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On the Geometric Series of Linear Positive Operators

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ABSTRACT. We study the existence and the norm of operators obtained as power series of linear positive operators with particularization to Bernstein operators. We also obtain a Voronovskaja-kind theorem.

Keywords: Positive linear operators, Geometric series of operators, Bernstein operators, Voronovskaja theorem

2010 Mathematics Subject Classification: 41A36, 41A25.

1. INTRODUCTION.

Let $L : C[0,1] \to C[0,1]$ be a positive linear operator. Denote by L^k, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the iterates of L, defined by $L^0 = I$, where I is the identical operator and $L^k = L \circ \ldots \circ L$, where L appears k times.
By geometric series of operator L we understand the series

$$G_L = \sum_{k=0}^{\infty} L^k. \tag{1.1}$$

The geometric series of operators were studied in [11], [1], [2], [3], [12]. The existence of these operators needs some restrictions of the domain of definition. It is necessary to consider some special subspaces of functions. Let $\psi : [0,1] \to \mathbb{R}$, $\psi(x) = x(1-x)$. The more simple is the space

$$\psi C[0,1] = \{f \in C[0,1] :\ \exists g \in C[0,1],\ f = \psi g\},$$

which is a Banach space if it is endowed with the norm

$$\|f\|_{\psi} := \sup_{x\in(0,1)} \frac{|f(x)|}{\psi(x)}, \tag{1.2}$$

where $f \in \psi C[0,1]$.
Denote by B_n the Bernstein operators. In [11] there is proved that operators $A_n : \psi C[0,1] \to \psi C[0,1]$, given by

$$A_n = \frac{1}{n} \sum_{k=0}^{\infty} (B_n)^k$$

are well defined and the following result is true:
Theorem A *For any $g \in C[0,1]$ we have*

$$\lim_{n\to\infty} \|A_n(\psi g) - 2F(g)\|_{\psi} = 0, \tag{1.3}$$

where

$$F(g)(x) = (1-x)\int_0^x t g(t)dt + x\int_x^1 (1-t)g(t)dt,\ x \in [0,1]. \tag{1.4}$$

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Note that $(F(g)(x))'' = -g(x)$, for $x \in [0,1]$. Because the convergence in norm $\|\cdot\|_\psi$ implies the convergence in sup-norm $\|\cdot\|$, ($\|f\| = \max_{x\in[0,1]} |f(x)|$), we have
Corollary A *For any $g \in C[0,1]$ we have*

$$\lim_{n\to\infty} \|A_n(\psi g) - 2F(g)\| = 0. \tag{1.5}$$

In [3], the geometric series are consider for a large class of operators, defined on an more extended space $C_\psi[0,1]$ given by

$$C_\psi[0,1] := \{f : [0,1] \to \mathbb{R} \mid \exists g \in B[0,1] \cap C(0,1) : \ f = \psi g\},$$

or equivalently by:

$$C_\psi[0,1] := \{f \in C[0,1] \mid \exists M > 0 : \ |f(x)| \le M\psi(x), \ x \in [0,1]\}.$$

Space $C_\psi[0,1]$ is also a Banach space with regard the norm $\|\cdot\|_\psi$, defined in (1.2), but is not a Banach space with respect the sup-norm, $\|\cdot\|$. Theorem A is generalized in this extended context and also an inverse Voronovskaja theorem is obtained.
A more general space is

$$C_0[0,1] = \{f \in C[0,1], \ f(0) = 0 = f(1)\},$$

endowed by the usual sup-norm $\|\cdot\|$. Clearly, $\Psi C[0,1] \subset C_\psi[0,1] \subset C_0[0,1]$, but the topologies are different.
In paper [12] the geometric series are considered for multidimensional Bernstein operators for a simplex, on the space of continuous functions which vanish at the vertexes. In the unidimensional case we obtain the space $C_0[0,1]$. The definition of geometric series of Bernstein operators is possible because the norms of operators B_n on space $C_0[0,1]$ are strictly less than 1.
The first aim of the present paper is to study the norm of the operators defined by geometric series and in the particular case the norm of the series of powers of Bernstein operators. This allow us to extend Theorem A on space $C_0[0,1]$. In the final section, we derive a Voronovskaja type theorem for the geometric series of Bernstein operators.
For a general theory on Bernstein operators see the papers [9], [6], [5]. For specific problems regarding Voronovskaja theorem we indicate the papers [4], [7]. For general methods of estimating the degree of approximation we mention [10] and [8].

2. Preliminaries

Lemma 2.1. *We have:*

i) *$C_0[0,1]$ is a Banach space with regard the norm $\|\cdot\|$.*
ii) *With regard to the norm $\|\cdot\|$ we have:*

$$\overline{\psi C[0,1]} = C_0[0,1].$$

Proof. i) It is immediate.
ii) If $f \in C_0[0,1]$, then $B_n(f) \in \psi C[0,1]$, for $n \in \mathbb{N}$, where B_n are the Bernstein operators. Since $\lim_{n\to\infty} \|f - B_n(f)\| = 0$, it follows $f \in \overline{\psi C[0,1]}$. The inverse inclusion follows since, obviously $\psi C[0,1] \subset C_0[0,1]$ and $C_0[0,1]$ is closed. ∎

Lemma 2.2. *If $L : C[0,1] \to C[0,1]$ satisfies condition $L(e_j) = e_j$, $j = 0, 1$, then*

$$L(C_0[0,1]) \subset C_0[0,1].$$

Proof. It is well known that an operator $L : C[0,1] \to C[0,1]$ which satisfies the given condition has the property $L(f)(0) = f(0)$ and $L(f)(1) = f(1)$, for any $f \in C[0,1]$. $\square$

Definition 2.1. *Denote by $\Lambda_0[0,1]$, the class of positive linear operators $L : C[0,1] \to C[0,1]$ which satisfy the following conditions:*

a) $L(e_j) = e_j$, *for* $j = 0, 1$;
b) $\|L\|_{\mathcal{L}(C_0[0,1],C_0[0,1])} < 1$.

Lemma 2.3. *For any $L \in \Lambda_0[0,1]$ we have:*

i) *operator $G_L : C_0[0,1] \to C_0[0,1]$, given in (1.1) is well defined if we consider the convergence with regard to the sup-norm $\|\cdot\|$;*
ii) *operator G_L is positive and linear;*
iii) *$(I - L) \circ G_L = I$, in the Banach space $(C_0[0,1], \|\cdot\|)$;*
iv) *$G_L \circ (I - L) = I$, in the Banach space $(C_0[0,1], \|\cdot\|)$.*

Proof. i) Because the series $\sum_{k=0}^{\infty} \|L^k\|_{\mathcal{L}(C_0[0,1],C_0[0,1])}$ is convergent it follows that for each $f \in C_0[0,1]$, series $\sum_{k=0}^{\infty} L^k(f)$ is convergent in space $(C_0[0,1], \|\cdot\|)$.
Point ii) is obvious. The proof of points iii) and iv) is standard. $\square$

3. The norm of operators G_L

In this section we give estimates of the norm $\|G_L\|_{\mathcal{L}(C_0[0,1],C_0[0,1])}$ for operators $L \in \Lambda_0[0,1]$.
In the next lemma, for $x \in (0,1)$ we make the following conventions. If $t = 1$, then $\int_x^t \frac{t-u}{u(1-u)}\,du = \int_x^1 \frac{du}{u}$ and if $t = 0$, then $\int_x^t \frac{t-u}{u(1-u)}\,du = \int_0^x \frac{du}{1-u}$.

Lemma 3.4. *For all $x \in (0,1)$ and $t \in [0,1]$ we have*

$$0 \le \int_x^t \frac{t-u}{u(1-u)}\,du \le \frac{(t-x)^2}{x(1-x)}. \tag{3.6}$$

Proof. The left inequality is clear. For the second one first we consider that $0 < x \le t \le 1$. For a fixed $t \in [0,1]$ we have

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{t-u}{u(1-u)}\right) = \frac{-u^2 + 2ut - t}{u^2(1-u)^2} \le -\frac{(t-u)^2}{u^2(1-u)^2} \le 0.$$

From this it follows relation (3.6). The case $0 \le t \le x < 1$ can be reduced to the case above. Indeed if we made the chang of variable $u_1 = 1 - u$ and denote $x_1 = 1 - x$, $t_1 = 1 - t$ then we obtain

$$\int_x^t \frac{t-u}{u(1-u)}\,du = \int_{x_1}^{t_1} \frac{t_1 - u_1}{u_1(1-u_1)}\,du_1 \le \frac{(t_1 - x_1)^2}{x_1(1-x_1)} = \frac{(t-x)^2}{x(1-x)}.$$

$\square$

Consider function $\Phi \in C_0[0,1]$, defined by

$$\Phi(x) = x\ln x + (1-x)\ln(1-x),\ x \in (0,1),\ \Phi(0) = 0,\ \Phi(1) = 0. \tag{3.7}$$

Theorem 3.1. *If $L \in \Lambda_0[0,1]$, then*

$$\|G_L\|_{\mathcal{L}(C_0[0,1],C_0[0,1])} \ge \frac{\|\Phi\|}{\alpha_L} = \frac{\ln 2}{\alpha_L}, \tag{3.8}$$

where

$$\alpha_L = \sup_{x\in(0,1)} \frac{L((e_1 - x)^2)(x)}{\psi(x)}. \tag{3.9}$$

Proof. For $x \in (0,1)$ and $t \in [0,1]$, the Taylor formula yields

$$\Phi(t) = \Phi(x) + \Phi'(x)(t-x) + \int_x^t (t-u)\Phi''(u)du.$$

Since $\Phi''(u) = \frac{1}{u(1-u)}$, $u \in (0,1)$, by taking into account Lemma 3.4 we obtain

$$\Phi(t) \quad \leq \quad \Phi(x) + \Phi'(x)(t-x) + \frac{(t-x)^2}{x(1-x)}.$$

Applying operator L we obtain

$$L(\Phi)(x) \leq \Phi(x) + \alpha_L.$$

We use the immediate equality $L((e_1-x)^2)(x) = L(e_2)(x) - e_2(x)$ and the equalities $L(\Phi)(0) - \Phi(0) = 0$ and $L(\Phi)(1) - \Phi(1) = 0$. Since function Φ is convex and L preserves linear functions we have $L(\Phi) - \Phi \geq 0$. From these we deduce that $\frac{1}{\alpha_L}(L(\Phi) - \Phi) \in C_0[0,1]$ and $\left\|(\alpha_L)^{-1}(L(\Phi)-\Phi)\right\| \leq 1$. Therefore

$$\|G_L\|_{\mathcal{L}(C_0[0,1],C_0[0,1])} \geq \|G_L((\alpha_L)^{-1}(L(\Phi)-\Phi))\|.$$

But using Lemma 2.3 - iv) we obtain

$$G_L(L(\Phi)-\Phi) = -\Phi.$$

Consequently we obtain relation (3.8). □

4. Convergence of geometric series of Bernstein operators in the space $C_0[0,1]$

Let B_n, $n \in \mathbb{N}$ be the Bernstein operators. It is clear that $B_n \in \Lambda_0[0,1]$, for any $n \in \mathbb{N}$, see [13]. From Lemma 2.3, G_{B_n} is well defined on space $C_0[0,1]$.

Theorem 4.2. *For $n \in \mathbb{N}$, $n \geq 2$ we have*

$$n \ln 2 \leq \|G_{B_n}\|_{\mathcal{L}(C_0[0,1],C_0[0,1])} \leq 1 + 3n\ln 2. \tag{4.10}$$

Proof. For simplicity let denote $G_n = G_{B_n}$. The left inequality follows from Theorem 3.1, by taking into account that $\alpha_{B_n} = \frac{1}{n}$, for $n \in \mathbb{N}$.
We pass to the right inequality. Let $x \in (0,1)$ we have

$$\Phi''(x) = \frac{1}{x(1-x)}, \quad \Phi^{(3)}(x) = \frac{2x-1}{x^2(1-x)^2}, \quad \Phi^{(4)}(x) = \frac{2(1-3\Psi(x))}{\Psi^3(x)}.$$

Since $\Phi^{(4)} \geq 0$, using the Taylor formula for $x \in (0,1)$, $t \in [0,1]$:

$$\begin{aligned}\Phi(t) &= \sum_{k=0}^{3} \frac{\Phi^{(k)}(x)(t-x)^k}{k!} + \int_x^t \frac{(t-u)^3}{3!}\Phi^{(4)}(u)du \\ &\geq \sum_{k=0}^{3} \frac{\Phi^{(k)}(x)(t-x)^k}{k!}\end{aligned}$$

We have $B_n((e_1 - xe_0)^3)(x) = \frac{1}{n^2}(1-2x)x(1-x)$. Applying operator B_n we obtain:

$$B_n(\Phi)(x) \geq \Phi(x) + \frac{1}{2n} - \frac{1}{6n^2} \cdot \frac{(1-2x)^2}{x(1-x)}.$$

Take here $x = \frac{k}{n}$, $1 \leq k \leq n-1$. We obtain, for $n \geq 2$:

$$\max_{1\leq k\leq n-1} \frac{1}{6n^2} \cdot \frac{\left(1-2\frac{k}{n}\right)^2}{\frac{k}{n}\left(1-\frac{k}{n}\right)} = \frac{1}{6n^2} \cdot \frac{\left(1-\frac{2}{n}\right)^2}{\frac{1}{n}\left(1-\frac{1}{n}\right)} \leq \frac{1}{6n}.$$

Hence

$$B_n(\Phi)\left(\frac{k}{n}\right) - \Phi\left(\frac{k}{n}\right) \geq \frac{1}{3n},\ 1 \leq k \leq n-1. \tag{4.11}$$

Since $G_n = I + G_n \circ B_n$ we obtain

$$\|G_n\|_{\mathcal{L}(C_0[0,1],C_0[0,1])} \leq 1 + \|G_n \circ B_n\|_{\mathcal{L}(C_0[0,1],C_0[0,1])}. \tag{4.12}$$

Fix $f_0 \in C_0[0,1]$ arbitrary such that, $f_0 \geq 0$ and $f_0\left(\frac{k}{n}\right) = 1$, $1 \leq k \leq n-1$. It is easy to see that

$$\|G_n \circ B_n\|_{\mathcal{L}(C_0[0,1],C_0[0,1])} = \|G_n(B_n(f_0))\|. \tag{4.13}$$

From relation (4.11) and since $f_0(0) = 0 = f_0(1)$ and $(B_n(\Phi) - \Phi)(0) = 0 = (B_n(\Phi) - \Phi)(1)$ it follows that

$$f_0\left(\frac{k}{n}\right) \leq 3n\left[B_n(\Phi)\left(\frac{k}{n}\right) - \Phi\left(\frac{k}{n}\right)\right],\ 0 \leq k \leq n$$

and from this we obtain

$$B_n(f_0) \leq 3nB_n(B_n(\Phi) - \Phi).$$

Applying operator G_n to this inequality we arrive to

$$G_n(B_n(f_0)) \leq 3nG_n \circ (B_n - I)(B_n(\Phi)).$$

By tacking into account Lemma 2.3 - iv) we get

$$G_n(B_n(f_0)) \leq -3nB_n(\Phi).$$

Now, since $f_0 \geq 0$ it follows $G_n(B_n(f_0)) \geq 0$ and from the inequality above we obtain

$$\|G_n(B_n(f_0))\| \leq 3n\|B_n(\Phi)\| \tag{4.14}$$

From relations (4.12), (4.13), (4.14) and inequality $\|B_n(\Phi)\| \leq \|\Phi\|$ we deduce relation (4.15). □

Lemma 4.5. *Let F be the operator defined in relation (1.4). We have:*

i) $F\left(\psi^{-1}\right) = -\Phi$.
ii) *If $f \in C[0,1]$, then $F(\psi^{-1}f)$ is well defined and $F(\psi^{-1}f) \in C_0[0,1]$.*

Proof. i) It follows by a simple direct calculus.
ii) Let $x \in (0,1)$. Then $0 \leq F(\psi^{-1}|f|)(x) \leq \|f\|F(\psi^{-1})(x) = -\|f\|\Phi(x) < \infty$. Since $F(\psi^{-1}|f|)$ is well defined it follows that $F(\psi^{-1}f)$ is well defined. Also, from the inequality above it follows that $F(\psi^{-1}f) \in C_0[0,1]$. □

According to notations used in the previous sections we have $A_n = \frac{1}{n}G_{B_n}$, $n \in \mathbb{N}$.

Theorem 4.3. *We have*

$$\lim_{n\to\infty} \|n^{-1}G_{B_n}(f) - 2F(\psi^{-1}f)\| = 0,\ \textit{for all } f \in C_0[0,1]. \tag{4.15}$$

Proof. Let $f \in C_0[0,1]$. Let $\varepsilon > 0$ be arbitrarily chosen. Since the space $\psi C[0,1]$ is dense in $C_0[0,1]$ (Lemma 2.1), we can find $g \in \psi C[0,1]$ such that $\|f - g\| < \varepsilon$. From Corollary A there is $n_\varepsilon \in \mathbb{N}$ such that $\|n^{-1}G_{B_n}(g) - 2F(\psi^{-1}g))\| < \varepsilon$, for $n \geq n_\varepsilon$. Then, for such index n we obtain

$$\begin{aligned}
&\|n^{-1}G_{B_n}(f) - 2F(\psi^{-1}f)\| \\
&\leq \|n^{-1}G_{B_n}(f-g)\| + \|n^{-1}G_{B_n}(g) - 2F(\psi^{-1}g)\| + \|2F(\psi^{-1}(f-g))\| \\
&\leq n^{-1}\|G_{B_n}\|_{\mathcal{L}(C_0[0,1],C_0[0,1])}\|f-g\| + \varepsilon + \|f-g\| \cdot \|2F(\psi^{-1})\| \\
&\leq (n^{-1} + 3\ln 2)\varepsilon + \varepsilon + 2\ln 2 \cdot \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the proof is finished. □

5. Voronovskaya type result

Recall that $A_n = \frac{1}{n}\sum_{k=0}^{\infty}(B_n)^k$, where B_n is the Bernstein operator of order n.

Theorem 5.4. *If $f \in C^4[0,1]$, then*

$$\lim_{n\to\infty} n(A_n(\psi f)(x) - F(f)(x)) = \frac{1}{2}\psi(x)f(x) - \frac{1}{3}F(f)(x), \tag{5.16}$$

uniformly with regard to $x \in [0,1]$.

Proof. Fix $f \in C^4[0,1]$ and denote $F(t) = F(f)(t)$, $t \in [0,1]$. Because $F''(t) = -f(t)$, we have from Taylor formula, for $s, t \in [0,1]$:

$$\begin{aligned} F(s) &= F(t) + F'(t)(s-t) - \frac{1}{2}f(t)(s-t)^2 - \frac{1}{6}f'(t)(s-t)^3 \\ &\quad -\frac{1}{24}f''(t)(s-t)^4 - \frac{1}{120}f'''(t)(s-t)^5 - R_5(t,s), \end{aligned} \tag{5.17}$$

where

$$R_5(t,s) = \frac{1}{5!}\int_t^s (s-u)^5 f^{IV}(u)du.$$

Denote $m_k(t) = B_n((e_1 - t)^k)(t)$, for $k = 0, 1, 2, \ldots$, $t \in [0,1]$. In [9] the following relation is given:

$$m_{s+1}(t) = \frac{\psi(t)}{n}\left[m_s'(t) + s m_{s-1}(t)\right],\ s = 1, 2, \ldots,\ t \in [0,1].$$

we obtain

$$\begin{aligned} m_2(t) &= \frac{1}{n}\psi(t) \\ m_3(t) &= \frac{1}{n^2}\psi(t)\psi'(t), \\ m_4(t) &= \frac{3}{n^2}\psi^2(t) + \frac{1}{n^3}\psi(t)(1-6\psi(t)), \\ m_5(t) &= \frac{10}{n^3}\psi^2(t)\psi'(t) + \frac{1}{n^4}(\psi(t)\psi'(t) - 12\psi^2(t)\psi'(t)), \\ m_6(t) &= \frac{15}{n^3}\psi^3(t) + \frac{1}{n^4}(24\psi^2(t) - 124\psi^3(t)) + \frac{1}{n^5}(\psi(t) - 28\psi^2(t) + 120\psi^3(t)). \end{aligned}$$

Applying operator B_n from relation (5.17) we obtain

$$\begin{aligned} (I - B_n)(F)(t) &= \frac{1}{2}f(t)m_2(t) + \frac{1}{6}f'(t)m_3(t) + \frac{1}{24}f''(t)m_4(t) + \frac{1}{120}f'''(t)m_5(t) \\ &\quad + B_n(R_5(t,\cdot))(t). \end{aligned}$$

Note that $F \in \psi C[0,1]$. Also $m_k \in \psi C[0,1]$, $k = 2,3,4,5$. From the above equality it follows that also $B_n(R_5(t,\cdot))(t) \in \psi C[0,1]$. So that we can apply operator $G_{B_n} = nA_n$ to the tems of the both side of above equality and from Lemma 2.3 - iv), we obtain

$$\begin{aligned} F(x) &= \frac{1}{2}A_n(f\psi)(x) + \frac{n}{6}A_n(f'm_3)(x) + \frac{n}{24}A_n(f''m_4)(x) + \frac{n}{120}A_n(f'''m_5)(x) \\ &\quad + nA_n(B_n(R_5(t,\cdot))(t))(x), \end{aligned}$$

Finally we obtain

$$\begin{aligned} n(A_n(f\psi)(x)-2F(x)) &= -\frac{1}{3}n^2A_n(f'm_3)(x)-\frac{1}{12}n^2A_n(f''m_4)(x)-\frac{n^2}{60}A_n(f'''m_5)(x)\\ &\quad -2n^2A_n(B_n(R_5(t,\cdot))(t))(x), \end{aligned}$$

Using Corollary A we obtain

$$\begin{aligned} -\frac{1}{3}n^2A_n(f'm_3)(x) &= -\frac{1}{3}A_n(f'\psi\psi')(x)\\ &= -\frac{2}{3}F(f'\psi')(x)+o(1);\\ -\frac{1}{12}n^2A_n(f''m_4)(x) &= -\frac{1}{4}A_n(f''\psi^2)(x)-\frac{1}{12n}A_n(f''\psi(1-6\psi))(x)\\ &= -\frac{1}{4}(2F(f''\psi)(x)+o(1))-\frac{1}{12n}(2F(f''(1-6\psi))(x)+o(1))\\ &= -\frac{1}{2}F(f''\psi)(x)+o(1);\\ -\frac{n^2}{60}A_n(f'''m_5)(x) &= -\frac{1}{6n}A_n(f'''\psi^2\psi')(x)-\frac{1}{60n^2}A_n(f'''(\psi\psi'-12\psi^2\psi'))(x)\\ &= -\frac{1}{6n}(2F(f'''\psi\psi')(x)+o(1))-\frac{1}{60n^2}(2F(f'''(\psi'-12\psi\psi'))+o(1))\\ &= o(1). \end{aligned}$$

In all these relations $o(1)$ is uniform with regard to $x\in[0,1]$. Also we have

$$|R_5(t,s)|\le\frac{\|f^{IV}\|}{5!}\int_t^s(s-u)^5du=\frac{(s-t)^6}{6!}\|f^{IV}\|.$$

Therefore

$$B_n(|R_5(t,\cdot)|)(t)\le\frac{1}{6!}m_6(t)\|f^{IV}\|.$$

It follows

$$\begin{aligned} &|-2n^2A_n(B_n(R_5(t,\cdot))(t))(x)|\le\frac{2n^2}{6!}\|f^{IV}\|A_n(m_6)(x)\\ \le\ &\frac{2\|f^{IV}\|}{6!}A_n\left(\frac{15}{n}\psi^3+\frac{1}{n^2}(24\psi^2-124\psi^3)+\frac{1}{n^3}(\psi-28\psi^2+120\psi^3\right)(x)\\ =\ &\frac{2\|f^{IV}\|}{6!}\left[2F\left(\frac{15}{n}\psi^2+\frac{1}{n^2}(24\psi-124\psi^2)+\frac{1}{n^3}(e_0-28\psi+120\psi^2\right)(x)+o(1)\right]\\ =\ &o(1). \end{aligned}$$

From the relation above we coclude that

$$\lim_{n\to\infty}n(A_n(f\psi)(x)-2F(x))=-\frac{2}{3}F(f'\psi')(x)-\frac{1}{2}F(f''\psi)(x). \tag{5.18}$$

Next integrating by parts we obtain

$$\begin{aligned} -\frac{2}{3}F(f'\psi')(x) &= -\frac{2}{3}\left[(1-x)\int_0^x t(1-2t)f'(t)dt+x\int_x^1(1-t)(1-2t)f'(t)dt\right]\\ &= \frac{2}{3}(1-x)\int_0^x(1-4t)f(t)dt+\frac{2}{3}x\int_x^1(4t-3)f(t)dt. \end{aligned}$$

$$\begin{aligned} -\frac{1}{2}F(f''\psi)(x) &= -\frac{1}{2}\left[(1-x)\int_0^x t^2(1-t)f''(t)dt + x\int_x^1 t(1-t)^2 f''(t)dt\right] \\ &= \frac{1}{2}(1-x)\int_0^x (2t-3t^2)f'(t)dt + \frac{1}{2}x\int_x^1 (1-4t+3t^2)f'(t)dt \\ &= \frac{1}{2}f(x)\psi(x) + (1-x)\int_0^x (3t-1)f(t)dt + x\int_x^1 (2-3t)f(t)dt. \end{aligned}$$

Hence

$$-\frac{2}{3}F(f'\psi')(x) - \frac{1}{2}F(f''\psi)(x) = \frac{1}{2}\psi(x)f(x) - \frac{1}{3}\left[(1-x)\int_0^x (1-t)f(t)dt + x\int_x^1 tf(t)dt\right].$$

$\square$

References

[1] U. Abel, *Geometric series of Bernstein-Durrmeyer operators*, East J. on Approx. Vol. 15, No. 4 (2009) 439–450.

[2] U. Abel, M. Ivan, R. Păltănea, *Geometric series of Bernstein operators revisited*, J. Math. Anal. Appl. Vol. 400. No. 1 (2013) 22-24.

[3] U. Abel, M. Ivan, R. Păltănea, *Geometric series of positive linear operators and the inverse Voronovskaya theorem on a compact interval*, J. Approx. Theory Vol. 184 (2014), 163-175.

[4] F. Altomare, S. Diomede, *Asymptotic formulae for positive linear operators: direct and converse results*, Jaen J. Approx. Vol. 2, No. 2 (2010) 255–287.

[5] J. Bustamante, *Bernstein Operators and Their Properties*, Birkhäuser, 2017.

[6] R. A. DeVore, G. G. Lorentz, *Constructive approximation*, Springer, Berlin, 1993.

[7] H Gonska, R. Păltănea, *General Voronovskaja and asymptotic theorems in simultaneous approximation*, Mediterranean J. Math. Vol. 7 (2010) 37-49.

[8] V. Gupta, G. Tachev, *Approximation with Positive Linear Operators and Linear Combinations*, Springer, 2017.

[9] G. G. Lorenz, *Bernstein polynomials*, Univ. Toronto Press, 1953.

[10] R. Păltănea, *Approximation Theory Using Positive Linear Operators*, Birkhäuser, Boston, 2004.

[11] R. Păltănea, *The power series of Bernstein operators*, Automation Computers Applied Mathematics Vol. 15, No. 1 2006, 7-14.

[12] I. Raşa, *Power series of Bernstein operators and approximation resolvents* Mediterr. J. Math. Vol. 9 (2012) 635-644.

[13] I. A. Rus, *Iterates of Bernstein operators, via contraction principle*, J. Math. Anal. Appl. Vol 292, No. 1 (2004) 259-261.

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On Some Bivariate Gauss-Weierstrass Operators

GRAŻYNA KRECH AND IRENEUSZ KRECH

ABSTRACT. The aim of the paper is to investigate the approximation properties of bivariate generalization of Gauss-Weierstrass operators associated with the Riemann-Liouville operator. In particular, the approximation error will be estimated by these operators in the space of functions defined and continuous in the half-plane $(0, \infty) \times \mathbb{R}$, and bounded by certain exponential functions.

Keywords: Gauss-Weierstrass operator, Linear operators, Approximation order

2010 Mathematics Subject Classification: 41A25, 41A36.

1. INTRODUCTION

Numerous issues related to positive linear integral operators were and still are the subject of research. The reason lays with their numerous applications in different domains of mathematics and physics. The classical Gauss-Weierstrass singular integral

$$W(f; x, t) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{4t}\right) f(y)dy, \tag{1.1}$$

has been studied systematically in the past. The integral W is a solution of the heat equation. The details can be found, for example, in [13]. Approximation properties of the operator W were given in many papers and monographs (see, for example, [13, 14, 18]). In [4], Anastassiou and Mezei investigated the smooth Gauss-Weierstrass singular integral operators (not in general positive) over the real line regarding their simultaneous global smoothness preservation property with respect to the L^p norm, by involving higher order moduli of smoothness. Some Lipschitz type results for the smooth Gauss-Weierstrass type singular integral operators were established in [17]. Approximation properties of the classical Gauss-Weierstrass integrals for functions of two variables in exponential weighted space were presented in [11] and a certain modification of these integrals which has a better order of approximation than the classical integrals was investigated in [19]. Khan and Umar (see [16]) gave a generalization of the Gauss-Weierstrass integrals and obtained the rate of convergence of the integral operator. In [5], Aral proposed a definition of the λ-Gauss Weierstrass singular integral with the kernel depending on a nonisotropic distance, its generalization, and gave some approximation properties of these integrals in certain function spaces. In [3], Anastassiou and Duman studied statistical L_p-approximation properties of the double Gauss-Weierstrass singular integral operators which do not need to be positive. Similar issues were also examined in the complex case in note [2]. Recently, various q-generalizations of Gauss-Weierstrass singular integral operators based on q-calculus (see [15]) and their approximation properties were investigated intensively (see, for example, [1, 6, 7, 8]).

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The aim of this paper is to study approximation properties of the family of bivariate Gauss-Weierstrass operators associated with the Riemann-Liouville operator (see [10]). This family is of the form

$$V_\alpha^t(f)(r,x) = V_\alpha(f;r,x,t) = \int_{\mathbb{R}}\int_0^\infty K_\alpha^t(r,x,s,y)f(s,y)dsdy,$$

where the kernel is defined by

$$K_\alpha^t(r,x,s,y) = \frac{(2t)^{-(\alpha+3/2)}}{\sqrt{2\pi}} e^{-\frac{r^2+s^2+(x-y)^2}{4t}} \left(\frac{rs}{2t}\right)^{-\alpha} I_\alpha\left(\frac{rs}{2t}\right) s^{2\alpha+1},$$

for $\alpha \geq -\frac{1}{2}$, $r > 0$, $x \in \mathbb{R}$, $t > 0$, and I_α is a modified Bessel function

$$I_\alpha(z) = \sum_{k=0}^\infty \frac{z^{\alpha+2k}}{2^{\alpha+2k}k!\Gamma(\alpha+k+1)}.$$

In paper [9], the operator V_α is considered for functions belonging to L^p, $1 \leq p \leq \infty$ and S, which is a space of infinitely differentiable functions, rapidly decreasing together with all their derivatives, even with respect to the first variable.
It is known (see [9, Proposition 3.4]) that the operator V_α is a positive linear operator from L^p into itself and for every $f \in L^p$, $1 \leq p \leq \infty$, we have

$$||V_\alpha^t(f)||_{L^p} \leq ||f||_{L^p}.$$

Moreover, for every $1 \leq p < \infty$, the family $(V_\alpha^t)_{t>0}$ is strongly continuous semigroup of operators on L^p and it is called Gauss semigroup associated with the Riemann-Liouville operator.
Armi and Rachdi proved that if $f \in S$, then V_α is a function of the class C^∞ on $(0,\infty)\times\mathbb{R}\times(0,\infty)$ and satisfies the following equations (see [9]):

$$\begin{aligned} &\frac{\partial u(r,x,t)}{\partial t} = \frac{\partial^2 u(r,x,t)}{\partial x^2} + \frac{2\alpha+1}{r}\frac{\partial u(r,x,t)}{\partial r} + \frac{\partial^2 u(r,x,t)}{\partial r^2}, \\ &\lim_{t\to 0^+} V_\alpha(f;r,x,t) = f(r,x) \quad \text{uniformly on} \quad (0,\infty)\times\mathbb{R}. \end{aligned} \tag{1.2}$$

An interesting fact related to the study of the operator V_α is the following remark. If $f(r,x) = f_1(r)f_2(x)$, then

$$V_\alpha(f;r,x,t) = W_\alpha(f_1;r,t)W(f_2;x,t), \tag{1.3}$$

where

$$W_\alpha(f_1;r,t) = \frac{1}{2t}\int_0^\infty r^{-\alpha}s^{\alpha+1}\exp\left(-\frac{r^2+s^2}{4t}\right) I_\alpha\left(\frac{rs}{2t}\right) f_1(s)\,ds$$

and W is defined by (1.1). Note that $W_{-\frac{1}{2}}$ is the classical Gauss-Weierstrass integral (1.1) and

$$W_{-\frac{1}{2}}(f_1;r,t) = \frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}}\exp\left(-\frac{(r-s)^2}{4t}\right)\tilde{f}_1(s)ds,$$

where

$$\tilde{f}_1(s) = \begin{cases} f_1(s) & \text{if } s \geq 0, \\ f_1(-s) & \text{if } s < 0. \end{cases}$$

It is worth mentioning that for $f(s) = s^{2k}$, $k \in \mathbb{N}$, the function $W_\alpha(f)$ is a polynomial called radial heat polynomial [12].
Some properties of the operator W_α, in particular, an estimation of the rate of convergence, were studied in [20].

In this work, we will investigate approximation properties of V_α in the space E_K, $K \geq 0$, consisting of all continuous functions f defined on the half-plane $(0,\infty)\times\mathbb{R}$, and such that

$$|f(r,x)| \leq Me^{K(r^2+x^2)}$$

for some $M>0$. The norm in E_K is given by

$$||f||_{E_K} = \sup_{(r,x)\in D} e^{-K(r^2+x^2)}|f(r,x)|,$$

where $D=\{(r,x) : r>0, x\in\mathbb{R}\}$. Observe that if $0\leq K_1\leq K_2$, then $E_{K_1}\subset E_{K_2}$ and $\|f\|_{K_2}\leq\|f\|_{K_1}$.
We shall prove that the operator V_α is bounded and maps E_K into $E_{K+\delta}$, where $\delta>0$. Moreover, we shall estimate an order of approximation by this operator.

2. Approximation properties

Applying the method used in [20], we can prove the following theorem.

Theorem 2.1. *Let $f\in E_K$.*

(a) *The function $V_\alpha(f)$ is of the class C^∞ in the set*

$$\Omega=\left\{(r,x,t); r>0, x\in\mathbb{R}, 0<t<\frac{1}{4K}\right\}$$

(if $K=0$, then $0<t<\infty$).

(b) *The function $V_\alpha(f)$ is a solution of the equation (1.2) in Ω and*

$$\lim_{(r,x,t)\to(r_0,x_0,0^+)} V_\alpha(f;r,x,t)=f(r_0,x_0)$$

for every $(r_0,x_0)\in\Omega$. Moreover, we have

$$\lim_{t\to 0^+} V_\alpha(f;r,x,t)=f(r,x)$$

in every closed subset in Ω.

In what follows, it will be useful to consider the functions:

$$\psi_{0,0}(r,x)=e^{K(r^2+x^2)}, \quad \psi_{0,i}(r,x)=x^ie^{K(r^2+x^2)},$$
$$\psi_{i,0}(r,x)=r^{2i}e^{K(r^2+x^2)} \quad \text{for} \quad i=1,2.$$

Using (see [20])

$$\int_0^\infty s^{\alpha+2b+1}\exp\left(-as^2\right)I_\alpha(\beta s)\,ds=\sum_{k=0}^\infty\frac{\beta^{\alpha+2k}\Gamma(\alpha+k+b+1)}{k!\Gamma(\alpha+k+1)a^{\alpha+k+b+1}2^{\alpha+2k+1}},$$

$\alpha\geq-\frac{1}{2}$, $b\geq 0$, $a>0$, $\beta>0$ and the equation (1.3), we have the following lemma.

Lemma 2.1. *Let $I=\left(0,\frac{1}{4K}\right)$ for $K>0$ and $I=(0,\infty)$ for $K=0$. For $t\in I$, we have*

$$\begin{aligned}
V_\alpha(\psi_{0,0};r,x,t) &= A,\\
V_\alpha(\psi_{0,1};r,x,t) &= Ax(1-4Kt)^{-1},\\
V_\alpha(\psi_{0,2};r,x,t) &= A\left[2x^2(1-4Kt)^{-2}+2t(1-4Kt)^{-1}\right],\\
V_\alpha(\psi_{1,0};r,x,t) &= A\left[r^2(1-4Kt)^{-2}+4t(\alpha+1)(1-4Kt)^{-1}\right],\\
V_\alpha(\psi_{2,0};r,x,t) &= A\left[r^4(1-4Kt)^{-4}+8tr^2(\alpha+2)(1-4Kt)^{-3}\right.\\
&\quad\left.+16t^2(\alpha+2)(\alpha+1)(1-4Kt)^{-2}\right],
\end{aligned}$$

where $A = (1-4Kt)^{-(\alpha+\frac{3}{2})} e^{\frac{K(r^2+x^2)}{1-4Kt}}$.

Theorem 2.2. *Let* $f \in E_K$*. If* $K > 0$*, then for every* $\delta > 0$ *and* $t \in (0, \frac{\delta}{4K(K+\delta)})$*, the operator* V_α *maps the space* E_K *in* $E_{K+\delta}$ *and*

$$\left\|V_\alpha^t(f)\right\|_{K+\delta} \le \left(1+\frac{\delta}{K}\right)^{\alpha+\frac{3}{2}} \|f\|_K. \tag{2.4}$$

If $K = 0$*, then* V_α *maps the space* E_0 *into itself and*

$$\left\|V_\alpha^t(f)\right\|_0 \le \|f\|_0. \tag{2.5}$$

Proof. By the positivity and linearity of V_α, we get

$$|V_\alpha(f;r,x,t)| \le V_\alpha(|f|;r,x,t) \le \|f\|_K V_\alpha(\psi_{0,0};r,x,t) = A\|f\|_K.$$

From above we have (2.5) for $K = 0$.
Let $K > 0$. If $\delta > 0$ and $t \in (0, \frac{\delta}{4K(K+\delta)})$, then $\frac{K}{1-4Kt} < K+\delta$. Hence

$$\begin{aligned}
\|V_\alpha^t\|_{K+\delta} &= \sup_{(r,x)\in D} e^{-(K+\delta)(r^2+x^2)}|V_\alpha(f;r,x,t)| \\
&\le \sup_{(r,x)\in D} e^{-\frac{K}{1-4Kt}(r^2+x^2)}|V_\alpha(f;r,x,t)| \\
&\le (1-4K)^{-(\alpha+\frac{3}{2})}\|f\|_K \le \left(1+\frac{\delta}{K}\right)^{\alpha+\frac{3}{2}} \|f\|_K,
\end{aligned}$$

which gives (2.4). □

3. Rate of convergence

In this section, we shall state an estimate of the rate of convergence of the integral V_α in terms of the modulus of continuity.
Let $\delta > 0$ and

$$\omega(f;E_K,\delta) = \sup_{\sqrt{(s-r)^2+(y-x)^2}\le\delta} |f(s,y)-f(r,x)|e^{-K(s^2+y^2)}, \quad K \ge 0.$$

Observe that

$$\omega(f;E_K,\delta_1) \le \omega(f;E_K,\delta_2) \quad \text{for} \quad 0 < \delta_1 \le \delta_2,$$

$$\omega(f;E_K,\lambda\delta) \le (1+\lambda)\omega(f;E_K,\delta) \quad \text{for} \quad \lambda > 0.$$

Theorem 3.3. *Let* $f \in E_K$*,* $K \ge 0$ *and* $A = (1-4Kt)^{-(\alpha+\frac{3}{2})} e^{\frac{K(r^2+x^2)}{1-4Kt}}$*. We have*

$$|V_\alpha(f;r,x,t) - f(r,x)| \le 2A\omega(f;E_K,\delta),$$

where

$$\begin{aligned}
\delta \;=\; &\Big\{x^2 - 2x^2(1-4Kt)^{-1} + x^2(1-4Kt)^{-2} + 2t(1-4Kt)^{-1} \\
&+ \big[r^4 - 2r^4(1-4Kt)^{-2} + r^4(1-4Kt)^{-4} - 8tr^2(\alpha+1)(1-4Kt)^{-1} \\
&+ 8tr^2(\alpha+2)(1-4Kt)^{-3} + 16t^2(\alpha+2)(\alpha+1)(1-4Kt)^{-2}\big]^{1/2}\Big\}^{1/2}
\end{aligned}$$

for $r > 0$*,* $x \in \mathbb{R}$*,* $0 < t < \frac{1}{4K}$ *and* $K > 0$.

If $K = 0$, *we have*

$$|V_\alpha(f; r, x, t) - f(r, x)| \leq 2\omega \left(f; E_0, \sqrt{2t + \sqrt{8tr^2 + 16t^2(\alpha + 2)(\alpha + 1)}} \right)$$

for $r > 0$, $x \in \mathbb{R}$, $t > 0$.

Proof. Let $\delta > 0$. Using the property of the modulus of continuity, we obtain

$$|f(s, y) - f(r, x)| \leq e^{K(s^2+y^2)} \omega \left(f; E_K, \sqrt{(s - r)^2 + (y - x)^2} \right)$$

for $f \in E_K$. From this, we get

$$\begin{aligned} &|f(s, y) - f(r, x)| \\ &\quad \leq \quad e^{K(s^2+y^2)} \left(1 + \frac{\sqrt{(s - r)^2 + (y - x)^2}}{\delta} \right) \omega (f; E_K, \delta) \\ &\quad \leq \quad e^{K(s^2+y^2)} \left(1 + \frac{(s - r)^2 + (y - x)^2}{\delta^2} \right) \omega (f; E_K, \delta) . \end{aligned}$$

In view of $(s - r)^2 \leq |s^2 - r^2|$, we can write

$$|f(s, y) - f(r, x)| \leq e^{K(s^2+y^2)} \left(1 + \frac{|s^2 - r^2| + (y - x)^2}{\delta^2} \right) \omega (f; E_K, \delta) .$$

The operator V_α is positive and linear (see also [9]), so

$$\begin{aligned} &|V_\alpha(f; r, x, t) - f(r, x)| \\ &\quad \leq \quad V_\alpha(|f - f(r, x)|; r, x, t) \\ &\quad \leq \quad \omega (f; E_K, \delta) \, V_\alpha \left(\psi_{0,0} + \frac{x^2\psi_{0,0} - 2x\psi_{0,1} + \psi_{0,2} + \phi\psi_{0,0}}{\delta^2}; r, x, t \right), \end{aligned}$$

where $\phi(s, y) = |s^2 - r^2|$. Observe that

$$\begin{aligned} V_\alpha (\phi\psi_{0,0}; r, x, t) \quad &\leq \quad \left\{ V_\alpha(\psi_{0,0}; r, x, t) V_\alpha(\phi^2\psi_{0,0}; r, x, t) \right\}^{1/2} \\ &= \quad \left\{ V_\alpha(\psi_{0,0}; r, x, t) \left[r^4 V_\alpha(\psi_{0,0}; r, x, t) \right. \right. \\ &\qquad \left. \left. - \; 2r^2 V_\alpha(\psi_{1,0}; r, x, t) + V_\alpha(\psi_{2,0}; r, x, t) \right] \right\}^{1/2} . \end{aligned}$$

Hence

$$\begin{aligned} &|V_\alpha(f; r, x, t) - f(r, x)| \\ &\quad \leq \quad \omega (f; E_K, \delta) \Big\{ V_\alpha (\psi_{0,0}; r, x, t) \\ &\qquad + \; \frac{1}{\delta^2} \left[x^2 V_\alpha (\psi_{0,0}; r, x, t) - 2x V_\alpha (\psi_{0,1}; r, x, t) + V_\alpha (\psi_{0,2}; r, x, t) \right] \\ &\qquad + \; \frac{1}{\delta^2} \left[V_\alpha(\psi_{0,0}; r, x, t) \left(r^4 V_\alpha(\psi_{0,0}; r, x, t) - 2r^2 V_\alpha(\psi_{1,0}; r, x, t) \right. \right. \\ &\qquad + \; \left. \left. V_\alpha(\psi_{2,0}; r, x, t) \right) \right]^{1/2} \Big\} . \end{aligned}$$

If $K = 0$, then from Lemma 2.1, we have

$$\begin{aligned} V_\alpha\left(\psi_{0,0}; r, x, t\right) &= 1, \\ V_\alpha\left(\psi_{0,1}; r, x, t\right) &= x, \\ V_\alpha\left(\psi_{0,2}; r, x, t\right) &= 2x^2 + 2t, \\ V_\alpha\left(\psi_{1,0}; r, x, t\right) &= r^2 + 4t(\alpha+1), \\ V_\alpha\left(\psi_{2,0}; r, x, t\right) &= r^4 + 8tr^2(\alpha+2) + 16t^2(\alpha+2)(\alpha+1). \end{aligned}$$

Hence, we conclude

$$|V_\alpha(f; r, x, t) - f(r, x)| \le 2\omega\left(f; E_0, \sqrt{2t + \sqrt{8tr^2 + 16t^2(\alpha+2)(\alpha+1)}}\right)$$

for $r > 0$, $x \in \mathbb{R}$, $t > 0$.
For $K > 0$, we obtain from Lemma 2.1 the following estimation

$$\begin{aligned} &|V_\alpha(f; r, x, t) - f(r, x)| \\ &\le A\omega\left(f; E_K, \delta\right) \\ &\quad \times \Big\{ 1 + \frac{1}{\delta^2}\left[x^2 - 2x^2(1-4Kt)^{-1} + x^2(1-4Kt)^{-2} + 2t(1-4Kt)^{-1}\right] \\ &\quad + \frac{1}{\delta^2}\big[r^4 - 2r^4(1-4Kt)^{-2} + r^4(1-4Kt)^{-4} - 8tr^2(\alpha+1)(1-4Kt)^{-1} \\ &\quad + 8tr^2(\alpha+2)(1-4Kt)^{-3} + 16t^2(\alpha+2)(\alpha+1)(1-4Kt)^{-2}\big]^{1/2} \Big\}. \end{aligned}$$

Setting

$$\begin{aligned} \delta = \; &\Big\{ x^2 - 2x^2(1-4Kt)^{-1} + x^2(1-4Kt)^{-2} + 2t(1-4Kt)^{-1} \\ &+ \big[r^4 - 2r^4(1-4Kt)^{-2} + r^4(1-4Kt)^{-4} - 8tr^2(\alpha+1)(1-4Kt)^{-1} \\ &+ 8tr^2(\alpha+2)(1-4Kt)^{-3} + 16t^2(\alpha+2)(\alpha+1)(1-4Kt)^{-2}\big]^{1/2} \Big\}^{1/2}, \end{aligned}$$

we get the assertion. □

References

[1] G. A. Anastassiou and A. Aral: *On Gauss-Weierstrass type integral operators*. Demonstratio Math. 43(4) (2010), 841–849.

[2] G. A. Anastassiou and O. Duman: *Statistical approximation by double complex Gauss-Weierstrass integral operators*. Appl. Math. Letters 24(4) (2011), 438–443.

[3] G. A. Anastassiou and O. Duman: *Statistical L_p-approximation by double Gauss-Weierstrass singular integral operators*. Comput. Math. Appl. 59(6) (2010), 1985–1999.

[4] G. A. Anastassiou and R. A. Mezei: *Global smoothness and uniform convergence of smooth Gauss-Weierstrass singular operators*. Math. Comput. Modelling 50(7-8) (2009), 984–998.

[5] A. Aral: *On a generalized λ-Gauss Weierstrass singular integral*. Fasc. Math. 35 (2005), 23–33.

[6] A. Aral: *On the generalized Picard and Gauss Weierstrass singular integrals*, J. Comput. Anal. Appl. 8(3) (2006), 246–261.

[7] A. Aral: *Pointwise approximation by the generalization of Picard and Gauss-Weierstrass singular integrals*. J. Concr. Appl. Math. 6 (2008), 327–339.

[8] A. Aral and S. G. Gal: *q-generalizations of the Picard and Gauss-Weierstrass singular integrals*. Taiwanese J. Math. 12(9) (2008), 2501–2515.

[9] B. Armi and L. T. Rachdi: *The Littlewood-Paley g-function associated with the Riemann-Liouville operator*. Ann. Univ. Paedagog. Crac. Stud. Math. 12 (2013), 31–58.

[10] C. Baccar, N. B. Hamadi and L. T. Rachdi: *Inversion formulas for Riemann-Liouville transform and its dual associated with singular partial differential operators*. Int. J. Math. Math. Sci. (2006), Art. ID 86238, 26.

[11] K. Bogalska, E. Gojka, M. Grudek and L. Rempulska: *The Picard and the Gauss-Weierstrass singular integrals of function of two variables*. Le Mathematiche LII (1997), 71–85.
[12] L. R. Bragg: *The radial heat polynomials and related functions*. Trans. Amer. Math. Soc. 119 (1965), 270–290.
[13] P. L. Butzer and R. J. Nessel: *Fourier Analysis and Approximation*. Vol 1, Birkhauser, Basel and Academic Press, New York 1971.
[14] B. Firlej and L. Rempulska: *On some singular integrals in Hölder spaces*. Mat. Nachr. 170 (1994), 93–100.
[15] F. H. Jackson: *On a q-definite integrals*. Quart. J. Pure Appl. Math. 41 (1910), 193–203.
[16] A. Khan and S. Umar: *On the order of approximation to a function by generalized Gauss-Weierstrass singular integrals*. Commun. Fac. Sci. Univ. Ank., Series A1 30 (1981), 55–62.
[17] R. A. Mezei: *Applications and Lipschitz results of approximation by smooth Picard and Gauss-Weierstrass type singular integrals*. Cubo 13(3) (2011), 17–48.
[18] R. N. Mohapatra and R. S. Rodriguez: *On the rate of convergence of singular integrals for Hölder continuous functions*. Math. Nachr. 149 (1990), 117–124.
[19] L. Rempulska and Z. Walczak: *On modified Picard and Gauss-Weierstrass singular integrals*. Ukrainian Math. J. 57(11) (2005), 1844-1852.
[20] E. Wachnicki: *On a Gauss-Weierstrass generalized integral*. Rocznik Naukowo-Dydaktyczny Akademii Pedagogicznej w Krakowie, Prace Matematyczne 17 (2000), 251–263.

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General Multivariate Iyengar Type Inequalities

GEORGE A. ANASTASSIOU

ABSTRACT. Here we give a variety of general multivariate Iyengar type inequalities for not necessarily radial functions defined on the shell and ball. Our approach is based on the polar coordinates in $\mathbb{R}^N$, $N \geq 2$, and the related multivariate polar integration formula. Via this method we transfer well-known univariate Iyengar type inequalities and univariate author's related results into general multivariate Iyengar inequalities.

Keywords: Iyengar inequality, Polar coordinates, Not necessarily radial function, Shell, Ball.

2010 Mathematics Subject Classification: 26D10, 26D15.

1. BACKGROUND

In the year 1938, Iyengar [5] proved the following interesting inequality:

Theorem 1.1. *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leq M_1$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leq \frac{M_1 (b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M_1}. \tag{1.1}$$

In 2001, X.-L. Cheng [4] proved that

Theorem 1.2. *Let $f \in C^2([a,b])$ and $|f''(x)| \leq M_2$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2 (f'(b)-f'(a)) \right| \tag{1.2}$$

$$\leq \frac{M_2}{24}(b-a)^3 - \frac{(b-a)}{16M_2}\Delta_1^2,$$

where

$$\Delta_1 = f'(a) - \frac{2(f(b)-f(a))}{(b-a)} + f'(b).$$

In 1996, Agarwal and Dragomir [1] obtained a generalization of (1.1):

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that for all $x \in [a,b]$ with $M > m$ we have $m \leq f'(x) \leq M$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right|$$

$$\leq \frac{(f(b)-f(a)-m(b-a))(M(b-a)-f(b)+f(a))}{2(M-m)}.$$

In [7], Qi proved the following:

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Theorem 1.4. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function such that for all $x\in[a,b]$ with $M>0$ we have $|f''(x)|\le M$. Then*

$$\left|\int_a^b f(x)\,dx-\frac{(f(a)+f(b))}{2}(b-a)+\frac{(1+Q^2)}{8}(f'(b)-f'(a))(b-a)^2\right|$$
$$\le\quad \frac{M(b-a)^3}{24}\left(1-3Q^2\right),$$

where

$$Q^2=\frac{\left(f'(a)+f'(b)-2\left(\frac{f(b)-f(a)}{b-a}\right)\right)^2}{M^2(b-a)^2-(f'(b)-f'(a))^2}.$$

In 2005, Zheng Liu, [6], proved the following:

Theorem 1.5. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that f' is integrable on $[a,b]$ and for all $x\in[a,b]$ with $M>m$ we have*

$$m\le\frac{f'(x)-f'(a)}{x-a}\le M\ \text{ and } m\le\frac{f'(b)-f'(x)}{b-x}\le M.$$

Then

$$\left|\int_a^b f(x)\,dx-\frac{(f(a)+f(b))}{2}(b-a)+\left(\frac{1+P^2}{8}\right)(f'(b)-f'(a))(b-a)^2\right.$$
$$\left.-\left(\frac{1+3P^2}{48}\right)(m+M)(b-a)^3\right|\le\frac{(M-m)(b-a)^3}{48}\left(1-3P^2\right),$$

where

$$P^2=\frac{\left(f'(a)+f'(b)-2\left(\frac{f(b)-f(a)}{b-a}\right)\right)^2}{\left(\frac{M-m}{2}\right)^2(b-a)^2-\left(f'(b)-f'(a)-\left(\frac{m+M}{2}\right)(b-a)\right)^2}.$$

Next we list some author's related results, (here $L_\infty([a,b])$ is the normed space of essentially bounded functions over $[a,b]$):

Theorem 1.6. ([3]) *Let $n\in\mathbb{N}$, $f\in AC^n([a,b])$ (i.e. $f^{(n-1)}\in AC([a,b])$, absolutely continuous functions). We assume that $f^{(n)}\in L_\infty([a,b])$. Then*

(i)

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}+(-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right| \tag{1.3}$$
$$\le\quad\frac{\left\|f^{(n)}\right\|_{L_\infty([a,b])}}{(n+1)!}\left[(t-a)^{n+1}+(b-t)^{n+1}\right],$$

for all $t\in[a,b]$,

(ii) at $t=\frac{a+b}{2}$, the right hand side of (1.3) is minimized, and we get:

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|$$
$$\le\quad\frac{\left\|f^{(n)}\right\|_{L_\infty([a,b])}}{(n+1)!}\frac{(b-a)^{n+1}}{2^n},$$

(iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$ *for all* $k = 0, 1, ..., n-1$, *then we obtain*

$$\left| \int_a^b f(x)\,dx \right| \le \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \tag{1.4}$$

$$\le \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \left(\frac{b-a}{N} \right)^{n+1} \left[j^{n+1} + (N-j)^{n+1} \right],$$

(v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, *from (1.4) we get:*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \tag{1.5}$$

$$\le \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \left(\frac{b-a}{N} \right)^{n+1} \left[j^{n+1} + (N-j)^{n+1} \right]$$

for $j = 0, 1, 2, ..., N \in \mathbb{N}$,

(vi) when $N = 2$ *and* $j = 1$, *(1.5) turns to*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \le \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}, \tag{1.6}$$

(vii) when $n = 1$ *(without any boundary conditions), we get from (1.6) that*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \le \|f'\|_{[a,b],\infty} \frac{(b-a)^2}{4},$$

a similar to Iyengar inequality (1.1).

We mention here $L_1([a,b])$ is the normed space of integrable functions over $[a,b]$).

Theorem 1.7. *([3]) Let* $f \in AC^n([a,b])$, $n \in \mathbb{N}$. *Then*

(i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \tag{1.7}$$

$$\le \frac{\left\| f^{(n)} \right\|_{L_1([a,b])}}{n!} \left[(t-a)^n + (b-t)^n \right],$$

for all $t \in [a,b]$,

(ii) at $t = \frac{a+b}{2}$, *the right hand side of (1.7) is minimized, and we get:*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right|$$

$$\le \frac{\left\| f^{(n)} \right\|_{L_1([a,b])}}{n!} \frac{(b-a)^n}{2^{n-1}},$$

(iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, ..., n-1$, we obtain

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{n!}\frac{(b-a)^n}{2^{n-1}},$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[j^{k+1}f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b)\right]\right| \tag{1.8}$$

$$\leq \frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{n!}\left(\frac{b-a}{N}\right)^n \left[j^n + (N-j)^n\right],$$

(v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, from (1.8) we get:

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{N}\right)\left[jf(a) + (N-j)f(b)\right]\right| \tag{1.9}$$

$$\leq \frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{n!}\left(\frac{b-a}{N}\right)^n \left[j^n + (N-j)^n\right],$$

for $j = 0, 1, 2, ..., N \in \mathbb{N}$,

(vi) when $N = 2$ and $j = 1$, (1.9) turns to

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)}{2}\left(f(a) + f(b)\right)\right| \leq \frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{n!}\frac{(b-a)^n}{2^{n-1}}, \tag{1.10}$$

(vii) when $n = 1$ (without any boundary conditions), we get from (1.10) that

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)\left(f(a) + f(b)\right)\right| \leq \left\|f'\right\|_{L_1([a,b])}(b-a).$$

We mention here $L_q([a,b])$ is the normed space of functions f such that $|f|^q$ is integrable over $[a,b]$)

Theorem 1.8. *([3]) Let $f \in AC^n([a,b])$, $n \in \mathbb{N}$; $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and $f^{(n)} \in L_q([a,b])$. Then*

(i)

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right| \tag{1.11}$$

$$\leq \frac{\left\|f^{(n)}\right\|_{L_q([a,b])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}}\left[(t-a)^{n+\frac{1}{p}} + (b-t)^{n+\frac{1}{p}}\right],$$

for all $t \in [a,b]$,

(ii) *at* $t = \frac{a+b}{2}$, *the right hand side of* (1.11) *is minimized, and we get:*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n+\frac{1}{p}\right) (p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}},$$

(iii) *if* $f^{(k)}(a) = f^{(k)}(b) = 0$, *for all* $k = 0, 1, ..., n-1$, *we obtain*

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n+\frac{1}{p}\right) (p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}},$$

which is a sharp inequality,

(iv) *more generally, for* $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N}\right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n+\frac{1}{p}\right) (p(n-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N}\right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}} \right], \tag{1.12}$$

(v) *if* $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, *from* (1.12) *we get:*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N}\right) [j f(a) + (N-j) f(b)] \right| \leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n+\frac{1}{p}\right) (p(n-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N}\right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}} \right], \tag{1.13}$$

for $j = 0, 1, 2, ..., N \in \mathbb{N}$,

(vi) *when* $N = 2$ *and* $j = 1$, (1.13) *turns to*

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{2} (f(a) + f(b)) \right| \leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n+\frac{1}{p}\right) (p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}}, \tag{1.14}$$

(vii) *when* $n = 1$ *(without any boundary conditions), we get from* (1.14) *that*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right) (f(a) + f(b)) \right| \leq \frac{\| f' \|_{L_q([a,b])}}{\left(1+\frac{1}{p}\right)} \frac{(b-a)^{1+\frac{1}{p}}}{2^{\frac{1}{p}}}.$$

We need

Remark 1.1. *We define the ball* $B(0, R) = \left\{ x \in \mathbb{R}^N : |x| < R \right\} \subseteq \mathbb{R}^N$, $N \geq 2$, $R > 0$, *and the sphere*

$$S^{N-1} := \left\{ x \in \mathbb{R}^N : |x| = 1 \right\},$$

where $|\cdot|$ *is the Euclidean norm. Let* $d\omega$ *be the element of surface measure on* S^{N-1} *and*

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$

is the area of S^{N-1}.
For $x \in \mathbb{R}^N - \{0\}$ *we can write uniquely* $x = r\omega$, *where* $r = |x| > 0$ *and* $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$.
Note that $\int_{B(0,R)} dy = \frac{\omega_N R^N}{N}$ *is the Lebesgue measure on the ball, that is the volume of* $B(0,R)$, *which exactly is* $Vol(B(0,R)) = \frac{\pi^{\frac{N}{2}} R^N}{\Gamma\left(\frac{N}{2}+1\right)}$.
Following [8, pp. 149-150, exercise 6], *and* [9, pp. 87-88, Theorem 5.2.2] *we can write for* $F : \overline{B(0,R)} \to \mathbb{R}$ *a Lebesgue integrable function that*

$$\int_{B(0,R)} F(x)\,dx = \int_{S^{N-1}} \left(\int_0^R F(r\omega)\, r^{N-1} dr \right) d\omega, \tag{1.15}$$

and we use this formula a lot.
Typically here the function $f : \overline{B(0,R)} \to \mathbb{R}$ *is not radial. A radial function* f *is such that there exists a function* g *with* $f(x) = g(r)$, *where* $r = |x|$, $r \in [0,R]$, *for all* $x \in \overline{B(0,R)}$.

We need

Remark 1.2. *Let the spherical shell* $A := B(0,R_2) - \overline{B(0,R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$, $x \in \overline{A}$. *Consider that* $f : \overline{A} \to \mathbb{R}$ *is not radial. A radial function* f *is such that there exists a function* g *with* $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, *for all* $x \in \overline{A}$. *Here* x *can be written uniquely as* $x = r\omega$, *where* $r = |x| > 0$ *and* $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$, *see ([8], p. 149-150 and [2], p. 421), furthermore for* $F : \overline{A} \to \mathbb{R}$ *a Lebesgue integrable function we have that*

$$\int_A F(x)\,dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega)\, r^{N-1} dr \right) d\omega. \tag{1.16}$$

Here

$$Vol(A) = \frac{\omega_N \left(R_2^N - R_1^N\right)}{N} = \frac{\pi^{\frac{N}{2}} \left(R_2^N - R_1^N\right)}{\Gamma\left(\frac{N}{2}+1\right)}.$$

In this article we derive general multivariate Iyengar type inequalities on the shell and ball of $\mathbb{R}^N$, $N \geq 2$, for not necessarily radial functions. Our results are based on Theorems 1.1-1.8.

2. Main Results

We present the following non-radial multivariate Iyengar type inequalities:
We start with

Theorem 2.9. *Let the spherical shell* $A := B(0,R_2) - \overline{B(0,R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$. *Consider* $f : \overline{A} \to \mathbb{R}$ *that is not necessarily radial, and that* $f \in C^1\left(\overline{A}\right)$. *Assume that* $\left|\frac{\partial f(s\omega)}{\partial s}\right| \leq M_1$, *for all* $s \in [R_1, R_2]$, *and for all* $\omega \in S^{N-1}$, *where* $M_1 > 0$.
Then

$$\left| \int_A f(y)\,dy - \frac{(R_2 - R_1)}{2} \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega)\,d\omega \right) \right|$$

$$\leq \frac{M_1 \pi^{\frac{N}{2}} (R_2 - R_1)^2}{2\Gamma\left(\frac{N}{2}\right)} - \frac{\int_{S^{N-1}} \left(f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1} \right)^2 d\omega}{4M_1}.$$

Proof. Here $f(s\omega)s^{N-1} \in C^1([R_1, R_2])$, $N \geq 2$, for all $\omega \in S^{N-1}$. By (1.1) we get

$$\begin{aligned} & \left| \int_{R_1}^{R_2} f(s\omega)s^{N-1}ds - \frac{1}{2}(R_2 - R_1)\left(f(R_1\omega)R_1^{N-1} + f(R_2\omega)R_2^{N-1}\right) \right| \\ \leq & \frac{M_1(R_2-R_1)^2}{4} - \frac{\left(f(R_2\omega)R_2^{N-1} - f(R_1\omega)R_1^{N-1}\right)^2}{4M_1} =: \lambda_1(\omega), \end{aligned}$$

for all $\omega \in S^{N-1}$.
Equivalently, we have

$$-\lambda_1(\omega) \leq \int_{R_1}^{R_2} f(s\omega)s^{N-1}ds - \frac{1}{2}(R_2-R_1)\left(f(R_1\omega)R_1^{N-1} + f(R_2\omega)R_2^{N-1}\right) \leq \lambda_1(\omega),$$

for all $\omega \in S^{N-1}$.
Hence it holds

$$\begin{aligned} - & \int_{S^{N-1}} \lambda_1(\omega)d\omega \leq \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(s\omega)s^{N-1}ds \right) d\omega \\ - & \frac{1}{2}(R_2-R_1)\left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega)d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega)d\omega \right) \\ \leq & \int_{S^{N-1}} \lambda_1(\omega)d\omega. \end{aligned}$$

That is (by (1.16))

$$\begin{aligned} - & \left[\frac{\pi^{\frac{N}{2}} M_1(R_2-R_1)^2}{2\Gamma\left(\frac{N}{2}\right)} - \frac{\int_{S^{N-1}} \left(f(R_2\omega)R_2^{N-1} - f(R_1\omega)R_1^{N-1}\right)^2 d\omega}{4M_1} \right] \\ \leq & \int_A f(y)dy - \frac{(R_2-R_1)}{2}\left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega)d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega)d\omega \right) \\ \leq & \frac{\pi^{\frac{N}{2}} M_1(R_2-R_1)^2}{2\Gamma\left(\frac{N}{2}\right)} - \frac{\int_{S^{N-1}} \left(f(R_2\omega)R_2^{N-1} - f(R_1\omega)R_1^{N-1}\right)^2 d\omega}{4M_1}, \end{aligned}$$

proving the claim. □

We continue with

Theorem 2.10. *Let the spherical shell $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$. Consider $f : \overline{A} \to \mathbb{R}$ that is not necessarily radial, and that $f \in C^2(\overline{A})$. Assume that $\left|\frac{\partial^2 f(s\omega)}{\partial s^2}\right| \leq M_2$, for all $s \in [R_1, R_2]$, and for all $\omega \in S^{N-1}$, where $M_2 > 0$.*
Set

$$\begin{aligned} \Delta_1(\omega) := & \left(f(s\omega)s^{N-1}\right)'(R_1) - \frac{2\left(f(R_2\omega)R_2^{N-1} - f(R_1\omega)R_1^{N-1}\right)}{R_2 - R_1} \\ + & \left(f(s\omega)s^{N-1}\right)'(R_2), \quad \forall\, \omega \in S^{N-1}. \end{aligned}$$

Then

$$\begin{aligned}
&\left|\int_A f(y)\,dy - \frac{(R_2-R_1)}{2}\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right)\right. \\
&+ \left.\frac{(R_2-R_1)^2}{8}\left[\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)'(R_2)\,d\omega - \int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)'(R_1)\,d\omega\right]\right| \\
&\le \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{M_2}{12}(R_2-R_1)^3 - \frac{(R_2-R_1)}{16M_2}\int_{S^{N-1}} \Delta_1^2(\omega)\,d\omega.
\end{aligned}$$

Proof. Here $f(s\omega)s^{N-1} \in C^2([R_1,R_2])$, $N \ge 2$, for all $\omega \in S^{N-1}$. By (1.2) we get

$$\begin{aligned}
&\left|\int_{R_1}^{R_2} f(s\omega)s^{N-1}ds - \frac{1}{2}(R_2-R_1)\left(f(R_1\omega)R_1^{N-1} + f(R_2\omega)R_2^{N-1}\right)\right. \\
&+ \left.\frac{1}{8}(R_2-R_1)^2\left(\left(f(s\omega)s^{N-1}\right)'(R_2) - \left(f(s\omega)s^{N-1}\right)'(R_1)\right)\right| \\
&\le \frac{M_2}{24}(R_2-R_1)^3 - \frac{(R_2-R_1)}{16M_2}\Delta_1^2(\omega) =: \lambda_2(\omega),
\end{aligned}$$

for all $\omega \in S^{N-1}$.
Equivalently, we have

$$\begin{aligned}
&- \lambda_2(\omega) \le \int_{R_1}^{R_2} f(s\omega)s^{N-1}ds - \frac{(R_2-R_1)}{2}\left(f(R_1\omega)R_1^{N-1} + f(R_2\omega)R_2^{N-1}\right) \\
&+ \frac{1}{8}(R_2-R_1)^2\left(\left(f(s\omega)s^{N-1}\right)'(R_2) - \left(f(s\omega)s^{N-1}\right)'(R_1)\right) \le \lambda_2(\omega),
\end{aligned}$$

for all $\omega \in S^{N-1}$.
Hence it holds

$$\begin{aligned}
&- \int_{S^{N-1}} \lambda_2(\omega)\,d\omega \le \int_{S^{N-1}}\left(\int_{R_1}^{R_2} f(s\omega)s^{N-1}ds\right)d\omega \\
&- \frac{(R_2-R_1)}{2}\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right) \\
&+ \frac{(R_2-R_1)^2}{8}\left[\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)'(R_2)\,d\omega - \int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)'(R_1)\,d\omega\right] \\
&\le \int_{S^{N-1}} \lambda_2(\omega)\,d\omega.
\end{aligned}$$

That is (by (1.16))

$$\begin{aligned}
&- \left[\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{M_2}{12}(R_2-R_1)^3 - \frac{(R_2-R_1)}{16M_2}\int_{S^{N-1}} \Delta_1^2(\omega)\,d\omega\right] \\
&\le \int_A f(y)\,dy - \frac{(R_2-R_1)}{2}\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right) \\
&+ \frac{(R_2-R_1)^2}{8}\left[\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)'(R_2)\,d\omega - \int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)'(R_1)\,d\omega\right] \\
&\le \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{M_2}{12}(R_2-R_1)^3 - \frac{(R_2-R_1)}{16M_2}\int_{S^{N-1}} \Delta_1^2(\omega)\,d\omega,
\end{aligned}$$

proving the claim. □

We give

Theorem 2.11. *Let the spherical shell $A := B(0,R_2) - \overline{B(0,R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$. Consider $f : \overline{A} \to \mathbb{R}$ that is not necessarily radial, and that $f \in C^1(\overline{A})$. Let $M > m$ and assume that $m \leq \frac{\partial f(s\omega)}{\partial s} \leq M$, for all $s \in [R_1, R_2]$, and for all $\omega \in S^{N-1}$.*
Then

$$\begin{aligned} & \left| \int_A f(y)\,dy - \frac{(R_2 - R_1)}{2} \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega)\,d\omega \right) \right| \\ \leq \; & \frac{1}{2(M-m)} \int_{S^{N-1}} \left[\left(f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1} - m(R_2 - R_1) \right) \right. \\ \times \; & \left. \left(M(R_2 - R_1) - f(R_2\omega) R_2^{N-1} + f(R_1\omega) R_1^{N-1} \right) \right] d\omega. \end{aligned}$$

Proof. Similar to the proof of Theorem 2.9 by using Theorem 1.3 and (1.16). □

We give

Theorem 2.12. *Let the spherical shell $A := B(0,R_2) - \overline{B(0,R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$. Consider $f : \overline{A} \to \mathbb{R}$ that is not necessarily radial, and that $f \in C^2(\overline{A})$. Assume that $\left| \frac{\partial^2 f(s\omega)}{\partial s^2} \right| \leq M_3$, for all $s \in [R_1, R_2]$, and for all $\omega \in S^{N-1}$, where $M_3 > 0$.*
Set

$$\begin{aligned} & Q_1^2(\omega) \\ := & \frac{\left[\left(f(s\omega) s^{N-1} \right)'(R_1) + \left(f(s\omega) s^{N-1} \right)'(R_2) - 2 \left(\frac{f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1}}{R_2 - R_1} \right) \right]^2}{\left[M_3^2 (R_2 - R_1)^2 - \left(\left(f(s\omega) s^{N-1} \right)'(R_2) - \left(f(s\omega) s^{N-1} \right)'(R_1) \right)^2 \right]}, \end{aligned}$$

for all $\omega \in S^{N-1}$.
Then

$$\begin{aligned} & \left| \int_A f(y)\,dy - \frac{(R_2 - R_1)}{2} \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega)\,d\omega \right) \right. \\ + & \left. \frac{(R_2 - R_1)^2}{8} \int_{S^{N-1}} \left(1 + Q_1^2(\omega) \right) \left(\left(f(s\omega) s^{N-1} \right)'(R_2) - \left(f(s\omega) s^{N-1} \right)'(R_1) \right) d\omega \right| \\ \leq & \frac{M_3 (R_2 - R_1)^3}{24} \int_{S^{N-1}} \left(1 - 3Q_1^2(\omega) \right) d\omega. \end{aligned}$$

Proof. Similar to the proof of Theorem 2.10 by using Theorem 1.4 and (1.16). □

We continue with

Theorem 2.13. *Here all as in Theorem 2.9, and let $M_1 > m_1$. Assume that*

$$m_1 \leq \frac{\left(f(s\omega) s^{N-1} \right)'(x) - \left(f(s\omega) s^{N-1} \right)'(R_1)}{x - R_1} \leq M_1,$$

and

$$m_1 \leq \frac{\left(f(s\omega) s^{N-1} \right)'(R_2) - \left(f(s\omega) s^{N-1} \right)'(x)}{R_2 - x} \leq M_1,$$

for all $x \in [R_1, R_2]$, for all $\omega \in S^{N-1}$.

Set

$$P_1^2(\omega)$$
$$= \frac{\left[\left(f(s\omega)s^{N-1}\right)'(R_1)+\left(f(s\omega)s^{N-1}\right)'(R_2)-2\left(\frac{f(R_2\omega)R_2^{N-1}-f(R_1\omega)R_1^{N-1}}{R_2-R_1}\right)\right]^2}{\left(\frac{M_1-m_1}{2}\right)^2(R_2-R_1)^2-\left[\left(f(s\omega)s^{N-1}\right)'(R_2)-\left(f(s\omega)s^{N-1}\right)'(R_1)-\left(\frac{m_1+M_1}{2}\right)(R_2-R_1)\right]^2},$$

for all $\omega \in S^{N-1}$.
Then

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega+R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right)\right.$$
$$+\frac{(R_2-R_1)^2}{8}\int_{S^{N-1}}\left(1+P_1^2(\omega)\right)\left(\left(f(s\omega)s^{N-1}\right)'(R_2)-\left(f(s\omega)s^{N-1}\right)'(R_1)\right)d\omega$$
$$\left.-\frac{(R_2-R_1)^3}{48}(m_1+M_1)\int_{S^{N-1}}\left(1+3P_1^2(\omega)\right)d\omega\right|$$
$$\leq \frac{(M_1-m_1)(R_2-R_1)^3}{48}\int_{S^{N-1}}\left(1-3P_1^2(\omega)\right)d\omega.$$

Proof. Similar to the proof of Theorem 2.10 by using Theorem 1.5 and (1.16). □

We present

Theorem 2.14. *Consider $f:\overline{A}\to\mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega)s^{N-1}\in AC^n([R_1,R_2])$ (i.e. $\left(f(s\omega)s^{N-1}\right)^{(n-1)}\in AC([R_1,R_2])$ absolutely continuous functions), for all $\omega\in S^{N-1}$, $N\geq 2$. We assume that $\left(f(s\omega)s^{N-1}\right)^{(n)}\in L_\infty([R_1,R_2])$, for all $\omega\in S^{N-1}$. There exists $K_1>0$ such that $\left\|\left(f(s\omega)s^{N-1}\right)^{(n)}\right\|_{L_\infty([R_1,R_2])}\leq K_1$, where $s\in[R_1,R_2]$, for all $\omega\in S^{N-1}$.*
Then

(i)

$$\left|\int_A f(y)\,dy-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)(t-R_1)^{k+1}\right.\right.$$
$$\left.\left.+(-1)^k\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)(R_2-t)^{k+1}\right]\right|$$
$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_1}{(n+1)!}\left[(t-R_1)^{n+1}+(R_2-t)^{n+1}\right], \tag{2.17}$$

for all $t\in[R_1,R_2]$,

(ii) at $t=\frac{R_1+R_2}{2}$, the right hand side of (2.17) is minimized, and we get:

$$\left|\int_A f(y)\,dy-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^{k+1}}\left[\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right.\right.$$
$$\left.\left.+(-1)^k\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right]\right|$$
$$\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_1}{(n+1)!}\frac{(R_2-R_1)^{n+1}}{2^{n-1}},$$

(iii) *if* $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1) = \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2) = 0$, *for all* $\omega \in S^{N-1}$, *(i.e.* $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for all* $k = 0,1,...,n-1$, *we obtain*

$$\left|\int_A f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_1}{(n+1)!} \frac{(R_2-R_1)^{n+1}}{2^{n-1}},$$

which is a sharp inequality,

(iv) *more generally, for* $j = 0,1,2,...,\overline{N} \in \mathbb{N}$, *it holds*

$$\text{(2.18)} \quad \left|\int_A f(y)\,dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left(\frac{R_2-R_1}{\overline{N}}\right)^{k+1}\left[j^{k+1}\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)\right.\right.$$
$$\left.\left. + \ (-1)^k\left(\overline{N}-j\right)^{k+1}\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)\right]\right|$$
$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_1}{(n+1)!}\left(\frac{R_2-R_1}{\overline{N}}\right)^{n+1}\left[j^{n+1} + \left(\overline{N}-j\right)^{n+1}\right],$$

(v) *if* $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1) = \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2) = 0$, *for all* $\omega \in S^{N-1}$, *(i.e.* $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for* $k = 1,...,n-1$, *from (2.18) we get:*

$$\text{(2.19)} \quad \left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{\overline{N}}\right)\left[jR_1^{N-1}\left(\int_{S^{N-1}} f(R_1\omega)\,d\omega\right)\right.\right.$$
$$\left.\left. + \ \left(\overline{N}-j\right)R_2^{N-1}\left(\int_{S^{N-1}} f(R_2\omega)\,d\omega\right)\right]\right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot$$
$$\times \ \frac{K_1}{(n+1)!}\left(\frac{R_2-R_1}{\overline{N}}\right)^{n+1}\left[j^{n+1} + \left(\overline{N}-j\right)^{n+1}\right],$$

for $j = 0,1,2,...,\overline{N} \in \mathbb{N}$,

(vi) *when* $\overline{N} = 2$ *and* $j = 1$, *(2.19) turns to*

$$\text{(2.20)} \quad \left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right)\right|$$
$$\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_1}{(n+1)!} \frac{(R_2-R_1)^{n+1}}{2^{n-1}},$$

(vii) *when* $n = 1$ *(without any boundary conditions), we get from (2.20) that*

$$\left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right)\right|$$
$$\leq \frac{\pi^{\frac{N}{2}}K_1}{2\Gamma\left(\frac{N}{2}\right)}(R_2-R_1)^2.$$

Proof. Similar to the proof of Theorem 2.9. We apply Theorem 1.6 along with (1.16). □

We continue with

Theorem 2.15. *Consider* $f : \overline{A} \to \mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega)s^{N-1} \in AC^n([R_1,R_2])$ *(i.e.* $\left(f(s\omega)s^{N-1}\right)^{(n-1)} \in AC([R_1,R_2])$ *absolutely continuous functions), for all* $\omega \in S^{N-1}$, $N \geq 2$. *Here there exists* $K_2 > 0$ *such that* $\left\|\left(f(s\omega)s^{N-1}\right)^{(n)}\right\|_{L_1([R_1,R_2])} \leq K_2$, *where* $s \in [R_1,R_2]$, *for all* $\omega \in S^{N-1}$.

Then

(i)

$$\left|\int_A f(y)\,dy - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)(t-R_1)^{k+1}\right.\right.$$
$$\left.\left.+\quad(-1)^k\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)(R_2-t)^{k+1}\right]\right|$$
$$\leq\quad\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_2}{n!}\left[(t-R_1)^n+(R_2-t)^n\right], \tag{2.21}$$

for all $t\in[R_1,R_2]$,

(ii) at $t=\frac{R_1+R_2}{2}$*, the right hand side of (2.21) is minimized, and we get:*

$$\left|\int_A f(y)\,dy - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^{k+1}}\left[\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right.\right.$$
$$\left.\left.+\quad(-1)^k\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right]\right|$$
$$\leq\quad\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_2}{n!}\frac{(R_2-R_1)^n}{2^{n-2}},$$

(iii) if $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)=\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)=0$*, for all* $\omega\in S^{N-1}$*, (i.e.* $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for all* $k=0,1,\ldots,n-1$*, we obtain*

$$\left|\int_A f(y)\,dy\right|\leq\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_2}{n!}\frac{(R_2-R_1)^n}{2^{n-2}},$$

which is a sharp inequality,

(iv) more generally, for $j=0,1,2,\ldots,\overline{N}\in\mathbb{N}$*, it holds*

$$\left|\int_A f(y)\,dy - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R_2-R_1}{\overline{N}}\right)^{k+1}\left[j^{k+1}\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)\right.\right.$$
$$\left.\left.+\quad(-1)^k\left(\overline{N}-j\right)^{k+1}\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)\right]\right|$$
$$\leq\quad\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_2}{n!}\left(\frac{R_2-R_1}{\overline{N}}\right)^n\left[j^n+\left(\overline{N}-j\right)^n\right], \tag{2.22}$$

(v) if $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)=\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)=0$*, for all* $\omega\in S^{N-1}$*, (i.e.* $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for* $k=1,\ldots,n-1$*, from (2.22) we get:*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{\overline{N}}\right)\left[jR_1^{N-1}\left(\int_{S^{N-1}}f(R_1\omega)\,d\omega\right)\right.\right.$$
$$\left.\left.+\quad\left(\overline{N}-j\right)R_2^{N-1}\left(\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right]\right|\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$
$$\times\quad\frac{K_2}{n!}\left(\frac{R_2-R_1}{\overline{N}}\right)^n\left[j^n+\left(\overline{N}-j\right)^n\right], \tag{2.23}$$

for $j=0,1,2,\ldots,\overline{N}\in\mathbb{N}$,

(vi) *when* $\overline{N}=2$ *and* $j=1$, (2.23) *turns to*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}}f(R_1\omega)\,d\omega+R_2^{N-1}\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right| \tag{2.24}$$
$$\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_2}{n!}\frac{(R_2-R_1)^n}{2^{n-2}},$$

(vii) *when* $n=1$ *(without any boundary conditions), we get from* (2.24) *that*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}}f(R_1\omega)\,d\omega+R_2^{N-1}\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right|$$
$$\leq \frac{2\pi^{\frac{N}{2}}K_2}{\Gamma\left(\frac{N}{2}\right)}(R_2-R_1).$$

Proof. Similar to the proof of Theorem 2.9. We apply Theorem 1.7 along with (1.16). □

We continue with

Theorem 2.16. *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}>1$. *Consider* $f:\overline{A}\to\mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega)s^{N-1}\in AC^n([R_1,R_2])$ *(i.e.* $\left(f(s\omega)s^{N-1}\right)^{(n-1)}\in AC([R_1,R_2])$ *absolutely continuous functions), for all* $\omega\in S^{N-1}$, $N\geq 2$. *We assume that* $\left(f(s\omega)s^{N-1}\right)^{(n)}\in L_q([R_1,R_2])$, *for all* $\omega\in S^{N-1}$. *There exists* $K_3>0$ *such that* $\left\|\left(f(s\omega)s^{N-1}\right)^{(n)}\right\|_{L_q([R_1,R_2])}\leq K_3$, *where* $s\in[R_1,R_2]$, *for all* $\omega\in S^{N-1}$.
Then

(i)

$$\left|\int_A f(y)\,dy-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)(t-R_1)^{k+1}\right.\right. \tag{2.25}$$
$$\left.\left.+\ (-1)^k\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)(R_2-t)^{k+1}\right]\right|$$
$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_3}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}}\left[(t-R_1)^{n+\frac{1}{p}}+(R_2-t)^{n+\frac{1}{p}}\right],$$

for all $t\in[R_1,R_2]$,
(ii) *at* $t=\frac{R_1+R_2}{2}$, *the right hand side of* (2.25) *is minimized, and we get:*

$$\left|\int_A f(y)\,dy-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^{k+1}}\right.$$
$$\left.\times\left[\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega+(-1)^k\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right]\right|$$
$$\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_3}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}}\frac{(R_2-R_1)^{n+\frac{1}{p}}}{2^{n-1-\frac{1}{q}}},$$

(iii) if $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1) = \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2) = 0$*, for all* $\omega \in S^{N-1}$*, (i.e.* $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for all* $k = 0,1,...,n-1$*, we obtain*

$$\left|\int_A f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_3}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}}\frac{(R_2-R_1)^{n+\frac{1}{p}}}{2^{n-1-\frac{1}{q}}},$$

which is a sharp inequality,

(iv) more generally, for $j = 0,1,2,...,\overline{N} \in \mathbb{N}$*, it holds*

$$\left|\int_A f(y)\,dy - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R_2-R_1}{\overline{N}}\right)^{k+1}\left[j^{k+1}\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)\right.\right. \tag{2.26}$$

$$\left.\left. + \quad (-1)^k\left(\overline{N}-j\right)^{k+1}\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)\right]\right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_3}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}}\left(\frac{R_2-R_1}{\overline{N}}\right)^{n+\frac{1}{p}}\left[j^{n+\frac{1}{p}}+\left(\overline{N}-j\right)^{n+\frac{1}{p}}\right],$$

(v) if $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1) = \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2) = 0$*, for all* $\omega \in S^{N-1}$*, (i.e.* $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for* $k = 1,...,n-1$*, from (2.26) we get:*

$$\left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{\overline{N}}\right)\left[jR_1^{N-1}\left(\int_{S^{N-1}}f(R_1\omega)\,d\omega\right)\right.\right. \tag{2.27}$$

$$\left.\left. + \quad \left(\overline{N}-j\right)R_2^{N-1}\left(\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right]\right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$

$$\times \quad \frac{K_3}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}}\left(\frac{R_2-R_1}{\overline{N}}\right)^{n+\frac{1}{p}}\left[j^{n+\frac{1}{p}}+\left(\overline{N}-j\right)^{n+\frac{1}{p}}\right],$$

for $j = 0,1,2,...,\overline{N} \in \mathbb{N}$*,*

(vi) when $\overline{N} = 2$ *and* $j = 1$*, (2.27) turns to*

$$\left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}}f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right| \tag{2.28}$$

$$\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_3}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}}\frac{(R_2-R_1)^{n+\frac{1}{p}}}{2^{n-1-\frac{1}{q}}},$$

(vii) when $n = 1$ *(without any boundary conditions), we get from (2.28) that*

$$\left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}}f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right|$$

$$\leq \frac{2^{\frac{1}{q}}\pi^{\frac{N}{2}}K_3}{\Gamma\left(\frac{N}{2}\right)\left(1+\frac{1}{p}\right)}(R_2-R_1)^{1+\frac{1}{p}}.$$

Proof. Similar to the proof of Theorem 2.9. We apply Theorem 1.8 along with (1.16). □

We continue with results on the ball. We present

Theorem 2.17. *Consider* $f : \overline{B(0,R)} \to \mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega)s^{N-1} \in AC([0,R])$*, for all* $\omega \in S^{N-1}$*,* $N \geq 2$*. We further assume* *that* $\frac{\partial f(s\omega)s^{N-1}}{\partial s} \in L_{\infty}([0,R])$*, for all* $\omega \in S^{N-1}$*. Suppose there exists* $M_1 > 0$ *such that* $\left\| \frac{\partial f(s\omega)s^{N-1}}{\partial s} \right\|_{\infty,(s\in[0,R])} \leq M_1$*, for all* $\omega \in S^{N-1}$*.*

Then

(i)

$$\left| \int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega \right) R^{N-1}(R-t) \right| \leq \frac{\pi^{\frac{N}{2}} M_1}{\Gamma\left(\frac{N}{2}\right)} \left[t^2 + (R-t)^2 \right], \tag{2.29}$$

for all $t \in [0,R]$*,*

(ii) at $t = \frac{R}{2}$*, the right hand side of (2.29) is minimized, and we get:*

$$\left| \int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega \right) \frac{R^N}{2} \right| \leq \frac{\pi^{\frac{N}{2}} M_1 R^2}{2\Gamma\left(\frac{N}{2}\right)},$$

(iii) if $f(R\omega) = 0$*, for all* $\omega \in S^{N-1}$*, (i.e.* $f(\cdot\omega)$ *vanishes on* $\partial B(0,R)$*), we obtain*

$$\left| \int_{B(0,R)} f(y)\,dy \right| \leq \frac{\pi^{\frac{N}{2}} M_1 R^2}{2\Gamma\left(\frac{N}{2}\right)},$$

which is a sharp inequality,

(iv) more generally, for $j = 0,1,2,...,\overline{N} \in \mathbb{N}$*, it holds*

$$\left| \int_{B\,(0,R)} f(y)\,dy - \frac{R^N}{\overline{N}} \left(\overline{N} - j\right) \int_{S^{N-1}} f(R\omega)\,d\omega \right| \leq \frac{\pi^{\frac{N}{2}} M_1}{\Gamma\left(\frac{N}{2}\right)} \left(\frac{R}{\overline{N}}\right)^2 \left[j^2 + \left(\overline{N} - j\right)^2 \right], \tag{2.30}$$

(v) when $\overline{N} = 2$ *and* $j = 1$*, (2.30) turns to*

$$\left| \int_{B(0,R)} f(y)\,dy - \frac{R^N}{2} \int_{S^{N-1}} f(R\omega)\,d\omega \right| \leq \frac{\pi^{\frac{N}{2}} M_1 R^2}{2\Gamma\left(\frac{N}{2}\right)}.$$

Proof. Same as the proof of Theorem 2.14, just set there $R_1 = 0$ and $R_2 = R$ and use (1.15). □

We continue with

Theorem 2.18. *Consider* $f : \overline{B(0,R)} \to \mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega)s^{N-1} \in AC([0,R])$*, for all* $\omega \in S^{N-1}$*,* $N \geq 2$*. Suppose there exists* $M_2 > 0$ *such that* $\left\| \frac{\partial f(s\omega)s^{N-1}}{\partial s} \right\|_{L_1([0,R])} \leq M_2$*, for all* $\omega \in S^{N-1}$*.*

Then

(i)

$$\left| \int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega \right) R^{N-1}(R-t) \right| \leq \frac{2\pi^{\frac{N}{2}} M_2 R}{\Gamma\left(\frac{N}{2}\right)}, \tag{2.31}$$

for all $t \in [0,R]$*,*

(ii) if $f(R\omega) = 0$*, for all* $\omega \in S^{N-1}$*, (i.e.* $f(\cdot\omega)$ *vanishes on* $\partial B(0,R)$*) from (2.31), we obtain*

$$\left| \int_{B(0,R)} f(y)\,dy \right| \leq \frac{2\pi^{\frac{N}{2}} M_2 R}{\Gamma\left(\frac{N}{2}\right)},$$

which is a sharp inequality,

(iii) more generally, for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$, it holds

$$\left| \int_{B\ (0,R)} f(y)\, dy - \frac{R^N}{\overline{N}} \left(\overline{N} - j \right) \int_{S^{N-1}} f(R\omega)\, d\omega \right| \leq \frac{2\pi^{\frac{N}{2}} M_2 R}{\Gamma\left(\frac{N}{2}\right)}, \tag{2.32}$$

(iv) when $\overline{N} = 2$ and $j = 1$, (2.32) turns to

$$\left| \int_{B(0,R)} f(y)\, dy - \frac{R^N}{2} \int_{S^{N-1}} f(R\omega)\, d\omega \right| \leq \frac{2\pi^{\frac{N}{2}} M_2 R}{\Gamma\left(\frac{N}{2}\right)}.$$

Proof. Same as the proof of Theorem 2.15, just set there $R_1 = 0$ and $R_2 = R$ and use (1.15). $\square$

We continue with

Theorem 2.19. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Consider $f : \overline{B(0,R)} \to \mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega) s^{N-1} \in AC([0,R])$, for all $\omega \in S^{N-1}$, $N \geq 2$. We further assume that $\frac{\partial f(s\omega)s^{N-1}}{\partial s} \in L_q([0,R])$, for all $\omega \in S^{N-1}$. Suppose there exists $M_3 > 0$ such that $\left\| \frac{\partial f(s\omega)s^{N-1}}{\partial s} \right\|_{L_q([0,R])} \leq M_3$, for all $\omega \in S^{N-1}$.*

Then

(i)

$$\left| \int_{B(0,R)} f(y)\, dy - \left(\int_{S^{N-1}} f(R\omega)\, d\omega \right) R^{N-1} (R - t) \right| \tag{2.33}$$

$$\leq \frac{2\pi^{\frac{N}{2}} M_3}{\Gamma\left(\frac{N}{2}\right)\left(1 + \frac{1}{p}\right)} \left[t^{1+\frac{1}{p}} + (R - t)^{1+\frac{1}{p}} \right],$$

for all $t \in [0, R]$,

(ii) at $t = \frac{R}{2}$, the right hand side of (2.33) is minimized, and we get:

$$\left| \int_{B(0,R)} f(y)\, dy - \left(\int_{S^{N-1}} f(R\omega)\, d\omega \right) \frac{R^N}{2} \right| \leq \frac{2^{\frac{1}{q}} \pi^{\frac{N}{2}} M_3 R^{1+\frac{1}{p}}}{\Gamma\left(\frac{N}{2}\right)},$$

(iii) if $f(R\omega) = 0$, for all $\omega \in S^{N-1}$, (i.e. $f(\cdot\omega)$ vanishes on $\partial B(0,R)$), we obtain

$$\left| \int_{B(0,R)} f(y)\, dy \right| \leq \frac{2^{\frac{1}{q}} \pi^{\frac{N}{2}} M_3 R^{1+\frac{1}{p}}}{\Gamma\left(\frac{N}{2}\right)},$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$, it holds

$$\left| \int_{B\ (0,R)} f(y)\, dy - \frac{R^N}{\overline{N}} \left(\overline{N} - j \right) \int_{S^{N-1}} f(R\omega)\, d\omega \right| \tag{2.34}$$

$$\leq \frac{2\pi^{\frac{N}{2}} M_3}{\left(1 + \frac{1}{p}\right)\Gamma\left(\frac{N}{2}\right)} \left(\frac{R}{\overline{N}} \right)^{1+\frac{1}{p}} \left[j^{1+\frac{1}{p}} + \left(\overline{N} - j \right)^{1+\frac{1}{p}} \right],$$

(v) when $\overline{N} = 2$ and $j = 1$, (2.34) turns to

$$\left| \int_{B(0,R)} f(y)\, dy - \frac{R^N}{2} \int_{S^{N-1}} f(R\omega)\, d\omega \right| \leq \frac{2^{\frac{1}{q}} \pi^{\frac{N}{2}} M_3 R^{1+\frac{1}{p}}}{\left(1 + \frac{1}{p}\right)\Gamma\left(\frac{N}{2}\right)}.$$

Proof. Same as the proof of Theorem 2.16, just set there $R_1 = 0$ and $R_2 = R$ and use (1.15). $\square$

REFERENCES

[1] R. P. Agarwal, S. S. Dragomir: *An application of Hayashi's inequality for differentiable functions*, Computers Math. Applic., 6 (1996), 95-99.

[2] G. A. Anastassiou: *Fractional Differentiation Inequalities*, Research Monograph, Springer, New York, 2009.

[3] G. A. Anastassiou: *General Iyengar type inequalities*, submitted, 2018.

[4] Xiao-Liang Cheng: *The Iyengar-type inequality*, Applied Math. Letters 14 (2001), 975-978.

[5] K. S. K. Iyengar: *Note on an inequality*, Math. Student 6, (1938), 75-76.

[6] Zheng Liu: *Note on Iyengar's inequality*, Univ. Beograd Publ. Elektrotechn. Fak., Ser. Mat. 16 (2005), 29-35.

[7] F. Qi: *Further generalizations of inequalities for an integral*, Univ. Beograd Publ. Elektrotechn. Fak., Ser. Mat. 8 (1997), 79-83.

[8] W. Rudin: *Real and Complex Analysis*, International Student edition, Mc Graw Hill, London, New York, 1970.

[9] D. Stroock: *A Concise Introduction to the Theory of Integration*, Third Edition, Birkhaüser, Boston, Basel, Berlin, 1999.

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A General Korovkin Result Under Generalized Convergence

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ABSTRACT. In this paper, the classic result of Korovkin about the convergence of sequences of functions defined from sequences of linear operators is reformulated in terms of generalized convergence. This convergence extends some others given in the literature. The operator of the sequence fulfill a shape preserving property more general than the positivity. This property is related with certain extension of the notion of derivative. This extended derivative is precisely the object of the approximation process. The study is completed by analysing the conditions for the existence of an asymptotic formula, from which some interesting consequences are derived as a local version of the shape preserving property. Finally, as applications of the previous results, the author use the following notion of generalized convergence, an extension of Nörlund-Cesáro summability given by V. Loku and N. L. Braha in 2017. A way to transfer a notion of generalized convergence to approximation theory by means of linear operators is showed.

Keywords: Korovkin results, asymptotic condition, generalized convergence.

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1. INTRODUCTION

The following sequence of positive linear operators is studied in [2]:

$$B_n^\tau f(t) = \sum_{k=0}^{n} \binom{n}{k} \tau(t)^k (1-\tau(t))^{n-k} (f \circ \tau^{-1})(k/n), \quad f \in C[0,1],\ t \in [0,1],$$

where τ is a function defined on $[0,1]$ infinitely differentiable, such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(t) > 0,\ t \in (0,1)$. The convergence of $B_n^\tau f$ towards f can be analyzed by using the classical result of Korovkin [11], according to which it suffices to check it for these three test functions $1, t, t^2$, or other three, say ψ_0, ψ_1, ψ_2 that form a Tchebychev System. In particular, the choice $1, \tau, \tau^2$ is the more convenient for B_n^τ.

Now, let L_n be a slight modification of the previous sequence of positive linear operators, $L_n f(t) = (1 + a_n) B_n^\tau f(t)$, where a_n does not converge to 0 in the classical sense. The aforementioned result of Korovkin allows to conclude that the approximation process defined by L_n is not convergent. That said, if a_n is convergent in some other sense, a question arises whether the sequence would be convergent under this other notion. This is a motivation for a long list of papers where the so called Korovkin theory has been extended by considering new notions of convergence. We mention a few, restricting our attention to sequences of linear operator defined on spaces of real continuous functions on a compact interval.

In 1970, J. P. King and J. J. Swetits [10] studied the almost convergence, introduced by Lorentz in 1948 [14]. In 2002, A. D. Gadjiev and C. Orhan[6] proceeded analogously with statistical convergence, a now classic concept that was conceived by H. Fast in 1957 [5]. More recently, we may mention some papers by V. Karakaya and A. Karaisa in 2015 [9], where they considered weighted $\alpha\beta$-statistical convergence, T. Acar and S. A. Mohiuddine in 2016 [1] dealt with

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statistical $(C, 1)(E, 1)$ summability, or finally, in 2016, D. Ali Karaisa, [8] worked with statistical $(N^\gamma, \alpha\beta)$ summability.
All the quoted papers dealt with positive linear operators and more importantly, for the proofs of their main results the same arguments of continuity and boundness were strongly used. Our main purpose with this work is to bring a sort of unification by proving a general qualitative Korovkin result, in such a way that this result can be applied whenever a concept of convergence is moved from mathematical analysis to Korovkin-type approximation theory. Moreover, we shall deal with a shape preserving property more general than the mere positivity, related to the preservation of the sign of certain generalized derivative.
This paper is organized as follows. In section 2, we will show some required notions and the notation will be set. In section 3, the qualitative Korovkin type result will be shown. Besides this, in the section 4, we will add the analysis of existence of the asymptotic condition by means of another Korovkin type result. In section 5, some consequences of the existence of an asymptotic condition will be given. In the last section, we show an example that shows the applicability of our result, by recovering the paper by V. Loku and N. L. Braha [12].

2. General settings

In this section, we will establish the framework, and present the required tools. Some notation will be set as well.
Let $\mathcal{S}$ be the usual linear space of all real sequences, and let $\mathcal{S}_0$ be a subspace of $\mathcal{S}$ closed under the usual sum and scalar multiplication.
Let $\mathcal{L}$ be a linear functional defined on $\mathcal{S}_0$ fulfilling the following properties:

(I) if x_n is convergent in the classical sense, then $\mathcal{L}(x_n) = \lim x_n$, where lim refers to the classic limit (as a consequence $\mathcal{L}(x_n) = \ell \quad \Leftrightarrow \quad \mathcal{L}(x_n - \ell) = 0$);
(II) if $x_n \leq y_n$ for every $n \in \mathbb{N}$, then $\mathcal{L}(x_n) \leq \mathcal{L}(y_n)$ for every $n \in \mathbb{N}$. In short, $x_n \leq y_n$ implies $\mathcal{L}(x_n) \leq \mathcal{L}(y_n)$;
(III) if a_n is non negative, $\lim a_n = 0$ and $\mathcal{L}(x_n) = \ell$, then $\mathcal{L}(a_n \cdot x_n) = 0$;
(IV) if $x_n \leq z_n \leq y_n$ and $\mathcal{L}(x_n) = \mathcal{L}(y_n) = \ell$, then $z_n \in \mathcal{S}_0$ and $\mathcal{L}(z_n) = \ell$.

We have assumed, and will assume from now onwards that $x_n \in \mathcal{S}_0$ whenever we write $\mathcal{L}(x_n)$. On the other hand, to fix ideas, notice that under the classical setting $\mathcal{S}_0$ is formed by all convergent sequences, and, under statistical convergence, our functional $\mathcal{L}$ coincides with the so noted $st - \lim$.
Recall the following properties for a Tschebyshev System, $T = \{\psi_0, \psi_1, \psi_2\}$, on an interval $[a, b]$:

P1: Given three points $x_1, x_2, x_3 \in [a, b]$ and three real numbers a_1, a_2, a_3, there exists only one T-polynomial (i.e. a function that belongs to the space spanned by ψ_0, ψ_1, ψ_2), such that $p_T(x_i) = a_i, \ \ i = 1, 2, 3$.
P2: For all $\alpha \in (a, b)$, we can find a T-polynomial, $p_{T,\alpha}$, such that, α is a double root of $p_{T,\alpha}$.

As it is usual $C^m[0, 1]$ is the set of the all functions m-times differentiable with continuous $m-th$ derivative. Notice that $C[0, 1]$ is simply the set of continuous function on $[0, 1]$ and $C^\infty[0, 1] = \cap_{i \in \mathbb{N}} C^i[0, 1]$. Now let $\tau \in C^\infty[0, 1]$, with $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(t) > 0, \ t \in (0, 1)$. In relation with the function τ, it is considered $e_{\tau,i}(t) = \tau(t)^i, \quad e^x_{\tau,i}(t) = (\tau(t) - \tau(x))^i$. Associated with the function τ the following differential operator is defined, see [13]

$$D^i_\tau f(t) := D^i \left(f \circ \tau^{-1} \right) (\tau(t)). \tag{2.1}$$

We notice that $D_\tau^0 = \mathbb{I}$. The previous definition is equivalent to this other:

$$D^{0,\tau'} = \mathbb{I}, \quad D^{1,\tau'} = \frac{1}{\tau'} D^1, \quad D^{i+1,\tau'} = D^{1,\tau'} \circ D^{i,\tau'},\ i \in \mathbb{N}.$$

This differential operator has been dealt by the author inside approximation theory in [7] and [3].
It it easy to observe that for $x \in (0,1)$,

$$(2.2) \qquad D_\tau^i e_{\tau,j} = \begin{cases} \frac{j!}{(j-i)!} e_{\tau,j-i}, & \text{if } j \geq i; \\ 0, & \text{if } j < i, \end{cases} \quad \text{and } D_\tau^i e_{\tau,j}^x = \begin{cases} \frac{j!}{(j-i)!} e_{\tau,j-i}^x, & \text{if } j \geq i; \\ 0, & \text{if } j < i. \end{cases}$$

3. QUALITATIVE KOROVKIN TYPE RESULT

Here is one of the main result of the paper, extension of the classical result of Korovkin.

Theorem 3.1. *Let $L_n : C^m[0,1] \to C^m[0,1]$ be a sequence of linear operators fulfilling the following shape preserving property:*

$$(3.3) \qquad \text{if } D_\tau^m f \geq 0 \text{ then } D_\tau^m L_n f \geq 0$$

Suppose we have three functions, $F_0, F_1, F_2 \in C^m[0,1]$ such that $T = \{D_\tau^m F_0, D_\tau^m F_1, D_\tau^m F_2\}$ is a Tschebyshev System on $C[0,1]$, then the following sentences are equivalent:

(i) $\mathcal{L}(D_\tau^m L_n F_i(x)) = D_\tau^m F_i(x),\ \ i = 0,1,2,$
(ii) $\mathcal{L}(D_\tau^m L_n f(x)) = D_\tau^m f(x)$ *for all function* $f \in C^m[0,1]$.

Proof.
$(ii) \Rightarrow (i)$ is trivial. We are going to prove the converse. First of all, we consider that $x \in (0,1)$. We define the function $\phi \in C^m[0,1]$ as $\phi(t) = f(t) - \frac{D_\tau^m f(x)}{D_\tau^m G(x)} G(t)$, where $D_\tau^m G$ is a T-polynomial of $T = \{D_\tau^m F_0, D_\tau^m F_1, D_\tau^m F_2\}$, a Tchebychev System on $[0,1]$, with $D_\tau^m G(x) \neq 0$.
$D_\tau^m \phi$, is continuous and it vanishes at x, then for all positive real number ϵ there exists $\delta > 0$ such that if $|t-x| < \delta$, then

$$-\epsilon \leq D_\tau^m \phi(t) \leq \epsilon.$$

On the other hand $D_\tau^m \phi$ is bounded on $[0,1]$, then there exists $M > 0$ such that

$$-M \leq D_\tau^m \phi(t) \leq M.$$

By property **P2** , for x, we can find two non negative $T-$polynomials, f_x, h, where the first function has a double root at x and the second function is greater than or equal to 1 on $[0,1]$. Let $k = \min_{|t-x|\geq\delta} f_x(t) > 0$ and $F_x, H \in C^m[0,1]$ such that $D_\tau^m F_x = f_x, D_\tau^m H = h$, then the following inequality is satisfied for $t \in [0,1]$

$$-\epsilon D_\tau^m H(t) - \frac{M}{k} D_\tau^m F_x(t) \leq D_\tau^m \phi(t) \leq \epsilon D_\tau^m H(t) + \frac{M}{k} D_\tau^m F_x(t),$$

or equivalently on $[0,1]$,

$$D_\tau^m \left(-\epsilon H - \frac{M}{k} F_x \right) \leq D_\tau^m \phi \leq D_\tau^m \left(\epsilon H + \frac{M}{k} F_x \right).$$

Applying the shape preserving property (3.3), linearity and then evaluating at x we have,

$$(3.4) \qquad -\epsilon D_\tau^m L_n H(x) - \frac{M}{k} D_\tau^m L_n F_x(x) \leq D_\tau^m L_n \phi(x) \leq \epsilon D_\tau^m L_n H(x) + \frac{M}{k} D_\tau^m L_n F_x(x).$$

Since ϵ is arbitrary, we can choose $\epsilon = \frac{1}{n}$. As F_x, H belong to space spanned by F_0, F_1, F_2, then we use the hypothesis (i) to get $\mathcal{L}(D^m_\tau L_n F_x(x)) = D^m_\tau F_x(x) = 0$ and $\mathcal{L}\left(D^m_\tau L_n H(x)\right) = D^m_\tau H(x) = h(x)$.
From (3.4) and property (III), we have that $\mathcal{L}(\epsilon D^m_\tau L_n H(x) + \frac{M}{k} D^m_\tau L_n F_x(x)) = 0$. Then from property (IV), we deduce that $\mathcal{L}(D^m_\tau L_n \phi(x)) = 0$ or equivalently $\mathcal{L}(D^m_\tau L_n f(x) - \frac{D^m_\tau f(x)}{D^m_\tau G(x)} D^m_\tau L_n G(x)) = 0$, so $\mathcal{L}(D^m_\tau L_n f(x)) = D^m_\tau f(x)$.
Now, we will prove the result for the end points of the interval $x = 0$ and $x = 1$. In this case, we define ϕ as $\phi(t) = f(t) - G(t)$, where $D^m_\tau G$ is a T-polynomial, with $D^m_\tau G(0) = D^m_\tau f(0), D^m_\tau G(1) = D^m_\tau f(1)$.
Again we use the continuity, in this case in 0 and 1, and the bound M of $D^m_\tau \phi$, as well as the fact that $D^m_\tau \phi$ vanishes at the endpoints of the interval. Then for all $\epsilon > 0$, there exist $\delta > 0$ such that for $0 \le t \le \delta,\ 1 - \delta \le t \le 1$

$$-\epsilon \le D^m_\tau \phi(t) \le \epsilon \text{ and } - M \le D^m_\tau \phi(t) \le M.$$

Now, we choose $F_{01} \in C^m[0,1]$, where $D^m_\tau F_{01}$ is a T-polynomial, $D^m_\tau F_{01}(0) = D^m_\tau F_{01}(1) = 0$ and $D^m_\tau F_{01} \ge 0$. Now, we take $k = \min\limits_{\delta \le x \le 1-\delta} D^m_\tau F_{01}(t) > 0$. Then we have the following inequalities on $[0,1]$

$$-\frac{M}{k} D^m_\tau F_{01} - \epsilon \le D^m_\tau \phi \le \epsilon + \frac{M}{k} D^m_\tau F_{01}.$$

Finally, we can end the proof with similar arguments to the other case. $\square$

4. Asymptotic condition

Once guaranteed the generalized convergence of the process, we are going to analyze the sequence $D^m_\tau L_n f(x) - D^m_\tau f(x)$ comparing it with another sequence of real numbers λ_n with $\mathcal{L}(\lambda_n) = 0$. The purpose is to obtain an asymptotic condition. Here it is the corresponding result. Again it is a Korovkin type result.

Theorem 4.2. *Let L_n be the sequence of linear operators as that of Section 3. Let $x \in (0,1)$ and let us assume that there exist a sequence λ_n of positive real numbers, with $\mathcal{L}(\lambda_n) = 0$ and three strictly positive functions w_0, w_1 and w_2 defined on $(0,1)$ with $w_i \in C^{2-i}(0,1)$ such that, for $s \in \{m, m+1, m+2, m+4\}$,*

$$\mathcal{L}\left(\frac{D^m_\tau L_n e^x_{\tau,s}(x) - D^m_\tau e^x_{\tau,s}(x)}{\lambda_n}\right) = w_2^{-1} D^1(w_1^{-1} D^1(w_0^{-1} D^m_\tau e^x_{\tau,s}))(x) \tag{4.5}$$

Then, for $f \in C^m(0,1)$, $m+2$ times differentiable in some neighborhood of x,

$$\mathcal{L}\left(\frac{D^m_\tau L_n f(x) - D^m_\tau f(x)}{\lambda_n}\right) = w_2^{-1} D^1(w_1^{-1} D^1(w_0^{-1} D^m_\tau f))(x). \tag{4.6}$$

Proof.
The proof similar to the one we can find in [3], with the proper changes. First of all, we apply the Taylors's formula to the function $D^m_\tau f \circ \tau^{-1}$ centered at a point $\tau(x)$ and evaluated at $\tau(t),\ t \in (0,1)$, i.e.:

$$D^m_\tau f \circ \tau^{-1}(\tau(t)) = \sum_{s=0}^{2} \frac{1}{s!} D^s(D^m_\tau f \circ \tau^{-1})(\tau(x))(\tau(t) - \tau(x))^s + h(\tau(t) - \tau(x))(\tau(t) - \tau(x))^2,$$

where h is a continuous function that vanishes at zero. Now using the definition of the differential operator (2.1) and the notation of Section 2, we have:

$$D_\tau^m f(t) = D_\tau^0(D_\tau^m f)(x)e_{\tau,0}^x(t) + D_\tau^1(D_\tau^m f)(x)e_{\tau,1}^x(t)$$
$$+\frac{1}{2}D_\tau^2(D_\tau^m f)(x)e_{\tau,2}^x(t) + h(\tau(t)-\tau(x))e_{\tau,2}^x(t).$$

Using (2.2), we can write

$$D_\tau^m f(t) = D_\tau^m\left(\sum_{s=0}^{2}\frac{1}{(m+s)!}D_\tau^s(D_\tau^m f)(x)e_{\tau,m+s}^x + H_x\right)(t)$$

with $H_x \in C^m(J)$ and $D_\tau^m H_x(t) = h(\tau(t)-\tau(x))e_{\tau,2}^x(t)$. Then, we apply the linear operator and evaluate at x to obtain

$$D_\tau^m L_n f(x) = D_\tau^m L_n\left(\sum_{s=0}^{2}\frac{1}{(m+s)!}D_\tau^s(D_\tau^m f)(x)e_{\tau,m+s}^x + H_x\right)(x).$$

By linearity,

$$D_\tau^m L_n f(x) = \sum_{s=0}^{2}\frac{1}{(m+s)!}D_\tau^s(D_\tau^m f)(x)D_\tau^m L_n e_{\tau,m+s}^x(x) + D_\tau^m L_n H_x(x).$$

Introducing this term, $D_\tau^m f(x) = \sum_{s=0}^{2}\frac{1}{(m+s)!}D_\tau^s(D_\tau^m f)(x)D_\tau^m e_{\tau,m+s}^x(x)$, to both sides of the equality and dividing by λ_n

$$\frac{D_\tau^m L_n f(x) - D_\tau^m f(x)}{\lambda_n} = \sum_{s=0}^{2}\frac{1}{(m+s)!}D_\tau^s(D_\tau^m f)(x)\frac{D_\tau^m L_n e_{\tau,m+s}^x(x) - D_\tau^m e_{\tau,m+s}^x(x)}{\lambda_n} +$$
$$\frac{D_\tau^m L_n H_x(x)}{\lambda_n}.$$

Now, we consider the hypothesis (4.5) for $m=0, m=1, m=2$. After some calculations,

$$\mathcal{L}\left(\sum_{s=0}^{2}\frac{1}{(m+s)!}D_\tau^s(D_\tau^m f)(x)\frac{D_\tau^m L_n e_{\tau,m+s}^x(x) - D_\tau^m e_{\tau,m+s}^x(x)}{\lambda_n}\right) =$$
$$w_2^{-1}D^1(w_1^{-1}D^1(w_0^{-1}D_\tau^m f))(x).$$

Finally, the proof of 4.6 will be finished if we prove that $\mathcal{L}\left(\frac{D_\tau^m L_n H_x(x)}{\lambda_n}\right) = 0$ and the proof will be finished.

To do this, we use continuity arguments on the function h to guarantee the existence of a neighborhood of x, say θ_x, for a given $\epsilon > 0$, such that for $t \in \theta_x$,

$$|h(\tau(t)-\tau(x))| < \epsilon.$$

Then, for all $t \in [0,1]$,

$$|D_\tau^m H_x(t)| = |h(\tau(t)-\tau(x))|e_{\tau,2}^x(t) \le \epsilon e_{\tau,2}^x(t) + \max\{0, |h(\tau(t)-\tau(x))| - \epsilon\}e_{\tau,2}^x(t).$$

Let us consider a function $W \in C^m[0,1]$ such that $D_\tau^m W(t) = \max\{0, |h(\tau(t)-\tau(x))| - \epsilon\}e_{\tau,2}^x(t)$. As $D_\tau^m W$ vanishes in θ_x, then, for a sufficiently large constant M, one has $|D_\tau^m W(t)| \le M D_\tau^m e_{\tau,m+4}^x(t)$. So, gathering the last inequalities we get,

$$|D_\tau^m H_x(t)| \le \frac{2\epsilon}{(m+2)!}D_\tau^m e_{\tau,m+2}^x(t) + M D_\tau^m e_{\tau,m+4}^x(t).$$

We use the shape preserving property (3.3), and divide by $\lambda_n > 0$, to obtain, after evaluating at the point x,

$$\left| \frac{D_\tau^m L_n H_x(x)}{\lambda_n} \right| \leq \frac{2\epsilon}{(m+2)!} \frac{D_\tau^m L_n e_{\tau,m+2}^x(x)}{\lambda_n} + M \frac{D_\tau^m L_n e_{\tau,m+4}^x(x)}{\lambda_n}.$$

As regards the hypothesis of the result for $s = m+2$ and $s = m+4$, after some calculations using (2.2), we can write respectively,

$$\mathcal{L}\left(\frac{D_\tau^m L_n e_{\tau,m+2}^x(x)}{\lambda_n} \right) = \frac{2\tau'(x)^2}{w_2(x) w_1(x) w_0(x)} > 0$$

and

$$\mathcal{L}\left(\frac{D_\tau^m L_n e_{\tau,m+4}^x(x)}{\lambda_n} \right) = 0.$$

Finally, properties (III) and (IV) and the fact that $\epsilon > 0$ was arbitrary, allow us to finish the proof. □

5. Further results

From now on, we will assume that the sequence of linear operators is endowed with an asymptotic condition of the type (4.6). We are going to deduce some consequences of the latter fact. First of all, the existence of an asymptotic condition allows us to establish a local version of the shape preserving property. We use the notation $a_n = o_{\mathcal{L}}(b_n)$ to refer to two sequences such that $a_n, b_n \in \mathcal{S}_0,\ \mathcal{L}(a_n) = \mathcal{L}(b_n) = \mathcal{L}\left(\frac{a_n}{b_n}\right) = 0$.

Lemma 5.1. *Let $h \in C^m[0,1]$ and $x \in (0,1)$. We assume that there exists a neighborhood N_x of x where $D_\tau^m h \geq 0$. Then,*

$$D_\tau^m L_n h(x) \geq 0 + o_{\mathcal{L}}(\lambda_n).$$

Proof. Let $x_1, x_2 \in N_x$ with $x_1 < x < x_2$ and let τ_1, τ_2 belong to the space spanned by $1, \tau^1, \ldots, \tau^m$ such that for $j = 1,2$ and $0 \leq i \leq m$, $D_\tau^i \tau_j(x_j) = D_\tau^i h(x_j)$ (notice that $\{1, \tau^1, \ldots, \tau^m\}$ is a Tchebychev system). Let $\widetilde{h} \in C^m[0,1]$ be defined as:

$$\widetilde{h}(t) = \begin{cases} \tau_1(t) & t < x_1 \\ h(t) & x_1 \leq t \leq x_2 \\ \tau_2(t) & x_2 < t. \end{cases}$$

Then, on $[0,1]$, $D_\tau^m \widetilde{h} \geq 0$ and on (x_1, x_2), $D_\tau^m(\widetilde{h} - h) = 0$. Indeed, it is enough to recall that for $i = 0, 1, \ldots, m-1$, $D_\tau^m \tau^i = 0$ and observe that $D_\tau^m \tau^m = m!$. Finally, using the existence of an asymptotic condition (4.6), yields $D_\tau^m L_n \widetilde{h}(x) - D_\tau^m L_n h(x) = o_{\mathcal{L}}(\lambda_n)$, and from (3.3)

$$0 \leq D_\tau^m L_n \widetilde{h}(x) = D_\tau^m L_n h(x) + o_{\mathcal{L}}(\lambda_n).$$

□

If $g \in C^m[0,1]$ is a solution on $(a,b) \subset [0,1]$ of the ordinary differential equation

$$w_2^{-1} D^1(w_1^{-1} D^1(w_0^{-1} D_\tau^m y)) \equiv 0, \tag{5.7}$$

by asymptotic condition (4.6) it is obvious that if $x \in (a,b)$, $D_\tau^m L_n f(x) - D_\tau^m f(x) = o_{\mathcal{L}}(\lambda_n)$, but the converse is also true, as we can see in the next result.

Theorem 5.3. *Let $a, b \in (0,1)$ with $a < b$. If $f \in C^m[0,1]$ satisfies $D_\tau^m L_n f(x) - D_\tau^m f(x) = o_{\mathcal{L}}(\lambda_n)$ at each point $x \in (a,b)$, then f is a solution of (5.7).*

Before the proof, we write some remarks. The ordinary differential equation (5.7) is of order $m+2$, with fundamental set of solutions $\{1, \tau, \ldots, \tau^{m-1}, y_1, y_2\}$. The change of variable $z = D_\tau^m v$ makes equation (5.7) become the following one of second order:

$$\frac{1}{w_2} D^1 \left(\frac{1}{w_1} D^1 \left(\frac{z}{w_0} \right) \right) \equiv 0. \tag{5.8}$$

The following lemma, whose proof can be found in, [7, Lemma 1] it is necessary for the proof of the theorem.

Lemma 5.2. *Let J be a bounded open subinterval of $[0,1]$. Let $g, h \in C(J)$ and $t_0, t_1, t_2 \in J$ such that $t_0 \in (t_1, t_2)$, $g(t_1) = g(t_2) = 0$ and $g(t_0) > 0$. Then there exist a real number $\alpha < 0$, a solution of the differential equation (5.8) on J, say z, and a point $x \in (t_1, t_2)$ such that for all $t \in [t_1, t_2]$, $\alpha h(t) + z(t) \geq g$, and at the point x, $\alpha h(x) + z(x) = g(x)$.*

Let us proceed to the proof of Theorem 5.3. Let $f \in C^m[0,1]$ and let z_0 be the unique solution of (5.8) such that $z_f(a) = D_\tau^m f(a)$ and $z_0(b) = D_\tau^m f(b)$ and suppose that there exists $x_0 \in (a,b), z_f(x_0) > D_\tau^m f(x_0)$(by linearity, one may proceed analogously if the other inequality is assumed). We apply Lemma 5.2 with $g = z_f - D_\tau^m f, h = D_\tau^m e_{\tau,m+2}^x,\ t_1 = a,\ t_2 = b,\ t_0 = x_0$. In this case, there exist $\alpha < 0$, z solution of (5.8) and $x \in (a,b)$ such that,

$$\begin{gathered} \alpha D_\tau^m e_{\tau,m+2}^x(t) + z(t) \geq z_f(t) - D_\tau^m f(t),\ t \in (a,b), \\ \alpha D_\tau^m e_{\tau,m+2}^x(x) + z(x) = z_f(x) - D_\tau^m f(x). \end{gathered} \tag{5.9}$$

Now if we consider $Z_f \in C^m[0,1],\ D_\tau^m Z_f = z_f$ and $Z \in C^m[0,1],\ D_\tau^m Z = z$, applying the localization Lemma 5.1 and dividing by λ_n, from (5.9) we obtain,

$$\alpha \frac{D_\tau^m L_n e_{\tau,m+2}^x(x) - D_\tau^m e_{\tau,m+2}^x(x)}{\lambda_n} + \frac{D_\tau^m L_n Z(x) - D_\tau^m Z(x)}{\lambda_n} \geq \\ \frac{D_\tau^m L_n Z_f(x) - D_\tau^m Z_f(x)}{\lambda_n} + \frac{o_{\mathcal{L}}(\lambda_n)}{\lambda_n}.$$

We use property (II) to get

$$\alpha \mathcal{L}\left(\frac{D_\tau^m L_n e_{\tau,m+2}^x(x) - D_\tau^m e_{\tau,m+2}^x(x)}{\lambda_n} \right) + \mathcal{L}\left(\frac{D_\tau^m L_n Z(x) - D_\tau^m Z(x)}{\lambda_n} \right) \geq \\ \mathcal{L}\left(\frac{D_\tau^m L_n Z_f(x) - D_\tau^m Z_f(x)}{\lambda_n} \right),$$

and finally we apply asymptotic condition (4.6) to obtain the following expression in contradiction with the hypothesis,

$$\alpha \frac{2\tau'(x)^2}{w_2(x) w_1(x) w_0(x)} > 0,$$

to conclude that $D_\tau^m f$ is a solution of (5.8), so f is a solution of (5.7). □

6. An example

As it was pointed out in the introductory section, in this section we apply the results of the paper to the notion of generalized convergence considered by V. Loku and N. L. Braha [12]. Let p_n be a non negative, non increasing real sequence. Let $N_n^p C_n^1(\cdot)$ be the linear transformation that assigns to each real sequence x_n this other

$$N_n^p C_n^1(x_n) = \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k \frac{1}{k} \sum_{v=1}^k x_v, \quad n \in \mathbb{N}.$$

The sequence x_n is said to be Nörlund-Cesáro summable by the weighted mean determined by p_n, or briefly $(N, p_n)(C, 1)$-summable if

$$\lim N_n^p C_n^1(x_n) = \ell.$$

In that case, the following notation is used: $N_n^p C_n^1 - \lim x_n = \ell$. Moreover, the set of all $(N, p_n)(C, 1)$-summable sequences is denoted by $N_n^p C_n^1$.
Let us now recover the sequence of operators $L_n f(t) = (1 + a_n)B_n^\tau f(t)$ with $a_n \in N_n^p C_n^1$ and $a_n \geq 1$. In order to prove the following statement, no Korovkin-type proof is needed.

Theorem 6.4. *Let $F_0, F_1, F_2 \in C^m[0,1]$ such that $\{D_\tau^m F_0, D_\tau^m F_1, D_\tau^m F_2\}$ is a Tschebyshev System on $C[0,1]$. Then the followings sentences are equivalent:*

(i) $N_n^p C_n^1 - \lim D_\tau^m L_n F_i(x) = D_\tau^m F_i(x), \quad i = 0, 1, 2$
(ii) $N_n^p C_n^1 - \lim D_\tau^m L_n f(x) = D_\tau^m f(x), \quad i = 0, 1, 2,$ *for all function* $f \in C^m[0,1]$.

Following the results of the paper, for the proof of the theorem we only have to check that the shape preserving property (3.3) is fulfilled, and that the linear functional $\mathcal{L}(x_n) = N_n^p C_n^1 - \lim x_n$, defined on $\mathcal{S}_0 = N_n^p C_n^1$ satisfies properties (I)-(VI). Moreover, all the results of the paper apply to this situation accordingly.
Finally, for the sake of completeness, I write a remark about a recent paper. In [15], the authors defined a new sequence of linear operators and proved a result under statistical convergence. The main theorem of the current paper shows an alternative approach to the problem.

References

[1] T. Acar and S. A. Mohiuddine, *Statistical $(C,1)(E,1)$ Summability and Korovkin's Theorem*, Filomat, 30:02, 2016, 387–393.
[2] D. Cárdenas-Morales, P. Garrancho and I. Raşa, *Bernstein-type operators which preserve polynomials*, Comput. Math. Appl., 62, 2011, 158–163.
[3] D. Cárdenas-Morales and P. Garrancho, *$B-$statistical A-summability in conservative approximation*, Math. Inequal. Appl.,19(3), 2016, 923–936.
[4] O. Duman, M. K. Khan and C. Orhan, *A-Statistical convergence of approximating operators*, Math. Inequal. Appl. 6, 2003, 689–699.
[5] H. Fast, *Sur la convergence statistique*, Colloq. Math., 2, 1951, 241–244.
[6] A. D. Gadjiev and C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math., 32, 2002, 129–138.
[7] P. Garrancho and D. Cárdenas-Morales, *A converse of asymptotic formulae in simultaneous approximation*, Appl. Math. and Comp., 217, 2010, 2676–2683.
[8] A. Karaisa, *Statistical $\alpha\beta-$Summability and Korovkin Type Approximation Theorem*, Filomat ,30:13, 2016, 3483–3491.
[9] V. Karakaya and A. Karaisa, *Korovkin type approximation theorems for weighted $\alpha\beta-$statistical convergence*, Bull. Math. Sci., 5, 2015, 159–169.
[10] J. P. King and J. J. Swetits, *Positive linear operators and summability*, Austral J. Math., 11, 1970, 281–291.
[11] P. P. Korovkin, *Linear operators and approximation theory*, Hindustan Publishing Corp., Delhi, India, 1960.
[12] V. Loku and N. L. Braha, *Tauberian Theorems by Weighted Summability Method*, Armenian J. of Math., 9(1), 2017, 35–42.
[13] A.-J. López-Moreno and F.-J. Muñoz-Delgado, *Asymptotic expansion of multivariate conservative linear operators*, J. Comput. Appl. Math, 150, 2003, 219–251.
[14] G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta. Math., 80, 1948, 167–190.
[15] H. Sharma, R. Maurya and C. Gupta, *Approximation properties of Kantorovich Type Modifications of p, q-Meyer-König-Zeller Operators*, Constr. Math. Anal., 1(1), 2018, 58–72.

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Set-Valued Additive Functional Equations

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ABSTRACT. In this paper, we introduce set-valued additive functional equations and prove the Hyers-Ulam stability of the set-valued additive functional equations by using the fixed point method.

Keywords: Hyers-Ulam stability; set-valued additive functional equation; fixed point.

2010 Mathematics Subject Classification: 47H10, 54C60, 39B52, 47H04.

1. INTRODUCTION AND PRELIMINARIES

Set-valued functions in Banach spaces have been developed in the last decades. The pioneering papers by Aumann [4] and Debreu [11] were inspired by problems arising in Control Theory and Mathematical Economics. We can refer to the papers by Arrow and Debreu [2], McKenzie [24], the monographs by Hindenbrand [18], Aubin and Frankowska [3], Castaing and Valadier [7], Klein and Thompson [22] and the survey by Hess [17].

The stability problem of functional equations originated from a question of Ulam [37] concerning the stability of group homomorphisms. Hyers [19] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [35] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [16] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach

The functional equation $f(x+y) = f(x) + f(y)$ is called an *additive functional equation*. In particular, every solution of the additive functional equation is said to be an *additive mapping*. The functional equation $2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$ is called a *Jensen additive functional equation*. In particular, every solution of the Jensen additive functional equation is said to be a *Jensen additive mapping*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [15, 16, 20, 36]).

Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Margolis and Diaz.

Theorem 1.1. [8, 12] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

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for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Rassias [21] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [9, 10, 13, 26, 32, 34]).

Let Y be a Banach space. We define the following:
2^Y : the set of all subsets of Y;
$C_b(Y)$: the set of all closed bounded subsets of Y;
$C_c(Y)$: the set of all closed convex subsets of Y;
$C_{cb}(Y)$: the set of all closed convex bounded subsets of Y.
On 2^Y we consider the addition and the scalar multiplication as follows:

$$C + C' = \{x + x' : x \in C, x' \in C'\}, \qquad \lambda C = \{\lambda x : x \in C\},$$

where $C, C' \in 2^Y$ and $\lambda \in \mathbb{R}$. Further, if $C, C' \in C_c(Y)$, then we denote by $C \oplus C' = \overline{C + C'}$.
It is easy to check that

$$\lambda C + \lambda C' = \lambda(C + C'), \qquad (\lambda + \mu)C \subseteq \lambda C + \mu C.$$

Furthermore, when C is convex, we obtain $(\lambda + \mu)C = \lambda C + \mu C$ for all $\lambda, \mu \in \mathbb{R}^+$.

For a given set $C \in 2^Y$, the distance function $d(\cdot, C)$ and the support function $s(\cdot, C)$ are respectively defined by

$$\begin{aligned} d(x, C) &= \inf\{\|x - y\| : y \in C\}, \qquad x \in Y, \\ s(x^*, C) &= \sup\{\langle x^*, x\rangle : x \in C\}, \qquad x^* \in Y^*. \end{aligned}$$

For every pair $C, C' \in C_b(Y)$, we define the Hausdorff distance between C and C' by

$$H(C, C') = \inf\{\lambda > 0 : C \subseteq C' + \lambda B_Y, \qquad C' \subseteq C + \lambda B_Y\},$$

where B_Y is the closed unit ball in Y.

The following proposition reveals some properties of the Hausdorff distance.

Proposition 1.1. *For every $C, C', K, K' \in C_{cb}(Y)$ and $\lambda > 0$, the following properties hold*
(a) $H(C \oplus C', K \oplus K') \leq H(C, K) + H(C', K')$;
(b) $H(\lambda C, \lambda K) = \lambda H(C, K)$.

Let $(C_{cb}(Y), \oplus, H)$ be endowed with the Hausdorff distance h. Since Y is a Banach space, $(C_{cb}(Y), \oplus, H)$ is a complete metric semigroup (see [7]). Debreu [11] proved that $(C_{cb}(Y), \oplus, H)$ is isometrically embedded in a Banach space as follows.

Lemma 1.1. [11] *Let $C(B_{Y^*})$ be the Banach space of continuous real-valued functions on B_{Y^*} endowed with the uniform norm $\|\cdot\|_u$. Then the mapping $j : (C_{cb}(Y), \oplus, H) \to C(B_{Y^*})$, given by $j(A) = s(\cdot, A)$, satisfies the following properties:*
(a) $j(A \oplus B) = j(A) + j(B)$;
(b) $j(\lambda A) = \lambda j(A)$;
(c) $H(A, B) = \|j(A) - j(B)\|_u$;
(d) $j(C_{cb}(Y))$ *is closed in* $C(B_{Y^*})$
for all $A, B \in C_{cb}(Y)$ and all $\lambda \geq 0$.

Let $f : \Omega \to (C_{cb}(Y), H)$ be a set-valued function from a complete finite measure space (Ω, Σ, ν) into $C_{cb}(Y)$. Then f is *Debreu integrable* if the composition $j \circ f$ is Bochner integrable (see [6]). In this case, the Debreu integral of f in Ω is the unique element $(D)\int_{\Omega} f d\nu \in C_{cb}(Y)$ such tha $j((D)\int_{\Omega} f d\nu)$ is the Bochner integral of $j \circ f$. The set of Debreu integrable functions from Ω to $C_{cb}(Y)$ will be denoted by $D(\Omega, C_{cb}(Y))$. Furthermore, on $D(\Omega, C_{cb}(Y))$, we define $(f+g)(\omega) = f(\omega) \oplus g(\omega)$ for all $f, g \in D(\Omega, C_{cb}(Y))$. Then we obtain that $((\Omega, C_{cb}(Y)), +)$ is an abelian semigroup.

Set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [5, 27, 28, 29, 30, 31, 33]).

Using the fixed point method, we prove the Hyers-Ulam stability of the following set-valued additive functional equations

$$H(F(x+y), F(x) \oplus F(y)) \le \varphi(x,y) \tag{1.1}$$

and

$$H\left(2F\left(\frac{x+y}{2}\right), F(x) \oplus F(y)\right) \le \varphi(x,y). \tag{1.2}$$

Throughout this paper, let X be a real normed space and Y a real Banach space.

2. Stability of the set-valued additive functional equation (1.1)

Using the fixed point method, we prove the Hyers-Ulam stability of the set-valued additive functional equation (1.1).

Definition 2.1. [23] *Let $F : X \to C_{cb}(Y)$. The set-valued additive functional equation is defined by*

$$F(x+y) = F(x) \oplus F(y)$$

for all $x, y \in X$. Every solution of the set-valued additive functional equation is called a set-valued additive mapping.

Definition 2.2. *Let $F : X \to C_{cb}(Y)$. The set-valued Jensen additive functional equation is defined by*

$$2F\left(\frac{x+y}{2}\right) = F(x) \oplus F(y)$$

for all $x, y \in X$. Every solution of the set-valued Jensen additive functional equation is called a set-valued Jensen additive mapping.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{2}\varphi(2x, 2y)$$

for all $x, y \in X$. Suppose that $F : X \to (C_{cb}(Y), H)$ is a mapping satisfying

$$H(F(x+y), F(x) \oplus F(y)) \le \varphi(x,y) \tag{2.3}$$

for all $x, y \in X$. Let r and M be positive real numbers with $r > 1$ and $\operatorname{diam} F(x) \le M\|x\|^r$ for all $x \in X$. Then there exists a unique set-valued additive mapping $A : X \to (C_{cb}(Y), H)$ such that

$$H(F(x), A(x)) \le \frac{L}{2-2L}\varphi(x,x) \tag{2.4}$$

for all $x \in X$.

Proof. Let $y = x$ in (2.3). Since $F(x)$ is convex, we get

$$H(F(2x), 2F(x)) \leq \varphi(x,x) \tag{2.5}$$

and so

$$H\left(F(x), 2F\left(\frac{x}{2}\right)\right) \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{2}\varphi(x,x) \tag{2.6}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g,f) = inf\{\mu \in (0,\infty) : \ H(g(x), f(x)) \leq \mu\varphi(x,x), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [14, Theorem 2.4] and [25, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, f \in S$ be given such that $d(g,f) = \varepsilon$. Then

$$H(g(x), f(x)) \leq \varepsilon\varphi(x,x)$$

for all $x \in X$. Hence

$$H(Jg(x), Jf(x)) \ = \ H\left(2g\left(\frac{x}{2}\right), 2f\left(\frac{x}{2}\right)\right) = 2H\left(g\left(\frac{x}{2}\right), f\left(\frac{x}{2}\right)\right) \leq L\varepsilon\varphi(x,x)$$

for all $x \in X$. So $d(g,f) = \varepsilon$ implies that $d(Jg, Jf) \leq L\varepsilon$. This means that

$$d(Jg, Jf) \leq Ld(g,f)$$

for all $g, f \in S$.

It follows from (2.6) that $d(F, JF) \leq \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.7}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.7) such that there exists a $\mu \in (0,\infty)$ satisfying

$$H(F(x), A(x)) \leq \mu\varphi(x,x)$$

for all $x \in X$;

(2) $d(J^n F, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n F\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(F, A) \leq \frac{1}{1-L}d(F, JF)$, which implies the inequality

$$d(F, A) \leq \frac{L}{2-2L}.$$

This implies that the inequality (2.4) holds.

$$\begin{aligned} H(A(x+y), A(x) \oplus A(y)) &= \lim_{n\to\infty} 2^n H\left(F\left(\frac{x+y}{2^n}\right), F\left(\frac{x}{2^n}\right) \oplus F\left(\frac{y}{2^n}\right)\right) \\ &\leq \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \end{aligned}$$

for all $x, y \in X$. Since $\operatorname{diam} F(x) \leq M\|x\|^r$ for all $x \in X$, $\operatorname{diam}\left(2^n F\left(\frac{x}{2^n}\right)\right) \leq \frac{2^n}{2^{rn}} M\|x\|^r$ for all $x \in X$ and so $A(x) = 2^n F\left(\frac{x}{2^n}\right)$ is a singleton set and $A(x+y) = A(x) \oplus A(y)$ for all $x, y \in X$. □

Corollary 2.1. *Let $p > 1$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $F : X \to (C_{cb}(Y), H)$ is a mapping satisfying*

$$H(F(x+y), F(x) \oplus F(y)) \leq \theta(||x||^p + ||y||^p) \tag{2.8}$$

for all $x, y \in X$. Let r and M be positive real numbers with $r > 1$ and $\operatorname{diam} F(x) \leq M\|x\|^r$ for all $x \in X$. Then there exists a unique set-valued additive mapping $A : X \to Y$ satisfying

$$H(F(x), A(x)) \leq \frac{2\theta}{2^p - 2} ||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $L := 2^{1-p}$ and

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. □

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $F : X \to (C_{cb}(Y), H)$ is a mapping satisfying (2.3). Let r and M be positive real numbers with $r < 1$ and $\operatorname{diam} F(x) \leq M\|x\|^r$ for all $x \in X$. Then there exists a unique set-valued additive mapping $A : X \to (C_{cb}(Y), H)$ such that

$$H(F(x), A(x)) \leq \frac{1}{2 - 2L}\varphi(x, x)$$

for all $x \in X$.

Proof. It follows from (2.5) that

$$H\left(F(x), \frac{1}{2}F(2x)\right) \leq \frac{1}{2}\varphi(x, x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.2. *Let $1 > p > 0$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $F : X \to (C_{cb}(Y), H)$ is a mapping satisfying (2.8). Let r and M be positive real numbers with $r < 1$ and $\operatorname{diam} F(x) \leq M\|x\|^r$ for all $x \in X$. Then there exists a unique set-valued additive mapping $A : X \to Y$ satisfying*

$$H(F(x), A(x)) \leq \frac{2\theta}{2 - 2^p} ||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking $L := 2^{p-1}$ and

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. □

3. Stability of the set-valued Jensen additive functional equation (1.2)

Using the fixed point method, we prove the Hyers-Ulam stability of the set-valued Jensen additive functional equation (1.2).

Theorem 3.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{2}\varphi(2x,2y)$$

for all $x, y \in X$. Suppose that $F : X \to (C_{cb}(Y), H)$ is a mapping satisfying $F(0) = \{0\}$ and

$$H\left(2F\left(\frac{x+y}{2}\right), F(x) \oplus F(y)\right) \le \varphi(x,y) \tag{3.9}$$

for all $x, y \in X$. Let r and M be positive real numbers with $r > 1$ and $\operatorname{diam} F(x) \le M\|x\|^r$ for all $x \in X$. Then there exists a unique set-valued Jensen additive mapping $A : X \to (C_{cb}(Y), H)$ such that

$$H(F(x), A(x)) \le \frac{1}{1-L}\varphi(x,0)$$

for all $x \in X$.

Proof. Let $y = 0$ in (3.9). Since $F(x)$ is convex, we get

$$H\left(F(x), 2F\left(\frac{x}{2}\right)\right) \le \varphi(x,0) \tag{3.10}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y),\ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g,f) = inf\{\mu \in (0,\infty) : \ H(g(x), f(x)) \le \mu\varphi(x,0),\ x \in X\},$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that (S,d) is complete (see [14, Theorem 2.4] and [25, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (3.10) that $d(F, JF) \le 1$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.3. *Let $p > 1$ and $\theta \ge 0$ be real numbers, and let X be a real normed space. Suppose that $F : X \to (C_{cb}(Y), H)$ is a mapping satisfying $F(0) = \{0\}$ and*

$$H\left(2F\left(\frac{x+y}{2}\right), F(x) \oplus F(y)\right) \le \theta(||x||^p + ||y||^p) \tag{3.11}$$

for all $x, y \in X$. Let r and M be positive real numbers with $r > 1$ and $\operatorname{diam} F(x) \le M\|x\|^r$ for all $x \in X$. Then there exists a unique set-valued Jensen additive mapping $A : X \to Y$ satisfying

$$H(F(x), A(x)) \le \frac{2^p\theta}{2^p - 2}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $L := 2^{1-p}$ and

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$.. $\square$

Theorem 3.5. *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. *Suppose that* $F : X \to (C_{cb}(Y), H)$ *is a mapping satisfying* $F(0) = \{0\}$ *and (3.9). Let* r *and* M *be positive real numbers with* $r < 1$ *and* $\operatorname{diam} F(x) \leq M\|x\|^r$ *for all* $x \in X$. *Then there exists a unique set-valued Jensen additive mapping* $A : X \to (C_{cb}(Y), H)$ *such that*

$$H(F(x), A(x)) \leq \frac{L}{1 - L}\varphi(x, 0)$$

for all $x \in X$.

Proof. It follows from (3.11) that

$$H\left(F(x), \frac{1}{2}F(2x)\right) \leq \frac{1}{2}\varphi(2x, 0) \leq L\varphi(x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.4. $\square$

Corollary 3.4. *Let* $0 < p < 1$ *and* $\theta \geq 0$ *be real numbers, and let* X *be a real normed space. Suppose that* $F : X \to (C_{cb}(Y), H)$ *is a mapping satisfying* $F(0) = \{0\}$ *and (3.11). Let* r *and* M *be positive real numbers with* $r < 1$ *and* $\operatorname{diam} F(x) \leq M\|x\|^r$ *for all* $x \in X$. *Then there exists a unique set-valued Jensen additive mapping* $A : X \to Y$ *satisfying*

$$H(F(x), A(x)) \leq \frac{2^p\theta}{2 - 2^p}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.5 by taking $L := 2^{p-1}$ and

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. $\square$

References

[1] T. Aoki: *On the stability of the linear transformation in Banach spaces*. J. Math. Soc. Japan **2** (1950), 64–66.

[2] K. J. Arrow and G. Debreu: *Existence of an equilibrium for a competitive economy*. Econometrica **22** (1954), 265–290.

[3] J. P. Aubin and H. Frankowska: *Set-Valued Analysis*. Birkhäuser, Boston, 1990.

[4] R. J. Aumann: *Integrals of set-valued functions*. J. Math. Anal. Appl. **12** (1965), 1–12.

[5] T. Cardinali, K. Nikodem and F. Papalini: *Some results on stability and characterization of K-convexity of set-valued functions*. Ann. Polon. Math. **58** (1993), 185–192.

[6] T. Cascales and J. Rodrigeuz: *Birkhoff integral for multi-valued functions*. J. Math. Anal. Appl. **297** (2004), 540–560.

[7] C. Castaing and M. Valadier: *Convex Analysis and Measurable Multifunctions*. Lect. Notes in Math. **580**, Springer, Berlin, 1977.

[8] L. Cădariu and V. Radu: *Fixed points and the stability of Jensen's functional equation*. J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).

[9] L. Cădariu and V. Radu: *On the stability of the Cauchy functional equation: a fixed point approach*. Grazer Math. Ber. **346** (2004), 43–52.

[10] L. Cădariu and V. Radu: *Fixed point methods for the generalized stability of functional equations in a single variable*. Fixed Point Theory Appl. **2008**, Art. ID 749392 (2008).

[11] G. Debreu: *Integration of correspondences*. Proceedings of Fifth Berkeley Symposium on Mathematical Statistics and Probability, Vol. II, Part I (1966), 351–372.
[12] J. Diaz and B. Margolis: *A fixed point theorem of the alternative for contractions on a generalized complete metric space*. Bull. Am. Math. Soc. **74** (1968), 305–309.
[13] Iz. EL-Fassi: *New stability results for the radical sextic functional equation related to quadratic mappings in* $(2, \beta)$*-Banach spaces*. J. Fixed Point Theory Appl. **20** (2018), no. 4, Art. 138, 17 pp.
[14] M. Eshaghi Gordji, C. Park and M. B. Savadkouhi: *The stability of a quartic type functional equation with the fixed point alternative*. Fixed Point Theory **11** (2010), 265–272.
[15] M. Eshaghi Gordji and M. B. Savadkouhi: *Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces*. Appl. Math. Letters **23** (2010), 1198–1202.
[16] P. Găvruta: *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*. J. Math. Anal. Appl. **184** (1994), 431–436.
[17] C. Hess: *Set-valued integration and set-valued probability theory: an overview,* in Handbook of Measure Theory. Vols. I, II, North-Holland, Amsterdam, 2002.
[18] W. Hindenbrand: *Core and Equilibria of a Large Economy*. Princeton Univ. Press, Princeton, 1974.
[19] D. H. Hyers: *On the stability of the linear functional equation*. Proc. Nat. Acad. Sci. USA **27** (1941), 222–224.
[20] G. Isac and Th. M. Rassias: *On the Hyers-Ulam stability of* ψ*-additive mappings*. J. Approx. Theory **72** (1993), 131–137.
[21] G. Isac and Th. M. Rassias: *Stability of* ψ*-additive mappings: Applications to nonlinear analysis*. Int. J. Math. Math. Sci. **19** (1996), 219–228.
[22] E. Klein and A. Thompson: *Theory of Correspondence*. Wiley, New York, 1984.
[23] K. Lee: *Stability of functional equations related to set-valued functions* (preprint).
[24] L. W. McKenzie: *On the existence of general equilibrium for a competitive market*. Econometrica **27** (1959), 54–71.
[25] D. Miheţ and V. Radu: *On the stability of the additive Cauchy functional equation in random normed spaces*. J. Math. Anal. Appl. **343** (2008), 567–572.
[26] M. Mirzavaziri and M. S. Moslehian: *A fixed point approach to stability of a quadratic equation*. Bull. Braz. Math. Soc. **37** (2006), 361–376.
[27] K. Nikodem: *On quadratic set-valued functions*. Publ. Math. Debrecen **30** (1984), 297–301.
[28] K. Nikodem: *On Jensen's functional equation for set-valued functions*. Radovi Mat. **3** (1987), 23–33.
[29] K. Nikodem: *Set-valued solutions of the Pexider functional equation*. Funkcialaj Ekvacioj **31** (1988), 227–231.
[30] K. Nikodem: K*-Convex and* K*-Concave Set-Valued Functions*. Zeszyty Naukowe Nr. **559**, Lodz, 1989.
[31] Y. J. Piao: *The existence and uniqueness of additive selection for* (α, β)-(β, α) *type subadditive set-valued maps*. J. Northeast Normal University **41** (2009), 38–40.
[32] S. Pinelas, V. Govindan and K. Tamilvanan: *Stability of a quartic functional equation*. J. Fixed Point Theory Appl. **20** (2018), no. 4, Art. 148, 10 pp.
[33] D. Popa: *Additive selections of* (α, β)*-subadditive set-valued maps*. Glas. Mat. Ser. III, **36 (56)** (2001), 11–16.
[34] V. Radu: *The fixed point alternative and the stability of functional equations*. Fixed Point Theory **4** (2003), 91–96.
[35] Th. M. Rassias: *On the stability of the linear mapping in Banach spaces*. Proc. Am. Math. Soc. **72** (1978), 297–300.
[36] L. Székelyhidi: *Superstability of functional equations related to spherical functions*. Open Math. **15** (2017), 427–432.
[37] S. M. Ulam: *Problems in Modern Mathematics*. Chapter VI, Science ed., Wiley, New York, 1940.

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