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Forcing linearity numbers for coatomic modules

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Abstract
We show that an integer $n \in \mathbb{N} \cup \{0\}$ is the forcing linearity number of a coatomic module over an arbitrary commutative ring with identity if and only if $n \in \{0,1,2,\infty\} \cup \{q+2 \,|\, q$ is a prime power$\}$.

Keywords: Homogeneous functions, Forcing linearity numbers, Coatomic modules
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1. Introduction

Throughout this paper R shall denote a commutative ring with identity and V a unital right R-module. Consider the set $M_R(V) := \{f : V \to V \,|\, f(vr) = f(v)r$ for all $r \in R, v \in V\}$. Under the operations of pointwise addition and composition of functions, $M_R(V)$ is a near-ring with identity, called the near-ring of homogeneous functions. Note that $M_R(V)$ contains the endomorphism ring $End_R(V)$. The question arises how much linearity is needed on a function $f \in M_R(V)$ to ensure that f is linear on all of V, i.e. $f \in End_R(V)$. More precisely, we say that a collection $\{W_i | i \in I\}$ of proper submodules forces linearity on V, if whenever $f \in M_R(V)$ and f is linear on each $W_i, i \in I$, then $f \in End_R(V)$. Thus $M_R(V) = End_R(V)$ if and only if the empty collection forces linearity on V. The smallest number of modules which force linearity on V gives rise to the forcing linearity number of V.

Definition 1.1. *[3] Let V be an R–module. The forcing linearity number $f\ln(V) \in \mathbb{N} \cup \{0,\infty\}$ of V is defined as follows:*

1. *If $M_R(V) = End_R(V)$, then $f\ln(V) = 0$.*
2. *If $M_R(V) \neq End_R(V)$, and there is some finite collection $\{W_i | 1 \le i \le n\}, n \in \mathbb{N}$, of proper submodules of V which forces linearity on V, but no collection of fewer than n proper submodules forces linearity, then we say that $f\ln(V) = n$.*
3. *If neither 1. or 2. holds, then we say that $f\ln(V) = \infty$.*

Forcing linearity numbers have been found for several classes of rings and modules, see for example [3], [4], [5] and their references. In section 2 we determine the forcing linearity number of coatomic modules over an arbitrary commutative ring R with identity. An R–module V is called coatomic, if every proper submodule is contained in a maximal submodule of V. For example a finitely generated module or a semisimple module over any ring is coatomic. For a commutative noetherian local ring, the coatomic modules have been characterized in [7].

2. Forcing linearity numbers of coatomic modules

For an R–module V and subsets S_1, S_2 of V let $(S_1 : S_2) = \{r \in R | S_2 r \subseteq S_1\}$. For $v \in V$ let $Ann(v) = \{r \in R | vr = 0\}$.

Theorem 2.1. *Let V be an R–module and let M,N be maximal submodules of V, $M \neq N$. The following are equivalent:*

1. *The collection $\{M,N\}$ does not force linearity.*
2. $\exists\, w \neq 0 \in V : (M:V) = (N:V) = Ann(w)$.

Proof. $1 \Rightarrow 2$: Since $\{M,N\}$ does not force linearity on V, there exists a function $f \in M_R(V)$ such that f is linear on the submodules M,N, but $f \notin End_R(V)$. Let $u,v \in V$ be such that $w := f(u+v) - f(u) - f(v) \neq 0$. Since $M \neq N$, and M,N are maximal, we have that $M+N = V$. For every $v \in V - M$, $(M:v) = (M:V)$, therefore $(M:V)$ and $(N:V)$ are maximal ideals. If $(M:V) \neq (N:V)$, then $(M:V)+(N:V) = R$, hence $r+s = 1$ for some $r \in (M:V)$, $s \in (N:V)$. Now $wr = f(ur+vr) - f(ur) - f(vr) = f(ur) + f(vr) - f(ur) - f(vr) = 0$, since f is linear on M. Similarly, $ws = 0$, hence $w = w.1 = w(r+s) = 0$, a contradiction. Thus $(M:V) = (N:V)$, and since $(M:V) \subseteq Ann(w)$ and $(M:V)$ is a maximal ideal, it follows that $(M:V) = Ann(w)$.
$2 \Rightarrow 1$: Let $v \in V - M$. Then $(M:v) = (M:V) = Ann(w)$ and $h : V/M \to Rw$, $h(vr/M) := wr$ is an isomorphism. Define a function $f : V \to V$ as follows: For $m \in M, n \in N$ let

$$f(m+n) := \begin{cases} h(n/M) & \text{if } m+n \notin M \cup N \\ 0 & \text{otherwise} \end{cases}$$

Since $M+N = V$, f is defined on V. We show that f is well–defined. Suppose $m_1+n_1 = m_2+n_2$, $m_1,m_2 \in M$, $n_1,n_2 \in N$. If $m_1+n_1 \in M \cup N$, then $f(m_1+n_1) = f(m_2+n_2) = 0$. If $m_1+n_1 \notin M \cup N$, then $n_1/M = n_2/M$, hence $f(m_1+n_1) = h(n_1/M) = h(n_2/M) = f(m_2+n_2)$. Next we show that f is homogeneous. Let $S := V - (M \cup N)$. If $m+n \in S$, then $(N:m) = (N:V)$ and $(M:n) = (M:V)$. By our assumption $(M:V) = (N:V) = Ann(w) \neq R$, hence $(N:m) = (M:n)$. If $r \notin (M:n)$, then $r \notin (N:m)$, which implies that $(m+n)r = mr+nr \in S$, hence $f((m+n)r) = h(nr/M) = h(n/M)r = f(m+n)r$. If $r \in (M:n)$, then $(m+n)r \notin S$, hence $f(m+n)r = h(n/M)r = h(nr/M) = h(0) = 0 = f((m+n)r)$. Now suppose $m+n \notin S$. Then $m+n \in M \cup N$, hence $(m+n)r \in M \cup N$ for all $r \in R$. Thus $f(m+n)r = 0 = f((m+n)r)$. It now follows that $f \in M_R(V)$. Since $f|M = f|N = 0$, f is linear on M and N. However, for $m \in M - N$ and $n \in N - M$, we have that $m+n \in S$, thus $f(m+n) = h(n/M) \neq 0$, since h is an isomorphism, whereas $f(m)+f(n) = 0$, so $f \notin End_R(V)$.Therefore the collection $\{M,N\}$ does not force linearity on V. □

For an $R-$module V let $Rad(V)$ denote the Jacobson radical of V and let $J := Rad(R)$. Recall that an $R-$module V is called local, if V contains a unique maximal submodule.

Theorem 2.2. *For a noncyclic coatomic module V, the following are equivalent:*

1. $f\ln(V) > 2$.
2. *$I := (Rad(V):V)$ is a maximal ideal and $I = Ann(w)$ for some $0 \neq w \in V$.*

Proof. $1 \Rightarrow 2$: Let $\mathbf{M}$ denote the collection of all maximal submodules of V. Since V is coatomic, $\mathbf{M} \neq \emptyset$. If there exist $M_1,M_2 \in \mathbf{M}$ such that $(M_1:V) \neq (M_2:V)$, then by Theorem 2.1 the collection $\{M_1,M_2\}$ forces linearity on V. Thus $(M_1:V) = (M_2:V)$ for all $M_1,M_2 \in \mathbf{M}$ and $I = \bigcap\{(M:V) |\, M \in \mathbf{M}\} = (M:V)$ for all $M \in \mathbf{M}$, hence $I = (Rad(V):V)$ is a maximal ideal. Like in the proof of Theorem 1, we see that $I = Ann(w)$ for some $w \neq 0$.
$2 \Rightarrow 1$: Suppose that V is a local module with unique maximal submodule M. Let $v \in V - M$. If $vR \neq V$, then vR is contained in a maximal submodule, which implies $vR \subseteq M$, a contradiction. Consequently $vR = V$ for all $v \in V - M$, which contradicts our assumption that V is noncyclic. Therefore there exist at least two maximal submodules. Suppose $f\ln(V) \leq 2$. Then there exists a collection of submodules $\{S_1,S_2\}$ which forces linearity on V. Since V is coatomic, there exist maximal submodules M_1,M_2 such that $S_1 \subseteq M_1$, $S_2 \subseteq M_2$. Without loss of generality we may assume that $M_1 \neq M_2$ (otherwise we can choose another maximal submodule, since V is not local). Then $\{M_1,M_2\}$ also forces linearity on V. We have $(Rad(V):V) \subseteq (M_1:V) \neq R$. By our assumptions $(Rad(V):V)$ is a maximal ideal, hence $(Rad(V):V) = (M_1:V) = (M_2:V)$. Also, $(Rad(V):V) = Ann(w)$ for some $0 \neq w \in V$. Therefore $\{M_1,M_2\}$ does not force linearity by Theorem 1, a contradiction. □

Theorem 2.3. *Let V be coatomic. Suppose $I := (Rad(V):V)$ is a maximal ideal of R and there exists $0 \neq w \in V$ such that $I = Ann(w)$. Then*

$$fln_R(V) = fln_{R/I}(V/Rad(V))$$

Proof. We first show that $f\ln_{R/I}(V/Rad(V)) \leq f\ln_R(V)$. Let $\{W_i | i \in I\}$ be a collection of proper submodules which forces linearity on V. Since V is coatomic, we may assume that each W_i, $i \in I$, is maximal. We show that the collection $\{W_i/Rad(V) |\ i \in I\}$ forces linearity on $V/Rad(V)$. Suppose that this is not the case. Then there exists a homogeneous function $f : V/Rad(V) \to V/Rad(V)$, which is linear on each submodule $W_i/Rad(V)$, $i \in I$, but not linear on $V/Rad(V)$. Let $\pi_M : V/Rad(V) \to V/M$ denote the projection of $V/Rad(V)$ onto V/M for a maximal submodule M. Since f is not linear, there exists a maximal submodule M of V such that $\pi_M f : V/Rad(V) \to V/M$ is not linear. Since I is a maximal ideal, $I = (M : V)$, hence $w(M : V) = 0$, which implies $V/M \simeq wR$. Thus we obtain a homogeneous map $f_1 : V/Rad(V) \to wR$, which is linear on each submodule $W_i/Rad(V)$, $i \in I$. If $g : V \to V$ is defined by $g(v) := f_1(v/Rad(V))$, then $g \in M_R(V)$ and linear on each W_i, $i \in I$, but not linear on V, a contradiction to our assumption that $\{W_i | i \in I\}$ forces linearity on V. For the reverse inequality suppose first that $f\ln_{R/I}(V/Rad(V)) \leq 1$. Since $V/Rad(V)$ is a vector space over the field R/I, it follows from Theorem 3.1 in [3] that $\dim_{R/I}(V/Rad(V)) = 1$. Note that $Rad(V)$ is a superfluous submodule, since V is coatomic. It follows that V is cyclic, hence $f\ln_{R/I}(V/Rad(V)) = 0 = f\ln(V)$. If $dim_{R/I}(V/Rad(V)) = 2$ or $f\ln_{R/I}(V/Rad(V)) \geq 2$ and R/I is infinite, we have that $f\ln_{R/I}(V/Rad(V)) = \infty$ by Theorem 3.1 in [3]. So suppose that $f\ln_{R/I}(V/Rad(V)) \geq 3$ and $|R/I| =: q \in \mathbb{N}$. By [3], 3.8 and 3.10, $f\ln_{R/I}(V/Rad(V)) = q+2$. Choose $\{r_1, ..., r_q\} \subseteq R$ such that $R/I = \{r_1/I, ..., r_q/I\}$. It suffices to give a collection of $q+2$ proper submodules which forces linearity on V. Let $\{b_i | i \in I\} \subseteq V$ be such that $\{b_i/Rad(V) | i \in I\}$ is a basis of the vector space $V/Rad(V)$. As we have seen above, $|I| \geq 3$, so we can choose pairwise different elements $i_1, i_2, i_3 \in I$. Let $\langle X \rangle$ denote the submodule generated by a subset $X \subseteq V$, and define $S_1 := \langle b_{i_1}, b_{i_2} \rangle + Rad(V)$, $S_2 := \langle b_{i_1} + b_{i_3} \rangle + \langle b_i | i \notin \{i_1, i_3\} \rangle + Rad(V)$, and for $r \in \{r_1, ..., r_q\}$ define $S_r := \langle b_{i_1} + rb_{i_2},\ b_{i_1} + b_{i_3} \rangle + \langle b_i | i \notin \{i_1, i_2, i_3\} \rangle + Rad(V)$. Note that all submodules are proper, since $Rad(V)$ is superfluous. Similarly as in Theorems 3.8, 3.10 in [3], one can prove that the collection $\{S_1, S_2\} \cup \{S_{r_i} |\ i \in \{1, ..., q\}\}$ forces linearity on V. □

For R local and J T-nilpotent, Theorem 2.3 has been proved in [4], Theorem 5.1. The following example shows that Theorem 2.3 is not true in general, if I is not the annihilator of some $0 \neq w \in V$.

Example 2.4. *Let $R := F[[x]]$ denote the ring of formal power series over a field F and let $V := R \times R$. Since R is local with radical $J = (x)$, $Rad(V) = VJ = (x) \times (x)$ and $I = (Rad(V) : V) = (x)$ is maximal. By [3], Corollary 2.4, $f\ln_R(V) = 1$. However, $f\ln_{R/I}(V/Rad(V)) = f\ln_F(F^2) = \infty$, by [3], Theorem 3.1.*

Theorem 2.5. *Let $n \in \mathbb{N} \cup \{0, \infty\}$. Then n is the forcing linearity number of a coatomic module over a commutative ring if and only if $n \in \{0, 1, 2, \infty\} \cup \{q+2 |\ q$ is a prime power$\}$.*

Proof. It is well-known that there exist coatomic modules V over a commutative ring R such that $f\ln_R(V) \in \{0, 1, 2, \infty\}$, see for example [5]. If V is a cyclic module, then $M_R(V) = End_R(V)$, hence $f\ln_R(V) = 0$. Now suppose $f\ln_R(V) > 2$. By Theorem 2.2, $I = (Rad(V) : V)$ is a maximal ideal and $I = Ann(w)$ for some $0 \neq w \in V$. By Theorem 2.3, $f\ln_R(V) = f\ln_{R/I}(V/Rad(V))$ and as we have remarked previously, $f\ln_{R/I}(V/Rad(V)) \in \{\infty\} \cup \{q+2 |$ q is a prime power$\}$. □

It is not known to the author, whether Theorem 2.5 is true for every module over a commutative ring.

There is a class of rings which have the property that every right module is coatomic, or which is easily seen to be equivalent, every nonzero right module has a maximal submodule.

Definition 2.6. *A ring R is called a right max–ring, if every right $R-$module is coatomic. See [6].*

Theorem 2.7. *[2] For a commutative ring R, the following are equivalent:*

1. *R is a max–ring.*
2. *J is T–nilpotent and R/J is von Neumann regular.*

Theorem 2.8. *Let V be a module over a commutative max–ring R. If R is not local, then $f\ln_R(V) \leq 2$.*

Proof. Suppose that R is not local, but $f\ln(V) > 2$. Since R is a max-ring, it follows from Theorem 2.7 and from [1], Proposition 18.3 that $Rad(V) = VJ$. By Theorem 2.2, $(Rad(V) : V) = (VJ : V)$ is a maximal ideal. We have $J \subseteq (VJ : V)$. Suppose that there exists an element $r \in (VJ : V) - J$. Then $r \notin M$ for some maximal ideal M of R. Let R_M, V_M denote the localisations of R, V at M. By [1], Proposition 18.3, $Rad(V_M) = V_M J_M$. Since R is a max-ring J is T-nilpotent, thus J_M is T-nilpotent. It follows from Theorem 2.5 that R_M is a max-ring, hence $Rad(V_M) = V_M J_M \neq V_M$. So let $w/1 \in V_M - Rad(V_M)$. From $r \in (VJ : V)$, $w/1 \cdot r/1 = wr/1 \in V_M J_M$. Since $r \notin M$, $r/1$ is invertible in R_M, hence $w/1 \in V_M J_M = Rad(V_M)$, a contradiction. It now follows that $J = (VJ : V)$ is a maximal ideal of R, which contradicts our assumption that R is not local. □

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Annihilation of $\mathrm{tor}_{\mathbb{Z}_p}(\mathscr{G}^{\mathrm{ab}}_{K,S})$ for real abelian extensions $K/\mathbb{Q}$

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Abstract
Let K be a real abelian extension of $\mathbb{Q}$. Let p be a prime number, S the set of p-places of K and $\mathscr{G}_{K,S}$ the Galois group of the maximal $S \cup \{\infty\}$-ramified pro-p-extension of K (i.e., unramified outside p and ∞). We revisit the problem of annihilation of the p-torsion group $\mathscr{T}_K := \mathrm{tor}_{\mathbb{Z}_p}(\mathscr{G}^{\mathrm{ab}}_{K,S})$ initiated by us and Oriat then systematized in our paper on the construction of p-adic L-functions in which we obtained a canonical ideal annihilator of $\mathscr{T}_K$ in full generality (1978–1981). Afterwards (1992–2014) some annihilators, using cyclotomic units, were proposed by Solomon, Belliard–Nguyen Quang Do, Nguyen Quang Do–Nicolas, All, Belliard–Martin. In this text, we improve our original papers and show that, in general, the Solomon elements are not optimal and/or partly degenerated. We obtain, whatever K and p, an universal non-degenerated annihilator in terms of p-adic logarithms of cyclotomic numbers related to L_p-functions at $s = 1$ of *primitive characters of* K (Theorem 9.4). Some computations are given with PARI programs; the case $p = 2$ is analyzed and illustrated in degrees 2, 3, 4 to test a conjecture.

Keywords: Class field theory, Abelian p-ramification; annihilation of p-torsion modules, p-adic L-functions, Stickelberger's elements, Cyclotomic units
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1. Introduction

Let $K/\mathbb{Q}$ be a real abelian extension of Galois group G_K. Let p be a prime number, S the set of p-places of K, and $\mathscr{G}_{K,S}$ the Galois group of the maximal S-ramified in the ordinary sense (i.e., unramified outside p and ∞, whence totally real if $p = 2$) pro-p-extension of K.

We revisit the classical problem of annihilation of the so-called $\mathbb{Z}_p[G_K]$-module $\mathscr{T}_K := \mathrm{tor}_{\mathbb{Z}_p}(\mathscr{G}^{\mathrm{ab}}_{K,S})$, as dual of $\mathrm{H}^2(\mathscr{G}_{K,S}, \mathbb{Z}_p(0))$. This was initiated by us [12] (1979) and improved by Oriat [22] (1981). Then in our paper [13] (1978/79) on the construction of p-adic L-functions (via an "arithmetic Mellin transform" from the "Spiegel involution" of suitable Stickelberger elements) we obtained incidentally a canonical ideal annihilator $\mathscr{A}_K$ of $\mathscr{T}_K$ in full generality, but our purpose, contrary to the present work, was the semi-simple case with p-adic characters and the annihilation of the isotopic components; this aspect has then been outdated by the "principal theorems" of Ribet–Mazur–Wiles–Kolyvagin–Greither (refer for instance to the bibliography of [15]), and many other contributions.

Afterwards some annihilators, using cyclotomic units, were proposed by Solomon [26] (1992), Belliard–Nguyen Quang Do [5] (2005), Nguyen Quang Do–Nicolas [21] (2011), All [1] (2013), Belliard–Martin [4] (2014), using techniques of Sinnott, Rubin, Thaine, Coleman, from Iwasawa's theory.

In this text, we translate into english some parts of the above 1978–1981's papers, written in french with tedious classical techniques, then we show that, in general, the Solomon elements Ψ_K are often degenerated regarding the annihilator $\mathscr{A}_K$, even for cyclic fields, and explain the origin of this gap due to trivialization of some Euler factors.

We obtain, whatever K and p (Theorem 9.4), an universal non-degenerated annihilator $\mathscr{A}_K$, in terms of p-adic logarithms of cyclotomic numbers, perhaps the best possible regarding these classical methods, but probably too general to cover all the possible Galois structures of $\mathscr{T}_K$, which raises the question of the existence of a better theorem than Stickelberger's one.

Indeed, if the semi-simple case is now completely solved, the non-semi-simple case is far to be known. Numerical experiments show in this case that the results are far to give the precise Galois structure of $\mathscr{T}_K$ (e.g., in direction of its Fitting ideal), moreover, it seems to us that many (all ?) papers are based on the classical reasoning with Kummer's theory and Leopoldt's Spiegel involution applied to Stickelberger's elements, even translated into Iwasawa's theory, without practical analysis of the results (e.g., with extensive numerical illustrations). So, there is some difficulties to compare these various contributions.

Thus, we perform some computations given with PARI programs [23] to analyse the quality of such annihilators, which is in general not addressed by papers dealing with Iwasawa's theory. We consider in a large part the case $p = 2$, illustrated in degrees 2, 3, 4 to test the Conjecture 5.7.

2. Notations and reminders on p-ramification theory

Let K be a real abelian number field of degree d, of Galois group G_K, and let $p \geq 2$ be a prime number; we denote by S the set of prime ideals of K dividing p. Let $\mathscr{G}_{K,S}$ be the Galois group of the maximal $S \cup \{\infty\}$-ramified pro-p-extension of K and let H_K^{pr} be the maximal abelian $S \cup \{\infty\}$-ramified pro-p-extension of K. To simplify, we put $\mathscr{G}_{K,S}^{\mathrm{ab}} =: \mathscr{G}_K$ and (e.g., [8, Chapter III, § (c)]):

$$\mathscr{T}_K := \mathrm{tor}_{\mathbb{Z}_p}(\mathscr{G}_K) = \mathrm{Gal}(H_K^{\mathrm{pr}}/K_\infty)$$

where $K_\infty = K\mathbb{Q}_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension of K; so:

$$\mathscr{G}_K \simeq \mathbb{Z}_p \bigoplus \mathscr{T}_K$$

since, in the abelian case, Leopoldt's conjecture is true.

We denote by F an extension of K such that H_K^{pr} is the direct compositum of K_∞ and F over K, then by $\mathscr{C}\ell_K^\infty$ the subgroup of the p-class group $\mathscr{C}\ell_K$ corresponding, by class field theory, to $\mathrm{Gal}(H_K/K_\infty \cap H_K)$, where H_K is the p-Hilbert class field. We have (where $\sim$ means "equality up to a p-adic unit"):

$$\#\mathscr{C}\ell_K^\infty \sim \frac{\#\mathscr{C}\ell_K}{[K_\infty \cap H_K : K]} \sim \#\mathscr{C}\ell_K \cdot \frac{[K \cap \mathbb{Q}_\infty : \mathbb{Q}]}{e_p} \cdot \frac{2}{\#(\langle -1 \rangle \cap \mathrm{N}_{K/\mathbb{Q}}(U_K))}, \tag{2.1}$$

where e_p is the ramification index of p in $K/\mathbb{Q}$ [8, Theorem III.2.6.4], and U_K is defined as follows:

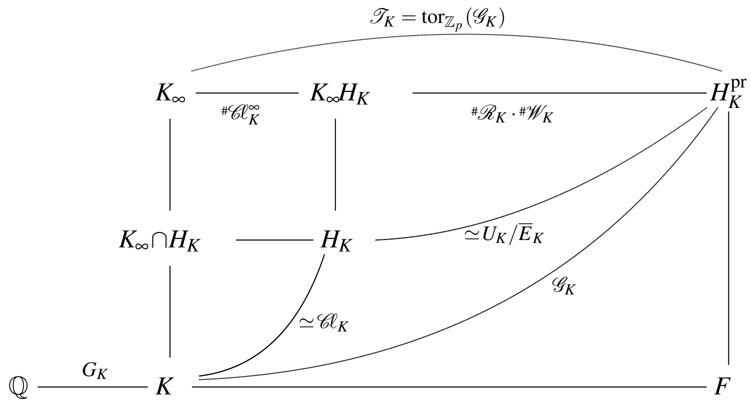
For each $\mathfrak{p} \mid p$, let $K_\mathfrak{p}$ be the $\mathfrak{p}$-completion of K and $\overline{\mathfrak{p}}$ the corresponding prime ideal of the ring of integers of $K_\mathfrak{p}$; then let:

$$U_K := \Big\{ u \in \bigoplus_{\mathfrak{p} \mid p} K_\mathfrak{p}^\times,\ u = 1 + x,\ x \in \bigoplus_{\mathfrak{p} \mid p} \overline{\mathfrak{p}} \Big\} \ \ \&\ \ W_K := \mathrm{tor}_{\mathbb{Z}_p}(U_K)$$

the $\mathbb{Z}_p$-module (of $\mathbb{Z}_p$-rank $d = [K : \mathbb{Q}]$) of principal local units at p and its torsion subgroup, respectively; by class field theory this gives in the diagram:

$$\mathrm{Gal}(H_K^{\mathrm{pr}}/H_K) \simeq U_K/\overline{E}_K \ \ \&\ \ \mathrm{Gal}(H_K^{\mathrm{pr}}/K_\infty H_K) \simeq \mathrm{tor}_{\mathbb{Z}_p}\big(U_K/\overline{E}_K\big),$$

where $\overline{E}_K$ is the closure of the group E_K of p-principal global units of K (i.e., units $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p}\mid p}\mathfrak{p}}$):



For any field k, let μ_k be the group of roots of unity of k of p-power order. Then $W_K = \bigoplus_{\mathfrak{p}\mid p} \mu_{K_\mathfrak{p}}$. We have the following exact sequence defining $\mathscr{W}_K$ and $\mathscr{R}_K$ via the p-adic logarithm log ([8, Lemma III.4.2.4] or [9, Lemma 3.1 & § 5]):

$$\begin{aligned} 1 \to \mathscr{W}_K := W_K/\mu_K &\longrightarrow \mathrm{tor}_{\mathbb{Z}_p}\big(U_K/\overline{E}_K\big) \\ &\xrightarrow{\ \log\ } \mathrm{tor}_{\mathbb{Z}_p}\big(\log(U_K)/\log(\overline{E}_K)\big) =: \mathscr{R}_K \to 0. \end{aligned} \tag{2.2}$$

The group $\mathscr{R}_K$ is called the *normalized p-adic regulator of K* and makes sense for any number field (see the above references in [9] for more details and the main properties of these invariants).

It is clear that the annihilation of $\mathscr{T}_K$ mainely concerns the group $\mathscr{R}_K$ since the p-class group is in general trivial (and so for p large enough) and because the regulator may be non-trivial with large valuations and unpredictible p (see [11] for some conjectures and [10] giving programs of fast computation of the *group structure of* $\mathscr{T}_K$ *for any number field* given by means of polynomials).

Definition 2.1. *A field K is said to be p-rational if the Leopoldt conjecture is satisfied for p in K and if the torsion group $\mathscr{T}_K$ is trivial ([14, Section III, § 2], then [8, Theorem IV.3.5], [10], and bibliographies for the history and properties of p-rationality).*

This has deep consequences in Galois theory over K since $\mathscr{T}_K$ is the dual of $\mathrm{H}^2(\mathscr{G}_{K,S}, \mathbb{Z}_p(0))$ [18].

3. Kummer theory and Spiegel involution

3.1 Kummer theory

We denote by $\mathbb{Q}_n$, $n \geq 0$, the nth stage in $\mathbb{Q}_\infty$ so that $[\mathbb{Q}_n : \mathbb{Q}] = p^n$. Let $n_0 \geq 0$ be defined by $K \cap \mathbb{Q}_\infty =: \mathbb{Q}_{n_0}$.

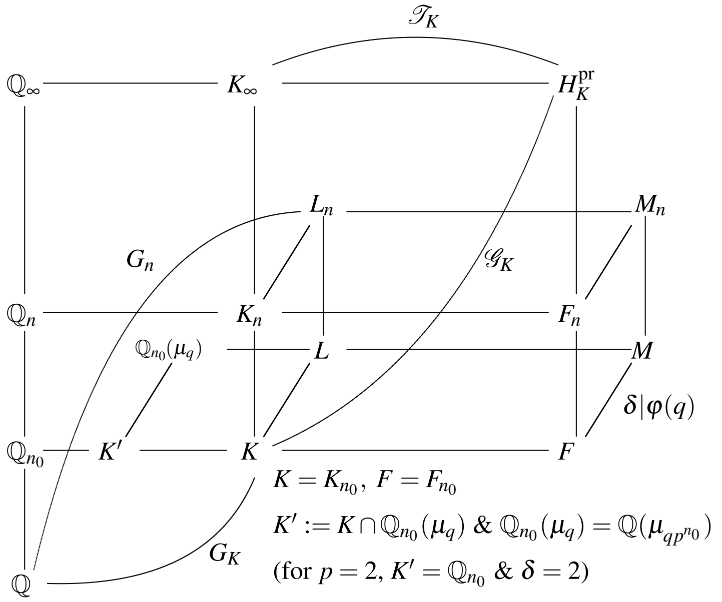
Let $n \geq n_0$. We denote by K_n the compositum $K\mathbb{Q}_n$ and by F_n the compositum $FK_n = F\mathbb{Q}_n$ (in other words, $K = K_{n_0}$, $F = F_{n_0}$). Then we have the *group isomorphism* $\mathrm{Gal}(F_n/K_n) \simeq \mathscr{T}_K$ for all $n \geq n_0$.

Put $q = p$ (resp. 4) if $p \neq 2$ (resp. $p = 2$). Let $L = K(\mu_q)$ and $M = F(\mu_q)$; then put $L_n := LK_n$ for all $n \geq n_0$.

Let $M_n := F_n(\mu_q)$ (whence $L = L_{n_0}$, $M = M_{n_0}$). For $p \neq 2$, the degrees $[L_n : K_n] = [M_n : F_n]$ are equal to a divisor δ of $p-1$ independent of $n \geq n_0$ (δ is even since K is real). For $p = 2$, $\delta = 2$. In any case, one has, for $n \geq n_0$:

$$L_n = K(\mu_{qp^n}).$$

All this is summarized by the following diagram:



Lemma 3.1. *Let f_K be the conductor of K. Then the conductor f_{L_n} of L_n $(n \geq n_0)$ is equal to* l.c.m. (f_K, qp^n). *Thus for n large enough (explicit), $f_{L_n} = qp^n f'$, with $p \nmid f'$. If $p \nmid f_K$, then $f_{L_n} = qp^n f_K$ for all $n \geq n_0 + e$.*

Proof. A classical formula (see, e.g., [8, Proposition II.4.1.1]). □

Lemma 3.2. *Let p^e, $e \geq 0$, be the exponent of $\mathscr{T}_K$. Then, for all $n \geq n_0 + e$, the restriction $\mathscr{T}_K \longrightarrow \mathrm{Gal}(F_n/K_n)$ is an isomorphism of G_K-modules and $\mathscr{T}_K \simeq \mathrm{Gal}(M_n/L_n)$.*

Proof. The abelian group $\mathscr{G}_K := \mathrm{Gal}(H_K^{\mathrm{pr}}/K)$ is normal in $\mathrm{Gal}(H_K^{\mathrm{pr}}/\mathbb{Q})$, then $(\mathscr{G}_K)^{p^{n-n_0}}$ is normal; but $(\mathscr{G}_K)^{p^{n-n_0}}$ fixes F_n which is Galois over $\mathbb{Q}$. In other words, G_K, as well as $\mathrm{Gal}(K_n/\mathbb{Q})$ or $\mathrm{Gal}(K_\infty/\mathbb{Q})$, operate by conjugation in the same way since $\mathscr{G}_K$ is abelian; if F is clearly non-unique, then F_{n_0+e} is canonical, being the fixed fiel of $\big(\mathscr{G}_K\big)^{p^e}$. Then $\mathrm{Gal}(M_n/L_n) \simeq \mathrm{Gal}(F_n/K_n)$ is trivialy an somorphism of G_K-modules. □

The use of the extension F is not strictly necessary but clarifies the reasoning which needs to work at any level $n \geq n_0 + e$ to preserve Galois structures.

The extension M_n/L_n (of exponent p^e) is a Kummer extension for the "exponent" qp^n since L_n contains the group μ_{qp^n} and since $n \geq n_0 + e$.

Let $G_n := \mathrm{Gal}(L_n/\mathbb{Q})$ and let, for $n \geq n_0 + e$,

$$\mathrm{Rad}_n := \{w \in L_n^\times,\ \sqrt[qp^n]{w} \in M_n\}$$

be the radical of M_n/L_n. Then we have the group isomorphism:

$$\mathrm{Rad}_n/L_n^{\times qp^n} \simeq \mathrm{Gal}(M_n/L_n).$$

In some sense, the group $\mathrm{Rad}_n/L_n^{\times qp^n}$ does not depend on $n \geq n_0 + e$ since the canonical isomorphism $\mathrm{Gal}(M_{n+h}/L_{n+h}) \simeq \mathrm{Gal}(M_n/L_n)$ gives $L_{n+h}(\sqrt[qp^n]{\mathrm{Rad}_n}) = M_{n+h}$; the map $\mathrm{Rad}_n/L_n^{\times qp^n} \xrightarrow{p^h} \mathrm{Rad}_{n+h}/L_{n+h}^{\times qp^{n+h}}$ is an isomorphism for any $h \geq 0$. In other words, as soon as $n \geq n_0 + e$, we have:

$$\mathrm{Rad}_n \subseteq L_n^{\times qp^{n-e}}\ \ \&\ \ \mathrm{Rad}_{n+h} = \mathrm{Rad}_n^{p^h} \cdot L_{n+h}^{\times qp^{n+h}}.$$

3.2 Spiegel involution

The structures of $(\mathbb{Z}/qp^n\mathbb{Z})[G_n]$-modules of the Galois group $\mathrm{Gal}(M_n/L_n)$ and $\mathrm{Rad}_n/L_n^{\times qp^n}$ are related via the "Spiegel involution" defined as follows: let $\omega_n : G_n \longrightarrow \mathbb{Z}/qp^n\mathbb{Z}$ be the *character of Teichmüller of level n* defined by:

$$\zeta^s = \zeta^{\omega_n(s)}, \text{ for all } s \in G_n \text{ and all } \zeta \in \mu_{qp^n}.$$

The Spiegel involution is the involution of $(\mathbb{Z}/qp^n\mathbb{Z})[G_n]$ defined by:

$$x := \sum_{s\in G_n} a_s \cdot s \;\mapsto\; x^* := \sum_{s\in G_n} a_s \cdot \omega_n(s) \cdot s^{-1}.$$

Thus, if s is the Artin symbol $\left(\frac{L_n}{a}\right)$, then $\left(\frac{L_n}{a}\right)^* \equiv a \cdot \left(\frac{L_n}{a}\right)^{-1} \pmod{qp^n}$. For the convenience of the reader we prove once again the very classical:

Lemma 3.3. *Let $n \geq n_0 + e$ where $p^{n_0} = [K \cap \mathbb{Q}_\infty : \mathbb{Q}]$ and p^e is the exponent of $\mathscr{T}_K$. The annihilators A_n of $\mathrm{Gal}(M_n/L_n)$ (thus of $\mathscr{T}_K$) in $(\mathbb{Z}/qp^n\mathbb{Z})[G_n]$ are the images of the annihilators S_n of $\mathrm{Rad}_n/L_n^{\times qp^n}$ by the Spiegel involution and inversely. An annihilator A_n of $\mathscr{T}_K$ only depends on its projection $A_{K,n}$ in $(\mathbb{Z}/qp^n\mathbb{Z})[G_K]$.*

Proof. To simplify, put $\overline{\mathrm{Rad}} := \mathrm{Rad}_n/L_n^{\times qp^n}$, $\mathscr{T} := \mathrm{Gal}(M_n/L_n) \simeq \mathscr{T}_K$. Let:

$$\begin{aligned} \lambda \;:\; \overline{\mathrm{Rad}} \times \mathscr{T} &\longrightarrow \mu_{qp^n} \\ (\overline{w}, \tau) &\longmapsto \big(\sqrt[qp^n]{w}\big)^{\tau-1}; \end{aligned}$$

then λ is a non-degenerated $\mathbb{Z}/qp^n\mathbb{Z}$-bilinear form such that:

$$\lambda(\overline{w}^s, \tau) = \lambda(\overline{w}, \tau^{s^*}), \text{ for all } s \in G_n,$$

where $s^* = \omega_n(s) \cdot s^{-1}$ (see e.g., [8, Corollary I.6.2.1]).
Let $S_n = \sum_{s\in G_n} a_s \cdot s \in (\mathbb{Z}/qp^n\mathbb{Z})[G_n]$; then, for all $(\overline{w}, \tau) \in \overline{\mathrm{Rad}} \times \mathscr{T}$ we have:

$$\lambda(\overline{w}^{S_n}, \tau) = \prod_{s\in G_n} \lambda(\overline{w}^s, \tau)^{a_s} = \prod_{s\in G_n} \lambda(\overline{w}, \tau^{s^*})^{a_s} = \lambda(\overline{w}, \tau^{S_n^*}).$$

So, if S_n annihilates $\overline{\mathrm{Rad}}$, then $\lambda(\overline{w}, \tau^{S_n^*}) = 1$ for all $\overline{w}$ & τ; since λ is non-degenerated, $\tau^{S_n^*} = 1$ for all $\tau \in \mathscr{T}$. Whence the annihilation of $\mathscr{T}$ by $A_n = S_n^*$ *(without any assumption on K nor on p)*, then by the projection $A_{K,n}$ since $\mathrm{Gal}(L_n/K)$ acts trivially on $\mathrm{Gal}(M_n/L_n)$. □

Remark 3.4. *(i) As we have mention, the radical* Rad_n *does not depend realy on the field L_n for $n \geq n_0 + e$; so, if we consider the radical of the maximal p-ramified abelian p-extension of L_n, of exponent qp^n:*

$$\mathrm{Rad}'_n := \{w' \in L_n^\times,\ L_n(\sqrt[qp^n]{w'})/L_n \textit{ is p-ramified}\},$$

we obtain a group whose p-rank tends to infinity with n; this is due mainely to the $\mathbb{Z}_p$-rank of the compositum of the $\mathbb{Z}_p$-extensions of L_n (totally imaginary) and from the less known $\mathscr{T}_{L_n}$ which contains $\mathscr{T}_{K_n}$. But since $\mathscr{T}_K$ is annihilated by $1 - s_\infty$, $\mathrm{Rad}_n/L_n^{\times qp^n}$ *is annihilated by $(1-s_\infty)^* = 1 + s_\infty$ which means that only the "minus part" of* $\mathrm{Rad}'_n/L_n^{\times qp^n}$ *is needed, which eliminates the huge "plus" part containing in particular all the units. Thus* Rad_n *is essentially given by the "relative" S'_n-units of L_n (S'_n being the set of p-places of L_n) and generators of some "relative" p-classes of L_n.*

(ii) In the case $p = 2$, let $\mathscr{T}_K^{\mathrm{res}} := \mathrm{tor}_{\mathbb{Z}_2}\big(\mathscr{G}_{K,S}^{\mathrm{res\,ab}}\big)$, where $\mathscr{G}_{K,S}^{\mathrm{res}}$ is the Galois group of the maximal abelian S-ramified (i.e., unramified outside 2 but possibly complexified) pro-2-extension of K and let $\mathrm{Rad}_n^{\mathrm{res}}$ *the corresponding radical $\{w \in L_n^\times,\ \sqrt[4\cdot 2^n]{w} \in M_n^{\mathrm{res}}\}$, where M_n^{res} is analogous to M_n for the restricted sense. We observe that in the restricted sense, we have the exact sequence [8, Theorem III.4.1.5] $0 \to (\mathbb{Z}/2\mathbb{Z})^d \longrightarrow \mathscr{T}_K^{\mathrm{res}} \longrightarrow \mathscr{T}_K \to 1$, then a dual exact sequence with radicals. As in [2], one may consider more general ray class fields and find results of annihilation with suitable Stickelberger or Solomon elements.*

4. Stickelberger elements and cyclotomic numbers

4.1 General definitions

Let $f \geq 1$ be any modulus and let $\mathbb{Q}^f$ be the corresponding cyclotomic field $\mathbb{Q}(\mu_f)$. [1] Let L be a subfield of $\mathbb{Q}^f$.

(i) We define (where all Artin symbols are taken over $\mathbb{Q}$):

$$\mathscr{S}_{\mathbb{Q}^f} := -\sum_{a=1}^{f} \Big(\frac{a}{f} - \frac{1}{2}\Big) \cdot \Big(\frac{\mathbb{Q}^f}{a}\Big)^{-1}$$

and the restriction:

$$\mathscr{S}_L := \mathrm{N}_{\mathbb{Q}^f/L}(\mathscr{S}_{\mathbb{Q}^f}) := -\sum_{a=1}^{f} \Big(\frac{a}{f} - \frac{1}{2}\Big) \cdot \Big(\frac{L}{a}\Big)^{-1}$$

to L of $\mathscr{S}_{\mathbb{Q}^f}$, where a runs trough the integers $a \in [1, f]$ prime to f. In this case, one must precise the relation between f and the conductor f_L of L.

We know that the properties of annihilation of ideal classes need to multiply $\mathscr{S}_L$ by an element of the ideal annihilator of the group μ_f (or μ_{2f}), which is generated by f (or $2f$) and the multiplicators:

$$\delta_c := 1 - c \cdot \Big(\frac{\mathbb{Q}^f}{c}\Big)^{-1},$$

for c odd, prime to f. This shall give integral elements in the group algebra.

(ii) Then we define in the same way:

$$\eta_{\mathbb{Q}^f} := 1 - \zeta_f \ \ \& \ \ \eta_L := \mathrm{N}_{\mathbb{Q}^f/L}(1 - \zeta_f), \ f \neq 1,$$

where ζ_f is a primitive fth root of unity for which we assume the coherent definitions $\zeta_f^{m'} = \zeta_m$ if $f = m' \cdot m$.

It is well known that if f is not a prime power, then η_f is a unit, otherwise, $\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}}(1 - \zeta_f) = \ell$ if $f = \ell^r$, $\ell \geq 2$ prime, $r \geq 1$.

Definition 4.1. *Since* $\frac{f-a}{f} - \frac{1}{2} = -\big(\frac{a}{f} - \frac{1}{2}\big)$, $\mathscr{S}_{\mathbb{Q}^f} = \mathscr{S}'_{\mathbb{Q}^f} \cdot (1 - s_\infty)$ *and* $\mathscr{S}_L = \mathscr{S}'_L \cdot (1 - s_\infty)$, *where* $s_\infty := \big(\frac{\mathbb{Q}^f}{-1}\big)$ *is the complex conjugation, and where:*

$$\mathscr{S}'_{\mathbb{Q}^f} := -\sum_{a=1}^{f/2} \Big(\frac{a}{f} - \frac{1}{2}\Big) \cdot \Big(\frac{\mathbb{Q}^f}{a}\Big)^{-1} \ \ \& \ \ \mathscr{S}'_L := -\sum_{a=1}^{f/2} \Big(\frac{a}{f} - \frac{1}{2}\Big) \cdot \Big(\frac{L}{a}\Big)^{-1}.$$

4.2 Norms of Stickelberger elements and cyclotomic numbers

Let $f \geq 1$ and $m \mid f$ be any modulus and let $\mathbb{Q}^f$ and $\mathbb{Q}^m \subseteq \mathbb{Q}^f$ be the corresponding cyclotomic fields. Let $\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}$ be the restriction map:

$$\mathbb{Q}[\mathrm{Gal}(\mathbb{Q}^f/\mathbb{Q})] \longrightarrow \mathbb{Q}[\mathrm{Gal}(\mathbb{Q}^m/\mathbb{Q})],$$

or the usual arithmetic norm in $\mathbb{Q}^f/\mathbb{Q}^m$. Consider as above:

$$\mathscr{S}_{\mathbb{Q}^f} := -\sum_{a=1}^{f} \Big(\frac{a}{f} - \frac{1}{2}\Big) \cdot \Big(\frac{\mathbb{Q}^f}{a}\Big)^{-1} \ \ \& \ \ \eta_{\mathbb{Q}^f} := 1 - \zeta_f \ (f \neq 1).$$

We have, respectively:

$$\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\mathscr{S}_{\mathbb{Q}^f}) = \prod_{\ell \mid f,\ \ell \nmid m} \Big(1 - \Big(\frac{\mathbb{Q}^m}{\ell}\Big)^{-1}\Big) \cdot \mathscr{S}_{\mathbb{Q}^m}, \tag{4.1}$$

$$\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\eta_{\mathbb{Q}^f}) = \big(\eta_{\mathbb{Q}^m}\big)^{\prod_{\ell \mid f,\ \ell \nmid m}\left(1 - \left(\frac{\mathbb{Q}^m}{\ell}\right)^{-1}\right)} \ \text{if } m \neq 1. \tag{4.2}$$

[1] Such modulus are conductors of the corresponding cyclotomic fields, except for an even integer not divisible by 4; but this point of view is essential to establish the functional properties of Stickelberger elements and cyclotomic numbers. So, if f is odd, we distinguish, by abuse, the notations $\mathbb{Q}^f$ and $\mathbb{Q}^{2f}$ despite their equality.

As we have explained in the previous footnote, if m is odd, then we have:

$$\mathrm{N}_{\mathbb{Q}^{2m}/\mathbb{Q}^m}(\mathscr{S}_{\mathbb{Q}^{2m}}) = \big(1-\big(\tfrac{\mathbb{Q}^m}{2}\big)^{-1}\big)\cdot\mathscr{S}_{\mathbb{Q}^m},\quad \mathrm{N}_{\mathbb{Q}^{2m}/\mathbb{Q}^m}(\eta_{\mathbb{Q}^{2m}}) = \eta_{\mathbb{Q}^m}^{\big(1-\big(\frac{\mathbb{Q}^m}{2}\big)^{-1}\big)},$$

where the "norms" $\mathrm{N}_{\mathbb{Q}^{2m}/\mathbb{Q}^m}$ are of course the identity. For instance one verifies immediately that $\mathscr{S}_{\mathbb{Q}^6} = \frac{1}{3}(1-s_\infty)$ and $\mathscr{S}_{\mathbb{Q}^3} = \frac{1}{6}(1-s_\infty)$, but since 2 is inert in $\mathbb{Q}^3/\mathbb{Q}$, $\big(1-\big(\frac{\mathbb{Q}^3}{2}\big)^{-1}\big) = 1-s_\infty$ and one must compute $(1-s_\infty)\mathscr{S}_{\mathbb{Q}^3} = \frac{1}{6}(1-s_\infty)^2 = \frac{1}{3}(1-s_\infty)$ as expected. We have $\mathscr{S}_{\mathbb{Q}^2} = 0$ and $\mathscr{S}_{\mathbb{Q}^1} = -\frac{1}{2}$.

If L (imaginary or real), of conductor f, is an extension of k, of conductor $m \mid f$, let $\mathscr{S}_L := \mathrm{N}_{\mathbb{Q}^f/L}(\mathscr{S}_{\mathbb{Q}^f})$ and $\eta_L := \mathrm{N}_{\mathbb{Q}^f/L}(\eta_{\mathbb{Q}^f})$, then:

$$\mathrm{N}_{L/k}(\mathscr{S}_L) = \prod_{\ell \mid f,\ \ell\nmid m}\Big(1-\Big(\frac{k}{\ell}\Big)^{-1}\Big)\cdot\mathscr{S}_k,$$

$$\mathrm{N}_{L/k}(\mathscr{S}'_L) \equiv \prod_{\ell \mid f,\ \ell\nmid m}\Big(1-\Big(\frac{k}{\ell}\Big)^{-1}\Big)\cdot\mathscr{S}'_k \pmod{(1+s_\infty)\cdot\mathbb{Q}[G_k]},$$

$$\mathrm{N}_{L/k}(\eta_L) = (\eta_k)^{\prod_{\ell \mid f,\ \ell\nmid m}\big(1-\big(\frac{k}{\ell}\big)^{-1}\big)} \text{ if } m\neq 1 \text{ (i.e., } k\neq\mathbb{Q}).$$

If $f = \ell^r$, ℓ prime, $r\geq 1$, then $\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}}(\eta_{\mathbb{Q}^f}) = \ell$, otherwise $\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}}(\eta_{\mathbb{Q}^f}) = 1$.

This implies that $\mathrm{N}_{L/k}(\mathscr{S}_L) = 0$ (resp. $\mathrm{N}_{L/k}(\eta_L) = 1$) as soon as there exists a prime $\ell \mid f$, $\ell \nmid m$, totally split in k. In particular, if k is real, the formula is valid for the infinite place and $\mathrm{N}_{L/k}(\mathscr{S}_L) = 0$ (of course, if $L\neq\mathbb{Q}$ is real, $S_L = 0$).

For the classical proofs, we consider by induction the case $f = \ell\cdot m$, with ℓ prime and examine the two cases $\ell \mid m$ and $\ell \nmid m$; the case of Stickelberger elements been crucial for our purpose, we give again a proof (a similar reasoning will be detailed for the Theorem 7.2).

To simplify, put $\mathscr{S}_{\mathbb{Q}^f} =: \mathscr{S}_f$, $\mathscr{S}_{\mathbb{Q}^m} =: \mathscr{S}_m$, and consider:

$$\mathscr{S}_f = -\sum_{a=1}^{f}\Big(\frac{a}{f}-\frac{1}{2}\Big)\cdot\Big(\frac{\mathbb{Q}^f}{a}\Big)^{-1},$$

for $f = \ell\cdot m$, $\ell\nmid m$, where a runs trough the integers $a\in[1,f]$ prime to f.

Put $a = b+\lambda\cdot m$, $b\in[1,m]$, $\lambda\in[0,\ell-1]$; since a must be prime to f, b is automatically prime to m but we must exclude $\lambda^*_b\in[0,\ell-1]$ such that:

$$b+\lambda^*_b\cdot m = b'_\ell\cdot\ell,\ \ b'_\ell\in[1,m]\ (b'_\ell \text{ is necessarily prime to } m).$$

We then have:

$$\begin{aligned}
&\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\mathscr{S}_f)\\
&\quad= -\sum_{a=1}^{f}\Big(\frac{a}{f}-\frac{1}{2}\Big)\cdot\Big(\frac{\mathbb{Q}^m}{a}\Big)^{-1} = -\sum_{b,\lambda\neq\lambda^*_b}\Big(\frac{b+\lambda\, m}{\ell m}-\frac{1}{2}\Big)\cdot\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\\
&\quad= -\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\sum_{\lambda\neq\lambda^*_b}\Big(\frac{b}{\ell m}+\frac{\lambda}{\ell}-\frac{1}{2}\Big)\\
&\quad= -\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\Big(\frac{\ell-1}{\ell}\frac{b}{m}-\frac{\ell-1}{2}\Big)-\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{1}{\ell}\Big(\frac{\ell(\ell-1)}{2}-\lambda^*_b\Big)\\
&\quad= -\Big(1-\frac{1}{\ell}\Big)\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{b}{m}+\frac{1}{\ell}\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\lambda^*_b.
\end{aligned}$$

Since the correspondence $b\mapsto b'_\ell$ is bijective on the set of elements prime to m in $[1,m]$, one has, with $\lambda^*_b = \frac{b'_\ell\cdot\ell-b}{m}$ and $\Big(\frac{\mathbb{Q}^m}{b}\Big) = \Big(\frac{\mathbb{Q}^m}{b'_\ell}\Big)\Big(\frac{\mathbb{Q}^m}{\ell}\Big)$:

$$\begin{aligned}
\frac{1}{\ell}\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\lambda^*_b &= \sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{b'_\ell}{m}-\frac{1}{\ell}\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{b}{m}\\
&= \Big(\frac{\mathbb{Q}^m}{\ell}\Big)^{-1}\sum_b\Big(\frac{\mathbb{Q}^m}{b'_\ell}\Big)^{-1}\frac{b'_\ell}{m}-\frac{1}{\ell}\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{b}{m}\\
&= \Big(\Big(\frac{\mathbb{Q}^m}{\ell}\Big)^{-1}-\frac{1}{\ell}\Big)\cdot\sum_b\Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\frac{b}{m}.
\end{aligned}$$

Thus we obtain:

$$\begin{aligned}\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\mathscr{S}_f) &= -\Big(1-\tfrac{1}{\ell}\Big)\sum_b \Big(\tfrac{\mathbb{Q}^m}{b}\Big)^{-1}\tfrac{b}{m} + \Big(\Big(\tfrac{\mathbb{Q}^m}{\ell}\Big)^{-1} - \tfrac{1}{\ell}\Big)\cdot\sum_b \Big(\tfrac{\mathbb{Q}^m}{b}\Big)^{-1}\tfrac{b}{m}\\ &= -\Big(1-\Big(\tfrac{\mathbb{Q}^m}{\ell}\Big)^{-1}\Big)\sum_b \Big(\tfrac{\mathbb{Q}^m}{b}\Big)^{-1}\tfrac{b}{m}.\end{aligned}$$

But $\frac{1}{2}\sum_b \Big(\frac{\mathbb{Q}^m}{b}\Big)^{-1}\Big(1-\Big(\frac{\mathbb{Q}^m}{\ell}\Big)^{-1}\Big)=0$; so replacing $\frac{b}{m}$ by $\frac{b}{m}-\frac{1}{2}$ we get:

$$\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\mathscr{S}_f) = \Big(1-\Big(\tfrac{\mathbb{Q}^m}{\ell}\Big)^{-1}\Big)\cdot\mathscr{S}_m.$$

Then it is easy to compute that if $\ell \mid m$, any $\lambda\in[0,\ell-1]$ is suitable, giving:

$$\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\mathscr{S}_f) = \mathscr{S}_m.$$

The case of cyclotomic elements η_f is exactly the same, replacing the additive setting by the multiplicative one.

4.3 Multiplicators of Stickelberger elements

The conductor of L_n, $n\ge n_0$, is $f_{L_n} = \mathrm{l.c.m.}\,(f_K, qp^n)$ (Lemma 3.1). So in general $f_{L_n} = qp^n\cdot f'$ with $p\nmid f'$, except if f_K is divisible by a large power of p in which case one must take n large enough in the practical computations (write $f_K = qp^{n_0+r}f'$, $r\ge 0$, and take $n\ge n_0+r$). In some formulas we shall abbreviate f_{L_n} by f_n.

Let c be an (odd) integer, prime to f_n, and let:

$$\mathscr{S}_{L_n}(c) := \Big(1-c\Big(\tfrac{L_n}{c}\Big)^{-1}\Big)\cdot\mathscr{S}_{L_n}. \tag{4.3}$$

Then $\mathscr{S}_{L_n}(c)\in\mathbb{Z}[G_n]$ as we have explain; indeed, we have:

$$\mathscr{S}_{L_n}(c) = \frac{-1}{f_n}\sum_a\Big[a\Big(\tfrac{L_n}{a}\Big)^{-1} - ac\Big(\tfrac{L_n}{a}\Big)^{-1}\Big(\tfrac{L_n}{c}\Big)^{-1}\Big] + \frac{1-c}{2}\sum_a\Big(\tfrac{L_n}{a}\Big)^{-1}.$$

Let $a'_c\in[1,f_n]$ be the unique integer such that $a'_c\cdot c\equiv a \pmod{f_n}$ and put $a'_c\cdot c = a+\lambda^n_a(c)f_n$, $\lambda^n_a(c)\in\mathbb{Z}$; then, using the bijection $a\mapsto a'_c$ in the second summation and the fact that $\Big(\frac{L_n}{a'_c}\Big)\Big(\frac{L_n}{c}\Big) = \Big(\frac{L_n}{a}\Big)$, this yields:

$$\begin{aligned}\mathscr{S}_{L_n}(c) &= \frac{-1}{f_n}\Big[\sum_a a\Big(\tfrac{L_n}{a}\Big)^{-1} - \sum_a a'_c\cdot c\Big(\tfrac{L_n}{a'_c}\Big)^{-1}\Big(\tfrac{L_n}{c}\Big)^{-1}\Big] + \frac{1-c}{2}\sum_a\Big(\tfrac{L_n}{a}\Big)^{-1}\\ &= \frac{-1}{f_n}\sum_a\Big[a-a'_c\cdot c\Big]\Big(\tfrac{L_n}{a}\Big)^{-1} + \frac{1-c}{2}\sum_a\Big(\tfrac{L_n}{a}\Big)^{-1}\\ &= \sum_a\Big[\lambda^n_a(c)+\frac{1-c}{2}\Big]\Big(\tfrac{L_n}{a}\Big)^{-1}\in\mathbb{Z}[G_n].\end{aligned}$$

Lemma 4.2. *We have the relations $\lambda^n_{f_n-a}(c)+\frac{1-c}{2} = -\big(\lambda^n_a(c)+\frac{1-c}{2}\big)$ for all $a\in[1,f_n]$ prime to f_n. Then:*

$$\mathscr{S}'_{L_n}(c) := \sum_{a=1}^{f_n/2}\Big[\lambda^n_a(c)+\frac{1-c}{2}\Big]\Big(\tfrac{L_n}{a}\Big)^{-1}\in\mathbb{Z}[G_n] \tag{4.4}$$

is such that $\mathscr{S}_{L_n}(c) = \mathscr{S}'_{L_n}(c)\cdot(1-s_\infty)$, whence $\mathscr{S}_{L_n}(c)^* = \mathscr{S}'_{L_n}(c)^*\cdot(1+s_\infty)$.

Proof. By definition, the integer $(f_n-a)'_c$ is in $[1,f_n]$ and congruent modulo f_n to $(f_n-a)\,c^{-1}\equiv -ac^{-1}\equiv -a'_c \pmod{f_n}$; thus $(f_n-a)'_c = f_n-a'_c$ and

$$\lambda^n_{f_n-a}(c) = \frac{(f_n-a)'_c\,c-(f_n-a)}{f_n} = \frac{(f_n-a'_c)\,c-(f_n-a)}{f_n} = c-1-\lambda^n_a(c),$$

whence $\lambda^n_{f_n-a}(c)+\frac{1-c}{2} = -\big(\lambda^n_a(c)+\frac{1-c}{2}\big)$ and the result. □

The multiplicator $\delta_c := \big(1-c\,\big(\frac{L_n}{c}\big)^{-1}\big)$ has a great importance since the image of δ_c by the Spiegel involution is $\delta^*_c := 1-\big(\frac{L_n}{c}\big) \pmod{qp^n}$; the order of the Artin symbol of c shall be crucial.

5. Annihilation of radicals and Galois groups

5.1 Annihilation of $\mathrm{Rad}_n/L_n^{\times qp^n}$

We begin with the classical property of annihilation of class groups of imaginary abelian fields by modified Stickelberger elements $\mathscr{S}_{L_n}(c) = \delta_c \cdot \mathscr{S}_{L_n}$. Before let's give two technical lemmas. Recall that $\mathscr{S}_{L_n}(c) = \mathscr{S}'_{L_n}(c)\cdot(1-s_\infty)$ and that, from § 4.2, the $\mathscr{S}_{L_n}$, $\mathscr{S}_{L_n}(c)$ and $\mathscr{S}'_{L_n}(c) \pmod{(1+s_\infty)\mathbb{Z}[G_n]}$ form coherent families in $\varprojlim_{n\geq n_0+e} \mathbb{Q}[G_n]$ for the "norm" since f_{L_n} and $f_{L_{n+h}}$ are divisible by the same prime numbers for all $h \geq 0$.

Lemma 5.1. *Let $\zeta \in \mu_{qp^n}$, $n \geq n_0+e$. If $\zeta \in \mathrm{Rad}_n$ (or $\mathrm{Rad}_n^{\mathrm{res}}$ when $p=2$) then $\zeta = 1$.*

Proof. If $\zeta \neq 1$ with $L_n(\sqrt[qp^n]{\zeta}) \subseteq M_n$ (or M_n^{res}), we would have $L_n(\sqrt[qp^n]{\zeta}) = L_{n+h}$, where $h \geq 1$ since $\sqrt[qp^n]{\zeta}$ is of order $\geq qp^{n+1}$ and since $\mu_{p^\infty} \cap L_n^\times = \mu_{qp^n}$, which is absurd because of the linear disjonction $L_{n+h} \cap M_n = L_n$ (or $L_{n+h} \cap M_n^{\mathrm{res}} = L_n$). □

Lemma 5.2. *Let $w_0 \in \mathrm{Rad}_n$ be real. Then $w_0^2 \in L_n^{\times qp^n}$.*

Proof. Since K is real, we know that $1-s_\infty$ annihilates the $(\mathbb{Z}/qp^n\mathbb{Z})[G_n]$-module $\mathrm{Gal}(M_n/L_n)$, thus $1+s_\infty$ annihilates $\mathrm{Rad}_n/L_n^{\times qp^n}$ and $w_0^{1+s_\infty} = w_0^2 \in L_n^{\times qp^n}$ (this does not work for the restricted sense since the minus part of $\mathscr{T}_K^{\mathrm{res}}$ is of order 2^d). □

Theorem 5.3. *Let p^e be the exponent of $\mathscr{T}_K := \mathrm{tor}_{\mathbb{Z}_p}(\mathcal{G}^{\mathrm{ab}}_{K,S})$ (p-ramification in the ordinary sense). For $p=2$, let $2^{e^{\mathrm{res}}}$ be the exponent of $\mathscr{T}_K^{\mathrm{res}} := \mathrm{tor}_{\mathbb{Z}_2}(\mathcal{G}^{\mathrm{res\,ab}}_{K,S})$, where $\mathcal{G}^{\mathrm{res}}_{K,S}$ is the Galois group of the maximal S-ramified in the restricted sense (i.e., unramified outside 2 but complexified) pro-2-extension of K and let $\mathrm{Rad}_n^{\mathrm{res}}$ be the corresponding radical.*

(i) $p>2$. For all $n \geq n_0+e$, the $(\mathbb{Z}/qp^n\mathbb{Z})[G_n]$-module $\mathrm{Rad}_n/L_n^{\times qp^n}$ is annihilated by $\mathscr{S}'_{L_n}(c)$. Thus, $\mathscr{S}'_{L_n}(c)^*$ annihilates $\mathscr{T}_K$.

(ii) $p=2$, ordinary sense. The annihilation occurs with $2\mathscr{S}_{L_n}(c)$ and with $4\mathscr{S}'_{L_n}(c)$. Thus $2\mathscr{S}_{L_n}(c)^*$ and $4\mathscr{S}'_{L_n}(c)^*$ annihilate $\mathscr{T}_K$.

(iii) $p=2$, restricted sense. For all $n \geq n_0+e^{\mathrm{res}}$, the $(\mathbb{Z}/4\cdot 2^n\mathbb{Z})[G_n]$-module $\mathrm{Rad}_n^{\mathrm{res}}/L_n^{\times 4\cdot 2^n}$ is annihilated by $2\mathscr{S}_{L_n}(c)$; thus $2\mathscr{S}_{L_n}(c)^*$ annihilates $\mathscr{T}_K^{\mathrm{res}}$.

Proof. Let $w \in \mathrm{Rad}_n$; since $L_n(\sqrt[qp^n]{w})/L_n$ is p-ramified, $(w) = \mathfrak{a}^{qp^n}\cdot\mathfrak{b}$ where $\mathfrak{a}$ is an ideal of L_n, prime to p, and $\mathfrak{b}$ is a product of prime ideals $\mathfrak{p}_n$ of L_n dividing p. Let $\mathfrak{p}_n \mid \mathfrak{b}$ and consider $\mathfrak{p}_n^{\mathscr{S}_{L_n}(c)}$; one can replace $\mathscr{S}_{L_n}(c)$ by its restriction to the decomposition field k (possibly $k=\mathbb{Q}$) of p in the abelian extension $L_n/\mathbb{Q}$, which gives rise to the Euler factor $1-\left(\frac{k}{p}\right)^{-1}$ since k, of conductor prime to p, is strictly contained in L_n of conductor $qp^n f'$ for $n \geq n_0+e$; so this factor is 0 and $\mathfrak{b}^{\mathscr{S}_{L_n}(c)} = 1$.

From the principality of the ideal $\mathfrak{a}^{\mathscr{S}_{L_n}(c)}$ (Stickelberger's theorem) there exists $\alpha_n \in L_n^\times$ and a unit ε_n of L_n such that:

$$w^{\mathscr{S}_{L_n}(c)} = \alpha_n^{qp^n}\cdot\varepsilon_n. \tag{5.1}$$

We see that $\varepsilon_n^{1+s_\infty}$ is the qp^nth power of a unit of L_n: consider $\varepsilon_n^{1+s_\infty}$ in (5.1) with the fact that $\mathscr{S}_{L_n}(c) = \mathscr{S}'_{L_n}(c)(1-s_\infty)$. Since the $\mathbb{Z}$-rank of the groups of units of L_n and L_n^+ (the maximal real subfield of L_n) are equal, a power ε_n^N of ε_n is a real unit; so $\varepsilon_n^{1-s_\infty}$ is a torsion element and $\varepsilon_n^2 = \varepsilon_n^{1+s_\infty}\varepsilon_n^{1-s_\infty}$ is equal, up to a qp^nth power, to a p-torsion element of the form $\zeta' \in \mathrm{Rad}_n$. Thus $\zeta' = 1$ (Lemma 5.1) and $\varepsilon_n^2 \in L_n^{\times qp^n}$.

(i) Case $p \neq 2$. We deduce from the above that $\varepsilon_n \in L_n^{\times p^{n+1}}$. We have $w^{\mathscr{S}'_{L_n}(c)(1-s_\infty)} = \beta_n^{p^{n+1}}$; but $\beta_n^{1+s_\infty} = 1$ (the property is also true for $p=2$ since the result is a totally positive root of unity in L_n^+, but the proof only works taking the square of the relation (5.1) using ε_n^2), and there exists $\gamma_n \in L_n^\times$ such that $\beta_n = \gamma_n^{1-s_\infty}$, and $w^{\mathscr{S}'_{L_n}(c)}\cdot\gamma_n^{-p^{n+1}} = w_0$, a real number in the radical, thus a p^{n+1}th power (Lemma 5.2) (as above, the proof for $p=2$ only works taking once again the square of this relation to get w_0^2). Other proof for any $p \geq 2$: since $\mathscr{T}_K$ is annihilated by $1-s_\infty$, $\mathrm{Rad}_n/L_n^{\times qp^n}$ is annihilated by $1+s_\infty$, thus $w^{1-s_\infty} \in w^2\cdot L_n^{\times qp^n}$ for all $w \in \mathrm{Rad}_n$, and $w^{\mathscr{S}_{L_n}(c)} = w^{2\mathscr{S}'_{L_n}(c)}$ up to $L_n^{\times qp^n}$.

(ii) Case $p=2$ in the ordinary sense (so $L_n^+ = K_n$). The result is obvious taking the square in the previous computations giving ε_n^2 instead of ε_n for the annihilation with $2\mathscr{S}_{L_n}(c)$, then w_0^2 for the annihilation with $4\mathscr{S}'_{L_n}(c)$.

(iii) Case $p=2$ in the restricted sense. The proof is in fact contained in the same relation $(w) = \mathfrak{a}^{4\cdot 2^n}\cdot\mathfrak{b}$, for all $w \in \mathrm{Rad}_n^{\mathrm{res}}$, where $\mathfrak{a}$ is an ideal of L_n, prime to 2, and $\mathfrak{b}$ is a product of prime ideals $\mathfrak{p}_n$ of L_n dividing 2, then the relation (5.1), $n \geq n_0+e$. □

5.2 Computation of $\mathscr{S}_{L_n}(c)^*$ or $\mathscr{S}'_{L_n}(c)^*$ – Annihilation of $\mathscr{T}_K$

From the expressions (4.3) and (4.4) of $\mathscr{S}_{L_n}(c)$, the image by the Spiegel involution is:

$$\mathscr{S}_{L_n}(c)^* \equiv \sum_{a=1}^{f_n} \Big[\lambda_a^n(c) + \frac{1-c}{2}\Big] a^{-1} \Big(\frac{L_n}{a}\Big) \pmod{qp^n},$$

which defines a coherent familly $(\mathscr{S}_{L_n}(c)^*)_n \in \varprojlim_{n \geq n_0+e} \mathbb{Z}/qp^n\mathbb{Z}[G_n]$ of annihilators of the Galois groups $\mathrm{Gal}(M_n/L_n) \simeq \mathscr{T}_K$. In the case $p \neq 2$, one may use equivalently $\mathscr{S}'_{L_n}(c)^*$ with the half summation.

Since the operation of $\mathrm{Gal}(L_n/K)$ on $\mathrm{Gal}(M_n/L_n)$ is trivial, by restriction of $\mathscr{S}_{L_n}(c)^*$ to K (see Lemma 3.3), one obtains a coherent familly of annihilators of $\mathscr{T}_K$ denoted $(\mathscr{A}_{K,n}(c))_n \in \varprojlim_{n \geq n_0+e} \mathbb{Z}/qp^n\mathbb{Z}[G_K]$, whose p-adic limit:

$$\mathscr{A}_K(c) := \lim_{n\to\infty} \mathscr{A}_{K,n}(c) = \lim_{n\to\infty} \sum_{a=1}^{f_n} \Big[\lambda_a^n(c) + \frac{1-c}{2}\Big] a^{-1} \Big(\frac{K}{a}\Big) \in \mathbb{Z}_p[G_K]$$

is a canonical annihilator of $\mathscr{T}_K$ that we shall link to p-adic L-functions; of course, it is sufficient to know its coefficients modulo the exponent p^e of $\mathscr{T}_K$ and in a programming point of view, the element $\mathscr{A}_{K,n_0+e}(c)$ annihilates $\mathscr{T}_K$, knowing that [10, Program I, § 3.2] gives the group structure of $\mathscr{T}_K$.

Remark 5.4. *Let* $\alpha_{L_n} := \sum_{a=1}^{f_n} a^{-1}\Big(\frac{L_n}{a}\Big) \equiv \Big[\sum_{a=1}^{f_n} \Big(\frac{L_n}{a}\Big)^{-1}\Big]^*$*; we have:*

$$\alpha_{L_n} := \sum_{a=1}^{f_n/2} a^{-1}\Big(\frac{L_n}{a}\Big) + (f_n-a)^{-1}\Big(\frac{L_n}{f_n-a}\Big) \equiv \sum_{a=1}^{f_n/2} a^{-1}\Big(\frac{L_n}{a}\Big)(1-s_\infty) \pmod{f_n}$$

which annihilates $\mathscr{T}_K$ and is such that $\mathrm{N}_{L_n/K}(\alpha_{L_n}) \equiv 0 \pmod{qp^n}$ since K is real. We shall neglect such expressions and use the symbol $\,\widetilde{\equiv}\,$, where $A \,\widetilde{\equiv}\, B \pmod{p^{n+1}}$ will mean $A = B + \mu\cdot p^{n+1} + \nu \cdot \sum_{a=1}^{f_n} a^{-1}\Big(\frac{K}{a}\Big)$, in the group algebra $\mathbb{Z}_p[G_K]$, μ, ν in $\mathbb{Z}_p$ (we put the modulus p^{n+1} instead of qp^n to cover, subsequently, the case $p=2$; moreover, p^{n+1} annihilates $\mathscr{T}_K$ since $n \geq n_0+e$). By abuse, we still denote $\mathscr{A}_K(c) = \lim_{n\to\infty} \sum_{a=1}^{f_n} \lambda_a^n(c)\, a^{-1}\big(\frac{K}{a}\big)$.

Thus, we have obtained:

Theorem 5.5. *Let c be any integer prime to $2p$ and to the conductor of K. Assume $n \geq n_0+e$ and let f_n be the conductor of L_n; for all $a \in [1, f_n]$, prime to f_n, let a'_c be the unique integer in $[1, f_n]$ such that $a'_c \cdot c \equiv a \pmod{f_n}$ and put $a'_c \cdot c - a = \lambda_a^n(c)\, f_n$, $\lambda_a^n(c) \in \mathbb{Z}$.*

Let $\mathscr{A}_{K,n}(c) := \sum_{a=1}^{f_n} \lambda_a^n(c)\, a^{-1}\Big(\frac{K}{a}\Big)$ and put $\mathscr{A}_{K,n}(c) = \mathscr{A}'_{K,n}(c)\cdot(1+s_\infty)$ where $\mathscr{A}'_{K,n}(c) = \sum_{a=1}^{f_n/2} \lambda_a^n(c)\, a^{-1}\Big(\frac{K}{a}\Big)$. Let $\mathscr{A}_K(c) := \lim_{n\to\infty} \Big[\sum_{a=1}^{f_n} \lambda_a^n(c)\, a^{-1}\Big(\frac{K}{a}\Big)\Big]$ and put $\mathscr{A}_K(c) =: \mathscr{A}'_K(c)\cdot(1+s_\infty)$.

(i) For $p \neq 2$, $\mathscr{A}'_K(c)$ annihilates the $\mathbb{Z}_p[G_K]$-module $\mathscr{T}_K$.

(ii) For $p=2$, the annihilation is true for $2\cdot\mathscr{A}_K(c)$ and $4\cdot\mathscr{A}'_K(c)$.

In practice, when the exponent p^e is known, one can use $n = n_0+e$ and the annihilators $\mathscr{A}_{K,n}(c)$ or $\mathscr{A}'_{K,n}(c)$, the annihilator limit $\mathscr{A}_K(c)$ being related to p-adic L-functions of primitive characters, thus giving the other approach than Solomon one, that we shall obtain in Theorem 9.4.

Remark 5.6. *We have proved in a seminar report (1977) that for $p=2$, $\mathscr{S}'_{L_n}(c)$ annihilates $\mathscr{C}\ell_{L_n}/\mathscr{C}\ell^0_{L_n}$, where $\mathscr{C}\ell_{L_n}$ is the 2-class group of L_n and where $\mathscr{C}\ell^0_{L_n}$ is generated by the classes of the the invariant ideals in L_n/K_n.*

This shows that some 2-classes may give an obstruction; but Rad_n is particular as we have explained in Remark 3.4. In [15], Greither gives suitable statements about Stickelberger's theorem for $p=2$, using the main theorems of Iwasawa's theory about the orders $\frac{1}{2}L_2(1,\chi)$ of the isotypic components.

From this, as well as some numerical experiments, and the roles of ε_n and w_0 in the above reasonings, we may propose the following conjecture:

Conjecture 5.7. *Let $p=2$ and let K be a real abelian number field linearly disjoint from the cyclotomic $\mathbb{Z}_2$-extension. Put $\mathscr{A}_K(c) = \mathscr{A}'_K(c)\cdot(1+s_\infty)$ (see formula of Theorem 5.5). Then $\mathscr{A}'_K(c)$ annihilates $\mathscr{T}_K$.*

If there exists, in the class of $\mathscr{A}'_K(c)$ modulo $\sum_{\sigma\in G_K}\sigma$, an element of the form $2\cdot\mathscr{A}''_K(c)$, $\mathscr{A}''_K(c) \in \mathbb{Z}_p[G_K]$, one may ask if $\mathscr{A}''_K(c)$ does annihilate $\mathscr{T}_K$. We shall give a counterexample for the annihilation of $\mathscr{T}_K$ by $\mathscr{A}''_K(c)$ (see § 6.5.5), but we ignore if this may be true under some assumptions.

5.3 Experiments for cyclic cubic fields with $p \equiv 1 \pmod 3$

To simplify we suppose f_K prime. The first part of the program gives a defining polynomial. A second part computes the p-adic valuation of $\#\mathscr{T}_K$ using [10, Program I, § 3.2] and gives $\mathscr{A}_K(c) = \Lambda_0 + \Lambda_1\sigma^{-1} + \Lambda_2\sigma^{-2}$ modulo a power of p, after the choice of c, prime to $2pf_K$, with an Artin symbol of order 3; in the program p^{ex} is the exponent p^e of $\mathscr{T}_K$ and fn the conductor of L_n. The parameter nt must be $>$ ex.

```
{p=7;nt=8;forprime(f=7,10^4,if(Mod(f,3)!=1,next);
for(bb=1,sqrt(4*f/27),if(vf==2 & Mod(bb,3)==0,next);A=4*f-27*bb^2;
if(issquare(A,&aa)==1,if(Mod(aa,3)==1,aa=-aa);
P=x^3+x^2+(1-f)/3*x+(f*(aa-3)+1)/27;K=bnfinit(P,1);Kpn=bnrinit(K,p^nt);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
h=component(component(component(K,8),1),2);L=List;ex=0;
i=component(matsize(Hpn),2);R=0;for(k=1,i-1,co=component(Hpn,i-k+1);
if(Mod(co,p)==0,R=R+1;val=valuation(co,p);if(val>ex,ex=val);
listinsert(L,p^val,1)));Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);if(vptor>1,S0=0;S1=0;S2=0;
pN=p*p^ex;nu=(f-1)/3;fn=pN*f;z=znprimroot(f);
zz=lift(z);t=lift(Mod((1-zz)/f,2*p));c=zz+t*f;
for(a=1,fn/2,if(gcd(a,fn)!=1,next);asurc=lift(a*Mod(c,fn)^-1);
lambda=(asurc*c-a)/fn;u=Mod(lambda*a^-1,pN);
a0=lift((a*z^0)^nu);a1=lift((a*z^2)^nu);a2=lift((a*z)^nu);
if(a0==1,S0=S0+u);if(a1==1,S1=S1+u);if(a2==1,S2=S2+u));
L0=lift(S0);L1=lift(S1);L2=lift(S2);
j=Mod(y,y^2+y+1);Y=L0+j*L1+j^2*L2;nj=valuation(norm(Y),p);
print(f," ",P," vptor=",vptor," T_K=",L," A= ",L0," ",L1," ",L2," ",nj)))))}
```

Let's give a partial table for $p = 7$ and 13, in which vptor $:= v_p(\#\mathscr{T}_K)$ (examples limited to vptor ≥ 2), and nj $= v_p\big(\mathrm{N}_{\mathbb{Q}(j)/\mathbb{Q}}(\Lambda_0 + \Lambda_1 \cdot j + \Lambda_2 \cdot j^2)\big)$; one sees that, as expected, all the examples give nj $=$ vptor since $\mathscr{T}_K$ is a finite $\mathbb{Z}_7[j]$-module which may be decomposed with two 7-adic characters:

f	P	vptor	T_K	coefficients	nj
313	x^3+x^2-104*x+371	2	[7,7]	[41, 41, 48]	2
577	x^3+x^2-192*x+171	2	[49]	[183, 17, 280]	2
823	x^3+x^2-274*x+61	3	[343]	[761, 419, 437]	3
883	x^3+x^2-294*x+1439	2	[7,7]	[14, 0, 35]	2
1051	x^3+x^2-350*x-2608	2	[49]	[4, 247, 309]	2
1117	x^3+x^2-372*x+2565	2	[7,7]	[7, 7, 42]	2
1213	x^3+x^2-404*x+629	2	[49]	[45, 313, 268]	2
1231	x^3+x^2-410*x-1003	2	[49]	[247, 73, 273]	2
1237	x^3+x^2-412*x+1741	2	[49]	[108, 336, 128]	2
1297	x^3+x^2-432*x-1345	2	[49]	[277, 62, 14]	2
1327	x^3+x^2-442*x-344	2	[49]	[217, 340, 251]	2
1381	x^3+x^2-460*x-1739	4	[343,7]	[1738, 2186, 2361]	4
1567	x^3+x^2-522*x-4759	2	[49]	[219, 137, 78]	2
(...)					
2203	x^3+x^2-734*x+408	2	[7,7]	[28, 28, 35]	2
2251	x^3+x^2-750*x-1584	2	[49]	[191, 274, 151]	2
2557	x^3+x^2-852*x+9281	3	[49,7]	[235, 3, 286]	3

For $f = 33199$, $P = x^3 + x^2 - 11066x + 238541$, $\mathscr{T}_K \simeq \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$, $h = 14$, and the annihilator is equivalent, modulo $1 + \sigma + \sigma^2$, to $A = \sigma - 2$.

For $f = 20857$, $P = x^3 + x^2 - 6952x + 210115$, $\mathscr{T}_K \simeq \mathbb{Z}/7^2\mathbb{Z} \times \mathbb{Z}/7^2\mathbb{Z}$, $h = 1$, and the annihilator is equivalent to $A = 7^2(\sigma - 3)$ where $\sigma - 3$ is invertible modulo 7.

For $f = 1381$, $\mathscr{T}_K \simeq \mathbb{Z}/7^3\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$, $h = 1$, $A = 1738 + 2186\,\sigma + 2361\,\sigma^2$ is equivalent to $7 \cdot (448 + 623\,\sigma)$ and $448 + 623\,\sigma$ operates on $\mathscr{T}_K^7 \simeq \mathbb{Z}/7^2\mathbb{Z}$ as $\sigma - 18$ modulo 7^2 where 18 is of order 3 modulo 7^2 as expected.

For $f = 39679$, $\mathscr{T}_K \simeq \mathbb{Z}/7^3\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$, $h = 7$, and one finds the annihilator $A = 7^2(\sigma - 4)$ where $\sigma - 4$ is not invertible $(\mathrm{N}_{\mathbb{Q}(j)/\mathbb{Q}}(j - 4) = 21)$.

For $p = 13$, the same program gives the following similar results:

```
f    P                  vptor  T_K      coefficients  nj
1033 x^3+x^2-344*x+1913 2    [169]     [311, 455, 919] 2
1459 x^3+x^2-486*x+2864 2    [13,13]   [101, 88, 153] 2
1483 x^3+x^2-494*x-2197 2    [169]     [911, 1868, 1628] 2
1543 x^3+x^2-514*x+4229 2    [169]     [1598, 603, 1866] 2
1747 x^3+x^2-582*x-4141 2    [169]     [1952, 505, 155] 2
3391 x^3+x^2-1130*x+14192 3 [169,13]  [803, 1765, 283] 3
4423 x^3+x^2-1474*x+10648 2 [169]     [52, 1213, 1888] 2
4933 x^3+x^2-1644*x-1827 2  [13,13]   [92, 79, 105] 2
5011 x^3+x^2-1670*x-4083 2  [169]     [602, 1673, 869] 2
5479 x^3+x^2-1826*x+13799 2 [13,13]   [93, 158, 28] 2
7321 x^3+x^2-2440*x-45824 2 [169]     [745, 409, 1546] 2
7963 x^3+x^2-2654*x+43944 2 [169]     [1805, 794, 860] 2
9319 x^3+x^2-3106*x-67649 2 [13,13]   [26, 52, 0]   2
```

6. Experiments and heuristics about the case $p = 2$

Conjecture 5.7 gives various possibilities of annihilation, depending on the choice of $\mathscr{A}_{K,n}(c)$, $\mathscr{A}'_{K,n}(c)$ or else, and of the degree of $K/\mathbb{Q}$, odd, even, or a 2th power. We shall give some illustrations with quadratic, quartic and cubic fields.

6.1 Quadratic fields

Although the order of $\mathscr{T}_K$ is known and given by $\frac{1}{2}L_2(1,\chi)$ (for $K \neq \mathbb{Q}(\sqrt{2})$), we give the computations for the quadratic fields K of conductor $f \geq 5$ with $\mathscr{A}'_{K,n}(c)$ ($a \in [1, f_n/2]$) instead of $\mathscr{A}_{K,n}(c)$ to test the conjecture; the computation of the Artin symbols is easily given by PARI with $\mathsf{kronecker(f,a)} = \pm 1$. The modulus $f_n = \mathrm{l.c.m.}(f_K, 4\cdot 2^n)$ is computed exactely and we take $n = e+2$.

From the annihilator $A' = a_0 + a_1 \cdot \sigma$ (in $(\mathsf{L}_0, \mathsf{L}_1)$), we deduce, modulo the norm, an equivalent annihilator denoted by abuse $A' = a_1 - a_0 \in \mathbb{Z}$.

One finds $A' \equiv 2 \cdot \#\mathscr{T}_K \pmod{2^{2+e}}$ for all $f \neq 8$ (only case with $K \cap \mathbb{Q}_\infty \neq \mathbb{Q}$) in this interval; then the class group is given (be careful to take nt large enought for the computation of the structure of $\mathscr{T}_K$):

```
{p=2;nt=18;bf=5;Bf=10^4;for(f=bf,Bf,v=valuation(f,2);M=f/2^v;
if(core(M)!=M,next);if((v==1||v>3)||(v==0 & Mod(M,4)!=1)||
(v==2 & Mod(M,4)==1),next);P=x^2-f;K=bnfinit(P,1);Kpn=bnrinit(K,p^nt);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
h=component(component(component(K,8),1),2);L=List;ex=0;
i=component(matsize(Hpn),2);for(k=1,i-1,co=component(Hpn,i-k+1);
if(Mod(co,p)==0,val=valuation(co,p);if(val>ex,ex=val);
listinsert(L,p^val,1)));Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);tor=p^vptor;S0=0;S1=0;w=valuation(f,p);
pN=p^2*p^ex;fn=pN*f/2^w;if(ex==0 & w==3,fn=p*fn);
for(cc=2,10^2,if(gcd(cc,p*f)!=1 || kronecker(f,cc)!=-1,next);c=cc;break);
for(a=1,fn/2,if(gcd(a,fn)!=1,next);asurc=lift(a*Mod(c,fn)^-1);
lambda=(asurc*c-a)/fn;u=Mod(lambda*a^-1,pN);
s=kronecker(f,a);if(s==1,S0=S0+u);if(s==-1,S1=S1+u));
L0=lift(S0);L1=lift(S1);A=L1-L0;if(A!=0,A=p^valuation(A,p));
print(f," P=",P," ",L0," ",L1," A=",A," tor=",tor," T_K=",L," Cl_K=",h))}

f_K=8   P=x^2-8   (1,0)      A'=1  tor=1   T_K=[]   Cl_K=[]
(...)
f_K=508 P=x^2-508 (223,479) A'=256 tor=128 T_K=[128] Cl_K=[]
(...)
f_K=1160 P=x^2-1160 (2,6)  A'=4  tor=2   T_K=[2]  Cl_K=[2,2]
f_K=1164 P=x^2-1164 (12,4) A'=8  tor=4   T_K=[4]  Cl_K=[4]
(...)
f_K=1185 P=x^2-1185 (1640,1640) A'=0 tor=1024 T_K=[2,512] Cl_K=[2]
f_K=1189 P=x^2-1189 (2,6)  A'=4  tor=2   T_K=[2]  Cl_K=[2]
```

```
(...)
f_K=1196 P=x^2-1196 (4,20) A'=16 tor=8  T_K=[8]  Cl_K=[2]
f_K=1201 P=x^2-1201 (7752,3656) A'=4096 tor=2048 T_K=[2048] Cl_K=[]
(...)
f_K=1209 P=x^2-1209 (4,4) A'=0  tor=4  T_K=[2,2] Cl_K=[2]
(...)
f_K=1217 P=x^2-1217 (16,48) A'=32 tor=16 T_K=[16] Cl_K=[]
f_K=1221 P=x^2-1221 (8,8) A'=0  tor=8  T_K=[2,4] Cl_K=[4]
(...)
f_K=1596 P=x^2-1596 (16,16) A'=0 tor=16 T_K=[8, 2] Cl_K=[4,2]
```

Remark 6.1. *(i) For $f = 1160$, one sees that $\#\mathscr{C}\ell_K^{\infty} = \frac{1}{2}\#\mathscr{C}\ell_K$ (indeed, -1 is norm in $K/\mathbb{Q}$, cf. (2.1)).*

(ii) It seems that for all the conductors, A' is of the form $2^h\,(1+\sigma)$ up to a 2-adic unit, where $h \geq 0$ takes any value and can exceed the exponent.

(iii) For f prime, the annihilator of $\mathscr{T}_K$, given by the Theorem 9.4, or by any Solomon's type element, is related to its order:

$$\tfrac{1}{2}\,L_2(1,\chi) \sim \tfrac{1}{2}\sum_{a=1}^{f}\chi(a)\cdot\log(1-\zeta_f^a) = \tfrac{1}{2}\cdot\big[\log(\eta_K) - \log(\eta_K^{\sigma})\big],$$

where $\eta_K = \mathrm{N}_{\mathbb{Q}^f/K}(1-\zeta_f)$ (here the character χ is primitive modulo f since $K = k_\chi$). The following program verifies (at least for these kind of prime conductors with trivial class group) that we have $\eta_K \cdot \varepsilon = \pm\sqrt{f}$, where ε is the fundamental unit of K or its inverse (the program gives in N_0 and N_1 the conjugates of η_K and gives ε in E):

```
{f=1201;N0=1;N1=1;X=exp(2*I*Pi/f);z=znprimroot(f);E=quadunit(f);zk=1;
for(k=1,(f-1)/2,zk=zk*z^2;N0=N0*(1-X^lift(zk));N1=N1*(1-X^lift(zk*z)));
print(N0*E," ",N1/E)}
```

We find $N_0\,\varepsilon = N_1\,\varepsilon^{-1} \approx 34.65544690 = \sqrt{1201}$, which implies that:

$$\tfrac{1}{2}\,L_2(1,\chi) \sim \tfrac{1}{2}\,(2\log(\varepsilon)) = \log(\varepsilon).$$

A direct computation gives $\log(\varepsilon) \sim 2^{12}$ as expected since $\#\mathscr{T}_K = 2^{11}$ with $\#\mathscr{R}_K \sim 2^{10}$ [9, Proposition 5.2] and $\#\mathscr{W}_K = 2$ since 2 splits in K. Same kind of result with $f = 1217$.

6.2 A familly of cyclic quartic fields of composite conductor

We consider a conductor f product of two prime numbers q_1 and q_2 such that $q_1 - 1 \equiv 2 \pmod 4$ and $q_2 - 1 \equiv 0 \pmod 8$. So there exists only one real cyclic quartic field K of conductor f which is found eliminating the imaginary and non-cyclic fields; the quadratic subfield of K is $k = \mathbb{Q}(\sqrt{q_2})$. The program is written with $\mathscr{A}'_{K,n}(c)$ and gives all information for k and K.

The following result may help to precise the annihilations (see [14, Theorem 2] or [8, Theorem IV.3.3, Exercise IV.3.3.1]):

Lemma 6.2. *Let k be a totally real number field and let K/k be a Galois p-extension with Galois group g of order p^r. Then we have the fixed point formula: $\#\mathscr{T}_K^{g} = \#\mathscr{T}_k \cdot p^h$, where ($\mathfrak{l} \nmid p$ being the ramified primes in K/k):*

$$h := \min\,(n_0 + r;\ldots,\nu_{\mathfrak{l}} + \varphi_{\mathfrak{l}} + \gamma_{\mathfrak{l}},\ldots) - (n_0 + r) + \sum_{\mathfrak{l}\nmid p} e_{\mathfrak{l}},$$

with:

- $p^{\nu_{\mathfrak{l}}} :=$ *p-part of $q^{-1}\log(\ell)$, where $\mathfrak{l} \cap \mathbb{Z} =: \ell\mathbb{Z}$,*
- $p^{\varphi_{\mathfrak{l}}} :=$ *p-part of the residue degree of ℓ in $K/\mathbb{Q}$,*
- $p^{\gamma_{\mathfrak{l}}} :=$ *p-part of the number of prime ideals $\mathfrak{L} \mid \mathfrak{l}$ in K/k,*
- $p^{e_{\mathfrak{l}}} :=$ *p-part of the ramification index of $\mathfrak{l}$ in K/k.*

In such famillies of cyclic quartic fields, $h = \sum_{\mathfrak{l}\nmid p} e_{\mathfrak{l}}$.

6.2.1 The program

In the present familly, $h = 2$ (resp. 3) if q is inert (resp. splits) in $k/\mathbb{Q}$.

```
{p=2;nt=18;forprime(qq=17,100,if(Mod(qq,8)!=1,next);Pk=x^2-qq;
k=bnfinit(Pk,1);kpn=bnrinit(k,p^nt);Hkpn=component(component(kpn,5),2);
Lk=List;i=component(matsize(Hkpn),2);
for(j=1,i-1,C=component(Hkpn,i-j+1);if(Mod(C,p)==0,
listinsert(Lk,p^valuation(C,p),1)));forprime(q=5,100,
if(valuation(q-1,2)!=2,next);f=q*qq;Q=polsubcyclo(f,4);
for(j=1,7,P=component(Q,j);K=bnfinit(P,1);C7=component(K,7);
S=component(C7,2);D=component(C7,3);
if(Mod(D,f)!=0 || S!=[4,0] || component(polgalois(P),2)!=-1,next);break);
Cl=component(component(component(K,8),1),2);Kpn=bnrinit(K,p^nt);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn=component(component(Kpn,5),2);L=List;ex=0;
i=component(matsize(Hpn),2);for(k=1,i-1,co=component(Hpn,i-k+1);
if(Mod(co,p)==0,val=valuation(co,p);if(val>ex,ex=val);
listinsert(L,p^val,1)));Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);if(vptor>0,S0=0;S1=0;S2=0;S3=0;
pN=p^2*p^ex;fn=pN*f;dqq=(qq-1)/4;dq=(q-1)/2;
z=znprimroot(q);zz=znprimroot(qq);for(cc=3,f,if(gcd(cc,p*f)!=1,next);
cz=lift((cc*z)^dq);czz=lift((cc*zz)^dqq);if(cz!=1 || czz!=1,next);
c=cc;break);cm1=Mod(c,fn)^-1;for(a=1,fn/2,if(gcd(a,fn)!=1,next);
asurc=lift(a*cm1);lambda=(asurc*c-a)/fn;u=Mod(lambda*a^-1,pN);
aqq0=lift((a*zz^0)^dqq);aqq1=lift((a*zz^1)^dqq);
aqq2=lift((a*zz^2)^dqq);aqq3=lift((a*zz^3)^dqq);
aq0=lift((a*z^0)^dq);aq1=lift((a*z^1)^dq);
if(aqq0==1 & aq0==1,S0=S0+u);if(aqq0==1 & aq1==1,S2=S2+u);
if(aqq1==1 & aq0==1,S1=S1+u);if(aqq1==1 & aq1==1,S3=S3+u);
if(aqq2==1 & aq0==1,S2=S2+u);if(aqq2==1 & aq1==1,S0=S0+u);
if(aqq3==1 & aq0==1,S3=S3+u);if(aqq3==1 & aq1==1,S1=S1+u));
L0=lift(S0);L1=lift(S1);L2=lift(S2);L3=lift(S3);Y=Mod(y,y^2+1);
ni=L0+Y*L1+Y^2*L2+Y^3*L3;Nni=valuation(norm(ni),2));V0=1;V1=1;V2=1;V3=1;
if(L0!=0,V0=2^valuation(L0,2));if(L1!=0,V1=2^valuation(L1,2));
if(L2!=0,V2=2^valuation(L2,2));if(L3!=0,V3=2^valuation(L3,2));
print();F=component(factor(f),1);
print("f=",F," Cl=",Cl," P=",P," tor=",2^vptor," Nni=",2^Nni);
print("A=",V0,"*",L0/V0," ",V1,"*",L1/V1," ",V2,"*",L2/V2," ",V3,"*",L3/V3);
print("q=",q," qq=",qq," T_k=",Lk," T_K=",L)))}

f=[5, 17] h=[2] P=x^4-x^3-23*x^2+x+86 tor=16 Nni=16
 A=[2*5, 4*1, 2*1, 1*0] q=5 qq=17 T_k=List([2]) T_K=[4, 2, 2]
f=[13, 17] h=[2] P=x^4-x^3-57*x^2+x+664 tor=32 Nni=32
 A=[2*1, 2*1, 2*3, 2*3] q=13 qq=17 T_k=[2] T_K=[4, 4, 2]
f=[17, 29] h=[2] P=x^4-x^3-125*x^2+x+3452 tor=16 Nni=16
 A=[4*3, 2*1, 1*0, 2*1] q=29 qq=17 T_k=[2] T_K=[4, 2, 2]
f=[17, 37] h=[10] P=x^4-x^3-159*x^2+x+5662 tor=16 Nni=16
 A=[4*1, 2*3, 8*1, 2*7] q=37 qq=17 T_k=[2] T_K=[4, 2, 2]
f=[17, 53] h=[2, 2] P=x^4-x^3-227*x^2+x+11714 tor=32 Nni=32
 A=[2*1, 2*5, 2*3, 2*7] q=53 qq=17 T_k=[2] T_K=[4, 4, 2]
f=[17, 61] h=[2] P=x^4-x^3-261*x^2+x+15556 tor=16 Nni=16
 A=[2*1, 8*1, 2*5, 4*3] q=61 qq=17 T_k=[2] T_K=[4, 2, 2]
f=[5, 41] h=[2] P=x^4-x^3-56*x^2-100*x+160 tor=256 Nni=32
 A=[2*13, 2*45, 2*59, 2*27] q=5 qq=41 T_k=[16] T_K=[32, 4, 2]
f=[13, 41] h=[2] P=x^4-x^3-138*x^2-264*x+1472 tor=256 Nni=32
 A=[2*13, 2*27, 2*51, 2*5] q=13 qq=41 T_k=[16] T_K=[32, 4, 2]
f=[29, 41] h=[2] P=x^4-x^3-302*x^2-592*x+8032 tor=1024 Nni=128
 A=[4*21, 4*5, 4*15, 4*15] q=29 qq=41 T_k=[16] T_K=[32, 8, 4]
f=[37, 41] h=[2] P=x^4-x^3-384*x^2-756*x+13280 tor=256 Nni=32
 A=[2*57, 2*7, 2*47, 2*33] q=37 qq=41 T_k=[16] T_K=[32, 4, 2]
f=[41, 53] h=[2] P=x^4-x^3-548*x^2-1084*x+27712 tor=512 Nni=64
 A=[4*23, 8*15, 4*5, 8*7] q=53 qq=41 T_k=[16] T_K=[32, 4, 4]
f=[41, 61] h=[2, 2] P=x^4-x^3-630*x^2-1248*x+36896 tor=8192 Nni=1024
```

```
  A=[32*3, 16*7, 1*0, 16*7] q=61 qq=41 T_k=[16] T_K=[32, 16, 16]
f=[5, 73] h=[2] P=x^4-x^3-100*x^2+187*x+1389 tor=8 Nni=8
  A=[1*5, 1*9, 1*15, 1*3] q=5 qq=73 T_k=[2] T_K=[4, 2]
f=[13, 73] h=[2] P=x^4-x^3-246*x^2+479*x+11171 tor=8 Nni=8
  A=[1*7, 1*13, 1*13, 1*15] q=13 qq=73 T_k=[2] T_K=[4, 2]
f=[29, 73] h=[2] P=x^4-x^3-538*x^2+1063*x+58767 tor=8 Nni=8
  A=[1*5, 1*7, 1*15, 1*5] q=29 qq=73 T_k=[2] T_K=[4, 2]
f=[37, 73] h=[2] P=x^4-x^3-684*x^2+1355*x+96581 tor=128 Nni=128
  A=[1*0, 16*1, 8*1, 8*1] q=37 qq=73 T_k=[2] T_K=[8, 8, 2]
f=[53, 73] h=[10] P=x^4-x^3-976*x^2+1939*x+200241 tor=8 Nni=8
  A=[1*15, 1*15, 1*5, 1*13] q=53 qq=73 T_k=[2] T_K=[4, 2]
f=[61, 73] h=[2] P=x^4-x^3-1122*x^2+2231*x+266087 tor=16 Nni=16
  A=[8*1, 2*3, 1*0, 2*1] q=61 qq=73 T_k=[2] T_K=[4, 2, 2]
f=[5, 89] h=[2, 2] P=x^4-x^3-122*x^2-217*x+1699 tor=16 Nni=16
  A=[1*0, 2*1, 8*1, 2*3] q=5 qq=89 T_k=[2] T_K=[4, 2, 2]
f=[13, 89] h=[2] P=x^4-x^3-300*x^2-573*x+13625 tor=8 Nni=8
  A=[1*1, 1*7, 1*11, 1*13] q=13 qq=89 T_k=[2] T_K=[4, 2]
f=[29, 89] h=[2] P=x^4-x^3-656*x^2-1285*x+71653 tor=8 Nni=8
  A=[1*11, 1*5, 1*1, 1*15] q=29 qq=89 T_k=[2] T_K=[4, 2]
f=[37, 89] h=[2] P=x^4-x^3-834*x^2-1641*x+117755 tor=8 Nni=8
  A=[1*9, 1*15, 1*3, 1*5] q=37 qq=89 T_k=[2] T_K=[4, 2]
f=[53, 89] h=[2] P=x^4-x^3-1190*x^2-2353*x+244135 tor=16 Nni=16
  A=[4*1, 2*5, 4*1, 2*7] q=53 qq=89 T_k=[2] T_K=[4, 2, 2]
f=[61, 89] h=[2] P=x^4-x^3-1368*x^2-2709*x+324413 tor=8 Nni=8
  A=[1*1, 1*9, 1*11, 1*11] q=61 qq=89 T_k=[2] T_K=[4, 2]
f=[5, 97] h=[2] P=x^4-x^3-133*x^2-479*x+36 tor=16 Nni=16
  A=[2*5, 8*1, 2*1, 4*3] q=5 qq=97 T_k=[2] T_K=[4, 2, 2]
f=[13, 97] h=[10] P=x^4-x^3-327*x^2-1255*x+2558 tor=16 Nni=16
  A=[4*1, 2*7, 8*1, 2*3] q=13 qq=97 T_k=[2] T_K=[4, 2, 2]
f=[29, 97] h=[2] P=x^4-x^3-715*x^2-2807*x+16914 tor=16 Nni=16
  A=[2*3, 8*1, 2*3, 4*3] q=29 qq=97 T_k=[2] T_K=[4, 2, 2]
f=[37, 97] h=[2] P=x^4-x^3-909*x^2-3583*x+28748 tor=16 Nni=16
  A=[4*3, 2*7, 1*0, 2*3] q=37 qq=97 T_k=[2] T_K=[4, 2, 2]
f=[53, 97] h=[2] P=x^4-x^3-1297*x^2-5135*x+61728 tor=64 Nni=64
  A=[8*3, 4*7, 16*1, 4*7] q=53 qq=97 T_k=[2] T_K=[8, 4, 2]
f=[61, 97] h=[2] P=x^4-x^3-1491*x^2-5911*x+82874 tor=32 Nni=32
  A=[2*7, 2*5, 2*5, 2*7] q=61 qq=97 T_k=[2] T_K=[4, 4, 2]
```

6.2.2 The case $f = 5 \cdot 73$

One may try to find a contradiction to Conjecture 5.7 with the $\mathscr{A}'_{K,n}(c)$ given by the above data. One sees that $\frac{1}{2}\mathscr{A}'_{K,n}(c)$ is not always in $\mathbb{Z}[G_K]$, but modulo the norm we have an annihilator of the form $2 \cdot \mathscr{A}''_{K,n}(c)$, and similarly we may ask under what condition $\mathscr{A}''_{K,n}(c)$ annihilates $\mathscr{T}_K$.

For $f = 5 \cdot 73$, $P = x^4 - x^3 - 100x^2 + 187x + 1389$, for which we have $\mathscr{T}_K \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathscr{T}_k \simeq \mathbb{Z}/2\mathbb{Z}$, $\mathsf{Cl} = 2$, $\mathscr{A}'_{K,n}(c) = 5 + 9\sigma + 15\sigma^2 + 3\sigma^3$, giving:

$$\mathscr{A}''_{K,n}(c) = \tfrac{1}{2}\big[5 + 9\sigma + 15\sigma^2 + 3\sigma^3 - 3(1 + \sigma + \sigma^2 + \sigma^3)\big] \equiv 1 - \sigma + 2\sigma^2 \pmod 4$$

without obvious contradiction since $\#\mathscr{T}_K^{g} = 8$ (i.e., $\mathscr{T}_K^{g} = \mathscr{T}_K$) and $\#\mathscr{T}_K^{G_K} = 4$ (Lemma 6.2). Moreover, we deduce from this that $\mathrm{N}_{K/k}(\mathscr{T}_K) = \mathscr{T}_k$.

6.3 Cyclic cubic fields of prime conductors

The following program gives, for $p = 2$ and for cyclic cubic fields of prime conductor f, the group structure of $\mathscr{T}_K$ in L (from [10, § 3.2]; recall that in all such programs, the parameter $\mathtt{nt}$ must be large enough regarding the exponent of $\mathscr{T}_K$), then the (conjectural) annihilator $\mathscr{A}'_{K,n}(c)$, reduced modulo $1 + \sigma + \sigma^2$; it is equal, up to an invertible element, to a power of 2 (2 is inert in $\mathbb{Q}(j)$):

```
{p=2;nt=12;forprime(f=10^4,2*10^4,if(Mod(f,3)!=1,next);P=polsubcyclo(f,3);
```

```
K=bnfinit(P,1);Kpn=bnrinit(K,p^nt);C5=component(Kpn,5);
Hpn0=component(C5,1);Hpn=component(C5,2);L=List;ex=0;
i=component(matsize(Hpn),2);for(k=1,i-1,co=component(Hpn,i-k+1);
if(Mod(co,p)==0,val=valuation(co,p);if(val>ex,ex=val);
listinsert(L,p^val,1)));Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);if(vptor>2,S0=0;S1=0;S2=0;pN=p^2*p^ex;
D=(f-1)/3;fn=pN*f;z=znprimroot(f);zz=lift(z);t=lift(Mod((1-zz)/f,p));
c=zz+t*f;for(a=1,fn/2,if(gcd(a,fn)!=1,next);asurc=lift(a*Mod(c,fn)^-1);
lambda=(asurc*c-a)/fn;u=Mod(lambda*a^-1,pN);
a0=lift((a*z^0)^D);a1=lift((a*z^2)^D);a2=lift((a*z)^D);
if(a0==1,S0=S0+u);if(a1==1,S1=S1+u);if(a2==1,S2=S2+u));
L0=lift(S0);L1=lift(S1);L2=lift(S2);L1=L1-L0;L2=L2-L0;
A=gcd(L1,L2);A=2^valuation(A,2);print(f," ",P," ", A," ",L)))}

f     P                     A   L
10399 x^3+x^2-3466*x+7703  4   [4,4]
10513 x^3+x^2-3504*x-80989 8   [8,8]
10753 x^3+x^2-3584*x-76864 4   [4,4]
10771 x^3+x^2-3590*x-26728 4   [4,4]
10903 x^3+x^2-3634*x+26248 8   [8,8]
10939 x^3+x^2-3646*x-46592 16  [16,16]
10957 x^3+x^2-3652*x-39364 4   [4,4]
11149 x^3+x^2-3716*x+39228 4   [2,2,2,2]
(...)
12757 x^3+x^2-4252*x+103001 4 [4,4]
13267 x^3+x^2-4422*x+96800 16 [16,16]
13297 x^3+x^2-4432*x+94064 4  [4,4]
13309 x^3+x^2-4436*x+100064 4 [4,4]
13591 x^3+x^2-4530*x-63928 8  [8,8]
```

6.4 Cyclic quartic fields of prime conductors

Let's give the same program for prime conductors $f \equiv 1 \pmod 8$, with the annihilator $\mathscr{A}_{K,n}(c)$:

```
{p=2;nt=18;d=4;forprime(f=5,500,if(Mod(f,2*d)!=1,next);P=polsubcyclo(f,d);
K=bnfinit(P,1);Kpn=bnrinit(K,p^nt);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);L=List;ex=0;
i=component(matsize(Hpn),2);for(k=1,i-1,co=component(Hpn,i-k+1);
if(Mod(co,p)==0,val=valuation(co,p);if(val>ex,ex=val);
listinsert(L,p^val,1)));Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);if(vptor>1,S0=0;S1=0;S2=0;S3=0;
pN=p^2*p^ex;D=(f-1)/d;fn=pN*f;z=znprimroot(f);zz=lift(z);
t=lift(Mod((1-zz)/f,p));c=zz+t*f;for(a=1,fn,if(gcd(a,fn)!=1,next);
asurc=lift(a*Mod(c,fn)^-1);lambda=(asurc*c-a)/fn;u=Mod(lambda*a^-1,pN);
a0=lift((a*z^0)^D);a1=lift((a*z^1)^D);a2=lift((a*z^2)^D);a3=lift((a*z^3)^D);
if(a0==1,S0=S0+u);if(a1==1,S1=S1+u);if(a2==1,S2=S2+u);if(a3==1,S3=S3+u));
L0=lift(S0);L1=lift(S1);L2=lift(S2);L3=lift(S3);Y=Mod(y,y^2+1);
ni=L0+Y*L1+Y^2*L2+Y^3*L3;Nni=valuation(norm(ni),2);
print(f," ",P," ",L0," ",L1," ",L2," ",L3," ",L," ",2^Nni)))}
```

One gets the following examples (with vptor > 1 and where 2^{Nni} is the norm of $L_0 - L_2 + (L_1 - L_3)\sqrt{-1}$ with $\mathscr{A}_{K,n}(c) = L_0 + L_1\sigma + L_2\sigma^2 + L_3\sigma^3$, given in $\mathsf{A} = [\mathsf{L0}, \mathsf{L1}, \mathsf{L2}, \mathsf{L3}]$); then the list L gives the structure of $\mathscr{T}_K$:

```
f   P                      A                 L       2^Nni
17  x^4+x^3-6*x^2-x+1      [4, 6, 0, 6]      [4]     16
41  x^4+x^3-15*x^2+18*x-4  [90, 28, 102, 100] [32] 16
73  x^4+x^3-27*x^2-41*x+2  [4, 4, 0, 0]      [2,2,2] 32
89  x^4+x^3-33*x^2+39*x+8  [4, 4, 0, 0]      [2,2,2] 32
97  x^4+x^3-36*x^2+91*x-61 [8, 10, 12, 2] [4]     16
113 x^4+x^3-42*x^2-120*x-64 [16, 28, 8, 12] [2,2,8] 64
```

```
137  x^4+x^3-51*x^2-214*x-236 [26, 8, 30, 16] [16] 16
193  x^4+x^3-72*x^2-205*x-49 [6, 0, 14, 12] [4]    16
233  x^4+x^3-87*x^2+335*x-314 [4, 0, 0, 4] [2,2,2] 32
241  x^4+x^3-90*x^2-497*x-739 [6, 0, 6, 4] [4]     16
257  x^4+x^3-96*x^2-16*x+256 [28, 20, 20, 60] [2,4,16] 128
281  x^4+x^3-105*x^2+123*x+236 [4, 4, 0, 0] [2,2,2] 32
313  x^4+x^3-117*x^2+450*x-324 [78, 12, 106, 108] [32] 16
337  x^4+x^3-126*x^2+316*x+104 [28, 12, 28, 28] [2,8,8] 256
353  x^4+x^3-132*x^2+684*x-928 [112, 60, 80, 68] [2,2,32] 64
401  x^4+x^3-150*x^2-25*x+625 [14, 4, 6, 8] [4]    16
409  x^4+x^3-153*x^2-230*x+548 [22, 8, 26, 24] [8] 16
433  x^4+x^3-162*x^2+839*x-1003 [2, 4, 10, 0] [4]  16
449  x^4+x^3-168*x^2-477*x+335 [10, 4, 10, 8] [4]  16
457  x^4+x^3-171*x^2+1114*x-2044 [76, 10, 28, 30] [32] 16
```

6.5 Detailed example of annihilation

The case of the cyclic quartic field K of conductor $f = 3433$ is particularly interesting:

6.5.1 Numerical data

We have $P = x^4 + x^3 - 1287x^2 - 12230x + 3956$ and $\mathscr{T}_K \simeq \mathbb{Z}/2^7\mathbb{Z}$, knowing that the quadratic subfield $k = \mathbb{Q}(\sqrt{3433})$ is such that $\mathscr{T}_k \simeq \mathbb{Z}/2^6\mathbb{Z}$:

```
{P=x^4+x^3-1287*x^2-12230*x+3956;K=bnfinit(P,1);p=2;nt=18;
Kpn=bnrinit(K,p^nt);Hpn=component(component(Kpn,5),2);L=List;
i=component(matsize(Hpn),2);for(k=1,i-1,c=component(Hpn,i-k+1);
if(Mod(c,p)==0,listinsert(L,p^valuation(c,p),1)));print("Structure: ",L)}
Structure: List([128])

{P=x^2-3433;K=bnfinit(P,1);p=2;nt=18;Kpn=bnrinit(K,p^nt);
Hpn=component(component(Kpn,5),2);L=List;i=component(matsize(Hpn),2);
for(k=1,i-1,c=component(Hpn,i-k+1);if(Mod(c,p)==0,
listinsert(L,p^valuation(c,p),1)));print("Structure: ",L)}
Structure: List([64])
```

The class group of K is trivial and its three fundamental units are:

```
[227193/338*x^3-6613325/338*x^2-93274465/338*x+14925255/169,
34349/169*x^3+1388772/169*x^2+10559389/169*x-3491425/169,
70276336974818125/338*x^3-677429229869394661/338*x^2
     -8323827298356088143/338*x+13065197272033438434/169]
```

6.5.2 Annihilation from $\mathscr{A}_{K,n}(c)$

We have computed $\mathscr{A}_{K,n}(c)$ and obtained:

$$\mathscr{A}_{K,n}(c) =: A_K \equiv 8\cdot 13 + 2\cdot 21\,\sigma + 16\cdot 7\,\sigma^2 + 2\cdot 23\,\sigma^3 \pmod{2^7}.$$

Let h be a group generator of $\mathscr{T}_K$ (order 2^7) and let h_0 be a generator of $\mathscr{T}_k$ (order 2^6); it is easy to prove that one may suppose $h^2 = j_{K/k}(h_0)$ (injectivity of the transfer map $j_{K/k}$) and $h_0^{\sigma^2} = h_0$. We put $j_{K/k}(h_0) =: h_0$ for simplicity. Then it follows that

$$h^{A_K} = h_0^{4\cdot 13 + 21\,\sigma + 8\cdot 7\,\sigma^2 + 23\,\sigma^3} = 1.$$

Since $h_0^{\sigma^2} = h_0$, we obtain $h^{A_K} = h_0^{(4\cdot 13+8\cdot 7)+(21+23)\,\overline{\sigma}} = h_0^{4\cdot 27 + 4\cdot 11\,\overline{\sigma}} = 1$; giving, modulo the norm $1+\overline{\sigma}$, $h_0^{4\cdot(27-11)} = h_0^{2^6} = 1$, as expected.

6.5.3 Annihilation from $\mathscr{A}'_{K,n}(c)$

There is (by accident ?) no numerical obstruction for an annihilation by $A'_K := \mathscr{A}'_{K,n}(c)$, with the same program replacing "$\mathsf{for}(\mathsf{a}=1,\mathsf{fn},\ldots)$" by "$\mathsf{for}(\mathsf{a}=1,\mathsf{fn}/2,\ldots)$". Then it follows that the program gives $h^{A'_K} = h^{4\cdot 13+21\,\sigma+8\cdot 15\,\sigma^2+23\,\sigma^3} = 1$. Since the restriction of A'_K to k is A'_k (no Euler factors), we get:

$$h_0^{A'_k} = h_0^{4\cdot 13+8\cdot 15+(21+23)\cdot\overline{\sigma}} = h_0^{4\cdot 43+4\cdot 11\,\overline{\sigma}} = 1$$

which is equivalent, modulo the norm, to the annihilation by $4\cdot 43 - 4\cdot 11 = 2^7$ for a cyclic group of order 2^6.

Now we may return to the annihilation of h; since $h^{1+\sigma^2} \in j_{K/k}(\mathscr{T}_k)$ we put $h^{1+\sigma^2} = h_0^t$. Then, with $u = 13$, $v = 21$, $w = 15$, $z = 23$, we have:

$$\begin{aligned} h^{4u+v\sigma+8w\sigma^2+z\sigma^3} &= h_0^{2u+4w\sigma^2} h^{(v+z\sigma^2)\sigma} \\ &= h_0^{2u+4w+23t\,\sigma} h^{(v-z)\sigma} = h_0^{2\cdot 43+23t\,\sigma} h^{-2\sigma} \\ &= h_0^{2\cdot 43+(23t-1)\,\sigma} = h_0^{2\cdot 43-23t+1} = h_0^{87-23t} = 1 \end{aligned}$$

so necessarily $87 - 23t \equiv 0 \pmod{2^6}$, giving $t \equiv 1 \pmod{2^6}$. So we can write:

$$h^{1+\sigma^2} = j_{K/k}(h_0).$$

6.5.4 Direct study of the G_K-module structure of $\mathscr{T}_K$

We consider $\mathscr{T}_K$ only given with the following information: h is a group generator such that $h^2 = h_0$, a generator of $j_{K/k}(\mathscr{T}_K)$; $h^\sigma = h^x$, $x \in \mathbb{Z}/2^7\mathbb{Z}$, whence $h_0^\sigma = h_0^x = h_0^{-1}$ giving $x \equiv -1 \pmod{2^6}$. The relation $h^{\sigma^2+1} = h^{x^2+1} = h^2 = h_0$ gives again $t = 1$ in the previous notation $h^{\sigma^2+1} = h_0^t$. Moreover, $h^{\sigma^2-1} = h^{x^2-1} = 1$, which is in accordance with Lemma 6.2 and gives $\mathscr{T}_K^g = \mathscr{T}_K$.

If we take into account these theoretical informations for the "annihilators" A_K and A'_K we find no contradiction, but we do not know if $x = -1$ or $x = -1 + 2^6$ (modulo 2^7). The prime 2 splits in k, is inert in K/k and the class number of K is 1; so we have $\mathscr{W}_K \simeq \mathscr{W}_k \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathscr{T}_K = \mathrm{tor}_{\mathbb{Z}_2}\big(U_K/\overline{E}_K\big)$; then the result about x depends on the exact sequence (2.2):

$$1 \to \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathscr{T}_K \simeq \mathbb{Z}/2^7\mathbb{Z} \xrightarrow{\ \log\ } \mathrm{tor}_{\mathbb{Z}_2}\big(\log(U_K)/\log(\overline{E}_K)\big) =: \mathscr{R}_K \to 0,$$

knowing the units and then the structure of the regulator $\mathscr{R}_K$.

6.5.5 About the case $f_K = 233$

The field K is defined by the polynomial $P = x^4 + x^3 - 87x^2 + 335x - 314$ for which $\mathscr{T}_K \simeq (\mathbb{Z}/2\mathbb{Z})^3$ and $\mathscr{T}_k \simeq \mathbb{Z}/2\mathbb{Z}$.

In this case an annihilator is $A_K = 4\cdot(1+\sigma^3)$, which shows that $A'_K = 2\cdot(1+\sigma^3)$ is also suitable. Then $A''_K = \frac{1}{2}A'_K$ should be equivalent to $1-\sigma$.

Since 2 splits completely in K, we have $\mathscr{T}_K = \mathscr{W}_K \simeq (\mathbb{Z}/2\mathbb{Z})^3$ and in the same way, $\mathscr{T}_k = \mathscr{W}_k \simeq \mathbb{Z}/2\mathbb{Z}$, for which the Galois structures are well-known: in particular, $1-\sigma$ *does not annihilate* $\mathscr{T}_K$ (the class of $(1,-1,1,-1)$ is invariant). Another proof: use Lemma 6.2 giving here $\#\mathscr{T}_K^{G_K} = 2$.

7. p-adic measures and annihilations

To establish (in Section 9) a link with the values of p-adic L-functions, $L_p(s,\chi)$, at $s = 1$, we shall refer to [13, Section II] using the point of view of explicit p-adic measures (from pseudo-measures in the sense of [24]) with a Mellin transform for the construction of $L_p(s,\chi)$ and the application to some properties of the λ invariants of Iwasawa's theory.

But since we only need the value $L_p(1,\chi)$, instead of $L_p(s,\chi)$, for $s \in \mathbb{Z}_p$, we can simplify the general setting, using a similar computation of $\mathscr{S}_{L_n}(c)^*$, directly in $\mathbb{Z}[G_n]$, given by Oriat in [22, Proposition 3.5].

7.1 Definition of $\mathscr{A}_{L_n}$ and $\mathscr{A}_{L_n}(c)$

Let $n \geq n_0 + e$, where $\mathscr{T}_K^{p^e} = 1$, and put $\varphi_n := \varphi(qp^n) = (p-1)\cdot p^n$ if $p \neq 2$, $\varphi_n = 2^{n+1}$ otherwise.

We consider (where c is odd and prime to f_n and where a runs trough the integers in $[1, f_n]$, prime to f_n):

$$\mathscr{A}_{L_n} := \frac{-1}{f_n\varphi_n}\sum_a a^{\varphi_n}\left(\frac{L_n}{a}\right) \ \ \&\ \ \mathscr{A}_{L_n}(c) := \left[1 - c^{\varphi_n}\left(\frac{L_n}{c}\right)\right]\mathscr{A}_{L_n}. \tag{7.1}$$

For now, these elements, or more precisely their restrictions to K, are not to be confused with the restrictions $\mathscr{A}_{K,n}(c)$ of $\mathscr{S}_{L_n}(c)^*$ defined in § 5.2, even we shall prove that they are indeed equal; but such an expression is more directly associated to L_p-functions. Then:

$$\begin{aligned}\mathscr{A}_{L_n}(c) &= \Big[1 - c^{\varphi_n}\Big(\frac{L_n}{c}\Big)\Big]\frac{-1}{f_n\varphi_n}\sum_a a^{\varphi_n}\Big(\frac{L_n}{a}\Big)\\ &\cong \frac{-1}{f_n\varphi_n}\Big[\sum_a a^{\varphi_n}\Big(\frac{L_n}{a}\Big) - \sum_a a^{\varphi_n}c^{\varphi_n}\Big(\frac{L_n}{a}\Big)\Big(\frac{L_n}{c}\Big)\Big]\\ &\text{(in the same way, use } a'_c \text{ such that}\\ &\qquad a'_c\cdot c \equiv a \pmod{f_n},\ 1\le a'_c\le f_n)\\ &\cong \frac{-1}{f_n\varphi_n}\Big[\sum_a a^{\varphi_n}\Big(\frac{L_n}{a}\Big) - \sum_a a'^{\,\varphi_n}_c c^{\varphi_n}\Big(\frac{L_n}{a'_c}\Big)\Big(\frac{L_n}{c}\Big)\Big]\\ &\cong \frac{1}{f_n\varphi_n}\sum_a\Big[(a'_c\cdot c)^{\varphi_n} - a^{\varphi_n}\Big]\Big(\frac{L_n}{a}\Big).\end{aligned}$$

Lemma 7.1. *We have* $(a'_c\cdot c)^{\varphi_n} - a^{\varphi_n} \equiv 0 \pmod{f_n\varphi_n}$.

Proof. By definition, $a'_c\cdot c = a + \lambda^n_a(c)f_n$ with $\lambda^n_a(c)\in\mathbb{Z}$. Consider:

$$\begin{aligned}A &:= \frac{(a'_c\cdot c)^{\varphi_n} - a^{\varphi_n}}{f_n\varphi_n}\\ &= \frac{[a^{\varphi_n} + \lambda^n_a(c)f_n\varphi_n a^{\varphi_n-1} + \lambda^n_a(c)^2 f_n^2\frac{\varphi_n(\varphi_n-1)}{2}a^{\varphi_n-2} + \cdots] - a^{\varphi_n}}{f_n\varphi_n}\\ &\equiv \lambda^n_a(c)a^{\varphi_n-1} + \lambda^n_a(c)^2 f_n\frac{(\varphi_n-1)}{2}a^{\varphi_n-2}\\ &\equiv \lambda^n_a(c)a^{\varphi_n-1} \equiv \lambda^n_a(c)a^{-1} \pmod{p^{n+1}},\end{aligned}$$

since $a^{\varphi_n}\equiv 1 \pmod{qp^n}$.

When $p=2$, one must take into account the term $\lambda^n_a(c)f_n\frac{\varphi_n-1}{2}a^{\varphi_n-2}\sim\frac{1}{2}\lambda^n_a(c)f_n$, in which case the congruence is with the modulus p^{n+1} (which is sufficient since for $n\ge n_0+e$, this modulus annihilates $\mathscr{T}_K$ for any p). □

We have obtained for all $n\ge n_0+e$:

$$\mathscr{A}_{L_n}(c) \cong \sum_{a=1}^{f_n}\lambda^n_a(c)\cdot a^{-1}\Big(\frac{L_n}{a}\Big) \cong \mathscr{S}_{L_n}(c)^*, \tag{7.2}$$

thus giving again, by restriction to K, the annihilator $\mathscr{A}_{K,n}(c)\in\mathbb{Z}_p[G_K]$ of $\mathscr{T}_K$ such that (for all $n\ge n_0+e$) $\mathscr{A}_{K,n}(c)\cong\sum_{a=1}^{f_n}\lambda^n_a(c)a^{-1}\Big(\frac{K}{a}\Big)$.

7.2 Normic properties of the $\mathscr{A}_{L_n}$ – Euler factors

Theorem 7.2. *[13, Proposition II.2 (iv)]. Let K be of conductor $f=m\ell$ where m is the conductor of a subfield k of K and where $\ell\ne p$ is a prime number. For $n\ge n_0$, let $L_n := K(\mu_{qp^n})$ and the analogous field l_n for k, of conductors f_n and m_n, respectively; recall that $\varphi_n=\varphi(qp^n)$.*

Let $\mathscr{A}_{L_n} := \frac{-1}{f_n\varphi_n}\sum_a^{f_n} a^{\varphi_n}\Big(\frac{L_n}{a}\Big)$ *and* $\mathscr{A}_{l_n} := \frac{-1}{m_n\varphi_n}\sum_b^{m_n} b^{\varphi_n}\Big(\frac{l_n}{b}\Big)$. *Then:*

$$\mathrm{N}_{L_n/l_n}(\mathscr{A}_{L_n}) \cong \Big(1-\ell^{\varphi_n}\frac{1}{\ell}\Big(\frac{l_n}{\ell}\Big)\Big)\mathscr{A}_{l_n},\ \ \textit{resp.},\ \ \mathrm{N}_{L_n/l_n}(\mathscr{A}_{L_n}) \cong \mathscr{A}_{l_n},$$

if $\ell\nmid m$, resp., $\ell\mid m$ (congruences modulo $p^{n+1}\mathbb{Z}_p[G_n]+(1-s_\infty)\mathbb{Z}_p[G_n]$).

Proof. Suppose first that $\ell\nmid m$, so $f_n = lm_n$.[2] Put $a=b+\lambda\, m_n$, $\lambda\in[0,\ell-1]$, $b\in[1,m_n]$ prime to m_n; since $a\in[1,f_n]$ is prime to f_n, b is prime to m_n and $\lambda\ne\lambda^*_b$ such that $b+\lambda^*_b m_n =: b'_\ell\cdot\ell$, $b'_\ell\in\mathbb{Z}$. Thus $a^{\varphi_n} = (b+\lambda\, m_n)^{\varphi_n} \equiv b^{\varphi_n} + b^{\varphi_n-1}\lambda\, m_n\varphi_n$

[2] For $\ell=2$ and m odd, $f=2m$ is not a conductor stricto sensu, but the following computations are exact and necessary with the modulus m_n and $f_n=2m_n$; then if $f=2^k\cdot m$ (m odd, $k\ge 2$), the second case of the theorem applies and shall give the Euler factor $\big(1-2^{\varphi_n}\frac{1}{2}\big(\frac{l_n}{2}\big)\big)\cong\big(1-\frac{1}{2}\big(\frac{l_n}{2}\big)\big)$. If $p\mid f$ and $p\nmid m$, there is no Euler factor for p since m_n and f_n are divisible by p; in other words, these computations and the forthcoming ones are, by nature, not "primitive" at p.

(mod $m_n \varphi_n p^{n+1}$). Then:

$$\begin{aligned}
\mathrm{N}_{L_n/l_n}(\mathscr{A}_{L_n}) &\mathrel{\overline{\overline{\equiv}}} \frac{-1}{\ell m_n \varphi_n} \cdot \sum_{b,\,\lambda \neq \lambda_b^*} \Big[b^{\varphi_n} + b^{\varphi_n - 1} \lambda\, m_n \varphi_n \Big] \Big(\frac{l_n}{b}\Big) \\
&\mathrel{\overline{\overline{\equiv}}} \frac{-(\ell-1)}{\ell m_n \varphi_n} \sum_b b^{\varphi_n} \Big(\frac{l_n}{b}\Big) - \frac{1}{\ell} \sum_{b,\,\lambda \neq \lambda_b^*} b^{\varphi_n - 1} \lambda \Big(\frac{l_n}{b}\Big) \\
&\mathrel{\overline{\overline{\equiv}}} \Big(1 - \frac{1}{\ell}\Big) \mathscr{A}_{l_n} - \frac{1}{\ell} \sum_{b,\,\lambda \neq \lambda_b^*} b^{\varphi_n - 1} \lambda \Big(\frac{l_n}{b}\Big) \\
&\mathrel{\overline{\overline{\equiv}}} \Big(1 - \frac{1}{\ell}\Big) \mathscr{A}_{l_n} - \frac{1}{\ell} \sum_b b^{\varphi_n - 1} \Big(\frac{l_n}{b}\Big) \Big(\sum_{\lambda \neq \lambda_b^*} \lambda \Big) \\
&\mathrel{\overline{\overline{\equiv}}} \Big(1 - \frac{1}{\ell}\Big) \mathscr{A}_{l_n} - \frac{1}{\ell} \sum_b b^{\varphi_n - 1} \Big(\frac{l_n}{b}\Big) \Big(\frac{\ell(\ell-1)}{2} - \lambda_b^* \Big).
\end{aligned}$$

We remark that $\lambda_b^* = \lambda_b^n(\ell)$ is relative to the writing $b'_\ell \cdot \ell = b + \lambda_b^n(\ell)\, m_n$ and that $b^{\varphi_n - 1} \equiv b^{-1} \pmod{p^{n+1}}$, whence using $\sum_b b^{-1}\big(\frac{l_n}{b}\big) \mathrel{\overline{\overline{\equiv}}} 0$:

$$\mathrm{N}_{L_n/l_n}(\mathscr{A}_{L_n}) \mathrel{\overline{\overline{\equiv}}} \Big(1 - \frac{1}{\ell}\Big) \mathscr{A}_{l_n} + \frac{1}{\ell} \sum_b \lambda_b^* \cdot b^{-1} \Big(\frac{l_n}{b}\Big).$$

But as we know (see relations 7.1 and (7.2)), $\sum_b \lambda_b^* b^{-1} \Big(\frac{l_n}{b}\Big) \mathrel{\overline{\overline{\equiv}}} \mathscr{A}_{l_n}(\ell)$; so $\mathrm{N}_{L_n/l_n}(\mathscr{A}_{L_n}) \mathrel{\overline{\overline{\equiv}}} \Big(1 - \frac{1}{\ell}\Big)\mathscr{A}_{l_n} + \frac{1}{\ell}\mathscr{A}_{l_n}(\ell)$: since $\mathscr{A}_{l_n}(\ell) \mathrel{\overline{\overline{\equiv}}} \Big(1 - \ell^{\varphi_n}\Big(\frac{l_n}{\ell}\Big)\Big)\mathscr{A}_{l_n}$, we get $\mathrm{N}_{L_n/l_n}(\mathscr{A}_{L_n}) \mathrel{\overline{\overline{\equiv}}} \Big(1 - \ell^{\varphi_n}\frac{1}{\ell}\Big(\frac{l_n}{\ell}\Big)\Big)\mathscr{A}_{l_n}$.

The case $\ell \mid m$ is obtained more easily from the same computations. □

Of course, for all $h \geq 0$ we get:

$$\mathrm{N}_{L_{n+h}/L_n}(\mathscr{A}_{L_{n+h}}) \mathrel{\overline{\overline{\equiv}}} \mathscr{A}_{L_n},$$

which expresses the coherence of the family $\big(\mathscr{A}_{L_n}\big)_n$ in the cyclotomic tower.

Corollary 7.3. *(i) Let K/k be an extension of fields of conductors f_K and f_k, respectively. Multiplying by $\Big[1 - c^{\varphi_n}\Big(\frac{l_n}{c}\Big)\Big] = \mathrm{N}_{L_n/l_n}\Big[1 - c^{\varphi_n}\Big(\frac{L_n}{c}\Big)\Big]$ to get elements in the algebras $(\mathbb{Z}/p^{n+1}\mathbb{Z})[\mathrm{Gal}(L_n/\mathbb{Q})]$ and $(\mathbb{Z}/p^{n+1}\mathbb{Z})[\mathrm{Gal}(l_n/\mathbb{Q})]$, one obtains $\mathrm{N}_{L_n/l_n}(\mathscr{A}_{L_n}(c)) \mathrel{\overline{\overline{\equiv}}} \prod_{\ell \mid f_K, \ell}$ $\frac{1}{\ell}\Big(\frac{l_n}{\ell}\Big)\Big)\mathscr{A}_{l_n}(c)$.*

(ii) Let $\mathscr{A}_{K,n}(c)$ and $\mathscr{A}_{k,n}(c)$ be the restrictions of $\mathscr{A}_{L_n}(c)$ and $\mathscr{A}_{l_n}(c)$ to K and k, respectively; then $\mathrm{N}_{K/k}(\mathscr{A}_{K,n}(c)) \mathrel{\overline{\overline{\equiv}}} \prod_{\ell \mid f_K,\, \ell \nmid p f_k}\Big(1 - \frac{1}{\ell}\Big(\frac{k}{\ell}\Big)\Big) \cdot \mathscr{A}_{k,n}(c)$.

(iii) The family $(\mathscr{A}_{K,n})_n = (\mathrm{N}_{L_n/K}(\mathscr{A}_{L_n}))_n$ defines a pseudo-measure denoted $\mathscr{A}_K$ by abuse, such that the measure $(\mathscr{A}_{K,n}(c))_n$ defines the element $\mathscr{A}_K(c) = \Big(1 - \Big(\frac{K}{c}\Big)\Big) \cdot \mathscr{A}_K \in \mathbb{Z}_p[G_K]$ and gives the main formula:

$$\mathrm{N}_{K/k}(\mathscr{A}_K(c)) \mathrel{\overline{\overline{\equiv}}} \prod_{\ell \mid f_K,\, \ell \nmid p f_k} \Big(1 - \frac{1}{\ell}\Big(\frac{k}{\ell}\Big)\Big) \cdot \mathscr{A}_k(c).$$

Remark 7.4. *(i) In a numerical point of view, we only need a minimal value of n, and we shall write (e.g., for $n = e$ when $K \cap \mathbb{Q}_\infty = \mathbb{Q}$):*

$$\mathscr{A}_{K,e}(c) \mathrel{\overline{\overline{\equiv}}} \sum_{\sigma \in G_K} \Big[\sum_{a,\,(\frac{K}{a}) = \sigma} \lambda_a^e(c)\, a^{-1} \Big] \cdot \sigma =: \sum_{\sigma \in G_K} \Lambda_\sigma^e(c) \cdot \sigma.$$

Then the next step shall be to interprete the limit, $\Lambda_\sigma(c)$, of the coefficients $\Lambda_\sigma^n(c) = \sum_{a,\,(\frac{K}{a}) = \sigma} \lambda_a^n(c)\, a^{-1}$, for $n \to \infty$, giving an equivalent annihilator, but with a more canonical interpretation.

(ii) In [12, 13, 22, 26, 28, 29, 5, 19, 27, 21, 1, 2, 4], some limits are expressed by means of p-adic logarithms of cyclotomic numbers/units of $\mathbb{Q}^f$ as expressions of the values at $s = 1$ of the p-adic L-functions of K (for instance, in [29, Theorem 2.1] a link between Stickelberger elements and cyclotomic units is given following Iwasawa and Coleman). But these results are obtained with various non-comparable techniques; this will be discussed later.

(iii) In the relation $\mathscr{A}_K(c) := \Big[1 - \Big(\frac{K}{c}\Big)\Big]\mathscr{A}_K$, the choice of c must be such that the integers $1 - \chi(c)$ be of minimal p-adic valuation for the characters χ of K. But $1 - \chi(c)$ is invertible if and only if $\chi(c)$ is not a root of unity of p-power order.

8. Remarks about Solomon's annihilators

We shall give two examples: one giving the same annihilator as our's, and another giving a Solomon annihilator in part degenerated, contrary to $\mathscr{A}_K(c)$.

8.1 Cubic field of conductor 1381 and Solomon's Ψ_K

We have (see the previous table of § 5.3) $P = x^3 + x^2 - 460x - 1739$ and the classical program gives the class number in h, the group structure of $\mathscr{T}_K$ (in L) and the units in E:

```
{P=x^3+x^2-460*x-1739;K=bnfinit(P,1);p=7;nt=8;Kpn=bnrinit(K,p^nt);r=1;
Hpn=component(component(Kpn,5),2);C8=component(K,8);E=component(C8,5);
h=component(component(C8,1),1);L=List;i=component(matsize(Hpn),2);
for(k=1,i-1,c=component(Hpn,i-k+1);if(Mod(c,p)==0,
listinsert(L,p^valuation(c,p),1)));print(L);print("h=",h," ",L," E=",E)}

h=1  List([343, 7])
E=[245/13*x^2-4606/13*x-21522/13, 147/13*x^2+3479/13*x+11272/13]
```

So, the class group is trivial, $\mathscr{T}_K = \mathscr{R}_K \simeq \mathbb{Z}/7^3\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$ and the cyclotomic units are the fundamental units. Then we shall use a definition of the automorphism σ to define the Galois operation on the units:

```
{P=x^3 + x^2 - 460*x - 1739;print(nfgaloisconj(P))}
[x, -1/13*x^2 - 2/13*x + 302/13,  1/13*x^2 - 11/13*x - 315/13]
```

From $\varepsilon = \frac{245}{13}x^2 - \frac{4606}{13}x - \frac{21522}{13}$ and $\sigma : x \mapsto -\frac{1}{13}x^2 - \frac{2}{13}x + \frac{302}{13}$, one gets:

```
Mod(245/13*(-1/13*x^2 - 2/13*x + 302/13)^2 -
4606/13*(-1/13*x^2 - 2/13*x + 302/13) - 21522/13,P)=
Mod(147/13*x^2 + 3479/13*x + 11259/13, x^3 + x^2 - 460*x - 1739)
```

which is ε^σ and the units are, on the $\mathbb{Q}$-base $\{1, x, x^2\}$:

$\varepsilon = \varepsilon_1 = \frac{245}{13}x^2 - \frac{4606}{13}x - \frac{21522}{13}$,
$\varepsilon^\sigma = \varepsilon_2 = \frac{147}{13}x^2 + \frac{3479}{13}x + \frac{11259}{13}$,
$\varepsilon^{\sigma^2} = \varepsilon_3 = -\frac{392}{13}x^2 + \frac{1127}{13}x + \frac{175948}{13}$.

The second unit given by PARI is $\frac{147}{13}x^2 + \frac{3479}{13}x + \frac{11272}{13} = -\varepsilon^{-\sigma^2}$. The order of ε modulo $p = 7$ is 114. We compute $A_i := \varepsilon_i^{114}$ modulo 7^6, $i = 1, 2, 3$), then $L_i := A_i - 1$:

```
{P=x^3+x^2-460*x-1739;
E1=Mod(245/13*x^2-4606/13*x-21522/13,P+Mod(0,7^6));
E2=Mod(147/13*x^2+3479/13*x+11259/13,P+Mod(0,7^6));
E3=Mod(-392/13*x^2+1127/13*x+175948/13,P+Mod(0,7^6));
L1=E1^114-1;L2=E2^114-1;L3=E3^114-1;
print(lift(L1)," ",lift(L2)," ",lift(L3))}
```

$L_1 = 17542x^2 + 48608x + 81879 = 7^2(358x^2 + 992x + 1671) = 7^2\alpha_1$,
$L_2 = 62867x^2 + 833x + 33761 = 7^2(1283x^2 + 17x + 689) = 7^2\alpha_2$,
$L_3 = 37240x^2 + 68208x + 2009 = 7^2(760x^2 + 1392x + 41) = 7^2\alpha_3$,
giving $\frac{1}{7}\log(\varepsilon_i) \equiv 7\alpha_i - \frac{1}{2}7^3\alpha_i^2 \pmod{7^4}$:
$\frac{1}{7}\log(\varepsilon) \equiv 791x^2 + 2142x + 378 = 7(113x^2 + 306x + 54) \pmod{7^4}$,
$\frac{1}{7}\log(\varepsilon^\sigma) \equiv 2121x^2 + 119x + 364 = 7(303x^2 + 17x + 52) \pmod{7^4}$,
$\frac{1}{7}\log(\varepsilon^{\sigma^2}) \equiv 1890x^2 + 140x + 1659 = 7(270x^2 + 20x + 237) \pmod{7^4}$.

So, the Solomon annihilator $\frac{1}{p}\sum_{\sigma \in G_K} \log(\varepsilon^\sigma) \cdot \sigma^{-1}$ of $\mathscr{T}_K$ is (modulo 7^3 and up to a 7-adic unit):

$$\Psi_K \equiv 7 \cdot \left[15x^2 + 12x + 5 + (9x^2 + 17x + 3)\sigma^{-1} + (25x^2 + 20x + 41)\sigma^{-2}\right].$$

Since the norm is a trivial annihilator, we can replace Ψ_K by

$$\begin{aligned}\Psi'_K &= \Psi_K - 7\cdot(15x^2+12x+5)(1+\sigma^{-1}+\sigma^{-2})\\ &\equiv 7\cdot\big[(43x^2+5x+47)\,\sigma^{-1}+(10x^2+8x+36)\big]\,\sigma^{-2} \pmod{7^3}.\end{aligned}$$

Then, $43x^2+5x+47$ is invertible p-adically (its norm is prime to 7) which gives the equivalent annihilator:

$$7\cdot\big[\sigma+(10x^2+8x+36)\cdot(43x^2+5x+47)^{-1}\equiv\sigma+31 \pmod{7^2}\big]$$

equivalent to the annihilator defined by $7\cdot(\sigma-18)$ modulo 7^3.

Our annihilator, given by the previous table, is $1738+2186\,\sigma^{-1}+2361\,\sigma^{-2}$ equivalent to $448+623\,\sigma\equiv 7\cdot(\sigma-18) \pmod{7^3}$. So $\sigma-18$ is an annihilator for the submodule $\mathscr{T}_K^7\simeq\mathbb{Z}/7^2\mathbb{Z}$, which is coherent since 18 is of order 3 modulo 7^3.

The perfect identity of the two results shows that no information has been lost for this particular case, whatever the method (but in the case of cyclic fields of prime degree, there is not any Euler factor).

8.2 Cyclic quartic field of conductor $37\cdot 45161$ and Solomon's Ψ_K

Let K be a real cyclic quartic field of conductor f such that the quadratic subfield k has conductor $m\mid f$, with for instance $f=\ell m$, ℓ prime split in $k/\mathbb{Q}$. We take $p\equiv 1 \pmod 4$, $p\nmid f$.

Put $\eta_f:=1-\zeta_f$, $\eta_m:=1-\zeta_m$, $\eta_K:=\mathrm{N}_{\mathbb{Q}^f/K}(\eta_f)$, $\eta_k:=\mathrm{N}_{\mathbb{Q}^m/k}(\eta_m)$.

Then we have the Solomon annihilator:

$$\Psi_K=\frac{1}{p}\sum_{\sigma\in G_K}\log(\eta_K^{\sigma})\cdot\sigma^{-1}.$$

Since, from the formula (4.2) (which applies since $m\neq 1$), one has $\mathrm{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\eta_f)=\eta_m^{(1-(\frac{\mathbb{Q}^m}{\ell})^{-1})}$, i.e., $\mathrm{N}_{K/k}(\eta_K)=\eta_k^{(1-(\frac{k}{\ell})^{-1})}=1$, we get (with $G_K=\{1,\sigma,\sigma^2,\sigma^3\}$):

$$\begin{aligned}\Psi_K &= \frac{1}{p}\big(\log(\eta_K)+\log(\eta_K^{\sigma})\cdot\sigma^{-1}+\log(\eta_K^{\sigma^2})\cdot\sigma^{-2}+\log(\eta_K^{\sigma^3})\cdot\sigma^{-3}\big)\\ &= \frac{1}{p}\big(\log(\eta_K)+\log(\eta_K^{\sigma})\cdot\sigma^{-1}-\log(\eta_K)\cdot\sigma^{-2}-\log(\eta_K^{\sigma})\cdot\sigma^{-3}\big)\end{aligned}$$

So, in this particular situation, one has:

$$\Psi_K=\frac{1}{p}\big(\log(\eta_K)+\log(\eta_K^{\sigma})\cdot\sigma^{-1}\big)\cdot(1-\sigma^2). \tag{8.1}$$

Suppose that $\mathscr{T}_K$ is equal to the transfer of $\mathscr{T}_k$ (many examples are available), then $\mathscr{T}_K$ is annihilated by $(1-\sigma^2)$, whatever the structure of $\mathscr{T}_k\simeq\mathscr{T}_K$; but one expects annihilators A_K such that $\mathrm{N}_{K/k}(A_K)=A_k$ be a non-trivial annihilator of $\mathscr{T}_k$.

For instance, define K by $x=\sqrt{\ell\sqrt{m}\,\frac{\sqrt{m}+a}{2}}$ where $m=a^2+b^2$, $b=2b'$. This gives the polynomial $P=x^4-\ell m x^2+\ell^2 m b'^2$. The following program gives many examples with non-trivial $\mathscr{T}_k$ (with m prime, $p=5$):

```
{p=5;forprime(m=1,10^5,if(Mod(m,20)!=1,next);P=x^2-m;K=bnfinit(P,1);nt=12;
Kpn=bnrinit(K,p^nt);Hpn=component(component(Kpn,5),2);L=List;
i=component(matsize(Hpn),2);R=0;for(k=1,i-1,c=component(Hpn,i-k+1);
if(Mod(c,p)==0,R=R+1;listinsert(L,p^valuation(c,p),1)));if(R>0,
print("m=",m," structure",L)))}
```

For $m=45161$, one obtains $\mathscr{T}_k\simeq\mathbb{Z}/5^5\mathbb{Z}$; then $a=205$, $b'=28$. Now we find some primes ℓ with the following program:

```
{p=5;m=45161;bprim=28;forprime(ell=7,10^3,if(Mod(ell,4)!=1,next);
if(kronecker(m,ell)!=1,next);P=x^4-ell*m*x^2+ell^2*m*bprim^2;
K=bnfinit(P,1);nt=12;Kpn=bnrinit(K,p^nt);Hpn=component(component(Kpn,5),2);
L=List;i=component(matsize(Hpn),2);
for(k=1,i-1,c=component(Hpn,i-k+1);if(Mod(c,p)==0,
listinsert(L,p^valuation(c,p),1)));
print("ell=",ell," m=",m," P=",P," structure",L))}
```

giving the following examples (for which $\mathscr{T}_k$ is a direct factor in $\mathscr{T}_K$):

```
ell=13 P=x^4-587093*x^2+5983651856 structure [3125]
ell=17 P=x^4-767737*x^2+10232398736 structure [3125,5,5]
ell=37 P=x^4-1670957*x^2+48471120656 structure [3125]
ell=997 P=x^4-45025517*x^2+35194105312016 structure [3125,25]
```

We consider the case $\ell = 37$, $P = x^4 - 1670957x^2 + 48471120656$ for wich PARI gives the following information that may be used by the reader:

```
nfgaloisconj(x^4-1670957*x^2+48471120656)=
[-x, x, -1/212380*x^3 + 43593/5740*x, 1/212380*x^3 - 43593/5740*x]

{P=x^4-1670957*x^2+48471120656;K=bnfinit(P,1);p=5;nt=8;Kpn=bnrinit(K,p^nt);
r=1; Hpn=component(component(Kpn,5),2);C8=component(K,8);E=component(C8,5);
h=component(component(C8,1),1);L=List;i=component(matsize(Hpn),2);R=0;
for(k=1,i-1,c=component(Hpn,i-k+1);if(Mod(c,p)==0,R=R+1;
listinsert(L,p^valuation(c,p),1)));print("h=",h," ",L);print("E=",E)}

h=2  List([3125])
```

Now, consider the annihilator $\mathscr{A}_{K,n}(c) =: A_K$; since $\mathscr{T}_K \simeq \mathscr{T}_k$, we get $\mathscr{T}_K^{A_K} \simeq \mathscr{T}_k^{\mathrm{N}_{K/k}(A_K)}$, where (see Corollary 7.3):

$$\mathrm{N}_{K/k}(\mathscr{A}_{K,n}(c)) \cong \Big(1 - \tfrac{1}{\ell}\Big(\tfrac{k}{\ell}\Big)\Big)\mathscr{A}_{k,n}(c).$$

Then $\ell = 37 \equiv 2 \pmod 5$ splits in k and $1 - \frac{1}{\ell}\Big(\frac{k}{\ell}\Big) = 1 - \frac{1}{\ell}$ is invertible modulo 5.

So A_K acts on $\mathscr{T}_K$ as $\mathscr{A}_{k,n}(c)$ on $\mathscr{T}_k$; we can use the program for quadratic fields and $p > 2$ (of course the bounds bf, Bf may be arbitrary):

```
{p=5;nt=8;bf=45161;Bf=45161;for(f=bf,Bf,v=valuation(f,2);M=f/2^v;
if(core(M)!=M,next);if((v==1||v>3)||(v==0 & Mod(M,4)!=1)||
(v==2 & Mod(M,4)==1),next);P=x^2-f;K=bnfinit(P,1);Kpn=bnrinit(K,p^nt);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
h=component(component(component(K,8),1),2);L=List;ex=0;
i=component(matsize(Hpn),2);for(k=1,i-1,co=component(Hpn,i-k+1);
if(Mod(co,p)==0,val=valuation(co,p);if(val>ex,ex=val);
listinsert(L,p^val,1)));Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);tor=p^vptor;S0=0;S1=0;pN=p*p^ex;fn=pN*f;
for(cc=2,10^2,if(gcd(cc,p*f)!=1 || kronecker(f,cc)!=-1,next);c=cc;break);
for(a=1,fn/2,if(gcd(a,fn)!=1,next);asurc=lift(a*Mod(c,fn)^-1);
lambda=(asurc*c-a)/fn;u=Mod(lambda*a^-1,pN);
s=kronecker(f,a);if(s==1,S0=S0+u);if(s==-1,S1=S1+u));
L0=lift(S0);L1=lift(S1);A=L1-L0;if(A!=0,A=p^valuation(A,p));
print(f," P=",P," ",L0," ",L1," A=",A," tor=",tor," T_K=",L," Cl_K=",h))}
```

giving the annihilator $A_k \equiv 10185 + 3935\,\overline{\sigma} \pmod{5^6}$ where $\overline{\sigma}$ generates $\mathrm{Gal}(k/\mathbb{Q})$; then, A_k is equivalent, modulo the norm, to the integer $10185 - 3935 \equiv 2 \cdot 5^5 \pmod{5^6}$, which is perfect since $\mathscr{T}_k \simeq \mathbb{Z}/5^5\mathbb{Z}$.

The class group of k being trivial, the fundamental unit ε is the cyclotomic one and is such that $\varepsilon^4 = 1 + 5^6 \cdot \alpha$, α prime to 5, which confirms that:

$$\Psi_k \sim \tfrac{1}{5}(\log(\varepsilon) + \log(\varepsilon^{\overline{\sigma}}) \cdot \overline{\sigma}) = \tfrac{1}{5}\log(\varepsilon)(1 - \overline{\sigma}) \tag{8.2}$$

equivalent (modulo the norm) to $\frac{2}{5}\log(\varepsilon)$ and $\Psi_k = A_k$ as expected. Meanwhile, the Solomon annihilator Ψ_K does not give Ψ_k by restriction, but 0.

9. About the annihilator $\mathscr{A}_K(c)$ and the primitive $L_p(1,\chi)$

9.1 Galois characters v.s. Dirichlet characters

Let f_K be the conductor of K. In most formulas, the characters χ of K must be primitive of conductor $f_\chi \mid f_K$, whence Dirichlet characters on $(\mathbb{Z}/f_\chi\mathbb{Z})^\times$ such that $\chi\big[\big(\frac{\mathbb{Q}^{f_\chi}}{a}\big)\big]$ makes sense for $a \in \mathbb{Z}$, prime to f_χ, but not necessarily for $\chi\big[\big(\frac{\mathbb{Q}^{f_K}}{a}\big)\big]$ if a prime ℓ divides both a and f_K but not f_χ. This is an obstruction to consider them as Galois characters over $\mathbb{Z}_p[G_K]$ for instance, whence defined on $(\mathbb{Z}/f_K\mathbb{Z})^\times$; so we shall introduce the corresponding Galois character of G_K, denoted $\psi_\chi =: \psi$. A Galois character ψ of G_K is also a character of $G_n = \mathrm{Gal}(L_n/\mathbb{Q})$ whose kernel fixes K, so $\psi(a)$ ($a \in [1, f_n]$ prime to f_n) is the image by ψ of the Artin symbol $\left(\frac{L_n}{a}\right)$ whence of $\left(\frac{K}{a}\right)$.

Any non-primitive writing $\psi(\mathscr{A}_K)$, for $\mathscr{A}_K \in \mathbb{Z}_p[G_K]$, may introduce a product of Euler factors. Indeed, let k_χ be the subfield fixed by the kernel of $\psi = \psi_\chi$ (then χ is a primitive character of k_χ but not necessarily of K); then, $\psi(\mathscr{A}_K) = \psi(\mathrm{N}_{K/k_\chi}(\mathscr{A}_K)) = \chi(\mathscr{E}_{k_\chi}) \cdot \chi(\mathscr{A}_{k_\chi})$ in which $\chi(\mathscr{E}_{k_\chi})$ may be non-invertible (or 0).

9.2 Expression of $\psi(\mathscr{A}_K(c))$

Let ψ be any Galois character of K considered as Galois character of $\mathrm{Gal}(L_n/\mathbb{Q})$, for $n \geq n_0 + e$. We then have the following result about the computation of the annihilator $\mathscr{A}_K(c) =: \sum_{\sigma\in G_K} \Lambda_\sigma(c)\cdot\sigma$ (given explicitely by the Theorem 5.5), without any hypothesis on K and p:

Lemma 9.1. *The expression $\psi(\mathscr{A}_K(c))$ is the product of the multiplicator $1 - \psi\big(\big(\frac{L_\infty}{c}\big)\big)$ by the non-primitive value $L_p(1,\psi)$. In other words, one has:*

$$\begin{aligned}\psi(\mathscr{A}_K(c)) &= (1-\psi(c))\cdot L_p(1,\psi)\\ &= (1-\psi(c))\cdot \prod_{\ell\mid f_K,\, \ell\nmid p f_\chi}\big(1-\chi(\ell)\ell^{-1}\big)\, L_p(1,\chi).\end{aligned}$$

Proof. This comes from the classical construction of p-adic L-functions [13, Propositions II.2, II.3, Définition II.3, II.4, and Remarques II.3, II.4], then [7, page 292]. Thus we obtain, using the computations of the § 7.1, the link between the limit (for $n \to \infty$):

$$\psi(\mathscr{A}_K(c)) = \sum_{\sigma\in G_K}\Lambda_\sigma(c)\cdot\psi(\sigma) \ \ \text{(cf. Remark 7.4 (i))},$$

of $\psi(\mathscr{A}_{L_n}(c)) = \psi(\mathscr{A}_{K,n}(c)) = \sum_{\sigma\in G_K}\Lambda_\sigma^n(c)\,\psi(\sigma)$, and the value at $s=1$ of the L_p-function of the *primitive character* χ associated to ψ. □

Remark 9.2. *Note that in the various calculations in § 7.1, $\varphi_n = \varphi(qp^n)$ when $n\to\infty$ plays the role of $1-s$ when $s\to 1$ in the construction of p-adic L_p-functions by reference to Bernoulli numbers.*

For all primitive Dirichlet character $\chi \neq 1$ of K, of modulus f_χ (or pf_χ if $p \nmid f_\chi$), and for all $p \geq 2$, we have the classical formulas of the value at $s=1$ of the p-adic L-functions (see for instance [30, Theorem 5.18]), where $\tau(\chi) = \sum_{(a,f_\chi)=1}\chi(a)\zeta_{f_\chi}^a$ is the primitive Gauss sum of χ:

$$L_p(1,\chi) = -\Big(1-\frac{\chi(p)}{p}\Big)\cdot\frac{\tau(\chi)}{f_\chi}\sum_{a\in[1,f_\chi],\,(a,f_\chi)=1}\chi^{-1}(a)\log(1-\zeta_{f_\chi}^a),$$

where the Euler factor $1-\chi(p)p^{-1}$ illustrates the fact that for L_p-functions, any character χ is considered modulo pf_χ when $p \nmid f_\chi$.

From the Coates formula [6] and classical computations (see also some details in [11, § 2.2]) we recall that $\#\mathscr{T}_K \sim [K\cap\mathbb{Q}_\infty : \mathbb{Q}]\cdot\prod_{\chi\neq 1}\frac{1}{2}L_p(1,\chi)$ (up to a p-adic unit), thus $\#\mathscr{T}_K \sim \prod_{\chi\neq 1}\frac{1}{2}L_p(1,\chi)$ if $K\cap\mathbb{Q}_\infty = \mathbb{Q}$ (i.e., $n_0 = 0$). Moreover, we know that in the semi-simple case, one obtains the orders of the isotypic components of $\mathscr{T}_K$ by means of the $\frac{1}{2}L_p(1,\chi)$; but the whole Galois structure of $\mathscr{T}_K$ is more precise that the set of those given by the components $\mathscr{T}_K^{e_\theta}$, where the e_θ are the corresponding p-adic idempotents.

Remark 9.3. *Let χ be a primitive Dirichlet character of conductor $f_\chi \neq 1$. We define the "modified Solomon element" of $\mathbb{Z}_p[G_{k_\chi}]$:*

$$\Psi_{k_\chi} := -\Big(1-\frac{\chi(p)}{p}\Big)\cdot\frac{\tau(\chi)}{f_\chi}\sum_{\tau\in G_{k_\chi}}\log(\eta_{k_\chi}^\tau)\cdot\tau^{-1}.$$

Whence $L_p(1,\chi) = \chi\big(\Psi_{k_\chi}\big)$ *(*$\chi \neq 1$ *primitive). Put:*

$$C_\chi := -\Big(1 - \frac{\chi(p)}{p}\Big)\cdot\frac{\tau(\chi)}{f_\chi}.$$

When $p \nmid f_\chi$*,* $\tau(\chi)$ *is invertible and* $C_\chi \cdot \log(\eta^\tau_{k_\chi}) \sim \frac{1}{p}\cdot\log(\eta^\tau_{k_\chi}) \sim \Psi_{k_\chi}$ *(the original Solomon element); when* $p \mid f_\chi$*, the factor* $\frac{1}{p}$ *in* C_χ *is replaced, ahead the logarithms, by the quotient* $\frac{1}{\tau(\chi)}$ *having the suitable p-valuations. For instance, if d is prime and p unramified,* $\frac{1}{p}\sum_{\sigma\in G_K}\log(\eta^\sigma_K)\cdot\sigma^{-1}$ *annihilates* $\mathscr{T}_K$*.*

9.3 The annihilator $\mathscr{A}_K(c)$ and the Ψ_{k_χ}

The following statement does not assume any hypothesis on K and p and gives again the known results of annihilation (e.g., semi-simple case, but also the point of view of [22]):

Theorem 9.4. *Let K be a real abelian number field, of degree d, of Galois group* G_K *and of conductor* f_K*. Let* $\mathscr{A}_K(c) = \lim\limits_{n\to\infty}\mathscr{A}_{K,n}(c) \in \mathbb{Z}_p[G_K]$ *annihilating* $\mathscr{T}_K$ *(cf. Theorem 5.5). Then we have (where each* χ *is the primitive Dirichlet character associated to the Galois character* ψ *of* G_K*):*

$$\mathscr{A}_K(c) = \frac{1}{d}\sum_{\sigma\in G_K}\Big[\sum_{\psi\neq 1}\psi^{-1}(\sigma)(1-\psi(c))\cdot\prod_{\ell\mid f_K,\,\ell\nmid pf_\chi}\Big(1-\frac{\chi(\ell)}{\ell}\Big)\cdot\chi(\Psi_{k_\chi})\Big]\cdot\sigma,$$

with $\Psi_{k_\chi} = -\Big(1-\frac{\chi(p)}{p}\Big)\frac{\tau(\chi)}{f_\chi}\sum_{\tau\in G_{k_\chi}}\log\big(\mathrm{N}_{\mathbb{Q}^{f_\chi}/k_\chi}(1-\zeta_{f_\chi})^\tau\big)\cdot\tau^{-1}$*.*

Thus, $\mathscr{T}_K$ *is annihilated by the ideal* $\mathfrak{A}_K$ *of* $\mathbb{Z}_p[G_K]$ *generated by the* $\mathscr{A}_K(c)$*,* $c\in\mathbb{Z}$*, prime to* $2pf_K$*.*

Proof. For all Galois character ψ of G_K, Lemma 9.1 leads to the identity:

$$\begin{aligned}\psi(\mathscr{A}_K(c)) &= \sum_{\sigma\in G_K}\Lambda_\sigma(c)\cdot\psi(\sigma)\\ &= (1-\psi(c))\cdot\prod_{\ell\mid f_K,\,\ell\nmid pf_\chi}\big(1-\chi(\ell)\ell^{-1}\big)\cdot L_p(1,\chi)\\ &= (1-\psi(c))\cdot\prod_{\ell\mid f_K,\,\ell\nmid pf_\chi}\big(1-\chi(\ell)\ell^{-1}\big)\cdot\chi(\Psi_{k_\chi})\end{aligned}$$

with $\psi_1(\mathscr{A}_K(c)) = 0$ for the unit character ψ_1.

Since the matrix $\big(\psi(\sigma)\big)_{\psi,\sigma}$ is invertible with inverse $\frac{1}{d}\Big(\psi^{-1}(\sigma)\Big)_{\sigma,\psi}$, this yields $\Lambda_\sigma(c) = \frac{1}{d}\sum_\psi\psi^{-1}(\sigma)\psi(\mathscr{A}_K(c)) = \frac{1}{d}\sum_\psi\psi^{-1}(\sigma)(1-\psi(c))\cdot L_p(1,\psi)$. Whence the result using the expression of $L_p(1,\psi)$ in Lemma 9.1. □

9.4 A cyclic quartic field K of conductor $37\cdot 45161$

We recall from § 8.2 that $m = 45161$ is totally ramified in K, that $\ell = 37$ splits in the quadratic subfield $k = \mathbb{Q}(\sqrt{m})$ and is ramified in K/k; then $p = 5$ totally splits in K. We have $\mathscr{T}_k \simeq \mathbb{Z}/5^5\mathbb{Z}$.

Denote the four characters by ψ_1, ψ_2, ψ_4 & ψ_4^{-1} (orders 1,2, 4, respectively) and let $G_K = \{1,\sigma^2,\sigma,\sigma^{-1}\}$ with σ of order 4. We shall put $\psi_4(\sigma) = i$, and so on by conjugation and the relation $\psi_2 = \psi_4^2$.

Then, using the modified Solomon elements Ψ_k, Ψ_K (expressions (8.1), (8.2)):

$$\Psi_k = 5^5\cdot u \quad\&\quad \Psi_K = \frac{v}{5}\big(\log(A)+\log(B)\,\sigma\big)(1-\sigma^2),$$

where u and v are p-adic units, A & $B = A^\sigma$ are the two independent units of K of relative norm 1.

We have to compute the coefficients $\psi^{-1}(\sigma)(1-\psi(c))$, which gives the array:

	ψ_1	ψ_2	ψ_4	ψ_4^{-1}
1	0	$1\cdot 2$	$1\cdot(1-i)$	$1\cdot(1+i)$
σ^2	0	$1\cdot 2$	$-1\cdot(1-i)$	$-1\cdot(1+i)$
σ	0	$-1\cdot 2$	$-i\cdot(1-i)$	$i\cdot(1+i)$
σ^{-1}	0	$-1\cdot 2$	$i\cdot(1-i)$	$-i\cdot(1+i)$

Then the terms $\prod_{\ell\mid f_K,\,\ell\nmid pf_\chi}\big(1-\chi(\ell)\ell^{-1}\big)\cdot\chi(\Psi_{k_\chi})$ have the following values, depending on the character ψ in the summation of the theorem:

- $5^5{\cdot}u$ for ψ_2, since $1-\chi_2(\ell)\ell^{-1} = 1-37^{-1} \sim 1$,
- $\frac{2v}{5}\big(\log(A)+i\log(B)\big)$ & $\frac{2v}{5}\big(\log(A)-i\log(B)\big)$, for ψ_4 & ψ_4^{-1}.

We obtain, up to a p-adic unit, using the coefficients of the above array:

$$\begin{aligned}
&\mathscr{A}_K(c) = \\
&\Big[\tfrac{v}{5}\big[\log(A)+\log(B)\big]+5^5{\cdot}u\Big]+\Big[\tfrac{v}{5}\big[-\log(A)-\log(B)\big]+5^5{\cdot}u\Big]\cdot\sigma^2+ \\
&\Big[\tfrac{v}{5}\big[-\log(A)+\log(B)\big]-5^5{\cdot}u\Big]\cdot\sigma+\Big[\tfrac{v}{5}\big[\log(A)-\log(B)\big]-5^5{\cdot}u\Big]\cdot\sigma^{-1} \\
&= 5^5u{\cdot}(1-\sigma)(1+\sigma^2) \\
&\qquad\qquad + v\Big[\tfrac{1}{5}\big[\log(A)+\log(B)\big]-\tfrac{1}{5}\big[\log(A)-\log(B)\big]\cdot\sigma\Big]\cdot(1-\sigma^2).
\end{aligned}$$

We give A, one of the two units of relative norm 1 (the other being $B = A^\sigma$):

```
377216797578975495402206020260112295002483855252847326395960961891321756
935656033880097414072613343385538964199960251752277854265043908282068622
071287/424760*x^3 -
863005972214749996449837366815586234260744443520807110375190268414267539
937539821074892103868728835668111842347981799323725052575447796376125480
7708541/7585*x^2 -
301058401703043815651487372068244675606729686675124486738439428208587682
003249385550605088262234049232685807258542997079887400411162925713036023
300228411/11480*x +
137753779960320144069066397981124894126287808388246384703621136571725449
454295610577594731673630502306081901547245942649393930683936045056394190
29007385081/410
```

So it is easy to compute A^4-1, congruent modulo 5^8 to:

$$5\cdot\alpha = 317056x^3+260605x^2+260934x+182595,$$

whence $\log(A)\sim 5\cdot\alpha$. The decompositions into prime ideals of 5 (which is totally split in $K/\mathbb{Q}$) and of $5\cdot\alpha$ give respectively for the 5-places:

```
[[5,[-3,-2,2,2]~,1,1,[3,4,1,1]~]1] [[5,[-3,0,2,-2]~,1,1,[2,0,4,1]~]1]
[[5,[-1,-2,-2,-2]~,1,1,[1,1,1,1]~]1] [[5,[0,-1,-2,2]~,1,1,[2,2,4,1]~]1]

[[5,[-3,-2,2,2]~,1,1,[3,4,1,1]~]2] [[5,[-3,0,2,-2]~,1,1,[2,0,4,1]~]1]
[[5,[-1,-2,-2,-2]~,1,1,[1,1,1,1]~]2] [[5,[0,-1,-2,2]~,1,1,[2,2,4,1]~]1]
```

Dividing by 5, we find that $\frac{1}{5}\log(A)\sim\pi_1\cdot\pi_2$ then $\frac{1}{5}\log(A^\sigma)\sim(\pi_1\cdot\pi_2)^\sigma =: \pi_3\cdot\pi_4$, where the π_i are integers with valuation 1 at the four prime ideals dividing 5; thus the coefficient:

$$\begin{aligned}
U-V\,\sigma &= \tfrac{1}{5}\log(A\,B)-\tfrac{1}{5}\log(A\,B^{-1}) \\
&\sim u\,\pi_1\cdot\pi_2+u'\,\pi_3\cdot\pi_4-(u\,\pi_1\cdot\pi_2-u'\,\pi_3\cdot\pi_4)\cdot\sigma,
\end{aligned}$$

of $1-\sigma^2$ in $\mathscr{A}_K(c)$ is such that:

$$U^2+V^2\equiv 2\,(u^2\,\pi_1^2\cdot\pi_2^2+u'^2\,\pi_3^2\cdot\pi_4^2) \pmod 5$$

is 5-adically invertible. So $\mathscr{A}_K(c) = 5^5u(1-\sigma)(1+\sigma^2)+w(1-\sigma^2)$, u,w invertible. This gives the optimal annihilation of both $\mathscr{T}_k$ (since $\mathscr{T}_K = j_{K/k}(\mathscr{T}_k)$), and the relative factor $\mathscr{T}_K^* = 1$, as kernel of the relative norm $1+\sigma^2$ in K/k, since the operation is given by $U-V\,\sigma$ which is invertible.

9.5 A cyclic quartic field K of conductor $2^2\cdot 16212\cdot 677$

Let $K=\mathbb{Q}(x)$ where $x=\sqrt{677\,\frac{1621+39\sqrt{1621}}{2}}$. This field is also defined by $P = x^4-1097417x^2+18573782725$. The conjugates of x are given by:

```
nfgaloisconj(P)=[-x, x, -1/132015*x^3+1571/195*x, 1/132015*x^3-1571/195*x]
```

We still consider the case $p=5$. The prime $\ell=677$ splits in the quadratic subfield $k=\mathbb{Q}(\sqrt{1621})$, the ramified prime 2 does not split in k; the class number of k is 1 and that of K is 4, so we obtain a trivial 5-class group and the following group structures giving, here, a non-trivial relative $\mathscr{T}_K^*$:

$$\mathscr{T}_k \simeq \mathbb{Z}/5^2\mathbb{Z}, \quad \mathscr{T}_K \simeq \mathbb{Z}/5^2\mathbb{Z}\times\mathbb{Z}/5^3\mathbb{Z}.$$

In k, the cyclotomic unit is the fundamental unit and is given by:

$$\varepsilon = \frac{119806883557}{26403}x^2 - \frac{3042847629386}{39};$$

we compute that $\frac{1}{5}\cdot\log(\varepsilon)\sim 5^2 \sim \Psi_k$ as expected since $\mathscr{T}_k=\mathscr{R}_k$.

The cyclotomic units A and $B=A^\sigma$ of K, of relative norm 1, are too large to be given here, but we can work with some representatives modulo a large power of 5. As in the previous example, we have to compute (up to 5-adic units since the Euler factors for 2 and 677 are invertible):

$$\Big[\tfrac{1}{5}\big[\log(A)+\log(B)\big]-\tfrac{1}{5}\big[\log(A)-\log(B)\big]\cdot\sigma\Big]\cdot(1-\sigma^2). \tag{9.1}$$

We see that $\log(A)$ is of the form $5\cdot\alpha$, where α is a 5-adic unit, and that $\frac{1}{5}\big[\log(A)-\log(B)\big]$ and $\frac{1}{5}\big[\log(A)+\log(B)\big]$ are 5-adically invertible, so we consider for instance:

$$C := \frac{\log(A)+\log(B)}{\log(A)-\log(B)} \equiv 13\cdot 5^2x^3+5^3x^2+19\cdot 5^2x+57 \pmod{5^4}$$

and we verify that, despite the denominators 5, $\frac{3}{5}\cdot x^3-\frac{1}{5}\cdot x$ is an integer of K (congruent to x^σ modulo 5 as given by $nfgaloisconj(P)$) so that:

$$C \equiv 5^3\cdot 3\cdot\Big(\frac{3}{5}x^3-\frac{1}{5}x+x^2\Big)+57 \pmod{5^4}.$$

Since the exponent of $\mathscr{T}_K$ is 5^3, we obtain that the coefficient $U-V\cdot\sigma$ (in (9.1)) is equal to $(57-\sigma)\cdot(1-\sigma^2)$; thus the whole annihilator is:

$$\mathscr{A}_K(c)\equiv 5^2\cdot u\cdot(1-\sigma)(1+\sigma^2)+v\cdot(57-\sigma)\cdot(1-\sigma^2) \pmod{5^4}.$$

So, on the factor $\mathscr{T}_k$ the annihilator $\mathscr{A}_K(c)$ acts as the order 5^2 of $\mathscr{T}_k$, and on the relative submodule $\mathscr{T}_K^*$, it acts as $57-\sigma$, which is very satisfactory since 57 is of order 4 modulo 5^3 (note that $57^2+1=5^3\cdot 26$).

These examples show that $\mathscr{A}_K(c)$ takes into account the whole structure of $\mathscr{T}_K$; but when the Euler factor is not a p-adic unit because of a prime $\ell\equiv 1 \pmod p$ which splits in k and is ramified in K/k, the annihilation is probably not optimal.

It should be usefull to know if the annihilators, given more recently in the literature, have best properties or not in this point of view, which is not easy since numerical tests are absent (to our knowledge).

9.6 Ideal of annihilation for arbitrary real abelian number fields

We do not make any assumption on p and G_K, nor on the decomposition of the primes $\ell \mid f_K$ in the real abelian extension $K/\mathbb{Q}$. If $K/\mathbb{Q}$ is cyclic, one can choose c (prime to $2pf_K$) such that for all $\psi\neq 1$, $1-\psi(c)$ is non-zero with minimal p-adic valuation; this valuation is 0 as soon as d is not divisible by p, taking $\big(\frac{K}{c}\big)$ as a generator of G_K. Since in the non-cyclic case, this is impossible, we can consider the augmentation ideal $\mathscr{I}_K=\big\langle 1-\big(\frac{K}{c}\big),\ c \text{ prime to } 2pf_K\big\rangle_{\mathbb{Z}[G_K]}$ of G_K and the ideal:

$$\mathscr{I}_K\cdot\mathscr{A}_K$$

which annihilates $\mathscr{T}_K$. It is clear, from Corollary 7.3, that the pseudo-measure $\mathscr{A}_K$ does not depend on $\mathscr{I}_K$ and that any choice of $\delta_K\in\mathscr{I}_K$ is such that $\delta_K\mathscr{A}_K\in\mathbb{Z}_p[G_K]$.

In a p-group G_K of p-rank r, $\delta_K=\sum_{i=1}^r\lambda_i\cdot(1-\sigma_i)$, where the generators σ_i are suitable Artin symbols of integers c_i prime to $2pf_K$; then the characters ψ may be written $\psi=\prod_{i=1}^r\psi_i$, with obvious definition of the ψ_i, so that $\psi(\delta_K)=\sum_{i=1}^r\psi(\lambda_i)\cdot(1-\psi_i(\sigma_i))=\sum_{i=1}^r\psi(\lambda_i)\cdot(1-\xi_i)$, where the ξ_i are roots of unity of p-power order. So we can minimize the p-adic valuations of the $\psi(\delta_K)$ to obtain the best annihilator.

For instance, if K is the compositum of two cyclic cubic fields and $p=3$, whatever the choice of $\delta_K=\lambda_1(1-\sigma_1)+\lambda_2(1-\sigma_2)$, λ_1,λ_2 prime to 3, where σ_1, σ_2 are two generators of G_K, then $\psi(\delta_K)\sim 1-j$ for 6 characters and $\psi(\delta_K)\sim 3$ for 2 other characters $\psi\neq 1$. So the result depends on the structures of the $\mathscr{T}_k$ of the 4 cubic subfields k of K.

Remark 9.5. *(i) Let k be a subfield of K and let $j_{K/k}$ be the "transfer map" $\mathscr{T}_k \to \mathscr{T}_K$. Then, for $\delta_K \mathscr{A}_K$, we get:*

$$(j_{K/k}(\mathscr{T}_k))^{\delta_K \mathscr{A}_K} = j_{K/k}(\mathscr{T}_k^{\mathrm{N}_{K/k}(\delta_K \mathscr{A}_K)}) \simeq \mathscr{T}_k^{\mathrm{N}_{K/k}(\delta_K \mathscr{A}_K)} = \mathscr{T}_k^{\mathscr{E}_k \cdot \delta_k \mathscr{A}_k};$$

indeed, this comes from the injectivity of the transfer since the Leopoldt conjecture is true in abelian extensions (see e.g., [8, Theorem IV.2.1]); then if the product of Euler factors $\mathscr{E}_k := \prod_{\ell \mid f_K, \ell \nmid p f_k} \Big(1 - \frac{1}{\ell}\Big(\frac{k}{\ell}\Big)\Big)$ is invertible (i.e., $\chi(\mathscr{E}_k)$ prime to p for all χ), this means that there is no loss of information by using the annihilation of $\mathscr{T}_K$ by the $\delta_K \mathscr{A}_K$, instead of that of $\mathscr{T}_k$ by the $\delta_k \mathscr{A}_k$; otherwise, it is not possible to eliminate the Euler factors "hidden" in $\delta_K \mathscr{A}_K$ when they are non-invertible (although they are never zero) unless to restrict ourselves to the use of the $\delta_k \mathscr{A}_k$ for $\mathscr{T}_k$, at the cost of a weaker information on the global Galois structure of $\mathscr{T}_K$.

(ii) The G_K-module $\mathscr{T}_K$ gives rise to the following submodules or quotients-modules which have interesting arithmetical meaning and are of course annihilated by the $\delta_K \mathscr{A}_K$: [3]

- *The submodule $\mathscr{C}\ell_K^\infty := \mathrm{Gal}(K_\infty H_K/K_\infty)$ isomorphic to a sub-module of $\mathscr{C}\ell_K$. Note that if p is unramified in $K/\mathbb{Q}$ and if (for $p = 2$) -1 is not a local norm at 2, then $\mathscr{C}\ell_K^\infty \simeq \mathscr{C}\ell_K$ (cf. (2.1)), which explains that, in general, one says that the p-class group is annihilated by the annihilators of $\mathscr{T}_K$.*
- *The module $\mathscr{W}_K$ and the normalized p-adic regulator $\mathscr{R}_K$ defining the exact sequence (2.2).*
- *The Bertrandias–Payan module $\mathscr{B}\mathscr{P}_K := \mathscr{T}_K/\mathscr{W}_K$ for which the fixed field H_K^{bp} by $\mathscr{W}_K$ in $H_K^{\mathrm{pr}}/K_\infty$ is the compositum of the p-cyclic extensions of K which are embeddable in p-cyclic extensions of arbitrary large degree.*

Then some "logarithmic objects" defined and studied by Jaulent (see [16], [17, § 2.3, Schéma] and [3]), in a theoretical and computational point of view:

- *The logarithmic class group $\widetilde{\mathscr{C}\ell}_K := \mathrm{Gal}(H_K^{\mathrm{lc}}/K_\infty)$ (H_K^{lc} is the maximal abelian locally cyclotomic pro-p-extension of K), defining the exact sequence $1 \to \widetilde{\mathscr{C}\ell}_K^{[p]} \to \widetilde{\mathscr{C}\ell}_K \to \mathscr{C}\ell_K^{S\infty} \to 1$ ($\mathscr{C}\ell_K^S := \mathscr{C}\ell_K/c\ell_K(S)$ is the p-group of S-classes of K and $\widetilde{\mathscr{C}\ell}_K^{[p]}$ the subgroup generated by S).*
- *The "logarithmic regulator" $\widetilde{\mathscr{R}}_K$ as quotient of the group of "semi-local logarithmic units" by the "global logarithmic units".*

10. Conclusion

This elementary study, especially with the help of numerical computations, shows that the broad generalizations of $\mathbb{Z}_p[G_K]$-annihilations, that come from values of partial ζ-functions, with various base fields (see, e.g., [19, 20, 21, 25] among many others), may be difficult to analyse, owing to the fact that the results are not so efficients (especially in the non semi-simple and/or the non-cyclic cases), and that some degeneracies may occur because of Euler factors as soon as the p-adic pseudo-measures that are used are of "Stickelberger's type" like Solomon's elements or cyclotomic units.

Moreover, Iwasawa's techniques give more elegant formalism but do not avoid the question of Euler factors.

Depending on whether one deals with imaginary or real fields, the suitable object to be annihilated is not defined in an unique way as shown by the context of the present paper about the G_K-module $\mathscr{T}_K$. Moreover, roughly speaking, some objects are relative to the values $L_p(0,\chi)$ (order of some component of the p-class group of some non-real "mirror field"), while some other are relative to the values $L_p(1,\chi)$ (groups $\mathscr{T}_K$), and it is well known that the points "$s = 0$" and "$s = 1$" are mysteriousely independent, giving sometimes abundant "Siegel zeros" near 1, as explained by Washington in many papers (see [11] and its bibliography), whence an unpredictible order of magnitude of the annihilators.

11. Note

All the programs of the paper may be found at:
https://www.dropbox.com/s/jb5nfc3l8gcn630/Georges%20Gras%20–%20Annihilation%20%28programs%29.pdf?dl=0

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Notes on UP-ideals in UP-algebras

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Abstract
The concept of UP-algebras was introduced and analyzed in 2017 by A. Iampan. In his article, he introduced the concept of UP-ideals in such algebras. In this article, we show that this concept can be determined in some other way than it was done in Iampan's article. In addition, we are more profoundly analyzing UP-ideals in UP-algebras and establish some of their additional properties.

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1. Introduction

The basic concepts of UP-algebra are taken from the text [2]. The author in his article has introduced and analyzed the concepts of UP-algebra, UP-subalgebra and UP-ideals. In the article [9] the authors introduced the concept of UP-filter in UP algebras. This latter concept has a non-standard attitude towards the concept of UP-ideals. In our recently published article [7] and in the forthcoming article [8] we introduce and analyze the concept of proper UP-filters in UP-algebra. A number of authors investigated the reflections of the UP-substructures in UP-algebras within the fuzzy environment (See articles [1, 4, 5, 10, 9]). In the article [1], the authors determined the concept of a strong UP ideals. Since according to Theorem 2.1 in [1], a subset J in a UP-algebra A is a strong UP-ideal in A if and only if $J = A$, it seems to us that this concept will not be of interest in the further researching of properties of UP-algebras.

In this article, we are more profoundly analyze UP-ideals in UP-algebras and establish some of their additional properties. We show that the concept of UP-ideals in a UP-algebra can be determined in some other way than it was done in Iampan's article. Finally, we have shown the theorem that we can look at as the Second theorem on isomorphisms between UP-algebras.

2. Preliminaries

First, let us recall the definition of UP-algebra.

Definition 2.1 ([2], Definition 1.3)**.** *An algebra* $A = (A, \cdot, 0)$ *of type* $(2,0)$ *is called a* UP- algebra *if it satisfies the following axioms:*

(UP - 1): $(\forall x,y,z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP - 2): $(\forall x \in A)(0 \cdot x = x)$,
(UP - 3): $(\forall x \in A)(x \cdot 0 = 0)$,
(UP - 4): $(\forall x,y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \Longrightarrow x = y)$.

On a UP-algebra $(A, \cdot, 0)$, we define a binary relation $' \leqslant '$ on A as follows:

$$(\forall x, y \in A)(x \leqslant y \iff x \cdot y = 0).$$

Second, in the following we give definition of the concept of UP-ideals of UP-algebra.

Definition 2.2 ([2], Definition 2.1)**.** *Let A be a UP-algebra. A subset J of A is called a* UP-ideal *of A if it satisfies the following properties:*
(1) $0 \in J$,
(2) $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in J \wedge y \in J) \Longrightarrow x \cdot z \in J)$.

Third, let's remind ourselves of UP-homomorphisms between UP-algebras.

Definition 2.3 ([2], Definition 4.1)**.** *A mapping $f : A \longrightarrow B$ between two UP-algebras is a UP-homomorphism if the following holds*

$$(\forall x, y \in A)(f(x \cdot y) = f(x) \cdot f(y)).$$

3. Main results

For this talk, the recognizable feature of the UP-ideal is given in the following theorem. Although the statements (3) and (4) of this theorem are proved in the text [2], we show extraordinarily simpler evidence of these claims here than was done in Iampan's article.

Theorem 3.1. *Let A be a UP-algebra and J a UP-ideal of A. Then*
(3) $(\forall x, y \in A)((x \cdot y \in J \wedge x \in J) \Longrightarrow y \in J)$,
(4) $(\forall x, y \in A)(y \in J \Longrightarrow x \cdot y \in J)$.

Proof. If we put $x = 0$, $y = x$ and $z = y$ in (2) we got (3) taking into account (UP - 2).
If we put $z = y$ in (2) we have $(x \cdot (y \cdot y) \in J \wedge y \in J) \Longrightarrow x \cdot y \in J$. Thus $0 = x \cdot 0 \in J$ and $y \in J$ implies $x \cdot y \in J$. So, the property (4) is proven taking into account (1) of Theorem 1.7 in [2], (1) and (UP - 3). □

The following property of UP-ideal follows immediately from (3):

Corollary 3.2. *Let J be a UP-ideal of UP-algebra A. Then*
(5) $(\forall x, y \in A)((x \leqslant y \wedge x \in J) \Longrightarrow y \in J)$.

Let us show (3) $\wedge$ (4) implies (1) $\wedge$ (2) if we assume $J \neq \emptyset$.

Theorem 3.3. *Let J be a non empty subset of UP-algebra for which the formulas* (3) *and* (4) *are valid. Then J is a UP-ideal in A.*

Proof. Let us suppose that for a nonempty subset J in A formulas (3) and (4) are valid.
Since the set J is nonempty, there exists at least one element y in J. If we put $x = y$ in (4), we get $y \cdot y \in J$. Then $0 \in J$ considering the statement (1) of the Theorem 1.7 in [2].
Let $x, y, z \in A$ be arbitrary elements such that $x \cdot (y \cdot z) \in J$ and $y \in J$. From $y \in J$ follows $x \cdot y \in J$ by (4). On the other hand, (UP-1) gives us $y \cdot z \leqslant (x \cdot y) \cdot (x \cdot z)$. If we assume that $\neg(yz \in J)$, we would have $(\neg(yz \in J) \wedge x \in J) \Longrightarrow \neg(x \cdot (y \cdot z) \in J)$ by the contraposition of (3). This is in a contradiction with the first hypothesis. Therefore, it must be $yz \in J$. Thus, from $y \cdot z \in J$ and $y \cdot z \leqslant (x \cdot y) \cdot (x \cdot z)$, we get $(x \cdot y) \cdot (x \cdot z) \in J$ by (5). Now, from last and $z \cdot y \in J$ we have $x \cdot z \in J$ by (3). So, (2) is proven. □

In addition to the previous criterion, we have the following possibility to check whether to nonempty subset J of a UP-algebra A is a UP-ideal in A or not.

Theorem 3.4. *A subset J of a UP-algebra A such that $0 \in J$ is a UP-ideal in A if and only if the following holds*
(6) $(\forall x, y, z \in A)((\neg(x \cdot z \in J) \wedge y \in J) \Longrightarrow \neg(x \cdot (y \cdot z) \in J))$.

Proof. Let J be a UP-ideal in UP-algebra A. Suppose $\neg(x \cdot z \in J)$ and $y \in J$ hold. If $x \cdot (y \cdot z) \in J$, we would have $x \cdot z \in J$, which is contradictory to the first hypothesis. So it has to be $\neg(x \cdot (y \cdot z) \in J)$. Therefore, (6) is proven.
Opposite, let (6) be holds. Suppose that hypothesis in the formula (2) are valid. If it were $\neg(x \cdot z \in J)$, then it would have $\neg(x \cdot (y \cdot z) \in J$ by (6) in contradiction with $x \cdot (y \cdot z) \in J$. So it has to be $x \cdot z \in J$. Therefore, (3) is proven. □

Corollary 3.5. *Let J be a UP-ideal in a UP-algebra A. Then*
(7) $(\forall x,y \in A)((x \in J \wedge \neg(y \in J) \Longrightarrow \neg(x \cdot y \in J))$.

Proof. Is we put $x = 0$, $y = x$ and $z = y$ in (6) we obtain (7). ☐

Theorem 3.6. *A subset J of a UP-algebra A such that* $0 \in J$ *is a UP-ideal in A if and only if the following holds*
(8) $(\forall x,y,z \in A)((\neg(x \cdot z \in J) \wedge x \cdot (y \cdot z) \in J) \Longrightarrow \neg(y \in J))$.

Proof. Let J be a UP-ideal in UP-algebra A. Suppose $\neg(x \cdot z \in J)$ and $x \cdot (y \cdot z) \in J$ hold. If $y \in J$, we would have $x \cdot z \in J$, which is contradictory to the first hypothesis. So it has to be $\neg(y \in J)$. Therefore, (8) is proven.

Opposite, let (8) be holds. Suppose that hypothesis in the formula (2) are valid. If it were $\neg(x \cdot z \in J)$, then it would have $\neg(y \in J)$ by (8) in contradiction with $y \in J$. So it has to be $x \cdot z \in J$. Therefore, (3) is proven. ☐

Corollary 3.7. *Let J be a UP-ideal in a UP-algebra A. Then*
(9) $(\forall x,y \in A)((x \cdot y \in J \wedge \neg(y \in J) \Longrightarrow \neg(x \in J))$.

Proof. Is we put $x = 0$, $y = x$ and $z = y$ in (8) we obtain (9). ☐

One part of the following theorem is proved in [2] (Theorem 2.6). We repeat this proof as it has some useful consequences.

Theorem 3.8. *The family* $\mathfrak{J}_A$ *of all UP-ideals in a UP-algebra A forms the completely lattice.*

Proof. Let $\{J_i\}_{i \in I}$ be a family of UP-ideals in a UP-algebra A.
(a) Obviously, the following is true $0 \in \bigcup_{i \in I} J_i$ and $0 \in \bigcap_{i \in I} J_i$.

(b) Let $x,y,z \in A$ arbitrary elements such that $x \cdot (y \cdot z) \in \bigcap_{i \in I} J_i$ and $y \in \bigcap_{i \in I} J_i$. Thus $x \cdot (y \cdot z) \in J_i$ and $y \in J_i$ for any $i \in I$. Then $x \cdot z \in J_i$ since J_i is a UP-ideal in A. Therefore, $x \cdot z \in \bigcap_{i \in I} J_i$. So, $\bigcap_{i \in I} J_i$ is a UP-ideal in A.

(c) Let $\mathfrak{X}$ be the family of all UP-filters of UP-algebra A contained the union $\bigcup_{i \in I} J_i$. The $\bigcap \mathfrak{X}$ is a UP-ideal in A by the first part of this proof.

(d) If we put $\sqcap_{i \in I} J_i = \bigcap_{i \in I} J_i$ and $\sqcup_{i \in I} J_i = \bigcap \mathfrak{X}$, then $(\mathfrak{J}, \sqcap, \sqcup)$ is a completely lattice. ☐

Corollary 3.9. *Let X be an arbitrary subset of UP-algebra A. Then there is the minimal UP-ideal* $\langle X \rangle$ *containing the set X. Specifically, for every element x in A there is the minimal UP-ideal in A that contains x.*

Proof. Let $F = \{J : J$ is a UP-ideal contains $X\}$. Then $\bigcap F$ is the minimal UP-ideal in A contains the subset X. Specifically, for $X = \{x\}$ we get the second part of this claim. ☐

Let $f : A \longrightarrow B$ be a UP-homomorphism between two UP-algebras. In [2] it has been shown (Theorem 4.5) that $Kerf$ is an UP-ideal and that $f(A)$ is an UP-subalgebra of algebra A. Without major difficulties, it can be proved that if J is an UP-ideal in UP-algebra A and $' \sim '$ the congruence on A determined by the ideal J ([2], Proposition 3.5), then $A/J \equiv A/\sim\ = \ \{[x]_J : x \in A\}$ is also UP-algebra with the internal operation $' \cdot '$ defined by

$$(\forall x,y \in A)([x]_J \cdot [y]_J = [x \cdot y]_J)$$

and the fixed element J. The following claims is proven by direct verification. Furthermore, without major difficulties, it can be shown ([8], Theorem 3.5) that if $f : A \longrightarrow B$ is a UP-homomorphism between UP-algebras, then there is an UP-isomorphism $g : A/Kerf \longrightarrow f(A)$ such that $f = g \circ \pi$, where π is the natural UP-epimorphism. We can look at this as the First theorem on isomorphisms between UP-algebras. In addition, if J and K are UP-ideals in UP-algebra A such that J is contained in K, then K/J is a UP-ideal in UP-algebra A/J ([8], Theorem 3.6). For more details, see articles [3, 6].

Now we can express the following theorem, so called the second isomorphism theorem. The proof of this theorem differs significantly from the proof of the analogous theorem in the article [6].

Theorem 3.10. *Let J and K be UP-ideals of a UP-algebra A such that* $J \subseteq K$. *Then* K/J *is a UP-ideal of UP-algebra* A/J *and the following holds*

$$(A/J)/(K/J) \cong A/K.$$

Proof. Since $K/J = \{[x]_J : x \in K\}$ is a UP-ideal in UP-algebra $A/J = \{[x]_J : x \in A\}$, then the factor-set $(A/J)/(K/J) = \{[[x]_J]_{K/J}] : [x]_J \in A/J\}$ can be correctly determined as the factor-algebra of the UP-algebra A/J by the UP-ideal K/J. If we define mapping $\varphi : (A/J)/(K/J) \longrightarrow A/K$ by the following way $\varphi([[x]_J]_{K/J} = [x]_K$ without major difficulties we can verify that this mapping is a UP isomorphism. Since it is obvious that φ is a UP-endomorphism, it is sufficient to check that φ is a UP-monomorphism. Let $[x]_K = \varphi([[x]_J]_{K/J})$ and $[y]_K = \varphi([[y]_J]_{K/J})$ be arbitrary elements of A/K such that $[x]_K = [y]_K$. This means $x \cdot y \in K$ and $y \cdot x \in K$. Thus $[x \cdot y]_J \in K/J$ and $[y \cdot x]_J \in K/J$. Then $[x]_J \cdot [y]_J \in K/J$ and $[y]_J \cdot [x]_J \in K/J$. Last equality means $[[x]_J]_{K/J} = [[y]_J]_{K/J}$. So, the UP-homomorphism φ is a UP-monomorphism. ☐

4. Conclusion

In this article analyzing the axioms by which the concept of UP-ideals in UP-algebras was determined we offered some possibilities for introduction of UP-ideals on different ways. This analysis enables us to gain a more serious insight into the properties of ideals in these algebraic. Finally, we have shown the Theorem that we can look at as the Second theorem on isomorphisms between UP-algebras.

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A note on the ”saturation” of poisson-exponential cumulative function in Hausdorff sense

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Abstract
In this paper we study the important ”saturation” characteristic for the Poisson–exponential cumulative distribution function in the Hausdorff sense. The results have independent significance in the study of issues related to lifetime analysis, insurance mathematics, biochemical kinetics, population dynamics and debugging theory. Numerical examples, illustrating our results are presented using programming environment Mathematica.

Keywords: Poisson–exponential cumulative distribution function (Pcdf), Hausdorff distance, Upper and lower bounds
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1. Introduction

The Poisson–exponential cumulative distribution function (Pcdf) is given by (see for instance [1]):

$$M(t;\lambda;\beta) = \frac{e^{\lambda e^{-\beta t}} - e^{\lambda}}{1-e^{\lambda}} \tag{1.1}$$

where $\beta > 0$; $\lambda > 0$.

For other extensions and estimations, see [2] – [3]. Some applications of the (Pcdf) to rainfall and aircraft data with zero occurrence can be found in [3].

In this note we study the saturation of the Poisson–exponential cumulative distribution family of functions (1) to asymptote $t = 1$ in the Hausdorff sense.

Definition 1.1. *[4] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f,g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,*

$$\rho(f,g) = \max\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A,x_A)$, $B = (t_B,x_B)$ in $\mathbb{R}^2$ is $||A-B|| = max(|t_A - t_B|, |x_A - x_B|)$.

We propose a software modules (intellectual properties) within the programming environment CAS Mathematica for the analysis.

2. Main Results

Without loosing of generality we will look at the following (Pcdf):

$$M^*(t) = \frac{e^{\lambda e^{-\beta t}} - e^{\lambda}}{1 - e^{\lambda}}, \tag{2.1}$$

with

$$t_0 = -\frac{1}{\beta} \ln\left(\frac{1}{\lambda} \ln\left(\frac{1+e^{\lambda}}{2}\right)\right); \quad M^*(t_0) = \frac{1}{2}. \tag{2.2}$$

To evaluate the important saturation characteristic d of the (Pcdf) to asymptote $t = 1$ in the Hausdorff sense we will use the following representation:

$$M^*(t_0 + d) = 1 - d. \tag{2.3}$$

The following theorem gives upper and lower bounds for d

Theorem 1. Let

$$p = -\frac{1}{2},$$

$$q = 1 - \frac{\beta}{1-e^{\lambda}} \frac{1+e^{\lambda}}{2} \ln\left(\frac{1+e^{\lambda}}{2}\right).$$

For the Hausdorff distance d the following inequalities hold for:

$$2.1q > e^{1.05} \approx 1.36079$$

$$d_l = \frac{1}{2.1q} < d < \frac{\ln(2.1q)}{2.1q} = d_r. \tag{2.4}$$

Proof. Let us examine the function:

$$F(d) = M^*(t_0 + d) - 1 + d. \tag{2.5}$$

From $F'(d) > 0$ we conclude that function F is increasing.
Consider the function

$$G(d) = p + qd. \tag{2.6}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.
Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 1).
In addition $G'(d) > 0$.
Further, for $2.1q > e^{1.05}$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

The model ((2)–(3)) for $\beta = 5$, $\lambda = 0.8$, $t_0 = 0.10302$ is visualized on Fig. 2. From the nonlinear equation (4) and inequalities (5) we have: $d = 0.177469$, $d_l = 0.114887$, $d_r = 0.248593$.

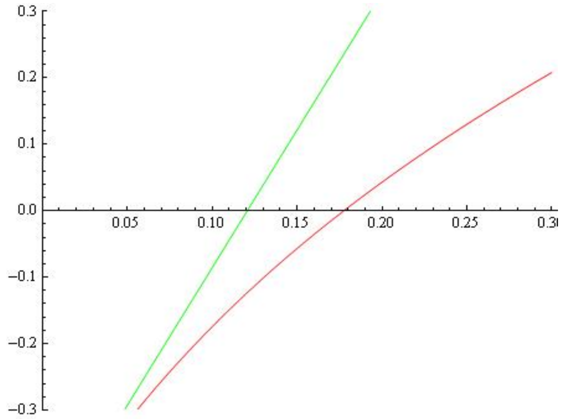

Figure 2.1. The functions $F(d)$ and $G(d)$.

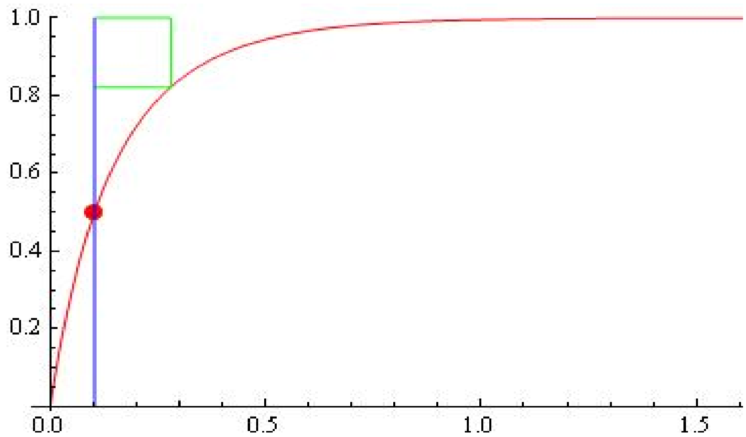

Figure 2.2. The model ((2)–(3)) for $\beta = 5$, $\lambda = 0.8$, $t_0 = 0.10302$; H–distance $d = 0.176469$, $d_l = 0.114887$, $d_r = 0.248593$.

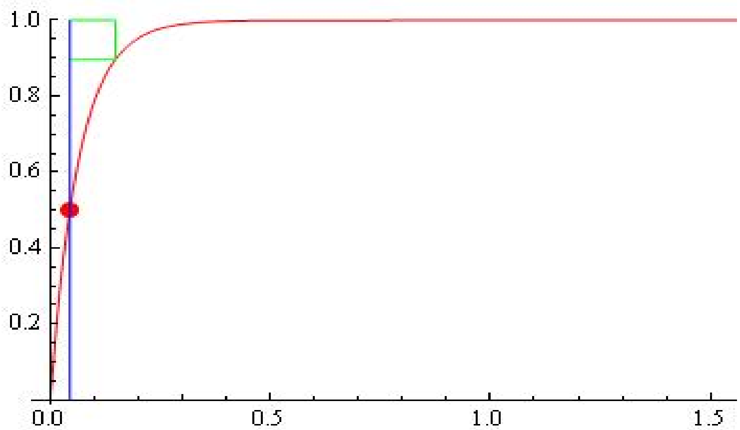

Figure 2.3. The model ((2)–(3)) for $\beta = 15$, $\lambda = 0.1$, $t_0 = 0.0445643$; H–distance $d = 0.10359$, $d_l = 0.0547734$, $d_r = 0.159092$.

```
Clear[λ];
Clear[β];
Manipulate[Dynamic@Show[Plot[f[t], {t, 0, 1},
    LabelStyle → Directive[Blue, Bold],
    PlotLabel → (Exp[λ * Exp[-β * t]] - Exp[λ]) / (1 - Exp[λ])],
   PlotRange → {0, 1}],
 {{λ, 0.1}, 0.01, 30, Appearance → "Open"},
 {{β, 0.1}, 0.01, 100, Appearance → "Open"},
 Initialization :→
  (f[t_] := (Exp[λ * Exp[-β * t]] - Exp[λ]) / (1 - Exp[λ]))]
```

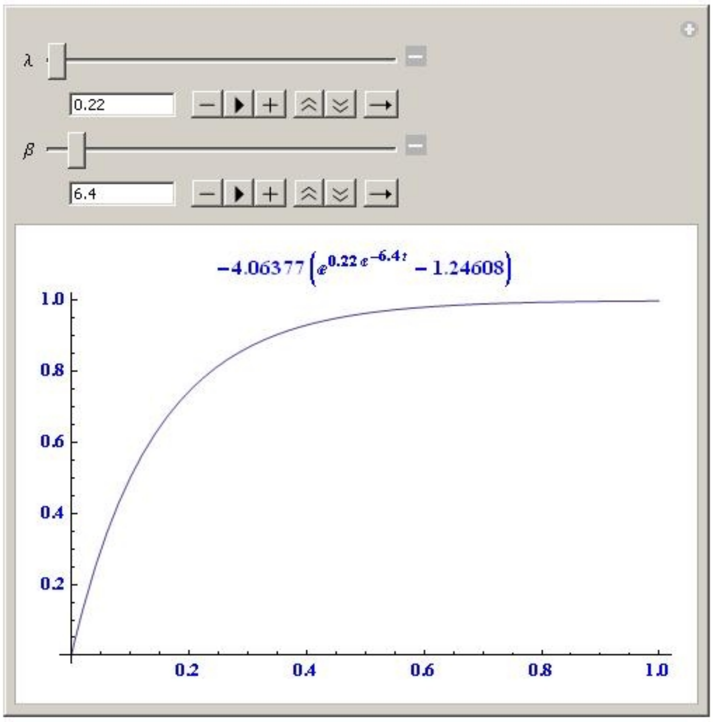

Figure 2.4. An example of the usage of dynamical and graphical representation for the family $M(t)$. For example $\lambda = 0.22$, $\beta = 6.4$. The plots are prepared using CAS Mathematica.

The model ((2)–(3)) for $\beta = 15$, $\lambda = 0.1$, $t_0 = 0.0445643$ is visualized on Fig. 3. From the nonlinear equation (4) and inequalities (5) we have: $d = 0.10359$, $d_l = 0.0547734$, $d_r = 0.159092$.

From the above examples, it can be seen that the proven bottom estimate (see Theorem 1) for the value of the Hausdorff distance is reliable when assessing the important characteristic - ”saturation”.

This characteristic (as we have already shown in our previous publications) has its equal participation together with the other two characteristics - ”confidence intervals” and ”confidence bounds” in the area of the Software Reliability Theory. Constructions of ”confidence curves” and ”confidence bounds” as basics elements from Software Reliability Theory are not easy to be calculated in comparison to estimations pointed in the theorem proven here.

Some software reliability models, can be found in [5]–[6].

Remark. Ramos, Percontini, Cordeiro and Silva [7] studied the following Burr XII–Negative–Binomial Distribution with applications to lifetime data:

$$M_1(t) = \frac{(1-\beta)^{-s} - \left(1 - \beta\left(1 + \left(\frac{t}{a}\right)^c\right)^{-k}\right)^{-s}}{(1-\beta)^{-s} - 1}$$

where $a,k,s,c > 0$ and $\beta \in (0,1)$.

The reader can get accurate bounds for the saturation feature using the technique described in this article.

For some approximation and modeling aspects see [8]–[21].

We hope that the results will be useful for specialists in this scientific area.

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A Rosetta Stone for information theory and differential equations

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Abstract
In this paper, we propose a dictionary between Partial Differential Equations and Information Theory. As a model case, we will discuss in detail the example of the Schrödinger Equation and Shannon Information Theory. Comments will be made in both the continuous and discrete case and in both the noiseless and noisy case.

Keywords: Information Theory, PDEs, Nyquist Bit Rate, Shannon Capacity, Strichartz Estimates
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1. Introduction

The Rosetta Stone is a stele inscribed with a decree issued at Memphis, Egypt, in 196 BC on behalf of King Ptolemy V. The Rosetta Stone has writings in both Egyptian and Greek, using the three different scripts popular in Egypt at that time: hieroglyphic, demotic and Greek. This particular feature of having written essentially the same text on itself in the three scripts, provided a fundamental dictionary to understand Egyptian hieroglyphs.

The search of dictionaries is not a phenomenon confined to Humanistic Sciences and Linguistics: it is ubiquitous also in Technical Sciences, such as Mathematics and Physics. One of the most famous of these dictionaries is due to Wu and Yang [49], who published a paper containing a list of correspondences between the mathematical terminology related to the theory of Connections on Riemannian Manifolds, and the physical terminology concerning the Yang-Mills Theory. This dictionary translates, one to the other, physical and mathematical terminologies. This list is referred to in the literature by some as the *Wu-Yang Dictionary* (See for example [50]). The relationship between these two fields was most likely clear to many others before, at least implicitly. However, the *Wu-Yang Dictionary* played an important role. Since then, the interactions between mathematics and physics have become more natural and fruitful.

In a recent series of seminars, Weinstein [47], [48] proposed new applications of Gauge Theory beyond those familiar in the Natural Sciences. In his view, Neoclassical Economics is an example of Gauge Theory and, for this reason, he proposes a dictionary between Economics and Gauge Theory. The terminology ”Rosetta Stone” in our title is inspired by his presentations.

In this paper, we propose a dictionary between Partial Differential Equations (PDEs) and Information Theory.

Information Theoretical methods have been already used to investigate the large time behavior of solutions of PDEs. Remarkable is the use of entropy (Shannon Entropy, Kullback-Leibler Entropy, Von Neumann Entropy, etc..) to study the asymptotics of dissipative and heat-type equations, both in the case of classical and quantum systems. For a detailed discussion, we refer to [21].

Beyond the typical use of instruments from Information Theory being applied to PDEs, our dictionary underlines that many instruments of Information Theory already have natural counterparts in PDEs. For the use of some Information methods in the context of PDEs and Optimal Transport, we refer to [45], [44], [35], [5]. Our model examples will be the Schrödinger Equation and Shannon Theory of Communication.

Several authors have studied the relationship between the Schrödinger Equation and Information Theoretical tools, like the Fisher Information. In particular, there is a program due to Frieden and his collaborators (see for example [16], [17], [18], [19], [20] and the reference therein), based on what they called Extreme Physical Information (EPI). EPI states that scientific laws can be derived through the Fisher Information and are ruled by differential equations and probability. They extended their approach to encompass existing laws of biology, cancer growth, chemistry, and economics. In our paper, we do not argue in favor or against the claim that Information is the leading principle of all physical laws, but we underline that the vocabulary of Information Theory can have a natural translation into the PDE language and vice versa.

The two pioneering achievements of Classical Information Theory are the following theorems due to Shannon [40] (See Section 2 or directly [40] for the precise definitions and terminology involved in the theorems). The first theorem treats the case of a noiseless channel.

Theorem 1.1 (Fundamental theorem for a noiseless channel)**.** *Let a source S have an entropy H and a channel K have a capacity C. Then, it is possible to encode the output O of S in such a way to transmit at the average rate $C/H-\varepsilon$ over the channel K. Here $0<\varepsilon\ll 1$ is an arbitrary constant. It is not possible to transmit at an average rate greater than C/H.*

In this theorem, Shannon demonstrates the existence of a limit to the efficiency of *Source Coding*. The entropy of a source H corresponds to the minimum binary digits to be used for its coding. Any discrepancy from this limit translates into a growing complexity. A second theorem deals with the noisy case.

Theorem 1.2 (Fundamental theorem for a channel with noise)**.** *Let a source S have an entropy H and a channel K have a capacity C. If $H\leq C$, there exists a coding system such that the output O of S can be transmitted over K with an arbitrarily small frequency of errors (also called* equivocation*). If $H>C$, it is possible to encode S so that the* equivocation *is less than $H-C+\varepsilon$. Here $0<\varepsilon\ll 1$ is an arbitrarily small arbitrary constant. There is no method of encoding which gives an equivocation less than $H-C$.*

For the precise definitions of *Entropy* and *Capacity*, we refer to Section 2.

Remark 1.3. *Since the source is characterized by its information transmission rate (according to Shannon's definition of entropy), this theorem explains that the transmission of this information requires a channel with $C>H$. If we try to transmit a message through a channel of lower capacity, any excess of source entropy with respect to channel capacity will imply an increased rate of error for the receiver. Viceversa, a regime where $C\simeq H$ or $C\gg H$, translates to an increment of the complexity.*

Since then, a huge amount of literature has been developed. We refer to [40], [26], [9] and [33] for more details on Classical Information Theory. We refer to [32] , [11] for different uses of Information Theoretical tools in different areas of mathematics and statistics. In particular, the theory has enlarged to Quantum Information Theory, see for example [38], [37], [36], [27], [29], [30], [31], [23], [28] and [22] for a theoretical background and some connections to Fisher Information Inequalities. We specify that the main emphasis of our paper is on Classical more than Quantum Information Theory, even if our model example is the Schrödinger Equation.

A fundamental question in Information Theory is: how fast can we send data? Data rate depends on the bandwidth B, the level of the signal S and the level of the noise N.

For a noiseless channel, Nyquist's Theorem says that the theoretical maximum bit rate is given by

$$C=2\times B\times\log_2 L,$$

with L the number of signal levels used to represent data, and the bit rate C is measured bits/second.

The Shannon-Hartley's Theorem says that highest data rate for a noisy channel is given by

$$C = B \times \log_2 \left(1 + \frac{S}{N}\right).$$

We believe that the speed of data transmission plays the role of the speed of propagation in dispersive equations. This idea motivates our analogy and the dictionary.

The remaining part of the paper is organized as follows. In Section 2, we introduce some notation and give some preliminary results. In particular, we introduce Information Theoretic concepts, Strichartz Estimates and briefly treat the ODE case. In Section 2.2, we connect the terminologies of PDE and Information Theory, proposing a dictionary between the two fields and revisiting Shannon Code Sourcing Theorem and Strichartz Estimates from a common point of view. In particular, we introduce the Schrödinger Equation, Keel and Tao's Strichartz Estimates and explicitly illustrate the dictionary. In Section 4, we describe some possible further connections between PDEs and Shannon Information Theory, like the ones between maximizers of Entropy and Strichartz Norms, Time Recovery and the role of symmetries. We conclude with Section 5, in which we treat the discrete case, the noisy case, and give the example of the Kinetic Transport Equation.

2. Notation and preliminaries

In this section, we introduce the Informational Theoretic Concepts as presented in [40], we introduce Strichartz Estimates, as presented for example in [42] and [24], and, at the end, we briefly treat the ODE case.

2.1 Information theoretic concepts

The most important concepts employed by Shannon in [40] are the ones of *Entropy* and *Capacity*. In this subsection, we introduce the precise definitions of these two concepts.

Definition 2.1 (Entropy)**.** *Suppose that* $X \in S := \{x_1, \dots, x_n\}$ *is a discrete random variable with pmf* $p(x) := P(X = x)$ *for every* $x \in S$ *and* $p(x) = 0$ *otherwise. Then, the* Entropy $H(X)$ of the Discrete Random Variable X *is defined as follows*

$$H(X) := \mathbb{E}_X[I(x)] = -\sum_{x \in S} p(x) \log p(x). \tag{2.1}$$

Here, $I(x) := -\log p(x)$ *is called* Self-Information *and it is the entropy contribution of the individual message* x. $\mathbb{E}_X$ *is the expected value, taken with respect to* $p(x)$.

In an analogous manner, we define the Differential Entropy $H(X)$ of a Continuous Random Variable X *with support* $x \in S \subset \mathbb{R}^n$ *by:*

$$H(X) := \mathbb{E}_X[I(x)] = -\int_{x \in S} p(x) \log p(x) dx. \tag{2.2}$$

Again, $\mathbb{E}_X$ *is the expected value, taken with respect to* $p(x)$.

Remark 2.2. *A property of the discrete entropy is that it is maximized when all the messages in the message space are equi-probable* $p(x) = 1/n$ *(most unpredictable case), which gives* $H(X) = \log n$.

Definition 2.3 (Joint Entropy)**.** *The* Joint Entropy of two Discrete Random Variables $X \in S := \{x_1, \dots, x_n\}$ and $Y \in T := \{y_1, \dots, y_n\}$ *is the entropy of their pairing* (X, Y)*:*

$$H(X,Y) := \mathbb{E}_{X,Y}[I(x,y)] = -\sum_{x,y \in S \times T} p(x,y) \log p(x,y). \tag{2.3}$$

Here, $I(x,y) := -\log p(x,y)$ *is the* Joint Self-Information*, which is the entropy contribution of the individual joint message* (x,y). $\mathbb{E}_{X,Y}$ *is the expected value, taken with respect to the joint pmf* $p(x,y)$.

In an analogous manner, we define the Joint Entropy $H(X,Y)$ of two Continuous Random Variables X and Y *with supports* $x \in S \subset \mathbb{R}^n$ *and with support* $y \in T \subset \mathbb{R}^n$ *by:*

$$H(X,Y) := \mathbb{E}_{X,Y}[I(x,y)] = -\int\int_{(x,y)\in S\times T} p(x,y)\log p(x,y)dxdy, \tag{2.4}$$

with $p(x,y)$ *the joint pdf. Again,* $\mathbb{E}_{X,Y}$ *is the expected value, taken with respect to the joint pdf* $p(x,y)$.

Remark 2.4. *Note that if X and Y are independent, then their joint entropy is the sum of their individual entropies.*

From now on, we will skip specify the support in order to enlighten the notation.

Definition 2.5 (Conditional Entropy)**.** *The* Conditional Entropy $H(X|Y)$ of a Discrete Random Variable X given random variable Y *is defined by:*

$$H(X|Y) := \mathbb{E}_Y[H(X|y)] = -\sum_{y\in Y} p(y)\sum_{x\in X} p(x|y)\log p(x|y) = -\sum_{x,y} p(x,y)\log\frac{p(x,y)}{p(y)}. \tag{2.5}$$

The Conditional Entropy $H(X|Y)$ of a Continuous Random Variable X given random variable Y *is defined by:*

$$H(X|Y) := \mathbb{E}[H(X|y)] = -\int\int_{x,y} p(x,y)\log\frac{p(x,y)}{p(y)}dxdy, \tag{2.6}$$

where

$$p(y) = \int_x p(x,y)dx.$$

Remark 2.6. *The* Conditional Entropy $H(X|Y)$ of a Random Variable X given random variable Y *is the average conditional entropy of* X *over* Y. *It is also called* Equivocation of X about Y.

Remark 2.7. *Consider for example the continuous case. Then*

$$H(Y|X) := \mathbb{E}_X[H(Y|x)] = -\int\int_{x,y} p(x,y)\log\frac{p(x,y)}{p(x)}dxdy, \tag{2.7}$$

where

$$p(x) = \int_y p(x,y)dy.$$

In some sense, the conditional entropy is "almost symmetric", because the only difference between $H(Y|X)$ *and* $H(X|Y)$ *is in the denominator of the log. The complete symmetry is recovered in the case when* $p(x) = p(y)$ *both in the dependent or independent case.*

Remark 2.8. *A basic property of the conditional entropy is that:*

$$H(X|Y) = H(X,Y) - H(Y).$$

This means that the information produced by X *given* Y *is the same as the information jointly produced by* X *and* Y *minus the information produced by* Y *alone.*

Definition 2.9. *[Mutual Information] The* Mutual Information of two Discrete Random Variables X and Y *is defined as:*

$$I(X;Y) := \mathbb{E}_{X,Y}[SI(X,Y)] = \sum_y\sum_x p(x,y)\log\left(\frac{p(x,y)}{p(x)\,p(y)}\right). \tag{2.8}$$

In the above formula, SI is called Specific Mutual Information. *Here* $p(x,y)$, $p(x)$ *and* $p(y)$ *are defined as in the previous definitions.*
The Mutual Information of two Continuous Random Variables X and Y *is defined as:*

$$I(X;Y) := \mathbb{E}_{X,Y}[SI(X,Y)] = \int_y\int_x p(x,y)\log\left(\frac{p(x,y)}{p(x)\,p(y)}\right)dxdy. \tag{2.9}$$

Again, SI is called Specific Mutual Information *and* $p(x,y)$, $p(x)$ *and* $p(y)$ *are defined as in the previous definitions.*

Remark 2.10. *The* Mutual Information $I(X;Y)$ *measures the amount of information that can be obtained about one random variable by observing another. It is a measure of the mutual dependence of two random variables and determines how similar the joint distribution* $p(X,Y)$ *is to the product of the marginal distributions* $p(X)p(Y)$*. It is important in communication theory because it can be used to maximize the amount of information shared between sent and received signals.*

Lemma 2.11 (Properties of the Discrete Entropy)**.** *Suppose* X *and* Y *are discrete random variables. Then, the following properties hold:*

- $I(X;Y) = H(X) - H(X|Y)$.
- $I(X;Y) = I(Y;X) = H(X) + H(Y) - H(X,Y)$.

Lemma 2.12 (Properties of the Continuous Entropy)**.** *Suppose* X *and* Y *are continuous random variables. Then, the following properties hold:*

- *If* X *is limited to a certain volume* V*, then* $H(X)$ *is a maximum and equal to* $\log V$ *when* $p(x)$ *is constant* $(p(x) = 1/V)$ *in* V*.*
- *We have*

  $$H(X;Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

  and

  $$H(Y|X) \leq H(Y).$$

- *Let* $X \in \mathbb{R}$ *be a random variable. The pdf* $p(x)$ *giving maximum entropy subject to the condition that the standard deviation of* X *is fixed to be* σ *is the Gaussian Distribution. Similarly in* n *dimensions, subject to the constraint of Variance-Covariance Matrix to be* Σ*. The entropy of a one-dimensional Gaussian distribution whose standard deviation is* σ *is given by* $H(x) = \frac{1}{2}\log\left[2\pi e\sigma^2\right]$*, while the* n*-dimensional counterpart is* $H(X) = \frac{1}{2}\log\left[(2\pi e)^n \det(\Sigma)\right]$*.*

We give the definition of discrete channel.

Definition 2.13. *We define a* channel *to be a triplet* $\{X, p(y|x), Y\}$ *consisting of an input random variable* X*, an output random variable* Y *and a conditional probability distribution* $p(y|x)$ *specifying the probability that we observe the output* $Y = y$ *given that* $X = x$*. The channel is said to be* memoryless *if the output distribution depends only on the input distribution and is conditionally independent of previous channel inputs and outputs.*

From now on, we will always consider memoryless channels.

Definition 2.14 (Capacity of a Channel)**.** *Consider the memoryless channel* $\{X, p(y|x), Y\}$*, as in Definition 2.13. Let* $I(X;Y)$ *be the* Mutual Information of Y and X *of Definition 2.9. The* Channel Capacity *is defined as*

$$C = \sup_{p_X(x)} I(X;Y), \tag{2.10}$$

where the supremum is taken over all possible pdfs $p_X(x)$ *of the input variable* X*.*

Remark 2.15. *The conditional distribution function of* Y *given* X*,* $p_{Y|X}(y|x)$ *is an intrinsic property of the channel. A single choice of* $p_X(x)$ *determines the joint pdf* $p_{X,Y}(x,y)$ *and so the* Mutual Information $I(X;Y)$*. Basically,* $I(X;Y)$ *depends on the channel and on the choice of the distribution of the input. The capacity then depends just on* $p_{Y|X}(y|x)$ *and so it is an intrinsic property of the channel.*

Remark 2.16. *There is one important difference between the continuous and discrete entropies. In the discrete case, the entropy measures in an absolute way the randomness of the chance variable. In the continuous case, the measurement is relative to the coordinate system.*

2.2 Strichartz estimates

In this section, we introduce the *Strichartz Estimates*. These estimates are very important in the context of PDEs and Harmonic Analysis, because they provide useful information concerning the dispersive behaviour of solutions to PDEs. Among the other things, one can give a more general characterization of the *Gaussian Distribution* by maximizing Strichartz Norms. We first introduce some characteristic quantities called *Admissible Exponents*.

Definition 2.17. *Fix $n \geq 1$. We call a set of exponents (q,r)* admissible *if $2 \leq q,r \leq +\infty$ and*

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}.$$

Remark 2.18. *These exponents are characteristic quantities of certain norms, called* Strichartz Norms*, naturally arising in the context of Dispersive Equations and can vary from an equation to another equation. We refer to [43] for more details.*

Here is the precise characterization of the Multivariate Normal Distribution, through Strichartz Estimates.

Theorem 2.19. *[42], [24], [7], [39] Suppose $n = 1$ or $n = 2$. Then, for every (q,r) and $(\tilde{q},\tilde{r})$ admissible and for every $u_0 \in L^2_x(\mathbb{R}^n)$ such that $||u_0||^2_{L^2(\mathbb{R}^n)} = 1$, we have*

$$\left|\left|e^{-it\Delta}u_0\right|\right|_{L^q_t L^r_x} \leq S(n,q,r), \tag{2.11}$$

where $S_h(n,q,r) = S_h(n,r)$ is the Sharp Homogeneous Strichartz Constant*, defined by*

$$S_h(n,r) := \sup\left\{||u||_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} : ||u||^2_{L^2_x(\mathbb{R}^n)} = 1\right\}, \tag{2.12}$$

and given by

$$S_h(n,r) = 2^{\frac{n}{4} - \frac{n(r-2)}{2r}} r^{-\frac{n}{2r}}. \tag{2.13}$$

Moreover, the inequality (2.11) becomes an equality if and only if $|u_0|^2$ is the pdf of a Multivariate Normal Distribution.

Recall that

$$\|f\|_{L^2(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f|^2 dx\right)^{1/2}$$

and

$$\|F\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |F(t,x)|^r dx\right)^{q/r} dt\right)^{1/q}.$$

Analogously, we can define l^p spaces of integrable sequences by substituting integration with summation.

Remark 2.20. *This characterization does not need the restriction of fixed variance as the one achieved using the* Entropy Functional *and so it is more general. The result is conjectured to be true for any dimension $n \geq 1$. See for example [39], where the optimal constant has been computed in any dimension $n \geq 1$, under the hypothesis that the maximizers are* Gaussians *also in dimension $n \geq 3$.*

2.3 The ODE case

Consider the following ODE:

$$\dot{x}(t) = Lx(t), \quad x(0) = x_0.$$

Here $x : I \subset \mathbb{R} \to V$ where I is an interval containing the origin $t = 0$, V is a vector space and x_0 represents the initial position. The operator $L : V \to V$ is taken to be linear and determines the behaviour of the solution $x(t) = e^{tL}x_0$ with its spectral properties. We define the operator norm of L as

$$\|L\|_{op} := \inf\{c > 0 : \|Lv\|_V \leq c\|v\|_V\}$$

and also the operator norm of e^{tL} as

$$\|e^{tL}\|_{op} := \inf\{c > 0 : \|e^{tL}v\|_V \leq c\|v\|_V\},$$

for any $t \in I$. Note that in our case, it is natural to take $x_0 = (x_0^1, \dots, x_0^n)$ with $n = dim(V)$ such that $x_0^i > 0$ for $i = 1, \dots, n$ and $\|x_0\|_V = \|x_0\|_{l^1} = \sum_{i=1}^n x_0^i$. We give here a restricted dictionary for ODEs:

Information Theory	Differential Equations
Source	time $t = 0$
Source Encoding	map $0 \mapsto x_0$
Transmitter	Initial Datum x_0
Channel	Propagator e^{tL}
Receiver	Solution $x(t) = e^{tL}x_0$
Decoder	map $x(t) \mapsto t$
Inference of the source	time t
Nyquist Bit Rate	C_{ODE}

Here, we used the notation: $C_{ODE} := \sup\{\|e^{tL}x_0\|_{L_t^q l_x^r}$ s.t $\|x_0\|_{l^1} = 1\}$. As we will explain later in the PDE case, the role of the source and the transmitter in the case when the equation is deterministic (noiseless case) as in all this paper, are basically identical. Since the ODEs we are considering are linear, there exists a unique solution for any initial datum. Therefore, there is a bijection between t and $x(t)$. In the case in which we add to the ODE a random component (noisy case), this bijection disappears. The *source* is the position $t = 0$, to which the *encoder* assigns the initial datum x_0. The encoded message x_0 is transmitted by the channel e^{tL} to the receiver $x(t) = e^{tL}x_0$ which can deduce the position t by uniqueness. A quantity which characterizes the speed of transmission of the channel is C_{ODE}.

Example 2.21. *Consider the simplest case of a noiseless linear system of differential equations in a vector space $V = \mathbb{R}^2$ and $F(u) = Lu$, for some linear $L : V \to V$, given by $L = -Id_{2\times 2}$. Take $I = [0, +\infty)$. The ODE is then $\frac{d}{dt}x(t) = -x(t)$. The channel "input" is the initial condition $x(0)$, the channel is the fundamental matrix e^{tL} and the channel "output" is the solution $x(t) = e^{tL}x(0) = e^{-t}x(0)$. The corresponding of the Nyquist Bit Rate in the dictionary is therefore:*

$$C_{ODE} = \sup_{x_0:\|x_0\|_{l^1}=1} \|x_0\|_{l^r} \int_0^{+\infty} e^{-qt} = \frac{1}{q},$$

when one component of x_0 is zero. Note that $1/q$ corresponds to the "rate" parameter of the corresponding exponential distribution.

Remark 2.22. *Consider the following ODE:*

$$\dot{x}(t) = F(x(t)), \quad x(0) = x_0.$$

Here $x : I \subset \mathbb{R} \to V$ where I is an interval containing the origin $t = 0$, V is a vector space and x_0 represents the initial position. Suppose that the operator $F : V \to V$ is nonlinear. This problem is more complicated than the case where $F = L$. One important point is that the existence is not guaranteed anymore for every time $t \in \mathbb{R}$. For example take $V = \mathbb{R}$ and $F(u) = u^2$. The corresponding ODE admits solutions blowing up in finite time. Even in the cases where the solutions exist for all times, uniqueness is not guaranteed. Consider for example $V = \mathbb{R}$ and $F(u) = \sqrt{|u|}$. The corresponding ODE admits multiple solutions with initial datum $x_0 = 0$. We will give further comments in Section 3.

3. The Rosetta Stone

In this section, we make explicit our proposed *Rosetta Stone* between Information Theory and Partial Differential Equations. We start with a general theorem of Keel and Tao on Strichartz Estimates [24], which generalizes Theorem 2.19 and then restrict our attention to a toy model, the Schrödinger Equation. For the purpose of translation, we rephrase their Strichartz Estimates into Information Theoretical terminology, connect maximizers of Entropy with Maximizers of Strichartz Norms and discuss a possible role of symmetries into encoding.

3.1 Keel and Tao's theorem

Let (X,dx) be a measure space and H be a Hilbert space. Consider the Banach space of functions $f : X \to \mathbb{C}$ with the following norm bounded:

$$\|f\|_p := \|f\|_{L^p(X)} = \left(\int_X |f(x)|^p dx\right)^{\frac{1}{p}}.$$

Consider the family of operators indexed by t given by

$$U(t) : X \to L^2(X)$$

and satisfying the following estimates:

- (*Energy Estimate*) for all t and all $f \in H$ we have:

$$\|U(t)f\|_{L^2(X)} \leq S\|f\|_H; \tag{3.1}$$

For some $\sigma > 0$, one of the following decay estimate holds

- for all $t \neq s$ and all $g \in L^1(X)$:

$$\|U(s)U(t)^*g\|_\infty \leq C|t-s|^{-\sigma}\|g\|_1 \tag{3.2}$$

or

- for all t,s and all $g \in L^1(X)$:

$$\|U(s)U(t)^*g\|_\infty \leq C(1+|t-s|)^{-\sigma}\|g\|_1. \tag{3.3}$$

Whenever one of these last two estimates holds together with the *Energy Estimate*, we have the following.

Definition 3.1. *We say that the exponent pair (q,r) is* σ-admissible *if $q,r \geq 2$, $(q,r,\sigma) \neq (2,\infty,1)$ and*

$$\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}.$$

Theorem 3.2. *[24] If $U(t)$ obeys the estimates (3.1) and one between (3.2) and (3.3), then the estimates*

-
$$\|U(t)f\|_{L^q_t L^r_x} \leq S\|f\|_H \tag{3.4}$$

-
$$\left\|\int dsU(s)^*F\right\|_H \leq \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \tag{3.5}$$

-
$$\left\|\int_{s<t} dsU(t)U(s)^*F\|_{L^q_t L^r_x}\right\| \leq \|F\|_{L^{q'}_t L^{r'}_x} \tag{3.6}$$

hold for all sharp admissible pairs (q,r) and $(\tilde{q},\tilde{r})$.

3.2 The case of the Schrödinger equation

Keel and Tao's Theorem is abstract and holds for very general propagators $U(t)$. In the following, we will concentrate on the case of the Schrödinger Equation

$$i\partial_t u(t,x) = \Delta u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \tag{3.7}$$

and give some further comments on other PDEs in later sections.

3.3 Conservation of mass and flow on the space of probability measures

It is well known that if $p_0(x) = |u_0|^2$ defines a probability distribution, then also $p_t(x) = |e^{it\Delta}u_0|^2$ defines a probability distribution. This is mainly a consequence of the property of $e^{it\Delta}$ of being a unitary operator.

Theorem 3.3. *Consider $\mathscr{P}(\mathbb{R}^n)$, the set of all probability distributions on $\mathbb{R}^n$ and $u : (0,\infty) \times \mathbb{R}^n \to \mathbb{C}$ a solution to (3.7). Then u induces a flow in the space of probability distributions.*

Remark 3.4. *This observation justifies our choice of the Schrödinger Equation as a toy model. In fact, not all PDEs possess the property of conserving the charge/mass/number of particles/etc. For example, both the heat and the wave equation do not possess this property.*

3.4 Fundamental solution for the linear Schrödinger equation using fourier transform

The solution of the Linear Schrödinger Equation

$$i\partial_t u(t,x) = \Delta u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^n,$$

with initial datum $u_0(x) = e^{-|x|^2} \in \mathscr{S}(\mathbb{R}^n)$ (Schwartz class) is given by

$$u(t,x) = (1-4it)^{-n/2} e^{-\frac{|x|^2}{1-4it}}. \tag{3.8}$$

This solution induces the probability density function:

$$p(t,x) = \left(\frac{\pi}{2}\right)^{-\frac{n}{2}} |1+16t^2|^{-n/2} e^{-\frac{2|x|^2}{1+16t^2}}. \tag{3.9}$$

Remark 3.5. *Note that if the initial datum is Gaussian, the solution is Gaussian for every time $t \in \mathbb{R}$. This implies that, as we see in later sections, the estimation of parameters from the final solutions can be naturally done in a parametric way. See Subsection 4.2.*

3.5 The information theoretic perspective of Strichartz estimates

In this section, we restate the Strichartz Estimate with Information Theoretical terminology. We propose the following dictionary:

Information Theory	**Differential Equations**
Source	time $t = 0$
Source Encoding	map $0 \mapsto u_0$
Transmitter	Initial Datum u_0
Channel	Propagator e^{tL}
Channel Encoding	External Potential
Receiver	Solution $u(t) = e^{tL}u_0$
Decoder	map $u(t) \mapsto t$
Inference of the source	time t
Nyquist Bit Rate	Strichartz Constant
Entropy	Strichartz Norms
Maximizer of the Entropy	Maximizer of Strichartz Norms
Linear Channel	$F = L$-Linear PDE
Nonlinear Channel	F Nonlinear-Nonlinear PDE
Gaussian	Gaussian

The *source* is the position $t = 0$, to which the *encoder* assigns the initial datum u_0. The message u_0 is transmitted by the channel e^{tL} to the receiver $u(t) = e^{tL}u_0$ which can deduce the position t by uniqueness. Before reaching the receiver, the message might be modified by the presence of an external potential (channel encoding). A quantity which characterizes the dispersion of the initial datum is the entropy of the source/a space-time norm, like the Strichartz Norm. The supremum over all possible sources gives the best Strichartz Constant, which is an intrinsic characteristic of the propagator and measures the maximal dispersive ability of the channel. Source encoding attempts to compress the data from a source in order to transmit it more efficiently. The best efficient way is when u_0 is a Gaussian pdf.

Remark 3.6. *Channel encoding adds extra data bits to make the transmission of data more robust to disturbances present on the transmission channel. This is basically the role played by a confining potential which tends to stabilize the wave.*

Remark 3.7. *This dictionary will be adjusted for the noisy case. First of all, the role played by the Nyquist Bit Rate is substituted by the Shannon Capacity. Then, for deterministic PDEs for which there is uniqueness, the map $t \to u(t)$ is a bijection, if we fix initial datum u_0. Therefore, the decoding is basically an identification map. In the noisy case, this cannot be true, because what the receiver gets is the noisy solution that the decoder needs to extract, so there cannot be a one to one correspondence between the noisy solution and the time t. In the noisy case, therefore, the noisy solution plays the role of the receiver and the denoised solution plays the role of the inference of the source. We refer to Section 5 for more details on the noisy case.*

We are now ready to restate Theorem 2.19 about Strichartz Estimates, using an Informational Theoretic point of view.

Theorem 3.8. *Let $t = 0$ be a source, whose transmitted signal u_0 has "entropy" H given by*

$$H := \left|\left|e^{-it\Delta}u_0\right|\right|_{L_t^q L_x^r}. \tag{3.10}$$

Consider the Channel $U(t) := e^{-it\Delta}$, whose Nyquist Bit Rate is

$$C := S(n,q,r) = \sup\left\{||u||_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} : ||u||^2_{L_x^2(\mathbb{R}^n)} = 1\right\}.$$

Then, it is possible to encode $t = 0$ by u_0 in such a way that the message can be transmitted at an average rate of at most C. The maximum rate $C = 2^{n/4-n(r-2)/(2r)}r^{n/(2r)}$ is reached when u_0 is Gaussian and measures the maximal speed of transmission of the channel $U(t)$. It is not possible to transmit at an average rate greater than the Strichartz Constant.

Some explanations are in order. Since we are in the noiseless case and by the uniqueness of the solution of the PDE (in this paper we are considering just linear PDEs), once you know $u(t,x)$ at any time t, then you know the solution at any previous and subsequent time. Therefore, any measure of relative entropy must be zero $H(X|Y) = H(Y|X) = 0$. No extra entropy is added during the flow, since the flow is deterministic. So, a good measure of mutual information must give $I(X,Y) = H(X) = H(Y)$. A reasonable such measure is indeed given by the Strichartz norms:

$$I(X,Y) = \left|\left|e^{-it\Delta}u_0\right|\right|_{L_t^q L_x^r}.$$

Therefore, a measure of speed of transmission is given by

$$S(n,q,r) = \sup_{u_0\in L^2(X)} I(X,Y).$$

The Sharp Strichartz Constant plays then the role of *Nyquist Bit Rate of the Channel*.

Remark 3.9. *Our parallel between Strichartz Norms and Entropy is also justified by the result in [3], where the authors prove a monotonicity property of the Strichartz Norms under the evolution of a certain quadratic heat flow that looks like the monotonicity property of the entropy. The entropy measures a level of uncertainty and it always increases. Similarly, the Strichartz Norms measure dispersion and they are always increasing under the heat flow.*

Remark 3.10. *The Strichartz norm quantifies the dispersion of the propagator and "translates" to the Nyquist Bit Rate. It is actually an intrinsic property of the propagator and measures the information rate at which the input can travel. The Schrödinger Strichartz constant is the optimal upper bound on the rate at which information can be transmitted over the Schrödinger channel.*

Remark 3.11. *The Gaussian maximizes both the Shannon Entropy and the Strichartz Norms, but, differently from the usual capacity measure, there is no need of power upper bounds and the Gaussian is an absolute maximizer of the mutual information.*

Remark 3.12. *In quantum mechanics, loss of information corresponds to the violation of the unitary property, which has to do with the conservation of probability. Under the flow of the equation that we are considering, unitarity is preserved and in fact, the measure of information that we are using is constant along the flow as we showed in Subsection 3.3.*

Remark 3.13. *The Gaussian is added to the dictionary to show a connection between maximizers, as we will see in next section. It shows, also, a kind of centrality that that distribution plays in several fields. It is a kind of "fix point" of the dictionary.*

Remark 3.14. *Since the equations treated in this paper are all linear, we have uniqueness for free. This simplifies the decoding procedure. In the case of non-uniqueness, the situation can become more involved. Another complication appears if the transmitter can send just some information about the initial datum and not all. In this case, the presence of the symmetry of the equation might play a role in the reconstruction of the signal. See Section 4.3.*

Remark 3.15. *It might be interesting to extend this notion to the nonlinear case. In that situation for a certain range of nonlinearities, the long time behaviour of the solution is still linear (see [43]) and so a similar dictionary seem to be suitable.*

4. Further connections between PDEs and Shannon information theory

In this section, we describe some possible further parallels between PDEs and Information Theory.

4.1 Maximizers of the entropy are maximizers of the Strichartz norm?

The *Principle of Maximum Entropy* states that, among distributions belonging to a particular class (e.g. fixed variance, supported on the half-line, etc...), you should select the distribution with the maximum entropy, because it is the most uninformative. By doing this, you minimize the possibility of adding extra bias and you follow the physical principle that many systems tend to stabilize towards maximal entropy configurations. The following is a well-known theorem of Boltzmann on maximizers of the Entropy.

Theorem 4.1 (Boltzmann's Theorem)**.** *Consider the following subset of $\mathbb{R}^n$:*

$$\mathscr{C} := \{g : \mathbb{R}^n \to \mathbb{R} \text{ such that } \mathrm{E}[g(x)] = c \text{ with } x \in S \subset \mathbb{R}^n\},$$

where $S \subset \mathbb{R}^n$ is a closed subset, $g(x) = (g_1(x), \dots, g_n(x))$ and $c = (c_1, \dots, c_n) \in \mathbb{R}^n$. Suppose there exists $g \in \mathscr{C}$, such that $supp(g) = S$ and such that

$$g \in argmax_{\mathscr{C}} H(X),$$

then

$$g(x) = c \exp\left(\sum_{j=1}^{n} \lambda_j g_j(x)\right) \qquad \text{for all } x \in S$$

with c, λ_j such that $\int_S g(x) = 1$ and such that $\mathrm{E}[g(x)] = c$. A viceversa holds.

Remark 4.2. *A similar version of this theorem works also in the discrete case.*

We can list several cases in which maximizers of the entropy are known:

- $\mathbb{R}^n$ *case*: The Univariate $N(\mu, \sigma^2)$ or Multivariate $N(\mu, \Sigma)$ Normal Distribution has maximum entropy among all real-valued distributions with specified mean μ and standard deviation σ and Var-Cov matrix $|\Sigma|$ respectively. Therefore, the assumption of normality imposes the minimal prior structural constraint beyond these moments. We saw that the entropy of the Univariate Normal Distribution whose standard deviation is σ is given by $H(x) = \frac{1}{2}\log\left[2\sigma^2\pi e\right]$, while the entropy of the Multivariate Normal Distribution with fixed $|\Sigma|$ is $H(X) = \frac{1}{2}\log\left[(2\pi e)^n \det(\Sigma)\right]$
- *Interval case*: The uniform distribution on the interval $[a,b]$ is the maximum entropy distribution among all continuous distributions which are supported in the interval $[a,b]$. The entropy of the uniform distribution in the interval $[a,b]$ is given by $\log[b-a]$.
- *Circular case*: The von Mises distribution [34] has pdf given by

$$f(\theta; \mu, \kappa) = \frac{e^{\kappa \cos(\theta - \mu)}}{2\pi I_0(\kappa)}.$$

The function $I_0(\kappa)$ denotes the modified Bessel function of order 0, the angle $\theta \in [0,2\pi]$ and μ and κ are the scale and concentration parameter, respectively. The entropy of the Von Mises distribution is given by

$$H = -\int_0^{2\pi} f(\theta;\mu,\kappa)\ln(f(\theta;\mu,\kappa))d\theta = \ln(2\pi I_0(\kappa)) - \kappa\frac{I_1(\kappa)}{I_0(\kappa)},$$

with the function $I_1(\kappa)$ denoting the modified Bessel function of order 1.

- *Half-line case*: If $X \in \mathbb{R}^+$, namely $p(x) = 0$ for $x \leq 0$, and the first moment of X is fixed to be a, $a = \int_0^\infty p(x)xdx$,, then the maximum entropy distribution is the exponential distribution with pdf $p(x) = \frac{1}{a}e^{-x/a}$ for $x > 0$ and $p(x) = 0$ otherwise. In this case, the entropy is equal to $\log[ea]$.

Remark 4.3. *Some classes of distributions do not contain a maximum entropy distribution. It is possible that a class contain distributions of arbitrarily large entropy (for example, the class of continuous distributions on $\mathbb{R}$ with mean 0 but arbitrary standard deviation), or that the Entropy is bounded above, but there is no distribution which attains the maximal entropy (for example, if you add too many constraints -See [9]-).*

In the case of $\mathbb{R}^n$, Strichartz Norms are maximized by Gaussians (conjectured in $n \geq 3$ and proved $n = 1,2$, see [39]). As far as we know, no work has been done to describe the maximizers of the Strichartz Norms on other types of domains. We have the following question.

Question:

We are wondering if the connection between Strichartz Norms and Entropy that we are proposing here is reflected also in the maximizers of these norms on domains different from $\mathbb{R}^n$. More precisely. If we fix the domain Ω, is it true that the class of functions which maximize some Strichartz Norms on Ω maximize also the Entropy? For which admissible exponents is this true?

Differently from the Entropy, computing the Strichartz Norm of distribution functions, apart from few particular cases, like the Gaussian (see [39]) is not exactly simple. Just for the sake of illustration, we discuss the case of the pdf of a uniform random variable between -1 and 1. We take as initial datum u_0 the function $u_0 = \frac{1}{2}\chi_{[-1,1]}$. If we compute its Fourier Transform, we get:

$$\hat{u}(0,\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u_0(x)e^{ix\xi}dx = \frac{1}{2\sqrt{2\pi}}\int_{-1}^{1} e^{ix\xi}dx = \frac{1}{2\sqrt{2\pi}}\frac{e^{ix\xi}}{i\xi}\Big|_{-1}^{1} = \frac{1}{2\sqrt{2\pi}}\frac{e^{+i\xi}-e^{-i\xi}}{i\xi}.$$

Using the propagator (similarly to what we did in Section 3.4), we get

$$u(t,x) = \int_{\mathbb{R}} e^{i|\xi|^2t+ix\xi}\frac{1}{2\sqrt{2\pi}}\frac{e^{-i\xi}-e^{i\xi}}{-i\xi}d\xi.$$

If now we take the space derivative of this function, we get:

$$\frac{\partial}{\partial x}u(t,x) = \int_{\mathbb{R}} e^{i|\xi|^2t+ix\xi}\frac{1}{2\sqrt{2\pi}}\left(e^{i\xi}-e^{-i\xi}\right)d\xi$$

$$= \frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}} e^{i|\xi|^2t+i(x+1)\xi}d\xi - \frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}} e^{i|\xi|^2t+i(x-1)\xi}d\xi.$$

Each of the two terms represent the Schrödinger Evolution of a Delta Function, with center at $+1$ and -1 respectively. Therefore, we get:

$$\frac{\partial}{\partial x}u(t,x) = \frac{\pi^{\frac{1}{2}}}{4t^{\frac{1}{2}}}e^{-i\frac{t}{|t|}\frac{\pi}{4}}e^{-i\frac{(x+1)^2}{4t}} - \frac{\pi^{\frac{1}{2}}}{4t^{\frac{1}{2}}}e^{-i\frac{t}{|t|}\frac{\pi}{4}}e^{-i\frac{(x-1)^2}{4t}}.$$

Now, by integrating in x, we find:

$$u(t,x) \propto e^{-i\frac{t}{|t|}}\left(\Phi\left(\frac{x+1}{2}\left(\frac{i}{t}\right)^{\frac{1}{2}}\right) - \Phi\left(\frac{x-1}{2}\left(\frac{i}{t}\right)^{\frac{1}{2}}\right)\right).$$

Here $\Phi(x)$ is the cumulative distribution function of the Standard Normal. Now, we should compute the Strichartz Norm of this function $u(t,x)$, but since already this cannot be computed in closed form, neither any space time integral can be. So, it seems hard to verify that an optimal bound is attained by a certain distribution, just by direct computation.

4.2 Time recovery: estimation of the source

One of the main goals of the *Theory of Communication* is to transmit a message and be able to recover it entirely or at least with the highest possible precision. In this section, we rephrase this in the context of PDEs, using the dictionary.

Given n output observations, it is simple to estimate the time t at which the signal has been sent. Suppose the input is again a Gaussian distribution $u_0(x) = e^{-|x|^2}$ and so that $p_0 \propto e^{-2|x|^2}$. Then

$$u(t,x) = (1-4it)^{-n/2} e^{-\frac{|x|^2}{1-4it}} \tag{4.1}$$

and so

$$p(t,x) = \left(\frac{\pi}{2}\right)^{-\frac{n}{2}} |1+16t^2|^{-n/2} e^{-\frac{2|x|^2}{1+16t^2}}, \tag{4.2}$$

as we showed in Subsection 3.4. A random process X_t with this distribution is a Normal Random Variable $X_t \simeq \mathcal{N}(0, \sigma_t^2)$ with $\sigma_t^2 = \frac{1+16t^2}{4}$. This automatically implies that: $t^2 = \frac{4\sigma_t^2 - 1}{16}$. Therefore the MLE estimator of the time t^2 is a linear transformation of the MLE of the variance: $\hat{t^2}_{MLE} = \frac{4\hat{\sigma_t^2}_{MLE}-1}{16}$, with $\hat{\sigma_t^2}_{MLE} := \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$.

Remark 4.4. *The correspondence between a solution $u(t,x)$ of the Schrödinger equation and $p(t,x)$ is not a bijection. In fact, once you know $p(t,x)$, every function $e^{i\theta}u(t,x)$ with $\theta \in [0,2\pi)$ gives rise to the same $p(t,x)$. This does not prevent to recover the time t, since the parameter θ does not have any effect on the variance.*

Similarly, suppose that different subsets of the data are observed at different instants: independently, we observe X_i, $i = 1,\dots,n$ at t_X and we observe Y_j, $j = 1,\dots,m$ at t_Y. We can estimate the signed distance T between the emission times of the signals X and Y in the following way. We first estimate the time when X_i's have been emitted:

$$\hat{t^2_X}_{MLE} = \frac{4\hat{\sigma^2_X}_{MLE} - 1}{16},$$

with

$$\hat{\sigma^2_X}_{MLE} := \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$$

and

$$\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i.$$

Then, we estimate the time at which the Y_j's have been emitted:

$$\hat{t^2_Y}_{MLE} = \frac{4\hat{\sigma^2_Y}_{MLE} - 1}{16},$$

with

$$\hat{\sigma^2_Y}_{MLE} := \frac{1}{m-1}\sum_{j=1}^m (Y_j - \bar{Y})^2$$

and

$$\bar{Y} = \frac{1}{m}\sum_{i=1}^m Y_i.$$

Then,

$$\hat{T} = \hat{t_Y}_{MLE} - \hat{t_X}_{MLE} = \left(\frac{4\hat{\sigma^2_Y}_{MLE} - 1}{16}\right)^{1/2} - \left(\frac{4\hat{\sigma^2_Y}_{MLE} - 1}{16}\right)^{1/2}.$$

We can also estimate the period if the same signal is sent in a periodic way at instants t_k, $k = 1, \dots, N$. First, we can estimate for each k:

$$\hat{t^2_{k\,MLE}} = \frac{4\hat{\sigma^2_{t_k MLE}} - 1}{16},$$

with

$$\hat{\sigma^2_{t_k MLE}} := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

and

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Then,

$$\hat{T} := \frac{1}{N-1}\sum_{k=1}^{N-1}(t_{i+1} - t^i) = \frac{1}{N-1}(t_N - t_1).$$

Remark 4.5. *In the case that the signal is not Gaussian, using the Central Limit Theorem, we can prove that these estimators are asymptotic estimators, for $t \in [0, T]$ with $0 < T < +\infty$.*

4.3 Symmetries and encoding/decoding

The purpose of source encoding is to decrease the dimension of the source data. In the context of the Schrödinger Equation, the source coding can be seen as a way to organize the particles. For example in the case of Gaussian initial data, this means giving to the Variance-Covariance Matrix a particular structure. The main point here is that, due to its symmetries and since it is possible to transmit the message at the optimal rate, the source encoding for the Schrodinger Equation, based on domain, co-domain and structure invariance do not affect the optimal transmission rate. This is because the symmetries of the propagator have counterparts in the symmetries of the Strichartz Norms. In this section, we summarize these symmetries, to make explicit what we mean. As explained in [15] (see also [39] following [15]), Strichartz Estimates are invariant by the following set of symmetries.

Lemma 4.6. *[15] Let $\mathscr{G}$ be the group of transformations generated by:*

- *space-time translations: $u(t,x) \mapsto u(t+t_0, x+x_0)$, with $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$;*
- *parabolic dilations: $u(t,x) \mapsto u(\lambda^2 t, \lambda x)$, with $\lambda > 0$;*
- *change of scale: $u(t,x) \mapsto \mu u(t,x)$, with $\mu > 0$;*
- *space rotations: $u(t,x) \mapsto u(t, Rx)$, with $R \in SO(n)$;*
- *phase shifts: $u(t,x) \mapsto e^{i\theta} u(t,x)$, with $\theta \in \mathbb{R}$;*
- *Galilean transformations:*

  $$u(t,x) \mapsto e^{\frac{i}{4}\left(|v|^2 t + 2v\cdot x\right)} u(t, x+tv),$$

  with $v \in \mathbb{R}^n$.

Then, if u solves equation (3.7) and $g \in \mathscr{G}$, also $v = g \circ u$ solves equation (3.7). Moreover, the constants $S_h(n,q,r)$ are left unchanged by the action of $\mathscr{G}$.

Not all these symmetries leave invariant the set of probability distributions $\mathscr{P}(\mathbb{R}^n)$. Therefore, we need to reduce the set of symmetries in our treatment and, in particular, we need to combine the scaling and the parabolic dilations in order to have all the family inside the space of probability distributions $\mathscr{P}(\mathbb{R}^n)$.

Lemma 4.7. *Consider $u_{\mu,\lambda} = \mu u(\lambda^2 t, \lambda x)$ such that $u(t,x) \in \mathscr{P}(\mathbb{R}^n)$ maximizes (2.11), then $\mu = \lambda^{n/2}$.*

Proof.

$$1 = ||u_\lambda||^2_{L^2(\mathbb{R}^n)} = \mu^2 \int_{\mathbb{R}^n} |u(\lambda^2 t, \lambda x)|^2 dx = \mu^2 \lambda^{-n} ||u||^2_{L^2(\mathbb{R}^n)} = \mu^2 \lambda^{-n},$$

so $\mu = \lambda^{n/2}$. □

Remark 4.8. *We notice that some of the symmetries can be seen just at the level of the generator of the family u, but not by the family of probability distributions $p_t(x)$. For example the phase shifts $u(t,x) \mapsto e^{i\theta} u(t,x)$, with $\theta \in \mathbb{R}$ give rise to the same probability distribution function because $|e^{i\theta} u(t,x)|^2 = |u(t,x)|^2$ and, partially, the Galilean transformations $u(t,x) \mapsto e^{\frac{i}{4}\left(|v|^2 t + 2v\cdot x\right)} u(t,x+tv)$, with $v \in \mathbb{R}^n$ reduces to a space translation with $x_0 = vt$, since $\left| e^{\frac{i}{4}\left(|v|^2 t + 2v\cdot x\right)} u(t,x+tv) \right|^2 = |u(t,x+tv)|^2$. In some sense, the parameter θ can be seen as a* latent variable.

Therefore, we have the complete set of probability distributions induced by the generator $u(t,x)$.

Theorem 4.9. *Consider $p_t(x) = |u(t,x)|^2$ a probability distribution function generated by $u(t,x)$ (see Subsection 3.4). Let $\mathscr{S}$ be the group of transformations generated by:*

- *inertial-space translations and time translations: $p(t,x) \mapsto p(t+t_0, x+x_0+vt)$, with $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$;*
- *scaling-parabolic dilations: $u(t,x) \mapsto \lambda^n u(\lambda^2 t, \lambda x)$, with $\lambda > 0$;*
- *space rotations: $u(t,x) \mapsto u(t,Rx)$, with $R \in SO(n)$;*

Then, if u solves equation (3.7) and $g \in \mathscr{S}$, also $v = g \circ u$ solves equation (3.7), $q_t(x) = |v(t,x)|^2$ is still a probability distribution for every $g \in \mathscr{S}$ and the constant $S_h(n,q,r)$ is left unchanged by the action of $\mathscr{S}$.

Remark 4.10. *Optimality and symmetry are two strictly related concepts in the calculus of variations. Usually, extremizers possess some extra structure (see for example [6]). This might suggest to use symmetries/properties of the channel to optimize a code. Strichartz Inequalities, Soboloev Inequalities and several others possess radial extremizers (see again [6]).*

5. Some final remarks

In this section, we collect some further comments about the discrete case, the noisy case and the Kinetic Transport Equation.

5.1 The discrete case

Up to now, we mainly treated the case of a continuous channel and so continuous PDEs. One can think about extending this machinery to the discrete case. A possible approach would be to consider the Continuous Schrödinger Equation and use as initial data a sum of delta functions. This approach fails for the following reason. Consider for instance the case of a single delta function $u_0(x) = \delta(x)$. This gives rise to the solution:

$$u(t,x) = \left(\frac{1}{\pi t}\right)^{\frac{1}{2}} e^{i\frac{x^2}{t}}.$$

ans so to the pdf

$$|u(t,x)|^2 = \left(\frac{1}{\pi t}\right)^{\frac{1}{2}},$$

which is constant in space. So, every Strichartz Norm of this function would be infinity. Moreover, the estimates cannot work even locally in space because of the following reason. Consider a subset of $\mathbb{R}^n$ with volume V. Then,

$$\|u(t,x)\|_{L^q_t L^r_x} = \int_t \left(\frac{1}{\pi t}\right)^{\frac{q}{2}} V^{\frac{1}{r}} dt,$$

while

$$\|u(t,x)\|_{L^2_x} = \left(\frac{1}{\pi t}\right)^{\frac{1}{2}} V^{\frac{1}{2}}.$$

But an inequality of the type

$$\int_t \left(\frac{1}{\pi t}\right)^{\frac{q}{2}} V^{\frac{1}{r}} dt \leq C \left(\frac{1}{\pi t}\right)^{\frac{1}{2}} V^{\frac{1}{2}},$$

cannot hold for a constant C independent of time, because the time variable appears at different powers in the left and right hand side of the inequality. The main reason for all of this is that the mass is conserved just for L^2 solutions and $\delta(x)$ has a regularity lower than L^2.

A second approach is to consider the *Discrete Schrödinger Equation*:

$$i\partial_t u_k(t) + h^{-2}\left(u_{k+h}(t) + u_{k-h}(t) - 2u_n(t)\right) = 0,$$

with $k \in \mathbb{Z}$ and $u_0 \in l^2$. This equation resembles the continuous model, when the step size h is small $0 < h \ll 1$. Now, the domain is itself discrete and so we cannot do anything but choosing an initial datum defined on a discrete set. Moreover, the quantity which is now conserved for this equation is the $\|\cdot\|_{l^2}$, namely the discrete version of the $\|\cdot\|_{L^2}$ norm. Using the main theorem of [24], the authors in [41] have been able to prove Strichartz Estimates:

$$\|u_n(t)\|_{L_t^q l_x^r} \leq C\|u_n(0)\|_{l_x^2},$$

for $(q,r) \geq 2$ and

$$\frac{1}{q} + \frac{n}{3r} \leq \frac{n}{6}.$$

In this setting, we can use the Rosetta Stone and translate this discrete PDE in the context of Information Theory, in particular using the Discrete Schrödinger Equation as a toy model for *Discrete Channels*.

Remark 5.1. *For the* Discrete Schrödinger Equation*, the problem of finding the Sharp Strichartz Constant is still open, and the problem of finding the maximizer is open, as well. Possibly, the dictionary will be somehow helpful to answer this question.*

5.2 The noisy case

In this subsection, we consider the *Stochastic Linear Schrödinger Equation*

$$i\partial_t u(t,x) = \Delta u(t,x) \circ d\beta, \quad (t \geq s, x) \in (0,\infty) \times \mathbb{R}^n,$$

with initial datum $u(s,x) = u_s(x)$. This equation has an explicit solution (see [10] for details).

Proposition 5.2. *[10] For any $s \leq T$ and $u_s(x) \in \mathscr{S}'(\mathbb{R}^n)$, there exists a unique solution of the* Stochastic Linear Schrödinger Equation*, almost surely in $C([s,T];\mathscr{S}'(\mathbb{R}^n))$. Its Fourier Transform in space is given by*

$$\hat{u}(t,\xi) = e^{-i|\xi|^2(\beta(t)-\beta(s))}\hat{u}_s(\xi), \quad t \geq s, \quad \xi \in \mathbb{R}^n.$$

Moreover, if $u_s \in H_x^\sigma$ for some $\sigma \in \mathbb{R}$, then $u(\cdot) \in C([0,T];H_x^\sigma)$ a.s. and $\|u(t)\|_{H^\sigma} = \|u_s\|_{H^\sigma}$, a.s. for $t \geq s$. If $u_s \in L_x^1$, then the explicit solution takes the form:

$$u(t) = U(t,s)u_s := \frac{1}{(4\pi i(\beta(t)-\beta(s)))^{n/2}} \int_{\mathbb{R}^n} exp\left(\frac{i|x-y|^2}{(4(\beta(t)-\beta(s)))}\right) u_s(y)dy$$

for $t \in [s,T]$.

Strichartz Estimates have been proved also in the stochastic case.

Theorem 5.3. *Let $2 \leq r < +\infty$ and $2 \leq p \leq +\infty$ be such that $\frac{2}{r} > n\left(\frac{1}{2}-\frac{1}{p}\right)$ or $r = +\infty$ and $p = 2$. Let ρ be such that $r' \leq \rho \leq r$. Then, there exists a constant $c = c(\rho,r,p) > 0$ such that, for any $s \in \mathbb{R}$, $T \geq 0$ and $f \in L_{\mathscr{P}}^\rho(\Omega;L^{r'}(s,s+T;L_x^{p'}))$, the following estimate holds:*

$$\left|\int_s U(\cdot,\sigma)f(\sigma)d\sigma\right| \leq c(\rho,r,p)T^\beta |f|_{L_{\mathscr{P}}^\rho(\Omega;L^{r'}(s,s+T;L_x^{p'}))}$$

with $\beta = \frac{2}{r} - \frac{n}{2}\left(\frac{1}{2}-\frac{1}{p}\right)$.

Remark 5.4. *For the precise definition of the function spaces in the previous theorem we refer to [10].*

Concerning the optimal Strichartz Constant in the stochastic case $c(\rho, r, p)$ and the function which realizes it, as far as we know, there are no results. Nevertheless, we can propose a dictionary in the same flavour of the deterministic case.

Information Theory	**Differential Equations**
Source	u_0
Source Encoding	map $u_0 \mapsto u_0 + noise$
Transmitter	Noisy Initial Datum $u_0 + noise$
Channel	Propagator e^{tL}
Channel Encoding	External Potential
Receiver	Noisy Solution $u(t) + noise = e^{tL}(u_0 + noise)$
Decoder	map $u(t) + noise \mapsto u(t)$
Inference of the source	$E[u(t) + noise]$
Shannon Capacity	Strichartz Constant
Entropy	Strichartz Norms
Maximizer of the Entropy	Maximizer of Strichartz Norms
Linear Channel	$F = L$-Linear PDE
Nonlinear Channel	F Nonlinear-Nonlinear PDE
Gaussian	Gaussian

The main difference with respect to the deterministic case is that Nyquist Bit Rate is substituted by Shannon Capacity. Moreover, in the stochastic case, we do not have a one to one correspondence between the encoded signal and the time $t = 0$, as well as between what the decoder sees and the time t. For this reason, the dictionary cannot be simplified using the bijection between t and $u(t)$, as in the deterministic case.

Remark 5.5. *Each couple of admissible exponents give a different measure of "capacity", non necessarily equivalent. For this reason, there exist different notions of capacity that might not be equivalent.*

If we try to compute the mutual information for a joint pdf $p_\lambda(x,y) = \lambda^a p(\lambda^b x, \lambda^c y)$*, using the constraints that also* $p(x)$*,* $p(y)$*,* $p(x,y)$ *need to be pdfs as well with the marginals of* $p_\lambda(x,y)$*, we conclude that* I *is invariant under this rescaling if and only if* $a = n(b+c)$*.*

Consider a generalized version of the mutual information:

$$I_{\alpha,\beta,\gamma,r}(X;Y) = \int_y \int_x p(x,y)^r \log\left(\frac{p(x,y)^\alpha}{p(x)^\beta p(y)^\gamma}\right) dxdy. \tag{5.1}$$

The scale invariance implies $r = 1$ *and so from now on* $I_{\alpha,\beta,\gamma}(X;Y) := I_{\alpha,\beta,\gamma,r}(X;Y)$*. The argument of the log needs to be scale free and so* $\alpha(b+c) = (\beta b + \gamma c)$*. This condition is trivially satisfied when* $\alpha = \beta = \gamma$*, which gives the original* $I(X;Y)$ *up to a constant scale. But for example, in the symmetric case* $c = b$ *and so* $a = 2nb$*, for* $\alpha = \frac{\beta+\gamma}{2}$ *we have the corresponding scale invariance of* $I_{\alpha,\beta,\gamma}(X;Y)$*. Note that fixed* $q := \frac{b}{b+c}$*, we have scale invariance whenever* $\alpha = q\beta + (1-q)\gamma$*. The choice of* q *is strictly related to the choice of the channel* $p(y|x)$*. This is reminiscent of the admissible exponents, which are strictly related to the linear propagator of the corresponding PDE.*

It would be interesting to study more deeply the properties of $I_{\alpha,\beta,\gamma}(X;Y)$ *in the context of Information Theory and see if the corresponding capacity*

$$C_{\alpha,\beta,\gamma} := \sup_{p_X(x)} I_{\alpha,\beta,\gamma}(X;Y)$$

plays any important role.

5.3 The example of the kinetic transport equation

In this subsection, we give another example of our proposed relationship between Information Theory and PDEs. We consider the *Kinetic Transport Equation*:

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = 0, \quad f(0,x,v) = f^0(x,v)$$

for $(t,x,v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$. This equation satisfies a similar set of Strichartz Estimates:

$$\|f\|_{L^q_t L^p_x L^r_v} \le C \|f^0\|_{L^a_{x,v}}$$

for a set of admissible exponents;

$$\frac{2}{q} = n\left(\frac{1}{r} - \frac{1}{p}\right), \quad \frac{1}{a} = \frac{1}{2}\left(\frac{1}{r} + \frac{1}{p}\right), \quad q > a, \quad p \ge a.$$

The distribution function $f(t,x,v)$ is a non negative function $f(t,x,v) \ge 0$ depending on the time $t \in \mathbb{R}$, on the position $x \in \mathbb{R}^n$ and on the velocity $v \in \mathbb{R}^n$ and it is required to be integrable in x and v, $\iint_{x,v} f(x,v,)dxdv < +\infty$ for every $t \in \mathbb{R}$. From a physical point of view, the density f describes the evolution of the system of particles and $\int_C f(t,x,v)dxdv$ represents the probability of finding particles in the position-velocity space region C, at a fixed instant t. Furthermore, the *Kinetic Transport Equation* admits several conservation laws and in particular the conservation of the number of the particles (see for example [12]). These properties put ourselves in the same framework of the Schrödinger Equation and therefore a similar Rosetta Stone might be produced in the case of the Kinetic Transport Equation, as well.

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The sum of the largest and smallest signless laplacian eigenvalues and some Hamiltonian properties of graphs

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Abstract
The signless Laplacian eigenvalues of a graph G are eigenvalues of the matrix $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of the degrees of the vertices in G and $A(G)$ is the adjacency matrix of G. Using a result on the sum of the largest and smallest signless Laplacian eigenvalues obtained by Das in [2], we in this note present sufficient conditions based on the sum of the largest and smallest signless Laplacian eigenvalues for some Hamiltonian properties of graphs.

Keywords: Signless Laplacian Eigenvalues, Hamiltonian Properties
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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. For a graph $G = (V(G), E(G))$, we use n to denote its order $|V(G)|$. A subset V_1 of the vertex set $V(G)$ is independent if no two vertices in V_1 are adjacent in G. The size of a maximum independent set is called the independence number of G and it is denoted by $\alpha(G)$. We use $G_1 \vee G_2$ to denote the the join of two disjoint graphs G_1 and G_2. The graph consists of p isolated vertices is denoted by pK_1. Let $D(G)$ be a diagonal matrix such that its diagonal entries are the degrees of vertices in a graph G. The signless Laplacian matrix of a graph G, denoted $Q(G)$, is defined as $D(G) + A(G)$, where $A(G)$ is the adjacency matrix of G. The eigenvalues $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$ of $Q(G)$ are called the signless Laplacian eigenvalues of G. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path.

In this note we present sufficient conditions based on the sum of the largest and smallest signless Laplacian eigenvalues for the Hamiltonian and traceable graphs. The main results are as follows.

Theorem 1.1. *Let G be a k-connected graph ($k \geq 2$) of order $n \geq 4$. If $q_1 + q_n \geq 3n - 2k - 4$, then G is Hamiltonian or G is $(k+1)K_1 \vee K_r$ with $2 \leq r \leq k$.*

Theorem 1.2. *Let G be a k-connected ($k \geq 1$) graph of order $n \geq 4$. If $q_1 + q_n \geq 3n - 2k - 6$, then G is traceable or G is $(k+2)K_1 \vee K_r$ with $1 \leq r \leq k$.*

2. Proofs

In order to prove Theorem 1.1 and Theorem 1.2, we need the following result obtained by Das as our lemma. Lemma 2.1 below is Theorem 3.2 on Page 995 in [2].

Lemma 2.1. *Let G be a connected graph on $n \geq 4$ vertices with independence number α. Then $q_1 + q_n + 2\alpha \leq 3n - 2$ with equality holding if and only if G is $\alpha K_1 \vee K_{n-\alpha}$.*

Proof of Theorem 1. Let G be a graph satisfying the conditions in Theorem 1.1. Suppose, to the contrary, that G is not Hamiltonian. Since $k \geq 2$, G has a cycle. Choose a longest cycle C in G and give an orientation on C. Since G is not Hamiltonian, there exists a vertex $u_0 \in V(G) - V(C)$. By Menger's theorem, we can find k pairwise disjoint (except for u_0) paths P_1, P_2, ..., P_k between u_0 and $V(C)$. Let v_i be the end vertex of P_i on C, where $1 \leq i \leq k$. Without loss of generality, we assume that the appearance of v_1, v_2, ..., v_k agrees with the orientation of C. We use v_i^+ to denote the successor of v_i along the orientation of C, where $1 \leq i \leq k$. Since C is a longest cycle in G, we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq k$ and the index $k+1$ is regarded as 1. Moreover, $S := \{u_0, v_1^+, v_2^+, \ldots, v_k^+\}$ is independent (otherwise G would have cycles which are longer than C). From Lemma 2.1, we have that

$$3n - 2 = 3n - 2k - 4 + 2(k+1) \leq q_1 + q_n + 2|S| \leq q_1 + q_n + 2\alpha \leq 3n - 2.$$

From Lemma 2.1 again, we have that $q_1 + q_n = 3n - 2k - 4$, S is a maximum independent set of size $\alpha = k+1$, and G is $(k+1)K_1 \vee K_{n-(k+1)}$. Notice that G is Hamiltonian if $n - (k+1) \geq (k+1)$. Thus $n - (k+1) \leq k$. Since G is k-connected with $k \geq 2$, G must be $(k+1)K_1 \vee K_r$ with $2 \leq r \leq k$. □

Proof of Theorem 2. Let G be a graph satisfying the conditions in Theorem 1.2. Suppose, to the contrary, that G is not traceable. Choose a longest path P in G and give an orientation on P. Let x and y be the two end vertices of P. Since G is not traceable, there exists a vertex $u_0 \in V(G) - V(P)$. By Menger's theorem, we can find k pairwise disjoint (except for u_0) paths P_1, P_2, ..., P_k between u_0 and $V(P)$. Let v_i be the end vertex of P_i on P, where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of v_1, v_2, ..., v_k agrees with the orientation of P. Since P is a longest path in G, $x \neq v_i$ and $y \neq v_i$, for each i with $1 \leq i \leq k$, otherwise G would have paths which are longer than P. We use v_i^+ to denote the successor of v_i along the orientation of P, where $1 \leq i \leq k$. Since P is a longest path in G, we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq k-1$. Moreover, $\{u_0, v_1^+, v_2^+, \ldots, v_k^+, x\}$ is independent (otherwise G would have paths which are longer than P). From Lemma 2.1, we have that

$$3n - 2 = 3n - 2k - 6 + 2(k+2) \leq q_1 + q_n + 2|S| \leq q_1 + q_n + 2\alpha \leq 3n - 2.$$

From Lemma 2.1 again, we have that $q_1 + q_n = 3n - 2k - 6$, S is a maximum independent set of size $\alpha = k+2$, and G is $(k+2)K_1 \vee K_{n-(k+2)}$. Notice that G is traceable if $n - (k+2) \geq (k+1)$. Thus $n - (k+2) \leq k$. Since G is k-connected with $k \geq 1$, G must be $(k+2)K_1 \vee K_r$ with $1 \leq r \leq k$. □

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A metaheuristic optimization algorithm for multimodal benchmark function in a GPU architecture

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Abstract
It is well known that the numerical solution of evolutionary systems and problems based on topological design requires a high computational power. In the last years, many parallel algorithms have been developed in order to improve its performance. Among them, genetic algorithms (GAs) are one of the most popular metaheuristic algorithms inspired by Darwin's evolution theory. From the High Performance Computing (HPC) point of view, the CUDA environment is probably the parallel computing platform and programming model that more heyday has had in recent years, mainly due to the low acquisition cost of graphics processing units (GPUs) compared to a cluster with similar functional characteristics. Consequently, the number of GPU-CUDAs present in the top 500 fastest supercomputers in the world is constantly growing. In this paper, a numerical algorithm developed in the NVIDIA CUDA platform capable of solving classical optimization functions usually employed as benchmarks is presented. The obtained results demonstrate that GPUs are a valuable tool for acceleration of GAs and may enable its use in much complex problems. Also, a sensitivity analysis is carried out in order to show the relative weight of each GA operator in the whole computational cost of the algorithm.

Keywords: CUDA environment, Genetic algorithm, Mathematical function optimization, GPU architecture
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1. Introduction

It is well known that Genetic Algorithms (GAs) are robust and efficient domain-independent searching techniques for global optimization problems inspired by the Darwinian evolution theory. Most classical GAs include three different operators: selection, mutation and crossover. In each iteration, individuals are evaluated using the objective (or fitness) function and a stopping criteria is also required in order to end the iterative process [5]. However, despite GA is very useful for many practical optimization problems, the execution time can become a limiting factor for some huge problems [4, 6, 9].

There are many possibilities to accelerate a GA code [1]. One of the most widespread strategies proposed in order to reduce the computational cost are the MPI-based parallelization techniques. In this case, a cluster computer is required [19, 20]. Probably, the main problem in the near future will be the platforms heterogeneity of data and applications. However, since the CPU maximum frequency seems to be reached [8], many-core parallel techniques appear to be an interesting option.

During the last decade, Graphic Processing Unit (GPU) has been used for computing acceleration due to the intrinsic vector-oriented design of the chip set. This gave race to a new programming paradigm: the General Purpose Computing on Graphics Processing Units (GPGPU) [11, 16, 12]. In general, the GPGPU acronym is used for all techniques able to develop algorithms extending computer graphics applications but running on a GPU. This paradigm is widely used for a very large range of applications, in computational fluid dynamics (CFD) problems [24], environmental applications [2], advection transport [7], numerical analysis [22], optimization techniques [21, 29], among others.

Later on, the GPGPU programming paradigm was replaced by the Compute Unified Device Architecture (CUDA) in 2007. CUDA has several advantages compared to GPGPU and CPU which includes faster memory sharing and read backs, minimal threads creation overhead, etc. [17]. Since GAs are inherently parallel, the CUDA computing paradigm provides an interesting framework, allowing a strong optimization in terms of processing performance and scalability [15, 25, 23].

In this paper an open source code written in the C++ programming language that allows the optimization of n-dimensional space functions using GA classical metaheuristic techniques is developed. In order to reduce the computational cost of metaheuristic optimization, a parallelization of a classic GA code using a GPU in a CUDA environment is also proposed. Moreover, a sensitivity analysis is performed in order to identify the relative computational cost of each GA operator.

Three different objective functions are employed, which are often used as benchmark functions in optimization problems: De Jong's function, Rastrigin's function and Ackley's function. The global optimum is the same and is known for all test cases: this allows to make comparisons from the GA performance point of view. The paralellized algorithm propoed in this work is capable of optimize n-dimensional functions using CUDA programming paradigm. The obtained results using a GeForce GTX 750 Ti GPU show that the proposed code is a valuable tool for accelerating GAs, reducing its computational cost by about 92%.

The article is structured as follows. Section 2 presents some characteristics of classical GAs as well as the main features of the GA proposed in this work. Section 3 describes the characteristics of the employed hardware and some key aspects of the parallelization strategy using CUDA. The results are shown in Section 4 while the conclusions and final discussions are presented in Section 5.

2. Classical genetic algorithm

Based on the Darwin's evolution theory, genetic algorithms are analogous to the natural selection laws and survival of strongest individuals. Therefore, the individuals with higher fitness in a population have a greater chance of survival than the others. Each individual is a candidate solution of the optimization problem and its quality is assessed by evaluating an objective (or fitness) function which has to be minimized or maximized.

2.1 General characteristics

Some features of classical genetic algorithms are [5, 14]:

- Individuals can be defined with arrays of integer, real or binary numbers as well as a combination.
- The iterative process is usually known as *elitist algorithm*. Thus, the better adapted individuals pass to the next generation without going through the crossover and mutation procedures.
- Continuous functions for individual definition allow imposing conditions on lower and upper bounds of the variables in order to fulfill the objective function domain.
- Partial renewal of the population in order to prevent the saturation with the best individuals. In this step, the mutation procedure is no longer required.

Figure 1.1a shows the pseudocode of the GA developed in this article. Also, the main features of the classical GA are included, being *pop*, *sel* and *shoot* three entities accounting population, selection operator and the randomized shooting procedure, respectively.

2.2 Characterization of individuals

In Fig. 2.1, a generic coordinate x_i is presented. Therefore, each generic coordinate of an individual is composed by *nvdec* genes and *nvbin* chromosomes. The first gene is used for sign assignation. Thus, the negative sign of a generic coordinate is adopted when the first digit is less than five, otherwise the obtained coordinate is positive. Form example, Fig. 2.2 shows the determination of a real number from a binary random matrix.

Although individuals are defined from a binary matrix of $nvbin * nvdec$ elements, in the GA code developed in this work the population is organized as a vector of size $psize * nvbin * nvdec$ in order to optimize memory access. According to this definition of individuals, the domain of the generic coordinate is setting automatically in $x_i \in \ ]-1,1[$. Thus, no further penalty functions are required in order to impose boundary conditions.

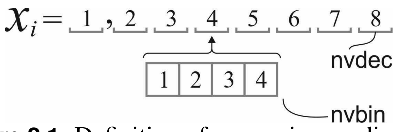

Figure 2.1. Definition of a generic coordinate x_i

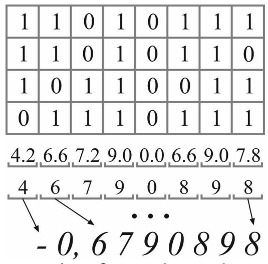

Figure 2.2. Example of a real number determination.

2.3 Initiation

At the beginning, the control variables should be defined, i.e. the population size (*psize*), the maximum number of generations (*ngen*), the number of genes (or variables) for each individual (*nvars*), which in this case is obtained by the product between the total amount of decimal digits plus one, for the sign assignation (*nvdec*) and the adopted quantity of binary digits (*nvbin*). Also, the mutation probability (*mutprob*) and the elite size (*nelit*) should be set at the beginning of the GA.

2.4 Initial population

The initial population is generated by a random algorithm specially designed to fulfil the boundary conditions regarding to the upper and lower bounds of the variables. Individuals are vectors of *nvars* dimension composed by eight integer digits (the first one used for sign assignation). Each integer digit is determined by a four digits binary number (see Fig. 2.1). Therefore, on the optimization problem of two three-dimensional functions (two independent variables), the individual dimension is $nvars = 2 * nvdec * nvbin = 128$ binary digits.

2.5 Cost functions

In a classical GA, the concept of fitness involves testing how “fit” a given individual is in comparison with other individuals using a cost function in order to obtain the best individual of the population. If a GA is used in unconstrained optimization problems, as in the case of this study, the fitness coincided with the actual cost function [26].

In the following, three different cost functions are considered. These functions are widely used to evaluate the performance of new optimization algorithms [15, 27, 28, 23]. Being n the problem dimension, the following cost functions are used:

- The simplest test function, the De Jong’s function, also known as the sphere model. It is continuos, convex and unimodal (see Fig. 2.3a):

$$f_1 = \sum_{i=1}^{n} x_i^2 \ . \tag{2.1}$$

- The Ackley’s function [3], widely used as a multimodal test function. It has many extreme values and a unique absolute minimum (see Fig. 2.3b):

$$f_2 = -a \exp\left[-b\left(\sum_{i=1}^{n} x_i^2/n\right)^{-\frac{1}{2}}\right] - \exp\left(\sum_{i=1}^{n}\cos(c\, x_i)/n\right) + a + \exp(1) \ , \tag{2.2}$$

  where, the adopted values of the constant parameters a, b and c are: 20, 0.2 and 2π, respectively.

- The Rastrigin's function, based on function f_1, Eq. (2.1), with the addition of a cosine modulation to produce many local minima. Therefore, is highly multimodal. However, the location of the minima are regularly distributed (see Fig. 2.3c):

$$f_3 = 10\,n + \sum_{i=1}^{n} x_i^2 - 10\,\cos(2\pi\,x_i)\;. \tag{2.3}$$

All the adopted cost functions share special characteristics. They all are continous functions with monotonic shape and their absolute minimum is perfectly determined at $x_i = 0$. This important feature allows an accurate evaluation of the GA performance analyzed in this work.

2.6 Selection

The selection process is also known as *Tournament* or *Ranking* selection method. In this method, a ranking-based competition is carried out in small groups of individuals (*ngroup*), randomly generated, in which the best of each group is selected according to its cost.

The main disadvantage of this procedure is that the worst individuals have nearly zero probability of being selected. In contrast, the population heterogeneity is better assessed in comparison with the classical *Simple Roulete* selection method [5, **?**].

2.7 Crossover

The crossover procedure is a crucial module for the GA performance. However, as the main purpose of this work is not focused on qualitatively improving convergence behavior of GA, an enhanced crossover technique is not required.

Furthermore, the GA parallelization in a GPGPU presented in this study allows computational cost of classical GA to be greatly reduced. Thus the simple well known *one-point crossover* method is used. All data beyond that point are swapped between the two parent individuals.

2.8 Mutation

Mutation is a genetic operator used to preserve the population diversity, behaving analogously to a biological mutation. Another purpose of this operator is to prevent the algorithm from being trapped into a local optimum, thus increasing GA search capability. The simplest mutation consists in modifying one or more gene values in a chromosome from its initial state according to a user-defined mutation probability during the evolutionary process. This probability should be set to a low value ($< 5\%$). Otherwise, the process could switch to a *primitive random search* [30].

In this research, the *Flip Bit* mutation operator was implemented. This technique is widely used and inverts the bits of a chosen genome according to its mutation probability (i.e. if the genome bit is 1, it is changed to 0 and viceversa).

3. Parallelization strategy in the CUDA environment

The main feature of graphics processing units is the ability to run a common process on a large number of cores working all together over different data. In order to improve the GPGPU paradigm, the CUDA environment has been developed by NVIDIA [18], allowing a more efficient programming. Therefore, there are two important issues that directly affect the performance of CUDA codes. The first one is related to a basic processing element: *threads*. Threads are labelled between 0 and *BlockDim*. The group of threads is called a *block*, and it contains a (recommended) 32 multiple number of threads.

The following aspect to be considered in the CUDA programming style is the device architecture. The minimum unit, where a single thread runs, is the *Streaming Processor* (SP). A *Streaming Multiprocessor* (SM) is a group of SPs.

The parallelization on GPU/CUDA architecture can be done on a *fine-grained* or *coarse-grained* level [18]. In order to reduce the data processing latency, a fine-grained parallelization strategy is adopted [10, 13], in which each data is assigned to each thread. This strategy in general improves GPU performance keeping busy at all times all multiprocessors cores, and consequently, the latency diminishes [25].

The coarsest granularity of synchronization occurs between commands in a queue or stream. Although this does provide synchronization across the entire kernel, it is very slow, as it often even involves a round trip to the CPU for API call completion. Finer-grained barriers can also be used to synchronize control (not data) within a thread block.

Figure 1.1a presents a standard GA flowchart. To qualitatively show the computational cost of each subroutine of this GA, the inner loops schemes required in each function of the GA are also depicted.

The flowchart of the CUDA GA algorithm implemented in this research is shown in Fig. 1.1b. The main difference observed between the classical sequential GA and the fully parallel CUDA-based GA is the absence of the inner loop in each GA subroutine. As will be seen later, this difference introduces a substantial improvement in the computational cost of the GA.

Table 1. Main features of GTX GeForce 750 Ti related to CUDA environment

CUDA Driver Version / Runtime Version:	7.0 / 6.5
Total amount of global memory:	2048 MBytes
(5) Multiprocessors, (128) CUDA Cores/MP:	640 CUDA Cores
L2 Cache Size:	2097152 bytes
Total amount of constant memory:	65536 bytes
Total amount of shared memory per block:	49152 bytes
Warp size:	32
Maximum number of threads per multiprocessor:	2048
Maximum number of threads per block:	1024

Furthermore, in the Appendix A the main CUDA parallelized functions developed in this work are presented. This section was included in order to illustrate a simple way to introduce a CUDA parallelized function in a classical GA.

The graphics card used in this work is the GeForce GTX 750 Ti, and its main features can be obtained from the *deviceQuery* application found in the installation folder of NVIDIA CUDA enviroment [17]. In Table 1, the main characteristics of the GPU employed in this work are summarized.

4. Numerical results

In this section, the numerical results obtained from the computational tests using the CUDA GA algorithm developed in this work are presented.

4.1 Sensitivity analysis on the computational cost of the different operations entailed by GA

Firstly, a sensitivity analysis similar to the proposed in [**?**] is carried out. In this case, the main objective is the determination of the relative computational cost of each GA subroutine. For that purpose, each subroutine is parallelized while keeping the traditional sequential scheme for the remaining of the code. In order to have a homogeneous basis of comparison, $ngen = 100$ and $psize = 320$ are adopted for this computational test as control variables.

Figure 4.1 shows the computational cost of each GA subroutine compared with the fully parallelized CUDA version considering only the De Jong's cost function. It can clearly be seen that the most expensive process is the cost function evaluation (*Func*), followed by the crossing operator (*Crossover*).

Furthermore, it can be observed that the computational cost of the fully parallelized GA (3.54 sec) is only the 7.66% of the computational cost of the sequential GA scheme (46.21 sec). Hence, the parallel implementation allows to achieve an 92% improvement.

The following sensitivity analysis test consists in evaluate the De Jong's cost function with different dimensions of the independent variable (x_i). The results obtained were presented in Fig. 4.2a in order to qualitatively show the non-linear dependence of the total speedup of the fully parallelized GA with the independent variable dimension. Furthermore, the table included in Fig. 4.2b shows mean values as well as standard deviation of both generations and time.

4.2 Effect of population size

The effect on population size on the computational cost of the whole GA optimization process is then analyzed. For that purpose, both sequential and parallel codes are used with different population sizes. Also, in order to optimize the memory access, the chosen population size (*psize*) is proportional to the *warp size* (which is equal to 32 for the graphic card used in this work) in each numerical test.

Figure 4.3 shows a comparison between the numerical results obtained from the classical sequential GA and the fully parallelized CUDA version, for different population sizes. In addition, a third order polynomial fitting equation is plotted. It can be noted that the non-linear coefficients (corresponding to x^3 and x^2) are very small in both cases, therefore a linear function adjustment is enough to fit the numerical test. Hence, the average speedup of the GA can be deduced through the ratio between the values given by the two curves plotted in Fig. 4.3. Thus, the speedup obtained is equal to 14.4, neglecting higher order terms. These results were obtained with a fixed number of iterations ($ngen = 100$) in order to show the average computational cost of the sequential code compared with the fully parallelized implementation.

Actually, in classical GA optimization problems the number of generations required for obtaining the optimal solution is not a deterministic variable. Therefore, optimized design and computational cost should be evaluated in terms of computation time statistics (mean and standard deviation). In this regard, Fig. 4.4 illustrates the behaviour of the mean computation time for each cost function defined in section 2.5 considering different population sizes. It can be observed that the computation time dispersion decreases as the population size increases.

4.3 Computational cost analysis of the optimization process

This section analyzes the computational cost required for the optimization benchmark problem. In order to obtain a probabilistic estimation, one hundred samples were obtained by running the fully parallelized GA code considering the following population sizes: 320, 640, 960, 1280, 1600, 1920, 2240.

In this analysis the number of generations is not set previously. Therefore, a stopping criteria for the iterative procedure must be adopted. Since the global minimum is known in all benchmark functions described in Section 2.5 ($x_i = 0$), the stopping criteria is $|f(x_i)| < tol$, being the adopted tolerance $tol = 10^{-6}$.

Figure 4.5 shows the numerical results of this computational test for the De Jong's function with different population sizes. Similar to the results discussed in the previous section, the points representing each global optimum, obtained in successive runs with the same population size are fairly close to a linear function. Furthermore, it can noted that the population size directly affects the average slope of the linear function. However, the mean values for each population size is better fitted by an exponential function (discontinuous line in Fig. 4.5). Furthermore, the statistics analysis of the fully parallel GA considering different population sizes for each cost functions defined in section 2.5 are presented in Table 2, 3 and 4 for De Jong, Ackley and Rastrigin's cost functions, respectively. Also, in the above mentioned tables $\overline{g}$, Sd_g are the mean and standard deviation of the generations number, respectively, as well as $\bar{t}$, Sd_t are the mean and standard deviation of computation time, respectively.

Table 2. Comparative study of the parallel algorithm considering different population sizes and De Jong's cost functions.

psize	$\overline{g}$	Sd_g	$\bar{t}$	Sd_t
320	1240.13	670.51	40.33	21.75
640	299.23	209.60	20.00	11.49
960	153.28	122.32	12.51	9.75
1280	74.08	54.36	8.38	5.87
1600	47.92	34.81	6.76	4.55
1920	38.54	28.82	6.54	4.44
2240	27.91	18.63	5.79	3.37
2560	40.59	14.67	9.41	3.09
2880	161.00	125.12	13.00	2.80

Table 3. Comparative study of the parallel algorithm considering different population sizes and Ackley's cost functions.

psize	$\overline{g}$	Sd_g	$\bar{t}$	Sd_t
320	2602.90	930.13	84.81	30.28
640	724.31	299.81	39.85	16.39
960	346.78	141.93	27.91	11.29
1280	202.62	83.81	22.21	11.00
1600	146.13	84.14	19.55	10.96
1920	131.43	61.64	20.78	9.46
2240	100.49	47.17	18.91	8.52
2560	98.46	39.04	21.56	8.20
2880	351.00	143.20	28.00	10.10

Table 4. Comparative study of the parallel algorithm considering different population sizes and Rastrigin's cost functions.

psize	$\bar{g}$	Sd_g	$\bar{t}$	Sd_t
320	1850.04	841.17	60.23	27.35
640	512.10	217.98	28.23	11.96
960	227.37	103.85	18.33	8.25
1280	128.94	60.58	14.10	6.68
1600	96.71	36.58	13.07	5.00
1920	73.03	27.07	11.81	4.16
2240	61.40	22.33	11.84	4.02
2560	52.43	15.26	11.86	3.20
2880	126.00	58.50	14.00	3.01

4.4 Effect of cost function complexity on computational cost

In this section, the effect of cost function complexity on the computational cost of the optimization process is studied for each test problem running 100 times the fully parallelized GA code considering the following population sizes: 320, 640, 960, 1280, 1600, 1920 and 2240. The numerical results are summarized in Table 5.

Table 5. Statistical data on the CUDA GA computational cost for the different test problems.

psize	De Jong's function			Ackley's function			Rastrigin's function		
	$\bar{t}$	Sd_t	FA	$\bar{t}$	Sd_t	FA	$\bar{t}$	Sd_t	FA
320	1262.56	710.36	0	2428.01	971.40	4	1840.53	804.28	42
640	299.23	209.60	0	750.02	314.20	0	519.47	205.38	42
960	153.28	122.32	0	338.75	154.05	0	232.11	86.19	28
1280	78.82	69.34	0	208.82	118.89	0	134.49	66.45	23
1600	47.92	34.81	0	162.56	104.04	0	100.24	41.52	21
1920	38.54	28.82	0	126.61	50.95	0	74.65	32.20	17
2240	27.91	18.63	0	92.05	50.93	0	59.47	19.50	13

$\bar{t}$, Sd_t: mean and standard deviation of computation time, respectively.
FA: Failed attempts.

Figure 4.6 shows the mean lapsed computational time required for the convergence of the fully parallelized GA code against the generations numbers considering each cost function adopted in Section 2.5.

It can be seen that the De Jong's function optimization is faster than the others. Although the optimization of the Rastrigin's function requires less CPU time than the Ackley's problem, it should be noted that this test case has a much higher number of failed attempts, where the target global optimum could not be reached (see Table 5).

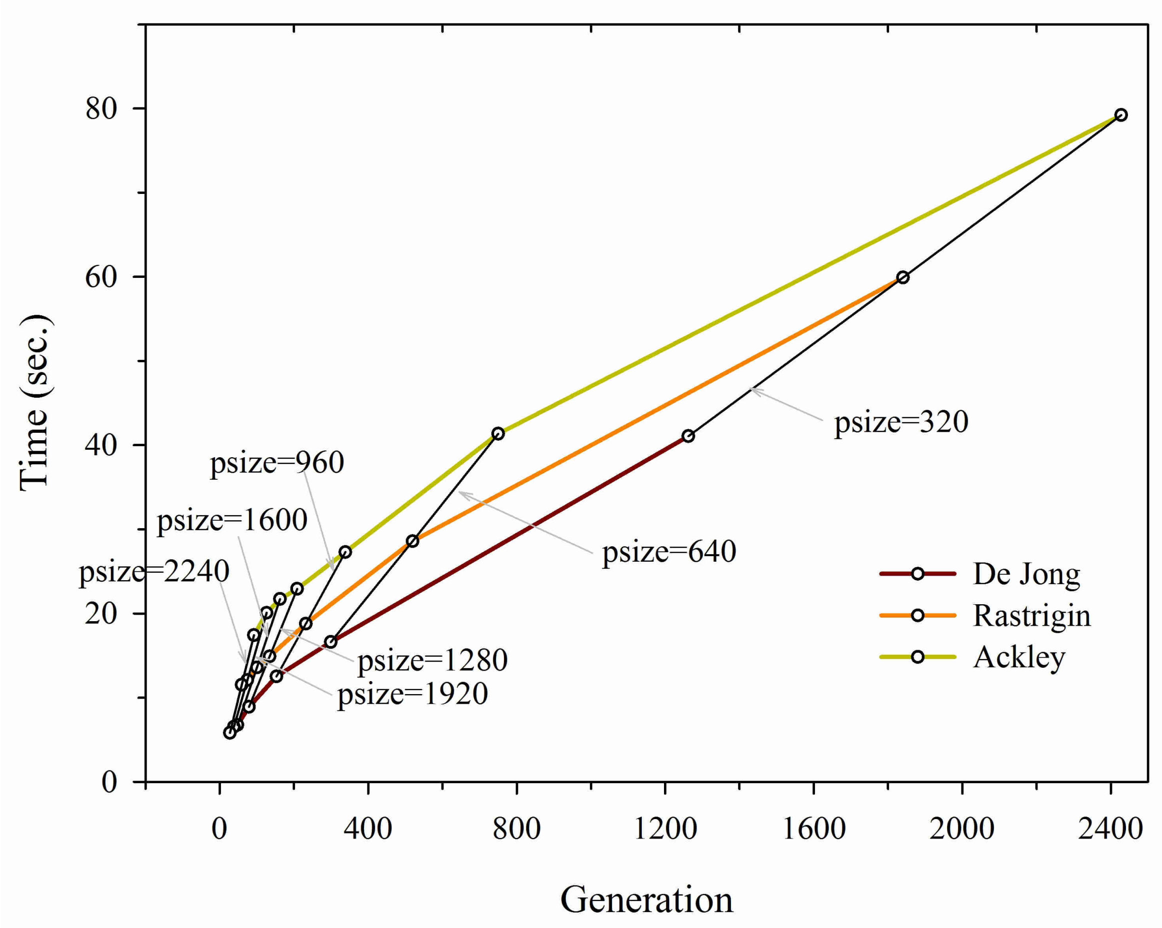

Figure 4.6. Comparison of computational cost of parallel CUDA GA for the different test problems.

5. Conclusions

A general GA code for global optimization implemented in the GPU CUDA environment was developed in this study. The code is capable of optimize n-dimensional functions which are frequently used as benchmark problems. Using CUDA programming paradigm, an interesting increment in the convergence rate of about 92% compared with the sequential code is obtained, and also, the speedup reported is about 13.05.

Furthermore, a sensitivity analysis performed on each GA operator allowed to determine its relative influence on the total computational cost of the algorithm and enables the programmer to focus on certain routines of the GA when the goal is not to fully parallelize the GA. The obtained numerical results show the efficiency and scalability of the proposed algorithm for the chosen cost functions. The presented GA is very general and may be extended to more complex problems or real-life applications in an straightforward manner, by rewriting the objective function.

6. Acknowledgment

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1. Parallelized functions

Main device functions parallelized in CUDA framework are presented in this appendix.

A.1 InitPop

This is the first device function, included in any classical GA, with the aim of generate the random initial population

```
__global__ void InitPop(int *d_Pop, int nvars, curandState* globalState){
int it = blockDim.x * blockIdx.x + threadIdx.x;
for(int i=0; i<nvars ; i++) {
float k = generate(globalState, i+it)*1;
d_Pop[it*nvars + i] = lroundf(k);
}
}
```

A.2 Func_n

This device subroutine contain precisely the cost functions adopted in this article for n-dimensional generic variables x_i.

```
__global__ void Func_n(float *d_Sol, int *d_Pop, int nvbin, int nvdec, int ndim){
int it = blockDim.x * blockIdx.x + threadIdx.x;
int function = 3 ; // 1=DeJong, 2=Ackley , 3= Rastrigin
int ipop=0, two=2, ten=10, bin[4];
float x=0, y=0, X=0, Y=0, sum = 0, PI=3.141592653, xi[2], lim = 9/(powf(2,nvbin)-1);
for(int ix=0; ix<ndim; ix++){
for (int inum=1; inum<nvdec; inum++){
for(int i = 0; i<nvbin; i++) bin[i] = d_Pop[it*nvbin*nvdec*ndim + nvbin*nvdec*ix +
    nvbin*inum + i];
X = lroundf(BinDec(bin, nvbin)*lim); x = x + powf(ten,(0-inum))*X;
}
xi[ix] = x, x=0, X=0;
}

for(int i=0; i<nvbin; i++) bin[i]=d_Pop[it*nvbin*nvdec*ndim+i];
if(BinDec(bin,nvbin) < (powf(two,nvbin)-1)*0.5) x = (-1)*x;

d_Sol[it]=0;
float a=20, b=0.2, c=2*PI, d=10, x1=0, x2=0;
switch(function) {
case 1: // De Jong's function
for(int ix=0; ix<ndim; ix++) d_Sol[it] = d_Sol[it] + xi[ix]*xi[ix];
break;
case 2: // Ackley's function
for(int ix=0; ix<ndim; ix++){
x1 = x1 + xi[ix]*xi[ix]; x2 = x2 + cos(c*xi[ix]);
}
d_Sol[it] = -a*expf(-b*sqrt(x1/ndim)) - expf(x2/ndim) + a + expf(1);
break;
case 3: // Rastrigin's function
for(int ix=0; ix<ndim; ix++) x1 = x1+xi[ix]*xi[ix]-d*cos(two*PI*xi[ix]);
d_Sol[it] = d*ndim + x1;
break;
}
}
```

A.3 GroupSelection

This device function generate the group selection for the Ranking selection method adopted in this classical genetic algorithm.

```
__global__ void GroupSelection(int *sel, int *group, int gsize, int ngroup){
int it = blockDim.x * blockIdx.x + threadIdx.x;
if(it<ngroup){
sel[it] = group[it*gsize];
for(int i=0; i<gsize; i++){
if (sel[it] < group[it*gsize + i] ) sel[it] = group[it*gsize + i];
}
```

```
}
}
```

A.4 CrossoverSingle

The crossover operation is easily performed in this device function.

```
__global__ void CrossoverSingle(int *NPop, int *Male, int *Female, int *Pop, int psize, int
    nvars, int ndim){
int it = blockDim.x * blockIdx.x + threadIdx.x ;
if(threadIdx.x < nvars/2){
NPop[it*2] = Pop[Male[blockIdx.x]*nvars + threadIdx.x*2];
NPop[it*2+1] = Pop[Female[blockIdx.x]*nvars + threadIdx.x*2+1];
}else{
NPop[it*2] = Pop[Female[blockIdx.x]*nvars + (threadIdx.x - nvars/2)*2];
NPop[it*2+1] = Pop[Male[blockIdx.x]*nvars + (threadIdx.x - nvars/2)*2+1];
}
}
```

A.5 Mutation

The mutation operation is performed in this device function.

```
__global__ void Mutation(int *Pop, float mutp, int nvec, int nvars, curandState*
    globalState){
int it = blockDim.x * blockIdx.x + threadIdx.x;
if (it < nvec) {
float ran = generate(globalState, it)*(1.0);
if(ran<mutp){
int ivar = lroundf(generate(globalState, it)*(nvars-0.5));
if(Pop[it*nvars+ivar]==0){
Pop[it*nvars+ivar] = 1;
}else{
Pop[it*nvars+ivar] = 0;
}
}
}
}
```

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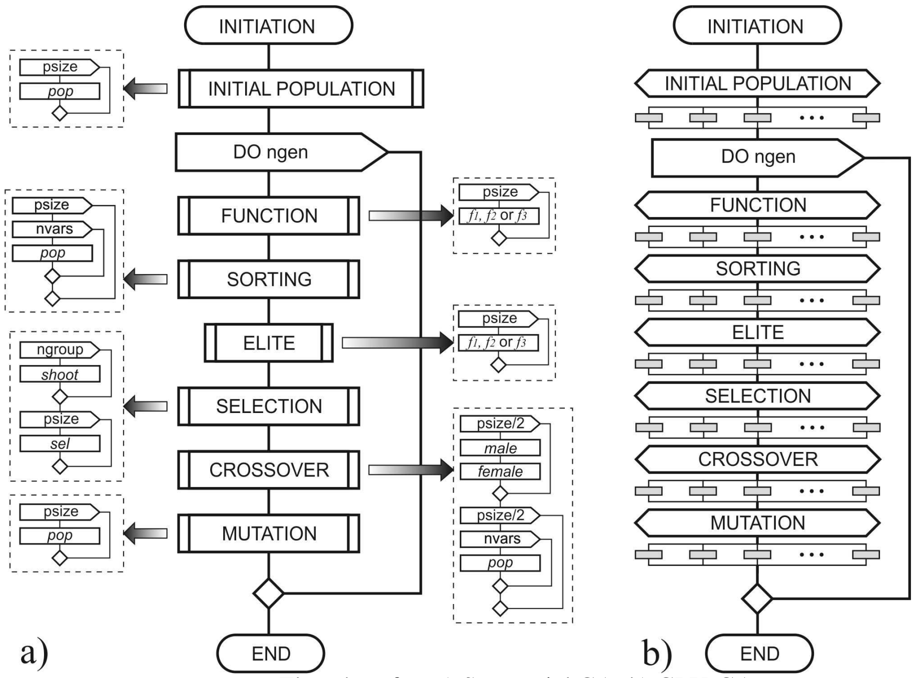


Figure 1.1. Flowchart for: a) Sequential GA; b) GPU-GA.

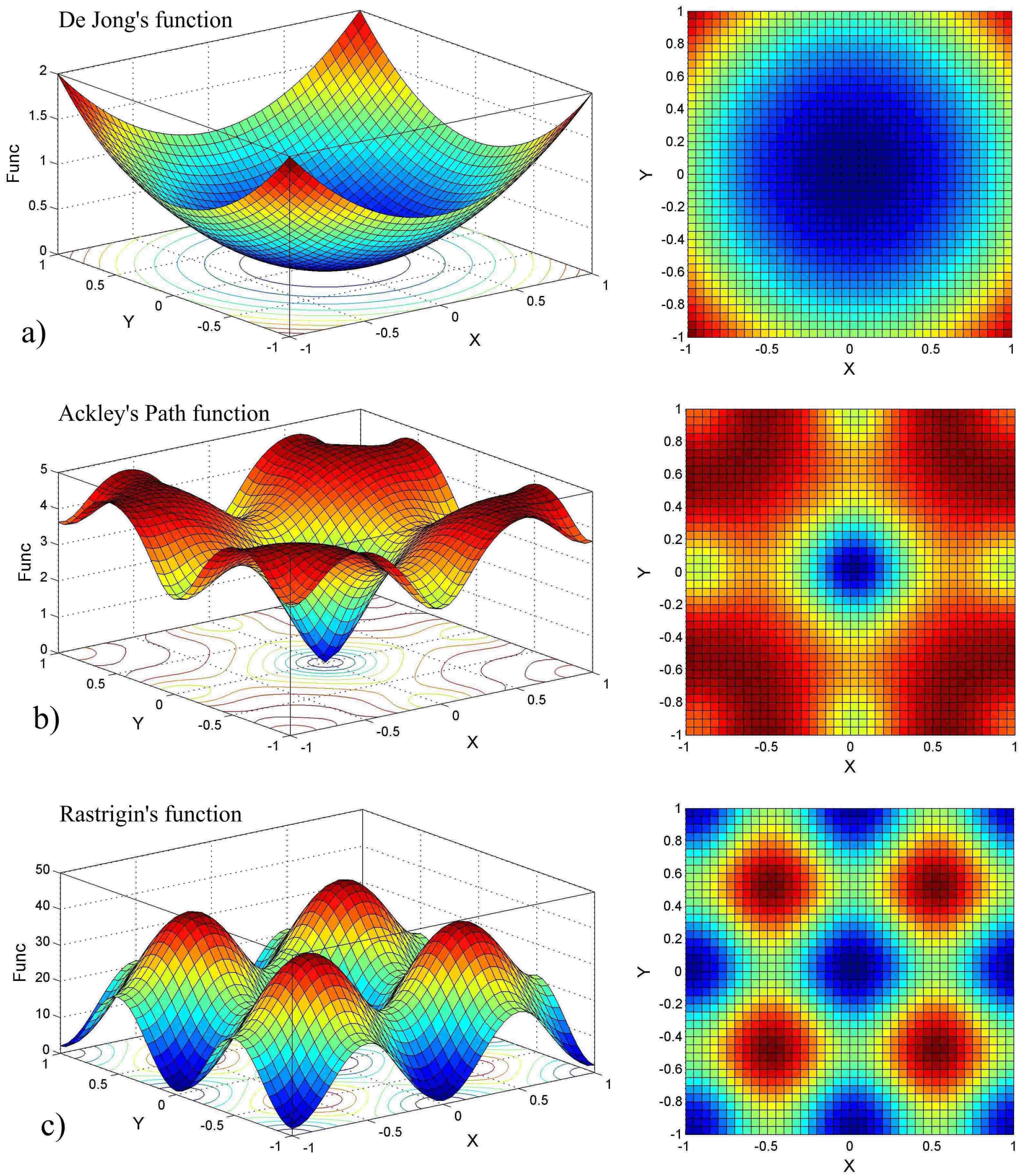


Figure 2.3. Representation of the cost function for two-dimensional optimization ($n = 2$): a) De Jong’s function; b) Ackley’s function; c) Rastrigin’s function.

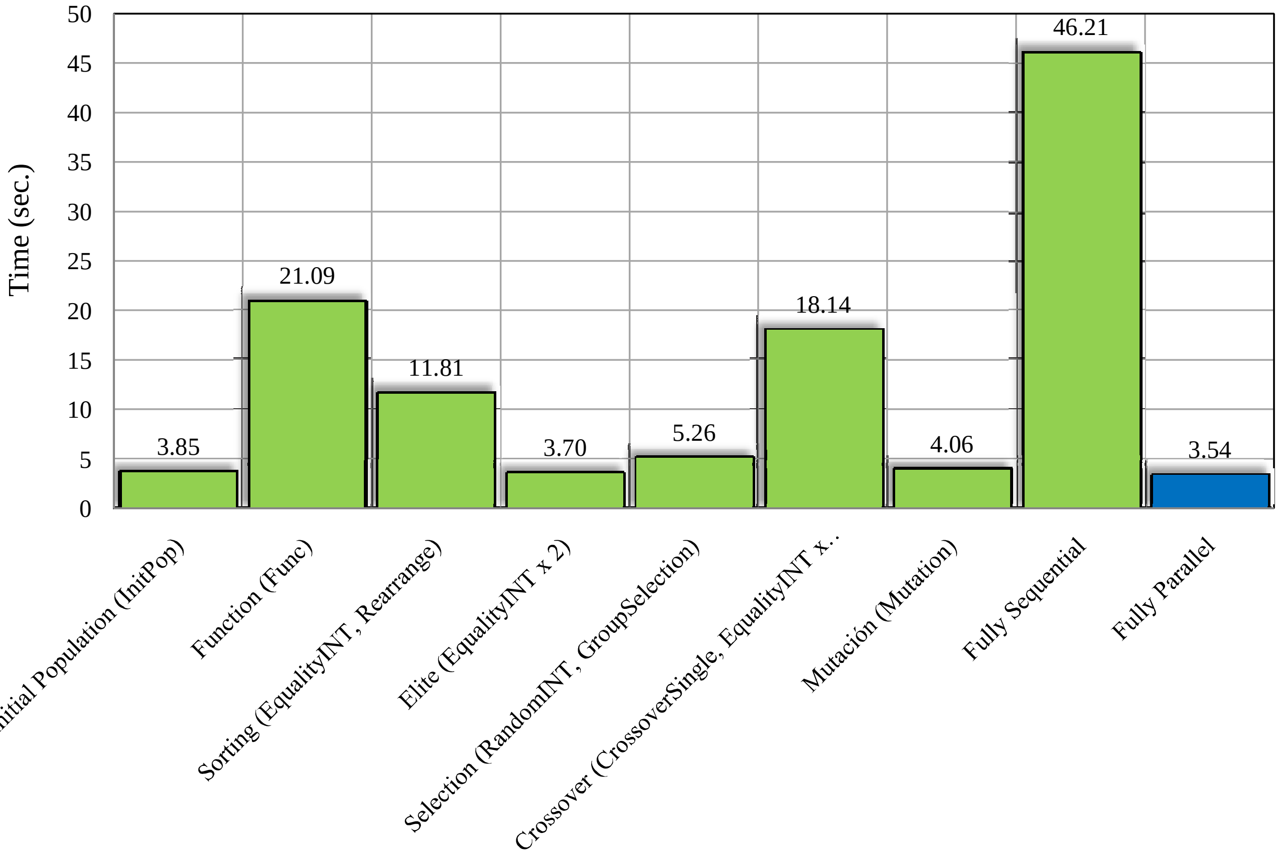
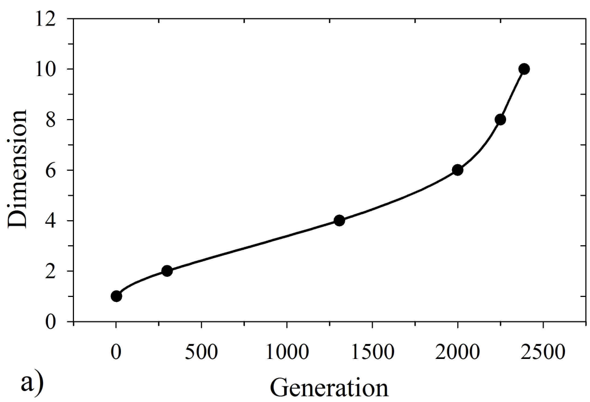


Figure 4.1. Sensitivity analysis on the computational cost of classical GA operations for De Jong's cost function ($ngen = 100$, $psize = 320$)



ndim	ngen	desvio	time	desvio
1	2,48	0,76	0,23	0,08
2	299,23	209,60	16,59	11,49
4	1307,49	517,97	72,99	28,76
6	2000,46	698,60	112,71	39,11
8	2250,00	1278,36	116,34	73,38
10	2389,49	1427,61	138,93	82,31

b)

Figure 4.2. Influence of the variable dimension on speedup of the fully parallelized GA: a) Qualitative representation; b) Data table

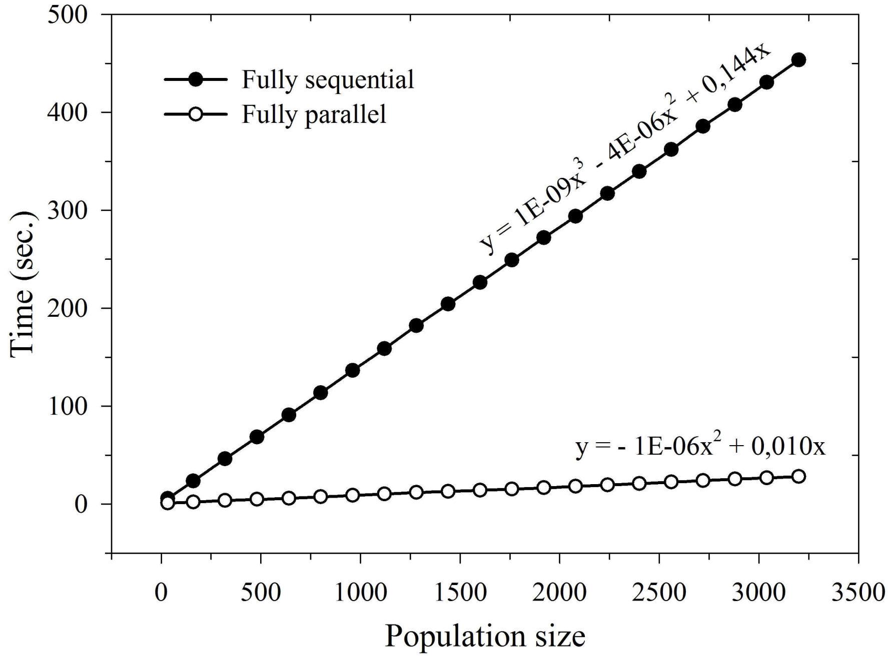
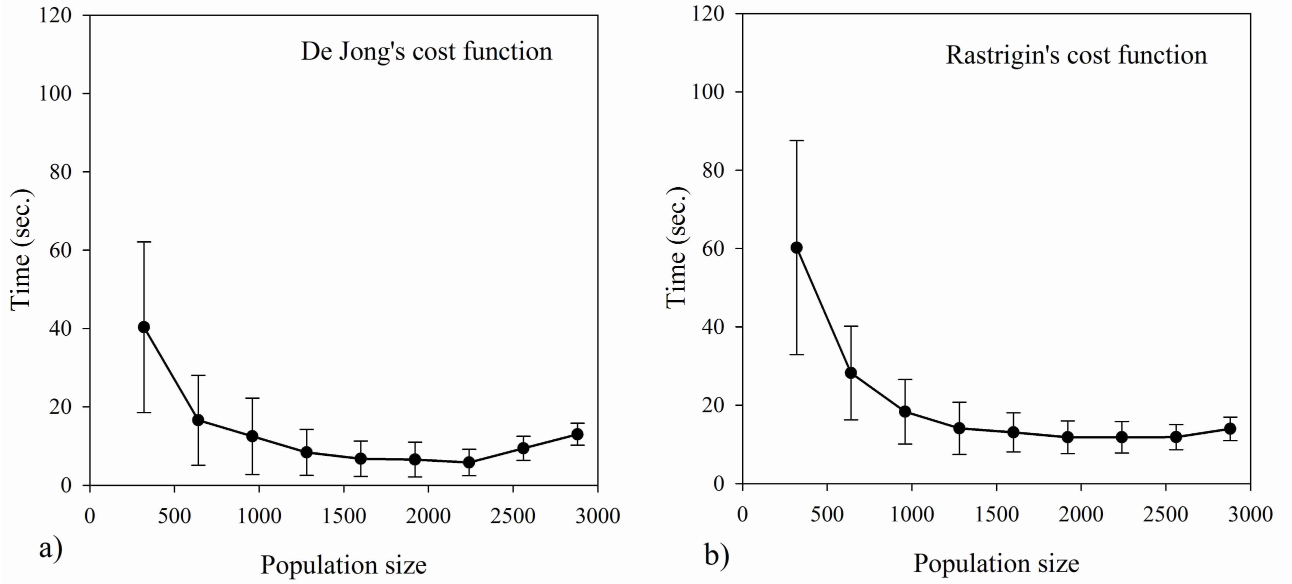
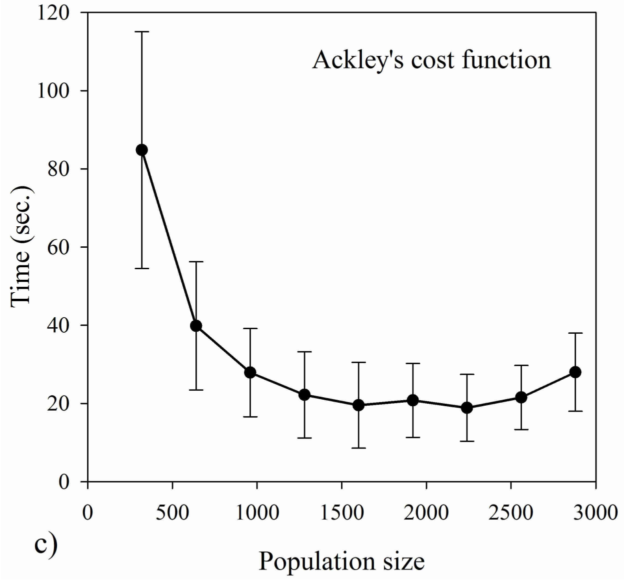


Figure 4.3. Comparison of the computational cost of both the fully parallelized and sequential GA implementations, assuming the De Jong's cost function with different population sizes.





psize	De Jong's function		Ackley's function		Rastrigin's function	
	time	desvio	time	desvio	time	desvio
320	40.33	21.75	84.81	30.28	60.23	27.35
640	16.59	11.49	39.85	16.39	28.23	11.96
960	12.51	9.75	27.91	11.29	18.33	8.25
1280	8.38	5.87	22.21	11.00	14.10	6.68
1600	6.76	4.55	19.55	10.96	13.07	5.00
1920	6.54	4.44	20.78	9.46	11.81	4.16
2240	5.79	3.37	18.91	8.52	11.84	4.02
2560	9.41	3.09	21.56	8.20	11.86	3.20
2880	13.02	2.80	28.04	9.99	14.00	3.01

d)

Figure 4.4. Mean speedup of the fully parallelized GA code vs population size. a) De Jong's function, b) Rastrigin's function, c) Ackley's function, d) Mean values and standard deviations.

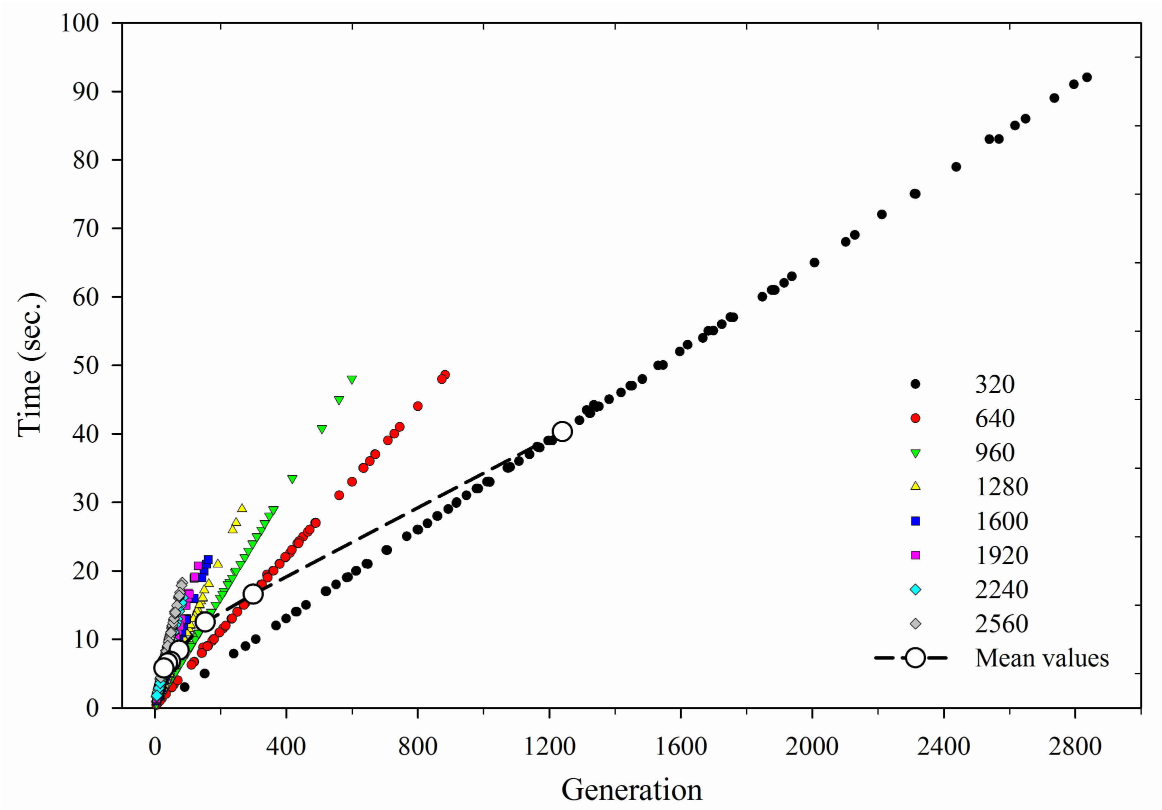


Figure 4.5. Actual computational cost of the fully parallelized CUDA GA implementation for the De Jong's function with different population sizes.

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Local convergence for composite Chebyshev-type methods

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Abstract
We replace Chebyshev's method for solving equations requiring the second derivative by a Chebyshev-type second derivative free method. The local convergence analysis of the new method is provided using hypotheses only on the first derivative in contrast to the Chebyshev method using hypotheses on the second derivative. This way we extend the applicability of the method. Numerical examples are also used to test the convergence criteria and to obtain error bounds and also the radius of convergence.

Keywords: Chebyshev method, Newton method, Local convergence, Fréchet derivative, Divided differences
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1. Introduction

Let $\mathscr{B}_1, \mathscr{B}_2$ be Banach spaces, $\Omega \subseteq \mathscr{B}_1$ be nonempty and convex set. Numerous problems can be written in the form

$$F(x) = 0, \tag{1.1}$$

using mathematical modeling, where $F : \Omega \longrightarrow \mathscr{B}_2$ is a continuously Fréchet differentiable operator. Analytical solutions x_* are not easy or impossible to find in general for equation (1.1). This leads researchers and practitioners to use iterative methods to generate a sequence approximating x_*.

Newton's method defined for $x_0 \in \Omega$ and for each $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \tag{1.2}$$

is the most popular method for solving equation (1.1). Newton's method converges quadratically under certain conditions [1, 2, 3, 11]. Higher convergence order methods have also been suggested such as the cubically convergent Chebyshev's method defined for each $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - (I + B_n)F'(x_n)^{-1}F(x_n), \tag{1.3}$$

where $B_n = F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n)$. If one considers a system of k equations in k unknowns, then $F'(x)$ is a matrix with k^2 evaluations whereas $F''(x)$ requires $\frac{k^2(k+1)}{2}$ evaluations. That is Chebyshev's method is expensive to implement. Moreover, the convergence requires conditions of the form [6, 7, 8, 9, 10, 11, 12]

$$\|F'(x^*)^{-1}F''(x)\| \leq a \text{ for each } x \in \Omega$$

and

$$\|F'(x^*)^{-1}(F''(x)-F''(y))\| \leq b \text{ for each } x,y\in\Omega.$$

These conditions limit the applicability of Chebyshev's method. As a motivational example, let us define function F on $X=[-\frac{1}{2},\frac{5}{2}]$ by

$$F(x)=\begin{cases} x^3\ln x^2+x^5-x^4, & x\neq 0\\ 0, & x=0\end{cases}$$

Choose $x^*=1$. We have that

$$\begin{aligned} F'(x) &= 3x^2\ln x^2+5x^4-4x^3+2x^2,\ F'(1)=3,\\ F''(x) &= 6x\ln x^2+20x^3-12x^2+10x\\ F'''(x) &= 6\ln x^2+60x^2-24x+22.\end{aligned}$$

Then, obviously function F does not have bounded third derivative in X. That is why we suggest the method defined for each $n=0,1,2,\dots$ by

$$\begin{aligned} y_n &= x_n-F'(x_n)^{-1}F(x_n)\\ z_n &= y_n-F'(x_n)^{-1}F(y_n)\\ x_{n+1} &= z_n-C_nF'(x_n)^{-1}F(z_n), \end{aligned}\tag{1.4}$$

where $C_n=2I-F'(x_n)^{-1}[z_n,y_n;F]$ and $[.,.;F]:\Omega\times\Omega\longrightarrow\mathscr{B}_2$ is a divided difference of order one.

The study of convergence of iterative algorithms is usually centered into two categories: semi-local and local convergence analysis. The semi-local convergence is based on the information around an initial point, to obtain conditions ensuring the convergence of these algorithms, while the local convergence is based on the information around a solution to find estimates of the computed radii of the convergence balls. Local results are important since they provide the degree of difficulty in choosing initial points.

Our local convergence analysis uses only hypotheses on the first Fréchet derivative, whereas the order of convergence is established using (COC) and (ACOC) (see Remark 2.2). Hence, we expand the applicability of method (1.4).

Section 2 contains the local convergence of method (1.4), whereas in the concluding Section 3, we provide numerical examples.

2. Local convergence

Let $\varphi_0: I_0\longrightarrow I_0$ be a continuous and increasing function with $\varphi_0(0)=0$, where $I_0=\mathbb{R}_+\cup\{0\}$. Suppose that equation

$$\varphi_0(t)=1. \tag{2.1}$$

has at least one positive solution. Denote by ρ_0 the smallest such solution. Let $\varphi:[0,\rho_0)\longrightarrow I_0$ be a continuous and increasing function with $\varphi(0)=0$. Define functions g_1 and h_1 on $[0,\rho_0)$ by

$$g_1(t)=\frac{\int_0^1\varphi((1-\theta)t)d\theta}{1-\varphi_0(t)}$$

and

$$h_1(t)=g_1(t)-1.$$

We have that $h_1(0)=-1$ and $h_1(t)\longrightarrow+\infty$ as $t\longrightarrow\rho_0^-$. It then follows from the intermediate value theorem that equation $h_1(t)=0$ has at least one solution in the interval $(0,\rho_0)$. Denote by r_1 the smallest such solution.

Suppose that

$$\varphi_0(g_1(t)t)=1. \tag{2.2}$$

has at least one positive solution. Denote by ρ_1 the smallest such solution. Let $\varphi_1 : [0,\rho_0) \longrightarrow I_0$ be continuous and increasing function. Moreover, define functions g_2 and h_2 on $[0,\rho)$ by

$$\begin{aligned} g_2(t) &= \left[\frac{\int_0^1 \varphi((1-\theta)g_1(t)t)d\theta}{1-\varphi_0(g_1(t)t)}\right. \\ &\quad \left.+\frac{(\varphi(t)+\varphi_0(g_1(t)t))\int_0^1 \varphi_1(\theta g_1(t)t)d\theta}{(1-\varphi_0(g_1(t)t))(1-\varphi_0(t))}\right] g_1(t) \end{aligned}$$

and

$$h_2(t) = g_2(t) - 1,$$

where $\rho = \min\{\rho_0,\rho_1\}$. We get that $h_2(0) = -1$ and $h_2(t) \longrightarrow +\infty$ as $t \longrightarrow \rho^-$. Denote by r_2 the smallest solution of equation $h_2(t) = 0$ in the interval $(0,\rho)$. Let $\varphi_2 : [0,\rho)\times[0,\rho) \longrightarrow I_0$ be a continuous and increasing function. Furthermore, define functions g_3 and h_3 on the interval $[0,\rho)$ by

$$\begin{aligned} g_3(t) &= \left[1+\left(1+\frac{\varphi_0(t)+\varphi_2(g_2(t)t,g_1(t)t)}{1-\varphi_0(t)}\right)\right. \\ &\quad \left.\times\frac{\int_0^1 \varphi_1(\theta g_2(t)t)d\theta}{1-\varphi_0(t)}\right] g_2(t) \end{aligned}$$

and

$$h_3(t) = g_3(t) - 1.$$

We obtain $h_3(t) = -1$ and $h_3(t) \longrightarrow +\infty$ as $t \longrightarrow \rho^-$. Denote by r_3 the smallest solution of equation $h_3(t) = 0$ in $(0,\rho)$. Define the radius of convergence r by

$$r = \min\{r_i\}\ , i = 1,2,3. \tag{2.3}$$

Then, for each $t \in [0,r)$ we have

$$0 \le \varphi_0(t) < 1 \tag{2.4}$$

$$0 \le \varphi_1(g_1(t)) < 1 \tag{2.5}$$

and

$$0 \le g_i(t) < 1. \tag{2.6}$$

Let $B(u,\tau), \bar{B}(u,\tau)$ stand for the open and closed balls in $\mathscr{B}_1$, respectively with center $u \in \mathscr{B}_1$ and of radius $\tau > 0$.

The local convergence of method (1.2) is based on the conditions (A):

(a1) $F : \Omega \subset \mathscr{B}_1 \longrightarrow \mathscr{B}_2$ is a continuously Fréchet-differentiable operator and $[.,.;F] : \Omega\times\Omega \longrightarrow \mathscr{L}(\mathscr{B}_1,\mathscr{B}_2)$ a divided difference of order one for F.

(a2) There exists $x_* \in \Omega$ such that $F(x_*) = 0$ and $F(x_*)^{-1} \in \mathscr{L}(\mathscr{B}_2,\mathscr{B}_1)$.

(a3) There exist a continuous and increasing function $\varphi_0 : I_0 \longrightarrow I_0$ such that for each $x \in \Omega$,

$$\|F'(x_*)^{-1}(F'(x)-F'(x_*))\| \le \varphi_0(\|x-x_*\|).$$

Set $\Omega_0 = \Omega \cap \bar{U}(x_*,\rho_0)$ where ρ_0 is given by (2.1).

(a4) There exist functions $\varphi : [0,\rho_1) \longrightarrow I_0, \varphi_1 : [0,\rho_1) \longrightarrow I_0,\ \varphi_2 : [0,\rho_1)^2 \longrightarrow I_0$ continuous, increasing with $\varphi(0) = \varphi_2(0,0) = 0$ such that for each $x,y,z \in \Omega_0$

$$\|F'(x_*)^{-1}(F'(x)-F'(y))\| \le \varphi(\|x-y\|),$$

$$\|F'(x_*)^{-1}F'(x)\| \le \varphi_1(\|x-x_*\|)$$

and

$$\|F'(x_*)^{-1}([y,z;F]-F'(x_*))\| \le \varphi_2(\|x-x_*\|,\|z-x_*\|).$$

(a5) There exist $\bar{r} \geq r$ such that

$$\int_0^1 \varphi_0(\theta \bar{r}) d\theta < 1.$$

Set $\Omega_1 = \Omega \cap \bar{B}(x_*, \bar{r})$. Next, the local convergence analysis of method (1.2) follows:

Theorem 2.1. *Suppose that the conditions (A) hold. Then, sequence $\{x_n\}$ generated for $x_0 \in B(x_*, r) - \{x_*\}$ by method (1.2) is well defined in $B(x_*, r)$, remains in $B(x_*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x_*, so that*

$$\|y_n - x_*\| \leq g_1(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| < \rho \tag{2.7}$$

$$\|z_n - x_*\| \leq g_2(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| \tag{2.8}$$

and

$$\|x_{n+1} - x_*\| \leq g_3(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|, \tag{2.9}$$

where functions $g_i, i = 1, 2, 3$ are defined previously and the radius r is given in (2.3). Moreover, x_* is the only solution of equation $F(x) = 0$ in Ω_1.

Proof. Inequations (2.7)-(2.9) are shown using mathematical induction. First, we shall show that iterates $\{x_n\}$ are well defined and inequation (2.7)-(2.9) are satisfied for $n = 0$. Let $x \in B(x_*, r) - \{x_*\}$. Using (2.1), (2.3) and (2.4), we have in turn that

$$\|F'(x^*)^{-1}(F'(x) - F'(x_*))\| \leq \varphi_0(\|x - x_*\|) \leq \varphi_0(r) < 1 \tag{2.10}$$

which together with the Banach Lemma on invertible operators [1, 4, 11] imply that $F'(x)^{-1} \in \mathscr{L}(\mathscr{B}_2, \mathscr{B}_1)$ and

$$\|F'(x)^{-1}F'(x^*)\| \leq \frac{1}{1 - \varphi_0(\|x - x_*\|)}. \tag{2.11}$$

Notice that (2.11) holds for $x = x_0$, since $x_0 \in B(x_*, r)$ and y_0, z_0 are well defined by the first and second sub-step of method (1.2) for $n = 0$. We have by the first substep of method (1.2) for $n = 0$

$$\begin{aligned} & y_0 - x_* \\ = \;& x_0 - x_* - F'(x_0)^{-1}F(x_0) \\ = \;& F'(x_0)^{-1}\int_0^1 (F'(x_* + \theta(x_0 - x_*)) - F'(x_0))(x_0 - x_*)d\theta. \end{aligned} \tag{2.12}$$

By (a1)-(a4), (2.3), (2.6) (for $i = 1$), (2.11) and (2.12), we get in turn that

$$\begin{aligned} & \|y_0 - x_*\| \\ \leq \;& \|F'(x_*)^{-1}F'(x_*)\| \|\int_0^1 F'(x_*)^{-1}(F'(x_* + \theta(x_0 - x_*)) - F'(x_0))d\theta\| \|x_0 - x_*\| \\ \leq \;& \frac{\int_0^1 \varphi((1-\theta)\|x - x_0)\|)d\theta}{1 - \varphi_0(\|x_0 - x_*\|)}\|x_0 - x_*\| \\ \leq \;& \|x_0 - x_*\| < r, \end{aligned} \tag{2.13}$$

which shows (2.7) for $n = 0, y_0 \in B(x_*, r)$ and (2.11) hold for $x = y_0$. That is

$$\begin{aligned} \|F'(y_0)^{-1}F'(x_*)\| \;&\leq\; \frac{1}{1 - \varphi_0(\|y_0 - x_*\|)} \\ &\leq\; \frac{1}{1 - \varphi_0(g_1(\|x_0 - x_*\|)\|x_0 - x_*\|)}. \end{aligned} \tag{2.14}$$

We can write

$$F(x_0) = F(x_0) - F(x_*) = \int_0^1 F'(x_* + \theta(x_0 - x_*))(x_0 - x_*)d\theta. \tag{2.15}$$

In view of (a4) (second condition) and (2.15), we obtain

$$\begin{aligned}\|F'(x_*)^{-1}F(x_0)\| &= \|\int_0^1 F'(x_*)^{-1}F'(x_*+\theta(x_0-x_*))d\theta(x_0-x_*)\| \\ &\le \int_0^1 \varphi_1(\theta\|x_0-x_*\|)d\theta\|x_0-x_*\|. \end{aligned} \tag{2.16}$$

Then, using the second substep of method (1.2), (2.3), (2.6) (for $i=2$), (2.13) (for $x_0=y_0$), (2.14) and (2.16) (for $y_0=x_0$), we have in turn from

$$\begin{aligned} z_0-x_* &= y_0-x_*-F'(y_0)^{-1}F(y_0) \\ &\quad +F'(y_0)^{-1}(F'(x_0)-F'(y_0))F'(x_0)^{-1}F(y_0), \end{aligned} \tag{2.17}$$

so

$$\begin{aligned} \|z_0-x_*\| &\le \|y_0-x_*\|+\|F'(y_0)^{-1}F'(x_*)\| \\ &\quad \times[\|F'(x_*)^{-1}(F'(x_0)-F'(x_*))\|+\|F'(x_*)^{-1}(F'(y_0)-F'(x_*))\|] \\ &\quad \times\|F'(x_0)^{-1}F'(x_*)\|\|F'(x_*)^{-1}F(y_0)\| \\ &\le \left[\frac{\int_0^1\varphi((1-\theta)\|y_0-x_*\|)d\theta}{1-\varphi_0(\|y_0-x_*\|)}\right. \\ &\quad \left.+\frac{(\varphi_0(\|x_0-x_*\|)+\varphi_0(\|y_0-x_*\|))\int_0^1\varphi_1(\theta\|y_0-x_*\|)d\theta}{(1-\varphi_0(\|y_0-x_*\|))(1-\varphi_0(\|x_0-x_*\|))}\right]\|y_0-x_*\| \\ &\le g_2(\|x_0-x_*\|)\|x_0-x_*\| \le \|x_0-x_*\| < r \end{aligned} \tag{2.18}$$

which shows (2.8) for $n=0$ and $z_0\in B(x_*,r)$. The third substep of method (1.4) together with (2.3), (2.6) (for $i=3$), (2.15) (for $x_0=z_0$), the third hypothesis in (a4) and (2.18), we get

$$\begin{aligned} \|x_1-x_*\| &\le \|z_0-x_*\| \\ &\quad +\|F'(x_0)^{-1}(2F'(x_0)-[z_0,y_0;F])F'(x_0)^{-1}F(z_0)\| \\ &\le \|z_0-x_*\|+[1+\|F'(x_0)^{-1}F'(x_*)[(F'(x_*)^{-1}(F'(x_0)-F'(x_*)) \\ &\quad +F'(x_*)^{-1}(F'(x_*)-[z_0,y_0;F])]\|]\|F'(x_0)^{-1}F'(x_*)\| \\ &\quad \times\|F'(x_*)^{-1}F(z_0)\| \\ &\le \left[1+\left(1+\frac{\varphi_0(\|x_0-x_*\|)+\varphi_2(\|z_0-x_*\|,\|y_0-x_*\|)}{1-\varphi_0(\|x_0-x_*\|)}\right)\right. \\ &\quad \left.\frac{\int_0^1\varphi_1(\theta\|z_0-x_*\|)d\theta}{1-\varphi_0(\|x_0-x_*\|)}\right]\|z_0-x_*\| \\ &\le g_3(\|x_0-x_*\|)\|x_0-x_*\| \le \|x_0-x_*\| < r, \end{aligned} \tag{2.19}$$

which shows (2.9) and $z_0\in B(x_*,r)$. The induction for inequation (2.7)-(2.9) is completed replacing x_0,y_0,z_0,x_1 by x_m,y_m,z_m,x_{m+1} in the preceding estimates. We then also have that

$$\|x_{m+1}-x_*\| \le q\|x_m-x_*\| < r \tag{2.20}$$

where $q=g_3(\|x_0-x^*\|)\in[0,1)$, leading to $\lim_{m\longrightarrow+\infty}x_m=x_*$ and $x_{m+1}\in B(x_*,r)$. The, uniqueness part is shown as follows: Let $Q=\int_0^1F'(x_*+\theta(y_*-x_*))d\theta$ for some $y_*\in\Omega_1$ with $F(y_*)=0$. The condition (a5) gives

$$\|F'(x_*)^{-1}(Q-F'(x_*))\| \le \int_0^1\varphi_0(\theta\|y_*-x_*\|)d\theta \le \int_0^1\varphi_0(\theta\bar{r})d\theta < 1, \tag{2.21}$$

so $Q^{-1}\in\mathscr{L}(\mathscr{B}_2,\mathscr{B}_1)$ and from the identity

$$0=F(y_*)-F(x_*)=Q(y_*-x_*), \tag{2.22}$$

we deduce that $x_*=y_*$.

□

Remark 2.2. *1. The second condition in (a4) can be dropped, since this condition follows from (a3), if we set*

$$\varphi_1(t) = 1 + \varphi_0(t).$$

2. *The results obtained here can be used for operators F satisfying autonomous differential equations [11] of the form*

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

3. *The radius r was shown by us to be the convergence radius of Newton's method [1, 2, 3, 4, 5]*

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{2.23}$$

under the conditions (a1)–(a4) for $\varphi_0(t) = L_0 t$ and $\varphi(t) = Lt$. It follows from the definition of r that the convergence radius r_1 of the method (1.4) cannot be larger than the convergence radius r_1 of the second order Newton's method (2.23). As already noted in [11] r_1 is at least as large as the convergence ball given by Rheinboldt [11]

$$r_R = \frac{2}{3L}. \tag{2.24}$$

In particular, for $L_0 < L$ we have that

$$r_R < r_1$$

and

$$\frac{r_R}{r_1} \to \frac{1}{3} \text{ as } \frac{L_0}{L} \to 0.$$

That is our convergence ball r_1 is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [13].

4. *It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [14]. Moreover, we can compute the computational order of convergence (COC) defined by*

$$\xi = \ln\left(\frac{|x_{n+1} - x_*|}{|x_n - x_*|}\right) / \ln\left(\frac{|x_n - x_*|}{|x_{n-1} - x_*|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}\right) / \ln\left(\frac{|x_n - x_{n-1}|}{|x_{n-1} - x_{n-2}|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F.

3. Numerical examples

In this Section the divided difference is given by $[x, y; F] = \int_0^1 F'(y + \theta(x - y))d\theta$.

Example 3.1. *Returning back to the example in the introduction, we have for $\varphi_0(t) = \varphi(t) = 147t, \varphi_1(t) = 1 + \varphi_0(t), \varphi_2(s,t) = \frac{1}{2}(\varphi_0(s) + \varphi_0(t))$. Using the definition of r we obtain*

$$r_1 = 0.0045, r_2 = 0.0029 = r, r_3 = 0.0039.$$

Example 3.2. *Let $\mathscr{X} = \mathscr{Y} = \mathbb{R}^3, \Omega = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on Ω for $w = (x, y, z)^T$ by*

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we have $\varphi_0(t) = (e-1)t, \varphi(t) = e^{\frac{1}{e-1}}t$, $\varphi_1(t) = 1 + \varphi_0(t), \varphi_2(s,t) = \frac{1}{2}(\varphi_0(s) + \varphi_0(t))$. *Using the definition of* r *we obtain*

$$r_1 = 0.4977, r_2 = 0.3731 = r, r_3 = 0.4951.$$

Example 3.3. *Let* $\mathscr{X} = \mathscr{Y} = C[0,1]$, *be the space of continuous functions on* $[0,1]$ *equipped with the max-norm. Let* $\Omega = \bar{U}(0,1)$. *Define* F *on* Ω *by*

$$F(\varphi)(x) = \varphi(x) - 10\int_0^1 x\theta\varphi(\theta)^3 d\theta.$$

We have that

$$[F'(\varphi(\xi))](x) = \xi(x) - 30\int_0^1 x\theta\varphi(\theta)^2 d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, $\varphi_0(t) = 15t$, $\varphi(t) = 30t, \varphi_1(t) = 1 + \varphi_0(t), \varphi_2(s,t) = \frac{1}{2}(\varphi_0(s) + \varphi_0(t))$. *We obtain*

$$r_1 = 0.0333, r_2 = 0.0197 = r, r_3 = 0.0350.$$

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