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On some α-admissible contraction mappings on Branciari b-metric spaces

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Abstract

In this paper α-admissible contraction mappings on Branciari b-metric spaces are defined. Conditions for the existence and uniqueness of fixed points for these mappings are discussed and related theorems are proved. Various consequences of these theorems are given and specific examples are presented.

Keywords: Fixed point, Branciari b-metric space, α-admissible contraction mappings, b-comparison functions
2010 MSC: 47H10, 54C60, 54H25, 55M20.

1. Introduction and Preliminaries

In this section we define some basic concepts and notions which are going to be used in the paper. The concept of b-metric spaces have been introduced by Czerwik [7] and Bakhtin [2].

Definition 1.1. [2, 7] Let X be a nonempty set and let $d : X \times X \to [0, +\infty)$ be a mapping satisfying the following conditions for all $x, y, z \in X$:

(M_b1) $d(x, y) = 0$ if and only if $x = y$;

(M_b2) $d(x, y) = d(y, x)$;

(M_b3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for some real number $s \geq 1$.

Then the mapping d is called a b-metric and the pair (X, d) is called a b-metric space(M_bS) with a constant $s \geq 1$.

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On the other hand, Branciari [3] proposed a generalization of the metric in which he replaced the triangular inequality by a rectangular inequality. This new metric has been referred to by different names such as generalized metric, rectangular metric and Branciari metric. Following the paper by Aydi et.al [1], we will call it Branciari metric.

Definition 1.2. [3] Let X be a nonempty set and let $d : X \times X \to [0, +\infty)$ be a function such that for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y, the following conditions are satisfied:

$(BM1)$ $d(x, y) = 0$ if and only if $x = y$;

$(BM2)$ $d(x, y) = d(y, x)$;

$(BM3)$ $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

The map d is called a Branciari metric and the pair (X, d) is called a Branciari metric space (BMS).

Combining the definitions of b-metric and Branciari metric, the so-called Branciari b-metric is defined as follows.

Definition 1.3. [8] Let X be a nonempty set and let $d : X \times X \to [0, +\infty)$ be a function such that for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y, the following conditions are satisfied:

(BM_b1) $d(x, y) = 0$ if and only if $x = y$;

(BM_b2) $d(x, y) = d(y, x)$;

(BM_b3) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ for some real number $s \geq 1$.

The map d is called a Branciari b-metric and the pair (X, d) is called a Branciari b-metric space (BM_bS) with a constant $s \geq 1$.

On a Branciari b-metric space we define and denote an open ball of radius r centered at $x \in X$ as

$$B_r(x, r) = \{y \in X : |d(x, y) < r\}.$$

However, such an open ball is not always an open set.

Let $\mathcal{P}$ be the collection of all subsets $\mathcal{Y}$ of X with the following property: For each $y \in \mathcal{Y}$ there exist $r > 0$ such that $B_r(y) \subseteq \mathcal{Y}$. Then $\mathcal{P}$ defines a topology for the BM_bS (X, d), which is not necessarily Hausdorff.

Convergent sequence, Cauchy sequence, completeness and continuity on Branciari b-metric space are defined as follows.

Definition 1.4. [8] Let (X, d) be a Branciari b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

1. A sequence $\{x_n\} \subset X$ is said to converge to a point $x \in X$ if, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$. The convergence is also represented as follows.

$$\lim_{n \to \infty} x_n = x \text{ or } x_n \to x \text{ as } n \to \infty.$$

2. A sequence $\{x_n\} \subset X$ is said to be a Cauchy sequence if, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0$, $p > 0$ or equivalently, if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.
3. (X, d) is said to be a complete Branciari b-metric space if every Cauchy sequence in X converges to some $x \in X$.
4. A mapping $T : X \to X$ on is said to be continuous with respect to the Branciari b-metric d if, for any sequence $\{x_n\} \subset X$ which converges to some $x \in X$, that is $\lim\limits_{n\to\infty} d(x_n, x) = 0$ we have $\lim\limits_{n\to\infty} d(Tx_n, Tx) = 0$.

It should be noted that the limit of a sequence in a BM_bS is not necessarily unique. In addition, a convergent sequence in a BM_bS is not necessarily a Cauchy sequence. Moreover, a Branciari b-metric is not necessarily continuous. The following example illustrates these facts.

Example 1.5. Let $A = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$, $B = \{0, 3\}$ and $X = A \cup B$. Define the function $d(x,y) : X \times X \to [0,\infty)$ such that $d(x,y) = d(y,x)$ in the following way.

$$d(x,y) = \begin{cases} 0 & \text{if} \quad x = y, \\ 4 & \text{if} \quad x, y \in A, \\ \dfrac{1}{n} & \text{if} \quad x \in A, y \in B, \\ 2 & \text{if} \quad x, y \in B. \end{cases}$$

Notice that

$$d(\frac{1}{2}, 1) = 4 > d(\frac{1}{2}, 0) + d(0,1) = \frac{3}{2},$$

so, $d(x,y)$ is not a metric. In addition,

$$d(\frac{1}{2}, 1) = 4 > d(\frac{1}{2}, 0) + d(0,3) + d(3,1) = \frac{7}{2},$$

hence, $d(x,y)$ is not a Branciari metric. Moreover,

$$d(\frac{1}{m}, \frac{1}{n}) = 4 > s[d(\frac{1}{n}, 0) + d(0, \frac{1}{m})] = s\frac{m+n}{mn},$$

for $n, m \in \mathbb{N}$ satisfying $\dfrac{4mn}{m+n} > s$. Therefore, $d(x,y)$ is not a b-metric as well. However, it is Branciari b-metric with $s = 2$. Indeed, then we have

$$d(\frac{1}{m}, \frac{1}{n}) = 4 \leq 2[d(\frac{1}{n}, 0) + d(0,3) + d(3, \frac{1}{m})] = 2(2 + \frac{m+n}{mn}).$$

Observe also that

$$\lim_{n\to\infty} d(\frac{1}{2n}, 0) = \lim_{n\to\infty} \frac{1}{2n} = 0,$$

and

$$\lim_{n\to\infty} d(\frac{1}{2n}, 3) = \lim_{n\to\infty} \frac{1}{2n} = 0,$$

that is, both 0 and 3 are limits of the sequence $\{\frac{1}{2n}\}$.

Another fact about this metric is that even though the sequence $\{\dfrac{1}{2n}\}$ is convergent, it is not a Cauchy sequence. Obviously,

$$\lim_{p\to\infty} d(x_n, x_{n+p}) = \lim_{p\to\infty} d(\frac{1}{2n}, \frac{1}{2n+2p}) = \lim_{n\to\infty} 4 = 4.$$

Finally, we note that although the open set $B_1(\frac{1}{3})$ contains 0, that is $B_1(\frac{1}{3}) = \{0, 3, \frac{1}{3}\}$, there is no positive r for which $B_r(0) \subset B_1(\frac{1}{3})$.

Regarding the above facts about Branciari b-metric, we need the following property of Branciari metric space, the proof of which can be found in [10].

Proposition 1.6. *[10] Let $\{x_n\}$ be a Cauchy sequence in a Branciari metric space (X,d) such that $\lim_{n\to\infty} d(x_n, x) = 0$, where $x \in X$. Then $\lim_{n\to\infty} d(x_n, y) = d(x,y)$, for all $y \in X$. In particular, the sequence $\{x_n\}$ does not converge to y if $y \neq x$.*

Remark 1.7. The Proposition 1.6 is valid if we replace Branciari metric space by a Branciari b-metric space.

Berinde [4] and Rus [11] defined and later modified a class of functions called comparison functions. These functions are being used by many authors to replace the usual contractive condition by a more general one. We next define the comparison and (b)-comparison functions.

An increasing function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying $\varphi^n(t) \to 0$, $n \to \infty$ for any $t \in [0, \infty)$ is called a comparison function, (CF) (see e.g. [4],[11])..

A (b)-comparison function, (BCF), (see e.g.[5],[6]) is a function $\varphi_b : [0, +\infty) \to [0, +\infty)$ satisfying the conditions

(b_1) φ_b is increasing,

(b_2) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} \nu_k$ such that $s^{k+1}\varphi_b^{k+1}(t) \le as^k\varphi_b^k(t) + \nu_k$, for $k \ge k_0$ and any $t \in [0, \infty)$.

for some $s \ge 1$.

In the sequel, we denote the class of comparison functions by Φ and the class of (b)-comparison functions by Φ_b.

Comparison and (b)-comparison functions satisfy the following properties.

Lemma 1.8. *(Berinde [4], Rus [11]) Any comparison function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfies the following:*

(1) *Every iterate φ^k of φ $k \ge 1$, is also a comparison function;*
(2) *φ is continuous at 0;*
(3) *$\varphi(t) < t$, for any $t > 0$.*

Lemma 1.9. *[6] A (b)-comparison function $\varphi_b : [0, +\infty) \to [0, +\infty)$ satisfies the following:*

(1) *the series $\sum_{k=0}^{\infty} s^k\varphi_b^k(t)$ converges for any $t \in [0, +\infty)$;*
(2) *the function $b_s : [0, +\infty) \to [0, +\infty)$ defined by $b_s(t) = \sum_{k=0}^{\infty} s^k\varphi_b^k(t)$, $t \in [0, \infty)$ is increasing and continuous at 0.*

Finally, we note that any (b)-comparison function is a comparison function.

We also need to recall the notion of α-admissibility introduced by Samet et al [12] (see also [9]).

Definition 1.10. A mapping $T : X \to X$ is called α-admissible if for all $x, y \in X$ we have

$$\alpha(x, y) \ge 1 \Rightarrow \alpha(Tx, Ty) \ge 1, \tag{1.1}$$

where $\alpha : X \times X \to [0, \infty)$ is a given function.

2. Existence and uniqueness theorems on complete Branciari *b*-metric spaces

In what follows, we define some classes of α-admissible contractions.

Definition 2.1. Let (X, d) be a Branciari b-metric space with a constant $s \ge 1$ and let $\alpha : X \times X \to [0, \infty)$ and $\varphi_b \in \Phi_b$ be two given functions.

(i) An $\alpha - \varphi_b$ contractive mapping $T : X \to X$ is of type (A) if it is α-admissible and satisfies

$$\alpha(x, y)d(Tx, Ty) \le \varphi_b (M(x, y)), \text{ for all } x, y \in X \tag{2.1}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

(ii) An $\alpha - \varphi_b$ contractive mapping $T : X \to X$ is of type (B) if it is α-admissible and satisfies

$$\alpha(x, y)d(Tx, Ty) \le \varphi_b (N(x, y)), \text{ for all } x, y \in X \tag{2.2}$$

where

$$N(x, y) = \max\{d(x, y), \frac{1}{2s}[d(x, Tx) + d(y, Ty)]\}.$$

Remark 2.2. Clearly, we have $d(x,y) \leq N(x,y) \leq M(x,y)$ for all $x, y \in X$.

We state and prove an existence theorem for fixed point of $\alpha - \varphi_b$ contractive mapping in class (A).

Theorem 2.3. *Let (X,d) be a complete Branciari b-metric space with a constant $s \geq 1$. Suppose that $T : X \to X$ is an $\alpha - \varphi_b$ contractive mapping of type (A) satisfying the following conditions.*

(i) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.

(ii) T is continuous.

Then T has a fixed point.

Proof. Regarding the condition (i), we choose $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$ and define the sequence $\{x_n\}$ as

$$x_{n+1} = Tx_n \text{ for } n \in \mathbb{N}.$$

First, we assume that any two consecutive members of the sequence $\{x_n\}$ are distinct, that is, $x_n \neq x_{n+1}$ for all $n \geq 0$. Otherwise, we would have $x_p = x_{p+1} = Tx_p$ for some $p \in \mathbb{N}$, which means that x_p is a fixed point of T.

Since T is α-admissible, the condition (i) implies

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1, \tag{2.3}$$

or, continuing in this way,

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}. \tag{2.4}$$

In a similar way, starting with

$$\alpha(x_0, x_2) = \alpha(x_0, T^2x_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_2) = \alpha(x_1, x_3) \geq 1, \tag{2.5}$$

we deduce

$$\alpha(x_n, x_{n+2}) \geq 1, \text{ for all } n \in \mathbb{N}. \tag{2.6}$$

The rest of the proof is done in 4 steps.

Step 1: We will prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.7}$$

For $x = x_n$ and $y = x_{n+1}$ with the use of (2.4), the contractive condition (2.1) becomes

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n) \leq \varphi_b(M(x_{n-1}, x_n)), \end{aligned} \tag{2.8}$$

for all $n \geq 1$, where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \end{aligned}$$

The first possibility, that is $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$ for some $n \geq 1$, implies

$$d(x_n, x_{n+1}) \leq \varphi_b(M(x_{n-1}, x_n)) = \varphi_b(d(x_n, x_{n+1})) < d(x_n, x_{n+1}),$$

since $d(x_n, x_{n+1}) > 0$ and $\varphi_b(t) < t$, which is not possible. Hence, for all $n \geq 1$ we must have $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$. Then the inequality (2.8) becomes

$$d(x_n, x_{n+1}) \leq \varphi_b(M(x_{n-1}, x_n)) \leq \varphi_b(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n), \text{ for all } n \geq 1. \tag{2.9}$$

Therefore, the sequence $\{d(x_{n-1}, x_n)\}$ is decreasing ,that is,

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), \text{ for all } n \geq 1. \tag{2.10}$$

Repeated application of (2.9) yields,

$$d(x_{n+1}, x_n) \leq \varphi_b^n(d(x_0, x_1)), \text{ for all } n \geq 1. \tag{2.11}$$

Taking limit as $n \to \infty$ in(2.11) and using the statement (1) of Lemma 1.9, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Step 2: At this step we prove that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.12}$$

Let $x = x_{n-1}$ and $x = x_{n+1}$ in (2.1) and take into account (2.6). This gives

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \alpha(x_{n-1}, x_{n+1}) d(Tx_{n-1}, Tx_{n+1}) \leq \varphi_b(M(x_{n-1}, x_{n+1})), \end{aligned} \tag{2.13}$$

for all $n \geq 1$, where

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &= \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\} \\ &= \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\} \end{aligned} \tag{2.14}$$

Regarding (2.10), $M(x_{n-1}, x_{n+1})$ can be either $d(x_{n-1}, x_{n+1})$ or $d(x_{n-1}, x_n)$.
Define $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_n, x_{n+1})$. Thus, from (2.13) we have

$$\begin{aligned} a_n &= d(x_n, x_{n+2}) \leq \varphi_b(M(x_{n-1}, x_{n+1})) \\ &= \varphi_b(\max\{a_{n-1}, b_{n-1}\}) < \max\{a_{n-1}, b_{n-1}\}, \text{ for all } n \geq 1 \end{aligned} \tag{2.15}$$

On the other hand, by (2.10) we also have

$$b_n \leq b_{n-1} \leq \max\{a_{n-1}, b_{n-1}\}.$$

As a result, we get

$$\max\{a_n, b_n\} \leq \max\{a_{n-1}, b_{n-1}\} \text{ for all } n \geq 1,$$

that is, the sequence $\{\max\{a_n, b_n\}\}$ is non increasing and hence, it converges to some $l \geq 0$. If $l > 0$, due to (2.7) we have

$$l = \lim_{n\to\infty} \max\{a_n, b_n\} = \max\{\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\} = \lim_{n\to\infty} a_n$$

Now, we let $n \to \infty$ in (2.15), so that we conclude

$$l = \lim_{n\to\infty} a_n < \lim_{n\to\infty} \max\{a_{n-1}, b_{n-1}\} = l,$$

which is a contradiction and hence, $l = 0$. Then, we conclude

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0,$$

that is, (2.12) is proved.

Step 3: We shall prove that for all $n \neq m$,

$$x_n \neq x_m. \tag{2.16}$$

Assume that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $n \neq m$. We already have $d(x_p, x_{p+1}) > 0$ for each $p \in \mathbb{N}$, hence, without loss of generality we may take $m > n+1$. Consider now

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, Tx_n) = d(x_m, Tx_m) \\ &= d(Tx_{m-1}, Tx_m) \leq \alpha(x_{m-1}, x_m) d(Tx_{m-1}, Tx_m) \\ &\leq \varphi_b(M(x_{m-1}, x_m)), \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} M(x_{m-1}, x_m) &= \max\{d(x_{m-1}, x_m), d(x_{m-1}, Tx_{m-1}), d(x_m, Tx_m)\} \\ &= \max\{d(x_{m-1}, x_m), d(x_{m-1}, x_m), d(x_m, x_{m+1})\} \\ &= \max\{d(x_{m-1}, x_m), d(x_m, x_{m+1})\} = d(x_{m-1}, x_m), \end{aligned} \tag{2.18}$$

because of (2.10). Then we have,

$$d(x_m, Tx_m) \leq \varphi_b(d(x_{m-1}, x_m)),$$

for all $m \in \mathbb{N}$. Hence,

$$d(x_m, Tx_m) \leq \varphi_b(d(x_{m-1}, x_m)) \leq \varphi_b^2(d(x_{m-2}, x_{m-1})) \leq \cdots \leq \varphi_b^{m-n}(d(x_n, x_{n+1})), \tag{2.19}$$

Combining (2.17) and (2.19) we get

$$d(x_n, x_{n+1}) = d(x_m, Tx_m) \leq \varphi_b^{m-n}(d(x_n, x_{n+1})). \tag{2.20}$$

Since every iterate of a comparison function is also a comparison function, then

$$\varphi_b^{m-n}(d(x_n, x_{n+1})) < d(x_n, x_{n+1}),$$

thus, the inequality (2.20) yields

$$d(x_n, x_{n+1}) \leq \varphi_b^{m-n}(d(x_n, x_{n+1})) < d(x_n, x_{n+1}), \tag{2.21}$$

which is not possible. Therefore, our initial assumption is incorrect and we should have $x_n \neq x_m$ for all $m \neq n$.

Step 4: At this step we will prove that $\{x_n\}$ is a Cauchy sequence, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0, \text{ for all } k \in \mathbb{N}. \tag{2.22}$$

The cases $k = 1$ and $k = 2$ are proved, respectively in (2.7) and (2.12). Assume that $k \geq 3$. We have two cases:

Case 1: Suppose that $k = 2m+1$ where $m \geq 1$. Regarding Step 3, we have $x_l \neq x_s$ for all $l \neq s$, so that we can apply the condition BM_b3 in Definition 1.3, together with (2.11) implies

$$\begin{aligned} d(x_n, x_{n+k}) &= d(x_n, x_{n+2m+1}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})] \\ &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ &+ s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})] \\ &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\ &+ s^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \ldots + s^{m+1}[d(x_{n+2m}, x_{n+2m+1})] \\ &\vdots \\ &\leq s[\varphi_b^n(d(x_0, x_1)) + \varphi_b^{n+1}(d(x_0, x_1))] + s^2[\varphi_b^{n+2}(d(x_0, x_1)) + \varphi_b^{n+3}(d(x_0, x_1))] \\ &+ s^3[\varphi^{n+4}(d(x_0, x_1) + \varphi^{n+5}(d(x_0, x_1))] + \ldots + s^m[\varphi^{n+2m}(d(x_0, x_1))] \\ &\leq s\varphi_b^n(d(x_0, x_1)) + s^2\varphi_b^{n+1}(d(x_0, x_1)) + s^3\varphi_b^{n+2}(d(x_0, x_1)) \\ &+ s^4\varphi_b^{n+3}(d(x_0, x_1)) + s^5\varphi^{n+4}(d(x_0, x_1) + \ldots + s^{2m+1}\varphi^{n+2m}(d(x_0, x_1)) \\ &= \frac{1}{s^{n-1}}\left[s^n\varphi_b^n(d(x_0, x_1)) + s^{n+1}\varphi_b^{n+1}(d(x_0, x_1) + s^{n+2}\varphi_b^{n+2}(d(x_0, x_1)))\right. \\ &+ \left.\ldots + s^{n+2m}\varphi_b^{n+2m}(d(x_0, x_1))\right]. \end{aligned}$$

Define

$$\mathcal{S}_n = \sum_{p=0}^{n} s^p \varphi_b^p(d(x_0, x_1)) \text{ for } n \geq 1. \tag{2.23}$$

Then, the inequality above becomes

$$d(x_n, x_{n+2m+1}) \leq \frac{1}{s^{n-1}} \left[\mathcal{S}_{n+2m} - \mathcal{S}_{n-1}\right], \; n \geq 1, m \geq 1.$$

By the initial assumption, $x_0 \neq x_1$ and by the Lemma 1.9, we observe that the series $\sum_{p=0}^{\infty} s^p \varphi_b^p(d(x_0, x_1))$ converges to some $\mathcal{S} \geq 0$. Thus,

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = \lim_{n\to\infty} d(x_n, x_{n+2m+1}) = 0. \tag{2.24}$$

Case 2. Suppose that $k = 2m$ where $m \geq 2$. We use again the condition BM_b3 in Definition 1.3, together with (2.11) so that,

$$\begin{aligned}
d(x_n, x_{n+k}) &= d(x_n, x_{n+2m}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})] \\
&\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\
&+ s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m})] \\
&\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\
&+ \ldots + s^{m-1}\left[d(x_{n+2m-4}, x_{n+2m-3}) + d(x_{n+2m-3}, x_{n+2m-2})\right. \\
&+ \left. d(x_{n+2m-2}, x_{n+2m})\right] \\
&\vdots \\
&\leq s[\varphi_b^n(d(x_0, x_1)) + \varphi_b^{n+1}(d(x_0, x_1))] + s^2[\varphi_b^{n+2}(d(x_0, x_1)) + \varphi_b^{n+3}(d(x_0, x_1))] \\
&+ \ldots + s^{m-1}[\varphi^{n+2m-4}(d(x_0, x_1) + \varphi^{n+2m-3}(d(x_0, x_1))] \\
&+ s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \\
&\leq s\varphi_b^n(d(x_0, x_1)) + s^2\varphi_b^{n+1}(d(x_0, x_1)) + s^3\varphi_b^{n+2}(d(x_0, x_1)) \\
&+ \ldots + s^{2m-3}\varphi_b^{n+2m-4}(d(x_0, x_1)) + s^{2m-2}\varphi^{n+2m-3}(d(x_0, x_1)) \\
&+ s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \\
&= \frac{1}{s^{n-1}}\left[s^n\varphi_b^n(d(x_0, x_1)) + s^{n+1}\varphi_b^{n+1}(d(x_0, x_1) + s^{n+2}\varphi_b^{n+2}(d(x_0, x_1)))\right. \\
&+ \left. \ldots + +s^{n+2m-3}\varphi_b^{n+2m-3}(d(x_0, x_1))\right] + s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \\
&= \sum_{p=n}^{n+2m-3} s^p \varphi_b^p(d(x_0, x_1)) + s^{m-1} d(x_{n+2m-2}, x_{n+2m}).
\end{aligned}$$

Using the notation in (2.23), we rewrite the inequality above as

$$d(x_n, x_{n+k}) = \frac{1}{s^{n-1}} \left[\mathcal{S}_{n+2m-3} - \mathcal{S}_{n-1}\right] + s^{m-1} d(x_{n+2m-2}, x_{n+2m}). \tag{2.25}$$

From (2.12) we have $\lim_{n\to\infty} s^{m-1} d(x_{n+2m-2}, x_{n+2m}) = 0$, and using the Lemma 1.9 we get

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, x_{n+k}) &= \lim_{n\to\infty} d(x_n, x_{n+2m}) \\
&\leq \lim_{n\to\infty} \left[\frac{1}{s^{n-1}}(\mathcal{S}_{n+2m-3} - \mathcal{S}_{n-1}) + s^{2m-1} d(x_{n+2m-2}, x_{n+2m})\right] = 0.
\end{aligned} \tag{2.26}$$

Therefore, for any $k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0,$$

that is, the sequence $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete Branciari b-metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{2.27}$$

By the condition (ii) of the hypothesis, T is continuous. Then, from (2.27) we have

$$\lim_{n\to\infty} d(Tx_n, Tu) = \lim_{n\to\infty} d(x_{n+1}, Tu) = 0,$$

that is, the sequence $\{x_n\}$ converges to Tu as well. But then, the Proposition 1.6 implies that $Tu = u$, that is, u is a fixed point of T. □

The Theorem 2.3 provides the existence of a fixed point. To have uniqueness we impose an additional requirement.

(U) For every pair x and y of fixed points of T, $\alpha(x, y) \geq 1$.

Theorem 2.4. *If we add the condition (U) to the statement of Theorem 2.3, the fixed point of the mapping is unique.*

Proof. The existence of a fixed point is proved in Theorem 2.3. Assume that the map T has two fixed points, say $x, y \in X$, such that $x \neq y$. The condition (U) implies that $\alpha(x, y) \geq 1$. If $d(x, y) > 0$ then the contractive condition (2.1) with the fixed points x and y yields

$$d(x, y) = \alpha(x, y)d(Tx, Ty) \leq \varphi_b(M(x, y)),$$

where,

$$M(x, y) = \max\{d(x, y), d(Tx, x), d(Ty, y)\} = d(x, y).$$

Since $\varphi_b(t) < t$ for $t > 0$, we have

$$d(x, y) \leq \varphi_b(d(x, y)) < d(x, y),$$

which is not possible. Therefore, $d(x, y) = 0$, or, $x = y$ which completes the proof of the uniqueness. □

The strong condition on continuity of the map T can be replaced by a weaker condition called α-regularity of the space. This condition reads as follows.

(RG) A Branciari b-metric space (X, d) is called α-regular if for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} d(x_n, x) = 0$ and satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

If we replace the continuity condition of the mapping T by the α-regularity of the space (X, d) we have the following result.

Theorem 2.5. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Suppose that $T : X \to X$ is an $\alpha - \varphi_b$ contractive mapping of type (A) and that the following conditions hold.*

(i) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.*

(ii) *(X, d) is α-regular, that is (RG) holds on (X, d).*

Then T has a fixed point. If, in addition the condition (U) holds on X, the fixed point is unique.

Proof. Starting with the element $x_0 \in X$ satisfying the condition (i), we construct the sequence of successive iterations $\{x_n\}$ as $x_n = Tx_{n-1}$, for $n \in \mathbb{N}$.

The convergence of this sequence can be shown exactly as in the proof of Theorem 2.3.

Let u be the limit of $\{x_n\}$, that is,

$$\lim_{n\to\infty} d(x_n, u) = 0.$$

We will show that u is a fixed point of T. For the sequence $\{x_n\}$ which converges to u we have from (2.4) that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$. Then, the α-regularity condition (RG) implies that

$$\alpha(x_n, u) \geq 1, \text{ for all } n \in \mathbb{N}_0.$$

The contractive inequality (2.1) with x_n and u becomes

$$d(Tx_n, Tu) \leq \alpha(x_n, u)d(Tx_n, Tu) \leq \varphi_b(M(x_n, u)), \tag{2.28}$$

where

$$M(x_n, u) = \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu)\}.$$

If $M(x_n, u) > 0$, then (2.28) implies

$$\begin{aligned} d(Tx_n, Tu) &\leq \alpha(x_n, u)d(Tx_n, Tu) \leq \varphi_b(M(x_n, u)) \\ &< M(x_n, u) = \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu)\}, \end{aligned} \tag{2.29}$$

whereupon, by letting $n \to \infty$ and regarding the Proposition 1.6, we obtain

$$d(u, Tu) = \lim_{n\to\infty} d(x_{n+1}, Tu) < \lim_{n\to\infty} \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu)\} = d(u, Tu), \tag{2.30}$$

which is a contradiction. Then we should have $M(x_n, u) = 0$, that is $d(u, Tu) = 0$, hence, u is a fixed point of T.

The proof of uniqueness is identical to the proof of Theorem 2.4. □

We present next some immediate consequences of the main results given in Theorems 2.3, 2.4 and 2.5. First, we observe that regarding the Remark 2.2, the existence and uniqueness of a fixed point of the contraction mappings of type (B) is easily concluded.

Theorem 2.6. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Suppose that $T : X \to X$ is an $\alpha - \varphi_b$ contractive mapping of type (B) satisfying the following:*

(i) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.
(ii) Either T is continuous or (X, d) satisfies (RG).

Then T has a fixed point.
If, in addition the condition (U) holds on X, the fixed point is unique.

Another result follows from the Remark 2.2.

Theorem 2.7. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Suppose that $\alpha(x, y) : X \times X \to [0, \infty)$ is a given mapping and that $T : X \to X$ is an α-admissible continuous mapping satisfying the conditions:*

(i) $\alpha(x, y)d(Tx, Ty) \leq \varphi_b(d(x, y))$, for all $x, y \in X$.
(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.
(iii) Either T is continuous or (X, d) satisfies (RG).

Then T has a fixed point. If, in addition the condition (U) holds on X, the fixed point is unique.

Taking $\alpha(x, y) = 1$ for all $x, y \in X$ in Theorem 2.3, we obtain the following corollary the proof of which is also obvious.

Corollary 2.8. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Suppose that $T : X \to X$ is a continuous mapping satisfying*

$$d(Tx, Ty) \leq \varphi_b(M(x, y)), \tag{2.31}$$

for all $x, y \in X$, where $\varphi_b \in \Psi_b$.

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Then T has a unique fixed point.

Corollary 2.9. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Suppose that $T : X \to X$ is a continuous mapping satisfying*

$$d(Tx, Ty) \leq \varphi_b\left(d(x, y)\right), \text{ for all } x, y \in X. \tag{2.32}$$

Then T has a unique fixed point.

The following result is obtained by choosing a particular (b)-comparison function as $\varphi_b(t) = \dfrac{k}{s}t$ with $0 < k < 1$.

Corollary 2.10. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Suppose that $\alpha : X \times X \to [0, \infty)$ is a given function and $T : X \to X$ is an α-admissible mapping satisfying the following.*

(i)

$$\alpha(x, y)d(Tx, Ty) \leq \frac{k}{s}M(x, y), \tag{2.33}$$

for all $x, y \in X$ and some $0 < k < 1$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

(ii) $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$ for some $x_0 \in X$.

(iii) Either T is continuous or (X, d) satisfies (RG). Then T has a fixed point in X. If, in addition, the condition (U) holds on X, the fixed point is unique.

As a final consequence, we give the following corollary.

Corollary 2.11. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$ and $T : X \to X$ be a continuous mapping. Suppose that for some constants $a, b, c \geq 0$ and $0 < k < 1$ with $a + b + c \leq \dfrac{k}{s}$ the inequality*

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty), \tag{2.34}$$

holds for all $x, y \in X$. Then T has a unique fixed point.

Proof. Observe that for all $x, y \in X$

$$ad(x, y) + bd(x, Tx) + cd(y, Ty) \leq \frac{k}{s}M(x, y)$$

where $0 < k < 1$. Then the proof follows from Corollary 2.10. □

We give an example to illustrate the theoretical results presented above.

Example 2.12. Suppose that $X = A \cup B$ where $A = \left\{\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{6}, \dfrac{1}{8}\right\}$ and $B = [1, 4]$. Define the mapping $d : X \times X \to [0, \infty)$ with $d(x, y) = d(y, x)$ as follows.

For $x, y \in B$, or $x \in A$ and $y \in B$, $d(x, y) = |x - y|$ and

$$d(\tfrac{1}{2}, \tfrac{1}{4}) = d(\tfrac{1}{6}, \tfrac{1}{8}) = 0.2$$

$$d(\tfrac{1}{2}, \tfrac{1}{6}) = d(\tfrac{1}{4}, \tfrac{1}{6}) = d(\tfrac{1}{4}, \tfrac{1}{8}) = 0.1$$

$$d(\tfrac{1}{2}, \tfrac{1}{8}) = 1$$

This mapping is a Branciari b-metric with $s = 2$. Let $T : X \to X$ be defined as

$$Tx = \begin{cases} \frac{x}{4} & \text{if } \quad x \in B, \\ \\ \frac{1}{6} & \text{if } \quad x \in A. \end{cases}$$

Then, the mapping T satisfies the condition

$$d(Tx, Ty) \leq \varphi_b \left(d(x, y) \right),$$

for all $x, y \in X$ where $\varphi_b(t) = \frac{t}{4}$ is a (b)-comparison function. Hence, by Corollary 2.11, T has a unique fixed point which is $x = \frac{1}{6}$.

3. Concluding Remarks

The main contributions of this study to Fixed point theory are the existence-uniqueness results given in Theorems 2.3, 2.4 and 2.5. These theorems provides existence and uniqueness conditions for a large class of contractive mappings on Branciari b-metric spaces. By taking $s = 1$ and/or $\alpha(x, y) = 1$ in all the theorems and corollaries, various existing results on Branciari b-metric and Branciari metric spaces can be obtained.

On the other hand, it should be mentioned that by choosing the function α in the definition of α-admissible mappings in a particular way, it is possible to obtain existence and uniqueness results for maps defined on partially ordered metric spaces.

Define a partial ordering $\preceq$ on a Branciari b-metric space (X, d). Let $T : X \to X$ be an increasing mapping. Then, we can easily proof the following fixed point theorem.

Theorem 3.1. *Let $(X, d, \preceq)$ be a complete Branciari b-metric space with a constant $s \geq 1$ on which a partial ordering $\preceq$ is defined. Suppose that $T : X \to X$ is an increasing mapping satisfying the following:*

(i)

$$d(Tx, Ty) \leq \varphi_b(M(x, y)),$$

for all x, y in X with $x \preceq y$ and some (b)-comparison function φ_b where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

(ii) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2 x_0$.

(iii) Either T is continuous or, for any increasing sequence $\{x_n\} \in X$ which converges to x we have $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then T has a fixed point.If, in addition any two fixed points of T are comparable, that is, $x \preceq y$ or $y \preceq x$, then the fixed point of T is unique.

Proof. Observe that all the conditions of Theorems 2.3, 2.4 and 2.5 hold if we choose the function α as

$$\alpha(x, y) = \begin{cases} 1 & \text{if } \quad x \preceq y \text{ or } y \preceq x \\ 0 & \text{if } \quad \text{otherwise} \end{cases}.$$

Then, the mapping T has a unique fixed point. □

In addition, all the consequent results of Theorems 2.3, 2.4 and 2.5 can be written on Branciari b-metric spaces with a partial ordering can be proved in a similar way.

4. Competing Interests

The authors declare that they have no competing interests.

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On some Banach lattice-valued operators: A Survey

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Abstract

In 1928, at the International Mathematical Congress held in Bologna (Italy), Frigyes Riesz introduced the notion of vector lattice on function spaces and, talked about linear operators that preserve the join operation, nowadays known in the literature as Riesz homomorphisms (see [32]). In this survey we review the behaviors of some non-linear join-preserving Riesz space-valued functions, and we show how existing addition dependent results can be proved in these environments mutatis mutandis. (We kindly refer the reader to the papers [1, 2, 3, 4, 6, 7, 8, 9, 10, 5] for more information.)

Keywords: Banach lattices, optimal measure, optimal average, dual Orlicz spaces, functional equation, functional inequality, Hyers-Ulam-Aoki type of stability.
2010 MSC: 47H10, 54C60, 54H25, 55M20.

1. Motivations, historical background and introduction

1.1. The motivations

By splitting Mathematics into two, the group of addition-related environments and the group of addition-free environments, we then ask the question to know whether there are addition-dependent environments and if any, what results they contain that can be proved in addition-free environments, the proofs being carried out mutatis-mutandis.

The collection of the present results aims to provide some answers to the above series of questions in the affirmative. In fact, we consider mappings whose target sets are lattices and show how existing addition-dependent results can be proved similarly in lattice environments. In the early 90's we substituted with the

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lattice join operation, the addition in the definition of measure as well as in the Lebesgue integral to obtain lattice-dependent operators which behave similarly as their counterparts in Measure Theory (sometimes under restraints), in the sense that existing major theorems in Measure Theory are also proved with the addition replaced by the join (or supremum). It is worth also to turn our interest to what make these two groups of environments different from each other, yet similar can their results be. The next targetted environment is the famous Cauchy functional equation. Replacing the addition by lattice operations the Hyers-Ulam stability problem can be posed. In this case also, the various solutions obtained for such problem are the same as their counterparts in the literature. Furthermore, on group structure separation theorem can also be proved when the target set is a lattice [10]. To illustrate the divergence of the above two groups, there are characterizations of various properties of measurable functions [3], as well as the characterization of an arbitrary infinite σ-algebra to be equinumerous with a power set [4].

2. Historical backgrounds and notations

2.1. About the convergence of function sequences

Augustin Louis Cauchy in 1821 published a faulty proof of the false statement that the pointwise limit of a sequence of continuous functions is always continuous. Joseph Fourier and Niels Henrik Abel found counter examples in the context of Fourier series. Dirichlet then analyzed Cauchy's proof and found the mistake: the notion of pointwise convergence had to be replaced by uniform convergence.
The concept of uniform convergence was probably first used by Christoph Gudermann. Later his pupil Karl Weierstrass coined the term *gleichmäßig konvergent* (German: uniform convergence) which he used in his 1841 paper *Zur Theorie der Potenzreihen*, published in 1894. Independently a similar concept was used by Philipp Ludwig von Seidel and George Gabriel Stokes but without having any major impact on further development. G.H. Hardy compares the three definitions in his paper *Sir George Stokes and the concept of uniform convergence* and remarks: *Weierstrass's discovery was the earliest, and he alone fully realized its far-reaching importance as one of the fundamental ideas of analysis.* For more materials about these facts we refer to [33] or
http://en.wikipedia.org/wiki/Uniform_convergence.

Ever since many other types of convergence have been brought to light. We can list some few of them: discrete and equal convergence introduced by Á. Császár and M. Laczkovich in 1975 (cf. [14, 15, 16]), topologically speaking the weak and strong convergence, the latest being at the origin of the so-called *Banach spaces*, which are very broad and interesting classes of functions, indeed.

2.2. Riesz spaces

A vector space over the field of real line endowed with a partial ordering is called a *Riesz space* if the following clauses are met:

1. the algebraic structure of the vector space and the ordering are compatible, i.e. the ordering is translation invariant and positive homogenious (referred to as *a vector lattice*),
2. every finite subset of the space has a least upper bound called the supremum.

It can be seen that a vector lattice is a Riesz space if and only if every pair of elements in the space has an infimum (cf. [11, Aliprantis and Burkinshaw, Lemma 1.2]). The next very important properties enjoyed by Riesz spaces are:

a. Every Riesz space is a distributive lattice.
b. The positive cone of any Riesz space is generating, i.e. every element of the space can be expressed as the differerence of two elements of the positive cone. (For more see [28].)

This last point means that working on the positive cone of a Riesz space is just as working on the whole space.

The notion of vector lattice was introduced by Frigyes Riesz on function spaces at the International Mathematical Congress in Bologna (1928), which was publised two years later (cf. [32]). Around the mid-thirties Riesz was relayed by Hans Freudenthal (cf. [20]) and L.V. Kantorovich (cf. [24, 25]) by simultaneously laying the strict axiomatic foundation of the theory of Riesz spaces. This new concept has grown very rapidly in the 1940s and early 50s, thanks to Japanese and Russian schools which were created to cultivate this young theory. (Cf. [11, Aliprantis and Burkinshaw] for more historical background.) At the earliest stages rather algebraic aspect of the theory was studied. The analytical aspect started with a series of articles by W.A.J. Luxemburg and A.C. Zaanen which can be found in the book by Aliprantis and Burkinshaw, reference [89]. Another aspect of the theory of Riesz spaces is topological (cf. [19, Fremlin]). We would also like to stress the important place supremum preserving linear operators (so-called Riesz homomorphisms) occupy in the literature.

2.3. **Notations.**

⋆ $\mathbb{N}$ denotes the set of positive integers.
⋆ $\mathbb{R}$ denotes the set of real numbers.
⋆ $\mathbb{R}_+$ denotes the set of non-negative real numbers.
⋆ $\chi(B)$ stands for the characteristic function of the set B.
⋆ $|B|$ designates the cardinality of the set B.
⋆ $\bigvee$ and $\vee$ (respectively, $\bigwedge$ and $\wedge$) stand for the maximum (respectively the minimum) operator.
⋆ $\mathcal{P} := \mathcal{P}_{<\infty} \cup \mathcal{P}_{\infty}$ will denote the set of all optimal measures defined on measurable space $(\Omega, \mathcal{F})$, with both Ω and $\mathcal{F}$ being infinite sets, where $\mathcal{P}_{<\infty}$ (resp. $\mathcal{P}_{\infty}$) denotes the set of all optimal measures whose generating systems are finite (resp. countably infinite).
⋆ For every $A \in \mathcal{F}$, we write $\overline{A}$ for the complement of A.
⋆ $A \subset B$ means set A is a proper subset of set B.
⋆ $A \subseteq B$ means set A is a subset of set B.
⋆ The power set of set A will be denoted by $\mathbb{P}(A)$ or 2^A.

We would like to note that our approach of dealing with Riesz spaces seems new. The results we present here are selected from [1, 2, 3, 4, 6, 8, 9, 7, 10, 5] and they all fall outside the scope of Riesz homomorphisms.

3. Optimal measures and the structure theorem

By replacing the addition in the definition of (probability) measure by the supremum we expect to obtain a non-additive set function which behaves almost like a (probability) measure. To this end normalizing properties and the continuity from below are necessary to have similar effects as in the case of measure.

3.1. Optimal measure

Definition 3.1 ([1], Definition 0.1)**.** A set function $p : \mathcal{F} \to [0, 1]$ will be called *optimal measure* if it satisfies the following three axioms:

Axiom 1. $p(\Omega) = 1$ and $p(\varnothing) = 0$.

Axiom 2. $p(B \cup E) = p(B) \vee p(E)$ for all measurable sets B and E.

Axiom 3. p is continuous from above, i.e. whenever $(E_n) \subset \mathcal{F}$ is a decreasing sequence, then $p\left(\bigcap\limits_{n=1}^{\infty} E_n\right) = \lim\limits_{n\to\infty} p(E_n) = \bigwedge\limits_{n=1}^{\infty} p(E_n)$.

The triple $(\Omega,\, \mathcal{F},\, p)$ will be referred to as an *optimal measure space*. For all measurable sets B and C with $B \subset C$, the identity

$$p\,(C\backslash B) = p\,(C) - p\,(B) + \min\left\{p\,(C\backslash B)\,,\, p\,(B)\right\} \tag{3.1}$$

holds, and especially for all $B \in \mathcal{F}$,

$$p\left(\overline{B}\right) = 1 - p\,(B) + \min\left\{p\,(B)\,,\, p\left(\overline{B}\right)\right\}.$$

In fact, it is obvious (via Axiom 2) that,

$$\begin{aligned} p\,(B) + p\,(C\backslash B) &= \max\left\{p\,(C\backslash B)\,,\, p\,(B)\right\} + \min\left\{p\,(C\backslash B)\,,\, p\,(B)\right\} \\ &= p\,(C) + \min\left\{p\,(C\backslash B)\,,\, p\,(B)\right\}. \end{aligned}$$

Lemma 3.2 ([1], Lemma 0.1)**.** *Let $(B_n) \subset \mathcal{F}$ be any sequence tending increasingly to a measurable set B, and p an optimal measure. Then $\lim\limits_{n\to\infty} p\,(B_n) = p\,(B)$.*

Proof. The lemma will be proved if we show that for some $n_0 \in \mathbb{N}$, the identity $p\,(B) = p\,(B_n)$ holds true whenever $n \geq n_0$. Assume that for every $n \in \mathbb{N}$, $p\,(B) \neq p\,(B_n)$, which is equivalent to $p\,(B_n) < p\,(B)$, for all $n \in \mathbb{N}$. This inequality, however, implies that $p\,(B) = p\,(B\backslash B_n)$ for each $n \in \mathbb{N}$. But since sequence $(B\backslash B_n)$ tends decreasingly to $\varnothing$, we must have that $p\,(B) = 0$, a contradiction which proves the lemma. □

It is clear that every optimal measure p is monotonic and σ-subadditive.

The following example was given in [1], Example 3.1 and its check was left as an exercise.

Example 3.3. The function $\Phi : 2^{\mathbb{N}} \to [0,\, 1]$ defined by $\Phi\,(A) = \frac{1}{\min A}$ is an optimal measure (where $\min \emptyset = \infty$ by convention).

Proof. The normalization properties are obvious. We show that Φ is a join homomorphism. In fact, let $A,\, B \in 2^{\mathbb{N}}$ be arbitrary. Then as $\min(A \cup B) = \min\{\min A; \min B\}$ it ensues that

$$\Phi(A \cup B) = \frac{1}{\min\{\min A; \min B\}} = \frac{1}{\min A} \vee \frac{1}{\min B} = \Phi(A) \vee \Phi(B).$$

To check the continuity from above, pick arbitrarily a sequence $(A_n) \subset 2^{\mathbb{N}}$ which tends decreasingly to some subset A of $\mathbb{N}$. Then for all natural numbers n and from the trivial identity $A_n = A \cup (A_n \setminus A)$ we have $\Phi(A_n) = \Phi(A) \vee \Phi(A_n \setminus A)$. But since sequence $(A_n \setminus A)_{n\in\mathbb{N}}$ tends deacreasingly to the empty set, it follows that $\lim\limits_{n\to\infty} \min(A_n \setminus A) = \infty$ which yields

$$\lim_{n\to\infty} \Phi(A_n \setminus A) = \lim_{n\to\infty} \frac{1}{\min(A_n \setminus A)} = 0.$$

Consequently,

$$\lim_{n\to\infty} \Phi(A_n) = \bigwedge_{n=1}^{\infty} \Phi(A_n) = \Phi(A) \vee \left(\bigwedge_{n=1}^{\infty} \Phi(A_n \setminus A)\right) = \Phi(A).$$

□

Example 3.4 ([1], Example 0.1)**.** Let $(\Omega,\, \mathcal{F})$ be a measurable space, $(\omega_n) \subset \Omega$ be a fixed sequence, and $(\alpha_n) \subset [0,\, 1]$ a given sequence tending decreasingly to zero. The function $p : \mathcal{F} \to [0,\, 1]$, defined by

$$p\,(B) = \max\left\{\alpha_n : \omega_n \in B\right\} \tag{3.2}$$

is an optimal measure.

Moreover, if $\Omega = [0,\, 1]$ and $\mathcal{F}$ is a σ-algebra of $[0,\, 1]$ containing the Borel sets, then every optimal measure defined on $\mathcal{F}$ can be obtained as in (3.2).

Proof of the moreover part. We first prove that if $B \in \mathcal{F}$ and $p(B) = c > 0$, then there is an $x \in B$ which satisfies $p(\{x\}) = c$. To do this let us show that there exists a nested sequence of intervals $I_0 \supset I_1 \supset I_2 \supset \ldots$ such that $|I_n| = 2^{-n}$ and $p(B \cap I_n) = c$, for every $n \in \mathbb{N} \cup \{0\}$. In fact, let $I_0 = [0, 1]$. If I_n has been defined then let $I_n = E \cup H$, where E and H are non-overlapping intervals with $|E| = |H| = 2^{-n-1}$. Obviously, we may choose $I_{n+1} = E$ or H. By the continuity from above we have $p\left(\bigcap_{n=1}^{\infty}(B \cap I_n)\right) = c > 0$. In particular, $B \cap \left(\bigcap_{n=1}^{\infty} I_n\right) \neq \varnothing$. This implies that $B \cap \left(\bigcap_{n=1}^{\infty} I_n\right) = \{x\}$ and $p(\{x\}) = c$. Fix $c > 0$. Then the set $\{x : p(\{x\}) \geq c\}$ is finite. Assume in the contrary that there is an infinite sequence $(x_k) \subset [0, 1]$ such that $p(\{x_k\}) \geq c$, $k \in \mathbb{N}$. Thus denoting $B_k = \{x_k, x_{k+1}, \ldots\}$, it is clear that $\bigcap_{k=1}^{\infty} B_k = \varnothing$; but this contradicts the fact that $p(B_k) \geq c$. Consequently, the set $E_n = \left\{x : p(\{x\}) \geq n^{-1}\right\}$ is finite for all $n \in \mathbb{N}$. Hence there is a sequence $(x_n) \subset [0, 1]$ such that $p(\{x_n\}) \downarrow 0$ (as $n \to \infty$) and every point $x \in [0, 1]$ with $p(\{x\}) \geq 0$ is contained in (x_n). Therefore, for all $B \in \mathcal{F}$, $p(B) = \max\{\alpha_n : x_n \in B\}$ which is just the above optimal measure. □

3.2. The structure of optimal measures

By a p-atom we mean a measurable set $H, p(H) > 0$ such that whenever $B \in \mathcal{F}$ and $B \subset H$, then $p(B) = p(H)$ or $p(B) = 0$.

Definition 3.5 ([2], Definition 1.1)**.** A p-atom H is decomposable if there exists a subatom $B \subset H$ such that $p(B) = p(H) = p(H \backslash B)$. If no such subatom exists, we shall say that H is indecomposable.

Lemma 3.6 ([2], Lemma 1.1)**.** *Any atom H can be expressed as the union of finitely many disjoint indecomposable subatoms of the same optimal measure as H.*

Proof. We say that a measurable set E is *good* if it an be expressed as the union of finitely many disjoint indecomposable subatoms. Let H be an atom and suppose that H is not good. Then H is decomposable. Set $H = B_1 \cup C_1$, where B_1 and C_1 are disjoint measurable sets with $p(B_1) = p(C_1) = p(H)$. Since H is not good, at least one of the two measurable sets B_1 and C_1 is not good; suppose, e.g. that B_1 is not good. Then B_1 is decomposable. Write $B_1 = B_2 \cup C_2$, where B_2 and C_2 are disjoint measurable sets with $p(B_2) = p(C_2) = p(H)$. Continuing this process for every $n \in \mathbb{N}$ we obtain two measurable sets B_n and C_n such that the C_n's are pairwise disjoint with $p(C_n) = p(H)$. This, however, is impossible since $E_n = \bigcup\limits_{k=n}^{\infty} C_k$ tends decreasingly to the empty set and hence, by Axiom 3, $p(E_n) \to p(\varnothing)$ as $n \to \infty$, which contradicts that $p(E_n) \geq p(C_n) = p(H) > 0$, $n \in \mathbb{N}$. □

An immediate consequent of Lemma 3.6 is as follows.

Remark 3.7 ([2], Remark 1.1)*.* Let H be any indecomposable p-atom and E any measurable set, with $p(E) > 0$. Then, either $p(H) = p(H \backslash E)$ and $p(H \cap E) = 0$, or $p(H) = p(H \cap E)$ and $p(H \backslash E) = 0$.

The Structure Theorem ([2], Theorem 1.2) Let $(\Omega, \mathcal{F}, p)$ be an optimal measure space. Then there exists a collection $\mathcal{H}(p) = \{H_n : n \in J\}$ of disjoint indecomposable p-atoms, where J is some countable (i.e. finite or countably infinite) index set, such that for every measurable set $B \in \mathcal{F}$ with $p(B) > 0$ we have

$$p(B) = \max\{p(B \cap H_n) : n \in J\}. \tag{3.3}$$

Moreover, if J is countably infinite, then the only limit point of the set $\{p(H_n) : n \in J\}$ is 0.

The proof was derived from the following lemmas, which we shall recollect without their proofs.

Lemma 3.8 ([2], Lemma 1.3)**.** *Let $E \in \mathcal{F}$ be with $p(E) > 0$, and $B_k \in \mathcal{F}$, $B_k \subset E$ $(k \in J)$, where J is any countable index set. Then*

$$p\left(\bigcup_{k \in J} B_k\right) < p(E) \quad \text{if and only if} \quad p(B_k) < p(E) \quad \text{for all } k \in J. \tag{3.4}$$

Lemma 3.9 ([2], Lemma 1.4). *For every sequence $(B_n) \subset \mathcal{F}$ and every optimal measure p we have*

$$p\left(\bigcup_{n=1}^{\infty} B_n\right) = \max\left\{p\left(B_n\right) : n \in \mathbb{N}\right\}.$$

Lemma 3.10 ([2], Lemma 1.5). *Every measurable set $E \in \mathcal{F}$ with $p(E) > 0$ contains an atom $H \subset E$ such that $p(E) = p(H)$.*

Lemma 3.11 ([2], Lemma 1.6). *Let $\mathcal{H} = \{H_n : n \in J\}$ be as above. Then for every measurable set $B \in \mathcal{F}$ with $p(B) > 0$, the identity(6.4)*

$$p\left(B \backslash \bigcup_{n \in J} (B \cap H_n)\right) = 0 \tag{3.5}$$

holds.

We are now in the position to prove the Structure Theorem.

Proof of the Structure Theorem. Let $\mathcal{G}$ be a set of pairwise disjoint atoms. It is clear that the collection of all such $\mathcal{G}$, denoted by Γ, is partially ordered by the set inclusion and every subset of Γ has an upper bound. Then, the Zorn lemma entails that Γ contains a maximal element, which we shall denote by $\mathcal{G}^*$. As we have done above, one can easily verify that the set

$$\left\{K \in \mathcal{G}^* : p(K) > n^{-1}\right\}$$

is finite. Hence $\mathcal{G}^* = \{K_j : j \in \nabla\}$, where ∇ is a countable index set. It is obvious that $p(K_j) \to 0$ as $j \to \infty$, whenever ∇ is a countably infinite set. Consequently, it ensues, via Lemma 3.6, that each atom $K_j \in \mathcal{G}^*$ can be expressed as the union of finitely many disjoint indecomposable subatoms of the same optimal measure as K_j. Finally, let us list these indecomposable atoms occurring in the decompositions of the elements of $\mathcal{G}^*$ as follows: $\mathcal{H} = \{H_n : n \in J\}$, where J is a countable index set. Now, via Lemma 3.9, the identity (3.5) and Axiom 2, one can easily observe that (3.3) holds for every set $B \in \mathcal{F}$, with $p(B) > 0$. It is also obvious that 0 is the only limit point of the set $\{p(H_n) : n \in J\}$ whenever J is a countably infinite set. This ends the proof of the theorem. □

To end the section, we need to point out that an elementary proof was given to the Structure Theorem in [17].

4. Lebesgue's type integral in lattice environments

In comparison with the mathematical expectation or Lebesgue integral, we define a non-linear functional (first for non-negative measurable simple functions and secondly for non-negative measurable functions) which provide us with many well-known results in measure theory. Their proofs are carried out similarly.

4.1. Optimal average

In the whole section we shall be dealing with an arbitrary but fixed optimal measure space $(\Omega, \mathcal{F}, p)$. Let

$$s = \sum_{i=1}^{n} b_i \chi(B_i)$$

be an arbitrary non-negative measurable simple function, where
$\{B_i : i = 1, \ldots, n\} \subset \mathcal{F}$ is a partition of Ω. Then the so-called optimal average of s is defined by

Definition 4.1 ([1], Definition 1.1)**.** The quantity

$$\bigvee_{\Omega} s\, dp := \bigvee_{i=1}^{n} b_i p\left(B_i\right)$$

will be called optimal average of s, and for $E \in \mathcal{F}$

$$\bigvee_{B} s\chi\left(E\right) dp := \bigvee_{i=1}^{n} b_i p\left(E \cap B_i\right)$$

as the optimal average of s on E, where $\chi\left(E\right)$ is the indicator function of the measurable set E. These quantities will be sometimes denoted respectively by $I\left(s\right)$ and $I_E\left(s\right)$.

As it is well-known, a measurable simple function can have many decompositions. The question thus arises (just as in the case of Lebesgue integral) whether or not the optimal average of a simple function depends on its decompositions. The following result gives a satisfactory answer to this question, making the definition of optimal average as deep as the Lebesgue integral is.

Theorem 4.2 ([1], Theorem 1.0)**.** *Let*

$$\sum_{i=1}^{n} b_i \chi\left(B_i\right) \quad \textit{and} \quad \sum_{k=1}^{m} c_k \chi\left(C_k\right)$$

be two decompositions of a measurable simple function $s \geq 0$*, where* $\{B_i : i = 1, \ldots, n\}$ *and* $\{C_k : k = 1, \ldots, m\} \subset \mathcal{F}$ *are partitions of* Ω*. Then*

$$\max\left\{b_i p\left(B_i\right) : i = 1, \ldots, n\right\} = \max\left\{c_k p\left(C_k\right) : k = 1, \ldots, m\right\}.$$

Proof. Since $B_i = \bigcup_{k=1}^{m}\left(B_i \cap C_k\right)$ and $C_k = \bigcup_{i=1}^{n}\left(B_i \cap C_k\right)$, Axiom 2 of optimal measure implies that

$$p\left(B_i\right) = \max\left\{p\left(B_i \cap C_k\right) : k = 1, \ldots, m\right\} \text{ and } p\left(C_k\right) = \max\left\{p\left(B_i \cap C_k\right) : i = 1, \ldots, n\right\}$$

Thus

$$\max\left\{c_k p\left(C_k\right) : k = 1, \ldots, m\right\} = \max\left\{\max\left\{c_k p\left(B_i \cap C_k\right) : i = 1, \ldots, n\right\} : k = 1, \ldots, m\right\}$$

and

$$\max\left\{b_i p\left(B_i\right) : i = 1, \ldots, n\right\} = \max\left\{\max\left\{b_i p\left(B_i \cap C_k\right) : k = 1, \ldots, m\right\} : i = 1, \ldots, n\right\}.$$

Clearly, if $B_i \cap C_k \neq \varnothing$, then $b_i = c_k$, or if $B_i \cap C_k = \varnothing$, then $p\left(B_i \cap C_k\right) = 0$. Thus, by the associativity and the commutativity, we obtain

$$\max\left\{b_i p\left(B_i\right) : i = 1, \ldots, n\right\} = \max\left\{c_k p\left(C_k\right) : k = 1, \ldots, m\right\}.$$

This completes the proof. □

Proposition 4.3 ([1], Proposition 2.0)**.** *Let* $f \geq 0$ *be any bounded measurable function. Then*

$$\sup_{s \leq f} \bigvee_{\Omega} s\, dp = \inf_{\overline{s} \geq f} \bigvee_{\Omega} \overline{s} dp,$$

where s *and* $\overline{s}$ *denote non-negative measurable simple functions.*

Proof. Let f be a measurable function such that $0 \le f \le b$ on Ω, where b is some constant. Let $E_k = \left(kbn^{-1} \le f \le (k+1)\,bn^{-1}\right)$, $k = 1, \ldots, n$. Clearly, $\{E_k : k = 1, \ldots, n\} \subset \mathcal{F}$ is a partition of Ω. Define the following measurable simple functions:

$$s_n = bn^{-1} \sum_{k=0}^{n} k\chi\left(E_k\right), \;\; \overline{s}_n = bn^{-1} \sum_{k=0}^{n} (k+1)\,\chi\left(E_k\right).$$

Obviously, $s_n \le f \le \overline{s}_n$. Then we can easily observe that

$$\sup_{s\le f} \mathop{\diagdown}\limits_{\Omega} s\,dp \ge \mathop{\diagdown}\limits_{\Omega} s_n dp = n^{-1} b \max\left\{kp\left(E_k\right) : k = 0, \ldots, n\right\}$$

and

$$\inf_{\overline{s}\ge f} \mathop{\diagdown}\limits_{\Omega} \overline{s} dp \le \mathop{\diagdown}\limits_{\Omega} \overline{s}_n dp = n^{-1} b \max\left\{(k+1)\,p\left(E_k\right) : k = 0, \ldots, n\right\}.$$

Hence

$$0 \le \inf_{\overline{s}\ge f} \mathop{\diagdown}\limits_{\Omega} \overline{s} dp - \sup_{s\le f} \mathop{\diagdown}\limits_{\Omega} s\,dp \le bn^{-1}.$$

The result follows by letting $n \to \infty$ in this last inequality. □

Definition 4.4 ([1], Definition 2.1)**.** The optimal average of a measurable function f is defined by

$$\mathop{\diagdown}\limits_{\Omega} |f|\,dp = \sup \mathop{\diagdown}\limits_{\Omega} s\,dp, \tag{4.1}$$

where the supremum is taken over all measurable simple functions $s \ge 0$ for which $s \le |f|$. The optimal average of f on any given measurable set E is defined by $\mathop{\diagdown}\limits_{E} |f|\,dp = \mathop{\diagdown}\limits_{\Omega} \chi\left(E\right)|f|\,dp$.

For convenience reasons at times we shall write $A\,|f|$ for the optimal average of the measurable function f.

Proposition 4.5 ([1], Proposition 2.1)**.** *Let $f \ge 0$ and $g \ge 0$ be any measurable simple functions, $b \in \mathbb{R}_+$ and $B \in \mathcal{F}$ be arbitrary. Then*

1. $A\left(b\mathbf{1}\right) = b$.
2. $A\left(\chi\left(B\right)\right) = p\left(B\right)$.
3. $A\left(bf\right) = bAf$.
4. $A\left(f\chi\left(B\right)\right) = 0$ *if* $p\left(B\right) = 0$.
5. $Af \le Ag$ *if* $f \le g$.
6. $A\left(f+g\right) \le Af + Ag$.
7. $A\left(f\chi\left(B\right)\right) = Af$ *if* $p\left(\overline{B}\right) = 0$.
8. $A\left(f \vee g\right) = Af \vee Ag$.

The almost everywhere notion in measure theory also makes sense in optimal measure theory.

Definition 4.6 ([1], Definition 2.2)**.** Let p be an optimal measure. A property is said to hold almost everywhere if the set of elements where it fails to hold is a set of optimal measure zero.

As an immediate consequent of the atomic structural behavior of optimal measures we can formulate the following.

Remark 4.7 ([2], Remark 2.1). If a function $f : \Omega \to \mathbb{R}$ is measurable, then it is constant almost everywhere on every indecomposable atom.

Proposition 4.8 ([2], Proposition 2.6)**.** *Let $p \in \mathcal{P}$ and f be any measurable function. Then*

$$\Big\backslash_{\Omega} |f|\, dp = \sup \left\{ \Big\backslash_{H_n} |f|\, dp : n \in J \right\},$$

where $\mathcal{H}(p) = \{H_n : n \in J\}$ is a p-generating countable system.
Moreover if $A|f| < \infty$, then $\Big\backslash_{\Omega} |f|\, dp = \sup \{c_n \cdot p(H_n) : n \in J\}$, where $c_n = f(\omega)$ for almost all $\omega \in H_n$, $n \in J$.

Proposition 4.9 (**Optimal Markov Inequality** ([1], Proposition 2.2))**.** *Let $f \geq 0$ be any measurable function. Then for every number $x > 0$ we have*

$$xp(f \geq x) \leq Af.$$

Proposition 4.10 ([1], Proposition 3.4)**.** *Let $f \geq 0$ be any bounded measurable function. Then for every $\varepsilon > 0$ there is some $\delta > 0$ such that $\Big\backslash_B f dp < \varepsilon$ whenever $B \in \mathcal{F}$, $p(B) < \delta$.*

Proof. By assumption $0 \leq f \leq b$ for some number $b > 0$. Then Proposition 4.5 entails, for the choice $0 < \delta < \varepsilon b^{-1}$, that $\Big\backslash_B f dp \leq bp(B) < \delta b < \varepsilon$. □

In the example below we shall show that Proposition 4.10 does not hold for unbounded measurable functions.

Example 4.11 ([1], Example 3.2)**.** Consider the measurable space $(\mathbb{N}, 2^{\mathbb{N}})$. Define the set function $p : 2^{\mathbb{N}} \to [0, 1]$ by $p(B) = \dfrac{1}{\min B}$. It is known from Example 3.3 that p is an optimal measure. Consider the following measurable function $f(\omega) = \omega$, $\omega \in \mathbb{N}$. Clearly, $Af \geq 1$. Let $s = \sum_{j=1}^{n} b_j \chi(B_j)$ be a measurable simple function with $0 \leq s \leq f$. Denote $\omega_j = \min B_j$ for $j = 1, \ldots, n$. Then $p(B_j) = \dfrac{1}{\omega_j}$ and $b_j \leq \omega_j$ for all $j = 1, \ldots, n$. Thus $\Big\backslash_{\Omega} s\, dp \leq 1$, and hence $\Big\backslash_{\Omega} f \leq 1$. Consequently, $\Big\backslash_{\Omega} f = 1$. On the one hand, there is no $\delta > 0$ such that $p(E) < \delta$ implies that $\Big\backslash_E f dp < 1$. Indeed, $\Big\backslash_{\{\omega\}} f dp = 1$ for every $\omega \in \mathbb{N}$, and $p(\{\omega\}) \to 0$ as $\omega \to \infty$.

4.2. The corresponding Radon-Nikodym Theorem in lattice environments

Definition 4.12 ([2], Definition 2.1)**.** By a quasi-optimal measure we a set function $q : \mathcal{F} \to \mathbb{R}_+$ satisfying Axioms 1-3, with the hypothesis $q(\Omega) = 1$ in Axiom 1 being replaced by the hypothesis $0 < q(\Omega) < \infty$.

Proposition 4.13 ([2], Proposition 2.1)**.** *If $f \geq 0$ is a bounded measurable function, then the set function $q_f : \mathcal{F} \to \mathbb{R}_+$,*

$$q_f(E) = \Big\backslash_{E} f dp,$$

is a quasi-optimal measure.

Definition 4.14 ([2], Definition 2.2)**.** We shall say that a quasi-optimal measure q is absolutely continuous relative to p (abbreviated $q \ll p$) if $q(B) = 0$ whenever $p(B) = 0$, $B \in \mathcal{F}$.

Proposition 4.15 ([2], Proposition 2.2)**.** *Let q be a quasi-optimal measure. Then $q \ll p$ if and only if for every $\varepsilon > 0$ there is some $\delta > 0$ such that $q(B) < \varepsilon$ whenever $p(B) < \delta$, $B \in \mathcal{F}$.*

The proof of Proposition 4.15 is similarly done as in the case of measure theory.

Lemma 4.16 ([2], Lemma 2.3)**.** *Let q be a quasi-optimal measure and $\mathcal{H}(p)$ be a p-generating system. If $q \ll p$, then*

$$\mathcal{H}(q) = \{H \in \mathcal{H}(p) : q(H) > 0\}$$

is a q-generating system.

Remark 4.17 ([3], Remark 2.1). Let $p, q \in \mathcal{P}, \mathcal{H}(p) = \{H_n : n \in J\}$ be a p-generating countable system and f any measurable function. Suppose that $q \ll p$ and $q(H) \leq p(H)$ for every $H \in \mathcal{H}(p)$. Then $\int_\Omega |f|\, dq \leq \int_\Omega |f|\, dp$, provided that $\int_\Omega |f|\, dp < \infty$.

This remark is immediate from Lemma 4.16 and Proposition 4.8.

Theorem 4.18 (**Optimal Radon-Nikodym** ([2], Theorem 2.4))**.** *Let q be a quasi-optimal measure such that $q \ll p$. Then there exists a unique measurable function $f \geq 0$ such that for every measurable set $B \in \mathcal{F}$,*

$$q(B) = \int_B f dp.$$

This measurable function, explicitly given in (4.2), will be called Optimal Radon-Nikodym derivative and denoted by $\dfrac{dq}{dp}$.

Proof. Let $\mathcal{H}(p) = \{H_n : n \in J\}$ be a p-generating countable system. Define the following non-negative measurable function

$$f = \max\left\{\frac{q(H_n)}{p(H_n)} \cdot \chi(H_n) : n \in J\right\}. \tag{4.2}$$

Fix an index $n \in J$ and let $B \in \mathcal{F}$, $p(B) > 0$. Then Remark 3.7 and the absolute continuity property imply that

$$\frac{q(H_n)}{p(H_n)} p(B \cap H_n) = \begin{cases} 0 & \text{if } p(B \cap H_n) = 0 \\ q(B \cap H_n), & \text{otherwise.} \end{cases}$$

Hence, by a simple calculation, one can observe that

$$\int_B f dp = \max\{q(B \cap H_n) : n \in J\}.$$

Consequently, Lemma 4.16 yields

$$\int_B f dp = \begin{cases} \max\{q(B \cap H_n) : q(H_n) > 0,\, n \in J\} & \text{if } q(B) > 0 \\ 0, & \text{otherwise,} \end{cases}$$

and thus (4.2) holds.

Let us show that the decomposition (4.2) is unique. In fact, there exist two measurable functions $f \geq 0$ and $g \geq 0$ satisfying (4.2) . Then for each set $B \in \mathcal{F}$, we have:

$$\int_B f dp = \int_B g dp.$$

Put $E_1 = (f < g)$ and $E_2 = (g < f)$. Obviously, E_1 and $E_2 \in \mathcal{F}$. If the inequality $p(E_1) > 0$ should hold, it would follow that

$$\bigvee_{E_1} g dp = \bigvee_{E_1} f dp < \bigvee_{E_1} g dp,$$

which is impossible. This contradiction yields $p(E_1) = 0$. We can similarly show that $p(E_2) = 0$. These last two equalities imply that $p(f \neq g) = 0$, i.e. the decomposition (4.2) is unique. The theorem is thus proved. □

5. Counterparts in lattice environments of well-known convergence theorems

5.1. Some convergence with respect to individual optimal measures

In this subsection we shall explore in lattice environments the counterparts of the monotone convergence theorem, the Fatou's lemma and the dominated convergence theorem well-known in Measure Theory. The results are related to an arbitrarily fixed optimal measure space $(\Omega, \mathcal{F}, p)$, unless otherwise stated.

Theorem 5.1 (**Optimal monotone convergence**, ([1], Theorem 3.1)**.**

1. *If (f_n) is an increasing sequence of non-negative measurable functions, then*

$$\lim_{n\to\infty} \bigvee_{\Omega} f_n dp = \bigvee_{\Omega} \left(\lim_{n\to\infty} f_n\right) dp.$$

2. *If (g_n) is a decreasing sequence of non-negative measurable functions with $g_1 \leq b$ for some $b \in (0, \infty)$, then*

$$\lim_{n\to\infty} \bigvee_{\Omega} g_n dp = \bigvee_{\Omega} \left(\lim_{n\to\infty} g_n\right) dp.$$

The following example shows why the optimal monotone convergence theorem fails to hold for all decreasing sequences of measurable functions.

Example 5.2 ([1], Example 3.1)**.** Let $\left(\mathbb{N}, 2^{\mathbb{N}}, p\right)$ be the optimal measure space we considered in Example 4.11. Define the following measurable function

$$g_n(\omega) = \begin{cases} 0 & \text{if } \omega < n \\ \omega & \text{if } \omega \geq n. \end{cases}$$

Obviously, sequence $(g_n)_{n\in\mathbb{N}}$ tends decreasingly to zero as $n \to \infty$. It will be enough to show that $\bigvee_{\mathbb{N}} g_n\, dp = 1$ for all $n \in \mathbb{N}$. In fact, it is clear by definition that $(g_n < n) = \{1, \ldots, n-1\}$ and $(g_n \geq n) = \{n, n+1, \ldots\}$, and so $\mathbb{N} = (g_n < n) \cup (g_n \geq n)$ for every fixed natural number $n \in \mathbb{N}$. We also know by definition that g_n assumes the value 0 on $\{1, \ldots, n-1\}$ and the value n on $\{n, n+1, \ldots\}$, for every fixed natural number $n \in \mathbb{N}$. Hence, by the considered optimal measure we trivially have

$$\bigvee_{\mathbb{N}} g_n\, dp = \bigvee_{\{n, n+1, \ldots\}} g_n\, dp = np(\{n, n+1, \ldots\}) = \frac{n}{\min(\{n, n+1, \ldots\})} = 1.$$

Lemma 5.3 (**Optimal Fatou** ([1], Lemma 3.2))**.** *If $(f_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ are sequences of non-negative measurable functions, then for every optimal measure p, we have that:*

1. $\bigvee_{\Omega} \left(\liminf_{n\to\infty} f_n\right) dp \leq \liminf_{n\to\infty} \bigvee_{\Omega} f_n dp;$

2. $\limsup_{n\to\infty} \bigvee_{\Omega} h_n dp \leq \bigvee_{\Omega} \left(\limsup_{n\to\infty} h_n\right) dp$ *, whenever $(h_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence.*

Theorem 5.4 (**Optimal Dominated Convergenc** ([1], Theorem 3.3))**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of non-negative measurable functions. Then $A\left(\lim_{n\to\infty} f_n\right) = Af$, where $\lim_{n\to\infty} f_n = f$ almost everywhere.*

6. Banach lattice induced by optimal measures

Throughout this section we shall deal with an arbitrary but fixed optimal measure space $(\Omega, \mathcal{F}, p)$, i.e. $(\Omega, \mathcal{F})$ is a measurable space and p an optimal measure.

6.1. The counterpart of the L^p-spaces ($p \in [1, \infty]$) in lattice environments

Definition 6.1. Let $f : \Omega \to \mathbb{R} \cup \{-\infty, \infty\}$ be any measurable function. We shall say that f belongs to:

1. $\mathcal{A}^\infty$ if $p(|f| \leq b) = 1$ for some constant $b \in (0, \infty)$.

2. $\mathcal{A}^\alpha$ if $\bigwedge_\Omega |f|^\alpha \, dp < \infty$, $\alpha \in [1, \infty)$.

For any $\alpha \in [1, \infty]$, the space $\mathcal{A}_\alpha$ endowed with the norm $\|\cdot\|_\alpha$, defined by

$$\|f\|_{\mathcal{A}^\alpha} := \begin{cases} \inf\{b \in (0, \infty) : p(|f| \leq b) = 1\}, & \text{if } f \in \mathcal{A}_\infty, \ \alpha = \infty \\ \sqrt[\alpha]{\bigwedge_\Omega |f|^\alpha \, dp}, & \text{if } f \in \mathcal{A}_\alpha, \ \alpha \in [1, \infty) \end{cases}$$

As in the case of L^p-spaces ($p \in [1, \infty]$) in Measure Theory, it can be similarly seen that $\|\cdot\|_\alpha$ is a semi-norm for every $\alpha \in [1, \infty]$.

Lemma 6.2 ([1], Lemma 4.1)**.**

1. $A|fg| \leq \|f\|_{\mathcal{A}^\alpha} \|g\|_{\mathcal{A}^\infty}$ *whenever* $f \in \mathcal{A}^1$ *and* $g \in \mathcal{A}^\infty$.

2. *Let* α *and* $\beta \in (1, \infty)$ *be such that* $\alpha^{-1} + \beta^{-1}$. *Then* $A|fg| \leq \|f\|_{\mathcal{A}^\alpha} \|g\|_{\mathcal{A}^\beta}$ *(called the optimal Hölder inequality), whenever* $f \in \mathcal{A}^\alpha$ *and* $g \in \mathcal{A}^\beta$.

3. $\|f + g\|_{\mathcal{A}^\alpha} \leq \|f\|_{\mathcal{A}^\alpha} + \|g\|_{\mathcal{A}^\alpha}$ *(called the optimal Minkowski inequality) whenever* $f \in \mathcal{A}^\alpha$ *and* $g \in \mathcal{A}^\alpha$, *with* $\alpha \in [1, \infty]$.

Theorem 6.3 ([1], Theorem 4.2)**.** *For each number* $\alpha \in [1, \infty]$, $\mathcal{A}^\alpha$ *is a Banach space (i.e. every Cauchy sequence in* $\mathcal{A}^\alpha$ *converges to a measurable function in* $\mathcal{A}^\alpha$*-norm).*

6.2. Orlicz-space and its dual in lattice environments

Let Φ be a *convex Young function*, i.e.

$$\Phi(x) = \int_0^x \varphi(t) \, dt, \ x \in \mathbb{R}_+,$$

where $\varphi : (0, \infty) \to (0, \infty)$ is a right-continuous and increasing function such that $\varphi(0) \geq 0$ and $\varphi(\infty) = \infty$.

The conjugate Young functions are defined as follows:

For $t \in (0, \infty)$ put $\psi(t) := \sup\{x > 0 : \varphi(x) < t\}$ and let $\psi(0) = 0$. It can be easily checked that ψ satisfies all the conditions imposed on φ and we trivially have $\psi(\varphi(x)) \leq x \leq \psi(\varphi(x) + 0)$, whenever $x \in (0, \infty)$.

The convex Young function

$$\Psi(x) := \int_0^x \psi(t) \, dt, \ x \in [0, \infty),$$

is said to be *conjugate* to Φ and the pair (Φ, Ψ) is referred to as *mutually conjugate convex Young functions.*

Every pair (Φ, Ψ) of mutually conjugate convex Young functions satisfies the fundamental Young inequality

$$xy \leq \Phi(x) + \Psi(y) \tag{6.1}$$

for all $x,\ y \in [0,\ \infty)$, and the Young equality

$$xy = \Phi(x) + \Psi(y) \tag{6.2}$$

if and only if $y \in [\varphi(x),\ \varphi(x+0)]$ or $x \in [\psi(y),\ \psi(y+0)]$. (For more about convex Young functions, see [26].)

We extend some basic results about the Orlicz L^{Φ} space in Measure Theory to the framework of Optimal Measure Theory, by generalizing the space $\mathcal{A}^{\alpha}$ to the space $\mathcal{A}^{\Phi}$, where Φ is a convex Young function. In the image of the dual space of the Orlicz L^{Φ} space some set of non-linear functionals $F : \mathcal{A}^{\Phi} \to [0,\ \infty]$, (called the laud space of $\mathcal{A}^{\Phi}$), is studied.

Definition 6.4 ([7], Definition 2.1)**.** We say that a measurable function f belongs to $\mathcal{A}^{\Phi}$ if there is a constant $c \in (0,\ \infty)$ such that

$$\bigg\backslash_{\Omega} \Phi\left(\frac{|f|}{c}\right) dp \le 1. \tag{6.3}$$

In the image of the Luxemburg norm define on $\mathcal{A}^{\Phi}$ the operator $\|\cdot\|_{\mathcal{A}^{\Phi}}$ by

$$\|f\|_{\mathcal{A}^{\Phi}} = \inf\left\{c \in (0,\ \infty) : \ \bigg\backslash_{\Omega} \Phi\left(\frac{|f|}{c}\right) dp \le 1\right\}, \tag{6.4}$$

and $\|f\|_{\mathcal{A}^{\Phi}} = \infty$ if there is no $c \in (0,\ \infty)$ such that (6.3) holds.

Note that if $\Phi(t) = \dfrac{t^{1+\alpha}}{1+\alpha}$, $t \in [0,\ \infty)$ and $\alpha \in (0,\ \infty)$, then $\mathcal{A}^{\Phi} = \mathcal{A}^{1+\alpha}$.

Theorem 6.5 ([7], Theorem 2.2)**.** *Let $\Phi : [0,\ \infty) \to [0,\ \infty)$ be any function and f a non-negative finite measurable function. Then the inequality*

$$\Phi\left(\bigg\backslash_{\Omega} f dp\right) \le \bigg\backslash_{\Omega} \Phi(f)\, dp$$

holds, and is referred to as the Optimal Jensen inequality, provided that Φ is a convex Young function. Furthermore, the inequality is reversed if Φ is a concave Young function.

We prepare the ground for the proof of Theorem 6.5.

Let $J \subset \mathbb{N}$ be an index set. Then the weighted supremum of a sequence $(b_n)_{n\in J} \subset [0,\ \infty)$ is defined by $\sup\limits_{n\in J} b_n\alpha_n$, where $(\alpha_n)_{n\in J} \subset [0,\ 1]$ is a prescribed sequence with 0 as its unique limit point if the index set is infinite (in symbol $|J| = \infty$).

Remark 6.6 ([7], Remark 3.1). For all $d \in \mathbb{R}$, $c \in (0,\ \infty)$ and $(b_n)_{n\in J} \subset [0,\ \infty)$, where J is an index set, then

$$\sup_{n\in J}(d + cb_n) = d + c\sup_{n\in J} b_n.$$

Remark 6.6 is obvious.

Lemma 6.7 ([7], Lemma 3.2)**.** *Let $J \subset \mathbb{N}$ be an index set and $\Phi : [0,\ \infty) \to [0,\ \infty)$ be any function. Consider two sequences $(b_n)_{n\in J} \subset [0,\ \infty)$ and $(\alpha_n)_{n\in J} \subset [0,\ 1]$ possessing 0 as its unique limit point if $|J| = \infty$. Then*

$$\Phi\left(\sup_{n\in J} b_n\alpha_n\right) \le \sup_{n\in J}\Phi(b_n)\,\alpha_n$$

provided that Φ is a convex Young function. Furthermore, the inequality is reversed if Φ is a concave Young function.

The Proof of Theorem 6.5. We note that the proof follows from the conjunction of both Proposition 2.1 in [3] and the above Lemma 6.7. □

Definition 6.8 ([7], Definition 2.3)**.** Let $\mathcal{A}_+^\Phi := \left\{f \in \mathcal{A}^\Phi : f \geq 0\right\}$. We say that a functional $F : \mathcal{A}_+^\Phi \to [0, \infty]$ belongs to $\widetilde{\mathcal{A}^\Phi}$ if the following conditions hold true simultaneously:

1. For all $f,\, h \in \mathcal{A}_+^\Phi$, and $\alpha,\, \beta \in [0, \infty)$ we have

$$F\left(\alpha f \vee \beta h\right) = \alpha F\left(f\right) \vee \beta F\left(h\right).$$

2. F is continuous from below, i.e. if $\left(f_n\right)_{n\in\mathbb{N}} \subset \mathcal{A}_+^\Phi$ is an increasing sequence, then

$$\lim_{n\to\infty} F\left(f_n\right) = F\left(\lim_{n\to\infty} f_n\right).$$

3. There is some constant $C > 0$ for which

$$F\left(f\right) \leq C \left\|f\right\|_{\mathcal{A}^\Phi}, \text{ whenever } f \in \mathcal{A}_+^\Phi.$$

We extend Definition 6.8 to the entire $\mathcal{A}^\Phi$ space as follows.

Definition 6.9 ([7], Definition 2.4)**.** A functional $F \circ |\cdot| : \mathcal{A}^\Phi \to [0, \infty]$ is said to belong to $\widetilde{\mathcal{A}^\Phi}$ if the following conditions hold true simultaneously:

1. For all $f,\, h \in \mathcal{A}^\Phi$, and $\alpha,\, \beta \in [0, \infty)$ we have

$$F\left(\alpha \left|f\right| \vee \beta \left|h\right|\right) = \alpha F\left(\left|f\right|\right) \vee \beta F\left(\left|h\right|\right).$$

2. F is non-negatively continuous from below, i.e. if $\left(f_n\right)_{n\in\mathbb{N}} \subset \mathcal{A}^\Phi$ is a non-negative increasing sequence, then

$$\lim_{n\to\infty} F\left(f_n\right) = F\left(\lim_{n\to\infty} f_n\right).$$

3. There is some constant $C > 0$ for which

$$F\left(\left|f\right|\right) \leq C \left\|f\right\|_{\mathcal{A}^\Phi}, \text{ whenever } f \in \mathcal{A}^\Phi.$$

The set $\widetilde{\mathcal{A}^\Phi}$ will thus be referred to as the "*laud*" *space* of $\mathcal{A}^\Phi$, in contrast with the "dual" space of L^Φ in Measure Theory.

The counterpart of Proposition IX-2-2 in the appendix of [30] can be stated as follows.

Theorem 6.10 ([7], Theorem 2.5)**.** *The following assertions hold.*

1. *The mapping* $\left\|\cdot\right\|_{\mathcal{A}^\Phi} : \mathcal{A}^\Phi \to [0, \infty)$ *defined by* (6.4) *is a norm.*
2. $\mathcal{A}^\Phi \subset \mathcal{A}^1$*, i.e. there exist some constant* $\delta > 0$ *such that*

$$\delta \left\|f\right\|_{\mathcal{A}^1} \leq \left\|f\right\|_{\mathcal{A}^\Phi},$$

whenever $f \in \mathcal{A}^\Phi$.
3. $\mathcal{A}^\Phi$ *is a Banach space, i.e. every Cauchy sequence in* $\mathcal{A}^\Phi$ *converges to a measurable function in* $\mathcal{A}^\Phi$*-norm.*
4. *If* $f \in \mathcal{A}^\Phi$ *and* $h \in \mathcal{A}^\Psi$*, then*

$$\left\|fh\right\|_{\mathcal{A}^1} \leq 2 \left\|f\right\|_{\mathcal{A}^\Phi} \cdot \left\|h\right\|_{\mathcal{A}^\Psi},$$

which shall be referred to as the Optimal Hölder Inequality.

5. *Given any $h \in \mathcal{A}^{\Psi}$, the mapping $F_h \circ |\cdot| : \mathcal{A}^{\Phi} \to [0, \infty)$ defined by*

$$F_h(|f|) = \bigvee_{\Omega} |fh| \, dp,$$

belongs to the laud space of $\mathcal{A}^{\Phi}$. Moreover, letting $\mathfrak{M}$ stand for the set of all measurable functions defined on $(\Omega, \mathcal{F})$, the quantity

$$\|h\|^*_{\mathcal{A}^{\Phi}} := \sup_{f \in \mathcal{A}^{\Phi} \setminus \{0\}} \frac{F_h(|f|)}{\|f\|_{\mathcal{A}^{\Phi}}} = \sup \left\{ F_h(|f|) : f \in \mathfrak{M}, \quad \bigvee_{\Omega} \Phi(|f|) \, dp \le 1 \right\} \tag{6.5}$$

defines a norm on the space $\mathcal{A}^{\Psi}$ which is equivalent to the norm $\|\cdot\|_{\mathcal{A}^{\Psi}}$, more precisely

$$\lambda \|h\|_{\mathcal{A}^{\Psi}} \le \|h\|^*_{\mathcal{A}^{\Phi}} \le 2 \|h\|_{\mathcal{A}^{\Psi}},$$

for some constant $\lambda \in (0, 2]$ and all $h \in \mathcal{A}^{\Psi}$.

6. *If $F \circ |\cdot| : \mathcal{A}^{\Phi} \to [0, \infty)$ is a mapping belonging to $\widetilde{\mathcal{A}^{\Phi}}$, then there is an $h \in \mathcal{A}^{\Psi}$ with $\|h\|_{\mathcal{A}^{\Psi}} \le C$ (the constant C being as in Definition 6.9) such that for all $f \in \mathcal{A}^{\Phi}$,*

$$F(|f|) = \bigvee_{\Omega} |fh| \, dp.$$

Before tackling the proof of Theorem 6.10 (which goes down the line of the proof given in [30] for Proposition IX-2-2), some essential results need to be mentioned with the proofs.

Remark 6.11 ([7], Remark 1.1). Let be given any optimal measure p with $\mathcal{H}(p) = \{H_n : n \in J\}$ its generating system and a measurable set $A \in \mathcal{F}$. Then $p(A) = 0$ if and only if $p(A \cap H) = 0$ for every $H \in \mathcal{H}(p)$.

Lemma 6.12 ([7], Lemma 3.5)**.** *Let y be a bounded measurable function and consider the quasi-optimal measure $q_y : \mathcal{F} \to [0, \infty)$,*

$$q_y(A) = \bigvee_{A} |y| \, dp.$$

Then $dq_y = |y| \, dp$ p-a.e. Moreover,

$$|y| = \max \left\{ \frac{q_y(H)}{p(H)} \cdot \chi_H : H \in \mathcal{H}(p), \, q_y(H) > 0 \right\}$$

on $\bigcup \mathcal{H}(p)$.

Remark 6.13 ([7], Remark 3.6). Given any convex Young function Φ, for every $f \in \mathcal{A}^{\Phi}$ we have

$$\|f\|_{\mathcal{A}^{\Phi}} \le \max \left\{ 1 \, ; \, \bigvee_{\Omega} \Phi(|f|) \, dp \right\}.$$

Remark 6.14 ([7], Remark 3.7). For every measurable function f we have that $\|f\|_{\mathcal{A}^{\Phi}} \le 1$ if and only if

$$\bigvee_{\Omega} \Phi(|f|) \, dp \le 1.$$

Remark 6.15 ([7], Remark 3.8). For any convex Young function Ψ and any measurable simple function of the form $h = b\chi_A$ where $A \in \mathcal{F}$ with $p(A) > 0$ we have

$$\|h\|_{\mathcal{A}^{\Psi}} = \frac{|b|}{\Psi^{-1}\left(\frac{1}{p(A)}\right)}.$$

Remarks 6.14 and 6.15 can be easily checked, so we shall omit their proofs.

The Proof of Theorem 6.10.

PART 1. Let f, h be any measurable functions. It is trivial that $\|f\|_{\mathcal{A}^\Phi} \geq 0$. We want to prove that if $\|f\|_{\mathcal{A}^\Phi} = 0$, then $p(|f| \neq 0) = 0$. In fact, suppose that $\|f\|_{\mathcal{A}^\Phi} = 0$ but $p(0 < |f| \leq \infty) = p(|f| \neq 0) > 0$. Then by Remark 6.11 a non-empty subset J_0 of the index set J exists such that $p(H_n \cap (0 < |f| \leq \infty)) > 0$, whenever $n \in J_0$ and $p(H_n \cap (0 < |f| \leq \infty)) = 0$ otherwise, where J is the index set of the generating system $\mathcal{H}(p) = \{H_n : n \in J\}$. Note that $\|f\|_{\mathcal{A}^\Phi} = \inf S$, where

$$S = \left\{ \delta > 0 : \bigvee_{\Omega} \Phi\left(\frac{|f|}{\delta}\right) dp \leq 1 \right\}.$$

From the assumption and the definition of the infimum there is a sequence $(\delta_k)_{k\in\mathbb{N}} \subset S$ such that $0 < \delta_k < \dfrac{1}{k}$ for all $k \in \mathbb{N}$. By applying the Optimal Jensen Inequality we can observe that

$$1 \geq \bigvee_{\Omega} \Phi\left(\frac{|f|}{\delta_k}\right) dp \geq \Phi\left(\bigvee_{\Omega} \frac{|f|}{\delta_k} dp\right).$$

Hence

$$\delta_k \Phi^{-1}(1) \geq \bigvee_{\Omega} |f| \, dp,$$

which implies, via Proposition 2.1 in [3], that

$$\sup_{n\in J_0} \bigvee_{H_n \cap (0<|f|\leq\infty)} |f| \, dp = \bigvee_{\Omega} |f| \, dp = 0. \tag{6.6}$$

Clearly, $p(|f| = \infty) = 0$, otherwise the left hand side of (6.6) would assume the value ∞, a contradiction. Then necessarily, $p(H_n \cap (0 < |f| < \infty)) = 0$ for every $n \in J_0$, which is impossible because of the assumption. By this absurdity we have thus proved that if $\|f\|_{\mathcal{A}^\Phi} = 0$, then $f = 0$, p-a.e. Note that its converse is obvious. We show the triangle inequality in the next step. In fact, via the monotonicity and the convexity, we observe that

$$\Phi\left(\frac{|f+h|}{\|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}}\right) \leq \Phi\left(\frac{|f|+|h|}{\|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}}\right) \leq$$
$$\leq \frac{\|f\|_{\mathcal{A}^\Phi}}{\|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}} \Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}}\right) + \frac{\|h\|_{\mathcal{A}^\Phi}}{\|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}} \Phi\left(\frac{|h|}{\|h\|_{\mathcal{A}^\Phi}}\right).$$

Hence

$$\bigvee_{\Omega} \Phi\left(\frac{|f+h|}{\|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}}\right) \leq \frac{\|f\|_{\mathcal{A}^\Phi}}{\|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}} \bigvee_{\Omega} \Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}}\right) dp +$$
$$+ \frac{\|h\|_{\mathcal{A}^\Phi}}{\|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}} \bigvee_{\Omega} \Phi\left(\frac{|h|}{\|h\|_{\mathcal{A}^\Phi}}\right) dp \leq 1,$$

since

$$\bigvee_{\Omega} \Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}}\right) dp \leq 1 \quad \text{and} \quad \bigvee_{\Omega} \Phi\left(\frac{|h|}{\|h\|_{\mathcal{A}^\Phi}}\right) dp \leq 1.$$

Consequently,

$$\|f+h\|_{\mathcal{A}^\Phi} \leq \|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}.$$

We leave to the reader the verification of the homogeneity axiom.

PART 2. We prove that $\delta_1 \|f\|_{\mathcal{A}^1} \leq \|f\|_{\mathcal{A}^\Phi}$ for some constant $\delta_1 > 0$ and all $f \in \mathcal{A}^\Phi$. In fact, let $u_0 \in (0, \infty)$ such that $\varphi(u_0) > 0$ and $u_0 + (\varphi(u_0))^{-1} \geq 1$. Making use of the inequality here below (proved in [30] on page 198)

$$\Phi(x) \geq (x - u_0)^+ \varphi(u_0), \ x \in [0, \infty),$$

we have

$$1 \geq \int_\Omega \Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}}\right) dp \geq \varphi(u_0) \int_\Omega \left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}} - u_0\right)^+ dp$$

and hence by Remark 6.6,

$$u_0 + \frac{1}{\varphi(u_0)} \geq \int_\Omega \left[u_0 + \left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}} - u_0\right)^+\right] dp \geq \int_\Omega \frac{|f|}{\|f\|_{\mathcal{A}^\Phi}} dp.$$

Whence, $\|f\|_{\mathcal{A}^1} \leq \left(u_0 + \frac{1}{\varphi(u_0)}\right) \|f\|_{\mathcal{A}^\Phi}$.

PART 3. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{A}^\Phi$ be any Cauchy sequence. Then we can extract from it a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that

$$\sum_{k=1}^{\infty} \left\|f_{n_{k+1}} - f_{n_k}\right\|_{\mathcal{A}^\Phi} < \infty$$

and hence by Part 2,

$$\sum_{k=1}^{\infty} \left\|f_{n_{k+1}} - f_{n_k}\right\|_{\mathcal{A}^1} < \infty.$$

Since $\mathcal{A}^1$ is a Banach space, the limit $\lim_{k\to\infty} f_{n_k} = f$ exists almost everywhere. Clearly, for every $k \in \mathbb{N}$,

$$f_{n_k} = f_{n_1} + \sum_{j=1}^{k-1} \left(f_{n_{j+1}} - f_{n_j}\right),$$

Write

$$S_{n_k} = |f_{n_1}| + \sum_{j=1}^{k-1} \left|f_{n_{j+1}} - f_{n_j}\right|, \ \ k \in \mathbb{N}.$$

Obviously,

$$\|S_{n_k}\|_{\mathcal{A}^\Phi} \leq \|f_{n_1}\|_{\mathcal{A}^\Phi} + \sum_{j=1}^{k-1} \left\|f_{n_{j+1}} - f_{n_j}\right\|_{\mathcal{A}^\Phi}, \ \ k \in \mathbb{N}.$$

Since $(S_{n_k})_{k\in\mathbb{N}}$ is an increasing sequence it ensues that

$$\|f\|_{\mathcal{A}^\Phi} \leq \liminf_{k\to\infty} \|S_{n_k}\|_{\mathcal{A}^\Phi} \leq \|f_{n_1}\|_{\mathcal{A}^\Phi} + \sum_{j=1}^{\infty} \left\|f_{n_{j+1}} - f_{n_j}\right\|_{\mathcal{A}^\Phi} < \infty.$$

Hence $f \in \mathcal{A}^\Phi$. Note that

$$\|f - f_{n_k}\|_{\mathcal{A}^\Phi} \leq \sum_{j=k+1}^{\infty} \left\|f_{n_{j+1}} - f_{n_j}\right\|_{\mathcal{A}^\Phi}$$

which yields

$$\lim_{k\to\infty} \|f - f_{n_k}\|_{\mathcal{A}^\Phi} = 0.$$

By the triangle inequality we have

$$\|f - f_n\|_{\mathcal{A}^\Phi} \leq \|f - f_{n_k}\|_{\mathcal{A}^\Phi} + \|f_n - f_{n_k}\|_{\mathcal{A}^\Phi} \to 0,$$

as $k \to \infty$ and $n \to \infty$.

PART 4. Let $f \in \mathcal{A}^{\Phi}$ and $h \in \mathcal{A}^{\Psi}$ be arbitrary such that $\|f\|_{\mathcal{A}^{\Phi}} > 0$ and $\|h\|_{\mathcal{A}^{\Psi}} > 0$. Then by applying the fundamental inequality (6.1) to $u = \frac{|f|}{\|f\|_{\mathcal{A}^{\Phi}}}$ and $v = \frac{|h|}{\|h\|_{\mathcal{A}^{\Psi}}}$ yields

$$\bigvee_{\Omega} |fh|\, dp \leq \|f\|_{\mathcal{A}^{\Phi}} \cdot \|h\|_{\mathcal{A}^{\Psi}} \left(\bigvee_{\Omega} \Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^{\Phi}}}\right) dp \quad + \quad \bigvee_{\Omega} \Phi\left(\frac{|h|}{\|h\|_{\mathcal{A}^{\Psi}}}\right) dp \right) \leq$$

$$\leq 2\, \|f\|_{\mathcal{A}^{\Phi}} \cdot \|h\|_{\mathcal{A}^{\Psi}}\,.$$

PART 5. To show that $\|\cdot\|^{*}_{\mathcal{A}^{\Phi}}$ is a norm we shall only verify the biconditional $\|h\|^{*}_{\mathcal{A}^{\Phi}} = 0$ if and only if $h = 0$, p-a.e. because the two other norm axioms can be easily checked. To this end we need to prove first that $\|h\|^{*}_{\mathcal{A}^{\Phi}} = 0$ implies $h = 0$, p-a.e. In fact, suppose (by the contrapositive) that there is some $H \in \mathcal{H}(p)$ for which the inequality $p(H \cap (|h| > 0)) > 0$ holds. Write $A := H \cap (|h| > 0)$. Consider the measurable function $f_{\delta} = \delta \chi_A$ with $\delta > 0$ such that

$$\bigvee_{\Omega} \Psi(f_{\delta})\, dp = \Psi(\delta)\, p(A) \leq 1.$$

This can be done, because Ψ is a convex Young function. Then

$$\|h\|^{*}_{\mathcal{A}^{\Phi}} \geq \bigvee_{\Omega} |h|\, f_{\delta} dp > 0.$$

Hence, $\|h\|^{*}_{\mathcal{A}^{\Phi}} = 0$ implies $h = 0$, p-a.e. Note that the converse conditional is straightforward.
By applying the Optimal Hölder Inequality, we observe from (6.5) that

$$\|h\|^{*}_{\mathcal{A}^{\Phi}} = \sup_{\left\{f \in \mathfrak{M}:\, \|f\|_{\mathcal{A}^{\Phi}} \leq 1\right\}} \bigvee_{\Omega} |fh|\, dp \leq 2\, \|h\|_{\mathcal{A}^{\Psi}}\,.$$

Next, we shall show the inequality $\lambda \|h\|_{\mathcal{A}^{\Psi}} \leq \|h\|^{*}_{\mathcal{A}^{\Phi}}$ for some constant $\lambda \in (0, 2]$ and all $h \in \mathcal{A}^{\Psi}$. In fact, assume the contrary, i.e. for every constant $\lambda \in (0, 2]$ we can find an $h \in \mathcal{A}^{\Psi}$ for which $\lambda \|h\|_{\mathcal{A}^{\Psi}} > \|h\|^{*}_{\mathcal{A}^{\Phi}}$. Now, choose $f_0 = \frac{\|h\|_{\mathcal{A}^{\Psi}}}{\rho p(H)} \chi_H$, where $H \in \mathcal{H}(p)$, $\rho > 0$ and $p(H \cap (|h| = \rho)) = p(H)$. Then $f_0 \in \mathcal{A}^{\Phi}$, via Remark 6.15. Consequently,

$$\lambda \|h\|_{\mathcal{A}^{\Psi}} > \|h\|^{*}_{\mathcal{A}^{\Phi}} \; = \sup_{\left\{f \in \mathfrak{M}:\, \|f\|_{\mathcal{A}^{\Phi}} \leq 1\right\}} \bigvee_{\Omega} |fh|\, dp \geq \bigvee_{\Omega} |f_0|\, |h|\, dp = \|h\|_{\mathcal{A}^{\Psi}}$$

so that $\lambda > 1$ for all $\lambda \in (0, 2]$. Letting $\lambda \to 0$ would entail $0 > 1$ which is absurd, indeed. Therefore, the inequality $\lambda \|h\|_{\mathcal{A}^{\Psi}} \leq \|h\|^{*}_{\mathcal{A}^{\Phi}}$ fulfils for some constant $\lambda \in (0, 2]$ and all $h \in \mathcal{A}^{\Psi}$.

PART 6. Let $F \circ |\cdot| \in \widetilde{\mathcal{A}^{\Phi}}$. Define the function $q : \mathcal{F} \to [0, \infty)$ by $q(A) = F(\chi_A)$. Via the assumption for every $A \in \mathcal{F}$,

$$q(A) \leq C\, \|\chi_A\|_{\mathcal{A}^{\Phi}}\,.$$

Consider the continuous function

$$\eta(t) = \begin{cases} \frac{1}{\Phi^{-1}\left(\frac{1}{t}\right)} & \text{whenever } t > 0 \\ 0 & \text{if } \; t = 0. \end{cases}$$

A simple calculus shows that

$$\mathop{\diagdown}\limits_{\Omega} \Phi\left(\frac{\chi_A}{\eta\left(p\left(A\right)\right)}\right)dp = \Phi\left(\frac{1}{\eta\left(p\left(A\right)\right)}\right)p\left(A\right) = 1.$$

Hence $q\left(A\right) \le C\eta\left(p\left(A\right)\right)$, whenever $A \in \mathcal{F}$. Consequently, $q \ll p$, i.e. q is absolutely continuous with respect to p. Then by Theorem 2.4 of [2],

$$h = \max\left\{\frac{q\left(H\right)}{p\left(H\right)} \cdot \chi_H : H \in \mathcal{H}\left(p\right),\, q\left(H\right) > 0\right\}$$

is the unique measurable function such that $dq = h \cdot dp$ almost everywhere. Consequently, for every measurable simple function $s = \sum\limits_{i=1}^{n} b_i\chi_{B_i} = \bigvee\limits_{i=1}^{n} b_i\chi_{B_i}$ we have

$$\bigvee_{i=1}^{n}\left|b_i\right|F\left(\chi_{B_i}\right) = \bigvee_{i=1}^{n}F\left(\left|b_i\right|\chi_{B_i}\right) = F\left(\bigvee_{i=1}^{n}\left|b_i\right|\chi_{B_i}\right) = \mathop{\diagdown}\limits_{\Omega} h\bigvee_{i=1}^{n}\left|b_i\right|\chi_{B_i}dp =$$

$$= \mathop{\diagdown}\limits_{\Omega} h\left|s\right|dp = F\left(\left|s\right|\right).$$

Next, we show that $\left\|h\right\|_{\mathcal{A}^{\Psi}} \le 2C$. To this end, let (s_n) be a sequence of non-negative measurable simple functions tending increasingly to h. Then by the Young equality (6.2) one can observe that

$$\Psi\left(\frac{s_n}{2C}\right) + \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) = \frac{s_n}{2C}\psi\left(\frac{s_n}{2C}\right).$$

On the one hand,

$$\mathop{\diagdown}\limits_{\Omega}\left[\Psi\left(\frac{s_n}{2C}\right) + \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right)\right]dp \ge \mathop{\diagdown}\limits_{\Omega}\max\left\{\Psi\left(\frac{s_n}{2C}\right) ; \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right)\right\}dp =$$

$$= \max\left\{\mathop{\diagdown}\limits_{\Omega}\Psi\left(\frac{s_n}{2C}\right)dp \quad ; \quad \mathop{\diagdown}\limits_{\Omega}\Phi\left(\psi\left(\frac{s_n}{2C}\right)\right)dp\right\}.$$

On the other hand we observe via Remark 6.13 that

$$\mathop{\diagdown}\limits_{\Omega}\frac{s_n}{2C}\psi\left(\frac{s_n}{2C}\right)dp \le \frac{1}{2C}\mathop{\diagdown}\limits_{\Omega}h\psi\left(\frac{s_n}{2C}\right)dp = \frac{1}{2C}F\left(\psi\left(\frac{s_n}{2C}\right)\right) \le$$

$$\le \frac{1}{2}\left\|\psi\left(\frac{s_n}{2C}\right)\right\|_{\mathcal{A}^{\Phi}} \le \frac{1}{2}\max\left\{1 ; \mathop{\diagdown}\limits_{\Omega}\Phi\left(\psi\left(\frac{s_n}{2C}\right)\right)dp\right\}.$$

Consequently,

$$\mathop{\diagdown}\limits_{\Omega}\Psi\left(\frac{s_n}{2C}\right)dp + \mathop{\diagdown}\limits_{\Omega}\Phi\left(\psi\left(\frac{s_n}{2C}\right)\right)dp \le$$

$$\le 2\max\left\{\mathop{\diagdown}\limits_{\Omega}\Psi\left(\frac{s_n}{2C}\right)dp ; \mathop{\diagdown}\limits_{\Omega}\Phi\left(\psi\left(\frac{s_n}{2C}\right)\right)dp\right\} \le$$

$$\le \max\left\{1 ; \mathop{\diagdown}\limits_{\Omega}\Phi\left(\psi\left(\frac{s_n}{2C}\right)\right)dp\right\}$$

$$\le 1 + \mathop{\diagdown}\limits_{\Omega}\Phi\left(\psi\left(\frac{s_n}{2C}\right)\right)dp.$$

This implies that

$$\int_{\Omega} \Psi\left(\frac{s_n}{2C}\right) dp \leq 1,\ n \in \mathbb{N}, \tag{6.7}$$

since $\int_{\Omega} \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) dp < \infty$. Finally, letting $n \to \infty$ in (6.7), the Optimal Monotone Convergence Theorem (cf. [1], Theorem 3.1/i) implies that $\int_{\Omega} \Psi\left(\frac{h}{2C}\right) dp \leq 1$. Therefore, $h \in \mathcal{A}^{\Psi}$.

□

7. Cauchy-type functional equation in lattice environments

The most famous functional equation by Cauchy and known as *linear functional equation* reads:

$$f(x+y) = f(x) + f(y), \tag{7.1}$$

where f is a real function.
We should point out that equation (7.1) has been investigated for many spaces and in various perspectives such as its stability which has been intensively considered in the literature. The stability problem was first posed by M. Ulam (see [36]) in the terms: "Give conditions in order for a linear mapping near an approximately linear mapping to exist." More precisely the problem can be formulated as follows:
Given two Banach algebras E and E', a transformation $f : E \to E'$ is called δ-linear if

$$\|f(x+y) - f(x) - f(y)\| < \delta, \tag{7.2}$$

for all $x, y \in E$.

The stability problem of equation (7.1) can be stated as follows. Does there exist for each $\varepsilon \in (0, 1)$ some $\delta > 0$ such that to each δ-linear transformation $f : E \to E'$ there corresponds a linear transformation $l : E \to E'$ satisfying the inequality $\|f(x) - l(x)\| < \varepsilon$ for all $x \in E$? This question was answered in the affirmative by Hyers [23] and then generalized by Aoki [12]. Ever since various problems of stability on various spaces have come to light. We shall list just few of them: [22, 31, 27, 29, 35].

7.1. Functional equation with both lattice operations

In the sequel $(\mathcal{X}, \wedge_{\mathcal{X}}, \vee_{\mathcal{X}})$ will denote a vector lattice and $(\mathcal{Y}, \wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}})$ a Banach lattice with $\mathcal{X}^+$ and $\mathcal{Y}^+$ their respective positive cones.

We recall that a functional $H : \mathcal{X} \to \mathcal{Y}$ is *cone-related* if $H(\mathcal{X}^+) = \{H(|x|) : x \in \mathcal{X}\} \subset \mathcal{Y}^+$ (see more about this notion in [6]).

In the image of the Cauchy functional equation we consider the following operator equation

$$T(|x|\, \Delta^*_{\mathcal{X}}\, |y|)\, \Delta^*_{\mathcal{Y}} T(|x|\, \Delta^{**}_{\mathcal{X}}\, |y|) = T(|x|)\, \Delta^{**}_{\mathcal{Y}} T(|y|) \tag{7.3}$$

to hold true for all $x, y \in \mathcal{X}$, where $\Delta^*_{\mathcal{X}}, \Delta^{**}_{\mathcal{X}} \in \{\wedge_{\mathcal{X}}, \vee_{\mathcal{X}}\}$ and $\Delta^*_{\mathcal{Y}}, \Delta^{**}_{\mathcal{Y}} \in \{\wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}}\}$ are fixed lattice operations.

Note that if in the special case the above four lattice operations are at the same time the supremum (join) or the infimum (meet), then the functional equation (7.3) is just a join-homomorphism or a meet-homomorphism. Moreover, if operations $\Delta^*_{\mathcal{X}}$ and $\Delta^{**}_{\mathcal{X}}$ are the same, then the left hand side of (7.3) is the maps of the meets or the joins, which are just in the image of (7.1).
Problem: Given lattice operations $\Delta^*_{\mathcal{X}}, \Delta^{**}_{\mathcal{X}} \in \{\wedge_{\mathcal{X}}, \vee_{\mathcal{X}}\}$ and $\Delta^*_{\mathcal{Y}}, \Delta^{**}_{\mathcal{Y}} \in \{\wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}}\}$, a vector lattice G_1, a vector lattice G_2 endowed with a metric $d(\cdot,\cdot)$ and a positive number ε, does there exist some $\delta > 0$ such that, if a mapping $F : G_1 \to G_2$ satisfies

$$d\left(F(|x|\, \Delta^*_{\mathcal{X}}\, |y|)\, \Delta^*_{\mathcal{Y}} F(|x|\, \Delta^{**}_{\mathcal{X}}\, |y|),\ F(|x|)\, \Delta^{**}_{\mathcal{Y}} F(|y|)\right) \leq \delta$$

for all x, $y \in G_1$, then an operation-preserving functional $T : G_1 \to G_2$ exists with the property that

$$d\left(T(x), F(x)\right) \leq \varepsilon$$

for all $x \in G_1$?
One can view this problem as a lattice version of the Ulam's stability problem formulated in [36]. We shall present here only one type of clauses leading to a unique solution.

Theorem 7.1 ([8], Theorem 2.1)**.** *Consider a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ for which there are numbers $\vartheta > 0$ and $\alpha \in (-\infty, 1)$ such that*

$$\left\| \frac{F\left(|x| \, \Delta^{*}_{\mathcal{X}} \, |y|\right) \Delta^{*}_{\mathcal{Y}} \, F\left(|x| \, \Delta^{**}_{\mathcal{X}} \, |y|\right)}{\tau} - F\left(\frac{|x|}{\tau}\right) \Delta^{**}_{\mathcal{Y}} F\left(\frac{|y|}{\tau}\right) \right\| \leq \frac{\vartheta}{4}\left(\|x\|^{\alpha} + \|y\|^{\alpha}\right) \tag{7.4}$$

for all x, $y \in \mathcal{X}$ and $\tau \in (0, \infty)$, where $\Delta^{*}_{\mathcal{X}}$, $\Delta^{**}_{\mathcal{X}} \in \{\wedge_{\mathcal{X}}, \vee_{\mathcal{X}}\}$ and $\Delta^{*}_{\mathcal{Y}}$, $\Delta^{**}_{\mathcal{Y}} \in \{\wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}}\}$ are fixed lattice operations. Then the sequence $\left(2^{-n} F\left(2^{n} |x|\right)\right)_{n \in \mathbb{N}}$ is a Cauchy sequence for every $x \in \mathcal{X}$. Moreover, let the functional $T : \mathcal{X} \to \mathcal{Y}$ be defined by

$$T\left(|x|\right) = \lim_{n \to \infty} 2^{-n} F\left(2^{n} |x|\right). \tag{7.5}$$

Then

(a.) *T is semi-homogeneous, i.e. $T\left(\gamma \, |x|\right) = \gamma T\left(|x|\right)$, for all $x \in \mathcal{X}$ and all $\gamma \in [0, \infty)$;*
(b.) *T is the unique cone-related functional satisfying both identity* (7.3) *and inequality*

$$\left\| T\left(|x|\right) - F\left(|x|\right) \right\| \leq \frac{2^{\alpha} \vartheta}{2 - 2^{\alpha}} \|x\|^{\alpha} \tag{7.6}$$

for every $x \in \mathcal{X}$.

Before we start the proof the following obvious remarks are worth being mentioned, as they will be used multiple times.

Remark 7.2 ([8], Remark 2.1)*.* If the conditions of Theorem 7.1 holds true, then $F\left(0\right) = 0$.

Remark 7.3 ([8], Remark 2.2)*.* Let Z be a set closed under the scalar multiplication, i.e. $bz \in Z$ whenever $b \in \mathbb{R}$ and $z \in Z$. Given a number $c \in \mathbb{R}$ let the function $\gamma : Z \to Z$ be defined by $\gamma\left(z\right) = cz$. Then $\gamma^{j} : Z \to Z$ the j-th iteration of γ is given by $\gamma^{j}\left(z\right) = c^{j} z$ for every counting number $j \geq 2$.

Proof of Theorem 7.1. First, if we choose $\tau = 2$, $y = x$ and replace x by $2x$ in inequality (7.4) then we obviously have

$$\left\| \frac{F\left(2\,|x|\right)}{2} - F\left(|x|\right) \right\| \leq \vartheta 2^{\alpha - 1} \|x\|^{\alpha}. \tag{7.7}$$

Next, let us define the following functions:

1.) $G : \mathcal{X} \to \mathcal{X}, \quad G\left(|x|\right) = 2\,|x|$.
2.) $\delta : \mathcal{X} \to [0, \infty), \quad \delta\left(|x|\right) = \vartheta 2^{\alpha-1} \|x\|^{\alpha}$.
3.) $\varphi : [0, \infty) \to [0, \infty), \quad \varphi\left(t\right) = 2^{-1} t$.
4.) $H : \mathcal{Y} \to \mathcal{Y}, \quad H\left(|y|\right) = 2^{-1} \,|y|$.
5.) $d\left(\cdot\,, \cdot\right) : \mathcal{Y} \times \mathcal{Y} \to [0, \infty), \quad d\left(y_1\,, y_2\right) = \|y_1 - y_2\|$.

We shall verify the fulfilment all the three conditons of the first Forti's theorem (cf. [18, Theorem 1]) as follows.

(I.) From inequality (7.7) we obviously have

$$d\left(H\left(F\left(G\left(|x|\right)\right)\right)\,,\, F\left(|x|\right)\right) = \left\| \frac{F\left(2\,|x|\right)}{2} - F\left(|x|\right) \right\| \leq \vartheta 2^{\alpha-1} \|x\|^{\alpha} = \delta\left(|x|\right).$$

(II.) $d\left(H\left(|y_1|\right)\,,\,H\left(|y_2|\right)\right)=2^{-1}\left\|y_1-y_2\right\|=\phi\left(d\left(y_1\,,\,y_2\right)\right)$ for all $y_1,\,y_2\in\mathcal{Y}$.

(III.) Clearly, on the one hand φ is a non-decreasing subadditive function on the positive half line, and on other hand by applying Remark 7.3 on both the iterations G^j and φ^j of G and φ respectively, one can observe that

$$\sum_{j=0}^{\infty}\varphi^j\left(\delta\left(G^j\left(|x|\right)\right)\right)=\vartheta 2^{\alpha-1}\left\|x\right\|^{\alpha}\sum_{j=0}^{\infty}2^{(\alpha-1)j}=\vartheta\left\|x\right\|^{\alpha}\frac{2^{\alpha}}{2-2^{\alpha}}<\infty.$$

Then in virtue of Forti's first theorem in [18] sequence $\left(H^n\left(F\left(G^n\left|x\right|\right)\right)\right)$ is a Cauchy sequence for every $x\in\mathcal{X}$ and thus so is sequence $\left(2^{-n}F\left(2^n\left|x\right|\right)\right)$ and furthermore, the mapping (7.5) is the unique functional which satisfies inequatility (7.6).

Next, we prove the validity of inequality (7.3). In fact, in (7.4) substitute x with 2^nx and y with 2^ny, and also let $\tau=1$. Then

$$\left\|F\left(2^n\left(|x|\,\Delta_{\mathcal{X}}^{*}\,|y|\right)\right)\Delta_{\mathcal{Y}}^{**}F\left(2^n\left(|x|\,\Delta_{\mathcal{X}}^{**}\,|y|\right)\right)-F\left(2^n\,|x|\right)\Delta_{\mathcal{Y}}^{**}F\left(2^n\,|y|\right)\right\|\leq\frac{\vartheta}{4}2^{n\alpha}\left(\|x\|^{\alpha}+\|y\|^{\alpha}\right).$$

Dividing both sides of this last inequality by 2^n yields

$$\begin{aligned}\left\|\frac{F\left(2^n\left(|x|\,\Delta_{\mathcal{X}}^{*}\,|y|\right)\right)\Delta_{\mathcal{Y}}^{**}F\left(2^n\left(|x|\,\Delta_{\mathcal{X}}^{**}\,|y|\right)\right)}{2^n}-\frac{F\left(2^n\,|x|\right)\Delta_{\mathcal{Y}}^{**}F\left(2^n\,|y|\right)}{2^n}\right\|&\leq\\ \leq\frac{\vartheta}{4}\left(\|x\|^{\alpha}+\|y\|^{\alpha}\right)2^{(\alpha-1)n}.&\end{aligned}\tag{7.8}$$

Taking the limit in (7.8) we have via (7.5) that

$$\left\|T\left(|x|\,\Delta_{\mathcal{X}}^{*}\,|y|\right)\Delta_{\mathcal{Y}}^{*}T\left(|x|\,\Delta_{\mathcal{X}}^{**}\,|y|\right)-T\left(|x|\right)\Delta_{\mathcal{Y}}^{**}T\left(|y|\right)\right\|=0$$

which is equivalent to

$$T\left(|x|\,\Delta_{\mathcal{X}}^{*}\,|y|\right)\Delta_{\mathcal{Y}}^{*}T\left(|x|\,\Delta_{\mathcal{X}}^{**}\,|y|\right)=T\left(|x|\right)\Delta_{\mathcal{Y}}^{**}T\left(|y|\right).$$

Because of Remark 7.2 identity $\gamma F\left(|x|\right)=F\left(\gamma\,|x|\right)$ is trivial on the one hand for $\gamma=0$ and all $x\in\mathcal{X}$, on the other hand for $x=0$ and all $\gamma\in[0,\,\infty)$. Without loss of generality let us thus fix arbitrarily a number $\gamma\neq 0$ and an $x\in\mathcal{X}\setminus\{0\}$. In (7.4) choose $y=x$, $\tau=\gamma^{-1}$ and change x to 2^nx. Then

$$\left\|\gamma F\left(2^n\,|x|\right)-F\left(\gamma 2^n\,|x|\right)\right\|\leq\frac{\vartheta}{2}\left\|x\right\|^{\alpha}2^{n\alpha}.$$

Divide both sides of this last inequality by 2^n to get

$$\left\|\gamma 2^{-n}F\left(2^n\,|x|\right)-2^{-n}F\left(\gamma 2^n\,|x|\right)\right\|\leq\frac{\vartheta}{2}\left\|x\right\|^{\alpha}2^{(\alpha-1)n}.\tag{7.9}$$

By taking the limit in (7.9) we have via (7.5) that

$$\left\|\gamma T\left(|x|\right)-T\left(\gamma\,|x|\right)\right\|=0$$

or equivalently,

$$T\left(\gamma\,|x|\right)=\gamma T\left(|x|\right)$$

for all $x\in\mathcal{X}$. We have thus shown the semi-homogeneity of operator T. We can conclude on the validity of the argument. □

Next, we shall provide an example showing that if in (7.4) the parameter τ is omitted and the power p of the norms equals the unity, then stability cannot always be guaranteed. We remind that in the addition environments Gajda in [21] and Găvruţa in [**?**] gave some interesting examples to show how stability fails when the power of the norms is equal to 1.

Example 7.4 ([8], Example 1)**.** Consider the Lipschitz-continuous function

$$F : [0,\, \infty) \to [0,\, \infty)\,,\, F\,(x) = \sqrt{x^2+1}.$$

Fix arbitrarily two numbers $x,\, y \in [0,\, \infty)$. Since F is an increasing function the very first equality in the chain of relations here below is valid, implying the subsequent relations in the chain:

$$\begin{aligned} |F\,(x \vee y) - (F\,(x) \wedge F\,(y))| &= |F\,(x \vee y) - F\,(x \wedge y)| \\ &= \left|\sqrt{(x \vee y)^2+1} - \sqrt{(x \wedge y)^2+1}\right| \\ &= \frac{(x \vee y)^2 - (x \wedge y)^2}{\sqrt{(x \vee y)^2+1} + \sqrt{(x \wedge y)^2+1}} = \\ |x-y| \cdot \frac{(x \vee y) + (x \wedge y)}{\sqrt{(x \vee y)^2+1} + \sqrt{(x \wedge y)^2+1}} &\le |x-y| \le x+y \end{aligned}$$

for all $x,\, y \in [0,\, \infty)$. Now, let $T : [0,\, \infty) \to [0,\, \infty)$ be a function such that $T\,(x) = xT\,(1)$ for all $x \in [0,\, \infty)$. Then a simple argument shows

$$\sup_{x \in (0,\, \infty)} \frac{|F\,(x) - T\,(x)|}{x} = \sup_{x \in (0,\, \infty)} \left|\sqrt{1+x^{-2}} - T\,(1)\right| = \infty.$$

7.2. Schwaiger's type functional equation

Schwaiger's theorem reads [34]:

Theorem 7.5 (Schwaiger's Stability Theorem)**.** *Given a real vector space E_1 and a real Banach space E_2, let $f : E_1 \to E_2$ be a mapping for which inequality*

$$\|f\,(x+\alpha y) - f\,(x) - \alpha f\,(y)\| \le b\,(\alpha) \tag{7.10}$$

is satisfied for all $\alpha \in \mathbb{R}$. Then there exists a unique linear function $g : E_1 \to E_2$ such that $f-g$ is bounded.

In the sequel $(\mathcal{X},\, \wedge_{\mathcal{X}},\, \vee_{\mathcal{X}})$ will denote a vector lattice and $(\mathcal{Y},\, \wedge_{\mathcal{Y}},\, \vee_{\mathcal{Y}})$ a Banach lattice with $\mathcal{X}^+$ and $\mathcal{Y}^+$ their respective positive cones.
Given two positive real numbers p and q consider the functional equation

$$T\,((\tau^q\,|x|) \vee |y|) = (\tau^p T\,(|x|)) \vee T\,(|y|) \tag{7.11}$$

for all $x,\, y \in \mathcal{X}$ and $\tau \in [0,\, \infty)$, where T maps $\mathcal{X}$ into $\mathcal{Y}$.

The following simple examples show that the functional equation (7.11) has at least one solution. This can easily checked from the monotonicity of the functions.

Example 7.6 ([9], Example 1)**.** The function $T_1 : [0,\, \infty) \to [0,\, \infty)$ defined by $T_1\,(x) = x$ is a solution of (7.11), for all $\tau,\, q,\, x,\, y \in [0,\, \infty)$ with the choice $p = q$.

Example 7.7 ([9], Example 2)**.** The function $T_2 : [0,\, \infty) \to [0,\, \infty)$ defined by $T_2\,(x) = \sqrt{x}$ is a solution of (7.11), for all $\tau,\, q,\, x,\, y \in [0,\, \infty)$ with the choice $p = \frac{q}{2} < q$.

Example 7.8 ([9], Example 3)**.** The function $T_3 : [0,\, \infty) \to [0,\, \infty)$ defined by $T_3\,(x) = x^2$ is a solution of (7.11), for all $\tau,\, q,\, x,\, y \in [0,\, \infty)$ with the choice $p = 2q > q$.

Example 7.9 ([9], Example 4)**.** Let $\mathcal{X} = B\,(M,\, \mathbb{R})$ be the space of all bounded real-valued functions defined on M. Then the functional $T : \mathcal{X} \to \mathcal{X}$, such that $T\,(|f|) = |f|^{\alpha}$, solves (7.11) for arbitrary positive numbers q and α with $p = q\alpha$.

Our essential goal in this part is to prove the stability of the functional equation (7.11) to be viewed as a counterpart of the Schwaiger type stability theorem (cf. [34]).

We recall that a functional $H : \mathcal{X} \to \mathcal{Y}$ is *cone-related* if $H(\mathcal{X}^+) = \{H(|x|) : x \in \mathcal{X}\} \subset \mathcal{Y}^+$ (see more about this notion in [6]).

Remark 7.10 ([9], Remark 1.1). Given two positive real numbers p and q, if a cone-related operator $T : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation (7.11), then

1.) $T(|x| \vee |y|) = T(|x|) \vee T(|y|)$ for all $x, y \in \mathcal{X}$ and $\tau = 1$;

2.)
$$T(\tau^q |x|) = \tau^p T(|x|) \tag{7.12}$$
for all $x \in \mathcal{X}$ and all $\tau \in [0, \infty) \setminus \{1\}$.

Proof. Note that by letting $\tau = 1$ in (7.11) shows that T is trivially a join-homomorphism. To show the second part we first prove that $T(0) = 0$. In fact, take $x = y = 0$ in (7.11). Then $T(0) = (\tau^p T(0)) \vee T(0)$. But since τ runs over the non-negative real line, by choosing $\tau = 2$ yields $T(0) = (2T(0)) \vee T(0)$, which is possible only if $T(0) = 0$. Consequently, (7.12) follows if we select $y = 0$ in (7.11). □

Theorem 7.11 ([9], Theorem 2.1). *Given a pair of positive real numbers (p, q), consider a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ for which there are numbers $\vartheta > 0$ and α with $q\alpha \in (0, p)$ such that*
$$\|F((\tau^q |x|) \vee |y|) - (\tau^p F(|x|)) \vee F(|y|)\| \le 2^{-p}\vartheta (\|x\|^\alpha + \|y\|^\alpha) \tag{7.13}$$
for all $x, y \in \mathcal{X}$ and all $\tau \in [0, \infty)$. Then the sequence $(2^{-np} F(2^{nq}|x|))_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in \mathcal{X}$. Let the functional $T : \mathcal{X} \to \mathcal{Y}$ be defined by
$$T(|x|) = \lim_{n\to\infty} 2^{-np} F(2^{nq}|x|). \tag{7.14}$$
Then

a.) *T is a solution of the functional equation (7.11);*

b.) *T is the unique cone-related functional which satisfies inequality*
$$\|T(|x|) - F(|x|)\| \le \frac{2^{q\alpha}\vartheta}{2^p - 2^{q\alpha}} \|x\|^\alpha \tag{7.15}$$
for every $x \in \mathcal{X}$.

Moreover, assume that $\mathcal{X}$ is a Banach lattice and F is continuous from below on the positive cone $\mathcal{X}^+$. Then in order that the limit operator T be continuous from below on $\mathcal{X}^+$, it is necessary and sufficient that
$$\lim_{n\to\infty}\lim_{k\to\infty} \frac{F(2^{nq}x_k)}{2^{np}} \le \lim_{k\to\infty}\lim_{n\to\infty} \frac{F(2^{nq}x_k)}{2^{np}}, \tag{7.16}$$
for any increasing sequence $(x_k) \subset \mathcal{X}^+$., provided that the limits exist.

Before we start the proof the following obvious remarks are worth being mentioned, as they will be used multiple times. The first will be checked and the second one can be found in [8] without proof.

Remark 7.12 ([9], Remark 2.1). If the condition of Theorem 7.11 hold true, then $F(0) = 0$.

Proof. In (7.13) choose $x = y = 0$ and observe that $\|F(0) - (\tau^p F(0)) \vee F(0)\| = 0$ so that $F(0) = (\tau^p F(0)) \vee F(0)$. But since τ runs over the non-negative real line, by choosing $\tau = 2$ yields $F(0) = (2F(0)) \vee F(0)$, which is possible only if $F(0) = 0$. □

Remark 7.13 ([9], Remark 2.2). Let Z be a set closed under the scalar multiplication, i.e. $bz \in Z$ whenever $b \in (0, \infty)$ and $z \in Z$. Given a number $c \in (0, \infty)$ let the function $\gamma : Z \to Z$ be defined by $\gamma(z) = cz$. Then $\gamma^j : Z \to Z$ the j-th iteration of γ is given by $\gamma^j(z) = c^j z$ for every counting number $j \ge 2$.

Proof of Theorem 7.11. First, we choose $\tau = 2^{-1}$, $y = 0$ and replacing x by $2^q x$ in (7.13) we obviously have

$$\left\| \frac{F\left(2^q |x|\right)}{2^p} - F\left(|x|\right) \right\| \leq \vartheta 2^{q\alpha - p} \left\| x \right\|^{\alpha}. \tag{7.17}$$

Next, let us define the following functions:

1.) $G : \mathcal{X} \to \mathcal{X}, \quad G\left(|x|\right) = 2^q |x|.$
2.) $\delta : \mathcal{X} \to [0, \infty), \quad \delta\left(|x|\right) = \vartheta 2^{q\alpha - p} \left\| x \right\|^{\alpha}.$
3.) $\varphi : [0, \infty) \to [0, \infty), \quad \varphi\left(t\right) = 2^{-p} t.$
4.) $H : \mathcal{Y} \to \mathcal{Y}, \quad H\left(|y|\right) = 2^{-p} |y|.$
5.) $d\left(\cdot, \cdot\right) : \mathcal{Y} \times \mathcal{Y} \to [0, \infty), \quad d\left(y_1, y_2\right) = \left\| y_1 - y_2 \right\|.$

We shall verify the fulfilment of all the three conditions of the first Forti's theorem (cf. [18, Theorem 1]) as follows.

(I.) From inequality (7.17) we obviously have

$$d\left(H\left(F\left(G\left(|x|\right)\right)\right), F\left(|x|\right)\right) = \left\| \frac{F\left(2^q |x|\right)}{2^p} - F\left(|x|\right) \right\| \leq \vartheta 2^{q\alpha - p} \left\| x \right\|^{\alpha} = \delta\left(|x|\right).$$

(II.) $d\left(H\left(|y_1|\right), H\left(|y_2|\right)\right) = 2^{-p} \left\| y_1 - y_2 \right\| = \varphi\left(d\left(y_1, y_2\right)\right)$ for all $y_1, y_2 \in \mathcal{Y}$.

(III.) Clearly, on the one hand φ is a non-decreasing subadditive function on the positive half line, and on other hand by applying Remark 7.13 on both the iterations G^j and φ^j of G and φ respectively, one can observe that

$$\sum_{j=0}^{\infty} \varphi^j\left(\delta\left(G^j\left(|x|\right)\right)\right) = \vartheta 2^{(q\alpha - p)} \left\| x \right\|^{\alpha} \sum_{j=0}^{\infty} 2^{(q\alpha - p)j} = \vartheta \left\| x \right\|^{\alpha} \frac{2^{q\alpha}}{2^p - 2^{q\alpha}} < \infty.$$

Then in virtue of Forti's first theorem in [18], sequence $\left(H^n\left(F\left(G^n |x|\right)\right)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in \mathcal{X}$ and thus so is sequence $\left(2^{-np} F\left(2^{nq} |x|\right)\right)_{n\in\mathbb{N}}$ and furthermore, the mapping (7.14) is the unique functional which satisfies inequatility (7.15). Next, we prove that the mapping T, defined in (7.14), satisfies the functional equation (7.11). In fact, in (7.13) substitute x with $2^{nq}x$ also y with $2^{nq}y$, and fix arbitarily $\tau \in [0, \infty)$. Then

$$\left\| F\left(2^{nq}\left(\left(\tau^q |x|\right) \vee |y|\right)\right) - \left(\tau^p F\left(2^{nq} |x|\right)\right) \vee F\left(2^{nq} |y|\right) \right\| \leq \vartheta 2^{-p} 2^{q\alpha n} \left(\left\| x \right\|^{\alpha} + \left\| y \right\|^{\alpha}\right).$$

Dividing both sides of this last inequality by 2^{np} yields

$$\left\| \frac{F\left(2^{nq}\left(\left(\tau^q |x|\right) \vee |y|\right)\right)}{2^{np}} - \frac{\left(\tau^p F\left(2^{nq} |x|\right)\right) \vee F\left(2^{nq} |y|\right)}{2^{np}} \right\| \leq \vartheta 2^{-p} 2^{(q\alpha - p)n} \left(\left\| x \right\|^{\alpha} + \left\| y \right\|^{\alpha}\right). \tag{7.18}$$

Taking the limit in (7.18) we have via (7.14) that for all $\tau \in [0, \infty)$ and all $x, y \in \mathcal{X}$

$$\left\| T\left(\left(\tau^q |x|\right) \vee |y|\right) - \left(\tau^p T\left(|x|\right)\right) \vee T\left(|y|\right) \right\| = 0$$

which is equivalent to (7.11).

The moreover part can be proved the same way the moreover parts of the theorems in [6] were, after we will have shown that the limits on both sides of (7.16) exist. In fact, on the one hand, the existence of the limit on the left hand side follows from the combination of the monotonicity of F and (7.14). On the other hand, because of (7.14) the inner limit on the right hand side equals $T\left(x_k\right)$ for every $k \in \mathbb{N}$. But since the limit operator T is a join-homomorphism, it is also isotonic or increasing. Consequently, $\left(T\left(x_k\right)\right)_{k\in\mathbb{N}}$ is a convergent sequence. We have thus proved that the limits on both sides of (7.16) exist.

Therefore, we can conclude on the validity of the argument. □

The example hereafter is to show that stability fails in some cases. if the range of the parameters p and q is retricted and the power α of the norms equals the ratio of p and q To end the section we shall provide some example showing that if in (7.13) parameter τ does not range over the whole non-negative half-line and the power α of the norms equals the ratio of p and q, then stability cannot always be guaranteed. A similar example can be found in [8].

Example 7.14 ([9], Example 5)**.** Fix arbitrarily three numbers $p,\, q,\, c \in (0,\, \infty)$ and consider the function

$$F : \mathbb{R} \to \mathbb{R},\, F\left(|x|\right) = c.$$

Then whenever $\tau \in (0,\, 1]$ we have:

$$\left|F\left(\left(\tau^q |x|\right) \vee |y|\right) - \left(\tau^p F\left(|x|\right)\right) \vee F\left(|y|\right)\right| = \left|c - \left(\tau^p c\right) \vee c\right| = 0 \leq |x|^\alpha + |y|^\alpha, \text{ where } \quad \alpha = \frac{p}{q}.$$

Since $|x| = \left(|x|^{\frac{1}{q}}\right)^q$, for any function $T : \mathbb{R} \to \mathbb{R}$ which solves (7.11) the following consecutive relations are true:

$$\sup_{|x| \in (0,\, \infty)} \frac{\left|F\left(|x|\right) - T\left(|x|\right)\right|}{|x|^\alpha} = \sup_{|x| \in (0,\, \infty)} \frac{\left|c - T\left(\left(|x|^{\frac{1}{q}}\right)^q\right)\right|}{|x|^\alpha} = \sup_{|x| \in (0,\, \infty)} \frac{\left|c - |x|^\alpha T\left(1\right)\right|}{|x|^\alpha} =$$

$$= \sup_{|x| \in (0,\, \infty)} \left|\frac{c}{|x|^\alpha} - T\left(1\right)\right| = \infty.$$

8. Concluding Remarks

We would like to pinpoint that Riesz spaces can offer a very fertile soil for proving addition dependent results in addition-free environments. We believe that this is yet to come to an end. So broad can be the spectrum of questions to ask and to answer that we judge not to cite any of them here.

9. Competing Interests

The author declares that he has no competing interests.

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Finding the Fixed Points Inside Large Mapping Sets: Integral Equations

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Abstract

Let $xf(t,x) > 0$ for $x \neq 0$ and let $A(t-s)$ satisfy some classical properties yielding a nice resolvent. Using repeated application of a fixed point mapping and induction we develop an asymptotic formula showing that solutions of the Caputo equation

$$^{c}D^{q}x(t) = -f(t,x(t)), \quad 0 < q < 1, \quad x(0) \in \Re, \quad x(0) \neq 0,$$

and more generally of the integral equation

$$x(t) = x(0) - \int_0^t A(t-s)f(s,x(s))ds, x(0) \neq 0,$$

all satisfy $x(t) \to 0$ as $t \to \infty$.

Keywords: compact maps repeated mappings integral equations fixed points limit sets
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1. Introduction

In this paper $\Re$ denotes the set of real numbers and $(\mathcal{B}, \|\cdot\|)$ denotes the Banach space of bounded continuous functions $\phi : [0,\infty) \to \Re$ with the supremum norm.

Integral equations of the form

$$x(t) = x(0) - \int_0^t A(t-s)f(s,x(s))ds$$

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are found throughout applied mathematics and Caputo fractional differential equations under the "spring condition" that $xf(t,x) > 0$ for $x \neq 0$ and A satisfies properties parallel to those found in heat transfer problems. The study of such problems as found in the literature can be very challenging and it is certainly true that parts of the study are very deep. But the thesis here is that qualitative properties of solutions can be so similar to those of the elementary ordinary differential equation

$$x' = -f(t,x)$$

that simpler attacks may be very enlightening and fruitful.

Here is a sketch of our work. After a crucial transformation the natural mapping defined by the equation will map a closed ball in a Banach space $(\mathcal{B}, \|\cdot\|)$ into itself and there will be a fixed point. But that is very crude and we would like further properties and the location of that fixed point. If we try to impose additional conditions on the closed ball such as requiring all functions to tend to zero at infinity then the result fails because of problems with compactness. Here we discover a way out.

If P is the natural mapping defined by the integral equation and if M is the ball of radius $|x(0)|$ then $PM =: M_1 \subset M$ so $P : M_1 \to M$ and if P has a fixed point it will also reside in PM. In fact, P will have a fixed point and we seek its properties. It turns out that if we continue and repeat the mapping then using mathematical induction we can find an asymptotic formula of the fixed point and its limit as $t \to \infty$ is very simply calculated. In the next section we extend the introduction and add explicit details.

2. A sketch of the study

The vehicle for explaining the theory introduced here will be the scalar integral equation

$$x(t) = x(0) - \int_0^t A(t-s)f(s,x(s))ds \tag{2.1}$$

where $x(0) \in \Re$, $x(0) \neq 0$, $f\colon [0,\infty) \times \Re \to \Re$ is continuous, and A satisfies the following conditions found in Miller [9, p. 209]:

A1) The function $A \in C(0,\infty) \cap L^1(0,1)$.

A2) $A(t)$ is positive and nonincreasing for $t > 0$.

A3) For each $T > 0$ the function $A(t)/A(t+T)$ is nonincreasing in t for $0 < t < \infty$.

Under these conditions the resolvent equation

$$R(t) = A(t) - \int_0^t A(t-s)R(s)ds \tag{2.2}$$

has a continuous solution $R : (0,\infty) \to (0,\infty)$ satisfying

$$0 < R(t) \leq A(t) \tag{2.3}$$

for $t > 0$; the strict positivity is found in [8]. If $A \in L^1(0,\infty)$ and $\alpha = \int_0^\infty A(s)ds$ then

$$\int_0^\infty R(s)ds = \alpha(1+\alpha)^{-1} < 1 \tag{2.4}$$

while if $\int_0^\infty A(s)ds = \infty$ then

$$\int_0^\infty R(s)ds = 1. \tag{2.5}$$

Finally, there is a nonlinear variation of parameters formula [9, pp. 191-193] which we used in [2] to show that for every $J > 0$ (2.1) can be mapped into the equivalent equation

$$x(t) = x(0)\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds \tag{2.6}$$

and the mapping is reversible. There are more details given in [5]. R changes with $J > 0$, but (2.5) still holds and R is still positive.

Conditions A1)–A3) are not contrived, but rather appear widely in the literature. The Caputo fractional differential equation (see [6])

$$ {}^cD^q x(t) = -f(t, x(t)), \quad x(0) \neq 0, \quad 0 < q < 1, \tag{2.7} $$

is known to invert [6, p. 86] as

$$ x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds \tag{2.8} $$

where Γ is the Euler gamma function. A selection of real-world problems modelled by (2.1) is found in [1], Section 5.

It is assumed that f satisfies what may be called the spring condition

$$ x \neq 0 \implies xf(t, x) > 0. \tag{2.9} $$

In view of (2.7) we take our pattern from the ordinary differential equation

$$ x'(t) = -f(t, x(t)), \quad x(0) \neq 0. \tag{2.10} $$

We can define a Liapunov function by

$$ V(x) = x^2 $$

so that if (2.10) has a solution, $x(t)$, then we can take the derivative of V along the solution using the chain rule and have

$$ \frac{dV(x(t))}{dt} = 2x(t)[-f(t, x(t))] < 0 \tag{2.11} $$

when $x \neq 0$. Clearly the solution then satisfies

$$ x^2(t) \leq x^2(0) \tag{2.12} $$

and if f is bounded away from zero for $x \neq 0$ then an integration of (2.11) drives $x(t)$ to zero.

Goal

Use the transformation and a fixed point map to show parallel properties for (2.1) transformed to (2.6) with a strengthened form of (2.9) holding.

3. The mappings

From now on our focus is on the transformed equation (2.6) which we now designate by

$$ x(t) = x(0)\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)\left[x(s) - \frac{f(s, x(s))}{J}\right] ds \tag{3.1} $$

with all the past continuity conditions on f and the conditions on R. But we will always assume that the integral of A is infinite so that we have

$$ \int_0^\infty R(s)ds = 1. \tag{3.2} $$

Everything will be based on showing that the natural mapping defined by (3.1) will map the closed ball in $\mathcal{B}$

$$ M = \{\phi : [0, \infty) \to \Re \colon |\phi(t)| \leq |x(0)|\} \tag{3.3} $$

into itself and then successive mappings by P will send all fixed points to zero as $t \to \infty$. The proof rests on a repeated fixed point map, induction, and a very simple lemma which now follows.

Lemma 3.1. *Let R be the resolvent with $\int_0^\infty R(s)ds = 1$ and let $\phi : [0,\infty) \to [0,\infty)$ with $0 \le \phi(t) < 1$. Suppose that for each $\epsilon > 0$ there exists $T > 0$ such that $t \ge T \implies 1-\epsilon \le \phi(t) < 1$. Then*

$$\int_0^t R(t-s)\phi(s)ds \to 1$$

as $t \to \infty$. In particular,

$$\int_0^t R(t-s)\int_0^s R(s-u)duds \to 1$$

as $t \to \infty$.

Proof. Clearly

$$\int_0^t R(t-s)\phi(s)ds < 1.$$

Now for an $\epsilon > 0$ and the corresponding T then $0 < T < t$ implies that

$$\int_0^t R(t-s)\phi(s)ds \ge \int_T^t R(t-s)(1-\epsilon)ds = (1-\epsilon)\int_0^{t-T} R(t-T-s)ds$$
$$= (1-\epsilon)\int_0^{t-T} R(u)du \to 1-\epsilon$$

as $t \to \infty$. As $\epsilon \to 0$ we find that

$$\int_0^t R(t-s)\phi(s)ds \to 1$$

as $t \to \infty$.

For the final conclusion take $\phi(t) = \int_0^t R(t-u)du$ and conclude that

$$\int_0^t R(t-s)\int_0^s R(s-u)duds \to 1$$

as $t \to \infty$. □

For our work below, we set

$$\begin{aligned} R_1(t) \quad &:= \int_0^t R(s)\,ds = \int_0^t R(t-s)\,ds, \quad t \ge 0, \\ R_i(t) \quad &= \int_0^t R(t-s)\,R_{i-1}(s)\,ds, \quad t \ge 0, \quad i = 2,3,.... \end{aligned}$$

Clearly

$$\lim_{t\to\infty} R_2(t) := \lim_{t\to\infty}\int_0^t R(t-s)\,R_1(s)\,ds = 1.$$

By repeated use of Lemma 3.1 we may see that for any integer $n > 0$ then

$$\lim_{t\to\infty} R_n(t) = \lim_{t\to\infty}\int_0^t R(t-s)\,R_{n-1}(s)\,ds = 1.$$

Note that by the definition of R_n and the fact that $R_1(t) < 1, t \ge 0$, for any such n we have

$$0 < ... \le R_{n+1}(t) \le R_n(t) \le ... \le R_1(t) < 1, \quad t \ge 0.$$

Refer now to (3.1) and refine the conditions on f as

$$0 \le 1 - \frac{f(t,x)}{Jx} \le 1 - \frac{k}{J}, \tag{3.4}$$

for $0 < k < J$ and for $|x| \le |x(0)|$.

Theorem 3.2. *Let M be defined by (3.3) and let (3.4) hold. Let $P : M \to \mathcal{B}$ be defined by $\phi \in M$ implies that*

$$(P\phi)(t) = x(0)\big[1 - \int_0^t R(s)ds\Big] + \int_0^t R(t-s)\phi(s)\Big[1 - \frac{f(s,\phi(s))}{J\phi(s)}\Big]ds. \tag{3.5}$$

Then $P : M \to M$ and if $M_1 = PM$ then $\phi \in M_1$ implies

$$\begin{aligned}|(P\phi)(t)| \le &|x(0)|\Big[1 - \int_0^t R(s)ds\Big] + |x(0)|\Big[1 - \frac{k}{J}\Big]\int_0^t R(s)ds \\ &= |x(0)|\Big[1 - \frac{k}{J}\int_0^t R(s)ds\Big].\end{aligned}$$

Finally, P has at least one fixed point ξ in M which, of course, also resides in M_1. Thus $P\xi = \xi$, $\xi \in M_1$, and ξ satisfies (3.1) on $[0, \infty)$.

Proof. Using the natural mapping defined by (3.1), if $\phi \in M$ then

$$\begin{aligned}|(P\phi)(t)| &\le |x(0)|\Big[1 - \int_0^t R(s)ds\Big] + \int_0^t R(t-s)|x(0)|\Big[1 - \frac{k}{J}\Big]ds \\ &= |x(0)| - |x(0)|\int_0^t R(s)ds + |x(0)|\int_0^t R(s)ds - |x(0)|\frac{k}{J}\int_0^t R(s)ds \\ &= |x(0)|\Big[1 - \frac{k}{J}\int_0^t R(s)ds\Big].\end{aligned}$$

There is a long list of papers dealing with existence of solutions of this equation. First, [3, p. 95,Theorem 4.1] as corrected in [4, p. 234] yields a fixed point $\xi \in M$ when $A(t-s) = (t-s)^{q-1}, 0 < q < 1$. The general case under A1)–A3) is proved in exactly the same way. The main points are that P maps M into an equicontinuous set, P is continuous, and M is a ball. The last two points are the same for the general A as for the $(t-s)^{q-1}$ case. More detail on equicontinuity and continuity is found in Dwiggins [7]. □

Theorem 3.3. *Under the same conditions if $P : M \to \mathcal{B}$ and $\phi \in M$, for any positive integer n we have*

$$\left|P^{(n)}(\phi)(t)\right| \le |x(0)|\left[1 - \frac{k}{J}\sum_{i=1}^{i=n}\left(1 - \frac{k}{J}\right)^{i-1} R_i(t)\right], \quad t \ge 0, \tag{3.6}$$

where

$$\begin{aligned}R_1(t) \ &: = \int_0^t R(s)\,ds = \int_0^t R(t-s)\,ds, \\ R_i(t) \ &: = \int_0^t R(t-s)\,R_{i-1}(s)\,ds, \quad i = 2, 3, \ldots.\end{aligned}$$

In particular, if ξ is a fixed point of P then

$$|\xi(t)| \le |x(0)|\left[1 - \frac{k}{J}\sum_{i=1}^{\infty}\left(1 - \frac{k}{J}\right)^{i-1} R_i(t)\right], \quad t \ge 0 \tag{3.7}$$

and

$$\lim_{t\to\infty}\xi(t) = 0. \tag{3.8}$$

Proof. Inequality (3.6) for $n = 1$ becomes

$$|P(\phi(t))| \le |x(0)|\left[1 - \frac{k}{J}R_1(t)\right], \quad t \ge 0,$$

which holds true by Theorem 3.2. To prove (3.6) by induction, we assume that (3.6) holds for some positive integer m, i.e., that

$$\left|P^{(m)}(\phi)(t)\right| \le |x(0)| \left[1 - \sum_{i=1}^{i=m} \left(1-\frac{k}{J}\right)^{i-1} \frac{k}{J} R_i(t)\right], \quad t \ge 0, \tag{3.9}$$

and we want to prove that (3.6) holds for $n = m+1$.

Employing (3.9) in the definition of P we have

$$\begin{aligned}
& \left|P^{(m+1)}(\phi)(t)\right| = \left|P\left(P^{(m)}(\phi)\right)(t)\right| \\
\le\ & |x(0)| \left[1 - \int_0^t R(s)\,ds\right] + \int_0^t R(t-s) \left|P^{(m)}(\phi)(s)\right| \left(1-\frac{k}{J}\right) ds \\
\le\ & |x(0)| \left[1 - R_1(t)\right] + \\
& + \int_0^t R(t-s) |x(0)| \left[1 - \sum_{i=1}^{i=m} \left(1-\frac{k}{J}\right)^{i-1} \frac{k}{J} R_i(s)\right] \left(1-\frac{k}{J}\right) ds \\
=\ & |x(0)| \left\{1 - R_1(t) + \left(1-\frac{k}{J}\right) \times \right. \\
& \left. \times \left[\int_0^t R(t-s)\,ds - \int_0^t R(t-s) \sum_{i=1}^{i=m} \left(1-\frac{k}{J}\right)^{i-1} \frac{k}{J} R_i(s)\,ds\right]\right\} \\
=\ & |x(0)| \left\{1 - R_1(t) + \left(1-\frac{k}{J}\right) R_1(t) \right. \\
& \left. - \sum_{i=1}^{i=m} \left(1-\frac{k}{J}\right)^{i} \frac{k}{J} \int_0^t R(t-s) R_i(s)\,ds\right\} \\
=\ & |x(0)| \left\{1 - \frac{k}{J} R_1(t) - \sum_{i=1}^{i=m} \left(1-\frac{k}{J}\right)^{i} \frac{k}{J} R_{i+1}(t)\right\} \\
=\ & |x(0)| \left\{1 - \frac{k}{J}\left(1-\frac{k}{J}\right)^{1-1} R_1(t) - \sum_{i=2}^{i=m+1} \left(1-\frac{k}{J}\right)^{i-1} \frac{k}{J} R_i(t)\right\},
\end{aligned}$$

i.e.,

$$\left|P^{(m+1)}(\phi)(t)\right| \le |x(0)| \left\{1 - \sum_{i=1}^{i=m+1} \left(1-\frac{k}{J}\right)^{i-1} \frac{k}{J} R_i(t)\right\}, \quad t \ge 0,$$

which is (3.6) for $n = m+1$, thus induction is completed and inequality (3.6) is proved.

Next, note that if ξ is a fixed point of P then $P(\xi) = \xi$, thus for any positive integer n we have $P^{(n)}(\xi) = \xi$, and so

$$|\xi(t)| \le |x(0)| \left[1 - \frac{k}{J} \sum_{i=1}^{n} \left(1-\frac{k}{J}\right)^{i-1} R_i(t)\right], \quad t \ge 0. \tag{3.10}$$

Since

$$0 \le \sum_{i=1}^{n} \left(1-\frac{k}{J}\right)^{i-1} R_i(t) \le \sum_{i=1}^{n} \left(1-\frac{k}{J}\right)^{i-1} < \infty$$

we see that the series of nonnegative terms at the right hand side of (3.10) converges uniformly so taking $n \to \infty$ in (3.10) leads to (3.7).

Before we prove (3.8) we recall that

$$\lim_{t\to\infty} R_1(t) := \lim_{t\to\infty} \int_0^t R(s)\,ds = 1,$$

so by use of Lemma 3.1 we take

$$\lim_{t\to\infty} R_2(t) := \lim_{t\to\infty} \int_0^t R(t-s)\, R_1(s)\, ds = 1.$$

By a simple induction we see that for any positive integer n we have

$$\lim_{t\to\infty} R_n(t) = \lim_{t\to\infty} \int_0^t R(t-s)\, R_{n-1}(s)\, ds = 1.$$

It follows that the limit of the right hand side of (3.6) exists and

$$\begin{aligned}\lim_{t\to\infty}\left[1-\frac{k}{J}\sum_{i=1}^{i=n}\left(1-\frac{k}{J}\right)^{i-1}R_i(t)\right] &= 1-\frac{k}{J}\sum_{i=1}^{i=n}\left(1-\frac{k}{J}\right)^{i-1}\\ &= 1-\frac{k}{J}\frac{1-\left(1-\frac{k}{J}\right)^n}{\frac{k}{J}},\end{aligned}$$

i.e.,

$$\lim_{t\to\infty}\left[1-\sum_{i=1}^{i=n}\left(1-\frac{k}{J}\right)^{i-1}\frac{k}{J}R_i(t)\right] = \left(1-\frac{k}{J}\right)^n.$$

Then (3.8) follows by observing that the series at the left hand side of (3.7) converges uniformly and that

$$\lim_{n\to\infty}\left(1-\frac{k}{J}\right)^n = 0.$$

□

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Study the topology of Branciari metric space via the structure proposed by Császár

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Abstract

In this paper, we topologically study the generalized metric space proposed by Branciari [3] via the weak structure proposed by Császár [9, 10], and compare convergent sequences in several different senses. We also introduce the concepts of available points and unavailable points on such structures. Besides, we define the continuous function on structures and investigate further characterizations of continuous functions.

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1. Introduction and Preliminaries

Branciari [3] introduced the concept of a generalized metric space where the triangle inequality is replaced by a rectangular inequality. Many authors studied the fixed point theory on such generalized metric space (cf. [1, 2, 3, 4, 5, 6, 7]). Recall the notion of Branciari metric space.

Definition 1.1. *[3] For a nonempty set X, let $d : X \times X \longrightarrow [0, \infty]$ be a map such that for any $x, y \in X$ and distinct $u, v \in X \setminus \{x, y\}$,*

$$\begin{array}{ll} (BMS1) & d(x,y) = 0 \text{ if and only if } x = y, \\ (BMS2) & d(x,y) = d(y,x), \\ (BMS3) & d(x,y) \le d(x,u) + d(u,v) + d(v,y). \end{array}$$

The map d is called a Branciari metric, and the pair (X, d) is called a Branciari metric space, abbreviated as BMS. The open ball and closed ball are defined respectively by

$$B(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}, \; B(x,\varepsilon] = \{y \in X : d(x,y) \le \varepsilon\}$$

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for all $x \in X$ *and* $\varepsilon > 0$.

A sequence $\{x_n\}$ *in* (X, d) *is convergent to* x *if* $d(x_n, x) \to 0$ *as* $n \to \infty$.

The story in this paper starts from the topology of BMS. In contrast to the metric space, the topology of BMS,

$$\mathcal{T} = \{S \subset X : \forall x \in S, \exists r > 0 \text{ s.t. } B(x,r) \subset S\} \cup \{\varnothing\},$$

is difficult to describe. In the topology space $(X, \mathcal{T})$, an open ball may not be open. Furthermore, a terrible fact is that $x_n \xrightarrow{\mathcal{T}} x$ (i.e., x_n converges to x with respect to the topology $\mathcal{T}$) can not guarantee that $x_n \to x$, i.e., $d(x_n, x) \to 0$ (see Example 2.2 for details).

To remedy this problem, an alternative way is to define a new topology $\widetilde{\mathcal{T}}$ generated by all open balls (as subbase). In this topology, the above problems are solved, that is, every open ball is open, and $x_n \xrightarrow{\widetilde{\mathcal{T}}} x$ implies $x_n \to x$.

However, a new phenomenon arises: $x_n \to x$ can not guarantee $x_n \xrightarrow{\widetilde{\mathcal{T}}} x$ (see Example 2.2).

In some sense, the topology equivalent to convergent sequences with respect to d has no equivalence relation with open balls. How can we directly study the convergent sequence $x_n \to x$ from topological view?

One way to overcome all the difficults is adopting the generalized topology proposed by Császár [9], which removes the intersection property of finite number of open sets. Let $\mathcal{T}' = \{\cup_{B \in \mathcal{B}_0} B : \mathcal{B}_0 \subset \mathcal{B}\}$, where $\mathcal{B} = \{B(x, r) : x \in X, r > 0\}$. Then $\mathcal{T}'$ is a generalized topology on X which contains all the open balls as its generalized topological base. With the aid of the generalized topology, we show an easy way to study the convergent sequence $x_n \to x$ using topological method in Sections 2 and 3.

The generalized topology was extended to weak structure by Császár [10], in which some families of sets (like $\beta(\omega)$, $\rho(\omega)$, $\sigma(\omega)$, $\pi(\omega)$, $\alpha(\omega)$) play very fundamental roles. There have been some further results about these families of sets, such as [11], [12]. In Section 4, we introduce the available points and unavailable points on structures (mentioned by Császár in the introduction of [10]), and define the interior points, accumulation points, isolated points of a set. With the help of these points, we define the interior operator and closure operator, which are equivalent to the corresponding concepts defined by Császár. We also establish the Kuratowski 7-sets theorem and some other results on structures. A main contribution is to characterize the continuity on structures, where Theorems 5.7 and 5.11 are commendatory results in Section 5.

2. Convergent sequences with respect to d, $\mathcal{T}$ and $\widetilde{\mathcal{T}}$

Theorem 2.1. *Let* (X, d) *be a BMS. Then we have:*

$$x_n \xrightarrow{\widetilde{\mathcal{T}}} x \;\Rightarrow\; x_n \to x \;\Rightarrow\; x_n \xrightarrow{\mathcal{T}} x.$$

The converse is false, i.e., $x_n \xrightarrow{\widetilde{\mathcal{T}}} x \;\not\Leftarrow\; x_n \to x \;\not\Leftarrow\; x_n \xrightarrow{\mathcal{T}} x$.

Proof. Suppose $x_n \xrightarrow{\widetilde{\mathcal{T}}} x$. Then for any $U \in \widetilde{\mathcal{T}}$ with $x \in U$, there exists $N > 0$ such that $x_n \in U$ for any $n > N$. Taking $U = B(x, \epsilon)$, we have $d(x_n, x) < \epsilon$ for $n > N$, which deduces that $x_n \to x$.

Assume $x_n \to x$. For any $V \in \mathcal{T}$ with $x \in V$, there exists $B(x, \epsilon) \subset V$. So, there is $N > 0$ such that $x_n \in B(x, \epsilon) \subset V$ for $n > N$. Accordingly, $x_n \xrightarrow{\mathcal{T}} x$.

See Example 2.2 for the counter-example of the converse. □

Example 2.2. *Let* $X = [0, 1]$ *and let* $d : [0, 1] \times [0, 1] \to [0, +\infty)$ *be a symmetric function defined by*

$$d(y,x) = d(x,y) = \begin{cases} |x-y|, & \textit{if } x \in [0,1] \cap \mathbb{Q} \textit{ and } y \in [0,1] \setminus \mathbb{Q}, \\ 1, & \textit{if } x \neq y,\ x, y \in [0,1] \cap \mathbb{Q} \textit{ or } x, y \in [0,1] \setminus \mathbb{Q}, \\ 0, & \textit{if } x = y. \end{cases}$$

It can be easily verified that (X, d) is a BMS.

We will prove that $\mathcal{T}$ is the standard Euclidean topology on $[0,1]$, and $\widetilde{\mathcal{T}}$ is the discrete topology on $[0,1]$. For $0 < r < 1$, keep

$$B(x,r) = \begin{cases} \{y \in [0,1] \setminus \mathbb{Q} : |y-x| < r\} \cup \{x\}, & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ \{y \in [0,1] \cap \mathbb{Q} : |y-x| < r\} \cup \{x\}, & \text{if } x \in [0,1] \setminus \mathbb{Q}, \end{cases}$$

in mind.

For any $U \in \mathcal{T} \setminus \{\varnothing\}$ and $x \in U$, there exists $r > 0$ such that $B(x,r) \subset U$. Without loss of generality, we may assume $x \in \mathbb{Q} \cap [0,1]$. Then $\{y \in [0,1] \setminus \mathbb{Q} : |y-x| < r\} \subset U$, i.e., $(x-r, x+r) \cap [0,1] \setminus \mathbb{Q} \subset U$. Thus, for any $y \in (x-r, x+r) \cap [0,1] \setminus \mathbb{Q}$, there exists $r_y \le r - |x-y|$ such that $\{z \in [0,1] \cap \mathbb{Q} : |z-y| < r_y\} \subset U$, i.e., $(y-r_y, y+r_y) \cap [0,1] \cap \mathbb{Q} \subset U$. Therefore,

$$\bigcup_{y \in (x-r,x+r) \cap [0,1] \setminus \mathbb{Q}} (y-r_y, y+r_y) \cap [0,1] \cap \mathbb{Q} \subset U,$$

i.e., $(x-r, x+r) \cap [0,1] \cap \mathbb{Q} \subset U$. Together with $(x-r, x+r) \cap [0,1] \setminus \mathbb{Q} \subset U$, we obtain $(x-r, x+r) \cap [0,1] \subset U$. On the other hand, for any $y \in (x-r, x+r) \cap [0,1]$, let $r' = r - |x-y| > 0$. Then $B(y, r') \subset (x-r, x+r) \cap [0,1]$, which implies that $(x-r, x+r) \cap [0,1] \in \mathcal{T}$. So, $\{(x-r, x+r) \cap [0,1] : x \in [0,1], r > 0\}$ forms a topological base of $\mathcal{T}$. This means that $\mathcal{T}$ is the Euclidean topology on $[0,1]$.

Since $B(x,r) \in \widetilde{\mathcal{T}}$, $\forall x \in [0,1]$ and $r > 0$, we have $B(x,r) \cap B(y,r') \in \widetilde{\mathcal{T}}$, $\forall x, y \in [0,1]$, $\forall r, r' > 0$. For $x \in [0,1] \cap \mathbb{Q}$ and $y \in [0,1] \setminus \mathbb{Q}$,

$$B(x,r) \cap B(y,r') = \begin{cases} \varnothing, & \text{if } r, r' \le |x-y|, \\ \{x\}, & \text{if } r \le |x-y| < r', \\ \{y\}, & \text{if } r' \le |x-y| < r, \\ \{x,y\}, & \text{if } |x-y| < r, r'. \end{cases}$$

Hence, $\{x\}, \{y\} \in \widetilde{\mathcal{T}}$. This deduces that every singleton set is an open set, which means that $\widetilde{\mathcal{T}}$ is a discrete topology.

Since $|\frac{1}{n} - 0| \to 0$ and $d(\frac{1}{n}, 0) = 1 \not\to 0$, we have $\frac{1}{n} \xrightarrow{\mathcal{T}} 0$ and $\frac{1}{n} \not\to 0$.

Since $d(\frac{\sqrt{2}}{n}, 0) = \frac{\sqrt{2}}{n} \to 0$ and $\{0\}$ is an open set in $\widetilde{\mathcal{T}}$, one has $\frac{\sqrt{2}}{n} \to 0$ and $\frac{\sqrt{2}}{n} \overset{\widetilde{\mathcal{T}}}{\not\to} 0$.

3. The second countability and the separability on BMS

We call a Branciari metric space a **strong separable** space if there exists a countable subset A of X such that for any $x \in X$, there is a Cauchy sequence $\{x_n\} \subset A$ with different terms such that $x_n \to x$, $n \to +\infty$ unless x is a isolated point in A. Here A is said to be a **strong dense** set.

To describe the second countability on generalized metric space, we replace 'topology' by 'generalized topology', in which every generalized open set is defined to be the union of a family of balls $B(x,r)$ in X.

We call $\mathcal{U}$ a generalized topological base of X, if any generalized open set V can be written as the union of some generalized open sets from $\mathcal{U}$.

A BMS is said to be a **generalized second countable** BMS if there is a generalized topological base with countable members.

Note that $\rho(x,y) := \inf_{z \in X} d(x,z) + d(z,y) \le d(x,y)$ and $\rho(x,y) \le \rho(x,z) + \rho(z,y)$, for any $x, y, z \in X$ (see [6]). For describing more properties, we introduce the (K) condition as follows:
(K) There is $k \in (0,1)$ such that $\rho(x,y) \ge k d(x,y)$ for any $x, y \in X$.

Theorem 3.1. *Let (X, d) be a Branciari metric space. We have the following results.*

(1) *If X is strong separable, then it must be a generalized second countable space.*

(2) *If X is generalized second countable, then X is separable.*

(3) *If X is generalized second countable with the condition (K), then X is strong separable.*

Proof. (1) Let $A \subset X$ be a countable strong dense subset, and $\mathcal{B} = \{B(a, q) : a \in A,\ q \in \mathbb{Q}^+\}$. We will show that each $B(x, r)$ can be covered by some elements in $\mathcal{B}$, where $r \in \mathbb{R}^+$ and $x \in X$.

If $y \in B(x, r) \cap A$ is an isolated point of A, then there exists $r' \in \mathbb{Q}^+$ such that $B(y, r') \cap A = \{y\}$. If $B(y, r') \setminus A \neq \varnothing$, then for any $z \in B(y, r') \setminus A$, there exists a Cauchy sequence $\{z_n\} \subset A$ satisfying $z_n \to z$ with $z_n \neq z$, $z_n \neq z_m$, $n \neq m$. So $d(y, z_n) \leq d(y, z) + d(z, z_m) + d(z_m, z_n) \to d(y, z) < r'$ as $n, m \to +\infty$. That is, $z_n \in B(y, r') \cap A$ for sufficiently large n, which contradicts with $B(y, r') \cap A = \{y\}$. Consequently, $B(y, r') \setminus A = \varnothing$, and thus $B(y, r') = \{y\} \subset B(x, r)$.

Next, we assume $y \in B(x, r)$ is not an isolated point of A, Let δ be a positive rational number with $\delta \leq r - d(x, y)$. Since A is strong dense in X, there exists a Cauchy sequence $\{x_n\}$ in A converging to y with $x_n \neq y$ and $x_n \neq x_m$ for any $n \neq m$. Thus, there is some $N \in \mathbb{N}$ such that $d(x_n, y) < \frac{\delta}{2}$ and $d(x_n, x_m) < \frac{\delta}{2}$ for all $n, m > N$. Let m be a natural number with $m > N$. Now for each $z \in B(x_m, \frac{\delta}{2})$, we show that $d(z, x) < r$.

Case I: $z \neq x_m$.

$$\begin{aligned} d(z, x) &\leq d(z, x_m) + d(x_m, y) + d(y, x) \\ &< \frac{\delta}{2} + \frac{\delta}{2} + d(y, x) = r. \end{aligned}$$

Case II: $z = x_m$.

$$\begin{aligned} d(x_m, x) &\leq d(x_m, x_n) + d(x_n, y) + d(y, x) \\ &< \frac{\delta}{2} + \frac{\delta}{2} + d(y, x) = r. \end{aligned}$$

This proves $y \in B(x_m, \frac{\delta}{2}) \subset B(x, r)$, and hence $\mathcal{B}$ is a countable base for X. Therefore, X is generalized second countable.

(2) Let $\mathcal{U} = \{U_1, U_2, \cdots\}$ be a topological base. Take $x_n \in U_n$, and let $A = \{x_n : n \in \mathbb{N}\}$. Now we show that A is a countable dense set in X. In fact, for any $x \in X$, and $m \in \mathbb{N}^+$, there is a U_{n_m} contained in $B(x, \frac{1}{m})$, so $x_{n_m} \in B(x, \frac{1}{m})$, i.e., $\lim\limits_{m \to +\infty} x_{n_m} = x$.

(3) We only need to show that, under the condition (K), $x_n \to x$ implies that $\{x_n\}$ is Cauchy. Indeed,

$$\begin{aligned} d(x_n, x_m) &\leq \frac{1}{k}\rho(x_n, x_m) \leq \frac{1}{k}(\rho(x_n, x) + \rho(x_m, x)) \\ &\leq \frac{1}{k}(d(x_n, x) + d(x_m, x)) \to 0, n, m \to +\infty. \end{aligned}$$

□

An analogous result of Theorem 3.1 on partial metric space was provided in [8].

4. Structures

Let X be a nonempty set and A be a subset of X. We denote by A^c the complement of A.

Definition 4.1. *Let X be a nonempty set and let $\mathcal{S}$ be a nonempty family of subsets of X, then $\mathcal{S}$ is called a structure on X. The elements of $\mathcal{S}$ are called open sets and the complements of open sets are called closed sets.*

Definition 4.2. *For $x \in X$, $\mathcal{S}_x := \{u \in \mathcal{S} : x \in u\}$ is said to be the open neighbourhood system of x.*

Definition 4.3. *We call points in* $X - \bigcup_{u\in\mathcal{S}} u$ *the unavailable points and in* $\bigcup_{u\in\mathcal{S}} u$ *the available points, denoted by* $\mathrm{UK}(X)$ *and* $\mathrm{K}(X)$*, respectively.*

Proposition 4.4. *For* $x \in X$*,* $x \in \mathrm{K}(X)$ *if and only if* $\mathcal{S}_x$ *is nonempty.*

Proof. Clearly, by Definitions 4.2 and 4.3, it is easy to see that $\mathcal{S}_x \neq \varnothing$ if and only if $x \in \bigcup_{u\in\mathcal{S}} u = \mathrm{K}(X)$. □

Definition 4.5. *For* $x \in A \subset X$*,* x *is called an interior point of* A *if there exists* $u \in \mathcal{S}_x$ *such that* $u \subset A$*. The interior of* A *is the union of all interior points of* A*, denoted by* $i(A)$*. If* A *has no interior points, we denote* $i(A) = \varnothing$*.*

Similar to the Lemma 2.2 in [10], we immediately get that $i(A)$ is the union of all open sets contained in A.

Definition 4.6. *For* $x \in \mathrm{K}(X)$ *and* $A \subset X$*, we call* x *an accumulation point of* A *if* $\forall u \in \mathcal{S}_x$*,* $u \bigcap A - \{x\} \neq \varnothing$*. We call* x *an isolated point of* A *if* $\exists u \in \mathcal{S}_x$*,* $u \bigcap A = \{x\}$*.*

Definition 4.7. *The derived set of* A *is the union of all accumulation points of* A*, denoted by* $d(A)$*. The closure of* A *is the union of all unavailable points of* X*, all accumulation points and all isolated points of* A*, denoted by* $c(A)$*.*

We simply use iA, dA and cA instead of $i(A)$, $d(A)$ and $c(A)$, respectively.

Remark 4.8. *It follows from Definition 4.7 that* $cA = dA \bigcup A \bigcup \mathrm{UK}(X)$*.*

Proposition 4.9. *(1)* $cA = \{x : \forall u \in \mathcal{S}_x,\ u \bigcap A \neq \varnothing\} \bigcup \mathrm{UK}(X)$*.*

(2) If A *is a closed set, then* $A = cA$*. If* A *is an open set, then* $A = iA$*.*

(3) If the union of any subfamily of $\mathcal{S}$ *always belongs to* $\mathcal{S}$*, then* A *is closed iff* $A = cA$ *and* A *is open iff* $A = iA$*.*

Proof. (1) It follows directly from Remark 4.8.

(2) We only show that A is closed $\Rightarrow A = cA$. Suppose that $cA - A \neq \varnothing$, then we pick $x \in cA - A$. Note that $A^c \in \mathcal{S}_x$, but $A^c \bigcap A = \varnothing$, which is a contradiction, so $cA = A$.

(3) We only need to show $\forall A \subset X,\ iA \in \mathcal{S}$.

For any $x \in iA$, there exists $u \in \mathcal{S}_x$ such that $u \subset A$. If $u - iA \neq \varnothing$, then picking $y \in u - iA$, we have $y \in A \bigcap (iA)^c$. So, $\forall v \in \mathcal{S}_y$, $v \not\subset A$. Note that $u \in \mathcal{S}_y$, so $u \not\subset A$. This is a contradiction.

Hence, we get $u - iA = \varnothing$, which means that $\forall x \in iA$, $\exists u_x \in \mathcal{S}_x$ such that $x \in u_x \subset iA$. Then $iA = \bigcup_{x\in iA} u_x$, where $u_x \in \mathcal{S}$. Thus $iA \in \mathcal{S}$. □

Remark 4.10. *Proposition 4.9 (1) is an equivalent definition of* cA*.*

If $\mathcal{S}$ *is closed under arbitrary union, then* iA *is the maximal open set contained in* A*.*

Proposition 4.11. *(1)* $c\varnothing = \mathrm{UK}(X)$*.*

(2) $A \subset cA$*.*

(3) $cA = ccA$*.*

(4) $A \subset B \Rightarrow cA \subset cB$*. If* $\mathcal{S}$ *is closed under finite intersection, then* $cA \bigcup cB = c(A \bigcup B)$*.*

Proof. (1) Suppose $c\varnothing \neq \mathrm{UK}(X)$. Then $\forall x \in c\varnothing - \mathrm{UK}(X), \forall u \in \mathcal{S}_x$, $u \bigcap \varnothing \neq \varnothing$, which is a contradiction. Consequently, $c\varnothing = \mathrm{UK}(X)$.

(2) It follows directly from Proposition 4.9 (1).

(3) $\forall x \in ccA$, $\forall u \in \mathcal{S}_x$, $u \bigcap cA \neq \varnothing$. Take $y \in u \bigcap cA$. Then $u \in \mathcal{S}_y$ and thus $u \bigcap A \neq \varnothing$. It follows that $x \in cA$. Accordingly, $ccA \subset cA$ and combining with (2), we get $cA = ccA$.

(4) $\forall x \in cA$, $\forall u \in \mathcal{S}_x$, $u \bigcap A \neq \varnothing$. Thus $u \bigcap B \neq \varnothing$, and then $x \in cB$, i.e., $cA \subset cB$. In consequence, $cA \bigcup cB \subset c(A \bigcup B)$.

Now assume that $\mathcal{S}$ is closed under finite intersection. Suppose $c(A\bigcup B) - cA\bigcup cB \neq \varnothing$. Let $x \in c(A\bigcup B) - cA\bigcup cB$. Then there exist $u_A, u_B \in \mathcal{S}_x$ such that $u_A \bigcap A = \varnothing$, $u_B \bigcap B = \varnothing$. Let $u = u_A \bigcap u_B \in \mathcal{S}_x$. Then $u\bigcap A = \varnothing$, $u\bigcap B = \varnothing$, and thus $u\bigcap(A\bigcup B) = (u\bigcap A)\bigcup(u\bigcap B) = \varnothing$.

This is a contradiction with $x \in c(A\bigcup B)$. Therefore, $cA\bigcup cB = c(A\bigcup B)$. □

Similarly, we have:

Proposition 4.12. *(1) $i\varnothing = \varnothing$. (2) $iA \subset A$. (3) $iA = iiA$.*
(4) $A \subset B \Rightarrow iA \subset iB$. If $\mathcal{S}$ is closed under finite intersection, then $iA\bigcap iB = i(A\bigcap B)$.

Proposition 4.13. *(1) $d\varnothing = \varnothing$.*
(2) $ddA \subset A\bigcup dA$.
(3) $A \subset B \Rightarrow dA \subset dB$. If $\mathcal{S}$ is closed under finite intersection, then $dA\bigcup dB = d(A\bigcup B)$.

Next we show the relations among these operators, $i(\cdot)$, $c(\cdot)$ and $d(\cdot)$.

Proposition 4.14. *(1) $(cA^c)^c = iA$, $(iA^c)^c = cA$.*
(2) If $x \in dA$, then $c(A - \{x\}) = cA$.
(3) $dA = \{x \in \mathrm{K}(X) : x \in c(A - \{x\})\}$.

Proof. (1) It has been shown in Theorem 2.1 [10].

(2) We only need to show $cA \subset c(A - \{x\})$. Assume there exists $y \in cA - c(A - \{x\}) \subset \mathrm{K}(X)$. Then let $u \in \mathcal{S}_y$ such that $u\bigcap(A - \{x\}) = \varnothing$. It follows that $u\bigcap A \subset \{x\}$, $u\bigcap A \neq \varnothing$, i.e., $u\bigcap A = \{x\}$. It is easy to see that $u \in \mathcal{S}_x$. Then by $x \in d(A)$, we get $u\bigcap A - \{x\} \neq \varnothing$, which is a contradiction.

(3) $\forall x \in d(A)$, $x \in cA = c(A - \{x\})$, so $d(A) \subset \{x \in \mathrm{K}(X) : x \in c(A - \{x\})\}$. On the other hand, if $x \in \mathrm{K}(X)\bigcap c(A - \{x\})$, then $x \in d(A - \{x\})\bigcup(A - \{x\})$, i.e., $x \in d(A - \{x\}) \subset d(A)$. Hence $\{x \in \mathrm{K}(X) : x \in c(A - \{x\})\} \subset d(A)$. □

Inspired by Proposition 4.14 (3), we can define a dual concept of derived set.

Definition 4.15. *$eA = \{x \in X : x \in i(A\bigcup\{x\})\}$ is called the dual derived set of A.*

From Proposition 4.14 (1), we know that c and i are dual operators. Moreover, the following result concludes that d and e are also dual operators (relative to $\mathrm{K}(X)$).

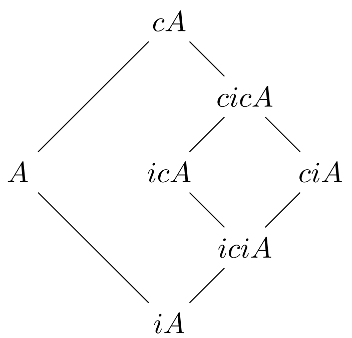
Proposition 4.16. *$eA = d(A^c)^c \cap \mathrm{K}(X)$ and $dA = e(A^c)^c \cap \mathrm{K}(X)$ hold for any subset A of X.*

Proof. We only need to prove $eA = d(A^c)^c \cap \mathrm{K}(X)$. For any $x \in e(A)$, we have $x \in \mathrm{K}(X)$ and $x \in i(A\bigcup\{x\})$. By Proposition 4.14 (1) we obtain $c(A^c - \{x\}) = c((A\bigcup\{x\})^c) = (i(A\bigcup\{x\}))^c$. Hence, $x \notin c(A^c - \{x\})$, that is, $x \notin d(A^c)$. Thus $e(A) \subset d(A^c)^c \cap \mathrm{K}(X)$.

On the other hand, for any $x \in d(A^c)^c \cap \mathrm{K}(X)$, we have $x \in \mathrm{K}(X)$ but $x \notin c(A^c - \{x\}) = (i(A\bigcup\{x\}))^c$, i.e., $x \in i(A\bigcup\{x\})$. So, $x \in eA$ and then $d(A^c)^c \cap \mathrm{K}(X) \subset eA$. □

The following theorem is a counterpart of Kuratowski 7-sets theorem.

Theorem 4.17. *Let $A \subset X$. The number of distinct sets which can be obtained from A by successively taking c and i (in any order) is at most 7. The inclusion relations of the 7 sets are $iA \subset A \subset cA$ and $iA \subset iciA \subset ciA \cap icA \subset ciA \cup icA \subset cicA \subset cA$, which can be written as a Hasse diagram as follows:*



Proof. It is easy to check the result by Theorem 2.1 and Proposition 2.6 in [10]. □

As a supplement of Theorem 2.1(c) in [12], we have:

Proposition 4.18. *Let A be a subset of X. Then the following statements are equivalent:*
(1) $cA = cicA$.
(2) For any open set u satisfying $u \bigcap A \neq \varnothing$, we have $u \bigcap icA \neq \varnothing$.

Proof. $\Leftarrow$: $\forall x \in cA \bigcap \mathrm{K}(X)$, $\forall u \in \mathcal{S}_x$, $u \bigcap A \neq \varnothing$. So $u \bigcap icA \neq \varnothing$ and hence $x \in cicA$. Therefore, $cA \subset cicA$. Since $cicA \subset cA$ (by Theorem 4.17), we have $cA = cicA$.

$\Rightarrow$: If there exists V such that $V \bigcap A \neq \varnothing$ and $V \bigcap icA = \varnothing$, then $icA \subset V^c$. Note that V^c is closed. So $cicA \subset V^c$, i.e., $cicA \bigcap V = \varnothing$. Since $cA \bigcap V \neq \varnothing$, there exists $x \in cA$ such that $x \notin cicA$. □

5. Continuous map, open map and closed map

Definition 5.1. *Let $x \in \mathrm{K}(X)$, $f(x) \in \mathrm{K}(Y)$. $f : X \to Y$ is said to be continuous at x if for all $v \in \mathcal{S}_{f(x)}$, there exists $u \in \mathcal{S}_x$ such that $f(u) \subset v$. We call f a continuous map, if it is continuous at every point in $\mathrm{K}(X) \bigcap f^{-1}(\mathrm{K}(Y))$.*

To get more properties of continuous mapping, we introduce the following concepts.

Definition 5.2. *In $(X, \mathcal{S})$, let $A \subset X$. We call A a generalized closed set if $cA = A$. We call A a generalized open set if $A = iA$.*

Definition 5.3. *Let $\mathcal{S}^\sim = \{iA : A \subset X\}$ and let $i_{\mathcal{S}^\sim} A$ denote the interior of A in $(X, \mathcal{S}^\sim)$. Similarly, the open neighborhood system of x in $(X, \mathcal{S}^\sim)$ is denoted by $\mathcal{S}_x^\sim$.*

Proposition 5.4. *$\mathcal{S}^\sim = \{A : A \subset iA\} = \{A : A = iA\}$ is a set of all generalized open sets in X.*

Proposition 5.5. *$iA = i_{\mathcal{S}^\sim} A$ is open in $(X, \mathcal{S}^\sim)$.*

Proof. We first show that if $\mathcal{S} \subset \mathcal{B}$, then $i_{\mathcal{S}} A \subset i_{\mathcal{B}} A$. Without loss of generality, we assume $iA \neq \varnothing$. Then $\forall x \in iA$, $\exists u \in \mathcal{S}_x \subset i_{\mathcal{B}} A$ and $u \subset A$. So x is an interior point of A in $(X, \mathcal{B})$. Hence $iA \subset i_{\mathcal{B}} A$.

Now we prove that if $\mathcal{B} = \mathcal{S}^\sim$, then $i_{\mathcal{S}^\sim} A \subset iA$. Assume that $i_{\mathcal{S}^\sim} A - iA \neq \varnothing$, then we take $x \in i_{\mathcal{S}^\sim} A - iA$. There exists $u \in \mathcal{S}^\sim$ such that $x \in u \subset A$. Take $v \subset X$ satisfying $u = iv$. By $x \in iv$, there exists $w \in \mathcal{S}$ such that $x \in w \subset iv = u \subset A$. So x is an interior point of A in $(X, \mathcal{S})$, and then $x \in iA$, which is a contradiction. Thus $i_{\mathcal{S}^\sim} A = iA$. $\forall A \subset X$, $i_{\mathcal{S}^\sim} A = iA \in \mathcal{S}^\sim$. So $i_{\mathcal{S}^\sim} A$ is open in $(X, \mathcal{S}^\sim)$. □

Proposition 5.6. *$\varnothing \in \mathcal{S}^\sim$ and $\mathcal{S}^\sim$ is closed under arbitrary union.*

Proof. Since $i\varnothing = \varnothing$, we have $\varnothing \in \mathcal{S}^\sim$. For any $\mathcal{B} \subset \mathcal{S}^\sim$ and $\forall A \in \mathcal{B}$,

$$A \subset \bigcup_{B \in \mathcal{B}} B \Rightarrow iA \subset i\left(\bigcup_{B \in \mathcal{B}} B\right) \Rightarrow \bigcup_{A \in \mathcal{B}} iA \subset i\left(\bigcup_{B \in \mathcal{B}} B\right).$$

By $A \in \mathcal{S}^\sim \Leftrightarrow A = iA$, we get $\bigcup_{A \in \mathcal{B}} A \subset i(\bigcup_{A \in \mathcal{B}} A) \subset \bigcup_{A \in \mathcal{B}} A$. So $i(\bigcup_{A \in \mathcal{B}} A) = \bigcup_{A \in \mathcal{B}} A$, and then $\bigcup_{A \in \mathcal{B}} A \in \mathcal{S}^\sim$. □

With the aid of operators, i and c, we can study specific structures. For a subset $A \subset X$, let $A \in \alpha(\mathcal{S})$ iff $A \subset iciA$; $A \in \sigma(\mathcal{S})$ iff $A \subset ciA$; $A \in \pi(\mathcal{S})$ iff $A \subset icA$; $A \in \beta(\mathcal{S})$ iff $A \subset cicA$; $A \in \rho(\mathcal{S})$ iff $A \subset ciA \cup icA$.

By the counterparts of Theorems 3.1 and 3.2 [10] on structures, and Theorem 4.17 and Proposition 5.6 in the present paper, we immediately get:

Theorem 5.7. *$\mathcal{S}^\sim$, $\alpha(\mathcal{S})$, $\sigma(\mathcal{S})$, $\pi(\mathcal{S})$, $\rho(\mathcal{S})$ and $\beta(\mathcal{S})$ are generalized topologies on X and they satisfy:*

$$\mathcal{S} \subset \mathcal{S}^\sim \subset \alpha(\mathcal{S}) \subset \sigma(\mathcal{S}) \cap \pi(\mathcal{S}) \subset \sigma(\mathcal{S}) \cup \pi(\mathcal{S}) \subset \rho(\mathcal{S}) \subset \beta(\mathcal{S}).$$

Proposition 5.8. *Assume $x \in \mathrm{K}(X)$ and $f(x) \in \mathrm{K}(Y)$. Then f is continuous at x if and only if for each $v \in \mathcal{S}_{f(x)}$, there exists $iu \in \mathcal{S}_x^\sim$ such that $f(iu) \subset v$.*

The following statements are equivalent: (1) f is continuous.

(2) The preimage of every open set is generalized open.

(3) The preimage of every generalized open set is generalized open.

(4) $f^{-1}(iB) \subset if^{-1}(B)$.

Proof. If $f(u) \subset v$, then $f(iu) \subset v$. On the other hand, since $x \in iu$, there exists $w \in \mathcal{S}_x$ such that $w \subset iu$. So $f(w) \subset v$.

(1) $\Rightarrow$ (2): For any open set v, we will prove that $f^{-1}(v)$ is generalized open. For all $x \in f^{-1}(v)$, since f is continous at x, there exists $u \in \mathcal{S}_x$ such that $f(u) \subset v$, which implies $u \subset f^{-1}(v)$. Therefore x is an interior point of $f^{-1}(v)$. This shows that $f^{-1}(v)$ is generalized open.

(2) $\Rightarrow$ (1): Assume that for all $x \in X$, $v \in \mathcal{S}_{f(x)}$. Since $f^{-1}(v)$ is generalized open, there exists $u \in \mathcal{S}_x$ such that $u \subset f^{-1}(v)$, i.e., $f(u) \subset v$. This implies that f is continuous.

(3) $\Rightarrow$ (2): Since open sets are generalized open, it is trivial.

(2) $\Rightarrow$ (3): For any generalized open set $iB \subset Y$, iB can be written as $iB = \bigcup v_i$, where v_i is open. Since $f^{-1}(iB) = \bigcup f^{-1}(v_i)$, and $f^{-1}(v_i)$ is generalized open for any i, we deduce that $f^{-1}(iB)$ is generalized open.

(4) $\Rightarrow$ (3): Let B be a generalized open set. Then $f^{-1}(B) = f^{-1}(iB) \subset if^{-1}(B)$. Hence $f^{-1}(B) = if^{-1}(B)$, and thus $f^{-1}(B)$ is generalized open.

(3) $\Rightarrow$ (4): Note that $f^{-1}(iB)$ is generalized open. So $f^{-1}(iB) = if^{-1}(iB) \subset if^{-1}(B)$. □

Proposition 5.5, 5.6 and 5.8 indicate that we can assume $\mathcal{S}$ is closed under arbitrary union if we only concentrate on continuity and interior. That is, in some sense, the generalized topology is enough.

Definition 5.9. *We say that $f : X \to Y$ is open, if for any open set $u \subset X$, $f(u)$ is generalized open.*

We say that $f : X \to Y$ is closed, if for any closed set $A \subset X$, $f(A)$ is generalized closed.

Theorem 5.10. *Let $f : X \to Y$ be a map. Then we have:*

(1) $\forall A \subset X$, $cf(A) \subset f(cA) \Leftrightarrow f$ is closed.

(2) $\forall A \subset X$, $f(cA) \subset cf(A) \Leftrightarrow f$ is continuous.

(3) $\forall A \subset X$, $f(iA) \subset if(A) \Leftrightarrow f$ is open.

(4) $\forall B \subset Y$, $cf^{-1}(B) \subset f^{-1}(cB) \Leftrightarrow f$ is continuous.

(5) $\forall B \subset Y$, $f^{-1}(cB) \subset cf^{-1}(B) \Leftrightarrow f$ is open.

(6) $\forall B \subset Y$, $if^{-1}(B) \subset f^{-1}(iB) \Leftrightarrow f$ is open.

(7) $\forall B \subset Y$, $f^{-1}(iB) \subset if^{-1}(B) \Leftrightarrow f$ is continuous.

Proof. Since the proofs are standard and similar, we only show (3) and (7).

(3). $\Rightarrow$: For any open set $V \subset X$, $f(V) = f(iV) \subset if(V) \subset f(V)$. So $f(V) = if(V)$, i.e., $f(V)$ is open. Thus, f is open.

$\Leftarrow$: If f is open, then $\forall A \subset X$, $f(iA) = if(iA) \subset if(A)$.

(7). $\Leftarrow$: If f is continuous, then $\forall B \subset Y$, $f^{-1}(iB)$ is generalized open. Note that $f^{-1}(iB) \subset f^{-1}(B)$. Thus $f^{-1}(iB) \subset if^{-1}(B)$.

$\Rightarrow$: For any open set $V \subset Y$, $f^{-1}(V) = f^{-1}(iV) \subset if^{-1}(V) \subset f^{-1}(V)$. So $f^{-1}(V) = if^{-1}(V)$, i.e., $f^{-1}(V)$ is generalized open. In consequence, f is continuous. □

Theorem 5.11. *Let $f : X \to Y$ be a surjection. Assume $\forall A \subset X$, $if(A) \subset f(iA)$. Then f is continuous.*

Proof. Consider the set

$$H = \{h : f(h(y)) = y, \forall y \in Y, \text{ where } h : Y \to X\}.$$

Clearly, H is nonempty since $f : X \to Y$ is a surjection. For any open set $B \subset Y$, $f^{-1}(B) = \bigcup_{h\in H} h(B)$, we only need to prove that $h(B)$ is generalized open. Accroding to $if(h(B)) \subset f(ih(B)) \subset f(h(B))$ and

$f(h(B)) = B = iB$, we get $f(ih(B)) = f(h(B))$. Since $f|_{h(B)}$ is an injection, we have $h(B) = ih(B)$ and thus $h(B)$ is generalized open. Therefore, $f^{-1}(B) = \bigcup_{h\in H} h(B)$ is generalized open. It follows from Proposition 5.8 (2) that f is continuous. □

Remark 5.12. *(1) The conditions of Theorem 5.11 are all necessary. In fact, if we remove the condition that f is a surjection, then Theorem 5.11 is false. Two examples are shown in Examples 5.13 and 5.14.*

(2) The converse of Theorem 5.11 is not true, that is, $if(A) \subset f(iA)$ is not always true when f is a continuous surjection (see Example 5.15).

Example 5.13. *Let $X = \{1, 2\}$, $T_X = \{\varnothing, \{1\}, X\}$, $Y = \{1, 2, 3\}$ and $T_Y = \{\varnothing, \{1\}, \{2, 3\}, Y\}$. Suppose $f : X \to Y$ satisfying $f(1) = 1$ and $f(2) = 2$. Then f is an injection.*

Note that $f^{-1}(\{2, 3\}) = \{2\}$ is not open, which means that f is not continuous.

Since $if(1) = \{1\} = f(i\{1\})$, $if(2) = \{2\}^o = \varnothing \subset f(i\{2\})$ and $if(\{1, 2\}) = i\{1, 2\} = \{1\} \subset \{1, 2\} = f(i\{1, 2\})$, we get that f satisfies $if(A) \subset f(iA), \forall A \subset X$.

Example 5.14. *Let $f(x) = \begin{cases} 0, & -1 \le x < 0, \\ 1, & 0 \le x \le 1. \end{cases}$ Then $f : [-1, 1] \to \mathbb{R}$ is not continuous. Note that $\forall A \subset [-1, 1]$, $f(A) \subset \{0, 1\}$. So $if(A) \subset i\{0, 1\} = \varnothing \subset f(iA)$.*

Example 5.15. *Let $X = \{1, 2\}$, $T_X = \{\varnothing, \{1\}, X\}$, $Y = \{1\}$ and $T_Y = \{\varnothing, Y\}$. Set $f : X \to Y$ with $f(1) = 1$ and $f(2) = 1$.*

Note that $f^{-1}(1) = \{1, 2\} = X$ is open, which deduces that f is continuous. Since $if(2) = i\{1\} = \{1\} \not\subset \varnothing = f(\varnothing) = f(i\{2\})$, $if(A) \subset f(iA)$ fails to hold.

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A NEW GENERALIZATION OF WARDOWSKI FIXED POINT THEOREM IN COMPLETE METRIC SPACES

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Abstract

The aim of this paper is to state and prove Wardowski type fixed point theorem in metric spaces. The paper includes an example which shows that our result is a proper extension of some known results.

Keywords: Wardowski type contraction, fixed point, metric space
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1. Introduction and Preliminaries

Starting from one of the fundamental results of fixed point theory known as the Banach contraction principle [5], several authors proved many interesting extensions and generalizations ([1]-[4], [6]-[18]).

In 2012, D. Wardowski [14], using functions $F : \mathbb{R}_+ \to \mathbb{R}$ proved a fixed point theorem concerning a new type of contractions, called $F-$contractions.

Let function $F : \mathbb{R}_+ \to \mathbb{R}$ such that:

$(F1)$ F is strictly increasing, that is, for all $x, y \in \mathbb{R}_+$ if $x < y$ then $F(x) < F(y)$;

$(F2)$ For each sequence $\{\alpha_n\}$ of positive numbers,

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if only if } \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

$(F3)$ There exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \left(\alpha^k F(\alpha)\right) = 0$

We denote by $\mathcal{F}$ the family of all that functions.

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Definition 1.1. *[14] Let (X, d) be a metric space. A map $T : X \to X$ is said to be an $F-$contraction on (X, d) if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)) \tag{1}$$

Theorem 1.2. *[14] Let (X, d) be a complete metric space and $T : X \to X$ be an $F-$contraction. Then T has a unique fixed point x^* and for all $x \in X$ the sequence $\{T^n x\}$ is convergent to x^*.*

Remark 1.3. *From $(F1)$ and (1) it follows that*

$$\begin{aligned} F(d(Tx, Ty)) &\leq F(d(x, y)) - \tau < F(d(x, y)) \Rightarrow \\ &\Rightarrow d(Tx, Ty) < d(x, y) \end{aligned}$$

for all $x, y \in X$ such that $Tx \neq Ty$. Also, T is a continuous operator.

Afterwards, Wardowski and Van Dung [15] have introduced the notion of a $F-$weak contraction, in this way.

Definition 1.4. *[15] Let (X, d) be a metric space. A map $T : X \to X$ is said to be a $F-$weak contraction on (X, d) if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$ satisfying $d(Tx, Ty) > 0$, the following holds:*

$$\tau + F\left(d(Tx, Ty)\right) \leq F\left(M(x, y)\right) \tag{2}$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

By using this notion, Wardowski and Van Dung [15] have demonstrated a fixed point theorem which generalizes the theorem 1.2 as follows.

Theorem 1.5. *[15] Let (X, d) be a complete metric space and $T : X \to X$ be a $F-$weak contraction. If T or F is continous, then T has a unique fixed point x^* and for all $x \in X$ the sequence $\{T^n x\}$ is convergent to x^*.*

Latter, Piri and Kumam [12] introduced a large class of functions by replacing the condition $(F3)$ in the definition of $F-$contraction with the following

$(F3')$ F is continous on $(0, \infty)$

and they denote the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfies the conditions $(F1)$, $(F2)$, and $(F3')$ by $\mathfrak{F}$.

With this assumptions, Piri and Kumam [12] proved the next fixed point theorem.

Theorem 1.6. *[12].Let (X, d) be a complete metric space and a mapping $T : X \to X$. Suppose there exists $F \in \mathfrak{F}$ and $\tau > 0$ such that, for all $x, y \in X$*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F\left(d(Tx, Ty)\right) \leq F(d(x, y)).$$

Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}$ converges to x^*.

In this paper, using the ideea from **[10],** we introduce a new type of $F-$contraction, and prove a fixed point theorem which generalizes some known results.

2. Main results

First, let $\mathcal{F}_E$ denote the familly of all functions $F:\mathbb{R}_+\to\mathbb{R}$ which satisfies the following conditions:
(F_E1) F is strictly increasing, that is, for all $x,y\in\mathbb{R}_+$, if $x<y$ then $F(x)<F(y)$;
(F_E2) There exists $\tau>0$ such that $\tau+\liminf\limits_{t\to t_0}F(t)>\limsup\limits_{t\to t_0}F(t)$, for every $t_0>0$.

Definition 2.1. *Let (X,d) be a metric space. A map $T:X\to X$ is said to be a F_E-contraction on (X,d) if there exists $F\in\mathcal{F}_E$ and $\tau>0$ such that for all $x,y\in X$*

$$d(Tx,Ty)>0\Rightarrow \tau+F\left(d\left(Tx,Ty\right)\right)\leq F\left(E(x,y)\right) \tag{3}$$

where

$$E(x,y)=d(x,y)+\left|d(x,Tx)-d\left(y,Ty\right)\right|. \tag{4}$$

Remark 2.2. *(1) Every F_E- contraction is an $F-$ contraction, but the inverse implication does not hold. (2) Not every $F-$ weak contraction is a F_E contraction .*

The following example shows that the statements from previous remark hold.

Example 2.3. *Let $X=\left[0,\frac{7}{10}\right]\cup\{1\}$ and $d(x,y)=|x-y|$, $x,y\in X$. Then (X,d) is a complete metric space. Define $T:X\to X$ by*

$$Tx=\begin{cases}\frac{x}{2}, & 0\leq x\leq\frac{7}{10}\\ \frac{1}{4}, & x=1\end{cases}$$

and choosing $F(\alpha)=\ln\alpha$, $\alpha\in(0,\infty)$ and $\tau=\ln 7$.

Since T is not continuous, T is not an $F-$contraction. In addition to that, for $x=\frac{1}{4}$ and $y=1$ we have

$$d\left(T\frac{1}{4},T1\right)=\left|\frac{1}{8}-\frac{1}{4}\right|=\frac{1}{8}>0$$

and

$$\begin{aligned}M\left(\frac{1}{4},1\right) &= \max\left\{d\left(\frac{1}{4},1\right),d\left(\frac{1}{4},T\frac{1}{4}\right),d\left(1,T1\right),\frac{d\left(1,T\frac{1}{4}\right)+d\left(\frac{1}{4},T1\right)}{2}\right\}\\ &= \max\left\{\frac{1}{8},\frac{3}{4},\frac{3}{4},\frac{7}{16}\right\}=\frac{3}{4}.\end{aligned}$$

Then,

$$\begin{aligned}\tau+F\left(d\left(T\frac{1}{4},T1\right)\right) &= \ln 7+\ln\left(\frac{1}{8}\right)=\ln\left(\frac{7}{8}\right)\\ &\geq \ln\left(\frac{3}{4}\right)=F\left(M\left(\frac{1}{4},1\right)\right)\end{aligned}$$

so T is not a $F-$weak contraction.

For $x\in\left[0,\frac{7}{10}\right]$ and $y=1$, we have

$$d\left(Tx,T1\right)=d\left(\frac{x}{2},\frac{1}{4}\right)=\frac{|2x-1|}{4}$$

and

$$\begin{aligned}E(x,1) &= d\left(x,1\right)+\left|d\left(x,Tx\right)-d\left(1,T1\right)\right|\\ &= 1-x+\left|\frac{x}{2}-\frac{3}{4}\right|=\frac{7-6x}{4}.\end{aligned}$$

Therefore,

$$\begin{aligned}\ln 7+\ln\left(d\left(Tx,T1\right)\right) &\le \ln\left(E(x,1)\right)\Leftrightarrow\\ \ln 7+\ln\left(\frac{|2x-1|}{4}\right) &\le \ln\left(\frac{7-6x}{4}\right)\Leftrightarrow\\ 7\cdot\frac{|2x-1|}{4} &\le \frac{7-6x}{4}.\end{aligned}$$

For $x\le\frac{1}{2}$,

$$7\cdot\frac{1-2x}{4}\le\frac{7-6x}{4}\Leftrightarrow 7-14x\le 7-6x\Leftrightarrow x\ge 0,$$

and for $x>\frac{1}{2}$

$$7\cdot\frac{2x-1}{4}\le\frac{7-6x}{4}\Leftrightarrow 14x-7\le 7-6x\Leftrightarrow x\le\frac{7}{10}$$

which prove that T *is a* F_E-*contraction.*

Now we state the main result of the paper.

Theorem 2.4. *Let* (X,d) *be a complete metric space and* $T:X\to X$ *be a* F_E- *contraction. Then* T *has a unique fixed point* x^* *and for all* $x_0\in X$ *the sequence* $\{T^n x_0\}$ *is convergent to* x^*.

Proof. Let $x_0\in X$ be arbitrary and fixed and we define $x_{n+1}=Tx_n=T^n x_0$ for all $n\in\mathbb{N}$. If there exists $n_0\in\mathbb{N}\cup\{0\}$ such that $x_{n_0+1}=x_{n_0}$, because $x_{n_0+1}=Tx_{n_0}$, we obtain that $Tx_{n_0}=x_{n_0}$, so x_{n_0} is a fixed point of T.

Now, we suppose that $x_{n+1}\neq x_n$ for all $n\in\mathbb{N}\cup\{0\}$. So, $d(x_n,x_{n+1})>0$, $(\forall)\,n\in\mathbb{N}\cup\{0\}$ and from (3) it follows that, for all $n\in\mathbb{N}$

$$\begin{aligned}d\left(x_n,x_{n+1}\right) &= d\left(Tx_{n-1},Tx_n\right)>0\Rightarrow\\ &\Rightarrow \tau+F\left(d\left(Tx_{n-1},Tx_n\right)\right)\le F\left(E(x_{n-1},x_n)\right)\\ &\Leftrightarrow \tau+F\left(d\left(x_n,x_{n+1}\right)\right)\le\\ &\le F\left(d(x_{n-1},x_n)+|d(x_{n-1},Tx_{n-1})-d(x_n,Tx_n)|\right.\Leftrightarrow\\ &\Leftrightarrow \tau+F\left(d\left(x_n,x_{n+1}\right)\right)\le\\ &\le F\left(d(x_{n-1},x_n)+|d(x_{n-1},x_n)-d(x_n,x_{n+1})|\right.\end{aligned}$$

or, if we denote by $d_n=d\left(x_{n-1},x_n\right)$, we have

$$\tau+F(d_{n+1})\le F\left(d_n+|d_n-d_{n+1}|\right). \tag{5}$$

If there exists $n\in\mathbb{N}$ such that $d_{n+1}\ge d_n$, then (5) becomes

$$\tau+F(d_{n+1})\le F\left(d_{n+1}\right)\Rightarrow\tau\le 0.$$

But, this is a contradiction, so, for $d_{n+1}<d_n$ we have

$$\tau+F\left(d_{n+1}\right)\le F\left(2d_n-d_{n+1}\right) \tag{6}$$

$$\Leftrightarrow F\left(d_{n+1}\right)\le F\left(2d_n-d_{n+1}\right)-\tau<F\left(2d_n-d_{n+1}\right)$$

and using (F_E1)

$$d_{n+1}<2d_n-d_{n+1}.$$

Therefore, the sequence $\{d_n\}$ is strictly increasing and bounded.

Now, let $d = \lim_{n\to\infty} d_n$ and we suppose that $d > 0$. Because $d_n \searrow d$ it results that $(2d_n - d_{n+1}) \searrow d$ and taking the limit as $n \to \infty$ in (6), we get

$$\tau + F(d+0) \leq F(d+0) \Rightarrow \tau \leq 0.$$

It is a contradiction, so

$$d = \lim_{n\to\infty} d_n = \lim_{n\to\infty} d\left(x_{n-1}, x_n\right) = 0. \tag{7}$$

In order to prove that $\{x_n\}$ is a Cauchy sequence in (X, d), we suppose the contrary, that is, there exists $\varepsilon > 0$ and the sequences $\{n(k)\}, \{m(k)\}$ of positive integers, with $n(k) > m(k) > k$ such that

$$d(x_{n(k)}, x_{m(k)}) \geq \varepsilon \text{ and } d\left(x_{n(k)-1}, x_{m(k)}\right) < \varepsilon \tag{8}$$

for any $k \in \mathbb{N}$.

Then, we have

$$\begin{aligned} \varepsilon &\leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \\ &< d(x_{n(k)}, x_{n(k)-1}) + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ and using (7) it follows

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{9}$$

Furthermore, using the triangle inequality, we obtain that

$$\begin{aligned} 0 &\leq \left| d\left(x_{n(k)+1}, x_{m(k)+1}\right) - d\left(x_{n(k)}, x_{m(k)}\right)\right| \\ &= d\left(x_{n(k)+1}, x_{n(k)}\right) + d\left(x_{m(k)}, x_{m(k)+1}\right) \end{aligned}$$

and

$$\begin{aligned} &\lim_{k\to\infty} \left| d\left(x_{n(k)+1}, x_{m(k)+1}\right) - d\left(x_{n(k)}, x_{m(k)}\right)\right| \\ = &\lim_{k\to\infty} \left[d\left(x_{n(k)+1}, x_{n(k)}\right) + d\left(x_{m(k)}, x_{m(k)+1}\right)\right] = 0. \end{aligned}$$

So,

$$\lim_{k\to\infty} d\left(x_{n(k)+1}, x_{m(k)+1}\right) = \lim_{k\to\infty} d\left(x_{n(k)}, x_{m(k)}\right) = \varepsilon. \tag{10}$$

On the other hand, because from (7)

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0,$$

there exists $N \in \mathbb{N}$ such that

$$d(x_{n(k)}, Tx_{n(k)}) < \frac{\varepsilon}{4} \text{ and } d(x_{m(k)}, Tx_{m(k)}) < \frac{\varepsilon}{4}, \ \ (\forall)\, k \geq N. \tag{11}$$

Assuming by contradiction, that there exists $l \in \mathbb{N}$ such that $d(x_{n(l)+1}, x_{m(l)+1}) = 0$, from (11) and (7) it follows that

$$\begin{aligned} \varepsilon &\leq d(x_{n(l)}, x_{m(l)}) \\ &\leq d(x_{n(l)}, x_{n(l)+1}) + d(x_{n(l)+1}, x_{m(l)+1}) + d(x_{m(l)+1}, x_{m(l)}) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned}$$

This is a contradiction. So we proved that the inequality occurs

$$d(Tx_{n(k)}, Tx_{m(k)}) = d(x_{n(k)+1}, x_{m(k)+1}) > 0 \tag{12}$$

for all $k \geq N$, and using (3), there exists $\tau > 0$ such that

$$\tau + F\left(d(Tx_{n(k)}, Tx_{m(k)})\right) \leq F\left(E(x_{n(k)}, x_{m(k)})\right)$$

for any k, where

$$\begin{aligned} E(x_{n(k)}, x_{m(k)}) &= d(x_{n(k)}, x_{m(k)}) + \left|d\left(x_{n(k)}, Tx_{n(k)}\right) - d\left(x_{m(k)}, Tx_{m(k)}\right)\right| \\ &= d(x_{n(k)}, x_{m(k)}) + \left|d\left(x_{n(k)}, x_{n(k)+1}\right) - d\left(x_{m(k)}, x_{m(k)+1}\right)\right|. \end{aligned}$$

Hence $\lim\limits_{k\to\infty} E(x_{n(k)}, x_{m(k)}) = \varepsilon$ and by (10) we have

$$\begin{aligned} \tau + \lim_{k\to\infty} \inf F\left(d(Tx_{n(k)}, Tx_{m(k)})\right) &\leq \liminf_{k\to\infty} F\left(E(x_{n(k)}, x_{m(k)})\right) \\ &\leq \limsup_{k\to\infty} F\left(E(x_{n(k)}, x_{m(k)})\right) \Leftrightarrow \\ &\Leftrightarrow \tau + F(\varepsilon+) \leq F(\varepsilon+) \end{aligned}$$

which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and by completeness of X there converges to some point $x^* \in X$.

Next, we show that x^* is a fixed point of T. We consider two cases:

(1) For any $n \in \mathbb{N}$ there exists $k_n > k_{n-1}$, $k_0 = 1$ and $x_{k_n+1} = Tx^*$. Then, $x^* = \lim\limits_{n\to\infty} x_{k_n+1} = Tx^*$, so x^* is fixed point of T.

(2) There exists $m \in \mathbb{N}$ such that for all $n \geq m$, $d(Tx_n, Tx^*) > 0$. Substituting $x = x_n$ and $y = x^*$ in (3), there exists $\tau > 0$ such that

$$\begin{aligned} \tau + F(d(Tx_n, Tx^*) &\leq F(E(x_n, x^*)) \Leftrightarrow \\ \tau + F(d(x_{n+1}, Tx^*)) &\leq F(d(x_n, x^*) + |d(x_n, Tx_n) - d(x^*, Tx^*)|) \Leftrightarrow \\ \tau + F(d(x_{n+1}, Tx^*)) &\leq F(d(x_n, x^*) + |d(x_n, x_{n+1}) - d(x^*, Tx^*)|). \end{aligned}$$

We suppose that $x^* \neq Tx^*$. letting $n \to \infty$, from(7) we obtain

$$\tau + \liminf_{t\to d(x^*,Tx^*)} F(t) < \liminf_{t\to d(x^*,Tx^*)} F(t) < \limsup_{t\to d(x^*,Tx^*)} F(t)$$

which contradicts (F_E2) of the hypothesis. Hence $Tx^* = x^*$.

Now, let us show that T must have only one fixed point. If there exists another point $y^* \in X$, $x^* = y^*$ such that $Ty^* = y^*$, then $d(x^*, y^*) = d(Tx^*, Ty^*) > 0$ and we get

$$\begin{aligned} \tau + F(d(Tx^*, Ty^*) &\leq F(E(x^*, y^*)) \Leftrightarrow \\ \tau + F(d(x^*, y^*)) &\leq F(d(x^*, y^*) + |d(x^*, Tx^*) - d(y^*, Ty^*)|) \Leftrightarrow \\ \tau + F(d(x^*, y^*)) &\leq F(d(x^*, y^*) + |d(x^*, x^*) - d(y^*, y^*)|). \Leftrightarrow \\ \tau + F(d(x^*, y^*)) &\leq F(d(x^*, y^*)) \end{aligned}$$

which is a contradiction. □

Example 2.5. *Let T be given as in Example 2.3. Since T is not a contraction, Theorem 1.2 is not applicable to T and because T is not a F-weak contraction, Theorem 1.6 can not be applied. On the other hand let F and τ be given as in Example 2.3. Then T is an F_E contraction, and Theorem 2.4 can be applicable to T and the unique fixed point of T is 0.*

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