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Contents

Second-Order Differential Operators with Non-Local Ventcel's Boundary Conditions

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ABSTRACT. Different boundary conditions have been introduced for second-order differential operators and the properties of the operators on the corresponding domains have been deeply investigated since the work of Feller. The aim of this paper is to study second-order differential operators satisfying a Ventcel's type boundary condition which involves simultaneously both the endpoints of a real interval. We study different general properties and a resolvent estimate for this kind of operators.

Keywords: Ventcel's Type Boundary Conditions, Second-Order Differential Operators, Boundary Conditions

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1. INTRODUCTION AND NOTATION

Let $-\infty \leq r_1 < r_2 \leq +\infty$, set $I :=]r_1, r_2[$ and consider the second-order differential operator

$$(1.1) \qquad Au(x) = \alpha(x)u''(x) + \beta(x)u'(x), \qquad x \in I,$$

where $\alpha, \beta \in C(\overline{I})$ and $\alpha(x) > 0$ for every $x \in I$.

Using the work of Feller [7], several authors have characterized the generation of a C_0-semigroup studying particular boundary conditions on the operator A, such as Ventcel's boundary conditions [5] on the corresponding domain

$$D_V(A) = \left\{ u \in C(\overline{I}) \cap C^2(I) \mid \lim_{x \to r_1} Au(x) = 0 \text{ and } \lim_{x \to r_2} Au(x) = 0 \right\},$$

or the maximal domain [9]

$$D_M(A) = \left\{ u \in C(\overline{I}) \cap C^2(I) \mid Au \in C(\overline{I}) \right\}.$$

We also point out that further characterizations have been obtained in $L^1(I)$ for different domains [3] and even for periodic functions [4].

In this paper, we consider a different boundary condition which involves the two endpoints simultaneously. Namely, we fix a real parameter $\rho \neq 0$ and define

$$(1.2) \qquad D_\rho(A) := \left\{ u \in D_M(A) \mid Au(r_1) + \rho Au(r_2) = 0 \right\}.$$

Obviously

$$D_V(A) \subset D_\rho(A) \subset D_M(A),$$

and this implies that $(A, D_\rho(A))$ is densely defined in $C(\overline{I})$.

Our aim is to study some properties of the operator $(A, D_\rho(A))$ in $C(\overline{I})$. These conditions are expressed in terms of Feller's classification of the boundary points and therefore we make the following recalls. Fix $x_0 \in I$ and define, for every $x \in I$,

$$W(x) := \exp\left(-\int_{x_0}^x \frac{\beta}{\alpha}\right), \; Q(x) := \frac{1}{\alpha(x)W(x)}\int_{x_0}^x W, \; R(x) := W(x)\int_{x_0}^x \frac{1}{\alpha W}.$$

According to the classification introduced by Feller [7] (see also [6, p. 396]), the endpoint r_2 is

(1.3)

a regular boundary	if	$Q \in L^1(]x_0, r_2[)$, $R \in L^1(]x_0, r_2[)$;
an exit boundary	if	$Q \notin L^1(]x_0, r_2[)$, $R \in L^1(]x_0, r_2[)$;
an entrance boundary	if	$Q \in L^1(]x_0, r_2[)$, $R \notin L^1(]x_0, r_2[)$;
a natural boundary	if	$Q \notin L^1(]x_0, r_2[)$, $R \notin L^1(]x_0, r_2[)$.

The classification of r_1 is similar with $]r_1, x_0[$ in place of $]x_0, r_2[$.

Boundary conditions involving simultaneously the two endpoints arise in many differential problems, such as oscillations under external and resistant forces or the working of kidneys in biology (see also [8, Sections IV.3 and V.2] and [7, Section 23] for some related discussions).

2. GENERAL PROPERTIES OF NON-LOCAL BOUNDARY CONDITIONS.

We begin with some preliminary properties of the operator $(A, D_\rho(A))$. Some straightforward details given in [7] and also in [5, 9] will be briefly outlined.

Lemma 2.1. *The operator $(A, D_\rho(A))$ is closed.*

Proof. Let $u \in D_\rho(A)$ and $[a, b] \subset]r_1, r_2[$. If $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that $|u'(x)| \le \varepsilon \|u''\|_{[a,b]} + K_\varepsilon \|u\|_{[a,b]}$ for every $x \in [a, b]$. Since $u \in C^2([a, b])$ and β is continuous, we can find $C_\varepsilon > 0$ such that $|\beta(x)u'(x)| \le \varepsilon \|u''\|_{[a,b]} + C_\varepsilon \|u\|_{[a,b]}$.

Since $\alpha > 0$ on $[a, b]$ there exists $C > 0$ such that $\|u''\|_{[a,b]} \le C(\|Au\|_{[a,b]} + \|u\|_{[a,b]})$. Now, consider a sequence (u_n) in $D_\rho(A)$ which converges uniformly to $u \in C([r_1, r_2])$ and such that (Au_n) converges uniformly to $v \in C([r_1, r_2])$. We have only to show that $u \in D_\rho(A)$ and $Au = v$.

If $[a, b] \subset]r_1, r_2[$, from the preceding estimates we have that (u_n'') is a Cauchy sequence in $C([a, b])$ and therefore it is uniformly convergent in $C([a, b])$ and analogously, we have also the uniform convergence in $C([a, b])$ of the sequence (u_n'). We conclude that $u \in C^2([a, b])$ and $Au = v$ on $[a, b]$. Since $[a, b]$ is arbitrarily chosen, we get $u \in C([r_1, r_2]) \cap C^2(]r_1, r_2[)$ and $Au = v$. Finally, the boundary condition $\lim_{x \to r_1} Au(x) + \rho \lim_{x \to r_2} Au(x) = 0$ is a consequence of the uniform convergence of (Au_n) to Au and $u_n \in D_\rho(A)$. $\square$

Let $\lambda > 0$, $x_0 \in I$ and denote by $\Sigma_\lambda(x_0)$ the set of all solutions $u \in C(\overline{I}) \cap C^2(I)$ of $\lambda u - Au = 0$ satisfying $u(x_0) = 1$. Moreover, let

$$A_{1,\lambda}(x_0) = \{u \in \Sigma_\lambda(x_0) \mid u \text{ is positive and increasing}\},$$

$$A_{2,\lambda}(x_0) = \{u \in \Sigma_\lambda(x_0) \mid u \text{ is positive and decreasing}\}.$$

From [7, Sections 8 and 9] and [5] there exist $\chi_{1,\lambda} \in A_{1,\lambda}(x_0)$ and $\chi_{2,\lambda} \in A_{2,\lambda}(x_0)$ such that

(1) If $u \in A_{1,\lambda}(x_0)$ we have $u \ge \chi_{1,\lambda}$ on $[r_1, x_0]$ and $u \le \chi_{1,\lambda}$ on $[x_0, r_2]$.
(2) If $u \in A_{2,\lambda}(x_0)$ we have $u \le \chi_{2,\lambda}$ on $[r_1, x_0]$ and $u \ge \chi_{2,\lambda}$ on $[x_0, r_2]$.

Moreover, if we put

$$\gamma_{i,\lambda} := \lim_{x \to r_i} \chi_{i,\lambda}(x)(= \inf \chi_{i,\lambda}), \quad M_{i,\lambda} := \lim_{x \to r_{3-i}} \chi_{i,\lambda}(x)(= \sup \chi_{i,\lambda}), \quad i = 1, 2,$$

we have ([7, Lemma 9.3] and [5, Lemma 5 and Lemma 7])

$$\gamma_{1,\lambda} > 0 \quad \Leftrightarrow \quad W \notin L^1(r_1, x_0) \quad \text{and} \quad \int_{r_1}^{x_0} W(x) \int_{r_1}^{x} \frac{1}{\alpha(t)W(t)} \, dt \, dx < +\infty,$$

$$\gamma_{2,\lambda} > 0 \quad \Leftrightarrow \quad W \notin L^1(x_0, r_2) \quad \text{and} \quad \int_{x_0}^{r_2} W(x) \int_{x}^{r_2} \frac{1}{\alpha(t)W(t)} \, dt \, dx < +\infty,$$

that is, r_1 (respectively r_2) is not an entrance boundary.
Finally, in [9, Lemma 6] it has been shown that

$$M_{1,\lambda} < +\infty \Leftrightarrow \int_{x_0}^{r_2} R(x) \, dx < +\infty, \quad M_{2,\lambda} < +\infty \Leftrightarrow \int_{r_1}^{x_0} R(x) \, dx < +\infty,$$

which means that r_1 (respectively r_2) is not entrance nor natural.
If $u_i \in A_{i,\lambda}(x_0)$, $i = 1, 2$, we can define the Green's function

$$(2.4) \qquad G_{u_1,u_2}(x,s) := \begin{cases} \frac{u_1(x)\, u_2(s)}{\alpha(s)\, v(s)}, & r_1 < x \leq s < r_2, \\ \frac{u_1(x)\, u_2(s)}{\alpha(s)\, v(s)}, & r_1 < s < x < r_2, \end{cases}$$

where $v(s) := u_1'(s)u_2(s) - u_2'(s)u_1(s)$. Then for every $f \in C(\overline{I})$, the function

$$(2.5) \qquad u_f(x) := \int_{r_1}^{r_2} G_{u_1,u_2}(x,s) \, f(s) \, ds, \qquad x \in]r_1, r_2[$$

is a solution of $\lambda u_f - A u_f = f$ satisfying $\lambda \|u_f\| \leq \|f\|$ and is positive whenever f is positive (see [7, Theorem 13.1]); if the endpoint r_i, $i = 1, 2$, is a regular or an exit point, we also have $\lim_{x \to r_i} u_f(x) = 0$ (see [8, #8, p. 32]).
Consequently the general solution of $\lambda u - A u = f$ (on I) is given by

$$(2.6) \qquad u_f(x; \lambda, c_1, c_2) = u_f(x) + c_1 \chi_{1,\lambda}(x) + c_2 \chi_{2,\lambda}(x), \quad x \in I.$$

If one of the endpoints is natural, then $D_\rho(A) = D_V(A)$. Indeed, from [7, Corollary to Theorem 13.1], the Ventcel's boundary is always satisfied at a natural endpoint and consequently the non local boundary condition (1.2) implies the Ventcel's boundary condition at the other endpoint too. In this case $(A, D_\rho(A))$ generates a C_0-semigroup if and only if the other endpoint is not an entrance boundary [5, Theorem 2].
In the case of two entrance boundary points, we cannot prescribe the values of Au at any endpoint and therefore we have the generation of a C_0-semigroup only on the maximal domain.
To avoid these well-known cases, in the sequel we shall require

$$(2.7) \qquad D_V(A) \underset{\neq}{\subseteq} D_\rho(A) \underset{\neq}{\subseteq} D_M(A) \, ,$$

which implies that each endpoint is not a natural boundary and that the two endpoints are not both entrance boundaries.
We shall find the most interesting properties in the case where one of the endpoints is an entrance boundary and the other a regular or exit boundary.
We shall require that the integral of R will be finite only on just one of the intervals $[r_1, x_0]$ and $[x_0, r_2]$, that is

$$(2.8) \qquad \int_{x_0}^{r_1} R(x) \, dx < +\infty, \quad \int_{x_0}^{r_2} R(x) \, dx = +\infty,$$

or

$$(2.9) \qquad \int_{x_0}^{r_1} R(x) \, dx = +\infty, \qquad \int_{x_0}^{r_2} R(x) \, dx < +\infty.$$

Remark 2.1. Assume that (2.7) is satisfied. Then, conditions (2.8)–(2.9) hold if and only if one boundary point is entrance and the other is regular or exit.
Indeed, changing the order of integration, we can write

$$\int_{x_0}^{r_1} R(x)\, dx = \int_{r_1}^{x_0} \frac{1}{\alpha(x)W(x)} \int_{r_1}^{x} W(t)\, dt\, dx$$

and

$$\int_{x_0}^{r_2} R(x)\, dx = \int_{x_0}^{r_2} \frac{1}{\alpha(x)W(x)} \int_{x}^{r_2} W(t)\, dt\, dx.$$

If $\int_{x_0}^{r_1} R(x)\, dx < +\infty$, we must have $W \in L(r_1, x_0)$; therefore, according to [7, Section 23], if $\frac{1}{\alpha W} \in L(r_1, x_0)$ then r_1 is a regular boundary point while if $\frac{1}{\alpha W} \notin L(r_1, x_0)$ then r_1 is an exit boundary point.
Conversely, it is clear that if r_1 is a regular or exit boundary point, then necessarily $\int_{x_0}^{r_1} R(x)\, dx < +\infty$ and thus condition $\int_{x_0}^{r_1} R(x)\, dx = +\infty$ implies that r_1 is an entrance or natural boundary point. An analogous reasoning holds at the point r_2.

If (2.8) holds, we must have

(2.10) $$\gamma_{1,\lambda} = 0, \quad M_{1,\lambda} = +\infty, \qquad \gamma_{2,\lambda} > 0, \quad M_{2,\lambda} < +\infty$$

and similarly, if (2.9) holds,

(2.11) $$\gamma_{1,\lambda} > 0, \quad M_{1,\lambda} < +\infty, \qquad \gamma_{2,\lambda} = 0, \quad M_{2,\lambda} = +\infty;$$

indeed, condition (2.8) is equivalent to $M_{1,\lambda} = +\infty$ and $M_{2,\lambda} < +\infty$ and (2.9) to $M_{1,\lambda} < +\infty$ and $M_{2,\lambda} = +\infty$.
Moreover, if (2.8) holds, again by (2.10) it follows that there exists no decreasing positive solution of $\lambda u - Au = 0$ which is independent of $\chi_{2,\lambda}$. Hence, as in [8, pp. 25–37], we can obtain a different expression of $\chi_{2,\lambda}$; namely, for $n \in \mathbf{N}$ and $x \in I$ we define recursively

$$u_0(x) = 1, \qquad u_{n+1}(x) = \int_{x_0}^{x} W(t) \int_{x_0}^{t} \frac{u_n(s)}{\alpha(s)W(s)}\, ds\, dt$$

and we put

(2.12) $$u(x, \lambda) := \sum_{n=0}^{+\infty} \lambda^n u_n(x).$$

A decreasing positive solution of $\lambda u - Au = 0$ is [8, pp. 26–27]

(2.13) $$u_+(x, \lambda) := u(x, \lambda) \int_x^{r_2} \frac{W(t)}{u(t, \lambda)^2}\, dt$$

and normalizing both solutions $\chi_{2,\lambda}$ and $u_+(\cdot, \lambda)$, we obtain

(2.14) $$\chi_{2,\lambda}(x) = M_{2,\lambda} \frac{u_+(x, \lambda)}{u_+(r_1, \lambda)}, \qquad x \in I.$$

Similarly, as a consequence of (2.9), it is possible to show that

(2.15) $$\chi_{1,\lambda}(x) = M_{1,\lambda} \frac{u_-(x, \lambda)}{u_-(r_2, \lambda)}, \qquad x \in\,]r_1, r_2[,$$

where

(2.16) $$u_-(x, \lambda) := u(x, \lambda) \int_{r_1}^{x} \frac{W(t)}{u(t, \lambda)^2}\, dt.$$

Taking into account the above discussion, we can state the following result.

Proposition 2.1. *Assume that condition (2.8) holds. Then, the map $\lambda \mapsto \gamma_{2,\lambda}/M_{2,\lambda}$ is continuous and strictly decreasing and further*

$$\lim_{\lambda \to 0^+} \frac{\gamma_{2,\lambda}}{M_{2,\lambda}} = 1, \quad \lim_{\lambda \to +\infty} \frac{\gamma_{2,\lambda}}{M_{2,\lambda}} = 0.$$

Analogously, if (2.9) holds, then the map $\lambda \mapsto \gamma_{1,\lambda}/M_{1,\lambda}$ is continuous and strictly decreasing and

$$\lim_{\lambda \to 0^+} \frac{\gamma_{1,\lambda}}{M_{1,\lambda}} = 1, \quad \lim_{\lambda \to +\infty} \frac{\gamma_{1,\lambda}}{M_{1,\lambda}} = 0.$$

Proof. We prove only the first part under the assumption that (2.8) holds. First, observe that the function $\lambda \mapsto \gamma_{2,\lambda}/M_{2,\lambda}$ is well-defined by (2.10) and takes its values in the interval $]0,1[$. Its continuity is a direct consequence of (2.12) and (2.13). To show the remaining properties, it is useful to consider the function $u_2(x,\lambda) = u_+(x,\lambda)/u_+(r_1,\lambda)$ and observe that $u_2(r_2,\lambda) = \gamma_{2,\lambda}/M_{2,\lambda}$. Since $u_2(x,\lambda)$ is a solution of $\lambda u - Au = 0$, for every $x \in I$ we have

$$\lambda = \frac{\alpha(x)u_2''(x,\lambda) + \beta(x)u_2'(x,\lambda)}{u_2(x,\lambda)} = \frac{u_2'(x,\lambda)}{u_2(x,\lambda)}\Big(\alpha(x)\frac{u_2''(x,\lambda)}{u_2'(x,\lambda)} + \beta(x)\Big).$$

If $\lambda \to 0^+$ we must have

$$\frac{u_2'(x,\lambda)}{u_2(x,\lambda)} \to 0 \quad \text{or} \quad \alpha(x)\frac{u_2''(x,\lambda)}{u_2'(x,\lambda)} + \beta(x) \to 0.$$

In the first case, we obtain

$$|u_2'(x,\lambda)| \leq \Big|\frac{u_2'(x,\lambda)}{u_2(x,\lambda)}\Big| \to 0$$

and hence $u_2'(x,\lambda) \to 0$ for every $x \in]r_1, r_2[$; then, it follows that $\lim_{\lambda \to 0^+} u_2(x,\lambda) = 1$ and in particular

$$\lim_{\lambda \to 0^+} \frac{\gamma_{2,\lambda}}{M_{2,\lambda}} = \lim_{\lambda \to 0^+} u_2(r_2,\lambda) = 1.$$

In the second case, we should have that $\lim_{\lambda \to 0^+} u_2''(x,\lambda)/u_2'(x,\lambda) = -\beta(x)/\alpha(x)$ from which $\lim_{\lambda \to 0^+} u_2'(x,\lambda) = cW(x)$ for a suitable constant c and consequently

$$\lim_{\lambda \to 0^+} u_2(x,\lambda) = c_1 + c_2 \int_{x_0}^{x} W(t)\, dt .$$

Since the term $\int_{x_0}^{x} W(t)\, dt$ is unbounded at r_2, the only possibility is that $c_2 = 0$ and consequently $c_1 = 1$ by evaluation at r_1; so, we obtain again $\lim_{\lambda \to 0^+} u_2(x,\lambda) = 1$.

If $\lambda \to +\infty$, by the equality $\lambda = \alpha(x)u_2''(x,\lambda)/u_2(x,\lambda) + \beta(x)u_2'(x,\lambda)/u_2'(x,\lambda)$, we deduce that necessarily $u_2(x,\lambda) \to 0$ at least at one point (and hence at every point by monotonicity). Otherwise, since the functions α and β are bounded, at every point we should have that $u_2'(x,\lambda) \to -\infty$ or $|u_2''(x,\lambda)| \to +\infty$. It follows that the set $X = \{x \in]r_1, r_2[\mid \lim_{\lambda \to +\infty} u_2'(x,\lambda) = -\infty\}$ is dense in $]r_1, r_2[$; for, if $[x,y] \cap X = \emptyset$ with $x < y$, then by the equality $u_2'(y,\lambda) - u_2'(x,\lambda) = u_2''(t_\lambda,\lambda)(y-x)$ we get a contradiction by taking a cluster point t for $(t_\lambda)_{\lambda>0}$ where we should have $\lim_{\lambda \to +\infty} |u_2''(t,\lambda)| = +\infty$. At this point, it is clear that $\lim_{\lambda \to +\infty} u_2(x,\lambda) = 0$ for every $x \in]r_1, r_2[$ and in particular, by the continuity of the map $\lambda \mapsto u_2(\cdot, \lambda)$, we conclude that

$$\lim_{\lambda \to +\infty} \frac{\gamma_{2,\lambda}}{M_{2,\lambda}} = \lim_{\lambda \to +\infty} u_2(r_2,\lambda) = 0.$$

Finally, we have to show that $(u_2(r_2,\lambda))_{\lambda>0}$ is decreasing.

Let $0 < \lambda < \mu$; then, for every $x \in]r_1, r_2[$ we have

$$\Big(\frac{u_2'(x,\mu)}{W(x)}\Big)'\frac{1}{u_2(x,\mu)} - \Big(\frac{u_2'(x,\lambda)}{W(x)}\Big)'\frac{1}{u_2(x,\lambda)} = \mu - \lambda > 0$$

and consequently we have

$$\Big(\frac{u_2'(x,\lambda)}{W(x)}\Big)'\frac{1}{u_2(x,\lambda)} < \Big(\frac{u_2'(y,\mu)}{W(y)}\Big)'\frac{1}{u_2(y,\mu)}$$

for x, y in a suitable interval $]r_2 - \delta_1, r_2[$. By contradiction, assume that $u_2(r_2, \lambda) < u_2(r_2, \mu)$; we can choose $\delta_2 > 0$ such that $u_2(x, \lambda) < u_2(y, \mu)$ for $x, y \in]r_2 - \delta_2, r_2[$. Now, let $\delta = \min\{\delta_1, \delta_2\}$; since $u_2(\cdot, \lambda)/W$ tends to 0 at r_2 for every $\lambda > 0$ (see, *e.g.*, [7, Section 11, III, p. 488]), for every $x \in]r_2 - \delta, r_2[$ we can write

$$\frac{u_2'(x,\lambda)}{W(x)} = \Big(\frac{u_2'(\cdot,\lambda)}{W}\Big)'(t)(x - r_2), \ \frac{u_2'(x,\mu)}{W(x)} = \Big(\frac{u_2'(\cdot,\mu)}{W}\Big)'(s)(x - r_2)$$

for suitable $t, s \in]r_2 - \delta, r_2[$ and hence, by the above inequalities, we obtain

$$u_2'(x,\lambda)/W(x) > u_2'(x,\mu)/W(x), i.e., u_2'(x,\lambda) > u_2'(x,\mu) \ .$$

We observe that the inequality $u_2'(x, \lambda) > u_2'(x, \mu)$ cannot hold for every $x \in]r_1, r_2[$, since $u_2'(r_1, \lambda) = u_2'(r_1, \mu) = 1$. Hence we could not have $u_2(r_2, \lambda) < u_2(r_2, \mu)$. Then, we can find $x_1 \in]r_1, r_2[$ for which $u_2'(x_1, \lambda) = u_2'(x_1, \mu)$ and $u_2'(x, \lambda) > u_2'(x, \mu)$ for every $x \in]x_1, r_2[$. Finally, consider the function $u = u_2(\cdot, \mu) - u_2(\cdot, \lambda)$; u is strictly decreasing and positive on $[x_1, r_2]$. Consequently, we have $u'(x_1) = 0$, $u''(x_1) \le 0$ and hence $Au(x_1) \le 0$; on the other hand $Au(x_1) = Au_2(x_1, \mu) - Au_2(x_1, \lambda) = \mu u_2(x_1, \mu) - \lambda u_2(x_1, \lambda) > \lambda(u_2(x_1, \mu) - u_2(x_1, \lambda)) = \lambda u(x_1) > 0$ which is a contradiction. So, the proof is complete. $\qquad\square$

Theorem 2.1. *Assume that $\rho \ne 0$ and that $(A, D_\rho(A))$ satisfies condition (2.7). Then the following statements are equivalent:*

    a) *There exists $\lambda_0 \ge 0$ such that $\lambda - A$ is injective for every $\lambda > \lambda_0$.*

    b) *$(A, D_\rho(A))$ satisfies condition (2.8) or (2.9).*

Moreover, if condition (2.8) holds, we can take $\lambda_0 = 0$ if and only if $\rho \in [-1, +\infty[$ while, if condition (2.9) holds, we can take $\lambda_0 = 0$ if and only if $\rho \in]-\infty, -1] \cup [0, +\infty[$.

Proof. For every $\lambda > 0$ the general solution in I of $\lambda u - Au = 0$ is given by (see (2.6)) $u_0(\cdot; \lambda, c_1, c_2) = c_1\chi_{1,\lambda} + c_2\chi_{2,\lambda}$, with $c_1, c_2 \in \mathbb{R}$.

We discuss the existence of a unique solution in $D_\rho(A)$ on the finiteness of $M_{1,\lambda}$ and $M_{2,\lambda}$.

The case $M_1 = M_2 = +\infty$ cannot occur otherwise both the endpoints are of entrance or natural type (see [9, Lemma 6]) and this is excluded by condition (2.7).

If condition (2.8) holds, we have $M_{1,\lambda} = +\infty$ and $M_{2,\lambda} < +\infty$ and the general solution in $\overline{I}$ of $\lambda u - Au = 0$ is $u_0 := c_2\chi_{2,\lambda}$ with $c_2 \in \mathbb{R}$. Moreover $u_0 \in D_\rho(A)$ if and only if $c_2(M_{2,\lambda} + \rho\gamma_{2,\lambda}) = 0$. The term $M_{2,\lambda} + \rho\gamma_{2,\lambda}$ may vanish only if $\rho < -1$ and in this case (see Proposition 2.1) there exists $\lambda_0 > 0$ such that $\rho = -M_{2,\lambda_0}/\gamma_{2,\lambda_0}$. It follows $\gamma_{2,\lambda}/M_{2,\lambda} < \gamma_{2,\lambda_0}/M_{2,\lambda_0}$ for every $\lambda > \lambda_0$ and therefore $M_{2,\lambda} + \rho\gamma_{2,\lambda} > 0$. Hence, for every $\lambda > \lambda_0$, we must have $c_2 = 0$ and we get that the unique solution of $\lambda u - Au = 0$ in $D_\rho(A)$ is $u_0 = 0$. If $\rho \ge -1$ we have $M_{2,\lambda} + \rho\gamma_{2,\lambda} > 0$ for every $\lambda > 0$ and this yields again $c_2 = 0$ and the unique solution $u_0 = 0$ of $\lambda u - Au = 0$ in $D_\rho(A)$.

If condition (2.9) holds, we have $M_{1,\lambda} < +\infty$ and $M_{2,\lambda} = +\infty$ and the general solution in $\overline{I}$ of $\lambda u - Au = 0$ is $u_0 := c_1\chi_{1,\lambda}$ with $c_1 \in \mathbb{R}$. Moreover $u_0 \in D_\rho(A)$ if and only if $c_1(\gamma_{1,\lambda} + \rho M_{1,\lambda}) = 0$. In this case the term $\gamma_{1,\lambda} + \rho M_{1,\lambda}$ may vanish only if $-1 < \rho < 0$ and in this case (Proposition 2.1) there exists just one value $\lambda_0 > 0$ such that $\rho = -\gamma_{1,\lambda_0}/M_{1,\lambda_0}$. It follows

$\gamma_{1,\lambda}/M_{1,\lambda} < \gamma_{1,\lambda_0}/M_{1,\lambda_0}$ for every $\lambda > \lambda_0$ and consequently $\gamma_{1,\lambda} + \rho M_{1,\lambda} > 0$. Thus $c_1 = 0$ and we get the unique solution $u_0 = 0$ of $\lambda u - Au = 0$ in $D_\rho(A)$. If $\rho \notin \,]-1,0[$ we have $\gamma_{1,\lambda} + \rho M_{1,\lambda} > 0$ for every $\lambda > 0$ and again $c_1 = 0$ which implies that $u_0 = 0$ is the unique solution of $\lambda u - Au = 0$ in $D_\rho(A)$.

If $M_{1,\lambda}$ and $M_{2,\lambda}$ are both finite, the general solution in $\overline{I}$ of $\lambda u - Au = 0$ is $u_0 := c_1\chi_{1,\lambda} + c_2\chi_{1,\lambda}$ with $c_1, c_2 \in \mathbb{R}$ and we have $u_0 \in D_\rho(A)$ if and only if $c_1(\gamma_{1,\lambda} + \rho M_{1,\lambda}) + c_2(M_{2,\lambda} + \rho\gamma_{2,\lambda}) = 0$. In this case we may always obtain infinite solutions in $D_\rho(A)$ by taking

$$c_1 := -\frac{M_{2,\lambda} + \rho\gamma_{2,\lambda}}{\gamma_{1,\lambda} + \rho M_{1,\lambda}}c_2$$

if $\rho \notin \,]-1,0[$, or

$$c_2 = -\frac{\gamma_{1,\lambda} + \rho M_{1,\lambda}}{M_{2,\lambda} + \rho\gamma_{2,\lambda}}c_1$$

if $\rho \geq -1$. This show that if $\lambda - A$ is injective, we necessarily have that just one of the numbers M_1 and M_2 must be finite and consequently we obtain the validity of condition (2.8) or (2.9). $\square$

Remark 2.2. In general, we cannot expect the validity of the dissipativity property even in the cases where $\lambda - A$ is injective for every $\lambda > 0$.

To show this, consider the operator

$$Au(x) = x(1-x)u''(x) + u'(x), \qquad x \in]0,1[$$

on the domain $D_{-1}(A)$. In this case, choosing $x_0 = 1/2$, we have $W(x) = x/(1-x)$ and $R(x) = (2x-1)/(1-x)$.

Hence A satisfies (2.8); moreover, for $u \in C^2([0,1])$ we have $u \in D_{-1}(A)$ if and only if $u'(0) = u'(1)$.

Thus, if $c > 2$ and $\lambda > 0$, the function $u(x) = c - x$ is in $D_{-1}(A)$ and satisfies $\lambda\|u\| = \lambda c$, while $Au(x) = 1$ and consequently $\|\lambda u - Au\| = \lambda c - 1$; hence, the operator $(A, D_{-1}(A))$ is not dissipative.

A similar discussion can be carried out to show that $(A, D_\rho(A))$ does not satisfy in general the positive minimum principle. If we consider the above example, the function $u(x) = x$ is positive on $[0,1]$ and vanishes at 0, while $Au(x) = -1 < 0$. $\square$

According to the work of Feller [7], if A satisfies condition (2.7) and $\rho \neq 0$, we may not expect that $(A, D_\rho(A))$ generates a (positive) contraction semigroup.

However we can also obtain the surjectivity of the operator $\lambda - A$ for every $\lambda > \lambda_0$.

Proposition 2.2. *Assume that $\rho \neq 0$ and that $(A, D_\rho(A))$ satisfies condition (2.7) and one the conditions (2.8) or (2.9). Then there exists $\lambda_0 \geq 0$ such that $\lambda - A$ is surjective for every $\lambda > \lambda_0$. Moreover, for every $\lambda > \lambda_0$ and every f and $f \in C(\overline{I})$, the unique solution $u \in D_\rho(A)$ of $(\lambda - A)u = f$ satisfies the estimate*

$$(2.17) \qquad\qquad\qquad \|u\| \leq \frac{1 + 2|\rho|}{\lambda - \lambda_0}\|f\| \,.$$

Finally, if condition (2.8) holds, we can take $\lambda_0 = 0$ if and only if $\rho \in [-1,+\infty[$ while, if condition (2.9) holds, we can take $\lambda_0 = 0$ if and only if $\rho \in \,]-\infty,-1] \cup [0,+\infty[$.

Proof. Let λ_0 as in Theorem 2.1 and fix $f \in C(\overline{I})$. If (2.8) holds, the function $\chi_{2,\lambda}$ is bounded and therefore the equation $\lambda u - Au = f$ admits infinite bounded solutions

$$(2.18) \qquad\qquad\qquad u = u_f + c\chi_{2,\lambda},$$

depending on the constant c, where u_f is defined by considering in (2.5) the Green's function $G_{\chi_{1,\lambda},\chi_{2,\lambda}}$ relative to $\chi_{1,\lambda}$ and $\chi_{2,\lambda}$.

Moreover, since $\lim_{x \to r_1} Au_f(x) = 0$, we have

$$\lim_{x \to r_1} Au(x) + \rho \lim_{x \to r_2} Au(x) = c\lambda M_{2,\lambda} + \rho Au_f(r_2) + c\rho\lambda\gamma_{2,\lambda}$$

and therefore we obtain a unique solution in $D_\rho(A)$ corresponding to

(2.19) $$c := -\frac{\rho Au_f(r_2)}{\lambda(M_{2,\lambda} + \rho\gamma_{2,\lambda})}.$$

Hence, the function

(2.20) $$u := u_f - \frac{\rho Au_f(r_2)}{\lambda(M_{2,\lambda} + \rho\gamma_{2,\lambda})}\chi_{2,\lambda}$$

is the unique solution of $(\lambda - A)u = f$ and this completes the proof of the surjectivity of $\lambda - A$. As regards to estimate (2.17), let $f \in C(\overline{I})$ and consider the unique solution $u := (\lambda - A)f$ the unique solution of $(\lambda - A)u = f$ in $D_\rho(A)$ given in (2.20). We already know that the operator A generates a C_0-semigroup of positive contractions on the following domain (see [9, Theorem 4])

$$D_{VM} := \{u \in C(\overline{I}) \cap C^2(I) \mid Au \in C(\overline{I}), \lim_{x \to r_1} Au(x) = 0\} .$$

Hence the equation $(\lambda - A)v = f$ has a unique solution $v := R(\lambda, A)f \in D_{VM}(A)$ and further $\|v\| \le \|f\|/\lambda$. From the equality $AR(\lambda, A) = \lambda R(\lambda, A) - I$ and since $M_{2,\lambda} + \rho\gamma_{2,\lambda} > 0$ for every $\lambda > \lambda_0$, we get

$$\|u\| \le \|v\| + \frac{|\rho| \, \|AR(\lambda, A)f\|}{\lambda(M_{2,\lambda} + \rho\gamma_{2,\lambda})}M_{2,\lambda} \le \frac{\|f\|}{\lambda} + \frac{|\rho| \, \|R(\lambda, A)f\|}{1 + \rho\gamma_{2,\lambda}/M_{2,\lambda}} + \frac{|\rho| \, \|f\|}{\lambda(1 + \rho\gamma_{2,\lambda}/M_{2,\lambda})}$$

$$\le \frac{\|f\|}{\lambda} + \frac{|\rho| \, \|f\|}{\lambda} + \frac{|\rho| \, \|f\|}{\lambda} = \frac{1 + 2|\rho|}{\lambda}\|f\| \le \frac{1 + 2|\rho|}{\lambda - \lambda_0}\|f\|$$

which completes the proof in the case where condition (2.8) holds.
An analogous reasoning can be carried out if condition (2.9) holds.
The last part is a consequence of Theorem 2.1.   □

Assume that $\rho \ne 0$ and that $(A, D_\rho(A))$ satisfies condition (2.7) and one of the conditions (2.8) or (2.9). From the preceding results it follows that $(A, D_\rho(A))$ is closed (Lemma 2.1) and by Theorem 2.1 and Proposition 2.2 we find $\lambda_0 \ge 0$ such that for every $\lambda > \lambda_0$ the operator $\lambda - A$ is invertible and its inverse satisfies the estimate

$$\|(\lambda - A)^{-1}\| \le \frac{1 + 2|\rho|}{\lambda - \lambda_0} .$$

The last estimate does not ensure that that the operator $(A, D_\rho(A))$ generates a C_0-semigroup in $C(\overline{I})$. Moreover, the better case can be obtained by setting $\rho = 0$ and in this case we have the generation of C_0-semigroup from the classical Hille-Yosida theorem.
Many authors have studied the possibility of approximating the solutions of evolution problems by using the iterates of suitable approximation processes (see [1, 2] for a complete description of these methods). Using the results in this paper we can approximate the solution of similar problems when different boundary conditions are assigned.

References

[1] F. Altomare, M. Campiti, *Korovkin-type Approximation Theory and its Applications*, De Gruyter Studies in Mathematics 17, W. De Gruyter, Berlin-New York, 1994.

[2] F. Altomare, M. Cappelletti Montano, V. Leonessa and I. Raşa, *Markov Operators, Positive Semigroups and Approximation Processes*, de Gruyter Studies in Mathematics 61, de Gruyter, Berlin/Boston, 2014.

[3] A. Attalienti, M. Campiti, *Semigroups generated by ordinary differential operators in $L^1(I)$*, Positivity 8 (1) (2004), 11–30.

[4] M. Campiti, S. P. Ruggeri, *Approximation of semigroups and cosine functions in spaces of periodic functions*, Applicable Analysis 86 (2) (2007), 167–186.

[5] Ph. Clément, C. A. Timmermans, *On C_0-semigroups generated by differential operators satisfying Ventcel's boundary conditions*, Indag. Math. 89 (1986), 379–387.

[6] K.-J. Engel, R. Nagel, *One-parameter semigroups for linear evolution equations,* Graduate Text in Mathematics 194, Springer, New York, 2000.

[7] W. Feller, *The parabolic differential equations and the associated semi-groups of transformations,* Annals of Math. 55 (3) (1952), 468–519.

[8] P. Mandl, *Analytical treatment of one-dimensional Markov processes,* Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 151, Springer-Verlag, Berlin-Heidelberg-New York, 1969.

[9] C. A. Timmermans, *On C_0-semigroups in a space of bounded continuous functions in the case of entrance or natural boundary points*. In: Gòmez-Fernandez J.A., Guerra-Vázquez F., Lòpez-Lagomasino G., Jiménez-Pozo M.A. (eds). *Approximation and Optimization*, Lecture Notes in Mathematics, 1354. Springer, Berlin, Heidelberg, 2006.

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Bivariate Generalized Exponential Sampling Series and Applications to Seismic Waves

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ABSTRACT. In this paper, we introduce the generalized exponential sampling series of bivariate functions and establish some pointwise and uniform convergence results, also in a quantitative form. Moreover, we study the pointwise asymptotic behaviour of the series. One of the basic tools is the Mellin–Taylor formula for bivariate functions, here introduced. A practical application to seismic waves is also outlined.

Keywords: Bivariate generalized exponential sampling series, Mellin–Taylor formulae, moments, magnitude.

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Dedicated to the memory of our close unforgettable Friend and Colleague Professor Domenico Candeloro, who survives in our hearts.

1. INTRODUCTION

The classical exponential sampling series of a function $f : \mathbb{R}^+ \to \mathbb{C}$, in one dimensional case, represents a tool of relevant interest for optical phenomena, for example the light scattering and Fraunhofer diffraction, see e.g. [17, 11, 18] and [23].

It is defined by

$$(E_{c,T}f)(x) := \sum_{k=-\infty}^{\infty} f(e^{k/T})\mathrm{lin}_{c/T}(e^{-k}x^T), \quad T > 0, \ x \in \mathbb{R}^+.$$

The lin_c-function for $c \in \mathbb{R}$, $\mathrm{lin}_c : \mathbb{R}^+ \to \mathbb{R}$, is defined, for $x \in \mathbb{R}^+ \setminus \{1\}$, by

$$\mathrm{lin}_c(x) = \frac{x^{-c}}{2\pi i}\frac{x^{\pi i} - x^{-\pi i}}{\log x} = x^{-c}\mathrm{sinc}(\log x) = \frac{x^{-c}}{2\pi}\int_{-\pi}^{\pi} x^{-it}dt,$$

with the continuous extension $\mathrm{lin}_c(1) := 1$.

Here, the "sinc" function, as usual, is defined by

$$\mathrm{sinc}(u) := \frac{\sin(\pi u)}{\pi u}, \ u \neq 0, \ \mathrm{sinc}(0) = 1.$$

The exponential sampling theorem for Mellin band-limited functions states that

$$(E_{c,T}f)(x) = f(x)$$

at every point x, and the series is absolutely and uniformly convergent on every compact interval of $\mathbb{R}^+$. A rigorous treatment of this theorem by a mathematical point of view was given in [14, 3, 10]. This theorem represents a Mellin version of the classical Shannon sampling theorem of Fourier analysis ([20, 27]).

Note that, the exponential sampling theorem was framed in the field of the Mellin transform theory, which was first introduced in [22] and then developed in a systematic way in [13].

Now, as it happens for the Fourier band-limited functions, the assumption that f is Mellin band-limited is very restrictive, due to the Mellin–Paley–Wiener theorem (see [4], [5]). Therefore, in a recent paper [7] it was studied a generalization of the one dimensional exponential sampling theorem where the $\lin_c$ function is replaced by an arbitrary function φ satisfying suitable assumptions in an analogous way as for the generalized sampling series of Fourier analysis (see [15, 16, 6, 21, 2, 1]). In this way, we obtained an approximate reconstruction of a not necessarily Mellin band-limited function f.

The aim of this paper is to introduce a multivariate version of the generalized exponential sampling theorem in order to obtain new interesting applications to the study of the seismic waves. In this respect, for a sake of simplicity, we limit ourselves to consider the two dimensional case, being the general case carried on analogously.

Our main theoretical results concern the pointwise and uniform convergence and the study of the pointwise order of approximation through a bivariate asymptotic Voronovskaja formula. Basic tools are a two dimensional Mellin–Taylor formula, established in Section 3 both in the local and global version, and a notion of logarithmic modulus of continuity here introduced as a generalization of one dimensional case (see [8, 9]).

In Section 6, we give two important examples of bivariate kernel functions satisfying the required assumptions, namely the bivariate Mellin splines and the Mellin–Fejer kernels.

The last section is devoted to the study of the magnitude of an earthquake through the behaviour of the seismic waves.

We wish to dedicate this paper to the memory of our very close friend and colleague Professor Domenico Candeloro who passed away in May. He was a fine mathematician who combined his deep mathematical culture with a great modesty, a trait of his character that makes him an unforgettable person.

2. PRELIMINARIES

Let us denote by $\mathbb{N}^2$, $\mathbb{N}_0^2$ and $\mathbb{Z}^2$ the sets of vectors $\mathrm{k} = (k_1, k_2)$ with k_1, k_2 positive integers, nonnegative integers and integers respectively and we set $\llbracket \mathrm{k} \rrbracket := k_1 + k_2$. Moreover, by $\mathbb{R}^2$ we will denote the two dimensional Euclidean space comprising all vectors (x_1, x_2) with $x_1, x_2 \in \mathbb{R}$.

Given $\mathrm{x} = (x_1, x_2), \mathrm{y} = (y_1, y_2) \in \mathbb{R}^2$ we will say that $\mathrm{x} > \mathrm{y}$ if and only if $x_i > y_i$ for $i = 1, 2$ and we will denote by $1 := (1, 1)$, $0 := (0, 0)$ and by $\mathbb{R}_+^2$ the space of all vectors $\mathrm{x} > 0$.

Given $\mathrm{x}, \mathrm{y} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ we put as usual: $\mathrm{x} + \mathrm{y} := (x_1 + y_1, x_2 + y_2)$ and $\alpha \mathrm{x} := (\alpha x_1, \alpha x_2)$. We will employ the following notations: $\mathrm{xy} := (x_1 y_1, x_2 y_2)$, $\dfrac{\mathrm{x}}{\mathrm{y}} := (\dfrac{x_1}{y_1}, \dfrac{x_2}{y_2})$ (for $y_1, y_2 \neq 0$),

$[\mathrm{x}] := (|x_1|, |x_2|)$, $\alpha^{\mathrm{x}} := (\alpha^{x_1}, \alpha^{x_2})$ with $\alpha > 0$, and $\mathrm{x}^{\mathrm{y}} := \prod\limits_{i=1}^{2} x_i^{y_i}$, $\log(\mathrm{x}) := (\log x_1, \log x_2)$ with $\mathrm{x} > 0$.

We set $\|\mathrm{x}\| := \sqrt{x_1^2 + x_2^2}$, and the Euclidean distance $d(\mathrm{x}, \mathrm{y}) := \|\mathrm{x} - \mathrm{y}\|$.

For $\mathrm{w} = (w_1, w_2) \in \mathbb{R}_+^2$, by $\mathrm{w} \to \infty$ we mean $\underline{w} := \min\{w_1, w_2\} \to +\infty$.

Let J be an interval, bounded or not. We will denote by $C(J)$ the space of all continuous and bounded functions on J, by $C_c(J)$ the space of all continuous functions with compact support. Moreover, for $m \in \mathbb{N}$, by $C^{(m)}(J)$ we denote the subspace of $C(J)$ comprising all functions f with the derivatives up to the order m in $C(J)$.

Now, we introduce the following notion of continuity. We will say that a function $f : J \to \mathbb{C}$ is log-uniformly continuous on J if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|f(\mathrm{x}) - f(\mathrm{y})| <$

ε, whenever $x, y \in \mathbb{R}_+^2$ with $\| \log(x) - \log(y) \| \le \delta(\varepsilon)$. We will denote by $\mathcal{C}(J)$ the space of all log-uniformly continuous and bounded functions on J.

Note that for compact intervals $J \subset \mathbb{R}_+^2$ the notion of log-uniform continuity is equivalent to the classical uniform continuity.

Finally, we will say that a function $f : J \to \mathbb{C}$ belongs to $C^{(m)}(J)$ locally at a point $x \in J$ if there is a neighbourhood I of x such that f is $(m-1)$-times differentiable on I and the derivative $f^{(m)}(x)$ exists.

3. TWO-DIMENSIONAL MELLIN–TAYLOR FORMULAE

We begin with the notion of partial derivatives of a function $f : \mathbb{R}_+^2 \to \mathbb{C}$ in the Mellin frame. The first partial Mellin derivative of f with respect to the variable x_i, $i = 1, 2$ at the point $x = (x_1, x_2)$ is given by

$$\Theta_{x_i} f(x) := x_i \frac{\partial f(x)}{\partial x_i}.$$

For a given $k = (k_1, k_2) \in \mathbb{N}_0^2$ we define the partial Mellin derivatives of order $r = \|k\| = k_1 + k_2$ at the point x as

$$(3.1) \qquad \Theta^r_{x_1^{k_1} x_2^{k_2}} f(x) := \Theta^{k_1}_{x_1}(\Theta^{k_2}_{x_2} f)(x).$$

We will put $\Theta^1_{x_i} f(x) := \Theta_{x_i} f(x)$ e $\Theta^0_{x_i} f(x) := f(x)$.

Note that for example,

$$\Theta^2_{x_i} = \Theta_{x_i}(\Theta_{x_i} f)(x) = x_i \frac{\partial f(x)}{\partial x_i} + x_i^2 \frac{\partial^2 f(x)}{\partial x_i^2}.$$

In order to extend the one-dimensional Mellin–Taylor formulae introduced in [8] to the bivariate case we will use the following notation. For a given $x = (x_1, x_2)$ and $t = (t_1, t_2)$ we set

$$(3.2) \qquad (\Theta_{x_1} \log t_1 + \Theta_{x_2} \log t_2)^m f(x_1, x_2)$$
$$:= \sum_{k=0}^m \binom{m}{k} \Theta^{m-k}_{x_1}(\Theta^k_{x_2} f)(x_1, x_2) \log^{m-k} t_1 \log^k t_2$$

with $m \in \mathbb{N}$ and $f \in C^{(m)}(\mathbb{R}_+^2)$ locally in (x_1, x_2).

For example for $m = 2$ we obtain

$$(\Theta_{x_1} \log t_1 + \Theta_{x_2} \log t_2)^2 f(x_1, x_2) =$$
$$\Theta^2_{x_1} f(x_1, x_2) \log^2 t_1 + 2\Theta_{x_1}(\Theta_{x_2} f)(x_1, x_2) \log t_1 \log t_2 + \Theta^2_{x_2} f(x_1, x_2) \log^2 t_2.$$

We have the following proposition.

Proposition 3.1. *Let* $f \colon \mathbb{R}_+^2 \to \mathbb{C}$ *be a function in* $C^{(m)}(\mathbb{R}_+^2)$ *with* $m \in \mathbb{N}$. *Then for* $x = (x_1, x_2) \in \mathbb{R}_+^2$ *and* $t = (t_1, t_2) \in \mathbb{R}_+^2$, *we have*

$$f(t_1 x_1, t_2 x_2) = f(x_1, x_2) + (\Theta_{x_1} \log t_1 + \Theta_{x_2} \log t_2) f(x_1, x_2) +$$
$$\frac{1}{2!}(\Theta_{x_1} \log t_1 + \Theta_{x_2} \log t_2)^2 f(x_1, x_2) + \ldots +$$
$$\frac{1}{(m-1)!}(\Theta_{x_1} \log t_1 + \Theta_{x_2} \log t_2)^{m-1} f(x_1, x_2) + R_m(t_1, t_2),$$

with Lagrange remainder

$$R_m(t_1, t_2) = \frac{1}{m!}(\Theta_{x_1}\log t_1 + \Theta_{x_2}\log t_2)^m f(\theta, \eta),$$

where (θ, η) is a suitable point in the segment L_{t_1, t_2} with end points (x_1, x_2), $(t_1 x_1, t_2 x_2)$.

Proof. We prove the case $m = 2$. For the general case one can apply (3.2).

Let us take the function $F(t) = f(t_1^{\log t}x_1, t_2^{\log t}x_2)$ with $t \in [1, e]$. Applying the one dimensional Mellin–Taylor formula with Lagrange remainder, we obtain

$$F(t) = F(1) + \Theta F(1)\log t + \frac{\Theta^2 F(\tilde{t})}{2}\log^2 t$$

with $\tilde{t} \in]1, e[$. We have

$$\Theta F(t) = \frac{\partial f}{\partial x_1}(t_1^{\log t}x_1, t_2^{\log t}x_2)x_1 t_1^{\log t}\log t_1 + \frac{\partial f}{\partial x_2}(t_1^{\log t}x_1, t_2^{\log t}x_2)x_2 t_2^{\log t}\log t_2$$

and for $t = 1$ we obtain

$$\Theta F(1) = \Theta_{x_1}f(x_1, x_2)\log t_1 + \Theta_{x_2}f(x_1, x_2)\log t_2 = (\Theta_{x_1}\log t_1 + \Theta_{x_2}\log t_2)f(x_1, x_2).$$

Analogously for $\Theta^2 F(t) = tF'(t) + t^2 F''(t)$, we have

$$\Theta^2 F(t) = \frac{\partial^2 f}{\partial x_1^2}(t_1^{\log t}x_1, t_2^{\log t}x_2)x_1^2 t_1^{2\log t}\log^2 t_1 + \frac{\partial^2 f}{\partial x_2^2}(t_1^{\log t}x_1, t_2^{\log t}x_2)x_2^2 t_2^{2\log t}\log^2 t_2 +$$

$$2\frac{\partial^2 f}{\partial y \partial x_1}(t_1^{\log t}x_1, t_2^{\log t}x_2)x_1 x_2(t_1 t_2)^{\log t}\log t_1 \log t_2 +$$

$$\frac{\partial f}{\partial x_1}(t_1^{\log t}x_1, t_2^{\log t}x_2)x_1 t_1^{\log t}\log^2 t_1 + \frac{\partial f}{\partial x_2}(t_1^{\log t}x_1, t_2^{\log t}x_2)x_2 t_2^{\log t}\log^2 t_2$$

and for $t = \tilde{t}$

$$\frac{\Theta^2 F(\tilde{t})}{2} = \frac{1}{2}\left\{\left(\frac{\partial^2 f}{\partial x_1^2}(\theta, \eta)\theta^2 + \frac{\partial f}{\partial x_1}(\theta, \eta)\theta\right)\log^2 t_1 + \left(\frac{\partial^2 f}{\partial x_2^2}(\theta, \eta)\eta^2 + \frac{\partial f}{\partial x_2}(\theta, \eta)\eta\right)\log^2 t_2\right.$$

$$\left. + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(\theta, \eta)\theta\eta\log t_1 \log t_2\right\},$$

with $(\theta, \eta) = (t_1^{\log \tilde{t}}x_1, t_2^{\log \tilde{t}}x_2) \in L_{t_1, t_2}$.

Now, using the definition of the partial Mellin derivative, we have the formulae

$$\frac{\partial^2 f}{\partial x_1^2}(\theta, \eta)\theta^2 = [\Theta_{x_1}^2 f(\theta, \eta) - \Theta_{x_1}f(\theta, \eta)],$$

$$\frac{\partial^2 f}{\partial x_2^2}(\theta, \eta)\eta^2 = [\Theta_{x_2}^2 f(\theta, \eta) - \Theta_{x_2}f(\theta, \eta)],$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(\theta, \eta)\theta\eta = \Theta_{x_1}(\Theta_{x_2}f)(\theta, \eta),$$

then

$$\frac{\Theta^2 F(\tilde{t})}{2} = \frac{1}{2}(\Theta_{x_1}\log t_1 + \Theta_{x_2}\log t_2)^2 f(\theta, \eta).$$

So the assertion follows. $\square$

By Proposition 3.1, we can deduce a local version of the Mellin–Taylor formula, namely a formula with the Peano remainder. It is based on the following proposition

Proposition 3.2. *Under the same assumptions and notations of Proposition 3.1 there holds*

$$\lim_{(t_1,t_2)\to(1,1)} \frac{(\Theta_{x_1}\log t_1 + \Theta_{x_2}\log t_2)^m f(\theta,\eta) - (\Theta_{x_1}\log t_1 + \Theta_{x_2}\log t_2)^m f(x_1,x_2)}{(\log^2 t_1 + \log^2 t_2)^{m/2}} = 0.$$

Proof. We consider the case $m = 2$, the general case is carried on in a similar way. Setting

$$I := \left|(\Theta_{x_1}\log t_1 + \Theta_{x_2}\log t_2)^2 f(\theta,\eta) - (\Theta_{x_1}\log t_1 + \Theta_{x_2}\log t_2)^2 f(x_1,x_2)\right|,$$

we have

$$\begin{aligned}
I \leq &\ |\Theta_{x_1}^2 f(\theta,\eta) - \Theta_{x_1}^2 f(x_1,x_2)|\log^2 t_1 \\
+ &\ 2|\Theta_{x_1}(\Theta_{x_2}f)(\theta,\eta) - \Theta_{x_1}(\Theta_{x_2}f)(x_1,x_2)||\log t_1 \log t_2| \\
+ &\ |\Theta_{x_2}^2 f(\theta,\eta) - \Theta_{x_2}^2 f(x_1,x_2)|\log^2 t_2,
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{I}{\log^2 t_1 + \log^2 t_2} \leq &|\Theta_{x_1}^2 f(\theta,\eta) - \Theta_{x_1}^2 f(x_1,x_2)| + |\Theta_{x_1}(\Theta_{x_2}f)(\theta,\eta) - \Theta_{x_1}(\Theta_{x_2}f)(x_1,x_2)| + \\
&|\Theta_{x_2}^2 f(\theta,\eta) - \Theta_{x_2}^2 f(x_1,x_2)|.
\end{aligned}$$

Taking into account that $(\theta,\eta) \in L_{t_1,t_2}$ the assertion follows from the assumption $f \in C^{(2)}(\mathbb{R}_+^2)$. $\square$

By Proposition 3.2, we can write the local form of the Mellin–Taylor formula as

$$\begin{aligned}
f(t_1 x, t_2 y) =&\ f(x_1,x_2) + (\Theta_{x_1}\log t_1 + \Theta_{x_2}\log t_2)f(x_1,x_2) + \\
&\ \frac{1}{2!}(\Theta_{x_1}\log t_1 + \Theta_{x_2}\log t_2)^2 f(x_1,x_2) + \ldots + \\
&\ \frac{1}{m!}(\Theta_{x_1}\log t_1 + \Theta_{x_2}\log t_2)^m f(x_1,x_2) + R_m(t_1,t_2),
\end{aligned}$$

with the Peano remainder

$$R_m(t_1,t_2) = H(t_1,t_2)(\log^2 t_1 + \log^2 t_2)^{m/2},$$

where $H(t_1,t_2)$ is a bounded function such that $\lim_{(t_1,t_2)\to(1,1)} H(t_1,t_2) = 0$.

Remark 3.1. Note that setting $\mathtt{t} = (t_1,t_2)$, $\mathtt{x} = (x_1,x_2)$ we can write

$$R_m(\mathtt{t}) = H(\mathtt{t})\|\log(\mathtt{tx}) - \log\mathtt{x}\|^m.$$

Moreover, it is not difficult to see that the local version of the Mellin–Taylor formula can be proved under the more general assumptions $f \in C^{(m)}(\mathbb{R}_+^2)$ locally at the point $\mathtt{x}$.

4. BIVARIATE GENERALIZED SAMPLING OPERATOR

Let $\varphi \colon \mathbb{R}_+^2 \to \mathbb{R}$ be a continuous function such that

$(\varphi.1)$ $\sum_{\mathtt{k}\in\mathbb{Z}^2} \varphi(e^{-\mathtt{k}}\mathtt{x}) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} \varphi(e^{-k_1}x_1, e^{-k_2}x_2) = 1$ for every $\mathtt{x} = (x_1,x_2) \in \mathbb{R}_+^2$;

$(\varphi.2)$ there holds

$$M_0(\varphi) := \sup_{\mathtt{x}\in\mathbb{R}_+^2} \sum_{\mathtt{k}\in\mathbb{Z}^2} |\varphi(e^{-\mathtt{k}}\mathtt{x})| < +\infty;$$

$(\varphi.3)$ $\lim_{r\to+\infty} \sum_{\|\mathtt{k}-\log(\mathtt{x})\|\geq r} |\varphi(e^{-\mathtt{k}}\mathtt{x})| = 0$, uniformly with respect to $\mathtt{x}$.

Let Φ be the class of all functions φ satisfying the above assumptions.

Let $\mathbf{j} = (j_1, j_2) \in \mathbb{N}_0^2$ and let $\nu = [\![\mathbf{j}]\!]$. For $\mathbf{x} \in \mathbb{R}_+^2$, we define the moments of order $\mathbf{j}$ of $\varphi \in \Phi$ as

$$m_{\mathbf{j}}^{\nu}(\varphi, \mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^2} \varphi(e^{-\mathbf{k}}\mathbf{x}) \log^{\mathbf{j}}(e^{\mathbf{k}}\mathbf{x}^{-1}) = \sum_{\mathbf{k} \in \mathbb{Z}^2} \varphi(e^{-\mathbf{k}}\mathbf{x})(\mathbf{k} - \log(\mathbf{x}))^{\mathbf{j}}$$

$$= \sum_{(k_1, k_2) \in \mathbb{Z}^2} \varphi(e^{-k_1}x_1, e^{-k_2}x_2)(k_1 - \log x_1)^{j_1}(k_2 - \log x_2)^{j_2}.$$

The absolute moments of order $\mathbf{j}$ of $\varphi \in \Phi$ are defined as

$$M_{\mathbf{j}}^{\nu}(\varphi, \mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^2} |\varphi(e^{-\mathbf{k}}\mathbf{x})|[\log(e^{\mathbf{k}}\mathbf{x}^{-1})]^{\mathbf{j}} = \sum_{\mathbf{k} \in \mathbb{Z}^2} |\varphi(e^{-\mathbf{k}}\mathbf{x})|[\mathbf{k} - \log(\mathbf{x})]^{\mathbf{j}}$$

$$= \sum_{(k_1, k_2) \in \mathbb{Z}^2} |\varphi(e^{-k_1}x_1, e^{-k_2}x_2)||k_1 - \log x_1|^{j_1} |k_2 - \log x_2|^{j_2}.$$

Finally, we set $M_{\mathbf{j}}^{\nu}(\varphi) := \sup_{\mathbf{x} \in \mathbb{R}_+^2} M_{\mathbf{j}}^{\nu}(\varphi, \mathbf{x})$.

Let $\varphi \in \Phi$. For any $\mathbf{w} > 0$, $\mathbf{w} = (w_1, w_2)$ and $f \colon \mathbb{R}_+^2 \to \mathbb{C}$, we define the generalized exponential series as

$$(4.3) \qquad (E_{\mathbf{w}}^{\varphi}f)(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^2} f(e^{\frac{\mathbf{k}}{\mathbf{w}}})\varphi(e^{-\mathbf{k}}\mathbf{x}^{\mathbf{w}}) = \sum_{(k_1, k_2) \in \mathbb{Z}^2} f(e^{\frac{k_1}{w_1}}, e^{\frac{k_2}{w_2}})\varphi(e^{-k_1}x_1^{w_1}, e^{-k_2}x_2^{w_2})$$

for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}_+^2$ and for any function $f \in \operatorname{dom} E_{\mathbf{w}}^{\varphi}$, being $\operatorname{dom} E_{\mathbf{w}}^{\varphi}$ the set of all functions f for which the series is absolutely convergent on every $\mathbf{x}$. Using the conditions of the class Φ, it is easy to see that the above operator is well defined as an absolutely convergent series, for any bounded function f. In particular $C(\mathbb{R}_+^2) \subset \operatorname{dom} E_{\mathbf{w}}^{\varphi}$, for any $\mathbf{w} > 0$.

We begin with the following pointwise convergence theorem.

Theorem 4.1. *Let $f \in C(\mathbb{R}_+^2)$ and $\varphi \in \Phi$. Then*

$$(4.4) \qquad \lim_{\mathbf{w} \to \infty} \sum_{\mathbf{k} \in \mathbb{Z}^2} f(e^{\frac{\mathbf{k}}{\mathbf{w}}})\varphi(e^{-\mathbf{k}}\mathbf{x}^{\mathbf{w}}) = f(\mathbf{x}), \quad for \quad \mathbf{x} \in \mathbb{R}_+^2.$$

Proof. Since $\varphi \in \Phi$, we have

$$\left| \sum_{\mathbf{k} \in \mathbb{Z}^2} f(e^{\frac{\mathbf{k}}{\mathbf{w}}})\varphi(e^{-\mathbf{k}}\mathbf{x}^{\mathbf{w}}) - f(\mathbf{x}) \right| \le \sum_{\mathbf{k} \in \mathbb{Z}^2} |f(e^{\frac{\mathbf{k}}{\mathbf{w}}}) - f(\mathbf{x})||\varphi(e^{-\mathbf{k}}\mathbf{x}^{\mathbf{w}})|.$$

For a fixed $\varepsilon > 0$, by the continuity of f at $\mathbf{x}$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\| \log(\mathbf{x}) - \log(e^{\frac{\mathbf{k}}{\mathbf{w}}}) \| = \| \log(\mathbf{x}) - \frac{\mathbf{k}}{\mathbf{w}} \| < \delta$, then $|f(\mathbf{x}) - f(e^{\frac{\mathbf{k}}{\mathbf{w}}})| < \varepsilon$. We write

$$\sum_{\mathbf{k} \in \mathbb{Z}^2} |f(e^{\frac{\mathbf{k}}{\mathbf{w}}}) - f(\mathbf{x})||\varphi(e^{-\mathbf{k}}\mathbf{x}^{\mathbf{w}})|$$

$$= \left\{ \sum_{\| \frac{\mathbf{k}}{\mathbf{w}} - \log(\mathbf{x}) \| < \delta} + \sum_{\| \frac{\mathbf{k}}{\mathbf{w}} - \log(\mathbf{x}) \| \ge \delta} \right\} |f(e^{\frac{\mathbf{k}}{\mathbf{w}}}) - f(\mathbf{x})||\varphi(e^{-\mathbf{k}}\mathbf{x}^{\mathbf{w}})| =: I_1 + I_2.$$

Now by assumption $(\varphi.2)$, we have immediately $I_1 \leq M_0(\varphi)\,\varepsilon$. As to I_2 by the boundedness of f and $(\varphi.3)$, taking into account that

$$\|\frac{\mathbf{k}}{\mathtt{w}} - \log(\mathbf{x})\| \leq \frac{\|\mathbf{k} - \log(\mathbf{x}^{\mathtt{w}})\|}{\underline{w}},$$

we have, for sufficiently large $\underline{w}$,

$$I_2 = \sum_{\|\frac{\mathbf{k}}{\mathtt{w}} - \log(\mathbf{x})\| \geq \delta} |f(e^{\frac{\mathbf{k}}{\mathtt{w}}}) - f(\mathbf{x})||\varphi(e^{-\mathbf{k}}\mathbf{x}^{\mathtt{w}})| \leq 2\|f\|_\infty \sum_{\|\mathbf{k} - \log(\mathbf{x}^{\mathtt{w}})\| \geq \delta\underline{w}} |\varphi(e^{-\mathbf{k}}\mathbf{x}^{\mathtt{w}})| < 2\|f\|_\infty \varepsilon,$$

and so the assertion follows. $\qquad\square$

Using essentially the same reasoning employed in the previous theorem, we can prove the following uniform convergence result.

Theorem 4.2. *Let $f \in \mathcal{C}(\mathbb{R}_+^2)$ and $\varphi \in \Phi$, then*

$$(4.5) \qquad \lim_{\mathtt{w}\to\infty} \Big\| \sum_{\mathbf{k}\in\mathbb{Z}^2} f(e^{\frac{\mathbf{k}}{\mathtt{w}}})\varphi(e^{-\mathbf{k}}\mathbf{x}^{\mathtt{w}}) - f(\mathbf{x}) \Big\|_\infty = 0.$$

5. ESTIMATION OF THE ERROR OF APPROXIMATION

We premise the following notion. The logarithmic modulus of continuity of $f \in \mathcal{C}(\mathbb{R}_+^2)$ is defined, for $\delta > 0$, by

$$\omega(f,\delta) := \sup\{|f(\mathbf{x}) - f(\mathbf{y})| : \mathbf{x}, \mathbf{y} \in \mathbb{R}_+^2, \quad \|\log(\mathbf{x}) - \log(\mathbf{y})\| \leq \delta\}.$$

This modulus satisfies all the properties of the one dimensional logarithmic modulus of continuity (see [9]). In particular, it is a monotone increasing function of $\delta > 0$ and the following inequality holds, for $\lambda > 0$

$$(5.6) \qquad \omega(f, \lambda\delta) \leq (1+\lambda)\omega(f,\delta).$$

We have the following theorem.

Theorem 5.3. *If $f \in \mathcal{C}(\mathbb{R}_+^2)$, $\varphi \in \Phi$, and*

$$D := \sup_{\mathbf{x}\in\mathbb{R}_+^2} \sum_{\mathbf{k}\in\mathbb{Z}^2} |\varphi(e^{-\mathbf{k}}\mathbf{x})|\|\mathbf{k} - \log\mathbf{x}\| < +\infty,$$

then for $\mathtt{w} > 0$ and $\delta > 0$, we have

$$\Big| \sum_{\mathbf{k}\in\mathbb{Z}^2} f(e^{\frac{\mathbf{k}}{\mathtt{w}}})\varphi(e^{-\mathbf{k}}\mathbf{x}^{\mathtt{w}}) - f(\mathbf{x}) \Big| \leq M_0(\varphi)\omega(f,\delta) + D\frac{\omega(f,\delta)}{\delta\underline{w}}.$$

Proof. Using that $\varphi \in \Phi$, (5.6) and the inequality

$$\|\frac{\mathbf{k}}{\mathtt{w}} - \log(\mathbf{x})\| \leq \frac{\|\mathbf{k} - \log(\mathbf{x}^{\mathtt{w}})\|}{\underline{w}},$$

we have

$$|E_{\mathtt{w}}^\varphi f(\mathbf{x}) - f(\mathbf{x})| \leq \omega(f,\delta)\Big(\sum_{\mathbf{k}\in\mathbb{Z}^2} |\varphi(e^{-\mathbf{k}}\mathbf{x}^{\mathtt{w}})| + \sum_{\mathbf{k}\in\mathbb{Z}^2} |\varphi(e^{-\mathbf{k}}\mathbf{x}^{\mathtt{w}})|\frac{\|\mathbf{k} - \log(\mathbf{x}^{\mathtt{w}})\|}{\delta\underline{w}} \Big)$$

$$\leq \omega(f,\delta)M_0(\varphi) + \frac{\omega(f,\delta)}{\delta\underline{w}}D$$

and so the assertion follows. $\qquad\square$

As a corollary we can prove

Corollary 5.1. *Under the assumptions of Teorem 5.3 there holds*

(5.7)
$$\Big| \sum_{\mathbf{k}\in\mathbb{Z}^2} f(e^{\frac{\mathbf{k}}{\underline{w}}})\varphi(e^{-\mathbf{k}}\mathbf{x^w}) - f(\mathbf{x})\Big| \leq C(\varphi)\omega\Big(f, \frac{1}{\underline{w}}\Big).$$

Proof. Applying Theorem 5.3 with $\delta = \dfrac{1}{\underline{w}}$, we obtain

$$|E_{\mathbf{w}}^{\varphi} f(\mathbf{x}) - f(\mathbf{x})| \leq \omega\Big(f, \frac{1}{\underline{w}}\Big) M_0(\varphi) + \omega\Big(f, \frac{1}{\underline{w}}\Big) D.$$

Setting $C(\varphi) = M_0(\varphi) + D$ we have the assertion.   □

Now, we obtain estimations of the order of approximation under some local regularity assumptions on the function f. In order to do that we will need further assumptions on the kernel function φ, i.e., there exists $\ell \in \mathbb{N}$ such that for every $\mathbf{j} \in \mathbb{N}_0^2$, $\|\mathbf{j}\| \leq \ell$

(φ.4) $m_{\mathbf{j}}^{\|\mathbf{j}\|}(\varphi, \mathbf{x}) =: m_{\mathbf{j}}^{\|\mathbf{j}\|}(\varphi)$ is independent of $\mathbf{x}$;
(φ.5) $M_{\mathbf{j}}^{\|\mathbf{j}\|}(\varphi) < +\infty$ and

$$\lim_{r\to+\infty} \sum_{\|\mathbf{k}-\log(\mathbf{x})\|>r} |\varphi(e^{-\mathbf{k}}\mathbf{x})| \, \|(\mathbf{k}-\log(\mathbf{x}))\|^{\ell} = 0,$$

uniformly with respect to $\mathbf{x}$.

Remark 5.2. Since for any $\mathbf{j} = (j_1, j_2) \in \mathbb{N}_0^2$ and any vector $\mathbf{v} = (v_1, v_2) \in \mathbb{R}_+^2$ we have $[\mathbf{v}]^{\mathbf{j}} \leq \|\mathbf{v}\|^{\mathbf{j}}$, we deduce immediately that assumption (φ.5) implies that

$$\lim_{r\to+\infty} \sum_{\|\mathbf{k}-\log(\mathbf{x})\|>r} |\varphi(e^{-\mathbf{k}}\mathbf{x})| \, [\mathbf{k}-\log(\mathbf{x})]^{\|\mathbf{j}\|} = 0,$$

uniformly with respect to $\mathbf{x}$, for every $\mathbf{j}$ with $\|\mathbf{j}\| \leq \ell$.

We denote by Φ_ℓ the set of functions φ satisfying conditions $(\varphi.1), (\varphi.4), (\varphi.5)$.

We have the following result, in which we assume $\ell = 2$.

Theorem 5.4. *Let $f: \mathbb{R}_+^2 \to \mathbb{C}$ be a function such that $f \in C^{(2)}(\mathbb{R}_+^2)$ locally at the point $\mathbf{x} = (x_1, x_2) \in \mathbb{R}_+^2$. If $\varphi \in \Phi_2$, then for $\mathbf{w} = (w_1, w_2) > 0$,*

$$(E_{(w_1, w_2)}^{\varphi} f)(x_1, x_2) - f(x_1, x_2) = \sum_{\nu=1}^{2} \sum_{\|\mathbf{h}\|=\nu} \Big(\frac{\Theta^\nu f(x_1, x_2}{\nu!} \frac{m_h^\nu(\varphi)}{\mathbf{w^h}}\Big) + o(\underline{w}^{-2}).$$

Proof. Since $\varphi \in \Phi_2$, applying the Mellin–Taylor formula of the second order with local remainder, we can write

$$(E^{\varphi}_{(w_1,w_2)}f)(x_1,x_2) - f(x_1,x_2) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} \varphi(e^{-k_1}x_1^{w_1}, e^{-k_2}x_2^{w_2})\left((\Theta_{x_1}\log(\frac{e^{\frac{k_1}{w_1}}}{x_1})+\right.$$

$$\Theta_{x_2}\log(\frac{e^{\frac{k_2}{w_2}}}{x_2}))f(x_1,x_2) + \frac{1}{2}(\Theta_{x_1}\log(\frac{e^{\frac{k_1}{w_1}}}{x_1}) + \Theta_{x_2}\log(\frac{e^{\frac{k_2}{w_2}}}{x_2}))^2 f(x_1,x_2)+$$

$$\left.H(\frac{e^{\frac{k_1}{w_1}}}{x_1}, \frac{e^{\frac{k_2}{w_2}}}{x_2})(\log^2(\frac{e^{\frac{k_1}{w_1}}}{x_1}) + \log^2(\frac{e^{\frac{k_2}{w_2}}}{x_2}))\right)$$

$$= \frac{1}{w_1}\Theta_{x_1}f(x_1,x_2)m^1_{(1,0)}(\varphi) + \frac{1}{w_2}\Theta_{x_2}f(x_1,x_2)m^1_{(0,1)}(\varphi) + \frac{1}{2}\frac{1}{w_1^2}\Theta^2_{x_1}f(x,y)m^2_{(2,0)}(\varphi)+$$

$$\frac{1}{2}\frac{1}{w_2^2}\Theta^2_{x_2}f(x_1,x_2)m^2_{(0,2)}(\varphi) + \frac{1}{w_1 w_2}\Theta_{x_1}(\Theta_{x_2}f)(x_1,x_2)m^2_{(1,1)}(\varphi)+$$

$$\sum_{(k_1,k_2)\in\mathbb{Z}^2} \varphi(e^{-k_1}x_1^{w_1}, e^{-k_2}x_2^{w_2})H\left(\frac{e^{\frac{k_1}{w_1}}}{x_1}, \frac{e^{\frac{k_2}{w_2}}}{x_2}\right)\left(\log^2\left(\frac{e^{\frac{k_1}{w_1}}}{x_1}\right) + \log^2\left(\frac{e^{\frac{k_2}{w_2}}}{x_2}\right)\right)$$

$$= \sum_{\nu=1}^{2}\sum_{|h|=\nu} \left(\frac{\Theta^\nu f(x_1,x_2)}{\nu!}\frac{m^\nu_h(\varphi)}{\mathbf{w^h}}\right)+$$

$$\sum_{(k_1,k_2)\in\mathbb{Z}^2} \varphi(e^{-k_1}x_1^{w_1}, e^{-k_2}x_2^{w_2})H\left(\frac{e^{\frac{k_1}{w_1}}}{x_1}, \frac{e^{\frac{k_2}{w_2}}}{x_2}\right)\left(\log^2\left(\frac{e^{\frac{k_1}{w_1}}}{x_1}\right) + \log^2\left(\frac{e^{\frac{k_2}{w_2}}}{x_2}\right)\right).$$

Here, $H(t_1,t_2)$ tends to zero, for $(t_1,t_2) \to (1,1)$. Thus, for a given, $\varepsilon > 0$ there is $\delta > 0$ such that

$$H\left(\frac{e^{k_1/w_1}}{x_1}, \frac{e^{k_2/w_2}}{x_2}\right) < \varepsilon,$$

whenever $\|\log(e^{\mathbf{k}/\mathbf{w}}) - \log(\mathbf{x})\| < \delta$.

Setting

$$R := \sum_{(k_1,k_2)\in\mathbb{Z}^2} \varphi(e^{-k_1}x_1^{w_1}, e^{-k_2}x_2^{w_2})H\left(\frac{e^{\frac{k_1}{w_1}}}{x_1}, \frac{e^{\frac{k_2}{w_2}}}{x_2}\right)\left(\log^2\left(\frac{e^{\frac{k_1}{w_1}}}{x_1}\right) + \log^2\left(\frac{e^{\frac{k_2}{w_2}}}{x_2}\right)\right),$$

taking into account that

$$\|\frac{\mathbf{k}}{\mathbf{w}} - \log(\mathbf{x})\| \le \frac{\|\mathbf{k} - \log(\mathbf{x^w})\|}{\underline{w}}$$

as in Theorem 4.1, we obtain

$$\underline{w}^2|R| \le \sum_{(k_1,k_2)\in\mathbb{Z}^2} |\varphi(e^{-k_1}x_1^{w_1}, e^{-k_2}x_2^{w_2})|\left|H\left(\frac{e^{\frac{k_1}{w_1}}}{x_1}, \frac{e^{\frac{k_2}{w_2}}}{x_2}\right)\right|\|\mathbf{k} - \log(\mathbf{x^w})\|^2$$

$$\le \left\{\sum_{\|\frac{\mathbf{k}}{w}-\log(\mathbf{x})\|<\delta} + \sum_{\|\mathbf{k}-\log(\mathbf{x^w})\|\ge\delta\underline{w}}\right\}|\varphi(e^{-k_1}x_1^{w_1}, e^{-k_2}x_2^{w_2})|\left|H\left(\frac{e^{\frac{k_1}{w_1}}}{x_1}, \frac{e^{\frac{k_2}{w_2}}}{x_2}\right)\right|\|\mathbf{k} - \log(\mathbf{x^w})\|^2$$

$$=: S_1 + S_2.$$

As to S_1, denoting $M_2(\varphi) := \max_{\|j\|=2} M_j^2(\varphi)$, we have easily $S_1 \leq 3M_2(\varphi)\varepsilon$, while for S_2, by $(\varphi.5)$ and the boundedness of H, we obtain $S_2 \leq \|H\|_\infty \, \varepsilon$. Thus the proof is complete. $\qquad\square$

6. SOME EXAMPLES

In [7] the one-dimensional generalized exponential sampling series was introduced, in which the generating (one-dimensional) kernel $\phi : \mathbb{R}^+ \to \mathbb{R}$ satisfies the assumptions:

$(\phi.1)$ $\sum\limits_{k \in \mathbb{Z}} \phi(e^{-k}x) = 1$ for every $x \in \mathbb{R}^+$,

$(\phi.2)$ $\sup\limits_{x \in \mathbb{R}^+} \sum\limits_{k \in \mathbb{Z}} |\phi(e^{-k}x)| < +\infty$;

$(\phi.3)$ $\lim\limits_{r \to +\infty} \sum\limits_{|k-\log(x)|>r} |\phi(e^{-k}x)| = 0$, uniformly with respect to x.

We will denote by Ψ the set comprising all functions $\phi : \mathbb{R}^+ \to \mathbb{R}$ satisfying $(\phi.1), (\phi.2)$ and $(\phi.3)$. Using a product of two such functions, we can construct examples of two-dimensional kernel $\varphi \in \Phi$. In this respect we have the following proposition.

Proposition 6.3. *If $\phi_1, \phi_2 \in \Psi$ are bounded, then $\Gamma(x_1, x_2) := \phi_1(x_1)\phi_2(x_2) \in \Phi$.*

Proof. Assumptions $(\varphi.1)$ and $(\varphi.2)$ are immediate, applying $(\phi.1)$ and $(\phi.2)$. As to $(\varphi.3)$, note that if $\|(k_1 - \log(x_1), k_2 - \log(x_2))\| > r$, then $r < \|(k_1 - \log(x_1), k_2 - \log(x_2))\| \leq |k_1 - \log(x_1)| + |k_2 - \log(x_2)|$. Therefore,

$$\lim_{r \to +\infty} \sum_{\|(k_1-\log(x_1),k_2-\log(y))\|>r} |\varphi_1(e^{-k_1}x)\varphi_2(e^{-k_2}x_2)|$$
$$\leq \lim_{r \to +\infty} \sum_{|k_1-\log(x_1)|+|k_2-\log(x_2)|>r} |\varphi_1(e^{-k_1}x)\varphi_2(e^{-k_2}x_2)|.$$

For a fixed $\mathbf{x} = (x_1, x_2) \in \mathbb{R}_+^2$ and $r > 0$, we set

$$A := \{(k_1, k_2) : |k_1 - \log(x_1)| + |k_2 - \log(x_2)| > r\};$$

$$B_1 := \{(k_1, k_2) : |k_1 - \log(x_1)| > \frac{r}{2}\}, \quad B_2 := \{(k_1, k_2) : |k_2 - \log(x_2)| > \frac{r}{2}\}$$

and $B := B_1 \cup B_2$. Since $A \subset B$, we have

$$\lim_{r \to +\infty} \sum_{(k_1,k_2)\in A} |\varphi_1(e^{-k_1}x_1)\varphi_2(e^{-k_2}x_2)| \leq \lim_{r \to +\infty} \sum_{(k_1,k_2)\in B} |\varphi_1(e^{-k_1}x_1)\varphi_2(e^{-k_2}x_2)|$$

$$\leq \lim_{r \to +\infty} \sum_{(k_1,k_2)\in B_1} |\varphi_1(e^{-k_1}x_1)\varphi_2(e^{-k_2}x_2)| + \lim_{r \to +\infty} \sum_{(k_1,k_2)\in B_2} |\varphi_1(e^{-k_1}x_1)\varphi_2(e^{-k_2}x_2)|.$$

By the boundedness of the functions ϕ_1 and ϕ_2 and by $(\phi.3)$, we obtain easily $(\varphi.3)$. $\qquad\square$

Making use of Proposition 6.3, we costruct some box-type kernel, using the product of two classical one-dimensional kernels.

Example 6.1. Denoting by r_+ the positive part of a number $r \in \mathbb{R}$, for $n \in \mathbb{N}$, we define the (one-dimensional) Mellin spline of order n, as (see [7, 10]

$$(6.8) \qquad B_n(x) := \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (\frac{n}{2} + \log x - j)_+^{n-1}, \quad (x \in \mathbb{R}^+).$$

These kernel functions are the Mellin version of the classical central B-splines see [25]. The functions B_n are compactly supported, and satisfy all the assumptions of the class Ψ (see [7]). Using these functions we can define, for $n, m \in \mathbb{N}$,

(6.9) $$B_{n,m}(x_1, x_2) := B_n(x_1)\, B_m(x_2).$$

By Proposition 6.3, the kernel $B_{n,m} \in \Phi$. In particular, for $n = m = 2$, we obtain

$$B_{2,2}(x_1, x_2) = \begin{cases} (1 + \log x_1)(1 + \log x_2), & e^{-1} < x_1, x_2 < 1 \\ (1 - \log x_1)(1 + \log x_2), & 1 < x_1 < e, e^{-1} < x_2 < 1 \\ (1 + \log x_1)(1 - \log x_2), & e^{-1} < x_1 < 1, 1 < x_2 < e \\ (1 - \log x_1)(1 - \log x_2), & 1 < x_1, x_2 < e \\ 0, & \text{otherwise.} \end{cases}$$

Example 6.2. Another interesting example, is given by the product of two one-dimensional Mellin–Fejér kernels, which are defined for any $\rho > 0$, $c \in \mathbb{R}$ and $x \in \mathbb{R}^+$ by (see [7, 10])

(6.10) $$F_\rho^c(x) := \begin{cases} \dfrac{x^{-c}}{2\pi}\, \rho \operatorname{sinc}^2(\dfrac{\rho}{\pi} \log \sqrt{x}), & x \neq 1 \\[2mm] \dfrac{\rho}{2\pi}, & x = 1 \end{cases}.$$

We have $F_\rho^c \in \Psi$ and using again Proposition 6.3, we see that the kernel

(6.11) $$F_{\rho_1, \rho_2}^{c_1, c_2}(x_1, x_2) = F_{\rho_1}^{c_1}(x_1)\, F_{\rho_2}^{c_2}(x_2) \qquad (x_1, x_2) \in \mathbb{R}_+^2,$$

with $\rho_1, \rho_2 > 0$, $c_1, c_2 \in \mathbb{R}$, belongs to the class Φ.

Remark 6.3. As remarked in [7], the one-dimensional Mellin–Fejér kernel does not satisfy the (one-dimensional) moment condition

$$\widetilde{M}_1(\phi) := \sup_{x \in \mathbb{R}^+} \sum_{k=-\infty}^{\infty} |\varphi(e^{-k}x)|\, |k - \log x| < +\infty.$$

Therefore, in [7, 10] a suitable modification of the kernel was studied, introducing the so-called Mellin–Jackson kernels, which satisfy the above condition. Thus, one can obtain other interesting examples, by considering the product of two Mellin–Jackson kernels.

7. SEISMIC WAVES AND EXPONENTIAL SAMPLING

Due to their nature and their way of propagation, seismic waves can be modeled using exponential functions. Seismic waves, due to the continuous and natural movements of the terrestrial plates, originate from a point at a certain depth into the ground, this point called hypocenter, and develop for several kilometers, being attenuated thanks to the elastic properties of the medium they cross. The projection of the hypocenter on the heart surface is called epicenter and it is the point of maximum amplitude of the seismic wave. For a given direction $\theta \in [-\pi, \pi]$ in the horizontal plane, the definition of amplitude $A(R, \theta)$ of a seismic wave, used in this work, is the variation, measured in mm, registered by a standard Wood-Anderson seismograph at a certain distance R from the epicenter. Connected with the definition of $A(R, \theta)$ is the formalization of the Local Magnitude $LM(R, \theta)$ [12], needed to measure the intensity of an earthquake according to the Richter scale [19, 24]:

(7.12) $$LM(R, \theta) := \log_{10} A(R, \theta) - \log_{10} A_0(R, \theta),$$

where $\log_{10} A_0(R, \theta)$ is a calibration function such that, for $R = 100\, Km$ and in whatever direction θ, $\log_{10} A_0(100, \theta) = 3$. We assume the model to be symmetric with respect to the epicenter that is located in the center of the axis ((0,0) coordinates).

The model derived in [12], based on the dataset provided by ISNet (Irpinia Seismic Network), has been used to test how the mathematical theory approximates real data. ISNet is a network of 27 stations located in the South of Italy, along the Apennines chain [26]. In the ISNet dataset the epicenter distance R has been substituted with the hypocenter distance, committing an error of less than 1%, approximation possible thanks to the reduced depth (<20 Km) of the hypocenter in the ISNet data. To approximate the real data the following model has been used:

$$(7.13) \qquad LM(R, \theta) = \log_{10} A(R, \theta) - \alpha \log_{10} R - kR - \beta$$

where $\alpha = -1.79$, $\beta = 0.58$, $k = 0$. The values of the parameters have been achieved considering a minimization criteria, according to the elastic structural parameters characterizing the area monitored by the ISNet network. In the light of the previous considerations, the model assumes the form (see [12] for all the details):

$$(7.14) \qquad LM(R, \theta) = \log_{10} A(R, \theta) + 1.79 \log_{10} R - 0.58$$

from which, in case of invertibility, we can write the inverse formulation:

$$A(R, \theta, LM) = 10^{(LM(R, \theta) - 1.79 \log_{10} R + 0.58)},$$

or equivalently in cartesian coordinates:

$$A(R_1, R_2, LM) = 10^{(LM(R_1, R_2) - 1.79 \log_{10} \arctan(R_1/R_2) + 0.58)},$$

where R_1 and R_2 are, respectively, the horizontal and vertical cartesian axis such that $R = \arctan(R_1/R_2)$, $R_1 = R\cos(\theta)$, $R_2 = R\sin(\theta)$. Fixed a value for $LM(R_1, R_2)$, it is possible to calculate $A(R_1, R_2)$ and to approximate it with an exponential sampling operator, defined as:

$$(7.15) \qquad (E^{\varphi}_{(w_1, w_2)} A)(R_1, R_2) = \sum_{(k_1, k_2) \in \mathbb{Z}^2} A(e^{\frac{k_1}{w_1}}, e^{\frac{k_2}{w_2}}) \varphi(e^{-k_1} R_1{}^{w_1}, e^{-k_2} R_2{}^{w_2}),$$

with R_1, R_2 integers.

Finally, a quantification of the reconstruction absolute mean error AME has been provided introducing the following error estimator

$$AME := \frac{1}{N_1 N_2} \sum_{R_1=0}^{N_1} \sum_{R_2=0}^{N_2} |(E^{\varphi}_{(w_1, w_2)} A)(R_1, R_2) - A(R_1, R_2)|.$$

In the previous expression N_1 and N_2 are the number of points in the two main cartesian axis directions, $N_1 \times N_2$ being the total number of the samples in the grid and $R_1 \in [0, N_1]$, $R_2 \in [0, N_2]$.

Chosen $LM = 2.7$ in the ISNet dataset, we achieve, for bivariate Mellin–Fejer kernels, the numerical values shown in table 1, where only a single row of the approximating matrix is reported for practical reasons. Other rows of the same matrix exhibit the same trend. Using Mellin Splines kernels, we achieve better approximation results (see table 2). In any case, increasing w_1, w_2, N_1, N_2 we observe as AME decreases, whatever kernel being used in the approximation formula.

A	N=15, w=5	N=30, w=10	N=60, w=20
918.4869	2016.2750	1420.2633	1182.1876
907.7103	1545.3763	1213.2350	1064.3751
890.3316	1319.0114	1090.6358	990.5768
867.1510	1169.1624	998.5332	932.5291
839.1579	1046.1715	926.9082	880.6653
807.4335	945.1902	861.7557	832.4209
773.0601	862.8483	803.7728	786.1427
737.0491	794.8578	752.2213	741.7739
700.2923	737.3727	704.8035	698.9664
663.5358	687.5171	660.3791	658.5373
627.3745	643.3295	618.9350	619.9301
592.2586	603.5319	580.7086	583.3547
558.5099	567.3041	545.7103	549.1905
526.3414	534.1130	513.6886	517.2242
495.8780	503.5975	484.2704	487.1994
467.1756	475.4960	457.0936	459.1393
440.2385	449.6028	431.8738	433.1069
415.0331	425.7436	408.4140	408.9969
391.4999	403.7617	386.5841	386.5923
369.5621	383.5117	366.2922	365.7069

TABLE 1. In the first column of the table the values of the 15th row of the real data for $LM = 2.7$. In the following columns, from left to right, the reconstructed data with bivariate Mellin–Fejer kernels with $N = N_1 = N_2, w = w_1 = w_2$ respectively equal to $15, 30, 60$ and $5, 10, 20$.

A	N=15, w=5	N=30, w=10	N=60, w=20
918.4869	919.7995	919.1974	918.8313
907.7103	908.9871	908.3652	908.0278
890.3316	891.5131	890.9341	890.6223
867.1510	867.3503	867.5869	867.3689
839.1579	838.9015	839.5563	839.3507
807.4335	806.7726	807.6950	807.5622
773.0601	772.5943	772.9821	773.1665
737.0491	736.8876	737.0027	737.0632
700.2923	699.0600	700.4001	700.3419
663.5358	663.8368	663.6755	663.5998
627.3745	626.2857	627.4080	627.3887
592.2586	592.0042	592.1529	592.2225
558.5099	557.9946	558.3477	558.4872
526.3414	525.8575	526.2417	526.3176
495.8780	495.8853	495.8670	495.8693
467.1756	467.2693	467.1553	467.1692
440.2385	440.3887	440.2862	440.2415
415.0331	415.0450	415.0472	415.0398
391.4999	392.0046	391.6462	391.5060
369.5621	370.4046	369.5769	369.5696

TABLE 2. In the first column of the table the values of the 15th row of the real data for $LM = 2.7$. In the following columns, from left to right, the reconstructed data with the bivariate Mellin-Splines $B_{2,2}$ with $N = N_1 = N_2, w = w_1 = w_2$ respectively equal to $15, 30, 60$ and $5, 10, 20$.

REFERENCES

[1] L. Angeloni, D. Costarelli, G. Vinti, *A characterization of the convergence in variation for the generalized sampling series*, Ann. Acad. Sci. Fenn. Math., 43(2), (2018), 755–767.

[2] L. Angeloni, G. Vinti, *Discrete operators of sampling type and approximation in φ-variation*, Math. Nachr., 291(4), (2018), 546–555.

[3] C. Bardaro, P.L. Butzer, I. Mantellini, *The exponential sampling theorem of signal analysis and the reproducing kernel formula in the Mellin transform setting*, Sampling Theory in Signal and Image Process., 13(1), (2014), 35–66.

[4] C. Bardaro, P.L. Butzer, I. Mantellini, G. Schmeisser, *On the Paley–Wiener theorem in the Mellin transform setting*, J. Approx. Theory, 207, (2016), 60–75.

[5] C. Bardaro, P.L. Butzer, I. Mantellini, G. Schmeisser, *A fresh approach to the Paley–Wiener theorem for Mellin transforms and the Mellin–Hardy spaces*, Math. Nachr., 290, (2017), 2759–2774.

[6] C. Bardaro, P.L. Butzer, R.L. Stens, G. Vinti, *Prediction by samples from the past with error estimates covering discontinuous signals*, IEEE Trans. Information Theory, 56(1), (2010), 614–633.

[7] C. Bardaro, L. Faina, I. Mantellini, *A generalization of the exponential sampling series and its approximation properties*, Math. Slovaca, 67(6), (2017), 1481–1496.

[8] C. Bardaro, I. Mantellini, *A note on the Voronovskaja theorem for Mellin–Fejer convolution operators*, Appl. Math. Lett., 24, (2011), 2064–2067.

[9] C. Bardaro, I. Mantellini, *Asymptotic behaviour of Mellin–Fejer convolution operators*, East J. Approx., 17(2), (2011), 181–201.

[10] C. Bardaro, I. Mantellini, G. Schmeisser, *Exponential sampling series: convergence in Mellin–Lebesgue spaces*, Results Math., 74, (2019), Art. 119.

[11] M. Bertero, E.R. Pike, *Exponential sampling method for Laplace and other dilationally invariant transforms I. Singular-system analysis. II. Examples in photon correction spectroscopy and Frauenhofer diffraction*, Inverse Problems, 7, (1991), 1–20; 21–41.

[12] A. Bobbio, M. Vassallo, G. Festa, *Local Magnitude estimation for the Irpinia Seismic Network*, Bull. Seismol. Soc. Am., 99, (2009), 2461–2470.

[13] P.L. Butzer, S. Jansche, *A direct approach to the Mellin transform*, J. Fourier Anal. Appl., 3, (1997), 325–375.

[14] P.L. Butzer, S. Jansche, *The exponential sampling theorem of signal analysis*, Atti Sem. Mat. Fis. Univ. Modena, Suppl. Vol. 46, (special issue dedicated to Professor Calogero Vint) (1998), 99–122.

[15] P.L. Butzer, R.L. Stens, *Prediction of non-bandlimited signals in terms of splines of low degree*, Math. Nachr., 132, (1987), 115–130.

[16] P.L. Butzer, R.L. Stens, *Linear prediction by samples from the past*. In: R.J. Marks II (ed.) *Advanced Topics in Shannon Sampling and Interpolation Theory*,Springer, New York, (1993), 157–183.

[17] D. Casasent (Ed), *Optical data processing*, Springer, Berlin, (1978), 241–282.

[18] F. Gori, *Sampling in optics* In: R.J. Marks II (ed.), *Advances Topics in Shannon Sampling and Interpolation Theory*,Springer, New York, (1993), 37–83.

[19] B. Gutenberg, C. F. Richter, *Discussion: Magnitude and energy of earthquakes*, Science, 83, 2147, (1936), 183–185.

[20] J.R. Higgins, *Sampling theory in Fourier and signal analysis, Foundations*, Oxford Univ. Press., Oxford, (1996).

[21] A. Kivinukk, G. Tamberg, *On window methods in generalized Shannon sampling operators*. In: G. Schmeisser and A. Zayed (Eds) *New perspectives on approximation and sampling theory*, 63–85, Appl. Numer. Harmon. Anal., Birkhaeuser Springer, Cham, (2014).

[22] R.G. Mamedov, *The Mellin transform and approximation theory* (in Russian), "Elm", Baku, (1991).

[23] N. Ostrowsky, D. Sornette, P. Parker, E.R. Pike, *Exponential sampling method for light scattering polydispersity analysis*, Opt. Acta, 28, (1994), 1059–1070.

[24] C. F. Richte, *An instrumental earthquake magnitude scale*, Bull. Seismol. Soc. Am., 25, (1935), 1–32.

[25] L.L. Schumaker, *Spline functions: basic theory*, John Wiley and Sons, New York, (1981).

[26] E. Weber, V. Convertito, G. Iannaccone, A. Zollo, A. Bobbio, L. Cantore, M. Corciulo, M. Di Crosta, L. Elia, C. Martino, A. Romeo, C. Satriano, *An advanced seismic network in the southern Apennines (Italy) for seismicity investigations and experimentation with earthquake early warning* Seism. Res. Lett., 78, (2007), 622–534.

[27] A.I. Zayed, *Advances in Shannon's Sampling Theory*, CRC Press, Boca Raton (1993).

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Research Article

Quadrature Formulas of Gaussian Type for Fast Summation of Trigonometric Series

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ABSTRACT. A summation/integration method for fast summing trigonometric series is presented. The basic idea in this method is to transform the series to an integral with respect to some weight function on $\mathbb{R}_+$ and then to approximate such an integral by the appropriate quadrature formulas of Gaussian type. The construction of these quadrature rules, as well as the corresponding orthogonal polynomials on $\mathbb{R}_+$, are also considered. Finally, in order to illustrate the efficiency of the presented summation/integration method two numerical examples are included.

Keywords: Summation, trigonometric series, Gaussian quadrature rule, weight function, orthogonal polynomial, three-term recurrence relation, convergence, Laplace transform.

2010 Mathematics Subject Classification: 65B10, 41A55, 65D32.

1. INTRODUCTION

Let $\mathcal{P}$ be the space of all polynomials and $\mathcal{P}_n$ its subspace of polynomials of degree at most n. In a joint paper with Walter Gautschi [9], we developed the Gauss-Christoffel quadratures on $(0, +\infty)$,

$$(1.1) \qquad \int_0^{+\infty} f(t) w_\nu(t) \, dt = \sum_{k=1}^N A_{\nu,k}^{(N)} f(\tau_{\nu,k}^{(N)}) + R_{N,\nu}(f) \quad (\nu = 1, 2),$$

with respect to the Bose-Einstein and Fermi-Dirac weights, which are defined by

$$(1.2) \qquad w_1(t) = \varepsilon(t) = \frac{t}{e^t - 1} \quad \text{and} \quad w_2(t) = \varphi(t) = \frac{1}{e^t + 1},$$

respectively. These N-point quadrature formulas are exact on the space of all algebraic polynomials of degree at most $2N - 1$, i.e., $R_{N,\nu}(\mathcal{P}_{2N-1}) = 0$, $\nu = 1, 2$.

The weight functions (1.2) and the corresponding quadratures (1.1) are widely used in solid state physics, e.g., the total energy of thermal vibration of a crystal lattice can be expressed in the form $\int_0^{+\infty} f(t) \varepsilon(t) \, dt$, where $f(t)$ is related to the phonon density of states. Also, integrals with the second weight function $\varphi(t)$ are encountered in the dynamics of electrons in metals.

In the same paper [9], we showed that these quadrature formulas can be used for summation of slowly convergent series of the form

$$(1.3) \qquad T = \sum_{k=1}^{+\infty} a_k \quad \text{and} \quad S = \sum_{k=1}^{+\infty} (-1)^k a_k.$$

In general, the basic idea in such the so-called *summation/integration* procedures is to transform the sum to an integral with respect to some weight function $w(t)$ on $\mathbb{R}$ or $\mathbb{R}_+ = [0, +\infty)$, and then to approximate this integral by a finite quadrature sum, i.e.,

$$\int_{\mathbb{R}} f(x)w(x)\,\mathrm{d}x \approx Q_N(f) = \sum_{\nu=1}^{N} A_\nu^{(N)} f\big(x_\nu^{(N)}\big),$$

where the function f is connected with a_k in some way, and $x_\nu^{(N)}$ and $A_\nu^{(N)}$, $\nu = 1, \ldots, N$, are nodes and weights of the quadrature rule $Q_N(f)$ (usually of Gaussian type), which is efficient for approximating a large class of functions with a relatively small number of quadrature nodes N.

As a transformation method of sums to integrals, we can use the Laplace transform as in [9] (see also [5, 8]) or some methods of complex contour integration as in our papers [12, 13] (see also [14, 15, 20, 17]). An account on summation/integration methods for the computation of slowly convergent power series and finite sums was given in [16].

In order to apply the quadrature rules (1.1) to the series T and S in (1.3), in the mentioned paper [9], we supposed that the general term of series is expressible in terms of the Laplace transform, or its derivative, of a known function. For example, let $a_k = F(k)$ and $F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} \mathrm{e}^{-st} f(t)\,\mathrm{d}t$ for $\mathrm{Re}\, s \geq 1$. Then

$$T = \sum_{k=1}^{+\infty} F(k) = \sum_{k=1}^{+\infty} \int_0^{+\infty} \mathrm{e}^{-kt} f(t)\,\mathrm{d}t = \int_0^{+\infty} \left(\sum_{k=1}^{+\infty} \mathrm{e}^{-kt} \right) f(t)\,\mathrm{d}t,$$

i.e.,

$$(1.4) \qquad T = \int_0^{+\infty} \frac{\mathrm{e}^{-t}}{1 - \mathrm{e}^{-t}}\, f(t)\,\mathrm{d}t = \int_0^{+\infty} \frac{t}{\mathrm{e}^t - 1} \frac{f(t)}{t}\,\mathrm{d}t = \int_0^{+\infty} \varepsilon(t) \frac{f(t)}{t}\,\mathrm{d}t.$$

Similarly, for "alternating" series, we have

$$(1.5) \qquad S = \sum_{k=1}^{+\infty} (-1)^k F(k) = \int_0^{+\infty} \frac{1}{\mathrm{e}^t + 1} (-f(t))\,\mathrm{d}t,$$

where the Fermi-Dirac weight function $\varphi(t)$ on $(0, +\infty)$ is appeared on the right-hand side in (1.5).

In this way, the summation of the series T and S is transformed to the integration problems with respect to the weight functions $w_1(t) = \varepsilon(t)$ and $w_2(t) = \varphi(t)$, respectively. An application of quadrature formulas (1.1) for $\nu = 1$ and $\nu = 2$ to the integrals in (1.4) and (1.5), respectively, provides an acceptable procedure for summation of slowly convergent series T and S.

In this paper, we consider the corresponding summation for the convergent trigonometric series

$$(1.6) \qquad C(x) = \sum_{k=1}^{+\infty} a_k \cos k\pi x \quad \text{and} \quad S(x) = \sum_{k=1}^{+\infty} a_k \sin k\pi x \quad (-1 < x < 1).$$

The corresponding series

$$A(x) = \sum_{k=1}^{+\infty} (-1)^{k-1} a_k \cos k\pi x \quad \text{and} \quad B(x) = \sum_{k=1}^{+\infty} (-1)^{k-1} a_k \sin k\pi x,$$

can be also considered, putting $x := x - 1$. Then $A(x) = -C(x-1)$ and $B(x) = -S(x-1)$.

The series (1.6) can be treated in the complex form

$$(1.7) \qquad C(x) + \mathrm{i}S(x) = \sum_{k=1}^{+\infty} a_k \mathrm{e}^{\mathrm{i}k\pi x}.$$

The paper is organized as follows. In Section 2, we present the transformation of (1.7) to the "weighted" integrals over $(0, +\infty)$. The construction of the corresponding quadrature formulas of Gaussian type for such integrals is given in Section 3. A simpler method for the sinus-series is presented in Section 4. Finally, in order to illustrate our methods, some numerical examples are given in Section 5.

2. TRANSFORMATION OF (1.7) TO "WEIGHTED" INTEGRALS

We consider the series (1.7) whose general term a_k is expressible in terms of the Laplace transform of a known function, i.e., let $a_k = F(k)$, where

$$(2.1) \qquad F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} \mathrm{e}^{-st} f(t)\,\mathrm{d}t, \quad \operatorname{Re} s \geq 1.$$

Then, we have

$$C(x) + \mathrm{i}S(x) = \sum_{k=1}^{+\infty} \mathrm{e}^{\mathrm{i}k\pi x} \int_0^{+\infty} \mathrm{e}^{-kt} f(t)\,\mathrm{d}t = \pi \int_0^{+\infty} \left(\sum_{k=1}^{+\infty} \mathrm{e}^{-k\pi(t-\mathrm{i}x)} \right) f(\pi t)\,\mathrm{d}t,$$

i.e.,

$$(2.2) \qquad C(x) + \mathrm{i}S(x) = \sum_{k=1}^{+\infty} a_k \mathrm{e}^{\mathrm{i}k\pi x} = \pi \int_0^{+\infty} \frac{\mathrm{e}^{\mathrm{i}\pi x}}{\mathrm{e}^{\pi t} - \mathrm{e}^{\mathrm{i}\pi x}} f(\pi t)\,\mathrm{d}t.$$

The obtained integral on the right-hand side in (2.2) is weighted with respect to the one-parametar "complex weight function"

$$(2.3) \qquad w(t; x) = \frac{\mathrm{e}^{\mathrm{i}\pi x}}{\mathrm{e}^{\pi t} - \mathrm{e}^{\mathrm{i}\pi x}}, \quad -1 < x < 1.$$

We note that

$$w(t; 0) = \frac{\varepsilon(\pi t)}{\pi t}, \quad w(t; 1/2) = -\varphi(2\pi t) + \frac{\mathrm{i}}{2\cosh \pi t}, \quad \text{and} \quad w(t; 1) = -\varphi(\pi t),$$

where $\varepsilon(t)$ and $\varphi(t)$ are given by (1.2). As we can see, only for $x = 0$ and $x = \pm 1$, the function $w(t; x)$ is real. Also, $w(t; -x) = \overline{w(t; x)}$, so that it is enough to consider only the case when $0 < x \leq 1$. The case $x = 0$ is not interesting because it leads to a numerical series.

Lemma 2.1. *The moments of the function* (2.3) *are given by*

$$(2.4) \qquad \mu_k(x) = \int_0^{+\infty} t^k w(t; x)\,\mathrm{d}t = \begin{cases} -\dfrac{1}{\pi} \operatorname{Log}(1 - \mathrm{e}^{\mathrm{i}\pi x}), & k = 0, \\[2ex] \dfrac{k!}{\pi^{k+1}} \operatorname{Li}_{k+1}(\mathrm{e}^{\mathrm{i}\pi x}), & k \in \mathbb{N}, \end{cases}$$

where Li_n *is the polylogarithm function defined by*

$$(2.5) \qquad \operatorname{Li}_n(z) = \sum_{\nu=1}^{+\infty} \frac{z^\nu}{\nu^n}.$$

Proof. In order to calculate the moments (2.4), i.e., the integrals

$$\mu_k(x) = \int_0^{+\infty} \frac{t^k \mathrm{e}^{\mathrm{i}\pi x}}{\mathrm{e}^{\pi t} - \mathrm{e}^{\mathrm{i}\pi x}} \, \mathrm{d}t, \quad k \geq 0,$$

we note that, for $a = \mathrm{e}^{\mathrm{i}\pi x}$, we have

$$\frac{a}{\mathrm{e}^{\pi t} - a} = \frac{a\mathrm{e}^{-\pi t}}{1 - a\mathrm{e}^{-\pi t}} = \sum_{\nu=1}^{+\infty} a^\nu \mathrm{e}^{-\nu\pi t} \quad \left(|a\mathrm{e}^{-\pi t}| = \mathrm{e}^{-\pi t} < 1 \right).$$

Then, we get

$$\mu_k(x) = \sum_{\nu=1}^{+\infty} a^\nu \int_0^{+\infty} t^k \mathrm{e}^{-\nu\pi t} \, \mathrm{d}t = \frac{k!}{\pi^{k+1}} \sum_{\nu=1}^{+\infty} \frac{a^\nu}{\nu^{k+1}},$$

which is the desired result, having in mind (2.5). $\qquad\square$

Remark 2.1. An analytic extension of the function Li_n is given by

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{\mathrm{e}^t - z} \, \mathrm{d}t \qquad (|\arg(1 - z)| < \pi).$$

The function Li_n is suitable for both symbolic and numerical calculation. It has a branch cut discontinuity in the complex z-plane running from 1 to ∞. This function is implemented in MATHEMATICA software as `PolyLog[n,z]` and it can be evaluated to arbitrary numerical precision.

Seperating the real and imaginary parts in (2.3), i.e.,

$$(2.6) \qquad w(t; x) = \frac{1}{2} \left\{ \frac{\cos \pi x - \mathrm{e}^{-\pi t}}{\cosh \pi t - \cos \pi x} + \mathrm{i} \frac{\sin \pi x}{\cosh \pi t - \cos \pi x} \right\},$$

and using (2.2), we obtain the following results:

Lemma 2.2. *We have*

$$C(x) = \sum_{k=1}^{+\infty} a_k \cos k\pi x = \frac{\pi}{2} \int_0^{+\infty} \frac{\cos \pi x - \mathrm{e}^{-\pi t}}{\cosh \pi t - \cos \pi x} f(\pi t) \, \mathrm{d}t,$$

$$S(x) = \sum_{k=1}^{+\infty} a_k \sin k\pi x = \frac{\pi}{2} \int_0^{+\infty} \frac{\sin \pi x}{\cosh \pi t - \cos \pi x} f(\pi t) \, \mathrm{d}t,$$

where $a_k = F(k)$ *and* $f(t) = \mathcal{L}^{-1}[F(s)]$.

Remark 2.2. Similar formulas as in Lemma 2.2 are mentioned in [22, p. 725].

The real and imaginary parts of $2w(t; x) = w_R(t; x) + \mathrm{i}w_I(t; x)$ for different values of x are presented in Figure 1.

As we can see, the imaginary part $t \mapsto w_I(t; x) = \mathrm{Im}\,(2w(t; x))$ is a positive function on $\mathbb{R}_+$ for each $0 < x < 1$, and all its moments are

$$\mu_k^I(x) = \int_0^{+\infty} t^k \frac{\sin \pi x}{\cosh \pi t - \cos \pi x} \, \mathrm{d}t = \begin{cases} 1 - x, & k = 0, \\ \dfrac{2k!}{\pi^{k+1}} \mathrm{Im}\left\{ \mathrm{Li}_{k+1}(\mathrm{e}^{\mathrm{i}\pi x}) \right\}, & k \in \mathbb{N}, \end{cases}$$

so that the orthogonal polynomials with respect to the inner product

$$(2.7) \qquad (p, q) = \int_0^{+\infty} p(t)q(t)w_I(t; x) \, \mathrm{d}t,$$
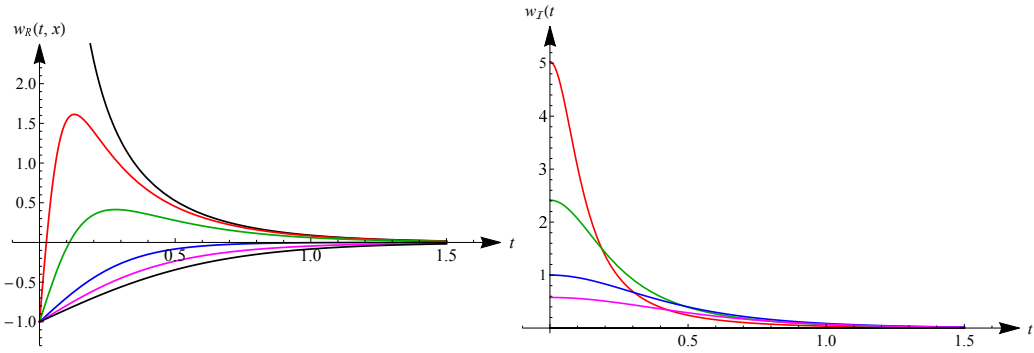
FIGURE 1. Real part (left) and imaginary part (right) of $t \mapsto 2w(t;x)$ on $[0, 1.5]$ for $x = 0$ (black line), $x = 1/8$ (red line), $x = 1/4$ (green line), $x = 1/2$ (blue line), $x = 2/3$ (magenta line), and $x = 1$ (black line)

as well as the corresponding quadrature formulas of Gaussian type exist for each $n \in \mathbb{N}$.

However, the real part $t \mapsto w_R(t;x) = \mathrm{Re}\,(2w(t;x))$ changes its sign at the point $t = \pi^{-1} \log(1/\cos \pi x) \in (0, +\infty)$, when $0 < x < 1/2$, while for $1/2 \le x \le 1$ this function is negative for each $t \in \mathbb{R}_+$. The moments of the function $w_R(t;x)$ are

$$\mu_k^R(x) = \int_0^{+\infty} t^k \frac{\cos \pi x - e^{-\pi t}}{\cosh \pi t - \cos \pi x}\, dt = \begin{cases} -\dfrac{2}{\pi} \log \left(2 \sin \dfrac{\pi x}{2} \right), & k = 0, \\[2ex] \dfrac{2k!}{\pi^{k+1}} \mathrm{Re}\left\{ \mathrm{Li}_{k+1}(e^{i\pi x}) \right\}, & k \in \mathbb{N}. \end{cases}$$

Regarding these facts a system of orthogonal polynomials with respect to $t \mapsto w_R(t;x)$ on $\mathbb{R}_+$ exists for each $1/2 \le x \le 1$. However, for $0 \le x < 1/2$ the existence is not guaranteed.

3. CONSTRUCTION OF POLYNOMIALS ORTHOGONAL WITH RESPECT TO THE WEIGHTS $t \mapsto w_I(t;x)$ AND $t \mapsto w_R(t;x)$ ON $\mathbb{R}_+$ AND CORRESPONDING GAUSSIAN RULES

As we mentioned in the previous section, the (monic) polynomials $p_k^I(t;x)$, $k = 0, 1, \ldots$, orthogonal with respect to the inner product (2.7) exist uniquely, as well as the corresponding quadrature formulas of Gaussian type

$$(3.1) \qquad \int_0^{+\infty} g(t) w_I(t;x)\, dt = \sum_{\nu=1}^N A_\nu^I g\bigl(\tau_\nu^I\bigr) + R_N(g;x),$$

where τ_ν^I $(\equiv \tau_\nu^I(N,x))$ and A_ν^I $(\equiv A_\nu^I(N,x))$ are their nodes and weight coefficients. The corresponding remainder term $R_N(g;x)$ vanishes for each $g \in \mathcal{P}_{2n-1}$. Some error estimates of Gaussian rules for certain classes of functions can be found in [11, Sect. 5.1.5].

The monic polynomials $p_k^I(t;x)$ satisfy the three-term recurrence relation

$$(3.2) \qquad p_{k+1}^I(t;x) = (t - \alpha_k^I(x)) p_k^I(t;x) - \beta_k^I(x) p_{k-1}^I(t;x), \quad k = 0, 1, \ldots,$$

with $p_0^I(t;x) = 1$ and $p_{-1}^I(t;x) = 0$.

The nodes τ_ν^I in the Gaussian quadrature rule (3.1) are eigenvalues of the symmetric tridiagonal Jacobi matrix (cf. [11, pp. 325–328])

$$(3.3) \qquad J_N(w_I(\,\cdot\,;x)) = \begin{bmatrix} \alpha_0^I(x) & \sqrt{\beta_1^I(x)} & & & \mathbf{O} \\ \sqrt{\beta_1^I(x)} & \alpha_1^I(x) & \sqrt{\beta_2^I(x)} & & \\ & \sqrt{\beta_2^I(x)} & \alpha_2^I(x) & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{N-1}^I(x)} \\ \mathbf{O} & & & \sqrt{\beta_{N-1}^I(x)} & \alpha_{N-1}^I(x) \end{bmatrix},$$

and the weight coefficients A_ν^I are given by $A_\nu^I = \beta_0^I(x)v_{\nu,1}^2$, $\nu = 1, \ldots, N$, where $v_{\nu,1}$ is the first component of the eigenvector $\mathbf{v}_\nu$ ($= [v_{\nu,1} \; \ldots \; v_{\nu,n}]^{\mathrm{T}}$) corresponding to the eigenvalue τ_ν^I and normalized such that $\mathbf{v}_\nu^{\mathrm{T}}\mathbf{v}_\nu = 1$. The most popular method for solving this eigenvalue problem is the Golub-Welsch procedure, obtained by a simplification of the QR algorithm [10].

Unfortunately, the coefficients in the three-term recurrence relation (3.2) are not known. They are known explicitly only for some narrow classes of orthogonal polynomials, including a famous class of the *classical orthogonal polynomials* (Jacobi, the generalized Laguerre, and Hermite polynomials). Orthogonal polynomials for which the recursion coefficients are not known are known as *strongly non–classical polynomials*. In the eighties of the last century, Walter Gautschi developed the so-called *constructive theory of orthogonal polynomials on* $\mathbb{R}$, including effective algorithms for numerically generating the recurrence coefficients for non-classical orthogonal polynomials, a detailed stability analysis of such algorithms as well as the corresponding software and several new applications of orthogonal polynomials (in particular see [4], [6], [7], as well as [18, 19, 20]).

On the other side, recent progress in symbolic computation and variable-precision arithmetic now makes it possible to generate the recurrence coefficients directly by using the original Chebyshev method of moments, but in a sufficiently high precision arithmetic. Such an approach allows us to overcome numerical instability in the map, in notation $\mathbf{K}_n : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, of the first $2n$ moments to $2n$ recursive coefficients,

$$\boldsymbol{\mu} = (\mu_0^I(x), \mu_1^I(x), \ldots, \mu_{2n-1}^I(x)) \mapsto \boldsymbol{\rho} = (\alpha_0^I(x) \ldots, \alpha_{n-1}^I(x), \beta_0^I(x), \ldots, \beta_{n-1}^I(x)),$$

which is a major construction problem. Respectively symbolic/variable-precision software for orthogonal polynomials is now available: Gautschi's package SOPQ in MATLAB and our MATHEMATICA package OrthogonalPolynomials (see [3] and [21]), which is downloadable from the web site http://www.mi.sanu.ac.rs/~gvm/.

The package OrthogonalPolynomials, beside the numerical construction of the recurrence coefficients, enables also the construction in a symbolic form for a reasonable value of n. For example, executing the following commands

```
<< orthogonalPolynomials`
muI[x_, n_]:=Table[If[k==0,1-x,
        2k!/Pi^(k+1) Im[PolyLog[k+1,Exp[I Pi x]]]],{k,0,2n-1}];
mom = muI[x,5];
{alI,beI}=aChebyshevAlgorithm[mom,Algorithm->Symbolic];
```

we obtain the first five coefficients $\alpha_k^I(x)$ and $\beta_k^I(x)$, $k = 0, 1, 2, 3, 4$, whose graphics are presented in Figure 2.
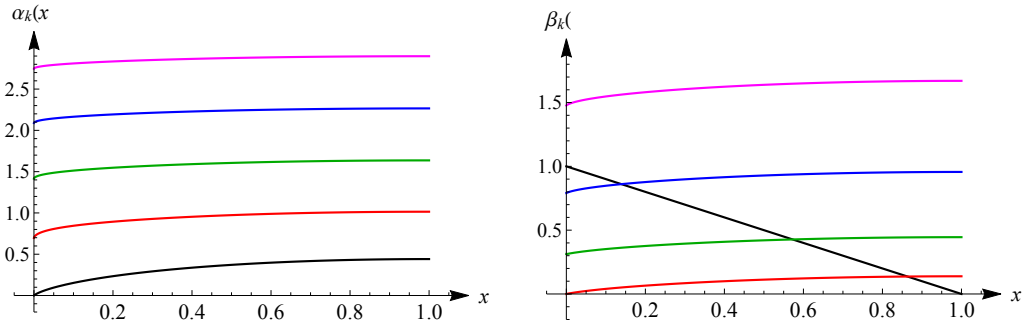
FIGURE 2. The coefficients $\alpha_k^I(x)$ (left) and $\beta_k^I(x)$ (right), for $k = 0$ (black line), $k = 1$ (red line), $k = 2$ (green line), $k = 3$ (blue line), and $k = 4$ (magenta line)
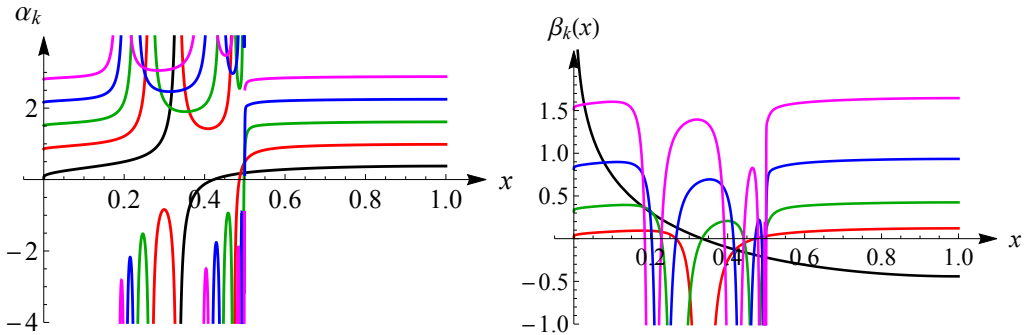
FIGURE 3. The coefficients $\alpha_k^R(x)$ (left) and $\beta_k^R(x)$ (right), for $k = 0$ (black line), $k = 1$ (red line), $k = 2$ (green line), $k = 3$ (blue line), and $k = 4$ (magenta line)

Using these coefficients, we can calculate Gaussian parameters (nodes and weights) for each $N \leq n = 5$ and each x. For larger values of n and a given x, it is more convenient to use the option for numerical construction in the function `aChebyshevAlgorithm`, instead of symbolic construction. A numerical example is given in Section 5.

In the same way, we can obtain the graphics of the coefficients $\alpha_k^R(x)$ and $\beta_k^R(x)$, $k = 0, 1, 2, 3, 4$, for the polynomials $p_k^R(t; x)$, $k = 0, 1, \ldots$, orthogonal with respect to the function $t \mapsto w_R(t; x)$ on $\mathbb{R}_+$ (see Figure 3). As we mention before, these polynomials exist uniquely for $1/2 \leq x \leq 1$, but for $0 \leq x < 1/2$ their existence is not guaranteed.

4. SOME CLASSES OF POLYNOMIALS ORTHOGONAL ON THE SEMIAXIS AND CORRESPONDING GAUSSIAN QUADRATURE RULES

There are orthogonal polynomials related to Bernoulli numbers, discovered as early as Stieltjes [23] and later extended by Touchard [24] and Carlitz [1] (for details see Chihara [2, pp. 191–193]. Carlitz defined polynomials

$$\Omega_k^{(\lambda)}(t) = \frac{(-1)^k (\lambda + 1)_k k!}{2^k \left(\frac{1}{2}\right)_k} F_k^\lambda (1 - \lambda + 2t),$$

where, as usual $(\lambda)_k$ is the well known Pochhammer symbol (or the raised factorial, since $(1)_k = k!$), defined by

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \lambda(\lambda + 1)\cdots(\lambda + k - 1),$$

and $F_k^\lambda(t) = {}_3F_2\big[-k, k+1, \frac{1}{2}(1 + \lambda + t); 1, \lambda + 1; 1\big]$ is the so-called Pasternak polynomial. These polynomials are orthogonal (but not positive-definite) on a line in the complex plane $L = (c - \mathrm{i}\infty, c + \mathrm{i}\infty)$, $-1 < c < 0$, with respect to the complex weight function $z \mapsto 1/(\sin(\pi z)\sin\pi(z - \lambda))$. However, taking $(\lambda - 1 + \mathrm{i}t)/2$ instead of t (see [2, p. 192]), we get the positive-definite monic polynomials

$$(4.1) \qquad G_k^{(\lambda)}(t) = (-\mathrm{i})^k \Omega_k^{(\lambda)}\left(\frac{\lambda - 1 + \mathrm{i}t}{2}\right), \quad -1 < \lambda < 1,$$

orthogonal with respect to the weight function

$$(4.2) \qquad t \mapsto w^G(t; \lambda) = \frac{1}{\cosh \pi t + \cos \pi \lambda} \quad \text{on} \quad \mathbb{R}.$$

These polynomials satisfy the three-term recurrence relation

$$G_{k+1}^{(\lambda)}(t) = t G_k^{(\lambda)}(t) - B_k(\lambda) G_{k-1}^{(\lambda)}(t), \quad G_0^{(\lambda)}(t) = 1, \; G_{-1}^{(\lambda)}(t) = 0,$$

where the recurrence coefficients are given by

$$(4.3) \qquad B_0(\lambda) = \int_{-\infty}^{+\infty} w^G(t; \lambda)\, \mathrm{d}t = \frac{2\lambda}{\sin \pi \lambda}, \quad B_k(\lambda) = \frac{k^2(k^2 - \lambda^2)}{4k^2 - 1}, \; k = 1, 2, \ldots \, .$$

Remark 4.1. When $\lambda \to 0$ these polynomials $G_k^{(\lambda)}$ reduce to orthogonal polynomials with respect to the logistic weight $t \mapsto 1/(\cosh \pi t + 1) = 2\mathrm{e}^{-\pi t}/(1 + \mathrm{e}^{-\pi t})^2$ (see [16, p. 49]).

As we can see, there is a connection between the weights $w_I(t; x)$ and $w^G(t; \lambda)$ on $\mathbb{R}_+$. Namely,

$$w_I(t; x) = \sin(\pi x)\, w^G(t; 1 - x), \quad 0 \le t < +\infty.$$

Using this fact and some results from [11, pp. 102–103], we can get the recurrence coefficients in an explicit form for polynomials $M_k(t; x)$ orthogonal with respect to the weight function

$$(4.4) \qquad t \mapsto w^M(t; x) = \frac{\sin \pi x}{\sqrt{t}\big(\cosh \pi \sqrt{t} - \cos \pi x\big)} \quad \text{on} \quad \mathbb{R}_+ \quad (0 < x < 1).$$

Here, $\sin(\pi x)$ is a constant factor and it can be omitted.

Theorem 4.1. *The polynomials* $\{M_k(t; x)\}_{k=0}^{+\infty}$ *orthogonal with respect to the weight function* $w^M(t; x)$, *given by* (4.4), *satisfy the three-term recurrence relation*

$$(4.5) \qquad M_{k+1}(t; x) = (t - \alpha_k^M(x)) M_k(t; x) - \beta_k^M(x) M_{k-1}(t; x), \quad k = 0, 1, \ldots,$$

with $M_0(t; x) = 1$ and $M_{-1}(t; x) = 0$. The recurrence coefficients are

(4.6)
$$
\begin{cases}
\alpha_0^M(x) = \dfrac{1}{3}x(2-x), \\[2mm]
\alpha_k^M(x) = \dfrac{32(k+1)k^3 - 8k^2(x-2)x - 4k(x-1)^2 + (x-2)x}{(4k-1)(4k+3)}, \quad k \in \mathbb{N}; \\[2mm]
\beta_0^M(x) = 2(1-x), \\[2mm]
\beta_k^M(x) = \dfrac{4k^2(2k-1)^2 \left(4k^2 - (1-x)^2\right)\left((2k-1)^2 - (1-x)^2\right)}{(4k-3)(4k-1)^2(4k+1)}, \quad k \in \mathbb{N}.
\end{cases}
$$

In terms of polynomials (4.1), these polynomials can be expressed in the form

(4.7)
$$
M_k(t; x) = G_{2k}^{(1-x)}\left(\sqrt{t}\,\right), \quad k = 0, 1, 2, \dots .
$$

Proof. According to (4.1), (4.2) and [11, Theorem 2.2.11], we conclude that $G_{2k}^{(1-x)}\left(\sqrt{t}\,\right)$ are monic polynomials orthogonal with respect to the weight function $t \mapsto w^G(\sqrt{t}; 1-x)/\sqrt{t}$ on $\mathbb{R}_+$, so that (4.7) holds.

Now, using [11, Theorem 2.2.12] we obtain the coefficients in the three-term recurrence relation (4.8). As usual, we put (cf. [11, p. 97])

$$
\beta_0^M(x) = \int_0^{+\infty} w^M(t; x)\, \mathrm{d}t = 2(1-x).
$$

Thus, we have $\alpha_0^M(x) = B_1(1-x) = x(2-x)/3$, as well as

$$
\alpha_k^M(x) = B_{2k}(1-x) + B_{2k+1}(1-x) \quad \text{and} \quad \beta_k^M(x) = B_{2k-1}(1-x)B_{2k}(1-x),
$$

where the coefficients B_k are given in (4.3). These formulas give the desired results. $\qquad\square$

Remark 4.2. A few first polynomials $M_k(t; x)$ are

$$
\begin{aligned}
M_0(t; x) &= 1, \\
M_1(t; x) &= t + \frac{1}{3}x(x-2), \\
M_2(t; x) &= t^2 + \frac{2}{7}\left(3x^2 - 6x - 10\right)t + \frac{3}{35}x\left(x^2 - 4\right)(x-4), \\
M_3(t; x) &= t^3 + \frac{5}{11}\left(3x^2 - 6x - 28\right)t^2 + \frac{1}{11}\left(5x^4 - 20x^3 - 80x^2 + 200x + 224\right)t \\
&\quad + \frac{5}{231}x\left(x^2 - 4\right)\left(x^2 - 16\right)(x-6), \\
M_4(t; x) &= t^4 + \frac{28}{15}\left(x^2 - 2x - 18\right)t^3 + \frac{14}{39}\left(3x^4 - 12x^3 - 100x^2 + 224x + 648\right)t^2 \\
&\quad + \frac{4}{2145}\left(105x^6 - 630x^5 - 4830x^4 + 23520x^3 + 54824x^2 - 158368x - 146112\right)t \\
&\quad + \frac{7}{1287}x\left(x^2 - 4\right)\left(x^2 - 16\right)\left(x^2 - 36\right)(x-8),
\end{aligned}
$$

etc.

As an additional result, which will not be of interest in our summation of trigonometric series, we can prove the following statement:

Theorem 4.2. *The polynomials $\{N_k(t;x)\}_{k=0}^{+\infty}$ orthogonal with respect to the weight function*

$$t \mapsto w^N(t;x) = \frac{\sin \pi x \sqrt{t}}{\cosh \pi \sqrt{t} - \cos \pi x} \quad on \quad \mathbb{R}_+ \quad (0 < x < 1)$$

satisfy the three-term recurrence relation

(4.8) $\qquad N_{k+1}(t;x) = (t - \alpha_k^N(x))N_k(t;x) - \beta_k^N(x)N_{k-1}(t;x), \quad k = 0, 1, \ldots,$

with $N_0(t;x) = 1$ and $N_{-1}(t;x) = 0$. The recurrence coefficients are

$$\alpha_0^N(x) = \frac{1}{5}\left(-3x^2 + 6x + 4\right),$$

$$\alpha_k^N(x) = \frac{32(k+3)k^3 - 8k^2\left(x^2 - 2x - 12\right) - 12k(x-3)(x+1) - 3x^2 + 6x + 4}{(4k+1)(4k+5)},$$

$$\beta_k^N(x) = \frac{4k^2(2k+1)^2\left(4k^2 - (1-x)^2\right)\left((2k+1)^2 - (1-x)^2\right)}{(4k-1)(4k+1)^2(4k+3)}$$

for each $k \in \mathbb{N}$. In terms of polynomials (4.1)*, the polynomials $N_k(t;x)$ can be expressed in the form $N_k(t;x) = G_{2k+1}^{(1-x)}(\sqrt{t})/\sqrt{t}$ for each $k \in \mathbb{N}_0$.*

The coefficient $\beta_0^N(x)$ may be arbitrary, because it multiplies $N_{-1}(t;x) = 0$, but usually, it is appropriate to take

$$\beta_0^N(x) = \int_0^{+\infty} w^N(t;x)\,\mathrm{d}t = \frac{8}{\pi^3}\,\mathrm{Im}\left\{\mathrm{Li}_3(\mathrm{e}^{\mathrm{i}\pi x})\right\}.$$

In the sequel, we consider the Gaussian quadrature formula with respect to the weight function $w^M(t;x)$ on $\mathbb{R}_+$,

(4.9) $\qquad \displaystyle\int_0^{+\infty} g(t)w^M(t;x)\,\mathrm{d}t = \sum_{\nu=1}^{N} A_\nu^{(N)}(x)g\big(\tau_\nu^{(N)}(x)\big) + R_N(g;x),$

where $R_N(g;x)$ is the corresponding remainder term ($g \in \mathcal{P}_{2n-1}$). As we mentioned in the previous section, the parameters of the quadrature formula (4.9), the nodes $\tau_\nu^{(N)}(x)$ and the weight coefficients $A_\nu^{(N)}(x)$, can be calculated very easy from the symmetric tridiagonal Jacobi matrix $J_N(w^M(\,\cdot\,;x))$ by the Golub-Welsch procedure. It is also implemented in the package `OrthogonalPolynomials` by the function `aGaussianNodesWeights`. Taking the recursion coefficients $\alpha_k^M(x)$ and $\beta_k^M(x)$, $k = 0, 1, \ldots, n-1$, defined before in (4.6), we can calculate nodes and weights in (4.9) for a given x and any $N \le n$.

For calculating values of the series $S(x)$, presented in the form

(4.10) $\qquad \displaystyle S(x) = \sum_{k=1}^{+\infty} a_k \sin k\pi x = \frac{\pi}{4}\int_0^{+\infty} \frac{\sin \pi x}{\sqrt{t}\left(\cosh \pi \sqrt{t} - \cos \pi x\right)} f\big(\pi\sqrt{t}\big)\,\mathrm{d}t,$

we use the quadrature rule (4.9). Thus, we approximate $S(x)$ by the quadrature sum $Q_N(f;x)$, where

(4.11) $\qquad\qquad\qquad\qquad \displaystyle Q_N(f;x) = \frac{\pi}{4}\sum_{\nu=1}^{N} A_\nu^{(N)}(x)f\big(\xi_\nu^{(N)}(x)\big)$

and $\xi_\nu^{(N)}(x) = \pi\sqrt{\tau_\nu^{(N)}(x)}$, $\nu = 1, \ldots, N$. The corresponding (relative) error is given by

$$(4.12) \qquad E_N(x) = \left| \frac{Q_N(f; x) - S(x)}{S(x)} \right|.$$

5. NUMERICAL EXAMPLES

Through two examples, we illustrate the efficiency of our methods. All computations were performed in Mathematica, Ver. 12, on MacBook Pro (15-inch, 2017), OS X 10.14.6.

Example 5.1. We consider the following series

$$C(x) = \sum_{k=1}^{+\infty} \frac{k}{4k^2 - 1} \cos k\pi x \quad \text{and} \quad S(x) = \sum_{k=1}^{+\infty} \frac{k}{4k^2 - 1} \sin k\pi x.$$

The sum of the first series is given by (cf. [22, p. 731])

$$C(x) = -\frac{1}{4} - \frac{1}{4} \cos \frac{\pi x}{2} \log\left|\tan \frac{\pi x}{4}\right|,$$

while for the sinus series, one can find that $S(x) = (\pi/8)\cos(\pi x/2)$.

Since

$$f(t) = \mathcal{L}^{-1}\left[\frac{s}{4s^2 - 1}\right] = \frac{1}{4} \cosh \frac{t}{2},$$

using (2.2) and the corresponding Gaussian rules with respect to the weights $w_R(t; x)$ and $w_I(t; x)$, we have

$$C(x) = \frac{\pi}{8} \sum_{\nu=1}^N A_\nu^R \cosh\left(\frac{\pi\tau_\nu^R}{2}\right) + R_N^R(x) \quad \text{and} \quad S(x) = \frac{\pi}{8} \sum_{\nu=1}^N A_\nu^I \cosh\left(\frac{\pi\tau_\nu^I}{2}\right) + R_N^I(x),$$

respectively.

TABLE 1. Relative errors $E_N^I(x)$ and $E_N^R(x)$, when $N = 5, 10, 15, 20$ and $x = 0.1(0.1)0.7$

rel. err.	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$	$x = 0.6$	$x = 0.7$
$E_5^I(x)$	1.10(−5)	2.03(−5)	2.84(−5)	3.54(−5)	4.13(−5)	4.62(−5)	5.00(−5)
$E_{10}^I(x)$	2.40(−10)	4.51(−10)	6.39(−10)	8.06(−10)	1.10(−15)	1.07(−9)	1.16(−9)
$E_{15}^I(x)$	4.78(−15)	9.07(−15)	1.29(−14)	1.64(−14)	1.10(−15)	2.19(−14)	2.39(−14)
$E_{20}^I(x)$	9.14(−20)	1.74(−19)	2.50(−19)	3.17(−19)	1.88(−15)	4.25(−19)	4.64(−19)
$E_5^R(x)$	2.88(−5)	6.78(−5)	9.34(−5)	2.67(−4)	1.09(−8)	2.57(−5)	3.58(−5)
$E_{10}^R(x)$	6.40(−10)	1.15(−9)	9.49(−10)	2.97(−9)	5.16(−17)	7.74(−10)	8.14(−10)
$E_{15}^R(x)$	1.22(−14)	2.25(−14)	4.91(−14)	8.44(−14)	2.23(−25)	1.16(−14)	1.66(−14)
$E_{20}^R(x)$	5.86(−19)	4.23(−19)	8.73(−19)	1.33(−18)	9.12(−34)	2.23(−19)	3.21(−19)

The relative errors in these quadrature sums are given by

$$E_N^R(x) = \left| \frac{R_N^R(x)}{C(x)} \right| = \left| \frac{1}{C(x)} \left(\frac{\pi}{8} \sum_{\nu=1}^N A_\nu^R \cosh\left(\frac{\pi\tau_\nu^R}{2}\right) - C(x) \right) \right|$$

and

$$E_N^I(x) = \left| \frac{R_N^I(x)}{S(x)} \right| = \left| \frac{1}{S(x)} \left(\frac{\pi}{8} \sum_{\nu=1}^N A_\nu^I \cosh\left(\frac{\pi\tau_\nu^I}{2}\right) - S(x) \right) \right|.$$

For getting recurrence parameters in the three-term recurrence relations for the polynomials $p_k^R(t;x)$ and $p_k^I(t;x)$, $k = 0, 1, \ldots$, we apply the procedure described Section 3. In order to save space the relative errors are given only at the points $x = j/10$, $j = 1, \ldots, 7$, for $N = 5, 10, 15$ and 20 nodes in the Gaussian rules (see Table 1). Numbers in parentheses indicate the decimal exponents, e.g., $1.10(-5)$ means 1.10×10^{-5}. For $N = 5$ we use the symbolic construction of recurrence coefficients, while for $N > 5$ we use numerical construction with the `WorkingPrecision -> 50`, with repetitions for each x.

As we mention before, the existence of the orthogonal polynomials $p_k^R(t;x)$, as well as the corresponding Gaussian formulas, are not guaranteed for $0 \le x < 1/2$. However, the obtained numerical results of $E_N^R(x)$, $N = 5(5)20$, for selected values of $x \in \{0.1, 0.2, 0.3, 0.4\}$ show the existence of such quadrature rules, as well as a fast convergence.

Example 5.2. Now we consider the series

$$S(x) = \sum_{k=1}^{+\infty} \frac{\sin k\pi x}{(1+k^2)^{1/2}}, \quad 0 < x < 1.$$

With $S_n(x)$ we denote the n-th partial sum, and by $e_n(x)$ its relative error, i.e.,

$$e_n(x) = \left| \frac{S_n(x) - S(x)}{S(x)} \right|.$$

The partial sums $S_n(x)$ are displayed in Figure 4 for $n = 5, 10$, and 50. As we can observe their convergence is very slow.
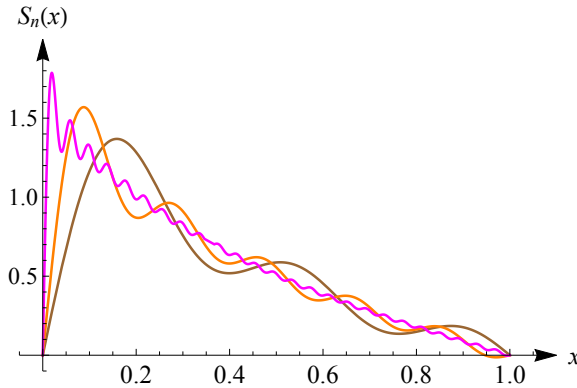
FIGURE 4. Partial sums $S_n(x)$ for $n = 5$ (brown line), $n = 10$ (orange line), and $n = 50$ (magenta line)

Now we apply our method for summing trigonometric series. Using Lemma 2.2, we can identify $F(s) = (1+s^2)^{-1/2}$ and $f(t) = J_0(t)$, where J_0 is the Bessel function. Then, according to (4.10) and (4.11), we have

$$S(x) = \sum_{k=1}^{+\infty} \frac{\sin k\pi x}{(1+k^2)^{1/2}} = \frac{\pi}{4} \int_0^{+\infty} \frac{\sin \pi x}{\sqrt{t}\left(\cosh \pi\sqrt{t} - \cos \pi x\right)} J_0\left(\pi\sqrt{t}\right) dt$$

$$\approx \frac{\pi}{4} \sum_{\nu=1}^{N} A_\nu^{(N)}(x) J_0\left(\xi_\nu^{(N)}(x)\right).$$

This quadrature process converges fast, because

$$t \mapsto J_0\big(\pi\sqrt{t}\big) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{\pi}{2}\right)^{2m} t^m$$

is an entire function.
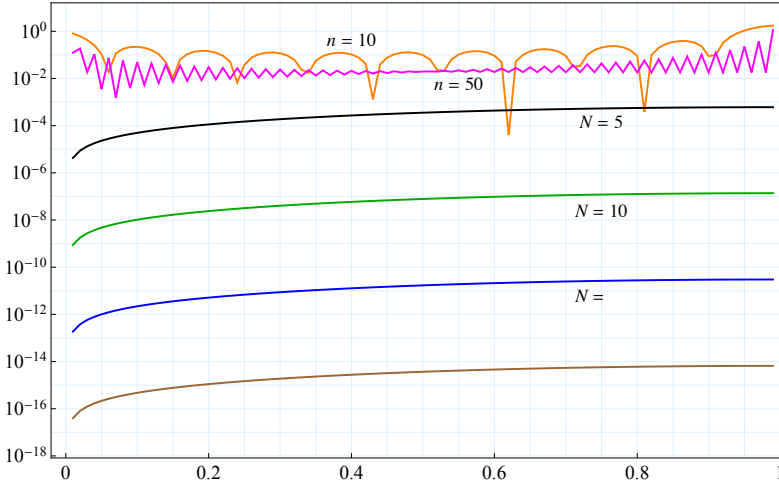
FIGURE 5. The relative errors $e_n(x)$ in the partial sums $S_n(x)$ for $n = 10$ and $n = 50$ and the relative errors $E_N(x)$ in the quadrature sums $Q_N(f; x)$ for $N = 5(5)20$

The relative errors $E_N(x)$ of quadrature approximation $Q_N(f; x)$ are given by (4.12), and presented in Figure 5 in log-scale for $N = 5, 10, 15$, and 20. Indeed, the convergence of $Q_N(f; x)$ is fast. As we can see, the five-point Gaussian formula gives three to five exact decimal digits (depending of x) of the sum $S(x)$, and the quadrature formula with $n = 20$ nodes gives an accuracy to more than 14 decimal digits.

TABLE 2. Relative errors $e_n(x)$ and $E_N(x)$, when $n = 100$ and 500 and $N = 5, 10, 20, 50$, for some selected values of x in $(0, 1)$

x	$e_{100}(x)$	$e_{500}(x)$	$E_5(x)$	$E_{10}(x)$	$E_{20}(x)$	$E_{50}(x)$
0.1	$2.49(-2)$	$4.99(-3)$	$4.96(-5)$	$1.03(-8)$	$4.72(-16)$	$4.78(-38)$
0.2	$1.51(-2)$	$3.03(-3)$	$1.13(-4)$	$2.39(-8)$	$1.10(-15)$	$1.13(-37)$
0.3	$1.12(-2)$	$2.41(-3)$	$1.88(-4)$	$4.03(-8)$	$1.88(-15)$	$1.93(-37)$
0.4	$1.06(-2)$	$2.12(-3)$	$2.70(-4)$	$5.87(-8)$	$2.76(-15)$	$2.84(-37)$
0.5	$9.87(-3)$	$1.97(-3)$	$3.54(-4)$	$7.80(-8)$	$3.68(-15)$	$3.81(-37)$
0.6	$9.44(-3)$	$1.89(-3)$	$4.35(-4)$	$9.66(-8)$	$4.59(-15)$	$4.76(-37)$
0.7	$9.18(-3)$	$1.84(-3)$	$5.06(-4)$	$1.13(-7)$	$5.39(-15)$	$5.61(-37)$
0.8	$9.04(-3)$	$1.81(-3)$	$5.61(-4)$	$1.26(-7)$	$6.03(-15)$	$6.28(-37)$
0.9	$8.96(-3)$	$1.79(-3)$	$5.96(-4)$	$1.34(-7)$	$6.44(-15)$	$6.71(-37)$

Also, the relative errors $e_n(x)$ in the partial sums $S_n(x)$ for $n = 10$ and $n = 50$ are shown in the same figure. Numerical values of the errors $e_n(x)$ in the partial sums $S_n(x)$

with $n = 100$ and $n = 500$ terms are given in the second and third column of Table 2 for equidistant values of x $(= 0.1, 0.2, \ldots, 0.9)$. We note that the number of exact digits in partial sums does not exceed three.

The numerical values of the corresponding relative errors $E_N(x)$ in the quadrature approximations $Q_N(f; x)$, with $N = 5, 10, 20, 50$ nodes, at the same values of x are given in the other columns of the same table. We note that the quadrature approximation $Q_{50}(f; x)$ has about 37 exact decimal digits! As an exact value of $S(x)$ we use one obtained by the Gaussian quadrature formula with $N = 100$ nodes.

6. Conclusion

In conclusion, we can say that the method presented in Section 3 is general for the both series $C(x)$ and $S(x)$, but for the sinus-series $S(x)$ the method presented in 4 is much simpler in applications, because in that case we have the recurrence relation (4.8), with recurrence coefficients in the explicit form (4.6).

7. Acknowledgment

References

[1] L. Carlitz: *Bernoulli and Euler numbers and orthogonal polynomials.* Duke Math. J. **26** (1959), 1–16.

[2] T. S. Chihara: *An Introduction to Orthogonal Polynomials.* Gordon and Breach, New York, 1978.

[3] A. S. Cvetković and G. V. Milovanović: *The Mathematica package "OrthogonalPolynomials".* Facta Univ. Ser. Math. Inform. **19** (2004), 17–36.

[4] W. Gautschi: *On generating orthogonal polynomials.* SIAM J. Sci. Statist. Comput. **3** (1982), 289–317.

[5] W. Gautschi: *A class of slowly convergent series and their summation by Gaussian quadrature.* Math. Comp. **57** (1991), 309–324.

[6] W. Gautschi: *Orthogonal polynomials: applications and computation.* Acta Numerica (1996), 45–119.

[7] W. Gautschi: *Orthogonal Polynomials: Computation and Approximation.* Clarendon Press, Oxford, 2004.

[8] W. Gautschi: *The spiral of Theodorus, numerical analysis, and special functions.* J. Comput. Appl. Math. **235** (2010), 1042–1052.

[9] W. Gautschi and G. V. Milovanović: *Gaussian quadrature involving Einstein and Fermi functions with an application to summation of series.* Math. Comp. **44** (1985), 177–190.

[10] G. H. Golub and J. H. Welsch: *Calculation of Gauss quadrature rule.* Math. Comp. **23** (1969), 221–230.

[11] G. Mastroianni and G. V. Milovanović: *Interpolation Processes – Basic Theory and Applications.* Springer Monographs in Mathematics, Springer – Verlag, Berlin – Heidelberg, 2008.

[12] G. V. Milovanović: *Summation of series and Gaussian quadratures.* In: Approximation and Computation (R.V.M. Zahar, ed.), pp. 459–475, ISNM Vol. 119, Birkhäuser Verlag, Basel–Boston–Berlin, 1994.

[13] G. V. Milovanović: *Summation of series and Gaussian quadratures, II.* Numer. Algorithms **10** (1995), 127–136.

[14] G. V. Milovanović: *Summation processes and Gaussian quadratures.* Sarajevo J. Math. **7** (20) (2011), 185–200.

[15] G. V. Milovanović: *Quadrature processes and new applications.* Bull. Cl. Sci. Math. Nat. Sci. Math. **38** (2013), 83–120.

[16] G. V. Milovanović: *Methods for computation of slowly convergent series and finite sums based on Gauss-Christoffel quadratures.* Jaen J. Approx. **6** (2014), 37–68.

[17] G. V. Milovanović: *On summation/integration methods for slowly convergent series.* Stud. Univ. Babeş-Bolyai Math. **61** (2016), 359–375.

[18] G. V. Milovanović: *Orthogonal polynomials on the real line*, In: Walter Gautschi: Selected Works with Commentaries, Chapter 11, Volume 2 (C. Brezinski, A. Sameh, eds.), pp. 3–16, Birkhäuser, Basel, 2014.

[19] G. V. Milovanović: *Computer algorithms and software packages*, In: Walter Gautschi: Selected Works with Commentaries, Chapter 23, Volume 3 (C. Brezinski, A. Sameh, eds.), pp. 9–10, Birkhäuser, Basel, 2014.

[20] G. V. Milovanović: *Construction and applications of Gaussian quadratures with nonclassical and exotic weight functions.* Stud. Univ. Babeş-Bolyai Math. **60** (2015), 211–233.

[21] G. V. MILOVANOVIĆ and A. S. CVETKOVIĆ: *Special classes of orthogonal polynomials and corresponding quadratures of Gaussian type.* Math. Balkanica **26** (2012), 169–184.

[22] A. P. PRUDNIKOV, YU. A. BRYCHKOV and O. I. MARICHEV: *Integrals and Series. Vol. 1. Elementary functions.* Gordon & Breach Science Publishers, New York, 1986.

[23] T. J. STIELTJES: *Sur quelques intégrals definies et leur développement en fractions continues.* Quart. J. Math. **24** (1890), 370–382 [Oeuvres. Vol. 2. Noordhoff, Groningen, 1918, pp. 378–394].

[24] J. TOUCHARD: *Nombres exponentiels et nombres de Bernoulli.* Canad. J. Math. **8** (1956), 305–320.

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CONSTRUCTIVE MATHEMATICAL ANALYSIS

Survey Article

From Sommerfeld Diffraction Problems to Operator Factorisation

Dedicated to Professor Erhard Meister on the occasion of his 90th anniversary

FRANK-OLME SPECK*

ABSTRACT. This article presents a brief survey devoted to the interplay between diffraction and operator theory, particularly between the so-called canonical diffraction problems (exemplified by Sommerfeld half-plane problems) on one hand and operator factorisation theory on the other hand. It is shown how operator factorisation concepts appear in a natural way from applications and how they can help to find solutions rigorously in case of well-posed problems or how to normalise problems by an adequate change of function spaces.

Keywords: Sommerfeld problem, diffraction, Wiener-Hopf method, Wiener-Hopf operator, factorisation, intermediate space, generalised inverse, operator relation.

2010 Mathematics Subject Classification: 47A68, 47B35.

1. INTRODUCTION

The original Sommerfeld diffraction problem has been solved in closed analytical form firstly by Arnold Sommerfeld in 1896 with the help of Fresnel integrals [84, 103]. An analytical solution with the so-called Wiener-Hopf technique was found in the 1940s, see [58, 86, 121]. It allowed a better understanding of the asymptotic behaviour of the solution, see also [66].

Another fourty years later, it has been proved to be well posed in the sense of Hadamard [47, 68] in certain Sobolev space settings [77, 107]. Moreover, the resolvent operator has been identified as an explicit formula based upon operator factorisation in the sense of Shinbrot in the context of a wider class of problems [33, 101, 106, 107]. We describe the corresponding basic ideas in the next two sections. It leads us to a detailed study of the Wiener-Hopf factorisation of matrix functions which will be discussed in Section 4. In Section 5, a general concept of operator factorisation in linear boundary value and transmission problems is briefly outlined. Section 6 is devoted to the choice of adequate function(al) spaces and symbol classes occurring from diffraction theory. In Section 7, we expose some logical sharp concepts of equivalence and reduction coming up with operator factorisation and with the aim to construct resolvent operators, i.e., inverses (or generalised inverses) of operators associated with linear boundary value and transmission problems. Section 8 presents a glance at concepts which are more general than operator factorisation, namely given by certain operator relations with applications in diffraction theory. In the last section, we demonstrate the usefulness of the preceding concepts once again by the generalisation of Sommerfeld problems from two to higher dimensions. At

the end of each section, the reader may find a short historical review of selected relevant publications.

The operator theoretical approach has the advantage of a compact presentation of results simultaneously for wide classes of diffraction problems which are analytically solvable by the Wiener-Hopf method and it gives a different and deeper understanding of the solution procedures. An interpretation in the physical sense, such as a consideration of the solution of a boundary value problem as the perturbed wave field in the diffraction of time-harmonic wave propagation, can be found in the cited literature, see [69, 70, 71, 72], for instance.

The main objective of this article is to demonstrate how real applications naturally guide us to operator factorisation concepts and how useful those are to simplify and to strengthen the reasoning in the applications. The exposition of these concepts will be accompanied by concrete problems closely connected with the Rawlins problem [92, 93].

2. FROM THE CLASSICAL WIENER-HOPF TECHNIQUE TO OPERATOR FACTORISATION

First, we consider the linear (non-singular) convolution equation on the real half-line

$$(2.1) \qquad \mathcal{W}f(x) \quad = \quad af(x) - \int_0^\infty K(x-y)\, f(y)\, dy \; = \; g(x) \quad , \quad x > 0.$$

Herein $a \in \mathbb{C}\setminus\{0\}, K \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R}_+), p \in [1, \infty]$ are (arbitrarily) given and $f \in L^p(\mathbb{R}_+)$ is unknown, $\mathbb{R}_+ = (0, \infty)$. It is well known that $\mathcal{W}$ defines a bounded linear operator acting in a Banach space $X = L^p(\mathbb{R}_+)$

$$(2.2) \qquad\qquad\qquad \mathcal{W} \quad : \quad L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+),$$

in brief $\mathcal{W} \in \mathcal{L}(L^p(\mathbb{R}_+))$, known as classical *Wiener-Hopf operator* (WHO) [42, 62, 79]. For simplicity, we abbreviate $L^p = L^p(\mathbb{R})$ and choose $a = 1$ and $p = 2$ in this discussion, focusing the scalar case (instead of systems of equations [43]).

Now, let us look at the steps of the Wiener-Hopf technique [70, 86] from the viewpoint of operator theory. Certainly one likes to find out whether the equation (2.1) is uniquely solvable and, in this case, the solution f depends continuously on the given function g, (i.e., whether (2.1) is well-posed) and also to obtain an explicit analytical formula for the solution, in brief to determine $W^{-1} \in \mathcal{L}(L^2(\mathbb{R}_+))$ if possible. To this end, we need some notation. We denote in (2.1) the convolution operator on the full real line by A and obtain with the help of the *Fourier transformation* $\mathcal{F}$ from the convolution theorem that

$$(2.3) \qquad\qquad A \quad = \quad I + K* \quad = \quad \mathcal{F}^{-1}\, \Phi \cdot \mathcal{F} \quad : \quad L^2 \to L^2 \,,$$

$$\Phi \quad = \quad 1 + \mathcal{F}K \quad \in \quad L^\infty(\mathbb{R}),$$

$$\mathcal{F}K(\xi) \quad = \quad \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} e^{ix\xi}\, K(x)\, dx \quad , \quad \xi \in \mathcal{R}\,.$$

Hence, the classical Wiener-Hopf operator can be written as

$$(2.4) \qquad\qquad \mathcal{W} \quad = \quad r_+ A\, \ell_0 \quad = \quad L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$$

where r_+ and ℓ_0 denote the restriction and zero extension operator, respectively:

$$(2.5) \qquad\qquad r_+ \; : \; L^2(\mathbb{R}) \to L^2(\mathbb{R}_+) \qquad , \qquad \ell_0 \; : \; L^2(\mathbb{R}_+) \to L^2(\mathbb{R}),$$

which are bounded linear operators. Obviously, we have

$$(2.6) \qquad\qquad r_+ \ell_0 \quad = \quad I_{L^2(\mathbb{R}_+)} \quad , \quad P \; = \; \ell_0\, r_+ \; \in \; \mathcal{L}(L^2),$$

where P is a projector acting in L^2, i.e., $P \in \mathcal{L}(L^2)$, $P^2 = P$, with image (or range) and kernel (or null space) given by

(2.7)
$$\begin{aligned} \operatorname{im} P &= L^2_+ = \{f \in L^2 \,:\, \operatorname{supp} f \subset \overline{\mathbb{R}_+}\}, \\ \ker P &= L^2_- = \{f \in L^2 \,:\, \operatorname{supp} f \subset \overline{\mathbb{R}_-}\}. \end{aligned}$$

We shall say that P projects onto $\operatorname{im} P$ along $\ker P$.

With the help of this basic notation the steps of the Wiener-Hopf technique can be described as follows. First, the equation (2.1) is extended by zero to an equation holding on the full line:

(2.8)
$$Af_+ = g_+ + g_-,$$

where $g_+ = \ell_0\, g \in L^2_+$ is known and $f_+ = \ell_0\, g \in L^2_+$, $g_- \in L^2_-$ are unknown provided (2.1) is solvable.

In what follows the *Fourier symbol* $\Phi = 1 + \mathcal{F}K$ of A plays a decisive role. Note that this function is also referred to as "Fourier transformed kernel" or simply as "kernel" in some papers such as [63] which has to be distinguished from the convolution kernel K of the integral operator A and from the kernel (or null space) of a linear operator like A or W (hence we avoid calling Φ a kernel).

Now, let us assume that $|\Phi|$ is bounded from below. Since the Fourier symbol Φ of A belongs to the *Wiener algebra* $\mathbb{C} \oplus \mathcal{F}L^1$ (with $\Phi(\infty) = 1$), it allows a factorisation [44, 62]

(2.9)
$$\Phi = \Phi_-\, \zeta^\kappa\, \Phi_+,$$

$$\kappa = \frac{1}{2\pi}\int_{\mathbb{R}} d\arg\Phi(\xi)d\xi \in \mathbb{Z} \quad,\quad \zeta(\xi) = \frac{\xi - i}{\xi + i} \;,\; \xi \in \mathbb{R},$$

$$\Phi_\pm = \exp\{\mathcal{F}\,\ell_0\, r_\pm \mathcal{F}^{-1}\, \log(\zeta^{-\kappa}\, \Phi)\}.$$

With the help of this factorisation one can continue considering the Fourier transformed equation of (2.8) (as common in the classical Wiener-Hopf technique [86]) or, alternatively, making directly use of the operator factorisation that results from (2.9):

(2.10)
$$A = A_-\, C\, A_+,$$

$$A_\pm = \mathcal{F}^{-1}\Phi_\pm \cdot \mathcal{F} \quad,\quad C = \mathcal{F}^{-1}\zeta^\kappa \cdot \mathcal{F} \quad:\quad L^2 \to L^2.$$

All the three factors A_-, C, A_+ are isomorphisms in L^2 (i.e., linear homeomorphisms, boundedly invertible operators), and the inverses are convolution operators with Fourier symbols $\Phi_-^{-1}, \zeta^{-\kappa}, \Phi_+^{-1}$ in the Wiener algebra and value 1 at infinity.

The next step in the Wiener-Hopf technique is a rearrangement of equation (2.8) which is different depending on whether $\kappa \geq 0$ or $\kappa \leq 0$ (and coinciding for $\kappa = 0$ where $C = I$), namely

(2.11)
$$\begin{aligned} CA_+ f_+ &= A_-^{-1} g_+ + A_-^{-1} g_- \quad,\quad \text{if } \kappa \geq 0, \\ A_+ f_+ &= C^{-1}A_-^{-1} g_+ + C^{-1}A_-^{-1} g_- \;,\; \text{if } \kappa \leq 0. \end{aligned}$$

Now, the classical Wiener-Hopf technique [86] works with holomorphy properties of the Fourier transformed terms of the equation and an additive decomposition of the first term on the right hand side which results in the same conclusion as applying the projector P on the two

equations, briefly

(2.12)
$$PCA_+ f_+ \;=\; PA_-^{-1} g_+ \;,\quad \text{if } \kappa \geq 0,$$

$$PA_+ f_+ \;=\; PC^{-1} A_-^{-1} g_+ \;,\quad \text{if } \kappa \leq 0.$$

This is just a consequence of the invariance properties of the factors: A_+ maps $\operatorname{im} P$ onto itself, A_- maps $\ker P$ onto itself. Furthermore, in case of $\kappa \geq 0$, the middle factor C maps $\operatorname{im} P$ into itself, whilst C^{-1} maps $\ker P$ into itself, and in case of $\kappa \leq 0$, it's the other way around. These properties altogether are equivalent to the following formulas, so-called invariance properties of the factors:

(2.13)
$$PA_+ P \;=\; A_+ P \;,\quad PA_+^{-1} P \;=\; A_+^{-1} P,$$

$$PA_- P \;=\; PA_- \;,\quad PA_-^{-1} P \;=\; PA_-^{-1},$$

$$PCP \;=\; CP \;,\quad PC^{-1} P \;=\; PC^{-1} \;,\quad \text{if } \kappa \geq 0,$$

$$PCP \;=\; PC \;,\quad PC^{-1} P \;=\; C^{-1} P \;,\quad \text{if } \kappa \leq 0.$$

If $\kappa \geq 0$, the equation (2.12) can be easily transformed with the help of the *factor properties* (2.13) into

(2.14)
$$f_+ \;=\; A_+^{-1} \, P \, C^{-1} \, P \, A_-^{-1} \, g_+ \;\in\; L_+^2.$$

In the original setting (2.1), we obtain

(2.15)
$$f \;=\; r_+ A_+^{-1} \, P \, C^{-1} \, P \, A_-^{-1} \, \ell_0 \, g \;\in\; L^2(\mathbb{R}_+).$$

However, this formula holds only if the equation (2.1) is solvable (as we started from the assumption that (2.1) holds). Looking carefully at (2.11) one can find (for $\kappa > 0$) a necessary solubility condition, namely $PA_-^{-1} g_+ \in \operatorname{im} PCL_+^2$, which can be verified to be sufficient, as well.

Instead of this argumentation one can prove that (2.15) presents a left inverse of the operator $\mathcal{W}$ in (2.4) for all $\kappa \geq 0$ by the help of (2.13). Similarly one verifies that it gives a right inverse of $\mathcal{W}$ in the case of $\kappa \leq 0$.

Denoting these one-sided inverses by $\mathcal{W}^-$ we have in the case $\kappa > 0$ a (non-trivial) projector onto the image of $\mathcal{W}$ given by $\mathcal{W}\mathcal{W}^-$ and in the case $\kappa < 0$ a (non-trivial) projector along the kernel of $\mathcal{W}$ given by $\mathcal{W}^- \mathcal{W}$. So, we obtain as a common operator theoretical interpretation:

Theorem 2.1. *Let $\mathcal{W}$ be given by (2.4) where $|\Phi|$ be bounded from below, i.e., $A \in \mathcal{L}(L^2(\mathbb{R}))$ is an isomorphism, and let $\mathcal{W}^-$ be the operator from (2.15). Then, we have for*

- $\kappa = 0$: *Equation (2.1) is uniquely solvable by (2.15) (where $C = I$) for any $g \in L^2(\mathbb{R}_+)$.*
- $\kappa > 0$: *Equation (2.1) is solvable for a certain $g \in L^2(\mathbb{R}_+)$ if and only if g satisfies the condition*

$$\mathcal{W}\mathcal{W}^- g \;=\; r_+ A_- PCPC^{-1} PA_-^{-1} \ell_0 g \;=\; g.$$

   In this case, (2.15) presents the unique solution.
- $\kappa < 0$: *Equation (2.1) is solvable for all $g \in L^2(\mathbb{R}_+)$. The general solution reads*

$$f \;=\; \mathcal{W}^- g \;+\; (I - \mathcal{W}^- \mathcal{W}) h \;,\quad h \in L^2(\mathbb{R}_+)$$

$$\mathcal{W}^- \mathcal{W} \;=\; r_+ A_+^{-1} PC^{-1} PCPA_+ \ell_0.$$

Remark 2.1. *Moreover one can conclude that $\mathcal{W}$ is a Fredholm operator, i.e., its defect numbers are finite: $\dim \ker \mathcal{W} < \infty$, $\operatorname{codim} \operatorname{im} \mathcal{W} < \infty$. The solubility conditions in the case $\kappa > 0$ can be written as κ orthogonality conditions using a suitable basis of L^2. The kernel of $\mathcal{W}$ can also be explicitly written as a $-\kappa$ - dimensional subspace of $L^2(\mathbb{R}_+)$ [42, 62, 79]. Details are given later in (7.91) - (7.94).*

Remark 2.2. *It is well known that the appearance of one-sided invertibility has a deeper background in the Theorem of Coburn, see* [10, *Section 2.6], which allows the conclusion that for certain classes of operators (such as classical scalar Toeplitz and Wiener-Hopf operators), the Fredholm property implies one-sided invertibility.*

Hence the question arises if, in a general setting, a factorisation of A with the factor properties (2.13) is not only sufficient for the one-sided invertibility of $\mathcal{W}$, but moreover if such a factorisation is necessary, as well. For clarity, let us first see the case where W is invertible and $C = I$ coming back to the general case later. To this end, we need the following notation.

Let X be a Banach space, $A \in \mathcal{L}(X)$ and $P \in \mathcal{L}(X)$ a projector, and put $Q = I - P$, $PX = \operatorname{im} P = \ker Q$, $QX = \operatorname{im} Q = \ker P$ for convenience. Then

$$(2.16) \qquad W \;=\; P\,A|_{PX} \;:\; PX \to PX$$

is referred to as *general Wiener-Hopf operator* (WHO). If A is an isomorphism, an operator pair A_+, $A_- \in \mathcal{L}(X)$ is said to be a *strong (right) WH factorisation* of A with respect to (X, P) if

$$(2.17) \qquad A \;=\; A_- A_+$$

and the first two lines of the relations (2.13) are satisfied, i.e., A_+ leaves $\operatorname{im} P$ invariant and A_- leaves $\ker P$ invariant. As a standard situation in this general setting we shall work only with Banach spaces; other convenient frameworks could be topological vector spaces or Hilbert spaces.

Remark 2.3. *The classical WHO* (2.4) *is not of the form* (2.16) *as* $L^2(\mathbb{R}_+)$ *is not a subspace of* $L^2(\mathbb{R})$. *But* $L^2(\mathbb{R}_+)$ *is isomorphic to the subspace* $L^2_+ \subset L^2(\mathbb{R})$, *see* (2.5) - (2.7). *With the operator P of* (2.6), *we therefore have*

$$W \;=\; \ell_0\, r_+\, A\, \ell_0\, r_+|_{L^2_+} \;=\; \ell_0\, \mathcal{W}\, r_+ \;:\; L^2_+ \to L^2_+\,,$$
$$\mathcal{W} \;=\; r_+\, P\, A|_{PX}\, \ell_0 \;=\; r_+\, W\, \ell_0 \;:\; L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)\,,$$

i.e., the classical WHO is equivalent to an operator of the form of a general WHO (2.16).

Remember that two bounded linear operators S, T acting in Banach spaces are said to be *equivalent*, if there are isomorphisms E, F such that

$$T \;=\; E\,S\,F\,,$$

briefly written as

$$T \;\sim\; S\,.$$

Theorem 2.2 (of Devinatz and Shinbrot). *Let the assumptions of* (2.16) *be fulfilled and A be an isomorphism. Then, the WHO W is boundedly invertible if and only if A admits a strong WH factorisation with respect to (X, P). In this case, the inverse of W is given by*

$$(2.18) \qquad W^{-1} \;=\; A_+^{-1}\, P\, A_-^{-1}|_{PX} \;:\; PX \to PX.$$

The proof is quite elementary, although not at all constructive. Sufficiency is verified with the help of the factor properties (2.13). Necessity is proved as follows: If W is invertible in $\mathcal{L}(PX)$, then $P\,A + Q$ is invertible in $\mathcal{L}(X)$ because of the well-known relation

$$P\,A\,P + Q \;=\; (I - P\,A\,Q)\,(P\,A + Q),$$

where $P\,A\,P + Q$ is obviously invertible and $(I - P\,A\,Q)^{-1} = I + P\,A\,Q$. Now put just

$$A \;=\; A_-\,A_+ \;=\; A_-\,(P\,A + Q)$$

and verify the factor properties, see [106] for details.

This result was a cornerstone for the study of general Wiener-Hopf operators as given by (2.16) and moreover for general WHOs in an asymmetric space setting [106, 112] defined as follows.

Let X, Y be Banach spaces, $A \in \mathcal{L}(X, Y)$ an isomorphism, $P_1 \in \mathcal{L}(X)$, $P_2 \in \mathcal{L}(Y)$ two projectors, and put $Q_j = I - P_j$, $P_1 X = \operatorname{im} P_1 = \ker Q_1$ etc. for convenience. Then

$$(2.19) \qquad\qquad W \quad = \quad P_2 \, A|_{P_1 X} \quad : \quad P_1 X \; \to \; P_2 Y$$

is referred to as a *general WHO* (*in asymmetric setting*).

The case of one-sided invertible WHOs can be seen as a special case of *generalised invertibility*, i.e., for $T \in \mathcal{L}(X, Y)$ there exists an operator T^- such that

$$(2.20) \qquad\qquad\qquad T \; T^- \; T \quad = \quad T.$$

This conception allows a unified discussion of many kinds of general WHOs and their realisations. It will be extended later in Section 7.

In what concerns Sommerfeld diffraction problems several modifications of the WHO (2.1) have to be considered, particularly:

- operators acting between different (Sobolev-like) functional spaces,
- matrix instead of scalar operators,
- more general Fourier symbol classes (rather than the Wiener algebra).

The need of these generalisations will be shown in the following Sections 3, 4, and 6, respectively.

Historical remarks.

Wiener-Hopf equations are named after Norbert Wiener and Eberhard Hopf who considered this kind of equations in 1931 (in German) [122]. Crucial progress in the solution of WH equations was presented in the fundamental paper by Mark Krein in 1958 (in Russian) [62] and for systems by Israel Gohberg and Mark Krein in the same year [43] (which was even published before the previous). Various function spaces were considered besides the Lebesgue spaces (with similar results) and also various symbol classes besides the Wiener algebra allowed similar results, because the Hilbert transformation is continuous in so-called decomposing algebras (including Hölder continuous functions). Other symbol classes (like the continuous functions) are not decomposing, the (formal) factors are not in the same class, see the books by I.C. Gohberg and I.A. Fel'dman [42] or Solomon Grigor'evich Mikhlin and Siegfried Prössdorf [79] and the most important article on generalised factorisation by I.B. Simonenko from 1968 [102].

The Wiener-Hopf technique (also called WH method or WH procedure) developed parallel to the previous advances in applications, mainly by British researchers, see the famous books of Ben Noble from 1958 [86], by Douglas Jones from 1964 [58], as well as the surveys on the classical WH method in applications by David Abrahams [2] and another one by Jane Lawrie and David Abrahams [63]. Further remarkable progress was obtained for instance in Canada by Albert Heins and Robert Allan Hurd (loc. cit.), in Russia, see the book of Lev Weinstein [121], and later in Turkey by Mithat Idemen [56], A. Hamit Serbest et al. [98].

General WHOs were introduced by Marvin Shinbrot in 1964 [101]. Theorem 2.5 appeared in a paper with Allan Devinatz in 1969 [33] firstly for separable Hilbert spaces with a rather complicated proof, a year after a more general result about one-sided invertibility of ring elements (instead of $PAP \in \mathcal{L}(X)$) that was already published by Grigorii Chebotarev in 1968 (in Russian) [29], under the name *abstract WHO*.

Some ideas about general WHOs appeared independently under the names truncation, projection, or compression of an operator, see the books [10, 44, 65], for instance.

Also, in the context of pseudo-differential operators (acting in Sobolev spaces) we can find similar ideas and related results, see [41, 117]. Some higher dimensional WH equations in applications have been addressed in [45, 75], for instance.

The connection between classical and general WHOs was firstly pointed out in [33, 75], see also [71, 72].

3. THE WH EQUATIONS IN SOMMERFELD HALF-PLANE PROBLEMS

A *Sommerfeld half-plane problem* is here referred to as to determine the solution u of a boundary value problem for the *Helmholtz equation*

$$(\Delta + k^2)\, u(x) \quad = \quad \frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^2 u(x)}{\partial x_2^2} + \frac{\partial^2 u(x)}{\partial x_3^2} + k^2 u(x) \quad = \quad 0$$

in the slit domain $\Omega = \mathbb{R}^3 \setminus \Sigma$, where the screen Σ is a half-plane $\Sigma = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0\,,\, x_2 = 0\}$ on which the field u is known, or its normal derivative or another linear boundary condition is prescribed on the two sides $\Sigma^{\pm}$ of the screen (corresponding with $x_1 \geq 0\,,\, x_2 = \pm 0$). The *wave number k* is assumed to have a positive imaginary part throughout this paper. With the argument that in certain cases (as for a plane wave incoming perpendicularly to the edge of the screen) the solution will not depend on the third variable, the problem is modified to be a two-dimensional one, briefly written as

(3.21)
$$\begin{aligned}
(\Delta + k^2)\, u &= 0 \quad \text{in } \Omega, \\
T_{0,\Sigma^{\pm}}\, u &= g \quad \text{on } \Sigma^{\pm}
\end{aligned}$$

in case of the *Sommerfeld-Dirichlet problem*, with the trace operator $T_{0,\Sigma^{\pm}}$. More precisely, we like to show that the problem is well-posed in a somehow reasonable space setting, and therefore we are looking for a weak solution in the energy space H^1 such that g has to be (arbitrarily) given in the trace space $H^{1/2}(\mathbb{R}_+) = r_+ H^{1/2}$ identifying the two banks $\Sigma^{\pm}$ of the screen with the half-line $\overline{\mathbb{R}_+}$. Moreover, an explicit solution in closed analytical form is wanted. See [77, 107] for more details.

We shall work mainly with the spaces of Bessel potentials H^s which coincide with the Sobolev spaces $W^{s,p}$ in the case $s \in \mathbb{Z}$, $p = 2$ and the Sobolev-Slobodeckij spaces for $s \in \mathbb{R}$, $p = 2$, all briefly referred to as *Sobolev spaces* [3, 41, 52, 119, 120, 124].

It has been shown [107] that problem (3.21) is solvable if and only if the difference of the traces of the normal derivative of u on the upper and lower bank of the full line $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$ (identified with $\mathbb{R}$), respectively, $f \in H^{-1/2}(\mathbb{R})$ satisfies the equation

(3.22)
$$r_+ A_\Phi f \quad = \quad 2g,$$

where $A_\Phi = \mathcal{F}^{-1} \Phi \cdot \mathcal{F}$, $\Phi(\xi) = (\xi^2 - k^2)^{-1/2}$, $\xi \in \mathbb{R}$, i.e., A_Φ is a convolution (or pseudo-differential) operator of order -1 and an isomorphism from $H^{-1/2}(\mathbb{R})$ onto $H^{1/2}(\mathbb{R})$. Furthermore, the functional f has to be supported on the positive half-line, which will be indicated by $f \in H_+^{-1/2}$ (analogously to the definition of L_+^2). It is important that the formulas (2.5) - (2.6) are valid for H^s instead of L^2 if and only if $|s| < 1/2$, see [41]. The different analytical nature of the spaces $H^{\pm 1/2}(\mathbb{R}_+)$ and $H_+^{\pm 1/2}$ is crucial. It is well-known [46, 52] that the spaces

$$\widetilde{H}^s(\mathbb{R}_+) \quad = \quad \{f \in H^s(\mathbb{R}_+) \; : \; r_+ f \in H_+^s\} \quad, \quad s \geq -\frac{1}{2},$$

equipped with the norm of H_+^s, satisfy

$$\widetilde{H}^s(\mathbb{R}_+) \quad = \quad H^s(\mathbb{R}_+) \ , \ |s| < \frac{1}{2},$$

$$\widetilde{H}^s(\mathbb{R}_+) \quad \overset{\subset}{\text{dense}} \quad H^s(\mathbb{R}_+) \ , \ s = \pm\frac{1}{2},$$

$$\widetilde{H}^s(\mathbb{R}_+) \quad \subset \quad C(\mathbb{R}_+) \ , \ s > \frac{1}{2}.$$

For these reasons it is not convenient to consider equation (3.22) as a WH integral equation of the first kind in L^2, but as a WH equation in asymmetric space setting with a WHO defined by

(3.23) $$W \quad = \quad r_+ \, A_\Phi|_{H_+^{-1/2}} \quad : \quad H_+^{-1/2} \ \to \ H^{1/2}(\mathbb{R}_+),$$

where the convolution $A_\Phi = \mathcal{F}^{-1} \, \Phi \cdot \mathcal{F} : H^{-1/2} \to H^{1/2}$ is invertible by $A_\Phi^{-1} = \mathcal{F}^{-1} \, \Phi^{-1} \cdot \mathcal{F}$.

Remark 3.4. *Note that the operator W does not have the form of a general WHO in asymmetric space setting (2.19), because $r_+ : H^{1/2} \to H^{1/2}(\mathbb{R}_+)$ is not a projector in $H^{1/2}$, nor is it equivalent to a WHO in $H^{1/2}(\mathbb{R}_+)$ in the same way as in the L^2 setting, because ℓ_0 does not act from $H^{1/2}(\mathbb{R}_+)$ into $H^{1/2}(\mathbb{R})$. However, any continuous extension operator $\ell^c : H^{1/2}(\mathbb{R}_+) \to H^{1/2}(\mathbb{R})$ with $P_2 = \ell^c \, r_+ = P_2^2 \in H^{1/2}(\mathbb{R})$ will serve this purpose and leads to more complicated formulas explained below.*

The square root function that appears in the Fourier symbol of A_Φ (with $\Im m \, k > 0$) is of great importance. We shall denote it by

(3.24) $$t(\xi) \quad = \quad (\xi^2 - k^2)^{1/2} \ , \quad \xi \in \mathbb{R},$$

and take the branch cut from k to $-k$ vertically via ∞ such that

$$t(\xi) \quad \approx \quad |\xi| \ , \quad \text{as } \xi \to \pm\infty.$$

Its factorisation (with corresponding branches)

(3.25) $$t(\xi) \quad = \quad (\xi - k)^{1/2} \, (\xi + k)^{1/2} \quad = \quad t_-^{1/2}(\xi) \, t_+^{1/2}(\xi), \quad \xi \in \mathbb{R},$$

is closely connected with the definition of the one-dimensional Sobolev spaces [41] and some important properties. Firstly, we have

$$H^s \quad = \quad A_{t^{-s}} \, L^2,$$

(3.26) $$H_+^s \quad = \quad A_{t_+^{-s}} \, L_+^2 \quad , \quad H_-^s \quad = \quad A_{t_-^{-s}} \, L_-^2,$$

$$H^s(\mathbb{R}_+) \quad = \quad r_+ \, A_{t_-^{-s}} \, L_+^2 \quad , \quad H^s(\mathbb{R}_-) \quad = \quad r_- \, A_{t_+^{-s}} \, L_-^2,$$

for all $s \in \mathbb{R}$ (in a distributional sense for $s < 0$). Note that in these formulas the wave number k takes over the common role of the imaginary unit i because of the assumption of $\Im m \, k > 0$ (the dependence on k is here suppressed for brevity). Moreover, they serve to construct projectors and extension operators (mentioned before) that are most convenient in the present context [22, 25].

To make the connection between (3.23) and (2.19) it is necessary to find convenient projectors and extension operators. This step is a little technical, however leads to a rigorous understanding of classical as general WHOs.

For any $s \in \mathbb{R}$, we find projectors in H^s abbreviating now (for cosmetic reasons) $P_+ = \ell_0 r_+ \in \mathcal{L}(L^2)$ and $P_- = I - P_+ = \ell_0 r_-$ by putting

$$(3.27) \qquad P_+^{(s)} = A_{t_+^{-s}} P_+ A_{t_+^s} \quad \text{onto} \quad H_+^s,$$

$$P_-^{(s)} = A_{t_-^{-s}} P_- A_{t_-^s} \quad \text{onto} \quad H_-^s,$$

$$\Pi_+^{(s)} = A_{t_-^{-s}} P_+ A_{t_-^s} \quad \text{along} \quad H_-^s,$$

$$\Pi_-^{(s)} = A_{t_+^{-s}} P_- A_{t_+^s} \quad \text{along} \quad H_+^s,$$

which are orthogonal (in the common sense) for $k = i$. Hence, $P_+^{(s)} + \Pi_-^{(s)} = I_{H^s}$ and $P_-^{(s)} + \Pi_+^{(s)} = I_{H^s}$ for any $s \in \mathbb{R}$. Suitable extension operators are (because of (3.26)) given by

$$\ell_+^{(s)} = A_{t_-^{-s}} P_+ A_{t_-^s} \ell^{(s)} \quad : \quad H^s(\mathbb{R}_+) \to H^s(\mathbb{R}),$$

$$\ell_-^{(s)} = A_{t_+^{-s}} P_- A_{t_+^s} \ell^{(s)} \quad : \quad H^s(\mathbb{R}_-) \to H^s(\mathbb{R}),$$

where $\ell^{(s)}$ stands for an arbitrary extension into H^s, see [41]. Obviously, we have the analogue of (2.6)

$$r_+ \ell_+^{(s)} = I_{H^s(\mathbb{R}_+)} \quad , \quad \Pi_+^{(s)} = \ell_+^{(s)} r_+ \in \mathcal{L}(H^s).$$

Now, we are in the position for a strict understanding of (3.23) in the sense of an asymmetric general WHO and its inversion in the spirit of Theorem 2.5. First, we see that W is equivalent to the operator

$$(3.28) \qquad \ell_+^{(1/2)} W = \Pi_+^{(1/2)} A_\Phi \big|_{\text{im } P_+^{(-1/2)}} \quad : \quad H_+^{-1/2} \to \ell_+^{(1/2)} H^{1/2}(\mathbb{R}_+),$$

which has the form of (2.19). Second, the idea of a strong factorisation (2.10) (with $C = I$) extends to the asymmetric case

$$(3.29) \qquad A_\Phi = A_{t_-^{-1/2}} \quad A_{t_+^{-1/2}}$$
$$: H^{1/2} \longleftarrow L^2 \longleftarrow H^{-1/2},$$

a factorisation into isomorphisms between the indicated spaces with factor properties analogous to the symmetric space case (2.17)

$$(3.30) \qquad P A_+ P_1 = A_+ P_1 \quad , \quad P_1 A_+^{-1} P = A_+^{-1} P,$$

$$P_2 A_- P = P_2 A_- \quad , \quad P A_-^{-1} P_2 = P A_-^{-1},$$

with the previous interpretation of operators, namely $P_1 = P_+^{(-1/2)}$, $P = \ell_0 r_+$, , $P_2 = \Pi_+^{(1/2)}$. It follows that the equation (3.22) is solved by the asymmetric analogue of (2.18):

$$(3.31) \qquad f = W^{-1} 2g = A_+^{-1} P A_-^{-1} \big|_{P_2 Y} \ell^{(1/2)} 2g.$$

Moreover, it turns out that the idea of Theorem 2.5 works for general WHOs in asymmetric space setting, as well, see [106, Chapter 2] or [113, Theorem 2.1]:

Theorem 3.3. *Let X, Y be Banach spaces, $A \in \mathcal{L}(X, Y)$ an isomorphism and $P_1 \in \mathcal{L}(X)$, $P_2 \in \mathcal{L}(Y)$ two projectors. Then the (general) WHO*

$$(3.32) \qquad W = P_2 A \big|_{P_1 X} \quad : \quad P_1 X \to P_2 Y$$

is boundedly invertible if and only if A admits a strong WH factorisation with respect to (X, Y, P_1, P_2):

$$(3.33) \qquad\qquad A \quad = \quad A_- \quad A_+$$
$$\qquad\qquad : \quad Y \longleftarrow Z \longleftarrow X,$$

where Z is a Banach space, $P \in \mathcal{L}(Z)$ a projector, and the relations (3.30) are satisfied. In this case, the inverse of W is given by

$$(3.34) \qquad\qquad W^{-1} \quad = \quad A_+^{-1} \, P \, A_-^{-1}|_{P_2 Y} \quad : \quad P_2 Y \to P_1 X.$$

The previous concept is applicable to various classes of diffraction problems that lead to systems of WH equations, as well. As a first example, we consider the *Rawlins problem* [77, 93, 107] where, instead of the Dirichlet condition on both banks of the screen (see (3.21)), a Dirichlet condition on the upper bank and a Neumann condition on the lower bank is prescribed, in brief

$$(3.35) \qquad (\Delta + k^2) \, u \quad = \quad 0 \quad \text{in } \Omega,$$

$$T_{0,\Sigma+} \, u \quad = \quad u(., x_2)|_{x_2 = +0} \quad = \quad g_0 \quad \in \quad H^{1/2}(\mathbb{R}_+),$$

$$T_{1,\Sigma-} \, u \quad = \quad \partial u / \partial x_2 \, u(., x_2)|_{x_2 = -0} \quad = \quad g_1 \quad \in \quad H^{-1/2}(\mathbb{R}_+).$$

The resulting 2×2 system of WH equations can be written (after some elementary substitutions) as

$$(3.36) \qquad\qquad r_+ A_\Phi \, f \quad = \quad g,$$

where A acts in a topological product (Banach) space (like a 2×2 matrix operator):

$$(3.37) \qquad A \quad = \quad \mathcal{F}^{-1} \, \Phi \cdot \mathcal{F} \quad : \quad H^{1/2} \times H^{-1/2} \quad \to \quad H^{1/2} \times H^{-1/2},$$

$$\Phi \quad = \quad \begin{pmatrix} -1 & t^{-1} \\ t & 1 \end{pmatrix},$$

$$f \quad \in \quad H_+^{1/2} \times H_+^{-1/2} \quad , \quad g \quad = \quad (g_0, g_1) \quad \in \quad H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+).$$

It turns out that a strong factorisation of A in the form (3.33) can be explicitly obtained with an intermediate space [77]

$$(3.38) \qquad\qquad Z \quad = \quad H^{1/4} \times H^{-1/4}.$$

This is an interpretation of the explicit factorisation of the matrix Φ, a spectacular result of Tony Rawlins in 1975 [91, 92], which will be discussed in a wider frame subsequently and leads us to the rigorous solution of a series of boundary-transmission problems and operator theoretical insights, presented below.

Hence, the problem (3.35) is well-posed in this space setting and the solution (resolvent operator) can be obtained by the help of the inversion formula of Theorem 3.2. Moreover, the asymptotical behavior of the solution near the edge of the screen is related to the order of the Sobolev spaces in (3.38):

$$\text{grad} \, u \quad \sim \quad |x|^{-3/4} \text{ as } x \to 0.$$

Further boundary-transmission problems can be studied by analogy when the two boundary conditions in (3.35) are replaced by an arbitrary linear combination of the two Dirichlet data $T_{0,\Sigma\pm}$ in the first place and an arbitrary linear combination of the two Neumann data $T_{1,\Sigma\pm}$ in the second place [77, 108]:

$$(\Delta + k^2) \, u \quad = \quad 0 \quad \text{in } \Omega,$$

$$a_0 u_0^+ + b_0 u_0^- \quad = \quad g_0 \quad , \quad a_1 u_1^+ + b_1 u_1^- \quad = \quad g_1 \quad \text{on } \Sigma.$$

They lead to a modification in the derived WH equation, namely replacing the Fourier symbol Φ in (3.37) by

$$(3.39) \qquad \sigma_\lambda \;\; = \;\; \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix},$$

where the parameter $\lambda \in \mathbb{C} \setminus \{0,1\}$ is an algebraic combination of the coefficients a_0, b_0, a_1, b_1 (under slight conditions on the coefficients that make the boundary operators to be of "normal type", see [108], formulas (1.5), (1.6)). We shall present the explicit factorisation in the next section and a complete discussion of all $\lambda \in \mathbb{C}$ in Section 5, after introducing some auxiliary material.

As a remarkable result the exponent $1/4$ in the intermediate space (3.39) has to be replaced by $\delta/2$, where $\delta \in (0,1]$ is simply computed from λ and determines the order of the singularity of $\operatorname{grad} u$ at the origin:

$$\delta \;\; = \;\; \Re e \left(\frac{i}{\pi} \log \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} \right) \quad , \quad \operatorname{grad} u \;\sim\; |x|^{\delta/2-1} \text{ as } x \to 0.$$

It turns out that, under the assumption $\lambda \notin [1,+\infty)$, all these problems are well-posed in the given setting and explicitly solvable by Theorem 3.2 with the help of a factorisation of the matrix (3.39), see [77, 108] for details.

The same holds true for transmission problems where a pair of functions $u = (u^+, u^-)$ in the upper and lower half-plane $\Omega^\pm = \{x \in \mathbb{R}^2 : \pm x_2 > 0\}$ satisfies

$$(3.40) \qquad (\Delta + k^2)\, u^\pm \;\; = \;\; 0 \quad \text{in } \Omega^\pm,$$

$$a_0 u_0^+ + b_0 u_0^- = g_0 \quad , \quad a_1 u_1^+ + b_1 u_1^- = g_1 \quad \text{on } \Sigma,$$

$$a_0' u_0^+ + b_0' u_0^- = g_0' \quad , \quad a_1' u_1^+ + b_1' u_1^- = g_1' \quad \text{on } \Sigma',$$

where $u_0^\pm$, $u_1^\pm$ are the traces of u and its normal derivative, respectively, on $x_2 = \pm 0$ and the coefficients are any complex numbers.

The Fourier symbol of the derived WH system is in this case given by

$$(3.41) \qquad \Phi \;\; = \;\; \frac{1}{a_0' b_1' + b_0' a_1'} \begin{pmatrix} a_0 b_1' + b_0 a_1' & -(a_0 b_0' + b_0 a_0')\, t^{-1} \\ -(a_1 b_1' + b_1 a_1')\, t & a_1 b_0' + b_1 a_0' \end{pmatrix}.$$

Sommerfeld half-plane problems with first and second kind boundary conditions can be seen as a special case of (3.40) where the third line is replaced by the homogeneous jump conditions

$$(3.42) \qquad u_0^+ - u_0^- \;\; = \;\; 0 \quad , \quad u_1^+ - u_1^- \;\; = \;\; 0 \quad \text{on } \Sigma'.$$

Another phenomenon appears when the two boundary conditions (or transmission conditions) have the same order, for instance the two Dirichlet data $T_{0,\Sigma^\pm}\, u = g^\pm \in H^{1/2}(\mathbb{R}_+)$ are given (DD) or two Neumann data $T_{1,\Sigma^\pm}\, u = g^\pm \in H^{-1/2}(\mathbb{R}_+)$ (NN) or two impedance data are given (II)

$$(3.43) \qquad T_{1,\Sigma^+}\, u \;+\; i\, p^+\, T_{0,\Sigma^+}\, u = g^+ \in H^{-1/2}(\mathbb{R}_+)$$

$$T_{1,\Sigma^-}\, u \;+\; i\, p^-\, T_{0,\Sigma^-}\, u = g^- \in H^{-1/2}(\mathbb{R}_+)$$

with different impedance numbers $p^\pm$, $\Im m\, p^\pm > 0$, and possibly different data in all cases. Obviously, it is necessary that the difference of the given data $g^+ - g^-$ (considered as function(al) on $\mathbb{R}_+$) has to be extensible by 0 to $\mathbb{R}_-$ within the data space $H^{1/2}$ or $H^{-1/2}$, respectively.

I.e., the following compatibility conditions are necessary for the solubility of the corresponding problem [76, 108]:

$$\begin{aligned}
(3.44) \qquad g^+ - g^- \quad &\in \quad \widetilde{H}^{1/2}(\mathbb{R}_+) \quad \text{in case } DD, \\
g^+ - g^- \quad &\in \quad \widetilde{H}^{-1/2}(\mathbb{R}_+) \quad \text{in case } NN \text{ and } II.
\end{aligned}$$

As a matter of fact these conditions arise automatically from an operator theoretical approach named minimal normalisation of WHOs of the form (2.1) in scales of Sobolev spaces [81, 82, 83].

Historical remarks.

Sommerfeld diffraction problems are named after Arnold Sommerfeld who found the first closed form solution of a problem of diffraction of a time-harmonic plane wave from a half-plane in terms of Fresnel integrals. It was part of his famous habilitation thesis, published 1896 (in German) in Mathematische Annalen [103, p. 366–371]; see also the annotated translation into English published as a book by R.J. Nagem, M. Zampolli, and G. Sandri [84, Section 8]. The formulation as a boundary value problem goes back to Henri Poincaré [88] who proposed a solution by series expansion in 1892, see [88], [84, p. 121] and [66, p. 145–146].

Several books and survey papers demonstrate the attention on Sommerfeld problems. For an overview about classical WH methods the reader may consult [2, 58, 63, 71, 72, 86] which report particularly on the pioneering work in diffraction theory by Edward Copson [31], Albert Heins [48, 49], Allan Hurd [54], Douglas Jones [57], Ernst Lüneburg [67], Anthony Rawlins [91, 92, 93], Thomas Senior [97], W.E. Williams [123], and others.

Formulations of Sommerfeld half-plane problems as boundary value and transmission (or interface) problems in Sobolev spaces started in the 1970s with Giorgio Talenti [117], continued in the 1980s by Erhard Meister and his co-authors, touching the more general theory of partial and pseudo-differential equations, for instance by Eli Shamir [99, 100], Mikhail Agranovich, Marko Vishik, and Gregory Eskin [3, 120, 41], Joseph Wloka [124], George Hsiao and Wolfgang Wendland [52], to mention some closely related work.

We may point out here particularly the first solutions of the Sommerfeld-Dirichlet and the Sommerfeld-Neumann problems by Edward Copson in 1946 [31] with the classical WH technique and the factorisation of the matrix coming up from the mixed Dirichlet-Neumann problem by Antony Rawlins in 1981 [91, 92, 93]. Erhard Meister and the author started to present resolvent operators in Sobolev spaces for much wider classes of Sommerfeld boundary value and transmission problems in 1985 [55, 77, 107, 108, 109]. They also stressed the importance of the intermediate space in operator factorisations, together with Luís Castro [23, 112] and the presence of compatibility conditions together with Francisco Teixeira and Ana Moura Santos [77, 82].

The bridge between WHOs and ΨDOs acting in Sobolev spaces was build by Gregory Eskin in the early 1970s [41]. He also introduced the most convenient notation (3.23) (avoiding the more complicated writing (3.28) presumably for cosmetic reasons) which allowed the consideration of the WHO in a scale of Sobolev spaces, its lifting (cf. Proposition 7.1) etc. The equivalence with operators of the form of a general WHO (2.19) was pointed out by the author in 1983 [105, 106] and systematically used in subsequent publications.

BVPs for systems of PDEs such as the Lamé equations (in crack problems) can be found in [2, 37, 78] for instance. They are, however, beyond the scope of this article.

4. CONSTRUCTIVE FACTORISATION OF NON-RATIONAL MATRIX FUNCTIONS

This topic can be considered as a separate area of research with components in algebra, analysis, matrix and operator theory, which is strongly related to applications. According to the

width of the area, we confine ourselves to a rough overview after focusing one distinguished class of matrix functions (3.39) as a prototype of those related to the theme of this survey article, which also gave a tremendous impact to factorisation theory [77, 91, 92].

Let us recall the WHO with symbol (3.39) written as

$$(4.45) \qquad W \quad = \quad r_+ A|_{P_1 X} \quad : \quad H_+^{1/2} \times H_+^{-1/2} \to H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+),$$

where $X = Y = H^{1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$, $A = \mathcal{F}^{-1}\sigma_\lambda \cdot \mathcal{F}$ with

$$(4.46) \qquad \sigma_\lambda \quad = \quad \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix},$$

and $t(\xi) = (\xi^2 - k^2)^{1/2}$, $\xi \in \mathbb{R}$, $\lambda \in \mathbb{C} \setminus \{0, 1\}$. The connection with a general WHO is given by (3.32) where, strictly speaking, $P_1 = P_+^{(1/2)} \otimes P_+^{(-1/2)} \sim \mathrm{diag}\,(P_+^{(1/2)},\ P_+^{(1/2)})$ and $P_2 = \Pi_+^{(1/2)} \otimes \Pi_+^{(-1/2)}$. A certain factorisation (with the common holomorphy properties and "low" increase at infinity) is derived with the help of Khrapkov's formulas and Daniele's trick [108]:

$$\sigma_{\lambda+} \quad = \quad (1 - \lambda^{-1})^{-1/4} \begin{pmatrix} c_+ & -s_+\sqrt{\lambda}/t \\ -c_+\xi/\sqrt{\lambda} - s_+t/\sqrt{\lambda} & s_+\xi/t + c_+ \end{pmatrix},$$

$$(4.47) \qquad \sigma_{\lambda-} \quad = \quad (1 - \lambda^{-1})^{-1/4} \begin{pmatrix} c_- - s_-\xi/t & -s_-\sqrt{\lambda}/t \\ -s_-t/\sqrt{\lambda} + c_-\xi/\sqrt{\lambda} & c_- \end{pmatrix},$$

where

$$c_\pm(\xi) \quad = \quad \cosh[C \log \gamma_\pm(\xi)],$$

$$s_\pm(\xi) \quad = \quad \sinh[C \log \gamma_\pm(\xi)],$$

$$\gamma_\pm(\xi) \quad = \quad \frac{\sqrt{k \pm \xi} + i\sqrt{k \mp \xi}}{\sqrt{2k}} \ , \ \xi \in \mathbb{R},$$

$$C \quad = \quad \frac{i}{\pi} \log \frac{\sqrt{\lambda} + 1}{\sqrt{\lambda} - 1} \quad .$$

Because of the asymptotic behavior of $\sigma_{\lambda\pm}$ at infinity, the corresponding factorisation of $A = \mathcal{F}^{-1}\sigma_\lambda \cdot \mathcal{F}$ represents a (so-called strong or canonical) Wiener-Hopf factorisation through a vector Sobolev space, provided $\lambda \notin [1, +\infty)$:

$$A_\lambda \quad = \quad A_{\lambda-} \quad A_{\lambda+} \quad = \quad \mathcal{F}^{-1}\sigma_{\lambda-}\cdot\mathcal{F} \quad \mathcal{F}^{-1}\sigma_{\lambda+}\cdot\mathcal{F},$$

$$(4.48) \qquad H^{1/2}\times H^{-1/2} \ \leftarrow \ Z \ \leftarrow \ H^{1/2} \times H^{-1/2}$$

$$Z \quad = \quad H^\vartheta(\mathbb{R}) \ , \quad \vartheta = (\vartheta_1, \vartheta_2) = (\frac{1}{2}(1 - \delta), \frac{1}{2}(\delta - 1)),$$

where $\delta = \Re e\, C = \frac{-1}{\pi} \arg \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} \in]0, 1]$. The crucial point is that $|\vartheta_j| < 1/2$ (cf. [112, Formula (6.4)]), briefly speaking: the Sobolev space orders $\vartheta_j \in (-1/2, 1/2)$ where r_+ is a left inverse of ℓ_0.

The asymptotic behavior of the gradient of the solution follows by analogy of the conclusions from (3.38), see [108] for details.

$$\mathrm{grad}\, u \quad \sim \quad |x|^{\delta/2-1} \text{ as } x \to 0.$$

The factorisation procedure consists of two steps: First the matrix belongs to the (commutative) so-called Daniele-Krapkov class which admit a meromorphic factorisation within the

same class in which the two factors have an algebraic behavior at infinity, however is not related to a generalised factorisation in the sense of Simonenko [10, 44, 102] and therefore does not directly imply a formula for the inverse of the WHO. The Daniele trick corrects this deficiency by twofold factorisation of a rational matrix function, related to the behavior of the previous at infinity, and a rearrangement of the first factorisation by adding factors in the middle of the first factorisation which compensate the increase [32, 77, 93, 107]. It turns out that the final factorisation can be considered as a factorisation into unbounded factors generating a bounded inverse and as a factorisation into bounded factors in the sense of (3.33), as well, provided $|\vartheta_j| < 1/2$ [77, 108]. To understand this technique in detail and to answer the question "What happens in case of $\lambda \in [0, +\infty)$?" completely, it is necessary to study the so-called lifting of WHOs in Sobolev spaces to WHOs in L^2 spaces presented in Section 6. Hence, we come back to this question later.

Now, let us have a wider look at the field of constructive matrix factorisation (related to Sommerfeld problems). The progress in this area is based upon many different ideas. We refer to the books [30, 65] and the survey papers [2, 12, 40, 59, 63, 95, 116, 117] and point out some basic ideas through relevant keywords that demonstrate the variety of facets in constructive matrix factorisation:

- factorisation of rational matrix functions by means of linear algebra [30],
- commutative matrix factorisation [48, 59],
- the Daniele-Khrapkov class [29, 32, 60],
- the Wiener-Hopf-Hilbert method [53, 54],
- understanding of a matrix factorisation as generalised factorisation in L^p spaces [90, 102],
- triangular matrix functions [65, 89],
- asymmetric space settings [106],
- generalised inverses for a unified approach [106],
- lifting [34, 41, 106, 108],
- separation of analytic and algebraic properties [77],
- classification of matrix functions with respect to the number of rationally independent entries [40, 90],
- reduction by rational transformation [40],
- meromorphic factorisation [14, 15, 56, 78],
- connection with corona problems and Riemann surfaces [8, 17],
- operator factorisation through an intermediate space [23, 113].

Proceeding beyond the classes of matrix functions which appear in Sommerfeld problems to further classes occurring in other canonical diffraction problems such as those for wave guides, wedges etc., one finds other species containing exponential terms, e.g., with an exhausting an still increasing amount of literature, see [8, 9], for instance.

Further features such as approximate factorisation and stability questions as well as numerical aspects exceed the scope of this exposition. In the present context, we refer to [1, 14, 61, 80].

Historical remarks.

This wide field of research developed mainly in the last three decades. We highlight only a few milestones which are most important in the present context. A first basic insight was that matrix functions with rational symbols and those with non-rational symbols are quite different in nature. Rational matrix functions have great importance in system theory [4] and admit a rather complete approach [30]. Non-rational Fourier symbols enter the scene with the square root function appearing in the representation formulas, see Proposition 5.4. Some

corresponding difficulties in matrix factorisation have been recognized already in the 1950s, see the book of Ben Noble [86].

The importance of the so-called Daniele-Khrapkov class in diffraction theory was recognized by A.A. Khrapkov in 1971. The pioneering results of Anthony Rawlins [92, 93, 94] and Vito Daniele in 1984 [32] released a tremendous investigation of constructive factorisation of matrix functions from this class and its applications, see for instance [16, 40, 63, 74, 77, 90, 95, 109].

The most important lines of research in view of the applications in diffraction theory are connected with the names of Abrahams, Daniele, Heins, Hurd, Jones, Lüneburg, Meister, Mishuris, Noble, Rawlins, Rogosin, Senior, Talenti, and Williams. Related theoretical work (existence questions, functional analytic aspects, etc.) can be found in the work of Böttcher, Câmara, Ehrhardt, Feldman, Gohberg, Krupnik, Litvinchuk, Prössdorf, dos Santos, Silbermann, Simonenko, Spitkovsky, and the author.

See the corresponding references for further studies of the history of matrix factorisation.

5. Operator factorisation in boundary value problems

Now, we demonstrate the connection between the solution of a boundary value problem (BVP) or a transmission problem (TRP) and operator factorisation starting with a quite basic contemplation [110], which leads us to an explicit representation of resolvent operators in the case of so-called canonical diffraction problems. Typically an elliptic linear boundary value problem is written in the form

$$(5.49) \qquad \begin{aligned} Au &= f \quad \text{in} \quad \Omega \qquad \text{(pde in nice domain)} \\ Bu &= g \quad \text{on} \quad \Gamma = \partial\Omega \quad \text{(boundary condition)}. \end{aligned}$$

I.e., we look for the general solution of the system (5.49), where the following are given: Ω is a Lipschitz domain in $\mathbb{R}^n$ (e.g.), $A \in \mathcal{L}(\mathcal{X}, Y_1), B \in \mathcal{L}(\mathcal{X}, Y_2)$ are bounded linear operators in Banach spaces of function(al)s living on Ω or $\Gamma = \partial\Omega$, respectively. The data (f, g) are arbitrarily given in a (known product) space $Y = Y_1 \times Y_2$ (denoting the topological product considered as a Banach space).

The situation becomes a bit more transparent if we consider the *operator associated with the boundary value problem*

$$(5.50) \qquad L = \left(\begin{array}{c} A \\ B \end{array} \right) : \mathcal{X} \to Y = Y_1 \times Y_2 \ ,$$

where the data space Y and the solution space $\mathcal{X}$ are usually assumed to be known (eventually modified later for practical reasons and in contrast to free boundary problems or certain inverse problems). It is clear that a linear boundary value problem in the abstract setting (5.49) is well-posed if and only if the operator L is an isomorphism. Thus the main problem is: Find (in a certain form) the inverse (resolvent) of the associated operator L (or a generalised inverse etc.). Associated operators of BVPs were systematically used, e.g., in the work of [13, 27, 28, 38, 39, 41, 82, 124]. Linear transmission problems can be considered in the same way, cf. [107, 113].

The classical idea to present the possible solution $u \in \mathcal{X}$ by surface and/or volume potentials can be seen as an operator factorisation:

If $\mathcal{K}$ is an isomorphism, then L is *equivalently reduced* to $T = \mathcal{K}L$ in the sense that the two operators are *(algebraically and topologically) equivalent*, i.e., that T is representable as

$$(5.51) \qquad T \quad = \quad E \quad L \quad F,$$

$$\begin{array}{ccc} & L = \begin{pmatrix} A \\ B \end{pmatrix} & \\ \mathcal{X} & \xrightarrow{\hspace{2cm}} & Y \\ \mathcal{K} \nwarrow & & \nearrow T \\ & Z & \end{array}$$

where E, F are linear homeomorphisms. Here $E = \mathcal{K}$ is a potential operator and $F = I$, $Z = Y$. Equation (5.51) defines an equivalence relation between classes of bounded linear operators in the genuine mathematical sense (reflexive, symmetric and transitive) and practically it includes the idea of a substitution in the solution and in the data space. For the existence of a relation (5.51), we write

$$(5.52) \qquad\qquad\qquad T \qquad \sim \qquad L.$$

Remark 5.5. *In potential theory, the reasoning is often based on a proof that a BVP is uniquely solvable if and only if a system of boundary integral equations is uniquely solvable, see [37, 52]. This would be a consequence of* (5.52)*, however there are other important operator relations which also imply the equivalence of the unique solubility and much more, cf. Section 8.*

Let us take a closer look at *canonical diffraction problems*. There is no strict definition for this kind of boundary value and transmission problems, however experts of the area arrived at a consensus about what it means in conferences [73, 98] dedicated to the one hundredth anniversary of Sommerfeld's famous paper [84, 103] as reported in the introduction of [27]: The word "canonical" stands (i) for the particular geometrical situations like axi-parallel, rectangular or circular configurations, (ii) for constant coefficients in the corresponding linear partial differential equations and in the boundary conditions, and (iii) for the importance of the problem to describe basic phenomena in linear time harmonic wave propagation, e.g., governed by the Helmholtz equation.

As a prototype, we focus again the Sommerfeld half-plane problems (3.40) with jump conditions (3.42) on Σ'. Let $\mathcal{H}^1(\Omega)$ denote the weak solutions [52] of the Helmholtz equation in the two-dimensional slit domain $\Omega = \mathbb{R}^2 \setminus \Sigma$ where $\Sigma = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\,, x_2 = 0\}$ is the half-line identified with $\mathbb{R}_+$ as in Section 3. The Sommerfeld half-plane problems with two *transmission conditions* of first and second kind, respectively, are briefly written as [108]

$$(5.53) \qquad\qquad u \quad \in \quad \mathcal{H}^1(\Omega),$$

$$B_1\, u \quad = \quad a_0 u_0^+ + b_0 u_0^- \quad = \quad g_0 \quad \in \quad H^{1/2}(\mathbb{R}_+),$$

$$B_2\, u \quad = \quad a_1 u_1^+ + b_1 u_1^- \quad = \quad g_1 \quad \in \quad H^{-1/2}(\mathbb{R}_+).$$

Remark 5.6. *Obviously we have a special case of the BVP* (5.49) *where $f = 0$, hence the associated operator is, instead of* (5.50)*, the following one:*

$$(5.54) \qquad\qquad L^0 \;\; = \;\; B|_{\ker A} \;\; : \;\; \mathcal{H}^1(\Omega) \rightarrow Y_2 = H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+)\,.$$

The relation between L and L^0 will be discussed later in Section 8.

Remark 5.7. *Note that we avoided the notation $u \in H^1(\Omega)$ here and before in the context of the slit domain $\Omega = \mathbb{R}^2 \setminus \Sigma$ (in Section 3), because otherwise we had $H^1(\Omega) = H^1(\mathbb{R}^2)|_\Omega$ according to the definition of Sobolev spaces which does not allow different traces of u on $\Sigma^\pm$. Hence, $\mathcal{H}^1(\Omega)$ strictly speaking denotes functions from $L^2(\Omega)$ for which any restriction to a proper sub-cone $\Omega' \subset \Omega$ is a $H^1(\Omega')$ solution of the Helmholtz equation in Ω'.*

The following representation formula for $u \in \mathcal{H}^1(\Omega)$ is well-known [77, 107].

Proposition 5.1. *A function* $u \in L^2(\mathbb{R}^2)$ *belongs to* $\mathcal{H}^1(\Omega)$ *if and only if*

$$u^{\pm} = u|_{\Omega^{\pm}} \in H^1(\Omega^{\pm}) \text{ and hence } u_0^{\pm} = u|_{x_2=\pm 0} \in H^{1/2}(\mathbb{R}),$$

$$u_0^+ - u_0^- = 0 \quad \text{and} \quad A_t(u_0^+ + u_0^-) = 0 \quad \text{on } \mathbb{R}_-,$$

(5.55)
$$u(x) = \mathcal{F}_{\xi \mapsto x_1}^{-1} e^{-x_2 t(\xi)} \widehat{u_0^+}(\xi) \, 1_+(x_2) + \mathcal{F}_{\xi \mapsto x_1}^{-1} e^{x_2 t(\xi)} \widehat{u_0^-}(\xi) \, 1_-(x_2),$$

$$x = (x_1, x_2) \in \Omega^{\pm},$$

where $u_0^{\pm}$ *are taken in the sense of the trace theorem,* $\widehat{u_0^{\pm}} = \mathcal{F} u_0^{\pm}$ *, and* $1_{\pm}$ *stand for the characteristic functions of* $\mathbb{R}_{\pm}$ *.*

We denote the *potential operator* in (5.55) as $\mathcal{K}$ and write briefly

(5.56)
$$\mathcal{K} \quad : \quad H^{1/2} \times H^{1/2} \quad \to \quad \mathcal{H}^1(\Omega^+) \times \mathcal{H}^1(\Omega^-),$$

$$u \quad = \quad \mathcal{K}(u_0^+, u_0^-).$$

It is easily shown that $\mathcal{K}$ is an isomorphism. To solve the problem (5.53), we now determine Φ from (3.41) - (3.42) (plugging (5.55) into the transmission conditions (5.53)) as

(5.57)
$$\begin{aligned}
\Phi &= \frac{1}{2} \begin{pmatrix} a_0 - b_0 & -(a_0 + b_0)\, t^{-1} \\ -(a_1 + b_1)\, t & a_1 - b_1 \end{pmatrix} \\[2mm]
&= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{a_1+b_1}{a_0+b_0} \end{pmatrix} \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix} \begin{pmatrix} a_0 + b_0 & 0 \\ 0 & -(a_0 + b_0) \end{pmatrix} \\[2mm]
&= \qquad\qquad E \qquad\qquad\qquad \sigma_\lambda \qquad\qquad\qquad F
\end{aligned}$$

with $\lambda = (a_1 - b_1)/(a_1 + b_1)$ by elementary transformation with invertible matrices $E, F \in \mathbb{C}^{2\times 2}$ provided $(a_0 \pm b_0) \neq 0$ and $(a_1 + b_1) \neq 0$.

We define $A_\Phi = \mathcal{F}^{-1} \Phi \cdot \mathcal{F} : X \to X$ $X = H^{1/2} \times H^{-1/2}$, $P_1 = P_+^{(1/2)} \otimes P_+^{(-1/2)}$, $P_2 = \Pi_+^{(1/2)} \otimes \Pi_+^{(-1/2)}$ (sometimes written as diagonal matrix operators) and carry out the factorisation of σ_λ as given in (4.47) $\sigma_\lambda = \sigma_{\lambda-} \sigma_{\lambda+}$, $A_{\sigma_\lambda} = A_{\sigma_{\lambda-}} A_{\sigma_{\lambda+}}$. Further put

(5.58)
$$W = P_2 A_\Phi|_{P_1 X}, \quad W^{-1} = F^{-1} A_{\sigma_{\lambda+}}^{-1} P A_{\sigma_{\lambda-}}^{-1}|_{P_2 X} E^{-1}$$

in accordance with Theorem 3.2, where $P = \ell_0 r_+$ (cf. also Section 7). The following result is taken from [108].

Theorem 5.4. *The BVP* (5.53) *is well posed, if and only if* $\lambda \notin [1, +\infty)$. *In this case, the operator* L^0 *associated to the (semi-homogeneous) BVP is equivalent to an invertible WHO* W *and the resolvent operator reads*

(5.59)
$$(L^0)^{-1} = \mathcal{K} W^{-1}.$$

A complete discussion in dependence of the parameter λ will be possible later in Section 7 after equivalent reduction of W to a WHO W_0 acting in L^2 spaces, the so-called lifting process.

In brief (operator language) the proof is based upon an equivalence relation

(5.60)
$$L^0 \quad \sim \quad W \quad = \quad L^0 \mathcal{K} \quad \sim \quad W_0$$

and a complete analysis of $W_0 \in \mathcal{L}(L^2(\mathbb{R}_+)^2)$ as a generalisation of the WHO in Section 2 which is only possible after some additional preparation in Section 6. Further concepts of operator factorisation in boundary value problems will be discussed in Section 7.

Historical remarks.

In the present sense, operators L associated with elliptic BVPs have been used most consequently in the book of Joseph Wloka [124]. In different context, it appeared already in the area of pseudo-differential operators, see [13] for instance.

The question of whether a BVP is well-posed goes back to Jacques Hadamard (1865-1963), see [47, 68]. Besides the unique solvability he asked for a continuous dependence of the solution from the data (here from f and g). Therefore a complete formulation of the linear BVP (5.49) needs at least topological vector spaces (for the sets of data and solutions). The restriction of the consideration to Banach or Hilbert spaces comes from practical reasons and physical motivation (the energy space). See [52, 124] for more details.

The formulation of an "equivalent reduction" of a BVP to a boundary integral equation, e.g., has many different forms in the literature, see for instance [52, Section 5.6.5] or [37]. In recent years, it reached a higher level of precision by use of operator relations [27, 77, 110, 113] and became subject of independent research [4, 5, 51].

6. On the Choice of Symbol Classes

In what concerns the classes of WHOs it is quite important and convenient to study not only Fourier symbols which are rational functions or which belong to a decomposing algebra but which are related to a decomposin algebra such as the Wiener algebra or the Banach algebra of *Hölder continuous functions* on the one-point compactification of the real line $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$:

$$(6.61) \qquad C^\mu(\dot{\mathbb{R}}) \;=\; \left\{ \phi \in C^\mu(\mathbb{R}) : \phi(\infty) = \lim_{\xi \to \infty} \phi(\xi) \text{ exists and} \right.$$
$$\left. \phi(\xi) - \phi(\infty) = \mathcal{O}\left(|\xi|^{-\mu}\right) \text{ as } \xi \to \infty \right\}, \quad \mu \in (0,1).$$

It is well known [44, 79] that $C^\mu(\dot{\mathbb{R}})$ forms a unital algebra, which is decomposing, i.e., the *Hilbert projectors* (also writable with the help of the Hilbert transformation which is here omitted)

$$(6.62) \qquad \mathcal{F}^{-1} \, P_\pm \, \mathcal{F} \;=\; \mathcal{F}^{-1} \, \ell_0 \, r_\pm \, \mathcal{F} \;=\; C_0^\mu(\dot{\mathbb{R}}) \to C_0^\mu(\dot{\mathbb{R}})$$

are continuous where $C_0^\mu(\dot{\mathbb{R}}) = \{\phi \in C^\mu(\dot{\mathbb{R}}) : \phi(\infty) = 0\}$ and we have an additive decomposition into complemented subspaces

$$(6.63) \qquad C_0^\mu(\dot{\mathbb{R}}) \;=\; C_+^\mu(\dot{\mathbb{R}}) \oplus C_-^\mu(\dot{\mathbb{R}}).$$

Roughly speaking these are the subspaces of $C_0^\mu(\dot{\mathbb{R}})$ functions which are holmorphically extensible into the upper and into the lower complex half-plane, respectively, with zero limit at infinity.

The regular functions of the algebra (6.61)

$$(6.64) \qquad \mathcal{G} \, C^\mu(\dot{\mathbb{R}}) \;=\; \{\phi \in C^\mu(\dot{\mathbb{R}}) \,,\; \phi(\xi) \neq 0 \text{ in } \dot{\mathbb{R}}\}$$

form a group which is inverse-closed, i.e., $\phi^{-1} \in \mathcal{G} \, C^\mu(\dot{\mathbb{R}})$ if $\phi \in \mathcal{G} \, C^\mu(\dot{\mathbb{R}})$. These functions allow a factorisation of the form (2.9) (after splitting $\phi(\infty)$) within $\mathcal{G} \, C^\mu(\dot{\mathbb{R}})$ with all the consequences (2.10) - (2.15).

If, instead of the situation before, there are two different limits $\phi(\pm\infty)$ at infinity, we have also a unital algebra denoted by

$$(6.65) \qquad C^\mu(\ddot{\mathbb{R}}) \quad = \quad \Big\{ \phi \in C^\mu(\mathbb{R}) : \phi(\pm\infty) = \lim_{\xi\to\pm\infty} \phi(\xi) \text{ exist and}$$

$$\phi(\xi) - \phi(\pm\infty) = \mathcal{O}\left(|\xi|^{-\mu}\right) \text{ as } \xi \to \pm\infty \Big\}, \quad \mu \in (0,1),$$

which is inverse-closed, as well. Unfortunately $C^\mu(\ddot{\mathbb{R}})$ is not decomposing, a factorisation like (2.10) with factors in $C^\mu(\ddot{\mathbb{R}})$ does not exist whenever $\phi(-\infty) \neq \phi(+\infty)$.

However, this algebra is crucial in the treatise of canonical diffraction problems since many Fourier symbols Φ in the related WH equations are generated by polynomials and square root functions like $t^{\pm 1}, t_+^{\pm 1/2}, t_-^{\pm 1/2}$, see the examples in Section 3. More precisely, all Fourier symbols that appear in Sommerfeld half plane problems (two-dimensional, only one wave number involved) belong to a certain algebra, the functions generated by polynomials and one square root function t which can be represented by

$$(6.66) \qquad\qquad \Phi \quad = \quad a_1\, Q_1 \,+\, a_2\, Q_2,$$

where a_j are scalar functions and Q_j are rational 2×2 matrix functions. These facts finally led to the intensive study of the algebra of matrix functions $C^\mu(\ddot{\mathbb{R}})^{n\times n}$, $n \in \mathbb{N}$, particularly in case of $n = 1$ and $n = 2$ [21, 26, 27, 83].

Let us have a look at the following factorisation theorem for scalar symbols which at the end is no more than a moderate modification of the formulas in Section 2 [34, 83].

Theorem 6.5. *Let $\Phi \in \mathcal{G}\, C^\mu(\ddot{\mathbb{R}})$. Then Φ admits a factorisation*

$$(6.67) \qquad\qquad \Phi \quad = \quad \Phi_-\, \zeta^\kappa\, \Phi_+ \quad where$$

$$\kappa \quad = \quad \max\{z \in \mathbb{Z} : z \leq \Re(\omega) + \tfrac{1}{2}\},$$

$$\omega \quad = \quad \frac{1}{2\pi i} \int_{\mathbb{R}} d\log\Phi(\xi) d\xi \in \mathbb{C},$$

$$\Psi \quad = \quad \zeta^{-\omega}\, \Phi^{-1}(+\infty)\, \Phi \quad , \quad \zeta(\xi) \quad = \quad \frac{\xi - i}{\xi + i} \quad , \quad \xi \in \mathbb{R},$$

$$\Psi_\pm \quad = \quad \exp\{\mathcal{F}\,\ell_0\, r_\pm \mathcal{F}^{-1}\, \log\Psi\} \quad \in \quad C_\pm^\mu(\ddot{\mathbb{R}}),$$

$$\Phi_- \quad = \quad \zeta_-^{\omega-\kappa} \Psi_- \quad , \quad \Phi_+ \quad = \quad \Phi(+\infty)\, \zeta_+^{\kappa-\omega} \Psi_+,$$

where $\zeta_\pm(\xi) = \xi \pm i\,, \xi \in \mathbb{R}$.

Remark 6.8. *Note that the complex winding number $\omega = \omega(\Phi)$ of Φ can be written as $\omega = \kappa + \eta + i\tau$ where $\kappa \in \mathbb{Z}\,, \eta \in (-1/2, +1/2]\,, \tau \in \mathbb{R}$. The function Ψ belongs to $C^\mu(\ddot{\mathbb{R}})\,, \Psi(\mp\infty) = 1$, and $\omega(\Psi) = 0$.*

Another observation is that $\zeta(\xi)$ may be replaced by $\zeta_k(\xi) = (\xi - k)/(\xi + k) = t_-(\xi)/t_+(\xi)$ which may have advantage in diffraction problems (cf. (7.76)).

Again, we formally define the operator factorisation corresponding with (2.10)

$$(6.68) \qquad\qquad A \quad = \quad A_-\, C\, A_+\,,$$

$$A_\pm \quad = \quad \mathcal{F}^{-1}\, \Phi_\pm \cdot \mathcal{F} \quad , \quad C \quad = \quad \mathcal{F}^{-1}\, \zeta^\kappa \cdot \mathcal{F} \quad : \quad L^2 \to L^2.$$

These formulas admit (at least) two different interpretations. Firstly, (6.67) represents a generalised factorisation in L^2 in the sense of [102], provided

$$(6.69) \qquad \Re(\omega) + \frac{1}{2} \quad \notin \quad \mathbb{Z},$$

see [23, 44, 79, 83]. I.e., $t_+^{-1}\,\Phi_+^{\pm 1} \in L^2(\mathbb{R}_+)$, $t_-^{-1}\,\Phi_-^{\pm 1} \in L^2(\mathbb{R}_-)$. $A_\pm$ in (6.68) must be regarded as unbounded operators in $L^2(\mathbb{R})$, densely defined and such that

$$(6.70) \qquad W^- \quad = \quad A_+^{-1}\,P\,C^{-1}\,P\,A_-^{-1}|_{L^2_+}$$

admits a bounded extension by continuity to $\mathcal{L}(L^2(\mathbb{R}))$. Depending on whether κ is positive or negative, this formula defines a right or a left inverse of $W = P\,A|_{PX}$ for $X = L^2$, $P = \ell_0\,r_+$, and an inverse of W for $\kappa = 0$, all this provided $\eta \neq 1/2$.

The second interpretation is a factorisation through an intermediate space [112]

$$(6.71) \qquad \begin{aligned} A \quad &= \quad A_- \quad C \quad A_+ \\ &: \quad X \longleftarrow Z \longleftarrow Z \longleftarrow X, \end{aligned}$$

where $X = L^2$ and $Z = H^\eta$ with the advantage to avoid unbounded operators. The consequences are the same as before but formulas are composed of bounded instead of unbounded operators.

In the case of $\eta = 1/2$ the operator W turns out to be not normally solvable. More precisely, its image is not closed. A normalisation can be achieved systematically by choosing a smaller image space imposing a compatibility condition (in the image space) and a corresponding norm such that the normalised operator is one-sided invertible and (6.70) becomes a one-sided inverse in the new space setting. See [82, 83] for details.

The generalisation of Theorem 6.1 to the systems case where Φ is a squared matrix function

$$(6.72) \qquad \Phi \quad \in \quad \mathcal{G}C^\mu(\ddot{\mathbb{R}})^{m \times m}$$

combines ideas of the factorisation of matrix functions with elements in decomposing algebras such as $\Phi \in C^\mu(\dot{\mathbb{R}})^{m \times m}$ [43], with the factorisation technique of Theorem 6.1, see [21]. The case $m = 2$ is needed for a complete analysis of (5.60).

In the next section, we shall see the abstract analogue of this factorisation (6.71) as a sufficient and necessary condition for the generalised invertibility of a general WHO in symmetric and in asymmetric space setting, respectively.

Historical remarks.

For a glance at the history of relevant function spaces we refer to the book by Hans Triebel [119]. The use of Sobolev-like spaces in BVPs started in the 1930s with the famous work of Sergei Sobolev, L.N. Slobodeckij, Beppo Levi, and Nachman Aronzjan and became a standard tool in the solubility theory for partial differential and pseudo-differential equations in the 1950s, see the work of Mikhail Agranovic, Nicolas Bourbaki, and Lars Hörmander. For general questions about smoothness and asymptotic behavior of solutions see [3, 120, 41, 52, 124].

The edge and radiation conditions [96, 104] are not relevant in the present context, where a solution u denotes a diffracted wave (without sources and sinks) in a suitable (energy) space.

In what concerns the most relevant classes of Fourier symbols we refer to [10, 42, 79]. In the applications the classes of matrix functions with entries from $C^\mu(\dot{\mathbb{R}})$ and $C^\mu(\ddot{\mathbb{R}})$ play a decisive role, see [43, 62] for the origin and [34, 83], respectively, for some relevant applications.

The idea of an intermediate space in WH factorisation (related to Sommerfeld problems) arose in [107] and gained further interest in [18, 23, 83, 112].

7. EQUIVALENCE AND REDUCTION VIA OPERATOR FACTORISATION

To reduce a problem or a system of equations $Tf = g$ can often be seen as to find a "simpler operator" S somehow related to the operator T such that conclusions for S imply conclusions for T. As before, we focus bounded linear operators in Banach spaces. The easiest case is the situation where the two operators are equivalent, i.e.,

$$(7.73) \qquad\qquad T \quad = \quad E \; S \; F,$$

where E and F are isomorphisms. In this case, many properties of S are transferred to properties of T, e.g., invertibility, the Fredholm property, explicit representation of (generalised) inverses, etc. [24] (and vice versa). Conclusions of that kind appeared already in Section 5. So, we shall speak about "operator relations" and their "transfer properties". Now, we outline four further examples of this rather strong equivalence relation which are most important in WHO theory and its applications:

- lifting from Sobolev to Lebesgue spaces [34, 41, 108],
- shifting of WHOs in a scale of Sobolev spaces [83],
- equivalence of a WHO with the truncation of a cross factor [105, 106],
- symmetrisation of general WHOs [11].

First, we describe the so-called *lifting of WHOs* from Sobolev to Lebesgue spaces in the scalar case. In view of (3.23) consider an operator of the form

$$(7.74) \qquad\qquad W \quad = \quad r_+ \; A_\Phi|_{H_+^r} \quad : \quad H_+^r \; \to \; H^s(\mathbb{R}_+),$$

where $A_\Phi : X = H^r \to Y = H^s$ is a bounded convolution operator, i.e., $t^{s-r}\Phi \in L^\infty$. Then we have as a consequence of (3.26):

Proposition 7.2. *Under these assumptions W is equivalent to a WHO acting in L^2 spaces:*

$$(7.75) \qquad\qquad W \quad \sim \quad W_0 \quad = \quad r_+ \; A_{\Phi_0}|_{L_+^2} \quad : \quad L_+^2 \; \to \; L^2(\mathbb{R}_+)$$
$$\Phi_0 \quad = \quad t_-^r \; \Phi \, t_+^{-s} \, .$$

The same idea works perfectly for matrix operators. As an example consider (4.45) again. There we get

$$W \quad = \quad r_+ A_{\sigma_\lambda}|_{H_+^{1/2} \times H_+^{-1/2}} \quad : \quad H_+^{1/2} \times H_+^{-1/2} \to H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+)$$
$$\sim \quad W_0 \quad = \quad r_+ A_{\sigma_{\lambda,0}}|_{L_+^2 \times L_+^2} \quad : \quad L_+^2 \times L_+^2 \to L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$$

with

$$\sigma_{\lambda,0} \quad = \quad \begin{pmatrix} t_-^{1/2} & 0 \\ 0 & t_-^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix} \begin{pmatrix} t_+^{-1/2} & 0 \\ 0 & t_+^{1/2} \end{pmatrix}$$

$$(7.76) \qquad\qquad = \quad \begin{pmatrix} \zeta_k^{1/2} & 1 \\ 1 & \lambda\,\zeta_k^{-1/2} \end{pmatrix}.$$

This allows a direct application of the theory of systems of WH equations in L^2 spaces [43, 79] and ends up with a complete discussion of properties of W in dependence of the parameter λ, see [108, Corollary 4.6], as follows:

(1) $\lambda = 0$: $\sigma_{\lambda,0}$ is triangular. W_0 decomposes into two single WH equations that are invertible;

(2) $\lambda = 1$: $\sigma_{\lambda,0}$ degenerates and $\mathrm{im}\, W_0$ is not closed;

(3) $\lambda \in (0,1)$: $\sigma_{\lambda,0}$ admits a bounded strong factorisation, W_0 is invertible;

(4) $\lambda \in (1, \infty) : \sigma_{\lambda, 0}$ admits a function theoretic factorisation but not a generalised factorisation in the L^2 setting, $\operatorname{im} W_0$ is not closed;

(5) $\lambda \notin [0, \infty) : \sigma_{\lambda, 0}$ admits an unbounded (generalised) factorisation with vanishing partial indices, W_0 is invertible.

As a remarkable fact we cannot find any non-invertible Fredholm operator in this class. All operators are either invertible or not normally solvable.

The preceding idea can also be used to discuss regularity properties of (7.75), say, i.e., to look at smoother solutions of the equation $W f = g$ (for smoother data g). A convenient way is to consider the restricted operator

$$(7.77) \qquad W_s \quad = \quad \operatorname{Rst} r_+ \ A_{\Phi_0}|_{H_+^s} \quad : \quad H_+^s \ \to \ H^s(\mathbb{R}_+) \quad , \quad \text{for } s > 0.$$

This makes sense because H^s is an invariant subspace of the convolution operator A_{Φ_0}. The consideration in larger spaces may be interesting, as well. Hence, we also define

$$(7.78) \qquad W_s \quad = \quad \operatorname{Ext} r_+ \ A_{\Phi_0}|_{H_+^s} \quad : \quad H_+^s \ \to \ H^s(\mathbb{R}_+) \quad , \quad \text{for } s < 0$$

by continuous extension since $H^s \subset L^2$, $H_+^s \subset L_+^2$, and $H^s(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$ are densely embedded for negative s. These operators (7.77) and (7.78) are referred to as *shifted operators* [83, 87].

Now, the point is that the shifted operator can also be lifted to the L^2 level and that lifted shifted WHO has a symbol $\Phi_{s,0} = t_-^s \, \Phi_0 \, t_+^{-s} = \zeta^s \, \Phi_0$ in the scalar case. Hence, if Φ_0 admits a factorisation like (6.67), then all lifted shifted Fourier symbols admit a "certain factorisation" of this kind, as well. This technique can be used, e.g., in the cases where the operator W_0 of (7.76) is not invertible. Namely, W can be normalised by a shift in the scale of Sobolev spaces, i.e., it becomes invertible as an operator restricted like

$$W^{(\varepsilon)} \quad : \quad H_+^{1/2+\varepsilon} \times H_+^{-1/2+\varepsilon} \ \to \ H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+)$$

with a suitable $\varepsilon \in (0, 1/2)$. For more details see [83] and [21] in the matrix case.

For the following let us recall some properties of generalised inverses T^- of $T \in \mathcal{L}(X, Y)$ as defined already in (2.20). It is well-known that the following assertions are equivalent:

- T is generalised invertible, i.e., $T T^- T = T$ for some $T^- \in \mathcal{L}(Y, X)$,
- T has a reflexive generalised inverse, i.e., $T^- T T^- = T^-$ holds additionally,
- $\ker T$ and $\operatorname{im} T$ are complemented subspaces of X and Y, respectively,
- there exist projectors $P_0 \in \mathcal{L}(X)$ and $P_1 \in \mathcal{L}(Y)$ onto $\ker T$ and onto $\operatorname{im} T$, respectively,
- there exist projectors $P_0 \in \mathcal{L}(X)$ and $P_1 \in \mathcal{L}(Y)$ such that, for arbitrary $g \in Y$, the equation $T f = g$ is solvable if and only if $P_1 g = g$ and, in this case, the general solution of the equation is given by $f = T^- g + P_0 h$, $h \in X$.

For the proof choose $T^- T T^-$ as a reflexive generalised inverse, and $P_0 = I - T^- T$, $P_1 = T T^-$ in the corresponding places, briefly speaking.

A third example for an equivalent reduction of a WHO by an operator equivalence (7.73) was touched already in (2.10). The following result is just an interpretation of those formulas, extended to the asymmetric case.

Proposition 7.3. *Let $W = P_2 A|_{P_1 X} : P_1 X \to P_2 Y$ be a general WHO with the assumptions of* (2.19). *Further let*

$$(7.79) \qquad\qquad A \quad = \quad A_- \quad C \quad A_+$$
$$: \quad Y \longleftarrow Z \longleftarrow Z \longleftarrow X,$$

be a factorisation into three isomorphisms with an intermediate Banach space Z, and a projector $P \in \mathcal{L}(Z)$ such that

(7.80) $$PA_+P_1 \;=\; A_+P_1 \quad, \quad P_2A_-P \;=\; P_2A_- .$$

Then W satisfies

(7.81) $$W \;=\; P_2\,A_-|_{PZ}\,PC|_{PZ}\,PA_-|_{P_1X} \;\sim\; PC|_{PZ} .$$

Various examples show that this idea of equivalent reduction yields direct results, provided the operator C has certain properties (as in (2.10)). We call an isomorphism $C \in \mathcal{L}(Z)$ a *cross factor* with respect to (Z, P) where $P \in \mathcal{L}(Z)$ is a projector if $C^{-1}PCP$ is idempotent, or equivalently if

(7.82) $$C^{-1}PCP = PC^{-1}PCP .$$

It follows that C splits the space Z twice into four subspaces with

(7.83)
$$
\begin{array}{ccccccccc}
 & & \overbrace{X_1 \;\dot{+}\; X_0}^{PZ} & & + & & \overbrace{X_2 \;\dot{+}\; X_3}^{QZ} & \\
Z & = & & & & & & \\
 & & \downarrow & & C \;\searrow\!\!\!\nwarrow & & \downarrow & \\
 & = & \underbrace{Y_1 \;\dot{+}\; Y_2}_{PZ} & & + & & \underbrace{Y_0 \;\dot{+}\; Y_3}_{QZ} & ,
\end{array}
$$

where $Q = I_Z - P$ and where C maps each X_j onto Y_j, j = 0, 1, 2, 3 , i.e., the complemented subspaces $X_0, X_1, ..., Y_3$ are images of corresponding projectors $p_0, p_1, ..., q_3$, namely $X_0 = p_0 Z = C^{-1}QCPZ$, $X_1 = p_1 Z = C^{-1}PCPZ, ..., Y_3 = q_3 Z = CQC^{-1}QZ$.

A WHO W which is a truncation of a cross factor, $W = PC|_{PX}$, allows very strong conclusions for the solution of the corresponding equation $W f = g$:

Proposition 7.4. *Let X be a Banach space, $P \in \mathcal{L}(X)$, $P^2 = P$, and $C \in \mathcal{GL}(X)$ a cross factor with respect to X, P. Then we have:*

- $W = PC|_{PX}$ *is generalised invertible,*
- $W^- = PC^{-1}|_{PX}$ *is a reflexive generalised inverse of W,*
- $\ker W = PC^{-1}QCPX = C^{-1}QCPX$,
- $\operatorname{im} W = PC^{-1}PCPX = C^{-1}PCPX$,
- *for any $g \in PX$ the equation $W f = g$ is solvable in PX if and only if $C^{-1}PCg = g$ and, in this case, the general solution of the equation is given by*

(7.84) $$f \;=\; W^- g + C^{-1}QCh \quad \text{with arbitrary} \quad h \in PX .$$

The proof of this result is elementary and can be seen as a consequence of the diagram (7.83). Moreover, the equivalence of a general WHO (2.16) in symmetric space setting with $A \in \mathcal{GL}(X)$ to a truncation of a cross factor (in the same space) is characteristic for the generalised invertibility of W:

Theorem 7.6. *The WHO (2.16) is generalised invertible if and only if A admits a cross factorisation (with respect to X and P), i.e.,*

(7.85) $$A \;=\; A_- \; C \; A_+,$$

where $A_\pm$ are strong WH factors (satisfying the first two lines of (2.13)) and C is a cross factor (with respect to X, P). In this case, we have

$$
\begin{aligned}
(7.86) \qquad W & = & P \; A_- \; P \; C \; P \; A_+|_{PX} \\
& = & P \, A_-|_{PX} \; P \, C|_{PX} \; P \, A_+|_{PX} \quad \sim \quad P \, C|_{PX}
\end{aligned}
$$

and a reflexive generalised inverse of W is given by

$$
(7.87) \qquad W^- \;\; = \;\; P \; A_+^{-1} \; P \; C^{-1} \; P \; A_-^{-1}|_{PX} \, .
$$

The proof of sufficiency is obvious, a verification of W^- to be a reflexive generalised inverse based on the factor properties of $A_\pm$ and C, however not very short. The proof of necessity is tricky, either by use of a space decomposition generated by a reflexive generalised inverse [105] or by a direct formula of a cross factorisation in terms of a generalised inverse, see [106, Chapter 6]. However, a corresponding result for general WHOs in asymmetric space setting $W = P_2 A|_{P_1 X}$ (see (2.19)) has been published only recently [11, 112], under an additional condition that guarantees the symmetrization of W:

Theorem 7.7. *Consider the general WHO (in asymmetric setting) (2.19). Then the following assertions are equivalent:*

(1) A admits a WH factorisation through an intermediate space (FIS)

$$
\begin{aligned}
(7.88) \qquad A & = & A_- & \quad C & \quad A_+ \\
& : & Y \longleftarrow Z & \longleftarrow Z & \longleftarrow X,
\end{aligned}
$$

where $A_\pm$ are strong WH factors, see (3.30), and C is a cross factor;
(2) W is generalised invertible and $P_1 \sim P_2$, i.e.,

$$
(7.89) \qquad \ker P_1 \;\; \cong \;\; \ker P_2 \quad and \quad \operatorname{im} P_1 \;\; \cong \;\; \operatorname{im} P_2 \, .
$$

In this case, a reflexive generalised inverse of W is given by

$$
(7.90) \qquad W^- \;\; = \;\; P_1 \; A_+^{-1} \; P \; C^{-1} \; P \; A_-^{-1}|_{P_2 Y} \, .
$$

This last result contains the idea to symmetrise a general WHO and (hence) the space setting itself, see [11] which is important in applications, since it simplifies the reasoning and enables the use of other results, e.g., about operators in Lebesgue spaces (by lifting, as mentioned before). However, the conditions of the last theorem are not necessary for a WHO to be generalised invertible, see the cross factorisation theorem in asymmetric settings in [106] and further going research in [115].

More important for applications is the stability of factorisations against a change of the underlying function spaces for normalisation and regularity results, see [83] for instance.

As an example that fits best to the present context consider the WHO (2.4) again where $C = \mathcal{F}^{-1} \zeta^\kappa \cdot \mathcal{F}$, $\kappa \in \mathbb{Z}$. It is not hard to prove the following facts.

- Formula (2.10) represents a cross factorisation provided Φ is invertible in the Wiener algebra,
- If $\kappa < 0$, the subspaces of (7.83) are

$$
\begin{aligned}
(7.91) \qquad X_0 & = & \ker P C|_{PX} & = & \operatorname{span} \{ \phi_j \, , \; j = 1, ..., -\kappa \} \\
\phi_j(x) & = & \mathcal{F}_{\xi \mapsto x}^{-1} \dfrac{(\xi - i)^{j-1}}{(\xi + i)^\kappa} \, , \; x \in \mathbb{R},
\end{aligned}
$$

and X_2 is not present, etc.

- If $\kappa > 0$, the subspaces of (7.83) are

(7.92)
$$X_2 \;=\; \ker QC|_{QX} \;=\; \operatorname{span}\{\psi_j\,,\,j=1,...,\kappa\}$$

$$\psi_j(x) \;=\; \mathcal{F}^{-1}_{\xi \mapsto x} \frac{(\xi+i)^{j-1}}{(\xi-i)^{\kappa}}\,,\, x \in \mathbb{R}\,,$$

and X_0 is not present, etc.

In the first case, the equation $W\,f \;=\; g$ is solvable for all $g \in L^2(\mathbb{R}_+)$ by

(7.93)
$$f \;=\; W^- g \;+\; \sum_{j=1}^{-\kappa} a_j\, A_+^{-1}\,\phi_j \;\text{ with }\; a_j \in \mathbb{C}\,.$$

In the second case, the equation $W\,f \;=\; g$ is solvable if and only if κ conditions are satisfied

(7.94)
$$\int_{\mathbb{R}_+} g\,P\,A_-^{-1}\,P\,C^{-1}\,\psi_j \;=\; 0 \;,\quad j=1,...,\kappa\,.$$

The solution reads $f \;=\; W^- g$ and is unique in this case.

These results are standard conclusions for Fredholm operators and their generalised inverses.

Finally, we like to remark that, in a symmetric space setting, every general WHO $W \;=\; PA|_{PX}$ which is generalised invertible can be presented as a truncation of a (suitable) cross factor [115]. This underlines the impression that cross factors are something natural, if we look for non-invertible Fredholm or other generalised invertible operators, see [112] for further examples.

Historical remarks.

The idea of equivalent reduction in the sense of (7.73) is a genuine element of WH factorisation and just an interpretation of the pioneering work mentioned in Section 2. The idea of the lifting process and its consequences was settled by Gregory Eskin and Roland Duduchava in the 1970s, see [34, 41].

For the history of generalised inverses see [85]. The generalised inversion of general WHOs was firstly studied by the author in [105, 106], where also the term "cross factor" was introduced (in symmetric and in asymmetric Banach space settings). Intermediate spaces in the context of generalised factorisation of matrix functions in the sense of I.B. Simonenko [102] were firstly studied by Luís Castro and the author [18, 23]. The complete proof of Theorem 7.5 was presented only in 2016 [11, 112].

The question whether a general WHO (2.19) in asymmetric setting can be equivalently reduced to a WHO (2.16) in symmetric setting was recently analysed in [11]. Further questions about equivalent reduction of general WHOs to more convenient forms have been answered in [115].

8. FROM OPERATOR FACTORISATION TO OPERATOR RELATIONS

The operator factorisations considered before can all be seen as equivalence relations in the sense of (7.73). However, there are other operator relations which play a crucial role in the present context and in general, as well. We point out two of them, again coming up from concrete applications.

It is well known that the diffraction problem where the trace of the diffracted field is given on a plane screen $\Sigma \in \mathbb{R}^3$ (equally on the two sides of the screen) leads to an (boundary integral)

equation

(8.95) $$r_\Sigma A_{t^{-1}} f \quad = \quad g,$$

where Σ is identified with a (Lipschitz) domain in $\mathbb{R}^2$, $t(\xi_1, \xi_2) = (\xi_1^2 + \xi_2^2 - k^2)^{1/2}$, $A_{t^{-1}} = \mathcal{F}^{-1} t^{-1} \cdot \mathcal{F}$, $\mathcal{F}$ denotes here the two-dimensional Fourier transformation, $g \in H^{1/2}(\Sigma)$ is given and $f \in H_\Sigma^{-1/2}$ is unknown, in analogy to the Sommerfeld-Dirichlet problem: f is an element of $H^{-1/2}(\mathbb{R}^2)$ supported on $\overline{\Sigma}$.

Assuming that the *complementary screen* $\Sigma' = \mathbb{R}^2 \setminus \overline{\Sigma}$ is also a Lipschitz domain, we consider the Sommerfeld-Neumann problem for Σ' and arrive by analogy at an equation

(8.96) $$r_{\Sigma'} A_t f_* \quad = \quad g_*,$$

where $g_* \in H^{-1/2}(\Sigma')$ is given and $f_* \in H_{\Sigma'}^{1/2}$ is unknown. Now, Lipschitz domains allow continuous extension operators [124], say

$$\ell_\Sigma^{(1/2)} \ : \ H^{1/2}(\Sigma) \to H^{1/2}(\mathbb{R}^2) \quad , \quad \ell_{\Sigma'}^{(-1/2)} \ : \ H^{-1/2}(\Sigma') \to H^{-1/2}(\mathbb{R}^2)$$

such that the projector $P_2 = \ell_\Sigma^{(1/2)} r_\Sigma$ acts in $H^{1/2}(\mathbb{R}^2)$ along $H_{\Sigma'}^{1/2}$ and $Q_1 = \ell_{\Sigma'}^{(-1/2)} r_{\Sigma'}$ acts in $H^{-1/2}(\mathbb{R}^2)$ along $H_\Sigma^{-1/2}$. In analogy to the half-line case in Section 3, we conclude that the operators in the foregoing equations (8.95) and (8.96) are equivalent to operators in the form of general WHOs

(8.97) $$r_\Sigma A_{t^{-1}}|_{H_\Sigma^{-1/2}} \sim W = P_2 A_{t^{-1}}|_{P_1 X},$$
$$r_{\Sigma'} A_t|_{H_{\Sigma'}^{1/2}} \sim W_* = Q_1 A_t|_{Q_2 Y},$$

where $X = H^{-1/2}$, $P_1 + Q_1 = I|_{H^{-1/2}}$ and $Y = H^{1/2}$, $P_2 + Q_2 = I|_{H^{1/2}}$, cf. (2.19). Using the space decompositions $X = P_1 X \oplus Q_1 X \cong P_1 X \times Q_1 X$ (identifying the direct sum with the topological product space considered as a Banach space written in matrix form) and $Y = P_2 Y \oplus Q_2 Y \cong P_2 Y \times Q_2 Y$, as well, we have

(8.98) $$A_{t^{-1}} \quad \sim \quad \begin{pmatrix} W & \cdot \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ \cdot & W_* \end{pmatrix}^{-1} \quad \sim \quad A_t^{-1},$$

where the dots stand for obvious entries $P_2 A_{t^{-1}}|_{Q_1 X}$ etc., i.e., W and W_* are *matricially coupled* (MC) operators [4, 5] and *associated WHOs* in the sense of [105, 106].

The MC relation between two bounded linear operators T, S in Banach spaces is often written in the form

(8.99) $$\begin{pmatrix} T & \cdot \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ \cdot & S \end{pmatrix}^{-1}.$$

It is well known that (8.99) is closely related to another operator relation which is called *equivalence after extension* (EAE) relation defined by

(8.100) $$\begin{pmatrix} T & 0 \\ 0 & I_{Z_1} \end{pmatrix} = E \begin{pmatrix} S & 0 \\ 0 & I_{Z_2} \end{pmatrix} F,$$

where Z_1, Z_2 are Banach spaces and E, F are isomorphisms acting between the corresponding spaces. In this case, one writes

(8.101) $$T \quad \overset{*}{\sim} \quad S.$$

The two relations (8.99) and (8.100) look quite different, however it turns out that they coincide:

Theorem 8.8 (of Bart and Tsekanovskii). *Two bounded linear operators acting in Banach spaces are matricially coupled if and only if they are equivalent after extension.*

There are some important facts to be pointed out:
- EAE is an equivalence relation in the genuine mathematical sense (reflexive, symmetric, and transitive) and such is MC;
- $T \sim S$ implies $T \overset{*}{\sim} S$, but not conversely (see the example below);
- The two relations (8.99) and (8.100) can be computed from each other [5, 114].

The following conclusion is most important in applications [24, Theorem 2.5].

Theorem 8.9. *If (8.100) is satisfied, and if S^- is a reflexive generalised inverse of S, then a reflexive generalised inverse of T is given by the formula*

$$(8.102) \qquad T^- \;=\; R_{11} \left(F^{-1} \begin{pmatrix} S^- & 0 \\ 0 & I_{Z_2} \end{pmatrix} E^{-1} \right),$$

where R_{11} denotes the restricted operator of the first matrix entry to the first component spaces.

An example for the ease of a MC relation appears in diffraction theory in the context of diffraction of time-harmonic waves from plane screens in $\mathbb{R}^3$. It turns out that the operators associated to certain problems for complementary plane screens are matricially coupled, which was denoted as "abstract Babinet principle" in [111]. Therefore, the construction of the resolvent operator for one of the problems implies a representation of the resolvent operator for the other one. Since the resolvent operators can be computed for certain convex screens, the preceeding idea yields the explicit solution of diffraction problems for a much wider class of plane screens, the so-called "polygonal-conical screens" [22].

Another application of the EAE relation is the following. A process commonly denoted by "equivalent reduction" is the step from a boundary value problem (BVP) to a semi-homogeneous BVP. This is not reflected by equivalence between the two associated operators but by equivalence after extension.

Consider the abstract BVP (5.1)-(5.2) and its associated operator (5.50). Moreover consider the semi-homogeneous (abstract) boundary value problem briefly written as

$$(8.103) \qquad L^0 u = \begin{pmatrix} A \\ B \end{pmatrix} u = \begin{pmatrix} 0 \\ g \end{pmatrix} \in \{0\} \times Y_2 \cong Y_2$$

with associated operator

$$(8.104) \qquad B|_{\ker A} \;:\; X_0 = \ker A \to Y_2.$$

By analogy, we may define $A|_{\ker B} : \ker B \to Y_1$.

Theorem 8.10. *Let $L = \begin{pmatrix} A \\ B \end{pmatrix} \in \mathcal{L}(\mathcal{X}, Y_1 \times Y_2)$ be a bounded linear operator in Banach spaces. Then*

$$AR = I \text{ for some } R \in \mathcal{L}(Y_1, \mathcal{X}) \qquad \text{implies} \qquad L \overset{*}{\sim} B|_{\ker A},$$
$$BR = I \text{ for some } R \in \mathcal{L}(Y_2, \mathcal{X}) \qquad \text{implies} \qquad L \overset{*}{\sim} A|_{\ker B}.$$

The proof (for the first case) is based upon the operator identity

$$(8.105) \qquad L = ETF = \begin{pmatrix} 0 & A|_{X_1} \\ I|_{Y_2} & B|_{X_1} \end{pmatrix} \begin{pmatrix} B|_{X_0} & 0 \\ 0 & I|_{X_1} \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}$$

which holds if A is right invertible: $AR = I|_{Y_1}$, say, with $X_0 = \ker A$, $X_1 = \operatorname{im} R$, $Q = RA$, $P = I - RA$. Formula (8.105) is an EAE relation between L and $B|_{X_0}$, more precisely an equivalence

after one-sided extension (EAOE) relation, since the operator L on the left side is not extended. For details and further consequences (transfer properties, normalization etc.) see [110]. As a consequence, the resolvent operator $(L^0)^{-1}$ of the semi-homogeneous BVP (8.103) implies a representation of the resolvent operator L^{-1} of the full problem by the help of formula (8.102).

For further operator relations, their interplay, and further applications see [6, 19, 24, 50, 51, 118].

Historical remarks.

The present concepts of equivalence and reduction are much stronger than those usually used in potential theory where, for instance elliptic BVPs are said to be equivalent to certain boundary integral equations if, roughly speaking, there are mappings which generate solutions from each other, see [52, Section 5.6.5] and [77, Section 2], for instance. Various other equivalence relations appear in the literature, e.g., defined by the fact that two operators are simultaneously Fredholm or not [7]. Those properties can be regarded as consequences of the MC and EAE relations and were denoted as "transfer properties" in [110, 115].

The MC and EAE relations were introduced by Harm Bart, Israel Gohberg, and Rien Kaashoek in the early 1980s [4], where the conclusion from MC to EAE was of great importance in system theory, see also [114]. The proof of the inverse conclusion (from EAE to MC) was obtained by Harm Bart and V.E. Tsekanovskii [5]. It gave a great impact to the study of operator relations in general, see [50, 51, 110, 118]. Its usefulness in the study of various classes of operators was demonstrated already by Luís Castro and the author in 1998 [19, 24, 25] and recently also for applications in diffraction theory, together with Roland Duduchava [22, 111, 113].

9. The step from 2 to 3 and to n dimensions

In this final section, we describe briefly an idea of how to study the Sommerfeld problems of Section 3 in higher dimensions, i.e., where Ω is three-dimensional or even n-dimensional which mathematically does not make a great difference to some extend. The FIS concept (see Theorem 7.5) improves to be very efficient, because the intermediate space turns out to be an anisotropic Sobolev space and the reasoning of an operator factorisation with bounded instead of unbounded operators applies as before in two dimensions.

Instead of a half-line (as in Section 3) consider the half-plane $\Sigma = \mathbb{R}^2_{1+} = \{x \in \mathbb{R}^2 : x_1 > 0\}$, the (orthogonal) projectors $P_+ = \ell_0 r_+ : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ onto $L^2_\Sigma = L^2_{\mathbb{R}^2_{1+}}$, $P_- = I - P_+$ and the Bessel potential operators [35, 36, 41] of order $s \in \mathbb{R}$:

$$\Lambda^s_+ = A_{\lambda^s_+} = \mathcal{F}^{-1} \lambda^s_+ \cdot \mathcal{F} \quad , \quad \lambda^s_+(\xi) = \left(\xi_1 + i\sqrt{\xi_2^2 + 1}\right)^s , \xi \in \mathbb{R}^2$$

$$\Lambda^s_- = A_{\lambda^s_-} = \mathcal{F}^{-1} \lambda^s_- \cdot \mathcal{F} \quad , \quad \lambda^s_-(\xi) = \left(\xi_1 - i\sqrt{\xi_2^2 + 1}\right)^s , \xi \in \mathbb{R}^2,$$

where $\mathcal{F}$ now denotes the two-dimensional Fourier transformation. For any $s \in \mathbb{R}$, we find the orthogonal projectors [25]

$$\begin{aligned}
P^{(s)}_+ &= \Lambda^{-s}_+ P_+ \Lambda^s_+ \quad \text{onto} \quad H^s_\Sigma \\
P^{(s)}_- &= \Lambda^{-s}_- P_- \Lambda^s_- \quad \text{onto} \quad H^s_{\Sigma'} \\
\Pi^{(s)}_+ &= \Lambda^{-s}_- P_+ \Lambda^s_- \quad \text{along} \quad H^s_{\Sigma'} \\
\Pi^{(s)}_- &= \Lambda^{-s}_+ P_- \Lambda^s_+ \quad \text{along} \quad H^s_\Sigma.
\end{aligned}$$

Hence $P^{(s)}_+ + \Pi^{(s)}_- = I_{H^s}$ and $P^{(s)}_- + \Pi^{(s)}_+ = I_{H^s}$.

As mentioned in Remark 6.2 one can work with $\sqrt{\xi_2^2 - k^2}$ instead of $\sqrt{\xi_2^2 + 1}$. In that case, the corresponding projectors are not anymore orthogonal in the usual sense.

Now, let us go to the higher-dimensional case ($m = n - 1 \geq 2$) where Σ is a half-space, cf. [20, 41, 45, 100, 106]. According to the traditional notation [41] x_n denotes now the "normal derivative direction" (corresponding to x_2 in Section 3) and $x_m = x_{n-1}$ denotes the "WH direction" (corresponding to x_1 in Section 3) whilst the remaining variables $x_1, ..., x_{m-1}$ play the role of parameters. Putting

$$X = Y = H^{1/2}(\mathbb{R}^m) \times H^{-1/2}(\mathbb{R}^m) \ , \ \Sigma = \mathbb{R}_+^m = \mathbb{R}^{m-1} \times]0, \infty[$$

and $t(\xi) = (\xi_1^2 + ... + \xi_m^2 - k^2)^{1/2}$, $\xi = (\xi', \xi_m) \in \mathbb{R}^m$, we can consider the same factorisation given by (4.47) replacing k by $(k^2 - \xi'^2)^{1/2}$, i.e., the previous factorisation as to be parameter-dependent of $\xi' \in \mathbb{R}^{m-1}$. It turns out that the factorisation (4.47) can be seen as a canonical FIS of A with an intermediate space which is an anisotropic vector Sobolev space

$$(9.106) \qquad Z \ = \ H^\vartheta(\mathbb{R}^m) \times H^{-\vartheta}(\mathbb{R}^m) \,,$$
$$H^\vartheta(\mathbb{R}^m) \ = \ \mathcal{F}(w_\vartheta L^2(\mathbb{R}^m)) \quad , \quad w_\vartheta(\xi) \ = \ (1 + |\xi'|^2)^{\vartheta_1/2} (1 + \xi_m^2)^{\vartheta_2/2},$$
$$\vartheta \ = \ (\vartheta_1, \vartheta_2) \ = \ (\frac{1}{2}(\delta - 1), \frac{1}{2}(1 - \delta)) \,,$$

see [113] for more details. Adding in the factorisation $\sigma_\lambda = \sigma_{\lambda-} \sigma_{\lambda+}$ (cf. (4.47)) a middle factor of the form $\mathrm{diag}\,(\zeta^{\kappa_1}, \zeta^{\kappa_2})$, where $\zeta(\xi) = (\xi_m - i |\xi'|)/(\xi_m + i |\xi'|)$, we find further examples with non-canonical FIS which are not Fredholm according to a non-trivial kernel (or co-kernel) that is translation invariant with respect to $x' = (x_1, ..., x_{m-1}) \in \mathbb{R}^{m-1}$, hence infinite dimensional. However, generalised inverses can be constructed as before according to the FIS interpretation.

One can find plenty of further boundary value and transmission problems in higher dimensions ($n \geq 3$), where the associated operators are not anymore Fredholm but generalised invertible, just adding one or more variables, see [28, 64, 81, 113], for instance.

Many problems in mathematical physics, for instance in elasticity theory, make sense only in three dimensions. They typically lead us to Wiener-Hopf equations, anisotropic function spaces and normalisation problems, see [37, 52, 78, 109], e.g.

Conclusion. The operator theoretical formulation allows a clear and compact description of results about the solution of linear BVPs, a simultaneous treatise of large classes of problems, and the solution of new classes of BVPs, as well.

For instance, the described ideas are applicable to other geometrical situations, particularly the conception of equivalent reduction via operator relations, see further work on diffraction by wedges or polygonal-conical screens etc. such as [20, 22] and [28].

References

[1] I. D. Abrahams: *The application of Padé approximants to Wiener-Hopf factorization.* IMA J. Appl. Math. **65** (2000), 257-281.

[2] I. D. Abrahams: *On the application of the Wiener-Hopf technique to problems in dynamic elasticity.* Wave Motion **36** (2002), 311-333.

[3] M. S. Agranovich, M. I. Vishik: *Elliptic problems with a parameter and parabolic problems of general type.* Russian Math. Surveys **19** (1964), 53-157. Translation from Usp. Mat. Nauk **19** (1964), 53-161, Russian.

[4] H. Bart, I. Gohberg and M. Kaashoek: *The coupling method for solving integral equations.* In: Topics in operator theory systems and networks, Workshop Rehovot/Isr. (1983), Oper. Theory Adv. Appl. **2**, Birkhäuser, Basel (1984), 39-73. Addendum in Integr. Equ. Op. Theory **8** (1985), 890-891.

[5] H. Bart, V. E. Tsekanovskii: *Matrical coupling and equivalence after extension.* In: Operator Theory and Complex Analysis, Eds: T. Ando et al., Oper. Theory Adv. Appl. **59**, Birkhäuser, Basel (1991), 143-160.

[6] H. Bart, V. E. Tsekanovskii: *Complementary Schur complements.* Linear Algebra Appl. **197** (1994), 651-658.

[7] M.A. Bastos, A.F. dos Santos and R. Duduchava: *Finite interval convolution operators on the Bessel potential spaces H_p^s.* Math. Nachr. **173** (1995), 49-63.

[8] M. A. Bastos, Yu. I. Karlovich, A. F. dos Santos and P. M. Tishin: *The corona theorem and the canonical factorization of triangular AP matrix functions – Effective criteria and explicit formulas.* J. Math. Anal. Appl. **223** (1998), 523-550.

[9] A. Böttcher, Yu. I. Karlovich and I. M. Spitkovsky: *Convolution Operators and Factorization of Almost Periodic Matrix Functions.* Oper. Theory Adv. Appl. **131**, Birkhäuser, Basel (2002).

[10] A. Böttcher, B. Silbermann: *Analysis of Toeplitz Operators.* Springer, Berlin (2006).

[11] A. Böttcher, F. -O. Speck : *On the symmetrization of general Wiener-Hopf operators.* J. Operator Theory **76** (2016), 335-349.

[12] A. Böttcher, I. M. Spitkovsky : *The factorization problem: some known results and open questions.*, In: Operator Theory, Operator Algebras and Applications Eds: A. Almeida et al., Oper. Theory Adv. Appl. **229**, Birkhäuser, Basel (2013), 101-122.

[13] L. Boutet de Monvel: *Boundary problems for pseudo-differential operators.* Acta Math. **126** (1971), 11-51.

[14] M. C. Câmara, A. B. Lebre and F. -O. Speck: *Meromorphic factorization, partial index estimates and elastodynamic diffraction problems.* Math. Nachr. **157** (1992), 291-317.

[15] M. C. Câmara, M. T. Malheiro: *Meromorphic factorization revisited and application to some groups of matrix functions.* Complex Anal. Oper. Theory **2** (2008), 299-326.

[16] M. C. Câmara, A. F. dos Santos and M. A. Bastos: *generalized factorization for Daniele-Khrapkov matrix functions – explicit formulas.* J. Math. Anal. Appl. **190** (1995), 295-328.

[17] M. C. Câmara, A. F. dos Santos and P. F. dos Santos: *Matrix Riemann-Hilbert problems and factorization on Riemann surfaces.* J. Funct. Anal. **255** (2008), 228-254.

[18] L. P. Castro: *The Characterization of the Intermediate Space in generalized Factorizations.* MSc thesis, Universidade Técnica de Lisboa (1994), vii + 90 p., Portuguese.

[19] L. P. Castro: *Relations between Singular Operators and Applications.* PhD thesis, Universidade Técnica de Lisboa (1998), xix + 163 p.

[20] L. P. Castro, R. Duduchava and F. -O. Speck: *Localization and minimal normalization of some basic mixed boundary value problems.* In: Factorization, Singular Operators and Related Problems, Eds: S. Samko et al., Kluwer, Dordrecht (2003), 73-100.

[21] L. P. Castro, R. Duduchava and F. -O. Speck: *Asymmetric factorizations of matrix functions on the real line.* In: Modern Operator Theory and Applications. The Igor Borisovich Simonenko Anniversary Volume Eds: Y.M. Erusalimskii et al., Oper. Theory Adv. Appl. **170**, Birkhäuser, Basel (2006), 53-74.

[22] L. P. Castro, R. Duduchava and F. -O. Speck: *Diffraction from polygonal-conical screens — an operator approach.* In: Operator Theory, Operator Algebras and Applications Eds: A. Bastos et al., Oper. Theory Adv. Appl. **242**, Birkhäuser, Basel (2014), 113-137.

[23] L. P. Castro, F. -O. Speck: *On the characterization of the intermediate space in generalized factorizations.* Math. Nachr. **176** (1995), 39-54.

[24] L. P. Castro, F. -O. Speck: *Regularity properties and generalized inverses of delta-related operators.* Z. Anal. Anwend. **17** (1998), 577-598.

[25] L. P. Castro, F. -O. Speck: *Relations between convolution type operators on intervals and on the half-line.* Integral Equations Oper. Theory **37** (2000), 169-207.

[26] L. P. Castro, F. -O. Speck: *Inversion of matrix convolution type operators with symmetry.* Port. Math. (N.S.) **62** (2005), 193-216.

[27] L. P. Castro, F. -O. Speck and F.S. Teixeira: *On a class of wedge diffraction problems posted by Erhard Meister.* In: Operator theoretical methods and applications to mathematical physics. The Erhard Meister memorial volume Eds. I. Gohberg et al., Oper. Theory Adv. Appl. **147** (2004), 213-240.

[28] L. P. Castro, F. -O. Speck and F.S. Teixeira: *Mixed boundary value problems for the Helmholtz equation in a quadrant.* Integr. Equ. Oper. Theory **56** (2006), 1-44.

[29] G. N. Chebotarev: *Several remarks on the factorization of operators in a Banach space and the abstract Wiener-Hopf equation.* Mat. Issled. **2** (1968), 215-218, Russian.

[30] K. F. Clancey, I. Gohberg : *Factorization of Matrix Functions and Singular Integral Operators.* Oper. Theory Adv. Appl. **3**, Birkhäuser, Basel (1981).

[31] E. T. Copson: *On an integral equation arising in the theory of diffraction.* Q. J. Math., Oxf. Ser. **17** (1946), 19-34.

[32] V. G. Daniele: *On the solution of two coupled Wiener-Hopf equations.* SIAM J. Appl. Math. **44** (1984), 667-680.

[33] A. Devinatz, M. Shinbrot : *General Wiener-Hopf operators.* Trans. AMS **145** (1969), 467-494.

[34] R. Duduchava : *Integral Equations with Fixed Singularities.* Teubner, Leipzig (1979).

[35] R. Duduchava, F. -O. Speck: *Bessel potential operators for the quarter-plane.* Appl. Anal. **45** (1992), 49-68.

[36] R. Duduchava, F. -O. Speck: *Pseudodifferential operators on compact manifolds with Lipschitz boundary.* Math. Nachr. **160** (1993), 149-191.

[37] R. Duduchava, W. L. Wendland: *The Wiener-Hopf method for systems of pseudodifferential equations with an application to crack problems.* Integr. Equ. Oper. Theory **23** (1995), 294-335.

[38] T. Ehrhardt, A. P. Nolasco and F. -O. Speck: *Boundary integral methods for wedge diffraction problems: the angle 2 pi/n, Dirichlet and Neumann conditions.* Operators and Matrices **5** (2011), 1-40.

[39] T. Ehrhardt, A. P. Nolasco and F. -O. Speck: *A Riemannn surface approach for diffraction from rational wedges.* Operators and Matrices **8** (2014), 301-355.

[40] T. Ehrhardt, F. -O. Speck: *Transformation techniques towards the factorization of non-rational 2x2 matrix functions.* Linear Algebra Appl. **353** (2002), 53-90.

[41] G. I. Eskin: *Boundary Value Problems for Elliptic Pseudodifferential Equations.* AMS, Providence (1981), Russian edition 1973.

[42] I. C. Gohberg, I. A. Fel'dman: *Convolution equations and projection methods for their solution.* Translations of Mathematical Monographs 41. American Mathematical Society, Providence, R. I., (1974).

[43] I. Z. Gohberg, M. G. Krein: *Systems of integral equations on a half-line with kernel depending on the difference of arguments.* AMS Trans. **14** (1960), 217-287, Russian edition 1958.

[44] I. Gohberg, N. Krupnik: *One-Dimensional Linear Singular Integral Equations I, II.* Birkhäuser, Basel (1992), German edition 1979, Russian edition 1973.

[45] L. S. Goldenstein, I. C. Gohberg: *On a multidimensional integral equation on a half-space whose kernel is a function on the difference of the arguments, and on a discrete analogue of this equation.* Sov. Math. Dokl. **1** (1960), 173-176.

[46] P. Grisvard: *Elliptic Problems in Nonsmooth Domains.* Pitman, London, (1985).

[47] J. Hadamard: *Sur les problèmes aux dérivées partielles et leur signification physique.* In: Princeton University Bulletin **13**, No. 4, (1902), 49-52.

[48] A. Heins: *Systems of Wiener-Hopf integral equations and their application to some boundary value problems in electromagnetic theory.* Proc. Symp. Appl. Math. **2** (1950), 76-81.

[49] A. Heins: *The Sommerfeld half-plane problem revisited. I: The solution of a pair of coupled Wiener-Hopf integral equations.* Math. Methods Appl. Sci. **4** (1982), 74-90.

[50] S. ter Horst, M. Messerschmidt and A. C. M. Ran: *Equivalence after extension for compact operators on Banach spaces.* J. Math. Anal. Appl. **431** (2015), 136-149.

[51] S. ter Horst, A. C. M. Ran: *Equivalence after extension and matricial coupling coincide with Schur coupling, on separable Hilbert spaces.* Linear Algebra Appl. **439** (2013), 793-805.

[52] G. C. Hsiao, W. L. Wendland: *Boundary Integral Equations.* Springer, Berlin (2008).

[53] R. A. Hurd: *The Wiener-Hopf-Hilbert method for diffraction problems.* Can. J. Phys. **54** (1976), 775-780.

[54] R. A. Hurd: *The explicit factorization of 2 x 2 Wiener-Hopf matrices.* Preprint 1040, Fachbereich Mathematik, Technische Hochschule Darmstadt (1987), 24 p.

[55] R. A. Hurd, E. Meister and F. -O. Speck :*Sommerfeld diffraction problems with third kind boundary conditions.* SIAM J. Math. Anal. **20** (1989), 589-607.

[56] M. Idemen: *A new method to obtain exact solutions of vector Wiener-Hopf equations.* Z. Angew. Math. Mech. **59** (1979), 656-658.

[57] D. S. Jones: *A simplifying technique in the solution of a class of diffraction problems.* Quart.J. Math. **3** (1952), 189-196.

[58] D. S. Jones: *The Theory of Electromagnetism.* Pergamon Press, Oxford (1964).

[59] D. S. Jones: *Commutative Wiener–Hopf factorization of a matrix.* Proc. Roy. Soc. London, (1984), 185-192.

[60] A. A. Khrapkov: *Certain cases of the elastic equilibrium of an infinite wedge with a nonsymmetric notch at the vertex, subjected to concentrated forces.* J. Appl. Math. Mech. **35** (1971), 625-637, English, translation from Prikl. Mat. Mekh. **35** (1971), 677-689, Russian.

[61] A. V. Kisil: *Stability analysis of matrix Wiener-Hopf factorization of Daniele-Khrapkov class and reliable approximate factorization.* Proc. A, R. Soc. Lond. **471** (2015), 15 p.

[62] M. G. Krein: *Integral equations on a half-line with kernel depending on the difference of arguments.* AMS Trans. **22** (1962), 163-288, Russian edition 1958.

[63] J. B. Lawrie, I. D. Abrahams: *A brief historical perspective of the Wiener-Hopf technique.* J. Eng. Math. **59** (2007), 351-358.

[64] A. B. Lebre, A. Moura Santos and F. -O. Speck: *Factorization of o class of matrices generated by Sommerfeld diffraction problems with oblique derivatives.* Math. Meth. Appl. Sciences **20** (1997), 1185-1198.

[65] G. S. Litvinchuk, I. M. Spitkovskii: *Factorization of Measurable Matrix Functions.* Oper. Mathematical Research **37**, Akademie-Verlag, Berlin (1987), and Theory Adv. Appl. **25**, Birkhäuser, Basel (1987).

[66] E. Lüneburg: *The Sommerfeld problem: Methods, generalization and frustrations.* In: E. Meister (ed.), *Modern Mathematical Methods in Diffraction Theory and its Applications in Engineering.* Proceedings of the Sommerfeld '96 workshop. Peter Lang, Europ. Verlag der Wissenschaften, Frankfurt am Main (1997). Methoden Verfahren Math. Phys. **42**, 145-162.

[67] E. Lüneburg, R. A. Hurd *On the diffraction problem on a half-plane with different face impedances.* Can. J. Phys. **62** (1984), 853-860.

[68] V. Maz'ya, T. Shaposhnikova: *Jacques Hadamard, a universal mathematician.* History of Mathematics **14**, American Mathematical Society, Providence, RI (1998).

[69] E. Meister: *Randwertaufgaben der Funktionentheorie. Mit Anwendungen auf singuläre Integralgleichungen und Schwingungsprobleme der mathematischen Physik.* Leitfäden der Angewandten Mathematik und Mechanik **59**. Teubner, Stuttgart (1983), German.

[70] E. Meister: *Integraltransformationen mit Anwendungen auf Probleme der mathematischen Physik.* Lang, Frankfurt (1983), German.

[71] E. Meister: *Some solved and unsolved canonical problems of diffraction theory.* In: Differential equations and mathematical physics, Proc. Int. Conf., Birmingham/Ala. (1986), Lect. Notes Math. **1285** (1987), 320-336. Short English, version of [72].

[72] E. Meister: *Einige gelöste und ungelöste kanonische Probleme der mathematischen Beugungstheorie.* Expo. Math. **5** (1987), 193-237, German.

[73] E. Meister (ed.): *Modern Mathematical Methods in Diffraction Theory and its Applications in Engineering.* Proceedings of the Sommerfeld'96 workshop at Freudenstadt, Germany. Methoden und Verfahren der Mathematischen Physik **42**. Peter Lang, Europ. Verlag der Wissenschaften, Frankfurt (1997).

[74] E. Meister, F. Penzel: *On the reduction of the factorization of matrix functions of Daniele- Khrapkov type to a scalar boundary value problem on a Riemann surface.* Complex Variables, Theory Appl. **18** (1992), 63-71.

[75] E. Meister, F. -O. Speck: *Some multidimensional Wiener-Hopf equations with applications.* In: Trends in applications of pure mathematics to mechanics, Vol. **2** (1979), Proceedings of a Symposium at Kozubnik Poland (1977), 217-262.

[76] E. Meister, F. -O. Speck: *Diffraction problems with impedance conditions.* Appl. Anal. **22** (1986), 193-211.

[77] E. Meister, F. -O. Speck: *Modern Wiener-Hopf methods in diffraction theory.* In: Ordinary and Partial Differential Equations, Vol. **2** Eds: B.D. Sleeman et al., Longman, London (1989), 130-171.

[78] E. Meister, F. -O. Speck: *The explicit solution of elastodynamical diffraction problems by symbol factorization.* Z. Anal. Anw. **8** (1989), 307-328.

[79] S. G. Mikhlin, S. Prössdorf: *Singular Integral Operators.* Springer, Berlin (1986), German edition: Akademie-Verlag, Berlin 1980.

[80] G. Mishuris, S. Rogosin: *Factorization of a class of matrix-functions with stable partial indices.* Math. Methods Appl. Sci. **39** (2016), 3791-3807.

[81] A. Moura Santos, F. -O. Speck: *Sommerfeld diffraction problems with oblique derivatives.* Math. Meth. Appl. Sciences **20** (1997), 635-652.

[82] A. Moura Santos, F. -O. Speck and F. S. Teixeira: *Compatibility conditions in some diffraction problems.* Pitman Research Notes in Mathematics Series **361**, Longman, London (1996), 25-38.

[83] A. Moura Santos, F. -O. Speck and F. S. Teixeira: *Minimal normalization of Wiener-Hopf operators in spaces of Bessel potentials.* J. Math. Anal. Appl. **225** (1998), 501-531.

[84] R. J. Nagem, M. Zampolli and G. Sandri: *Mathematical Theory of Diffraction.* Birkhäuser, Boston (2004).

[85] M. Z. Nashed, L. B. Rall: *Annotated bibliography on generalized inverses and applications.* In: Generalized Inverses and Applications (Ed: M.Z. Nashed), Academic Press, New York (1976), 771-1041.

[86] B. Noble: *Methods Based on the Wiener-Hopf Technique.* Pergamon Press, London (1958).

[87] F. Penzel, F. -O. Speck: *Asymptotic expansion of singular operators on Sobolev spaces.* Asymptotic Anal. **7** (1993), 287-300.

[88] H. Poincaré: *Sur la polarisation par diffraction.* Acta Math. **16** (1892), 297-339, French.

[89] L. Primachuk, S. Rogosin *Factorization of triangular matrix-functions of an arbitrary order.* Lobachevskii Journal of Mathematics **39** (2018), 809–817.

[90] S. Prössdorf, F. -O. Speck *A factorisation procedure for two by two matrix functions on the circle with two rationally independent entries.* Proc. R. Soc. Edinb., Sect. A **115** (1990), 119-138.

[91] A. D. Rawlins: *The solution of a mixed boundary value problem in the theory of diffraction by a semi-infinite plane.* Proc. Roy. Soc. London A **346** (1975), 469-484.

[92] A. D. Rawlins: *The explicit Wiener-Hopf factorisation of a special matrix.* Z. Angew. Math. Mech. **61** (1981), 527-528.

[93] A. D. Rawlins: *The solution of a mixed boundary value problem in the theory of diffraction.* J. Eng. Math. **18** (1984), 37-62.

[94] A. D. Rawlins, W. E. Williams: *Matrix Wiener-Hopf factorisation.* Q. J. Mech. Appl. Math. **34** (1981), 1-8.

[95] S. Rogosin, G. Mishuris *Constructive methods for factorization of matrix-functions.* IMA J. Appl. Math. **81** (2016), 365-391.

[96] A. F. dos Santos, F. S. Teixeira: *The Sommerfeld problem revisited: Solution spaces and the edge conditions.* J. Math. Anal. Appl. **143** (1989), 341-357.

[97] T. B. A. Senior: *Diffraction by a semi-infinite metallic sheet.* Proc. R. Soc. Lond., Ser. A **213** (1952), 436-458.

[98] A. H. Serbest (ed.), S. R. Cloude (ed.): *Direct and inverse electromagnetic scattering.* Proceedings of the workshop, September 24–30, (1995), Gebze, Turkey. Mathematics Series **361**. Longman, Harlow (1996).

[99] E. Shamir: *Mixed boundary value problems for elliptic equations in the plane.* The L^p theory. Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. **17** (1963), 117-139.

[100] E. Shamir *Elliptic systems of singular integral operators. I. The half-space case.* Trans. AMS **127** (1967), 107-124.

[101] M. Shinbrot: *On singular integral operators.* J. Math. Mech. **13** (1964), 395-406.

[102]] I. B. Simonenko: *Some general questions in the theory of the Riemann boundary problem.* Izv. Akad. Nauk SSSR, Ser. Mat. **32** (1968), 1138-1146, Russian. Math. USSR, Izv. **2** (1970), 1091-1099, English.

[103] A. Sommerfeld: *Mathematische Theorie der Diffraction.* Math. Ann. **47** (1896), 317-374, German. Annotated translation in English see [84].

[104] A. Sommerfeld: *Partial differential equations in physics.* Academic Press, New York (1949).

[105] F. -O. Speck: *On the generalized invertibility of Wiener-Hopf operators in Banach spaces.* Integr. Equ. Oper. Theory **6** (1983), 458-465.

[106] F. -O. Speck: *General Wiener-Hopf Factorization Methods.* Pitman, London (1985).

[107] F. -O. Speck: *Mixed boundary value problems of the type of Sommerfeld half-plane problem.* Proc. Royal Soc. Edinburgh **104 A** (1986), 261-277.

[108] F. -O. Speck: *Sommerfeld diffraction problems with first and second kind boundary conditions.* SIAM J. Math. Anal. **20** (1989), 396-407.

[109] F. -O. Speck: *In memory of Erhard Meister.* In: Operator Theoretical Methods and Applications to Mathematical Physics I, Eds. I. Gohberg et al., Oper. Theory Adv. Appl. **147**, Birkhäuser, Basel (2004), 27-46.

[110] F. -O. Speck: *On the reduction of linear systems related to boundary value problems.* In: Operator theory, pseudo-differential equations, and mathematical physics, Eds. Yu.I. Karlovich et al., The Vladimir Rabinovich anniversary volume. Oper. Theory Adv. Appl. **228**, Birkhäuser, Basel (2013), 391-406.

[111] F. -O. Speck: *Diffraction from a three-quarter-plane using an abstract Babinet principle.* Z. Angew. Math. Mech. **93** (2013), 485-491.

[112] F. -O. Speck: *Wiener-Hopf factorization through an intermediate space.* Integr. Equ. Oper. Theory **82** (2015), 395-415.

[113] F. -O. Speck: *A class of interface problems for the Helmholtz equation in $\mathbb{R}^n$.* Math. Meth. Appl. Sciences **40** (2017), 391-403.

[114] F. -O. Speck: *Paired operators in asymmetric space setting.* In: Large Truncated Toeplitz Matrices, Toeplitz Operators, and Related Topics, Eds. D. Bini et al., The Albrecht Böttcher Anniversary Volume. Oper. Theory Adv. Appl. **259**, Birkhäuser, Basel (2017), 681-702.

[115] F. -O. Speck: *On the reduction of general Wiener-Hopf operators.* In: Operator Theory, Analysis, and the State Space Approach, Eds. H. Bart et al., In Honor of Rien Kaashoek. Oper. Theory Adv. Appl. **271**, Birkhäuser, Basel (2018), 399-419.

[116] I. M. Spitkovsky, A. M. Tashbaev: *On the problem of effective factorization of matrix functions.* Izv. Vyssh. Uchebn. Zaved., Mat. **4** (1989), 69-76, Russian. Sov. Math. **33** (1989), 85-93, English.

[117] G. Talenti: *Sulle equazioni integrali di Wiener-Hopf.* Boll. Unione Mat. Ital., IV. Ser. **7**, Suppl. al Fasc. 1 (1973), 18-118, Italian.

[118] D. Timotin: *Schur coupling and related equivalence relations for operators on a Hilbert space.* Linear Algebra Appl. **452** (2014), 106-119.

[119] H. Triebel: *Theory of function spaces.* Akademische Verlagsgesellschaft, Leipzig (1983).

[120] M. I. Vishik, G. I. Eskin: *Equations in convolutions in a bounded region.* Russian Math. Surveys **20** (1965), 85-151. Translation from Usp. Mat. Nauk **20** (1965), 89-152.

[121] L. A. Weinstein: *The Theory of Diffraction and the Factorization Method.* Golem Press, Boulder, Colorado (1969).

[122] N. Wiener, E. Hopf: *Über eine Klasse singulärer Integralgleichungen.* Sitzungsber. Preuß. Akad. Wiss., Phys.-Math. Kl. **30-32** (1931), 696-706, German.

[123] W. E. Williams: *Recognition of some readily "Wiener–Hopf" factorizable matrices.* IMA J. Appl. Math. **32** (1984), 367–378.

[124] J. Wloka: *Partial Differential Equations.* Cambridge University Press, (1987).

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Research Article

Existence Results for Systems of Quasi-Variational Relations

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ABSTRACT. The existence of solutions for a system of variational relations, in a general form, is studied using a fixed point result for contractions in metric spaces. As a particular case, sufficient conditions for the existence of solutions of a system of quasi-equilibrium problems are given.

Keywords: Variational relations problems, system of equilibrium problems, fixed points.

2010 Mathematics Subject Classification: 47H09, 58E99.

1. INTRODUCTION AND PRELIMINARIES

For each $i \in I = \{1, \ldots, n\}$, let X_i be a nonempty subset of a complete metric space (E_i, d_i) and $X = \prod_{i \in I} X_i$ a subset of the product space $E = \prod_{i \in I} E_i$. Let $S_i, Q_i : X \to 2^{X_i}$ be two set-valued maps with nonempty values. Let $R_i(x, y_i)$ be a relation between $x \in X$ and $y_i \in X_i$.

The general system of quasi-variational relations that we consider in this paper is:

$$(SQVR) \quad \text{Find } \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X \text{ such that for each } i \in I,$$
$$\bar{x}_i \in S_i(\bar{x}) \text{ and } R_i(\bar{x}, y_i) \text{ holds for all } y_i \in Q_i(\bar{x}).$$

Variational relations problems were considered for the first time by D.T. Luc in [11], as a general model that encompasses optimization problems, equilibrium problems or variational inclusion problems. Several authors continued the study of variational relations problems, see for instance the papers [10], [12], [9], [2], [1] and the references therein. Existence results for the solutions of variational relations problems are obtained mostly in two ways: by applying intersection results for set valued mappings (see [11]) or by using various fixed points theorems (see [11], [7], [4]).

The system $(SQVR)$ was introduced by L.J. Lin and Q.H. Ansari in [8], where the existence of a solution was established using a maximal element theorem for a family of set-valued maps. The same system was studied in [5] by a factorization method, that followed the ideas from [6].

In this paper, we will give sufficient conditions for the existence of solutions of the system $(SQVR)$, using a fixed-point theorem for set-valued mappings that are Reich-type contractions. The general result obtained for the system of variational relations will be applied in the last section to a system of equilibrium problems.

In the rest of this section, we present some notations and results needed in the paper.

The metric on the product space will be defined by $d : E \times E \to \mathbb{R}_+$,

$$d(x, y) = d_1(x_1, y_1) + \cdots + d_n(x_n, y_n),$$

for $x = (x_1, \ldots, x_n) \in E$ and $y = (y_1, \ldots, y_n) \in E$.

For any nonempty sets $A, B \subset E$ and $x \in E$, denote by

$$
\begin{aligned}
D(x, B) &= \inf_{b \in B} d(x, b) \text{ and} \\
H(A, B) &= \max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}.
\end{aligned}
$$

$H(A, B)$ is the generalized Hausdorff functional of A and B. Similarly, we will denote by $H_i(A_i, B_i)$ the Hausdorff distance induced by d_i, for A_i and B_i subsets of E_i.

Lemma 1.1. *For $x = (x_1, \ldots, x_n)$, $A = A_1 \times \cdots \times A_n$ and $B = B_1 \times \cdots \times B_n$, we have*

$$
\begin{aligned}
D(x, B) &= D_1(x_1, B_1) + \cdots + D_n(x_n, B_n), \\
H(A, B) &\leq H_1(A_1, B_1) + \cdots + H_n(A_n, B_n).
\end{aligned}
$$

Lemma 1.2. *(a) If $A, B \subset E$ are such that for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leq c$ and for each $b \in B$ there exists $a \in A$ such that $d(a, b) \leq c$, then $H(A, B) \leq c$.*

(b) If $A, B \subset E$ and $\varepsilon > 0$, then for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \varepsilon$.

There is a vast literature on the existence of fixed points of generalized contractions, both single-valued and set-valued (see for instance [3], [14]). We will use the following:

A set-valued mapping $F : E \to 2^E$ is said to be a *Reich - type contraction* if there exist $a, b, c \geq 0$, with $a + b + c < 1$ such that $H(F(x), F(y)) \leq ad(x, y) + bD(x, F(x)) + cD(y, F(y))$, for each $x, y \in E$.

Theorem 1.1 ([13]). *Let (E, d) be a complete metric space and let $F : E \to 2^E$ be a Reich-type contraction. Suppose also that $F(x)$ is a closed set, for every $x \in E$. Then, F has at least a fixed point.*

2. AN EXISTENCE RESULT FOR A SYSTEM OF VARIATIONAL RELATIONS

We give in what follows sufficient conditions for the existence of solutions of the system $(SQVR)$ formulated in the previous section.

For $x = (x_1, \ldots, x_n) \in X$ and $i \in I$ fixed, we denote

$$
\Gamma_i(x) = \{z_i \in S_i(x) \mid R_i(x_1, \ldots, z_i, \ldots, x_n; t_i) \text{ holds for all } t_i \in Q_i(x)\}
$$

and we define the function $\Gamma : X \to 2^X$ by $\Gamma(x) = \Gamma_1(x) \times \cdots \times \Gamma_n(x)$. It is easy to see that any fixed point of Γ is a solution of (SQVR).

Theorem 2.2. *Suppose that for any $i \in I$, the set X_i is nonempty, closed and:*

(i) for any $x \in X$, $\Gamma_i(x)$ is nonempty;

(ii) there exists $q_i \in]0, 1[$ such that, for every $x^1, x^2 \in X$, if $z_i^1 \in \Gamma_i(x^1)$, there exists $z_i^2 \in \Gamma_i(x^2)$ such that

$$
d_i(z_i^1, z_i^2) \leq q_i H_i(S_i(x^1), S_i(x^2));
$$

(iii) there exist $a_i, b_i, c_i \in]0, 1[$, with $\max_{i \in I} a_i + \max_{i \in I} b_i + \max_{i \in I} c_i < 1$ such that, for every $x^1, x^2 \in X$,

$$
H_i(S_i(x^1), S_i(x^2)) \leq a_i d_i(x_i^1, x_i^2) + b_i D_i(x_i^1, S_i(x^1)) + c_i D_i(x_i^2, S_i(x^2));
$$

(iv) for any $x \in X$, the set $S_i(x)$ is closed;

(v) the relation R_i is closed in the $i-th$ variable, that is: for any sequence $(z_i^k)_{k \in \mathbb{N}} \subset X_i$ such that $z_i^k \to z_i$ when $k \to \infty$, if $R_i(x_i, \ldots, z_i^k, \ldots x_n; t_i)$ holds, then $R_i(x_i, \ldots, z_i, \ldots x_n; t_i)$ holds too.

Then, (SQVR) admits at least a solution.

Proof. We will prove that $\Gamma : X \to 2^X$ is a Reich-type contraction and we will use Theorem 1.1 to obtain the existence of a fixed point of Γ. Since X is closed and (E, d) is complete, the space (X, d) is complete too.

For each $i \in I$ and $x \in X$, hypotheses (iv) and (v) imply that $\Gamma_i(x)$ is closed. Then $\Gamma(x)$ is closed too.

Let $x^1 = (x_1^1, \ldots, x_n^1) \in X$ and $x^2 = (x_1^2, \ldots, x_n^2) \in X$. Let $z_i^1 \in \Gamma_i(x^1)$. According to (ii), there exists $z_i^2 \in \Gamma_i(x^2)$ such that

$$(2.1) \qquad d_i(z_i^1, z_i^2) \le q_i H_i(S_i(x^1), S_i(x^2)).$$

Similarly, for any $z_i^2 \in \Gamma_i(x^2)$ there exists $z_i^1 \in \Gamma_i(x^1)$ such that (2.1) holds. From Lemma 1.2, we have

$$(2.2) \qquad H_i(\Gamma_i(x^1), \Gamma_i(x^2)) \le q_i H_i(S_i(x^1), S_i(x^2)).$$

Further, using Lemma 1.1, (2.2), (iii), and the inclusion $\Gamma_i(x) \subseteq S_i(x)$, for any $x \in X$, follows

$$
\begin{aligned}
H(\Gamma(x^1), \Gamma(x^2)) \ &\le \ \sum_{i=1}^n H_i(\Gamma_i(x^1), \Gamma_i(x^2)) \le \sum_{i=1}^n q_i H_i(S_i(x^1), S_i(x^2)) \\
&\le \ \sum_{i=1}^n (q_i a_i d_i(x_i^1, x_i^2) + q_i b_i D_i(x_i^1, S_i(x^1)) + q_i c_i D_i(x_i^2, S_i(x^2))) \\
&\le \ qa\, d(x^1, x^2) + qb \sum_{i=1}^n D_i(x_i^1, \Gamma_i(x^1)) + qc \sum_{i=1}^n D_i(x_i^2, \Gamma_i(x^2)) \\
&= \ qa\, d(x^1, x^2) + qb\, D(x^1, \Gamma(x^1)) + qc\, D(x^2, \Gamma(x^2)),
\end{aligned}
$$

where $q = \max_{i \in I} q_i$, $a = \max_{i \in I} a_i$, $b = \max_{i \in I} b_i$, $c = \max_{i \in I} c_i$ and $qa + qb + qc < 1$. Applying Reich's theorem follows the existence of a fixed point for Γ and consequently of a solution of (SQVR). $\qquad\square$

By making a change in hypothesis (ii), we can obtain a second existence result:

Theorem 2.3. *Suppose that for any $i \in I$, the set X_i is nonempty, closed and:*
(i) for any $x \in X$, $\Gamma_i(x)$ is nonempty;
(ii) there exists $q_i \in\,]0, 1[$ such that, for every $x^1, x^2 \in X$, for every $z_i^1 \in \Gamma_i(x^1)$ and $z_i^2 \in \Gamma_i(x^2)$,

$$d_i(z_i^1, z_i^2) \le q_i H_i(S_i(x^1), S_i(x^2));$$

(iii) there exist $a_i, b_i, c_i \in\,]0, 1[$, with $\max_{i \in I} a_i + \max_{i \in I} b_i + \max_{i \in I} c_i < 1$ such that, for every $x^1, x^2 \in X$,

$$H_i(S_i(x^1), S_i(x^2)) \le a_i d_i(x_i^1, x_i^2) + b_i D_i(x_i^1, S_i(x^1)) + c_i D_i(x_i^2, S_i(x^2));$$

Then, (SQVR) admits a solution.

Proof. It can be noticed that for any $x \in X$ and $i \in I$, the set $\Gamma_i(x)$ contains only one element. Indeed, if $\zeta_i, \xi_i \in \Gamma_i(x)$, according to (ii), we get

$$d_i(\zeta_i, \xi_i) \le q_i H_i(S_i(x), S_i(x)) = 0,$$

so $\zeta_i = \xi_i$. Since $\Gamma_i(x)$ is a singleton, it is a closed set. The rest of the proof is the same as for Theorem 2.2. $\qquad\square$

Starting with another definition for the "partial" problem, we can obtain a new existence result, with different conditions.

For $x = (x_1, \ldots, x_n) \in X$ and $i \in I$ fixed, we denote

$$T_i(x) = \{z_i \in X_i \mid \quad z_i \quad \in S_i(x_1, \ldots, z_i, \ldots, x_n) \text{ and } R_i(x_1, \ldots, z_i, \ldots, x_n; \theta_i)$$
$$\text{holds for all } \theta_i \in Q_i(x_1, \ldots, z_i, \ldots, x_n)\}$$

and we define the function $T : X \to 2^X$ by $T(x) = T_1(x) \times \cdots \times T_n(x)$. It is easy to see that any fixed point of T is a solution of (SQVR).

Theorem 2.4. *Suppose that for any $i \in I$, the set X_i is nonempty, closed and:*

(i) for any $x \in X$, $T_i(x)$ is nonempty;

(ii) there exists $q_i \in]0, 1[$ such that, for every $x^1, x^2 \in X$, if $z_i^1 \in T_i(x^1)$, there exists $z_i^2 \in T_i(x^2)$ such that

$$d_i(z_i^1, z_i^2) \leq q_i H_i(S_i(x^1), S_i(x^2));$$

(iii) there exist $a_i, b_i, c_i \in]0, 1[$, with $\max_{i \in I} a_i + \max_{i \in I} b_i + \max_{i \in I} c_i < 1$ such that, for every $x^1, x^2 \in X$,

$$H_i(S_i(x^1), S_i(x^2)) \leq a_i d_i(x_i^1, x_i^2) + b_i D_i(x_i^1, S_i(x^1)) + c_i D_i(x_i^2, S_i(x^2));$$

(iv) for any sequence $(z_i^k)_{k \in \mathbb{N}} \subset X_i$ such that $z_i^k \to z_i$ when $k \to \infty$, if $z_i^k \in S_i(x_1, \ldots, z_i^k, \ldots, x_n)$ for any $k \in \mathbb{N}$, then $z_i \in S_i(x_1, \ldots, z_i, \ldots, x_n)$;

(v) for any sequence $(z_i^k)_{k \in \mathbb{N}} \subset X_i$ such that $z_i^k \to z_i$ when $k \to \infty$, if $R_i(x_i, \ldots, z_i^k, \ldots x_n; \theta_i)$ holds for any $\theta_i \in Q_i(x_1, \ldots, z_i^k, \ldots, x_n)$, then the relation $R_i(x_i, \ldots, z_i, \ldots x_n; t_i)$ holds for any $t_i \in Q_i(x_1, \ldots, z_i, \ldots, x_n)$.

Then, (SQVR) admits at least a solution.

Proof. Hypotheses (iv) and (v) imply that for every $x \in X$, $T(x)$ is closed. The rest of the proof is identical to the one of Theorem 2.2. $\qquad\square$

3. AN EXISTENCE RESULT FOR A SYSTEM OF QUASI-EQUILIBRIUM PROBLEMS

As a particular case of the system of quasi-variational relations, we consider

$$(SQEP) \quad \text{Find } \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X \text{ such that for each } i \in I,$$
$$\bar{x}_i \in S_i(\bar{x}) \text{ and } f_i(\bar{x}, t_i) \geq 0 \text{ for all } t_i \in S_i(\bar{x}).$$

The relation $R_i(x, t_i)$ holds iff $f_i(x, t_i) \geq 0$. In this section, we denote

$$\gamma_i(x) = \{z_i \in S_i(x) \mid f_i(x_1, \ldots, z_i, \ldots, x_n; t_i) \geq 0, \text{ for all } t_i \in S_i(x)\}$$

As a consequence of Theorem 2.3, we obtain:

Theorem 3.5. *Suppose that for any $i \in I$, the set X_i is nonempty, closed and:*

(a) for any $x \in X$, $\gamma_i(x)$ is nonempty;

(b) there exist $a_i, b_i, c_i \in]0, 1[$, with $\max_{i \in I} a_i + \max_{i \in I} b_i + \max_{i \in I} c_i < 1$ such that, for every $x^1, x^2 \in X$,

$$H_i(S_i(x^1), S_i(x^2)) \leq a_i d_i(x_i^1, x_i^2) + b_i D_i(x_i^1, S_i(x^1)) + c_i D_i(x_i^2, S_i(x^2));$$

(c) there exists $m_i > 0$ such that for every $x = (x_1, \ldots, x_n) \in X$ and $t_i \in X_i$,

$$f_i(x_1, \ldots, x_i, \ldots, x_n; t_i) + f_i(x_1, \ldots, t_i, \ldots, x_n; x_i) \leq -m_i d_i(x_i, t_i);$$

(d) f_i is lipschitz in the last variable, that is there exists $L_i > 0$ such that for every $x \in X$ and $t_i, \theta_i \in X_i$,

$$|f_i(x; t_i) - f_i(x; \theta_i)| \leq L_i d_i(t_i, \theta_i),$$

(e) f_i is lipschitz in the $i-th$ variable, that is there exists $\lambda_i > 0$ such that for every $x \in X$ and $\zeta_i, \xi_i, t_i \in X_i$,

$$|f_i(x_1, \ldots, \zeta_i, \ldots x_n; t_i) - f_i(x_1, \ldots, \xi_i, \ldots x_n; t_i)| \leq \lambda_i d_i(\zeta_i, \xi_i);$$

(f) $L_i + \lambda_i < m_i$.
Then, (SQEP) admits a solution.

Proof. To apply Theorem 2.3, we just need to verify hypothesis (ii). Let $\varepsilon > 0$. Let $x^1, x^2 \in X$ and $z_i^1 \in \gamma(x^1)$, $z_i^2 \in \gamma(x^2)$.

Since $z_i^1 \in S_i(x^1)$, from Lemma 1.2, there exists $t_i^2 \in S_i(x^2)$ such that

(3.3) $$d_i(z_i^1, t_i^2) \leq H_i(S_i(x^1), S_i(x^2)) + \varepsilon.$$

Similarly, since $z_i^2 \in S_i(x^2)$, there exists $t_i^1 \in S_i(x^1)$ such that

(3.4) $$d_i(z_i^2, t_i^1) \leq H_i(S_i(x^1), S_i(x^2)) + \varepsilon.$$

From the definitions of $\gamma_i(x^1)$ and $\gamma_i(x^2)$, we get

(3.5) $$f_i(x_1^1, \ldots, z_i^1, \ldots, x_n^1; t_i^1) \geq 0 \text{ and } f_i(x_1^2, \ldots, z_i^2, \ldots, x_n^2; t_i^2) \geq 0.$$

From condition (c), we have

$$d_i(z_i^1, z_i^2) \leq -\frac{1}{m_i} f_i(x_1^1, \ldots, z_i^1, \ldots, x_n^1; z_i^2) - \frac{1}{m_i} f_i(x_1^1, \ldots, z_i^2, \ldots, x_n^1; z_i^1),$$

$$d_i(z_i^1, z_i^2) \leq -\frac{1}{m_i} f_i(x_1^2, \ldots, z_i^1, \ldots, x_n^2; z_i^2) - \frac{1}{m_i} f_i(x_1^2, \ldots, z_i^2, \ldots, x_n^2; z_i^1).$$

Next, adding these two inequalities, using (3.5) and hypothesis (d) follows that

$$
\begin{aligned}
d_i(z_i^1, z_i^2) \quad \leq \quad & -\frac{1}{2m_i} f_i(x_1^1, \ldots, z_i^1, \ldots, x_n^1; z_i^2) + \frac{1}{2m_i} f_i(x_1^1, \ldots, z_i^1, \ldots, x_n^1; t_i^1) \\
& - \frac{1}{2m_i} f_i(x_1^1, \ldots, z_i^2, \ldots, x_n^1; z_i^1) - \frac{1}{2m_i} f_i(x_1^2, \ldots, z_i^2, \ldots, x_n^2; z_i^1) \\
& + \frac{1}{2m_i} f_i(x_1^2, \ldots, z_i^2, \ldots, x_n^2; t_i^2) - \frac{1}{2m_i} f_i(x_1^2, \ldots, z_i^1, \ldots, x_n^2; z_i^2) \\
\leq \quad & \frac{L_i}{2m_i} d_i(z_i^2, t_i^1) + \frac{L_i}{2m_i} d_i(z_i^1, t_i^2) \\
& - \frac{1}{2m_i} f_i(x_1^1, \ldots, z_i^2, \ldots, x_n^1; z_i^1) - \frac{1}{2m_i} f_i(x_1^2, \ldots, z_i^1, \ldots, x_n^2; z_i^2).
\end{aligned}
$$

On the other hand, $z_i^1 \in \gamma(x^1)$ implies that $f_i(x_1^1, \ldots, z_i^1, \ldots, x_n^1; z_i^1) \geq 0$. Similarly, we have $f_i(x_1^2, \ldots, z_i^2, \ldots, x_n^2; z_i^2) \geq 0$. So it follows, using also condition (e), the previous inequality, (3.3) and (3.4) that

$$
\begin{aligned}
d_i(z_i^1, z_i^2) \quad \leq \quad & \frac{L_i}{2m_i} d_i(z_i^2, t_i^1) + \frac{L_i}{2m_i} d_i(z_i^1, t_i^2) \\
& - \frac{1}{2m_i} f_i(x_1^1, \ldots, z_i^2, \ldots, x_n^1; z_i^1) - \frac{1}{2m_i} f_i(x_1^2, \ldots, z_i^1, \ldots, x_n^2; z_i^2) \\
& + \frac{1}{2m_i} f_i(x_1^1, \ldots, z_i^1, \ldots, x_n^1; z_i^1) + \frac{1}{2m_i} f_i(x_1^2, \ldots, z_i^2, \ldots, x_n^2; z_i^2) \\
\leq \quad & \frac{L_i}{m_i} H_i(S_i(x^1), S_i(x^2)) + \frac{L_i \varepsilon}{m_i} + \frac{\lambda_i}{m_i} d_i(z_i^1, z_i^2).
\end{aligned}
$$

From here, we get

$$(1 - \frac{\lambda_i}{m_i})d_i(z_i^1, z_i^2) \leq \frac{L_i}{m_i}H_i(S_i(x^1), S_i(x^2)) + \frac{L_i\varepsilon}{m_i}.$$

When $\varepsilon \to 0$, the inequality becomes

$$d_i(z_i^1, z_i^2) \leq \frac{L_i}{m_i - \lambda_i}H_i(S_i(x^1), S_i(x^2)),$$

so $q_i = \frac{L_i}{m_i - \lambda_i} \in]0, 1[$ as needed. $\square$

We mention that sufficient conditions for the non-emptiness of the sets $\Gamma_i(x)$ or $T_i(x)$ can be given, for instance, by using intersection theorems of Ky Fan type (see [5], [4]).

REFERENCES

[1] R. P. Agarwal, M. Balaj and D. O'Regan: *Variational relation problems in a general setting*. Journal of Fixed Point Theory and Applications **18** (2016), 479–493.

[2] M. Balaj: *Systems of variational relations with lower semicontinuous set-valued mappings*. Carpathian Journal of Mathematics **31** (2015), 269–275.

[3] A. Granas and J. Dugundji: Fixed Point Theory, Springer-Verlag, Berlin, 2003.

[4] D. Inoan: *Variational relations problems via fixed points of contraction mappings*. Journal of Fixed Point Theory and Applications **19** (2017), 1571–1580.

[5] D. Inoan: *Factorization of quasi-variational relations systems*. Acta Mathematica Vietnamica **39** (2014), 359-365.

[6] G. Kassay, J. Kolumbán and Z. Páles: *Factorization of Minty and Stampacchia variational inequality systems*. European J. Oper. Res. **143** (2002), 377-389.

[7] A. Latif and D. T. Luc: *Variational relation problems: existence of solutions and fixed points of contraction mappings*. Fixed Point Theory and Applications (2013) Article id. 315, 1–10.

[8] L-J. Lin and Q. H. Ansari: *Systems of quasi-variational relations with applications*. Nonlinear Anal. **72** (2010), 1210–1220.

[9] L-J. Lin, M. Balaj and Y. C. Ye: *Quasi-variational relation problems and generalized Ekeland's variational principle with applications*. Optimization **63** (2014), 1353–1365.

[10] L-J. Lin and S-Y. Wang: *Simultaneous variational relation problems and related applications*. Computers and Mathematics with Applications **58** (2009), 1711–1721.

[11] D. T. Luc: *An abstract problem in variational analysis*. J. Optim. Theory Appl. **138** (2008), 65–76.

[12] Y. J. Pu and Z. Yang: *Variational relation problem without the KKM property with applications*. J. Math. Anal. Appl. **393** (2012), 256–264.

[13] S. Reich: *Fixed point of contractive functions*. Boll. Un. Mat. Ital. **5** (1972), 26-42.

[14] I. A. Rus, A. Petrusel and G. Petrusel: Fixed Point Theory, Cluj University Press, Cluj-Napoca, 2008.

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