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Praface

Since 2012, series of IECMSA conferences have been held regularly every year and featured many distinguished participants from the across the globe. After the following six very successful international conferences; IECMSA-2012 Prishtine-Kosovo, IECMSA-2013 Sarajevo-Bosnia and Herzegovina, IECMSA-2014 Vienna-Austria, IECMSA-2015 Athens-Greece, IECMSA-2016 Belgrade-Serbia, and IECMSA-2017 Budapest-Hungary, we have successfully completed 7th International Eurasian Conference on Mathematical Sciences and Applications. On the basis of the impact of these remarkable conferences, IECMSA-2018 Kyiv-Ukraine has witnessed significant growth. The scientific committee of IECMSA-2018 accepted 184 oral (63%) and 16 poster (50%) presentations. The authors of submitted presentations come from 44 countries. Authors of accepted presentations are from 26 countries. The scientific program of the conference features invited talks, followed by contributed oral and poster presentations in parallel sessions.

The main aim of the conference includes almost all active research areas in pure, applied and inter-disciplinary mathematics reflecting the applications in the areas of sciences and engineering.

This volume contains the proceedings of the selected contributions of the participants of the 7th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2018) scheduled during August 28-31, 2018 in Kyiv, Ukraine.

The selection of papers included in this volume is based on a rigorous peer review process by the committee of experts in various disciplines. Every submitted paper was first screened by the members of the editorial board and once it clears the initial screening, it was sent for peer review to at least two potential reviewers in the related area of expertise from the pool of potential reviewers. The paper is accepted if at least two reviewers recommend it for acceptance. We thank all the invited speakers and the authors who made their valuable contributions towards the success of the conference IECMSA-2018. We are very much grateful to the members of the program committee for their continuous guidance and support which led to the selection of the contributed talks and the papers published in this volume.

See you in future conferences,

Prof. Dr. Murat TOSUN

Editor in Chief

IECMSA-2018

Contents

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Circling-Point Curve in Minkowski Plane

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Abstract: The purpose of this paper is to study the circling-point curve and its degenerate cases at the initial position of motion in Minkowski plane. The first part of the paper is devoted to the determination Bottema's instantaneous invariants and trajectory of origin with respect to these invariants in Minkowski plane. The intersection points of the circling-point curve and inflection curve are called Ball points. Here the number and also the geometric location of Ball points in Minkowski plane have been determined. The fundamental geometric property of a trajectory of each point in a plane is its curvature function κ. Under consideration $\kappa = \kappa' = \kappa'' = 0$, the existence conditions of Ball points in Minkowski plane have been given.

Keywords: Circling-point curve, Ball point, Instantaneous Invariants, Burmester Theory.

1 Introduction

Oene Bottema (1901-1992), Dutch mathematician devised the method of instantaneous invariants in instantaneous kinematics. Various geometric and kinematic properties of Euclidean planar and spatial motions are introduced with respect to the instantaneous invariants. The concept of instantaneous invariants is characterizing the trajectory of any point on a moving rigid body with arbitrary degrees [1–3]. In the meantime, Veldkamp has called the aforementioned invariants as B-invariants [4] and has handled the application of B-invariants to Burmester theory [4–6]. Burmester theory deals with the formulation of special locus curves as inflection circle, circling point curve, twice circling curve, and their intersection points as Ball and Burmester point for planar or spatial motions. Although this analytical method is preferred in a great amount of study of the kinematics, there have been few investigations on non-Euclidean planar kinematics [7, 8].

In consideration of these studies, we investigate the circling-point curve and its degenerate cases of the motion of Minkowski planes and give the existence conditions of Ball points in Minkowski plane.

2 Preliminaries

The Minkowski plane L is the plane R^2 endowed with the Lorentzian scalar product given by $\langle u, w\rangle = u_1 w_1 - u_2 w_2$, where $u = (u_1, u_2)$ and $w = (w_1, w_2)$. The norm of a vector U is defined by $\|u\| = \sqrt{|\langle u, u\rangle|}$. Let L_m and L_f be two coincident Minkowski planes, L_m moving with respect to L_f. The motion can be represented by

$$\begin{array}{c} X(\varphi) = x\cosh\varphi + y\sinh\varphi + a(\varphi) \\ Y(\varphi) = x\sinh\varphi + y\cosh\varphi + b(\varphi) \end{array}$$

such that Cartesian frames of reference xoy and XOY are located in L_m and L_f, respectively. The position corresponding to $\varphi = 0$ of L_m will be named zero-position. The value for zero-position of the nth $(n = 0, 1, 2, \ldots)$ derivative of a function f of φ with respect to φ will be denoted by f_n.

The derivatives $a_n,\ b_n\ (n = 0, 1, 2, \ldots)$ are known as Bottema's instantaneous invariants of the motion [2, 3]. It is well-known that the canonical relative system can be constructed by choose of

$$a = b = a_1 = b_1 = a_2 = 0 \quad \text{and} \quad b_2 = -1.$$

So, the instantaneous invariants $a_k\,(k = 3, 4, \ldots, n)$, $b_k\,(k = 2, 3, \ldots, n)$ completely characterize the infinitesimal properties of motion of Minkowski planes up to the $n-\mathrm{th}$ order as

$$\begin{array}{llll} X = x, & X_1 = y, & X_2 = x, & X_3 = y + a_3, \\ Y = y, & Y_1 = x, & Y_2 = y - 1, & Y_3 = x + b_3, \end{array} \tag{1}$$

at the zero-position [7, 8].

The non-null trajectory of the points satisfying $\kappa = 0$ is the inflection circle where $X' \neq \pm Y'$ in the Minkowski plane. Then the equation of the inflection circle can be obtained from $X'' : Y'' = X' : Y'$ since the curvature function is

$$\kappa = \frac{X'Y'' - X''Y'}{\left|(X')^2 - (Y')^2\right|^{\frac{3}{2}}}. \tag{2}$$

If we substitute the equalities of (1) into (2) at zero position we get the equation of the inflection circle during planar motion of L_m with respect to L_f as follows

$$x^2 - y^2 + y = 0. \tag{3}$$

where $(x, y) \neq (0, 0)$, $x \neq \mp y$ or $y \neq 0$ [7, 8].

3 The Trajectory of Origin of Minkowski Plane

The trajectory of the point $(0, 0)$ of the Minkowski plane L_m, which is coincident with the pole, can be given by

$$X = \sum_{n=3}^{\infty} \frac{a_n}{n!}\varphi^n, \quad Y = \frac{-1}{2}\varphi^2 + \sum_{n=3}^{\infty} \frac{b_n}{n!}\varphi^n \tag{4}$$

for sufficiently small values of $|\varphi|$ at the zero-position with respect to canonical relative systems.

Case 1. Let $a_3 \neq 0$. If ε is a sufficiently small positive number, then the trajectory described through the time interval $[-\varepsilon, \varepsilon]$ has a cusp at the pole of zero-position since $\lim\limits_{\varphi \to 0} |\kappa| = \infty$ and the tangent of the trajectory is pole normal.

Case 2. Let $a_3 = 0,\ a_4 \neq 0$. In this case $a_2 b_3 - a_3 b_2 = 0$ and $a_2 b_4 - a_4 b_2 \neq 0$. So two branches of the trajectory stay at the same side of the tangent. If ε is a sufficiently small positive number, then the trajectory described through the time interval $[-\varepsilon, \varepsilon]$ has a ramphoid cusp at the pole of the zero-position. In this case the curvature is obtained as

$$\begin{aligned} \kappa = \tfrac{a_4}{3} + \left(\tfrac{5a_4b_3}{12} + \tfrac{a_5}{8}\right)\varphi + \left(\tfrac{-a_4\,{b_3}^2}{8} + \tfrac{a_4\,b_4}{6} + \tfrac{7a_5\,b_3}{48} + \tfrac{a_6}{30}\right)\varphi^2 \\ + \left(\tfrac{7a_4\,b_5}{144} - \tfrac{a_4\,{b_3}^2}{12} - \tfrac{a_4\,b_3\,b_4}{24} + \tfrac{a_5\,b_4}{18} - \tfrac{a_5\,{b_3}^2}{16} + \tfrac{3a_6\,b_3}{80} + \tfrac{a_7}{144}\right)\varphi^3 + \ldots \end{aligned}$$

The successive curvatures of the trajectory at the pole are

$$\kappa_0 = \tfrac{a_4}{3}, \tag{5}$$

$$\kappa_1 = \frac{5a_4b_3}{12} + \frac{a_5}{8}, \tag{6}$$

$$\kappa_2 = \frac{-a_4{b_3}^2}{4} + \frac{a_4b_4}{3} + \frac{7a_5b_3}{24} + \frac{a_6}{15}, \tag{7}$$

$$\kappa_3 = \frac{7a_4b_5}{24} - \frac{a_4{b_3}^2}{2} - \frac{a_4b_3b_4}{4} + \frac{a_5b_4}{3} - \frac{3a_5{b_3}^2}{8} + \frac{9a_6b_3}{40} + \frac{a_7}{24}.$$

Case 3. Let $a_3 = a_4 = 0$. For sufficiently small values of ε, the trajectory described through the time interval $[-\varepsilon, \varepsilon]$ has cusp or ramphoid cusp, provided that the smallest value of n, where $a_n \neq 0$, is odd or even, respectively. In this case the curvature is given by

$$\kappa = 0 + \tfrac{a_5}{8}\varphi + \left(\tfrac{7a_5b_3}{48} + \tfrac{a_6}{30}\right)\varphi^2 + \left(\tfrac{a_5b_4}{18} - \tfrac{a_5{b_3}^2}{16} + \tfrac{3a_6b_3}{80} + \tfrac{a_7}{144}\right)\varphi^3 + \ldots$$

that is, the successive curvatures at pole are

$$\kappa_0 = 0,$$

$$\kappa_1 = \frac{a_5}{8},$$

$$\kappa_2 = \frac{7a_5b_3}{24} + \frac{a_6}{15},$$

$$\kappa_3 = \frac{a_5b_4}{3} - \frac{3a_5{b_3}^2}{8} + \frac{9a_6b_3}{40} + \frac{a_7}{24}.$$

4 Circling-Point Curve of Motions in Minkowski Plane

Definition 1. *The locus of the points with constant non-null trajectory curvature at the zero-position of the Minkowski plane L_m is called circling-point curve or cubic of stationary curvatures and denoted by cp.*

This means that the locus of the points satisfying $\kappa' = 0$ where $\left(X'\right)^2 - \left(Y'\right)^2 \neq 0$ is the circling-point curve in Minkowski plane. The differentiation of the equation (2) is

$$\kappa' = \frac{\left(X'Y''' - X'''Y'\right)\left(\left(X'\right)^2 - \left(Y'\right)^2\right) - 3\left(X'Y'' - X''Y'\right)\left(X'X'' - Y'Y''\right)}{\left|(X')^2 - (Y')^2\right|^{\frac{3}{2}}}.$$

In this regard, if we consider the equations of (1) and the last equation together, one can prove the following theorem.

Theorem 1. *In Minkowski plane the equation of the circling-point curve cp of the original motion L_m/L_f is*

$$\left(x^2 - y^2\right)(a_3x - b_3y) + 3x\left(x^2 - y^2 + y\right) = 0 \tag{8}$$

where $(x, y) \neq (0, 0)$ or $x \neq \mp y$.

If we recall the equation (6) for the case of $a_3 = 0$ and $a_4 \neq 0$, we can prove the following theorem.

Theorem 2. *The trajectory of the points different from the origin is the circling-point curve if and only if is*

$$10a_4b_3 + 3a_5 = 0.$$

in case of $a_3 = 0$ and $a_4 \neq 0$.

The graphics of the circling point curves for special cases in the Minkowski plane are drawn hereinafter and further detailed analysis of the graphics enables us to compare them with each other.

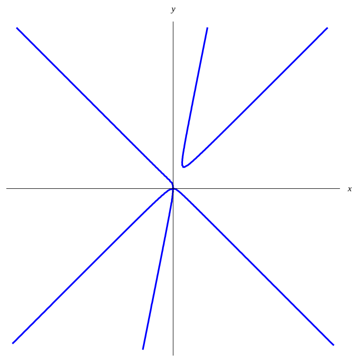


Fig. 1: The circling point curve *cp* for $a_3 = 2$ and $b_3 = 1$.

The circling point curve *cp* has node point at the pole. At the same time, tangents of the circling point curve *cp* are pole tangent and pole normal. Consequently, the cubic curve *cp* is a strophoid in Minkowski plane. Now let us investigate the degenerate cases of the circling point curve *cp*.

i. If $a_3 \neq -3$ and $b_3 = 0$ the equation of the circling-point curve *cp* in Minkowski plane is

$$x\left((a_3 + 3)\left(x^2 - y^2\right) + 3y\right) = 0. \tag{9}$$

This geometrically means that *cp* consists of the pole normal and the circle, which is donated by Γ, with the imaginary radius $\frac{3i}{2(a_3+3)}$. The center of Γ is $\left(0, \frac{3}{2(a_3+3)}\right)$ at the pole normal, see Figure 2a.

In addition, if $a_3 = 0$ when $b_3 = 0$, then the equation (9) becomes $x^2 - y^2 + y = 0$, that is, the circling point curve *cp* coincides with the inflection circle in the case of $a_3 = 0$ and $b_3 = 0$.

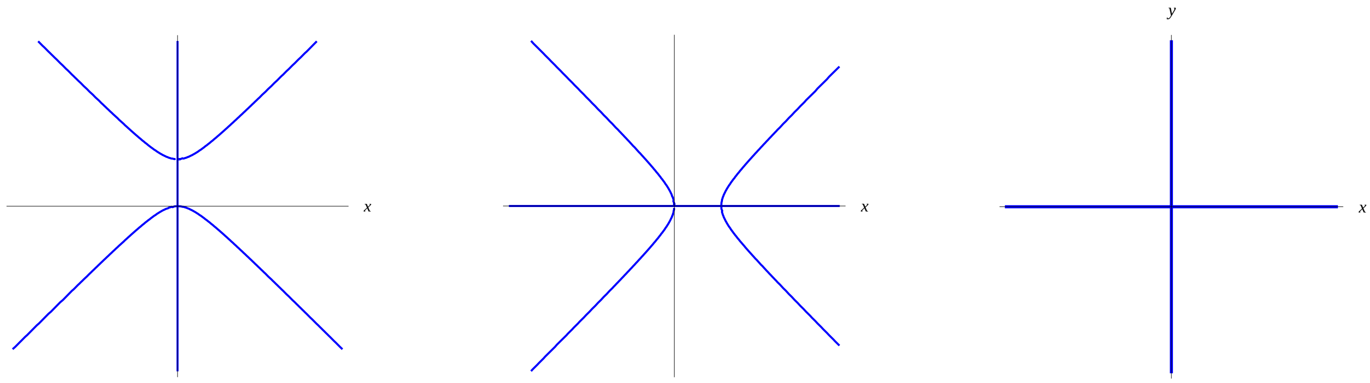
ii. If $a_3 = -3$ and $b_3 \neq 0$ the equation of the circling-point curve *cp* in Minkowski plane is

$$y\left(b_3\left(x^2 - y^2\right) - 3x\right) = 0.$$

Thus, the circling-point curve *cp* consists of pole tangent and the circle, which is donated by Γ_0, with the real radius $\frac{3}{2b_3}$. The center of Γ_0 is $\left(\frac{3}{2b_3}, 0\right)$ at the pole tangent, see Figure 2b.

iii. If $a_3 = -3$ and $b_3 = 0$, the equation of the circling-point curve *cp* is $xy = 0$. The curve consists of pole tangent and pole normal, see Figure 2c.

The circles Γ and Γ_0 are the circles of curvature of the circling-point curve *cp* at its node. From here the geometrical interpretation of the invariants a_3 and b_3 can be given as in the following theorem.



(a) $a_3 = 2,\ b_3 = 0$ (b) $a_3 = -3,\ b_3 = 1$ (c) $a_3 = -3,\ b_3 = 0$

Fig. 2: The circling-point curves in Minkowski plane

Theorem 3. *a_3 equals $3/2$ times the curvature of that branch of cp that touches the pole tangent and similarly b_3 equals $3/2$ times the curvature of that branch of cp that touches the pole normal.*

The equation of the real asymptote of the circling-point curve cp is obtained as

$$\left((a_3+3)^2 - {b_3}^2\right)(b_3 y - (a_3+3)\,x) + 3\,(a_3+3)\,b_3 = 0. \tag{10}$$

The real asymptotes of the circling-point curve cp drawn in the Figure 1 can be seen in the undermentioned figure.

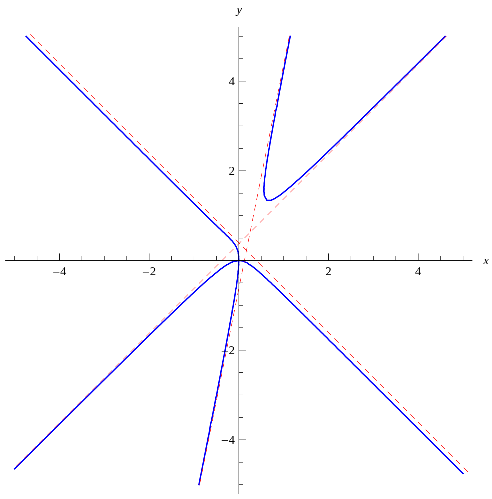


Fig. 3: The real asymptotes of the circling point curve cp for $a_3 = 2$ and $b_3 = 1$.

Furthermore, we can obtain a parametric representation of and irreducible curve cp by putting $y = ux$. This parametric equation is

$$x = \frac{3u}{(u^2-1)\,(-b_3 u + a_3 + 3)}, \quad y = \frac{3u^2}{(u^2-1)\,(-b_3 u + a_3 + 3)}. \tag{11}$$

If we substitute the equation (11) into the equation (10) we find parameter-value $u = \frac{(a_3+3)}{b_3}$. This parameter-value corresponds to the point of intersection of cp with its asymptote.

In the case of $a_3 \neq -3$ and $b_3 = 0$, the equation (11) takes the form

$$x = \frac{3u}{(u^2-1)\,(a_3+3)}, \quad y = \frac{3u^2}{(u^2-1)\,(a_3+3)},$$

which is the parametric representation of the circle Γ.

In a similar vein, if $a_3 = -3$ and $b_3 \neq 0$, under consideration the equation (11)the parametric representation of Γ_0 is given by

$$x = \frac{3}{b_3\,(1-u^2)}, \quad y = \frac{3u}{b_3\,(1-u^2)}.$$

5 Ball Points in Minkowski Plane

Definition 2. *The intersection points of the circling-point curve and inflection curve are called Ball points and denoted by Bl points.*

From this definition and the equations (3) and (8) the coordinates of a Bl point in Minkowski Plane is found as

$$\left(\frac{a_3 b_3}{a_3^2 - b_3^2}, \quad \frac{a_3^2}{a_3^2 - b_3^2}\right). \tag{12}$$

The pole is not a Bl point if $a_3 \neq 0$. Therefore we may draw the conclusion that in the case of $a_3 \neq 0$ and $a_3 \neq \pm b_3$ there is only one point in the zero position given by (12).

From the equation (12), if $a_3 = 0,\ b_3 \neq 0$ we cannot directly say that the origin is Bl point. Therefore in the case of $a_3 = 0,\ b_3 \neq 0$, if $a_4 = a_5 = 0$ we know that $\kappa_0 = \kappa_1 = 0$ is satisfied from the equations (5) and (6). From here if $a_3 = a_4 = a_5 = 0$ the origin is Bl point. Providing that $a_3 = 0,\ b_3 \neq 0$, there is no Bl point if and only if $a_4 \neq 0$ or $a_5 \neq 0$ (because of $\kappa_0 \neq 0$ or $\kappa_1 \neq 0$). Finally, we can say that there is no Bl point if $a_3 = 0,\ b_3 \neq 0,\ a_4^2 + a_5^2 \neq 0$. On the other hand if $a_3 = b_3 = 0$ the circling point curve splits up into the inflection circle and the pole normal. In the case of $a_4^2 + a_5^2 \neq 0$ any point on the inflection circle with the possible exception of the origin is a Bl point of the zero position, the origin being a Bl point too, if $a_4 = a_5 = 0$ at the same time.

The aforementioned analysis of Bl points in Minkowski plane is outlined in the following table.

Conditions	Bl point(s)
$a_3 \neq 0, a_3 \neq \pm b_3$	$\left(\frac{a_3 b_3}{a_3^2 - b_3^2}, \quad \frac{a_3^2}{a_3^2 - b_3^2}\right)$
$a_3 = a_4 = a_5 = 0, b_3 \neq 0$	the origin
$a_3 = 0,\ b_3 \neq 0,\ a_4^2 + a_5^2 \neq 0$	none
$a_3 = b_3 = 0, a_4^2 + a_5^2 \neq 0$	the points on the inflection circle with the exception of the origin
$a_3 = a_4 = a_5 = b_3 = 0$	all points of the inflection circle

As a consequence, if $a_3 \neq 0$ and $a_3 \neq \pm b_3$ the Bl point of the zero position is in the parametric representation (11) of *cp* indicated by the parameter value $u = a_3/b_3$.

6 Ball Points with Excess in Minkowski Plane

Definition 3. *If we have for a Ball point of a given position*

$$\kappa = \kappa' = ... = \kappa^{(r+1)} = 0,\ \kappa^{(r+2)} \neq 0$$

this point is called a Ball point with excess r and denoted by Bl_r point.

In the case of $a_3 \neq 0$, the zero position has a Bl point. Under this consideration the following theorem can be given.

Theorem 4. *In the case $a_3 \neq 0$, the Bl point is a Bl_1 point if and only if*

$$a_4 b_3 - a_3 b_4 = a_3.$$

Proof: From the equation (2), $\kappa = \kappa' = \ \kappa'' = 0$ if and only if $X_1 Y_4 - X_4 Y_1 = 0$. If we substitute the equation (1) into $X_1 Y_4 - X_4 Y_1 = 0$ we get

$$x^2 - y^2 + a_4 x - b_4 y = 0.$$

If the Bl_1 point has the coordinates (x_0, y_0) this last equation takes form of

$$x_0^2 - y_0^2 + a_4 x_0 - b_4 y_0 = 0. \tag{13}$$

In virtue of Bl_1 point is also on the inflection circle, the common solution of $x_0^2 - y_0^2 + y_0 = 0$ and the equation (4) gives us

$$a_4 x_0 + (-b_4 - 1)\, y_0 = 0. \tag{14}$$

Substituting the equation (12) into the equation (14) completes the proof. □

This relation represents a necessary and sufficient condition for the Bl point of the zero position to be a Bl_1 point for the case of $a_3 \neq 0$. In the zero position if $a_3 = a_4 = a_5 = 0$, $b_3 \neq 0$ the origin is the only Bl point. From the equation (7) this point is a Bl_1 point if and only if $a_6 = 0$. In the case of $a_3 = b_3 = 0$, $a_4^2 + a_5^2 \neq 0$ any point of the inflection circle with the exception of the origin is a Bl point of the zero position. From the equation (13) and the equation (14) it follows that all these points are Bl_1 points if and only if $a_4 = 0$, $b_4 = -1$ whereas in the case $a_4 \neq 0$ the only Bl_1 point of the zero position is given by:

$$\left(\frac{(b_4 + 1)\, a_4}{{a_4}^2 - (b_4 + 1)^2}, \quad \frac{{a_4}^2}{{a_4}^2 - (b_4 + 1)^2}\right).$$

In the case $a_3 = a_4 = a_5 = b_3 = 0$ any point of the inflection circle is a Bl point of the zero point.

If $b_4 = -1$ at the same time, all these points with exception of the origin are Bl_1 points, the origin being in this case is a Bl_1 point if moreover $a_6 = 0$. If, however, $b_4 \neq -1$ there is no Bl_1 point unless $a_6 = 0$ in which case the origin is the only Bl_1 point of the zero position. From here, we give conditions of being a Bl_1 point in Minkowski plane in the following table.

Condition(s)	Bl_1 point(s)
$a_3 = a_4 b_3 - a_3 b_4 \neq 0$, $a_3 \neq \pm b_3$	$\left(\frac{a_3 b_3}{a_3^2 - b_3^2}, \ \frac{a_3^2}{a_3^2 - b_3^2}\right)$
$a_3 = b_3 = 0, \ a_4 \neq 0$	$\left(\frac{a_4(b_4+1)}{a_4^2 - (b_4+1)^2}, \ \frac{a_4^2}{a_4^2 - (b_4+1)^2}\right)$
$a_3 = a_4 = a_5 = a_6 = 0$, $a_4^2 - (b_4+1)^2 \neq 0$	Origin

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The Norm Of Certain Matrix Operators On The New Block Sequence Space

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Abstract: The purpose of the this study is to introduce the sequence space

$$\ell_p(E,B(r,s)) = \left\{ x=(x_n)\in\omega : \sum_{n=1}^{\infty}\left|\sum_{j\in E_n} rx_j + \sum_{j\in E_{n+1}} sx_j\right|^p < \infty \right\},$$

where $E=(E_n)$ is a partition of finite subsets of the positive integers, $r,s\in\mathbb{R}\backslash\{0\}$ and $p\geq 1$. The topological and algebraical properties of this space are examined. Furthermore, we establish some inclusion relations. Finally, the problem of finding the norm of certain matrix operators such as Copson and Hilbert from ℓ_p into $\ell_p(E,B(r,s))$ is investigated.

Keywords: Capson operators, Hilbert operators, Matrix domain, Sequence spaces.

1 Introduction

By a sequence space, we understand a linear subspace of ω, the space of all real valued sequences $x=(x_n)$. The domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x=(x_k)\in\omega : Ax\in X\}, \tag{1}$$

which is a sequence space. If A is triangle, then one can easily observe that the sequence spaces X_A and X are linearly isomorphic, i.e., $X_A\cong X$. In the past, several authors studied matrix transformations on the sequence spaces that are the matrix domains of triangle matrices in classical spaces ℓ_p, ℓ_∞, c and c_0. For instance, some matrix domains of the difference operator were studied in [1–8]. In these studies, the matrix domains are obtained by triangle matrices, hence these spaces are normed sequence spaces. For more details on the domain of triangle matrices in some spaces, the reader may refer to Chapter 4 of [9]. The matrix domains given in this paper specify by a certain non-triangle matrix, so we should not expect that related spaces are normed sequence spaces.

In this study, we define the sequence space $\ell_p(E,B(r,s))$ and investigate some topological and algebraical properties of this space and derive inclusion relations concerning with its. Moreover, we shall consider the inequality of the form

$$\|Ax\|_{p,E,B(r,s)} \leq U\|x\|_p\ ,$$

for all the sequence $x\in\ell_p$. The costant U not depending on x and we seek the smallest possible value of U. In the study, we examine the problem of finding the upper bound of certain matrix operators from ℓ_p into $\ell_p(E,B(r,s))$ and we consider certain matrix operators such as Copson and Hilbert.

Let $E=(E_n)$ be a partition of finite subsets of the positive integers such that

$$\max E_n < \min E_{n+1}, \tag{2}$$

for $n=1,2,....$ Foroutannia defined the sequence space $\ell_p(E)$ by

$$\ell_p(E) = \left\{ x=(x_n)\in\omega : \sum_{n=1}^{\infty}\left|\sum_{j\in E_n} x_j\right|^p < \infty \right\} \quad (1\leq p<\infty),$$

with the semi-norm $\|.\|_{p,E}$, which is defined in the following way :

$$\|x\|_{p,E} = \left(\sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p \right)^{1/p}.$$

It is significant that in the special case $E_n = \{n\}$ for $n = 1, 2, ...$, we have $\ell_p(E) = \ell_p$ and $\|x\|_{p,E} = \|x\|_p$. For more details on the sequence space $\ell_p(E)$, the reader may refer to [10].

2 The Block Sequence Space $\ell_p(E, B(r, s))$ of Non-Absolute Type

Suppose $E = (E_n)$ is a partition of finite subsets of the positive integers that satisfies the condition (2). We define the sequence space $\ell_p(E, B(r, s))$ by

$$\ell_p(E, B(r,s)) = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} r x_j + \sum_{j \in E_{n+1}} s x_j \right|^p < \infty \right\},$$

with the semi-norm $\|.\|_{p,E,B(r,s)}$, which is defined in the following way :

$$\|x\|_{p,E,B(r,s)} = \left(\sum_{n=1}^{\infty} \left| \sum_{j \in E_n} r x_j + \sum_{j \in E_{n+1}} s x_j \right|^p \right)^{1/p}. \tag{3}$$

It should be noted that the function $\|.\|_{p,E,B(r,s)}$ can not be norm, since $x = (x_j) = \{(-1)^{j+1}\}_{j=1}^{\infty}$ and $E = \{2n-1, 2n\}$ for all n , then $\|x\|_{p,E,B(r,s)} = 0$ while $x \neq 0$.

It is also significant that in the special case $r = 1$ and $s = -1$, we have $\ell_p(E, B(r,s)) = \ell_p(E, \Delta)$ [11].

If the infinite matrix $A = \{a_{nk}\}$ is defined by

$$a_{nk} = \begin{cases} r \ , & \text{if } k \in E_n \\ s \ , & \text{if } k \in E_{n+1} \\ 0 \ , & \text{otherwise} \end{cases}$$

with the notation (1), we can redefine the space $\ell_p(E, B(r, s))$ as follows:

$$\ell_p(E, B(r,s)) = (\ell_p)_A.$$

Throughout this paper, the cardinal number of the set E_k is denoted by $|E_k|$.

Now we are beginning with the following theorem which is essential in the study.

Theorem 1. *Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of the positive integers that satisfies the condition (2). The set $\ell_p(E, B(r, s))$ becomes a vector space with coordinatewise addition and scalar multiplication, which is a complete semi-normed space by $\|.\|_{p,E,B(r,s)}$ defined by (3).*

It can easily checked that the absolute property does not hold on the space $\ell_p(E, B(r, s))$, that is $\|x\|_{p,E,B(r,s)} \neq \||x|\|_{p,E,B(r,s)}$ for at least one sequence in the space $\ell_p(E, B(r, s))$, and this says that $\ell_p(E, B(r, s))$ is a sequence space of nonabsolute type, where $|x_k| = (|x_k|)$.

Theorem 2. *Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of the positive integers that satisfies the condition (2). If*

$$M = \left\{ x = (x_n) : \sum_{j \in E_n} r x_j + \sum_{j \in E_{n+1}} s x_j = 0, \forall n \right\},$$

then we have $\ell_p(E, B(r, s))/M \simeq \ell_p$.

Note that the mapping defined in Theorem 2, T is not injective, while $\|Tx\|_p = \|x\|_{p,E,B(r,s)}$ for all $x \in \ell_p(E, B(r, s))$.

Let us derive some inclusion relations concerning with the space $\ell_p(E, B(r, s))$.

Result 1. *Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of positive integers that satisfies the condition (2). If $\sup_n |E_n| < \infty$, then $\ell_p \subset \ell_p(E)$. Moreover if $|E_n| > 1$ for an infinite number of n, then the inclusion is strict.*

Theorem 3. *Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of positive integers that satisfies the condition (2). Then $\ell_p(E) \subset \ell_p(E, B(r, s))$, furthermore the inclusion is strictly holds.*

Combining Lemma 1 and Theorem 3, we get the following corollary.

Corollary 1. *Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of positive integers that satisfies the condition (2). If $\sup_n |E_n| < \infty$, then $\ell_p \subset \ell_p(E, B(r, s))$. Moreover if $|E_n| > 1$ for an infinite number of n, then the inclusion is strict.*

Theorem 4. *Let $E = (E_n)$ be a partition of finite subsets of positive integers that satisfies the condition (2). Except the case $p = 2$, the space $\ell_p(E, B(r, s))$ is not a semi-inner product space.*

Definition 1. *Let X be a semi-normed space with a semi-norm g. A sequence (b_n) of elements of the semi-normed space X is called a Schauder basis (or briefly basis) for X iff, for each $x \in X$ there exists a unique sequence of scalars (α_n) such that*

$$\lim_{n\to\infty} g\left(x - \sum_{k=1}^{n} \alpha_k b_k\right) = 0.$$

The series $\sum_{k=1}^{n} \alpha_k b_k$ which has the sum x, is then called the expansion of x with respect to (b_n) and written as $x = \sum_{k=1}^{n} \alpha_k b_k$. In the following, we give a sequence of points of the space $\ell_p(E, B(r,s))$ which forms a basis for the space $\ell_p E, B(r,s)$.

Theorem 5. *Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of the positive integers that satisfies the condition (2). If the sequence $b^{(k)}(r,s) = \{b_j{}^{(k)}(r,s)\}_{j\in\mathbb{N}}$ is defined such that*

$$\sum_{j\in E_n} b_j{}^{(k)}(r,s) = \begin{cases} 0 & , \ if \ \ n < k \\ \frac{1}{r}(-\frac{s}{r})^n & , \ if \ \ n \geq k \end{cases}$$

and the remaining elements are zero, for $k = 1, 2, \ldots$. Then, the sequence $\left\{b^{(k)}(r,s)\right\}_{k\in\mathbb{N}}$ is a basis for the space $\ell_p(E, B(r,s))$ and any $x \in \ell_p(E, B(r,s))$ has a unique representation of the form

$$x = \sum_k \alpha_k b^{(k)}(r,s) \ ,$$

where $\alpha_k = \sum_{j\in E_k} x_j$ for $k = 1, 2, \ldots$.

3 The Norm of Matrix Operators from ℓ_p into $\ell_p(E, B(r,s))$

In this section, the problem of finding the norm of certain matrix operators such as Copson and Hilbert from ℓ_p into $\ell_p(E, B(r,s))$ is considered, where $p \geq 1$.

Theorem 6. *Let $A = (a_{n,k})$ be a matrix operator and $E = (E_n)$ be a partition that satisfies condition (2). If*

$$M = \sup_k \sum_{n=1}^{\infty} \left| \sum_{i\in E_n} r a_{i,k} + \sum_{i\in E_{n+1}} s a_{i,k} \right| < \infty,$$

then A is a bounded operator from ℓ_1 into $\ell_1(E, B(r,s))$ and $\|A\|_{1,E,B(r,s)} = M$.

In particular if

$$\sum_{i\in E_n} r a_{i,k} + \sum_{i\in E_{n+1}} s a_{i,k} \geq 0$$

and $r + s = 0$ for all n, k, then

$$\|A\|_{1,E,B(r,s)} = \sup_k \sum_{i\in E_1} r a_{i,k}$$

The Copson operator C is defined by $y = Cx$, where

$$y_n = \sum_{k=n}^{\infty} \frac{x_k}{k}, (\forall n).$$

It is given by the Copson matrix:

$$c_{n,k} = \begin{cases} \dfrac{1}{k} & , \ if \ \ n \leq k \\ 0 & , \ if \ \ n > k. \end{cases}$$

Corollary 2. *Let C be the Copson operator and $E = (E_n)$ be a partition that satisfies condition (2). If*

$$\sum_{i\in E_n} r c_{i,k} + \sum_{i\in E_{n+1}} s c_{i,k} \geq 0$$

for all n, k and $r + s = 0$, then C is a bounded operator from ℓ_1 into $\ell_1(E, B(r,s))$ and $\|C\|_{1,E,B(r,s)} = r$.

Corollary 3. *Suppose that C is the Copson operator, $rc_{n,k} + sc_{n+1,k} \geq 0$ for all n, k, $r + s = 0$ and $E = \{n\}$ for all n. Then C is a bounded operator from ℓ_1 into $\ell_1(B(r,s))$ and $\|C\|_{1,B(r,s)} = r$.*

Recall that the Hilbert operator H defined by the matrix:

$$h_{n,k} = \frac{1}{n+k} \quad , \qquad (n,k = 1,2,...).$$

Corollary 4. *Let H be the Hilbert operator and $E = (E_n)$ be a partition that satisfies the condition (2). If*

$$\sum_{i \in E_n} rh_{i,k} + \sum_{i \in E_{n+1}} sh_{i,k} \geq 0$$

for all n, k and $r + s = 0$, then H is a bounded operator from ℓ_1 into $\ell_1(E, B(r,s))$ and

$$\|H\|_{1,E,B(r,s)} = r\left(\frac{1}{2} + ... + \frac{1}{\max E_1 + 1}\right).$$

Corollary 5. *If H is the Hilbert operator, $rh_{n,k} + sh_{n+1,k} \geq 0$ for all n, k and $r + s = 0$, then H is a bounded operator from ℓ_1 into $\ell_1(E, B(r,s))$ and $\|H\|_{1,B(r,s)} = \frac{r}{2}$.*

Theorem 7. *([12], Theorem 275) Let $p > 1$ and $B = (b_{n,k})$ be a matrix operator with $b_{n,k} \geq 0$ for all n, k. Suppose that K and R are two strictly positive numbers such that*

$$\sum_{n=1}^{\infty} b_{n,k} \leq K, \text{ for all } k, \qquad \sum_{k=1}^{\infty} b_{n,k} \leq R \text{ for all } n,$$

(bounds for column and row sums respectively). Then

$$\|B\|_p \leq R^{(p-1)/p} \,.\, K^{1/p}.$$

Result 2. *If $A = (a_{n,k})$ and $B = (b_{n,k})$ are two matrix operators such that*

$$b_{n,k} = \sum_{i \in E_n} ra_{i,k} + \sum_{i \in E_{n+1}} sa_{i,k},$$

then

$$\|A\|_{p,E,B(r,s)} = \|B\|_p.$$

Hence, if B is a bounded operator on ℓ_p, then A will be a bounded operator from ℓ_p into $\ell_p(E, B(r,s))$.

Theorem 8. *Let C is the Copson matrix operator $p > 1$, $r > 0$ and $r + s = 0$. If N is a positive integer and $E_n = \{nN - N + 1, nN - N + 2, ..., nN\}$ for all n, then C is a bounded operator from ℓ_p into $\ell_p(E, B(r,s))$ and*

$$\|C\|_{p,E,B(r,s)} \leq r\left(N + \frac{N-1}{N+1} + \frac{N-2}{N+2} + ... + \frac{1}{2N-1}\right)^{\frac{(p-1)}{p}}.$$

Theorem 9. *Suppose that $p > 1$, $r > 0$, $r + s = 0$, N is a positive integer and $E_n = \{nN - N + 1, nN - N + 2, ..., nN\}$ for all n. If H is the Hilbert matrix operator, then it is a bounded operator from ℓ_p into $\ell_p(E, B(r,s))$ and*

$$\|H\|_{p,E,B(r,s)} \leq r\left(\frac{1}{2} + \frac{2}{3} + ... + \frac{N}{N+1} + ... + \frac{1}{2N}\right)^{\frac{(p-1)}{p}} \left(\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2N}\right)^{\frac{1}{p}}.$$

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Compactness of Matrix Operators on the Banach Space $\ell_p(T)$

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Abstract: In this study, by using the Hausdorff measure of non-compactness, we obtain the necessary and sufficient conditions for certain matrix operators on the spaces $\ell_p(T)$ and $\ell_\infty(T)$ to be compact, where $1 \leq p < \infty$.

Keywords: Compact operators, Hausdorff measure of non-compactness, Sequence spaces.

1 Introduction

By ω, we denote the space of all real sequences. Any subset of ω is called a sequence space. Let Ψ, ℓ_∞, c and c_0 denote the sets of all finite, bounded, convergent and null sequences, respectively and $\ell_p = \{u = (u_n) \in \omega : \sum_n |u_n|^p < \infty\}$ for $1 \leq p < \infty$. Throughout the study, we assume that $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

A B-space is a complete normed space. A topological sequence space in which all coordinate functionals π_k, $\pi_k(u) = u_k$, are continuous is called a K-space. A BK-space is defined as a K-space which is also a B-space, that is, a BK-space is a Banach space with continuous coordinates. A BK-space $\lambda \supset \psi$ is said to have AK if every sequence $u = (u_k) \in \lambda$ has a unique representation $u = \sum_k u_k e^{(k)}$, where $e^{(k)}$ is the sequence whose only non-zero term is 1 in the nth place for each $k \in \mathbb{N}$. For example, the space ℓ_p $(1 \leq p < \infty)$ is a BK-space with the norm $\|u\|_p = (\sum_k |u_k|^p)^{1/p}$ and c_0 and ℓ_∞ is a BK-space with the norm $\|u\|_\infty = \sup_k |u_k|$. Also, the BK-spaces c_0 and ℓ_p have AK but c and ℓ_∞ do not have AK.

The β-dual of a sequence space λ is defined by

$$\lambda^\beta = \{z = (z_k) \in \omega : zu = (z_k u_k) \in cs \text{ for all } u = (u_k) \in \lambda\}.$$

Let $\mathcal{A}$ be the sequence of n^{th} row of an infinite matrix $\mathcal{A} = (\mathfrak{a}_{nk})$ with real numbers $\mathfrak{a}_{nk}$ for each $n \in \mathbb{N}$. For a sequence $u = (u_k) \in \omega$, the $\mathcal{A}$-transform of u is the sequence $\mathcal{A}u = (\mathcal{A}_n(u))$, where

$$\mathcal{A}_n(u) = \sum_{n=0}^{\infty} \mathfrak{a}_{nk} u_k$$

provided that the series is convergent for each $n \in \mathbb{N}$.

(λ, μ) stands for the class of all infinite matrices from a sequence space λ into another sequence space μ. Hence, $\mathcal{A} \in (\lambda, \mu)$ if and only if $\mathcal{A}_n \in \lambda^\beta$ for all $n \in \mathbb{N}$.

Let λ be a normed space and S_λ be the unit sphere in λ. For a BK-space $\lambda \supset \psi$ and $z = (z_k) \in \omega$, we use the notation

$$\|z\|_\lambda^* = \sup_{u \in S_\lambda} \left| \sum_k z_k u_k \right|$$

under the assumption that the supremum is finite. In this case observe that $z \in \lambda^\beta$.

Lemma 1. *[1, Theorem 1.29] $\ell_1^\beta = \ell_\infty$, $\ell_p^\beta = \ell_q$ and $\ell_\infty^\beta = \ell_1$, where $1 < p < \infty$. If $\lambda \in \{\ell_1, \ell_p, \ell_\infty\}$, then $\|z\|_\lambda^* = \|z\|_{\lambda^\beta}$ holds for all $z \in \lambda^\beta$, where $\|.\|_{\lambda^\beta}$ is the natural norm on λ^β.*

By $\mathcal{B}(\lambda, \mu)$, we denote the set of all bounded (continuous) linear operators from λ to μ.

Lemma 2. *[1, Theorem 1.23 (a)] Let λ and μ be BK-spaces. Then, for every $\mathcal{A} \in (\lambda, \mu)$, there exists a linear operator $L_\mathcal{A} \in \mathcal{B}(\lambda, \mu)$ such that $L_\mathcal{A}(u) = \mathcal{A}u$ for all $u \in \lambda$.*

Lemma 3. *[1] Let $\lambda \supset \psi$ be a BK-space and $\mu \in \{c_0, c, \ell_\infty\}$. If $\mathcal{A} \in (\lambda, \mu)$, then*

$$\|L_{\mathcal{A}}\| = \|\mathcal{A}\|_{(\lambda,\mu)} = \sup_n \|\mathcal{A}_n\|_\lambda^* < \infty.$$

The Hausdorff measure of noncompactness of a bounded set Q in a metric space λ is defined by

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \subset \cup_{i=1}^n B(x_i, r_i), x_i \in \lambda, r_i < \varepsilon, n \in \mathbb{N}\},$$

where $B(x_i, r_i)$ is the open ball centered at x_i and radius ε for each $i = 1, 2, ..., n$.

The following theorem is useful to compute the Hausdorff measure of non-compactness in ℓ_p for $1 \leq p < \infty$.

Theorem 1. *[2] Let Q be a bounded subset in ℓ_p for $1 \leq p < \infty$ and $P_r : \ell_p \to \ell_p$ be the operator defined by $P_r(u) = (u_0, u_1, u_2, ..., u_r, 0, 0, ...)$ for all $u = (u_k) \in \ell_p$ and each $r \in \mathbb{N}$. Then, we have*

$$\chi(Q) = \lim_r \left(\sup_{u \in Q} \|(I - P_r)(u)\|_{\ell_p} \right),$$

where I is the identity operator on ℓ_p.

Let λ and μ be Banach spaces. Then, a linear operator $L : \lambda \to \mu$ is is said to be compact if the domain of L is all of λ and $L(Q)$ is a totally bounded subset of μ for every bounded subset Q in λ. Equivalently, we say that L is compact if its domain is all of λ and for every bounded sequence $u = (u_n)$ in λ, the sequence $(L(u_n))$ has a convergent subsequence in μ.

The idea of compact operators between Banach spaces is closely related to the Hausdorff measure of non-compactness. For $L \in \mathcal{B}(\lambda, \mu)$, the Hausdorff measure of non-compactness of L denoted by $\|L\|_\chi$ is given by

$$\|L\|_\chi = \chi(L(S_\lambda))$$

and we have

$$L \text{ is compact if and only if } \|L\|_\chi = 0.$$

Several authors have studied compact operators on the sequence spaces and given very important results related to the Hausdorff measure of non-compactness of a linear operator. For example [3]-[9].

The main purpose of this study is to obtain necessary and sufficient conditions for some matrix operators to be compact. For this purpose, we use the Banach spaces $\ell_p(T)$ and $\ell_\infty(T)$ introduced in [10] as

$$\ell_p(T) = \left\{ u = (u_n) \in \omega : \sum_n \left| t_n u_n - \frac{1}{t_n} u_{n-1} \right|^p < \infty \right\} \quad (1 \leq p < \infty)$$

and

$$\ell_\infty(T) = \left\{ u = (u_n) \in \omega : \sup_n \left| t_n u_n - \frac{1}{t_n} u_{n-1} \right| < \infty \right\}.$$

Here, the difference matrix matrix $T = (t_{nk})$ is defined by

$$t_{nk} = \begin{cases} t_n & , \quad k = n \\ -\frac{1}{t_n} & , \quad k = n - 1 \\ 0 & , \quad k > n \text{ or } 0 \leq k < n - 1, \end{cases}$$

where $t_n > 0$ for all $n \in \mathbb{N}$ and $t = (t_n) \in c \backslash c_0$.

Note that we use the sequence $v = (v_n)$ for the T-transform of a sequence $u = (u_n)$, that is,

$$v_n = T_n(u) = \begin{cases} t_0 u_0 & , \quad n = 0 \\ t_n u_n - \frac{1}{t_n} u_{n-1} & , \quad n \geq 1 \end{cases} \quad (n \in \mathbb{N}).$$

2 Compact Operators on the Spaces $\ell_p(T)$ and $\ell_\infty(T)$

For a sequence $a = (a_k) \in \omega$, we define a sequence $\tilde{a} = (\tilde{a}_k)$ as $\tilde{a}_k = \sum_{j=k}^\infty t_k \prod_{i=k}^j \frac{1}{t_i^2} a_j$ for all $k \in \mathbb{N}$.

We need the following results in the sequel.

Lemma 4. *Let $a = (a_k) \in (\ell_p(T))^\beta$, where $1 \leq p \leq \infty$. Then $\tilde{a} = (\tilde{a}_k) \in \ell_q$ and*

$$\sum_k a_k u_k = \sum_k \tilde{a}_k v_k \tag{1}$$

for all $u = (u_k) \in \ell_p(T)$.

Lemma 5. *The following statements hold.*

(a) $\|a\|^*_{\ell_1(T)} = \sup_k |\tilde{a}_k| < \infty$ *for all* $a = (a_k) \in (\ell_1(T))^\beta$.
(b) $\|a\|^*_{\ell_p(T)} = \left(\sum_k |\tilde{a}_k|^q\right)^{1/q} < \infty$ *for all* $a = (a_k) \in (\ell_p(T))^\beta$, *where* $1 \leq p \leq \infty$.
(c) $\|a\|^*_{\ell_\infty(T)} = \sum_k |\tilde{a}_k| < \infty$ *for all* $a = (a_k) \in (\ell_\infty(T))^\beta$.

Proof: We only prove part (a) and the others can be proved analogously. Choose $a = (a_k) \in (\ell_1(T))^\beta$. Then, by Lemma 4, we have $\tilde{a} = (\tilde{a}_k) \in \ell_\infty$ and (1) holds. Since $\|u\|_{\ell_1(T)} = \|v\|_{\ell_1}$ holds, we obtain that $u \in S_{\ell_1(T)}$ if and only if $v \in S_{\ell_1}$. Hence, we deduce that $\|a\|^*_{\ell_1(T)} = \sup_{u \in S_{\ell_1(T)}} |\sum_k a_k u_k| = \sup_{v \in S_{\ell_1}} |\sum_k \tilde{a}_k v_k| = \|\tilde{a}\|^*_{\ell_1}$. From Lemma 1, it follows that $\|a\|^*_{\ell_1(T)} = \|\tilde{a}\|^*_{\ell_1} = \|\tilde{a}\|_{\ell_\infty} = \sup_k |\tilde{a}_k|$. $\square$

Throughout this section, we use the matrix $\tilde{\mathcal{A}} = (\tilde{\mathfrak{a}}_{nk})$ defined by an infinite matrix $\mathcal{A} = (\mathfrak{a}_{nk})$ via

$$\tilde{\mathfrak{a}}_{nk} = \sum_{j=k}^{\infty} t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \mathfrak{a}_{nj}$$

for all $n, k \in \mathbb{N}$ under the assumption that the series is convergent.

Lemma 6. *Let* λ *be a sequence space. If* $\mathcal{A} \in (\ell_p(T), \lambda)$, *then* $\tilde{\mathcal{A}} \in (\ell_p, \lambda)$ *and* $\mathcal{A}u = \tilde{\mathcal{A}}v$ *for all* $u \in \ell_p(T)$, *where* $1 \leq p \leq \infty$.

Lemma 7. *If* $\mathcal{A} \in (\ell_1(T), \ell_p)$, *then we have*

$$\|L_{\mathcal{A}}\| = \|\mathcal{A}\|_{(\ell_1(T), \ell_p)} = \sup_k \left(\sum_n |\tilde{\mathfrak{a}}_{nk}|^p\right)^{1/p} < \infty,$$

where $1 \leq p \leq \infty$.

Lemma 8. *[11, Theorem 3.7] Let* $\lambda \supset \psi$ *be a BK-space. Then, the following statements hold.*
(a) $\mathcal{A} \in (\lambda, \ell_\infty)$, *then* $0 \leq \|L_{\mathcal{A}}\|_\chi \leq \limsup_n \|\mathcal{A}_n\|^*_\lambda$.
(b) $\mathcal{A} \in (\lambda, c_0)$, *then* $\|\mathcal{A}_S\|_\chi \leq \limsup_n \|\mathcal{A}_n\|^*_\lambda$.
(c) If λ *has AK or* $\lambda = \ell_\infty$ *and* $\mathcal{A} \in (\lambda, c)$, *then*

$$\frac{1}{2} \limsup_n \|\mathcal{A}_n - \alpha\|^*_\lambda \leq \|L_{\mathcal{A}}\|_\chi \leq \limsup_n \|\mathcal{A}_n - \alpha\|^*_\lambda,$$

where $\alpha = (\alpha_k)$ *and* $\alpha_k = \lim_n \mathfrak{a}_{nk}$ *for all* $k \in \mathbb{N}$.

Theorem 2.

1. *For* $\mathcal{A} \in (\ell_1(T), \ell_\infty)$,

$$0 \leq \|L_{\mathcal{A}}\|_\chi \leq \limsup_n \left(\sup_k |\tilde{\mathfrak{a}}_{nk}|\right)$$

holds.
2. *For* $\mathcal{A} \in (\ell_1(T), c)$,

$$\frac{1}{2} \limsup_n \left(\sup_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k|\right) \leq \|L_{\mathcal{A}}\|_\chi \leq \limsup_n \left(\sup_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k|\right)$$

holds.
3. *For* $\mathcal{A} \in (\ell_1(T), c_0)$,

$$\|L_{\mathcal{A}}\|_\chi = \limsup_n \left(\sup_k |\tilde{\mathfrak{a}}_{nk}|\right)$$

holds.
4. *For* $\mathcal{A} \in (\ell_1(T), \ell_1)$,

$$\|L_{\mathcal{A}}\|_\chi = \lim_m \left(\sup_k \sum_{n=m}^{\infty} |\tilde{\mathfrak{a}}_{nk}|\right)$$

holds.

Corollary 1.

1. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_1(T), \ell_\infty)$ *if*

$$\lim_n \left(\sup_k |\tilde{\mathfrak{a}}_{nk}|\right) = 0.$$

2. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_1(T), c)$, *if and only if*

$$\lim_n \left(\sup_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k|\right) = 0.$$

3. *$L_{\mathcal{A}}$ is compact for $\mathcal{A} \in (\ell_1(T), c_0)$ if and only if*

$$\lim_n \left(\sup_k |\tilde{\mathfrak{a}}_{nk}| \right) = 0.$$

4. *$L_{\mathcal{A}}$ is compact for $\mathcal{A} \in (\ell_1(T), \ell_1)$ if and only if*

$$\lim_m \left(\sup_k \sum_{n=m}^{\infty} |\tilde{\mathfrak{a}}_{nk}| \right) = 0.$$

Lemma 9. *Let $\lambda \supset \psi$ be a BK-space. If $\mathcal{A} \in (\lambda, \ell_1)$, then*

$$\lim_r \left(\sup_{N \in \mathcal{K}_r} \left\| \sum_{n \in N} \mathcal{A}_n \right\|_{\lambda}^{*} \right) \leq \|L_{\mathcal{A}}\|_{\chi} \leq 4 \lim_r \left(\sup_{N \in \mathcal{K}_r} \left\| \sum_{n \in N} \mathcal{A}_n \right\|_{\lambda}^{*} \right)$$

and $L_{\mathcal{A}}$ is compact if and only if $\lim_r \left(\sup_{N \in \mathcal{K}_r} \| \sum_{n \in N} \mathcal{A}_n \|_{\lambda}^{*} \right) = 0$, where $\mathcal{K}_r$ is the subcollection of $\mathcal{K}$ consisting of subsets of $\mathbb{N}$ with elements that are greater than r.

Theorem 3. *Let $1 < p < \infty$.*

1. *For $\mathcal{A} \in (\ell_p(T), \ell_\infty)$,*

$$0 \leq \|L_{\mathcal{A}}\|_{\chi} \leq \limsup_n \left(\sum_k |\tilde{\mathfrak{a}}_{nk}|^q \right)^{1/q}$$

holds.

2. *For $\mathcal{A} \in (\ell_p(T), c)$,*

$$\frac{1}{2} \limsup_n \left(\sum_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k|^q \right)^{1/q} \leq \|L_{\mathcal{A}}\|_{\chi} \leq \limsup_n \left(\sum_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k|^q \right)^{1/q}$$

holds.

3. *For $\mathcal{A} \in (\ell_p(T), c_0)$,*

$$\|L_{\mathcal{A}}\|_{\chi} = \limsup_n \left(\sum_k |\tilde{\mathfrak{a}}_{nk}|^q \right)^{1/q}$$

holds.

4. *For $\mathcal{A} \in (\ell_p(T), \ell_1)$,*

$$\lim_m \|\mathcal{A}\|_{(\ell_p(T), \ell_1)}^{(m)} \leq \|L_{\mathcal{A}}\|_{\chi} \leq 4 \lim_m \|\mathcal{A}\|_{(\ell_p(T), \ell_1)}^{(m)}$$

holds, where $\|\mathcal{A}\|_{(\ell_p(T), \ell_1)}^{(m)} = \sup_{N \in \mathcal{K}_m} \left(\sum_k | \sum_{n \in N} \tilde{\mathfrak{a}}_{nk}|^q \right)^{1/q}$.

Corollary 2. *Let $1 < p < \infty$.*

1. *$L_{\mathcal{A}}$ is compact for $\mathcal{A} \in (\ell_p(T), \ell_\infty)$ if*

$$\lim_n \left(\sum_k |\tilde{\mathfrak{a}}_{nk}|^q \right)^{1/q} = 0.$$

2. *$L_{\mathcal{A}}$ is compact for $\mathcal{A} \in (\ell_p(T), c)$ if and only if*

$$\lim_n \left(\sum_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k|^q \right)^{1/q} = 0.$$

3. *$L_{\mathcal{A}}$ is compact for $\mathcal{A} \in (\ell_p(T), c_0)$ if and only if*

$$\lim_n \left(\sum_k |\tilde{\mathfrak{a}}_{nk}|^q \right)^{1/q} = 0.$$

4. *$L_{\mathcal{A}}$ is compact for $\mathcal{A} \in (\ell_p(T), \ell_1)$ if and only if*

$$\lim_m \|\mathcal{A}\|_{(\ell_p(T), \ell_1)}^{(m)} = 0,$$

where $\|\mathcal{A}\|_{(\ell_p(T), \ell_1)}^{(m)} = \sup_{N \in \mathcal{K}_m} \left(\sum_k | \sum_{n \in N} \tilde{\mathfrak{a}}_{nk}|^q \right)^{1/q}$.

Theorem 4.

1. *For* $\mathcal{A} \in (\ell_\infty(T), \ell_\infty)$,

$$0 \leq \|L_{\mathcal{A}}\|_\chi \leq \limsup_n \sum_k |\tilde{\mathfrak{a}}_{nk}|$$

holds.

2. *For* $\mathcal{A} \in (\ell_\infty(T), c)$,

$$\frac{1}{2} \limsup_n \sum_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k| \leq \|L_{\mathcal{A}}\|_\chi \leq \limsup_n \sum_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k|$$

holds.

3. *For* $\mathcal{A} \in (\ell_\infty(T), c_0)$,

$$\|L_{\mathcal{A}}\|_\chi = \limsup_n \sum_k |\tilde{\mathfrak{a}}_{nk}|$$

holds.

Corollary 3.

1. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_\infty(T), \ell_\infty)$ *if*

$$\lim_n \sum_k |\tilde{\mathfrak{a}}_{nk}| = 0.$$

2. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_\infty(T), c)$, *if and only if*

$$\lim_n \sum_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k| = 0.$$

3. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_\infty(T), c_0)$ *if and only if*

$$\lim_n \sum_k |\tilde{\mathfrak{a}}_{nk}| = 0.$$

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Helicoidal Surfaces Which Have the Timelike Axis in Minkowski Space with Density

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Abstract: In this paper, we study the prescribed curvature problem in manifold with density. We consider the Minkowski 3-space with a positive density function. For a given plane curve and an axis in the plane in Minkowski 3-space, a helicoidal surface can be constructed by the plane curve under helicoidal motions around the axis. Also we give examples of helicoidal surface with weighted Gaussian curvature.

Keywords: Helicoidal, Manifold with density, Minkowski space, Weighted curvature.

1 Introduction

Helicoidal surface is a natural generalization of a rotation surface. A few works have been done with helicoidal surafaces under some given certain conditions [1–4]. Recently, the popular question is whether a helicoidal surface can be constructed when its curvatures are prescribed. Several researchers worked on this problem and obtained useful results. Firstly, Baikoussis et. al have studied helicoidal surfaces with prescribed mean and Gaussian curvature in $\mathbb{R}^3$ [5]. Then, Beneki et. al [6] and Ji et. al [7] have studied the similar work in $\mathbb{R}^3_1$. This problem is extended to manifolds with density. Helicoidal surfaces with prescribed mean and Gaussian curvature in $\mathbb{R}^3$ with density have been studied by Dae Won Yoon et. al [8]. Furthermore, Yıldız et. al have constructed the type I^+ helicoidal surfaces with prescribed weighted curvatures in $\mathbb{R}^3_1$ with density [9].

A manifold with a positive density function ψ used to weight the volume and the hypersurface area. In terms of the underlying Riemannian volume dV_0 and area dA_0, the new, weighted volume and area are given by $dV = \psi dV_0$ and $dA = \psi dA_0$, respectively. One of the most important examples of manifolds with density, with applications to probability and statistics, is Gauss space with density $\psi = e^{a\left(-x^2-y^2-z^2\right)}$ for $a \in \mathbb{R}, (x, y, z) \in \mathbb{R}^3$ [10]. For more details on manifolds with density, see [10–16].

In the Minkowski $3-$space with density e^{φ}, the weighted Gaussian curvature is given with

$$G_{\varphi} = G - \triangle\varphi$$

where G is the Gaussian curvature of the surface and $\triangle$ is the Laplacian operator [17].

In this paper, we study helicoidal surfaces which have the timelike axis in the Minkowski $3-$space R^3_1 with density e^{φ}, where $\varphi = x^2$. Firstly, we construct a helicodial surface with prescribed weighted Gaussian curvature. Finally, we give examples to illustrate.

2 Preliminaries

The Minkowski $3-$space $\mathbb{R}^3_1$ is the real vector space $\mathbb{R}^3$ provided with the standart flat metric given by

$$ds^2 = -dx^2 + dy^2 + dz^2$$

where (x, y, z) is a rectangular coordinate system of $\mathbb{R}^3_1$.

For a given plane curve and an axis in the plane in $\mathbb{R}^3_1$, a helicoidal surface can be constructed by the plane curve under helicoidal motions g_t : $\mathbb{R}^3_1 \to \mathbb{R}^3_1, t \in \mathbb{R}$ around the axis. So, a helicoidal surface is non-degenerate and invariant under $g_t, t \in \mathbb{R}$ for which one parameter subgroup of rigid motions is in $\mathbb{R}^3_1$. There exist four kinds of helicoidal surfaces in $\mathbb{R}^3_1$ which are defined by Beneki et. al [6] and these are called type I, type II, type III , type IV. In this study, type III^+ is considered which has the timelike axis of revolution and the profile curve in $xy-$plane. In addition, the helicoidal surface is called type III^+ since the discriminant of the first fundamental form $u^2\left(1 - g'^2\right) - c^2$ is positive [6].

Let γ be a C^2-curve on $xy-$plane of type $\gamma(u) = (g(u), u, 0)$ where $u \in I$ for an open interval $I \subset \mathbb{R} - \{0\}$. By using helicoidal motion on γ, we can obtain the helicoidal as

$$X(u,v) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{bmatrix} \begin{bmatrix} g(u) \\ u \\ 0 \end{bmatrix} + \begin{bmatrix} cv \\ 0 \\ 0 \end{bmatrix}$$

with $x-$axis and a pitch $c \in \mathbb{R}$. So the parametric equation can be given in the form

$$X(u,v) = (g(u) + cv, u\cos v, u\sin v).$$

It is straightforward to see that the Gaussian curvature G is

$$G = \frac{u^3 g' g'' - c^2}{\left[u^2\left(1 - g'^2\right) - c^2\right]^2}$$

where $u^2\left(1 - g'^2\right) - c^2 > 0$ [6]. We assume that M is the surface in $\mathbb{R}^3_1$ with density e^{φ}, where $\varphi = x^2$. By considering density function, we can calculate the weighted Gaussian curvature G_φ as

$$G_\varphi = \frac{u^3 g' g'' - c^2}{\left(u^2\left(1 - g'^2\right) - c^2\right)^2} - 2. \tag{1}$$

3 Helicoidal Surfaces with Prescribed Gaussian Curvature

Let's solve the ordinary differential equation (1), which is second-order nonlinear ordinary differantial equation. If we take

$$\Psi = \frac{-u^2 {g'}^2 - c^2}{\left(u^2\left(1 - g'^2\right) - c^2\right)} \tag{2}$$

then we obtain

$$G_\varphi = -\frac{1}{2u}\Psi' - 2$$

equivalently,

$$\Psi' = -2uG_\varphi - 4u. \tag{3}$$

The general solution of the equation (3) becomes

$$\Psi = -2u^2 - 2\int uG_\varphi du + c_1 \tag{4}$$

where $c_1 \in \mathbb{R}$. Combining the equation (2) and the equation (4), we get

$$u^2\left(-1 - 2u^2 - 2\int uG_\varphi du + c_1\right) {g'}^2(u) = \left(u^2 - c^2\right)\left(-2u^2 - 2\int uG_\varphi du + c_1\right) + c^2.$$

It follows that

$$g(u) = \mp\int \frac{1}{u}\left[\frac{\left(u^2 - c^2\right)\left(-2u^2 - 2\int uG_\varphi du + c_1\right) + c^2}{-1 - 2u^2 - 2\int uG_\varphi du + c_1}\right]^{\frac{1}{2}} du + c_2 \tag{5}$$

where $c_2 \in \mathbb{R}$.

Conversely, for a given $c \in \mathbb{R}$ and a smooth function $G_\varphi(u)$ defined on an open interval $I \subset \mathbb{R}^+$ and an arbitrary $u_0 \in I$, there exists an open subinterval $I' \subset I$ containing u_0 and an open interval $J \subset \mathbb{R}$ containing

$$\hat{c}_1 = \left(2 + 2u^2 + 2\int uG_\varphi du\right)(u_0)$$

such that

$$F(u, c_1) = -1 - 2u^2 - 2\int uG_\varphi du > 0$$

is defined on $I' \times J$ and it is easily seen F is positive. Thus, two-parameter family of the curves can be given as

$$\gamma(u, G_\varphi(u), c, c_1, c_2) = \left(\mp\int \frac{1}{u}\left[\frac{\left(u^2 - c^2\right)\left(-2u^2 - 2\int uG_\varphi du + c_1\right) + c^2}{-1 - 2u^2 - 2\int uG_\varphi du + c_1}\right]^{\frac{1}{2}} du + c_2, u, 0\right)$$

where $(u, c_1) \in I' \times J$; $c_2 \in \mathbb{R}$, $c \in \mathbb{R}$ and G_φ is smooth function.

Therefore, we have proved the following theorem.

Theorem 1. *Let $\gamma(u)$ be a profile curve of the helicoidal surface given with $X(u,v) = (g(u) + cv, u\cos v, u\sin v)$ in $\mathbb{R}^3_1$ with density e^{x^2} and $G_\varphi(u)$ be the weighted Gaussian curvature at $(g(u), u, 0)$. Then, there exists two-parameter family of the helicoidal surface given by the curves*

$$\gamma(u, G_\varphi(u), c, c_1, c_2) = \left(\mp \int \frac{1}{u} \left[\frac{(u^2 - c^2)\left(4u^2 - 2\int uG_\varphi du + c_1\right) + c^2}{-1 + 4u^2 - 2\int uG_\varphi du + c_1} \right]^{\frac{1}{2}} du + c_2, u, 0 \right)$$

here, c_1 and c_2 are constants. Conversely, for a given smooth function $G_\varphi(u)$, one can obtain the two-parameter family of curves $\gamma(u, G_\varphi(u), c, c_1, c_2)$ being the two-parameter family of helicoidal surfaces, accepting $G_\varphi(u)$ as the weighted Gaussian curvature c as a pitch.

Example Consider a helicoidal surface with the weighted Gaussian curvature

$$G_\varphi(u) = -\frac{8}{15} - \frac{u^2}{3} + \frac{\arctan\left(\sqrt{15}u\right)}{30\sqrt{15}u}$$

in R^3_1 with density e^{x^2}. By using the equation (5), we obtain

$$g(u) = 4u$$

for $c = 1, c_1 = 0, c_2 = 0$ and the parametrization of the surface as follows

$$X(u,v) = (4u + v, u\cos v, u\sin v).$$

The figure of the surface of the domain

$$\begin{cases} 2 < u < 5 \\ -10 < v < 10 \end{cases}$$

is given in Figure 1.

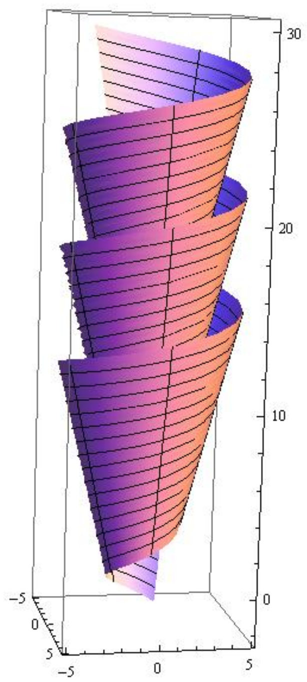

Fig. 1: The helicoidal surface with the weighted Gaussian curvature

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Further Results for Elliptic Biquaternions

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Abstract: In this study, we show that the elliptic biquaternion algebra is algebraically isomorphic to the 2×2 total elliptic matrix algebra and so, we get a faithful 2×2 elliptic matrix representation of an elliptic biquaternion. Also, we investigate the similarity and the Moore-Penrose inverses of elliptic biquaternions by means of these matrix representations. Moreover, we establish universal similarity factorization equality (USFE) over the elliptic biquaternion algebra which reveals a deeper relationship between an elliptic biquaternion and its elliptic matrix representation. This equality and these representations can serve as useful tools for discussing many problems concerned with the elliptic biquaternions, especially for solving various elliptic biquaternion equations.

Keywords: Elliptic biquaternion, Generalized inverse, matrix representation, Universal similarity factorization equality.

1 Introduction

Sir W. R. Hamilton introduced the set of quaternions in 1843 [1], that was one of the his best contribution made to mathematical science. The set of quaternions can be represented as

$$H = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} : \; q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

where the quaternion bases 1, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ satisfy the multiplication laws

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \;\; \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \;\; \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \;\; \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

W.R. Hamilton introduced complex quaternion algebra ten years later from discovery of quaternions, in 1853 [2]. The set of complex quaternions is defined by

$$H_{\mathbb{C}} = \{Q = Q_0 + Q_1\mathbf{i} + Q_2\mathbf{j} + Q_3\mathbf{k} : \;\; Q_0, Q_1, Q_2, Q_3 \in \mathbb{C}\}$$

where $1, \mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are exactly the same in quaternions. There can be found some studies related to quaternions in [3–10].

A fundamental fact (see e.g., [3–6]) is that complex quaternion algebra is isomorphic to the 2×2 total complex matrix algebra $M_2(\mathbb{C})$ by means of the isomorphism

$$\psi : \; H_{\mathbb{C}} \to M_2(\mathbb{C}), \quad \psi(a_o + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) = \begin{bmatrix} a_o + a_1 i & -a_2 - a_3 i \\ a_2 - a_3 i & a_o - a_1 i \end{bmatrix}.$$

Based on this isomorphism, any complex quaternion $x \in H_{\mathbb{C}}$ has a faithful complex matrix representation $\psi(x) \in M_2(\mathbb{C})$.

USFE over an algebra can serve as a precious material for investigating various problems concerned with this algebra and their applications. There can be found some studies which include USFE over various algebras in [11–15].

Recently, we have introduced the set of elliptic biquaternions and presented various studies related to elliptic biquaternions. We refer the readers to [16–20].

This article is organized as follows. In section 2, we recall the fundamental concepts of elliptic matrices and review the elliptic biquaternions and their matrices to disambiguate the ensuing sections. In section 3, 2×2 elliptic matrix representations of elliptic biquaternions are introduced. In section 4, the similarity of elliptic biquaternions is investigated and USFE for elliptic biquaternions is established. In section 5, the Moore-Penrose inverses of elliptic biquaternions are discussed with the aid of their aforementioned matrix representations.

Throughout this paper, the following notations are used. $\mathbb{C}$, $\mathbb{C}_p$, $H\mathbb{C}_p$, $M_{m\times n}(\mathbb{C})$, $M_{m\times n}(\mathbb{C}_p)$ and $M_{m\times n}(H\mathbb{C}_p)$ denote the complex number field, the elliptic number field, the elliptic biquaternion algebra, the set of all $m \times n$ complex matrices, the set of all $m \times n$ elliptic matrices and the set of all $m \times n$ elliptic biquaternion matrices, respectively. For convenience, the set of all square matrices on $\mathbb{C}_p$ is denoted by $M_n(\mathbb{C}_p)$.

2 Preliminaries

In this section, we recall some necessary properties of elliptic matrices. Also, we give some notions about elliptic biquaternions and their matrices. For more details see [16, 20, 21].

In the set of elliptic matrices $M_{m\times n}(\mathbb{C}_p)$ including $m\times n$ matrices with elliptic number entries, the scalar multiplication is defined as

$$\lambda A = \lambda\left[a_{ij}\right] = \left[\lambda a_{ij}\right] \in M_{m\times n}(\mathbb{C}_p)$$

where $\lambda \in \mathbb{C}_p$ and $A = \left[a_{ij}\right] \in M_{m\times n}(\mathbb{C}_p)$. Also, the ordinary matrix addition and multiplication are defined in this set. Let an elliptic matrix $A = \left[a_{ij}\right] \in M_{m\times n}(\mathbb{C}_p)$ be given. In that case, the complex conjugate of A is defined as $\overline{A} = \left[a_{ij}^*\right] \in M_{m\times n}(\mathbb{C}_p)$ where a_{ij}^* is the usual complex conjugation of $a_{ij} \in \mathbb{C}_p$. Also, the conjugate transpose of A is defined as $A^* = \left(\overline{A}\right)^T \in M_{n\times m}(\mathbb{C}_p)$.

On the other hand, the square elliptic matrices A and B with the same dimension over $\mathbb{C}_p$ are said to be similar, if there exists an invertible elliptic matrix P satisfying $P^{-1}AP = B$, [21].

The set of elliptic biquaternions is represented as

$$H\mathbb{C}_p = \{Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \,:\, A_0, A_1, A_2, A_3 \in \mathbb{C}_p\}$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are the quaternionic units which satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,\ \ \mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

The operations of addition, multiplication and scalar multiplication are given as

$$\begin{aligned}
Q + R &= (A_0 + B_0) + (A_1 + B_1)\,\mathbf{i} + (A_2 + B_2)\,\mathbf{j} + (A_3 + B_3)\,\mathbf{k} \\
QR &= [(A_0B_0) - (A_1B_1) - (A_2B_2) - (A_3B_3)] \\
&\quad + [(A_0B_1) + (A_1B_0) + (A_2B_3) - (A_3B_2)]\,\mathbf{i} \\
&\quad + [(A_0B_2) - (A_1B_3) + (A_2B_0) + (A_3B_1)]\,\mathbf{j} \\
&\quad + [(A_0B_3) + (A_1B_2) - (A_2B_1) + (A_3B_0)]\,\mathbf{k} \\
\lambda Q &= (\lambda A_0) + (\lambda A_1)\,\mathbf{i} + (\lambda A_2)\,\mathbf{j} + (\lambda A_3)\,\mathbf{k}
\end{aligned}$$

where $\lambda \in \mathbb{C}_p$ and $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $R = B_0 + B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k} \in H\mathbb{C}_p$. Also, the following equations

$$\begin{aligned}
Q^* &= {A_0}^* + {A_1}^*\mathbf{i} + {A_2}^*\mathbf{j} + {A_3}^*\mathbf{k} \\
\overline{Q} &= A_0 - A_1\mathbf{i} - A_2\mathbf{j} - A_3\mathbf{k} \\
Q^\dagger &= \left(\overline{Q}\right)^* = {A_0}^* - {A_1}^*\mathbf{i} - {A_2}^*\mathbf{j} - {A_3}^*\mathbf{k}
\end{aligned}$$

state the complex conjugate, quaternion conjugate and Hermitian conjugate of Q, respectively. Here the stars given as superscript on $A_0,\ A_1,\ A_2$ and A_3 indicate the usual complex conjugation. If $Q^\dagger = Q$, Q is said to be Hermitian, [16].

As can be seen easily, the meanings of the symbols; star and dagger given as superscript and over bar vary according to terms which they are applied to. We need to warn the readers about these cases for the rest of the paper.

Another thing that can be of importance is the inner product of two elliptic biquaternions. The inner product of Q and R is defined in the following way:

$$\langle Q, R\rangle = \frac{1}{2}\left(\overline{Q}R + \overline{R}Q\right) = \frac{1}{2}\left(Q\overline{R} + R\overline{Q}\right) = A_0B_0 + A_1B_1 + A_2B_2 + A_3B_3.$$

On the other hand, the semi-norm of Q is expressed as follows:

$$N_Q = \langle Q, Q\rangle = {A_0}^2 + {A_1}^2 + {A_2}^2 + {A_3}^2 = Q\,\overline{Q} = \overline{Q}\,Q \in \mathbb{C}_p.$$

When $N_Q \neq 0$, Q has a multiplicative inverse such that $Q^{-1} = \overline{Q}/N_Q$, [16].

The set of all $m\times n$ type matrices with elliptic biquaternion entries is denoted by $M_{m\times n}(H\mathbb{C}_p)$. The ordinary matrix addition and multiplication are defined in this matrix set. Also, the scalar multiplication is expressed as in the following:

$$QA = Q\left[a_{ij}\right] = \left[Qa_{ij}\right] \in M_{m\times n}(H\mathbb{C}_p)$$

where $Q \in H\mathbb{C}_p$ and $A = \left[a_{ij}\right] \in M_{m\times n}(H\mathbb{C}_p)$. For $A = \left[a_{ij}\right] \in M_{m\times n}(H\mathbb{C}_p)$, the Hermitian conjugate of A is defined as $A^\dagger = \left[{a_{ji}}^\dagger\right] \in M_{n\times m}(H\mathbb{C}_p)$ where ${a_{ji}}^\dagger$ is the Hermitian conjugate of $a_{ji} \in H\mathbb{C}_p$, [20].

3 Elliptic Matrix Representations of Elliptic Biquaternions

In this section, we get 2×2 elliptic matrix representations of elliptic biquaternions and give some properties which are satisfied by these representations and elliptic biquaternions.

Let us consider the matrix set $M_2(\mathbb{C}_p)$ which can be represented as

$$M_2(\mathbb{C}_p) = \left\{ \begin{bmatrix} x & y \\ z & t \end{bmatrix} : \ x, y, z, t \in \mathbb{C}_p \right\}.$$

In the following lemma, we show that this matrix set can be represented as in a somewhat different form which are used to define the required isomorphism.

Result 1. *The set of* 2×2 *elliptic matrices can be represented as*

$$M_2(\mathbb{C}_p) = \left\{ \begin{bmatrix} X_0 + \frac{1}{\sqrt{|p|}} I X_1 & -X_2 - \frac{1}{\sqrt{|p|}} I X_3 \\ X_2 - \frac{1}{\sqrt{|p|}} I X_3 & X_0 - \frac{1}{\sqrt{|p|}} I X_1 \end{bmatrix} : X_i = x_i + I x_i{}' \in \mathbb{C}_p, \ 0 \le i \le 3 \right\}. \tag{1}$$

Proof: Let $A = \begin{bmatrix} z_1 + I z_1{}^{\#} & z_2 + I z_2{}^{\#} \\ z_3 + I z_3{}^{\#} & z_4 + I z_4{}^{\#} \end{bmatrix}$ be an arbitrary 2×2 elliptic matrix where $z_1, \ z_1{}^{\#}, \ z_2, \ z_2{}^{\#}, \ z_3, \ z_3{}^{\#}, \ z_4$ and $z_4{}^{\#}$ are real numbers. Then, we can write

$$A = \begin{bmatrix} \left(x_0 - \sqrt{|p|}\, x_1{}'\right) + I\left(x_0{}' + \frac{x_1}{\sqrt{|p|}}\right) & \left(-x_2 + \sqrt{|p|}\, x_3{}'\right) + I\left(-x_2{}' - \frac{x_3}{\sqrt{|p|}}\right) \\ \left(x_2 + \sqrt{|p|}\, x_3{}'\right) + I\left(x_2{}' - \frac{x_3}{\sqrt{|p|}}\right) & \left(x_0 + \sqrt{|p|}\, x_1{}'\right) + I\left(x_0{}' - \frac{x_1}{\sqrt{|p|}}\right) \end{bmatrix} \tag{2}$$

such that

$$x_0 = \frac{z_1 + z_4}{2}, \quad x_0{}' = \frac{z_1{}^{\#} + z_4{}^{\#}}{2}, \quad x_1 = \frac{\sqrt{|p|}\left(z_1{}^{\#} - z_4{}^{\#}\right)}{2}, \quad x_1{}' = \frac{z_4 - z_1}{2\sqrt{|p|}} \in \mathbb{R}$$

$$x_2 = \frac{z_3 - z_2}{2}, \quad x_2{}' = \frac{z_3{}^{\#} - z_2{}^{\#}}{2}, \quad x_3 = -\frac{\sqrt{|p|}\left(z_2{}^{\#} + z_3{}^{\#}\right)}{2}, \quad x_3{}' = \frac{z_2 + z_3}{2\sqrt{|p|}} \in \mathbb{R}.$$

It can be easily seen that the arbitrary 2×2 elliptic matrix in (2) is equal to the matrix

$$\begin{bmatrix} X_0 + \frac{1}{\sqrt{|p|}} I X_1 & -X_2 - \frac{1}{\sqrt{|p|}} I X_3 \\ X_2 - \frac{1}{\sqrt{|p|}} I X_3 & X_0 - \frac{1}{\sqrt{|p|}} I X_1 \end{bmatrix}$$

where $X_i = x_i + I x_i{}' \in \mathbb{C}_p$, $0 \le i \le 3$.

Conversely, it is clear that the matrix given in (1) is a 2×2 elliptic matrix. □

Let us take into account the function

$$\begin{aligned} \sigma \ : \ & H\mathbb{C}_p \ \to \ M_2(\mathbb{C}_p) \\ Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \to \sigma(Q) = & \begin{bmatrix} A_0 + \frac{1}{\sqrt{|p|}} I A_1 & -A_2 - \frac{1}{\sqrt{|p|}} I A_3 \\ A_2 - \frac{1}{\sqrt{|p|}} I A_3 & A_0 - \frac{1}{\sqrt{|p|}} I A_1 \end{bmatrix}. \end{aligned}$$

The function σ comprises the properties

$$\sigma(Q + R) = \sigma(Q) + \sigma(R), \ \ \sigma(QR) = \sigma(Q)\,\sigma(R)$$

where Q and R are any elliptic biquaternions. Also it is bijection. So, σ is a linear isomorphism.

Corollary 1. *For an arbitrary* 2×2 *elliptic matrix* A*,* $Q \in H\mathbb{C}_p$ *satisfying the equality* $\sigma(Q) = A$ *is existence and uniqueness.*

Proof: The proof is obvious from the linear isomorphism σ and Lemma 1. □

Definition 1. *Let* $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in H\mathbb{C}_p$ *be an arbitrary elliptic biquaternion where* A_0*,* A_1*,* A_2*,* $A_3 \in \mathbb{C}_p$*, in that case the elliptic matrix*

$$\sigma(Q) = \begin{bmatrix} A_0 + \frac{1}{\sqrt{|p|}} I A_1 & -A_2 - \frac{1}{\sqrt{|p|}} I A_3 \\ A_2 - \frac{1}{\sqrt{|p|}} I A_3 & A_0 - \frac{1}{\sqrt{|p|}} I A_1 \end{bmatrix}$$

which corresponds to Q *is called* 2×2 *elliptic matrix representation of* Q*.*

Next two theorems include some properties which are satisfied by elliptic biquaternions and their 2×2 elliptic matrix representations.

Theorem 1. *Let $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $R = B_0 + B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k} \in H\mathbb{C}_p$ and $\lambda \in \mathbb{C}_p$ be given. In this case*

1. $\det(\sigma(Q)) = N_Q = {A_0}^2 + {A_1}^2 + {A_2}^2 + {A_3}^2$,
2. *Q is invertible if and only if $\sigma(Q)$ is invertible, then* $\sigma\left(Q^{-1}\right) = (\sigma(Q))^{-1}$ *and* $Q^{-1} = \frac{1}{4}E_2(\sigma(Q))^{-1}{E_2}^{\dagger}$,
3. $Q = R \Leftrightarrow \sigma(Q) = \sigma(R)$,
4. $\sigma(Q+R) = \sigma(Q) + \sigma(R)$, $\sigma(QR) = \sigma(Q)\sigma(R)$, $\sigma(\lambda Q) = \sigma(Q\lambda) = \lambda\sigma(Q)$, $\sigma(1) = I_2$,
5. $Q = \frac{1}{4}E_2\sigma(Q){E_2}^{\dagger}$,

where $E_2 = \begin{bmatrix} 1 - \frac{1}{\sqrt{|p|}}I\mathbf{i} & \mathbf{j} + \frac{1}{\sqrt{|p|}}I\mathbf{k} \end{bmatrix} \in M_{1\times 2}(H\mathbb{C}_p)$.

Proof: The proof of 3 and 4 are obvious due to the aforementioned linear isomorphism σ. On the other hand, the proof of 5 can be completed by direct calculation. Now, we will prove 1 and 2.

1. We know that $\sigma(Q) = \begin{bmatrix} A_0 + \frac{1}{\sqrt{|p|}}IA_1 & -A_2 - \frac{1}{\sqrt{|p|}}IA_3 \\ A_2 - \frac{1}{\sqrt{|p|}}IA_3 & A_0 - \frac{1}{\sqrt{|p|}}IA_1 \end{bmatrix}$. Then we obtain

$$\begin{aligned} \det(\sigma(Q)) &= {A_0}^2 - {A_1}^2\frac{I^2}{|p|} + {A_2}^2 - {A_3}^2\frac{I^2}{|p|} \\ &= {A_0}^2 - {A_1}^2\frac{p}{(-p)} + {A_2}^2 - {A_3}^2\frac{p}{(-p)} \\ &= {A_0}^2 + {A_1}^2 + {A_2}^2 + {A_3}^2. \end{aligned}$$

2. For an elliptic biquaternion Q, we know that Q is invertible if and only if $N_Q \neq 0$. Therefore, by means of the first property in this theorem, we can write

$$Q\,is\,invertible \Leftrightarrow N_Q \neq 0 \Leftrightarrow \det(\sigma(Q)) \neq 0 \Leftrightarrow \sigma(Q)\ is\,invertible.$$

Suppose that Q and $\sigma(Q)$ are invertible. In this case, from the inverse property, the equality

$$QQ^{-1} = Q^{-1}Q = 1$$

is satisfied. Then, by means of the third and fourth properties in this theorem, the equalities

$$\sigma(Q)\,\sigma\left(Q^{-1}\right) = \sigma\left(QQ^{-1}\right) = \sigma(1) = I_2$$

and

$$\sigma\left(Q^{-1}\right)\sigma(Q) = \sigma\left(Q^{-1}Q\right) = \sigma(1) = I_2$$

are obtained. It means that $(\sigma(Q))^{-1} = \sigma\left(Q^{-1}\right)$. Therefore, by considering the fifth property in this theorem, we obtain $Q^{-1} = \frac{1}{4}E_2(\sigma(Q))^{-1}{E_2}^{\dagger}$. □

Theorem 2. *Let $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in H\mathbb{C}_p$ be given. In this case*

1. $\sigma\left(\overline{Q}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(\sigma(Q))^T\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *where $\overline{Q}$ is the quaternion conjugate of Q,*
2. $\sigma(Q^*) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\overline{\sigma(Q)}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *where Q^* is the complex conjugate of Q,*
3. $\sigma\left(Q^{\dagger}\right) = \left(\overline{\sigma(Q)}\right)^T = (\sigma(Q))^*$ *where $Q^{\dagger}$ is the Hermitian conjugate of Q.*

Proof: 2 and 3 can be easily shown, Now, we will prove 1.

1. For $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, we can write $\overline{Q} = A_0 - A_1\mathbf{i} - A_2\mathbf{j} - A_3\mathbf{k}$. In this case, we get

$$\sigma\left(\overline{Q}\right) = \begin{bmatrix} A_0 - \frac{1}{\sqrt{|p|}} I A_1 & A_2 + \frac{1}{\sqrt{|p|}} I A_3 \\ -A_2 + \frac{1}{\sqrt{|p|}} I A_3 & A_0 + \frac{1}{\sqrt{|p|}} I A_1 \end{bmatrix}.$$

On the other hand, it is clear that

$$(\sigma(Q))^T = \begin{bmatrix} A_0 + \frac{1}{\sqrt{|p|}} I A_1 & A_2 - \frac{1}{\sqrt{|p|}} I A_3 \\ -A_2 - \frac{1}{\sqrt{|p|}} I A_3 & A_0 - \frac{1}{\sqrt{|p|}} I A_1 \end{bmatrix}.$$

Then, by directly multiplying we obtain

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} (\sigma(Q))^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} A_0 - \frac{1}{\sqrt{|p|}} I A_1 & A_2 + \frac{1}{\sqrt{|p|}} I A_3 \\ -A_2 + \frac{1}{\sqrt{|p|}} I A_3 & A_0 + \frac{1}{\sqrt{|p|}} I A_1 \end{bmatrix} = \sigma\left(\overline{Q}\right).$$

□

4 Similarity of Elliptic Biquaternions and USFE for Elliptic Biquaternions

In this section, we investigate the similarity of elliptic biquaternions with the aid of their elliptic matrix representations and establish universal similarity factorization equality for elliptic biquaternions.

4.1 Similarity of elliptic biquaternions

One of the natural questions concerned with elliptic biquaternions is the similarity of two elliptic biquaternions. By analogy with the classic quaternion case, the next definition is given.

Definition 2. *For $Q, R \in H\mathbb{C}_p$, if there exists an invertible elliptic biquaternion X such that $X^{-1}QX = R$, Q and R are called similar elliptic biquaternions. This case is denoted by $Q \sim R$.*

By considering Definition 2, a simple result, which characterizes the similarity of two elliptic biquaternions, can be given as follows.

Theorem 3. *Let $Q, R \in H\mathbb{C}_p$ be given. In this case,*

$$Q \sim R \Leftrightarrow \sigma(Q) \sim \sigma(R). \tag{3}$$

Proof: $Q \sim R$ if and only if there is an invertible elliptic biquaternion X such that $X^{-1}QX = R$. Then, we have

$$\begin{aligned} Q \sim R &\Leftrightarrow \sigma\left(X^{-1}QX\right) = \sigma(R) \\ &\Leftrightarrow \sigma\left(X^{-1}\right)\sigma(Q)\sigma(X) = \sigma(R) \\ &\Leftrightarrow (\sigma(X))^{-1}\sigma(Q)\sigma(X) = \sigma(R) \\ &\Leftrightarrow \sigma(Q) \sim \sigma(R) \end{aligned}$$

from Theorem 1 (2), (3) and (4). □

As a consequence of Theorem 3, we can give the following theorem.

Theorem 4. *Let $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in H\mathbb{C}_p$ be given where $Q \notin \mathbb{C}_p$.*

1. *If ${A_1}^2 + {A_2}^2 + {A_3}^2 \neq 0$, in that case $Q \sim A_0 + \gamma(Q)\,\mathbf{i}$ where $\gamma(Q)$ is an elliptic number satisfying the equality $\gamma^2(Q) = {A_1}^2 + {A_2}^2 + {A_3}^2$.*
2. *If ${A_1}^2 + {A_2}^2 + {A_3}^2 = 0$, in that case $Q \sim A_0 - \frac{1}{2}\mathbf{j} + \frac{1}{2\sqrt{|p|}} I\mathbf{k}$.*

Proof: For a given elliptic biquaternion $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in H\mathbb{C}_p$, we have its 2×2 elliptic matrix representation $\sigma(Q)$. We can calculate its characteristic polynomial as follows:

$$|\lambda I_2 - \sigma(Q)| = \begin{vmatrix} \lambda - A_0 - \dfrac{I}{\sqrt{|p|}}A_1 & A_2 + \dfrac{I}{\sqrt{|p|}}A_3 \\ -A_2 + \dfrac{I}{\sqrt{|p|}}A_3 & \lambda - A_0 + \dfrac{I}{\sqrt{|p|}}A_1 \end{vmatrix} = (\lambda - A_0)^2 + {A_1}^2 + {A_2}^2 + {A_3}^2.$$

For ${A_1}^2 + {A_2}^2 + {A_3}^2 \neq 0$, we can get the roots of the above characteristic polynomial as $\lambda_{1,2} = A_0 \pm \frac{1}{\sqrt{|p|}}I\gamma(Q)$. Thus, we immediately have

$$\sigma(Q) \sim \begin{bmatrix} A_0 + \frac{1}{\sqrt{|p|}}I\gamma(Q) & 0 \\ 0 & A_0 - \frac{1}{\sqrt{|p|}}I\gamma(Q) \end{bmatrix} = \sigma(A_0 + \gamma(Q)\,\mathbf{i}). \tag{4}$$

For ${A_1}^2 + {A_2}^2 + {A_3}^2 = 0$, we can get the roots of the characteristic polynomial of $\sigma(Q)$ as $\lambda_{1,2} = A_0$. Then, considering the Jordan canonical form of $\sigma(Q)$, we can write the following

$$\sigma(Q) \sim \begin{bmatrix} A_0 & 1 \\ 0 & A_0 \end{bmatrix} = \sigma\left(A_0 - \frac{1}{2}\mathbf{j} + \frac{1}{2\sqrt{|p|}}I\mathbf{k}\right). \tag{5}$$

If we apply Theorem 3 to (4) and (5), we can easily prove the first part and second part of this theorem, respectively. □

4.2 USFE for elliptic biquaternions

There is a deeper relationship between an elliptic biquaternion Q and its elliptic matrix representation $\sigma(Q)$ which appears with USFE over the elliptic biquaternion algebra.

In [11], Tian presents a general result on the universal similarity factorization of elements over any algebra as follows:
Let A be an algebra over an arbitrary field F and $M_n(A)$ be the matrix algebra which includes all $n \times n$ matrices with elements in A. Also, let $\{\tau_{ij}\}$ be the basis of A that satisfies the following rules

$$\tau_{i\,j}\tau_{s\,t} = \begin{cases} \tau_{i\,t} & j = s \\ 0 & j \neq s \end{cases}, \quad i, j, s, t = 1, ..., n. \tag{6}$$

In this case, any $Q = \sum\limits_{i,j=1}^{n} a_{i\,j}\tau_{i\,j} \in A \quad (a_{i\,j} \in F)$ satisfies the following USFE

$$P \begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & Q \end{bmatrix} P^{-1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{7}$$

where P has the following independent form

$$P = P^{-1} = \begin{bmatrix} \tau_{11} & \tau_{21} & \cdots & \tau_{n1} \\ \tau_{12} & \tau_{22} & \cdots & \tau_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ \tau_{1n} & \tau_{2n} & \cdots & \tau_{nn} \end{bmatrix}. \tag{8}$$

By basing on the general result indicated above, we establish USFE for elliptic biquaternions as follows.

Theorem 5. *Let $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in H\mathbb{C}_p$ be given. In this case, the elliptic biquaternion matrix $\begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}$ satisfies the following USFE*

$$P \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} P^{-1} = \begin{bmatrix} A_0 + \frac{1}{\sqrt{|p|}}IA_1 & -A_2 - \frac{1}{\sqrt{|p|}}IA_3 \\ A_2 - \frac{1}{\sqrt{|p|}}IA_3 & A_0 - \frac{1}{\sqrt{|p|}}IA_1 \end{bmatrix} = \sigma(Q) \in M_2(\mathbb{C}_p) \tag{9}$$

where P is in the following independent form:

$$P = P^{-1} = \frac{1}{2}\begin{bmatrix} 1 - \frac{1}{\sqrt{|p|}}I\mathbf{i} & \mathbf{j} + \frac{1}{\sqrt{|p|}}I\mathbf{k} \\ -\mathbf{j} + \frac{1}{\sqrt{|p|}}I\mathbf{k} & 1 + \frac{1}{\sqrt{|p|}}I\mathbf{i} \end{bmatrix} \in M_2(H\mathbb{C}_p). \tag{10}$$

Proof: Let $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in H\mathbb{C}_p$ be an arbitrary elliptic biquaternion and let us consider the elliptic biquaternions

$$\tau_{11} = \frac{1}{2} - \frac{I}{2\sqrt{|p|}}\mathbf{i},\ \ \tau_{12} = -\frac{1}{2}\mathbf{j} + \frac{I}{2\sqrt{|p|}}\mathbf{k},\ \ \tau_{21} = \frac{1}{2}\mathbf{j} + \frac{I}{2\sqrt{|p|}}\mathbf{k},\ \ \tau_{22} = \frac{1}{2} + \frac{I}{2\sqrt{|p|}}\mathbf{i}. \tag{11}$$

It is clear that the system $\{\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}\}$ is a base of elliptic biquaternion algebra from the equalities

$$\langle \tau_{st}\,,\, \tau_{pq} \rangle = \begin{cases} 1\,, \ \ (s = p) \wedge \ (t = q)\,, \ \ s,t,p,q = 1,2 \\ \\ 0\,, \ \ (s \neq p) \vee \ (t \neq q)\,, \ \ s,t,p,q = 1,2 \end{cases}$$

and

$$Q = \left(A_0 + \frac{A_1 I}{\sqrt{|p|}}\right)\tau_{11} + \left(-A_2 - \frac{A_3 I}{\sqrt{|p|}}\right)\tau_{12} + \left(A_2 - \frac{A_3 I}{\sqrt{|p|}}\right)\tau_{21} + \left(A_0 - \frac{A_1 I}{\sqrt{|p|}}\right)\tau_{22}.$$

For the case $n = 2$, it is easy to verify that these new bases in (11) satisfy the multiplication rules in (6). Then, if we consider the last equality above and (11) in (7) and in (8) by keeping the case $n = 2$ in mind, we get (9) and (10). □

If Lemma 1 is considered, by means of USFE for elliptic biquaternions, it can be said that every 2×2 elliptic matrix is uniformly similar to the diagonal matrix $diag\,(Q, Q)$ where Q is the elliptic biquaternion which corresponds to this 2×2 elliptic matrix.

5 Moore-Penrose Inverses of Elliptic Biquaternions

In this section, we define the Moore-Penrose inverse of any elliptic matrix and show that it always exists uniquely. Afterwards, we give the similar definition for elliptic biquaternions as well. Then, the existence and uniqueness of the Moore Penrose inverse for an elliptic biquaternion Q are determined by the matrix $\sigma\,(Q) \in M_2\,(\mathbb{C}_p)$.

Definition 3. *Let an arbitrary elliptic matrix $A \in M_{m\times n}\,(\mathbb{C}_p)$ be given. If the equations*

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA \tag{12}$$

have a common solution $X \in M_{n\times m}\,(\mathbb{C}_p)$, in this case this solution is called Moore-Penrose inverse of A. It is showed with $X = A^+$.

Theorem 6. *Let $A \in M_{m\times n}\,(\mathbb{C}_p)$ be given. In this case the Moore-Penrose inverse of A is existence and uniqueness.*

Proof: We define a function between the space of $m \times n$ elliptic matrices and the space of $m \times n$ complex matrices as follows:

$$\delta : M_{m\times n}\,(\mathbb{C}_p) \to M_{m\times n}\,(\mathbb{C})$$

$$\begin{bmatrix} a_{11} + Ib_{11} & \dots & a_{1n} + Ib_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + Ib_{m1} & \cdots & a_{mn} + Ib_{mn} \end{bmatrix} \to \begin{bmatrix} a_{11} + i\left(b_{11}\sqrt{|p|}\right) & \dots & a_{1n} + i\left(b_{1n}\sqrt{|p|}\right) \\ \vdots & \ddots & \vdots \\ a_{m1} + i\left(b_{m1}\sqrt{|p|}\right) & \cdots & a_{mn} + i\left(b_{mn}\sqrt{|p|}\right) \end{bmatrix}.$$

As can be seen easily, the function δ is bijection and so we can write

$$A = B \ \Leftrightarrow\ \delta\,(A) = \delta\,(B) \tag{13}$$

for $A,\ B \in M_{m\times n}\,(\mathbb{C}_p)$. Also, it comprises the following properties

$$\delta\,(A + B) = \delta\,(A) + \delta\,(B)\,, \quad \delta\,(AB) = \delta\,(A)\,\delta\,(B) \tag{14}$$

and

$$\delta\left(A^*\right) = \left(\delta\,(A)\right)^* \tag{15}$$

where A^* is the conjugate transpose of $A \in M_{m\times n}\,(\mathbb{C}_p)$ and $(\delta\,(A))^*$ is the conjugate transpose of $\delta\,(A) \in M_{m\times n}\,(\mathbb{C})$. From (13) and (14), it is clear that δ is an isomorphism.

Thanks to (13), (14) and (15), the elliptic matrix equation system (12) is equivalent to the following complex matrix equation system:

$$\begin{aligned} \delta\,(A)\,\delta\,(X)\,\delta\,(A) &= \delta\,(A)\,, & \delta\,(X)\,\delta\,(A)\,\delta\,(X) &= \delta\,(X)\,, \\ (\delta\,(A)\,\delta\,(X))^* &= \delta\,(A)\,\delta\,(X)\,, & (\delta\,(X)\,\delta\,(A))^* &= \delta\,(X)\,\delta\,(A)\,. \end{aligned} \tag{16}$$

According to the complex matrix theory (see [22] for more details) the four equations

$$\delta\,(A)\,Y\delta\,(A) = \delta\,(A)\,, \quad Y\delta\,(A)\,Y = Y, \quad (\delta\,(A)\,Y)^* = \delta\,(A)\,Y, \quad (Y\delta\,(A))^* = Y\delta\,(A)$$

have a unique common solution $Y = (\delta\,(A))^+$ which is called the Moore-Penrose inverse of $\delta\,(A)$. Thus, if we take into account the system (16) we can immediately obtain $\delta\,(X) = (\delta\,(A))^+$. From the definition of isomorphism δ, it is clear that the matrix $X \in M_{m\times n}\,(\mathbb{C}_p)$ which satisfies $\delta(X) = (\delta(A))^+$ is existence and uniqueness. In this case, with the aid of the equalities (13), (14) and (15), we conclude that the elliptic matrix X, which is indicated above, is the unique solution of the elliptic matrix equation system (12). $\square$

Definition 4. *Let an elliptic biquaternion $Q \in H\mathbb{C}_p$ be given. If the equations*

$$QXQ = Q, \quad XQX = X, \quad (QX)^\dagger = QX, \quad (XQ)^\dagger = XQ \tag{17}$$

have a common solution $X \in H\mathbb{C}_p$, in this case this solution is called Moore-Penrose inverse of Q. It is showed with $X = Q^+$.

Theorem 7. *Let $Q \in H\mathbb{C}_p$. In that case, its Moore-Penrose inverse Q^+ is existence and uniqueness. Also Q^+ satisfies the following equalities*

$$\sigma\left(Q^+\right) = (\sigma(Q))^+, \quad Q^+ = \frac{1}{4}E_2(\sigma(Q))^+{E_2}^\dagger$$

where $E_2 = \left[1 - \frac{1}{\sqrt{|p|}}I\mathbf{i} \quad \mathbf{j} + \frac{1}{\sqrt{|p|}}I\mathbf{k}\right] \in M_{1\times 2}(H\mathbb{C}_p)$.

Proof: If we consider Theorem 1 (3), (4) and Theorem 2 (3), we can easily see that the elliptic biquaternion equation system (17) is equivalent to the following elliptic matrix equation system:

$$\begin{aligned} &\sigma(Q)\,\sigma(X)\,\sigma(Q) = \sigma(Q)\,, &&\sigma(X)\,\sigma(Q)\,\sigma(X) = \sigma(X)\,, \\ &(\sigma(Q)\,\sigma(X))^* = \sigma(Q)\,\sigma(X)\,, &&(\sigma(X)\,\sigma(Q))^* = \sigma(X)\,\sigma(Q)\,. \end{aligned} \tag{18}$$

According to Definition 3 and Theorem 6, the four equations

$$\begin{aligned} &\sigma(Q)\,Y\sigma(Q) = \sigma(Q)\,, &&Y\sigma(Q)\,Y = Y, \\ &(\sigma(Q)\,Y)^* = \sigma(Q)\,Y, &&(Y\sigma(Q))^* = Y\sigma(Q) \end{aligned}$$

have a unique common solution $Y = (\sigma(Q))^+ \in M_2(\mathbb{C}_p)$ which is called the Moore-Penrose inverse of $\sigma(Q)$. Thus, if we take into account the system (18) we can write $\sigma(X) = (\sigma(Q))^+$. From Corollary 1, we know that the elliptic biquaternion $X \in H\mathbb{C}_p$ which satisfies $\sigma(X) = (\sigma(Q))^+$ is existence and uniqueness. In this case, with the aid of Theorem 1 (3), (4) and Theorem 2 (3), we conclude that the elliptic biquaternion X, which is indicated above, is the unique solution of the system (17). According to Definition 4, we denote this X by $X = Q^+$. Thus, it is clear that $\sigma\left(Q^+\right) = (\sigma(Q))^+$. From this last equality and Theorem 1 (5), $Q^+ = \frac{1}{4}E_2(\sigma(Q))^+{E_2}^\dagger$ can be easily obtained. $\square$

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One Parameter Elliptical Planar Motion

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Abstract: In order to describe the elliptical planar motion, two moving and one fixed elliptical planes have been considered. Thus, one parameter elliptical planar motion is defined by the help of these planes. The absolute, relative and sliding velocities have been obtained and the relation between these velocities have been proven. Also, pole points of the elliptical planar motion have been derived. Finally, some results have been given regarding to the absolute, relative, sliding velocities and the pole points of elliptical planar motion.

Keywords: Elliptical planar motion, Elliptical plane, Kinematics.

1 Introduction

Motion is mathematically described as a change in the position of an object or a point with respect to time. The rate of motion in a specific direction gives us velocity. The velocities of the planar motion is always measured with respect to a coordinate systems (frames of reference). Although spherical motions are of great interest to researchers, planar motion has an important place in kinematics. Because many objects in engineering are relatively flat and thin or symmetrical. That is, the motion of these objects is considered to be approximately planar motion.
Müller initially studied one parameter planar motions and obtained the relationships between absolute, relative and sliding velocities and accelerations in the Euclidean plane E^2. Moreover, he gave the Euler-Savary formula which gives the relationship between the curvatures of the trajectory curves [1].
Blaschke and Müller have introduced one parameter planar motions in terms of complex numbers [2]. In [3], it was demonstrated that the relation between complex velocities and pole points can be obtained with the help of moving coordinate system for the one paremeter motion in the complex plane.
Pereira and Ersoy have introduced elliptical harmonic motion by using elliptical numbers. Also, they have found the relationships between the absolute, the relative and the sliding velocities and accelerations for this motion. Furthermore, the canonical relative system of the motion has been deïňĄned and EulerâĂŞSavary formula has been obtained [4].
Özdemir has given the generation of elliptical rotations by the help of the elliptic scalar product and elliptic vector product for a given ellipsoid. For this purpose, an elliptical ortogonal matrix and an elliptical skew symmetric matrix have been defined for this elliptic inner product. Thereby, he has examined the motion of a point on the ellipsoid using elliptical rotation matrices [5].
This paper is organized as follows. In the ïňĄrst part, basic concepts have been represented as if elliptic inner product, elliptical norm of a vector, elliptical rotation matrix etc. In the second part, one parameter elliptical planar motion has been introduced by the help of the elliptically orthogonal systems of $\{O; \vec{e}_1, \vec{e}_2\}$ and $\{O'; \vec{e}'_1, \vec{e}'_2\}$. These orthogonal systems represent the moving elliptical plane E and the ïňĄxed elliptical plane E', respectively. Furthermore, an elliptically orthogonal, relative system $\left\{B; \overrightarrow{h_1}, \overrightarrow{h_2}\right\}$ has been considered. Thus, the theorems and results have been given regarding to the velocities and pole points of this motion.

2 Basic Concepts

Let us consider a for an ellipse in the form

$$(E) : a_1 x^2 + a_2 y^2 = 1, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a_1 = \frac{1}{a^2},\ a_2 = \frac{1}{b^2}$ and $a, b \in R$ (see [5]). The elliptic inner product or B-inner product for the vectors $\vec{u} = (u_1, u_2)$, $\vec{w} = (w_1, w_2) \in R^2$

$$B(\vec{u}, \vec{w}) = a_1 u_1 w_1 + a_2 u_2 w_2$$

where $a_1,\, a_2 \in R^+$. This scalar product is positive definite and also can be written as $B(\vec{u}, \vec{w}) = u^t \Omega w$ where the associated matrix Ω is defined as follows

$$\Omega = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}.$$

Thus, the real vector space R^2 equipped with the elliptic inner product will be denoted by ${R^2}_{a_1,a_2}$ and the number $\Delta = \sqrt{\det \Omega}$ will be called "constant of the scalar product" [5].
The elliptical norm of a vector $\vec{u} \in R^2$ is defined to be $\|\vec{u}\|_B = \sqrt{B(\vec{u}, \vec{u})}$. Moreover, two vectors $\vec{u}$ and $\vec{w}$ are called B-orthogonal or elliptically orthogonal vectors if $B(\vec{u}, \vec{w}) = 0$. In addition to that if their norms become 1, then these vectors are called elliptically orthonormal or B-orthonormal. The cosine of the angle between two vectors $\vec{u}$ and $\vec{w}$ is defined as,

$$\cos\theta = \frac{B(\vec{u}, \vec{w})}{\|\vec{u}\|_B \|\vec{w}\|_B}$$

where θ is compatible with the parameters of the angular parametric equations of ellipse [5]. Let T be a B-skew symmetric matrix. Then, an elliptical rotation matrix in the space $R^2_{a_1,a_2}$ is defined by

$$R^B_\theta = \begin{bmatrix} \cos\theta & \frac{\sqrt{a_2}}{\sqrt{a_1}}\sin\theta \\ -\frac{\sqrt{a_1}}{\sqrt{a_2}}\sin\theta & \cos\theta \end{bmatrix}.$$

3 One Parameter Planar Elliptical Motion

Let E_1 and E be moving and E' be fixed elliptical planes and $\left\{B; \overrightarrow{h_1}, \overrightarrow{h_2}\right\}$, $\{O; \vec{e}_1, \vec{e}_2\}$, $\{O'; \vec{e}_1', \vec{e}_2'\}$ represent their orthogonal coordinate systems, respectively.
Therefore, the following scalar products can be written for the vectors

$$B(\vec{e}_i, \vec{e}_j) = \begin{cases} a_i, & if\, i = j \\ 0, & if\, i \neq j \end{cases}$$

and

$$B(\vec{e}_i', \vec{e}_j') = \begin{cases} a_i, & if \quad i = j \\ 0, & if\, i \neq j \end{cases}$$

The motion of the moving relative plane E_1 with respect to other moving plane E is given by the following relation

$$\begin{aligned} \overrightarrow{h_1} &= \cos\theta\, \overrightarrow{e_1} + \frac{\sqrt{a_2}}{\sqrt{a_1}}\sin\theta\, \overrightarrow{e_2} \\ \overrightarrow{h_2} &= -\frac{\sqrt{a_1}}{\sqrt{a_2}}\sin\theta\, \overrightarrow{e_1} + \cos\theta\, \overrightarrow{e_2} \end{aligned} \tag{1}$$

and

$$\overrightarrow{OB} = \overrightarrow{b} = b_1\overrightarrow{h_1} + b_2\overrightarrow{h_2} \tag{2}$$

Here θ is elliptical rotation angle and $\overrightarrow{OB}$ represents the vector from the origin of the moving coordinate system to the origin of other moving relative coordinate system.
Similarly, the motion of the moving relative plane E_1 with respect to fixed plane E' is given by

$$\begin{aligned} \overrightarrow{h_1} &= \cos\theta\, \overrightarrow{e_1}' + \frac{\sqrt{a_2}}{\sqrt{a_1}}\sin\theta\, \overrightarrow{e_2}' \\ \overrightarrow{h_2} &= -\frac{\sqrt{a_1}}{\sqrt{a_2}}\sin\theta\, \overrightarrow{e_1}' + \cos\theta\, \overrightarrow{e_2}' \end{aligned} \tag{3}$$

and

$$\overrightarrow{O'B} = \overrightarrow{b'} = b_1'\overrightarrow{h_1} + b_2'\overrightarrow{h_2} \tag{4}$$

Here θ' is elliptical rotation angle and $\overrightarrow{O'B}$ represents the vector from the origin of the fixed coordinate system to the origin of moving relative coordinate system.
Taking the differentials of the equations (1) and (2) and rearranging for the motion of E_1/E, we obtain

$$\begin{aligned} d\overrightarrow{h_1} &= \frac{\sqrt{a_2}}{\sqrt{a_1}} d\theta\, \overrightarrow{h_2} \\ d\overrightarrow{h_2} &= -\frac{\sqrt{a_1}}{\sqrt{a_2}} d\theta\, \overrightarrow{h_1} \\ d\,\overrightarrow{b} &= \left(db_1 - \frac{\sqrt{a_1}}{\sqrt{a_2}} b_2\, d\theta\right)\overrightarrow{h_1} + \left(db_2 + \frac{\sqrt{a_2}}{\sqrt{a_1}} b_1 d\theta\right)\overrightarrow{h_2}. \end{aligned}$$

Similarly, taking the differentials of the equations (3), (4) and rearranging for the motion of E_1/E', we find

$$
\begin{aligned}
&d'\overrightarrow{h_1} = \frac{\sqrt{a_2}}{\sqrt{a_1}} d\theta' \, \overrightarrow{h_2} \\
&d'\overrightarrow{h_2} = -\frac{\sqrt{a_1}}{\sqrt{a_2}} d\theta' \, \overrightarrow{h_1} \\
&d'\overrightarrow{b} = \left(db_1{}' - \frac{\sqrt{a_1}}{\sqrt{a_2}} b_2{}' d\theta' \right) \overrightarrow{h_1} + \left(db_2{}' + \frac{\sqrt{a_2}}{\sqrt{a_1}} b_1{}' d\theta' \right) \overrightarrow{h_2}.
\end{aligned}
$$

For the sake of shortness let us use

$$
\begin{aligned}
&\lambda = d\theta, \quad \sigma_1 = db_1 - \frac{\sqrt{a_1}}{\sqrt{a_2}} b_2 \, d\theta, \quad \sigma_2 = db_2 + \frac{\sqrt{a_2}}{\sqrt{a_1}} b_1 d\theta \\
&\lambda' = d\theta', \quad \sigma_1{}' = db_1{}' - \frac{\sqrt{a_1}}{\sqrt{a_2}} b_2{}' d\theta', \quad \sigma_2{}' = db_2{}' + \frac{\sqrt{a_2}}{\sqrt{a_1}} b_1{}' d\theta'
\end{aligned}
$$

The derivative equations of the motion E_1/E become

$$
\begin{aligned}
&d\overrightarrow{h_1} = \frac{\sqrt{a_2}}{\sqrt{a_1}} \lambda \, \overrightarrow{h_2} \\
&d\overrightarrow{h_2} = -\frac{\sqrt{a_1}}{\sqrt{a_2}} \lambda \, \overrightarrow{h_1} \\
&d\,\overrightarrow{b} = \sigma_1 \overrightarrow{h_1} + \sigma_2 \overrightarrow{h_2}.
\end{aligned}
$$

Similarly, the derivative equations of the motion E_1/E' become

$$
\begin{aligned}
&d'\overrightarrow{h_1} = \frac{\sqrt{a_2}}{\sqrt{a_1}} \lambda' \, \overrightarrow{h_2} \\
&d'\overrightarrow{h_2} = -\frac{\sqrt{a_1}}{\sqrt{a_2}} \lambda' \, \overrightarrow{h_1} \\
&d'\,\overrightarrow{b} = \sigma_1{}' \overrightarrow{h_1} + \sigma_2{}' \overrightarrow{h_2}.
\end{aligned}
$$

Here $\sigma_1, \sigma_2, \sigma_1{}', \sigma_2{}'$ are Pfaffian forms of the motion.
Let we us a point $X = (x_1, x_2)$ according to the relative moving coordinate system to analyze the elliptical motions on the elliptical plane. Since the vector equations

$$
\begin{aligned}
\overrightarrow{OB} &= \overrightarrow{b} = b_1 \overrightarrow{h_1} + b_2 \overrightarrow{h_2} \\
\overrightarrow{BX} &= \overrightarrow{x} = x_1 \overrightarrow{h_1} + x_2 \overrightarrow{h_2}
\end{aligned}
$$

and

$$
\overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{BX}
$$

can be written as above, we have

$$
\overrightarrow{x} = (b_1 + x_1) \, \overrightarrow{h_1} + (b_2 + x_2) \, \overrightarrow{h_2}.
$$

So differential of X with respect to E is

$$
\overrightarrow{OB} = \overrightarrow{b} = b_1 \overrightarrow{h_1} + b_2 \overrightarrow{h_2} \tag{5}
$$

Therefore the relative velocity vector of X with respect to E is

$$
\overrightarrow{V_r} = \frac{d\,\overrightarrow{x}}{dt}.
$$

and also differential of X with respect to E' is

$$
d'\overrightarrow{x} = \left(dx_1 + \sigma_1{}' - \frac{\sqrt{a_1}}{\sqrt{a_2}} x_2 \, \lambda' \right) \overrightarrow{h_1} + \left(dx_2 + \sigma_2{}' + \frac{\sqrt{a_2}}{\sqrt{a_1}} x_1 \, \lambda' \right) \overrightarrow{h_2}. \tag{6}
$$

Thus, the absolute velocity vector of X with respect to E' is

$$
\overrightarrow{V_a} = \frac{d'\,\overrightarrow{x}}{dt}.
$$

If $\overrightarrow{V_r} = 0$ or $\overrightarrow{V_a} = 0$ then the point X is fixed in the planes E and E', respectively. Thus, the conditions that the points to be fixed in elliptical planes E and E' become

$$
dx_1 = -\sigma_1 + \frac{\sqrt{a_1}}{\sqrt{a_2}} x_2 \, \lambda, \quad dx_2 = -\sigma_2 - \frac{\sqrt{a_2}}{\sqrt{a_1}} x_1 \, \lambda \tag{7}
$$

and

$$
dx_1 = -\sigma_1{}' + \frac{\sqrt{a_1}}{\sqrt{a_2}} x_2 \, \lambda', \quad dx_2 = -\sigma_2{}' - \frac{\sqrt{a_2}}{\sqrt{a_1}} x_1 \, \lambda'.
$$

respectively. Substituting equation (7) into equation (6) and considering that the sliding velocity of the point X is $\overrightarrow{V_f} = \frac{d_f \overrightarrow{x}}{dt}$, we have

$$d_f \overrightarrow{x} = \left[\left(\sigma_1{}' - \sigma_1\right) - x_2 \frac{\sqrt{a_1}}{\sqrt{a_2}} \left(\lambda' - \lambda\right) \right] \overrightarrow{h_1} + \left[\left(\sigma_2{}' - \sigma_2\right) + x_1 \frac{\sqrt{a_2}}{\sqrt{a_1}} \left(\lambda' - \lambda\right) \right] \overrightarrow{h_2}. \tag{8}$$

Thus, the following theorem can be given.

Theorem 1. *Let X be a moving point on the plane E_1 and $\overrightarrow{V_r},\ \overrightarrow{V_a},\ \overrightarrow{V_f}$ be the relative, absolute and sliding velocities of X under the one-parameter planar motions, respectively. Then, the relation between the velocities is given as below:*

$$\overrightarrow{V_a} = \overrightarrow{V_r} + \overrightarrow{V_f}.$$

Proof: The proof can be easily seen by considering the equations (5), (6) and (8). □

Result 1. *In the case of $a_1 = 1,\ a_2 = 1$, the relative, absolute and sliding velocities are found as*

$$\begin{aligned} d\vec{x} &= \left(dx_1 + \sigma_1 - x_2\lambda\right) \overrightarrow{h_1} + \left(dx_2 + \sigma_2 + x_1\lambda\right) \overrightarrow{h_2} \\ d'\overrightarrow{x} &= \left(x_1 + dx_1 + \sigma_1{}' - x_2\,\lambda'\right) \overrightarrow{h_1} + \left(x_2 + dx_2 + \sigma_2{}' + x_1\,\lambda'\right) \overrightarrow{h_2} \\ d_f \overrightarrow{x} &= \left[\left(\sigma_1{}' - \sigma_1\right) - x_2 \left(\lambda' - \lambda\right)\right] \overrightarrow{h_1} + \left[\left(\sigma_2{}' - \sigma_2\right) + x_1 \left(\lambda' - \lambda\right)\right] \overrightarrow{h_2} \end{aligned}$$

respectively [1].

Theorem 2. *The pole point P of the one parameter elliptical planar motion E/E' is obtained by*

$$p_1 = -\frac{\sqrt{a_1}}{\sqrt{a_2}} . \frac{\left(\sigma_2{}' - \sigma_2\right)}{\left(\lambda' - \lambda\right)}, \qquad p_2 = \frac{\sqrt{a_2}}{\sqrt{a_1}} \cdot \frac{\left(\sigma_1{}' - \sigma_1\right)}{\left(\lambda' - \lambda\right)}$$

where $\overrightarrow{BP} = \overrightarrow{p} = p_1\,\overrightarrow{h_1} + p_2\,\overrightarrow{h_2}$.

Proof: In a one parameter elliptical motion, pole points of the motion are characterized for cases that the sliding velocity vector becomes zero. Namely, $d_f\,\overrightarrow{x} = 0$. It will be taken into account $\theta \neq 0$ and $\theta' \neq 0$ in order to avoid the pure rotation motion. Then, considering that the equality of (8) equals zero

$$\left[\left(\sigma_1{}' - \sigma_1\right) - x_2 \frac{\sqrt{a_1}}{\sqrt{a_2}} \left(\lambda' - \lambda\right) \right] \overrightarrow{h_1} + \left[\left(\sigma_2{}' - \sigma_2\right) + x_1 \frac{\sqrt{a_2}}{\sqrt{a_1}} \left(\lambda' - \lambda\right) \right] \overrightarrow{h_2} = 0$$

is found. From this last equation, coordinates of the pole point P for the one parameter elliptical motion E/E' are obtained by

$$\begin{aligned} x_1 = p_1 &= -\tfrac{\sqrt{a_1}}{\sqrt{a_2}} . \tfrac{\left(\sigma_2{}' - \sigma_2\right)}{\left(\lambda' - \lambda\right)}, \\ x_2 = p_2 &= \tfrac{\sqrt{a_2}}{\sqrt{a_1}} \cdot \tfrac{\left(\sigma_1{}' - \sigma_1\right)}{\left(\lambda' - \lambda\right)}. \end{aligned}$$

□

Result 2. *In the case of $a_1 = 1,\ a_2 = 1$, pole points of the one parameter elliptical planar motion E/E' correspond to pole points of the one parameter planar motion on the Euclidean plane [1].*

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On Some New Paranormed Lucas Sequence Spaces and Lucas Core

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Abstract: The sequence spaces $c_0(\hat{L}), c(\hat{L}), \ell_\infty(\hat{L})$ and $\ell_p(\hat{L})$ have been recently introduced and studied by Karakaş and Karabudak. The aim of this paper is to extend the results of Karakaş and Karabudak to the paranormed case and is to work the spaces $c_0(\hat{L},p), c(\hat{L},p), \ell_\infty(\hat{L},p)$ and $\ell(\hat{L},p)$. Furthermore, Lucas core of a complex-valued sequence has been introduced, and we prove some inclusion theorems related to this new type of core.

Keywords: Lucas numbers, Lucas core, Matrix transformations, Paranormed sequence spaces.

1 Introduction

In mathematics, the Fibonacci numbers are the numbers in the following integer sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots.$$

The sequence (f_n) of Fibonacci numbers is given by the linear recurrence relations

$$f_0 = 0, f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2}, n \geq 2.$$

This sequence has many interesting properties and applications in arts, sciences and architecture. For example, the ratio sequence of Fibonacci numbers converges to the golden ratio which is important in sciences and arts.

Similar to the Fibonacci numbers, each Lucas number is defined to be the sum of its two immediate previous terms, thereby forming a Fibonacci integer sequence. The first two Lucas numbers are $L_0 = 2$ and $L_1 = 1$ as opposed to the first two Fibonacci numbers $f_0 = 0$ and $f_1 = 1$. Though closely related in definition, Lucas and Fibonacci numbers exhibit distinct properties. The Lucas numbers may thus be defined as follows:

$$L_n = \begin{cases} 2 & , \quad n = 0, \\ 1 & , \quad n = 1, \\ L_{n-1} + L_{n-2} & , \quad n > 1. \end{cases}$$

The sequence of Lucas number is:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots.$$

The ratio of the successive both Fibonacci and Lucas numbers is as known golden ratio. There are many applications of golden ratio in many places of mathematics and physics, in general theory of high energy particle theory [1]. Also, some basic properties of Lucas numbers [1] are given as follows:

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{(Binet's formula for Lucas numbers)}$$

$$L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n \quad \text{and} \quad \sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2 \quad \text{(Additional identities)}$$

$$\lim_{n\to\infty} \frac{L_n}{L_{n-1}} = \frac{1+\sqrt{5}}{2} = \varphi \qquad \text{(Golden ratio)}$$

Lucas numbers was first used by Karakaş and Karabudak [2] in the theory of summability. Let L_n be the nth Lucas number for every $n \in \mathbb{N}$. Then, the infinite Lucas matrix $\widehat{L} = (\widehat{L}_{nk})$ is defined by

$$\widehat{L}_{nk} = \begin{cases} \dfrac{L_{k-1}^2}{L_n \cdot L_{n-1} + 2} & , \quad 1 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

where $n, k \in \mathbb{N}$ [2]. Recently, a lot of papers have been studying by many researchers on Lucas and Fibonacci sequences. For instance, see [3–12].

Assume here and after that (p_k) be a bounded sequences of strictly positive real numbers with $\sup p_k = H$ and $L = \max\{1, H\}$ and by $\mathcal{F}$ and $\mathbb{N}_k$, we shall denote the collection of all finite subsets of $\mathbb{N}$ and the set of all $n \in \mathbb{N}$ such that $n \geq k$, respectively. Then, the paranormed sequence spaces $\ell_\infty(p), c(p), c_0(p)$ and $\ell(p)$ were defined by Maddox [13] (see also Maddox [14] and Nakano [15]) as follows:

$$\begin{aligned} \ell_\infty(p) &= \{x = (x_k) \in w : \sup_{k \in \mathbb{N}} |x_k|^{p_k} < \infty\}, \\ c(p) &= \{x = (x_k) \in w : \lim_{k \to \infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{R}\}, \\ c_0(p) &= \{x = (x_k) \in w : \lim_{k \to \infty} |x_k|^{p_k} = 0\}, \\ \ell(p) &= \left\{x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty\right\}, \end{aligned}$$

which are the complete spaces paranormed by

$$g_1(x) = \sup_{k \in \mathbb{N}} |x_k|^{p_k/L} \Longleftrightarrow \inf p_k > 0 \text{ and } g_2(x) = \left(\sum_k |x_k|^{p_k}\right)^{1/L},$$

respectively. We shall assume throughout that $p_k^{-1} + (p_k^{'})^{-1} = 1$ provided $1 < \inf p_k < H < \infty$.

It is well known that paranormed spaces have more general properties than normed spaces. Recently, there have been many studies on both normed and paranormed sequence spaces. The reader can look at the articles on this subject [16–20, 22–32].

In this work, we generalize the normed sequence spaces defined by Karakaş and Karabudak [2] to paranormed spaces. Let μ denote any of the spaces c_0, c, ℓ_∞ and ℓ_p. We prove that $\mu(\hat{L}, p)$ is linearly paranorm isomorphic to $\mu(p)$ and determine the $\alpha-, \beta-$ and $\gamma-$duals of the $\mu(\hat{L}, p)$. Furthermore, Lucas core of a complex-valued sequence has been introduced, and we prove some inclusion theorems related to this new type of core.

2 The Paranormed Sequence Spaces $c_0(\widehat{L}, p)$, $c(\widehat{L}, p)$, $\ell_\infty(\widehat{L}, p)$ and $\ell(\widehat{L}, p)$

In this section, we define the new sequence spaces $c_0(\widehat{L}, p)$, $c(\widehat{L}, p)$, $\ell_\infty(\widehat{L}, p)$ and $\ell(\widehat{L}, p)$ by using the sequences of Lucas numbers, and prove that these sequence spaces are the complete paranormed linear metric spaces and compute their $\alpha-, \beta-$ and $\gamma-$ duals.

We define the sequence spaces $c_0(\widehat{L}, p)$, $c(\widehat{L}, p)$, $\ell_\infty(\widehat{L}, p)$ and $\ell(\widehat{L}, p)$ by

$$\begin{aligned} c_0(\widehat{L}, p) &= \left\{x = (x_k) \in w : \lim_{n \to \infty} \left| \frac{1}{L_n L_{n+1} + 2} \sum_{i=1}^n L_{i-1}^2 x_i \right|^{p_n} = 0\right\}, \\ c(\widehat{L}, p) &= \left\{x = (x_k) \in w : \exists l \in \mathbb{C} \ni \lim_{n \to \infty} \left| \frac{1}{L_n L_{n+1} + 2} \sum_{i=1}^n L_{i-1}^2 x_i - l \right|^{p_n} = 0\right\}, \\ \ell_\infty(\widehat{L}, p) &= \left\{x = (x_k) \in w : \sup_{n \in \mathbb{N}} \left| \frac{1}{L_n L_{n+1} + 2} \sum_{i=1}^n L_{i-1}^2 x_i \right|^{p_n} < \infty\right\}, \\ \ell(\widehat{L}, p) &= \left\{x = (x_k) \in w : \sum_n \left| \frac{1}{L_n L_{n+1} + 2} \sum_{i=1}^n L_{i-1}^2 x_i \right|^{p_n} < \infty\right\}. \end{aligned}$$

In the case $(p_n) = e = (1, 1, 1, ...)$, the sequence spaces $c_0(\widehat{L}, p)$, $c(\widehat{L}, p)$, $\ell_\infty(\widehat{L}, p)$ and $\ell(\widehat{L}, p)$ are, respectively, reduced to the sequence spaces $c_0(\widehat{L})$, $c(\widehat{L})$, $\ell_\infty(\widehat{L})$ and $\ell(\widehat{L})$ which are introduced by Karakaş and Karabudak [2].

Define the sequence $y = (y_k)$, which will be frequently used as the $\widehat{L}-$transform of a sequence $x = (x_k)$, i.e.,

$$y_k = \widehat{L}_k(x) = \frac{1}{L_k . L_{k-1} + 2} \sum_{i=1}^k L_{i-1}^2 . x_i \quad ; \quad (k \in \mathbb{N}_0).$$

Theorem 1. *The following statements hold:*

(i) The sequence spaces $c_0(\widehat{L},p), c(\widehat{L},p)$ and $\ell_\infty(\widehat{L},p)$ are the complete linear metric spaces paranormed by g, defined by

$$g(x) = \sup_{k\in\mathbb{N}} \left| \frac{1}{L_k.L_{k-1}+2} \sum_{i=1}^{k} L_{i-1}^2.x_i \right|^{p_k/L}.$$

(ii) $\ell(\widehat{L},p)$ is a complete linear metric space paranormed by

$$g^*(x) = \left(\sum_k \left| \frac{1}{L_k.L_{k-1}+2} \sum_{i=1}^{k} L_{i-1}^2.x_i \right|^{p_k} \right)^{1/L}.$$

Therefore, one can easily check that the absolute property does not hold on the spaces $c_0(\widehat{L},p), c(\widehat{L},p)$, $\ell_\infty(\widehat{L},p)$ and $\ell(\widehat{L},p)$ that is $h(x) \neq h(|x|)$ for at least one sequence in those spaces, and this says that $c_0(\widehat{L},p), c(\widehat{L},p)$, $\ell_\infty(\widehat{L},p)$ and $\ell(\widehat{L},p)$ are the sequence spaces of non-absolute type; where $|x| = (|x_k|)$.

Theorem 2. *The sequence spaces $c_0(\widehat{L},p), c(\widehat{L},p)$,$\ell_\infty(\widehat{L},p)$ and $\ell(\widehat{L},p)$ are linearly isomorphic to the spaces $c_0(p), c(p), \ell_\infty(p)$ and $\ell(p)$, respectively, where $0 < p_k \leq H < \infty$.*

Theorem 3. *The matrix $D = (d_{nk})$ is defined by*

$$d_{nk} = \begin{cases} \tilde{\Delta}\left[\dfrac{a_k}{L_{k-1}^2}\right](L_kL_{k-1}+2) & , \quad (0 \leq k \leq n-1) \\ \dfrac{L_nL_{n-1}+2}{L_{n-1}^2}a_n & , \quad (k=n) \\ 0 & , \quad (k>n) \end{cases}$$

for all $k,n \in \mathbb{N}$ and $M \in \mathbb{N}_2$. Let $K^* = \{k \in \mathbb{N} : 0 \leq k \leq n\} \cap K$ for $K \in \mathcal{F}$ and $M \in \mathbb{N}_2$. Define the sets $\widehat{L}_6(p)$, $\widehat{L}_7$, $\widehat{L}_8(p)$, $\widehat{L}_9$, $\widehat{L}_{10}(p)$, $\widehat{L}_{11}(p)$, $\widehat{L}_{12}(p)$, $\widehat{L}_{13}(p)$ as follows:

$$\begin{aligned}
\widehat{L}_6(p) &= \bigcup_{M>1} \left\{ a=(a_k) \in w : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} |d_{nk}|\, M^{-1/p_k} < \infty \right\}, \\
\widehat{L}_7 &= \left\{ a=(a_k) \in w : \lim_{n\to\infty} |d_{nk}| \text{ exists for each } k \in \mathbb{N} \right\}, \\
\widehat{L}_8(p) &= \bigcup_{M>1} \left\{ a=(a_k) \in w : \exists(\alpha_k) \in \mathbb{R} \ni \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} |d_{nk}-\alpha_k|\, M^{-1/p_k} < \infty \right\}, \\
\widehat{L}_9 &= \left\{ a=(a_k) \in w : \exists\alpha \in \mathbb{R} \ni \lim_{n\to\infty} \left| \sum_{k=0}^{n} d_{nk} - \alpha \right| = 0 \right\}, \\
\widehat{L}_{10}(p) &= \bigcap_{M>1} \left\{ a=(a_k) \in w : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} |d_{nk}|\, M^{-1/p_k} < \infty \right\}, \\
\widehat{L}_{11}(p) &= \bigcap_{M>1} \left\{ a=(a_k) \in w : \exists(\alpha_k) \in \mathbb{R} \ni \lim_{n\to\infty} \sum_{k=0}^{n} |d_{nk}-\alpha_k|\, M^{1/p_k} = 0 \right\}, \\
\widehat{L}_{12}(p) &= \bigcup_{M>1} \left\{ a=(a_k) \in w : \sup_n \sup_{k\in K^*} |d_{nk}M^{-1}|^{p_k} < \infty \right\}, \\
\widehat{L}_{13}(p) &= \bigcup_{M>1} \left\{ a=(a_k) \in w : \sup_n \sum_{k\in K^*} |d_{nk}M^{-1}|^{p'_k} < \infty \right\}.
\end{aligned}$$

Then,

(i) $\{c_0(\widehat{L},p)\}^\beta = \widehat{L}_6(p) \cap \widehat{L}_7 \cap \widehat{L}_8(p)$,

(ii) $\{c(\widehat{L},p\}^\beta = \{c_0(\widehat{L},p)\}^\beta \cap \widehat{L}_9$,

(iii) $\{\ell_\infty(\widehat{L},p)\}^\beta = \widehat{L}_{10}(p) \cap \widehat{L}_{11}(p)$,

(iv) $\{\ell(\widehat{L},p)\}^\beta = \widehat{L}_{12}(p) \cap \widehat{L}_{13}(p)$.

3 Lucas Core

Following Knopp, a core theorem is characterized a class of matrices for which the core of the transformed sequence is included by the core of the original sequence. For example Knopp Core Theorem [[33], p.138] states that $K-core(Ax) \subseteq K \ -core(x)$ for all real valued sequences x whenever A is a positive matrix in the class $(c:c)_{reg}$.

Now, let us write

$$y_n(x) = \widehat{L}_n(x) = \frac{1}{L_nL_{n-1}+2}\sum_{k=1}^{n} L_{k-1}^2 x_k; \quad (k \in \mathbb{N}_0).$$

Then we can define $\widehat{L}-$ core of a complex sequence as follows:

Let H_n be the least closed convex hull containing $y_n(x), y_{n+1}(x), , \ldots$. Then, $\hat{L}-$ core of x is the intersection of all H_n, i.e.,

$$\hat{L}-core(x) = \bigcap_{n=1}^{\infty} H_n.$$

Now, we may give some inclusion theorems. For brevity, in what follows we write $\tilde{e}_{nk}$ in place of

$$\frac{1}{L_nL_{n-1}+2}\sum_{k=1}^{n} L_{k-1}^2 x_k.$$

Theorem 4. *Let $B \in (c : c(\hat{L}))_{reg}$. Then, $\hat{L}-core(Bx) \subseteq K-core(x)$ for all $x \in \ell_\infty$ if and only if*

$$\lim_n \sum_k |\tilde{e}_{nk}| = 1. \tag{1}$$

Theorem 5. *Let $B \in (st \cap \ell_\infty : c(\hat{L}))_{reg}$. Then, $\hat{L}-core(Bx) \subseteq st-core(x)$ for all $x \in \ell_\infty$ if and only if (1) holds.*

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An Algorithm for Solutions to the Elliptic Quaternion Matrix Equation $AX = B$

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Abstract: In this paper, the existence of solution to the elliptic quaternion matrix equations $AX = B$ is characterized and solutions of this matrix equation are derived by means of real representations. Also, our results are illustrated with an example.

Keywords: Elliptic quaternion, Elliptic quaternion matrix, Real representation of elliptic quaternion matrices.

1 Introduction and Preliminaries

The well-known concept of the real quaternion was first introduced by Hamilton in 1843 [1]. It has four components, i.e.

$$q = q_r + q_i i + q_j j + q_k k$$

where $q_r, q_i, q_j, q_k \in R$ and i, j and k satisfy

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ ik = -ki = -j,\ jk = -kj = i.$$

The real quaternion algebra plays an important role in matrix analysis, quantum physics, kinematics, differential geometry, game development, image and signal processing etc. Thus, there are number of studies associated with real quaternions [2, 3]. Since the multiplication of real quaternions is non-commutative, all results about the complex numbers cannot be generalized in real quaternions. This problem restricts the applications of real quaternions. In addition, this can increase the complexity of many processes.

The concept of commutative quaternions was first introduced by Schütte and Wenzel [4]. The major difference between commutative quaternions and real quaternions is commutativeness of the multiplication, which are commutative for commutative quaternions. There are many studies on commutative quaternions in literature. Catoni et al. studied the functions of commutative quaternions variable and obtained generalized Cauchy-Riemann conditions [5]. Pei et. al introduced digital signal and image processing using commutative quaternions. For color image processing, they defined a simplified polar form of commutative quaternions to represent the color image and showed that this representation is useful to process color images in the brightness-hue-saturation color space [4]. In [6], Isokawa et al. investigated two types of multistate Hopfield neural networks based on commutative quaternions. Moreover, Kosal and Tosun investigated some algebraic properties of commutative quaternion matrices by means of complex representation of commutative quaternion matrices [7]. In [8], Kosal et al. constructed some explicit expression of the solution of the Kalman-Yakubovich-conjugate commutative quaternion matrix equations, by means of real representation of a commutative quaternion matrices. In [9], Kosal and Tosun studied some equivalence relations and related to results over the commutative quaternions and their matrices. In this sense, they defined consimilarity, semisimilarity and consemisimilarity over the commutative quaternions algebra and their matrix algebra and determined the equalities of these equivalence relations. In [10], Kosal and Tosun established universal similarity factorization equalities over the commutative quaternions and their matrices. Based on these equalities, real matrix representations of commutative quaternions and their matrices have been derived, and their algebraic properties and fundamental equations have been determined.

In this study, the existence of solution to the elliptic quaternion matrix equations $AX = B$ is characterized and solutions of this matrix equation are derived by means of real representations. Elliptic quaternions are generalized form of commutative quaternions and so complex numbers [5]. Thus, the obtained results extend, generalize and complement some known commutative quaternions matrices and complex matrices results from the literature.

A set of elliptic quaternions is denoted by [5]

$$H_p = \{a = a_0 + a_1 i + a_2 j + a_3 k :\ a_0, a_1, a_2, a_3 \in R,\ i, j, k \notin R\}$$

where

$$i^2 = k^2 = \alpha,\ j^2 = 1,\ ij = ji = k,\ jk = kj = i,\ ki = ik = \alpha j,\ \alpha < 0.$$

There are three types of conjugate of $a = a_0 + a_1 i + a_2 j + a_3 k \in H_p$. They are

$$
\begin{aligned}
{}^{1}\bar{a} &= a_0 - a_1 i + a_2 j - a_3 k, \\
{}^{2}\bar{a} &= a_0 + a_1 i - a_2 j - a_3 k, \ . \\
{}^{3}\bar{a} &= a_0 - a_1 i - a_2 j + a_3 k
\end{aligned}
$$

and norm of a is defined

$$
\begin{aligned}
\|a\| &= \sqrt[4]{\left| a \left({}^{1}\bar{a}\right) \left({}^{2}\bar{a}\right) \left({}^{3}\bar{a}\right) \right|} \\
&= \sqrt[4]{\left[(a_0 + a_2)^2 - \alpha (a_1 + a_3)^2 \right] \left[(a_0 - a_2)^2 - \alpha (a_1 - a_3)^2 \right]}.
\end{aligned}
$$

Addition, multiplication and scalar multiplication of the elliptic quaternions $a = a_0 + a_1 i + a_2 j + a_3 k, b = b_0 + b_1 i + b_2 j + b_3 k \in H_p$ and $\lambda \in R$ are defined by

$$
a + b = (a_0 + b_0) + (a_1 + b_1)\, i + (a_2 + b_2)\, j + (a_3 + b_3)\, k,
$$

$$
\begin{aligned}
pq = {} & (a_0 b_0 + \alpha a_1 b_1 + a_2 b_2 + \alpha a_3 b_3) + (a_1 b_0 + a_0 b_1 + a_3 b_2 + a_2 b_3)\, i \\
& + (a_0 b_2 + a_2 b_0 + \alpha a_1 b_3 + \alpha a_3 b_1)\, j + (a_3 b_0 + a_0 b_3 + a_1 b_2 + a_2 b_1)\, k,
\end{aligned}
$$

and

$$
\lambda a = \lambda \left(a_0 + a_1 i + a_2 j + a_3 k \right) = \lambda a_0 + \lambda a_1 i + \lambda a_2 j + \lambda a_3 k
$$

respectively.
If $a = a_0 + a_1 i + a_2 j + a_3 k \in H_p$ and $\|a\| \neq 0$ then a has multiplicative inverses. Multiplicative inverse of a is given by

$$
a^{-1} = \frac{\left({}^{1}\bar{a}\right)\left({}^{2}\bar{a}\right)\left({}^{3}\bar{a}\right)}{\|a\|^4}.
$$

2 Elliptic Quaternion Matrices

The set of $H_p^{m \times n}$ denotes all $m \times n$ type matrices with elliptic quaternion entries. For $A = \left(a_{ij}\right),\ B = \left(b_{ij}\right) \in H_p^{m \times n},\ C = \left(c_{jk}\right) \in H_p^{n \times l}$ and $\lambda \in R$, the ordinary matrix addition, scalar multiplication and multiplication are defined by

$$
A + B = \left(a_{ij}\right) + \left(b_{ij}\right) = \left(a_{ij} + b_{ij}\right) \in H_p^{m \times n},
$$

$$
\lambda A = \lambda \left(a_{ij}\right) = \left(\lambda a_{ij}\right) \in H_p^{m \times n}
$$

and

$$
AC = \left(\sum_{j=1}^{n} a_{ij} c_{jk} \right) \in H_p^{m \times l}
$$

respectively.
There are three types' of conjugate of $A = \left(a_{ij}\right) \in H_p^{m \times n}$. They are

$$
{}^{1}\bar{A} = \left({}^{1}\overline{a_{ij}}\right) \in H_p^{m \times n},\ {}^{2}\bar{A} = \left({}^{2}\overline{a_{ij}}\right) \in H_p^{m \times n}\, and\, {}^{3}\bar{A} = \left({}^{3}\overline{a_{ij}}\right) \in H_p^{m \times n}.
$$

A matrix $A^T \in H_p^{n \times m}$ is transpose of $A \in H_p^{m \times n}$. Also $A^{*_s} = \left({}^{s}\overline{A}\right)^T \in H_p^{m \times n},\ s = 1, 2, 3$, is called conjugate transpose according to the s^{th} conjugate of $A \in H_p^{m \times n}$.

Theorem 1. *Let A and B be elliptic quaternion matrices of appropriate sizes. Then followings are satisfied:*

1. $\left({}^{s}\bar{A}\right)^T = {}^{s}\overline{\left(A^T\right)}$,
2. $(AB)^{*_s} = B^{*_s} A^{*_s}$,
3. $(AB)^T = B^T A^T$,
4. ${}^{s}\overline{(AB)} = \left({}^{s}\bar{A}\right)\left({}^{s}\bar{B}\right)$,
5. *If A^{-1} and B^{-1} exist then* $(AB)^{-1} = B^{-1} A^{-1}$,
6. *If A^{-1} exists* $(A^{*_s})^{-1} = \left(A^{-1}\right)^{*_s}$,
7. $\left({}^{s}\bar{A}\right)^{-1} = {}^{s}\overline{\left(A^{-1}\right)}$.

This theorem can also be easily proved by direct calculation.

3 Real Representation of Elliptic Quaternion Matrices

Let $A = A_0 + A_1 i + A_2 j + A_3 k \in H_p^{m\times n}$ where $A_0, A_1, A_2, A_3 \in R^{m\times n}$. For $X \in H_p^{n\times n}$, we will define the linear transformations

$$\eta_A(X) = AX.$$

Then, we get real representations of elliptic quaternion matrix $A = A_0 + A_1 i + A_2 j + A_3 k \in H_p^{m\times n}$

$$\eta_p(A) = \begin{pmatrix} A_0 & \alpha A_1 & A_2 & \alpha A_3 \\ A_1 & A_0 & A_3 & A_2 \\ A_2 & \alpha A_3 & A_0 & \alpha A_1 \\ A_3 & A_2 & A_1 & A_0 \end{pmatrix} \in R^{4m\times 4n}.$$

Theorem 2. *Let $A, B \in H_p^{m\times n}$, $C \in H_p^{n\times p}$ and $\lambda \in R$. Then following identities are satisfied:*

1. $A = B \Leftrightarrow \eta_p(A) = \eta_p(B)$, $\eta_p(A+B) = \eta_p(A) + \eta_p(B)$,
2. $\eta_p(AC) = \eta_p(A)\eta_p(C)$, $\eta_p(\lambda A) = \eta_p(A\lambda) = \lambda\eta_p(A)$,
3. $A = \frac{1}{2-2\alpha} E_{4m}\eta_p(A) E_{4n}^{*_1}$ *where* $E_{4t} = \begin{pmatrix} I_t & iI_t & jI_t & kI_t \end{pmatrix} \in H^{t\times 4t}$,
4. *If A is a nonsingular matrix of size m then*

$$\eta_p\left(A^{-1}\right) = \eta_p^{-1}(A), \ A^{-1} = \frac{1}{2-2\alpha} E_{4m}\eta_p^{-1}(A) E_{4m}^{*_1},$$

5. $\eta_p(A) = R_{4m}^{-1}\eta_p(A) R_{4n}$, $\eta_p(A) = S_{4m}^{-1}\eta_p(A) S_{4n}$ *and* $\eta_p(A) = T_{4m}^{-1}\eta_p(A) T_{4n}$ *where*

$$Q_{4t} = \begin{pmatrix} 0 & \alpha I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha I_t \\ 0 & 0 & I_t & 0 \end{pmatrix}, \quad S_{4t} = \begin{pmatrix} 0 & 0 & I_t & 0 \\ 0 & 0 & 0 & I_t \\ I_t & 0 & 0 & 0 \\ 0 & I_t & 0 & 0 \end{pmatrix}, \quad T_{4t} = \begin{pmatrix} 0 & 0 & 0 & \alpha I_t \\ 0 & 0 & I_t & 0 \\ 0 & \alpha I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \end{pmatrix}.$$

4 On Solutions to the Elliptic Quaternion Matrix Equation $AX = B$

Now, we consider the solution of the

$$AX = B \tag{1}$$

by means of the real representation, where $A \in H_p^{m\times n}$, $B \in H_p^{m\times p}$. We define the real representation of the matrix equation (1) by

$$\eta_p(A) Y = \eta_p(B). \tag{2}$$

Proposition 1. *The equation (1) has a solution X if and only if the equation (2) has a solution $Y = \eta_p(X)$.*

Theorem 3. *The equation (2) has a solution $Y \in R^{4n\times 4p}$ if and only if the equation (1) has a solution $X \in H_p^{n\times p}$. In that case, if $Y \in R^{4n\times 4p}$ is a solution of (2), then the matrix*

$$X = \frac{1}{8-8\alpha}\left(I_m \, iI_m \, jI_m \, kI_m\right)\left(Y + Q_{4m}^{-1} Y Q_{4p} + S_{4m}^{-1} Y S_{4p} + T_{4m}^{-1} Y T_{4p}\right)\begin{pmatrix} I_p \\ -iI_p \\ jI_p \\ -kI_p \end{pmatrix} \tag{3}$$

is a solution of (1).

Proof:
We show that if the real matrix

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{pmatrix}, \ Y_{uv} \in R^{n\times p}, \ u, v = 1, 2, 3, 4 \tag{4}$$

is a solution to (2), then the matrix given in (3) is a solution to (1). Since $Q_m^{-1} Y Q_n = Y, \quad R_m^{-1} Y R_n = Y, \quad S_m^{-1} Y S_n = Y$, we have

$$\eta_p(A) Q_{4m}^{-1} Y Q_{4p} = \eta_p(B), \ \eta_p(A) R_{4m}^{-1} Y R_{4p} = \eta_p(B), \ \eta_p(A) S_{4m}^{-1} Y S_{4p} = \eta_p(B).$$

This equation shows that if Y is a solution to (2), then $Q_{4m}^{-1} Y Q_{4p}, \quad R_{4m}^{-1} Y R_{4p}$ and $S_{4m}^{-1} Y S_{4p}$ are also solutions to (2). Thus the following real matrix:

$$Y' = \frac{1}{4}\left(Y + Q_{4m}^{-1} Y Q_{4p} + R_{4m}^{-1} Y R_{4p} + S_{4m}^{-1} Y S_{4p}\right) \tag{5}$$

is a solution to (2). Now substituting (4) in (5) and the simplifying the expression, we easily get

$$Y' = \begin{pmatrix} Z_0 & \alpha Z_1 & Z_2 & \alpha Z_3 \\ Z_1 & Z_0 & Z_3 & Z_2 \\ Z_2 & \alpha Z_3 & Z_0 & \alpha Z_1 \\ Z_3 & Z_2 & Z_1 & Z_0 \end{pmatrix},$$

where

$$\begin{aligned} Z_0 &= \tfrac{1}{4}\left(Y_{11}+Y_{22}+Y_{33}+Y_{44}\right), \quad Z_1 = \tfrac{1}{4}\left(\tfrac{Y_{12}}{\alpha}+Y_{21}+\tfrac{Y_{34}}{\alpha}+Y_{43}\right), \\ Z_2 &= \tfrac{1}{4}\left(Y_{13}+Y_{24}+Y_{31}+Y_{42}\right), \quad Z_3 = \tfrac{1}{4}\left(\tfrac{Y_{14}}{\alpha}+Y_{23}+\tfrac{Y_{32}}{\alpha}+Y_{41}\right). \end{aligned}$$

Thus we obtain

$$X = Z_1 + Z_2 i + Z_3 j + Z_4 k = \frac{1}{8-8\alpha}\left(I_m \, iI_m \, jI_m \, kI_m\right)\left(Y + Q_{4m}^{-1} Y Q_{4p} + S_{4m}^{-1} Y S_{4p} + T_{4m}^{-1} Y T_{4p}\right)\begin{pmatrix} I_p \\ -iI_p \\ jI_p \\ -kI_p \end{pmatrix}.$$

□

5 Numerical Algorithms

Based on the discussions in the previous section, in this section we provide numerical algorithms for solving elliptic quaternion matrix equation $AX = B$.

1. Input $A_0,\ A_1,\ A_2,\ A_3$ and $B_0,\ B_1,\ B_2,\ B_3$.
2. Form $\eta_p(A)$ and $\eta_p(B)$.
3. Compute Y and $Y' = \frac{1}{4}\left(Y + Q_{4m}^{-1} Y Q_{4p} + R_{4m}^{-1} Y R_{4p} + S_{4m}^{-1} Y S_{4p}\right)$.
4. Calculate

$$X = \frac{1}{8-8\alpha}\left(I_m \, iI_m \, jI_m \, kI_m\right)\left(Y + Q_{4m}^{-1} Y Q_{4p} + S_{4m}^{-1} Y S_{4p} + T_{4m}^{-1} Y T_{4p}\right)\begin{pmatrix} I_p \\ -iI_p \\ jI_p \\ -kI_p \end{pmatrix}.$$

6 Numerical Examples

Let us for solve the elliptic quaternion matrix equation

$$\begin{pmatrix} 1 & 1+i \\ j & k \end{pmatrix} X = \begin{pmatrix} 1+2i+j & 2+j+k \\ -1+j+2k & -1+i+2j \end{pmatrix}.$$

Under consideration of the Theorem 3, real representation of given matrix equation is

$$\begin{pmatrix} 1 & 1 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \alpha \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha \\ 1 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} Y = \begin{pmatrix} 1 & 2 & 2\alpha & 0 & 1 & 1 & 0 & \alpha \\ -1 & -1 & 0 & \alpha & 1 & 2 & 2\alpha & 0 \\ 2 & 0 & 1 & 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 & 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & \alpha & 1 & 2 & 2\alpha & 0 \\ 1 & 2 & 2\alpha & 0 & -1 & -1 & 0 & \alpha \\ 0 & 1 & 1 & 1 & 2 & 0 & 1 & 2 \\ 2 & 0 & 1 & 2 & 0 & 1 & -1 & -1 \end{pmatrix}$$

If we solve this equation, we have

$$Y = \begin{pmatrix} 1 & 2 & 2\alpha & 0 & -1 & -1 & -2\alpha & -\alpha \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 2 & 0 & 1 & 2 & -2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ -1 & -1 & -2\alpha & -\alpha & 1 & 2 & 2\alpha & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & -1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, we obtain

$$\begin{aligned} X &= \tfrac{1}{8-8\alpha}\left(I_m \, iI_m \, jI_m \, kI_m\right)\left(Y + Q_{4m}^{-1} Y Q_{4p} + S_{4m}^{-1} Y S_{4p} + T_{4m}^{-1} Y T_{4p}\right)\begin{pmatrix} I_p \\ -iI_p \\ jI_p \\ -kI_p \end{pmatrix} \\ &= \begin{pmatrix} 1+2i-j-2k & 2-j-k \\ 2j & 2j \end{pmatrix}. \end{aligned}$$

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An Evaluation of Mathematics Achievement of High School Students with Mathematics Anxiety

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Abstract: In this study, we evaluate the relationship between anxiety and success regarding math classes by measuring, through Mathematics Anxiety Scale, the anxiety levels of first, second, and third grade public high schoolers for math classes. This study has been applied to the students of 9-A, 10-B, 11-A classes in the 75th Year Anatolian High School and the students of FL9-C, FL10-F, FL11-A classes in the Ayseabla College Science High School. During the collection of data, the scale that has been used to measure the anxiety of students is Erol's "Mathematics Anxiety Scale", [1]. The collected data has been inspected by t-testing, analysis of variance and especially correlation analysis. The purpose of the analysis is a contrast and comparison of the math anxiety and the success of the students based on their grade levels. The sample of the research consists of 30 female and 33 male students of 9-C, 10-F, 11-A classes in 75th Year Anatolian High School and 31 female and 27 male students of FL-9A, FL-10B, FL-11A in Ayseabla College Science High School. As a consequence, the sample consists of 121 students. The *Math Anxiety Scale* has been implemented to 121 of these students. The Math Anxiety scale, which has been implemented during this study, has a Cronbach Alpha coefficient of 0.91, [2]. The scale consists of 4 questions; each question has 4 Likert type answers. The highest attainable score is 180 whereas the lowest attainable score is 45. According to this standard, the scores between 45-68 is considered to be low, 69-109 is normal, 109-128 is anxious, and 129-180 is very anxious [3]. The results of this study show that there is not a noticeable difference between the grade type and the anxiety levels. However, a correlation has been discovered between the anxiety levels and success rates of the students in general. Consequently, as a result of the conclusions that have been reached through the evaluation of this correlation, certain proposals have been developed.

Keywords: Math anxiety scale, Interest in Math.

1 Introduction

In the realization of learning, besides individual characteristics such as age, intelligence, motivation, past experiences, internal communication, the state of anxiety is one of the most important factors affecting learning. Learner [4] concluded that, as a result of his research on mathematics, the fear and anxiety of mathematics makes the child think clearly, make an organization among the information and make relationships difficult and hamper. The state of anxiety at the intellectual, behavioral and physiological levels will decrease the attention of the students on the course and this will make learning difficult. In order for learning to take place, a permanent and effective communication between the billions of cells in the brain is needed. Anxiety prevents the establishment of this communication within the brain and adversely affects the mental activities of reasoning and abstract thinking. Anxiety level may affect the concentration and interest of the student who is constantly working in the classroom because of the fear of being successful in the exam and cannot provide the concentration of the student who is tired of studying, and cannot spend the duration of the study effectively [5]. Thus, various sizes of anxiety; future concerns, reading anxiety, test anxiety, math anxiety, and so on occurs. Mathematics Anxiety, the main theme of this study, is an important sensory factor that affects students negatively and causes negative attitudes towards mathematics [6]. Attitude refers to a person's positive or negative feelings about showing certain behavior. According to Tavşancıl [7], the tendency to react positively or negatively on learned against certain objects, situations, institutions, concepts or other people is called attitude. The information learned on any subject, even if they have fallen over time, is easily unforgettable [8]. Cognitive factors causing math anxiety through affective factors are related to lack of knowledge of teachers and authoritarian instructors, lack of mathematical background of students, necessity of memorizing of formulas, the prevalence of non-real-life problem applications, strict, difficult and time-bound examinations of concrete materials mathematics, normative books and the difficulty of some subjects [9]. Affective factors are related to personality types, mathematics avoidance, negative attitudes towards mathematics, lack of trust, level of mathematics achievement, negative school experiences, gender bias, family and teacher behaviors [10].

Mathematics-related researches have also shown that people are associated with negative experiences and training experienced in educational life. Teachers' negative attitudes and unrealistic lesson practices, time-limited mathematics testing, fear of doing something wrong in the classroom, asking teachers something, inadequate group work and the chance to work, material and manipulation can be important reasons for increasing math anxiety.

The methods used in mathematics education and the structure of mathematical concepts can be defined as situational reasons. Important reasons for creating mathematics anxiety are the teaching strategies used by mathematics teachers and teachers [11]. The behaviors of an authoritarian mathematics teacher may have negative effects such as failure, inadequacy and inadequate feeling on the students [12]. Mathematical concepts that have no connection in real life and therefore abstract content can increase anxiety [11]. The negative effects of math anxiety on mathematics learning may cause the individual to fail to escape mathematics naturally [13], [14]. While the negative effects of math anxiety on mathematics learning are so important, it is possible to cope with math anxiety by knowing and eliminating the causes. In this study, the differences and relation between Mathematical Anxiety Levels and Mathematics Achievements will be evaluated according to the grade levels of high school students from two different school types.

1.1 Scope and importance of the research

This research aims to determine the anxiety and mathematics achievement levels of high school students who have an important place in the education system of our country. In order to achieve this basic goal, the following questions are sought:

- What are the math anxiety levels and mathematics achievements of the students in private and public high schools in Turkey?
- Are there any differences between math anxiety levels and mathematics achievement of the students in terms the type of high schools they are enrolled in Turkey?

This research is important in terms of the effect of factors such as the type of school, gender, which affect the formation of anxiety in mathematics anxiety and mathematics achievement levels of high school students and thus guidance on reducing or completely eliminating the anxiety which may negatively affect mathematics achievement.

1.2 Counting and limitations of research

It was accepted that the students studying in the state and private schools participating in the research reflect their true feelings and thoughts while answering the questions in the measurement tools. This research is limited to data on the 2017-2018 academic year.

2 Method

In this part of the research; research model, sample and data collection tool, and analysis are emphasized.

2.1 Research model

In this study; State Anatolian High School and Private High School1, High SchooL2 and High School3. "Mathematics Anxiety" emotions and thoughts of the class students have been tried to be measured. In this context, research is a survey model research.

2.2 Data collection tool

In order to determine the anxiety levels of mathematics students of 75th Year Anatolian high school (State Anatolian High School) and Ayseabla College(Private), who are the sample of the research, Mathematics Anxiety Scale which was developed by Richardson and Suinn (1972) (Scale named as Math Anxiety Rating Scale-MARS-A) and adapted to Turkish culture by Erol [1], was used. Math anxiety scale which is a 4-item Likert-type scale of 45 items and the validity and reliability study was conducted, the Cronbach Alpha coefficient was determined as 0.91 for this study and it was accepted that the scale was reliable [15].

The highest score is 180 and the lowest score is 45. 45-68 Low, 69-108 Normal, 109-128 anxiety, 129-180 High anxiety was determined according to the scores obtained from this scale [3]. Student success grades; at the end of the semester the teacher was taken. No measurements have been made.

2.3 Sampling, data collection

A sample of 121 students, of which 58 students in the Ayseabla College and a total of 63 students in the 75th Year Anatolian High School, were selected. Mathematical Anxiety Scale was applied to the general mathematics course before applying; In the analysis of the data, it was examined whether the levels of math anxiety and success differed by using frequency distribution and variance analysis. Then the t-test was used to evaluate the relationship between gender and the difference between anxiety and success.

2.4 Analysis process

The data were entered into the SPSS program and the frequency and percentiles of the question items of the scale were used according to the school type variable levels, and the variance analysis (ANOVA) was used for the difference between the school type grade level anxiety scores and the spring year math achievement scores, and the correlation between the math anxiety scores and the math achievement scores were examined. Then the t-test was used to evaluate the relationship between gender and the difference between anxiety and success.

3 Results and Findings

In this section, the findings and comments are determined in accordance with the purpose of the research. The distribution of the school type, the grade level and the characteristics of the students are shown in Table 1.

Table 1 Sampling Characteristics

School	9th grade	10th grade	11th grade
75th Year Anatolian High School	12 Females	10 Females	8 Females
	9 Males	12 Males	12 Males
Ayse Abla College	9 Females	12 Females	10 Females
	13 Females	5 Females	9 Females

3.1 *Differences between school types*

75th Year Anatolian High School has been investigated by analyzing the ANOVA test in which the grade level math anxiety scores of high school students have been differentiated and the results are presented in Table 2. (9A;N=21,Mean=103,1905),(10B;N=22,Mean=95,8182) ((11A;N=20,Mean=90.6500)

Table 2 75th Year Anatolian High School, Class Level Mathematics Anxiety Score ANOVA Analysis Results

ANOVA					
Anxiety Score	**Sum of Squares**	**df**	**Mean Square**	**F**	**Sig.**
Between Groups	446,685	2	223,343	,499	,610
Between Groups	26869,061	60	447,818		
Total	27315,746	62			

75th Year Anatolian High School 9-C, 10-F, 11-A students' Anxiety Points in the analysis of the difference between the average; Firstly, the homogeneity of the variance of the groups was analyzed and the result; Test of Homogeneity of Variances was found to be $p > \alpha 0,875 > 0,05$ and variance, homogeneity was observed. The above findings were obtained. According to the findings; $0,610 > 0.05, p > \alpha$, Red hypothesis (Ho), hypothesis (Ho). In the evaluation, there was no difference between the 75th year Anatolian High School Grade Anxiety Grade scores. High school students' private high school Ayseabla College Science High School grade level math anxiety scores were analyzed by ANOVA test and the results are shown in Table 3. (FL9A;N=21,Mean=81.5714),(FL10A;N=19,Mean=78.1579) (FL11A;N=18,Mean=65.1111)

Table 3 Ayseabla College Grade Level Math Anxiety Score ANOVA Analysis Results

ANOVA					
Anxiety Score	**Sum of Squares**	**df**	**Mean Square**	**F**	**Sig.**
Between Groups	1153,947	2	576,974	0,868	,426
Between Groups	22376,122	56	406,839		
Total	23530,069	58			

Ayseabla college class levels (Science High School 9-A, Science High School 10-B, Science High School 11-A class) and the Mathematics Anxiety Score Scale that were obtained according to the results of Mathematics Anxiety Score Scale were used. First of all, the homogeneity of the variance of the groups was analyzed and the results of the Test of Homogeneity of Variances were found to be $p > \alpha\ 0,129 > 0,05$, and when the homogeneity of variance was observed, ANOVA was applied. $0,426 > 0.05, p > \alpha$, Reject Alternate Hypothesis (Ha), Accept Null hypothesis (Ho). Assessment: There is no statistical difference among Ayseabla College Class Levels Anxiety Points.

3.2 *Differences between genders*

The t-test results of the 75th year Anatolian High School students' mathematics anxiety levels were investigated in Table 4.

Table 4 t-Test Results of the Differences between the Gender Groups of the 75th Year Anatolian High School

	Gender	**N**	**Mean**	**Std. Deviation**	**Std. Error Mean**	**t**	**Sig(p)**
Math Anxiety	Females	26	93,2692	15,69855	3,07874	-1,532	0,196
	Males	37	100,6216	20,61870	3,38969		

According to Table 4, there is no difference between the gender groups in the 75th year Anatolian high school in terms of math anxiety $(0,196 > 0,05; p > \alpha)$.

3.3 t-Test Results of Math Anxiety Differences between Gender Groups

The t-test results of Ayseabla College Science High School students' mathematics anxiety levels were investigated in Table 5.

Table 5 Ayseabla College t-Test Results Related to Differences between Gender Groups

	Gender	**N**	**Mean**	**Std. Deviation**	**Std. Error Mean**	**t**
Math Anxiety	Females	29	82,8276	19,72864	3,66352	-2,469
	Males	29	96,0000	20,88232	3,87775	

Table 8 Ayseabla College Science High School Mathematics Achievement Turkey (LSD) multiple comparison tests

Multiple Comparisons						
Tukey HSD						
(I)Grade Level	(J)Grade Level	Mean Difference(I-J)	Std. Error	Sig.	95% Confidence Interval	
					Lower Bound	Upper Bound
FL9A	FL10A	3,41353	6,69124	,867	-12,7040	19,5311
	FL11A	16,46032*	6,78812	,048	,1094	32,8112
FL10B	FL9A	-3,41353	6,69124	,867	-19,5311	12,7040
	FL11A	13,04678	6,95106	,155	-3,6966	29,7902
FL11A	FL9C	-16,46032*	6,78812	,048	-32,8112	-,1094
	FL10F	-13,04678	6,95106	,155	-29,7902	3,6966
*The mean difference is significant at the 0.05 level						

According to Table 5, there is no difference in terms of math anxiety among the gender groups of the 75th year Anatolian high schools $(0,198 > 0,05; p > \alpha)$.

3.3 Differences between school type mathematics achievement grade levels

The variance analysis and the levels of the differences in the mathematics achievement level of the students who make up the school type sample according to the grade level they have studied are also investigated by Turkey (LSD) multiple comparison tests and the results are given. The Mathematics Achievement Score of 75th Year Anatolian High School students sample was analyzed by ANOVA test, and the results are presented in Table 6. Student success grades; At the end of the semester the math teacher evaluated. Evaluation was used. No measurements.

Table 6 Results of Analysis of Variance Related to Differences between Mathematics Achievement Points

ANOVA	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	2642,141	2	1321,071	2,965	,069
Within Groups	28072,359	3	445,593		
Total	30714,500	5			

9C, 10F, 11A CLASSES

In the sample, the homogeneity of the variance of the groups was analyzed and the result; Test of Homogeneity of Variances was found to be $p > \alpha$ $0,254 > 0,05$ and variance, homogeneity was observed.

The above findings were obtained. According to Table 6, there is no difference in terms of the 75th year Anatolian High School Grade Math Levels. ($0.069 > 0.05$ i.e. $p > \alpha$, Reject H_a, Accept H_0.

The mathematics achievement scores of the students in Ayseabla College of Science High School which the grade level achievement scores were taken from the course lecturer were analyzed through ANOVA test whether the scores were differentiated or not and the results were presented in Table 7.

Table 7 Results of analysis of variance related to differences between mathematics achievement points.

ANOVA			
	Sum of Squares	**df**	**Mean Square**
Between Groups	2849,656	2	1424,828
Within Groups	24563,447	55	446,608
Total	27413,103	57	

FL 9A, FL 10B, FL 11A CLASSES

In the sample, the homogeneity of the variance of the groups was analyzed and the result; Test of Homogeneity of Variances was found to be $p > \alpha$ $0,473 > 0,05$ and variance, homogeneity was observed. The above findings were obtained. According to Table 6, there is a difference in Ayseabla College Mathematics Achievement Points. ($0.049 < 0.05$, $p < \alpha$, H_a Accept, H_0 Reject). The levels of the differences were investigated with the help of Turkey (LSD) multiple comparison test and the results are given in Table **??**.

According to Table **??**; FL9A-FL11A ($p < \alpha, 0,048 < 0,05$) and FL11A-FL9C ($p < \alpha, 0,048 < 0,05$) There is a difference in success.

3.4 The relationship between math anxiety scores and mathematics achievement at the school level

A relationship between the grade level mathematics anxiety scores and mathematics achievement in both School types is analyzed by Correlation Analysis and explained in Table 9.

According to Table 8, there was no correlation between 75th year Anatolian High School and Ayseabla College Class Levels Mathematics Anxiety Scores and Mathematics Achievement Score $(0,379 > 0,05, 0,375 > 0,05, 0.86 > 0,05, 0,902 > 0,05, 0,847 > 0,05, 0,243 > 0.05)$

Table 9 School type Math anxiety and mathematics achievement Correlation scores table

	75thYear Anatolian High School			Ayseabla College		
	AL9C	**AL10F**	**AL11A**	**FL9A**	**FL10B**	**FL11B**
Per Cor (Rxy)	0,208	-0,210	0,034	-0,028	0,046	0,0281
Sig.(2 Tailed) (p)	0,379	0,375	0,886	0,902	0,847	0,243
	$0,379 > 0,05$	$0,375 > 0,05$	$0,886 > 0,05$	$0,902 > 0,05$	$0,847 > 0,05$	$0,243 > 0,05$

3.5 *Type of school mathematics anxiety status evaluation table*

In the evaluation of the Mathematical Anxiety Scale m, the highest score is 180 and the lowest score is 45. 45-68 Low, 69-108 Normal, 109-128 anxiety, 129-180 High anxiety was determined according to the scores obtained from this scale (Erktin, Donmez, Ozel, 2006). The evaluation of the school type sample is explained in Table 10.

Table 10 School type *Math Anxiety Scale* evaluation table

	75thYear Anatolian High School			Ayseabla College		
ANXIETY SITUATION	**AL9C**	**AL10F**	**AL11A**	**FL9A**	**FL10B**	**FL11B**
45-68 Low	1	-	3	2	6	-
69-108 Normal	12	16	14	13	8	18
109-128 With Anxiety	6	6	2	3	3	2
129-180 High Concerned Anxiety	2	-	1	4	-	1
Total	21	22	20	22	17	19

In the descriptive evaluation of the Mathematics Anxiety Scale scores; It is seen that the next grade level in which the anxiety exists in the first grade level of both school types is decreasing.

4 Conclusion, Discussion

According to the research results; 75th Year Anatolian High School and Ayseabla colleges Science High School Grade High School students did not differ in terms of math anxiety levels ($F(3) = 0,4999, 0,610 > 0.05, p > \alpha$). There is no statistically significant difference between the scores of mathematics grades of high school students according to their grade level. So, students' concerns about mathematics do not change significantly according to the level of the class they study; normal, 109-128 anxious, 129-180 high concerned; Anxious and High Anxiety Rate: It is seen in Table 11 that students' prejudices towards mathematics are broken as the grade level increases, and this result is seen in Table 11.

Table 11 Anxious and High Anxious

75thYear Anatolian High School			Ayseabla College		
AL9C	**AL10F**	**AL11A**	**FL9A**	**FL10B**	**FL11B**
%38	%27	%15	%31	%18	%5

Another result obtained from the study; the mathematics achievement level of the sample which applied both types of mathematics Mathematical Anxiety Scale was not significant in the 75th Year Anatolian High School while there was a significant difference between the grade levels of Ayseabla Science High School.

The difference from the FL11A subgroup was verified in the Tukey result. This can be interpreted as the increase in the level of mathematics achievement as the class level increases In both types of schools, Scale could not be applied because senior students were reported in the preparation of university exam. It may be thought that the inclusion of this sample will change the outcome of the data. Another finding found in the study was that there was no statistically significant difference between the levels of anxiety related to mathematics according to the gender variable ($t - 2,469, 0,198 > 0,05$, so $p > \alpha$).

This finding showed that the gender variable had a significant effect on students' math anxiety. However, the mean score of the female students about mathematics ($= 82.8$) was higher than that of male students ($= 96$). From here, it can be said that male students are slightly more anxious than female students.

Another finding reached in the study is; Correlation analysis between school type grade level anxiety score and mathematics achievement was not found to be significant. Correlation analysis rates are; 75thYear Anatolian High School ($9C$; $0,379 > 0,05$ $-10F$; $0,375 > 0,05$ $-11A$; $0.86 > 0,05$), Ayseabla College Science High School ($FL9A$;, $0,902 > 0,05$ $-FL10B$, $0,847 > 0,05$ $-FL11A$ $0,243 > 0,05$) The relationship of anxiety and success is not the case in two school types. This is an important finding for the purpose of the study. In the frequency analysis of the sample; with a high frequency (Question19; 55%). The most important cause of the students to worry about the mathematics exams were determined as unannounced. In this case, it is emphasized that the anxiety against the course increases in making an unannounced examination.

Another question item with high frequency is question 15, its frequency is 80%. In simple mathematical operations, he was asked if he doesn't sound like he couldn't calculate money, for example. This situation can be defined as a lack of self-confidence in the use and implementation of mathematics in daily life. Another question is; Question12 is the percentage and the frequency percentage is 65%. I'm afraid to explain even the problems I can solve. This situation can be defined as lack of self-confidence against mathematics.

4.1 Suggestions

The student fails when he/she is worried, and when he/she fails, the student is afraid of the lesson and the failure is realized. The students studying in the small class are found to be more anxious than the students in the higher level.

This situation was determined in the research findings that these students may cause a new school, new friends and different teachers to have a new system in their transition from primary to secondary education and this may cause anxiety.

The best constructive recommendation for this situation; It can be suggested to plan and implement adaptation programs, especially the 9th-grade students, general contents of the courses, the introduction of teachers, the school environment and the school life, the examination system and reducing anxiety. Orientation programs can be prepared. In addition, as the grade level increases, the increase in mathematics achievement can be interpreted as the focus on the lessons as a channel to the preparation of the university exam. In item frequency evaluation of the scale; the most common cause of the students to worry about the mathematics exams were determined unquestionably (Question19; 55%). For this situation that causes the student to be taken into consideration, before the examination of the students' readiness to take into account, if necessary, determining the dates of the exam with the students, will be shown to have a positive effect on the reduction of anxiety. The problem of simple mathematical operations, for example, not to deduct from the money to calculate the top of the sound (Question15; 80%), the suggestion of the student is not kept away from simple mathematical calculations in daily life and the practical calculation of the acquisition of the family to be sent to shopping, for example.

The percentage frequency of Question 12 in scale is 65%. It is stated that I am afraid to explain even the problems I can solve. For this situation, the lack of self-esteem for mathematics operations is the fact that it was proposed to be introduced in the previous years in the family and that it should be started from the pre-school.

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