as vertex set of a rooted tree.

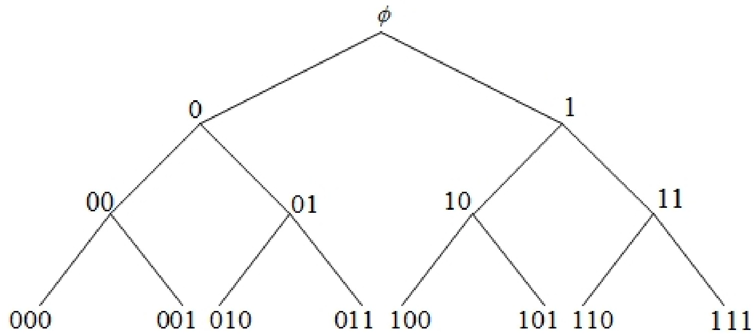

Figure 1. The first three levels of the binary rooted tree X^* for $X = \{0, 1\}$.

The set $X^n = \{v \in X^* \mid |v| = n\}$ is called the nth level of X^*. The empty word $\emptyset$ is the root of the tree X^*. Two words are connected by an edge if and only if they are of the form v, vx where $v \in X^*$ and $x \in X$.

A map $f : X^* \to X^*$ is an endomorphism of the tree X^* if it preserves the root and adjacency of the vertices. An automorphism is a bijective endomorphism. The group of all automorphisms of the tree X^* is denoted by $Aut(X^*)$.

If G is a subgroup of the automorphism group $Aut(X^*)$ of the rooted tree X^*, then for $v \in X^*$, the subgroup

$$G_v = \{g \in G \mid g(v) = v\}$$

is called the vertex stabilizer where $g(v)$ is the image of v under the action of g. The nth level stabilizer is the subgroup

$$St_G(n) = \bigcap_{v \in X^n} G_v.$$

We need a useful way to express the automorphisms the rooted tree X^* and to perform computations with them. For this aim, we give a definition and a proposition from [7].

Definition 2.1 ([7])**.** Let H be a group acting (from the right) by permutations on a set X and let G be an arbitrary group. Then the (permutational) wreath product $G \wr H$ is the semi-direct product $G^X \rtimes H$, where H acts on the direct power G^X by the respective permutations of the direct factors.

If $|X| = d$, then the elements of the wreath product are given by the forms $(g_1, g_2, \ldots, g_d)h$ for $g_i \in G$ and $h \in H$. The multiplication in the wreath product is given by

$$(g_1, g_2, \ldots, g_d)\alpha(h_1, h_2, \ldots, h_d)\beta = (g_1 h_{\alpha(1)}, g_2 h_{\alpha(2)}, \ldots, g_d h_{\alpha(d)})\alpha\beta$$

where $g_i, h_i \in G, \alpha, \beta \in H$ and $\alpha(i)$ is the image of i under the action of α.

Let $g : X^* \to X^*$ be an endomorphism of the rooted tree X^*. Then, $g : vX^* \to g(v)X^*$ is a morphism of the rooted trees where $v \in X^*$. The subtrees vX^* and $g(v)X^*$ are naturally isomorphic to the whole tree X^*. Identifying vX^* and $g(v)X^*$ with X^* we get an endomorphism $g|_v : X^* \to X^*$. It is uniquely determined by the condition

$$g(vw) = g(v)g|_v(w).$$

We call the endomorphism $g|_v$ the restriction of g in v (for details see [7]).

Proposition 2.1 ([7]). *Denote by $S(X)$ the symmetric group of all permutations of X. Fix some indexing $\{x_1, x_2, \ldots, x_d\}$ of X. Then we have an isomorphism*

$$\psi : Aut(X^*) \to Aut(X^*) \wr S(X),$$

given by

$$\psi(g) = (g|_{x_1}, g|_{x_2}, \ldots, g|_{x_d})\alpha,$$

where α is the permutation equal to the action of g on $X \subset X^*$.

Thus, $g \in Aut(X^*)$ is identified with the image $\psi(g) \in Aut(X^*) \wr S(X)$ and it is written as

$$g = (g|_{x_1}, g|_{x_2}, \ldots, g|_{x_d})\alpha.$$

The adding machine group: Let a be the transformation on X^* defined by the wreath recursion

$$a = (\underbrace{1, 1, \ldots, 1}_{p-1 \text{ times}}, a)\sigma$$

where $\sigma = (012 \ldots (p-1))$ is a permutation in S_p on $X = \{0, 1, 2, \ldots, (p-1)\}$.

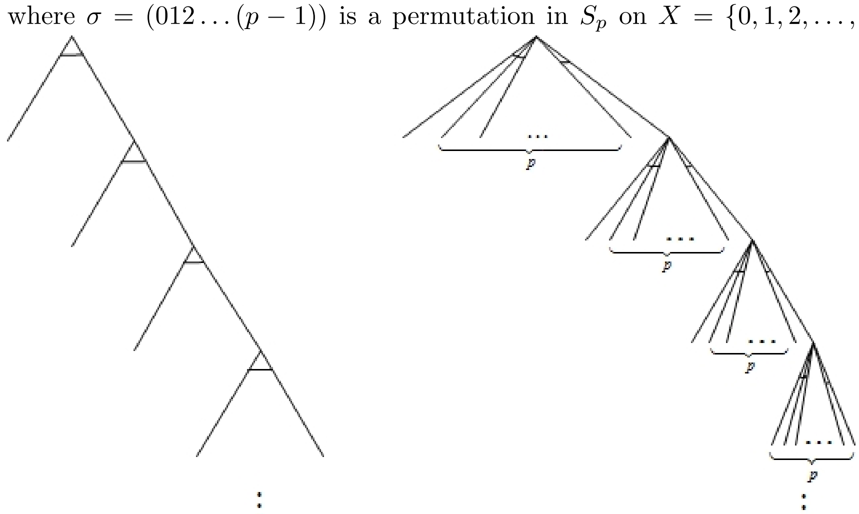

Figure 2. Portrait of the transformation a for $X = \{0, 1\}$ and $X = \{0, 1, \ldots, p-1\}$

The transformation a generates an infinite cyclic group on X^*. This group is called the adding machine group and we denote this group by A. For example, using permutational wreath product we obtain that

$$\begin{aligned} a^p &= (1, \ldots, 1, a)\sigma(1, \ldots, 1, a)\sigma \ldots (1, \ldots, 1, a)\sigma \\ &= (a, a, \ldots, a)\sigma^p \\ &= (a, a, \ldots, a) \end{aligned}$$

(for details see [2], [7]).

The Metric Space $(Aut(X^*), d)$: In the following definition, we equip the automorphism group of the $p-$ary rooted tree X^* with a natural metric where $X = \{0, 1, 2, \ldots, p-1\}$. This metric is also used in [1].

Definition 2.2. The metric function $d : Aut(X^*) \times Aut(X^*) \to \mathbb{R}$ can be defined by

$$d(g_1, g_2) = \begin{cases} \frac{1}{p^k} & \text{for } g_1^{-1}g_2 \in St_{Aut(X^*)}(k) \text{ and } g_1^{-1}g_2 \notin St_{Aut(X^*)}(k+1), \\ 0 & \text{for } g_1 = g_2 \end{cases}$$

where $g_1, g_2 \in Aut(X^*)$. In other words, if g_1 and g_2 agree on all vertices of the level k but do not agree at least one vertex of the level $(k+1)$ of the tree X^*, then the distance between g_1 and g_2 is $\frac{1}{p^k}$.

$(Aut(X^*), d)$ is a compact metric space and is a topological group. It is obvious that $\overline{A}$, the closure of A, is a subgroup of $Aut(X^*)$.

3. An Isometry between the Group of $p-$adic Integers and the Closure of Adding machine group

Now we give a formula for the distance between two elements of the adding machine group. Notice that this expression is similar to the distance between two $p-$adic integers.

Proposition 3.1. *For $a^n, a^m \in A$, the distance $d(a^n, a^m)$ can be defined by*

$$\begin{array}{rcl} d \;:\; A \times A & \to & A \\ (a^n, a^m) & \mapsto & d(a^n, a^m) = \begin{cases} 0 & \textit{for } n = m, \\ \frac{1}{p^k} & \textit{for } n - m = tp^k \end{cases} \end{array}$$

where $t, k \in \mathbb{Z}$, p is prime number and $(p, t) = 1$.

Proof. First we compute $St_A(1)$. Using permutational wreath product we obtain that

$$\begin{aligned} a^p &= (1, 1, \ldots, a)\sigma(1, 1, \ldots, a)\sigma \ldots (1, 1, \ldots, a)\sigma \\ &= (a, a, \ldots, a). \end{aligned}$$

This shows that $St_A(1) = \langle a^p \rangle$. Moreover, we get

$$\begin{aligned} a^{p^2} &= a^p a^p \ldots a^p \\ &= (a, a, \ldots, a)(a, a, \ldots, a) \ldots (a, a, \ldots, a) \\ &= (a^p, a^p, \ldots, a^p) \end{aligned}$$

We have $a^{p^2} \in St_A(2)$ because $a^p \in St_A(1)$. Therefore, it is obtained that $St_A(2) = \langle a^{p^2} \rangle$. By proceeding in a similar manner, we compute $St_A(k) = \langle a^{p^k} \rangle$.

So, elements of A which are in $St_A(1)$ but are not in $St_A(2)$ can be expressed as

$$St_A(1) - St_A(2) = \{a^{tp} : (p, t) = 1\}$$

and by using the induction method, it is easily seen that

$$St_A(k) - St_A(k+1) = \{a^{tp^k} : (p, t) = 1\}.$$

Let us take arbitrary $a^n, a^m \in A$. If $n = m$, then it is $a^n = a^m$ and $d(a^n, a^m) = 0$. If $n \neq m$, then there exists a unique expression $n - m = tp^k$ such that $(p, t) = 1$. Then we obtain

$$a^{-m}a^n = a^{n-m} = a^{tp^k} \in St_A(k) - St_A(k+1)$$

and thus it is $d(a^n, a^m) = \frac{1}{p^k}$. $\square$

Proposition 3.2. *Let $\sum_{i\geq 0}\alpha_i p^i \in \mathbb{Z}_p$. Then, the sequence*

$$a^{\alpha_0}, a^{\alpha_0+\alpha_1 p}, a^{\alpha_0+\alpha_1 p+\alpha_2 p^2}, \ldots$$

is convergent.

Proof. For any $\varepsilon > 0$, there is a positive integer n_0 such that $\frac{1}{p^{n_0}} < \varepsilon$. If $k > l$ and $k, l \geq n_0$, then it is obtained that

$$d(a^{\alpha_0+\alpha_1 p+\ldots+\alpha_k p^k}, a^{\alpha_0+\alpha_1 p+\ldots+\alpha_l p^l}) = \frac{1}{p^l} < \varepsilon$$

from Proposition 3.1. Thus, it is a Cauchy sequence. Since $Aut(X^*)$ is a complete metric space, this sequence is convergent. $\square$

Now we give our main proposition:

Proposition 3.3. *We define*

$$\varphi \quad : \quad \mathbb{Z}_p \quad \to \quad \overline{A}$$

such that $\varphi(\sum_{i\geq 0}\alpha_i p^i)$ is the limit of the sequence $a^{\alpha_0}, a^{\alpha_0+\alpha_1 p}, a^{\alpha_0+\alpha_1 p+\alpha_2 p^2}, \ldots$. Then, φ is both an isometry and a group isomorphism.

Proof. From Proposition 3.2, φ is well-defined. Now we show that φ is an isometry. In other words, we show that $d_p(\alpha, \beta) = d(\varphi(\alpha), \varphi(\beta))$ for every $\alpha, \beta \in \mathbb{Z}_p$. Let $\alpha = \sum_{i\geq 0}\alpha_i p^i$ and $\beta = \sum_{i\geq 0}\beta_i p^i$.

If $d_p(\alpha, \beta) = 0$, then we obtain $d(\varphi(\alpha), \varphi(\beta)) = 0$ since $\alpha_i = \beta_i$ for $i = 0, 1, 2, \ldots$.

If $d_p(\alpha, \beta) = \frac{1}{p^k}$, then $\alpha_i = \beta_i$ for $i < k$ and $\alpha_k \neq \beta_k$. We must show that $d(\varphi(\alpha), \varphi(\beta)) = \frac{1}{p^k}$. Because $\varphi(\alpha)$ and $\varphi(\beta)$ are the limits of the sequences

$$a^{\alpha_0}, a^{\alpha_0+\alpha_1 p}, a^{\alpha_0+\alpha_1 p+\alpha_2 p^2}, \ldots \quad \text{and} \quad a^{\beta_0}, a^{\beta_0+\beta_1 p}, a^{\beta_0+\beta_1 p+\beta_2 p^2}, \ldots$$

respectively, it is written the equality

$$\lim_{k\to\infty}(a^{\alpha_0+\alpha_1 p+\ldots+\alpha_k p^k}, a^{\beta_0+\beta_1 p+\ldots+\beta_k p^k}) = (\varphi(\alpha), \varphi(\beta)).$$

Since any metric function is continuous, we obtain that

$$d(a^{\alpha_0}, a^{\beta_0}), d(a^{\alpha_0+\alpha_1 p}, a^{\beta_0+\beta_1 p}), \ldots \to d(\varphi(\alpha), \varphi(\beta)).$$

From Proposition 3.1, we get

$$0, 0, ..., 0, \frac{1}{p^k}, \frac{1}{p^k}, \ldots, \frac{1}{p^k}, \ldots \to \frac{1}{p^k}.$$

This shows that $d(\varphi(\alpha), \varphi(\beta)) = \frac{1}{p^k}$. Namely, φ is an isometry map.

Moreover, φ is injective since φ is an isometry map.

Now we show that φ is surjective. Let b be an arbitrary element of $\overline{A}$. Thus, there exists a sequence

$$a^{n_0}, a^{n_1}, \ldots, a^{n_k}, \ldots \to b$$

whose elements are in A. Furthermore, every integer n_k can be expressed in $\mathbb{Z}_p$ as

$$\begin{array}{rcl} n_0 & = & \alpha_0^0 + \alpha_1^0 p + \alpha_2^0 p^2 + \ldots \\ n_1 & = & \alpha_0^1 + \alpha_1^1 p + \alpha_2^1 p^2 + \ldots \\ & \vdots & \\ n_k & = & \alpha_0^k + \alpha_1^k p + \alpha_2^k p^2 + \ldots \\ & \vdots & \end{array} \tag{3.1}$$

At least one of the numbers $0, 1, 2, ..., (p-1)$ occurs infinitely many times in the sequence $(\alpha_0^k)_k$. We choose one of them and denote it by β_0. Let $(\alpha_1^{k_l})_l$ be a subsequence of $(\alpha_1^k)_k$ such that $\alpha_0^{k_l} = \beta_0$ for $l = 0, 1, 2, \ldots$. Similarly, we denote by β_1, any one of the numbers that appears infinitely many times in the sequence $(\alpha_1^{k_l})_l$. Proceeding in this manner, we obtain a sequence

$$a^{\beta_0}, a^{\beta_0+\beta_1 p}, \ldots, a^{\beta_0+\beta_1 p+\ldots+\beta_k p^k}, \ldots.$$

From Proposition 3.2, this sequence is convergent. Now we show this sequence converges to b. Due to the construction of (3.1), there exists a subsequence (n_{k_s}) of the sequence (n_k) whose $p-$adic expression of term sth such that

$$\beta_0 + \beta_1 p + \beta_2 p^2 + \ldots + \beta_s p^s + \gamma_{s+1} p^{s+1} + \gamma_{s+2} p^{s+2} + \ldots.$$

Owing to the fact that

$$\lim_{s\to\infty} d(a^{\beta_0+\beta_1 p+\ldots+\beta_s p^s}, a^{n_{k_s}}) = 0$$

and from the triangle inequality, the sequence $(a^{\beta_0+\beta_1 p+\ldots+\beta_k p^k})$ converges to b. This shows that $\varphi(\sum_{i\geq 0} \beta_i p^i) = b$ and hence φ is surjective.

Finally, we prove that φ is a homomorphism. In other words, we prove that

$$\varphi(\alpha + \beta) = \varphi(\alpha)\varphi(\beta)$$

for every $\alpha, \beta \in \mathbb{Z}_p$. Let

$$\alpha = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots,$$
$$\beta = \beta_0 + \beta_1 p + \beta_2 p^2 + \ldots$$

and

$$\alpha + \beta = \gamma_0 + \gamma_1 p + \gamma_2 p^2 + \ldots.$$

From the definition of φ, we have

$$a^{\gamma_0}, a^{\gamma_0+\gamma_1 p}, a^{\gamma_0+\gamma_1 p+\gamma_2 p^2}, \ldots \to \varphi(\alpha + \beta).$$

Moreover, it follows that

$$a^{(\alpha_0+\beta_0)}, a^{(\alpha_0+\beta_0)+(\alpha_1+\beta_1)p}, a^{(\alpha_0+\beta_0)+(\alpha_1+\beta_1)p+(\alpha_2+\beta_2)p^2}, \ldots \to \varphi(\alpha)\varphi(\beta)$$

due to the fact that $Aut(X^*)$ is a topological group,

$$a^{\alpha_0}, a^{\alpha_0+\alpha_1 p}, a^{\alpha_0+\alpha_1 p+\alpha_2 p^2}, \ldots \to \varphi(\alpha)$$

and

$$a^{\beta_0}, a^{\beta_0+\beta_1 p}, a^{\beta_0+\beta_1 p+\beta_2 p^2}, \ldots \to \varphi(\beta).$$

In $\mathbb{Z}_p$, we have

$$\begin{array}{rcl}
\alpha_0 + \beta_0 & = & \gamma_0 + \overline{\gamma_0} p + 0p^2 + 0p^3 + 0p^4 + \ldots \\
(\alpha_0 + \beta_0) + (\alpha_1 + \beta_1)p & = & \gamma_0 + \gamma_1 p + \overline{\gamma_1} p^2 + 0p^3 + 0p^4 + 0p^5 + \ldots \\
& \vdots & \\
(\alpha_0 + \beta_0) + \ldots + (\alpha_k + \beta_k)p^k & = & \gamma_0 + \gamma_1 p + \ldots + \gamma_k p^k + \overline{\gamma_k} p^{k+1} + 0p^{k+2} \\
& & + 0p^{k+3} + +0p^{k+4} + \ldots. \\
& \vdots &
\end{array}$$

Let

$$x = (\alpha_0 + \beta_0) + \ldots + (\alpha_k + \beta_k)p^k$$

and

$$y = \gamma_0 + \gamma_1 p + \ldots + \gamma_k p^k + \overline{\gamma_k} p^{k+1} + 0p^{k+2} + 0p^{k+3} + \ldots.$$

Then, we have

$$d(a^x, a^y) = \begin{cases} \frac{1}{p^k} & \text{if } \overline{\gamma_k} \neq 0, \\ 0 & \text{if } \overline{\gamma_k} = 0. \end{cases}$$

It follows that $\varphi(\alpha + \beta) = \varphi(\alpha)\varphi(\beta)$ since

$$d(a^{\alpha_0+\beta_0}, a^{\gamma_0}), d(a^{\alpha_0+\beta_0+(\alpha_1+\beta_1)p}, a^{\gamma_0+\gamma_1 p}), \ldots \to d(\varphi(\alpha)\varphi(\beta), \varphi(\alpha+\beta))$$

and

$$\lim_{k\to\infty} d(a^x, a^y) = 0.$$

Hence, the proof is completed. $\square$

Consequently, the group of $p-$adic integers $\mathbb{Z}_p$ can be isometrically embedded into the metric space $Aut(X^*)$ since $\overline{A} \subseteq Aut(X^*)$.

Example 3.1. We show $\varphi(-1)$ for $p = 2$ in Figure **??**. It is well-known that

$$-1 = 1 + 1.2^1 + 1.2^2 + \ldots + 1.2^k + \ldots \in \mathbb{Z}_2.$$

Due to the definition of φ, $\varphi(-1)$ is the limit of the sequence

$$a^1, a^{1+1.2^1}, a^{1+1.2^1+1.2^2}, \ldots$$

in A for $X = \{0, 1\}$. This limit equals to $a^{-1} = (a^{-1}, 1)\sigma$ because of Proposition 3.1.

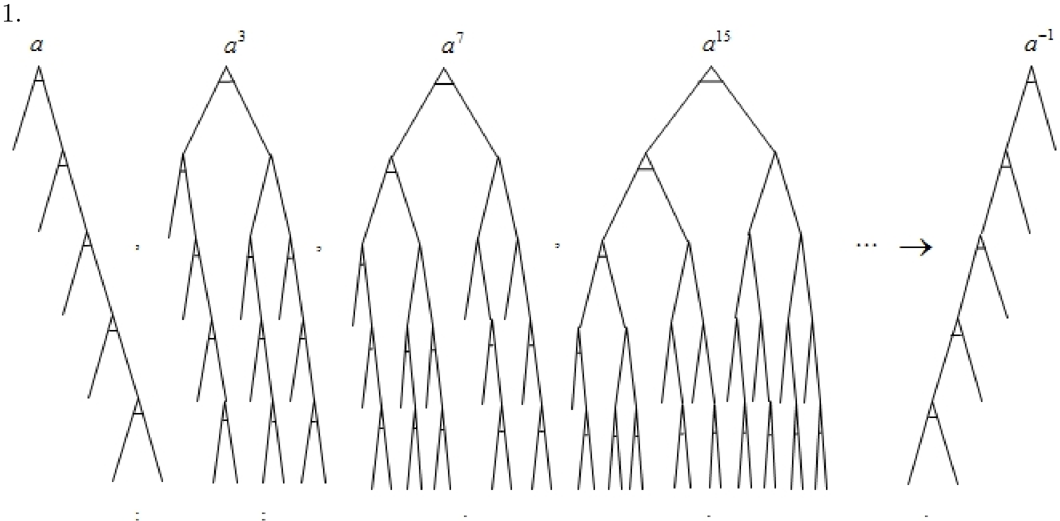

Figure 3. The image of $-1 \in Z_2$ under the map φ.

References

[1] G. N. Arzhantseva and Z. Sunic, *Construction of elements in the closure of Grigorchuk group*, arXiv:math/0607778v2.

[2] L. Bartholdi, R. Grigorchuk and V. Nekrashevych, *From Fractal Groups to Fractal Sets*, Fractals in Graz, (Peter Grabner and Wolfgang Woess, eds.), Birkhëuser Verlag, Basel, (2003), 25-118.

[3] H. Bass, M.V. Otera-Espinar, D. Rockmore and C. Tresser, *Cyclic Renormalition and Automorphism Groups and Rooted Trees*, Lecture Notes in Math., Springer-Verlag, Berlin, 1996.

[4] F. Q. Gouvêa, *$p-$adic Numbers*, Springer-Verlag, Berlin, 1997.
[5] R. Grigorchuk, V.V. Nekrashevich and V.I. Sushchanskii, *Automata, Dynamical systems and infinite groups*, Proc. Steklov Inst. Math., 231 (2000), 134–214.
[6] J. E. Holly, *Pictures of Ultrametric Spaces, the $p-$adic numbers, and valued fields*, Amer. Math. Monthly, 108 (2001), 721–728.
[7] V. Nekrashevych, *Self-similar Groups, Mathematical Surveys and Monographs*, Amer. Math. Soc., Providence, RI, 117, 2005.
[8] A. M. Robert, *A Course in $p-$adic Analysis*, Springer, 2000.
[9] W. H. Schikhof, *Ultrametric Calculus an Introduction to $p-$adic Calculus*, Cambridge University Press, New York, 1984.

Anadolu University, Faculty of Science, Department of Mathematics, Eskişehir-TURKEY
E-mail address: mustafasaltan@anadolu.edu.tr

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GENERALIZED RELATIVE ORDER OF FUNCTIONS ANALYTIC IN THE UNIT DISC

RATAN KUMAR DUTTA

Abstract. In this paper we consider generalized relative order of a function analytic in the unit disc with respect to an entire function and prove several theorems.

1. Introduction, Definitions and Notation

Let $f(z)$ be analytic in the unit disc $U : \{z : |z| < 1\}$ and

$$T_f(r) = T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} log^+|f(re^{i\theta})|d\theta$$

is Nevanlinna characteristic function of $f(z)$. If

$$T(r, f) = (1 - r)^{-\mu} \text{ for all } r \text{ in } 0 < r_0(\mu) < r < 1,$$

then the greatest lower bound of all such numbers μ is called Nevanlinna order [5] (Juneja and Kapoor 1985) of f. Thus the Nevanlinna order $\rho(f)$ of f is given by

$$\rho(f) = limsup_{r\to 1}\frac{log\, T(r, f)}{-log(1 - r)}.$$

In [1] Banerjee and Dutta introduce the idea of relative order of a function analytic in the unit disc with respect to an entire function.

Definition 1.1. [1] If f be analytic in U and g be entire, then the relative order of f with respect to g, denoted by $\rho_g(f)$ is defined by

$$\rho_g(f) = inf\{\mu > 0 : T_f(r) < T_g\left[\left(\frac{1}{1-r}\right)^{\mu}\right] \text{ for all } 0 < r_0(\mu) < r < 1\}.$$

Note 1.1. When $g(z) = exp\ z$, then Definition 1.1 coincides with the definition of Nevanlinna order of f.

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Notation 1.2. [3] $log^{[0]}x = x$, $exp^{[0]}x = x$ and for positive integer m, $log^{[m]}x = log(log^{[m-1]}x)$, $exp^{[m]}x = exp(exp^{[m-1]}x)$.

In [2] Datta and Jerin introduce the idea of generalized relative order.

Definition 1.2. [2] Let $T_f(r) = T(r, f)$ denote the Nevanlinna's characteristic function of f. The relative generalized Nevanlinna order $\rho_g^p(f)$ of an analytic function f in U with respect to another entire function g are defined in the following way:

$$\rho_g^p(f) = limsup_{r\to 1} \frac{log^{[p]}T_g^{-1}T_f(r)}{-log(1-r)} \ .$$

Definition 1.3. [1] An entire function g is said to have the property (A), if for any $\sigma > 1, \lambda > 0$ and for all $r, 0 < r < 1$ sufficiently close to 1

$$\left[G\left(\left(\frac{1}{1-r}\right)^{\lambda}\right)\right]^2 < G\left(\left(\left(\frac{1}{1-r}\right)^{\lambda}\right)^{\sigma}\right)$$

where $G(r) = max_{|z|=r}|g(z)|$.

The function $g(z) = exp\ z$ has the property (A) where as $g(z) = z$ has not.

In this paper we consider the definition of generalized relative order of a function analytic in the unit disc U with respect to an entire function and obtain the sum and product theorems. Also we show that the relative order of a function analytic in U with respect to an entire and to the derivative of the entire are same. We do not explain the standard notations and definitions of the theory of entire and meromorphic functions as those are available in [4], [5], [6] and [7]. Throughout we shall assume that f, f_1, f_2 etc, to be function analytic in U and g, g_1, g_2 etc, are non constant entire.

2. Known Lemmas

Lemma 2.1. [1] *Let g be an entire function which has the property (A). Then for any positive integer n and for all $\sigma > 1, \lambda > 0$,*

$$\left[G\left(\left(\frac{1}{1-r}\right)^{\lambda}\right)\right]^n < G\left(\left(\left(\frac{1}{1-r}\right)^{\lambda}\right)^{\sigma}\right)$$

holds for all $r, 0 < r < 1$, sufficiently close to 1.

Lemma 2.1 follows from Definition 1.3.

Lemma 2.2. [1] *If g is entire then*

$$T_g\left(\frac{1}{1-r}\right) \le logG\left(\frac{1}{1-r}\right) \le 3T_g\left(\frac{2}{1-r}\right)$$

for all r, $0 < r < 1$, sufficiently close to 1.

3. Preliminary Theorem

Theorem 3.1. *Let f be analytic in U of generalized relative order $\rho_g^p(f)$ where g is entire. Let $\epsilon > 0$ be arbitrary. Then*

$$T_f(r) = O\left(logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_g^p(f)+\epsilon}\right)\right)$$

holds for all $r,\ 0 < r < 1$, sufficiently close to 1.
Conversely, if for an analytic f in U and entire g having the property (A),

$$T_f(r) = O\left(logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k+\epsilon}\right)\right)$$

holds for all $r,\ 0 < r < 1$, sufficiently close to 1, and

$$T_f(r) = O\left(logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k-\epsilon}\right)\right)$$

does not hold for all $r,\ 0 < r < 1$, sufficiently close to 1, then $k = \rho_g^p(f)$.

Proof. From the definition of generalized relative order, we have

$$\begin{aligned}
T_f(r) &< T_g\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_g^p(f)+\epsilon}\right) \text{ for } 0 < r_0 < r < 1, \text{ say} \\
\text{or, } T_f(r) &< logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_g^p(f)+\epsilon}\right) \text{ for } 0 < r_0 < r < 1, \quad \text{by Lemma 2.2.} \\
\text{So, } T_f(r) &= O\left(logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_g^p(f)+\epsilon}\right)\right).
\end{aligned}$$

Conversely, if

$$T_f(r) = O\left(logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k+\epsilon}\right)\right)$$

holds for all $r,\ 0 < r < 1$, sufficiently close to 1, then

$$\begin{aligned}
T_f(r) &< [\alpha]logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k+\epsilon}\right),\ \alpha > 1 \\
&= \frac{1}{3}log\left[G\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k+\epsilon}\right)\right]^{[3\alpha]} \\
&\le \frac{1}{3}logG\left(\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k+\epsilon}\right)^{\sigma}\right) \text{ by Lemma 2.1, for any } \sigma > 1 \\
&\le T_g\left(2\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k+\epsilon}\right)^{\sigma}\right), \text{ by Lemma 2.2.}
\end{aligned}$$

$$\therefore \quad logT_g^{-1}T_f(r) \leq \log 2 + \log\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k+\epsilon}\right)^{\sigma}$$

$$\leq \sigma \exp^{[p-2]}\left(\frac{1}{1-r}\right)^{k+\epsilon} + O(1).$$

$$\therefore \quad log^{[2]}T_g^{-1}T_f(r) \leq \exp^{[p-3]}\left(\frac{1}{1-r}\right)^{k+\epsilon} + O(1).$$

So

$$limsup_{r\to 1-}\frac{log^{[p]}T_g^{-1}T_f(r)}{-log(1-r)} \leq k+\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$limsup_{r\to 1-}\frac{log^{[p]}T_g^{-1}T_f(r)}{-log(1-r)} \leq k. \tag{3.1}$$

Again there exists a sequence $\{r_n\}$ of values of r tending to 1_- for which

$$T_f(r) \geq logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k-\epsilon}\right)$$

$$\geq T_g\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k-\epsilon}\right), \text{ by Lemma 2.2}$$

and so

$$\frac{log^{[p]}T_g^{-1}T_f(r)}{-log(1-r)} \geq k-\epsilon \tag{3.2}$$

for $r = r_n \to 1_-$.
Since $\epsilon > 0$ is arbitrary, combining (3.1) and (3.2), we obtain $k = \rho_g^p(f)$.
This proves the theorem. □

4. Sum and Product Theorems

Theorem 4.1. *Let f_1 and f_2 be analytic in the unit disc U having generalized relative orders $\rho_g^p(f_1)$ and $\rho_g^p(f_2)$ respectively, where g is entire having the property (A). Then*

$$\begin{array}{lrcl} (a) & \rho_g^p(f_1 \pm f_2) & \leq & max\{\rho_g^p(f_1), \rho_g^p(f_2)\} \text{ and} \\ (b) & \rho_g^p(f_1.f_2) & \leq & max\{\rho_g^p(f_1), \rho_g^p(f_2)\}. \end{array}$$

The same inequality holds for the quotient. The equality holds in (b) if $\rho_g^p(f_1) \neq \rho_g^p(f_2)$.

Proof. We may suppose that $\rho_g^p(f_1)$ and $\rho_g^p(f_2)$ both are finite, because if one of them or both are infinite, the inequalities are evident. Let $\rho_1 = \rho_g^p(f_1)$ and $\rho_2 = \rho_g^p(f_2)$ and $\rho_1 \leq \rho_2$. For arbitrary $\epsilon > 0$ and for all r, $0 < r < 1$, sufficiently close to 1, we have

$$T_{f_1}(r) < T_g\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_1+\epsilon}\right) \leq logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_1+\epsilon}\right)$$

and

$$T_{f_2}(r) < T_g\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right) \leq logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right), \quad \text{using Lemma 2.2.}$$

Now for all $r,\ 0 < r < 1$, sufficiently close to 1,

$$\begin{aligned}
T_{f_1\pm f_2}(r) &\leq T_{f_1}(r) + T_{f_2}(r) + O(1)\\
&\leq logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_1+\epsilon}\right) + logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right) + O(1)\\
&\leq 3logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right)\\
&= \frac{1}{3}\ log\left[G\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right)\right]^9\\
&\leq \frac{1}{3}\ logG\left(\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right)^{\sigma}\right) \text{ by Lemma 2.1, for any } \sigma > 1\\
&\leq T_g\left(2\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right)^{\sigma}\right) \text{ by Lemma 2.2.}
\end{aligned}$$

$$\begin{aligned}
\therefore \quad logT_g^{-1}T_{f_1\pm f_2}(r) &\leq log2 + log\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right)^{\sigma}\\
&\leq \sigma\exp^{[p-2]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon} + O(1).
\end{aligned}$$

$$\therefore \quad log^{[2]}T_g^{-1}T_{f_1\pm f_2}(r) \leq \exp^{[p-3]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon} + O(1).$$

$$\begin{aligned}
\therefore \quad \rho_g^p(f_1\pm f_2) &= limsup_{r\to 1-}\frac{log^{[p]}T_g^{-1}T_{(f_1\pm f_2)}(r)}{-log(1-r)}\\
&\leq \rho_2 + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$\rho_g^p(f_1\pm f_2) \leq \rho_2 \leq max\{\rho_g^p(f_1), \rho_g^p(f_2)\}$$

which proves (a).
For (b), since,

$$T_{f_1.f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r),$$

we obtain similarly as above

$$\rho_g^p(f_1.f_2) \leq max\ \{\rho_g^p(f_1), \rho_g^p(f_2)\}.$$

Let $f = f_1.f_2$ and $\rho_g^p(f_1) < \rho_g^p(f_2)$. Then applying (b), we have $\rho_g^p(f) \leq \rho_g^p(f_2)$.
Again since $f_2 = f/f_1$, applying the first part of (b), we have

$$\rho_g^p(f_2) \leq max\ \{\rho_g^p(f), \rho_g^p(f_1)\}.$$

Since $\rho_g^p(f_1) < \rho_g^p(f_2)$, we have

$$\rho_g^p(f) = \rho_g^p(f_2) = max\{\rho_g^p(f_1), \rho_g^p(f_2)\}$$

when $\rho_g^p(f_1) \neq \rho_g^p(f_2)$.
This prves the theorem. □

5. Relative order with respect to the derivative of an entire function

Theorem 5.1. *If f is analytic in the unit disc U and g be transcendental entire having the property (A), then $\rho_g^p(f) = \rho_{g'}^p(f)$ where $g^{'}$ denotes the first derivative of g.*

To prove the theorem we require the following lemmas.

Lemma 5.1. [1] *If g be a transcendental entire, then for all $r,\ 0 < r < 1$, sufficiently close to 1 and for any $\lambda > 0$*

$$T_{g'}\left(\left(\frac{1}{1-r}\right)^{\lambda}\right) \leq 2\,T_g\left(2\left(\frac{1}{1-r}\right)^{\lambda}\right) + O\left(T_g\left(2\left(\frac{1}{1-r}\right)^{\lambda}\right)\right)$$

Lemma 5.2. [1] *Let g be a transcendental entire function, then for all $r,\ 0 < r < 1$, sufficiently close to 1 and $\lambda > 0$*

$$T_g\left(\left(\frac{1}{1-r}\right)^{\lambda}\right) < C\,\left[T_{g'}\left(2\left(\frac{1}{1-r}\right)^{\lambda}\right) + log\left(\frac{1}{1-r}\right)^{\lambda}\right]$$

where C is a constant which is only dependent on $g(0)$.

Proof of the Theorem 5.1:

Proof. From Lemma 5.1 and Lemma 5.2 we obtain for $r,\ 0 < r < 1$, sufficiently close to 1

$$T_{g'}\left(\left(\frac{1}{1-r}\right)^{\lambda}\right) < [K]\,T_g\left(2\left(\frac{1}{1-r}\right)^{\lambda}\right) \tag{5.1}$$

and

$$T_g\left(\left(\frac{1}{1-r}\right)^{\lambda}\right) < [K^{'}]\,T_{g'}\left(2\left(\frac{1}{1-r}\right)^{\lambda}\right) \tag{5.2}$$

where $K,\ K^{'} > 0$ and $\lambda > 0$ be any number.
From the definition of $\rho_{g'}^p(f)$, we get for arbitrary $\epsilon > 0$,

$$T_f(r) < T_{g'}\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_{g'}(f)+\epsilon}\right)$$

for all $r,\ 0 < r < 1$, sufficiently close to 1.
From (5.1) and by Lemma 2.1 and Lemma 2.2, for all $r,\ 0 < r < 1$, sufficiently close to 1

$$\begin{aligned}
T_f(r) &< [K]T_g\left(2\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho^p_{g'}(f)+\epsilon}\right)\\
&\leq [K]logG\left(2\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho^p_{g'}(f)+\epsilon}\right)\\
&= \frac{1}{3}log\left[G\left(2\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho^p_{g'}(f)+\epsilon}\right)\right]^{3[k]}\\
&\leq \frac{1}{3}log\left(G\left(2\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho^p_{g'}(f)+\epsilon}\right)^{\sigma}\right) \text{ for any } \sigma > 1\\
&\leq T_g\left(2^{\sigma+1}\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho^p_{g'}(f)+\epsilon}\right)^{\sigma}\right).
\end{aligned}$$

$$\therefore\quad \rho^p_g(f) = limsup_{r\to 1-}\ \frac{log^{[p]}T_g^{-1}T_f(r)}{-log(1-r)} \leq \rho^p_{g'}(f)+\epsilon.$$

Since $\epsilon > 0$ is arbitrary, so $\rho^p_g(f) \leq \rho^p_{g'}(f)$.
Using (5.2) we obtain similarly $\rho^p_{g'}(f) \leq \rho^p_g(f)$. So, $\rho^p_g(f) = \rho^p_{g'}(f)$.
This proves the theorem. □

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References

[1] D. Banerjee and R. K. Dutta, Relative order of functions analytic in the unit disc, Bull. Cal. Math. Soc., Vol. 101, No. 1, (2009), pp 95-104.
[2] S. K. Datta and E. Jerin: Further Results on the Generalised Growth Properties of Functions Analytic in a Unit Disc, International Journal of Contemporary Mathematical Sciences, Vol. 5, No. 23 (2010), pp. 1137-1143.
[3] D. Sato, On the rate of growth of entire functions of fast growth, Bull. Amer. Math. Soc., Vol. 69, (1963), pp. 411-414.
[4] W. K. Hayman, *Meromorphic functions*, The Clarendon Press, Oxford, 1964.
[5] O. P. Juneja and G. P. Kapoor, *Analytic functions -growth aspects*, Pitman advanced publishing program, 1985.
[6] G. Valiron, *Lectures on the general theory of Integral functions*, Chelsea Publishing Company, 1949.
[7] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers and Science Press, Beijing, 2003.

Department of Mathematics, Patulia High School, Patulia, Kolkata-119, West Bengal, India
E-mail address: ratan_3128@yahoo.com

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SOME RESULTS ON PSEUDO RICCI SYMMETRIC ALMOST α-COSYMPLECTIC f-MANIFOLDS

YAVUZ SELIM BALKAN AND NESIP AKTAN

Abstract. In this study, we consider pseudo Ricci symmetric almost α-cosymplectic f-manifolds. We get some results on pseudo Ricci symmetric α-cosymplectic f-manifolds and almost α-cosymplectic f-manifolds verifying (κ, μ, ν)-nullity conditions.

1. Introduction

The notion of f-structure, which is a generalization of almost complex structures and almost contact structures, was firstly introduced by Yano in 1963 [1]. In 1971, Goldberg and Yano defined that globally framed f-structures are f-structures and globally framed manifolds are f pk.-manifolds [2]. Later many authors studied f pk.-manifolds. In 2006, Falcitelli and Pastore defined $(2n+s)$-dimensional almost Kenmotsu f-manifolds [3]. Öztürk et. all defined almost α-cosymplectic f-manifolds [4].

The notion of an almost cosymplectic manifold was introduced by Goldberg and Yano in 1969 [9]. The simplest examples of such manifolds are those being the products (possibly local) of almost Kählerian manifolds and the real line $\mathbb{R}$ or the circle S^1. In particular, cosymplectic manifolds in the sense of Blair [10] are of this type. However the class of almost cosymplectic manifolds is much more wider. There are already many known examples (among others, compact, homogeneous) of such manifolds which are not products (even locally). See Cordero et al. [11] Chinea and Gonzalez [12] and Olszak ([13],[14]).

The topology of cosymplectic manifolds was studied by Blair and Goldberg [15], Chinea et al. ([16], [17]) and others. Most of the results of Libermann [18], Lichnerowicz [19], Fujimoto and Muto [20] also have applications in characterizing of topological and analytical properties of almost cosymplectic manifolds (these authors have used a different terminology).

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Curvature properties of almost cosymplectic manifolds were studied mainly by Golberg and Yano [9] ,Olszak ([13], [14]), Kirichenko [21] and Endo [22]. We relate some of them in a historical order.

As a generalization of Chaki's pseudosymmetric and pseudo Ricci sym-metric manifolds ([28] and [29]), the notion of weakly symmetric and weakly Ricci-symmetric manifolds were introduced by Tamassy and Binh ([30] and [31]). These type manifolds were studied with different structures by several authors ([30], [32] and [33]). Recently in [34], Özgür studied weakly symmetric Kenmotsu manifolds. The notion of special weakly Ricci symmetric manifolds was introduced and studied by Singh, and Khan in [35]. Aktan studied special weakly symmetric Kenmotsu manifolds [36].

In this paper, we have studied some geometric properties of pseudo Ricci-symmetric almost α-cosymplectic f-manifold.

2. Preliminaries

Let M be a real $(2n+s)$-dimensional smooth manifolds. M admits an f-structure [1] if there exists a non-null smooth $(1,1)$ tensor field φ, of the tangent bundle TM, satisfying $\varphi^3+\varphi=0,\ rank\varphi=2n$. An f-structure is a generalization of almost complex $(s=0)$ and almost contact $(s=1)$ structure. In the latter case, M is orientable [25]. Corresponding to two complementary projection operators P and Q applied to TM, defined by $P=-\varphi^2$ and $Q=\varphi^2+I$, where I is the identity operator, there exist two complementary distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ such that $\dim(\mathcal{D})=2n$ and $\dim(\mathcal{D})=s$. The following relations hold

$$\varphi P=P\varphi=\varphi,\quad \varphi Q=Q\varphi=0,\quad \varphi^2P=-P,\quad \varphi^2Q=0. \tag{2.1}$$

Thus, we have an almost complex distribution $\left(\mathcal{D},\ J=\varphi_{|_D},\ J^2=-I\right)$ and φ acts on $\mathcal{D}^\perp$ as a null operator. It follows that

$$TM=\mathcal{D}\oplus\mathcal{D}^\perp,\quad \mathcal{D}\cap\mathcal{D}^\perp=\{0\}. \tag{2.2}$$

Assume that $\mathcal{D}_p^\perp$ is spanned by s globally defined orthonormal vectors $\{\xi_i\}$ at each point $p\in M$, $(1\le i,j,...\le s)$, with its dual set $\{\eta_i\}$. Then one obtains

$$\varphi^2=-I+\sum_{i=1}^{s}\eta^i\otimes\xi_i. \tag{2.3}$$

In above case, M is called a globally framed manifold (or simply an f-manifold) ([1], [26] and [27]) and we denote its framed structure by $M(\varphi,\xi_i)$. From the conditions one has

$$\varphi\xi_i=0,\quad \eta^i\circ\varphi=0,\quad \eta^i(\xi_j)=\delta_i^j. \tag{2.4}$$

Now, we consider compatible Riemannian metric g on M with an f-structure such that

$$\begin{gathered} g(\phi X,Y)+g(X,\phi Y)=0,\\ g(\phi X,\phi Y)=g(X,Y)-\sum_{i=1}^{s}\eta^i(X)\eta^i(Y),\\ g(X,\xi_i)=\eta^i(X). \end{gathered} \tag{2.5}$$

In the above case, we say that M is a metric f-manifold and its associated structure will be denoted by $M(\phi,\xi_i,\eta^i,g)$.

A framed structure $M(\phi,\xi_i)$ is said to be normal [27] if the torsion tensor N_ϕ of ϕ is zero i.e., if

$$N_\phi = N + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0, \tag{2.6}$$

where N denotes the Nijenhuis tensor field of ϕ.

Define a 2-form Φ on M by $\Phi(X,Y) = g(X,\phi Y)$, for any $X,Y \in \Gamma(TM)$. The Levi-Civita connection ∇ of a metric f-manifold satisfies the following formula [26]:

$$\begin{aligned} 2g\left((\nabla_X\phi)Y,Z\right) &= 3d\Phi(X,\phi Y,\phi Z) - 3d\Phi(X,Y,Z) \\ &\quad + g\left(N(Y,Z),\phi X\right) + N_j^2(Y,Z)\eta^j(X) \\ &\quad + 2d\eta^j(\phi Y,X)\eta^j(Z) - 2d\eta^j(\phi Z,X)\eta^j(Y), \end{aligned} \tag{2.7}$$

where the tensor field N_j^2 is defined by $N_j^2(X,Y) = \left(L_{\phi X}\eta^j\right)Y - \left(L_{\phi Y}\eta^j\right)X = 2d\eta^j(\phi X,Y) - 2d\eta^j(\phi Y,X)$, for each $j \in \{1,...,s\}$.

Throughout this paper we denote by $\overline{\eta} = \eta^1 + \eta^2 + ... + \eta^s$, $\overline{\xi} = \xi^1 + \xi^2 + ... + \xi^s$ and $\overline{\delta}_i^j = \delta_i^1 + \delta_i^2 + ... + \delta_i^s$.

Definition 2.1. [4] Let $M\left(\varphi,\xi_i,\eta^i,g\right)$ be $(2n+s)$-dimensional a metric f-manifold. For each η^i, $(1 \le i \le s)$ 1-forms and each Φ 2-forms, if $d\eta^i = 0$ and $d\Phi = 2\alpha\overline{\eta} \wedge \Phi$ satisfy,then M is called almost α-cosymplectic f-manifold.

The manifold is called generalized almost Kenmotsu f-manifold for $\alpha = 1$ [4].

Let M be an almost α-cosymplectic f-manifold. Since the distribution $\mathcal{D}$ is integrable, we have $L_{\xi_i}\eta^j = 0$, $[\xi_i,\xi_j] \in \mathcal{D}$ and $[X,\xi_j] \in \mathcal{D}$ for any $X \in \Gamma(\mathcal{D})$. Then the Levi-Civita connection is given by:

(2.8)

$$2g\left((\nabla_X\varphi)Y,Z\right) = 2\alpha\left(\sum_{j=1}^{s}\left(g\left(\varphi X,Y\right)\xi_j - \eta^j\left(Y\right)\varphi X\right),Z\right) + g\left(N\left(Y,Z\right),\varphi X\right),$$

for any $X,Y,Z \in \Gamma(TM)$. Putting $X = \xi_i$ we obtain $\nabla_{\xi_i}\varphi = 0$ which implies $\nabla_{\xi_i}\xi_j \in \mathcal{D}^\perp$ and then $\nabla_{\xi_i}\xi_j = \nabla_{\xi_j}\xi_i$, since $[\xi_i,\xi_j] = 0$.

We put $A_iX = -\nabla_X\xi_i$ and $h_i = \dfrac{1}{2}\left(L_{\xi_i}\varphi\right)$, where L denotes the Lie derivative operator.

Proposition 2.1. [4] *For any $i \in \{1,...,s\}$ the tensor field A_i is a symmetric operator such that*

1) $A_i(\xi_j) = 0$, *for any* $j \in \{1,...,s\}$.

2) $A_i \circ \varphi + \varphi \circ A_i = -2\alpha\varphi$.

3) $tr(A_i) = -2\alpha n$.

Proposition 2.2. [5] *For any $i \in \{1,...,s\}$ the tensor field h_i is a symmetric operator and satisfies*

i) $h_i\xi_j = 0$, *for any* $j \in \{1,...,s\}$.

ii) $h_i \circ \varphi + \varphi \circ h_i = 0$.

iii) $trh_i = 0$.

iv) $tr\varphi h_i = 0$.

Proposition 2.3. [4] ∇_{φ} *satisfies the following relation:*

$$(\nabla_{X\varphi})\,Y + (\nabla_{\varphi X}\varphi)\,\varphi Y = \sum_{i=1}^{s}\left[-\alpha\left(\eta^{i}(Y)\,\varphi X + 2g(X,\varphi Y)\,\xi_{i}\right) - \eta^{i}(Y)\,h_{i}X\right]. \tag{2.9}$$

Definition 2.2. [4] Let M be an almost α-cosymplectic f-manifold, κ,μ and ν are real constants. We say that M verifies the (κ,μ,ν)-nullity condition if and only if for each $i\in\{1,...,s\}$, $X,Y\in\Gamma(TM)$ the following identity holds

$$\begin{aligned} R(X,Y)\,\xi_{i} = \;& \kappa\left(\overline{\eta}(X)\,\varphi^{2}Y - \overline{\eta}(Y)\,\varphi^{2}X\right) \\ & +\mu\left(\overline{\eta}(Y)\,h_{i}X - \overline{\eta}(X)\,h_{i}Y\right) \\ & +\nu\left(\overline{\eta}(Y)\varphi h_{i}X - \overline{\eta}(X)\varphi h_{i}Y\right). \end{aligned} \tag{2.10}$$

Lemma 2.1. [4] *Let M be an almost α-cosymplectic f-manifold verifying (κ,μ,ν)-nullity condition. Then we have*

(i) $h_{i}\circ h_{j} = h_{j}\circ h_{i}$ *for each* $i,j\in\{1,2,...,s\}$

(ii) $\kappa\leq-\alpha^{2}$

(iii) If $\kappa<-\alpha^{2}$ *then, for each* $i\in\{1,2,...,s\}$, h_{i} *has eingen values* $0, \pm\sqrt{-(\kappa+\alpha^{2})}$.

Proposition 2.4. [4] *Let M be an almost α-cosymplectic f-manifold verifying (κ,μ,ν)-nullity condition. Then*

$$h_{1} = ... = h_{s}. \tag{2.11}$$

Remark 2.1. [4] Throughout all this paper whenever (2.10) holds we put $h := h_{1} = ... = h_{s}$. Then (2.10) becomes

$$\begin{aligned} R(X,Y)\,\xi_{i} = \;& \kappa\left(\overline{\eta}(X)\,\varphi^{2}Y - \overline{\eta}(Y)\,\varphi^{2}X\right) \\ & +\mu\left(\overline{\eta}(Y)\,hX - \overline{\eta}(X)\,hY\right) \\ & +\nu\left(\overline{\eta}(Y)\varphi hX - \overline{\eta}(X)\varphi hY\right). \end{aligned} \tag{2.12}$$

Furthermore, using (2.12), the symmetry properties of the curvature tensor and the symmetry of φ^{2} and h, we get

$$\begin{aligned} R(\xi_{i},X)Y = \;& \kappa\left(\overline{\eta}(Y)\varphi^{2}X - g\left(X,\varphi^{2}Y\right)\overline{\xi}\right) \\ & +\mu\left(g\left(X,hY\right)\overline{\xi} - \overline{\eta}(Y)hX\right) \\ & +\nu\left(g\left(\varphi hX,Y\right)\overline{\xi} - \overline{\eta}(Y)\varphi hX\right) \end{aligned} \tag{2.13}$$

Remark 2.2. Let M be an almost α-cosymplectic f-manifold verifying (κ,μ,ν)-nullity condition, with $\kappa\neq-\alpha^{2}$. We denote by $\mathcal{D}_{+}$ and $\mathcal{D}_{-}$ the n-dimensional distributions of the eigenspaces of $\lambda=\sqrt{-(\kappa+\alpha^{2})}$ and $-\lambda$, respectively. We have that $\mathcal{D}_{+}$ and $\mathcal{D}_{-}$ are mutually orthogonal. Moreover, since φ anticommutes with h, we have $\varphi(\mathcal{D}_{+})=\mathcal{D}_{-}$ and $\varphi(\mathcal{D}_{-})=\mathcal{D}_{+}$. In other words, $\mathcal{D}_{+}$ is a Legendrian distribution and $\mathcal{D}_{-}$ is the conjugate Legendrian distribution of $\mathcal{D}_{+}$.

3. Curvature Properties

Let $M\left(\varphi,\xi_i,\eta^i,g\right)$ be a $(2n+s)$-dimensional almost Kenmotsu f-manifold. We consider the $(1,1)$-tensor fields defined by

$$l_{ij}\left(X\right)=R_{X\xi_i}\xi_j$$

for each $X\in\Gamma\left(TM\right),\ i,j\in\{1,...,s\}$ and put $l_i=l_{ii}$.

Proposition 3.1. [4] *Let M be an almost α-cosymplectic f-manifold. Then we have*

$$\begin{aligned}(3.1)\qquad R\left(X,Y\right)\xi_i &= \alpha^2\sum_{k=1}^{s}\left[\eta^k\left(Y\right)\varphi^2X-\eta^k\left(X\right)\varphi^2Y\right]\\ &\quad -\alpha\sum_{k=1}^{s}\left[\eta^k\left(X\right)\left(\varphi\circ h_k\right)Y-\eta^k\left(Y\right)\left(\varphi\circ h_k\right)X\right]\\ &\quad +\left(\nabla_Y\left(\varphi\circ h_i\right)\right)X-\left(\nabla_X\left(\varphi\circ h_i\right)\right)Y,\end{aligned}$$

for each $X\in\Gamma\left(TM\right)$.

Lemma 3.1. [4] *For an almost α-cosymplectic f-manifold with the f-structure $\left(\varphi,\xi_i,\eta^i,g\right)$, the following relations hold*

(3.2)

$$l_{ji}\left(X\right)=\sum_{k=1}^{s}\delta_j^k\left[\alpha^2\varphi^2X+\alpha\left(\varphi\circ h_k\right)X\right]+\alpha\left(\varphi\circ h_i\right)X-\left(h_i\circ h_j\right)X+\varphi\left(\nabla_{\xi_j}h_i\right)X,$$

$$(3.3)\qquad R\left(\xi_j,X\right)\xi_i-\varphi R\left(\xi_j,\varphi X\right)\xi_i=2\left[-\alpha^2\varphi^2X+\left(h_i\circ h_j\right)X\right],$$

$$(3.4)\qquad \left(\nabla_{\xi_j}h_i\right)X=-\varphi l_{ji}\left(X\right)-\alpha^2\varphi X-\alpha h_iX-\alpha h_jX-\left(\varphi\circ h_i\circ h_j\right)X,$$

$$(3.5)\qquad S\left(X,\xi_i\right)=-2n\alpha^2\sum_{k=1}^{s}\eta^k\left(X\right)-\left(div\left(\varphi\circ h_i\right)\right)X,$$

$$(3.6)\qquad S\left(\xi_i,\xi_j\right)=-2n\alpha^2-tr\left(h_j\circ h_i\right),$$

$$(3.7)\qquad \left(\nabla_{\xi_i}h_i\right)X=-\varphi l_{ii}\left(X\right)-\alpha^2\varphi X-2\alpha h_iX-\left(\varphi\circ h_i^2\right)X$$

for each $X\in\Gamma\left(TM\right)$.

Remark 3.1. Let M be an almost α-cosymplectic f-manifold verifying (κ,μ,ν)-nullity condition. Then for each $i,j\in\{1,...,s\}$ we have

$$(3.8)\qquad l_{ji}=-\kappa\varphi^2+\mu h+\nu\varphi h.$$

It follows that all the l_{ji}'s coincide. We put $l=l_{ji}$.

Lemma 3.2. *Let M be an almost α-cosymplectic f-manifold verifying (κ,μ,ν)-nullity condition. Then for each $i\in\{1,...,s\}$ and $X\in\Gamma\left(TM\right)$ we have*

$$(3.9)\qquad \left(\nabla_{\xi_i}h\right)X=-\mu\varphi hX+\nu hX-2\alpha hX,$$

$$(3.10)\qquad l\varphi-\varphi l=2\mu h\varphi+2\nu h,$$

$$(3.11)\qquad l\varphi+\varphi l=2\kappa\varphi,$$

$$(3.12)\qquad Q\xi_i=2n\kappa\overline{\xi}.$$

Proof. From (3.7), using (3.8) and (**??**) we get (3.9). (3.10) and (3.11) follow directly from (3.8) using $h \circ \varphi = -\varphi \circ h$. For the proof of (3.12) we fix $x \in M$ and $\{E_1, ..., E_{2n+s}\}$ a local φ-basis around x with $E_{2n+1} = \xi_1, ..., E_{2n+s} = \xi_s$. Then using (2.13) and $trace\,(h) = 0$ and $trace\,(\varphi h) = 0$ we get $Q\xi_i = \sum_{j=1}^{2n} R_{\xi_i E_j} E_j = \sum_{j=1}^{2n} \kappa g\left(\varphi^2 E_j, E_j\right) \overline{\xi} = \kappa \sum_{j=1}^{2n} \delta_{jj} \overline{\xi}$. □

4. On Pseudo Ricci-Symmetric Almost α-Cosymplectic f-Manifolds

Definition 4.1. A $(2n+s)$-dimensional almost α-cosymplectic f-manifold (M, g) is called a pseudo Ricci-symmetric manifold $(SWRS)_{(2n+s)}$ if

$$(\nabla_X S)(Y, Z) = 2\beta(X) S(Y, Z) + \beta(Y) S(X, Z) + \beta(Z) S(Y, X), \tag{4.1}$$

for any $X, Y, Z \in \Gamma(TM)$, where β is a 1-form and is defined by

$$\beta(X) = g(X, \rho), \tag{4.2}$$

where ρ is associated vector field. This notion was introduced by Chaki [29].

Theorem 4.1. *If a pseudo Ricci-symmetric almost α-cosymplectic f-manifold admits a cyclic parallel Ricci tensor then the* 1*-form β must be vanish.*

Proof. Let (4.1) and (4.2) be satisfied in an almost α-cosymplectic f-manifold. Taking the cyclic sum in (4.1) we get,

$$\begin{aligned} &(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) \\ &= 4\left(\beta(X) S(Y, Z) + \beta(Y) S(X, Z) + \beta(Z) S(X, Y)\right) \end{aligned} \tag{4.3}$$

Let M admit a cyclic Ricci tensor. Then (4.3) reduces to

$$\left(\beta(X) S(Y, Z) + \beta(Y) S(X, Z) + \beta(Z) S(X, Y)\right) = 0. \tag{4.4}$$

Taking $Z = \xi_i$ in (4.4) and using (3.5) and (4.2), we obtain

$$\begin{aligned} &\beta(X)\left[-2n\alpha^2 \textstyle\sum_{k=1}^{s} \eta^k(Y) - (div(\varphi \circ h_i)) Y\right] \\ &+\beta(Y)\left[-2n\alpha^2 \textstyle\sum_{k=1}^{s} \eta^k(X) - (div(\varphi \circ h_i)) X\right] \\ &+\eta_i(\rho) S(X, Y) = 0. \end{aligned} \tag{4.5}$$

Now putting $Y = \xi_i$ in (4.5) and using (2.4), (3.5) (4.2) and $(div(\varphi \circ h_i))\xi_i$, we get

$$\begin{aligned} &\beta(X)\left[-2n\alpha^2\right] \\ &+\eta_i(\rho)\left[-2n\alpha^2 \textstyle\sum_{k=1}^{s} \eta^k(X) - (div(\varphi \circ h_i)) X\right] \\ &+\eta_i(\rho)\left[-2n\alpha^2 \textstyle\sum_{k=1}^{s} \eta^k(X) - (div(\varphi \circ h_i)) X\right] = 0. \end{aligned} \tag{4.6}$$

Taking $X = \xi_i$ (4.6) and using (2.4), (3.5) (4.2) and $(div(\varphi \circ h_i))\xi_i$, we have

$$\eta_i(\rho) = 0. \tag{4.7}$$

So by use of (4.7) in (4.6), since $\alpha \neq 0$, we get $\beta(X) = 0$, for any vector field X on M. This completes the proof of the theorem. □

Theorem 4.2. *If a pseudo Ricci-symmetric almost α-cosymplectic f-manifold verifying (κ, μ, ν)-nullity condition, admits a cyclic parallel Ricci tensor then the* 1*-form β must be vanish.*

Proof. Let (4.1) and (4.2) be satisfied in an almost α-cosymplectic f-manifold verifying (κ, μ, ν)-nullity condition, . Taking the cyclic sum in (4.1) we get,

$$\begin{aligned} &(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) \\ &= 4\left(\beta(X) S(Y, Z) + \beta(Y) S(X, Z) + \beta(Z) S(X, Y)\right) \end{aligned} \tag{4.8}$$

Let M admit a cyclic Ricci tensor. Then (4.8) reduces to

$$(\beta(X)S(Y,Z)+\beta(Y)S(X,Z)+\beta(Z)S(X,Y))=0. \tag{4.9}$$

Taking $Z=\xi_i$ in (4.9) and using (3.12) and (4.2), we obtain

$$\beta(X)[2n\kappa\overline{\eta}(Y)]+\beta(Y)[2n\kappa\overline{\eta}(X)]+\eta_i(\rho)S(X,Y)=0. \tag{4.10}$$

Now putting $Y=\xi_i$ in (4.10) and using (2.4), (3.12) and (4.2), we get

$$\beta(X)[2n\kappa]+\eta_i(\rho)[2n\kappa\overline{\eta}(X)]+\eta_i(\rho)[2n\kappa\overline{\eta}(X)]=0. \tag{4.11}$$

Taking $X=\xi_i$ (4.11) and using (2.4), (3.12) and (4.2), we have

$$\eta_i(\rho)=0. \tag{4.12}$$

So by use of (4.12) in (4.11), we get $\beta(X)=0$, for any vector field X on M. This completes the proof of the theorem. □

Theorem 4.3. *A pseudo Ricci-symmetric almost α-cosymplectic f-manifold can not be an Einstein manifold if the 1-form $\beta=0$.*

Proof. For an Einstein manifold $(\nabla_X S)(Y,Z)$ and $S(Y,Z)=kg(Y,Z)$ $(k\in\mathbb{R})$ then (4.1) gives

$$2\beta(X)S(Y,Z)+\beta(Y)S(X,Z)+\beta(Z)S(Y,X)=0. \tag{4.13}$$

Taking $Z=\xi_i$ in (4.13) and using (3.5) and (4.2), we obtain

$$\begin{aligned}&2\beta(X)\left[-2n\alpha^2\textstyle\sum_{k=1}^{s}\eta^k(Y)-(div(\varphi\circ h_i))Y\right]\\&+\beta(Y)\left[-2n\alpha^2\textstyle\sum_{k=1}^{s}\eta^k(X)-(div(\varphi\circ h_i))X\right]\\&+\eta_i(\rho)S(Y,X)=0.\end{aligned} \tag{4.14}$$

Now putting $Y=\xi_i$ in (4.14) and using (2.4), (3.5) (4.2) and $(div(\varphi\circ h_i))\xi_i$, we get

$$\begin{aligned}&2\beta(X)\left[-2n\alpha^2\right]\\&+\eta_i(\rho)\left[-2n\alpha^2\textstyle\sum_{k=1}^{s}\eta^k(X)-(div(\varphi\circ h_i))X\right]\\&+\eta_i(\rho)\left[-2n\alpha^2\textstyle\sum_{k=1}^{s}\eta^k(X)-(div(\varphi\circ h_i))X\right]=0\end{aligned} \tag{4.15}$$

Taking $X=\xi_i$ in (4.15) and using (2.4), (3.5) (4.2) and $(div(\varphi\circ h_i))\xi_i$, we have

$$\eta_i(\rho)=0. \tag{4.16}$$

So by use of (4.16) in (4.15), since $\alpha\neq 0$, we get $\beta(X)=0$, for any vector field X on M. This completes the proof of the theorem. □

Theorem 4.4. *A pseudo Ricci-symmetric almost α-cosymplectic f-manifold verifying (κ,μ,ν)-nullity condition,can not be an Einstein manifold if the 1-form $\beta\neq 0$.*

Proof. From the previous theorem, taking $Z=\xi_i$ in (4.13) and using (3.5) and (4.2), we obtain

$$2\beta(X)[2n\kappa\overline{\eta}(Y)]+\beta(Y)[2n\kappa\overline{\eta}(X)]+\eta_i(\rho)S(Y,X)=0. \tag{4.17}$$

Now putting $Y=\xi_i$ in (4.17) and using (2.4), (3.12) and (4.2), we get

$$2\beta(X)[2n\kappa]+\eta_i(\rho)[2n\kappa\overline{\eta}(X)]+\eta_i(\rho)[2n\kappa\overline{\eta}(X)]=0 \tag{4.18}$$

Taking $X=\xi_i$ in (4.18) and using (2.4), (3.12) and (4.2), we have

$$\eta_i(\rho)=0. \tag{4.19}$$

So by use of (4.19) in (4.18), we get $\beta(X)=0$, for any vector field X on M. This completes the proof of the theorem. □

Theorem 4.5. *The Ricci tensor of a pseudo Ricci-symmetric almost α-cosymplectic f-manifold is parallel.*

Proof. Taking $Z=\xi_i$ in (4.1), we get

$$(\nabla_X S)(Y,\xi_i) = 2\beta(X) S(Y,\xi_i) + \beta(Y) S(X,\xi_i) + \beta(\xi_i) S(Y,X). \tag{4.20}$$

The left-hand side can be written in the form

$$(\nabla_X S)(Y,\xi_i) = \nabla_X S(Y,\xi_i) - S(\nabla_X Y,\xi_i) - S(Y,\nabla_X \xi_i). \tag{4.21}$$

Then, in view of (3.5), (4.2) and (4.21), equation (4.20) becomes

$$\begin{aligned} &\nabla_X S(Y,\xi_i) - S(\nabla_X Y,\xi_i) - S(Y,\nabla_X \xi_i) = \\ &2\beta(X)\left[-2n\alpha^2 \textstyle\sum_{k=1}^{s} \eta^k(Y) - (div(\varphi\circ h_i))Y\right] \\ &+\beta(Y)\left[-2n\alpha^2 \textstyle\sum_{k=1}^{s} \eta^k(X) - (div(\varphi\circ h_i))X\right] \\ &+\eta_i(\rho) S(Y,X). \end{aligned} \tag{4.22}$$

Now putting $Y=\xi_i$ in (4.22) and using (2.3), (2.4) (**??**) (3.5), (4.2) and $(div(\varphi\circ h_i))\xi_i$, we get

$$\begin{aligned} &0 = 2\beta(X)\left[-2n\alpha^2\right] \\ &+\eta_i(\rho)\left[-2n\alpha^2 \textstyle\sum_{k=1}^{s} \eta^k(X) - (div(\varphi\circ h_i))X\right] \\ &+\eta_i(\rho)\left[-2n\alpha^2 \textstyle\sum_{k=1}^{s} \eta^k(X) - (div(\varphi\circ h_i))X\right]. \end{aligned} \tag{4.23}$$

Taking $X=\xi_i$ in (4.23)

$$\eta_i(\rho) = 0. \tag{4.24}$$

Using (4.24) in (4.23)

$$\beta(X) = 0 \tag{4.25}$$

for any vector field X on M. Hence in view of (4.25) in (4.1), we obtain $\nabla_X S = 0$, which is proves the result. □

Theorem 4.6. *The Ricci tensor of a pseudo Ricci-symmetric almost α-cosymplectic f-manifold is parallel.*

Proof. Taking $Z=\xi_i$ in (4.1), we get

$$(\nabla_X S)(Y,\xi_i) = 2\beta(X) S(Y,\xi_i) + \beta(Y) S(X,\xi_i) + \beta(\xi_i) S(Y,X). \tag{4.26}$$

The left-hand side can be written in the form

$$(\nabla_X S)(Y,\xi_i) = \nabla_X S(Y,\xi_i) - S(\nabla_X Y,\xi_i) - S(Y,\nabla_X \xi_i). \tag{4.27}$$

Then, in view of (3.12), (4.2) and (4.27), equation (4.26) becomes

$$\begin{aligned} &\nabla_X S(Y,\xi_i) - S(\nabla_X Y,\xi_i) - S(Y,\nabla_X \xi_i) = \\ &2\beta(X)\left[2n\kappa\overline{\eta}(Y)\right] + \beta(Y)\left[2n\kappa\overline{\eta}(X)\right] \\ &+\eta_i(\rho) S(Y,X). \end{aligned} \tag{4.28}$$

Now putting $Y=\xi_i$ in (4.28) and using (2.3), (2.4) (**??**) (3.12) and (4.2) we get

$$0 = 2\beta(X)\left[2n\kappa\right] + \eta_i(\rho)\left[2n\kappa\overline{\eta}(X)\right] + \eta_i(\rho)\left[2n\kappa\overline{\eta}(X)\right]. \tag{4.29}$$

Taking $X=\xi_i$ in (4.29)

$$\eta_i(\rho) = 0. \tag{4.30}$$

Using (4.30) in (4.29)

$$\beta(X) = 0 \tag{4.31}$$

for any vector field X on M. Hence in view of (4.31) in (4.1), we obtain $\nabla_X S = 0$, which is proves the result. $\square$

References

[1] Yano, K., On a structure defined by a tensor field of type $(1,1)$ satisfying $f^3 + f = 0$, Tensor N. S., 14, 99-109, (1963).

[2] Golberg, S, and Yano, K., Globally framed f-manifolds, Illinois J. Math. 15, 456-474, (1971).

[3] Falcitelli, M. and Pastore, A. M., f-structure of Kenmotsu Type, Mediterr. J. Math. 3 549-564, (2006).

[4] Öztürk H., Aktan N., Murathan C., Vanlı Turgut A., Almost α-cosymplectic f-manifolds, Annals of the Alexandru Ioan Cuza University-Mathematics, in press.

[5] Blair D., Kouforgiorgos T. and Papantoniou B. J., Contact metric manifolds satisfying a nullity condition, Israel J. Math., 91 (1995), 189-214.

[6] Blair D. E., Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics, 203. Birkhâuser Boston, Inc., Boston, MA, 2002.

[7] Olszak Z., Locally conformal almost cosymplectic manifolds, Coll. Math., 57 (1989), 73–87.

[8] Blair D. E. and Goldberg S. I., Topology of almost contact manifolds, J. Differential geometry, 1(1967), 347-354.

[9] Goldberg S.I. and Yano K. Integrability of almost cosymplectic structures, Pasific J. Math., 31 (1969), 373-382.

[10] Blair D. E., The theory of quasi-Sasakian structures, J. Diff. Geometry, 1 (1967), 331-345.

[11] Cordero L. A., Fernandez M. and De Leon M., Examples of compact almost contact manifolds admitting neither Sasakian nor cosymplectic structures, Atti Sem. Mat. Univ. Modena, 34 (1985-86), 43-54.

[12] Chinea D. and Gonzalez C., An example of almost cosymplectic homogeneous manifold, in: Lect. Notes Math. Vol. 1209, Springer-Verlag, Berlin-Heildelberg-New York, (1986), 133-142.

[13] Olszak Z., On almost cosymplectic manifolds, Kodai Math. J., 4 (1981), 239-250.

[14] Olszak Z., Almost cosymplectic manifolds with Kählerian leaves, Tensor N. S., 46 (1987), 117-124.

[15] Blair D. E., Goldberg S. I., Topology of almost contact manifolds, J. Diff. Geometry, 1 (1967), 347-354.

[16] Chinea D., De Leon M., Marrero J. C., Topology of cosymplectic manifolds, J. Math. Pures Appl., 72 (1993), 567-591.

[17] Chinea D., De Leon M., Marrero J. C., Coeffective cohomology on almost cosymplectic manifolds, Bull. Sci. Math., 119 (1995), 3-20.

[18] Libermann M. P., Sur les automorphismes infinitesimaux des structures symplectiques et des structures de contact, in: Colloque de Geometrie Differentielle Globale (Bruxelles, 1958), Centre Belge de Recherche Mathematiques Louvain, (1959), 37-59.

[19] Lichnerowicz A., Theoremes de reductivite sur des algebres d'automorphismes, Rend. Mat., 22 (1963), 197-244.

[20] Fujimoto A. and Muto H., On cosymplectic manifolds, Tensor N. S., 28 (1974), 43-52.

[21] Kirichenko V. F., Almost cosymplectic manifolds satisfying the axiom of Φ-holomorphic planes (in Russian), Dokl. Akad. Nauk SSSR, 273 (1983), 280-284.

[22] Endo H., On Ricci curvatures of almost cosymplectic manifolds, An. Ştiint. Univ. "Al. I. Cuza" Iaşi, Mat., 40 (1994), 75-83.

[23] Boeckx E., A full classification of contact metric (κ, μ)-spaces, Illinois J. Math., 44 (1) (2000), 212-219.

[24] Koufogiorgos Th.and Tsichlias C., On the existence of a new class of contact metric manifolds, Canad. Math. Bull., 43 (2000), 440-447.

[25] Yano, K. and Kon, M., Structures on Manifolds, Series in Pure Math, Vol 3, World Sci, 1984.

[26] Blair, D.E., Geometry of manifolds with structural group $U(n) \times O(s)$, J. Diff . Geom., 4(1970), 155-167.

[27] Goldberg, S.I. and Yano, K., On normal globally framed f-manifolds, Tohoku Math. J., 22(1970), 362-370.

[28] Chaki M. C., On pseudo symmetric manifolds, An. Stiint. Univ. Al. I. Cuza Iasi Sect. I a Mat., 33 (1) (1987), 53-58.

[29] Chaki M. C.,On pseudo Ricci symmetric manifolds, Bulgar. J. Phys., 15 (6) (1988), 526-531.

[30] Tamassy L. and Binh T. Q., On weak symmetries of Einstein and Sasakian manifolds,International Conference on Differential Geometry and its Applications (Bucharest,1992). Tensor (N.S.), 53 (1993), Commemoration Volume I, 140-148.

[31] Tamassy L. and Binh T. Q., On weakly symmetric and weakly projective symmetric Riemannian manifolds, Differential geometry and its applications (Eger, 1989), 663-670, Colloq. Math. Soc. Janos Bolyai, 56, North-Holland, Amsterdam, 1992.

[32] De U. C., Shaikh A. A. and Biswas S., On weakly symmetric contact metric manifolds, Tensor (N.S.), 64 (2) (2003), 170-175.

[33] Özgür C., On weak symmetries of Lorentzian para-Sasakian manifolds, Rad. Mat., 11 (2) (2002/03), 263-270.

[34] Özgür C., On weakly symmetric Kenmotsu manifolds, Differ. Geom. Dyn. Syst., 8 (2006), 204-209.

[35] Singh H. and Khan Q., On special weakly symmetric Riemannian manifolds, Publ. Math. Debrecen, Hungary 58 (2001), 523-536.

[36] Aktan N., Görgülü A. and Özüsağlam E., On Special Weakly Symmetric Kenmotsu Manifolds, Sarajevo Journal of Mathematics, Vol.3 (15) (2007), 93-97.

(Yavuz Selim Balkan) Duzce University, Faculty of Art and Sciences, Department of Mathematics, Duzce/TURKEY

E-mail address: `y.selimbalkan@gmail.com`

(Nesip Aktan) Duzce University, Faculty of Art and Sciences, Department of Mathematics, Duzce/TURKEY

E-mail address: `nesipaktan@gmail.com`

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GENERALIZATION OF DIFFERENT TYPE INTEGRAL INEQUALITIES VIA FRACTIONAL INTEGRALS FOR FUNCTIONS WHOSE SECOND DERIVATIVES ABSOLUTE VALUES ARE QUASI-CONVEX

İMDAT İŞCAN

ABSTRACT. In this paper, the author establish some new estimates on Hermite-Hadamard type and Simpson type inequalities via Riemann Liouville fractional integral for functions whose second derivatives in absolute values at certain power are quasi-convex.

1. INTRODUCTION

The following definition for convex functions is well known in the mathematical literature:

Definition 1.1. A function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is said to be convex on $[a,b]$ if

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated as follow:

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function defined on the interval I of real numbers and $a,b\in I$ with $a<b$. Then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

The following inequality is one of the best-known results in the literature as Simpson's inequality:

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Theorem 1.1. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup\limits_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int\limits_{a}^{b}f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^{4}.$$

In recent years many authors were established an error estimations for both the Hermite-Hadamard inequality and the Simpson's inequality, for refinements, counterparts, generalizations and new inequalities for them see [2, 3, 4, 6, 7, 8, 10, 11].

We recall that the notion of quasi-convex function generalizes the notion of convex function. More exactly, a function $f:[a,b]\subset\mathbb{R}\rightarrow\mathbb{R}$ is said to be quasi-convex on $[a,b]$ if

$$f(tx+(1-t)y)\leq max\{f(x),f(y)\}$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex ([5]).

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.2. Let $f\in L[a,b]$. The Riemann-Liouville integrals $J_{a^{+}}^{\alpha}f$ and $J_{b^{-}}^{\alpha}f$ of oder $\alpha>0$ with $a\geq0$ are defined by

$$J_{a^{+}}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int\limits_{a}^{x}(x-t)^{\alpha-1}f(t)dt,\ x>a$$

and

$$J_{b^{-}}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int\limits_{x}^{b}(t-x)^{\alpha-1}f(t)dt,\ x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha)=\int\limits_{0}^{\infty}e^{-t}t^{\alpha-1}dt$ and $J_{a^{+}}^{0}f(x)=J_{b^{-}}^{0}f(x)=f(x)$.

In the case of $\alpha=1$, the fractional integral reduces to the classical integral. For some recent results connected with fractional integral ineqalities, see [9, 10, 11, 12, 13].

In [4], Barani et al. obtained the following theorems related to the right-hand side of (1.1) for functions whose second derivatives in absolute values at certain power are quasiconvex.

Theorem 1.2. *Let $f:I\rightarrow\mathbb{R}$ be a twice differentiable function on I° such that $f''\in L[a,b]$, where $a,b\in I^{\circ}$ with $a<b$. If $|f''|^{q}$ is quasi-convex on $[a,b]$ for $q\geq1$,*

then the following inequality holds

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \tag{1.2}$$
$$\leq \frac{(b-a)^2}{24}\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\}.$$

Theorem 1.3. *Let $f: I\to\mathbb{R}$ be a twice differentiable function on I° such that $f''\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f''|^{p/p-1}$ is quasi-convex on $[a,b]$, for $p>1$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \tag{1.3}$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\},$$

where $1/p+1/q=1$.

In [2], Alomari et al. established the following result connected with Simpson's type inequalities for twice differentiable functions:

Theorem 1.4. *Let $f': I\subset[0,\infty)\to\mathbb{R}$ be an absolutely continuous function on I° and $a,b\in I^\circ$ with $a<b$, such that $f''\in L[a,b]$. If $|f''|^q$ is quasi-convex on $[a,b]$, $q\geq 1$, then the following inequality holds*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \tag{1.4}$$
$$\leq \frac{(b-a)^2}{162}\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\}.$$

In [10], Sarıkaya et al. established some results connected with the left-hand side of (1.1) as follows:

Theorem 1.5. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable function on I° such that $f''\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f''|^q$ is quasi-convex on $[a,b]$ for $q\geq 1$, then the following inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{24}\left(\max\left\{\left|f''(a)\right|^q,\left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}. \tag{1.5}$$

Theorem 1.6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L[a,b]$, where $a, b \in I^{\circ}$ with $a < b$. If $|f''|^q$ is quasi-convex on $[a,b]$ for $q > 1$, then the following inequality holds:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)} \int_a^b f(x)dx \right| \tag{1.6}$$
$$\leq \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}} \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q \right\} \right)^{\frac{1}{q}},$$

where $1/p + 1/q = 1$.

We will establish some new results using the following Lemma:

Lemma 1.1. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a,b]$, where $a, b \in I$ with $a < b$. Then for all $x \in [a,b]$, $\lambda \in [0,1]$ and $\alpha > 0$ we have:*

$$(1-\lambda)\left[\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{b-a}\right] f(x) + \lambda \left[\frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a}\right]$$
$$+ \left(\frac{1}{\alpha+1} - \lambda\right)\left[\frac{(b-x)^{\alpha+1} - (x-a)^{\alpha+1}}{b-a}\right] f'(x) - \frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha} f(a) + J_{x^+}^{\alpha} f(b)\right]$$
$$= \frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)} \int_0^1 t\left((\alpha+1)\lambda - t^{\alpha}\right) f''(tx + (1-t)a)\, dt$$
$$+ \frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)} \int_0^1 t\left((\alpha+1)\lambda - t^{\alpha}\right) f''(tx + (1-t)b)\, dt. \tag{1.7}$$

A simple proof of equality can be given by performing an twice integration by parts in the integrals from the right side and changing the variable (see [6]).

The main aim of this article is to establish a generalization of Hermite Hadamard-type and Simpson-type inequalities via fractional integrals for functions whose absolute values of second derivatives are quasi-convex. By using the integral equality (1.7), the author establish some new inequalities of the Simpson-like and the Hermite-Hadamard-like type for these functions.

2. Main Results

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, the interior of I, throughout this section we will take

$$S_f(x, \lambda, \alpha; a, b)$$
$$= (1-\lambda)\left[\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{b-a}\right] f(x) + \lambda \left[\frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a}\right]$$
$$+ \left(\frac{1}{\alpha+1} - \lambda\right)\left[\frac{(b-x)^{\alpha+1} - (x-a)^{\alpha+1}}{b-a}\right] f'(x) - \frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha} f(a) + J_{x^+}^{\alpha} f(b)\right]$$

where $a, b \in I$ with $a < b$, $x \in [a,b]$, $\lambda \in [0,1]$, $\alpha > 0$ and Γ is Euler Gamma function.

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is quasi-convex on $[a,b]$ for some fixed $q \geq 1$, then for $x \in [a,b]$, $\lambda \in [0,1]$ and $\alpha > 0$, the following inequality for fractional integrals holds*

$$\begin{aligned} &|S_f(x,\lambda,\alpha;a,b)| \\ \leq\ & C_1(\alpha,\lambda)\left\{\frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\max\left\{|f''(x)|^q,|f''(a)|^q\right\}\right)^{\frac{1}{q}}\right. \\ &\left.+\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\max\left\{|f''(x)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}\right\}, \end{aligned} \tag{2.1}$$

where

$$C_1(\alpha,\lambda)=\begin{cases} \frac{\alpha[(\alpha+1)\lambda]^{\frac{\alpha+2}{\alpha}}+1}{\alpha+2}-\frac{(\alpha+1)\lambda}{2}, & 0\leq\lambda\leq\frac{1}{\alpha+1} \\ \frac{(\alpha+1)(\alpha+2)\lambda-2}{2(\alpha+2)}, & \frac{1}{\alpha+1}<\lambda\leq 1 \end{cases}.$$

Proof. From Lemma 1.1, property of the modulus and using the power-mean inequality we have

$$\begin{aligned} |S_f(x,\lambda,\alpha;a,b)| &\leq \frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\int_0^1 |t|\,|(\alpha+1)\lambda-t^\alpha|\,|f''(tx+(1-t)a)|\,dt \\ &+\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\int_0^1 |t|\,|(\alpha+1)\lambda-t^\alpha|\,|f''(tx+(1-t)b)|\,dt \\ \leq\ & \frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\int_0^1 t\,|(\alpha+1)\lambda-t^\alpha|\,dt\right)^{1-\frac{1}{q}} \\ &\times\left(\int_0^1 t\,|(\alpha+1)\lambda-t^\alpha|\,|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}} \\ &+\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\int_0^1 t\,|t(\alpha+1)\lambda-t^\alpha|\,dt\right)^{1-\frac{1}{q}} \\ &\times\left(\int_0^1 t\,|(\alpha+1)\lambda-t^\alpha|\,|f''(tx+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}. \end{aligned} \tag{2.2}$$

Since $|f''|^q$ is quasi-convex on $[a,b]$ we get

$$\begin{aligned} \int_0^1 t\,|(\alpha+1)\lambda-t^\alpha|\,|f''(tx+(1-t)a)|^q\,dt &\leq \int_0^1 t\,|(\alpha+1)\lambda-t^\alpha|\max\left\{|f''(x)|^q,|f''(a)|^q\right\}dt \\ &= \max\left\{|f''(x)|^q,|f''(a)|^q\right\} \end{aligned} \tag{2.3}$$

$$\int_0^1 t\left|(\alpha+1)\lambda-t^{\alpha}\right|\left|f''(tx+(1-t)b)\right|^q dt \leq \int_0^1 t\left|(\alpha+1)\lambda-t^{\alpha}\right|\max\left\{\left|f''(x)\right|^q,\left|f''(b)\right|^q\right\}dt$$

$$(2.4) \qquad = \max\left\{\left|f''(x)\right|^q,\left|f''(a)\right|^q\right\},$$

where we use the fact that

$$(2.5) \qquad C_1(\alpha,\lambda)=\int_0^1 t\left|(\alpha+1)\lambda-t^{\alpha}\right|dt$$

$$=\begin{cases}\begin{array}{c}(\alpha+1)\lambda\int_0^{[(\alpha+1)\lambda]^{\frac{1}{\alpha}}} tdt-\int_0^{[(\alpha+1)\lambda]^{\frac{1}{\alpha}}} t^{\alpha+1}dt \\ -(\alpha+1)\lambda\int_{[(\alpha+1)\lambda]^{\frac{1}{\alpha}}}^1 tdt+\int_{[(\alpha+1)\lambda]^{\frac{1}{\alpha}}}^1 t^{\alpha+1}dt\end{array}, & 0\leq\lambda\leq\frac{1}{\alpha+1} \\ (\alpha+1)\lambda\int_0^1 tdt-\int_0^1 t^{\alpha+1}dt, & \frac{1}{\alpha+1}<\lambda\leq 1\end{cases}$$

$$=\begin{cases}\frac{\alpha[(\alpha+1)\lambda]^{\frac{\alpha+2}{\alpha}}+1}{\alpha+2}-\frac{(\alpha+1)\lambda}{2}, & 0\leq\lambda\leq\frac{1}{\alpha+1} \\ \frac{(\alpha+1)(\alpha+2)\lambda-2}{2(\alpha+2)}, & \frac{1}{\alpha+1}<\lambda\leq 1\end{cases},$$

Hence, If we use (2.3), (2.4) and (2.5) in (2.2), we obtain the desired result. This completes the proof. □

Corollary 2.1. *In Theorem 2.1, if we take $q=1$, then we have*

$$\begin{aligned}&\left|S_f(x,\lambda,\alpha;a,b)\right| \\ \leq\ &\left\{\frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\max\left\{\left|f''(x)\right|,\left|f''(a)\right|\right\}\right)\right. \\ &\left.+\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\max\left\{\left|f''(x)\right|,\left|f''(b)\right|\right\}\right)\right\}.\end{aligned}$$

Corollary 2.2. *In Theorem 2.1, if we take $x=\frac{a+b}{2}$, then we have*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}}{(b-a)^{\alpha-1}}S_f\left(\frac{a+b}{2},\lambda,\alpha;a,b\right)\right| \\ =\ &\left|(1-\lambda)f\left(\frac{a+b}{2}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{\Gamma(\alpha+1)2^{\alpha-1}}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]\right| \\ \leq\ &\frac{(b-a)^2}{8(\alpha+1)}C_1(\alpha,\lambda)\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Corollary 2.3. *In Theorem 2.1, if we take* $x = \frac{a+b}{2}$ *and* $\lambda = \frac{1}{3}$*, then we have*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}}{(b-a)^{\alpha-1}} S_f\left(\frac{a+b}{2}, \frac{1}{3}, \alpha; a, b\right)\right| \\
= &\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{\Gamma(\alpha+1)\, 2^{\alpha-1}}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)\right]\right| \\
\leq &\frac{(b-a)^2}{8(\alpha+1)} C_1\left(\alpha, \frac{1}{3}\right)\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Remark 2.1. In Corollary 2.3, if we choose $\alpha = 1$, we have the following Simpson type inequality

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{(b-a)}\int_a^b f(x)dx\right| \\
\leq &\frac{(b-a)^2}{162}\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

which is the same of the inequality (1.4).

Corollary 2.4. *In Theorem 2.1, if we take* $x = \frac{a+b}{2}$ *and* $\lambda = 0$*, then we have*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}}{(b-a)^{\alpha-1}} S_f\left(\frac{a+b}{2}, 0, \alpha; a, b\right)\right| \\
= &\left|f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)\, 2^{\alpha-1}}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)\right]\right| \\
\leq &\frac{(b-a)^2}{8(\alpha+1)(\alpha+2)}\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Remark 2.2. In Corollary 2.4, if we choose $\alpha = 1$, we have the following midpoint inequality

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)} \int_a^b f(x)dx \right| \\ \leq\ & \frac{(b-a)^2}{48} \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(a)|^q \right\} \right)^{\frac{1}{q}} \right. \\ &\left. + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

which is better than the inequality (1.5).

Corollary 2.5. *In Theorem 2.1, if we take $x = \frac{a+b}{2}$ and $\lambda = 1$, then we have*

$$\begin{aligned} &\left| \frac{2^{\alpha-1}}{(b-a)^{\alpha-1}} S_f\left(\frac{a+b}{2}, 1, \alpha; a, b\right) \right| \\ =\ & \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)\, 2^{\alpha-1}}{(b-a)^\alpha} \left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) \right] \right| \\ \leq\ & \frac{\alpha(\alpha+3)(b-a)^2}{16(\alpha+1)(\alpha+2)} \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(a)|^q \right\} \right)^{\frac{1}{q}} \right. \\ &\left. + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Remark 2.3. In Corollary 2.5, if we choose $\alpha = 1$, we have the following midpoint inequality

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{(b-a)} \int_a^b f(x)dx \right| \\ \leq\ & \frac{(b-a)^2}{24} \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(a)|^q \right\} \right)^{\frac{1}{q}} \right. \\ &\left. + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

which is the same of the inequality (1.2).

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is quasi-convex on $[a,b]$ for some fixed $q > 1$, then for $x \in [a,b]$, $\lambda \in [0,1]$ and $\alpha > 0$, the following inequality for* *fractional integrals holds*

$$
\begin{aligned}
(2.6)\qquad & \left|S_f(x,\lambda,\alpha;a,b)\right| \\
\leq\ & C_2^{\frac{1}{p}}(\alpha,\lambda,p)\left\{\frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\
& \left.+\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\},
\end{aligned}
$$

where $p=\frac{q}{q-1}$,

$$
\begin{aligned}
& C_2(\alpha,\lambda,p) \\
= & \left\{\begin{array}{ll}
\frac{1}{p(\alpha+1)+1}, & \lambda=0 \\
\left[\begin{array}{c}\frac{[(\alpha+1)\lambda]^{\frac{1+(\alpha+1)p}{\alpha}}}{\alpha}\beta\left(\frac{1+p}{\alpha},1+p\right) \\ +\frac{[1-(\alpha+1)\lambda]^{p+1}}{\alpha(p+1)}\cdot{}_2F_1\left(1-\frac{1+p}{\alpha},1;p+2;1-(\alpha+1)\lambda\right)\end{array}\right], & 0<\lambda\leq\frac{1}{\alpha+1} \\
\frac{[(\alpha+1)\lambda]^{\frac{1+(\alpha+1)p}{\alpha}}}{\alpha}\beta\left(\frac{1}{(\alpha+1)\lambda};\frac{1+p}{\alpha},1+p\right), & \frac{1}{\alpha+1}<\lambda\leq 1
\end{array}\right. ,
\end{aligned}
$$

${}_2F_1$ *is Hypergeometric function defined by*

$$
{}_2F_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt,\ c>b>0,\ |z|<1 \text{ (see [1])},
$$

β *is Euler Beta function defined by*

$$
\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\quad x,y>0,
$$

and

$$
\beta(a,x,y)=\int_0^a t^{x-1}(1-t)^{y-1}\,dt,\quad 0<a<1,\ x,y>0,
$$

is incomplete Beta function.

Proof. From Lemma 1.1, property of the modulus and using the Hölder inequality we have

$$
\begin{aligned}
& \left|S_f(x,\lambda,\alpha;a,b)\right|\leq\frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\int_0^1|t|\left|(\alpha+1)\lambda-t^\alpha\right|\left|f''(tx+(1-t)a)\right|dt \\
& +\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\int_0^1|t|\left|(\alpha+1)\lambda-t^\alpha\right|\left|f''(tx+(1-t)b)\right|dt \\
\leq\ & \frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\int_0^1 t^p\left|(\alpha+1)\lambda-t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
(2.7)\ & +\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\int_0^1 t^p\left|(\alpha+1)\lambda-t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}
$$

Since $|f''|^q$ is quasi-convex on $[a,b]$ we get

$$\int_0^1 |f''(tx+(1-t)a)|^q\,dt \le \max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q\right\} \tag{2.8}$$

$$\int_0^1 |f''(tx+(1-t)b)|^q\,dt \le \max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(b)|^q\right\} \tag{2.9}$$

and

$$\int_0^1 t^p\,|(\alpha+1)\lambda - t^\alpha|^p\,dt \tag{2.10}$$

$$= \begin{cases} \int_0^1 t^{(\alpha+1)p}dt & \lambda=0 \\ \int_0^{[(\alpha+1)\lambda]^{\frac{1}{\alpha}}} t^p\,[(\alpha+1)\lambda - t^\alpha]^p\,dt + \int_{[(\alpha+1)\lambda]^{\frac{1}{\alpha}}}^1 t^p\,[t^\alpha-(\alpha+1)\lambda]^p\,dt, & 0<\lambda\le\frac{1}{\alpha+1} \\ \int_0^1 t^p\,[(\alpha+1)\lambda - t^\alpha]^p\,dt, & \frac{1}{\alpha+1}<\lambda\le 1 \end{cases}$$

$$= \begin{cases} \frac{1}{p(\alpha+1)+1}, & \lambda=0 \\ \left[\begin{array}{c} \frac{[(\alpha+1)\lambda]^{\frac{1+(\alpha+1)p}{\alpha}}}{\alpha}\beta\left(\frac{1+p}{\alpha},1+p\right) \\ +\frac{[1-(\alpha+1)\lambda]^{p+1}}{\alpha(p+1)}\cdot{}_2F_1\left(1-\frac{1+p}{\alpha},1;p+2;1-(\alpha+1)\lambda\right)\end{array}\right], & 0<\lambda\le\frac{1}{\alpha+1} \\ \frac{[(\alpha+1)\lambda]^{\frac{(\alpha+1)p+1}{\alpha}}}{\alpha}\beta\left(\frac{1}{(\alpha+1)\lambda};\frac{1+p}{\alpha},1+p\right), & \frac{1}{\alpha+1}<\lambda\le 1 \end{cases}$$

Hence, If we use (2.8), (2.9) and (2.10) in (2.7), we obtain the desired result. This completes the proof. □

Corollary 2.6. *In Theorem 2.2, if we take $x=\frac{a+b}{2}$, then we have*

$$\begin{aligned} &|S_f(x,\lambda,\alpha;a,b)| \\ \le\ & C_2^{\frac{1}{p}}(\alpha,\lambda,p)\frac{(b-a)^{\alpha+1}}{(\alpha+1)2^{\alpha+2}}\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(a)|^q\right\}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Corollary 2.7. *In Theorem 2.2, if we take $x=\frac{a+b}{2}$ and $\lambda=\frac{1}{3}$, then we have*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(\alpha+1)2^{\alpha-1}}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]\right| \\ \le\ & C_2^{\frac{1}{p}}\left(\alpha,\frac{1}{3},p\right)\frac{(b-a)^2}{8(\alpha+1)}\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(a)|^q\right\}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Corollary 2.8. *In Theorem 2.2, if we take $x = \frac{a+b}{2}$, $\lambda = \frac{1}{3}$ and $\alpha = 1$ then we have*

$$\begin{aligned}
&\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{(b-a)} \int_a^b f(x)dx \right| \\
\leq\ & \frac{(b-a)^2}{16} C_2^{\frac{1}{p}} \left(1, \frac{1}{3}, p\right) \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(a)\right|^q \right\} \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(b)\right|^q \right\} \right)^{\frac{1}{q}} \right\},
\end{aligned}$$

where

$$C_2\left(1, \frac{1}{3}, p\right) = \left(\frac{2}{3}\right)^{1+2p} \beta(1+p, 1+p) + \left(\frac{1}{3}\right)^{1+p} . {}_2F_1\left(-p, 1; p+2; \frac{1}{3}\right).$$

Corollary 2.9. *In Theorem 2.2, if we take $x = \frac{a+b}{2}$ and $\lambda = 0$, then we have*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1) 2^{\alpha-1}}{(b-a)^\alpha} \left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) \right] \right| \\
\leq\ & \frac{(b-a)^2}{16} \left(\frac{1}{p(\alpha+1)+1} \right)^{\frac{1}{p}} \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(a)\right|^q \right\} \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(b)\right|^q \right\} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Remark 2.4. In Corollary 2.9, if we choose $\alpha = 1$, we have the following midpoint inequality which is better than the inequality (1.6)

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)} \int_a^b f(x)dx \right| \\
\leq\ & \frac{(b-a)^2}{16} \left(\frac{1}{2p+1} \right)^{\frac{1}{p}} \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(a)\right|^q \right\} \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(b)\right|^q \right\} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Corollary 2.10. *In Theorem 2.2, if we take $x = \frac{a+b}{2}$ and $\lambda = 1$, then we have*

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1) 2^{\alpha-1}}{(b-a)^\alpha} \left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) \right] \right| \\
\leq\ & \frac{(b-a)^2}{16} C_2^{\frac{1}{p}} (\alpha, 1, p) \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(a)\right|^q \right\} \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(b)\right|^q \right\} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

where

$$C_2(\alpha,1,p)=\frac{(1+\alpha)^{\frac{(1+\alpha)(1+p)-\alpha}{\alpha}}}{\alpha}\beta\left(\frac{1}{1+\alpha};\frac{1+p}{\alpha},1+p\right).$$

Remark 2.5. In Corollary 2.10, if we choose $\alpha=1$, we have the following trapezoid inequality which is the same of the inequality (1.3)

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)dx\right|\\ \leq\ &\frac{(b-a)^2}{4}\left(\beta(1+p,1+p)\right)^{\frac{1}{p}}\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\},\end{aligned}$$

where

$$\beta(1+p,1+p)=2\beta\left(\frac{1}{2};1+p,1+p\right)=2^{-2p-1}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{\Gamma\left(\frac{3}{2}+p\right)},\ \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

References

[1] M. Abramowitz, I.A. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* Dover, New York, 1965.

[2] M. Alomari and M. Darus, On some inequalities of Simpson-type via quasi-convex functions with applications, *Tran. J. Math. Mech.* 2 (2010), 15-24.

[3] M. W. Alomari, M. Darus and S. S. Dragomir, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are quasi-convex, *Tamkang J. of Math.*, 41(4) (2010), 353-359.

[4] A. Barani, S. Barani and S.S. Dragomir, Refinements of Hermite-Hadamard type inequality for functions whose second derivative absolute values are quasi convex, RGMIA *Res. Rep. Col.*, 14 (2011).

[5] D.A. Ion, Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, *Annals of University of Craiova Math. Comp. Sci. Ser.*, 34 (2007), 82-87.

[6] I. Iscan, Generalization of different type integral inequalities for s-convex functions via fractional integrals, *Applicable Analysis*, accepted for publication, arXiv:1304.3897.

[7] I. Iscan, Hermite-Hadamard type inequalities for functions whose derivatives are$(\alpha,m)-$convex, *Int. J. of Eng. and Appl. Sci.*, 2(3) (2013), 53–62.

[8] I. Iscan, On generalization of some integral inequalities for quasi-convex functions and their applications, *Int. J. of Eng. and Appl. Sci.*, 3(1) (2013), 37-42.

[9] M.Z. Sarikaya, H. Ogunmez, On new inequalities via Riemann–Liouville fractional integration, *Abstr. Appl. Analysis*, 2012 (2012), Article ID 428983, 10 pages, doi:10.1155/2012/428983.

[10] M. Z. Sarikaya, A. Saglam, H. Yildirim, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex, arXiv:1005.0451 (2010).

[11] M.Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.* (2012), Online, doi:10.1016/j.mcm.2011.12.048.

[12] M.Z. Sarikaya and H. Yaldiz, On weighted Montogomery identities for Riemann-Liouville fractional integrals, *Konuralp J. of Math.*, 1(1) (2013) 48-53.

[13] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals, *Comp. Math. Appl.*, 63(7) (2012), 1147-1154.

Giresun University, Science and Art Faculty, Department of Mathematics, Giresun-TURKEY
E-mail address: imdat.iscan@giresun.edu.tr

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SIMILAR CURVES WITH VARIABLE TRANSFORMATIONS

MOSTAFA F EL-SABBAGH[1] AND AHMAD T ALI[2,3]

Abstract. In this paper, we define a new family of curves and call it a *family of similar curves with variable transformation* or briefly *SA-curves*. Also we introduce some characterizations of this family and we give some theorems. This definition introduces a new classification of a space curve. Also, we use this definition to deduce the position vectors of plane curves, general helices and slant helices, as examples of a similar curves with variable transformation.

1. Introduction

In the local differential geometry, we think of a curve as a geometric set of points, or locus. Intuitively, we are thinking of a curve as the path traced out by a particle moving in $\mathbf{E}^3$. So, investigating position vector of the curve is a classical aim to determine behavior of the particle (curve).

From the view of differential geometry, a *straight line* is a geometric curve with the curvature $\kappa(s) = 0$. A *plane curve* is a family of geometric curves with torsion $\tau(s) = 0$. Helix is a geometric curve with non-vanishing constant curvature κ and non-vanishing constant torsion τ [8]. The helix may be called a *circular helix* or *W-curve* [22]. It is known that straight line ($\kappa(s) = 0$) and circle ($\kappa(s) = a$, $\tau(s) = 0$) are degenerate-helices examples [15]. In fact, circular helix is the simplest three-dimensional spirals [4, 10].

A curve of constant slope or *general helix* in Euclidean 3-space $\mathbf{E}^3$ is defined by the property that the tangent makes a constant angle with a fixed straight line called the axis of the general helix. A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (see [25] for details) says that: *A necessary and sufficient condition that a curve be a general helix is that the ratio*

$$\frac{\tau}{\kappa}$$

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is constant along the curve, where κ and τ denote the curvature and the torsion, respectively. General helices or *inclined curves* are well known curves in classical differential geometry of space curves [19] and we refer to the reader for recent works on this type of curves [5, 12, 20, 26].

Izumiya and Takeuchi [14] have introduced the concept of *slant helix* by saying that the normal lines make a constant angle with a fixed straight line. They characterize a slant helix if and only if the *geodesic curvature* of the principal image of the principal normal indicatrix

$$\sigma = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\Big(\frac{\tau}{\kappa}\Big)'$$

is a constant function. Kula and Yayli [16] have studied spherical images of tangent indicatrix and binormal indicatrix of a slant helix and they showed that the spherical images are spherical helices. Recently, Kula et al. [17] investigated the relation between a general helix and a slant helix. Moreover, they obtained some differential equations which are characterizations for a space curve to be a slant helix.

A family of curves with constant curvature but non-constant torsion is called Salkowski curves and a family of curves with constant torsion but non-constant curvature is called anti-Salkowski curves [23]. Monterde [21] studied some characterizations of these curves and he proved that the principal normal vector makes a constant angle with fixed straight line. A unit speed curve of *constant precession* in Euclidean 3-space $\mathbf{e}^3$ is defined by the property that its (Frenet) Darboux vector $W = \tau\,\mathbf{T} + \kappa\,\mathbf{B}$ revolves about a fixed line in space with constant angle and constant speed. Kula and Yayli [16] proved that the geodesic curvature of the spherical image of the principal normal indicatrix of a curve of constant precession is a constant function. So that: Salkowski curves, anti-Salkowski curves and curves of constant presession are the important examples of slant helices.

Many important results in the theory of curves in $\mathbf{E}^3$ were initiated by G. Monge and G. Darboux pioneered the moving frame idea. Thereafter, F. Frenet defined his moving frame and his special equations which play important role in mechanics and kinematics as well as in differential geometry [9].

In this work, we define a new family of curves and we call it a family of *similar curves with variable transformation* or in brief *SA-curves*. Also, we introduce some characterizations of this family and give some theorems. This definition introduces a new classification of a space curve. In the last of this paper, we use this definition to deduce the position vectors of some important special curves. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling as well as other applications of interest.

2. Preliminaries

In Euclidean space $\mathbf{E}^3$, it is well known that to each unit speed curve with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are respectively, the tangent, the principal normal and the binormal vector fields [13]. We consider the usual metric in Euclidean 3-space $\mathbf{E}^3$, that is,

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbf{E}^3$. Let $\psi : I \subset \mathbb{R} \to \mathbf{E}^3$, $\psi = \psi(s)$, be an arbitrary curve in $\mathbf{E}^3$. The curve ψ is said to be of unit speed (or parameterized by the arc-length) if $\langle \psi'(s), \psi'(s) \rangle = 1$ for any $s \in I$. In particular, if $\psi(s) \neq 0$ for any s, then it is possible to re-parameterize ψ, that is, $\alpha = \psi(\phi(s))$ so that α is parameterized by the arc-length. Thus, we will assume throughout this work that ψ is a unit speed curve.

Let $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ be the moving frame along ψ, where the vectors $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$ are mutually orthogonal vectors satisfying $\langle \mathbf{T}, \mathbf{T} \rangle = \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 1$. The Frenet equations for ψ are given by ([25])

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix}. \tag{2.1}$$

If $\tau(s) = 0$ for all $s \in I$, then $\mathbf{B}(s)$ is a constant vector V and the curve ψ lies in a 2-dimensional affine subspace orthogonal to V, which is isometric to the Euclidean 2-space $\mathbf{E}^2$.

3. Position vector of a space curve

The problem of the determination of parametric representation of the position vector of an arbitrary space curve according to its intrinsic equations is still open in the Euclidean space $\mathbf{E}^3$ [11, 18]. This problem is not easy to solve in general case. However, this problem is solved in three special cases only, Firstly, in the case of a plane curve ($\tau = 0$). Secondly, in the case of a helix (κ and τ are both non-vanishing constant). Recently, Ali [6, 7] adapted fundamental existence and uniqueness theorem for space curves in Euclidean space $\mathbf{E}^3$ and constructed a vector differential equation to solve this problem in the case of a general helix ($\frac{\tau}{\kappa}$ is constant) and in the case of a slant helix

$$\frac{\tau(s)}{\kappa(s)} = \pm \frac{m \int \kappa(s) ds}{\sqrt{1 - m^2 \Big(\int \kappa(s) ds \Big)^2}}, \tag{3.1}$$

where $m = \frac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ and ϕ is the constant angle between the axis of a slant helix and the principal normal vector. However, this problem is not solved in other cases of space curves.

Now we describe this problem within the following theorem:

Theorem 3.1. *Let $\psi = \psi(s)$ be an unit speed curve parameterized by the arclength s. Suppose $\psi = \psi(\theta)$ is another parametric representation of this curve by the parameter $\theta = \int \kappa(s) ds$. Then, the tangent vector $\mathbf{T}$ satisfies a vector differential equation of third order as follows:*

$$\Big(\frac{1}{f(\theta)} \mathbf{T}''(\theta) \Big)' + \Big(\frac{1 + f^2(\theta)}{f(\theta)} \Big) \mathbf{T}'(\theta) - \frac{f'(\theta)}{f(\theta)} \mathbf{T}(\theta) = 0, \tag{3.2}$$

where $f(\theta) = \frac{\tau(\theta)}{\kappa(\theta)}$.

The Proof: Let $\psi = \psi(s)$ be a unit speed curve. If we write this curve in another parametric representation $\psi = \psi(\theta)$, where $\theta = \int \kappa(s)ds$, we have new Frenet equations as follows:

$$\begin{bmatrix} \mathbf{T}'(\theta) \\ \mathbf{N}'(\theta) \\ \mathbf{B}'(\theta) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & f(\theta) \\ 0 & -f(\theta) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(\theta) \\ \mathbf{N}(\theta) \\ \mathbf{B}(\theta) \end{bmatrix}, \tag{3.3}$$

where $f(\theta) = \frac{\tau(\theta)}{\kappa(\theta)}$. If we substitute the first equation of the new Frenet equations (3.3) to second equation of (3.3), we have

$$\mathbf{B}(\theta) = \frac{1}{f(\theta)}\Big[\mathbf{T}''(\theta) + \mathbf{T}(\theta)\Big]. \tag{3.4}$$

Substituting the above equation in the last equation from (3.3), we obtain a vector differential equation of third order (3.2) as desired.

The equation (3.2) is not easy to solve in general case. If one solves this equation, the natural representation of the position vector of an arbitrary space curve can be determined as follows:

$$\psi(s) = \int \mathbf{T}(s)\, ds + C, \tag{3.5}$$

or in parametric representation

$$\psi(\theta) = \int \frac{1}{\kappa(\theta)}\, \mathbf{T}(\theta)\, d\theta + C, \tag{3.6}$$

where $\theta = \int \kappa(s)ds$ and C is a constant vector.

4. Similar curves with variable transformations

Definition 4.1. Let $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ be two regular curves in $\mathbf{E}^3$ parameterized by arclengths s_α and s_β with curvatures κ_α and κ_β, torsion τ_α and τ_β and Frenet frames $\{\mathbf{T}_\alpha, \mathbf{N}_\alpha, \mathbf{B}_\alpha\}$ and $\{\mathbf{T}_\beta, \mathbf{N}_\beta, \mathbf{B}_\beta\}$. $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ are called similar curves with variable transformation λ^α_β if there exist a variable transformation

$$s_\alpha = \int \lambda^\alpha_\beta(s_\beta)\, ds_\beta \tag{4.1}$$

of the arclengths such that the tangent vectors are the same for the two curves i.e.,

$$\mathbf{T}_\beta(s_\beta) = \mathbf{T}_\alpha(s_\alpha), \tag{4.2}$$

for all corresponding values of parameters under the transformation λ^α_β.

Here λ^α_β is arbitrary function of the arclength s_β. It is worth noting that $\lambda^\alpha_\beta\, \lambda^\beta_\alpha = 1$. All curves satisfy equation (4.2) are called a family of similar curves with variable transformations. If we integrate the equation (4.2) we have the following important theorem:

Theorem 4.1. *The position vectors of the family of similar curves with variable transformations can be written in the following form:*

$$\psi_\beta(s_\beta) = \int \mathbf{T}_\beta\Big(s_\beta(s_\alpha)\Big)\, ds_\beta = \int \mathbf{T}_\alpha\Big(s_\alpha\Big)\, \lambda_\beta^\alpha\, ds_\alpha. \tag{4.3}$$

Theorem 4.2. *Let $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ be two regular curves in $\mathbf{E}^3$. Then $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ are similar curves with variable transformation if and only if the principal normal vectors are the same for all curves*

$$\mathbf{N}_\beta(s_\beta) = \mathbf{N}_\alpha(s_\alpha), \tag{4.4}$$

under the particular variable transformation

$$\lambda_\alpha^\beta = \frac{ds_\beta}{ds_\alpha} = \frac{\kappa_\alpha}{\kappa_\beta} \tag{4.5}$$

of the arc-lengths.

The Proof: ($\Rightarrow$) Let $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ be two regular similar curves with variable transformation in $\mathbf{E}^3$. Differentiating the equation (4.2) with respect to s_β we have

$$\kappa_\beta(s_\beta)\mathbf{N}_\beta(s_\beta) = \kappa_\alpha(s_\alpha)\mathbf{N}_\alpha(s_\alpha)\frac{ds_\alpha}{ds_\beta}. \tag{4.6}$$

The above equation leads to the two equations (4.4) and (4.5).

($\Leftarrow$) Let $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ be two regular curves in $\mathbf{E}^3$ satisfying the two equations (4.4) and (4.5). If we multiplying equation (4.4) by $\kappa_\beta(s_\beta)$ and integrate the result with respect to s_β we have

$$\int \kappa_\beta(s_\beta)\, \mathbf{N}_\beta(s_\beta)\, ds_\beta = \int \kappa_\beta(s_\beta)\, \mathbf{N}_\beta(s_\beta)\, \frac{ds_\beta}{ds_\alpha}\, ds_\alpha. \tag{4.7}$$

From the equations (4.4) and (4.5), equation (4.7) takes the form

$$\mathbf{T}_\beta(s_\beta) = \int \kappa_\beta(s_\beta)\, \mathbf{N}_\beta(s_\beta)\, ds_\beta = \int \kappa_\alpha(s_\alpha)\, \mathbf{N}_\alpha(s_\alpha)\, ds_\alpha = \mathbf{T}_\alpha(s_\alpha). \tag{4.8}$$

The proof is completed.

Theorem 4.3. *Let $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ be two regular curves in $\mathbf{E}^3$. Then $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ are similar curves with variable transformation if and only if the binormal vectors are the same, i.e.,*

$$\mathbf{B}_\beta(s_\beta) = \mathbf{B}_\alpha(s_\alpha), \tag{4.9}$$

under arbitrary variable transformation $s_\beta = s_\beta(s_\alpha)$ of the arclengths.

The Proof: ($\Rightarrow$) Let $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ be two regular similar curves with variable transformations in $\mathbf{E}^3$. Then there exists a variable transformation of the arclengths such that the tangent vectors and the principal normal vectors are the same (definition 4.1 and theorem 4.2). From equations (4.2) and (4.4) we have

$$\mathbf{B}_\beta(s_\beta) = \mathbf{T}_\beta(s_\beta) \times \mathbf{N}_\beta(s_\beta) = \mathbf{T}_\alpha(s_\alpha) \times \mathbf{N}_\alpha(s_\alpha) = \mathbf{B}_\alpha(s_\alpha). \tag{4.10}$$

($\Leftarrow$) Let $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ be two regular curves in $\mathbf{E}^3$ which the same binormal vector under the arbitrary variable transformation $s_\beta = s_\beta(s_\alpha)$ of the arclengths. If we differentiate the equation (4.9) with respect to s_β we have

$$-\tau_\beta(s_\beta)\mathbf{N}_\beta(s_\beta) = -\tau_\alpha(s_\alpha)\mathbf{N}_\alpha(s_\alpha)\frac{ds_\alpha}{ds_\beta}. \tag{4.11}$$

The above equation leads to the following two equations

$$\mathbf{N}_\beta(s_\beta) = \mathbf{N}_\alpha(s_\alpha) \quad \text{and} \quad \frac{ds_\beta}{ds_\alpha} = \frac{\tau_\alpha}{\tau_\beta} \tag{4.12}$$

From equations (4.9) and (4.12) we have

$$\mathbf{T}_\beta(s_\beta) = \mathbf{N}_\beta(s_\beta) \times \mathbf{B}_\beta(s_\beta) = \mathbf{N}_\alpha(s_\alpha) \times \mathbf{B}_\alpha(s_\alpha) = \mathbf{T}_\alpha(s_\alpha). \tag{4.13}$$

The proof is complete.

Theorem 4.4. *Let $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ be two regular curves in $\mathbf{E}^3$. Then $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ are two similar curves with variable transformation if and only if the ratios of torsion and curvature are the same for all curves*

$$\frac{\tau_\beta(s_\beta)}{\kappa_\beta(s_\beta)} = \frac{\tau_\alpha(s_\alpha)}{\kappa_\alpha(s_\alpha)}, \tag{4.14}$$

under the particular variable transformations ($\lambda_\alpha^\beta = \frac{ds_\beta}{ds_\alpha} = \frac{\kappa_\alpha}{\kappa_\beta}$) keeping equal total curvatures, i.e.,

$$\theta_\beta(s_\beta) = \int \kappa_\beta \, ds_\beta = \int \kappa_\alpha \, ds_\alpha = \theta_\alpha(s_\alpha). \tag{4.15}$$

of the arclengths.

The Proof: Let $\psi_\alpha(s_\alpha)$ and $\psi_\alpha(s_\beta)$ are two similar curves with variable transformation. Then from (4.5) and second equation of (4.12), we obtain the equation (4.14) under the variable transformations (4.15), which leads from (4.5) by integration.

($\Leftarrow$) Let $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ be two curves such that the equation (4.14) is satisfied under the variable transformation (4.15) of the arclengths. From theorem (3.1), the tangent vectors $\mathbf{T}_\alpha(s_\alpha)$ and $\mathbf{T}_\beta(s_\beta)$ of the two curves satisfy vector differential equations of third order as follows:

$$\Big(\frac{1}{f_\alpha(\theta_\alpha)}\mathbf{T}''_\alpha(\theta_\alpha)\Big)' + \Big(\frac{1+f_\alpha^2(\theta_\alpha)}{f_\alpha(\theta_\alpha)}\Big)\mathbf{T}'_\alpha(\theta_\alpha) - \frac{f'_\alpha(\theta_\alpha)}{f_\alpha(\theta_\alpha)}\mathbf{T}_\alpha(\theta_\alpha) = 0, \tag{4.16}$$

$$\Big(\frac{1}{f_\beta(\theta_\beta)}\mathbf{T}''_\beta(\theta_\beta)\Big)' + \Big(\frac{1+f_\beta^2(\theta_\beta)}{f_\beta(\theta_\beta)}\Big)\mathbf{T}'_\beta(\theta_\beta) - \frac{f'_\beta(\theta_\beta)}{f_\beta(\theta_\beta)}\mathbf{T}_\beta(\theta_\beta) = 0, \tag{4.17}$$

where $f_\alpha(\theta_\alpha) = \frac{\tau_\alpha(\theta_\alpha)}{\kappa_\alpha(\theta_\alpha)}$, $f_\beta(\theta_\beta) = \frac{\tau_\beta(\theta_\beta)}{\kappa_\beta(\theta_\beta)}$, $\theta_\alpha = \int \kappa_\alpha(s_\alpha)\, ds_\alpha$ and $\theta_\beta = \int \kappa_\beta(s_\beta)\, ds_\beta$.

The equation (4.14) leads to

$$f_\beta(\theta_\beta) = f_\alpha(\theta_\alpha), \tag{4.18}$$

under the variable transformation $\theta_\beta = \theta_\alpha$. So that the two equations (4.16) and (4.17) under the equation (4.14) and the transformation (4.15) are the same. Hence the solution is the same, i.e., the tangent vectors are the same which completes the proof of the theorem.

5. New classifications of curves

In this section, we will apply our definition of similar curves with variable transformations to deduce the position vectors of some special curves. First we can call the two curves $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ similar curves with variable transformation if and only if there exists an arbitrary function $\lambda^\alpha_\beta = \frac{ds_\alpha}{ds_\beta}$ such that the curvature and torsion of the curve ψ_β are the curvature and torsion of the curve ψ_α multiplied by this arbitrary function i,e.,

$$\kappa_\beta = \kappa_\alpha\,\lambda^\alpha_\beta, \quad \tau_\beta = \tau_\alpha\,\lambda^\alpha_\beta. \tag{5.1}$$

Class 1. If the curve is straight line then the curvature is $\kappa = 0$. Under the variable transformation λ the curvature does not change. So we have the following lemma:

Lemma 5.1. *The straight line alone forms a family of similar curves with variable transformation.*

Class 2. If the curve is a plane curve then the torsion is $\tau_\alpha = 0$. Under the variable transformation λ the torsion does not change. So we have the following lemma:

Lemma 5.2. *The family of plane curves forms a family of similar curves with variable transformations.*

We can deduce the position vector of a plane curve using the definition of similar curves with variable transformation as follows:

The simplest example of a plane curve is a circle of radius 1. The natural representation of this circle can be written in the form:

$$\psi_\alpha(u) = \Big(\sin[u], -\cos[u], 0\Big), \tag{5.2}$$

where $s_\alpha = u$ is the arclength of the circle and the curvature is $\kappa_\alpha(u) = 1$. The tangent vector of this circle takes the form:

$$\mathbf{T}_\alpha(u) = \Big(\cos[u], \sin[u], 0\Big). \tag{5.3}$$

From theorem (4.1) we can write any plane curve as the following:

$$\psi_\beta(s) = \int \Big(\cos\Big[u[s]\Big], \sin\Big[u[s]\Big], 0\Big)ds. \tag{5.4}$$

where $s_\beta = s$. From the equation (5.1), we have

$$ds_\alpha = \lambda^\alpha_\beta\,ds_\beta = \frac{\kappa_\beta}{\kappa_\alpha}\,ds_\beta. \tag{5.5}$$

or

$$s_\alpha(s_\beta) = \int \frac{\kappa_\alpha}{\kappa_\beta}\,ds_\beta. \tag{5.6}$$

If we put the curvature $\kappa_\beta = \kappa(s)$ $(s_\beta = s)$, we have

$$u(s) = \int \kappa(s)\,ds. \tag{5.7}$$

Then the position vector of the plane curve with arbitrary curvature $\kappa(s)$ takes the following form:

$$\psi(s) = \int \Big(\cos\Big[\int \kappa(s)\, ds \Big], \sin\Big[\int \kappa(s)\, ds \Big], 0 \Big) ds. \tag{5.8}$$

which is the position vector of a plane curve introduced in [18].

Class 3. If the curve ψ_α is a general helix ($\frac{\tau_\alpha}{\kappa_\alpha} = m$), where m is a constant, in the form $m = \cot[\phi]$ and ϕ is the angle between the tangent vector and the axis of the helix. Then any similar curve ψ_β with this helix has the the property $\frac{\tau_\beta}{\kappa_\beta} = m$. So that we have the following lemma:

Lemma 5.3. *The family of general helices with fixed angle ϕ between the axis of a general helix and the tangent vector forms a family of similar curves with variable transformations.*

We can deduce the position vector of a general helix using the definition of similar curves with variable transformations as follows:

The simplest example of a general helix is a circular helix or W-curve. The natural representation of a circular helix is:

$$\psi_\alpha(u) = \Big(\sqrt{1-n^2}\, \sin[u], -\sqrt{1-n^2}\, \cos[u], n\, u \Big), \tag{5.9}$$

where u is the arclength of the circular helix and $n = \cos[\phi]$, where ϕ is the constant angle between the tangent vector and the axis of a circular helix. The curvature of this circular helix is $\kappa_\alpha(u) = \sqrt{1-n^2}$. The tangent vector of this curve takes the form:

$$\mathbf{T}_\alpha(u) = \Big(\sqrt{1-n^2} \cos[u], \sqrt{1-n^2} \sin[u], n \Big). \tag{5.10}$$

From theorem (4.1) we can write any general helix as the following:

$$\psi_\beta(s) = \int \Big(\sqrt{1-n^2} \cos\Big[u(s) \Big], \sqrt{1-n^2} \sin\Big[u(s) \Big], n \Big) ds. \tag{5.11}$$

From equation (5.6) we have

$$u(s) = \int \frac{\kappa(s)}{\sqrt{1-n^2}}\, ds. \tag{5.12}$$

where $\kappa_\beta = \kappa(s)$, ($s_\beta = s$). Then the position vector of the general helix with arbitrary curvature $\kappa(s)$ takes the following form:

$$\psi = \int \Big(\sqrt{1-n^2} \cos\Big[\int \frac{\kappa(s)}{\sqrt{1-n^2}} ds \Big], \sqrt{1-n^2} \sin\Big[\int \frac{\kappa(s)}{\sqrt{1-n^2}} ds \Big], n \Big) ds. \tag{5.13}$$

which is the position vector of a general helix introduced in [6].

Class 4. If the curve is a slant helix then the relation (3.1) between the torsion and curvature is satisfied. Let ψ_α and ψ_β be two slant helices such that the transformation (4.15) is satisfied. Using the relation (3.1) and (4.15), it is easy to prove that:

$$\frac{\tau_\beta}{\kappa_\beta} = \frac{m\, \theta_\beta}{\sqrt{1 - m^2 \theta_\beta^2}} = \frac{m\, \theta_\alpha}{\sqrt{1 - m^2 \theta_\alpha^2}} = \frac{\tau_\alpha}{\kappa_\alpha},$$

where m is a constant value, $m = \cot[\phi]$ and ϕ is the angle between the principal normal vector and the axis of a slant helix. So that we have the following lemma:

Lemma 5.4. *The family of a slant helices with fixed angle ϕ between the axis of a slant helix and the principal normal vector forms a family of similar curves with variable transformation.*

Now, we can deduce the position vector of a slant helix using the definition of similar curves with variable transformations as follows:

The simplest example of a slant helix is Salkowski curve [7, 21, 23]. The explicit parametric representation of a Salkowski curve $\psi_\alpha(u) = \big(\psi_1(u), \psi_2(u), \psi_3(u)\big)$ takes the form:

$$\left\{\begin{array}{l} \psi_1(u) = -\frac{n}{4m}\Big[\frac{n-1}{2n+1}\cos[(2n+1)t] + \frac{n+1}{2n-1}\cos[(2n-1)t] - 2\cos[t]\Big], \\ \psi_2(u) = -\frac{n}{4m}\Big[\frac{n-1}{2n+1}\sin[(2n+1)t] - \frac{n+1}{2n-1}\sin[(2n-1)t] - 2\sin[t]\Big], \\ \psi_3(u) = \frac{n}{4m^2}\cos[2nt], \end{array}\right. \tag{5.14}$$

where $t = \frac{1}{n}\arcsin(mu)$, $m = \frac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ and ϕ is the constant angle between the axis of a slant helix and the principal normal vector. The curvature of the above curve is 1 and the torsion is

$$\tau(u) = \tan[nt] = \frac{m\,u}{\sqrt{1 - m^2\,u^2}}.$$

It is worth noting that: the variable t is a parameter while the variable u is the natural parameter.

The tangent and the principal normal vectors of the Salkowski curve (5.14) take the forms:

(5.15)

$$\mathbf{T}_\alpha(u) = -\Big(n\cos[t]\sin[nt] - \sin[t]\cos[nt], n\sin[t]\sin[nt] + \cos[t]\cos[nt], \frac{n}{m}\sin[nt]\Big).$$

$$\mathbf{N}_\alpha(u) = \Big(\sqrt{1-n^2}\cos[t], \sqrt{1-n^2}\sin[t], n\Big), \tag{5.16}$$

It is easy to write the tangent vector (5.15) in the simple form:

$$\mathbf{T}_\alpha(u) = \int \mathbf{N}(u)du = \int \Big(\sqrt{1-n^2}\cos[t], \sqrt{1-n^2}\sin[t], n\Big)du. \tag{5.17}$$

From theorem (4.1) we can write any slant helix as the following:

$$\psi_\beta(s) = \int \Big[\int \Big(\sqrt{1-n^2}\cos[t], \sqrt{1-n^2}\sin[t], n\Big)du\Big]ds. \tag{5.18}$$

From equation (5.6) we have

$$u(s) = \int \kappa(s)\,ds, \quad du = \kappa(s)\,ds, \tag{5.19}$$

where $\kappa_\beta = \kappa(s)$, $(s_\beta = s)$. Substituting equation (5.19) in (5.18) we obtain the position vector of a similar curve $\psi_\beta(s) = \big(\psi_1(s), \psi_2(s), \psi_3(s)\big)$of a slant helix with arbitrary curvature $\kappa(s)$ as follows:

$$\left\{\begin{array}{l} \psi_1(s) = \frac{n}{m}\int\left[\int\kappa(s)\cos\left[\frac{1}{n}\arcsin\Big(m\int\kappa(s)ds\Big)\right]ds\right]ds, \\ \psi_2(s) = \frac{n}{m}\int\left[\int\kappa(s)\sin\left[\frac{1}{n}\arcsin\Big(m\int\kappa(s)ds\Big)\right]ds\right]ds, \\ \psi_3(s) = n\int\left[\int\kappa(s)ds\right]ds, \end{array}\right. \tag{5.20}$$

which is the position vector of a slant helix introduced in [7].

Finally, we hope that we can introduce new classes of similar curves and deduce the position vectors of these classes in future work.

References

[1] O'Neill, B., *Semi Riemannian geometry with applications to relativity*, Academic Press, Inc. New York, 1983.

[2] Hacısalihoğlu, H. H., *Diferensiyel geometri*, Cilt I-II, Ankara Üniversitesi, Fen Fakültesi Yayınları, 2000.

[3] A. Görgülü and A. C. Çöken, The Euler theorem for parallel pseudo-Euclidean hypersurfaces in pseudo-Euclideanspace $\mathbb{E}_1^{n+1}$, Journ. Inst. Math. and Comp. Sci. (Math. Series) Vol:6, No.2 (1993), 161-165.

[4] A.T. Ali and M. Turgut, Position vector of a time-like slant helix in Minkowski 3-space, J. Math. Anal. Appl. **365** (2010) 559–569.

[5] A.T. Ali, Position vectors of spacelike general helices in Minkowski 3-space, Nonl. Anal. Theory Meth. Appl. **73** (2010) 1118–1126.

[6] A.T. Ali, Position vectors of general helices in Euclidean 3-space, Bull. Math. Anal. Appl. **3(2)**, (2010), 198–205.

[7] A.T. Ali, Position vectors of slant helices in Euclidean 3-space, J. Egyptian Math. Soc. **20**, (2012), 1–6.

[8] M. Barros, General helices and a theorem of Lancret, Proc. Amer. Math. Soc. **125** (1997), 1503–1509.

[9] C. Boyer, *History of Mathematics*, Wiley, New york, 1968.

[10] C. Camci, K. Ilarslan, L. Kula and H.H. Hacisalihoglu, Harmonic curvatures and generalized helices in $\mathbf{E}^n$, Chaos, Solitons and Fractals **40** (2009), 2590–2596.

[11] L.P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces*, Ginn and Co., 1909.

[12] H. Gluck, Higher curvatures of curves in Euclidean space, Amer. Math. Monthly **73** (1996), 699–704.

[13] H.H. Hacisalihoglu, *Differential Geometry*, Ankara University, Faculty of Science Press, 2000.

[14] S. Izumiya and N. Takeuchi, New special curves and developable surfaces, Turk. J. Math. **28** (2004), 531–537.

[15] W. Kuhnel, *Differential Geometry: Curves - Surfaces - Manifolds*, Wiesdaden: Braunchweig; 1999.

[16] L. Kula and Y. Yayli, On slant helix and its spherical indicatrix, Appl. Math. Comp. **169** (2005), 600–607.

[17] L. Kula, N. Ekmekci, Y. Yayli and K. Ilarslan, Characterizations of slant helices in Euclidean 3-space, Tur. J. Math. **33** (2009), 1–13.

[18] M.M. Lipschutz, *Schum's Outline of Theory and Problems of Differential Geometry*, McGraw-Hill Book Company, New York, 1969.

[19] R.S. Milman and G.D. Parker, *Elements of Differential Geometry*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1977.

[20] J. Monterde, Curves with constant curvature ratios, Bulletin of Mexican Mathematic Society, 3a serie vol. **13** (2007), 177–186.

[21] J. Monterde, Salkowski curves revisted: A family of curves with constant curvature and non-constant torsion, Comput. Aided Geomet. Design **26** (2009), 271–278.

[22] M. Petrovic-Torgasev and E. Sucurovic, W-curves in Minkowski space-time, Novi Sad J. Math. **32(2)** (2002), 55–65.

[23] E. Salkowski, Zur transformation von raumkurven, Mathematische Annalen **66** (1909), 517–557.

[24] P.D. Scofield, Curves of constant precession, Amer. Math. Monthly **102** (1995), 531–537.

[25] D.J. Struik, *Lectures in Classical Differential Geometry*, Addison,-Wesley, Reading, MA, 1961.

[26] M. Turgut and S. Yilmaz, Contributions to classical differential geometry of the curves in $\mathbf{E}^3$, Scientia Magna **4** (2008), 5–9.

1- Mathematics Department, Faculty of Science, Minia University, Minia, Egypt.

2- Al-Azhar University, Faculty of Science, Mathematics Department, Nasr City, 11884, Cairo, Egypt.

3- King Abdul Aziz University, Faculty of Science, Department of Mathematics, PO Box 80203, Jeddah, 21589, Saudi Arabia.

E-mail address: `elsabbaghmostafa@yahoo.com and atali71@yahoo.com`

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A NOTE ON CROSSED MODULES OF LEIBNIZ ALGEBRAS

AHMET FARUK ASLAN

Abstract. In this paper we give the relation between precrossed modules and crossed modules of Leibniz algebras. Also construct the coproduct object in the category of crossed L-modules on Leibniz algebras.

1. Introduction

Leibniz algebras were first introduced by Loday in [5]. They can be thought as a generalisation of Lie algebras. The main difference is that the bracket of Leibniz algebra is non-skew-symmetric. They have many applications in some branches of Mathematics and Physics.We refer the references given in [3] for a survey about Leibniz algebras.

Crossed modules were introduced by Whitehead in [9] as a model for connected homotopy 2-types. After then, crossed modules used in many branches of mathematics such as category theory, cohomology of algebraic structures, differential geometry and in physics. This makes the crossed modules one of the fundamental algebraic gadget. For some different usage, crossed modules were defined in different categoriess such as Lie algebras, commutative algebras etc.([7],[4]). The (pre)crossed modules are generalisations of Leibniz algebras. This is why the subject is important. At this vein, in the paper, we construct the relation between pre-crossed and crossed modules of Leibniz algebras and construct the coproducts in the category of crossed L-modules on Leibniz algebras.

2. Preliminaries

Definition 2.1. A Leibniz algebra L is a $\mathbb{k}$-vector space equipped with a bilinear map $[-,-] : L \times L \longrightarrow L$, satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

for all $x, y, z \in L$.

If $[x,x] = 0$, for all $x \in L$, then the Leibniz identity becomes to the Jacobi identity, so L will be a Lie algebra.

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Definition 2.2. A morphism of Leibniz algebras is a $\Bbbk$-linear map $f : L \longrightarrow L'$ such that
$$f[x,y] = [f(x), f(y)]$$
for all $x, y \in L$.

By this definition we have the category of Leibniz algebras which will be denoted by **Lbnz** in this work.

Definition 2.3. Let L be a Leibniz algebra and I be a subalgebra (a vector subspace of L closed under the bracket operation). If $[x,y], [y,x] \in I$, for all $x \in L$ and $y \in I$, then I is called a two-sided ideal of L and this is denoted by $I \trianglelefteq L$.

If $I \trianglelefteq L$ then, the quotient I/L inherits a Leibniz structure with the operations induced from L.

Definition 2.4. An abelian Leibniz algebra is a Leibniz algebra with the trivial bracket.

Example 2.1. Every $\Bbbk$-vector space has an abelian Leibniz algebra structure.

Definition 2.5. Let L and L' be Leibniz algebras. A Leibniz action from L over L' consist of $\Bbbk$-bilinear maps
$$\begin{array}{ll} \lambda : L \times L' \longrightarrow L', & (x,m) \longmapsto {}^{x}m \\ \rho : L' \times L \longrightarrow L', & (m,x) \longmapsto m^{x} \end{array}$$
satisfying
$$\begin{array}{lll} 1. & {}^{x}[m,n] & = [{}^{x}m, n] - [{}^{x}n, m] \\ 2. & [m, {}^{x}n] & = [m^{x}, n] - [m,n]^{x} \\ 3. & [m, n^{x}] & = [m,n]^{x} - [m^{x}, n] \\ 4. & {}^{x}({}^{y}m) & = {}^{[x,y]}m - ({}^{x}m)^{y} \\ 5. & ({}^{x}m^{y}) & = ({}^{x}m)^{y} - {}^{[x,y]}m \\ 6. & m^{[x,y]} & = (m^{x})^{y} - (m^{y})^{x} \end{array}$$
for all $x, y \in L, m, n \in L'$.

Now we will recall the definition of crossed modules on Leibniz algebras from [7].

Definition 2.6. A precrossed module on Leibniz algebras is a Leibniz algebra homomorphism $\partial : L_1 \longrightarrow L_0$ with an action of L_0 on L_1 such that
$$\partial({}^{l_0}l_1) = [l_0, \partial(l_1)], \quad \partial(l_1^{l_0}) = [\partial(l_1), l_0],$$
for all $l_0 \in L_0, l_1 \in L_1$. This is a crossed module if in addition it satisfy the Peiffer identities
$$l_1^{\partial(l_1')} = [l_1, l_1'], \quad {}^{\partial(l_1')}l_1 = [l_1', l_1],$$
for all $l_1, l_1' \in L_1$.

Definition 2.7. Let $\partial : L_1 \longrightarrow L_0$, $\delta : M_1 \longrightarrow M_0$ be (pre)crossed modules. A morphism of (pre)crossed modules is a pair (f_1, f_0) of Leibniz homomorphisms $f_1 : L_1 \longrightarrow M_1$, $f_2 : L_0 \longrightarrow M_0$ such that
$$f_0\partial = \delta f_1, \quad f_1({}^{l_0}l_1) = {}^{f_0(l_0)}f_1(l_1), \quad f_1(l_1^{l_0}) = (f_1(l_1))^{f_0(l_0)},$$
for all $l_0 \in L_0, l_1 \in L_1$.

Consequently, we have the category of precrossed modules and category of crossed modules which will denoted by **PXLbnz**, **XLbnz**, respectively.

Example 2.2. Let I be a two-sided ideal of a Leibniz algebra L. Then

$$inc.: I \longrightarrow L$$

is a crossed module with the conjugate action of L on I defined by $\quad {}^{l}i = [l, i]$, $i^{l} = [i, l]$, for all $i \in I, l \in L$.

Example 2.3. For any Leibniz algebra L, $L \longrightarrow 0$ is a precrossed module, which is called a trivial precrossed module.

Remark 2.1. We have the full inclusion

$$\mathbf{Lbnz} \subseteq \mathbf{PXLbnz}$$

where a Leibniz algebra L is identified with the trivial precrossed module $L \longrightarrow 0$. So precrossed modules can be thought as a generalisation of Leibniz algebras. We have the functor

$$inc.: \mathbf{Lbnz} \longrightarrow \mathbf{PXLbnz}$$

Also we have the forgetful functor

$$U: \mathbf{XLbnz} \longrightarrow \mathbf{PXLbnz}$$

which forgets the Peiffer identities.

3. From precrossed modules to crossed modules

In this section we will define the Peiffer ideals of Leibniz algebras and give a functor from precrossed modules to crossed modules. The group case of this work was given in [1].

Definition 3.1. A commutator in a Leibniz algebra is defined by $[x, y]$, for $x, y \in L$. The commutator ideal of L is the two-sided ideal generated by all commutators of L.

Definition 3.2. Let $\partial : L_1 \longrightarrow L_0$ be a precrossed module. A left Peiffer commutator is defined by

$$\langle a, b\rangle_l = {}^{\partial(a)}b - [a, b]$$

and the right Peiffer commutator is defined by

$$\langle a, b\rangle_r = b^{\partial(a)} - [b, a]$$

for all $a, b \in L_1$. The two-sided ideal generated by the set $\{\langle a, b\rangle_l , \langle a, b\rangle_r \mid a, b \in L_1\}$ is called the Peiffer ideal of L_1 and is denoted by $P_2(\partial)$.

Theorem 3.1. *Let $\partial : L_1 \longrightarrow L_0$ be a precrossed module. Then*

$$\partial^{cr} : L_1^{cr} = L_1/P_2(\partial) \longrightarrow L_0$$

is a crossed module where ${}^{l_0}(\overline{l_1}) = \overline{{}^{l_0}l_1}, \partial^{cr}(\overline{l_1}) = \partial(l_1)$ *for all* $l_0 \in L_0, l_1 \in L_1$.

Proof. We will only look the Peiffer conditions. Since ${}^{\partial(l_1)}m_1 \equiv [l_1, m_1]$, $(m_1)^{\partial(l_1)} \equiv [m_1, l_1]$ modulo $P_2(\partial)$, we have

$${}^{\partial_2(\overline{m_1})}(\overline{l_1}) = {}^{\partial_2(m_1)}(\overline{l_1}) = \overline{({}^{\partial_2(m_1)}l_1)} = \overline{[m_1, l_1]},$$

and by a similar way

$$(\overline{l_1})^{\overline{\partial_2(m_1)}} = \overline{[l_1, m_1]},$$

for all $l_1, m_1 \in L_1$ as required. $\square$

Remark 3.1. As a consequence of Theorem 14, we have the functor

$$()^{cr} : \mathbf{PXLbnz} \longrightarrow \mathbf{Lbnz}$$

defined by

$$()^{cr}(\partial : L_1 \longrightarrow L_0) = (\partial^{cr} : L_1^{cr} \longrightarrow L_0)$$

for any precrossed module $\partial : L_1 \longrightarrow L_0$.

4. Some properties of the categories of crossed L-modules

In this section we will define the full subcategory $\mathbf{XLbnz}/L$ of $\mathbf{XLbnz}$ and construct coproduct object in this subcategory.

Let L be a fixed Leibniz algebra. We define a subcategory of $\mathbf{XLbnz}$ whose objects are crossed modules with same base L. We will denote this category by $\mathbf{XLbnz}/L$. The objects of this category will be called as crossed L-modules.

Proposition 4.1. *a) Two morphisms with same source and target have equaliser in* $\mathbf{XLbnz}/L$
b) $\mathbf{XLbnz}/L$ *has pullbacks*
c) $inc : 0 \longrightarrow L$ *is the terminal object in* $\mathbf{XLbnz}/L$

Proof. Follows from a direct calculation similar to commutative algebra case given in [6]. □

Corollary 4.1. $\mathbf{XLbnz}/L$ *is finitely complete.*

Proof. It is an obvious result of proposition 16. □

Remark 4.1. Let $(N, \partial), (M, \delta)$ be crossed L-modules. M has a Leibniz action on N, via the homomorphism δ,thanks to the action of L on N. This action gives rise to the semi-direct product $N \rtimes M$ where the bracket is defined as follows;

$$[(n, m)(n_0, m_0)] = ([n, n_0] + {}^{\delta(m)}n_0 + n^{\delta(m_0)}, [m, m_0])$$

for all $m, m_0 \in M, n, n_0 \in N$. It is obvious that L has an action on $N \rtimes M$ defined by

$${}^{l}(n, m) = ({}^{l}n, {}^{l}m) \quad , \quad (n, m)^{l} = (n^{l}, m^{l})$$

for all $l \in L, n \in N, m \in M$.

Define $\alpha : N \rtimes M \longrightarrow L$ by $\alpha(n, m) = \partial(n) + \delta(m)$. $\alpha : N \rtimes M \longrightarrow L$ is a precrossed L-module with the action of L on $N \rtimes M$ defined above. Indeed

$$\begin{aligned} \alpha({}^{l}(n, m)) &= \alpha({}^{l}n, {}^{l}m) \\ &= \partial({}^{l}n) + \delta({}^{l}m) \\ &= [l, \partial(n)] + [l, \delta(m)] \\ &= [l, \partial(n) + \delta(m)] \\ &= [l, \alpha(n, m)] \end{aligned}$$

for all $l \in L, (n, m) \in N \rtimes M$, as required. By a similar calculation we have $\alpha((n, m)^{l}) = [\alpha(n, m), l]$, for all $l \in L, (n, m) \in N \rtimes M$.

But $\alpha : N \rtimes M \longrightarrow L$ do not satisfy the Peiffer identities in general. By theorem 14

$$\alpha^{cr} : (N \rtimes M)/P_2(\alpha) \longrightarrow L$$

is a crossed module.

Corollary 4.2. $\alpha^{cr} : (N \rtimes M)/P_2(\alpha) \longrightarrow L$ *is the coproduct of the crossed L-modules* $\partial : N \longrightarrow L$, $\delta : M \longrightarrow L$.

Proof. Direct checking. Details can be found in [6], for commutative algebra case. □

References

[1] Baues, H.J., Combinatorial homotophy and 4-dimensional complexes, *Walter de Gruyter*, 15, 380 p., (2000).
[2] Casas, J.M., Invariantes de módulos cruzados en álgebras de Lie, *Ph. D. Thesis*, Universidade de Santiago de Compostela (Spain). (1991).
[3] Casas, J.M., Khmaladze, E., Ladra, M., Crossed modules for Leibniz n-algebras, *Forum Mathematicum.*, 20, 841-858. (2008).
[4] Kassel, c., Loday, J.L., Extensions centrales d'algébres de Lie, *Ann. Ins. Fourier (grenoble)*, 33, 119-142. (1982).
[5] Loday, J.L., Cyclic Homology, *Springer-Verlag*, (1992), Second Edition (1998).
[6] Loday, J.L., 'Un version non commutative des algébres de Lie: les algébres de Leibniz ', *Enseign. Math.* , 39 , 269-293 , (1992).
[7] Porter, T., Homology of commutative algebras and an invariant of Simis and Vas- conceles, *J.Algebra.*, 99, 2. (1987).
[8] Shammu, N.M.,Algebraic and an Categorical Structure of Category of Crossed Modules of Algebras, *Ph. D. Thesis*, U.C.N.W. (1992).
[9] Whitehead, J.H.C., Combinatorial Homotophy II, *Bull. Amer. Math. Soc.*, 55, 453-496. (1949).

Department of Mathematics-Computer, Eskisehir Osmangazi University, Eskisehir, Turkey

E-mail address: afaslan@ogu.edu.tr