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A NEW APPROACH TO STATISTICALLY QUASI CAUCHY SEQUENCES

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Abstract. A sequence (α_k) of points in $\mathbb{R}$, the set of real numbers, is called *ρ-statistically p quasi Cauchy* if

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : |\Delta_p \alpha_k| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$, where $\boldsymbol{\rho} = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$, and $\Delta_p\alpha_{k+p} = \alpha_{k+p} - \alpha_k$ for each positive integer k. A real-valued function defined on a subset of $\mathbb{R}$ is called ρ-statistically p-ward continuous if it preserves ρ-statistical p-quasi Cauchy sequences. ρ-statistical p-ward compactness is also introduced and investigated. We obtain results related to ρ-statistical p-ward continuity, ρ-statistical p-ward compactness, p-ward continuity, continuity, and uniform continuity.

1. Introduction

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics but also in other branches of sciences involving mathematics especially in computer science, information theory, biological science.

The idea of statistical convergence was formerly given under the name "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 in [39]. The concept was formally introduced by Fast [26] and later was reintroduced by Schoenberg [34], and also independently by Buck [2]. Although statistical convergence was introduced over nearly the last eighty years, it has become an active area of research for thirty years with the contributions by several authors, Salat ([33]), Fridy [27], Caserta and Kocinac [24], Maio and Kocinac ([28]), Caserta, Maio and Kocinac ([25]), Patterson and Savas ([32],Mursaleen ([29]), Cakalli and Khan ([17]), Yildiz ([37], and [38]).

2010 *Mathematics Subject Classification.* Primary: 40A05 ; Secondaries: 26A15, 40A30 .

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A sequence (α_k) of points in $\mathbb{R}$ is called *ρ-statistically convergent* to an element ℓ of $\mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : |\alpha_k - \ell| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$, where $\boldsymbol{\rho} = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty$, and $(\Delta\rho_n)$ is a bounded sequence ([14]). This is denoted by $st_\rho - \lim_{k\to\infty} \alpha_k = \ell$. We note that such sequences were introduced without the assumption of boundedness of the downward difference sequence of $\boldsymbol{\rho}$ in [30], and was called quasi statistical convergence. The sequential method $st_\rho - \lim$ is a regular sequential method since any convergent sequence is ρ-statistically convergent ([30, page 13].

A sequence (α_k) of points in $\mathbb{R}$, the set of real numbers, is called *ρ-statistically quasi Cauchy* if

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : |\Delta\alpha_k| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$, where $\Delta\alpha_k = \alpha_{k+1} - \alpha_k$ for each positive integer k ([14]).

Using the idea of continuity of a real function in terms of sequences in the sense that a function preserves a certain kind of sequences, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: slowly oscillating continuity ([7]), quasi-slowly oscillating continuity ([23]), ward continuity ([12]), δ-ward continuity ([8]), statistical ward continuity ([10]), and N_θ-ward continuity ([4]) which enabled some authors to obtain conditions on the domain of a function for some characterizations of uniform continuity (see [36, Theorem 6],[3, Theorem 1 and Theorem 2],[23, Theorem 2.3], [3, Theorem 1], and [21, Theorem 5].

The purpose of this paper is to introduce and investigate the concept of ρ-statistical p-ward continuity of a real function, and prove interesting theorems.

2. Results

Now we introduce the concept of ρ-statistically p quasi Cauchyness.

Definition 2.1. *A sequence (α_k) of points in $\mathbb{R}$, the set of real numbers, is called ρ-statistically p quasi Cauchy if*

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : |\Delta_p\alpha_k| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$, where $\Delta_p\alpha_{k+p} = \alpha_{k+p} - \alpha_k$ for each positive integer k, p is a fixed positive integer.

Any quasi-Cauchy sequence is ρ-statistically p-quasi-Cauchy, but the converse is not always true. Any ρ-statistically convergent sequence is ρ-statistically p-quasi-Cauchy. There are ρ-statistically p-quasi-Cauchy sequences which are not ρ-statistically convergent. The sum of two ρ-statistical p-quasi-Cauchy sequences is ρ-statistically p-quasi-Cauchy, but the product of two ρ-statistical p-quasi-Cauchy sequences need not be ρ-statistically p-quasi-Cauchy.

Now we give the definition of ρ-statistical p-ward compactness.

Definition 2.2. *A subset E of $\mathbb{R}$ is called ρ-statistically p-ward compact if any sequence of points in E has a ρ-statistical p-quasi-Cauchy subsequence.*

First, we note that any finite subset of $\mathbb{R}$ is ρ-statistically p-ward compact, the union of two ρ-statistically p-ward compact subsets of $\mathbb{R}$ is ρ-statistically p-ward compact and the intersection of any family of ρ-statistically p-ward compact subsets of $\mathbb{R}$ is ρ-statistically p-ward compact. Any G-sequentially compact subset of $\mathbb{R}$ is ρ-statistically p-ward compact for a regular subsequential method G (see [6], and [9]). Furthermore any subset of a ρ-statistically p-ward compact set is ρ-statistically p-ward compact, any bounded subset of $\mathbb{R}$ is ρ-statistically p-ward compact, any slowly oscillating compact subset of $\mathbb{R}$ is ρ-statistically p-ward compact (see [7] for the definition of slowly oscillating compactness). These observations suggest to us the following.

Theorem 2.1. *A subset E of $\mathbb{R}$ is bounded if and only if it is ρ-statistically p-ward compact.*

Proof. If E is a bounded subset of $\mathbb{R}$, then any sequence of points in E has a convergent subsequence which is also ρ-statistically p-quasi-Cauchy. Conversely, suppose that E is not bounded. If it is not bounded below, then pick an element α_1 of E less than 0. Then we can choose an element α_2 of E such that $\alpha_2 < -p-\rho_1+\alpha_1$. Similarly we can choose an element α_3 of E such that $\alpha_3 < -p-\rho_2+\alpha_2$. We can inductively choose α_k satisfying $\alpha_{k+1} < -p-\rho_k+\alpha_k$ for each $k \in \mathbb{N}$. Hence $\alpha_k - \alpha_{k+p} > p+\rho_k$ for each $k \in \mathbb{N}$. Thus $|\alpha_{k+p}-\alpha_k| > p+\rho_k$ for each $k \in \mathbb{N}$. Then the sequence (α_k) does not have any ρ-statistically p-quasi Cauchy subsequence. If E is unbounded above, then we can find a β_1 greater than 0. Then we can pick a β_2 such that $\beta_2 > \rho_1+p+\beta_1$. We can successively find for each $k \in \mathbb{N}$ a β_{k+1} such that $\beta_{k+1} > \rho_k+p+\beta_k$. Then $\beta_{k+p}-\beta_k > p+\rho_k$ for each $k \in \mathbb{N}$. Thus $|\beta_{k+p}-\beta_k| > p+\rho_k$ for each $k \in \mathbb{N}$. Then the sequence (β_k) does not have any ρ-statistical p-quasi Cauchy subsequence. Thus E is not ρ-statistically p-ward compact. This completes the proof. □

Corollary 2.2. *A subset E of $\mathbb{R}$ is ρ-statistically p-ward compact if and only if it is both upward and downward statistically compact.*

Proof. The proof follows from the preceding theorem and [13, Theorem 3.3 and Theorem 3.6]. □

Theorem 2.3. *If a function f is uniformly continuous on a subset E of $\mathbb{R}$, then $(f(\alpha_k))$ is ρ-statistically p-quasi Cauchy whenever (α_k) is a quasi-Cauchy sequence of points in E.*

Proof. Take any p-quasi-Cauchy sequence (α_k) of points in E, and let ε be any positive real number. By uniform continuity of f, there exists a $\delta > 0$ such that $|f(\alpha)-f(\beta)| < \varepsilon$ whenever $|\alpha-\beta| < \delta$ and $\alpha, \beta \in E$. Since (α_k) is a p-quasi-Cauchy sequence, there exists a positive integer k_0 such that $|\alpha_{k+p}-\alpha_k| < \delta$ for $k \geq k_0$. Thus

$$|\{k \leq n : |f(\alpha_{k+p}) - f(\alpha_k)| \geq \varepsilon\}| \leq k_0.$$

Hence

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : |f(\alpha_{k+p}) - f(\alpha_k)| \geq \varepsilon\}| = 0$$

Thus $(f(\alpha_k))$ is a ρ-statistically p-quasi Cauchy sequence. This completes the proof of the theorem. □

Definition 2.3. *A function defined on a subset E of $\mathbb{R}$ is called ρ-statistically p-ward continuous if it preserves ρ-statistically p-quasi-Cauchy sequences, i.e. $(f(\alpha_n))$ is a ρ-statistically p-quasi-Cauchy sequence whenever (α_n) is.*

We see that the sum of two ρ-statistically p-ward continuous functions is ρ-statistically p-ward continuous, and cf is ρ-statistically p-ward continuous whenever c is a constant real number and f is a ρ-statistically p-ward continuous function.

Theorem 2.4. *If f is ρ-statistically p-ward continuous on a subset E of $\mathbb{R}$, then it is ρ-statistically continuous on E.*

Proof. Assume that f is a ρ-statistically p-ward continuous function on E. Let (α_n) be any ρ-statistically convergent sequence with $st_\rho - \lim_{k\to\infty} \alpha_k = \ell$. Then the sequence

$$(\alpha_1, \alpha_1, ..., \alpha_1, \ell, \ell, ..., \ell, \alpha_2, \alpha_2, ..., \ell, \ell, ..., \alpha_n, \alpha_n, ..., \ell, \ell, ...)$$

is ρ-statistically convergent to ℓ, where the same terms repeat p times. Hence it is ρ-statistically quasi-Cauchy, so is ρ-statistically p-quasi-Cauchy. As f is ρ-statistically p-ward continuous, the sequence

$$(f(\alpha_1), f(\alpha_1), ..., f(\alpha_1), f(\ell), f(\ell), ..., f(\ell), f(\alpha_2), f(\alpha_2), ..., f(\ell), f(\ell), ..., f(\alpha_n), f(\alpha_n), ..., f(\ell), f(\ell), ...)$$

is ρ-statistically p-quasi-Cauchy. Hence it follows that the sequence $(f(\alpha_n))$ is ρ-statistically converges to $f(\ell)$. This completes the proof of the theorem. □

Related to G-continuity we have the following result.

Corollary 2.5. *If f is ρ-statistically p-ward continuous, then it is G-continuous for any regular subsequential method G.*

The preceding corollary ensures that ρ-statistically p-ward continuity implies either of the following continuities; ordinary continuity, statistical continuity, lacunary statistical continuity ([5]), strongly lacunary continuity ([4]), λ-statistical continuity, I-sequential continuity for any non trivial admissible ideal I of $\mathbb{N}$ ([16]).

It is well known that any continuous function on a compact subset E of $\mathbb{R}$ is uniformly continuous on E. For ρ-statistically p-ward continuous functions we have the following.

Theorem 2.6. *Let E be a ρ-statistically p-ward compact subset E of $\mathbb{R}$ and let $f : A \longrightarrow \mathbb{R}$ be a ρ-statistically p-ward continuous function on E. Then f is uniformly continuous on E.*

Proof. Suppose that f is not uniformly continuous on E so that there exists an $\varepsilon_0 > 0$ such that for any $\delta > 0$ $x, y \in E$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$. For each positive integer n, there are α_n and β_n such that $|\alpha_n - \beta_n| < \frac{1}{n}$, and $|f(\alpha_n) - f(\beta_n)| \geq \varepsilon_0$. Since E is ρ-statistically p-ward compact, there exists a ρ-statistical p-quasi-Cauchy subsequence (α_{n_k}) of the sequence (α_n). It is clear that the corresponding subsequence (β_{n_k}) of the sequence (β_n) is also ρ-statistically p-quasi-Cauchy, since $(\beta_{n_{k+p}} - \beta_{n_k})$ is a sum of three ρ-statistical null sequences, i.e.

$$\beta_{n_{k+p}} - \beta_{n_k} = (\beta_{n_{k+p}} - \alpha_{n_{k+p}}) + (\alpha_{n_{k+p}} - \alpha_{n_k}) + (\alpha_{n_k} - \beta_{n_k}).$$

Then the sequence

$$(a_{n_1}, a_{n_1}, ..., a_{n_1}, \beta_{n_1}, \beta_{n_1}, ..., \beta_{n_1}, \alpha_{n_2}, \alpha_{n_2}, ..., \alpha_{n_2}, \beta_{n_2}, \beta_{n_2}, ..., \beta_{n_2}, , ..., \alpha_{n_k}, \alpha_{n_k}, ..., \alpha_{n_k}, \beta_{n_k}, \beta_{n_k}, ..., \beta_{n_k}, ...)$$

is ρ-statistical p-quasi-Cauchy since the sequence $(\alpha_{n_k} - \beta_{n_k})$ is ρ-statistically convergent to 0. But the transformed sequence is not ρ-statistically p-quasi-Cauchy. Thus f does not preserve ρ-statistical p-quasi-Cauchy sequences. This contradiction completes the proof of the theorem. □

Corollary 2.7. *If a function f is ρ-statistically p-ward continuous on a bounded subset E of $\mathbb{R}$, then it is uniformly continuous on E.*

Proof. The proof follows from Theorem 2.6 and Theorem 2.1. □

Theorem 2.8. *ρ-statistical p-ward continuous image of any ρ-statistically p-ward compact subset of $\mathbb{R}$ is ρ-statistically p-ward compact.*

Proof. Assume that f is a ρ-statistically p-ward continuous function on a subset E of $\mathbb{R}$, and A is a ρ-statistically p-ward compact subset of E. Let (β_n) be any sequence of points in $f(A)$. Write $\beta_n = f(\alpha_n)$ where $\alpha_n \in A$ for each positive integer n. ρ-statistically ward compactness of A implies that there is a subsequence $(\gamma_k) = (\alpha_{n_k})$ of (α_n) with $st_\rho - \lim_{k\to\infty} \Delta\gamma_k = 0$. Write $(t_k) = (f(\gamma_k))$. As f is ρ-statistically p-ward continuous, $(f(\gamma_k))$ is ρ-statistically p-quasi-Cauchy. Thus we have obtained a subsequence (t_k) of the sequence $(f(\alpha_n))$ with $st_\rho - \lim_{k\to\infty} \Delta^p t_k = 0$. Thus $f(A)$ is ρ-statistically p-ward compact. This completes the proof of the theorem. □

Corollary 2.9. *ρ-statistically p-ward continuous image of any compact subset of $\mathbb{R}$ is ρ-statistically ward compact.*

The proof follows from the preceding theorem.

Corollary 2.10. *ρ-statistically p-ward continuous image of any bounded subset of $\mathbb{R}$ is bounded.*

The proof follows from Theorem 2.1 and Theorem 2.8.

Corollary 2.11. *ρ-statistical p-ward continuous image of a G-sequentially compact subset of $\mathbb{R}$ is ρ-statistically p-ward compact for any subsequential regular method G.*

It is a well known result that uniform limit of a sequence of continuous functions is continuous. This is also true in case of ρ-statistical p-ward continuity, i.e. uniform limit of a sequence of ρ-statistical p-ward continuous functions is ρ-statistically p-ward continuous.

Theorem 2.12. *If (f_n) is a sequence of ρ-statistically p-ward continuous functions on a subset E of $\mathbb{R}$ and (f_n) is uniformly convergent to a function f, then f is ρ-statistically p-ward continuous on E.*

Proof. Let ε be a positive real number and (α_k) be any ρ-statistical p-quasi-Cauchy sequence of points in E. By the tuniform convergence of (f_n) there exists a positive integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in E$ whenever $n \geq N$. As f_N is ρ-statistically p-ward continuous on E, we have

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : f_N(\alpha_{k+p}) - f_N(\alpha_k)| \geq \frac{\varepsilon}{3}\}| = 0.$$

On the other hand we have
$\{k \leq n : |f(\alpha_{k+p}) - f(\alpha_k)| \geq \varepsilon\} \subset \{k \leq n : |f(\alpha_{k+p}) - f_N(\alpha_{k+p})| \geq \frac{\varepsilon}{3}\}$
$\cup \{k \leq n : |f_N(\alpha_{k+p}) - f_N(\alpha_k)| \geq \frac{\varepsilon}{3}\} \cup \{k \leq n : |f_N(\alpha_k) - f(\alpha_k)| \geq \frac{\varepsilon}{3}\}$

Now it follows from this inclusion that
$\lim_{n\to\infty}\frac{1}{\rho_n}|\{k\leq n: |f(\alpha_{k+p})-f(\alpha_k)|\geq\varepsilon\}|$

$$\leq \lim_{n\to\infty}\frac{1}{\rho_n}|\{k\leq n: |f(\alpha_{k+p})-f_N(\alpha_{k+p})|\geq\frac{\varepsilon}{3}\}|+\lim_{n\to\infty}\frac{1}{\rho_n}|\{k\leq n: |f_N(\alpha_{k+p})-f_N(\alpha_k)|\geq\frac{\varepsilon}{3}\}|$$

$+\lim_{n\to\infty}\frac{1}{\rho_n}|\{k\leq n: |f_N(\alpha_k)-f(\alpha_k)|\geq\frac{\varepsilon}{3}\}|=0+0+0=0.$
This completes the proof of the theorem. □

Theorem 2.13. *The set of all ρ-statistically p-ward continuous functions on a subset E of $\mathbb{R}$ is a closed subset of the set of all continuous functions on E, i.e. $\overline{\Delta\rho^pSWC(E)}=\Delta\rho^pSWC(E)$ where $\Delta\rho^pSWC(E)$ is the set of all ρ-statistically p-ward continuous functions on E, $\overline{\Delta\rho^pSWC(E)}$ denotes the set of all cluster points of $\Delta\rho^pSWC(E)$.*

Proof. f be any element in $\overline{\Delta\rho^pSWC(E)}$. Then there exists a sequence of points in $\Delta\rho^pSWC(E)$ such that $\lim_{k\to\infty}f_k=f$. To show that f is ρ-statistically p-ward continuous, take any ρ-statistical p-quasi-Cauchy sequence (α_k) of points in E. Let $\varepsilon>0$. Since (f_k) converges to f, there exists an N such that for all $x\in E$ and for all $n\geq N$, $|f(x)-f_n(x)|<\frac{\varepsilon}{3}$. As f_N is ρ-statistically p-ward continuous, we have $\lim_{n\to\infty}\frac{1}{\rho_n}|\{k\leq n: f_N(\alpha_{k+p})-f_N(\alpha_k)|\geq\frac{\varepsilon}{3}\}|=0$. On the other hand,
$\{k\leq n: |f(\alpha_{k+p})-f(\alpha_k)|\geq\varepsilon\}\subset\{k\leq n: |f(\alpha_{k+p})-f_N(\alpha_{k+p})|\geq\frac{\varepsilon}{3}\}$
$\cup\{k\leq n: |f_N(\alpha_{k+p})-f_N(\alpha_k)|\geq\frac{\varepsilon}{3}\}\cup\{k\leq n: |f_N(\alpha_k)-f(\alpha_k)|\geq\frac{\varepsilon}{3}\}$
Now it follows from this inclusion that
$\lim_{n\to\infty}\frac{1}{\rho_n}|\{k\leq n: |f(\alpha_{k+p})-f(\alpha_k)|\geq\varepsilon\}|$

$$\leq \lim_{n\to\infty}\frac{1}{\rho_n}|\{k\leq n: |f(\alpha_{k+p})-f_N(\alpha_{k+p})|\geq\frac{\varepsilon}{3}\}|+\lim_{n\to\infty}\frac{1}{\rho_n}|\{k\leq n: |f_N(\alpha_{k+p})-f_N(\alpha_k)|\geq\frac{\varepsilon}{3}\}|$$

$+\lim_{n\to\infty}\frac{1}{\rho_n}|\{k\leq n: |f_N(\alpha_k)-f(\alpha_k)|\geq\frac{\varepsilon}{3}\}|=0+0+0=0.$
This completes the proof of the theorem. □

Corollary 2.14. *The set of all ρ-statistically p-ward continuous functions on a subset E of $\mathbb{R}$ is a complete subspace of the space of all continuous functions on E.*

Proof. The proof follows straightforward from the preceding theorem.

□

3. Conclusion

The results in this paper not only generalize results studied in [10], and [11] as a special case, i.e. $\rho_n=n$ for each $n\in\mathbb{N}$, but also includes results which are also new for the special case. It turns out that the set of uniformly continuous functions includes the set of ρ-statistical ward continuous functions on bounded sets. We suggest to investigate ρ-statistically p-quasi-Cauchy sequences of fuzzy points or soft points (see [19], for the definitions and related concepts in fuzzy setting, and see [1] related concepts in soft setting). We also suggest to investigate ρ-statistically p-quasi-Cauchy double sequences (see for example [18] for the definitions and related concepts in the double sequences case). For another further study, we suggest to investigate ρ-statistically p-quasi-Cauchy sequences in abstract metric spaces (see [22], [31], [20], and [35]).

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LACUNARY STATISTICAL p-QUASI CAUCHY SEQUENCES

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Abstract. In this paper, we introduce a concept of lacunary statistically p-quasi-Cauchyness of a real sequence in the sense that a sequence (α_k) is lacunary statistically p-quasi-Cauchy if $\lim_{r\to\infty}\frac{1}{h_r}|\{k\in I_r : |\alpha_{k+p}-\alpha_k|\geq\varepsilon\}|=0$ for each $\varepsilon>0$. A function f is called lacunary statistically p-ward continuous on a subset A of the set of real numbers $\mathbb{R}$ if it preserves lacunary statistically p-quasi-Cauchy sequences, i.e. the sequence $(f(\alpha_n))$ is lacunary statistically p-quasi-Cauchy whenever $\boldsymbol{\alpha}=(\alpha_n)$ is a lacunary statistically p-quasi-Cauchy sequence of points in A. It turns out that a real valued function f is uniformly continuous on a bounded subset A of $\mathbb{R}$ if there exists a positive integer p such that f preserves lacunary statistically p-quasi-Cauchy sequences of points in A.

1. Introduction

Throughout this paper, $\mathbb{N}$, and $\mathbb{R}$ will denote the set of positive integers, and the set of real numbers, respectively. p will always be a fixed element of $\mathbb{N}$. The boldface letters such as $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\zeta}$ will be used for sequences $\boldsymbol{\alpha}=(\alpha_n)$, $\boldsymbol{\beta}=(\beta_n)$, $\boldsymbol{\zeta}=(\zeta_n)$, ... of points in $\mathbb{R}$. A function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is continuous if and only if it preserves convergent sequences. Using the idea of continuity of a real function in this manner, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: ward continuity ([15], [4]), p-ward continuity ([23]), δ-ward continuity ([18]), δ^2-ward continuity ([3]), statistical ward continuity, ([19]), λ-statistical ward continuity ([36]), ρ-statistical ward continuity ([6], [25]), slowly oscillating continuity ([12, 59, 35]), quasi-slowly oscillating continuity ([42]), Δ-quasi-slowly oscillating continuity ([16]), arithmetic continuity ([60], [5]), upward and downward statistical continuities ([24]), lacunary statistical ward continuity ([7], [66]), lacunary statistical δ ward continuity ([31]), lacunary statistical δ^2 ward continuity ([64]), N_θ-ward continuity ([22], [30], [48], [8], [48], [47]), N_θ-δ-ward continuity, and ([8]) , which enabled some authors to obtain interesting results.

In [45] Fridy and Orhan introduced the concept of lacunary statistically convergence in the sense that a sequence (α_k) of points in $\mathbb{R}$ is called lacunary statistically convergent, or S_θ-convergent, to an element L of $\mathbb{R}$ if $\lim_{r\to\infty}\frac{1}{h_r}|\{k\in I_r : |\alpha_k-L|\geq\varepsilon\}|=0$ for every positive real number ε where $I_r=(k_{r-1},k_r]$ and $k_0=0$,

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$h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$ and $\theta = (k_r)$ is an increasing sequence of positive integers (see also [46], [56] [11], [57], and [44]). In this case we write $S_\theta - lim\alpha_k = L$. The set of lacunary statistically convergent sequences of points in R is denoted by S_θ. In the sequel, we will always assume that $\lim inf_r\, q_r > 1$. A sequence (α_k) of points in R is called lacunary statistically quasi-Cauchy if $S_\theta - lim\Delta\alpha_k = 0$, where $\Delta\alpha_k = \alpha_{k+1} - \alpha_k$ for each positive integer k. The set of lacunary statistically quasi-Cauchy sequences will be denoted by ΔS_θ.

The purpose of this paper is to introduce lacunary statistically p-quasi-Cauchy sequences, and prove interesting theorems.

2. Variations on lacunary statistical ward compactness

The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to 0 and lacunary statistically tending to zero, and more generally speaking, than that the distance between p-successive terms is lacunary statistically tending to zero, by p-successive terms we mean α_{k+p} and α_k. Nevertheless, sequences which satisfy this weaker property are interesting in their own right.

Before giving our main definition we recall basic concepts. A sequence (α_n) is called quasi Cauchy if $\lim_{n\to\infty} \Delta\alpha_n = 0$, where $\Delta\alpha_n = \alpha_{n+1} - \alpha_n$ for each $n \in \mathbb{N}$ ([4], [15]). The set of all bounded quasi-Cauchy sequences is a closed subspace of the space of all bounded sequences with respect to the norm defined for bounded sequences ([51]). A sequence (α_k) of points in $\mathbb{R}$ is slowly oscillating if $\lim_{\lambda\to 1^+} \overline{\lim}_n \max_{n+1\leq k\leq[\lambda n]} |\alpha_k - \alpha_n| = 0$, where $[\lambda n]$ denotes the integer part of λn ([41]). A sequence (α_k) is quasi-slowly oscillating if $(\Delta\alpha_k)$ is slowly oscillating. A sequence (α_n) is called statistically convergent to a real number L if $\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : |\alpha_k - L| \geq \varepsilon\}| = 0$ for each $\varepsilon > 0$ ([43], [14], [32], and [9]). Recently in [23] it was proved that a real valued function is uniformly continuous whenever it is p-ward continuous on a bounded subset of $\mathbb{R}$. Now we introduce the concept of a lacunary statistically p-quasi-Cauchy sequence.

Definition 2.1. *A sequence (α_k) of points in $\mathbb{R}$ is called lacunary statistically p-quasi-Cauchy if $S_\theta - lim_{k\to\infty}\Delta_p\alpha_k = 0$, i.e. $\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : |\Delta_p\alpha_k| \geq \varepsilon\}| = 0$ for each $\varepsilon > 0$, where $\Delta_p\alpha_k = \alpha_{k+p} - \alpha_k$ for every $k \in \mathbb{N}$.*

We will denote the set of all lacunary statistically p-quasi-Cauchy sequences by Δ_p^θ. The sum of two lacunary statistically p-quasi-Cauchy sequences is lacunary statistically p-quasi-Cauchy, the product of a lacunary statistically p-quasi-Cauchy sequence and a constant real number is lacunary statistically p-quasi-Cauchy, so that the set of all lacunary statistically p-quasi-Cauchy sequences Δ_p^θ is a vector space. We note that a sequence is lacunary statistically quasi-Cauchy when $p = 1$, i.e. lacunary statistically 1-quasi-Cauchy sequences are lacunary statistical quasi-Cauchy sequences. It follows from the inclusion
$\{k \in I_r : |\alpha_{k+p} - \alpha_k| \geq \varepsilon\} \subseteq$
$\subseteq \{k \in I_r : |\alpha_{k+p} - \alpha_{k+p-1}| \geq \frac{\varepsilon}{p}\} \cup \{k \in I_r : |\alpha_{k+p-1} - \alpha_{k+p-2}| \geq \frac{\varepsilon}{p}\} \cup ...$
$\cup \{k \in I_r : |\alpha_{k+2} - \alpha_{k+1}| \geq \frac{\varepsilon}{p}\} \cup \{k \in I_r : |\alpha_{k+1} - \alpha_k| \geq \frac{\varepsilon}{p}\}$
that any lacunary statistically quasi-Cauchy sequence is also lacunary statistically p-quasi-Cauchy, but the converse is not always true as it can be seen by considering the the sequence (α_k) defined by $(\alpha_k) = (0, 1, 0, 1, ..., 0, 1, ...)$ is lacunary statistically 2-quasi Cauchy which is not lacunary statistically quasi Cauchy. More examples

can be seen in [51, Section 1.4]. It is clear that any Cauchy sequence is in $\bigcap_{\infty}^{p=1} \Delta_p^{\theta}$, so that each Δ_p^{θ} is a sequence space containing the space $\mathcal{C}$ of Cauchy sequences. It should also be noted that $\mathcal{C}$ is a proper subset of Δ_p^{θ} for each $p \in \mathbb{N}$.

Definition 2.2. *A subset A of $\mathbb{R}$ is called lacunary statistically p-ward compact if any sequence of points in A has a lacunary statistically p-quasi-Cauchy subsequence.*

We note that this definition of lacunary statistically p-ward compactness cannot be obtained by any summability matrix in the sense of [13] (see also [10], and [20]).

Since any lacunary statistically quasi-Cauchy sequence is lacunary statistically p-quasi-Cauchy we see that any lacunary statistically ward compact subset of $\mathbb{R}$ is lacunary statistically p-ward compact for any $p \in \mathbb{N}$. A finite subset of $\mathbb{R}$ is lacunary statistically p-ward compact, the union of finite number of lacunary statistically p-ward compact subsets of $\mathbb{R}$ is lacunary statistically p-ward compact, and the intersection of any family of lacunary statistically p-ward compact subsets of $\mathbb{R}$ is lacunary statistically p-ward compact. Furthermore any subset of a lacunary statistically p-ward compact set of $\mathbb{R}$ is lacunary statistically p-ward compact and any bounded subset of $\mathbb{R}$ is lacunary statistically p-ward compact. These observations above suggest to us the following.

Theorem 2.1. *A subset A of $\mathbb{R}$ is bounded if and only if there exists a $p \in \mathbb{N}$ such that A is lacunary statistically p-ward compact.*

Proof. The bounded subsets of $\mathbb{R}$ are lacunary statistically p-ward compact, since any bounded sequence of points in a bounded subset of $\mathbb{R}$ is bounded and any bounded sequence has a convergent subsequence which is lacunary statistically p-quasi-Cauchy for any $p \in \mathbb{N}$. To prove the converse, suppose that A is not bounded. If it is unbounded above, pick an element α_1 of A greater than p. Then we can find an element α_2 of A such that $\alpha_2 > 2p + \alpha_1$. Similarly, choose an element α_3 of A such that $\alpha_3 > 3p + \alpha_2$. So we can construct a sequence (α_j) of numbers in A such that $\alpha_{j+1} > (j+1)p + \alpha_j$ for each $j \in \mathbb{N}$. Then the sequence (α_j) does not have any lacunary statistically p-quasi-Cauchy subsequence. If A is bounded above and unbounded below, then pick an element β_1 of A less than $-p$. Then we can find an element β_2 of A such that $\beta_2 < -2p + \beta_1$. Similarly, choose an element β_3 of A such that $\beta_3 < -3p + \beta_2$. Thus one can construct a sequence (β_i) of points in A such that $\beta_{i+1} < -(i+1)p + \beta_i$ for each $i \in \mathbb{N}$. Then the sequence (α_i) does not have any lacunary statistically p-quasi-Cauchy subsequence. Thus this contradiction completes the proof of the theorem. □

It follows from Theorem 2.1 that lacunary statistically p-ward compactness of a subset of A of $\mathbb{R}$ coincides with either of the following kinds of compactness: p-ward compactness ([23, Theorem 2.3]), statistical ward compactness ([19, Lemma 2]), λ-statistical ward compactness ([36, Theorem 1]), ρ-statistical ward compactness ([6, Theorem 1]), strongly lacunary ward compactness ([22, Theorem 3.3]), slowly oscillating compactness ([17, Theorem 3]), lacunary statistical ward compactness (see [7]), and [17, Theorem 3]), ideal ward compactness ([29, Theorem 8]), Abel ward compactness ([**?**, Theorem 5]).

If a closed subset of $\mathbb{R}$ is lacunary statistically p-ward compact for a positive integer p, then any sequence of points in A has a (P_n, s)-absolutely almost convergent subsequence (see [27], [37], [52], [62], [2], [65], and [63]).

Corollary 2.2. A subset of R is statistically p-ward compact if and only if it is statistically q-ward compact for any $p, q \in \mathbb{N}$.

Corollary 2.3. A subset of R is statistically p-ward compact if and only if it is both statistically upward half compact and statistically downward half compact.

Proof. The proof follows from [24, Corollary 3.9], so is omitted. □

Corollary 2.4. A subset of $\mathbb{R}$ is lacunary statistically p ward compact for a $p \in \mathbb{N}$ if and only if it is both lacunary statistically upward half compact and lacunary statistically downward half compact.

Proof. The proof follows from [33, Theorem 1.3 and Theorem 1.9], so is omitted. □

3. Variations on lacunary statistical ward continuity

In this section, we investigate connections between uniformly continuous functions and lacunary statistically p-ward continuous functions. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous if and only if it preserves lacunary statistically convergent sequences. Using this idea, we introduce lacunary statistical p-ward continuity.

Definition 3.1. *A function f is called lacunary statistically p-ward continuous on a subset A of $\mathbb{R}$ if it preserves lacunary statistically p-quasi-Cauchy sequences, i.e. the sequence $(f(\alpha_n))$ is lacunary statistically p-quasi-Cauchy whenever (α_n) is lacunary statistically p-quasi-Cauchy of points in A.*

We see that this definition of lacunary statistically p-ward continuity can not be obtained by any summability matrix A (see [10]).

We note that the sum of two lacunary statistically p-ward continuous functions is lacunary statistically p-ward continuous, and for any constant $c \in \mathbb{R}$, cf is lacunary statistically p-ward continuous whenever f is a lacunary statistically p-ward continuous function, so that the set of all lacunary statistically p ward continuous functions is a vector space. The composite of two lacunary statistically p-ward continuous functions is lacunary statistically p-ward continuous, but the product of two lacunary statistically p-ward continuous functions need not be lacunary statistically p-ward continuous as it can be seen by considering product of the lacunary statistically p-ward continuous function $f(x) = x$ with itself. If f is a lacunary statistically p-ward continuous function, then $|f|$ is also lacunary statistically p-ward continuous since

$$|\{k \in I_r : |f(\alpha_{k+p}) - f(\alpha_k)| \geq \varepsilon\}| \subseteq |\{k \in I_r : ||f(\alpha_{k+p})| - |f(\alpha_k)|| \geq \varepsilon\}|$$

which follows from the inequality $||f(\alpha_{k+p})| - |f(\alpha_k)|| \leq |f(\alpha_{k+p}) - f(\alpha_k)|$. If f, and g are lacunary statistically p-ward continuous, then $max\{f, g\}$ is also lacunary statistically p-ward continuous, which follows from the equality $max\{f, g\} = \frac{1}{2}\{|f - g| + |f + g|\}$.

Theorem 3.1. *If f is lacunary statistically p-ward continuous on a subset A of $\mathbb{R}$ for some $p \in \mathbb{N}$, then it is lacunary statistically ward continuous on A.*

Proof. If $p = 1$, then it is obvious. So we would suppose that $p > 1$. Take any lacunary statistically p-ward continuous function f on A. Let (α_k) be any lacunary statistical quasi-Cauchy sequence of points in A. Write

$$(\xi_i) = (\alpha_1, \alpha_1, ..., \alpha_1, \alpha_2, \alpha_2, ..., \alpha_2, ..., \alpha_n, \alpha_n, ..., \alpha_n, ...),$$

where the same term repeats p times. The sequence

$$(\alpha_1, \alpha_1, ..., \alpha_1, \alpha_2, \alpha_2, ..., \alpha_2, ..., \alpha_n, \alpha_n, ..., \alpha_n, ...)$$

is also lacunary statistically quasi-Cauchy so it is lacunary statistically p-quasi-Cauchy. By the lacunary statistically p-ward continuity of f, the sequence

$$(f(\alpha_1), f(\alpha_1), ..., f(\alpha_1), f(\alpha_2), f(\alpha_2), ..., f(\alpha_2), ..., f(\alpha_n), f(\alpha_n), ..., f(\alpha_n), ...)$$

is lacunary statistically p-quasi-Cauchy, where the same term repeats p-times. Thus the sequence

$$(f(\alpha_1), f(\alpha_1), ..., f(\alpha_1), f(\alpha_2), f(\alpha_2), ..., f(\alpha_2), ..., f(\alpha_n), f(\alpha_n), ..., f(\alpha_n), ...)$$

is also lacunary statistically p quasi-Cauchy. It is easy to see that $S_\theta - lim(f(\alpha_{n+p}) - f(\alpha_n)) = 0$, which completes the proof of the theorem.

□

Corollary 3.2. *If f is lacunary statistically p-ward continuous on a subset A of $\mathbb{R}$, then it is continuous on A in the ordinary case.*

Proof. The proof follows immediately from [19, Theorem 3] so is omitted. □

Theorem 3.3. *Lacunary statistical p-ward continuous image of any lacunary statistically p-ward compact subset of $\mathbb{R}$ is lacunary statistically p-ward compact.*

Proof. Let f be a lacunary statistically p-ward continuous function, and A be a lacunary statistically p-ward compact subset of $\mathbb{R}$. Take any sequence $\boldsymbol{\beta} = (\beta_n)$ of terms in $f(E)$. Write $\beta_n = f(\alpha_n)$ where $\alpha_n \in E$ for each $n \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_n)$. Lacunary statistically p-ward compactness of A implies that there is a lacunary statistically p-quasi-Cauchy subsequence $\boldsymbol{\xi} = (\xi_k) = (\alpha_{n_k})$ of $\boldsymbol{\alpha}$. Since f is lacunary statistically p-ward continuous, $(t_k) = f(\boldsymbol{\xi}) = (f(\xi_k))$ is lacunary statistically p-quasi-Cauchy. Thus (t_k) is a lacunary statistically p-quasi-Cauchy subsequence of the sequence $f(\boldsymbol{\alpha})$. This completes the proof of the theorem. □

Corollary 3.4. *Lacunary statistical p-ward continuous image of any G-sequentially connected subset of R is G-sequentially connected for a regular subsequential method G.*

Proof. The proof follows from the preceding theorem, so is omitted (see [21] and [50] for the definition of G-sequential connectedness and related concepts). □

Theorem 3.5. *If f is uniformly continuous on a subset A of $\mathbb{R}$, then $(f(\alpha_n))$ is lacunary statistically p-quasi-Cauchy whenever (α_n) is a p-quasi-Cauchy sequence of points in A.*

Proof. Let (α_n) be any p-quasi-Cauchy sequence of points in A. Take any $\varepsilon > 0$. Uniform continuity of f on A implies that there exists a $\delta > 0$, depending on ε, such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$ and $x, y \in A$. For this $\delta > 0$, there exists an $N = N(\delta)$ such that $|\Delta_p \alpha_n| < \delta$ whenever $n > N$. Hence $|\Delta_p f(\alpha_n)| < \varepsilon$ if $n > N$. Thus $\{k \in I_r : |\Delta_p f(\alpha_k)| \geq \varepsilon\} \subseteq \{1, 2, ..., N\}$. Therefore $\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |\Delta_p f(\alpha_k)| \geq \varepsilon\}| \leq \lim_{n\to\infty} \frac{1}{n} |\{k \leq N : k \in \mathbb{N}\}| = 0$. It follows from this that $(f(\alpha_n))$ is a lacunary statistically p-quasi-Cauchy sequence. This completes the proof of the theorem. □

Corollary 3.6. *If f is slowly oscillating continuous on a bounded subset A of $\mathbb{R}$, then $(f(\alpha_n))$ is lacunary statistically p-quasi-Cauchy whenever (α_n) is a p quasi-Cauchy sequence of points in A.*

Proof. If f is a slowly oscillating continuous function on a bounded subset A of $\mathbb{R}$, then it is uniformly continuous on A by [38, Theorem 2.3]. Hence the proof follows from Theorem 3.5. □

It is well-known that any continuous function on a compact subset A of $\mathbb{R}$ is uniformly continuous on A. We have an analogous theorem for a lacunary statistically p-ward continuous function defined on a lacunary statistically p-ward compact subset of $\mathbb{R}$.

Theorem 3.7. *If a function is lacunary statistically p-ward continuous on a lacunary statistically p-ward compact subset of $\mathbb{R}$, then it is uniformly continuous on A.*

Proof. Suppose that f is not uniformly continuous on A so that there exist an $\epsilon_0 > 0$ and sequences (α_n) and (β_n) of points in A such that $|\alpha_n - \beta_n| < 1/n$ and $|f(\alpha_n) - f(\beta_n)| \geq \epsilon_0$ for all $n \in \mathbb{N}$. Since A is lacunary statistically p-ward compact, there is a subsequence (α_{n_k}) of (α_n) that is lacunary statistically p-quasi-Cauchy. On the other hand, there is a subsequence $(\beta_{n_{k_j}})$ of (β_{n_k}) that is lacunary statistically p-quasi-Cauchy as well. It is clear that the corresponding sequence $(a_{n_{k_j}})$ is also lacunary statistically p-quasi-Cauchy, since
$\{j \in I_r : |\alpha_{n_{k_{j+p}}} - \alpha_{n_{k_j}}| \geq \varepsilon\} \subseteq \{j \in I_r : |\alpha_{n_{k_{j+p}}} - \beta_{n_{k_{j+p}}}| \geq \frac{\varepsilon}{3}\} \cup \{j \in I_r : |\beta_{n_{k_{j+p}}} - \beta_{n_{k_j}}| \geq \frac{\varepsilon}{3}\} \cup \{j \in I_r : |\beta_{n_{k_j}} - \alpha_{n_{k_j}}| \geq \frac{\varepsilon}{3}\}$
for every $n \in \mathbb{N}$, and for every $\varepsilon > 0$. Hence it is easy to establishe a contradiction. thus this completes the proof of the theorem. □

Corollary 3.8. *If a function defined on a bounded subset of $\mathbb{R}$ is lacunary statistically p-ward continuous, then it is uniformly continuous.*

We note that when the domain of a function is restricted to a bounded subset of $\mathbb{R}$, lacunary statistically p-ward continuity implies not only ward continuity, but also slowly oscillating continuity.

4. Conclusion

In this paper, we introduce lacunary statistically p-quasi Cauchy sequences, and investigate conditions for a lacunary statistically p ward continuous real function to be uniformly continuous, and prove some other results related to these kinds of continuities and some other kinds of continuities. It turns out that lacunary statistically p-ward continuity implies uniform continuity on a bounded subset of $\mathbb{R}$. The results in this paper not only involves the related results in [7] as a special case for $p = 1$, but also some interesting results which are also new for the special case $p = 1$. The lacunary statistically p-quasi Cauchy concept for $p > 1$ might find more interesting applications than statistical quasi Cauchy sequences to the cases when statistically quasi Cauchy does not apply. For a further study, we suggest to investigate lacunary statistically p-quasi-Cauchy sequences of soft points and lacunary statistically p-quasi-Cauchy sequences of fuzzy points. However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work (for example see [1], [28], [40], and [49]). We also suggest to investigate lacunary statistically p-quasi-Cauchy double sequences of points in $\mathbb{R}$ (see [55], [54], [39], and [34] for the related definitions in the double case). For another further study, we suggest to investigate lacunary statistically p-quasi-Cauchy sequences in abstract metric spaces (see [26], [53], [35], [58], and [61]).

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ABEL STATISTICAL DELTA QUASI CAUCHY SEQUENCES OF REAL NUMBERS

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Abstract. In this paper, we investigate the concept of Abel statistical delta quasi Cauchy sequences. A real function f is called Abel statistically delta ward continuous it preserves Abel statistical delta quasi Cauchy sequences, where a sequence (α_k) of points in $\mathbb{R}$ is called Abel statistically delta quasi Cauchy if $\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2\alpha_k|\geq\varepsilon} x^k = 0$ for every $\varepsilon > 0$, where $\Delta^2\alpha_k = \alpha_{k+2} - 2\alpha_{k+1} + \alpha_k$ for every $k \in \mathbb{N}$. Some other types of continuities are also studied and interesting results are obtained.

1. Introduction

Throughout this paper, $\mathbb{N}$, and $\mathbb{R}$ will denote the set of positive integers, and the set of real numbers, respectively. The boldface letters such as $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\zeta}$ will be used for sequences $\boldsymbol{\alpha} = (\alpha_n)$, $\boldsymbol{\beta} = (\beta_n)$, $\boldsymbol{\zeta} = (\zeta_n)$, ... of points in $\mathbb{R}$. A real function f is continuous if and only if it preserves Abel statistical convergence, i.e. for each point ℓ in the domain, $Abel_{st} - \lim_{n\to\infty} f(\alpha_n) = f(\ell)$ whenever $Abel_{st} - \lim_{n\to\infty} \alpha_n = \ell$.

Using the idea of continuity of a real function in this manner, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: ward continuity ([12], [5]), p-ward continuity ([19]), δ-ward continuity ([15]), δ^2-ward continuity ([4]), statistical ward continuity, ([16]), λ-statistical ward continuity ([29]), ρ-statistical ward continuity ([6], [21]), slowly oscillating continuity ([10, 44, 28]), quasi-slowly oscillating continuity ([31]), Δ-quasi-slowly oscillating continuity ([13]), upward and downward statistical continuities ([20]), lacunary statistical ward continuity ([7], [47], and [48]), lacunary statistical δ ward continuity ([25]), lacunary statistical δ^2 ward continuity ([46]), N_θ-ward continuity ([18], [24], [36], [8], [36], [37]), and N_θ-δ-ward continuity ([8]), which enabled some authors to obtain interesting results.

The purpose of this paper is to introduce and investigate the concept of Abel statistical δ-ward continuity of a real function, and prove interesting theorems.

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2. Abel statistical δ quasi Cauchy sequences

A sequence (α_k) is called statistically convergent to an element ℓ of $\mathbb{R}$ if $\lim_{n\to\infty}\frac{1}{n}|\{k \leq n : |\alpha_k - \ell| \geq \varepsilon\}| = 0$ for each $\varepsilon > 0$ (see [34], [14], [21], and [26]).

A sequence (α_k) of real numbers is called Abel convergent (or Abel summable) to ℓ if the series

$$\Sigma_{k=0}^{\infty}\alpha_k x^k$$

is convergent for $0 \leq x < 1$ and $\lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty}\alpha_k x^k = \ell$ ([1], [3], and [35]). In this case, we write $Abel-\lim \alpha_k = \ell$. The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to 0 and specially speaking, than that the distance between successive terms is Abel convergent to zero. Nevertheless, sequences which satisfy this weaker property, i.e. Abel quasi Cauchy sequences satisfying $Abel-lim\Delta\ \alpha_k = 0$, are interesting in their own right. In other words, a sequence (α_k) of points in $\mathbb{R}$ is called Abel quasi-Cauchy if $(\Delta\alpha_k)$ is Abel convergent to 0, i.e. the series

$$\sum_{k=0}^{\infty}\Delta\alpha_k x^k$$

is convergent for $0 \leq x < 1$ and

$$\lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty}\Delta\alpha_k x^k = 0$$

where $\Delta\alpha_k = \alpha_{k+1} - \alpha_k$.

Recently the concept of Abel statistical convergence of a sequence is investigated in [43] in the sense that a sequence (α_k) is called Abel statistically convergent to a real number L if $4\lim_{x\to 1^-}(1-x)\sum_{k:|\alpha_k - L|\geq\varepsilon} x^k = 04$ for every $\varepsilon > 0$, and denoted by $Abel_{st} - lim\ \alpha_k = L$.

A sequence (α_k) of points in $\mathbb{R}$ is called Abel statistically quasi Cauchy if

$$\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta\alpha_k|\geq\varepsilon} x^k = 0$$

for every $\varepsilon > 0$ ([30]).

Now we introduce the concept of Abel statistically δ quasi Cauchyness in the following:

Definition 2.1. *A sequence of points in a subset A of $\mathbb{R}$ is called Abel statistically δ quasi Cauchy if*

$$\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2\alpha_k|\geq\varepsilon} x^k = 0$$

for every $\varepsilon > 0$, where $\Delta^2\alpha_k = \alpha_{k+2} - 2\alpha_{k+1} + \alpha_k$ for every $k \in \mathbb{N}$.

Any Abel statistically quasi-Cauchy sequence is Abel statistically δ quasi Cauchy, but the converse is not always true. Any quasi-Cauchy sequence is Abel statistically δ quasi Cauchy, but the converse is not always true. Any Abel statistically convergent sequence is Abel statistically δ quasi Cauchy. There are Abel statistically δ quasi Cauchy sequences which are not Abel statistically quasi Cauchy. Since the set of all convergent sequences c is a proper subset of $Abel_{st}^{\delta}$, and $Abel_{st}$ is a proper subset of $Abel_{st}^{\delta^2}$, the set of Abel statistical δ quasi Cauchy sequences, one can easily find that $c \subset \Delta \subset Abel_{st}^{\delta} \subset Abel_{st}^{\delta^2}$, where c, Δ, $\Delta Abel_{st}$, and $\Delta^2 Abel_{st}$, denote the set of convergent sequences, the set of quasi Cauchy sequences, the set of Abel statistically quasi Cauchy sequences, and the set of Abel statistically δ quasi Cauchy sequences.

Theorem 2.1. *The sum of two Abel statistical δ quasi-Cauchy sequences is Abel statistical δ quasi-Cauchy.*

Proof. Let (α_k) and (β_k) be Abel statistical δ quasi-Cauchy sequences of of points in A. Then $\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2\alpha_k|\geq\varepsilon} x^k = 0$ and $\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2\beta_k|\geq\varepsilon} x^k = 0$ for every $\varepsilon > 0$. Then $\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2(\alpha_k+\beta_k)|\geq\varepsilon} x^k \leq \lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2\alpha_k|\geq\varepsilon} x^k + \lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2\beta_k|\geq\varepsilon} x^k$. This completes the proof of the theorem. □

Now we give the definition of Abel statistical δ ward compactness.

Definition 2.2. *A subset A of $\mathbb{R}$ is called Abel statistically δ ward compact if any sequence of points in A has an Abel statistical δ quasi-Cauchy subsequence.*

First, we note that any finite subset of $\mathbb{R}$ is Abel statistically δ ward compact, the union of two Abel statistically δ ward compact subsets of $\mathbb{R}$ is Abel statistically δ ward compact and the intersection of any family of Abel statistically δ ward compact subsets of $\mathbb{R}$ is Abel statistically δ ward compact. Any G-sequentially compact subset of $\mathbb{R}$ is Abel statistically δ ward compact for a regular subsequential method G (see [11], [17]). Furthermore any subset of an Abel statistically δ ward compact set is Abel statistically δ ward compact, any bounded subset of $\mathbb{R}$ is Abel statistically δ ward compact, any slowly oscillating compact subset of $\mathbb{R}$ is Abel statistically δ ward compact (see [10] for the definition of slowly oscillating compactness).

Theorem 2.2. *If a function f is uniformly continuous on a subset A of $\mathbb{R}$, then $(f(\alpha_k))$ is Abel statistical δ quasi-Cauchy whenever (α_k) is a quasi-Cauchy sequence of points in A.*

Proof. Take any quasi-Cauchy sequence (α_k) of points in A, and let ε be any positive real number. By uniform continuity of f, there exists a $\delta > 0$ such that $|f(\alpha) - f(\beta)| < \varepsilon$ whenever $|\alpha - \beta| < \delta$ and $\alpha, \beta \in E$. Since (α_k) is a quasi-Cauchy sequence, there exists a positive integer k_0 such that $|\alpha_{k+1} - \alpha_k| < \delta$ for $k \geq k_0$. Thus

$$\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2\alpha_k|\geq\varepsilon} x^k = 0.$$

This completes the proof of the theorem. □

Definition 2.3. *A function defined on a subset A of $\mathbb{R}$ is called Abel statistically δ ward continuous if it preserves Abel statistical δ quasi-Cauchy sequences, i.e. $(f(\alpha_n))$ is an Abel statistical δ quasi-Cauchy sequence whenever (α_n) is.*

We note that Abel statistical δ ward continuity cannot be obtained by any sequential method G ([9], [17]). The composition of two Abel statistical δ ward continuous functions is Abel statistical δ ward continuous.

Theorem 2.3. *If f is Abel statistically δ ward continuous on a subset A of $\mathbb{R}$, then it is Abel statistically ward continuous on A.*

Proof. Let (α_n) be any sequence with $Abel_{st}-\lim_{k\to\infty}\Delta\alpha_k=0$. Then the sequence

$$(\alpha_1,\alpha_1,\alpha_2,\alpha_2,...,\alpha_n,\alpha_n,...)$$

is Abel statistical δ quasi-Cauchy hence, by the hypothesis, the sequence

$$(f(\alpha_1),f(\alpha_1),f(\alpha_2),f(\alpha_2),...,f(\alpha_n),f(\alpha_n),...)$$

is Abel statistical δ quasi-Cauchy . It follows from this that

$$(f(\alpha_1),f(\alpha_2),...,f(\alpha_n),...)$$

is Abel statistical quasi-Cauchy. This completes the proof of the theorem. □

Corollary 2.4. *Any Abel statistically δ ward continuous on a subset A of $\mathbb{R}$ is ordinary continuous on A.*

Theorem 2.5. *The sum of two Abel statistical δ ward continuous functions is Abel statistical δ ward continuous.*

Proof. The proof of this theorem follows easily, so is omitted. □

If c is a constant real number and f is an Abel statistically δ ward continuous function, then cf is Abel statistically δ ward continuous. Thus the set of Abel statistical δ ward continuous functions is a vector subspace of the vector space of continuous functions. Maximum of two Abel statistical δ ward continuous functions is Abel statistical δ ward continuous, and minimum of two Abel statistical δ ward continuous functions is Abel statistical δ ward continuous, which follow from $max\{f,g\}=\frac{1}{2}(f+g+|f-g|)$ and $min\{f,g\}=\frac{1}{2}(f+g-|f-g|)$, respectively.

Theorem 2.6. *Abel statistically δ ward continuous image of any Abel statistically δ ward compact subset of $\mathbb{R}$ is Abel statistically δ ward compact.*

Proof. Assume that f is a Abel statistically δ ward continuous function on a subset A of $\mathbb{R}$, and B is an Abel statistically δ ward compact subset of A. Let (β_n) be any sequence of points in $f(B)$. Write $\beta_n=f(\alpha_n)$ where $\alpha_n\in A$ for each positive integer n. Abel statistically δ ward compactness of B implies that there is a subsequence $(\gamma_k)=(\alpha_{n_k})$ of (α_n) with $Abel_st-\lim_{k\to\infty}\Delta^2\gamma_k=0$. Write $(t_k)=(f(\gamma_k))$. As f is Abel statistically δ ward continuous, $(f(\gamma_k))$ is Abel statistically δ quasi-Cauchy. Thus $f(B)$ is Abel statistically δ ward compact. This completes the proof of the theorem. □

Corollary 2.7. *Abel statistically δ ward continuous image of any compact subset of $\mathbb{R}$ is Abel statistically δ ward compact.*

Corollary 2.8. *Abel statistically δ ward continuous image of a G-sequentially compact subset of $\mathbb{R}$ is Abel statistically δ ward compact for any subsequential regular method G.*

3. Conclusion

In this paper, we obtain results related to Abel statistically δ ward continuity, Abel statistically δ ward compactness, ward continuity, continuity, and uniform continuity. We suggest to investigate Abel statistically δ quasi-Cauchy sequences of fuzzy points or soft points (see [23], [38] for the definitions and related concepts in fuzzy setting, and see [2], and [33] for the soft setting). We also suggest to investigate Abel statistically δ quasi-Cauchy double sequences (see for example [27],

[32], and [40] for the definitions and related concepts in the double sequences case). For another further study, we suggest to investigate Abel statistically δ Cauchy sequences of points in an abstract metric space ([39], [45], [44], [22], [41], and [28]).

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MORE ON TRANSLATIONALLY SLOWLY VARYING SEQUENCES

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Abstract. We define and study an equivalence relation in the class $\mathsf{Tr}(\mathsf{SV_s})$ of translationally slowly varying positive real sequences and its relations with selection principles and game theory. We also prove a game-theoretic result for translationally rapidly varying sequences.

1. Introduction

Throughout the paper $\mathbb{N}$ will denote the set of natural numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{S}$ the set of sequences of positive real numbers.

The theory of regular variation, including in particular slow variation, was initiated in 1930 by J. Karamata [8]. Nowadays this branch of asymptotic analysis of divergent processes is known as *Karamata's theory of regular variation.* Another kind of variation, called *rapid variation*, was introduced and first studied in 1970 by de Haan [7]. These two theories are developed for functions and sequences and have various applications in several mathematical disciplines: number theory, differential and difference equations, probability theory, q-calculus, and so on. For more information about the theory of regular variation and the theory of rapid variation we refer the reader to the book [1]. In this article we are interested in two classes of sequences related to slow and rapid variations.

We recall first the definitions of slowly and rapidly varying sequences.

Definition 1.1. ([1, 2, 12]) A sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$ is *slowly varying* (respectively, *rapidly varying*) if for each $\lambda > 0$ (respectively, $\lambda > 1$) the following is satisfied:

$$\lim_{n\to\infty} \frac{c_{[\lambda n]}}{c_n} = 1, \tag{1.1}$$

(respectively,

$$\lim_{n\to\infty} \frac{c_{[\lambda n]}}{c_n} = \infty), \tag{1.2}$$

where for $x \in \mathbb{R}$, $[x]$ denotes the greatest integer part of x.

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The classes of slowly varying and rapidly varying sequences are denoted by $\mathsf{SV_s}$ and $\mathsf{R_{s,\infty}}$, respectively.

In what follows we work with the following two classes of sequences.

Definition 1.2. ([3, 11]) A sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$ is *translationally slowly varying* (respectively, *translationally rapidly varying*) if for each $\lambda \geq 1$ the following asymptotic condition is satisfied:

$$\lim_{n\to\infty} \frac{c_{[n+\lambda]}}{c_n} = 1 \tag{1.3}$$

(respectively,

$$\lim_{n\to\infty} \frac{c_{[n+\lambda]}}{c_n} = \infty). \tag{1.4}$$

$\mathsf{Tr(SV_s)}$ denotes the class of translationally slowly varying sequences, and $\mathsf{Tr(R_{s,\infty})}$ denotes the class of translationally rapidly varying sequences (see [2, 3, 4, 5]).

Observe that $R_{s,\infty} \cap \mathsf{Tr(SV_s)} \neq \emptyset$, $R_{s,\infty} \setminus \mathsf{Tr(SV_s)} \neq \emptyset$, $\mathsf{Tr(SV_s)} \setminus \mathsf{R_{s,\infty}} \neq \emptyset$, and $\mathsf{Tr(R_{s,\infty})} \subset \mathsf{R_{s,\infty}}$.

In this paper we define and study a new equivalence relation in the class $\mathsf{Tr(SV_s)}$, in particular its relations with selection principles and game theory. We also provide a game-theoretic result concerning the class $\mathsf{Tr(R_{s,\infty})}$.

2. Results

We begin this section with definitions of concepts we use in this article.

Definition 2.1. Sequences $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ and $\mathbf{d} = (d_n)_{n\in\mathbb{N}}$ from $\mathbb{S}$ are *mutually translationally slowly equivalent*, denoted by

$$c_n \overset{ts}{\sim} d_n, \text{ as } n \to \infty,$$

if

$$\lim_{n\to\infty} \frac{c_{[n+\lambda]}}{d_n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{d_{[n+\lambda]}}{c_n} = 1 \tag{2.1}$$

hold for each $\lambda \geq 1$.

Definition 2.2. Sequences $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ and $\mathbf{d} = (d_n)_{n\in\mathbb{N}}$ from $\mathbb{S}$ are *mutually translationally rapidly equivalent*, denoted by

$$c_n \overset{tr}{\sim} d_n, \text{ as } n \to \infty,$$

if

$$\lim_{n\to\infty} \frac{c_{[n+\lambda]}}{d_n} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{d_{[n+\lambda]}}{c_n} = \infty \tag{2.2}$$

hold for each $\lambda \geq 1$.

Theorem 2.1. *Let sequences $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ and $\mathbf{d} = (d_n)_{n\in\mathbb{N}}$ be elements from $\mathbb{S}$. If $c_n \overset{ts}{\sim} d_n$, as $n \to \infty$, then $\mathbf{c} \in \mathsf{Tr(SV_s)}$ and $\mathbf{d} \in \mathsf{Tr(SV_s)}$.*

Proof. For $\lambda \geq 1$ we have

$$\lim_{n\to\infty} \frac{c_{[n+\lambda]}}{c_n} = \lim_{n\to\infty} \left(\frac{c_{n+1}}{c_n}\right)^{[\lambda]}$$

if the limit on the right side exists. Further, since $c_n \overset{ts}{\sim} d_n$, we have

$$\lim_{n\to\infty} \frac{c_{n+2}}{c_n} = \lim_{n\to\infty} \left(\frac{c_{n+2}}{d_{n+1}} \cdot \frac{d_{n+1}}{c_n}\right) = \lim_{n\to\infty} \frac{c_{n+2}}{d_{n+1}} \cdot \lim_{n\to\infty} \frac{d_{n+1}}{c_n} = 1.$$

Therefore

$$1 = \lim_{n\to\infty}\left(\frac{c_{n+2}}{c_{n+1}}\cdot\frac{c_{n+1}}{c_n}\right) = \lim_{k\to\infty}\left(\frac{c_{k+1}}{c_k}\right)^2,$$

hence

$$\lim_{n\to\infty}\frac{c_{n+1}}{c_n} = 1.$$

This means that

$$\lim_{n\to\infty}\frac{c_{[n+\lambda]}}{c_n} = 1 \quad \text{for each } \lambda \geq 1 \ ,$$

i.e. $\mathbf{c} \in \mathsf{Tr}(\mathsf{SV_s})$.

Similarly we prove $\mathbf{d} \in \mathsf{Tr}(\mathsf{SV_s})$. □

In a similar way, by suitable modifications in the proof, we prove the following result.

Theorem 2.2. *Let* $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ *and* $\mathbf{d} = (d_n)_{n\in\mathbb{N}}$ *be sequences in* $\mathbb{S}$. *If* $c_n \overset{tr}{\sim} d_n$, *as* $n \to \infty$, *then* $\mathbf{c} \in \mathsf{Tr}(\mathsf{R_{s,\infty}})$ *and* $\mathbf{d} \in \mathsf{Tr}(\mathsf{R_{s,\infty}})$.

Theorem 2.3. *Relation* $\overset{ts}{\sim}$ *is an equivalence relation on* $\mathsf{Tr}(\mathsf{SV_s})$.

Proof. 1. (Reflexivity) Let $\mathbf{c} \in \mathsf{Tr}(\mathsf{SV_s})$. Then $\lim_{n\to\infty}\frac{c_{[n+\lambda]}}{c_n} = 1$ for each $\lambda \geq 1$, that is $c_n \overset{ts}{\sim} c_n$ as $n \to \infty$, and so reflexivity holds.

2.(Symmetry) It follows from the definition of relation $\overset{ts}{\sim}$.

3. (Transitivity) Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$, $\mathbf{d} = (d_n)_{n\in\mathbb{N}}$ and $\mathbf{e} = (e_n)_{n\in\mathbb{N}}$ be elements from $\mathsf{Tr}(\mathsf{SV_s})$ such that $c_n \overset{ts}{\sim} d_n$, $n \to \infty$, and $d_n \overset{ts}{\sim} e_n$, $n \to \infty$. Then we have

$$\lim_{n\to\infty}\frac{c_{n+2}}{e_n} \quad = \quad \lim_{n\to\infty}\frac{c_{n+2}}{d_{n+1}}\cdot\lim_{n\to\infty}\frac{d_{n+1}}{e_n} = 1.$$

We conclude

$$1 = \lim_{n\to\infty}\left(\frac{c_{n+2}}{e_{n+1}}\cdot\frac{e_{n+1}}{e_n}\right).$$

Because of $\mathbf{e} \in \mathsf{Tr}(\mathsf{SV_s})$, we obtain

$$\lim_{n\to\infty}\frac{c_{n+1}}{e_n} = 1.$$

It follows from here that for each $\lambda \geq 1$ it holds

$$\lim_{n\to\infty}\frac{c_{[n+\lambda]}}{e_n} = 1.$$

In a similar way one proves

$$\lim_{n\to\infty}\frac{e_{[n+\lambda]}}{c_n} = 1,\ \lambda \geq 1,$$

which means $c_n \overset{ts}{\sim} e_n$. □

Remark. Let a sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ belong to the class $\mathsf{Tr}(\mathsf{SV_s})$ and let $\mathbf{d} = (d_n)_{n\in\mathbb{N}} \in \mathbb{S}$ be such that $c_n \overset{ts}{\sim} d_n$. Then

$$\lim_{n\to\infty}\frac{c_n}{d_n} = \lim_{n\to\infty}\left(\frac{c_n}{c_{n+1}}\cdot\frac{c_{n+1}}{d_n}\right) = 1$$

and we conclude that sequences $\mathbf{c}$ and $\mathbf{d}$ are strongly asymptotically equivalent (see, for instance. [1, 6]), i.e. $\lim_{n\to\infty}\frac{c_n}{d_n} = 1$.

Recall the definition of selection principles, which we need in what follows (see [9, 10]).

Definition 2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be subfamilies of the set $\mathbb{S}$. The symbol $\alpha_i(\mathcal{A},\mathcal{B})$, $i \in \{2,3,4\}$, denotes the following selection hypotheses: for each sequence $(A_n)_{n\in\mathbb{N}}$ of elements from $\mathcal{A}$ there is an element $B \in \mathcal{B}$ such that:

(1) $\alpha_2(\mathcal{A},\mathcal{B})$: the set $\mathrm{Im}(A_n) \cap \mathrm{Im}(B)$ is infinite for each $n \in \mathbb{N}$;
(2) $\alpha_3(\mathcal{A},\mathcal{B})$: the set $\mathrm{Im}(A_n) \cap \mathrm{Im}(B)$ is infinite for infinitely many $n \in \mathbb{N}$;
(3) $\alpha_4(\mathcal{A},\mathcal{B})$: the set $\mathrm{Im}(A_n) \cap \mathrm{Im}(B)$ is nonempty for infinitely many $n \in \mathbb{N}$,

where Im denotes the image of the corresponding sequence.

The following infinitely long game is related to α_2 (see [9, 10]).

Definition 2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subfamilies of $\mathbb{S}$. The symbol $\mathsf{G}_{\alpha_2}(\mathcal{A},\mathcal{B})$ denotes the following infinitely long game for two players, I and II, who play a round for each natural number n. In the first round I chooses an arbitrary element $\mathbf{A_1} = (A_{1,j})_{j\in\mathbb{N}}$ from $\mathcal{A}$, and II chooses a subsequence $y_{r_1} = (A_{1,r_1(j)})_{j\in\mathbb{N}}$ of the sequence A_1. At the k^{th} round, $k \geq 2$, I chooses an arbitrary element $A_k = (A_{k,j})_{j\in\mathbb{N}}$ from $\mathcal{A}$ and II chooses a subsequence $y_{r_k} = (A_{k,r_k(j)})_{j\in\mathbb{N}}$ of the sequence A_k, such that $\mathrm{Im}(r_{k(j)}) \cap \mathrm{Im}(r_{p(j)}) = \emptyset$ is satisfied, for each $p \leq k-1$. II wins a play

$$A_1, y_{r_1}; \ldots; A_k, y_{r_k}; \ldots$$

if and only if all elements from $Y = \bigcup_{k\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} A_k, r_{k(j)}$, with respect to second index, form a subsequence $\mathbf{y} = (y_m)_{m\in\mathbb{N}} \in \mathcal{B}$.

A strategy σ for the player II is a *coding strategy* if II remembers only the most recent move by I and by II before deciding how to play the next move.

Observe, that if II has a winning strategy in the game $\mathsf{G}_{\alpha_2}(\mathcal{A},\mathcal{B})$, then the selection principle $\alpha_2(\mathcal{A},\mathcal{B})$ is true. Also, $\alpha_2(\mathcal{A},\mathcal{B}) \Rightarrow \alpha_3(\mathcal{A},\mathcal{B}) \Rightarrow \alpha_4(\mathcal{A},\mathcal{B})$.

Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$. Then we define

$$[\mathbf{c}]_{ts} = \{\mathbf{d} = (d_n)_{n\in\mathbb{N}} \in \mathbb{S} : c_n \overset{ts}{\sim} d_n, n \to \infty\} \tag{2.3}$$

as the equivalence class of $\mathbf{c}$ in $\mathsf{Tr}(\mathsf{SV_s})$.

Theorem 2.4. *For a fixed element* $\mathbf{c} \in \mathsf{Tr}(\mathsf{SV_s})$*, the player* II *has a winning coding strategy in the game* $\mathsf{G}_{\alpha_2}([\mathbf{c}]_{ts}, [\mathbf{c}]_{ts})$*,*

Proof. (1^{st} round): Let σ be the strategy of the player II. The player I chooses a sequence $\mathbf{x_1} = (x_{1,n})_{n\in\mathbb{N}} \in [\mathbf{c}]_{ts}$ arbitrary. Then the player II chooses the subsequence $\sigma(\mathbf{x_1}) = (x_{1,k_1(n)})_{n\in\mathbb{N}}$ of the sequence $\mathbf{x_1}$, where $\mathrm{Im}(k_1)$ is the set of natural numbers greater of or equal to $n_1 \in \mathbb{N}$ which are divisible by 2 and not divisible by 2^2, and $1 - \frac{1}{2} \leq \frac{c_n}{x_{m,n}} \leq 1 + \frac{1}{2}$ holds for each $n \geq n_1$.

(m^{th} round, $m \geqslant 2$): The player I chooses a sequence $\mathbf{x_m} = (x_{m,n})_{n\in\mathbb{N}} \in [\mathbf{c}]_{ts}$. Then the player II chooses the subsequence

$$\sigma(\mathbf{x_m}, (x_{m-1,k_{m-1}(n)})_{n\in\mathbb{N}}) = (x_{m,k_m(n)})_{n\in\mathbb{N}}$$

of the sequence $\mathbf{x_m}$, so that $\mathrm{Im}(k_m)$ is the set of natural numbers greater of or equal to $n_m \in \mathbb{N}$, which are divisible by 2^m, and not divisible by 2^{m+1}, and $1 - \frac{1}{2^m} \leq \frac{c_n}{x_{m,n}} \leq 1 + \frac{1}{2^m}$ holds for each $n \geq n_m$.

Consider now the set $Y = \bigcup_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} x_{m,k_m(n)}$ in $\mathbb{S}$ indexed by the second index $k_m(n)$. This set we can consider as the subsequence of the sequence $\mathbf{y} = (y_i)_{i\in\mathbb{N}}$ given by:

$$y_i = \begin{cases} x_{m,k_m(n)}, & \text{if } i = k_m(n) \text{ for some } m,n \in \mathbb{N}; \\ c_i, & \text{otherwise.} \end{cases}$$

By the construction $\mathbf{y} \in \mathbb{S}$. Also, the intersection of $\mathbf{y}$ and $\mathbf{x_m}$, $m \in \mathbb{N}$, is an infinite set.

Let us prove that $y_m \overset{ts}{\sim} c_m$, as $m \to \infty$. Let $\varepsilon > 0$. Let m be the smallest natural number such that $\frac{1}{2^m} \leq \varepsilon$. For each $k \in \{1, 2, ..., m-1\}$ there is $n_k^* \in \mathbb{N}$, so that $1 - \varepsilon \leq \frac{c_i}{x_{k,n}} \leq 1 + \varepsilon$ for each $n \geq n_k^*$. Set $n^* = \max\{n_1^*, n_2^*, \ldots, n_{m-1}^*\}$. For each $i \geq n^*$ we have $1 - \varepsilon \leq \frac{c_i}{y_i} \leq 1 + \varepsilon$. Therefore, $\lim_{n\to\infty} \frac{c_i}{y_i} = 1$. It follows

$$\lim_{i\to\infty} \frac{c_{i+1}}{y_i} = \lim_{i\to\infty} \left(\frac{c_{i+1}}{c_i} \cdot \frac{c_i}{y_i} \right) = 1$$

because $\mathbf{c} \in \mathsf{Tr}(\mathsf{SV_s})$. In a similar way we prove

$$\lim_{i\to\infty} \frac{y_{i+1}}{c_i} = 1.$$

One concludes that for each $\lambda \geq 1$

$$\lim_{i\to\infty} \frac{y_{[i+\lambda]}}{c_i} = \lim_{i\to\infty} \frac{c_{[i+\lambda]}}{y_i} = 1$$

i.e. $\mathbf{y} = (y_i)_{i\in\mathbb{N}} \in [\mathbf{c}]_{ts}$. The theorem is proved. □

Corollary 2.5. *The selection principle* $\alpha_2([\mathbf{c}]_{ts}, [\mathbf{c}]_{ts})$ *holds for each fixed element* $\mathbf{c} \in \mathsf{Tr}(\mathsf{SV_s})$. *Consequently,* $\alpha_3([\mathbf{c}]_{ts}, [\mathbf{c}]_{ts})$ *and* $\alpha_4([\mathbf{c}]_{ts}, [\mathbf{c}]_{ts})$ *also hold.*

We end the paper by proving a result about mutually translationally rapidly equivalent sequences.

Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$. Then we define

$$[\mathbf{c}]_{tr} = \{\mathbf{d} = (d_n)_{n\in\mathbb{N}} \in \mathbb{S} : c_n \overset{tr}{\sim} d_n, n \to \infty\}. \tag{2.4}$$

Theorem 2.6. *The player* II *has a winning coding strategy in the game* $\mathsf{G}_{\alpha_2}([\mathbf{c}]_{tr}, [\mathbf{c}]_{tr})$, *for any fixed element* $\mathbf{c} \in \mathsf{Tr}(\mathsf{R_{s,\infty}})$.

Proof. Let σ be the strategy of II.

(m^{th} round, $m \geq 1$): The player I chooses a sequence $\mathbf{x_m} = (x_{m,n})_{n\in\mathbb{N}} \in [\mathbf{c}]_{tr}$. Then the player II chooses the subsequence

$$\sigma(\mathbf{x_m}, (x_{m-1,k_{m-1}(n)})_{n\in\mathbb{N}}) = (x_{m,k_m(n)})_{n\in\mathbb{N}}$$

of the sequence $\mathbf{x_m}$, so that $\mathrm{Im}(k_m)$ is the set of natural numbers greater of or equal to n_m, which are divisible with 2^m, and not divisible with 2^{m+1}, $n_m \in \mathbb{N}$, and $\frac{c_{n+1}}{x_{m,n}} \geq 2^m$ and $\frac{x_{m,n+1}}{c_n} \geq 2^m$ for each $n \geq n_m$. Let $\lambda \geq 1$. Since $\mathbf{c} \in \mathsf{Tr}(\mathsf{R_{s,\infty}})$, we have $\frac{c_{n+1}}{c_n} \geq 1$ for sufficiently large n. Then

$$\frac{c_{[n+\lambda]}}{x_{m,n}} = \frac{c_{[n+\lambda]}}{c_{[n+\lambda]-1}} \cdot \frac{c_{[n+\lambda]-1}}{c_{[n+\lambda]-2}} \cdots \frac{c_{n+1}}{x_{m,n}} \geq 2^m$$

for each $n \geq n_m$. Since $x_{m,n} \overset{tr}{\sim} c_n$, as $n \to \infty$, we have $\mathbf{x_m} \in \mathsf{Tr}(\mathsf{R_{s,\infty}})$ (Theorem 2.2). In a similar way we prove $\frac{x_{m,[n+\lambda]}}{c_n} \geq 2^m$ for all $n \geq n_m$.

Form the set $Y = \bigcup_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} x_{m,k_m(n)}$ of positive real numbers indexed by the second index. This set is a subsequence of the sequence $\mathbf{y} = (y_i)_{i\in\mathbb{N}}$ defined by:

$$y_i = \begin{cases} x_{m,k_m(n)}, & \text{if } i = k_m(n) \text{ for some } m, n \in \mathbb{N}; \\ c_i, & \text{otherwise.} \end{cases}$$

Evidently, $\mathbf{y} \in \mathbb{S}$ and the intersection of $\mathbf{y}$ and $\mathbf{x_m}$, $m \in \mathbb{N}$, is an infinite set.

We prove $y_m \overset{tr}{\sim} c_m$, as $m \to \infty$. Let $M > 0$. Choose the smallest $m \in \mathbb{N}$ such that $2^m > M$. For each $k \in \{1, 2, ..., m-1\}$ there is $n_k^* \in \mathbb{N}$, so that $\frac{c_{[n+\lambda]}}{x_{k,n}} \geq M$ and $\frac{x_{k,[n+\lambda]}}{c_n} \geq M$ for each $\lambda \geq 1$ and each $n \geq n_k^*$. Let $n^* = \max\{n_1^*, \ldots, n_{m-1}^*\}$. Therefore, the inequalities $\frac{c_{[i+\lambda]}}{y_i} \geq M$ and $\frac{y_{[i+\lambda]}}{c_i} \geq M$ hold for each $\lambda \geq 1$ and each $i \geq n^*$. As M was arbitrary, one concludes $y_i \overset{tr}{\sim} c_i$, as $i \to \infty$. In other words, $\mathbf{y} \in [\mathbf{c}]_{tr}$. □

Corollary 2.7. *The selection principle $\alpha_2([\mathbf{c}]_{tr}, [\mathbf{c}]_{tr})$ holds for each fixed element $\mathbf{c} \in \mathsf{Tr}(\mathsf{R}_{\mathsf{s},\infty})$, and thus $\alpha_3([\mathbf{c}]_{tr}, [\mathbf{c}]_{tr})$ and $\alpha_4([\mathbf{c}]_{tr}, [\mathbf{c}]_{tr})$ hold.*

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ON THE EXISTENCE AND MULTIPLICITY OF POSITIVE RADIAL SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATION ON BOUNDED ANNULAR DOMAINS VIA FIXED POINT INDEX

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Abstract. In this paper, we study the existence and multiplicity of positive radial solutions for a class of local elliptic boundary value problem defined on bounded annular domains. The existence and multiplicity of positive radial solutions are obtained by means of fixed point index theory. We include an example to illustrate our results.

1. Introduction

In this paper, we are interested in the existence of radial positive solutions to the following boundary value problem (BVP)

$$\begin{cases} -\triangle u(x) = f(|x|, u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega = \left\{x \in \mathbb{R}^N : R_0 < |x| < R_1,\ N \geq 3\right\}$ with $0 < R_0 < R_1$ is an annulus in $\mathbb{R}^N$ and $f \in C([0,1] \times [0,\infty), [0,\infty))$.

The study of such problems is motivated by a lot of physical applications starting from the well-known Poisson-Boltzmann equation (see [2, 26, 34]), also they serve as models for some phenomena which arise in fluid mechanics, such as the exothermic chemical reactions or autocatalytic reactions (see [31], Section 5.11.1). The nonlinearity f in applications always has a special form and here we assume only the continuity of f and some inequalities at some points for the values of this function. However, we know that in the integrand should stay a superposition of u with a given function (usually the exponent of u in applications) instead of u alone, but we treat this paper as the first step in this direction. The method we use is typical for local BVP. We shall formulate an equivalent fixed point problem and look for its solution in the cone of nonnegative function in an appropriate Banach space. The most popular fixed point theorem in a cone is the cone-compression

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and cone-expansion theorem due to M. Krasnosel'skii [25] which we use in the form taken from [12], [19]. We also point out the fact that problems of type (1.1) when equation does not contain parameter λ, are connected with the classical boundary value theory of Bernstein [1] (see also the studies of Granas, Gunther and Lee [17] for some extensions to nonlinear problems).

The existence and uniqueness of positive radial solutions for equations of type (1.1) when equation does not contain parameter λ, were obtained in [5], [27], [36]. Wang [36] proved that if $f:(0,\infty)\to(0,\infty)$ satisfies $\lim\limits_{z\to 0}\frac{f(z)}{z}=\infty$ and $\lim\limits_{z\to\infty}\frac{f(z)}{z}=0$ then problem (1.1) when equation does not contain parameter λ, has a positive radial solution in $\Omega=\left\{x\in\mathbb{R}^N,\ N>2\right\}$. That result was extended for the systems of elliptic equations by Ma [24]. We quote also the research of Ovono el al. [32] where the diffusion at each point depends on all the values of the solutions in a neighborhood of this point and Chipot et al. [13] considred the solvability of a class of nonlocal problems which admit a formulation in term of quasi-variational inequalities. There is a wide literature that deals with existence multiplicity results for various second-order, fourth-order and higher-order boundary value problems by different approaches, see [8, 9, 10, 11, 12, 14, 29, 30].

In 2011, Bohneure et al. [6] studied the existence of positive increasing radial solutions for superlinear Neumann problem in the unit ball B in $\mathbb{R}^N,\ N\geq 2$,

$$\begin{cases} -\Delta u+u=a\left(|x|\right)f\left(u\right), & in\ \ B,\\ u>0, & in\,B,\\ \partial_t u=0, & on\ \partial B,\end{cases}$$

where $a\in C^1\left([0,1],\mathbb{R}\right),\ a(0)>0$ is nondecreasing, $f\in C^1\left([0,1],\mathbb{R}\right),\ f(0)=0,\ \lim\limits_{s\to 0^+}\frac{f(s)}{s}=0$ and $\lim\limits_{s\to+\infty}\frac{f(s)}{s}>\frac{1}{a(0)}$.

In 2011, Hakimi and Zertiti [22] studied the nonexistence of radial positive solutions for a nonpositone problem when the nonliearity is superlinear and has more than one zero,

$$\begin{cases} -\triangle u\left(x\right)=\lambda f\left(u\left(x\right)\right), & x\in\Omega,\\ u\left(x\right)=0, & x\in\partial\Omega,\end{cases}$$

where $f\in C\left([0,+\infty),\mathbb{R}\right)$.

In 2014, Sfecci [35] obtained the existence result by introduced the $lim\,sup$ type of nonresonance condition with respect to the first positive eigenvalue λ_1 provided $\lim\limits_{|u|\to\infty}sup\frac{2F(u)}{u^2}<\lambda_1$ with a double $lim\,inf$ condition like the following one $\lim\limits_{u\to-\infty}sup\frac{2F(u)}{u^2}<\frac{\pi^2}{4\rho^2}$ *and* $\lim\limits_{u\to+\infty}inf\frac{2F(u)}{u^2}<\frac{\pi^2}{4\rho^2}$ for the following Neumann problems defined on the ball $B_R=\left\{x\in\mathbb{R}^N,\ |x|<R\right\}$,

$$\begin{cases} -\triangle u\left(x\right)=f\left(u\left(x\right)\right)+e\left(|x|\right), & in\ B_R,\\ u\left(x\right)=0, & on\ \partial B_R,\end{cases}$$

where $f\in C\left(\mathbb{R},\mathbb{R}\right)$, $e\in C\left([0,R],\mathbb{R}\right)$, F is a primitive of f and $\Omega=(-2\rho,2\rho)\subset\mathbb{R}$.

In 2014, Butler et. al, [7] studied the positive radial solutions to the BVP

$$\begin{cases} -\Delta u+u=\lambda a\left(|x|\right)f\left(u\right), & x\in\Omega,\\ \frac{\partial u}{\partial\eta}+\overline{c}\left(u\right)u=0, & |x|=r_0,\\ u\left(x\right)\to 0, & |x|\to\infty,\end{cases}$$

where $f \in C([0,\infty),\mathbb{R})$, $\Omega = \left\{x \in \mathbb{R}^N : N > 2, |x| > r_0 \, with \, r_0 > 0\right\}$, λ is a positive parameter, $a \in C([r_0,\infty),\mathbb{R}^+)$ such that $\underset{r\to\infty}{lim}\, a(r) = 0$, $\frac{\partial}{\partial u}$ is the outward normal derivative and $\overline{c} \in C([0,\infty),(0,\infty))$.

In 2003, Stanzy [34], by using the norm-type cone expansion and compression theorem proved that problem (1.1) has at least one positive radial solution under the following conditions

(B_1) for any $M > 0$ there exist a function $p_M \in C((1,+\infty),\mathbb{R}^+)$ with

$$\int_1^\infty s\left(1 - s^{2n}\right) p_M(s)\, ds < \infty,$$

such that

$$0 \le f(s,u) \le p_M(s), \, for\ any\ (s,u) \in (1,\infty) \times [0,M],$$

(B_2) there exist a set $B \in ((1,+\infty),\mathbb{R}^+)$ of positive measure such that

$$\underset{u\to+\infty}{lim} \frac{f(s,u)}{u} = +\infty, \ uniformly\ with\ respect\ to\ s \in B,$$

(B_3) there exist a function $p \in C((1,+\infty),\mathbb{R}^+)$ with $\int_1^\infty s\left(1 - s^{2n}\right) p(s)\, ds < \infty$ such that

$$\underset{u\to 0^+}{lim} \frac{f(s,u)}{up(s)} = 0, \ uniformly\ with\ respect\ to\ s \in B.$$

In 2006, Han [21], replacing the conditions listed above (B_1), (B_2) and (B_3) by the weaker ones

$$\underset{u\to 0^+}{lim\ inf}\ \underset{s\in[c,d]}{min} \frac{f(s,u)}{u} > \xi, \quad \underset{u\to 0^+}{lim\ sup} \frac{f(s,u)}{up(s)} < \eta,$$

uniformly with respect to $s \in (1,+\infty)$ for suitable positive numbers ξ and η, the authors proved that problem (1.1) still has at least one positive radial solution.

In 2014, Wu [37], studied problem (1.1) under some conditions concerning the first eigenvalues corresponding to the relevant linear operators, they obtained several existence theorems on multiple positive radial solutions of (1.1) in an exterior domain.

Inspired and motivated by the works mentioned above, we deal with existence and multiplicity of radial positive solutions to the BVP (1.1), our approach is based on fixed point index theory. The paper is organized as follows. In Section 2, we changes problem (1.1) into a sigular two-point boundary value problem and we will state all the lemmas which will be used to prove our main results in the later section. Setion 3 is devoted to the existence and multiplicity of positive solutions and positive radial solutions for BVP (1.1) and we give an example to illustrate our results.

2. Preliminaries

We shall consider the Banach space $E = C[0,1]$ equipped with sup norm $\|u\| = \underset{0\le t\le 1}{max} |u(t)|$ and $C^+[0,1]$ is the cone of nonnegative functions in $C[0,1]$.

Definition 2.1. *Anonempty closed and convex set $P \subset E$ is called a cone of E if it satisfies*

(i) $u \in P$, $r > 0$ *implies* $ru \in P$,
(ii) $u \in P$, $-u \in P$ *implies* $u = \theta$, *where* θ *denotes the zero element of* E.

Definition 2.2. *A cone P is said to be normal if there exists a positive number N called the normal constant of P, such that $\theta \le u \le v$ implies $\|u\| \le N\|v\|$.*

We are interested in finding radial solutions for problem (1.1). We proceed as in introduction. Since we are looking for the existence of nonnegative radial solutions $u(x) = z(|x|)$ of the problem (1.1), where $z : \mathbb{R}^+ \to \mathbb{R}$, one can substitute $v(t) = z\left(\frac{A}{B-t}\right)^{\frac{1}{n-2}}$ for $t \in [0,1]$, $n \ge 3$, thus reducing the BVP (1.1) to the following singular two-point BVP

$$\begin{cases} -v''(t) = g(t, v(t)), \; t \in (0,1), \\ v(0) = v(1) = 0, \end{cases} \tag{2.1}$$

where

$$g(t,v) = \phi(t) f\left(\left(\frac{A}{B-t}\right)^{\frac{1}{n-2}}, v\right), \tag{2.2}$$

$$A = \frac{(R_0R_1)^{n-2}}{R_1^{n-2} - R_0^{n-2}} \text{ and } B = \frac{R_1^{n-2}}{R_1^{n-2} - R_0^{n-2}}, \tag{2.3}$$

and

$$\phi(t) = \left(\frac{R_1^{-(n-2)} - R_0^{-(n-2)}}{n-2}\right)^2 \left(R_1^{-(n-2)} - \left(R_1^{-(n-2)} - R_0^{-(n-2)}\right)t\right), \; n \ge 3. \tag{2.4}$$

we can reformulate g as

$$g(t,v) = \phi(t) f\left(\left(\frac{A}{B-t}\right)^{\frac{1}{n-2}}, v\right),$$

where

$$\phi(t) = \left(\frac{R_1^{-(n-2)} - R_0^{-(n-2)}}{n-2}\right)^2 \left[\frac{1}{A^{\frac{2n-2}{n-2}}\left(R_1^{n-2} - R_0^{n-2}\right)^{\frac{2n-2}{n-2}}}\right]\left[\frac{A}{B-t}\right]^{\frac{2(n-1)}{n-2}}$$

We observe that the existence of radial positive solutions of (1.1) is equivalent to the existence of positive solutions of the problem (2.1).

In arriving our results, we need the following six preliminary lemmas. The first one is well known.

Lemma 2.1. *Let $y(\cdot) \in C[0,1]$. If $u \in C^2[0,1]$, then the BVP (2.1) has a unique solution*

$$v(t) = \int_0^1 G(t,s)\, y(s)\, ds, \tag{2.5}$$

where

$$G(t,s)=\begin{cases} s(1-t), & 0\le t\le s\le 1,\\ t(1-s), & 0\le s\le t\le 1.\end{cases} \tag{2.6}$$

Lemma 2.2. *For any* $(t,s)\in[0,1]\times[0,1]$, *we have*

$$0<G(t,s)\le G(s,s)=s(1-s).$$

Proof. The proof is evident, we omit it. □

Lemma 2.3. *(see* [23]*) For* $y(t)\in C^{+}[0,1]$. *Then the unique solution* $u(t)$ *of BVP* (2.1) *is nonnegative and satisfies*

$$\min_{R_0\le t\le R_1} v(t)\ge c\,\|v\|,$$

where $c=\min\{R_0,1-R_1\}$ *and* $[R_0,R_1]\subset(0,1)$.

If we let

$$P=\left\{v\in C^{+}[0,1]:\quad v(t)\ge 0,\ for\ t\in[0,1]\right\},$$

and

$$Q=\left\{v\in C^{+}[0,1]:\quad \min_{R_0\le t\le R_1} v(t)\ge c\,\|v\|\right\},$$

then it is easy see that P and Q are cones in $E=C[0,1]$.
Let $\Omega_r=\{u\in E:\|u\|<r\}$ be the open ball of radius r in E and the operator $A:E\to E$ define by

$$(Av)(t)=\int_0^1 G(t,s)\,g(s,v(s))\,ds,\ t\in[0,1]. \tag{2.7}$$

Define a set H by

$$H=\left\{h\in C\left((0,1),\mathbb{R}^{+}\right):\ h\ne 0,\ \int_0^1 t(1-t)\,h(t)\,dt<+\infty\right\}. \tag{2.8}$$

Now, we define an integral operators $T_h:E\to E$ for $h\in H$ by

$$(T_hv)(t)=\int_0^1 G(t,s)\,h(s)\,v(s)\,ds,\ for\ v\in E. \tag{2.9}$$

We have the following lemma.

Lemma 2.4. *For any* $h\in H$ *we have*
(i) T_h *is a completely continuous linear operator and the specteral radius* $r(T_h)\ne 0$ *and* T_h *has a positive eigenfunction* φ_{1h} *corresponding to its first eigenvalue* $\lambda_{1h}=(r(T_h))^{-1}$,
(ii) $T_h(P)\subset Q$,
(iii) there exist $\delta_1,\ \delta_2>0$, *such that*

$$\delta_1G(t,s)\le\varphi_{1h}(s)\le\delta_2G(s,s),\ t,\ s\in[0,1], \tag{2.10}$$

(iv) define a functional J_h *by* $J_h(v)=\int_0^1 h(t)\,\varphi_{1h}(t)\,v(t)\,dt$ *for* $v\in E$. *Then* $J_h(T_hv)=\lambda_{1h}^{-1}J_h(v)$ *for* $v\in E$,

(v) let

$$P_0 = \left\{ v \in P \,:\, J_h(v) \geq \lambda_{1h}^{-1} \delta_1 \|v\| \right\}, \tag{2.11}$$

then P_0 is a cone in E and $T_h(P) \subset P_0$ where δ_1 is defined by (2.10).

To prove Lemma 2.4, we need the following lemmas.

Lemma 2.5. *(see [24]) Suppose that E is a Banach space, $T_n : E \to E$, $n \in \mathbb{N}^*$ are completely continuous operators, $T : E \to E$ and*

$$\lim_{n\to+\infty} \max_{\|u\|<r} \|T_n u - Tu\| = 0, \quad \forall r > 0, \tag{2.12}$$

then T is completely continuous operator.

Lemma 2.6. *(see [25]) Suppose that E is a Banach space, $T : E \to E$ is completely continuous linear operators and $T(P) \subset P$. If there exist $\psi \in E \setminus (-P)$ and a constant $\mu > 0$ such that $\mu T\psi \geq \psi$, then the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 (r(T))^{-1}$.*

Proof. Proof of Lemma 2.4. It follows from the definition of H that for any $v \in E$

$$|(T_h v)(t)| \leq \int_0^1 G(t,s)\, h(s)\, |v(s)|\, ds,$$

$$\leq \|v\| \int_0^1 G(t,s)\, h(s)\, ds < +\infty. \tag{2.13}$$

Obviously, $T_h(P) \subset P$ and $T_h : E \to E$ is a positive linear operators.
We will show tha $T_h : E \to E$ is completely continuous. For any natural number $n \geq 2$, let

$$h_n(t) = \begin{cases} \inf\limits_{t\leq s\leq \frac{1}{n}} h(s), & 0 \leq t \leq \frac{1}{n}, \\ h(t), & \frac{1}{n} \leq t \leq \frac{n-1}{n}, \\ \inf\limits_{\frac{n-1}{n}\leq s\leq 1} h(s), & \frac{n-1}{n} \leq t \leq 1. \end{cases} \tag{2.14}$$

Then $h_n : [0,1] \to [0,\infty)$ is continuous and $h_n(t) \leq h(t)$ for all $t \in (0,1)$.
Let

$$(T_{h_n} v)(t) = \int_0^1 G(t,s)\, h_n(s)\, v(s)\, ds. \tag{2.15}$$

Now, we show that $T_{h_n} : E \to E$ is completely continuous. For any $r > 0$ and $v \in \Omega_r$, according to (2.14), (2.15) and the absolute continuity of integral, we have

$$\lim_{n\to+\infty} \|T_{h_n} v - Tv\| = \lim_{n\to+\infty} \max_{t\in[0,1]} \left| \int_0^1 G(t,s)\,(h_n(s) - h(s))\, v(s)\, ds \right|$$

$$\leq \|v\| \lim_{n\to+\infty} \left| \int_0^1 G(s,s)\,(h_n(s) - h(s))\, ds \right|$$

$$\leq \|v\| \lim_{n\to+\infty} \int_{e(n)} G(s,s)\,(h(s) - h_n(s))\, ds$$

$$\leq \|v\| \lim_{n\to+\infty} \int\limits_{e(n)} G(s,s)\, h(s)\, ds = 0, \tag{2.16}$$

where $e(n) = \left[0, \frac{1}{n}\right] \cup \left[\frac{n-1}{n}, 1\right]$.
Therefore, by Lemma 2.5, $T_{h_n} : E \to E$ is a completely continuous operator. It is obvious that there exists $t_1 \in (0,1)$ such that $G(t_1,t_1)\, h(t_1) > 0$. Thus there is $[a_1, b_1] \subset (0,1)$ such that $t_1 \in (a_1, b_1)$ and $G(t,s)\, h(s) > 0$ for all t, $s \in [a_1, b_1]$. Take $\zeta \in P$ such that $\zeta(t_1) > 0$ and $\zeta(t) = 0$ for all $t \notin [a_1, b_1]$. Then, for $t \in [a_1, b_1]$

$$(T_h\zeta)(t) = \int\limits_0^1 G(t,s)\, h(s)\, \zeta(s)\, ds$$

$$\geq \int\limits_{a_1}^{b_1} G(t,s)\, h(s)\, \zeta(s)\, ds > 0. \tag{2.17}$$

So, there exist a constant $\mu > 0$ such that $\mu(T_h\zeta)(t) \geq \zeta(t)$ for all $t \in [0,1]$. From Lemma 2.6, we have that the spectral radius $r(T_h) \neq 0$ and T_h has a positive eigenfunction corresponding to its first eigenvalue $\lambda_{1h}(r(T_h))^{-1}$.
(ii) To prove $T_h(P) \subset Q$, we only need to show

$$\min_{t\in[R_0,R_1]} (T_hv)(t) \geq \min\{R_0, 1-R_1\}\, \|T_hv\| \quad \text{for} \quad v \in P. \tag{2.18}$$

In fact, for every $v \in P$, from $0 < G(t,s) \leq G(s,s) = s(1-s)$ for t, $s \in [0,1]$, we have

$$(T_hv)(t) = \int\limits_0^1 G(t,s)\, h(s)\, v(s)\, ds$$

$$\leq \int\limits_0^1 s(1-s)\, h(s)\, v(s)\, ds,$$

so, for any $v \in P$, we have

$$\|T_hv\| \leq \int\limits_0^1 s(1-s)\, h(s)\, v(s)\, ds. \tag{2.19}$$

Notice that, for $t \in [R_0, R_1]$,

$$G(t,s) = \begin{cases} s(1-t) \geq s(1-R_1), & s \leq t, \\ t(1-s) \geq R_0(1-s), & t \leq s. \end{cases} \tag{2.20}$$

Thus, for $(t,s) \in [R_0,R_1] \times [0,1]$, we have

$$G(t,s) \geq \min\{R_0, 1-R_1\}\, s(1-s). \tag{2.21}$$

It follows, from (2.19) and (2.21) that for all $v \in P$

$$(T_hv)(t) = \int\limits_0^1 G(t,s)\, h(s)\, v(s)\, ds$$

$$\geq min\left\{R_0, 1-R_1\right\}\int\limits_0^1 s\left(1-s\right)h\left(s\right)v\left(s\right)ds$$

$$\geq min\left\{R_0, 1-R_1\right\}\left\|T_h v\right\|, \; t\in\left[R_0, R_1\right]. \tag{2.22}$$

So, (2.18) holds. Thus, T_h maps P into Q.
(iii) Since φ_{1h} is a positive eigenfunction of T_h, we know from the maximum principle (see [18]) that $\varphi_{1h}\left(t\right)>0$ for all $t\in\left(0,1\right)$.
Note that $G\left(0,s\right)=G\left(1,s\right)=0$ for $s\in\left(0,1\right)$, we have $\varphi_{1h}\left(0\right)=\varphi_{1h}\left(1\right)=0$.
This impleis that $\varphi_{1h}'\left(0\right)>0$ and $\varphi_{1h}'\left(1\right)<0$ (see [18]).
Define a function Φ_h on $[0,1]$ by

$$\Phi_h\left(s\right)=\begin{cases}\varphi_{1h}'\left(0\right), & s=0,\\ \frac{\varphi_{1h}(s)}{s(1-s)}, & s\in\left(0,1\right),\\ -\varphi_{1h}'\left(1\right), & s=1.\end{cases} \tag{2.23}$$

Then, it is easy to see that Φ_h continuous on $[0,1]$ and $\Phi_h\left(s\right)>0$ for all $s\in[0,1]$.
So, there exist $\delta_1,\ \delta_2>0$, such that

$$\delta_1 G\left(t,s\right)\leq\delta_1 s\left(1-s\right)\leq\varphi_{1h}\left(s\right)\leq\delta_2 s\left(1-s\right)\leq\delta_2 G\left(s,s\right), \tag{2.24}$$

for all $t,\ s\in[0,1]$.
(iv) From (2.10), for all $v\in E$, we have

$$J_h\left(v\right)=\int\limits_0^1 h\left(t\right)\varphi_{1h}\left(t\right)v\left(t\right)dt$$

$$\leq\delta_2\int\limits_0^1 t\left(1-t\right)h\left(t\right)v\left(t\right)dt<+\infty.$$

So, $J:E\rightarrow\mathbb{R}$ is well defined.
For all $v\in E$, we have

$$J_h\left(T_h v\right)=\int\limits_0^1 h\left(t\right)\varphi_{1h}\left(t\right)\left(\int\limits_0^1 G\left(t,s\right)h\left(s\right)v\left(s\right)ds\right)dt$$

$$=\int\limits_0^1 h\left(s\right)v\left(s\right)\left(\int\limits_0^1 G\left(s,t\right)h\left(t\right)\varphi_{1h}\left(t\right)dt\right)ds$$

$$=\int\limits_0^1 h\left(s\right)v\left(s\right)\left(r_{1h}\varphi_{1h}\left(s\right)\right)ds$$

$$=\lambda_{1h}^{-1}J_h\left(v\right), \tag{2.25}$$

for $v\in E$. Then $J_h\left(T_h v\right)=\lambda^{-1}J_h\left(v\right)$ for $v\in E$.
(v) It is easy to verify that P_0 is a cone in E. It follows from (2.10) and (2.25) that

$$J_h\left(T_h v\right)=\lambda_{1h}^{-1}\int\limits_0^1 h\left(s\right)\varphi_{1h}\left(s\right)v\left(s\right)ds$$

$$\geq \delta_1 \lambda_{1h}^{-1} \int_0^1 h(s)\, G(t,s)\, v(s)\, ds$$
$$= \delta_1 \lambda_{1h}^{-1} (T_h v)(t)\,,\ for\ v \in P. \tag{2.26}$$

The proof is completed. □

3. Existence results

3.1. **Positive solutions of singular two-point boundary value problems.** The following Lemma is a well-known result of the fixed point index theory, which will play an important role in the proof of our main results.

Lemma 3.1. (*see* [18])*Let Ω be a bounded open set in E with $\theta \in \Omega$, $A : P \cap \overline{\Omega} \to P$ a completely continuous operator, where θ denotes the null element of E. Assume that A has no fixed point on $P \cap \partial\Omega$.*

(i) (Homotopy invariance) If $u \neq \mu Au$ for all $\mu \in [0,1]$ and $u \in P \cap \partial\Omega$, then the fixed point index $i(A, P \cap \Omega, P) = 1$,

(ii) (omitting a direction) if there exists an element $\psi_0 \in P \setminus \{\theta\}$ such that $u \neq Au + \mu\psi_0$ for all $u \in P \cap \partial\Omega$ and $\mu \geq 0$, then $i(A, P \cap \Omega, P) = 0$,

(iii) (cone expansion) if $\|Au\| \geq \|u\|$ for all $u \in P \cap \partial\Omega$, then $i(A, P \cap \Omega, P) = 0$,

(iv) (additivity) suppose Ω_1 is an open subset of Ω with $\theta \in \Omega_1$ and $u \neq Au$ for $u \in P \cap \partial\Omega_1$, then

$$i(A, P \cap \Omega, P) = i(A, P \cap \Omega_1, P) + i\left(A, P \cap \left(\Omega \setminus \overline{\Omega}\right), P\right),$$

(v) $i(A, P \cap \Omega, P) \neq 0$, then A has at least one fixed point in $P \cap \Omega$.

Denote

$$M_1 = \left(\min_{t \in [R_0, R_1]} \int_{R_0}^{R_1} G(t,s)\, ds \right)^{-1},\ \eta = \left(\max_{t \in [0,1]} \int_{R_0}^{R_1} G(t,s)\, ds \right)^{-1}. \tag{3.1}$$

The following conditions holds.

(H_1) $g \in C((0,1) \times \mathbb{R}^+, \mathbb{R}^+)$ and for any $M > 0$ there exists a function $h_M \in H$ such that

$$g(t,v) \leq h_M(t)\,,\ \forall (t,v) \in (0,1) \times [0,M]\,, \tag{3.2}$$

(H_2) there exists a function $h \in H$ such that

$$\lim_{v \to 0^+} sup \frac{g(t,v)}{h(t)\, v} < \lambda_{1h},\ uniformly\ with\ respect\ to\ t \in (0,1)\,, \tag{3.3}$$

(H_3) there exists a function $h \in H$ such that

$$\lim_{v \to +\infty} sup \frac{g(t,v)}{h(t)\, v} < \lambda_{1h},\ uniformly\ with\ respect\ to\ t \in (0,1)\,, \tag{3.4}$$

(H_4) $\lim\limits_{v \to 0^+} inf \min\limits_{t \in [R_0, R_1]} \frac{g(t,v)}{v} > M_1$,

(H_5) $\lim\limits_{v \to +\infty} inf \min\limits_{t \in [R_0, R_1]} \frac{g(t,v)}{v} > M_1$,

(H_6) there exists a number $l > 0$ such that

$$g(t,v) > \eta l, \quad for\ (t,v) \in [R_0, R_1] \times [min\{R_0, 1-R_1\} l, l], \tag{3.5}$$

where η defined in (3.1),
(H_7) there exists a function $h \in H$ such that

$$\lim_{v\to 0^+} inf \frac{g(t,v)}{h(t)v} > \lambda_{1h},\ uniformly\ with\ respect\ to\ t \in (0,1), \tag{3.6}$$

(H_8) there exists a function $h \in H$ with $h(t) \neq 0$ for $t \in [R_0, R_1]$ and $q \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$g(t,v) \geq h(t)q(v), \quad \forall (t,v) \in (0,1) \times \mathbb{R}^+, \tag{3.7}$$

$$\lim_{v\to\infty} inf \frac{q(v)}{v} > \lambda_{1h}. \tag{3.8}$$

Lemma 3.2. *Assume (H_1) holds. Then $A : Q \to Q$ is a completely continuous operator.*

Proof. The proof is similar to that of Lemma 3.1 in [21]. □

Lemma 3.3. *assume (H_1) holds.*

(i) If (H_2) holds. Then $i(A, Q \cap \Omega_r, Q) = 1$ for sufficiently small positive number r.

(ii) If (H_3) holds. Then $i(A, Q \cap \Omega_R, Q) = 1$ for sufficiently large positive number R.

(iii) If (H_4) holds. Then $i(A, Q \cap \Omega_r, Q) = 0$ for sufficiently small positive number r.

(iv) If (H_5) holds. Then $i(A, Q \cap \Omega_R, Q) = 0$ for sufficiently large positive number R.

(v) If (H_6) holds. Then $i(A, Q \cap \Omega_l, Q) = 0$.

(vi) If (H_7) holds. Then $i(A, Q \cap \Omega_r, Q) = 0$ for sufficiently small positive number r.

(ii) If (H_8) holds. Then $i(A, Q \cap \Omega_R, Q) = 0$ for sufficiently large positive number R.

Proof. (i) By (H_2) there exists $r > 0$ such that

$$g(t,v) \leq \lambda_{1h} h(t) v,\ \forall (t,v) \in (0,1) \times [0,r]. \tag{3.9}$$

Define $S_h v = \lambda_{1h} T_h v$ for $v \in E$, then $S_h : E \to E$ is a bounded linear operator with $S_h(P) \subset Q$ and the spectral radial $r(S_h) = 1$. For every $v \in Q \cap \partial\Omega_r$, it follows from (3.9) that for $t \in [0,1]$,

$$(Av)(t) = \int_0^1 G(t,s)\, g(s, v(s))\, ds$$

$$\leq \lambda_{1h} \int_0^1 G(t,s)\, h(s)\, v(s)\, ds$$

$$\leq \lambda_{1h} (T_h v)(t) = (S_h v)(t). \tag{3.10}$$

So,

$$Av \leq S_h v, \quad \forall v \in Q \cap \partial\Omega_r. \tag{3.11}$$

If there exist $v_1 \in Q \cap \partial\Omega_r$ and $\mu_1 \in [0,1]$ such that $v_1 = \mu_1 A v_1$, then it is easy to see that $\mu_1 \in (0,1)$.
Thus $\tau_1 = \mu_1^{-1} > 1$ and $\tau_1 v_1 = Av_1 \leq S_h v_1$. By induction, we have $\tau_1^n v_1 = Av_1 \leq S_h^n v_1$, $n = 1, 2, \ldots$. Then $\tau_1^n v_1 = S_h^n v_1 \leq \|S_h\| \|v_1\|$ and taking the sepremum on $[0,1]$ gives $\tau_1^n \leq \|S_h^n\|$. By the spectral radius formula, we have

$$r(S_h) = \lim_{n\to+\infty} \sqrt[n]{\|S_h^n\|} \geq \tau_1 > 1, \tag{3.12}$$

which is contradiction.
According to the homotopy property invarience of fixed point index, we have $i(A, Q \cap \Omega_r, Q) = 1$.
(ii) By (H_3) there exists $\sigma > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$g(t,v) \leq \varepsilon_0 \lambda_{1h} h(t)\, v, \; \forall (t,v) \in (0,1) \times [\sigma, +\infty). \tag{3.13}$$

From (H_1) there is $h_\sigma \in H$ such that $g(t,v) \leq h_\sigma(t)$ for all $(t,v) \in (0,1) \times [0,\sigma]$. Hence

$$g(t,v) \leq \varepsilon_0 \lambda_{1h} h(t)\, v + h_\sigma(t), \; \forall (t,v) \in (0,1) \times [0, +\infty). \tag{3.14}$$

Define $S_h v = \varepsilon_0 \lambda_{1h} T_h v$, for $v \in E$, then $S_h : E \to E$ is a bounded linear operator with $S_h(P) \subset Q$. Let $C_1 = \int_0^1 t(1-t)\, h_\sigma(t)\, dt < +\infty$. Set

$$W = \{v \in Q : v = \rho A v, \; \rho \in [0,1]\}. \tag{3.15}$$

Next, we prove that W is bounded. For any $v \in W$. From (3.14), we have

$$\begin{aligned}
v(t) = \rho (Av)(t) &\leq (Av)(t) \\
&= \int_0^1 G(t,s)\, g(s, v(s))\, v(s)\, ds \\
&\leq \varepsilon_0 \lambda_{1h} \int_0^1 G(t,s)\, h(s)\, v(s) ds + \int_0^1 G(t,s)\, h_\sigma(s)\, ds \\
&\leq \varepsilon_0 \lambda_{1h} (T_h v)(t) + C_1 \\
&= (S_h v)(t) + C_1, \; t \in [0,1].
\end{aligned}$$

Thus

$$((I - S_h) v)(t) \leq C_1, \quad \forall v \in W, \; t \in [0,1]. \tag{3.16}$$

Since λ_{1h} is the first eigenvalue of S_h, $r(S_h)^{-1} > 1$. therefore, the inverce operator $(I - S_h)^{-1}$ exists and

$$(I - S_h)^{-1} = I + S_h + S_h^2 + \cdots + S_h^n + \cdots \tag{3.17}$$

It follows from $T_h(P) \subset Q$ that $(I - S_h)^{-1}(P) \subset Q$. Hence, we have from (3.16) that

$$v(t) \leq (I - S_h)^{-1} C_1, \; \forall v \in W, \; t \in [0,1] \tag{3.18}$$

that is W is bounded. Choose $R > \{\rho, supW\}$, then $v \neq \sigma Av$ for all $\sigma \in [0,1]$ and $v \in Q \cap \Omega_R$. By the homotopy property invarience of fixed point index, we have $i(A, Q \cap \Omega_R, Q) = 1$.
$(iii) - (v)$ have been proved in [21], so we skip it.
(vi) By (H_7) there exist $r > 0$ such that

$$g(t,v) \geq \lambda_{1h} h(t) v, \ \ \forall (t,v) \in (0,1) \times [0,r]. \tag{3.19}$$

For any $v \in Q \cap \Omega_r$, we have

$$\begin{aligned} (Av)(t) &= \int_0^1 G(t,s) g(s, v(s)) ds \\ &\geq \lambda_{1h} \int_0^1 G(t,s) h(s) v(s) ds \\ &= \lambda_{1h} (T_h v)(t), \ t \in [0,1]. \end{aligned} \tag{3.20}$$

Without loss of generality, we can suppose that A has no fixed point on $Q \cap \partial\Omega_r$. Suppose that there exist $v_1 \in Q \cap \partial\Omega_r$ and $\mu_1 \geq 0$ such that $v_1 = Av_1 + \mu_1 \varphi_{1h}$. Then $\mu_1 > 0$ and $v_1 = Av_1 + \mu_1 \varphi_{1h} \geq \mu_1 \varphi_{1h}$. Let

$$\mu^* = sup \{\rho > 0 \, : \, v_1 \geq \rho \varphi_{1h}\}. \tag{3.21}$$

Then $\mu^* \geq \mu_1 > 0$ and $v_1 \geq \mu^* \varphi_{1h}$.
Since T_h is a positive linear operator, we have

$$\lambda_{1h} T_h v_1 \geq \mu^* \lambda_{1h} T_h \varphi_{1h}. \tag{3.22}$$

Hence, by (3.20) we have

$$v_1 = Av_1 + \mu_1 \varphi_{1h} \geq \lambda_{1h} T_h v_1 + \mu_1 \varphi_{1h} \geq \mu^* \varphi_{1h} + \mu_1 \varphi_{1h}, \tag{3.23}$$

which is contradiction. Thus according to the homotopy property of omitting a direction for fixed point index, we have $i(A, Q \cap \Omega_r, Q) = 0$.
(vii) From (3.8) there exist there exists $\sigma > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$q(v) \geq (1 + \varepsilon_0) \lambda_{1h} v, \ \forall v \in [\sigma, +\infty). \tag{3.24}$$

Since q is bounded on $[0, \sigma]$, there is a constant $C_2 > 0$ such that

$$q(v) \geq (1 + \varepsilon_0) \lambda_{1h} v - C_2, \ \forall v \in [0, \sigma]. \tag{3.25}$$

Thus

$$q(v) \geq (1 + \varepsilon_0) \lambda_{1h} v - C_2, \ \forall v \in [0, +\infty).$$

Hence, by (3.7), we have

$$g(t,v) \geq (1 + \varepsilon_0) \lambda_{1h} v h(t) - C_2 h(t), \ \forall (t,v) \in (0,1) \times [0, +\infty). \tag{3.26}$$

Let $C_3 = \int_0^1 h(t) \varphi_{1h}(t) \left(\int_0^1 G(t,s) h(s) ds \right) dt < +\infty$. Then $C_3 > 0$ is a finite constant. Take

$$R > C_3 \left(\varepsilon_0 min \{R_0, 1 - R_1\} \int_{R_0}^{R_1} h(t) \varphi_{1h}(t) dt \right)^{-1}. \tag{3.27}$$

Suppose that there exsist $v_1 \in Q \cap \Omega_R$ and $\mu_1 \geq 0$ such that $v_1 = Av_1 + \mu_1 \varphi_{1h}$. Then

$$\begin{aligned}
J_h(v_1) &= J(Av_1) + \mu_1 J(\varphi_{1h}) \\
&\geq J(Av_1) \\
&\geq \int_0^1 h(t)\,\varphi_{1h}(t) \left(\lambda_{1h}(1+\varepsilon_0) \int_0^1 G(t,s)\,h(s)\,v_1(s)\,ds - C_2 T_h(1) \right) dt \\
&= \lambda_{1h}(1+\varepsilon_0)\,J_h(T_h v_1) - C_3 \\
&= (1+\varepsilon_0)\,J_h(v_1) - C_3. \qquad (3.28)
\end{aligned}$$

Hence

$$J_h(v_1) \leq C_3 \varepsilon_0^{-1}.$$

On the other hand

$$\begin{aligned}
J_h(v_1) &= \int_0^1 h(t)\,\varphi_{1h} v_1(t)\,dt \\
&\geq \int_{R_0}^{R_1} h(t)\,\varphi_{1h} v_1(t)\,dt \\
&\geq R \min\{R_0, 1-R_1\} \int_{R_0}^{R_1} h(t)\,.\varphi_{1h}\,dt. \qquad (3.29)
\end{aligned}$$

By the maximum principle, $\varphi_{1h}(t) > 0$ for all $t \in (0,1)$. By $h(t) \neq 0$ for $t \in [R_0, R_1]$, we have

$$\int_{R_0}^{R_1} h(t)\,\varphi_{1h}\,dt > 0.$$

Thus, from (3.28) and (3.29), we have

$$\begin{aligned}
R &\leq \left(min\{R_0, 1-R_1\} \int_{R_0}^{R_1} h(t)\,\varphi_{1h}\,dt \right)^{-1} J_h(v_1) \\
&\leq C_3 \left(min\{R_0, 1-R_1\} \int_{R_0}^{R_1} h(t)\,\varphi_{1h}\,dt \right)^{-1}. \qquad (3.30)
\end{aligned}$$

This is contradiction. So, by the property of omitting a direction for fixed point index, we have $i(A, Q \cap \Omega_R, Q) = 0$. The is completed. □

Now, we are in position to present our main results of this subsection.

Theorem 3.4. *Assume $(H_1) - (H_3)$ and (H_6) hold. Then the singular boundary value problem (2.1) has at least two positive solutions.*

Proof. According to Lemma 3.3, we can choose sufficiently small positive number r and sufficiently large positive number R satisfying $0 < r < l < R$, $i(A, P \cap \Omega_r, P) = 1$, $i(A, P \cap \Omega_R, P) = 1$. From $i(A, P \cap \Omega_l, P) = 0$ and additivity property of the fixed point index, we obtain

$$i\left(A, P \cap \left(\Omega_l \setminus \overline{\Omega_r}\right), P\right) = 0 - 1 = -1,$$

$$i\left(A, P \cap \left(\Omega_R \setminus \overline{\Omega_l}\right), P\right) = 1 - 0 = 1.$$

Hence, A has at least two fixed points, one in $\Omega_l \setminus \overline{\Omega_r}$ and another in $\Omega_R \setminus \overline{\Omega_l}$. That is the singular boundary value problem (2.1) has at least two positive solution. The proof is completed. □

Theorem 3.5. *If (H_1) and one of the following conditions are satisfied, then the singular boundary value problem (2.1) has at least one positive solution.*
(i) (H_2) and (H_5) holds,
(ii) (H_2) and (H_6) holds,
(iii) (H_2) and (H_8) holds,
(iv) (H_3) and (H_4) holds,
(v) (H_3) and (H_6) holds,
(vi) (H_3) and (H_7) holds.

Proof. By the property of the fixed point index, we only need to choose suitable positive numbers r and R. This completes the proof. □

We present an example to illustrate the applicability of the results shown before.

Example 3.1. *Let*

$$g(t,v) = \begin{cases} \frac{1}{t(t-1)}\left(\frac{cvl}{384}\right), & t \in (0,1),\ v \in \left[0, \frac{1}{8}l\right], \\ \frac{1}{t(t-1)}\left(\frac{cvl}{192} \times \frac{l-4v}{l} + \frac{16l(8v-l)}{l}\right), & t \in (0,1),\ v \in \left[\frac{1}{8}l, \frac{1}{4}l\right], \\ 16l, & t \in (0,1),\ v \in \left[\frac{1}{4}l, l\right], \\ 16l + t\sqrt{v-l}, & t \in (0,1),\ v \in [l, +\infty), \end{cases}$$

where $c,\ l > 0$. Obviously, $g(t,v) \le h(t)\psi(v)$ for all $(t,v) \in (0,1) \times \mathbb{R}^+$, where $h(t) = \frac{1}{t(t-1)}$ and

$$\psi(v) = \begin{cases} \left(\frac{cvl}{384}\right), & t \in (0,1),\ v \in \left[0, \frac{1}{8}l\right], \\ \left(\frac{cvl}{192} \times \frac{l-4v}{l} + \frac{16cvl(8v-l)}{l}\right), & t \in (0,1),\ v \in \left[\frac{1}{8}l, \frac{1}{4}l\right], \\ 16cvl, & t \in (0,1),\ v \in \left[\frac{1}{4}l, l\right], \\ 16cvl + t\sqrt{v-l}, & t \in (0,1),\ v \in [l, +\infty), \end{cases}$$

Since $\lambda = \frac{32}{3} < 16$, if $\lim\limits_{v \to 0^+} \frac{\psi(v)}{v} = \frac{cl}{348} < \lambda_{1h}$ and $\lim\limits_{v \to +\infty} \frac{\psi(v)}{v} = 16cl < \lambda_{1h}$, then g satisfies all the conditions of Theorem 3.4, thus we infer that the singular boundary value problem (2.1) has at least two positive solutions.

3.2. **Positive radial solutions of elliptic boundary value problems.**

Define a set

$$K = \left\{p \in C\left((R_0, R_1), \mathbb{R}^+\right) : p \ne 0,\right.$$

$$\left.\int_{R_0}^{R_1}\left(\frac{Bs^{n-2}-A}{s^{n-2}}\right)\left(1-\frac{Bs^{n-2}-A}{s^{n-2}}\right)\left(\frac{(n-2)\,As^{n-3}}{s^{2(n-2)}}\right)p\,(s)\,ds<+\infty\right\},$$

where A and B are defined above

Denote $c=\left(\frac{A}{B-R_0}\right)^{n-2}$ and $d=\left(\frac{A}{B-R_1}\right)^{n-2}$.

For $p\in K$, let

$$h\,(t)=\phi\,(s)\,p\left(\left(\frac{A}{B-t}\right)^{\frac{1}{n-2}}\right),$$

we can reformulate h as

$$h\,(t)=\phi\,(t)\,p\left(\left(\frac{A}{B-t}\right)^{\frac{1}{n-2}}\right),$$

where

$$\phi\,(t)=\left(\frac{R_1^{-(n-2)}-R_0^{-(n-2)}}{n-2}\right)^2\left[\frac{1}{A^{\frac{2n-2}{n-2}}\left(R_1^{n-2}-R_0^{n-2}\right)^{\frac{2n-2}{n-2}}}\right]\left[\frac{A}{B-s}\right]^{2(n-1)}.$$

For convinience, we let

$$\Delta=\left(\frac{R_1^{-(n-2)}-R_0^{-(n-2)}}{n-2}\right)^2\left[\frac{1}{A^{\frac{2n-2}{n-2}}\left(R_1^{n-2}-R_0^{n-2}\right)^{\frac{2n-2}{n-2}}}\right].$$

Then $h\in H$. As in (2.9) and Lemma 2.4, h confirms an operator T_h and its first eigenvalue λ_{1h}. To emphasize their relation with p, we use the notations h_p, λ_{1h_p} and φ_{1h_p}.

According to (2.2), we formulate the following conditions which correspond to those in Section 3.1.

(C_1) $f\in C\left((R_0,R_1)\times\mathbb{R}^+,\mathbb{R}^+\right)$ and for any $M>0$ there exist a function $p_M\in K$ such that

$$f\,(s,u)\le p_M\,(s)\,,\ \forall\,(s,u)\in(R_0,R_1)\times[0,M]\,,$$

(C_2) there exist a function $p\in K$ such that

$$\lim_{u\to0^+}sup\frac{f\,(s,u)}{p\,(s)\,u}<\lambda_{1h_p},\ uniformly\,with\,respect\,to\,t\in(R_0,R_1)\,,$$

(C_3) there exist a function $p\in K$ such that

$$\lim_{u\to+\infty}sup\frac{f\,(s,u)}{p\,(s)\,u}<\lambda_{1h_p},\ uniformly\,with\,respect\,to\,t\in(R_0,R_1)\,,$$

(C_4) $\lim\limits_{u\to0^+}inf\min\limits_{s\in[c,d]}\frac{f(s,u)}{u}>c^{2-2n}\Delta M_1$,

(C_5) $\lim\limits_{u\to+\infty}inf\min\limits_{s\in[c,d]}\frac{f(s,u)}{u}>c^{2-2n}\Delta M_1$,

(C_6) there exist a number $l>0$ such that

$$f\,(s,u)>\Delta\lambda l,\quad for\ \,(s,u)\in[c,d]\times[min\left\{R_0,1-R_1\right\}l,l]\,,$$

(C_7) there exist a function $p\in K$ such that

$$\lim_{u\to0^+}inf\frac{f\,(s,u)}{p\,(s)\,u}>\lambda_{1h_p},\ uniformly\,with\,respect\,to\,s\in(R_0,R_1)\,,$$

(C_8) there exist a function $p \in K$ with $p(s) \neq 0$ for $s \in (c,d)$ and $q \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$f(s,u) \geq p(s)\,q(u)\,, \quad \forall (s,u) \in (R_0, R_1) \times \mathbb{R}^+,$$

$$\lim_{u \to +\infty} inf \frac{q(u)}{u} > \lambda_{1h_p}.$$

Now, we are ready to state our main results for the elliptic BVP (1.1).

Theorem 3.6. *Assume $(C_1) - (C_3)$ and (C_6) hold. Then the singular boundary value problem (1.1) has at least two positive solution.*

Proof. The proof is similar to proof of Theorem 4.1 in [21] and from the proof of Theorem 3.4. □

Theorem 3.7. *If (C_1) and one of the following conditions are satisfied, then the singular boundary value problem (1.1) has at least one positive solution.*

(i) (C_2) and (C_5) holds,
(ii) (C_2) and (C_6) holds,
(iii) (C_2) and (C_8) holds,
(iv) (C_3) and (C_4) holds,
(v) (C_3) and (C_6) holds,
(vi) (C_3) and (C_7) holds.

Proof. The proof is similar to proof of Theorem 4.1 in [21] and from the proof of Theorem 3.5. □

Conclusion

In this contribution, we studied the existence and multiplicity of radial positive solutions for elliptic BVP (1.1) in the ball. The interest of such problem came from the lack of the existence of the multiple solutions by using bifurcation theory for shown that many local branches of solutions existe while, among them, only one is global and has no bifurcation point implies a considerable difficult to prove the existence of bifurcation point interior the ball. The main scope of these paper is the imposing some conditions on the nonlinearity f to prove the multiplicity of the solutions of problems (1.1) in smooth domains via fixed point index theory.

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