Advances in the Theory of Nonlinear Analysis and its Applications 2 (2018) No. 1, 1–10
Available online at www.atnaa.org
Review Article

Advances in the Theory of Nonlinear Analysis
and its Applications
ISSN: 2587-2648
Peer-Reviewed Scientific Journal

Some problems in the fixed point theory

Ioan A. Rus[a]

[a] *Department of Mathematics, Babeş-Bolyai University, Kogălniceanu Street nr. 1, 400084 Cluj-Napoca, Romania.*

Abstract

In this paper we present some of my favorite problems, all the time open, in the fixed point theory. These problems are in connection with the following two:

- Which properties have the fixed point equations for which an iterative algorithm is convergent ?
- Let us have a fixed point theorem, T, and an operator f (single or multivalued) which does not satisfy the conditions in T. In which conditions the operator f has an invariant subset Y such that the restriction of f to Y, $f\big|_Y$, satisfies the conditions of T ?

Keywords: ordered set, L-space, metric space, Banach space, Picard operator, weakly Picard operator, fixed point, fixed point structure, iterative algorithm, retraction-displacement condition, well-posedness of fixed point problem, Ostrowski property, global asymptotic stability, open problem, conjecture.
2010 MSC: 47H10, 54C60, 65J15.

1. Introduction

In this paper we present some problems, all the time open problems, in the fixed point theory. These problems are in connection with the following two research directions:

(I) Which properties have the fixed point equations for which an iterative algorithm is convergent ?

(II) Let us have a fixed point theorem, T, and an operator f (single or multivalued) which does not satisfy the conditions in the theorem T. In which conditions the operator f has an invariant subset Y such that the restriction of f to Y, $f\big|_Y$, satisfies the conditions of T ?

Throughout this paper, the standard notations and terminology are used. See for example, [33], [37] and [49]. For the basic fixed point theorems, see: [13], [19], [3], [9], [49] and [55].

Email address: `iarus@math.ubbcluj.ro` (Ioan A. Rus)

Received December 12, 2017, Accepted December 28, 2017.

2. Picard and weakly Picard operators

Let $(X, \to)$ be an L-space ((X, τ)-topological space, $\xrightarrow{\tau}$; (X, d)-metric space, $\xrightarrow{d}$; $(X, \|\cdot\|)$-normed space, $\xrightarrow{\|\cdot\|}$, $\rightharpoonup$; ...) and $f : X \to X$ be an operator.

By definition, f is a weakly Picard operator if the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ at its limit (which may depend on x) is a fixed point of f. If f is a weakly Picard operator, then we consider the operator $f^\infty : X \to X$, defined by, $f^\infty(x) := \lim\limits_{n\to\infty} f^n(x)$.

We remark that the operator f^∞ is a set retraction on the fixed point set of f, F_f.

If f is a weakly Picard operator and $F_f = \{x^*\}$, then by definition f is called Picard operator. If f is a Picard operator, we have that,

$$F_f = F_{f^n} = \{x^*\}, \text{ for all } n \in \mathbb{N}^*$$

and if f is a weakly Picard operator, then,

$$F_f = F_{f^n} \neq \emptyset, \text{ for all } n \in \mathbb{N}^*.$$

In the case of a metric space and of a contraction we have the following result.

Theorem 2.1 (see [47]). *Let (X, d) be a complete metric space and $f : X \to X$ be an l-contraction. Then we have:*

(i) f is a Picard operator ($F_f = \{x^*\}$).

(ii) $d(x, x^*) \leq \psi(d(x, f(x)))$, for all $x \in X$, where $\psi(t) = \frac{t}{1-l}$, $t \geq 0$.

(iii) If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty,$$

then, $y_n \to x^*$ as $n \to \infty$.

(iv) If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that

$$d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty,$$

then, $y_n \to x^*$ as $n \to \infty$.

From this result, the following problem rises:

Problem 2.2. Let (X, d) be a complete metric space and $f : X \to X$ be an operator. Which metric conditions on f imply a similar conclusion as that of Theorem 2.1 ?

Let us consider another result:

Theorem 2.3 (see [48]). *Let (X, d) be a complete metric space and $f : X \to X$ be an operator. We suppose that:*

(1) There exists $l \in]0, 1[$ such that,

$$d(f^2(x), f(x)) \leq l d(x, f(x)), \text{ for all } x \in X,$$

i.e., f is a graphic contraction.

(2) $\lim\limits_{n\to\infty} f(f^n(x)) = f(\lim\limits_{n\to\infty} f^n(x))$, for all $x \in X$.

Then we have:

(*i*) *f is a weakly Picard operator.*

(*ii*) $d(x, f^\infty(x)) \le \frac{1}{1-l} d(x, f(x))$, *for all* $x \in X$.

(*iii*) *For* $x^* \in F_f$, *let* $X_{x^*} := \{x \in X \mid f^n(x) \to x^* \text{ as } n \to \infty\}$. *Let* $\{y_n\}_{n\in\mathbb{N}}$ *be a sequence in* X_{x^*} *such that*

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty.$$

Then, $y_n \to x^*$ *as* $n \to \infty$.

(*iv*) *Let* $\{y_n\}_{n\in\mathbb{N}}$ *be a sequence in* X_{x^*}, $x^* \in F_f$. *If* $l < \frac{1}{3}$ *and*

$$d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty,$$

then, $y_n \to x^*$ *as* $n \to \infty$.

This result suggests the following problem:

Problem 2.4 (see [48])**.** Which metric conditions imposed on an operator f imply a similar conclusion as that in Theorem 2.3 ?

For a better understanding of the above problems, let us consider the following considerations:

(*a*) A weakly Picard operator $f : (X, d) \to (X, d)$ satisfies a retraction-displacement condition (see [8]) if there exists an increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(0) = 0$ and continuous in 0, such that

$$d(x, f^\infty(x)) \le \psi(d(x, f(x))), \text{ for all } x \in X.$$

This condition is useful in studying the data dependence of the fixed point, and of Ulam stability of the fixed point equations (see [44]).

So, conclusions (*ii*) in Theorems 2.1 and 2.3 are retraction-displacement conditions for the operator f.

(*b*) Conclusions (*iii*) in Theorems 2.1 and 2.3 can be formulated as follows: The fixed point problem for the operator f is well posed.

(*c*) Conclusions (*iv*) in Theorems 2.1 and 2.3 can be formulated as follows: The operator f has the Ostrowski property.

Problem 2.5. To study similar problems in the case of multivalued operators.

References for Problems 2.2 - 2.5: [47], [48], [39], [50], [52], [8], [28], [31], [32], [49], [51], [56], [57], [54], . . .

Problem 2.6. To study similar problems in the case of a convergent iterative algorithm.

References: [42], [27], [7], [6], [25], [26], . . .

3. Conjecture on global asymptotic stability

Let $(X, \to)$ be an L-space and $f : X \to X$ be an operator. A fixed point x^* of f is by definition globally asymptotically stable if f is a Picard operator, i.e., $f^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$.

In 1976, J.P. LaSalle presented (see [20]) the following conjecture:

Conjecture 1 (LaSalle's Conjecture)**.** *Let* $f : \mathbb{R}^m \to \mathbb{R}^m$ *be such that:*

(*i*) *there exists* $x^* \in \mathbb{R}^m$ *with* $f(x^*) = x^*$*;*

(ii) $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$;

(iii) *the spectral radius of the differential of* f *at* x, $\rho(df(x)) < 1$, *for all* $x \in \mathbb{R}^m$.

Then, x^* *is globally asymptotically stable.*

Papers on this conjecture were given by (see [46]): A. Cima - A. Gasull - F. Mañosas (1995, 1999, 2001, 2011, 2014), G. Meisters (1996), A.G. Aksoy - M. Martelli (2001), A. Castañeda - V. Guiñez (2012), D. Cheban (2014), ... The results are as follow:

(a) counterexamples to LaSalle Conjecture;

(b) classes of functions for which LaSalle Conjecture is a theorem;

(c) to study the dynamic generated by a function $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$, with $\rho(df(x)) < 1$, for all $x \in \mathbb{R}^m$.

We have the following remark: Let $(X, \to)$ be an L-space and $f : X \to X$ be an operator. The following statements are equivalent:

(i) f is a Picard operator;

(ii) for all $k \in \mathbb{N}^*$, f^k is a Picard operator;

(iii) there exists $k \in \mathbb{N}^*$ such that f^k is a Picard operator.

Starting from this general remark, in [46] the following conjecture is presented.

Problem 3.1 (a conjecture). Let X be a real Banach space, $\Omega \subset X$ be an open, convex subset and $f : \Omega \to \Omega$ be an operator. We suppose that:

(i) $f \in C^1(\Omega, X)$;

(ii) $\rho(df^k(x)) < 1$, for all $x \in \Omega$ and all $k \in \mathbb{N}^*$;

(iii) $F_f \neq \emptyset$.

Then, f is a Picard operator.

In connection with the above conjecture the following problems arise:

Problem 3.2. In which conditions we have that:

$$\rho(df(x)) < 1, \text{ for all } x \in \Omega \;\Rightarrow\; \rho(df^k(x)) < 1, \text{ for all } x \in \Omega \text{ and all } k \in \mathbb{N}^*?$$

Problem 3.3. In which conditions we have that:

$$\rho(df(x)) < 1, \text{ for all } x \in \Omega \;\Rightarrow\; f \text{ is nonexpansive with respect to an equivalent norm on } X?$$

We remember that if $(X, \|\cdot\|)$ is a complex Banach space and $f : X \to X$ is a bounded linear operator with the spectrum $\sigma(f)$, then (see [17], [5], [14], [4], ...)

$$\rho(f) = \sup_{\lambda \in \sigma(f)} |\lambda| = \lim_{n\to\infty} \|f^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}^*} \|f^n\|^{\frac{1}{n}} = \inf_{|\cdot| \sim \|\cdot\|} |f|.$$

If X is a real Banach space and $f : X \to X$ is a bounded linear operator, $X_{\mathbb{C}}$ the complexification of X, $f_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$ the complexification of f, then by definition, $\rho(f) := \rho(f_{\mathbb{C}})$.

References: [46], [20], [4], [25], [26], ...

4. Nonexpansive operators and graphic contractions

Problem 4.1. Let $(X, \|\cdot\|)$ be a (real or complex) Banach space. Which nonexpansive operators $f : X \to X$ are graphic contractions ?

Commentaries: If f is a graphic contraction then $\inf_{x\in X} \|x - f(x)\| = 0$. If $\Omega \subset X$ is an invariant subset of f and f is a graphic contraction then, $\inf_{x\in X} \|x - f(x)\| = 0$. On the other hand, in the case of nonexpansive operators we have the following Goebel-Karlovitz Lemma (see [12]): Let $\Omega \subset X$ be a convex, closed and bounded subset. Let $D \subset \Omega$ be a weakly compact, convex, minimal invariant set for a nonexpansive operator $f : \Omega \to \Omega$. If for a sequence $\{x_n\}_{n\in\mathbb{N}}$, $\lim_{n\to\infty} \|x_n - f(x_n)\| = 0$, then for any $z \in D$, we have that, $\lim_{n\to\infty} \|z - x_n\| = diam(D)$.

So, the above problem is a hard one.

Problem 4.2. Let X be an ordered Banach space. Which increasing, linear and nonexpansive operators $f : X \to X$ are graphic contractions ?

Problem 4.3. Let X be a Banach space. Which multivalued nonexpansive operators $T : X \to P(X)$ are graphic contractions ?

References: [36], [40], [43], [45], [1], [2], [10], [16], [19], [18], [30], [39], [49], ...

5. Abstract and concrete Gronwall lemmas

Let $(X, \to, \leq)$ be an ordered L-space and $f : X \to X$ be an operator. The following results are well known (see [38]:

Lemma 5.1 (Abstract Gronwall Lemma for Picard operators). *We suppose that:*

(i) *f is a Picard operator ($F_f = \{x^*\}$);*

(ii) *f is an increasing operator.*

Then we have that:

(a) *$x \in X$, $x \leq f(x) \Rightarrow x \leq x^*$;*

(b) *$x \in X$, $x \geq f(x) \Rightarrow x \geq x^*$.*

Lemma 5.2 (Abstract Gronwall Lemma for weakly Picard operators). *We suppose that:*

(i) *f is a weakly Picard operator;*

(ii) *f is an increasing operator*

Then we have that:

(a) *$x \in X$, $x \leq f(x) \Rightarrow x \leq f^\infty(x)$;*

(b) *$x \in X$, $x \geq f(x) \Rightarrow x \geq f^\infty(x)$.*

The above abstract Gronwall lemmas are very usefully for giving some concrete Gronwall lemmas. On the other hand a large number of concrete Gronwall lemmas are obtained by direct proofs. The following problems are arising:

Problem 5.3. In which Gronwall lemmas the upper bounds are fixed points of the corresponding operator ?

Problem 5.4. If there are found solutions for the Problem 5.3, which of them are consequences of some abstract Gronwall lemmas ?

References: [38], [35], [21], [11], [22], [23], [33], [39], [49], ...

6. Invariant subsets with fixed point property

For a rigorous formulation of a problem (II), from Introduction, we recall a few basic notions and examples of the fixed point structure theory (see [37]).

Let $\mathcal{C}$ be a class of structured sets (ordered sets, ordered linear spaces, topological spaces, metric spaces, Hilbert spaces, Banach spaces, ordered Banach spaces, generalized metric spaces, ...). Let Set^* be the class of nonempty sets and if X is a nonempty set, then, $P(X) := \{Y \subset X \mid Y \neq \emptyset\}$. We also shall use the following notations:

$P(\mathcal{C}) := \{U \in P(X) \mid X \in \mathcal{C}\}$,
$\mathbb{M}(U,V) := \{f : U \to V \mid f \text{ is an operator}\}$,
$\mathbb{M}(U) := \mathbb{M}(U,U)$,
$S : \mathcal{C} \multimap Set^*$, $X \mapsto S(X) \subset P(X)$,
$M : D_M \subset P(\mathcal{C}) \times P(\mathcal{C}) \multimap \mathbb{M}(P(\mathcal{C}), P(\mathcal{C}))$, $(U,V) \mapsto M(U,V) \subset \mathbb{M}(U,V)$

By a fixed point structure (f.p.s.) on $X \subset \mathcal{C}$ we understand a triple $(X, S(X), M)$ with the following properties:

(i) $U \in S(X) \Rightarrow (U,U) \in D_M$;

(ii) $U \in S(X)$, $f \in M(U) \Rightarrow F_f \neq \emptyset$;

(iii) M is such that:

$$(Y,Y) \in D_M,\ Z \in P(Y),\ (Z,Z) \in D_M\ \Rightarrow M(Z) \supset \{f\big|_Z \mid f \in M(Y)\}.$$

Here are some examples of f.p.s.

Example 6.1 (The f.p.s. of progressive operators)**.** Let $\mathcal{C}$ be the class of partially ordered sets. For $(X, \leq) \in \mathcal{C}$, let

$$S(X) := \{Y \in P(X) \mid (Y, \leq) \text{ has at least a maximal element}\}$$

and

$$M(Y) := \{f : Y \to Y \mid x \leq f(x),\ \text{for all } x \in Y\}.$$

Then, $(X, S(X), M)$ is a f.p.s.

Example 6.2 (The Tarski's f.p.s.)**.** Let $\mathcal{C}$ be the class of partially ordered sets. For $(X, \leq) \in \mathcal{C}$, let

$$S(X) := \{Y \in P(X) \mid (Y, \leq) \text{ is a complete lattice}\}$$

and

$$M(Y) := \{f : Y \to Y \mid f \text{ is an increasing operator}\}.$$

Then, $(X, S(X), M)$ is a f.p.s.

Example 6.3 (The f.p.s. of contractions)**.** Let $\mathcal{C}$ be the class of complete metric spaces. Let

$$S(X) := \{Y \in P(X) \mid Y \text{ is closed}\}$$

and

$$M(Y) := \{f : Y \to Y \mid f \text{ is a contraction}\}.$$

Then, $(X, S(X), M)$ is a f.p.s.

Example 6.4 (The f.p.s. of Schauder)**.** Let $\mathcal{C}$ be the class of Banach spaces. Let

$$S(X) := \{Y \in P(X) \mid Y \text{ is compact and convex}\}$$

and

$$M(Y) := \{f : Y \to Y \mid f \text{ is continuous}\}.$$

Then, $(X, S(X), M)$ is a f.p.s.

Now, our problem (II) takes the following form:

Problem 6.5. Let $(X, S(X), M)$ be a f.p.s. on $X \in \mathcal{C}$ and $f : A \to A$ be an operator with $A \subset X$. In which conditions there exists $Y \subset A$ such that

(a) $Y \in S(X)$;

(b) $f(Y) \subset Y$;

(c) $f\big|_Y \in M(Y)$?

We have a similar problem in the case of multivalued operators.
References: [37], [41], [29], [49], ...

7. Strict fixed point problems

Let X be a nonempty set and $T : X \to P(X)$ be a multivalued operator. Let $F_T := \{x \in X \mid x \in T(x)\}$ be the set of fixed point of T and $(SF)_T := \{x \in X \mid T(x) = \{x\}\}$ be the strict fixed point set of T.
We have the following result (see [33], p.87):
Let (X, d) be a metric space and $T : X \to P(X)$ be a multivalued l-contraction. If, $(SF)_T \neq \emptyset$, then,

$$F_T = (SF)_T = \{x^*\}.$$

The following problem is arising:

Problem 7.1. For which multivalued generalized contractions we have that

$$(SF)_T \neq \emptyset \;\Rightarrow\; F_T = (SF)_T = \{x^*\} \text{ ?}$$

Problem 7.2. Let $(X, S(X), M^\circ)$ be a multivalued fixed point structure (see [37]) on $X \in \mathcal{C}$. Let $Y \in S(X)$ and $T \in M^\circ(Y)$. In which conditions we have that

$$F_T = (SF)_T?$$

Commentaries:

(1) Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that:

  (a) $F_f = F_g$;
  (b) $x \leq f(x) \leq g(x)$, for all $x \in \mathbb{R}$.

  Let $T : \mathbb{R} \to P(\mathbb{R})$ be defined by,

$$T(x) := \{tf(x) + (1-t)g(x) \mid t \in [0, 1]\}.$$

  Then we have that, $F_T = (SF)_T$.

(2) Let (X,d) be a metric space, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ be a partition of X, and for each $\lambda \in \Lambda$, $T_\lambda : X_\lambda \to P(X_\lambda)$ be a multivalued contraction with respect to the Pompeiu-Hausdorff functional. We suppose that, $(SF)_{T_\lambda} \neq \emptyset$, for all $\lambda \in \Lambda$.

Let $T : X \to P(X)$ be defined by, $T(x) = T_\lambda(x)$, if $x \in X_\lambda$, $\lambda \in \Lambda$.

It is clear that, $F_T = (SF)_T \neq \emptyset$.

(3) Let $(X, S(X), M)$ be a fixed point structure of progressive operators on a partially ordered set $(X, \leq)$. Let $Y \in S(X)$ and $f, g \in M(Y)$. We suppose that:

(a) $f(x) \leq g(x)$, for all $x \in Y$;

(b) $x < f(x)$, for each nonmaximal element $x \in Y$.

Let $T : Y \to P(Y)$ be a multivalued operator defined by,

$$T(x) := \{y \in Y \mid f(x) \leq y \leq g(x)\}.$$

Then, $F_T = (SF)_T \neq \emptyset$.

References: [34], [53], [28], [49], [31], ...

8. Commutative pairs of operators with coincidence property

Problem 8.1. Which are the f.p.s. $(X, S(X), M)$, $X \in \mathcal{C}$, with the following property:

$$Y \in S(X),\ f, g \in M(Y),\ f \circ g = g \circ f \ \Rightarrow \ \text{there exists } x \in Y \text{ such that } f(x) = g(x)?$$

Commentaries:

(1) In the case of Tarski's fixed point structure we have that, $F_f \cap F_g \neq \emptyset$.

(2) In the case of Schauder's fixed point structure, the Problem 8.1 takes the following form:

Conjecture 2 (Horn's Conjecture)**.** *Let X be a Banach space, $Y \subset X$, compact and convex subset and $f, g : Y \to Y$ be two continuous operators. If $f \circ g = g \circ f$, then there exists $x \in Y$ such that $f(x) = g(x)$.*

(3) The Horn's Conjecture includes:

Conjecture 3 (Schauder-Browder-Nussbaum Conjecture)**.** *Let X be a Banach space, $Y \subset X$ be a bounded, closed and convex subset and $f : Y \to Y$ be a continuous operator. If there exists $n_0 \in \mathbb{N}^*$ such that f^{n_0} is compact, then $F_f \neq \emptyset$.*

References: [37], [41], [15], [24], [18], [49], ...

References

[1] O. Agratini, I.A. Rus, *Iterates of some bivariate approximation process via weakly Picard operators*, Nonlinear Anal. Forum, 8 (2003), 159-168.

[2] O. Agratini, I.A. Rus, *Iterates of a class of discrete linear operators via contraction principle*, Comment. Math. Univ. Carolinae, 44 (2003), No. 3, 555-563.

[3] J. Andres, L. Górniewicz, *Topological Principles for Boundary Value Problems*, Kluwer, 2003.

[4] J. Appell, E. De Pascale, A. Vignoli, *Nonlinear Spectral Theory*, Walter de Gruyter, 2004.

[5] G.R. Belitskii, Yu.I. Lyubich, *Matrix Norm and their Applications*, Birkhäuser, 1988.

[6] V. Berinde, *Iterative Approximation of Fixed Points*, Springer, 2007.
[7] V. Berinde, Şt. Măruşter, I.A. Rus, *An abstract point of view on iterative approximation of fixed points of nonself operators*, J. Nonlinear and Convex Anal., 15 (2014), No. 5, 851-865.
[8] V. Berinde, A. Petruşel, I.A. Rus, M.A. Şerban, *The retraction-displacement condition in the theory of fixed point equation with a convergent iterative algorithm*, In: T.M. Rassias and V. Gupta (Eds.), Mathematical Analysis, Approximation Theory and Their Applications, Springer, 2016, 75-106.
[9] R.F. Brown, M. Furi, L. Górniewicz, B. Jiang (Eds.), *Handbook of Topological Fixed Point Theory*, Springer, 2005.
[10] T. Cătinaş, D. Otrocol, I.A. Rus, *The iterates of positive linear operators with the set of constant functions as the fixed point set*, Carpathian J. Math., 32 (2016), No. 2, 165-172.
[11] C. Crăciun, N. Lungu, *Abstract and concrete Gronwall lemmas*, Fixed Point Theory, 10 (2009), No. 2, 221-228.
[12] K. Goebel, *Problems I left behind*, In: Proc. 10^{th} IC-FPTA, 9-20, 2012, Cluj-Napoca.
[13] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer, 2003.
[14] R.B. Hollmess, *A formula for the spectral radius of an operator*, Amer. Math. Monthly, 75 (1968), 163-166.
[15] W.A. Horn, *Some fixed point theorems for compact maps and flows in Banach spaces*, Trans. Amer. Math. Soc., 149 (1970), 391-404.
[16] J. Jachymski, *Convergence of iterates of linear operators and the Kelisky-Rivlin type theorems*, Studia Math., 195 (2009), No. 2, 99-112.
[17] L.V. Kantorovich, G.P. Akilov, *Functional Analysis*, Pergamon Press, 1982.
[18] W.A. Kirk, *Metric fixed point theory: old problems and new directions*, Fixed Point Theory, 11 (2010), No. 1, 45-58.
[19] W.A. Kirk, B. Sims (Eds.), *Handbook of Metric Fixed Point Theory*, Kluwer, 2001.
[20] J.P. LaSalle, *The Stability of Dynamical Systems*, SIAM, 1976.
[21] N. Lungu, S.A. Ciplea, *Optimal Gronwall lemmas*, Fixed Point Theory, 18 (2017), No. 1, 293-304.
[22] N. Lungu, I.A. Rus, *On a functional Volterra-Fredholm integral equation via Picard operators*, J. Math. Ineq., 3 (2009), No. 4, 519-527.
[23] N. Lungu, I.A. Rus, *Gronwall inequalities via Picard operators*, An. Şt. Univ. "Al. I. Cuza" (Iaşi), Mat., 58 (2012), f.2, 269-278.
[24] R.D. Nussbaum, *The fixed point index and fixed point theorems for k-set-contractions*, Ph.D. Dissertation, Univ. of Chicago, 1969.
[25] J.M. Ortega, *Numerical Analysis*, Acad. Press, New York, 1972.
[26] J.M. Ortega, W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Acad. Press, New York, 1970.
[27] A. Petruşel, I.A. Rus, *An abstract point of view on iterative approximation schemes of fixed points for multivalued operators*, J. Nonlinear Sci. Appl., 6 (2013), 97-107.
[28] A. Petruşel, I.A. Rus, M.A. Şerban, *Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem for a multivalued operator*, J. Nonlinear and Convex Anal., 15 (2014), No. 3, 493-513.
[29] A. Petruşel, I.A. Rus, M.A. Şerban, *Fixed point structures, invariant operators, invariant partitions, and applications to Carathéodory integral equations*, In: P.M. Pardalos and T.M. Rassias (Eds.), Contributions in Mathematics and Engineering, Springer, 2016, ...
[30] A. Petruşel, I.A. Rus, M.A. Şerban, *Nonexpansive operators as graphic contractions*, J. Nonlinear Anal. and Convex Anal., 17 (2016), No. 7, 1409-1415.
[31] A. Petruşel, I.A. Rus, *Multivalued Picard and weakly Picard operators*, In: Fixed Point Theory and Applications (E. Llorens Fuster, J. Garcia Falset and B. Sims (Eds.)), Yokohama Publ., 2004, 207-226.
[32] A. Petruşel, I.A. Rus, J.-C. Yao, *Well-posedness in the generalized sense of the fixed point problems for multivalued operators*, Taiwanese J. Math., 11 (2007), No. 3, 903-914.
[33] I.A. Rus, *Generalized Contractions and Applications*, Cluj Univ. Press, Cluj-Napoca, 2001.
[34] I.A. Rus, *Strict fixed point theory*, Fixed Point Theory, 4 (2003), No. 2, 177-183.
[35] I.A. Rus, *Fixed points, upper and lower fixed points: abstract Gronwall lemmas*, Carpathian J. Math., 20 (2004), No. 1, 125-134.
[36] I.A. Rus, *Iterates of Bernstein operators, via contraction principle*, J. Math. Anal. Appl., 292 (2004), 259-264.
[37] I.A. Rus, *Fixed Point Structure Theory*, Cluj Univ. Press, Cluj-Napoca, 2006.
[38] I.A. Rus, *Gronwall lemmas: ten open problems*, Sci. Math. Jpn., 70 (2009), No. 2, 221-228.
[39] I.A. Rus, *Picard operators and applications*, Sci. Math. Jpn., 58 (2003), No. 1, 191-219.
[40] I.A. Rus, *Fixed point and interpolation point set of a positive linear operator on* $C(\overline{D})$, Stud. Univ. Babeş-Bolyai, Math., 55 (2010), No. 4, 243-248.
[41] I.A. Rus, *Five open problems in fixed point theory in terms of fixed point structures (I): singlevalued operators*, In: Proc. 10^{th} IC-FPTA, 39-60, 2012, Cluj-Napoca.
[42] I.A. Rus, *An abstract point of view on iterative approximation of fixed points: impact on the theory of fixed point equations*, Fixed Point Theory, 13 (2012), No. 1, 179-192.
[43] I.A. Rus, *Heuristic introduction to weakly Picard operator theory*, Creat. Math. Inform., 23 (2014), No. 2, 243-252.
[44] I.A. Rus, *Results and problems in Ulam stability of operatorial equations and inclusions*, In: T.M. Rassias (Ed.), Handbook of Functional Equations: Stability Theory, Springer, 2014, 323-352.
[45] I.A. Rus, *Iterates of increasing linear operators, via Maia's fixed point theorem*, Stud. Univ. Babeş-Bolyai, Math., 60 (2015),

No. 1, 91-98.
[46] I.A. Rus, *Remarks on a LaSalle conjecture on global asymptotic stability*, Fixed Point Theory, 17 (2016), No. 1, 159-172.
[47] I.A. Rus, *Some variants of contraction principle, generalizations and applications*, Stud. Univ. Babeş-Bolyai, Math., 61 (2016), No. 3, 343-358.
[48] I.A. Rus, *Relevant classes of weakly Picard operators*, An. Univ. Vest Timişoara, Mat.-Inform., 54 (2016), No. 2, 3-19.
[49] I.A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, Cluj Univ. Press, Cluj-Napoca, 2008.
[50] I.A. Rus, A. Petruşel, M.A. Şerban, *Weakly Picard operators: equivalent definitions, applications and open problems*, Fixed Point Theory, 7 (2006), No. 1, 3-22.
[51] I.A. Rus, A. Petruşel, A. Sîntămărian, *Data dependence of the fixed point setof some multivalued weakly Picard operators*, Nonlinear Anal., 52 (2003), 1947-1959.
[52] I.A. Rus, M.A. Şerban, *Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem*, Carpathian J. Math., 29 (2013), No. 2, 239-258.
[53] A. Sîntămărian, *Metrical strict fixed point theorems for multivalued mappings*, Seminar on Fixed Point Theory, Preprint No. 3, 1997, Babeş-Bolyai Univ., Cluj-Napoca, 27-30.
[54] M.A. Şerban, *Saturated fibre contraction principle*, Fixed Point Theory, 18 (2017), No. 2, 729-740.
[55] T. van der Walt, *Fixed and Almost Fixed Points*, Math. Centrum, Amsterdam, 1963.
[56] J. Wang, Y. Zhou, M. Medved, *Picard and weakly Picard operators techniques for nonlinear differential equations in Banach spaces*, J. Math. Anal. Appl., 389(2012), 261-274.
[57] D.Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Sci. Publ. Co., 2014.

Advances in the Theory of Nonlinear Analysis and its Applications 2 (2018) No. 1, 11–32
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications
ISSN: 2587-2648
Peer-Reviewed Scientific Journal

Fractional Relaxation Equations and a Cauchy Formula for Repeated Integration of the Resolvent

Leigh C. Becker[a], Ioannis K. Purnaras[b]

[a] *Department of Mathematics, Christian Brothers University, 650 E. Parkway South, Memphis, TN 38104, USA*
[b] *Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece*

Abstract

Cauchy's formula for repeated integration is shown to be valid for the function

$$R(t) = \lambda\Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q)$$

where λ and q are given positive constants with $q \in (0,1)$, Γ is the Gamma function, and $E_{q,q}$ is a Mittag-Leffler function. The function R is important in the study of Volterra integral equations because it is the unique continuous solution of the so-called *resolvent equation*

$$R(t) = \lambda t^{q-1} - \lambda \int_0^t (t-s)^{q-1}R(s)\,ds$$

on the interval $(0,\infty)$. This solution, commonly called the *resolvent*, is used to derive a formula for the unique continuous solution of the Riemann-Liouville fractional relaxation equation

$$D^q x(t) = -ax(t) + g(t) \qquad (a > 0)$$

on the interval $[0,\infty)$ when g is a given polynomial. This formula is used to solve a generalization of the equation of motion of a falling body. The last example shows that the solution of a fractional relaxation equation may be quite elementary despite the complexity of the resolvent.

Keywords: Cauchy's formula for repeated integration, fractional differential equations, Mittag-Leffler functions, relaxation equations, resolvents, Riemann-Liouville operators, Volterra integral equations
2010 MSC: 34A08, 34A12, 45D05, 45E10

Email addresses: `lbecker@cbu.edu` (Leigh C. Becker), `ipurnara@uoi.gr` (Ioannis K. Purnaras)

Received January 01, 2018, Accepted January 08, 2018.

1. Introduction

In classical physics, the ordinary differential equation

$$x'(t) = -ax(t) + g(t) \tag{1.1}$$

is sometimes called the *relaxation equation* (cf. [6, p. 138], [11]) when the constant a is positive. A generalization of (1.1) in a fractional calculus setting is the *fractional relaxation equation*

$$D^q x(t) = -ax(t) + g(t), \tag{1.2}$$

where D^q denotes a fractional differential operator of order q with $q \in (0,1)$ (cf. [6, p. 138]; [12, p. 292]; [17, p. 224]).

This paper is a study of (1.2) when $g(t)$ is a polynomial. For given $a > 0$ and g, we will prove that this equation has a unique continuous solution on the half-closed interval $[0, \infty)$ and that necessarily $x(0) = 0$. Furthermore, in Section 7, we will derive a formula that expresses this solution as a sum involving two-parameter Mittag-Leffler functions (cf. (7.13)). Moreover, we will show that each term of this sum can also be expressed as a convolution integral involving the solution of the integral equation

$$R(t) = \lambda t^{q-1} - \lambda \int_0^t (t-s)^{q-1} R(s)\, ds, \tag{R_λ}$$

where λ is a positive constant related to the value of the constant a. In fact, it is well-established that (R_λ) has a unique continuous solution on the interval $(0, \infty)$ whenever λ and q are positive constants with $q \in (0,1)$. A proof of this for a more general version of equation (R_λ) can be found in the 1971 monograph by Miller [14, Ch. IV].

There is also the recent paper [3] that investigates (R_λ) directly. Not only is the existence and uniqueness of a continuous solution of (R_λ) on $(0, \infty)$ proven there but also a formula for it is derived, namely

$$R(t) = \lambda \Gamma(q) t^{q-1} E_{q,q}(-\lambda \Gamma(q) t^q), \tag{1.3}$$

where $E_{\alpha,\beta}$ $(\alpha, \beta > 0)$ denotes the two-parameter Mittag-Leffler function:

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{1.4}$$

We use the following established terminology: *resolvent equation* refers to equation (R_λ) and *resolvent* refers to its solution (1.3).

For given constants $\lambda > 0$ and $q \in (0,1)$, important characteristics of the resolvent (1.3) are:

(i) For all $t > 0$, $0 < R(t) \le \left(\dfrac{q}{q + \lambda t^q} \right) \lambda t^{q-1}$.

(ii) $R(t) \to \infty$ as $t \to 0^+$ and $R(t) \to 0$ as $t \to \infty$.

(iii) The graph of R is decreasing and concave upward on $(0, \infty)$. In fact, R is *completely monotone* on $(0, \infty)$. That is, $R(t)$ is infinitely differentiable on $(0, \infty)$ and $(-1)^k R^{(k)}(t) \ge 0$ for all $t > 0$ and for $k = 0, 1, 2, \ldots$.

(iv) For all $t > 0$,
$$\frac{1}{1 + \frac{q}{\lambda t^q}} \le \int_0^t R(s)\, ds \le 1 - e^{-\frac{\lambda t^q}{q}}.$$

(v) $\int_0^\infty R(s)\, ds = 1$.

(vi) For given $\lambda > 0$, R is the unique continuous solution on $(0, \infty)$ of the initial value problem

$$D^q x(t) = -\lambda\Gamma(q)x(t), \quad \lim_{t\to 0^+} t^{1-q}x(t) = \lambda.$$

Items (i) and (ii) are proved in [3, Cor. 4.6, Thm. 7.3]. In [14, Thm. 7.2], Miller states that the solution of the above-mentioned general version is completely monotone; for a proof of this, he references [7] (cf. [14, p. 243]). A proof for the less general (1.3) that is based on the complete monotonicity of $E_{q,q}(-t)$ for $t \geq 0$ is given in [3, pp. 29–30]. Item (iv) is proved in [3, Thm. 4.5]. Clearly (iv) implies (v). A proof of (vi) is found in [3, Thm. 5.2].

The resolvent (1.3) is also expressed in terms of classical functions of mathematical physics in [3] and [5] for the following values of q. For $q = 1/2$, it is shown in [3, (6.12)] that

$$\begin{aligned} R(t) &= \lambda\Gamma(\tfrac{1}{2})t^{-1/2}E_{\frac{1}{2},\frac{1}{2}}(-\lambda\Gamma(\tfrac{1}{2})t^{1/2}) \\ &= \frac{\lambda}{\sqrt{t}} - \pi\lambda^2 e^{\pi\lambda^2 t}\big(1 - \operatorname{erf}(\lambda\sqrt{\pi t})\big), \end{aligned} \tag{1.5}$$

where $\operatorname{erf}(\cdot)$ is the error function (cf. (4.14)). In [5, (5.7)], after adjusting the notation there to be in accord with this paper, we find for $q = 1/3$ the formula

$$\begin{aligned} R(t) = \frac{\lambda}{(\sqrt[3]{t})^2} &- \frac{\sqrt{3}\,\sigma}{2\pi\lambda\sqrt[3]{t}} \\ &+ \sigma^3 e^{-\sigma t}\left[1 + \frac{1}{\Gamma(\frac{1}{3})}\gamma(\tfrac{1}{3}, -\sigma t) + \frac{\sqrt{3}}{2\pi}\,\Gamma(\tfrac{1}{3})\gamma(\tfrac{2}{3}, -\sigma t)\right] \end{aligned} \tag{1.6}$$

where $\sigma := \big[\lambda\Gamma(\frac{1}{3})\big]^3$ and $\gamma(\cdot,\cdot)$ denotes the lower incomplete gamma function (cf. (4.20)). In Figure 1 of Section 4 the solid [resp. dashed] concave-upward curve is the graph of (1.5) [resp. (1.6)].

2. Riemann-Liouville Operators

For a function f that is (Riemann) integrable, we employ the integral operator J defined by

$$Jf(t) := \int_0^t f(s)\,ds.$$

Furthermore, for $n \in \mathbb{N}$ (set of natural numbers), let the operator J^n denote the n-fold iterate of J; that is,

$$J^n := JJ^{n-1} \quad \text{for} \quad n \geq 1$$

where $J^0 := I$, the identity operator. For example, taking $n = 2$ and applying J^2 to an integrable function f, we have

$$J^2 f(t) = \int_0^t Jf(s)\,ds = \int_0^t \left(\int_0^s f(u)\,du\right) ds = \int_0^t \left(\int_0^{t_2} f(t_1)\,dt_1\right) dt_2$$

or

$$J^2 f(t) = \int_0^t dt_2 \int_0^{t_2} f(t_1)\,dt_1.$$

In general,

$$J^n f(t) = \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_3} dt_2 \int_0^{t_2} f(t_1)\,dt_1. \tag{2.1}$$

This particular iterated integral can be expressed in terms of a single integral with a weighted integrand as in (2.2) below. It is known as *Cauchy's formula for repeated integration* (cf. [16, p. 38]). This formula is found in Abramowitz and Stegun's handbook [1, (25.4.58)]; and in some textbooks, such as [8, p. 487], it appears as an exercise. We omit its proof here because it is basically the mathematical induction argument that is used in the proof of Theorem 3.2 in Section 3.

Theorem 2.1 (Cauchy's formula for repeated integration)**.** *Let $n \in \mathbb{N}$. If f is integrable on $[0,T]$, then*

$$J^n f(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s)\, ds \tag{2.2}$$

for $t \in [0,T]$.

Now let us extend the values of n in (2.2) from $\mathbb{N}$ to $\mathbb{R}^+$ (set of strictly positive real numbers) by replacing $(n-1)!$ with $\Gamma(n)$, where Γ denotes the Gamma function. This leads to the well-known definition of the Riemann-Liouville integral operator of order n.

Definition 2.2. For any $n \in \mathbb{R}^+$, J^n denotes the integral operator

$$J^n f(t) := \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} f(s)\, ds, \tag{2.3}$$

where f denotes a function for which the integral exists. J^n is called the *Riemann-Liouville fractional integral operator of order* n. Furthermore, J and J^0 denote the operators

$$J := J^1 \text{ and } J^0 := I \tag{2.4}$$

where I denotes the identity operator.

Just as the integral operator J^n can be defined for all values of $n \in \mathbb{R}^+$, the same is true of D^n, namely, the classical ordinary differential operator of order $n \in \mathbb{N}$. That is, for an n-times differentiable function f,

$$Df(t) := \frac{d}{dt} f(t),\ D^2 f(t) := \frac{d^2}{dt^2} f(t),\ \dots\ , D^n f(t) := \frac{d^n}{dt^n} f(t).$$

This can be expressed recursively as follows:

$$D^n := DD^{n-1} \quad \text{for} \quad n \geq 2,$$

where $D^1 := D$ and $D^0 := I$, the identity operator.

In the following extension of the definition of D^n, we employ the *floor function* $\lfloor \cdot \rfloor$, where $\lfloor n \rfloor$ denotes the largest integer less than or equal to n.

Definition 2.3. For a given $n \in \mathbb{R}^+$, D^n denotes the differential operator

$$D^n f := D^m J^{m-n} f \tag{2.5}$$

where $m = \lfloor n \rfloor + 1$ and f denotes a function for which the right-hand side exists. For $n = 0$, $D^n f := f$. D^n is called the *Riemann-Liouville fractional differential operator of order* n (cf. [6, p. 27]).

Remark 2.4. The symbol D^n on the left-hand side of (2.5) denotes the fractional differential operator of order n whereas D^m on the right-hand side denotes the ordinary differential operator d^m/dt^m since $m \in \mathbb{N}$. If $n \in \mathbb{N}$, then $m = \lfloor n \rfloor + 1 = n + 1$; so

$$D^n f = D^{n+1} J^1 f = D^n DJf = D^n If = D^n f.$$

Thus the definition of the operator D^n is well-defined.

Remark 2.5. Combining (2.3) and (2.5), we obtain the form

$$D^n f(t) = \frac{1}{\Gamma(m-n)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-n-1} f(s)\, ds \tag{2.6}$$

that is found in well-known works such as [10, (2.1.10) on p. 70].

Example 2.6. Let $n = q$ where $0 < q < 1$. Then

$$m = \lfloor q \rfloor + 1 = 1 \quad \text{and} \quad m - n = 1 - q.$$

Consequently,

$$D^q f(t) = D^1 J^{1-q} f(t) = DJ^{1-q} f(t)$$

or

$$D^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} f(s)\, ds. \tag{2.7}$$

3. Cauchy's Formula for the Resolvent

Theorem 3.1. *Let $n \in \mathbb{R}^+$. Let R be the resolvent, namely, the unique continuous solution of* (R_λ) *on* $(0, \infty)$. *Then*

$$\int_0^t \int_0^s (s-u)^{n-1} R(u)\, du\, ds = \int_0^t \int_u^t (s-u)^{n-1} R(u)\, ds\, du$$

for all $t > 0$.

Proof. See the proof of Theorem 4.3 in [3], where the Tonelli-Hobson test ([2, p. 415], [15, p. 93]) is employed; and note that the proof is valid not only for $n \in (0,1)$ but for all $n > 0$. □

We now use this theorem to show that Cauchy's formula for repeated integration can be applied to the resolvent $R(t)$, notwithstanding the singularity at $t = 0$.

Theorem 3.2. *For $n \in \mathbb{N}$,*

$$J^n R(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} R(s)\, ds \tag{3.1}$$

for $t \geq 0$.

Proof. For a given $t \geq 0$, it is well-known (cf. [14, Ch. IV]) that the resolvent integral function

$$JR(t) = \int_0^t R(s)\, ds \tag{3.2}$$

exists. Furthermore, it is proven in [3, Thm. 9.5] that

$$JR(t) = 1 - E_q(-\lambda \Gamma(q) t^q)$$

where E_α, for $\alpha \in \mathbb{R}^+$, denotes the one-parameter Mittag-Leffler function, which is defined by

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{3.3}$$

Since $E_q(z)$ is an entire function of z (cf. [6, Thm. 4.1]) in the complex plane, it follows that

$$J^n R(t) = \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} R(t_1)\, dt_1$$

exists and is continuous on $[0, \infty)$ for each $n \in \mathbb{N}$.

Note that (3.1) simplifies to (3.2) when $n = 1$. We complete the proof using mathematical induction to establish that (3.1) is true for all $n \in \mathbb{N}$. Suppose that (3.2) is also true when $n = k$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} J^{k+1} R(t) = JJ^k R(t) &= \int_0^t J^k R(s)\, ds \\ &= \int_0^t \left[\frac{1}{(k-1)!} \int_0^s (s-u)^{k-1} R(u)\, du \right] ds. \end{aligned}$$

Interchanging the order of integration and appealing to Theorem 3.1, we have

$$\begin{aligned} J^{k+1}R(t) &= \frac{1}{(k-1)!}\int_0^t \left(\int_u^t (s-u)^{k-1} ds\right) R(u)\,du \\ &= \frac{1}{(k-1)!}\int_0^t \left[\frac{(s-u)^k}{k}\right]_{s=u}^{s=t} R(u)\,du \\ &= \frac{1}{(k-1)!}\int_0^t \left[\frac{(t-u)^k}{k}\right] R(u)\,du = \frac{1}{k!}\int_0^t (t-u)^k R(u)\,du, \end{aligned}$$

which is precisely (3.1) when $n = k+1$. Thus, as (3.1) is true for $n = 1$, it must be true for all $n \in \mathbb{N}$. □

The following result will be needed in the next section to prove Lemma 4.2. Although the proof is straightforward, it can be found in a number of places (e.g., [6, p. 28]).

Lemma 3.3. *Let $q \in (0,1)$ and $p > -1$. If $p \neq q-1$, then*

$$D^q t^p = \frac{\Gamma(p+1)}{\Gamma(p-q+1)} t^{p-q} \tag{3.4}$$

for $t > 0$. If $p = q-1$, then $D^q t^p = 0$ for $t > 0$.

4. Solution of a fractional relaxation equation

The following proof is an adaptation of a proof in [6, Thm. 2.14].

Theorem 4.1. *Let $n \in \mathbb{R}_0^+$, where $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$. If a function f is continuous and absolutely integrable on an interval $(0,T]$, then*

$$D^n J^n f(t) = f(t) \tag{4.1}$$

for all $t \in (0,T]$.

Proof. This is trivially true for $n = 0$ since by definition $J^0 := I$ and $D^0 := I$. It is also true for $n = 1$ because by the Fundamental Theorem of Calculus

$$D^1 J^1 f(t) = DJf(t) = \frac{d}{dt}\int_0^t f(s)\,ds = f(t)$$

for $0 < t \leq T$. It follows from this and an induction argument that (4.1) is true for all $n \in \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Now consider (4.1) for a given $n > 0$ when it is not a positive integer. Then, by Definition 2.3,

$$D^n J^n f = D^m J^{m-n} J^n f$$

where $m = \lfloor n \rfloor + 1$. Since f by hypothesis is continuous and absolutely integrable on $(0,T]$ and $m+n \geq 1$, we have

$$J^{m-n} J^n f(t) = J^{(m-n)+n} f(t) = J^m f(t)$$

for $0 \leq t \leq T$ by [4, Lemma 4.8]. As a result, since $m \in \mathbb{N}$,

$$D^n J^n f(t) = D^m J^m f(t) = f(t)$$

for $0 < t \leq T$. □

The following result relates solutions of (1.2) to those of a Volterra integral equation when $g(t) \equiv b$, a constant. It will be extended to all polynomials in Section 7.

Lemma 4.2. *Let a, b, and q be constants with $a > 0$, $b \in \mathbb{R}$, and $q \in (0,1)$. If there is a continuous solution of the fractional relaxation equation*

$$D^q x(t) = -ax(t) + b \tag{4.2}$$

on the interval $[0,\infty)$, then it is also a solution of

$$x(t) = \beta t^q - \lambda \int_0^t (t-s)^{q-1} x(s)\,ds \tag{4.3}$$

on $[0,\infty)$ when β and λ have the values

$$\beta = \frac{b}{\Gamma(q+1)} \quad \textit{and} \quad \lambda = \frac{a}{\Gamma(q)}. \tag{4.4}$$

Conversely, let $\beta \in \mathbb{R}$ and $\lambda > 0$ and suppose there is a continuous solution of the integral equation (4.3) on $[0,\infty)$. Then it is also a continuous solution of (4.2) on $[0,\infty)$ when

$$a = \lambda\Gamma(q) \quad \textit{and} \quad b = \beta\Gamma(q+1). \tag{4.5}$$

Proof. Let $\beta \in \mathbb{R}$ and $\lambda > 0$ be given constants. Then let a and b be defined by (4.5). Suppose there is a continuous function $x(t)$ that satisfies the integral equation (4.3) on $[0,\infty)$. Expressing this in terms of the Riemann-Liouville integral operator (2.3), we obtain

$$x(t) = \beta t^q - \lambda\Gamma(q) \cdot \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s)\,ds = \beta t^q - aJ^q x(t).$$

Applying the Riemann-Liouville differential operator D^q to this and using Theorem 4.1, we get

$$D^q x(t) = \beta\Gamma(q+1) - ax(t) = b - ax(t)$$

since $D^q t^q = \Gamma(1+q)$ (cf. Lemma 3.3). In other words, the function $x(t)$ must also be a solution of (4.2) on $[0,\infty)$. Note from (4.3) that $x(0) = 0$.

Now let $a > 0$ and $b \in \mathbb{R}$ be given constants. Then define constants $\beta \in \mathbb{R}$ and $\lambda > 0$ by (4.4) and suppose $x(t)$ is a continuous function satisfying (4.2) on $[0,\infty)$. And so

$$DJ^{1-q} x(t) = -ax(t) + b$$

since $D^q = DJ^{1-q}$. For a fixed $t > 0$, let $\eta \in (0,t)$. The integration

$$\int_\eta^t \frac{d}{ds} J^{1-q} x(s)\,ds = \int_\eta^t (-ax(s) + b)\,ds$$

yields

$$J^{1-q} x(t) - J^{1-q} x(\eta) = -a \int_\eta^t x(s)\,ds + b(t-\eta). \tag{4.6}$$

Lemma 3.1 in [3, p. 5] implies that

$$\lim_{\eta\to 0^+} J^{1-q} x(\eta) = \frac{1}{\Gamma(1-q)} \lim_{\eta\to 0^+} \int_0^\eta (\eta-s)^{-q} x(s)\,ds = 0. \tag{4.7}$$

Because of this and the continuity of x on $[0,\infty)$, we obtain

$$J^{1-q} x(t) = -a \int_0^t x(s)\,ds + bt = -aJx(t) + bt$$

upon taking the limit of both sides of (4.6) as $\eta \to 0^+$. Then the application of D^{1-q} yields

$$D^{1-q}J^{1-q}x(t) = -aD^{1-q}Jx(t) + bD^{1-q}t,$$

which because of Theorem 4.1 and (2.5) simplifies to

$$x(t) = -aDJ^qJx(t) + bD^{1-q}t. \tag{4.8}$$

It follows from [4, Lemma 4.8] that

$$DJ^qJx(t) = DJJ^qx(t) = J^qx(t)$$

and from Lemma 3.3 that

$$D^{1-q}t = \frac{1}{\Gamma(q+1)}t^q.$$

Therefore, we conclude from (4.8) that any continuous solution of (4.2) on $[0,\infty)$ must also be a solution of

$$\begin{aligned} x(t) &= -aJ^qx(t) + \frac{b}{\Gamma(q+1)}t^q \\ &= -\frac{a}{\Gamma(q)}\int_0^t (t-s)^{q-1}x(s)\,ds + \beta t^q = \beta t^q - \lambda\int_0^t (t-s)^{q-1}x(s)\,ds. \end{aligned}$$

Moreover, we see from this integral equation that $x(0) = 0$. □

Remark 4.3. Observe in the statement of Lemma 4.2 that no initial condition accompanies the fractional differential equation (4.2). At first this may appear to be an oversight until we realize from the proof that positing the existence of a continuous solution $x(t)$ of (4.2) for $t \geq 0$ implies $x(0) = 0$.

With the next theorem we complete what was initiated with Lemma 4.2 and that is to show that (4.2) and (4.3) do in fact share the same continuous solution on $[0,\infty)$. But first let us dispose of the special case $b = 0$.

Lemma 4.4. *There is one and only one continuous solution of*

$$D^qx(t) = -ax(t) \quad (a > 0) \tag{4.9}$$

on $[0,\infty)$*; it is the trivial solution* $x(t) \equiv 0$.

Proof. It follows from Lemma 4.2 that any continuous solution of (4.9) on $[0,\infty)$ must also be a continuous solution of

$$x(t) = -\lambda\int_0^t (t-s)^{q-1}x(s)\,ds$$

where $\lambda = a/\Gamma(q)$. But the only solution of this integral equation is $x(t) \equiv 0$ (cf. [3, p. 15]). □

Theorem 4.5. *For given constants* $a > 0$, $b \in \mathbb{R}$, *and* $q \in (0,1)$, *the fractional relaxation equation* (4.2) *has one and only one continuous solution on* $[0,\infty)$, *namely*

$$x(t) = \frac{b}{a}\int_0^t R(s)\,ds = \frac{b}{a}\left[1 - E_q(-at^q)\right], \tag{4.10}$$

where R *denotes the resolvent corresponding to* $\lambda = a/\Gamma(q)$. *This is also the unique continuous solution of the integral equation* (4.3) *on* $[0,\infty)$ *when* β *and* λ *have the values given by* (4.4).

Proof. First consider the integral equation (4.3) where $\beta \in \mathbb{R}$ and $\lambda > 0$ are given constants. From [3, Thm. 8.3] we know that if a function f is continuous on the interval $[0, \infty)$, then

$$x(t) = f(t) - \lambda \int_0^t (t-s)^{q-1} x(s)\, ds$$

has a unique continuous solution on $[0, \infty)$. Moreover, this solution is given by the linear variation of parameters formula:

$$x(t) = f(t) - \int_0^t R(t-s) f(s)\, ds.$$

Taking $f(t) = \beta t^q$, this becomes

$$x(t) = \beta t^q - \int_0^t R(t-s)\beta s^q\, ds = \beta t^q - \beta \int_0^t (t-u)^q R(u)\, du. \tag{4.11}$$

In other words, this is the unique continuous solution of (4.3) on $[0, \infty)$.

But we can simplify (4.11) as follows: integrating the resolvent equation (R_λ) and interchanging the order of integration (cf. Thm. 3.1), we obtain

$$\int_0^t R(s)\, ds = \frac{\lambda}{q} t^q - \frac{\lambda}{q} \int_0^t (t-u)^q R(u)\, du.$$

Thus,

$$\int_0^t (t-u)^q R(u)\, du = t^q - \frac{q}{\lambda} \int_0^t R(s)\, ds.$$

Substituting this into (4.11) and defining a and b by (4.5), we get

$$\begin{aligned} x(t) &= \beta t^q - \beta \left[t^q - \frac{q}{\lambda} \int_0^t R(s)\, ds \right] = \beta q \cdot \frac{1}{\lambda} \int_0^t R(s)\, ds \\ &= \frac{bq}{\Gamma(q+1)} \cdot \frac{\Gamma(q)}{a} \int_0^t R(s)\, ds = \frac{b}{a} \int_0^t R(s)\, ds. \end{aligned}$$

In [3, Thm. 9.5], we find the formula

$$\int_0^t R(s)\, ds = 1 - E_q(-at^q).$$

Therefore (4.10) is the unique continuous solution of (4.3). Moreover, Lemma 4.2 implies that it is also the unique continuous solution of (4.2) on $[0, \infty)$. □

Remark 4.6. We have shown that there is one and only one continuous solution $x(t)$ of the fractional relaxation equation (4.2) on the half-closed interval $[0, \infty)$. Moreover, from (4.10) we see that $x(0) = 0$. Thus, the initial value problem

$$D^q x(t) = -ax(t) + b, \quad x(0) = x_0$$

has no continuous solution on $[0, \infty)$ unless $x_0 = 0$.

Also, observe that if we formally let $q = 1$ in (4.10), then it simplifies to

$$x(t) = \frac{b}{a}\left[1 - E_1(-at)\right] = \frac{b}{a}\left(1 - e^{-at}\right) \tag{4.12}$$

since $E_1(z) = e^z$ (cf. (3.3)). Note that this is the unique continuous solution of the classical initial value problem

$$x'(t) = -ax(t) + b, \quad x(0) = 0.$$

Corollary 4.7. *If $b \neq 0$, then solution* (4.10) *has the following properties:*

(i) $x(0) = 0$.
(ii) $\lim_{t\to\infty} x(t) = b/a$.
(iii) *If $b > 0$ ($b < 0$), then $x(t)$ is strictly increasing (decreasing) on $[0, \infty)$ and $x(t) > 0$ ($x(t) < 0$) for all $t > 0$.*
(iv) *If $b > 0$ ($b < 0$), then $x(t)$ is concave downward (upward) on $(0, \infty)$.*
(v) *If $b > 0$, then the derivative $x'(t)$ is completely monotone on $(0, \infty)$, whereas $-x'(t)$ is completely monotone on $(0, \infty)$ if $b < 0$.*

Proof. Properties (i) and (ii) follow from (4.10) from which we see that $x(0) = 0$ and

$$\lim_{t\to\infty} x(t) = \frac{b}{a} \lim_{t\to\infty} \int_0^t R(s)\, ds = \frac{b}{a} \int_0^\infty R(s)\, ds = \frac{b}{a}.$$

Since the derivative of (4.10) is

$$x'(t) = \frac{b}{a} \frac{d}{dt} \int_0^t R(s)\, ds = \frac{b}{a} R(t), \tag{4.13}$$

it follows that $x'(t) > 0$ if $b > 0$. And so $x(t)$ is strictly increasing on $[0, \infty)$. This together with $x(0) = 0$ implies that $x(t) > 0$ for $t > 0$. Likewise, if $b < 0$, then $x(t)$ is strictly decreasing on $[0, \infty)$ and $x(t) < 0$ for $t > 0$. This concludes the proof of (iii).

To prove (iv), we use the result stated in Section 1 that the resolvent R is a completely monotone function on $(0, \infty)$. Thus,

$$x''(t) = \frac{b}{a} R'(t).$$

And so $x''(t) \leq 0$ if $b > 0$ and $x''(t) \geq 0$ if $b < 0$.

Finally, (v) follows from (4.13) and the complete monotonicity of R. □

Example 4.8. We illustrate some of the properties of solutions of (4.2) that are enumerated in Corollary 4.7 by choosing two different values of q and graphing the corresponding solutions (4.10). For both values, let $a = b = 1$.

First let $q = 1/2$. Then (4.10) is

$$x(t) = 1 - E_{1/2}(-\sqrt{t}).$$

According to [10, (1.8.6)],

$$E_{1/2}(z) = e^{z^2} [1 + \operatorname{erf}(z)]$$

where $\operatorname{erf}(z)$ is the *error function*:

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2}\, du. \tag{4.14}$$

Thus,

$$x(t) = 1 - e^t \left[1 + \operatorname{erf}(-\sqrt{t})\right] = 1 - e^t + e^t \operatorname{erf}(\sqrt{t}) \tag{4.15}$$

is the unique continuous solution of

$$D^{1/2} x(t) = -x(t) + 1 \tag{4.16}$$

on $[0, \infty)$. The graph of (4.15) is the solid concave-downward curve in Figure 1. (All the graphs in this paper were created with Maple$^{\text{TM}}$ 17.)

Now let $q = 1/3$. By (4.10) the unique continuous solution of

$$D^{1/3} x(t) = -x(t) + 1 \tag{4.17}$$

on $[0, \infty)$ is

$$x(t) = 1 - E_{1/3}(-\sqrt[3]{t}). \tag{4.18}$$

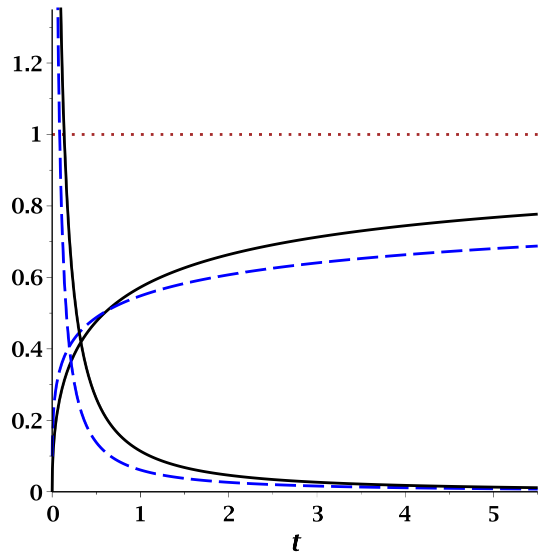

Figure 1: Solutions (1.5), (1.6), (4.15) and (4.21).

The second term can be calculated with the help of formula (1.8.5) in [10]: for $m = 2, 3, 4, \dots,$

$$E_{1/m}(z) = e^{z^m}\left[1 + m\int_0^z e^{-u^m}\left(\sum_{k=1}^{m-1}\frac{u^{k-1}}{\Gamma(k/m)}\right)du\right].$$

Consequently,

$$\begin{aligned} E_{1/3}(t) &= e^{t^3}\left[1 + 3\int_0^t e^{-u^3}\left(\frac{1}{\Gamma(1/3)} + \frac{u}{\Gamma(2/3)}\right)du\right] \\ &= e^{t^3}\left[1 + \frac{3}{\Gamma(1/3)}\int_0^t e^{-u^3}\,du + \frac{3}{\Gamma(2/3)}\int_0^t u\,e^{-u^3}\,du\right]. \end{aligned} \tag{4.19}$$

We can also express the solution (4.18) in terms of the *lower incomplete gamma function* $\gamma(a, z)$, namely

$$\gamma(a, z) := \int_0^z u^{a-1}e^{-u}\,du. \tag{4.20}$$

Changing the variable of integration to $z = u^3$, we obtain

$$\int_0^t e^{-u^3}\,du = \frac{1}{3}\int_0^{t^3} z^{-2/3}e^{-z}\,dz = \frac{1}{3}\gamma(1/3, t^3).$$

Likewise, the same change of variable yields

$$\int_0^t u\,e^{-u^3}\,du = \frac{1}{3}\int_0^{t^3} z^{-1/3}e^{-z}\,dz = \frac{1}{3}\gamma(2/3, t^3).$$

Thus,

$$E_{1/3}(t) = e^{t^3}\left[1 + \frac{\gamma(1/3, t^3)}{\Gamma(1/3)} + \frac{\gamma(2/3, t^3)}{\Gamma(2/3)}\right].$$

Therefore, an alternative form of (4.18) is

$$x(t) = 1 - e^{-t}\left[1 + \frac{\gamma(1/3, -t)}{\Gamma(1/3)} + \frac{\gamma(2/3, -t)}{\Gamma(2/3)}\right]. \tag{4.21}$$

Its graph is the concave-downward dashed curve in Figure 1.

5. Generalization of the equation of motion of a falling body

Consider the vertical downward motion of a body of mass m when it is released from rest above the ground. Besides the force due to gravity and the resisting force (*drag force*) exerted on the body as it moves through the air, assume that all other forces acting on the body are negligible. Also, let us suppose that the drag force is proportional to the velocity v of the body. Since the direction of the drag force is opposite that of the velocity of the body (downward, which we take as negative), the drag force F_d points upward. Thus $F_d = -kmv$, where the proportionality constant $k > 0$. Since the gravitational force acting on the body is $F_g = -mg$, Newton's second law of motion yields

$$m\frac{dv}{dt} = F_g + F_d = -mg - kmv.$$

Hence, we obtain the familiar classical equation of motion

$$\frac{dv}{dt} = -kv - g, \quad v(0) = 0 \tag{5.1}$$

that is found in most undergraduate physics textbooks, such as [13, p. 68]. Solving (5.1) by either separating variables or using the integrating factor e^{kt}, we obtain the solution

$$v(t) = -\frac{g}{k} + \frac{g}{k}e^{-kt}. \tag{5.2}$$

Now suppose we generalize the equation of motion (5.1) by replacing the classical differential operator d/dt with the Riemann-Liouville operator D^q. But this does not make complete sense due to the dimensional inconsistency of the units, where we see from (5.1) that k has the dimension of inverse time. However, we can rectify this with the replacement

$$\frac{d}{dt} \to k^{1-q}D^q$$

suggested by Rosales et al. in [18, p. 519]. (Actually they use the Caputo fractional derivative; however, since $q \in (0, 1)$ and the initial condition is $v(0) = 0$, the Caputo and Riemann-Liouville derivatives are equivalent.) Consequently, the fractional generalization of (5.1) is

$$k^{1-q}D^q v = -kv - g, \quad v(0) = 0$$

or

$$D^q v = -k^q v - k^{q-1}g, \quad v(0) = 0. \tag{5.3}$$

From Theorem 4.5 we see that the unique continuous solution of (5.3) is

$$v(t) = -\frac{g}{k} + \frac{g}{k}E_q(-(kt)^q). \tag{5.4}$$

This agrees with the velocity formula in [18]. Note that by formally letting $q = 1$, (5.4) simplifies to (5.2) because of (3.3).

6. Repeated integration of the resolvent

Lemma 6.1. *Let $R(t)$ be the resolvent, namely, the unique continuous solution of* (R_λ) *on* $(0, \infty)$*. Then*

$$\int_0^t R(s)\,ds = \frac{\lambda}{q}\left[t^q - \int_0^t (t-u)^q R(u)\,du\right] \tag{6.1}$$

for $t \geq 0$*.*

Proof. Integrating (R_λ) and interchanging the order of integration (cf. Thm. 3.1), we get

$$\begin{aligned}\int_0^t R(s)\,ds &= \lambda\int_0^t s^{q-1}\,ds - \lambda\int_0^t\int_0^s (s-u)^{q-1}R(u)\,du\,ds\\ &= \frac{\lambda}{q}t^q - \lambda\int_0^t\left(\int_u^t (s-u)^{q-1}\,ds\right)R(u)\,du\\ &= \frac{\lambda}{q}t^q - \frac{\lambda}{q}\int_0^t (t-u)^q R(u)\,du.\end{aligned}$$

□

Theorem 6.2. *Let $n\in\mathbb{N}$. The nth repeated integral of the resolvent $R(t)$, namely*

$$J^nR(t) = \int_0^t dt_n\int_0^{t_n} dt_{n-1}\cdots\int_0^{t_2} R(t_1)\,dt_1,$$

is given by the formula

$$J^nR(t) = \frac{\lambda\Gamma(q)}{\Gamma(q+n)}\left[t^{q+n-1} - \int_0^t (t-u)^{q+n-1}R(u)\,du\right] \tag{6.2}$$

for $t\geq 0$.

Proof. It follows from Lemma 6.1 that formula (6.2) holds for $n=1$. Let us show via a proof by induction that it holds for all $n\in\mathbb{N}$.

Suppose for some $k\in\mathbb{N}$ that (6.2) holds for $n=k$. Then

$$\begin{aligned}J^{k+1}R(t) &= \int_0^t J^kR(s)\,ds\\ &= \frac{\lambda\Gamma(q)}{\Gamma(q+k)}\int_0^t\left[s^{q+k-1} - \int_0^s (s-u)^{q+k-1}R(u)\,du\right]ds\\ &= \frac{\lambda\Gamma(q)}{\Gamma(q+k)}\left[\frac{t^{q+k}}{q+k} - \int_0^t\int_0^s (s-u)^{q+k-1}R(u)\,du\,ds\right].\end{aligned}$$

Interchanging the order of integration as in Theorem 3.1, we obtain

$$\begin{aligned}J^{k+1}R(t) &= \frac{\lambda\Gamma(q)}{\Gamma(q+k)}\left[\frac{t^{q+k}}{q+k} - \int_0^t\left(\int_u^t (s-u)^{q+k-1}\,ds\right)R(u)\,du\right]\\ &= \frac{\lambda\Gamma(q)}{\Gamma(q+k)}\left[\frac{1}{q+k}t^{q+k} - \frac{1}{q+k}\int_0^t (t-u)^{q+k}R(u)\,du\right]\\ &= \frac{\lambda\Gamma(q)}{\Gamma(q+k+1)}\left[t^{q+k} - \int_0^t (t-u)^{q+k}R(u)\,du\right].\end{aligned}$$

This shows that (6.2) holds for $n=k+1$ if it holds for $n=k$. Therefore, by induction, (6.2) holds for all $n\in\mathbb{N}$. □

Corollary 6.3. *Let $m\in\mathbb{N}_0$, $\lambda>0$, and $q\in(0,1)$. Let $R(t)$ be the resolvent corresponding to the parameter λ, i.e., the unique continuous solution of* (R_λ). *Then*

$$\int_0^t (t-s)^{q+m}R(s)\,ds = t^{q+m} - \frac{1}{\lambda}\cdot\frac{\Gamma(q+m+1)}{\Gamma(q)\Gamma(m+1)}\int_0^t (t-s)^m R(s)\,ds. \tag{6.3}$$

Proof. From Theorems 3.2 and 6.2, we have two different formulas for $J^nR(t)$. As a result, setting $n=m+1$ in (3.1) and (6.2), we get

$$\frac{\lambda\Gamma(q)}{\Gamma(q+m+1)}\left[t^{q+m} - \int_0^t (t-s)^{q+m}R(s)\,ds\right] = \frac{1}{m!}\int_0^t (t-s)^m R(s)\,ds.$$

Now solve this for the integral on the left-hand side. □

7. Solution of (1.2) for a given polynomial g

The first result of this section generalizes Lemma 4.2.

Lemma 7.1. *Let $a > 0$, $b \in \mathbb{R}$, $m \in \mathbb{N}_0$, and $q \in (0,1)$. If there is a continuous solution of*

$$D^q x(t) = -ax(t) + bt^m \tag{7.1}$$

on $[0,\infty)$, then it is also a solution of

$$x(t) = \beta t^{q+m} - \lambda \int_0^t (t-s)^{q-1} x(s)\, ds \tag{7.2}$$

on $[0,\infty)$ when

$$\beta = \frac{bm!}{\Gamma(q+m+1)} \quad \text{and} \quad \lambda = \frac{a}{\Gamma(q)}. \tag{7.3}$$

Conversely, let $\beta \in \mathbb{R}$ and $\lambda > 0$ and suppose there is a continuous solution of (7.2) on $[0,\infty)$. Then it is also a continuous solution of (7.1) on $[0,\infty)$ when

$$a = \lambda\Gamma(q) \quad \text{and} \quad b = \frac{\beta}{m!}\Gamma(q+m+1). \tag{7.4}$$

Proof. Equation (7.2), written in terms of the Riemann-Liouville integral operator, is

$$x(t) = \beta t^{q+m} - \lambda\Gamma(q) J^q x(t).$$

Applying the differential operator D^q, we get

$$\begin{aligned} D^q x(t) &= \beta D^q t^{q+m} - \lambda\Gamma(q) D^q J^q x(t) = \beta\, \frac{\Gamma(q+m+1)}{\Gamma(m+1)} t^m - ax(t) \\ &= \frac{\beta}{m!}\Gamma(q+m+1) t^m - ax(t) = -ax(t) + bt^m, \end{aligned}$$

where we have used Theorem 4.1, Lemma 3.3, and (7.4). Thus, if a continuous solution of (7.2) exists for $t \geq 0$, it must also be a solution of (7.1) when a and b have the values given by (7.4).

Conversely, suppose there exists a continuous solution $x(t)$ of (7.1) on $[0,\infty)$; hence

$$DJ^{1-q}x(t) = -ax(t) + bt^m.$$

Integrating, as in the proof of Lemma 4.2, we have

$$\int_\eta^t \frac{d}{ds} J^{1-q} x(s)\, ds = \int_\eta^t (-ax(s) + bs^m)\, ds$$

or

$$J^{1-q}x(t) - J^{1-q}x(\eta) = \int_\eta^t (-ax(s) + bs^m)\, ds. \tag{7.5}$$

Taking the limit of both sides as $\eta \to 0^+$, we obtain

$$J^{1-q}x(t) = \int_0^t (-ax(s) + bs^m)\, ds = -aJx(t) + \frac{b}{m+1} t^{m+1}$$

since $J^{1-q}x(\eta) \to 0$ (cf. (4.7)). Applying D^{1-q}, we get

$$D^{1-q}J^{1-q}x(t) = -aDJ^qJx(t) + \frac{b}{m+1} DJ^q t^{m+1}$$

or

$$x(t) = -aJ^q x(t) + \frac{b}{m+1} \cdot \frac{\Gamma(m+2)}{\Gamma(m+q+1)} t^{m+q}$$

because of Lemma 3.3 and Theorem 4.1. Since $m \in \mathbb{N}_0$, this simplifies to

$$x(t) = -aJ^q x(t) + \frac{bm!}{\Gamma(m+q+1)} t^{m+q}.$$

We conclude that if a continuous solution $x(t)$ of (7.1) exists, then it must also be a solution of

$$x(t) = \frac{bm!}{\Gamma(m+q+1)} t^{m+q} - \frac{a}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s)\, ds. \qquad \square$$

In the next theorem we prove that (7.1) does have a unique continuous solution on $[0, \infty)$. Moreover, with the following formula, which is found in [17, p. 25], we show how to express it in terms of a Mittag-Leffler function.

Lemma 7.2. *Let $\gamma \in \mathbb{R}$ and $\alpha, \beta, p \in \mathbb{R}^+$. Then*

$$\int_0^t (t-s)^{p-1} E_{\alpha,\beta}(\gamma s^\alpha) s^{\beta-1}\, ds = \Gamma(p) t^{p+\beta-1} E_{\alpha,\beta+p}(\gamma t^\alpha) \tag{7.6}$$

for $t > 0$.

Proof. Let us use (1.4) to write the integrand as the sum

$$(t-s)^{p-1} E_{\alpha,\beta}(\gamma s^\alpha) s^{\beta-1} = (t-s)^{p-1} \left(\sum_{k=0}^{\infty} \frac{(\gamma s^\alpha)^k}{\Gamma(k\alpha+\beta)} \right) s^{\beta-1} = \sum_{k=0}^{\infty} g_k(s),$$

where

$$g_k(s) := (t-s)^{p-1} \frac{\gamma^k}{\Gamma(k\alpha+\beta)}\, s^{k\alpha+\beta-1}.$$

Using an integration formula in [4, (4.4)] (or [6, p. 229]), we find for $t > 0$ that

$$\begin{aligned} \int_0^t |g_k(s)|\, ds &= \frac{|\gamma|^k}{\Gamma(k\alpha+\beta)} \int_0^t (t-s)^{p-1} s^{k\alpha+\beta-1}\, ds \\ &= \frac{|\gamma|^k}{\Gamma(k\alpha+\beta)}\, t^{p+k\alpha+\beta-1} \frac{\Gamma(p)\Gamma(k\alpha+\beta)}{\Gamma(p+k\alpha+\beta)} = \Gamma(p) t^{p+\beta-1} \frac{(|\gamma| t^\alpha)^k}{\Gamma(k\alpha+\beta+p)} < \infty. \end{aligned}$$

It then follows from a generalization of Levi's theorem for series ([2, p. 269]) that

$$\begin{aligned} \int_0^t (t-s)^{p-1} E_{\alpha,\beta}(\gamma s^\alpha) s^{\beta-1}\, ds &= \int_0^t \sum_{k=0}^{\infty} g_k(s)\, ds = \sum_{k=0}^{\infty} \int_0^t g_k(s)\, ds \\ &= \Gamma(p) t^{p+\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma t^\alpha)^k}{\Gamma(k\alpha+\beta+p)} = \Gamma(p) t^{p+\beta-1} E_{\alpha,\beta+p}(\gamma t^\alpha). \qquad \square \end{aligned}$$

With the integration formulas involving the resolvent that we found in Section 6.1 and the variation of parameters formula that was used earlier in the proof of Theorem 4.5, we can now establish the existence of continuous solutions of equations (7.1) and (7.2) on $[0, \infty)$ and their uniqueness.

Theorem 7.3. *Let $a > 0$, $b \in \mathbb{R}$, $m \in \mathbb{N}_0$, and $q \in (0,1)$. The fractional relaxation equation (7.1) has the unique continuous solution*

$$x(t) = \frac{b}{a}\int_0^t (t-s)^m R(s)\,ds = bm!\,t^{q+m}E_{q,q+m+1}(-at^q) \tag{7.7}$$

on $[0,\infty)$, where R denotes the resolvent corresponding to $\lambda = a/\Gamma(q)$. It is also the unique continuous solution of the integral equation (7.2) on $[0,\infty)$ when β and λ have the values given by (7.3).

Proof. First consider the integral equation (7.2) for given values of $\beta \in \mathbb{R}$ and $\lambda > 0$. By the variation of parameters formula, the function

$$x(t) = \beta t^{q+m} - \int_0^t R(t-s)\beta s^{q+m}\,ds = \beta t^{q+m} - \beta\int_0^t (t-s)^{q+m}R(s)\,ds$$

is the unique continuous solution of (7.2) on $[0,\infty)$. Because of Corollary 6.3, this solution can be simplified as follows: first define a and b by (7.4). Then

$$\begin{aligned} x(t) &= \beta t^{q+m} - \beta\left[t^{q+m} - \frac{1}{\lambda}\cdot\frac{\Gamma(q+m+1)}{\Gamma(q)\Gamma(m+1)}\int_0^t (t-s)^m R(s)\,ds\right] \\ &= \frac{\beta}{\lambda}\cdot\frac{\Gamma(q+m+1)}{\Gamma(q)\Gamma(m+1)}\int_0^t (t-s)^m R(s)\,ds \\ &= \frac{bm!}{\Gamma(q+m+1)}\cdot\frac{\Gamma(q)}{a}\cdot\frac{\Gamma(q+m+1)}{\Gamma(q)\Gamma(m+1)}\int_0^t (t-s)^m R(s)\,ds \\ &= \frac{b}{a}\int_0^t (t-s)^m R(s)\,ds. \end{aligned} \tag{7.8}$$

In short, we have shown that (7.8) is the unique continuous solution of (7.2) on $[0,\infty)$ if a and b have the values given by (7.4). Furthermore, we can see from Lemma 7.1 that it is also the unique continuous solution of the fractional differential equation (7.1) on $[0,\infty)$.

Finally, let us show how to express (7.8) in terms of a Mittag-Leffler function. From (1.3) we find that the resolvent of (R_λ) corresponding to $\lambda = a/\Gamma(q)$ is

$$R(t) = \lambda\Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q) = at^{q-1}E_{q,q}(-at^q).$$

Hence, from (7.8) we have

$$x(t) = \frac{b}{a}\int_0^t (t-s)^m R(s)\,ds = b\int_0^t (t-s)^m s^{q-1}E_{q,q}(-as^q)\,ds.$$

Then, by setting $p = m+1$, $\alpha = \beta = q$, and $\gamma = -a$ in Lemma 7.2, we obtain

$$x(t) = b\Gamma(m+1)t^{(m+1)+q-1}E_{q,q+m+1}(-at^q) = bm!t^{m+q}E_{q,q+m+1}(-at^q)$$

for $t > 0$. Note this formula is also valid for $t = 0$ since from (7.8) we see that $x(0) = 0$. □

Remark 7.4. According to (7.7), the solution of (7.1) when $m = 0$ is

$$x(t) = \frac{b}{a}\int_0^t R(s)\,ds = b\,t^q E_{q,q+1}(-at^q).$$

From equations (9.7) and (9.8) in [3], we see that

$$t^q E_{q,q+1}(-at^q) = \frac{1}{a}\left[1 - E_q(-at^q)\right].$$

Thus,

$$x(t) = \frac{b}{a}\left[1 - E_q(-at^q)\right],$$

which is precisely what was stated in Theorem 4.5.

Remark 7.5. If we disregard the hypothesis that $q \in (0,1)$ and set $q = 1$, then (7.7) becomes

$$x(t) = bm!\, t^{1+m} E_{1,2+m}(-at).$$

Can this formal substitution be justified? In [6, p. 69] and [17, p. 18], we find the formulas:

$$E_{1,1}(x) = e^x \text{ and } E_{1,n}(x) = \frac{1}{x^{n-1}} \left(e^x - \sum_{k=0}^{n-2} \frac{x^k}{k!} \right) \text{ for } n = 2, 3, \ldots.$$

Setting $n = 2 + m$ and $x = -at$, we obtain

$$\begin{aligned} x(t) &= bm!\, t^{1+m} \frac{1}{(-at)^{m+1}} \left(e^{-at} - \sum_{k=0}^{m} \frac{(-at)^k}{k!} \right) \\ &= (-1)^{m+1} \frac{bm!}{a^{m+1}}\, e^{-at} - (-1)^{m+1} \frac{bm!}{a^{m+1}} \sum_{k=0}^{m} \frac{(-at)^k}{k!} \\ &= (-1)^{m+1} \frac{bm!}{a^{m+1}}\, e^{-at} + \frac{b}{a} \sum_{k=0}^{m} (-1)^{m+2+k} \frac{m!}{a^{m-k} k!} t^k. \end{aligned}$$

Thus,

$$x(t) = \frac{b}{a} \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{k!\, a^{m-k}} t^k + \frac{b}{a} (-1)^{m+1} \frac{m!}{a^m} e^{-at}. \tag{7.9}$$

Writing out some of the terms of (7.9), we have

$$x(t) = \frac{b}{a} \left[(-1)^m \frac{m!}{a^m} + (-1)^{m-1} \frac{m!}{a^{m-1}} t + \cdots + t^m \right] + (-1)^{m+1} \frac{bm!}{a^{m+1}}\, e^{-at}.$$

Now note that when we evaluate this at $t = 0$, we get

$$x(0) = (-1)^m \frac{bm!}{a^{m+1}} + (-1)^{m+1} \frac{bm!}{a^{m+1}} = 0,$$

which agrees with the value of (7.7) at $t = 0$. Also note that the fractional differential operator D^q is defined to be the ordinary first-order differential operator D when $q = 1$ (cf. Def. 2.3). So the formal substitution $q = 1$ in (7.7) suggests that (7.9) is the solution of the classical initial value problem

$$x'(t) = -ax(t) + bt^m, \quad x(0) = 0. \tag{7.10}$$

Let us see if this is truly the case.

By the classical variation of parameters formula, the solution of (7.10) is

$$x(t) = e^{-at} x(0) + \int_0^t e^{-a(t-s)} bs^m \, ds, \tag{7.11}$$

which simplifies to

$$x(t) = be^{-at} \int_0^t s^m e^{as} \, ds$$

since $x(0) = 0$. Integrating by parts or consulting a table of integrals, such as [9, 2.321], we find that

$$\int s^m e^{as} \, ds = e^{as} \sum_{k=0}^{m} (-1)^k \frac{k!}{a^{k+1}} \binom{m}{k} s^{m-k}.$$

Hence,

$$
\begin{aligned}
x(t) = be^{-at}\int_0^t s^m e^{as}\,ds &= b\sum_{k=0}^{m}(-1)^k\frac{k!}{a^{k+1}}\binom{m}{k}t^{m-k} - b(-1)^m\frac{m!}{a^{m+1}}e^{-at}\\
&= \frac{b}{a}\sum_{k=0}^{m}(-1)^k\frac{m!}{(m-k)!a^k}t^{m-k} + \frac{b}{a}(-1)^{m+1}\frac{m!}{a^m}e^{-at}.
\end{aligned}
$$

With an appropriate change in the index of summation, we see that this is equivalent to (7.9). In sum, we have shown that (7.9) does in fact solve the classical initial value problem (7.10).

Because of Corollary 6.3 and Theorem 7.3, we can express the convolution of t^p for $p > -1$ and the resolvent R in terms of two-parameter Mittag-Leffler functions. This is the content of the next result.

Corollary 7.6. *Let $m \in \mathbb{N}_0$. Let R be the resolvent of* (R_λ)*; that is,*

$$R(t) = \lambda\Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q)$$

where $\lambda > 0$ and $0 < q < 1$. Then for $p > -1$,

$$
\begin{aligned}
&\int_0^t (t-s)^p R(s)\,ds\\
&\qquad = \begin{cases} t^{q-1} - \Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q) & \textit{if} \quad p = q-1\\ \lambda\Gamma(q)m!\,t^{q+m}E_{q,q+m+1}(-\lambda\Gamma(q)t^q) & \textit{if} \quad p = m\\ t^{q+m} - \Gamma(q+m+1)t^{q+m}E_{q,q+m+1}(-\lambda\Gamma(q)t^q) & \textit{if} \quad p = m+q. \end{cases}
\end{aligned}
$$

Proof. Suppose $p = q-1$. Then it follows from (R_λ) and (1.3) that

$$
\begin{aligned}
\int_0^t (t-s)^{q-1}R(s)\,ds &= t^{q-1} - \frac{1}{\lambda}R(t) = t^{q-1} - \frac{1}{\lambda}\left[\lambda\Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q)\right]\\
&= t^{q-1} - \Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q).
\end{aligned}
$$

Now suppose $p = m$ where $m \in \mathbb{N}_0$. Then from Theorem 7.3 we have

$$
\begin{aligned}
\int_0^t (t-s)^m R(s)\,ds &= am!\,t^{q+m}E_{q,q+m+1}(-at^q)\\
&= \lambda\Gamma(q)m!\,t^{q+m}E_{q,q+m+1}(-\lambda\Gamma(q)t^q)
\end{aligned}
$$

since $a = \lambda\Gamma(q)$.

Finally consider the case when $p = m+q$. Then it follows from the previous case and Corollary 6.3 that

$$
\begin{aligned}
&\int_0^t (t-s)^{m+q}R(s)\,ds\\
&\qquad = t^{q+m} - \frac{1}{\lambda}\cdot\frac{\Gamma(q+m+1)}{\Gamma(q)m!}\cdot\lambda\Gamma(q)m!\,t^{q+m}E_{q,q+m+1}(-\lambda\Gamma(q)t^q)\\
&\qquad = t^{q+m} - \Gamma(q+m+1)t^{q+m}E_{q,q+m+1}(-\lambda\Gamma(q)t^q).
\end{aligned}
$$

□

Our final result employs Theorem 7.3 to obtain the unique continuous solution of

$$D^q x(t) = -ax(t) + g(t) \tag{1.2}$$

on the interval $[0,\infty)$ when $g(t)$ is a given polynomial.

Theorem 7.7. *Let $q \in (0,1)$ and $a > 0$. Let $n \in \mathbb{N}_0$ and $b_m \in \mathbb{R}$ for $m = 0,1,2,\dots,n$. The fractional relaxation equation*

$$D^q x(t) = -ax(t) + \sum_{m=0}^{n} b_m t^m \tag{7.12}$$

has one and only one continuous solution on $[0,\infty)$, namely,

$$x(t) = \sum_{m=0}^{n} b_m m! t^{q+m} E_{q,q+m+1}(-at^q). \tag{7.13}$$

Proof. For $m = 0,1,\dots,n$, let x_m denote the continuous solution of

$$D^q x(t) = -ax(t) + b_m t^m$$

on $[0,\infty)$, whose existence and uniqueness was established with Theorem 7.3. It is clear from (2.7) that D^q is a linear operator. Consequently,

$$\begin{aligned} D^q\left(\sum_{m=0}^{n} x_m(t)\right) &= \sum_{m=0}^{n} D^q x_m(t) = \sum_{m=0}^{n}(-ax_m(t) + b_m t^m) \\ &= -a\sum_{m=0}^{n} x_m(t) + \sum_{m=0}^{n} b_m t^m = -a\sum_{m=0}^{n} x_m(t) + g(t) \end{aligned}$$

where

$$g(t) := \sum_{m=0}^{n} b_m t^m. \tag{7.14}$$

Thus $x(t) := \sum_{m=0}^{n} x_m(t)$ is a continuous solution of (7.12) on $[0,\infty)$.

As for uniqueness, suppose that $y(t)$ is also a continuous solution. Applying the operator D^q to

$$z(t) := x(t) - y(t),$$

we get

$$D^q z(t) = -ax(t) + g(t) - [-ay(t) + g(t)] = -a\,[x(t) - y(t)] = -az(t)$$

for $t \geq 0$. It follows from Lemma 4.4 that $z(t) \equiv 0$. In other words, $y(t) \equiv x(t)$ on $[0,\infty)$.

Finally, we obtain (7.13) from (7.7). $\square$

Example 7.8. The equation

$$D^{1/2}x(t) = -x(t) + 1 - 3t - 2t^2 + t^3 \tag{7.15}$$

has the unique continuous solution

$$x(t) = t^3 - \frac{16}{5\sqrt{\pi}}t^{5/2} + t^2 - \frac{8}{3\sqrt{\pi}}t^{3/2} - t + \frac{2}{\sqrt{\pi}}t^{1/2} \tag{7.16}$$

on the interval $[0,\infty)$.

Proof. Referring to (7.12) and (7.14), we have $q = 1/2$, $a = 1$, and

$$g(t) = \sum_{m=0}^{3} b_m t^m = 1 - 3t - 2t^2 + t^3 \tag{7.17}$$

where

$$b_0 = 1,\ b_1 = -3,\ b_2 = -2,\ b_3 = 1.$$

Accordingly, we see from (7.13) that

$$\begin{aligned}x(t) &= \sum_{m=0}^{3} b_m m! t^{\frac{1}{2}+m} E_{\frac{1}{2},\frac{1}{2}+m+1}(-\sqrt{t}) \\ &= t^{\frac{1}{2}} E_{\frac{1}{2},\frac{3}{2}}(-\sqrt{t}) - 3t^{\frac{3}{2}} E_{\frac{1}{2},\frac{5}{2}}(-\sqrt{t}) - 4t^{\frac{5}{2}} E_{\frac{1}{2},\frac{7}{2}}(-\sqrt{t}) + 6t^{\frac{7}{2}} E_{\frac{1}{2},\frac{9}{2}}(-\sqrt{t})\end{aligned}$$

is the unique continuous solution of (7.15) on $[0, \infty)$.

Each of these four terms can be expressed as a finite sum of powers of t and a constant multiple of $e^t \operatorname{erf}(\sqrt{t})$. For instance, consider the second term. From (1.4) we have

$$E_{\frac{1}{2},\frac{5}{2}}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\frac{1}{2}k + \frac{5}{2})}. \tag{7.18}$$

By the Cauchy-Hadamard formula for convergence and Stirling's formula for the Gamma function, the power series (1.4) defining $E_{\alpha,\beta}(z)$ converges absolutely for all z in the complex plane (cf. [6, p. 68]), and a fortiori for all real values of z. Consequently, we can rearrange the terms of (7.18) as follows:

$$E_{\frac{1}{2},\frac{5}{2}}(t) = \sum_{k=2}^{\infty} \frac{t^{2k-3}}{\Gamma(k+1)} + \sum_{k=2}^{\infty} \frac{t^{2k-4}}{\Gamma(k+\frac{1}{2})}.$$

In [1, (6.1.12)] we find the formula

$$\Gamma\left(k + \tfrac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2k-1)}{2^k} \Gamma\left(\tfrac{1}{2}\right).$$

Thus,

$$E_{\frac{1}{2},\frac{5}{2}}(t) = \sum_{k=2}^{\infty} \frac{t^{2k-3}}{k!} + \frac{1}{\sqrt{\pi}} \sum_{k=2}^{\infty} \frac{2^k t^{2k-4}}{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2k-1)}.$$

It then follows that

$$\begin{aligned}t^{\frac{3}{2}} E_{\frac{1}{2},\frac{5}{2}}(-\sqrt{t}) &= -\sum_{k=2}^{\infty} \frac{t^k}{k!} + \frac{1}{\sqrt{\pi}} \sum_{k=2}^{\infty} \frac{2^k t^{k-\frac{1}{2}}}{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2k-1)} \\ &= 1 + t - \sum_{k=0}^{\infty} \frac{t^k}{k!} - \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{2^k t^{k-\frac{1}{2}}}{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2k-1)} \\ &= 1 + t - e^t - \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} + \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{2^{k-1} t^{k-\frac{1}{2}}}{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2k-1)}.\end{aligned}$$

Changing the index of summation, we have

$$t^{\frac{3}{2}} E_{\frac{1}{2},\frac{5}{2}}(-\sqrt{t}) = 1 + t - e^t - \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{2^n t^{n+\frac{1}{2}}}{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n+1)}.$$

Employing the series expansion

$$e^{t^2} \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{2^n t^{2n+1}}{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n+1)}$$

found in [1, (7.1.6)], we see that

$$-3t^{\frac{3}{2}} E_{\frac{1}{2},\frac{5}{2}}(-\sqrt{t}) = -3 - 3t + 3e^t + \frac{6}{\sqrt{\pi}} t^{\frac{1}{2}} - 3e^t \operatorname{erf}(\sqrt{t}). \tag{7.19}$$

Similar calculations yield the following:

$$t^{\frac{1}{2}} E_{\frac{1}{2},\frac{3}{2}}(-\sqrt{t}) = 1 - e^t + e^t \operatorname{erf}(\sqrt{t}), \tag{7.20}$$

$$-4t^{\frac{5}{2}} E_{\frac{1}{2},\frac{7}{2}}(-\sqrt{t}) = -4 - 4t - 2t^2 + 4e^t + \frac{16}{3\sqrt{\pi}} t^{\frac{3}{2}} - \frac{8}{\sqrt{\pi}} t^{\frac{1}{2}} - 4e^t \operatorname{erf}(\sqrt{t}), \tag{7.21}$$

and

$$\begin{aligned} 6t^{\frac{7}{2}} E_{\frac{1}{2},\frac{9}{2}}(-\sqrt{t}) = 6 + 6t + 3t^2 + t^3 - 6e^t - \frac{16}{5\sqrt{\pi}} t^{\frac{5}{2}} - \frac{8}{\sqrt{\pi}} t^{\frac{3}{2}} \\ - \frac{12}{\sqrt{\pi}} t^{\frac{1}{2}} + 6e^t \operatorname{erf}(\sqrt{t}). \end{aligned} \tag{7.22}$$

Adding together the terms (7.19)–(7.22), we obtain (7.16). □

As in Remark 7.5, let us compare the solution (7.16) of the fractional relaxation equation (7.15) with the solution of the initial value problem

$$y'(t) = -y(t) + 1 - 3t - 2t^2 + t^3, \quad y(0) = 0. \tag{7.23}$$

Applying the variation of parameters formula or simply multiplying the differential equation by the integrating factor e^t and then integrating by parts and using the initial condition $y(0) = 0$, we obtain the solution

$$y(t) = -6 + 7t - 5t^2 + t^3 + 6e^{-t}. \tag{7.24}$$

The graph of the solution $y(t)$ (dashed curve) is shown in Figure 2. The solid curve is the graph of the solution (7.16) of the fractional relaxation equation (7.15). The curve that begins at $(0, 1)$ (dotted curve) is the graph of the polynomial (7.17).

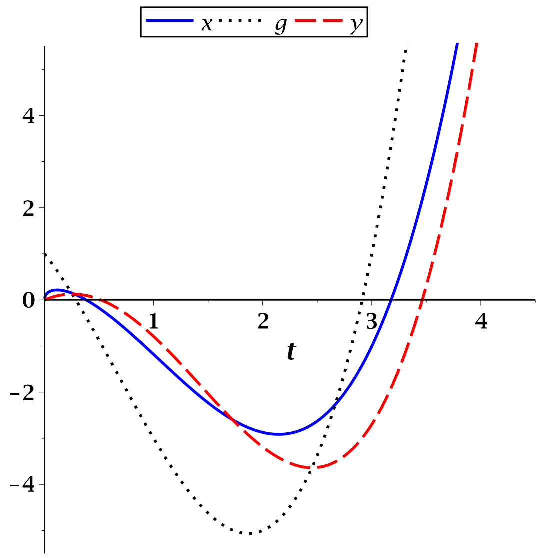

Figure 2: Graphs of (7.16), (7.17), and (7.24).

References

[1] M. Abramowitz and I. A. Stegun, (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 2nd printing, National Bureau of Standards, Applied Mathematical Series 55, 1964.

[2] T. M. Apostol, *Mathematical Analysis*, second ed., Addison-Wesley, Reading MA, 1974.

[3] L. C. Becker, Properties of the resolvent of a linear Abel integral equation: implications for a complementary fractional equation, *Electron. J. Qual. Theory Differ. Equ.*, No. 64 (2016), 1–38.

[4] L. C. Becker, T. A. Burton, and I. K. Purnaras, Complementary equations: A fractional differential equation and a Volterra integral equation, *Electron. J. Qual. Theory Differ. Equ.*, No. 12 (2015), 1–24.

[5] L. C. Becker, Resolvents for weakly singular kernels and fractional differential equations, *Nonlinear Anal.: TMA* **75** (2012), 4839–4861.

[6] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Heidelberg, 2010.

[7] A. Friedman, On integral equations of Volterra type, *J. d'Analyse Math.*, Vol. XI (1963), 381–413.

[8] W. Fulks, *Advanced Calculus*, 2nd ed., John Wiley & Sons, New York, 1969.

[9] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Seventh Edition, Elsevier Academic Press, 2007.

[10] A. Kilbas, H. Srivastava and J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies **204**, Elsevier, 2006.

[11] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Chaos, Solitons and Fractals*, Vol. 7, No. 9 (1996), 1461–1477.

[12] F. Mainardi and R. Gorenflo, On Mittag-Leffler-type functions in fractional evolution processes, *J. Comput. Appl. Math.*, **118** (2000), 283–299.

[13] J. B. Marion, *Classical Dynamics of Particles and Systems*, Academic Press, New York, 1965.

[14] Miller, R. K., *Nonlinear Volterra Integral Equations*, Benjamin, Menlo Park, CA, 1971.

[15] I. P. Natanson, *Theory of Functions of a Real Variable*, Vol. II, Frederick Ungar Publishing Co., New York, 1961.

[16] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Dover, Mineola, NY, 2006.

[17] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, Vol. 198, Academic Press, San Diego, 1999.

[18] J. J. Rosales, M. Guía, F. Gómez, F. Aguilar, and J. Martinez, Two dimensional fractional projectile motion in a resisting medium, *Cent. Eur. J. Phys.*, 12(7) (2014), 517–520.

Advances in the Theory of Nonlinear Analysis and its Applications 2 (2018) No. 1, 33–41
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

Fixed point theorem for a kind of Ćirić type contractions in complete metric spaces

Tomonari Suzuki[a]

[a] *Department of Basic Sciences, Faculty of Engineering, Kyushu Institute of Technology, Tobata, Kitakyushu 804-8550, Japan*

Abstract

We prove a fixed point theorem for a kind of Ćirić type contractions in complete metric spaces. In order to demonstrate the assumption of the fixed point theorem, we give an example. We also clarify the mathematical structure of some fixed point theorem proved by Mınak-Helvacı-Altun and Wardowski-Dung independently.

Keywords: Fixed point, F-contraction, contractive condition
2010 MSC: 54H25, 54E50

1. Introduction

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

In 2012, Wardowski in [10] introduced the concept of F-contraction and proved the following fixed point theorem.

Theorem 1.1 (Theorem 2.1 in Wardowski [10]). *Let (X, d) be a complete metric space and let T be a F-contraction on X, that is, there exist a function F from $(0, \infty)$ into $\mathbb{R}$ and real numbers $\tau \in (0, \infty)$ and $k \in (0, 1)$ satisfying the following:*

(F1) *F is strictly increasing.*
(F2) *For any sequence $\{\alpha_n\}$ of positive numbers, $\lim_n \alpha_n = 0 \Leftrightarrow \lim_n F(\alpha_n) = -\infty$.*
(F3) $\lim[t^k F(t) : t \to +0] = 0$.
(F4) *$\tau + F \circ d(Tx, Ty) \leq F \circ d(x, y)$ for any $x, y \in X$ with $Tx \neq Ty$.*

Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for all $x \in X$.

Email address: `suzuki-t@mns.kyutech.ac.jp` (Tomonari Suzuki)

Received January 01, 2018, Accepted January 21, 2018.

In 2014, Mınak, Helvacı and Altun in [6] and Wardowski and Dung in [11] independently proved the following fixed point theorem.

Theorem 1.2 (Theorem 2.2 in [6], Theorem 2.4 in [11]). *Let (X, d) be a complete metric space and let T be a mapping on X. Assume that there exist a function F from $(0, \infty)$ into $\mathbb{R}$ and real numbers $\tau \in (0, \infty)$ and $k \in (0, 1)$ satisfying* (F1)–(F3) *and the following*:

(F5) $\tau + F \circ d(Tx, Ty) \leq F \circ L(x, y)$ *for any $x, y \in X$ with $Tx \neq Ty$, where L is defined by*

$$L(x, y) = \max\Big\{d(x, y), \frac{d(x, Ty) + d(Tx, y)}{2}, d(x, Tx), d(y, Ty)\Big\}. \tag{1.1}$$

Assume also either of the following:

(F6) *T is continuous.*

(F7) *F is continuous.*

Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for all $x \in X$.

We assume (F6) or (F7) additionally. So, we note that Theorem 1.2 is not a generalization of Theorem 1.1. Also F appears in both sides of (F5). So, we do not understand the mathematical structure of Theorem 1.2 easily.

Motivated by the above, in this paper, we clarify the mathematical structure of Theorem 1.2. Indeed, in the case of (F6), we can prove Theorem 1.2 by using the known result (Theorem 4.1). Also we can weaken the assumption of (F7) (see Theorem 2.1). In both cases, we do not use F. Finally we give an example (Example 5.1), which implies that we cannot generalize Theorem 1.1 with using L. Also, Example 5.1 tells that the assumption of the new fixed point theorem (Theorem 2.1) is reasonably weak.

2. Main Result

In this section, we prove the following fixed point theorem.

Theorem 2.1. *Let (X, d) be a complete metric space and let T be a mapping on X. Define a function L from $X \times X$ into $[0, \infty)$ by (1.1). Assume that there exists a function φ from $[0, \infty)$ into itself satisfying the following*:

(i) $\varphi(t) < t$ *for any $t \in (0, \infty)$.*

(ii) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon < t < \varepsilon + \delta \quad \text{implies} \quad \varphi(t) \leq \varepsilon.$$

(iii) $d(Tx, Ty) \leq \varphi \circ L(x, y)$.

Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for all $x \in X$.

Remark 2.2.

- *Define a subset Q of $[0, \infty)^2$ by*

$$Q = \big\{\big(L(x, y), d(Tx, Ty)\big) : x, y \in X\big\}. \tag{2.1}$$

Then Q satisfies Condition $C(0, 1, 0)$ (see Section 3).

- *Since we do not assume the nondecreasingness of φ, we cannot prove this theorem by using Theorem 5 in [9].*

Proof of Theorem 2.1. Since $L(x, y) = 0$ implies $d(Tx, Ty) = 0$, without loss of generality, we may assume $\varphi(0) = 0$. By (i), we can easily prove that (ii) is equivalent to the following:

(ii') For any $\varepsilon > 0$, there exists $\delta > 0$ such that $t < \varepsilon + \delta$ implies $\varphi(t) \le \varepsilon$.

We will show the following:

$$x \neq y \quad \Rightarrow \quad d(Tx, Ty) < L(x, y), \tag{2.2}$$
$$d(Tx, Ty) \le L(x, y). \tag{2.3}$$

Indeed, if $x \neq y$ holds, then $L(x, y) > 0$ holds. We have by (i) and (iii)

$$d(Tx, Ty) \le \varphi \circ L(x, y) < L(x, y).$$

Thus (2.2) holds. If $x = y$ holds, then we have

$$d(Tx, Ty) = 0 \le L(x, y).$$

Combining this with (2.2), we obtain (2.3).

We next show the following:

$$x \neq Tx \quad \Rightarrow \quad d(Tx, T^2x) < d(x, Tx), \tag{2.4}$$
$$d(Tx, T^2x) \le d(x, Tx), \tag{2.5}$$
$$L(x, Tx) = d(x, Tx). \tag{2.6}$$

Indeed we have

$$\begin{aligned} L(x, Tx) &= \max\left\{d(x, Tx), \frac{d(x, T^2x) + d(Tx, Tx)}{2}, d(x, Tx), d(Tx, T^2x)\right\} \\ &= \max\left\{d(x, Tx), \frac{d(x, T^2x)}{2}, d(Tx, T^2x)\right\} \\ &= \max\left\{d(x, Tx), \frac{d(x, Tx) + d(Tx, T^2x)}{2}, d(Tx, T^2x)\right\} \\ &= \max\{d(x, Tx), d(Tx, T^2x)\}. \end{aligned}$$

If $x \neq Tx$ holds, then we have by (2.2)

$$d(Tx, T^2x) < L(x, Tx) = \max\{d(x, Tx), d(Tx, T^2x)\}.$$

So we obtain (2.4). Using (2.4), we can prove (2.5) and (2.6).

Fix $u \in X$ and define a sequence $\{u_n\}$ in X by $u_n = T^n u$ for $n \in \mathbb{N}$. From (2.5), $\{d(u_n, u_{n+1})\}$ is nonincreasing. So $\{d(u_n, u_{n+1})\}$ converges to some $\varepsilon_1 \ge 0$. Arguing by contradiction, we assume $\varepsilon_1 > 0$. From (ii'), there exists $\delta_1 > 0$ satisfying the following:

- $t < \varepsilon_1 + \delta_1$ implies $\varphi(t) \le \varepsilon_1$.

From the definition of ε_1, we can choose $\nu \in \mathbb{N}$ satisfying

$$L(u_\nu, u_{\nu+1}) = d(u_\nu, u_{\nu+1}) < \varepsilon_1 + \delta_1.$$

Then we have

$$0 < \varepsilon_1 \le d(u_{\nu+1}, u_{\nu+2}) \le \varphi \circ L(u_\nu, u_{\nu+1}) \le \varepsilon_1$$

and hence by (2.4),

$$\varepsilon_1 \le d(u_{\nu+2}, u_{\nu+3}) < d(u_{\nu+1}, u_{\nu+2}) = \varepsilon_1,$$

which implies a contradiction. Therefore we obtain $\varepsilon_1 = 0$. That is, $\lim_n d(u_n, u_{n+1}) = 0$ holds. Fix $\varepsilon_2 > 0$. Then from (ii'), there exists $\delta_2 > 0$ satisfying the following:

- $t < \varepsilon_2 + 2\,\delta_2$ implies $\varphi(t) \le \varepsilon_2$.

Let $\ell \in \mathbb{N}$ be large enough to satisfy $d(u_\ell, u_{\ell+1}) < \delta_2$. We will show

$$d(u_\ell, u_{\ell+j}) < \varepsilon_2 + \delta_2 \tag{2.7}$$

for $j \in \mathbb{N}$ by induction. It is obvious that (2.7) holds when $j = 1$. We assume that (2.7) holds for some $j \in \mathbb{N}$. Then we have by (2.5)

$$\begin{aligned}
&d(u_\ell, u_{\ell+j+1}) + d(u_{\ell+1}, u_{\ell+j}) \\
&\le d(u_\ell, u_{\ell+j}) + d(u_{\ell+j}, u_{\ell+j+1}) + d(u_{\ell+1}, u_\ell) + d(u_\ell, u_{\ell+j}) \\
&< 2\,\varepsilon_2 + 4\,\delta_2.
\end{aligned}$$

Hence

$$\begin{aligned}
&L(u_\ell, u_{\ell+j}) \\
&= \max\left\{d(u_\ell, u_{\ell+j}), \frac{d(u_\ell, u_{\ell+j+1}) + d(u_{\ell+1}, u_{\ell+j})}{2}, d(u_\ell, u_{\ell+1}), d(u_{\ell+j}, u_{\ell+j+1})\right\} \\
&< \varepsilon_2 + 2\,\delta_2
\end{aligned}$$

holds. So we have

$$d(u_{\ell+1}, u_{\ell+j+1}) \le \varphi \circ L(u_\ell, u_{\ell+j}) \le \varepsilon_2$$

and hence

$$d(u_\ell, u_{\ell+j+1}) \le d(u_\ell, u_{\ell+1}) + d(u_{\ell+1}, u_{\ell+j+1}) < \delta_2 + \varepsilon_2.$$

Thus, (2.7) holds with $j := j + 1$. So, by induction, (2.7) holds for every $j \in \mathbb{N}$. Since $\varepsilon_2 > 0$ is arbitrary, we obtain

$$\lim_{n\to\infty} \sup_{m>n} d(u_n, u_m) = 0,$$

which implies that $\{u_n\}$ is Cauchy. Since X is complete, $\{u_n\}$ converges to some $z \in X$. Arguing by contradiction, we assume $\tau := d(z, Tz) > 0$. Since $\{u_n\}$ converges to z, we can choose $\mu \in \mathbb{N}$ satisfying

$$\max\{d(z, u_\mu), d(z, u_{\mu+1}), d(u_\mu, u_{\mu+1})\} < \min\{\tau - \varphi(\tau), \tau/2\}.$$

We have

$$\begin{aligned}
&L(z, u_\mu) \\
&= \max\left\{d(z, u_\mu), \frac{d(z, u_{\mu+1}) + d(Tz, u_\mu)}{2}, d(z, Tz), d(u_\mu, u_{\mu+1})\right\} \\
&= \max\left\{\frac{\tau}{2}, \frac{\tau/2 + d(Tz, z) + d(z, u_\mu)}{2}, \tau, \frac{\tau}{2}\right\} \\
&= \max\left\{\tau, \frac{\tau/2 + \tau + \tau/2}{2}\right\} \\
&= \tau.
\end{aligned}$$

Hence

$$d(Tz, u_{\mu+1}) \le \varphi \circ L(z, u_\mu) = \varphi(\tau)$$

holds. We have

$$\tau = d(z, Tz) \le d(z, u_{\mu+1}) + d(u_{\mu+1}, Tz) < \tau - \varphi(\tau) + \varphi(\tau) = \tau,$$

which implies a contradiction. Therefore we have shown that z is a fixed point of T.

Let $w \in X$ be a fixed point of T. Then we have

$$L(z, w) = \max\left\{d(z, w), \frac{d(z, w) + d(z, w)}{2}, d(z, z), d(w, w)\right\} = d(z, w)$$

and hence

$$d(z, w) = d(Tz, Tw) \le \varphi \circ L(z, w) = \varphi \circ d(z, w),$$

which implies $d(z, w) = 0$, thus, the fixed point z is unique. □

3. Contractive Conditions

In this section, we study contractive conditions.

Definition 3.1. *Let X be a nonempty set and let p and d be functions from $X \times X$ into $[0, \infty)$. Let T be a mapping on X.*

(1) *T is said to be a CJM contraction [2, 4, 5] if the following hold:*
  (1-i) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) \le \varepsilon$.*
  (1-ii) *$d(Tx, Ty) < p(x, y)$ for any $x, y \in X$ with $d(Tx, Ty) > 0$.*

(2) *T is said to be of New-type [9] if there exists a function φ from $[0, \infty)$ into itself satisfying the following:*
  (2-i) *$\varphi(0) = 0$.*
  (2-ii) *$\varphi(t) < t$ for any $t \in (0, \infty)$.*
  (2-iii) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon < t < \varepsilon + \delta$ implies $\varphi(t) \le \varepsilon$.*
  (2-iv) *$d(Tx, Ty) \le \varphi \circ p(x, y)$ for all $x, y \in X$.*

(3) *T is said to be a Browder contraction [1] if there exists a function φ from $[0, \infty)$ into itself satisfying the following:*
  (3-i) *φ is nondecreasing and right continuous.*
  (3-ii) *$\varphi(t) < t$ for any $t \in (0, \infty)$.*
  (3-iii) *$d(Tx, Ty) \le \varphi \circ p(x, y)$ for all $x, y \in X$.*

In order to concentrate on contractive conditions, we consider subsets of $[0, \infty)^2$, see [3]. We give definitions which are strongly connected with contractive conditions in Definition 3.1.

Definition 3.2. *Let Q be a subset of $[0, \infty)^2$.*

(1) *Q is said to be CJM if the following hold:*
  (1-i) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $u \le \varepsilon$ holds for any $(t, u) \in Q$ with $t < \varepsilon + \delta$.*
  (1-ii) *$u < t$ holds for any $(t, u) \in Q$ with $u > 0$.*

(2) *Q is said to be of New-type if there exists a function φ from $[0, \infty)$ into itself satisfying the following:*
  (2-i) *$\varphi(0) = 0$.*
  (2-ii) *$\varphi(t) < t$ for any $t \in (0, \infty)$.*
  (2-iii) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon < t < \varepsilon + \delta$ implies $\varphi(t) \le \varepsilon$.*
  (2-iv) *$u \le \varphi(t)$ for all $(t, u) \in Q$.*

(3) *Q is said to be a Browder if there exists a function φ from $[0, \infty)$ into itself satisfying the following:*
  (3-i) *φ is nondecreasing and right continuous.*
  (3-ii) *$\varphi(t) < t$ for any $t \in (0, \infty)$.*
  (3-iii) *$u \le \varphi(t)$ for all $(t, u) \in Q$.*

The following obviously holds. See also Proposition 6 in [7].

Proposition 3.3. *Let X be a nonempty set and let p and d be functions from $X \times X$ into $[0, \infty)$. Let T be a mapping on X. Define a subset Q of $[0, \infty)^2$ by*

$$Q = \big\{ \big(p(x, y), d(Tx, Ty)\big) : x, y \in X \big\}. \tag{3.1}$$

Then the following hold:

(i) *T is a CJM contraction iff Q is CJM.*
(ii) *T is of New-type iff Q is of New-type.*
(iii) *T is a Browder contraction iff Q is Browder.*

Very recently, the concept of Condition $\mathrm{C}(p, q, r)$ was introduced in [8]. Using this concept, we can compare contractive conditions quite easily.

Definition 3.4 ([8]). *Let Q be a subset of $[0,\infty)^2$.*

(1) *Q is said to satisfy Condition $C(0,0,0)$ if the following hold:*
 (1-i) *$u < t$ for any $(t,u) \in Q$ with $u > 0$.*
 (1-ii) *There does not exist $\tau > 0$ and a sequence $\{(t_n, u_n)\}$ in Q satisfying $\tau < t_n$, $\tau < u_n$ and $\lim_n t_n = \lim_n u_n = \tau$.*

(2) *Q is said to satisfy Condition $C(0,0,1)$ if the following hold:*
 (2-i) *Q satisfies Condition $C(0,0,0)$.*
 (2-ii) *There does not exist $\tau > 0$ and a sequence $\{(t_n, u_n)\}$ in Q satisfying $\tau < t_n$, $u_n = \tau$ and $\lim_n t_n = \tau$.*

(3) *Q is said to satisfy Condition $C(0,0,2)$ if the following hold:*
 (3-i) *Q satisfies Condition $C(0,0,0)$.*
 (3-ii) *There does not exist $\tau > 0$ and a sequence $\{(t_n, u_n)\}$ in Q satisfying $\tau < t_n$, $u_n \le \tau$ and $\lim_n t_n = \lim_n u_n = \tau$.*

(4) *Q is said to satisfy Condition $C(0,1,0)$ if the following hold:*
 (4-i) *Q satisfies Condition $C(0,0,0)$.*
 (4-ii) *There does not exist $\tau > 0$ and a sequence $\{(t_n, u_n)\}$ in Q satisfying $t_n = \tau$, $u_n < \tau$ and $\lim_n u_n = \tau$.*

(5) *Q is said to satisfy Condition $C(1,0,0)$ if the following hold:*
 (5-i) *Q satisfies Condition $C(0,0,0)$.*
 (5-ii) *There does not exist $\tau > 0$ and a sequence $\{(t_n, u_n)\}$ in Q satisfying $t_n < \tau$, $u_n < \tau$ and $\lim_n t_n = \lim_n u_n = \tau$.*

(6) *Let $(p,q,r) \in \{0,1\}^2 \times \{0,1,2\}$. Then Q is said to satisfy Condition $C(p,q,r)$ if Q satisfies Conditions $C(p,0,0)$, $C(0,q,0)$ and $C(0,0,r)$.*

Remark 3.5. *The expressions on the above conditions are a little different from those in [8]. Of course, both are essentially the same.*

The following was essentially proved in [8].

Proposition 3.6 ([8]). *Let Q be a subset of $[0,\infty)^2$. Then the following hold:*

(i) *Q is CJM iff Q satisfies Condition $C(0,0,0)$.*
(ii) *Q is of New-type iff Q satisfies Condition $C(0,1,0)$.*
(iii) *Q is Browder iff Q satisfies Condition $C(1,1,2)$.*

We prove the following lemma, which plays an important role in this paper.

Lemma 3.7. *Let X be a nonempty set and let p and d be functions from $X \times X$ into $[0,\infty)$. Let T be a mapping on X. Assume that there exist a nondecreasing function F from $(0,\infty)$ into $\mathbb{R}$ and a real number $\tau \in (0,\infty)$ satisfying*

$$d(Tx,Ty) > 0 \quad \Rightarrow \quad \tau + F \circ d(Tx,Ty) \le F \circ p(x,y)$$

for any $x, y \in X$. Define a subset Q of $[0,\infty)^2$ by (3.1). Then the following hold:

(i) *Q satisfies Condition $C(1,0,0)$.*
(ii) *If F is right continuous, then Q satisfies Condition $C(1,0,1)$.*
(iii) *If F is left continuous, then Q satisfies Condition $C(1,1,0)$.*
(iv) *If F is continuous, then Q satisfies Condition $C(1,1,2)$.*

Remark 3.8. *(iii) and (iv) were essentially proved in [7]. See Remark below the proof of Theorem 17 in [7]. For the sake of completeness, we give a proof. We note that the proof below is much simpler than that in [7].*

Proof of Lemma 3.7. We first show (i). Let $(t, u) \in Q$ satisfy $u > 0$. Then there exist $x, y \in X$ satisfying $p(x, y) = t$ and $d(Tx, Ty) = u$. From the assumption, we have $\tau + F(u) \leq F(t)$, which implies $u < t$. Arguing by contradiction, we assume that there exist $\upsilon > 0$ and sequences $\{t_n\}$ and $\{u_n\}$ in (υ, ∞) satisfying $(t_n, u_n) \in Q$ and $\lim_n(t_n, u_n) = (\upsilon, \upsilon)$. Then we have

$$\tau + \lim_{t \to \upsilon+0} F(t) = \tau + \lim_{n \to \infty} F(u_n) \leq \lim_{n \to \infty} F(t_n) = \lim_{t \to \upsilon+0} F(t).$$

Since $\lim_{t \to \upsilon+0} F(t) \in \mathbb{R}$, we obtain a contradiction. Thus, Q satisfies Condition C$(0, 0, 0)$. Also, arguing by contradiction, we assume that there exist $\upsilon > 0$ and sequences $\{t_n\}$ and $\{u_n\}$ in $(0, \upsilon)$ satisfying $(t_n, u_n) \in Q$ and $\lim_n(t_n, u_n) = (\upsilon, \upsilon)$. Then we have

$$\tau + \lim_{t \to \upsilon-0} F(t) = \tau + \lim_{n \to \infty} F(u_n) \leq \lim_{n \to \infty} F(t_n) = \lim_{t \to \upsilon-0} F(t),$$

which implies a contradiction. Therefore we have shown that Q satisfies Condition C$(1, 0, 0)$.

In order to prove (ii), we assume that F is right continuous. Arguing by contradiction, we assume that there exist $\upsilon > 0$ and a sequence $\{t_n\}$ in (υ, ∞) satisfying $(t_n, \upsilon) \in Q$ and $\lim_n t_n = \upsilon$. Then we have

$$\tau + F(\upsilon) \leq \lim_{n \to \infty} F(t_n) = \lim_{t \to \upsilon+0} F(t) = F(\upsilon),$$

which implies a contradiction. Therefore we obtain (ii).

In order to prove (iii), we assume that F is left continuous. Arguing by contradiction, we assume that there exist $\upsilon > 0$ and a sequence $\{u_n\}$ in $(0, \upsilon)$ satisfying $(\upsilon, u_n) \in Q$ and $\lim_n u_n = \upsilon$. Then we have

$$\tau + F(\upsilon) = \tau + \lim_{t \to \upsilon-0} F(t) = \tau + \lim_{n \to \infty} F(u_n) \leq F(\upsilon),$$

which implies a contradiction. Therefore we obtain (iii).

In order to prove (iv), we assume that F is continuous. Arguing by contradiction, we assume that there exist $\upsilon > 0$ and sequences $\{t_n\}$ and $\{u_n\}$ in $(0, \infty)$ satisfying $(t_n, u_n) \in Q$ and $\lim_n(t_n, u_n) = (\upsilon, \upsilon)$. Then we have

$$\tau + F(\upsilon) = \tau + \lim_{n \to \infty} F(u_n) \leq \lim_{n \to \infty} F(t_n) = F(\upsilon),$$

which implies a contradiction. Therefore we obtain (iv). □

4. Proof of Theorem 1.2

In this section, in order to clarify the mathematical structure of Theorem 1.2, we give a proof of Theorem 1.2.

Theorem 4.1 (Theorem 2 in Jachymski [4])**.** *Let (X, d) be a complete metric space and let T be a continuous mapping on X. Define L by* (1.1). *Assume the following*:

(i) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon < L(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) \leq \varepsilon$.*
(ii) *$L(x, y) > 0$ implies $d(Tx, Ty) < L(x, y)$.*

Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for all $x \in X$.

Now we give a proof of Theorem 1.2.

Proof of Theorem 1.2. Define a subset Q of $[0, \infty)^2$ by (2.1).

We first assume (F6). Then by Lemma 3.7, Q satisfies Condition C$(1, 0, 0)$. So Q satisfies Condition C$(0, 0, 0)$. By Proposition 3.6, Q is CJM. Thus, all the assumption of Theorem 4.1 holds. By Theorem 4.1, we obtain the desired result.

We next assume (F7). Then by Lemma 3.7, Q satisfies Condition C$(1, 1, 2)$. In particular, Q satisfies Condition C$(0, 1, 0)$. By Proposition 3.6, Q is of New-type. Using Theorem 2.1, we obtain the desired result. □

Remark 4.2. *We do not need* (F3).

5. Example

The following example tells that the assumption of Theorem 2.1 is reasonably weak. Also the example implies that we cannot generalize Theorem 1.1 with using L.

Example 5.1. *Put $X = [0,1]$ and let d be as usual. Define a mapping T on X by*

$$Tx = \begin{cases} 1 & \text{if } x = 0 \\ x/2 & \text{if } x > 0. \end{cases}$$

Define L and Q by (1.1) *and* (2.1), *respectively. Then the following assertions hold*:

(i) *(X, d) is a complete metric space.*
(ii) *T does not have a fixed point.*
(iii) *Q does not satisfy Condition $C(0,1,0)$.*
(iv) *Q satisfies Condition $C(1,0,2)$.*

Proof. (i) and (ii) are obvious. Let us prove (iii) and (iv). For $x \in X \setminus \{0\}$, we have

$$L(x,0) = d(0,T0) = 1,$$
$$d(Tx,T0) = d(x/2,1) = 1 - x/2.$$

Define a sequence $\{(t_n, u_n)\}$ in Q by

$$t_n := L(2^{-n}, 0) = 1,$$
$$u_n := d(T2^{-n}, T0) = 1 - 2^{-n-1}.$$

Then, since $\{u_n\}$ converges to 1, Q does not satisfy Condition C$(0,1,0)$. For $x, y \in X \setminus \{0\}$, we have

$$d(Tx,Ty) = (1/2)\,d(x,y) \le (1/2)\,L(x,y).$$

So Q satisfies Condition C$(1,0,2)$. □

Acknowledgement

The author is supported in part by JSPS KAKENHI Grant Number 16K05207 from Japan Society for the Promotion of Science.

References

[1] F. E. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Nederl. Akad. Wetensch. Proc. Ser. A 71=Indag. Math. 30 (1968), 27–35. MR0230180

[2] Lj. B. Ćirić, *A new fixed-point theorem for contractive mappings*, Publ. Inst. Math. (Beograd), 30 (1981), 25–27. MR0672 538

[3] M. Hegedüs and T. Szilágyi, *Equivalent conditions and a new fixed point theorem in the theory of contractive type mappings*, Math. Japon., 25 (1980), 147–157. MR0571276

[4] J. Jachymski, *Equivalent conditions and the Meir-Keeler type theorems*, J. Math. Anal. Appl., 194 (1995), 293–303. MR1 353081

[5] J. Matkowski, *Fixed point theorems for contractive mappings in metric spaces*, Časopis Pěst. Mat., 105 (1980), 341–344. MR0597909

[6] G. Mınak, A. Helvacı and İ. Altun, *Ćirić type generalized F-contractions on complete metric spaces and fixed point results*, Filomat, 28 (2014), 1143–1151. MR3360088

[7] T. Suzuki, *Discussion of several contractions by Jachymski's approach*, Fixed Point Theory Appl., 2016, 2016:91. MR3548 683

[8] T. Suzuki, *Characterizations of contractive conditions by using convergent sequences*, Fixed Point Theory Appl., 2017, 2017:30.

[9] T. Suzuki, *A generalization of Hegedüs-Szilágyi's fixed point theorem in complete metric spaces*, Fixed Point Theory Appl., 2018, 2018:1.

[10] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., 2012, 2012:94. MR2949666

[11] D. Wardowski and N. V. Dung, *Fixed points of F-weak contractions on complete metric spaces*, Demonstr. Math., 47 (2014), 146–155. MR3200192

Advances in the Theory of Nonlinear Analysis and its Applications 2 (2018) No. 1, 42–61
Available online at www.atnaa.org
Review Article

Advances in the Theory of Nonlinear Analysis and its Applications
ISSN: 2587-2648
Peer-Reviewed Scientific Journal

Survey: On some Midpoint-type algorithms

Giuseppe Marino[a,b], Roberta Zaccone[a]

[a] *Universitá della Calabria, Dipartimento di Matematica e Informatica, Arcavacata di Rende (CS), 87036, Italy*
[b] *King Abdulaziz University, Department of Mathematics, P.O. Box 80203, Jeddah 21589, Saudi Arabia.*

Abstract

We introduce iterative methods approximating fixed points for nonlinear operators defined on infinite-dimensional spaces. The starting points are the Implicit and Explicit Midpoint Rules, which generate polygonal functions approximating a solution for an ordinary differential equation in finite-dimensional spaces. The purpose is to determine suitable conditions on the mapping and the underlying space, in order to get strong convergence of the generated sequence to a common solution of a fixed point problem and a variational inequality. The authors contributions appear in the papers [34], [60], [61].

Keywords: polygonal functions, Implicit Midpoint Rules, Explicit Midpoint Rules, strong convergence
2010 MSC: 47H10

---

1. Introduction

Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space, $C \subset X$ a nonempty and closed set, $T : C \to C$ a nonlinear operator with $Fix(T) = \{z \in C : Tz = z\} \neq \emptyset$.

A classical problem in Metric Fixed Point Theory can be formulated as:
Examine the conditions under which the equation $x = Tx$ may be solved by successive approximations:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = Tx_n, \quad n \geq 0. \end{cases} \tag{1.1}$$

Recall that a mapping $T : C \to C$ is said L-Lipschitzian if there exists a constant $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in E.$$

In particular,

Email addresses: `giuseppe.marino@unical.it` (Giuseppe Marino), `zaccone@mat.unical.it` (Roberta Zaccone)

Received February 03, 2018, Accepted March 21, 2018.

if $L < 1$ then T is called contraction;

if $L = 1$ then T is said nonexpansive.

The first result, dealing with the convergence of sequence (1.1), is the well known Banach Principle ([5]). It holds for contractions defined on a complete metric space. Nevertheless, under the same hypothesis, if the mapping T is nonexpansive, then it is not guaranteed neither the existence nor the uniqueness of a fixed point; moreover, the sequence of iterates (1.1) may fail to converge to the fixed point even if it exists (see [38, example 2.1] and [18, examplemple 6.4])

Recognition of fixed point existence results for nonexpansive mappings has a significant line of research in the works of Browder [8], Gohde [37] and Kirk [51] published in 1965. We recall the following results, which were proved independently:

Theorem 1.1 (Browder-Gohde's theorem)**.** *If C is a bounded, closed and convex subset of a uniformly convex Banach space X and if $T : C \to C$ is nonexpansive, then T has a fixed point.*

Theorem 1.2 (Kirk's theorem)**.** *Let C be a weakly-compact, convex subset of a Banach space X. Assume that C has the normal structure property , then any nonexpansive mapping $T : C \to C$ has a fixed point.*

Nonexpansive mappings, besides being a generalization of contractions, represent a class of interest for its connection with

- Evolution inclusions: $0 \in \frac{du}{dt} + T(t)u$, where $T(t)$ is, in general, set-valued and accretive or dissipative and minimally continuous.
- Convex minimization problems: let C a closed and convex subset of a real Hilbert space H, $\phi : C \to \mathbb{R}$ a convex and Fréchet differentiable function, finding $x_0 \in C$ such that

  $$\phi(x_0) = \min_{x \in C} \phi(x),$$

  that is equivalent to solve

  $$\langle \nabla\phi(x_0), y - x_0\rangle \geq 0, \quad \forall y \in C,$$

  can be treated as the fixed point problem

  $$x_0 = P_C(x_0 - \frac{1}{L_F}\nabla\phi(x_0)),$$

  where $\frac{1}{L_F}$ is the Lipschitz constant for $\nabla\phi$.

These facts promoted the development of two basic research directions:

- Study of suitable assumptions regarding the structure of the underlying space X and/or restrictions on T to ensure the existence of at least a fixed point;
- Construction of iterative methods for approximating the fixed points of T.

Historically, one of the most investigated methods approximating fixed points of nonexpansive mappings dates back to 1953 and is known as Mann's method, in light of Mann [58]. Let C be a nonempty, closed and convex subset of a Banach space X, Mann's scheme is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(x_n), \quad n \geq 0, \end{cases} \tag{1.2}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is a real control sequence in $(0, 1)$.

Given a nonexpansive mapping T with at least a fixed point, from [72] it is known that if $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = +\infty$ and C is a subset of a uniformly convex Banach space with Fréchet differentiable norm, then the sequence generated by (1.2) weakly converges to a fixed point of T. This result has been generalized in 2001 [30, Theorem 6.4], under the same hypotheses of the parameters sequence, to a more general setting of Banach spaces.

However the convergence is not strong, in general, even in a Hilbert space setting, as shows the celebrated counterexample in [35]. Since then, many modifications to the original Mann's algorithm have been provided in order to get strong convergence (see the books [1], [6] and the papers [16], [21], [40], [48], [50], [62], [65], [66], [81], [92], [93] with references therein). In detail, we mention the schemes obtained by:

Ishikawa ([46]):

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(\beta_n x_n + (1-\beta_n)Tx_n), \quad n \geq 0. \end{cases}$$

Halpern, ([39]):

$$\begin{cases} x_0, u \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n \quad n \geq 0. \end{cases}$$

Moudafi, ([64]):

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n \quad n \geq 0, \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \in (0,1)$ and $f : C \to C$ is a contraction.

More recently there exist other attempts to give iterative methods for nonexpansive mappings arising from a different perspective. In detail, consider an initial value problem for ODE's (Ordinary Differential Equations) of the type

$$\begin{cases} x'(t) = \Phi(x(t)) \\ x(t_0) = x_0 \end{cases} \tag{1.3}$$

Most of equations of type (1.3) cannot be solved in closed form, therefore numerical integration becomes an important tool in order to get informations about the solution trajectory. To this regard, we recall the Midpoint Numerical Rules. Given a time interval $[t_0, T]$, these procedures compute for each positive integer N:

- The step-size $h = \frac{T-t_0}{N}$,
- The time nodes $\{t_n = t_0 + nh\}_{n=0}^{N}$,
- Approximate values $\{y_n\}_{n=0}^{N}$ of the solution $x(t)$, $y_n \approx x(t_n)$;
- The polygonal $Y_N(t)$, connecting each pair of consecutive points (t_n, y_n), (t_{n+1}, y_{n+1}), for $n = 0, 1, \ldots, N$.

A midpoint numerical rule differs from another one for the way in which approximate values $\{y_n\}_{n=0}^{N}$ of the solution are given. Therefore we count:

Implicit Midpoint Rule (IMR):

$$\begin{cases} y_0 = x_0 \\ y_{n+1} = y_n + h\Phi\left(\frac{y_n + y_{n+1}}{2}\right), \quad n = 0, \cdots, N-1. \end{cases} \tag{1.4}$$

Explicit Midpoint Rule (EMR):

$$\begin{cases} y_0 = x_0, \\ \bar{y}_{n+1} = y_n + h\Phi(y_n), \\ y_{n+1} = y_n + h\Phi(\frac{y_n+\bar{y}_{n+1}}{2}), \quad n = 0, \cdots, N-1. \end{cases} \tag{1.5}$$

In both cases, the following theorem holds:

Theorem 1.3. *[67] If Φ is a Lipschitz continuous and sufficiently smooth function, then the sequence $\{Y_N\}_{N\in\mathbb{N}}$ converges to the exact solution of (1.3), as $N \to \infty$, uniformly on $t \in [t_0, T]$, for any fixed $T > 0$.*

It can be noticed that if $\Phi = I - g$, with I identity operator, then finding the critical points for

$$\begin{cases} x'(t) = \Phi(x(t)) \\ x(t_0) = x_0 \end{cases}$$

is equivalent to solve the fixed point problem for g, $x = g(x)$. This fact motivated M. A. Alghamdi, M. A. Alghamdi, N. Shahzad and H-K Xu, in [3], to introduce a fixed point iteration for nonexpansive mappings starting from formal analogy with the IMR scheme. The proposed method is implicit and is a Mann-type scheme, named Implicit Midpoint Rule for nonexpansive mappings:

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-t_n)x_n + t_n T(\frac{x_n+x_{n+1}}{2}), \quad n \geq 0, \end{cases} \tag{1.6}$$

where $T : H \to H$ is a nonexpansive mapping and $(t_n)_{n\in\mathbb{N}}$ is a sequence in $(0,1)$.

For this procedure, Alghamdi et al. proved the following weak convergence result:

Theorem 1.4. *Let H be a Hilbert space and $T : H \to H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$.*

Let $(x_n)_{n\in\mathbb{N}}$ be the sequence generated by

$$\begin{cases} x_0 \in H \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(\frac{x_n+x_{n+1}}{2}), \quad n \geq 0, \end{cases}$$

with $(\alpha_n)_{n\in\mathbb{N}} \in (0,1)$ satisfying the conditions

- $\alpha_{n+1}^2 \leq a\alpha_n, \quad \forall n \geq 0$ *and some* $a > 0$,
- $\liminf_{n\to\infty} \alpha_n > 0$.

The sequence $(x_n)_{n\in\mathbb{N}}$ weakly converges to a fixed point of T.

Our purpose is to provide a variation to (1.6) in order to get strong convergence. The modification line is analogous to that adopted in 2015 by N. Hussain, G. Marino, L. Muglia, L. Alamri in their work [43]: the proposed algorithm differs from scheme (1.6) for the introduction of a term $\alpha_n\mu_n(u - x_n)$ that can also be infinitesimal. The framework is still that of a Hilbert space H. The obtained scheme is given by

$$\begin{cases} x_0, u \in H, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(\frac{x_n+x_{n+1}}{2}) + \alpha_n\mu_n(u - x_n), \quad n \geq 0, \end{cases} \tag{1.7}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ are sequences in $(0,1]$ and $T : H \to H$ is a nonexpansive mapping.

We show that, under suitable conditions on the parameters $(\alpha_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$, the sequence $(x_n)_{n\in\mathbb{N}}$, generated by (1.7), converges strongly to the fixed point of T nearest to u.

A different situation occurs if the starting point is the less known numerical EMR. Through formal analogy with scheme (1.5), a recursive procedure for the fixed point problem $x = Tx$ is obtained and it is given by

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = (1 - t_n)x_n + t_n T(x_n) \\ x_{n+1} = (1 - t_n)x_n + t_n T(\frac{x_n + \bar{x}_{n+1}}{2}), \quad n \geq 0, \end{cases} \tag{1.8}$$

with $(t_n)_{n\in\mathbb{N}} \in (0, 1)$ and $T : H \to H$ is a nonexpansive mapping.

We designate it with Explicit Midpoint Rule for nonexpansive mappings.

Moreover, if the midpoint $\frac{x_n + \bar{x}_{n+1}}{2}$ in the evaluation of T in (1.8) is replaced with any convex combination between x_n and $\bar{x}_{n+1}$, then scheme (1.8) is named General Explicit Midpoint Rule for nonexpansive mappings.

We provide for the latter scheme the same formal modification as for the IMR for nonexpansive mappings, following [43]. The proposed method is given by

$$\begin{cases} x_0, u \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n, \quad n \geq 0 \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(s_n x_n + (1 - s_n)\bar{x}_{n+1}) + \alpha_n \mu_n (u - x_n), \quad n \geq 0, \end{cases} \tag{1.9}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}$ are sequences in $(0, 1]$.

Even in this case, we show that the sequence generated by (1.9) strongly converges to the fixed point of T closest to u.

Inspired by work [91] of H. K. Xu, M. A. Alghamdi, N. Shahzad and the paper [49] of Y. Ke and C. Ma, we propose another explicit iterative method, starting from the EMR scheme for nonexpansive mappings. It is called Generalized Viscosity Explicit Midpoint Rule (GVEMR) and is given by

$$\begin{cases} x_0 \in C \\ \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n, \quad n \geq 0, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(s_n x_n + (1 - s_n)\bar{x}_{n+1}), \quad n \geq 0. \end{cases} \tag{1.10}$$

Iteration (1.10) is obtained, from (1.8), introducing a viscosity term $f \in \Pi_C$ and replacing the midpoint of $[x_n, \bar{x}_{n+1}]$ with a generic point of the same interval in the evaluation of T.

The purpose is to approximate fixed points of quasi-nonexpansive mappings in Hilbert spaces. We recall that a mapping $T : C \to C$, with C a nonempty subset of a Banach space X, is said to be quasi-nonexpansive if T has at least a fixed point and verify

$$\|Tx - q\| \leq \|x - q\|, \quad \forall q \in Fix(T), \forall x \in C.$$

This class of mappings, besides for including the class of nonexpansive operators with at least a fixed point, is of interest for the researchers because they can be discontinuous (see, for examples, the pioneering works [25], [27] and more recently the monograph [18]). In literature can be found several works dealing with the fixed points approximation of a quasi-nonexpansive operators (see, for instance, [82], [26], [83], [56], [85]).

In a second stage, strong convergence results are proved for the class of quasi-nonexpansive mappings in the more general setting of p-uniformly convex Banach spaces, for $1 < p < \infty$, with new techniques with respect to those employed in a Hilbert spaces framework.

The first main result is applicable to l_p spaces; the second one, using the concept of ψ -expansive mappings (see [33] and references therein), is applicable to L_p spaces which fail to have a weakly continuous duality mapping.

2. Preliminaries

Throgouth the next sections, will be denoted with

H, a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$;

X, a real Banach space with norm $\| \cdot \|$;

X^*, the dual space of X with duality pairing $\langle x, x^* \rangle = x^*(x)$ for each $x \in X$ and $x^* \in X^*$;

C, a closed and convex subset of H or X;

$T : C \to C$, a nonexpansive or a quasi-nonexpansive mapping;

$f : C \to C$, a θ-contraction for a certain $\theta \in [0, 1)$;

$Fix(T)$, the fixed points set of T;

$P_{Fix(T)}$, the metric projection of C onto $Fix(T)$;

$Q : C \to Fix(T)$, a sunny nonexpansive retraction;

J_ϕ, the duality mapping associated to the gauge function ϕ;

J, the normalized duality mapping;

$\to$, the strong convergence;

$\rightharpoonup$, the weak convergence.

First of all we recall that, in a Hilbert space H, for each $x, y \in H$ and $\lambda \in [0, 1]$, the following inequalities hold:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle \tag{2.1}$$

$$\|\lambda x + (1 - \lambda)y\|^2 \le \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.2}$$

The framework of the convergence results in Theorem 3.17 and Theorem 3.22 is constituted by p-uniformly convex Banach spaces. About the matter, we need to recall the following definitions:

Definition 2.1. [38] A normed space X is called uniformly convex if for any $\epsilon \in (0, 2]$ there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \ge \epsilon$, then $\|\frac{x+y}{2}\| \le 1 - \delta$.

Definition 2.2. [38] The modulus of convexity of a Banach space X is the function $\delta_X : (0, 2] \to (0, 1]$ defined by

$$\delta_X(\epsilon) = \inf \left\{ 1 - \|\frac{x + y}{2}\|\} : \|x\| \le 1, \|y\| \le 1, \|x - y\| \ge \epsilon \right\}.$$

We mention a characterization for uniformly convex Banach spaces using the notion of modulus of convexity:

Theorem 2.3. *[18] A normed space X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for each $\epsilon \in (0, 2]$.*

Definition 2.4. [18]
Let $p > 1$ be a real number. Then X is said to be p-uniformly convex if there is a constant $c > 0$ such that

$$\delta_X(\epsilon) \ge c\epsilon^p.$$

From the definition, it follows that a p-uniformly convex Banach space is uniformly convex.

Example 2.5. [18]
If $X = L_p$ (or l_p), $1 < p < \infty$, then

1. $\delta_X(\epsilon) \ge \frac{1}{2^{p+1}}\epsilon^2$, if $1 < p < 2$,

2. $\delta_X(\epsilon) \geq \epsilon^p$, if $2 \leq p < \infty$.

In particular, for such class of Banach spaces, we mention that for all $x, y \in X$ and $\lambda \in [0,1]$, the following inequality is verified ([86]):

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - \lambda(1-\lambda)c\|x-y\|^p \tag{2.3}$$

for a certain positive constant c, for all $x, y \in X$ and $0 \leq \lambda \leq 1$.
The concept of duality mapping appeared for first time in the work of Beurling and Livingston ([7]).

Definition 2.6. [18] A continuous and strictly increasing function $\phi : [0,+\infty) \to [0,+\infty)$ such that

$$\phi(0) = 0$$

$$\lim_{t\to\infty} \phi(t) = +\infty,$$

is called a gauge function (or weight function).

Definition 2.7. [18] Given a gauge function ϕ, the mapping $J_\phi : X \to 2^{X^*}$ defined by

$$J_\phi(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\|x^*\|; \phi(\|x\|) = \|x^*\|\}$$

is called the duality mapping with gauge function ϕ.

If the gauge ϕ is given by $\phi(t) = t^{p-1}$, $1 < p < +\infty$ for all $t \in [0,+\infty)$, then $J_\phi = J_p$ is known as pth generalized duality mapping; in particular, for $p = 2$ $J_2 = J$ is called normalized duality mapping.
When we deal with $j_\phi(x)$ we mean a (single-valued) selection of $J_\phi(x)$.

Lemma 2.8. *[22] Let ϕ a gauge function and $\Phi(t) = \int_0^t \phi(s)\,ds$, then Φ is a convex function.*

Definition 2.9. [22] The subdifferential of a proper functional $g : X \to (-\infty, \infty)$ is a map designed with

$$\partial g : X \to 2^{X^*}$$

and defined by

$$\partial g(x) = \{x^* \in X^* : g(y) \geq g(x) + \langle y - x, x^*\rangle, \forall y \in X\}$$

The duality mapping J_ϕ, associated to ϕ, can be also described in the following way:

Theorem 2.10. *[4] If J_ϕ is the duality mapping associated to a gauge ϕ, then*

$$J_\phi x = \partial\Phi(\|x\|) \quad \forall x \in X.$$

Thus a subdifferential inequality holds:

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, j_\phi(x+y)\rangle, \quad j_\phi(x+y) \in J_\phi(x+y) \tag{2.4}$$

The following definition is due to Browder:

Definition 2.11. [9]
The duality mapping J_ϕ is said to be (sequentially) weak continuous if it is single-valued and maps weakly convergent sequences in X to *weak** convergent sequences in X^*, that is, if $x_n \rightharpoonup x$ in X, then $J_\phi(x_n) \rightharpoonup^* J_\phi(x)$ in X^*.

Example 2.12. [90] For each $1 < p < \infty$, the generalized duality map J_p of l_p is weakly continuous, instead that of L_p fails to be weakly continuous.

Example 2.13. [90] Let H be a real (infinite dimensional) Hilbert space. Then J_p is weakly continuous if and only if $p = 2$.

For the fixed points set of a quasi-nonexpansive mapping the following result holds:

Theorem 2.14. *[27, Theorem 1]*
If C is a closed, convex subset of a strictly convex normed linear space, and $T : C \to C$ is quasi-nonexpansive, then $Fix(T) = \{z \in C : Tz = z\}$ is a nonempty, closed and convex set in which T is continuous.

Definition 2.15. [38] A nonempty subset K of $C \subset X$ is said to be a retract of C if there exists a continuous mapping $Q : C \to K$ with $K = Fix(Q)$. Any such mapping Q is a retraction of C onto K

It is known (see [22]) that a Banach space X is smooth if and only if each duality mapping J_ϕ is single-valued. In such spaces, a characterization for a sunny nonexpansive retraction is given by:

Lemma 2.16. *[71, Lemma 2.7]*
Let X be a smooth Banach space and let C a nonempty subset of X. Let $Q : X \to C$ a retraction and let J be the normalized duality map on X. The the following are equivalent:

1. *Q is sunny and nonexpansive,*
2. *$\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle$ for all $x, y \in X$,*
3. *$\langle x - Qx, J(y - Qx)\rangle \leq 0$ for all $x \in X$ and $y \in C$.*

Hence, there is at most one sunny nonexpansive retraction on C.

Remark 2.17. Previous lemma holds even if the normalized duality map J is replaced with the duality map J_ϕ associated to a gauge function ϕ.

Let us recall the definition of ψ-expansive mapping (see papers [31], [32], [33], [54], [80] and references therein).

Definition 2.18. [33] A mapping $A : D(A) \subset X \to X$ is said to be ψ-expansive if there exists a function $\psi : [0, +\infty) \to [0, +\infty)$ such that for every $x, y \in D(A)$, the inequality $\|Ax - Ay\| \geq \psi(\|x - y\|)$ holds, with ψ satisfying

- $\psi(0) = 0$;
- $\psi(r) > 0 \quad \forall r > 0$;
- Either ψ is continuous or it is nondecreasing.

Finally, we recall that

Definition 2.19. [12] Let X be a real Banach space and C a nonempty and closed subset of X. A mapping $T : C \to C$ is said to be demiclosed (at y), if for any $(x_n)_{n\in N}$, in C, the conditions $x_n \rightharpoonup x$ and $Tx_n \to y$ imply $Tx = y$.

For a nonexpansive mapping defined on a uniformly convex Banach space, the following holds:

Lemma 2.20. *([12, Theorem 3]) Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X, and let $T : C \to X$ be a nonexpansive mapping. Then $I - T$ is demiclosed, that is*

$$(x_n)_{n\in\mathbb{N}} \subset C, \quad x_n \rightharpoonup x, \quad (I - T)x_n \to y \implies (I - T)x = y.$$

3. Convergence Results

Let's start recalling iteration (1.7):

$$\begin{cases} x_0, u \in H, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(\frac{x_n+x_{n+1}}{2}) + \alpha_n\mu_n(u - x_n), \quad n \geq 0, \end{cases}$$

We can notice that this method is well defined. Indeed, if $T : H \to H$ is a nonexpansive mapping, y, z, w are given points in H and $\alpha \in (0,1)$, then the mapping $\tilde{T} : H \to H$ defined by

$$\tilde{T}x = \alpha y + (1-\alpha)T(\frac{z+x}{2}) + w$$

is a contraction with constant $\frac{1-\alpha}{2}$. Therefore $\tilde{T}$ has a unique fixed point. Hence we prove the following result:

Theorem 3.1. *[61, Theorem 3.2] Let H be a real Hilbert space and $T : H \to H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume that the sequences $(\alpha_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}} \in (0,1]$ satisfy the conditions*

(1) $\lim_{n\to\infty} \alpha_n = 0,$

(2) $\sum_{n=0}^{\infty} \alpha_n\mu_n = +\infty,$

(3) $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n\mu_n} = 0,$

(4) $\lim_{n\to\infty} \frac{|\mu_n - \mu_{n-1}|}{\mu_n} = 0.$

Then the sequence $(x_n)_{n\in\mathbb{N}}$, generated by (**??**)*, strongly converges to the point $q_u \in Fix(T)$ nearest to u, that is $\|u - q_u\| = \min_{x\in Fix(T)} \|u - x\|$.*

A possible choise of parameters satisfying the hypotheses of Theorem 3.1 is given by

$$\alpha_n = \mu_n = \frac{1}{\sqrt{n}}$$

.

Remark 3.2. We point out that if $u = 0 \in H$, under the same hypotheses of Theorem 3.1, we get that the sequence $(x_n)_{n\in\mathbb{N}}$ generated by

$$\begin{cases} x_0 \in H, \\ x_{n+1} = \alpha_n x_n + \alpha_n T(\frac{x_n+x_{n+1}}{2}) - \alpha_n\mu_n x_n \quad n \geq 0 \end{cases}$$

strongly converges to the point $q \in Fix(T)$ nearest to $0 \in H$, that is, the fixed point of T with minimum norm $\|q\| = \min_{x\in Fix(T)} \|x\|$.

A particular case of Theorem 3.1 is obtained for $\mu_n = 1$. The resulting algorithm is a Halpern-type iteration, for which we claim that:

Corollary 3.3. *[61, Corollary 3.4] Let H be a real Hilbert space and $T : H \to H$ a nonexpansive maping with $Fix(T) \neq \emptyset$. If the sequence $(\alpha_n)_{n\in\mathbb{N}}, \subset (0,1]$ satisfies the conditions*

(1) $\lim_{n\to\infty} \alpha_n = 0,$

(2)′ $\sum_{n=0}^{\infty} \alpha_n = +\infty$,

(3)′ $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$.

Then the sequence $(x_n)_{n\in\mathbb{N}}$ *generated by*

$$\begin{cases} x_0, u \in H, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)T(\frac{x_n + x_{n+1}}{2}), \quad n \geq 0 \end{cases} \tag{3.1}$$

strongly converges to the point $q_u \in Fix(T)$ *nearest to* u*, that is*
$\|u - q_u\| = \min_{x\in Fix(T)} \|u - x\|$

Remark 3.4. Even in this case, it is considered the eventuality $u = 0 \in H$. Therefore, we get that, under the same assumptions of Corollary 3.3, the sequence $(x_n)_{n\in\mathbb{N}}$ generated by

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\alpha_n)T(\frac{x_n + x_{n+1}}{2}), \quad n \geq 0 \end{cases}$$

strongly converges to the point $q \in Fix(T)$ nearest to $0 \in H$, that is the fixed point of T with minimum norm $\|q\| = \min_{x\in Fix(T)} \|x\|$.

For the sequence generated by (1.9)

$$\begin{cases} x_0, u \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \quad n \geq 0 \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}) + \alpha_n\mu_n(u - x_n), \quad n \geq 0, \end{cases}$$

we prove the following:

Theorem 3.5. *[61, Theorem 4.2] Let* H *be a real Hilbert space and* $T : H \to H$ *a nonexpansive mapping with* $Fix(T) \neq \emptyset$*. Under the assumptions* (1), (2), (3), (4) *of Theorem 3.1, if the sequences* $(\alpha_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}, (s_n)_{n\in}$ $(0, 1]$ *satisfy also the hypotheses*

(5) $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{\alpha_n\mu_n} = 0$

(6) $\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n\mu_n} = 0$

(7) $\limsup_{n\to\infty} \beta_n(1 - s_n) + s_n > 0$,

then $(x_n)_{n\in\mathbb{N}}$ *generated by* (1.9) *strongly converges to the point* ${x_u}^* \in Fix(T)$ *nearest to* u*, that is*

$$\|u - {x_u}^*\| = \min_{x\in Fix(T)} \|u - x\|$$

An example of control sequences satisfying conditions $(1) - (7)$ is given by

$$\alpha_n = s_n = \mu_n = \frac{1}{\sqrt{n}}, \quad \beta_n = \frac{n}{n+1}$$

.

Remark 3.6. In case $u = 0$, under the same assumptions of Theorem 3.5, we obtain strong convergence of the sequence $(x_n)_{n\in\mathbb{N}}$, generated by

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \quad n \geq 0 \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}) - \alpha_n \mu_n x_n, \quad n \geq 0, \end{cases}$$

to the point $x* \in Fix(T)$ nearest to $0 \in H$, that is the fixed point of T with minimum norm $\|x^*\| = \min_{x\in Fix(T)} \|x\|$.

As in the previous case, in the eventuality $\mu_n = 1$, for all $n \in \mathbb{N}$, we have the following convergence result for a Halpern-type method:

Corollary 3.7. *[61, Corollary 4.4] Let H be a real Hilbert space and $T : H \to H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume that conditions $(1), (2)', (3)'$ of Corollary 3.3 hold and that the sequences $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}$ satisfy also the hypotheses:*

$(5)'$ $\lim_{n\to\infty} \dfrac{|s_n - s_{n-1}|}{\alpha_n} = 0,$

$(6)'$ $\lim_{n\to\infty} \dfrac{|\beta_n - \beta_{n-1}|}{\alpha_n} = 0,$

(7) $\limsup_{n\to\infty} \beta_n(1-s_n) + s_n > 0,$

then the sequence $(x_n)_{n\in\mathbb{N}}$ generated by

$$\begin{cases} x_0, u \in H \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \quad n \geq 0 \\ x_{n+1} = \alpha_n u + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}), \quad n \geq 0, \end{cases} \tag{3.2}$$

strongly converges to the point $x_u{}^* \in Fix(T)$ nearest to u.

Remark 3.8. For $u = 0$, under the same assumptions of Corollary 3.7, we obtain strong convergence of the sequence $(x_n)_{n\in\mathbb{N}}$, generated by

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \quad n \geq 0, \\ x_{n+1} = (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}), \quad n \geq 0, \end{cases}$$

to the point $x* \in Fix(T)$ nearest to $0 \in H$, that is the fixed point of T with minimum norm $\|x^*\| = \min_{x\in Fix(T)} \|x\|$.

Let us consider the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0,$

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty,$

(iii) $\limsup_{n\to\infty} \beta_n(1-\beta_n)(1-s_n) > 0.$

A strong convergence result of the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (1.10) to a fixed point of a quasi-nonexpansive operator is proved in the framework of Hilbert spaces:

Theorem 3.9. *[60, Theorem 3.2] Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to C$ be a quasi-nonexpansive mapping, with $I - T$ demiclosed at 0, and $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Let $(\alpha_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ be three sequences in $(0, 1)$, satisfying the conditions $(i), (ii), (iii)$. Then, the sequence $(x_n)_{n\in\mathbb{N}}$, defined in (**??**), strongly converges to $\bar{q} \in Fix(T)$, which is the unique solution in $Fix(T)$ of the variational inequality (VI)*

$$\langle \bar{q} - f(\bar{q}), \bar{q} - x\rangle \leq 0, \quad \forall x \in Fix(T). \tag{3.3}$$

Remark 3.10. Conditions

- T is quasi-nonexpansive,
- $I - T$ is demiclosed at 0,

appearing in the previous theorem, are not related. One could consider the following examples to confirm this:

Example 3.11. [41, Example 2.3]
Let $H = \mathbb{R}$, $C = [0, +\infty)$ and $T : C \to C$ a mapping defined by

$$Tx = \begin{cases} \frac{2x}{x^2+1} & x \in (1, +\infty) \\ 0 \quad x \in [0, 1] \end{cases}$$

It results that

- $Fix(T) = \{0\}$;
- T is discontinuous;
- T is quasi-nonexpansive, indeed if $x \in [0, 1]$ then $|Tx - 0| = 0 \leq |x - 0|$; while, if $x \in (1, +\infty)$ then $|Tx - 0| = \frac{2x}{1+x^2} \leq 1 \leq |x - 0|$;
- Considering the sequence $x_n = 1 + \frac{1}{n} \in (1, 2)$, it results that $x_n \to 1 \notin Fix(T)$ and $|x_n - Tx_n| \to 0$, thus $I - T$ is not demiclosed at 0.

Example 3.12. [18, Example 8.2]
Let $H = \mathbb{R}$, $C = [0, 1]$ and $T : C \to H$ defined by $Tx = 1 - x^{\frac{2}{3}}$.

- $Fix(T) = \{q\}$, with $q \in (0, 1)$;
- T is a continuous pseudo-contraction, since $I - T$ is monotone; therefore $I - T$ is demiclosed at 0 (see [93, Demi-closedness Principle]);
- T is not quasi-nonexpansive since:
  - if $x = 0$, then $|Tx - q| \leq |x - q|$ implies $1 - q \leq q$, and hence $q \geq \frac{1}{2}$
  - if $x = 1$, then $|Tx - q| \leq |x - q|$ implies $q \leq 1 - q$, and hence $q \leq \frac{1}{2}$

Thus it must be $q = \frac{1}{2}$, that is a contradiction.

Remark 3.13. There exist mappings $T : C \to C$, with $Fix(T)$ nonempty, which are quasi-nonexpansive and such that $I - T$ is demiclosed at 0. Among these, in addition to nonexpansive mappings, let us mention

- Nonspreading mappings, introduced by Kohsaka and Takahashi in 2008:

  Definition 3.14. [52] Let X be a smooth, strictly convex and reflexive Banach space, let J be the duality mapping of X and let C a nonempty, closed and convex subset of X. Then, a mapping $S : C \to C$ is said to be nonspreading if

  $$\Phi(Sx, Sy) + \Phi(Sy, Sx) \leq \Phi(Sx, y) + \Phi(Sy, x),$$

  for all $x, y \in C$, where $\Phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in X$.

Particularly, if X is a Hilbert space, it is known that $\Phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Then a nonspreading mapping $S : C \to C$ in a Hilbert space is defined as follows:

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2,$$

for all $x, y \in C$.

- L-hybrid mappings, introduced by Aoyam et al. in 2010:

  Definition 3.15. [2] Let $T : H \to H$ be a mapping and L a nonnegative number. We will say that T is L-hybrid, signified as $T \in \mathcal{H}_L$, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + L\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in H.$$

Theorem 3.9 holds for these classes of mappings.

Remark 3.16. In Theorem 3.9, no additional assumption has been formulated for the limit of the sequence $(s_n)_{n\in\mathbb{N}}$, hence it could be that $\lim_{n\to\infty} s_n = 0$.

Theorem 3.9 has been improved by Garcia-Falset, Marino and Zaccone in [34] in the more general setting of p-uniformly convex Banach spaces, in which inequalities analogous to (2.1) and (2.2), valid in Hilbert spaces, hold (see Xu's paper [86] for a detailed survey).

Theorem 3.17. *[34, Theorem 3.2] Let X be a p-uniformly convex Banach space, with $1 < p < +\infty$, having a weakly sequentially continuous duality mapping J_ϕ. Let C be a nonempty, closed and convex subset of X, $T : C \to C$ a quasi-nonexpansive mapping, such that $I - T$ is demiclosed at 0,1 and $f : C \to C$ a θ-contraction, for a certain $\theta \in [0, 1)$.*

Let $(x_n)_{n\in\mathbb{N}}$ be generated by (1.10). Assume that the sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$ satisfy the conditions $(i), (ii), (iii)$. If $Fix(T)$ is the sunny nonexpansive retract of C, with $Q : C \to Fix(T)$ sunny nonexpansive retraction, then $(x_n)_{n\in\mathbb{N}}$ strongly converges to $\bar{q} = Q(f(\bar{q}))$. Further $\bar{q}$ is the unique solution in $Fix(T)$ of the variational inequality

$$\langle f(\bar{q}) - \bar{q}, j_\phi(x - \bar{q})\rangle \leq 0, \quad \forall x \in Fix(T) \tag{3.4}$$

Remark 3.18. In order to have a strong convergence result in the setting of a p-uniformly Banach space X, we assume that X has weakly sequentially continuous duality mapping J_ϕ, for a certain gauge function ϕ. This hypothesis, compared to the weak continuity for the normalized duality map J, that is frequently assumed in trying to extend some results from the setting of Hilbert spaces to that of Banach spaces (see [47], [78] and other works), allows us to include, for instance, also the sequential spaces l_p. Indeed l_p spaces, for $p \neq 2$, fail to have weakly continuous map J, but they have generalized duality map J_p weakly sequentially continuous (for a detailed survey, see [90]).

Concerning the hypotheses assumed for the control sequence in Theorem (3.17), as well as in Theorem 3.9, no additional assumption has been formulated for the limit of sequence $(s_n)_{n\in\mathbb{N}} \subset (0, 1)$, that, for instance, may converge to zero. (see Example 4.2).

In light of Theorem (3.17), we include the following corollary:

Corollary 3.19. *[34, Corollary 3.4] Let X be a p-uniformly convex Banach space, for $1 < p < \infty$, having a weakly sequentially continuous duality mapping J_ϕ. Let C be a nonempty, closed and convex subset of X, $T : C \to C$ a quasi-nonexpansive mapping, such that $I - T$ is demiclosed at 0 and $Fix(T) = \{q\}$. Let $f : C \to C$ a θ-contraction, for a certain $\theta \in [0, 1)$.*

Let $(x_n)_{n\in\mathbb{N}}$ be the sequence generated by (1.10).

If the sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$ satisfy the conditions $(i), (ii), (iii)$, then $(x_n)_{n\in\mathbb{N}}$ strongly converges to q.

Remark 3.20. The assert follows from Theorem (3.17), considered that $Fix(T) = \{q\}$ is a sunny nonexpansive retract of C, with sunny nonexpansive retraction the constant function $Q(x) = q$.

If $X = H$ is a real Hilbert space, then X is 2-uniformly convex, with normalized duality mapping weakly sequentially continuous. Let C be a closed and convex subset of H. From [27, Theorem 1], it is known that if $T : C \to C$ is a quasi-nonexpansive then $Fix(T)$ is nonempty, closed and convex. Therefore the metric projection $P_{Fix(T)}$ is a sunny nonexpansive retraction from C onto $Fix(T)$. Since there is at most one sunny nonexpansive retraction (Lemma 2.16), we have that $Q \equiv P_{Fix(T)}$.

These considerations motivate the result that follows.

Corollary 3.21. *[34, Corollary 3.5] Let H be real Hilbert space. Let C be a nonempty, closed and convex subset of H, $T : C \to C$ a quasi-nonexpansive mapping, such that $I - T$ is demiclosed at 0, and $f : C \to C$ a θ-contraction, for a certain $\theta \in [0, 1)$.*

Let $(x_n)_{n\in\mathbb{N}}$ be generated by 1.10. Assume that the sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$ satisfy the conditions $(i), (ii)$ and (iii). Then $(x_n)_{n\in\mathbb{N}}$ strongly converges to $q \in Fix(T)$. Further q is the unique solution in $Fix(T)$ of the variational inequality

$$\langle f(q) - q, x - q\rangle \leq 0, \quad \forall x \in Fix(T).$$

Therefore Theorem 3.9 can be considered as a particular case of Theorem 3.17.

For the eventuality in which the Banach space X fails to have weakly sequentially continuous duality map J_ϕ, as occurs for L_p spaces, we establish a strong convergence result for the GVEMR, assuming the additional assumption that $I - T$ is ψ-expansive (for more details on this type of mapping, see [33] and references therein).

Theorem 3.22. *[34, Theorem 3.7] Let X be a p-uniformly convex Banach space, for $1 < p < \infty$, and $C \subset X$ a nonempty, closed and convex set. Let $T : C \to C$ a quasi-nonexpansive mapping such that $I - T$ is ψ-expansive, and $f : C \to C$ a θ-contraction for a certain $\theta \in (0, 1)$.*

Let $(x_n)_{n\in\mathbb{N}}$ the sequence generated by (1.10). If the parameters sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ satisfy the conditions $(i), (iii)$, then $(x_n)_{n\in\mathbb{N}}$ strongly converges to to the unique fixed point of T.

4. Examples

Inspired to [44, Example 2] by Iemoto and Takahashi, we give an example for the convergence result stated in Theorem 3.9 for the class of nonespreading operators:

Example 4.1. Let H be a Hilbert space. Assume that

$$\begin{aligned} B_1 &= \{x \in H \quad s.t. \quad \|x\| \leq 1\}, \\ B_2 &= \{x \in H \quad s.t. \quad \|x\| \leq 2\}, \\ B_3 &= \{x \in H \quad s.t. \quad \|x\| \leq 3\}. \end{aligned}$$

The mapping defined as

$$Sx = \begin{cases} 0 \quad if \quad x \in B_2 \\ P_{B_1}(x) \quad if \quad x \in B_3 \setminus B_2, \end{cases}$$

with P_{B_1} the metric projection of H onto B_1, is a nonspreading operator with $Fix(S) = \{0\}$, hence it is quasi-nonexpansive. It is known that $I - S$ is demiclosed at 0.

Moreover S is discontinuous, hence it is not nonexpansive.

Let us choose $H = \mathbb{R}$, $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n-1}{2n}$, $s_n = \frac{1}{n}$, $f(x) = \frac{x}{2}$, $x_0 = 3$.

The operator S can be written as

$$Sx = \begin{cases} 0 \quad if \quad x \in [-2,2] \\ P_{B_1}(x) \quad if \quad x \in [-3,3] \setminus [-2,2], \end{cases}$$

with $B_1 = [-1,1]$, $B_2 = [-2,2]$, $C = [-3,3]$.
While the sequence generated by (1.10) is given by

$$\bar{x}_{n+1} = \frac{n}{2(n+1)} x_n + \frac{n+2}{2(n+1)} S x_n, \quad n \geq 0$$

$$x_{n+1} = \frac{x_n}{2(n+1)} + \left(\frac{n}{n+1}\right) S \left(\frac{x_n}{n+1} + \frac{n}{n+1}\bar{x}_{n+1}\right), \quad n \geq 0.$$

Therefore the sequence $(x_n)_{n\in\mathbb{N}}$ is given by

$$\begin{aligned} x_0 &= 3, \\ x_1 &= \frac{3}{2}, \\ x_2 &= \frac{3}{8}, \\ &\cdots \\ x_{n+1} &= \frac{3}{(n+1)!2^{n+1}} \end{aligned}$$

that quickly converges to 0.

For Theorem (3.17) we include the following:

Example 4.2. Consider the real Banach space $X = l_p$, for $1 < p < +\infty$ endowed with the norm $\|x\| = \|x\|_p = \left[\sum_{n=1}^{\infty} |x_i|^p\right]^{\frac{1}{p}}$, for $x = (x_1, x_2, \cdots, x_n, \cdots)$.
Set

$$B_1 = \{x \in l_p : \|x\| \leq 1\}$$

$$B_2 = \{x \in l_p : \|x\| \leq 2\}$$

$$B_3 = \{x \in l_p : \|x\| \leq 3\}$$

Let $T : B_3 \to B_3$ the mapping defined as

$$Tx = \begin{cases} 0_{l_p} \quad x \in B_2 \\ P_{B_1}(x) \quad x \in B_3 \setminus B_2 \end{cases}$$

where $P_{B_1}(x)$ is the metric projection of X onto B_1.
Let $f : B_3 \to B_3$ the map defined as $f(x) = \frac{1}{2}x$.
Let us put $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{2(n+1)}$, $s_n = \frac{1}{n+1}$.
We point out the following considerations:

- l_p, with $1 < p < \infty$, is a p-uniformly convex Banach space with weakly continuous duality mapping J_p,
- $Fix(T) = \{0\}$,
- T is quasi-nonexpansive,
- T is not nonexpansive since it is discontinuous,

- $I - T$ is demiclosed at 0.

  Indeed if we consider $x_n \rightharpoonup x \in B_3 \setminus B_2$ then we have that

  $$\|x_n - Tx_n\| \geq |\|x_n\| - \|Tx_n\| \geq \|x_n\| - 1,$$

  so that $\liminf_{n\to+\infty} \|x_n - Tx_n\| \geq \|x\| - 1 \geq 1$.

  We approach to the same conclusion if $x_n \rightharpoonup x \in B_2$, with $x \neq 0$, considering that $\liminf_{n\to+\infty} \|x_n - Tx_n\| \geq \|x\| > 0$.

- f is a $\frac{1}{2}$-contraction,

- sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$ satisfy conditions (i), (ii) and (iii) of Theorem 3.17.

Fixed $x_0 = (3, 0, 0, \cdots)$, then algorithm (1.10) is given by

$$\begin{cases} \bar{x}_{n+1} = \frac{n}{2(n+1)} x_n + \frac{n+2}{2(n+1)} Tx_n, & n \geq 0 \\ x_{n+1} = \frac{x_n}{2(n+1)} + \frac{n}{n+1} T\left(\frac{x_n}{n+1} + \frac{n}{n+1}\bar{x}_{n+1}\right) & n \geq 0 \end{cases}$$

It generates the sequence:

$$\begin{aligned} x_0 &= (3, 0, 0, \cdots), \\ x_1 &= \frac{1}{2}(3, 0, 0, \cdots), \\ x_2 &= \frac{1}{8}(3, 0, 0, \cdots), \\ &\cdots \\ x_{n+1} &= \frac{1}{(n+1)!2^{n+1}}(3, 0, 0, \cdots), \end{aligned}$$

that converges to 0, as $n \to +\infty$.

For Theorem (3.22), we give the following example in a p-uniformly convex Banach space that fails to have a weakly continuous duality mapping:

Example 4.3. Let $X = L_p([0, 1])$, with $1 < p < +\infty$, endowed with the norm

$$\|x\| = \|x\|_p = \Big[\int_{[0,1]} |x(s)|^p \, ds\Big]^{\frac{1}{p}}$$

and

$$B_1 = \{x \in L_p[0, 1] : \|x\| \leq 1\}$$
$$B_2 = \{x \in L_p[0, 1] : \|x\| \leq 2\}$$

Let $T : B_2 \to B_2$ the map defined as

$$Tx = \begin{cases} 0 & x \in B_1 \\ -x & x \in B_2 \setminus B_1 \end{cases} \tag{4.1}$$

and $f : B_2 \to B_2$ such that $f(x) = \frac{x}{2}$.

If we set $\alpha_n = \frac{1}{(n+1)^2}$, $\beta_n = \frac{n}{2(n+1)}$, $s_n = \frac{1}{n+1}$ then algorithm (1.10) becomes

$$\begin{cases} \bar{x}_{n+1} = \frac{n}{2(n+1)} x_n + \frac{n+2}{2(n+1)} Tx_n, & n \geq 0 \\ x_{n+1} = \frac{x_n}{2(n+1)^2} + \left(\frac{n^2+2n}{(n+1)^2}\right) T\left(\frac{x_n}{n+1} + \frac{n}{n+1}\bar{x}_{n+1}\right) & n \geq 0 \end{cases}$$

We observe that

- L_p is a p-uniformly convex Banach space that fails to have a weakly continuous duality mapping,
- $Fix(T) = \{0\}$,
- T is quasi-nonexpansive,
- T is discontinuous so it is not nonexpansive,
- $I - T$ is ψ-expansive. Indeed:
  - if $x, y \in B_1$, then $\|(I-T)x - (I-T)y\| = \|x-y\|$,
  - if $x, y \in B_2$, then $\|(I-T)x - (I-T)y\| = 2\|x-y\|$,
  - if $x \in B_1$ and $y \in B_2$, then

$$\begin{aligned}\|(I-T)x-(I-T)y\| &= \|x-2y\|\\ &\geq 2\|y\|-\|x\|\\ &\geq 1\end{aligned}$$

Since

$$\|x-y\| \leq 3 \quad \forall x \in B_1, y \in B_2,$$

then $\|(I-T)x - (I-T)y\| \geq \frac{1}{3}\|x-y\|$.
Therefore $I - T$ is ψ-expansive with $\psi(t) = \frac{t}{3}$, for $t \in [0, +\infty)$,

- f is a $\frac{1}{2}$-contraction,
- The sequences α_n, β_n and s_n satisfy conditions 1) and 2) of the preceding theorem.

  Fixed the constant function $x_0 = 2$ in $L_p([0,1])$, then the other terms of the sequence $(x_n)_{n\in\mathbb{N}}$ are given by

$$x_0 = 2$$
$$x_1 = 1$$
$$x_2 = \frac{1}{8}$$
$$x_3 = \frac{1}{144}$$
$$\cdots$$

It can be noticed that, for $n \geq n_0 \in \mathbb{N}$, the term $T\left(\frac{x_n}{n} + \frac{n-1}{n}\bar{x}_{n+1}\right)$ vanishes since $\|\frac{x_n}{n} + \frac{n-1}{n}\bar{x}_{n+1}\| \leq 1$, then $x_{n+1} = \frac{x_n}{2n^2}$ and $\lim_{n\to+\infty} \|\frac{x_n}{2n^2}\| = 0$.

References

[1] R.P. Agarwal, D. O'Regan, D.R. Sahu, Fixed Point Theory for Lipschitzian-type Mappings with Applications, Series Topological Fixed Point Theory and Its Applications, Springer, New York, 2009.

[2] K. Aoyama, S. Iemoto, F. Kohsaka, W. Takahashi, Fixed point and ergodic theorems for λ-hybrid mappings in Hilbert spaces, J. Nonlinear Convex Anal., 11 (2010) 335-343.

[3] M.A. Alghamdi, M.A. Alghamdi, N. Shahzad, H.K. Xu, The implicit midpoint rule for nonexpansive mappings, Fixed Point Theory Appl. 2014, 96 (2014).

[4] E. Asplund, Positivity of duality mappings, Bull. Amer. Math. Soc, 73 (1967), 200-203.

[5] S. Banach Sur les opérations dans les ensembles abstraits et leur application aux équations integrals , Fundamenta Mathematicae, (1922).

[6] V. Berinde, Iterative Approximation of Fixed Points, second ed., in: Lecture Notes in Mathematics, vol. 1912, Springer, Berlin, 2007.

[7] A. Beurling and A. E. Livingston, A theorem on duality mappings in Banach spaces, Ark. Mat. 4 (1961), 405-411.

[8] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. U.S.A., 54 (1965) 1041-1044.

[9] F.E. Browder, Fixed point theorems for nonlinear semicontractive mappings in Banach paces, Arch. Rational Mech. Anal. 21 (1966) 259-269.
[10] Browder, F.E., Convergence of approximants to fixed points of nonexpansive nonlinear maps in Banach spaces, Arch. Rat. Mech. Anal., 24 (1967) 82-90.
[11] F. E. Browder, Nonlnear mappings of nonexpansive and accretive type in Banach spaces, Bull. Amer. Math. Soc. 73 (1967) 875-882.
[12] F.E. Browder, Semicontractive and semiaccretive nonlinear mappings in Banach spaces, Bull. Amer. Math. Soc. 74 (1968) 660-665.
[13] F. E. Browder and W. V. Petryshyn, The solution by iteration of nonlinear functional equations in Banach spaces, Bull. Amer. Math. Sot. 72 (1966) 571-575.
[14] F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20 (1967) 197-228.
[15] R. E. Bruck, Asymptotic behavior of nonexpansive mappings, Contemporary Math. 18 (1983) 1-47.
[16] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Problems, 20 (2004) 103-120.
[17] B. Carnahan, A. H. Luther, and J. Wilkes, Applied Numerical Methods, John Wiley and Sons, New York, 1969.
[18] C. Chidume, Geometric Properties of Banach Spaces and Nonlinear Iterations, vol. 1965 of Lecture Notes in Mathematics, Springer, London, UK, 2009.
[19] C. E. Chidume and C. O. Chidume, Iterative approximation of fixed points of nonexpansive mappings, Journal of Mathematical Analysis and Applications, vol. 318 (2006) no. 1, 288-295.
[20] Chidume, C.E., Mutangadura, S.A., An example on the Mann iteration method for Lipschitz pseudocontractions, Proc. Amer. Math. Soc., 129 (2001) No.8, 2359-2363.
[21] F. Cianciaruso, G. Marino, A. Rugiano, B. Scardamaglia, On strong convergence of viscosity type method using averaged type mappings, J. Nonlinear Convex Anal., 16 (2015) 1619-1640.
[22] I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer, Dordrecht, 1990;
[23] K. Deimling, Zeros of accretive operators, Manuscripta Math., 13 (1974) 365- 374.
[24] F. Deutsch, I. Yamada, Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings, Numer. Funct. Anal. Optim. 19 (1998) 33–56.
[25] J. B. Diaz and F. T. Metcalf, On the set of subsequential limit points of successive approximations, Trans. Amer. Math. Soc. 135 (1969), 459-485.
[26] W.G. Dotson, On the Mann iterative process, Trans. Amer. Math. Soc. 149 (1970) 65-73.
[27] W. G. Dotson, Jr., Fixed points of quasi-nonexpansive mappings, J. Austral. Math. Soc. 13 (1972) 167-170.
[28] M. Edelstein, A remark on a theorem of M. A. Krasnoselski, Amer. Math. Monthly 73 (1966) 509-510.
[29] R. L. Franks and R. P. Marzec, A theorem on mean value iterations, Proc. Amer. Math. Soc. 30 (1971) 324-326.
[30] J.Garcia-Falset, W. Kaczor, T. Kuczumow, S. Reich, Weak convergence theorems for asymptotically nonexpansive mappings and semigroups, Nonlinear Anal., 43 (2001) 377-401.
[31] J. Garcia-Falset, Existence of fixed points for the sum of two operators, Math. Nachr. 283 (12) (2010) 1726-1757.
[32] J. Garcia-Falset, K. Latrach, E. Moreno-Gálvez, M. A Taoudi, Shaefer-Krasnoselskii fixed points theorems using a usual measure of weak noncompactness, J. Diff. Equ. 352 (2012) 3436-3452.
[33] J. Garcia-Falset, O. Muniz-Perez, Fixed point theory for 1-set contractive and pseudocontractive mappings, Applied Mathematics and Computation, 219 (2013), 6843-6855.
[34] J. Garcia-Falset, G. Marino, R. Zaccone, An explicit midpoint algorithm in Banach spaces, J. Nonlinear and Convex Anal., 18 (2017) No: 11, 1933–1952.
[35] A. Genel and J. Lindenstrauss, An example concerning fixed points, Israel J. Math., 22 (1975) 81-86.
[36] M.K. Ghosh, L. Debnath, Convergence of Ishikawa iterates of quasi-nonexpansive mappings, J. Math. Anal. Appl., 207 (1997) 96-103.
[37] D. Gohde, Zum Prinzip der kontraktiven Abbildung, (German) Math. Nachr., 30 (1965) 251-258.
[38] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Stud. Adv. Math., vol. 28, Cambridge Univ. Press, 1990.
[39] B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc., 73 (1967) 957-961 .
[40] S. He, C. Yang , Solving the variational inequality problem defined on intersection of finite level sets, Abstr. Appl. Anal. Vol. 2013 (2013), Article ID 942315, 8 pages;
[41] Z. He, W-S Du, Nonlinear algorithms approach to split common solution problems, Fixed Point Theory Appl. 2012, Article ID 130 (2012).
[42] Hicks, T.L., Kubicek, J.R., On the Mann iteration process in Hilbert space, J. Math. Anal. Appl., 59 (1977) 498-504.
[43] N. Hussain, G. Marino, L. Muglia, L. Alamri, On some Mann's type iterative algorithms, Fixed Point Theory Appl 2015 (2015), Article ID 17;
[44] S. Iemoto and W. Takahashi, Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space, Nonlinear Anal., 71 (2009) 2082-2089.
[45] S. Ishikawa, Fixed points and iterations of a nonexpansive mapping in a Banach space, Proc. Amer. Math. Soc. 59 (1976) 65-71.
[46] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1974) 147-150.

[47] J.S. Jung, Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 302 (2005) 509-520.
[48] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. Optim., 13 (2003) 938-945.
[49] Y. Ke. C. Ma, The generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces, Fixed Point Theory Appl. 2015, 190 (2015), 21 pages;
[50] T.H. Kim, H.K. Xu, Strong convergence of modified Mann iterations, Nonlinear Anal., 61 (2005) 51-60.
[51] W. A. Kirk, A fixed point theorem for mappings which do not increase distance, Amer. Math. Monthly, 72 (1965) 1004-1006.
[52] F. Kohsaka, W. Takahashi, Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces, Archiv der Mathematik, 91 (2008) No. 2, 166-177.
[53] M. A. Krasnoselskii, Two observations about the method of successive approximations, Usp. Math. Nauk., 10(1955) 123-127.
[54] K. Latrach, M. Aziz Taoudi, A. Zeghal, Some fixed point theorems of Schauder and the Krasnoselskii type and application to nonlinear transport equations, J. Diff. Equ., 221 (2006) 256-271.
[55] P. L. Lions, Approximation de points fixes de contractions, C.R. Acad. Sci. Ser. A-B Paris, 284 (1977) 1357-1359.
[56] P.E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal., 16 (2008) 899-912.
[57] P.E. Maingé, The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces, Computers and Mathematics with Applications, 59 (2010) No. 1, 74-79.
[58] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc., 4 (1953) 506-510.
[59] G. Marino, H.K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl., 318 (2006) No.1, 43-52.
[60] G. Marino, B. Scardamaglia, R. Zaccone A general viscosity explicit midpoint rule for quasi-nonexpansive mappings, J. Nonlinear and Convex Anal., 18 (2017) No.1, 137-148.
[61] G. Marino, R. Zaccone, On strong convergence of some midpoint type methods for nonexpansive mappings, J. Nonlinear Var. Anal., 1 (2017) No. 2, 159-174.
[62] C. Martinez-Yanes, H.K. Xu, Strong convergence of the CQ method for fixed point iteration processes, Nonlinear Anal., 64 (2006) 2400-2411.
[63] D. P. Milman, On some criteria for the regularity of spaces of the type (B) (Russian), Dokl. Akad. Nauk SSSR 20 (1938)
[64] A. Moudafi, Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl., 241 (2000) 46-55.
[65] K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl., 279 (2003) 372-379.
[66] J.G. O'Hara, P. Pillay, H.K. Xu, Iterative approaches to finding nearest common fixed points of nonexpansive mappings in Hilbert spaces, Nonlinear Anal., 54 (2003) 1417-1426.
[67] E. Hairer, S. P. Nørsett and G. Wanner, Solving Ordinary Differential Equations I: Nonstiff Problems, Springer Series in Computational Mathematics, 2nd edn. (Springer-Verlag, 1993).
[68] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Sot., 73 (1967) 591-597.
[69] W. V. Petryshyn, Construction of fixed points of demicompact mappings in Hilbert space, Math. Anal. Appl., 14 (1966) 276-284.
[70] B. J. Pettis, A proof that every uniformly convex space is reflexive, Duke Math. J. 5 (1939) no. 2, 249–253.
[71] S. Reich, Asymptotic behavior of contractions in Banach spaces, J. Math. Anal. Appl. 44 (1973) 57-70.
[72] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 67 (1979) 274-276.
[73] S. Reich, A limit theorem for projections, Linear and Multilinear Algebra, 13 (1983) no. 3, 281-290.
[74] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl., 75 (1980) 287-292.
[75] H. Schaefer, Uber die Methode sukzessive Approximationen, Jahre Deutsch Math. Verein 59 (1957) 131-40.
[76] N. Shioji and W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, Proc. Amer. Math. Soc., 125 (1997) No.12, 3641-3645.
[77] Y. Song and X. Chai, Halpern iteration for firmly type nonexpansive mappings, Nonlinear Analysis: Theory, Methods And Applications, 71(2009) No. 10, 4500-4506.
[78] Y. Song, R. Chen, Viscosity approximation methods for nonexpansive nonself-mappings, J. Math. Anal. Appl., 321 (2006) 316-326;
[79] T. Suzuki, A sufficient and necessary condition for Halpern-type strong convergence to fixed points of nonexpansive mappings, Proceedings of the American Mathematical Society, 135 (2007) No. 1, 99-106.
[80] M.A. Taoudi, N. Salhi, B. Ghribi, Integrable solutions of a mixed type operator equation, Appl. Math. Comput., 216 (2010) 1150-1157.
[81] M. Tian, A general iterative algorithm for nonexpansive mappings in Hilbert spaces, Nonlinear Anal., 73 (2010) 689-694.
[82] F. Tricomi, Un teorema sulla convergenza delle successioni formate delle successive iterate di una funzione di una variabile reale, Giorn. Mat. Battaglini, 54 (1916) 1-9.
[83] W. V. Petryshyn and T. E. Williamson, Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mapping, J. Math. Anal. Appl., 43 (1973) 459-497.
[84] R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math., 58 (1992) 486-491.

[85] K. Wongchan and S. Saejung, On the strong convergence of viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces, Abstr. Appl. Anal., 2011 (2011), Article ID 385843, 9 pages

[86] H. K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal., 16 (1991) 1127-1138.

[87] H.K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. 66 (2002) 240-256.

[88] H.K. Xu, Another control condition in an iterative method for nonexpansive mappings, Bull. Austral. Math. Soc. 65 (2002) 109-113.

[89] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. 298 (2004) 279–291.

[90] H. K. Xu, T. H. Kim, X. Yin, Weak continuity of the normalized duality map, Journal of Nonlinear and Convex Analysis, 15 (2014) no 3, 595-604.

[91] H. K. Xu, M. A. Alghamdi, N. Shahzad, The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces, Fixed Point Theory Appl. 41 (2015) 12 pages..

[92] C. Yang, S. He, Convergence of explicit iterative algorithms for solving a class of variational inequalities, Wseas Trans. Math., 13 (2014) 830-839;

[93] H. Zhou, Convergence theorems of common fixed points for a finite family of Lipschitz pseudocontractions in Banach spaces, Nonlinear Analysis, 68 (2008) 2977-2983.

Advances in the Theory of Nonlinear Analysis and its Applications 2 (2018) No. 1, 62–69
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications
ISSN: 2587-2648
Peer-Reviewed Scientific Journal

Local convergence for a Chebyshev-type method in Banach space free of derivatives

Ioannis K. Argyros[a], Santhosh George[b]

[a]Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA
[b]Department of Mathematical and Computational Sciences, National Institute of Technology, Karnataka, 575 025, , India

Abstract

This paper is devoted to the study of a Chebyshev-type method free of derivatives for solving nonlinear equations in Banach spaces. Using the idea of restricted convergence domain, we extended the applicability of the Chebyshev-type methods. Our convergence conditions are weaker than the conditions used in earlier studies. Therefore the applicability of the method is extended. Numerical examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

Keywords: Chebyshev-type method restricted convergence domain radius of convergence local convergence
2010 MSC: 65D10, 49M15, 74G20, 41A25

1. Introduction

Let $F : \Omega \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a Fréchet differentiable operator between the Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$. Due to the wide applications, finding a solution for equation

$$F(x) = 0 \tag{1}$$

is an important problem in applied mathematics and computational sciences. Convergence analysis of iterative methods require assumptions on the Fréchet derivatives of the operator F. That restricts the applicability of these methods.

Email addresses: `iargyros@cameron.edu` (Ioannis K. Argyros), `sgeorge@nitk.ac.in` (Santhosh George)

Received March 02, 2018, Accepted March 25, 2018.

In this paper we study the seventh convergence order Chebyshev-type method [13]:

$$\begin{aligned} y_n &= x_n - A_n^{-1}F(x_n), \\ z_n &= y_n - B_nF(y_n), \\ x_{n+1} &= z_n - C_nF(z_n), \end{aligned} \tag{2}$$

where

$$\begin{aligned} A_n &= [w_n, x_n; F], \\ B_n &= (3I - A_n^{-1}([y_n, x_n; F] + [y_n, w_n; F]))A_n^{-1}, \\ C_n &= [z_n, x_n; F]^{-1}([w_n, x_n; F] + [y_n, x_n; F] - [z_n, x_n; F])A_n^{-1}, \\ w_n &= x_n + \gamma F(x_n), \ \ \gamma \in \mathbb{R}, \end{aligned}$$

$[., .; F]$ denotes a divided difference of order one on Ω^2 and $x_0 \in \Omega$ is an initial point. Throughout this paper $L(\mathcal{B}_2, \mathcal{B}_1)$ denotes the set of bounded linear operators between $\mathcal{B}_1$ and $\mathcal{B}_2$.

The study of convergence of iterative algorithms is involving categories: semi-local and local convergence analysis. The semi-local convergence is based on the information around an initial point, to derive conditions ensuring the convergence of these algorithms, while the local convergence is based on the information around a solution to get estimates of the computed radii of the convergence balls. Local results are important since they tell us about the degree of difficulty in choosing initial points.

The above method was studied in [13]. Convergence analysis in [13] is based on the assumptions on the Fréchet derivative F up to the order seven. In this study, we use only assumptions on the first Fréchet derivative of the operator F in our convergence analysis, so the the method (2) can be applied to solve equations but the earlier results cannot be applied [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] (see Example 3.2).

The rest of the paper is structured as follows. In Section 2 we present the local convergence analysis of the method (2). We also provide a radius of convergence, computable error bounds and a uniqueness result. Numerical examples are given in the last section.

2. Local convergence

We need a definition concerning the monotonicity of functions.

Definition 2.1. *Let $T : D \subseteq \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function. We say T is nondecreasing on Ω, if for each $(a_1, a_2), (a_3, a_4) \in D$ with $a_1 \leq a_3, a_2 \leq a_4$,*

$$T(a_1, a_2) \leq T(a_3, a_4). \tag{1}$$

Moreover, T is increasing on D, if $a_1 \leq a_3$ and $a_2 < a_4$ or $a_1 < a_3$ and $a_2 \leq a_4$ or $a_1 < a_3$ and $a_2 < a_4$ imply $T(a_1, a_2) < T(a_2, a_4)$.

Let us introduce some parameters and scalar functions to be used in the local convergence of method (2) that follows. Let $\gamma \in \mathbb{R}$ and $\delta \geq 0$ be parameters and let function $\omega_0 : [0, +\infty) \times [0, +\infty) \longrightarrow [0, +\infty)$ be continuous and nondecreasing with $\omega_0(0, 0) = 0$. Define parameter r_0 by

$$r_0 = \sup\{t \in [0, +\infty) : \omega_0(\delta t, t) < 1\}. \tag{2}$$

Let $v_0 : [0, r_0) \longrightarrow [0, +\infty), \omega_1 : [0, r_0) \times [0, r_0) \longrightarrow [0, +\infty)$ be continuous and nondecreasing functions. Define functions g_1 and h_1 on the interval $[0, r_0)$ by

$$g_1(t) = \frac{\omega_1(|\gamma|v_0(t)t, t)}{1 - \omega_0(\delta t, t)}$$

and

$$h_1(t) = g_1(t) - 1.$$

Suppose that

$$\omega_1(0,0) < 1. \tag{3}$$

Suppose that

$$h_1(t) \longrightarrow \text{ a positive number or } +\infty \text{ as } t \longrightarrow r_0^-. \tag{4}$$

We have by (3) that

$$h_1(0) = \frac{\omega_1(0,0)}{1-\omega_0(0,0)} - 1 < 0. \tag{5}$$

Then, by (4), (5) and the intermediate value theorem equation $h_1(t) = 0$ has solutions in the interval $(0, r_0)$. Denote by r_1 the smallest such zero. Let $v : [0, r_0) \longrightarrow [0, +\infty), \omega_2 : [0, r_0) \longrightarrow [0, +\infty)$ and $\omega_3 : [0, r_0) \times [0, r_0) \longrightarrow [0, +\infty)$ be continuous and nondecreasing functions. Define functions β, g_2, h_2 on $[0, r_0)$ by

$$\beta(t) = \frac{1 + \omega_0(\delta t, t) + \omega_2((\delta + g_1(t)t)t) + \omega_3((\delta + g_1(t))t, |\gamma| v_0(t)t) v(g_1(t)t)}{(1-\omega_0(\delta t, t))^2}$$

$$g_2(t) = (1 + \beta(t) v(g_1(t)t)) g_1(t)$$

and

$$h_2(t) = g_2(t) - 1.$$

Suppose that

$$(1 + \beta(0) v(0)) \omega_1(0,0) < 1 \tag{6}$$

and

$$h_2(t) \longrightarrow \text{ a positive number or } +\infty \text{ as } t \longrightarrow r_0^- \tag{7}$$

We get by (6) that $h_2(0) < 0$. So, by the intermediate value theorem equation $h_2(t) = 0$ has solutions in the interval $(0, r_0)$. Denote by r_2 the smallest solution of $h_2(t) = 0$ in the interval $(0, r_0)$. Define functions p_1 and h_{p_1} on the interval $[0, r_0)$ by

$$p_1(t) = \omega_0(g_2(t)t, g_1(t)t)$$

and

$$h_{p_1}(t) = p_1(t) - 1.$$

We have by the definition of function w_0 that $h_{p_1}(0) < 0$. Suppose that

$$h_{p_1}(t) \longrightarrow \text{ a positive number or } +\infty \text{ as } t \longrightarrow r_0^-. \tag{8}$$

Denote by r_{p_1} the smallest solution of equation $h_{p_1}(t) = 0$ on the interval $(0, r_0)$. Define functions φ, g_3, h_3 on the interval $[0, r_{p_1})$ by

$$\varphi(t) = \frac{1 + \omega_2((\delta + g_2(t)t) + \omega_0(g_2(t)t, t)}{(1 - p_1(t))(1 - \omega_0(\delta t, t))}$$

$$g_3(t) = (1 + \varphi(t) v(g_2(t)t)) g_2(t)$$

and

$$h_3(t) = g_3(t) - 1.$$

Suppose that

$$(1 + (1 + \omega_2(0)) v(0)) (1 + \beta(0) v(0)) \omega_1(0,0) < 1, \tag{9}$$

and

$$h_3(t) \longrightarrow \text{ a positive number or } +\infty \text{ as } t \longrightarrow r_{p_1}^-. \tag{10}$$

We have that $h_3(0) < 0$. Denote by r_3 the smallest solution of equation $h_3(t) = 0$ in the interval $(0, r_0)$. Define the radius of convergence r by

$$r = \min\{r_i\} \;\; i = 1, 2, 3. \tag{11}$$

Then, for each $t \in [0, r)$

$$0 \leq g_i(t) < 1 \tag{12}$$

$$0 \leq p(t) < 1 \tag{13}$$

and

$$0 \leq p_1(t) < 1. \tag{14}$$

Finally, define R^* by

$$R^* = \max\{r, \delta r\}. \tag{15}$$

Some alternatives to the aforementioned conditions are:
Equation

$$w_0(\delta t, t) = 1$$

has positive solutions. Denoted by r_0 the smallest such solution. Functions $v_0, \omega_1, v, \omega_2$ and ω_3 defined on the same intervals as before are increasing. Then, clearly conditions (4), (7), (8) and (10) hold.

We can show the local convergence analysis of method (2).

Theorem 2.2. *Let $F : \Omega \subset \mathcal{B}_1 \to \mathcal{B}_2$ be a continuously Fréchet differentiable operator and let $[.,.;F] : \Omega \times \Omega \longrightarrow L(\mathcal{B}_1, \mathcal{B}_2)$be a divided difference of order one on $\Omega \times \Omega$ for F. Suppose: there exists $x^* \in \Omega$ and function $\omega_0 : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ continuous and nondecreasing with $\omega_0(0, 0) = 0$ such that for each $x, y \in \Omega$,*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1); \tag{16}$$

and

$$\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| \leq \omega_0(\|x - x^*\|, \|y - x^*\|). \tag{17}$$

Let $\Omega_0 = \Omega \cap B(x^*, r_0)$. There exist $\gamma \in \mathbb{R}, \delta \geq 0$, functions $v_0, v, \omega_2 : [0, r_0) \to [0, +\infty)$, $\omega_1, \omega_3 : [0, r_0) \times [0, r_0) \to [0, +\infty)$ such that for each $x, y, z \in \Omega_0$

$$\|I + \gamma[x, x^*; F]\| \leq \delta, \tag{18}$$

$$\|[x, x^*; F]\| \leq v_0(\|x - x^*\|), \tag{19}$$

$$\|F'(x^*)^{-1}[x, x^*; F]\| \leq v(\|x - x^*\|), \tag{20}$$

$$\|F'(x^*)^{-1}([x, y; F] - [y, x^*; F])\| \leq \omega_1(\|x - y\|, \|y - x^*\|), \tag{21}$$

$$\|F'(x^*)^{-1}([x, y; F] - [z, y; F])\| \leq \omega_2(\|x - z\|), \tag{22}$$

$$\|F'(x^*)^{-1}([x, y; F] - [z, x; F])\| \leq \omega_3(\|x - z\|, \|y - x\|), \tag{23}$$

$$\bar{B}(x^*, R^*) \subseteq \Omega, \tag{24}$$

(4), (7), (8) and (9) hold. Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (2) is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{25}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| \tag{26}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{27}$$

where the functions $g_i, i = 1, 2, 3$ are defined previously. Furthermore, if there exists for $R_1 \geq r$ such that

$$\omega_0(R_1, 0) < 1 \;\text{ or }\; \omega_0(0, R_1) < 1, \tag{28}$$

then the limit point x^* is the only solution of equation $F(x) = 0$ in $\Omega_1 := \Omega \cap B(x^*, R_1)$.

Proof. The proof is induction based. By hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, the definition of w_0, A_0, r the fact that ω_0 is nondecreasing, we have that

$$\begin{aligned}
& \|F'(x^*)^{-1}(A_0 - F'(x^*))\| \\
\leq\ & (\text{by } (17))\, \omega_0(\|w_0 - x^*\|, \|x_0 - x^*\|) \\
\leq\ & (\text{by } (2))\, \omega_0(\|x_0 - x^* + \gamma[x_0, x^*; F](x_0 - x^*)\|, \|x_0 - x^*\|) \\
\leq\ & \omega_0(\|(I + \gamma[x_0, x^*; F])(x_0 - x^*)\|, \|x_0 - x^*\|) \\
\leq\ & (\text{by } (1) \text{ and } (2))\, \omega_0(\delta r, r) < 1.
\end{aligned} \tag{29}$$

In view of (29) and the Banach perturbation lemma [2, 3], we get that A_0 is invertible and

$$\|A_0^{-1} F'(x^*)\| \leq \frac{1}{1 - \omega_0(\delta\|x_0 - x^*\|, \|x_0 - x^*\|)}. \tag{30}$$

We also have that y_0 is well defined by the first substep of method (2) for $n = 0$. We can write by method (2) and (16) that

$$\begin{aligned}
y_0 - x^* &= (\text{by } (2))\, x_0 - x^* - A_0^{-1}F(x_0) \\
&= (\text{by } (10))\, A_0^{-1}(A_0(x_0 - x^*) - [x_0, x^*; F](x_0 - x^*)) \\
&= A_0^{-1}F'(x^*)[F'(x^*)^{-1}([u_0, x_0; F] - [x_0, x^*; F])](x_0 - x^*).
\end{aligned} \tag{31}$$

By the first substep of method (2) for $n = 0$, the definition of r, g_1, the fact that w_1 is nondecreasing, we obtain in turn that

$$\begin{aligned}
& \|y_0 - x^*\| \\
\leq\ & (\text{by } (2))\, \|A_0^{-1}F'(x^*)\| \|F'(x^*)^{-1}([w_0, x_0; F] - [x_0, x^*; F])\| \|x_0 - x^*\| \\
\leq\ & (\text{by } (21) \text{ and } (30))\, \frac{\omega_1(\|w_0 - x_0\|, \|x_0 - x^*\|)\|x_0 - x^*\|}{1 - \omega_0(\delta\|x_0 - x^*\|, \|x_0 - x^*\|)} \\
\leq\ & (\text{by } (2) \text{ and } (19))\, \frac{\omega_1(|\gamma|v_0(\|x_0 - x_0\|)\|x_0 - x^*\|}{1 - \omega_0(\delta\|x_0 - x^*\|, \|x_0 - x^*\|)}\|x_0 - x^*\| \\
=\ & g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq (\text{by } (7) \text{ for } i = 1)\, \|x_0 - x^*\| < r,
\end{aligned} \tag{32}$$

which shows (25) for $n = 0$ and $y_0 \in B(x^*, r)$. We need an estimate on $\|B_0 F'(x^*)\|$. By the definition of B_0, β and the fact that functions $\omega_0, \omega_2, \omega_3$ are nondecreasing, we have in turn that

$$\begin{aligned}
& \|B_0 F'(x^*)\| \\
=\ & \|A_0^{-1}(3A_0 - [y_0, x_0; F] - [y_0, w_0; F])A_0^{-1}\| \\
\leq\ & \|A_0^{-1}F'(x^*)\|^2[\|F'(x^*)^{-1}F'(x^*)\| \\
& +\|F'(x^*)^{-1}([w_0, w_0; F] - F'(x^*))\| \\
& +\|F'(x^*)^{-1}([w_0, x_0; F] - [y_0, x_0; F])\| \\
& +\|F'(x^*)^{-1}([w_0, x_0; F] - [y_0, w_0; F])\|] \\
\leq\ & (\text{by } (22), (23, (32)) \\
& \frac{1 + \omega_0(\|x_0 - x^*\|, \|x_0 - x^*\|) + \omega_2(\|w_0 - y_0\|) + \omega_3(\|w_0 - y_0\|, \|x_0 - w_0\|)}{(1 - \omega_0(\delta\|x_0 - x^*\|, \|x_0 - x^*\|))^2} \\
\leq\ & \beta(\|x_0 - x^*\|).
\end{aligned} \tag{33}$$

By the second substep of method (2), the fact that function v is nondecreasing , β is nonnegative and the definition of g_2 we get in turn that

$$\begin{aligned}
& \|z_0 - x^*\| \\
\leq\ & \text{(by the triangle inequality) } \|y_0 - x^*\| + \|B_0 F'(x^*)\| \|F'(x^*)^{-1} F(y_0)\| \\
\leq\ & \text{(by (33)) } (1 + \beta(\|y_0 - x^*\|) v(\|y_0 - x^*\|)) \|y_0 - x^*\| \\
\leq\ & \text{(by (32) and (33))} \\
& (1 + \beta(\|x_0 - x^*\|) v(g_1(\|x_0 - x^*\| \|x_0 - x^*\|)) g_1(\|x_0 - x^*\|) \|x_0 - x^*\| \\
=\ & \text{(by the definition of function } g_2) \, g_2(\|x_0 - x^*\|) \|x_0 - x^*\| \\
\leq\ & \text{(by (12) (for i=2)) } \|x_0 - x^*\| < r, \qquad (34)
\end{aligned}$$

which shows (26) for $n = 0$ and $z_0 \in B(x^*, r)$. We must show $[z_0, y_0; F]^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$. We get that

$$\begin{aligned}
& \|F'(x^*)^{-1}([z_0, y_0; F] - F'(x^*))\| \\
\leq\ & \text{(by (17)) } \omega_0(\|z_0 - x^*\|, \|y_0 - x^*\|) \\
\leq\ & \text{(by (32) and (34)) } \omega_0(g_2(\|x_0 - x^*\|) \|x_0 - x^*\|, g_1(\|x_0 - x^*\|) \|x_0 - x^*\|) \\
=\ & \text{(by the definition of function } p_1) \, p_1(\|x_0 - x^*\|) \qquad (35) \\
\leq\ & \text{(by (14)) } p_1(r) < 1,
\end{aligned}$$

so

$$\|[z_0, y_0; F]^{-1} F'(x^*)\| \leq \frac{1}{1 - p_1(\|x_0 - x^*\|)}. \qquad (36)$$

To obtain an estimate on $\|C_0 F'(x^*)\|$,

$$\begin{aligned}
& \|F'(x^*)^{-1}(([w_0, x_0; F] - [z_0, x_0; F]) + (]y_0, x_0; F] - F'(x^*)) + F'(x^*))\| \\
\leq\ & \text{(by (17) and (22)) } 1 + \omega_2(\|w_0 - z_0\|) + \omega_0(\|y_0 - x^*\|, \|x_0 - x^*\|) \\
\leq\ & \text{(by the triangle inequality)} \\
& 1 + \omega_2(\|w_0 - x^*\| + \|z_0 - x^*\|) + \omega_0(\|y_0 - x^*\|, \|x_0 - x^*\|),
\end{aligned}$$

so by the definition of φ

$$\begin{aligned}
\|C_0 F'(x^*)\| \ \leq\ & \text{(by (31) and (36))} \\
& \frac{1 + \omega_2(\|w_0 - x^*\|, \|z_0 - x^*\|) + \omega_0(\|y_0 - x^*\|, \|x_0 - x^*\|)}{(1 - p_1(\|x_0 - x^*\|))(1 - \omega_0 \delta(\|x_0 - x^*\|, \|x_0 - x^*\|))} \\
\leq\ & \varphi(\|x_0 - x^*\|) \qquad (37)
\end{aligned}$$

leading by the third substep of method (2) (by (11), (12) (for $i = 2$), and (37)) to the estimate

$$\begin{aligned}
& \|x_1 - x^*\| \\
\leq\ & \text{(by the triangle inequality) } \|z_0 - x^*\| + \|C_0 F'(x^*)\| \|F'(x^*)^{-1} F(z_0)\| \\
\leq\ & \text{(by (20) and (37)) } (1 + \varphi(\|x_0 - x^*\|) v(\|z_0 - x^*\|)) \|z_0 - x^*\| \\
\leq\ & \text{(by (34)) } (1 + \varphi(\|x_0 - x^*\|) v(g_2(\|x_0 - x^*\|) \|x_0 - x^*\|) \\
& \times g_2(\|x_0 - x^*\|) \|x_0 - x^*\| \\
=\ & \text{(by the definition of } g_3) \, g_3(\|x_0 - x^*\|) \|x_0 - x^*\| \qquad (38) \\
\leq\ & \text{(by (12) for } i = 3) \, \|x_0 - x^*\| < r,
\end{aligned}$$

which shows (27) and $x_1 \in U(x^*, r)$. The induction for (25)– (27) is completed in an analogous way, if we replace x_0, y_0, z_0, u_0, x_1 by $x_k, y_k, z_k, u_k, x_{k+1}$, respectively, in the previous estimates. Then, it follows from the estimate

$$\|x_{k+1} - x^*\| \leq c \|x_k - x^*\| < r, \qquad (39)$$

where $c = g_3(\|x_0 - x^*\|) \in [0, 1)$, that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. Let $y^* \in \Omega_1$ with $F(y^*) = 0$. Define Q by $Q = [y^*, x^*; f]$. Then, we get that

$$\begin{aligned} \|F'(x^*)^{-1}(Q - F'(x^*))\| &\le (\text{by } (17))\, \omega_0(0, \|y^* - x^*\|) \\ &\le (\text{by } (28))\, \omega_0(0, R_1) < 1, \end{aligned} \tag{40}$$

so Q is invertible. Then, from the identity $0 = F(y^*) - F(x^*) = Q(y^* - x^*)$, we conclude that $x^* = y^*$. □

Remark 2.3. *Method (2) is not changing if we use the new instead of the old conditions [13]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC) [14]*

$$\xi = \frac{ln \frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \qquad \text{for each } n = 1, 2, \dots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{ln \frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{ln \frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \qquad \text{for each } \ n = 0, 1, 2, \dots$$

instead of the error bounds obtained in Theorem 2.2.

3. Numerical Examples

The numerical examples are presented in this section. We choose

$$[x, y; F] = \int_0^1 F'(y + \theta(x - y))d\theta.$$

Example 3.1. *Let $X = \mathbb{R}^3, \Omega = \bar{U}(0, 1), x^* = (0, 0, 0)^T$. Define function F on Ω for $q = (x, y, z)^T$ by*

$$F(q) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(q) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the (18)-(23) conditions, we get $\omega_0(s,t) = \frac{L_0}{2}(s+t), \omega_1(s,t) = \frac{Ls+L_0t}{2}, \omega_2(t) = \frac{1}{2}e^{\frac{1}{L_0}}t, \omega_3(s,t) = \frac{L}{2}(s+t), v_0(t) = v(t) = \frac{1}{2}(1+e^{\frac{1}{L_0}}), r_0 = \frac{1}{L_0}, \delta = 1 + \frac{1}{2}|\gamma|(1+e^{\frac{1}{L_0}}), L_0 = e - 1$ and $L = e$. The parameters are

$$r_1 = 0.2010,\ r_2 = 0.0830,\ r_3 = 0.0639 = r.$$

Example 3.2. *Let $X = C[0, 1], \Omega = \bar{B}(x^*, 1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [7, 11] defined by*

$$x(s) = \int_0^1 K(s, t)\frac{x(t)^2}{2}dt,$$

where the kernel K is the Green's function defined on the interval $[0, 1] \times [0, 1]$ by

$$K(s, t) = \begin{cases} (1 - s)t, & t \le s \\ s(1 - t), & s \le t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1), where $F : C[0,1] \longrightarrow C[0,1])$ is defined by

$$F(x)(s) = x(s) - \int_0^1 K(s,t)\frac{x(t)^2}{2}dt.$$

Notice that [5, 7, 8]

$$\| \int_0^1 K(s,t)dt\| \leq \frac{1}{8}.$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^1 K(s,t)x(t)dt,$$

so since $F'(x^*(s)) = I$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \frac{1}{8}\|x - y\|.$$

We can choose $\omega_0(t,s) = \omega_1(t,s) = \omega_3(s,t) = \frac{t+s}{16}$, $\omega_2(t) = \frac{1}{16}t, v(t) = \frac{9}{16}$ and $\delta = 1 + |\gamma|\frac{9}{16}$. The parameters are

$$r_1 = 0.5805,\ r_2 = 0.2623,\ r_3 = 0.1463 = r.$$

References

[1] S. Amat, M.A. Hernández, N. Romero, Semilocal convergence of a sixth order iterative method for quadratic equations, Applied Numerical Mathematics, 62 (2012), 833-841.

[2] I.K. Argyros, Computational theory of iterative methods. Series: Studies in Computational Mathematics, 15, Editors: C.K.Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[3] I. K Argyros, A semilocal convergence analysis for directional Newton methods. Math. Comput. 80 (2011), 327–343.

[4] I.K. Argyros, S. Hilout, Weaker conditions for the convergence of Newton's method. J. Complexity 28 (2012) 364–387.

[5] I. K. Argyros and Said Hilout, Computational methods in nonlinear analysis. Efficient algorithms, fixed point theory and applications, World Scientific, 2013.

[6] I.K. Argyros and H. Ren, Improved local analysis for certain class of iterative methods with cubic convergence, Numerical Algorithms, 59(2012), 505-521.

[7] J. M. Gutiérrez, A.A. Magreñán and N. Romero, On the semi-local convergence of Newton-Kantorovich method under center-Lipschitz conditions, Applied Mathematics and Computation, 221 (2013), 79-88.

[8] L.V. Kantorovich, G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.

[9] A. A. Magrenan, Different anomalies in a Jarratt family of iterative root finding methods, Appl. Math. Comput. 233, (2014), 29-38.

[10] A. A. Magrenan, A new tool to study real dynamics: The convergence plane, Appl. Math. Comput. 248, (2014), 29-38.

[11] A.N .Romero, J.A. Ezquerro, M .A. Hernandez, Approximacion de soluciones de algunas equacuaciones integrals de Hammerstein mediante metodos iterativos tipo. Newton, XXI Congresode ecuaciones diferenciales y aplicaciones Universidad de Castilla-La Mancha (2009)

[12] J.R. Sharma, P.K. Guha and R. Sharma, An efficient fourth order weighted-Newton method for systems of nonlinear equations, Numerical Algorithms, 62, 2, (2013), 307-323.

[13] J.R. Sharma, H. Arora, A novel derivative free algorithm with seventh order convergence for solving systems of nonlinear equations, Numer. Algor., 67, (2014), 917–933.

[14] S. Weerakoon, T.G.I. Fernando, A variant of Newton's method with accelerated third-order convergence, Appl. Math. Lett. 13, (2000), 87-93.