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Praface

The ICOMAA conference, organized for the first time in 2018, helds its second conference in 2019 with participants from all over the world. The targeted success was achieved as a result of the returns received for the two conferences and the studies produced. As a result of these achievements, both the richness of the invited speaker staff composed of world-renowned scientists as well as the increase in the scientific quality of the study indicate that the new 2020 conference should be held. There are participants from 11 different countries in their presentations at the 2019 conference. In addition, a successful conference was held in 2019 with 8 invited speakers from 5 different countries.

The main objective of the conference is to bring scientists and researchers together with recent studies in pure and applied mathematics. In addition, inter-scientific communication is provided with the applications of mathematics in the major engineering fields. This volume contains the proceedings of the selected contributions of the participants of the 2th International Conference on Mathematical Advances and Applications (ICOMAA-2019) scheduled during May 3-5, 2019 in Istanbul, Turkey.

The selection of papers included in this volume is based on a rigorous peer review process by the committee of experts in various disciplines. Every submitted paper was first screened by the members of the editorial board and once it clears the initial screening, it was sent for peer review to at least two potential reviewers in the related area of expertise from the pool of potential reviewers. The paper is accepted if at least two reviewers recommend it for acceptance. We thank all the invited speakers and the authors for their valuable contributions towards the success of the conference ICOMAA-2019. We are very much grateful to the members of the program committee for their continuous guidance and support which led to the selection of the contributed talks and the papers published in this volume.

Hope to see you at the next conference,

Assoc. Prof. Dr. Yusuf ZEREN

Editor in Chief

ICOMAA-2019

Editor in Chief

Managing Editors

Editorial Board of Conference Proceedings of Science and Technology

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Health Efficiency Measurement In Turkey By Using Data Envelopment Analysis

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Özet: A performance indicator of the relative activity measurements are classified into two groups, parametric and nonparametric methods. Nonparametric methods measure the distance between the value of the efficiency obtained from the calculation and the distance from the efficiency limit by using techniques of linear programming. Data Envelopment Analysis (DEA) which is frequently used in nonparametric measurement methods. Data Envelopment Analysis is a method used to measure the relative effectiveness of economic or non-profit organizations that convert the input called the decision-making unit into output. In this study, health performances of provinces were examined by using statistical yearbook published by Ministry of Health. CCR and BCC models, which are the basic models of data envelopment analysis, were examined according to the cases of constant returns to scale and variable returns to scale.

Keywords: Data Envelopment Analysis, Health performance, Efficiency, Turkey

1 Introduction

The main problem of many countries is the limited resources to meet the needs of the growing population. Therefore, it is very important to use the resources in the most appropriate way and to obtain the maximum output. The ability to use resources effectively determines the level of productivity. Being productive has become important for the health sector like other sectors. As it is known, hospitals have high input costs. There is intense competition in the health services market. In this competitive environment, hospitals need to reduce their costs and get more output in order to survive. In such a case, the efficiency levels of the hospitals should be determined and the input variables that should be reduced and the output variables that should be increased should be determined if they are below the effective limit. The desire to be productive and efficient production is not only the problem of the undeveloped and developing countries. Especially in health systems and health institutions, most of the efforts to increase efficiency are carried out by developed countries. Evaluating health institutions in terms of efficiency has a very recent history. The number of studies in this field is rapidly increasing in order to examine the health performances of countries and formulate the necessary health policies. Due to the unique characteristics of health services, the number of inputs and outputs is quite high. In addition, these input and output factors do not have the same unit of measurement. Data Envelopment Analysis (DEA), which is a parameterless method, was used to analyze all these variables together, to make comparisons, to determine the most effective, and to determine what should be done for those who fall below the effective limit. DEA is a technique that determines the efficiency levels of decision-making units with similar characteristics according to the input and output amounts. DEA is applied in healthcare institutions, military areas, schools, banks and similar institutions. DEA limits the most effective decision-making unit and evaluates other decision-making units according to this limit. With DEA program, it is possible to calculate which variable to change in order to become effective. In this context, efficiency levels of public hospital associations, factors causing inefficiency and the steps to be effective for those who are below the effective limit have been determined. Therefore, in this study, it is aimed to calculate the most appropriate input composition to be used in DEA using health indicators of our provinces. As a model; Data-Envelopment Analysis's input-oriented model is used under the assumption of variable returns to scale. The aim of the study is to determine which variables should be used more effectively in the health variables of the provinces and the necessary health performance policy.

2 Data envelopment analysis

If there are more than one decision points for a decision maker, it is important to estimate the efficiencies of these decision points and to shape the decision based on these efficiencies. Indeed the efficiency ranking of the decision points are important for the decision maker and the decision maker wants to know how scenarios that will increase the efficiency of decision points of those with less efficiency than the others will affect the overall efficiency of the decision. Here, the Data Envelopment Analysis can be defined as a linear programming-based method that utilizes similar inputs to obtain an output or outputs to evaluate the relative efficiency of responsible decision points. The primary characteristic that sets the Data Envelopment Analysis apart from other similar purpose methods is that it enables evaluation in cases where there are many inputs and outputs. As a result of the analysis, information regarding the efficiency value of each decision point, how to increase the efficiency of decision points that are not efficient using which input/output ratios (scenarios), and decision points that can be used as a reference are obtained.

The Data Envelopment Analysis first took form in 1957 by Farrell with the Boundary Production Function suggestion that was put forward in reply to the Mean Performance scale [1]. Based on Farrell's piecewise linear convex envelopment approach to effective boundary predictions, Charnes, Cooper and Rhodes introduced the Data Envelopment Analysis (DEA) technique to the literature with their study in 1978 [2]. Linear programming is a mathematical technique that aims to determine the most optimal out of the different alternatives for the efficient use of limited resources for a certain objective. This technique is used more often in the solution of optimal resource distribution problems.

DEA, which is a mathematical programming based approach, is a method based on linear programming principles that is used to transform the input also known as a decision making unit (DMU) to an output and to measure the relative efficiency of establishments or financial institutions [3]. DEA is a linear programming process that can be defined as the boundary analysis of multiple inputs and multiple outputs. The process requires no prior determination of weighting for the inputs and outputs, and aims to determine the efficient and inefficient decision making units. In the method, which is based on the distribution of weights to the inputs and outputs of each individual DMU, after the weights have been determined the calculated efficiency scores of the DMU's cannot exceed 1.

DEA is a non-parametric linear programming based technique. In DEA, an assumption regarding the production function is not required as in parametric methods. Another feature is that it considers the boundaries rather than the central tendency; meaning that DMU's are compared not with units with mean efficiency values but rather units with full efficiency.

Data Envelopment Analysis is a very effective tool when used correctly. The advantages of Data Envelopment Analysis can be listed as follows:

- Data Envelopment Analysis is capable of processing multiple inputs and multiple outputs.
- Data Envelopment Analysis does not require a functional form that associates inputs and outputs, except for the linear form.
- With Data Envelopment Analysis, decision-making units whose activities are calculated are compared to those with relatively full effectiveness.
- Inputs and outputs may have very different units. In this case, it is not necessary to use various assumptions and make transformations in order to measure them in the same way.

Disadvantages of Data Envelopment Analysis can be listed as follows:

- Data Envelopment Analysis is very sensitive to measurement error.
- Data Envelopment Analysis is sufficient to measure the performance of decision points, but does not provide clues about the interpretation of this assessment on the basis of absolute effectiveness.
- Since Data Envelopment Analysis is a non-parametric technique, it is difficult to apply statistical hypothesis tests to the results.
- Since a separate linear programming model is required for each decision point, the solution of large-scale problems with Data Envelopment Analysis can be time consuming in terms of calculation.

2.1 Application phases of the data envelopment analysis

The main phases in the implementation of the Data Envelopment Analysis is as given below:

2.1.1 Selection of decision making units: A decision making unit is defined as a unit that analyzes the efficiency of homogenous elements such as businesses, institutions, companies and establishments that, in the DEA, produce similar outputs by use of similar inputs. Two factors affect the selection of DMU's. The first of these are that DMU's must be homogenous units that have similar aims and undertake the same task. The other is the number of DMU's. According to Ramanathan [4], the number of DMU's must be at least 2 or 3 times the total number of input and outputs. Another view states that there must be at least $m + s + 1$ DMU's, where m is number of inputs, s is the number of outputs. The number of DMU's is dependent on the aim of the DEA study. If the number of DMU's increases, the number of DMU's that determine the effective boundary also increases.

2.1.2 Selection of inputs and outputs: Input can be stated as the necessary personnel, consumable resources, capital and cash necessary for any economic or institutional system to perform its services or to enable production. Output is the products and services that result from the projects and activities of the system in question.

One of the most frequently encountered problems in the implementation of the DEA is the selection of the inputs and outputs. The selection of input and outputs are related to the personal perspective of the decision maker, there are no special rules defined for the selection of input and outputs. However, there are certain suggested rules. First of all, the input and outputs relevant to the study must be extensively listed. Subsequently, input and outputs of appropriate levels must be selected and integrated to the variable solution system. The decomposition ability of the DEA increases as a result of this process. Another problem that is encountered in the selection of input and outputs is the classification of which variable is to be an input and which variable is to be an output. Additionally, a variable can be both an input and an output. In this case, one way of variable classification is related to whether the variable in question increases the DMU's performance or not. If the variable in question increases the performance of the DMU it is used as an output, otherwise it is used as an input.

2.1.3 Selection of the data envelopment analysis model: After the selection of DMU's, inputs and outputs, the decision maker must select the DEA model most suited according the data structure and production planning. If there are uncontrollable input factors, it is more appropriate to prefer the output oriented models in which the amount of input is constant. On the other hand, if the outputs are not selected to show the best performance but rather determined according to the objectives of the decision maker, it is more appropriate to select input oriented models in which the amount of output is constant. If the inputs and outputs are desired to be specifically determined in the analysis, then the multiplicative models must be used; if the relationship between DMU's are desired to be determined, the envelopment models must be used. Another model selection criterion is whether the performance of the DMU's are dependent on the economy of scale. If the performance of the DMU's are not dependent on the economy of scale, the assumption of constant return to scale is appropriate. In other situations, variable return to scale is more appropriate.

2.1.4 Measurement of efficiency with the data envelopment analysis model: After the most appropriate DEA model for the observation set is determined, the selected model is solved for each DMU and the results regarding the efficiency values, idle variable values and the reference DMU's consisting of the efficient DMU's are obtained. The efficiency value for each DMU is between 0 and 1. The DMU's whose efficiency score is equal to 1 form the best group of the implementation.

2.1.5 Determination of the reference set: In DEA, the DMU's are compared with each other and the efficient and inefficient DMU's are determined. The set that consists of the efficient units is called the reference set. The main assumption in DEA, is that inefficient DMU's try to regularly regulate resource consumption and become efficient by taking reference to efficient DMU's. The strength of the efficient DMU's that are in the reference set is dependent on how much these units are taken reference by inefficient units.

2.1.6 Determining an objective for inefficient decision making units: Determining an objective for inefficient DMU's in order to make them efficient is one of DEA's important features. These objectives are defined as the weighted mean of the inefficient DMU's and the efficient DMU's that make up the reference set.

2.1.7 Evaluating the results of the model: As a result, a general evaluation is done considering the entire input and outputs for each DMU.

2.2 Data envelopment analysis models

DEA is a combination of ideas, thoughts and methods that are intertwined with several models. Charnes Cooper Rhodes (CCR) and Banker Charnes Cooper (BCC) models are the main two models of this method. These models can be studied in two groups as "input oriented" and "output oriented". Input and output oriented DEA models are very similar in principle. Input oriented DEA models investigate the most optimal combination of inputs in order to produce a determined output combination in the most efficient way. Output oriented DEA models investigate the highest obtainable output combination from a determined input combination.

2.2.1 Charnes, Cooper, Rhodes (CCR) models: It is the model developed by Charnes, Cooper and Rhodes in 1978 [2]. It forms the basis of the DEA. The analysis is conducted under the assumption of constant return to scale. This model makes a general evaluation of total efficiency. The total efficiency value; consists of technical efficiency and scale efficiency.

The objective function and constraints of the input oriented CCR model is as follows:

$$E_k = \max \sum_{r=1}^{s} u_r y_{rk}$$

s.t.

$$\begin{aligned}
&\sum_{i=1}^{m} v_i x_{ik} = 1\\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad (j = 1, \ldots, n)\\
&u_1, u_2, \ldots, u_s \geq 0\\
&v_1, v_2, \ldots, u_m \geq 0
\end{aligned}$$

The scores obtained from the solving of the model are the relative efficiency scales. This score being 1 means that the DMU whose efficiency is being analyzed is efficient, and the score being less than 1 means that the DMU is not efficient.

The inefficient DMU is rendered efficient in order to match the DMU's that make up the reference set. As forming reference sets in this model is difficult, the enveloping model, meaning a dual model is developed. The use of which and what amount of input and/or output for the DMU studied is observed in the enveloping model. Additionally in this method, the determination of the reference set is easier in comparison to the weighted method.

The objective function and constraints of the input oriented envelopment CCR model is provided below:

$$E_k = \min \alpha - \varepsilon \sum_{i=1}^{m} s_i^- - \varepsilon \sum_{i=1}^{s} s_r^+$$

s.t.

$$\begin{aligned}
&\sum_{j=1}^{n} X_{ij}\lambda_j + s_i^- - \alpha X_{ik} = 0, \quad i = 1, 2, \ldots, m\\
&\sum_{j=1}^{n} Y_{rj}\lambda_j - s_r^+ - Y_{rk} = 0, \quad r = 1, 2, \ldots, s\\
&\lambda_j \geq 0, j = 1, 2, \ldots, n\\
&s_i^- \geq 0, i = 1, 2, \ldots, m\\
&s_r^+ \geq 0, I = 1, 2, \ldots, s
\end{aligned}$$

Here;
α: contraction coefficient that determines how much the inputs of can be reduced for DMU k whose relative efficiency is being measured
λ_j: density value for DMU j
s_i^-: residual value for input i of DMU k
s_r^+: residual value for output r of DMU k

If the evaluated DMU is efficient, the relative efficiency level $E_k = 1$, no changes are made in the input and output vectors $\left(\alpha = 1, s^- = 0, s^+ = 0\right)$. Additionally it is found in its own reference set and $\lambda_k = 1$. If the evaluated DMU is inefficient, the contraction coefficient α that determines the level of efficiency is less than 1. This means that the input vectors can be reduced radially.

Output oriented CCR model

It investigates the highest amount of output attainable from a determined input combination. Ek being equal to 1 means that DMU k is efficient, whereas being greater than 1 means that it is inefficient.

The objective function and constraints of the output oriented CCR model is provided below:

$$E_k = \min \sum_{i=1}^{m} v_i x_{ik}$$

s.t.

$$\begin{aligned}
&\sum_{r=1}^{S} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad (j = 1, \ldots, n) \\
&\sum_{r=1}^{s} u_r y_{rk} = 1 \\
&u_1, u_2, \ldots, u_s \geq 0 \\
&v_1, v_2, \ldots, v_m \geq 0
\end{aligned}$$

As the identification of the reference sets for inefficient DMU's in this model is time-consuming, the envelopment model has been developed. The output oriented CCR envelopment model is obtained by taking the duality of the output oriented DEA model. With the help of this model, it is easy to determine which inefficient DMU's leave residuals in their inputs and outputs and define the DMU's that the inefficient DMU must refer to in order to become efficient.

$$E_k = \max \beta + \varepsilon \sum_{i=1}^{m} s_i^- + \varepsilon \sum_{r=1}^{s} s_r^+$$

s.t.

$$\begin{aligned}
&\sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- - x_{ik} = 0, \quad i = 1, 2, \ldots, m \\
&\sum_{i=1}^{n} y_{rj}\lambda_j - s_r^+ - \beta y_{rk} = 0, \quad r = 1, 2, \ldots, s \\
&\lambda_j \geq 0, \quad j = 1, 2, \ldots, n \\
&s_i^-, s_r^+ \geq 0
\end{aligned}$$

Here;

β: expansion coefficient that determines how much the outputs of can be increased for DMU k whose relative efficiency is being measured

λ: density value for DMU j

s_i^-: residual value for input i. of DMU k

s_r^+: residual value for output r. of DMU k

If the evaluated DMU is efficient, the relative efficiency level E_k is equal to 1. The efficiency level of inefficient DMU's are greater than 1. If the measured DMU is inefficient, the expansion coefficient β that determines the level of efficiency is greater than 1. This means that the output vectors can ben increased radially.

2.2.2 Banker, Charnes, Cooper (BCC) models: This model, which was put forward by R.D. Banker, A. Charnes and W.W. Cooper in 1984 and is depicted with the first letters of the surnames of these people, is based on the assumption of variable return to scale [5]. A return to scale type can also be identified for all DMU's using the BCC model. The BCC boundary is always below the CCR boundary. Therefore the CCR efficiency value is equal to or less than the BCC efficiency value.

The only difference of the BCC model from the CCR model is that under the variable return to scale assumption, the total sum of the λ_j values (the value that provides the necessary information to establish the possible efficient input output combination for an inefficient decision point) obtained from the solution of the linear program for each DMU is equal to 1 [5]. The BCC model is formed by placing the u0 variable in the input oriented model and v0 variable in the output oriented model of the CCR model. Due to the u0 and v0 variables, the BCC model is based on the assumption of variable return to scale.

Input oriented BCC model

$$E_k = \max \sum_{r=1}^{s} u_r y_{rk} - u_0$$

s.t.

$$\sum_{i=1}^{m} v_i x_{ik} = 1$$
$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - u_0 \leq 0, \quad j = 1, \ldots, n$$
$$u_1, u_2, \ldots, u_s \geq 0$$
$$v_1, v_2, \ldots, v_m \geq 0$$
$$u_0, \text{ free}$$

The input oriented BCC envelopment model is obtained with the mathematical formulation given below by taking the duality of the input oriented weighted model. Additionally as u_0 is in the objective function of the input oriented weighted BCC model, the input oriented envelopment BCC model has a $\sum_{j=1}^{n} \lambda_j = 1$ convexity constraint.

$$E_k = \min \alpha - \varepsilon \sum_{i=1}^{m} s_i^- - \varepsilon \sum_{r=1}^{s} s_r^+$$

s.t.

$$\sum_{i=1}^{n} x_{ij}\lambda_j + s_i^- - \alpha x_{ik} = 0$$
$$\sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ - y_{rk} = 0$$
$$\sum_{j=1}^{n} \lambda_j = 1$$
$$\lambda_j \geq 0, s_i^- \geq 0, s_r^+ \geq 0$$

As a result of solving this model, if the DMU in question is efficient, the relative efficiency measure E_k is equal to 1, and no change is made to the input and output vectors $\left(\alpha = 1, s^- = 0, s^+ = 0\right)$. Additionally it is found in its own reference set and $\lambda_k = 1$. If the evaluated DMU is inefficient, the contraction coefficient α that determines the level of efficiency is less than 1. This means that the input vectors can be reduced radially.

Output oriented BCC model

$$E_k = \min \sum_{i=1}^{m} v_i x_{ik} - v_0$$

s.t.

$$\sum_{r=1}^{s} u_r y_{rk} = 1$$
$$\sum_{i=1}^{m} v_i x_{ij} - \sum_{i=1}^{s} u_r y_{rj} - v_0 \geq 0, \quad j = 1, \ldots, n$$
$$u_1, u_2, \ldots, u_s \geq 0$$
$$v_1, v_2, \ldots, v_m \geq 0$$
$$v_0, \text{ free}$$

The output oriented envelopment BCC model is obtained with the mathematical formulation given below by taking the duality of the output oriented weighted model.

Additionally as v_0 is in the objective function of the output oriented weighted BCC model, the input oriented envelopment BCC model has a $\sum_{j=1}^{n} \lambda_j = 1$ convexity constraint.

The output oriented envelopment BCC model is given below:

$$E_k = \max \beta + \varepsilon \sum_{i=1}^{m} s_i^- + \varepsilon \sum_{r=1}^{s} s_r^+$$

Inputs	Total number of doctors	Number of nurses	The number of other health staff
Outputs	Number of operations	Bed occupancy rate	Crude death rate

Tablo 1 Input and Output Variables

Provinces	CCR Score	BCC Score	Provinces	CCR Score	BCC Score
Adana	0,9025	0,9423	Konya	0,7953	0,8094
Adıyaman	0,5188	0,5202	Kütahya	0,7668	0,8256
Afyonkarahisar	0,8071	0,8083	Malatya	0,7779	0,7785
Ağrı	0,5758	0,6242	Manisa	0,6839	0,807
Amasya	0,769	0,8055	Kahramanmaraş	0,746	0,747
Ankara	0,7658	0,9363	Mardin	0,7393	0,7415
Antalya	0,8821	0,9281	Muğla	0,5704	0,5795
Artvin	0,8279	0,961	Muş	0,6378	0,6968
Aydın	0,7124	0,7135	Nevşehir	0,8652	0,879
Balıkesir	0,7941	**1**	Niğde	0,5884	0,5886
Bilecik	0,8811	**1**	Ordu	0,9515	0,9544
Bingöl	0,5299	0,5795	Rize	0,8468	0,8481
Bitlis	0,8643	0,8755	Sakarya	0,8252	0,8262
Bolu	0,7576	0,7605	**Samsun**	0,7808	**1**
Burdur	0,8542	0,8638	Siirt	0,7006	0,7451
Bursa	0,7805	**1**	Sinop	0,8983	0,9662
Çanakkale	0,7023	0,7027	Sivas	0,595	0,5951
Çankırı	0,8736	**1**	Tekirdağ	0,8351	0,8428
Çorum	0,5532	0,6014	Tokat	0,7016	0,7022
Denizli	0,911	0,9142	Trabzon	0,7382	0,7451
Diyarbakır	0,6618	0,664	Tunceli	0,7244	0,9394
Edirne	0,7022	0,7042	Şanlıurfa	0,8879	0,8948
Elazığ	0,6308	0,6396	Uşak	0,9293	0,9495
Erzincan	0,6824	0,6838	Van	0,7028	0,7224
Erzurum	0,9384	0,94	Yozgat	0,4472	0,4545
Eskişehir	0,8442	0,8512	Zonguldak	0,7892	0,7908
Gaziantep	**1**	**1**	Aksaray	0,8955	0,9049
Giresun	0,6793	0,6904	**Bayburt**	**1**	**1**
Gümüşhane	0,8644	**1**	Karaman	0,9255	0,9377
Hakkari	0,6728	0,7536	Kırıkkale	0,7648	0,783
Hatay	0,833	0,8402	Batman	0,9849	0,9865
Isparta	0,8846	0,8992	Şırnak	0,7549	0,791
Mersin	0,7709	0,7754	**Bartın**	0,9629	**1**
İstanbul	**1**	**1**	**Ardahan**	**1**	**1**
İzmir	0,7695	0,9687	Iğdır	0,777	0,8799
Kars	0,6599	0,6942	**Yalova**	**1**	**1**
Kastamonu	0,6709	0,761	Karabük	0,8075	0,8591
Kayseri	0,7186	0,7194	Kilis	1	1
Kırklareli	0,6395	0,6489	Osmaniye	0,9743	0,9777
Kırşehir	0,5448	0,5572	Düzce	0,9263	0,9365
Kocaeli	0,762	0,8276			

Tablo 2 Efficiency Scores

s.t.

$$\sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- - x_{ik} = 0$$

$$\sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ - \beta y_{rk} = 0$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \geq 0, s_i^- \geq 0, s_r^+ \geq 0$$

If the evaluated DMU is efficient, the relative efficiency level E_k is equal to 1. The efficiency level of inefficient DMU's are greater than 1. If the measured DMU is inefficient, the expansion coefficient β that determines the level of efficiency is greater than 1. This means that the output vectors can ben increased radially.

The efficiency values in the BCC model are equal or greater than the CCR model. The reason for this is that the BCC model obtains a local technical efficiency value, whereas the CCR model obtains a general technical efficiency value.

Provinces	Input Reduction Rate	Reference Province and Reference Rate	Provinces	Input Reduction Rate	Reference Province and Reference Rate
1	0.05	27 (0,94) 34 (0,03) 77 (0,03)	42	0.19	27 (0,99) 34 (0,01)
2	0.47	27 (0,07) 79 (0,93)	43	0.17	27 (0,09) 77 (0,76) 79 (0,15)
3	0.19	27 (0,23) 79 (0,77)	44	0.22	27 (0,40) 79 (0,60)
4	0.38	27 (0,01) 79 (0,99)	45	0.19	10 (0,24) 27 (0,22) 34 (0,01) 55 (0,13) 77 (0,40)
5	0.19	27 (0,03) 77 (0,29) 79 (0,68)	46	0.25	27 (0,36) 79 (0,64)
6	0.06	27 (0,39) 34 (0,42) 77 (0,19)	47	0.25	27 (0,16) 79 (0,84)
7	0.07	27 (0,91) 34 (0,04) 77 (0,05)	48	0.42	27 (0,27) 69 (0,66) 77 (0,07)
8	0.03	11 (0,46) 69 (0,20) 74 (0,01) 77 (0,33)	49	0.30	27 (0,01) 79 (0,99)
9	0.29	27 (0,39) 69 (0,18) 77 (0,43)	50	0.12	27 (0,02) 79 (0,98)
12	0.42	69 (0,45) 79 (0,55)	51	0.41	27 (0,02) 69 (0,53) 77 (0,05) 79 (0,40)
13	0.13	27 (0,05) 79 (0,95)	52	0.04	27 (0,30) 77 (0,06) 79 (0,65)
14	0.24	27 (0,05) 77 (0,46) 79 (0,49)	53	0.16	27 (0,04) 77 (0,76) 79 (0,20)
15	0.13	27 (0,02) 79 (0,98)	54	0.17	27 (0,28) 69 (0,05) 77 (0,67)
17	0.29	27 (0,11) 77 (0,81) 79 (0,08)	56	0.26	27 (0,02) 79 (0,98)
19	0.40	27 (0,04) 77 (0,63) 79 (0,33)	57	0.03	69 (0,49) 77 (0,35) 79 (0,15)
20	0.09	27 (0,53) 69 (0,23) 77 (0,24)	58	0.40	27 (0,17) 69 (0,05) 77 (0,51) 79 (0,28)
21	0.34	27 (0,53) 79 (0,47)	59	0.15	27 (0,31) 69 (0,38) 77 (0,31)
22	0.30	27 (0,11) 77 (0,30) 79 (0,59)	60	0.29	27 (0,15) 79 (0,85)
23	0.36	27 (0,20) 79 (0,80)	61	0.25	27 (0,40) 77 (0,22) 79 (0,38)
24	0.31	27 (0,04) 69 (0,91) 77 (0,04)	62	0.06	69 (1,00)
25	0.06	27 (0,47) 79 (0,53)	63	0.10	27 (0,53) 79 (0,47)
26	0.15	27 (0,46) 77 (0,22) 79 (0,32)	64	0.05	27 (0,08) 77 (0,28) 79 (0,64)
28	0.31	27 (0,07) 77 (0,13) 79 (0,80)	65	0.28	27 (0,25) 79 (0,75)
30	0.25	69 (0,60) 79 (0,40)	66	0.55	27 (0,03) 79 (0,97)
31	0.16	27 (0,59) 77 (0,29) 79 (0,12)	67	0.21	27 (0,21) 77 (0,24) 79 (0,55)
32	0.10	27 (0,26) 77 (0,29) 79 (0,45)	68	0.09	27 (0,05) 79 (0,95)
33	0.23	27 (0,65) 77 (0,21) 79 (0,14)	70	0.06	27 (0,02) 79 (0,98)
35	0.03	10 (0,02) 34 (0,31) 77 (0,67)	71	0.21	27 (0,08) 69 (0,52) 77 (0,40)
36	0.31	27 (0,05) 69 (0,95)	72	0.01	27 (0,17) 79 (0,83)
37	0.24	69 (0,06) 77 (0,94)	73	0.21	27 (0,04) 69 (0,18) 79 (0,79)
38	0.28	27 (0,58) 79 (0,42)	76	0.12	69 (0,90) 79 (0,10)
39	0.35	27 (0,06) 69 (0,65) 79 (0,29)	78	0.14	27 (0,02) 77 (0,41) 79 (0,58)
40	0.44	27 (0,01) 69 (0,80) 77 (0,20)	80	0.02	27 (0,14) 79 (0,86)
41	0.17	10 (0,15) 16 (0,09) 27 (0,56) 55 (0,09) 77 (0,11)	81	0.06	27 (0,11) 69 (0,71) 77 (0,18)

Tablo 3 Inefficient Provinces, Their Input Variable Reduction Rates and the Provinces They Must Take as Reference

3 Application

This study consists of the annual health statistics of 2017 whose data sets are published by the Ministry of Health [6]. The aim of the study is to measure health performance using health statistics of Turkey. The data set is comprised of 81 provinces. The data envelopment analysis of the provincial health performance was done with the EMS program. The input oriented BCC model of the Data Envelopment Analysis was used.

The input and output variables used in the study are provided in Table 1.

Table 2 gives the total (CCR) and technical (BCC) efficiency scores of the provinces. Provinces that are efficient in CCR are also efficient in BCC. According to the CCR model, 6 provinces (Gaziantep, İstanbul, Bayburt, Ardahan, Yalova and Kilis); and according to the BCC model 13 provinces (Balıkesir, Bilecik, Bursa, Çankırı, Samsun, Bartın In addition, Table 2 shows how much inactive provinces should reduce each input. For example; Ankara needs to reduce its inputs by 7% in order to become active (given the BCC score). The mean efficiency score of the provinces was 0.82. The province with the lowest activity score and the farthest from the activity limit was Adıyaman. In order for Adıyaman to be active, its inputs must be reduced by 48%. Because the outputs obtained in response to the inputs used do not meet each other. Gümüşhane, Gaziantep, İstanbul, Bayburt, Ardahan, Yalova and Kilis) were found to be efficient.

It is possible to increase the efficiency levels of provinces that were not found to be efficient by saving on the inputs or by increasing the output levels. Information regarding the over-consumed input ratios of provinces and the level an inefficient province should take reference from which efficient province is provided in Table 3. Additionally, the last column of Table 3 gives the provinces that the inefficient provinces should take reference from.

4 Result

In this study was applied by selecting data envelopment analysis method which is one of the most widely used performance measurement methods, which can measure multiple input and output variables and carry the analysis results of decision making units to numerical values. DEA enables health managers to view production and service processes and make more effective decisions by using existing input and output variables. As a result of the analysis with DEA, the amount of input variables that are required to decrease the input variables were determined in order to become effective in the provinces below the efficiency limit. It has been determined which active provinces should take reference as well as the actions that inefficient provinces need to do in order to become effective.This study aims to evaluate the health performance of our provinces and the Data Envelopment Analysis method which is one of the non-parametric measurement methods is used. It was concluded that many of our provinces are not at a good level in terms of health performance. In terms of technical efficiency, 13 provinces were efficient whereas 68 provinces were below the efficiency limit and the mean was determined to be 0.82. As the input oriented model was preferred to evaluate performance, the conclusion was that the input variables used in the model must be reduced by an average of 0.17. Although the low amount of resources allocated to the Turkish health sector is known and accepted, this study has revealed the importance of resource consumption in hospitals and the necessity that they must be examined. Proper employment policies, managers placing emphasis on optimal use of resources and an effective auditing mechanism will prevent the wasting of resources. Finally, our recommendation; the efficiency levels of health care providers should be continuously examined by the Ministry of Health and related persons and organizations. The reasons for ineffectiveness should be identified and preventive actions should be initiated. It depends on the correct acquisition of activity and financial data in order to make accurate performance measurements in health institutions. Although the data is collected and analyzed regularly, these data must be properly analyzed and have the human resources to take the necessary steps.

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Coefficient Bounds for a Subclass of m-fold Symmetric Bi-univalent Functions Involving Hadamard Product and Differential Operator

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Abstract: In this study, we construct a new subclass of m-fold symmetric bi-univalent functions using by Hadamard product and generalized Salagean differential operator in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. We establish upper bounds for the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ belonging to this new class. The results presented here generalize some of the earlier studies.

Keywords: Bi-univalent functions, Coefficient estimates, m-fold symmetric functions.

1 Introduction

Let A be the family of analytic functions, normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the following form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

in the open unit disk D. We also denote by S the subclass of functions in A which are univalent in U (see for details [4]).

According to the *Koebe-One Quarter Theorem* [4], it provides that the image of U under every univalent function $f \in A$ contains a disk of radius $1/4$. Thus every univalent function $f \in A$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$ and $f\left(f^{-1}(w)\right) = w$ $\left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right)$, where

$$F(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions in U given by (1). The detailed information about the class of Σ was given in the references [2], [6], [7] and [10].

The Hadamard product or convolution of two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in A$, denoted by $f * g$, is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \quad (z \in U).$$

For $\delta \geq 1$ and $f \in A$, Al-Obodi [1] introduced the following differential operator:

$$\begin{aligned} &D_\delta^0 f(z) = f(z), \\ &D_\delta^1 f(z) = (1-\delta) f(z) + \delta z f'(z) = D_\delta f(z), \\ &\vdots \\ &D_\delta^n f(z) = (1-\delta) D_\delta^{n-1} f(z) + \delta z \left(D_\delta^{n-1} f(z)\right)' = D\left(D_\delta^n f(z)\right) \quad \left(z \in U,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\right). \end{aligned} \tag{3}$$

If f is given by (1), we see that

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k$$

with $D_\delta^n f(0) = 0$. It is worthy mentioning that when $\delta = 1$ in (3), we have the differential operator of Salagean [9].

Let m be a positive integer. A domain E is said to be m-fold symmetric if a rotation of E about the origin through an angle $2\pi/m$ carries E on itself. It follows that, a function f analytic in U is said to be m-fold symmetric if

$$f(e^{2\pi i/m}z) = e^{2\pi i/m}f(z).$$

A function is said to be *m*-fold symmetric if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in U, \ \ m \in \mathbb{N}). \tag{4}$$

Let S_m the class of *m*-fold symmetric univalent functions in U, which are normalized by the series expansion (4). In fact, the functions in the class S are *one*-fold symmetric. Analogous to the concept of *m*-fold symmetric univalent functions, we here introduced the concept of *m*-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an *m*-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of f is given as in (4) and the series expansion for f^{-1}, which has been recently proven by Srivastava et al. [9], is given as follows:

$$\begin{aligned} F(w) = f^{-1}(w) = \quad & w \ - a_{m+1}w^{m+1} + \left[\left(m+1 \right) a_{m+1}^2 - a_{2m+1} \right) \right] w^{2m+1} \\ & - \left[\tfrac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1} \right] w^{3m+1} + \cdots . \end{aligned}$$

We denote by Σ_m the class of *m*-fold symmetric bi-univalent functions in U. For $m = 1$, the formula (4) coincides with the formula (2) of the class Σ. Some examples of *m*-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1 - z^m} \right)^{\frac{1}{m}}, \ \ \left[-\log(1 - z^m) \right]^{\frac{1}{m}}, \ \ \left[\frac{1}{2} \log \left(\frac{1 + z^m}{1 - z^m} \right)^{\frac{1}{m}} \right].$$

The coefficient problem for m-fold symmetric analytic bi-univalent functions is one of the favourite subjects of Geometric Function Theory in these days, (see, e.g., [3], [5], [11], [12]).

Here, the aim of this study is to determine upper coefficients bounds $|a_{m+1}|$ and $|a_{2m+1}|$ belonging to the newly defined subclass.

Firstly, in order to derive our main results, we require the following lemma.

Lemma 1. *(See [8]) If a function $p \in P$ is given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in U),$$

then $|c_i|$ for each $i \in \mathbb{N}$, where the Caratheodory class P is the family of all functions p analytic in U for which $\Re(p(z)) > 0$ and $p(0) = 1$.

2 Coefficient bounds for the functions class $\Sigma_m^{t,n,\delta}(\tau, \alpha, \lambda)$

Definition 1. *A function f given by (4) is said to be in the class*

$$\Sigma_m^{t,n,\delta}(\tau, \alpha, \lambda) \quad (\tau \in \mathbb{C}\backslash\{0\}, 0 < \alpha \leq 1, \lambda > 0, t, n \in \mathbb{N}_0, t > n, \delta \geq 1, z, w \in U)$$

if the following conditions are satisfied:

$$f \in \Sigma_m, \quad \left| \arg \left(1 + \frac{1}{\tau} \left[(1-\alpha) \frac{D_\delta^n (f * g)(z)}{D_\delta^t (f * h)(z)} + \alpha \frac{(D_\delta^n (f * g)(z))'}{\left(D_\delta^t (f * h)(z) \right)'} - 1 \right] \right) \right| < \frac{\alpha\pi}{2} \tag{5}$$

and

$$\left| \arg \left(1 + \frac{1}{\tau} \left[(1-\alpha) \frac{D_\delta^n (F * g)(w)}{D_\delta^t (F * h)(w)} + \alpha \frac{(D_\delta^n (F * g)(w))'}{\left(D_\delta^t (F * h)(w) \right)'} - 1 \right] \right) \right| < \frac{\alpha\pi}{2}, \tag{6}$$

where $g(z) = z + \sum_{k=1}^{\infty} g_{mk+1} z^{mk+1}$, $h(z) = z + \sum_{k=1}^{\infty} h_{mk+1} z^{mk+1}$ and the function F is extension of f^{-1} to U.

We start by finding the estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for the functions in the $\Sigma_m^{t,n,\delta}(\tau, \alpha, \lambda)$.

Theorem 1. *Let the function f given by (4) be in the class $\Sigma_m^{t,n,\delta}(\tau,\alpha,\lambda)$. Then*

$$|a_{m+1}| \leq \frac{2\,|\tau|\,\lambda}{\sqrt{|A|}}$$

and

$$|a_{2m+1}| \leq \frac{2\,|\tau|\,\lambda}{(1+2m\alpha)\,|(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}|}+\frac{2(m+1)\tau^2\lambda^2}{|A|},$$

where

$$A=\tau\lambda(1+m)(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}\right]$$
$$-2\tau\lambda(1+2m\alpha+m^2\alpha)\left[(1+m\delta)^{t+n}h_{m+1}g_{m+1}-(1+m\delta)^{2t}h_{m+1}^2\right]-(\lambda-1)(1+m\alpha)^2\left[(1+m\delta)^n g_{m+1}-(1+m\delta)^t h_{m+1}\right]^2.$$

Proof: Suppose that $\Sigma_m^{t,n,\delta}(\tau,\alpha,\lambda)$. From the conditions (5) and (6), we can write

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{D_\delta^n(f*g)(z)}{D_\delta^t(f*h)(z)}+\alpha\frac{\left(D_\delta^n(f*g)(z)\right)'}{\left(D_\delta^t(f*h)(z)\right)'}-1\right]=[p(z)]^\lambda, \tag{7}$$

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{D_\delta^n(F*g)(w)}{D_\delta^t(F*h)(w)}+\alpha\frac{\left(D_\delta^n(F*g)(w)\right)'}{\left(D_\delta^t(F*h)(w)\right)'}-1\right]=[q(w)]^\lambda, \tag{8}$$

where $F=f^{-1}$, p,q in P and have the following forms

$$p(z)=1+p_m z^m+p_{2m}z^{2m}+\cdots,$$

$$q(w)=1+q_m w^m+q_{2m}w^{2m}+\cdots.$$

Clearly, we deduce that

$$[p(z)]^\lambda=1+\lambda p_m z^m+\left(\lambda p_{2m}+\frac{\lambda(\lambda-1)}{2}p_m^2\right)z^{2m}+\cdots,$$

$$[q(w)]^\lambda=1+\lambda q_m w^m+\left(\lambda q_{2m}+\frac{\lambda(\lambda-1)}{2}q_m^2\right)w^{2m}+\cdots.$$

Additionaly,

$$1+\tfrac{1}{\tau}\left[(1-\alpha)\tfrac{D_\delta^n(f*g)(z)}{D_\delta^t(f*h)(z)}+\alpha\tfrac{\left(D_\delta^n(f*g)(z)\right)'}{\left(D_\delta^t(f*h)(z)\right)'}-1\right]=1+\frac{(1+m\alpha)}{\tau}\left[(1+m\delta)^n g_{m+1}-(1+m\delta)^t h_{m+1}\right]a_{m+1}z^m+$$
$$\frac{\left\{(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}\right]a_{2m+1}-(1+2m\alpha+m^2\alpha)\left[(1+m\delta)^{t+n}h_{m+1}g_{m+1}-(1+m\delta)^{2t}h_{m+1}^2\right]a_{m+1}^2\right\}}{\tau}z^{2m}+\cdots$$

and

$$1+\tfrac{1}{\tau}\left[(1-\alpha)\tfrac{D_\delta^n(F*g)(w)}{D_\delta^t(F*h)(w)}+\alpha\tfrac{\left(D_\delta^n(F*g)(w)\right)'}{\left(D_\delta^t(F*h)(w)\right)'}-1\right]=1-\frac{(1+m\alpha)}{\tau}\left[(1+m\delta)^n g_{m+1}-(1+m\delta)^t h_{m+1}\right]a_{m+1}w^m+$$
$$\frac{\left\{(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}\right]\left[(1+m)a_{m+1}^2-a_{2m+1}\right]-(1+2m\alpha+m^2\alpha)\left[(1+m\delta)^{t+n}h_{m+1}g_{m+1}-(1+m\delta)^{2t}h_{m+1}^2\right]a_{m+1}^2\right\}}{\tau}w^{2m}$$

Now, equating the coefficients in (7) and (8), we have

$$(1+m\alpha)\left[(1+m\delta)^n g_{m+1}-(1+m\delta)^t h_{m+1}\right]=\tau\lambda p_m, \tag{9}$$

$$(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}\right]a_{2m+1}$$
$$-(1+2m\alpha+m^2\alpha)\left[(1+m\delta)^{t+n}h_{m+1}g_{m+1}-(1+m\delta)^{2t}h_{m+1}^2\right]a_{m+1}^2=\tau\left(\lambda p_{2m}+\tfrac{\lambda(\lambda-1)}{2}p_m^2\right), \tag{10}$$

$$m(1-\lambda)\left[2a_{2m+1}-(\lambda m+1)a_{m+1}^2\right]=\tau\left(\lambda p_{2m}+\frac{\lambda(\lambda-1)}{2}p_m^2\right)$$

and

$$-(1+m\alpha)\left[(1+m\delta)^n g_{m+1}-(1+m\delta)^t h_{m+1}\right]=\tau\lambda q_m, \tag{11}$$

$$\begin{aligned}&(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}\right]\left[(1+m)a_{m+1}^2-a_{2m+1}\right]\\&-(1+2m\alpha+m^2\alpha)\left[(1+m\delta)^{t+n}h_{m+1}g_{m+1}-(1+m\delta)^{2t}h_{m+1}^2\right]a_{m+1}^2=\tau\left(\lambda q_{2m}+\tfrac{\lambda(\lambda-1)}{2}q_m^2\right).\end{aligned}\tag{12}$$

From (9) and (11), we obtain

$$p_m=-q_m,\tag{13}$$

$$2(1+m\alpha)^2\left[(1+m\delta)^n g_{m+1}-(1+m\delta)^t h_{m+1}\right]a_{m+1}^2=\tau^2\lambda^2(p_m^2+q_m^2).\tag{14}$$

Next, by adding Eqs. (10) and (12), we obtain

$$\begin{aligned}&\left\{(1+m)(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}\right]\right.\\&\left.-2(1+2m\alpha+m^2\alpha)\left[(1+m\delta)^{t+n}h_{m+1}g_{m+1}-(1+m\delta)^{2t}h_{m+1}^2\right]\right\}a_{m+1}^2=\tau\left(\lambda\left(p_{2m}+q_{2m}\right)+\tfrac{\lambda(\lambda-1)}{2}(p_m^2+q_m^2)\right).\end{aligned}$$

Therefore, from (14), we get

$$a_{m+1}^2=\frac{\tau^2\lambda^2\left(p_{2m}+q_{2m}\right)}{A},\tag{15}$$

where

$$\begin{aligned}&A=\tau\lambda(1+m)(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}\right]\\&-2\tau\lambda(1+2m\alpha+m^2\alpha)\left[(1+m\delta)^{t+n}h_{m+1}g_{m+1}-(1+m\delta)^{2t}h_{m+1}^2\right]-(\lambda-1)(1+m\alpha)^2\left[(1+m\delta)^n g_{m+1}-(1+m\delta)^t h_{m+1}\right]^2.\end{aligned}$$

Now taking the absolute value of (15) and appying Lemma 1 for the coefficients p_{2m} and q_{2m}, we have the following inequality

$$|a_{m+1}|\le\frac{2\,|\tau|\,\lambda}{\sqrt{|A|}}.$$

Next, so as to obtain solution of the coefficient bound on $|a_{2m+1}|$, we subtract (12) from (10). We thus have

$$(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}\right]\left[2a_{2m+1}-(1+m)a_{m+1}^2\right]=\tau\left(\lambda\left(p_{2m}-q_{2m}\right)+\frac{\lambda(\lambda-1)}{2}(p_m^2-q_m^2)\right).\tag{16}$$

Also using (15) in (16) we obtain that

$$a_{2m+1}=\frac{\tau\lambda\left(p_{2m}-q_{2m}\right)}{2(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}\right]}+\frac{(m+1)\tau^2\lambda^2\left(p_{2m}+q_{2m}\right)}{2A}.\tag{17}$$

Taking the absolute value of (17) and applying Lemma 1.1 again for coefficients p_{2m}, p_m and q_{2m}, q_m we get the desired result. This completes the proof of Theorem 1. □

3 Concluding remark

Various choices of the functions h, g as mentioned above and by specializing on the parameters m, τ, t, n, δ we state some interesting results analogous to Theorem 1. The details involved may be left as an exercise for the interested reader.

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On $(P,Q)-$Lucas Polynomial Coefficients for a New Class of Bi-Univalent Functions Associated with q-Analogue of Ruscheweyh Operator

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Abstract: Recently, Fibonacci polynomials, Chebyshev polynomials, Lucas polynomials, Pell polynomials, LucasâĂŞLehmer polynomials, orthogonal polynomials and other special polynomials became more and more important in the field of Geometric Function Theory. The Theory of Geometric Functions and that of Special Functions are usually considered as very different fields. In this study, by using Lucas polynomials of the second kind, subordination and Ruschewey differential operator,these different fields were connected and a new class of bi-univalent functions was introduced. Also coefficient estimates were obtained for this new class.

Keywords: (P,Q)-Lucas polynomials, Coefficient bounds, Bi-univalent functions, Ruscheweyh differential operator, Subordination

1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \tag{1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$, and let $S = \{f \in A : f \text{ is univalent in } U\}$.

The Koebe one-quarter theorem [11] states that the range of every function $f \in S$ contains the disc of radius $\{w : |w| < \frac{1}{4}\}$. Thus every such function $f \in S$ has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

Definition 1. *If both f and f^{-1} are univalent in U, then a function $f \in A$ is said to be bi-univalent in U. We say that f is in the class Σ for such functions.*

Some functions in the class Σ are given in [23]. In 1986, Brannan and Taha [9] introduced certain subclasses of the bi-univalent function class similar to the familiar subclasses of starlike and convex functions of order. In 2012, Ali et al. [22] widen the result of Brannan and Taha by using subordination. The estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor Maclaurin series expansion (1) were found in several recent studies (see [1]-[6], [17], [19]-[20]) and still an interest to many researchers.

Definition 2. *For analytic functions f and g , f is said to be subordinate to g, denoted*

$$f(z) \prec g(z), \tag{3}$$

if there is an analytic function w such that

$$w(0) = 0\ ,\ |w(z)| < 1 \text{ and } f(z) = g(w(z))$$

Definition 3. *([14, 15]) For $q \in (0,1)$, the q-derivative of function $f \in \mathcal{A}$ is defined by*

$$\partial_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, z \neq 0 \tag{4}$$

and

$$\partial_q f(0) = f'(0).$$

Thus we have

$$\partial_q f(z) = 1 + \sum_{k=2}^{\infty} [k,q]\, a_k z^{k-1} \tag{5}$$

where $[k,q]$ is given by

$$[k,q] = \frac{1-q^k}{1-q}, \quad [0,q] = 0 \tag{6}$$

and the q-fractional is defined by

$$[k,q]! = \begin{cases} \prod\limits_{m=1}^{k} [m,q]\,, & k \in \mathbb{N} \\ 1, & k = 0 \end{cases}. \tag{7}$$

Also, the $q-$generalized Pochhammer symbol for $\mathfrak{p} \geq 0$ is given by

$$[\mathfrak{p},q]_k = \begin{cases} \prod\limits_{m=1}^{k} [\mathfrak{p}+m-1,q]\,, & k \in \mathbb{N} \\ 1, & k = 0 \end{cases}.$$

In addition, as $q \to 1$, we have $[k,q] \to k$. If we choose the function $g(z) = z^k$, then we have

$$\partial_q g(z) = \partial_q z^k = [k,q]\, z^{k-1} = g'(z),$$

where g' is the ordinary derivative.

Now, we point out the q-analogue of Ruscheweyh operator:

Definition 4. *[10] Let $f \in \mathcal{A}$. The $q-$analogue of Ruscheweyh operator is defined by*

$$\mathcal{R}_q^{\mu} f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\mu-1,q]!}{[k,q]!\,[k-1,q]!} a_k z^k, \tag{8}$$

where $[k,q]!$ is given by equation (7).

From the definition we observe that if $q \to 1$, we have

$$\lim_{q\to 1} \mathcal{R}_q^{\mu} f(z) = z + \lim_{q\to 1} \sum_{k=2}^{\infty} \frac{[k+\mu-1,q]!}{[\mu,q]!\,[k-1,q]!} a_k z^k = z + \sum_{k=2}^{\infty} \frac{[k+\mu-1]!}{[\mu]!\,[k-1]!} a_k z^k = \mathcal{R}^{\mu} f(z), \tag{9}$$

where $\mathcal{R}_q^{\mu} f(z)$ is Ruscheweyh differential operator defined in [29].

Some of special polynomials, for example Fibonacci polynomials, Lucas polynomials, Chebyshev polynomials, Pell polynomials, Lucas–Lehmer polynomials, orthogonal polynomials and the other special polynomials, are of great importance in several papers from a theoretical point of view (see, for example [7, 8, 12, 13, 18, 24–28]).

Definition 5. *[16] Let calP(x) and calQ(x) are polynomials with real coefficients. The $($calP,calQ$)$ Lucas polynomials $L_{\mathcal{P},\mathcal{Q},m}(x)$ are defined by the reccurence relation*

$$L_{\mathcal{P},\mathcal{Q},m}(x) = \mathcal{P}(x)L_{\mathcal{P},\mathcal{Q},m-1}(x) + \mathcal{Q}(x)L_{\mathcal{P},\mathcal{Q},m-2}(x) \quad (m \geq 2), \tag{10}$$

from which the first few Lucas polynomials can be found as

$$\begin{aligned} L_{\mathcal{P},\mathcal{Q},0}(x) &= 2, \\ L_{\mathcal{P},\mathcal{Q},1}(x) &= \mathcal{P}(x), \\ L_{\mathcal{P},\mathcal{Q},2}(x) &= \mathcal{P}^2(x) + 2\mathcal{Q}(x), \\ L_{\mathcal{P},\mathcal{Q},3}(x) &= \mathcal{P}^3(x) + 3\mathcal{P}(x)\mathcal{Q}(x) \end{aligned} \tag{11}$$

In this article, we aim at introducing a new class of bi-univalent functions defined through the $(P, Q)-$Lucas polynomials of the second kind.

Definition 6. *[16] Let $\mathcal{G}_{\{L_m(x)\}}(z)$ be the generating function of the $(P, Q)-$Lucas polynomial sequence $L_{\mathcal{P},\mathcal{Q},m}(x)$. Then*

$$\mathcal{G}_{\{L_m(x)\}}(z) = \sum_{m=0}^{\infty} L_{\mathcal{P},\mathcal{Q},m}(x)z^m = \frac{2 - \mathcal{P}(x)z}{1 - \mathcal{P}(x)z - \mathcal{Q}(x)z^2}. \tag{12}$$

2 The class $\mathbf{Q}^{\Sigma}(q, \mu; x)$

We begin this section by defining the class calQ$^{\Sigma}(q, \mu; x)$ and by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this class.

Definition 7. *The function f is said to be in the class $Q^{\Sigma}(q, \mu; x)$ if the following conditions are satisfied:*

$$\frac{z\partial_q(\mathcal{R}_q^{\mu} f(z))}{\mathcal{R}_q^{\mu} f(z)} \prec \mathcal{G}_{\{L_{\mathcal{P},\mathcal{Q},m}(x)\}}(z) - 1$$

and

$$\frac{w\partial_q(\mathcal{R}_q^{\mu} f(w))}{\mathcal{R}_q^{\mu} f(w)} \prec \mathcal{G}_{\{L_{\mathcal{P},\mathcal{Q},m}(x)\}}(w) - 1 \tag{13}$$

where $\mathcal{R}_q^{\mu} f(z)$ is Ruscheweyh differential operator defined in [29].

Theorem 1. *Let f given by (1) be in the class $Q^{\Sigma}(q, \mu; x)$. Then,*

$$|a_2| \leq \frac{|\mathcal{P}(x)| \sqrt{2 |\mathcal{P}(x)|}}{\sqrt{q\,[q+1, q] \left| \left\{ 2\,[\mu+2]_q - [\mu+1]_q\,(3q+1) \right\} \mathcal{P}^2(x) - 4q\,[\mu+1]_q\, \mathcal{Q}(x) \right|}} \tag{14}$$

and

$$|a_3| \leq \frac{\mathcal{P}^2(x)}{(1+\delta)^{2m}\,(1+\zeta)^2} + \frac{|\mathcal{P}(x)|}{(1+2\delta)^m\,(1+2\zeta)}. \tag{15}$$

Proof: Let $f \in \mathcal{Q}^{\Sigma}(q, \mu; x)$. Then from Definition 7, for some analytic functions Ω, Λ such that $\Omega(0) = \Lambda(0) = 0$ and $|\Omega(z)| < 1, |\Lambda(w)| < 1$ for all $z, w \in U$, we can write

$$\frac{z\partial_q(\mathcal{R}_q^{\mu} f(z))}{\mathcal{R}_q^{\mu} f(z)} = \mathcal{G}_{\{L_{\mathcal{P},\mathcal{Q},m}(x)\}}(\Omega(z)) - 1 \tag{16}$$

and

$$\frac{w\partial_q(\mathcal{R}_q^{\mu} f(w))}{\mathcal{R}_q^{\mu} f(w)} = \mathcal{G}_{\{L_{\mathcal{P},\mathcal{Q},m}(x)\}}(\Lambda(w)) - 1 \tag{17}$$

or equivalently

$$\frac{z\partial_q(\mathcal{R}_q^{\mu} f(z))}{\mathcal{R}_q^{\mu} f(z)} = -1 + L_{\mathcal{P},\mathcal{Q},0}(x) + L_{\mathcal{P},\mathcal{Q},1}(x)\Omega(z) + L_{\mathcal{P},\mathcal{Q},2}(x)\Omega^2(z) + \cdots \tag{18}$$

and

$$\frac{w\partial_q(\mathcal{R}_q^{\mu} f(w))}{\mathcal{R}_q^{\mu} f(w)} = -1 + L_{\mathcal{P},\mathcal{Q},0}(x) + L_{\mathcal{P},\mathcal{Q},1}(x)\Lambda(w) + L_{\mathcal{P},\mathcal{Q},2}(x)\Lambda^2(w) + \cdots \tag{19}$$

From the equalities (18)and (19), we obtain that

$$\frac{z\partial_q(\mathcal{R}_q^{\mu} f(z))}{\mathcal{R}_q^{\mu} f(z)} = 1 + L_{\mathcal{P},\mathcal{Q},1}(x)l_1 z + \left[L_{\mathcal{P},\mathcal{Q},1}(x)l_2 + L_{\mathcal{P},\mathcal{Q},2}(x)l_1^2 \right] z^2 + \cdots \tag{20}$$

and

$$\frac{w\partial_q(\mathcal{R}_q^{\mu} f(w))}{\mathcal{R}_q^{\mu} f(w)} = 1 + L_{\mathcal{P},\mathcal{Q},1}(x)r_1 w + \left[L_{\mathcal{P},\mathcal{Q},1}(x)r_2 + L_{\mathcal{P},\mathcal{Q},2}(x)r_1^2 \right] w^2 + \cdots \tag{21}$$

It is known before that if for $z, w \in U$,

$$\Omega(z) = \left| \sum_{i=1}^{m} l_i z^i \right| < 1$$

and

$$\Lambda(w)=\left|\sum_{i=1}^{m} r_i w^i\right|<1$$

than

$$|l_i|<1$$

and

$$\Lambda(w)=|r_i|<1$$

where $i\in\mathbb{N}$. Also, we can write

$$\frac{z\partial_q(\mathcal{R}_q^{\mu}f(z))}{\mathcal{R}_q^{\mu}f(z)}=1+q\left[\mu+1\right]_q a_2z+\left\{q\left[\mu+1\right]_q\left[\mu+2\right]_q a_3-q\left[\mu+1\right]_q^2 a_2^2\right\}z^2+\cdots,$$

and

$$\frac{w\partial_q(\mathcal{R}_q^{\mu}f(w))}{\mathcal{R}_q^{\mu}f(w)}=1-q\left[\mu+1\right]_q a_2w+\left\{-q\left[\mu+1\right]_q\left[\mu+2\right]_q a_3+q\left[\mu+1\right]_q\left(2\left[\mu+2\right]_q-\left[\mu+1\right]_q\right)a_2^2\right\}w^2+\cdots.$$

Now, comparing the corresponding coefficients in (20)and (21), we get

$$q\left[\mu+1\right]_q a_2 = L_{\mathcal{P},\mathcal{Q},1}(x)l_1, \tag{22}$$

$$q\left[\mu+1\right]_q\left[\mu+2\right]_q a_3-q\left[\mu+1\right]_q^2 a_2^2 = L_{\mathcal{P},\mathcal{Q},1}(x)l_2+L_{\mathcal{P},\mathcal{Q},2}(x)l_1^2, \tag{23}$$

$$-q\left[\mu+1\right]_q a_2 = L_{\mathcal{P},\mathcal{Q},1}(x)r_1, \tag{24}$$

$$\begin{aligned}&-\ q\left[\mu+1\right]_q\left[\mu+2\right]_q a_3+q\left[\mu+1\right]_q\left(2\left[\mu+2\right]_q-\left[\mu+1\right]_q\right)a_2^2\\ &\quad= L_{\mathcal{P},\mathcal{Q},1}(x)r_2+L_{\mathcal{P},\mathcal{Q},2}(x)r_1^2.\end{aligned} \tag{25}$$

From (22)and (24)

$$l_1=-r_1, \tag{26}$$

$$2q^2\left[\mu+1\right]_q^2 a_2^2=L_{\mathcal{P},\mathcal{Q},1}^2(x)\left(l_1^2+r_1^2\right). \tag{27}$$

Adding (23)and (25) we get

$$2q^{\mu+2}\left[\mu+1\right]_q a_2^2=L_{\mathcal{P},\mathcal{Q},1}(x)\left(l_2+r_2\right)+L_{\mathcal{P},\mathcal{Q},2}(x)\left(l_1^2+r_1^2\right) \tag{28}$$

By using (27) in (28) we have

$$\left[2L_{\mathcal{P},\mathcal{Q},1}^2(x)q^{\mu+2}\left[\mu+1\right]_q-2L_{\mathcal{P},\mathcal{Q},2}(x)q^2\left[\mu+1\right]_q^2\right]a_2^2=L_{\mathcal{P},\mathcal{Q},1}^3(x)\left(l_2+r_2\right) \tag{29}$$

which gives

$$|a_2|\leq\frac{|\mathcal{P}(x)|\sqrt{|\mathcal{P}(x)|}}{\sqrt{\left|\left[q^{\mu+2}\left[\mu+1\right]_q-q^2\left[\mu+1\right]_q^2\right]\mathcal{P}^2(x)-q^2\left[\mu+1\right]_q^2\mathcal{Q}(x)\right|}}.$$

Also, by subtructing (25) from (23) , we get

$$\left(2q\left[\mu+1\right]_q\left[\mu+2\right]_q\right)\left(a_3-a_2^2\right)=L_{\mathcal{P},\mathcal{Q},1}(x)\left(l_2-r_2\right). \tag{30}$$

Then, by using (26) and (27) in (30), we have

$$a_3=\frac{L_{\mathcal{P},\mathcal{Q},1}^2(x)\left(l_1^2+r_1^2\right)}{2q^2\left[\mu+1\right]_q^2}+\frac{L_{\mathcal{P},\mathcal{Q},1}(x)\left(l_2-r_2\right)}{2q\left[\mu+1\right]_q\left[\mu+2\right]_q}$$

and by the help of (9), we conclude that

$$|a_3|\leq\frac{\mathcal{P}^2(x)}{q^2\left[\mu+1\right]_q^2}+\frac{|\mathcal{P}(x)|}{q\left[\mu+1\right]_q\left[\mu+2\right]_q}.$$

□

Remark 1. *Choosing $\mu=0$ in Theorem 8, we obtain following corollary*

Corollary 1. *Let* $f \in \mathcal{Q}^{\Sigma}(q, 0; x) = \mathcal{Q}^{\Sigma}(q; x)$. *Then,*

$$|a_2| \leq \frac{|\mathcal{P}(x)| \sqrt{|\mathcal{P}(x)|}}{\sqrt{q^2 \left|-2\mathcal{Q}(x)\right|}} \tag{31}$$

and

$$|a_3| \leq \frac{\mathcal{P}^2(x)}{q^2} + \frac{|\mathcal{P}(x)|}{q(1+q)}. \tag{32}$$

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Stability of an Iterative Algorithm

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Abstract: We prove that iterative algorithm (1.7) of [7] is weak w^2-stable w.r.t. an operator T in the class of weak contraction mappings.

Keywords: Fixed point, Iterative algorithm, Stability.

1 Introduction

Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a mapping. An element x in C is said to be a fixed point of T if $Tx = x$.

Definition 1. *([1]) Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be weak-contraction if there exist $\delta \in [0, 1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx), \text{ for all } x, y \in M.$$

Theorem 1. *([1]) Let (M, d) be a complete metric space and $T : M \to M$ a weak-contraction for which there exist $\delta \in [0, 1)$ and $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \delta d(x, y) + L_1 d(x, Tx), \text{ for all } x, y \in M. \tag{1}$$

Then, T has a unique fixed point.

Definition 2. *([2]) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in C. We say that these sequences are equivalent if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Definition 3. *([3]) Let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence produced by operator T*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = f(T, x_n), \ n \in \mathbb{N}, \end{cases}$$

where f is a function. Assume that $\{x_n\}_{n=1}^{\infty}$ converges to a $p^* = Tp^*$. If for any equivalent sequence $\{y_n\}_{n=1}^{\infty} \subseteq C$ of $\{x_n\}_{n=1}^{\infty}$,

$$\lim_{n\to\infty} \|y_{n+1} - f(T, y_n)\| = 0 \Rightarrow \lim_{n\to\infty} y_n = p^*,$$

then the iterative sequence $\{x_n\}_{n=1}^{\infty}$ is said to be weak w^2-stable w.r.t. T.

Over the past few decades, many research papers are devoted to the study of stability of various well-known iterative algorithms for different classes of operators, see, e.g. [3–6] and references therein.

Recently, Karakaya et al. [7] introduced a three-step iterative algorithm as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = Ty_n, \\ y_n = (1 - \alpha_n) z_n + \alpha_n T z_n, \\ z_n = T x_n, \ n \in \mathbb{N}, \end{cases} \tag{2}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0, 1]$.

Karakaya et al. [7] showed that iterative algorithm (2) strongly converges to the fixed points of weak-contraction mappings. More precisely, they proved the following result.

Theorem 2. *([7]) Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a weak-contraction satisfying condition (1). Let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence generated by (2) with real sequence $\{\alpha_n\}_{n=1}^{\infty} \subseteq [0, 1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}_{n=1}^{\infty}$ converges to a unique fixed point p^* of T.*

2 Main result

Here, we prove that iterative sequence generated by (2) is weak w^2-stable w.r.t. a weak-contraction mapping T satisfying condition (1).

Theorem 3. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ with $p^* = Tp^*$ a weak-contraction satisfying condition (1). Let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence generated by (2) with real sequence $\{\alpha_n\}_{n=1}^{\infty} \subseteq [0,1]$ satisfying $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. Let $\{r_n\}_{n=1}^{\infty}$ be an equivalent sequence of $\{x_n\}_{n=1}^{\infty}$ and $\epsilon_n = \|r_{n+1} - Ts_n\|$, $s_n = (1-\alpha_n)p_n + \alpha_n Tp_n$, $p_n = Tr_n$ for all $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} \epsilon_n = 0$. Then, the sequence $\{x_n\}_{n=1}^{\infty}$ is weak w^2-stable w.r.t. T.*

Proof: From (1) and (2), we have

$$\begin{aligned}
\left\|r_{n+1} - p^*\right\| &= \|r_{n+1} - x_{n+1}\| + \left\|x_{n+1} - p^*\right\| \\
&\leq \|r_{n+1} - Ts_n\| + \|Ts_n - x_{n+1}\| + \left\|x_{n+1} - p^*\right\| \\
&\leq \epsilon_n + \|Ts_n - Ty_n\| + \left\|x_{n+1} - p^*\right\| \\
&\leq \epsilon_n + \delta \|s_n - y_n\| + L\|y_n - Ty_n\| + \left\|x_{n+1} - p^*\right\| \\
&\leq \epsilon_n + \delta \|(1-\alpha_n)(p_n - z_n) + \alpha_n (Tp_n - Tz_n)\| + L\left\|y_n - p^*\right\| \\
&\quad + L\left\|p^* - Ty_n\right\| + \left\|x_{n+1} - p^*\right\| \\
&\leq \epsilon_n + \delta\{(1-\alpha_n)\|p_n - z_n\| + \alpha_n\|Tp_n - Tz_n\|\} \\
&\quad + (1+\delta)L\left\|y_n - p^*\right\| + \left\|x_{n+1} - p^*\right\| \\
&\leq \epsilon_n + \delta\{(1-\alpha_n)\|Tr_n - Tx_n\| + \alpha_n\|Tp_n - Tz_n\|\} \\
&\quad + (1+\delta)L\left\|(1-\alpha_n)z_n + \alpha_n Tz_n - p^*\right\| + \left\|x_{n+1} - p^*\right\| \\
&\leq \epsilon_n + \delta\{(1-\alpha_n)\|Tr_n - Tx_n\| + \alpha_n\|Tp_n - Tz_n\|\} \\
&\quad + (1+\delta)L[1-\alpha_n(1-\delta)]\left\|z_n - p^*\right\| + \left\|x_{n+1} - p^*\right\| \\
&\leq \epsilon_n + \delta^2[1-\alpha_n(1-\delta)]\|r_n - x_n\| \\
&\quad + \delta L\left\{1-\alpha_n + (1-\alpha_n)\delta + \alpha_n\delta + \alpha_n\delta^2 + \alpha_n\delta(1+\delta)\right\}\left\|x_n - p^*\right\| \\
&\quad (1+\delta)L[1-\alpha_n(1-\delta)]\delta\left\|x_n - p^*\right\| + \left\|x_{n+1} - p^*\right\| \qquad (3)
\end{aligned}$$

Since sequences $\{x_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are equivalent, therefore we have $\lim_{n\to\infty}\|r_n - x_n\| = 0$. By Theorem 2, we have $\lim_{n\to\infty}\|x_n - p^*\| = 0$. Now taking limit on both sides of (3) and then using the assumption $\lim_{n\to\infty}\epsilon_n = 0$ leads to $\lim_{n\to\infty}\|r_{n+1} - p^*\| = 0$. Thus $\{x_n\}_{n=1}^{\infty}$ is weak w^2-stable w.r.t. T. □

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A Convergence Result for a Three-Step Iterative Algorithm

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Abstract: We prove under some mild conditions that iterative algorithm (1.7) of [1] converges strongly to the fixed point of a member in the class of weak contraction mappings.

Keywords: Fixed point, Iterative algorithm, Strong convergence.

1 Introduction

Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a mapping. An element x in C is said to be a fixed point of T if $Tx = x$.

Iterative approximation of fixed points has become a useful tool for solving many problems which arise in various branches of science and engineering.

Recently, Karakaya et al. [1] introduced a three-step iterative algorithm as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n) z_n + \alpha_n T z_n, \\ z_n = Tx_n, n \in \mathbb{N}, \end{cases} \tag{1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0, 1]$.

Definition 1. *([2]) Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be weak-contraction if there exist $\delta \in [0, 1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta d(x, y) + L d(y, Tx), \text{ for all } x, y \in M.$$

Theorem 1. *([2]) Let (M, d) be a complete metric space and $T : M \to M$ a weak-contraction for which there exist $\delta \in [0, 1)$ and $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \delta d(x, y) + L_1 d(x, Tx), \text{ for all } x, y \in M. \tag{2}$$

Then, T has a unique fixed point.

Karakaya et al. [1] showed that iterative algorithm (1) strongly converges to the fixed points of weak-contraction mappings. More precisely, they proved the following result.

Theorem 2. *([1]) Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a weak-contraction satisfying condition (2). Let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence generated by (1) with real sequence $\{\alpha_n\}_{n=1}^{\infty} \subseteq [0, 1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}_{n=1}^{\infty}$ converges to a unique fixed point p^* of T.*

2 Main result

Theorem 3. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ with $p^* = Tp^*$ a weak-contraction satisfying condition (2). Let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence generated by (1) with real sequence $\{\alpha_n\}_{n=1}^{\infty} \subseteq [0, 1]$. Then, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a unique fixed point p^* of T.*

Proof: The following inequality was obtained in ([1], Theorem 2.1):

$$\left\| x_{n+1} - p^* \right\| \leq \left\| x_1 - p^* \right\| \delta^{2n} \prod_{i=1}^{n} \left[1 - \alpha_i \left(1 - \delta \right) \right], \text{ for all } n \in \mathbb{N}. \tag{3}$$

As $\delta \in [0, 1)$ and $\{\alpha_n\}_{n=1}^{\infty} \subseteq [0, 1]$ implies $1 - \alpha_n (1 - \delta) < 1$ for all $n \in \mathbb{N}$, so inequality (3) becomes

$$\left\| x_{n+1} - p^* \right\| \leq \delta^{2n} \left\| x_1 - p^* \right\|, \text{ for all } n \in \mathbb{N}. \tag{4}$$

Taking limit on both sides of inequality (4), we have $\lim_{n \to \infty} \|x_n - p^*\| = 0$. □

3 Conclusion

Theorem 2 was proven under the condition $\sum_{n=1}^{\infty} \alpha_n = \infty$. In Theorem 3, we remove this condition. Therefore, Theorem 3 is an improvement of Theorem 2.

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Amalgam Spaces With Variable Exponent

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Abstract: Let $1 \leq s < \infty$ and $1 \leq r(.) \leq \infty$ where $r(.)$ is a variable exponent. In this study, we consider the variable exponent amalgam space $\left(L^{r(.)}, \ell^s\right)$. Moreover, we present some examples about inclusion properties of this space. Finally, we obtain that the space $\left(L^{r(.)}, \ell^s\right)$ is a Banach Function space.

Keywords: Amalgam space, Banach space, Variable exponent.

1 Introduction

The amalgam of L^p and l^q on the real line is the space $(L^p, l^q)(\mathbb{R})$ (or briefly (L^p, l^q)) consisting of functions f which are locally in L^p and have l^q behavior at infinity. Several authors studied special cases of amalgams on some sets including $\mathbb{R}$ and a locally compact abelian group G. The first appearance of amalgam spaces can be traced to Wiener [13]. A generalization Wiener's definition was given by Feichtinger in [6], and it can be found a good summary of some results about amalgam spaces in [10], [11]. For a historical background of classical amalgams we refer [7]. The variable exponent Lebesgue spaces $L^{p(\cdot)}$ and the classical Lebesgue spaces L^p have many common properties but a significant difference between these spaces is that $L^{p(\cdot)}$ is not invariant under translation in general, see [4], [12]. Recently, there are many interesting and important papers appeared in variable exponent amalgam space $\left(L^{r(\cdot)}, \ell^s\right)$ such as Aydin [1], Aydin and Gurkanli [3], Gurkanli and Aydin [9].

2 Main results

Definition 1. *For a measurable function $r(.) : \mathbb{R} \to [1, \infty)$ (called a variable exponent on $\mathbb{R}$), we put*

$$r^- = \underset{x\in\mathbb{R}}{essinf} r(x), \qquad r^+ = \underset{x\in\mathbb{R}}{esssup} r(x).$$

Also the convex modular function $\varrho_{r(.)}$ is defined as

$$\varrho_{r(.)}(f) = \int_{\mathbb{R}} |f(x)|^{r(x)}\, dx.$$

The variable exponent Lebesgue space $L^{r(\cdot)}(\mathbb{R})$ is defined as the set of all measurable functions f on $\mathbb{R}$ such that $\varrho_{r(.)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{r(.)} = \inf\left\{\lambda > 0 : \varrho_{r(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

Let $r^+ < \infty$. Then $f \in L^{r(\cdot)}(\mathbb{R})$ if and only if $\varrho_{r(.)}(f) < \infty$, that is, the norm topology is equivalent to modular topology. The space $L^{r(\cdot)}(\mathbb{R})$ is a Banach space with respect to $\|.\|_{r(.)}$. Moreover, it is well known that if we take $r(.) = r$ (const.), then the space $L^{r(\cdot)}(\mathbb{R})$ coincides with the classical Lebesgue space $L^r(\mathbb{R})$, see [12]. In this paper, we will assume that $r^+ < \infty$.

Definition 2. *Let $1 \leq r(.), s < \infty$ and $J_k = [k, k+1)$, $k \in \mathbb{Z}$. The variable exponent amalgam space $\left(L^{r(\cdot)}, \ell^s\right)$ is a normed space defined as*

$$\left(L^{r(\cdot)}, \ell^s\right) = \left\{f \in L^{r(\cdot)}_{loc}(\mathbb{R}) : \|f\|_{\left(L^{r(\cdot)}, \ell^s\right)} < \infty\right\},$$

where

$$\|f\|_{\left(L^{r(\cdot)}, \ell^s\right)} = \left(\sum_{k\in\mathbb{Z}} \|f\chi_{J_k}\|^s_{r(.)}\right)^{\frac{1}{s}}.$$

It is well known that $\left(L^{r(.)},\ell^{s}\right)$ *does not depend on the particular choice of* J_k*. This follows* J_k *can be equal to* $[k,k+1)$*,* $[k,k+1]$ *or* $(k,k+1)$*. Thus, we have same amalgam spaces* $\left(L^{r(.)},\ell^{s}\right)$*.*

Theorem 1. *The space* $\left(L^{r(.)},\ell^{s}\right)$ *is a Banach space with respect to the norm* $\|.\|_{\left(L^{r(.)},\ell^{s}\right)}$*.*

Proof: Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\left(L^{r(.)},\ell^{s}\right)$. Then given $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that if $n,m\geq N$, then we have

$$\|f_n-f_m\|_{\left(L^{r(.)},\ell^{s}\right)}=\left(\sum_{k\in\mathbb{Z}}\|f_n-f_m\|_{r(.),J_k}^{s}\right)^{\frac{1}{s}}<\varepsilon. \tag{1}$$

Hence, for any fixed k, we get

$$\|f_n-f_m\|_{r(.),J_k}<\varepsilon \ \ (n,m\geq N).$$

Thus $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^{r(.)}(J_k)$ for $k\in\mathbb{Z}$. Let us define $f=\sum_{k\in\mathbb{Z}}f^k\chi_{J_k}$ where $f^k\in L^{r(.)}(J_k)$. Now, we will show that $f\in\left(L^{r(.)},\ell^{s}\right)$. Using Fatou's Lemma (applied to the right-hand series viewed as integral over the integers),we obtain

$$\begin{aligned}\|f\|_{\left(L^{r(.)},\ell^{s}\right)}^{s} &= \sum_{k\in\mathbb{Z}}\left\|f^k\right\|_{r(.),J_k}^{s}=\sum_{k\in\mathbb{Z}}\lim_{n\to\infty}\|f_n\|_{r(.),J_k}^{s}\\ &\leq \lim_{n\to\infty}\inf\|f_n\|_{\left(L^{r(.)},\ell^{s}\right)}^{s}.\end{aligned} \tag{2}$$

Since $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence (hence $\{f_n\}_{n\in\mathbb{N}}$ is bounded in norm), the last quantity is finite. Therefore, the left side of (2) is finite, that is, $f\in\left(L^{r(.)},\ell^{s}\right)$. By (1), we have

$$\|f_m-f\|_{r(.),J_k}^{s}=\lim_{n\to\infty}\|f_m-f_n\|_{r(.),J_k}^{s}$$

and

$$\begin{aligned}\|f_m-f\|_{\left(L^{r(.)},\ell^{s}\right)}^{s} &= \sum_{k\in\mathbb{Z}}\lim_{n\to\infty}\|f_m-f_n\|_{r(.),J_k}^{s}\\ &\leq \lim_{n\to\infty}\inf\sum_{k\in\mathbb{Z}}\|f_m-f_n\|_{r(.),J_k}^{s}\\ &< \varepsilon\end{aligned}$$

for $m\geq N$. Thus the Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f, which is desired result. □

Now, we will show that $L^{r(.)}\neq\left(L^{r(.)},\ell^{s}\right)$ and that these two spaces are not translation invariant in general. Also, we will prove new two examples which are associated with this.

Example 1. *Let* $r(.):\mathbb{R}\to[0,\infty)$ *be a function such that for* $k\in\mathbb{Z}$

$$r(x)=\left\{\begin{array}{ll}1, & x\in A_k=[2k-1,2k)\\ 2, & x\in B_k=[2k-2,2k-1)\end{array}\right..$$

Hence, we have $r^{+}<\infty$ *and* $A_k\cap B_k=\phi$ *for all* $k\in\mathbb{Z}$*. Also let us define a function* f *as*

$$f(x)=\left\{\begin{array}{ll}0, & x\in A_k, k\in\mathbb{N}\\ \frac{1}{k}, & x\in B_k, k\in\mathbb{N},(k\neq0)\\ 0, & x<0\ (x\notin A_k\cup B_k)\end{array}\right.$$

Therefore, we have

$$\begin{aligned}\varrho_{r(.)}(f) &= \int_{\mathbb{R}}|f(x)|^{r(x)}dx=\sum_{k=1}^{\infty}\int_{J_k}|f(x)|^{r(x)}dx\\ &= \sum_{k=1}^{\infty}\int_{J_k\cap B_k}|f(x)|^{r(x)}dx\\ &= \sum_{k=1}^{\infty}\int_{B_k}\frac{1}{k^2}dx=\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty.\end{aligned}$$

This follows that $f \in L^{r(\cdot)}(\mathbb{R})$. Now, we will show that $f \notin \left(L^{r(\cdot)}, \ell^1\right)$. By using the definition of $\|.\|_{\left(L^{r(\cdot)},\ell^1\right)}$, we obtain

$$\begin{aligned} \|f\|_{\left(L^{r(\cdot)},\ell^1\right)} &= \sum_{k\in Z} \left\|f\chi_{[k,k+1)}\right\|_{r(.)} = \sum_{k=1}^{\infty} \left\|f\chi_{[2k-2,2k-1)}\right\|_{r(.)} \\ &= \sum_{k=1}^{\infty} \left\|f\chi_{[2k-2,2k-1)}\right\|_{2} \\ &= \sum_{k=1}^{\infty} \left(\int_{2k-2}^{2k-1} \frac{1}{k^2} dx\right)^{\frac{1}{2}} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty. \end{aligned}$$

Therefore, we have $f \notin \left(L^{r(\cdot)}, \ell^1\right)$.

Example 2. *Let $r(.) : \mathbb{R} \to [0, \infty)$ be a function such that for $k \in \mathbb{Z}$*

$$r(x) = \begin{cases} 1, & x \in A_k = [2k+1, 2(k+1)) \\ 2, & x \in B_k = [2k, 2k+1) \end{cases}.$$

Then, we define the space as

$$L^{r(\cdot)}(\mathbb{R}) = \left\{f : f = f_1 + f_2,\ f_1 \in L^1(\mathbb{R}), f_2 \in L^2(\mathbb{R}), \textit{supp} f_1 = \cup_{k\in\mathbb{Z}} A_k \textit{ and supp} f_2 = \cup_{k\in\mathbb{Z}} B_k\right\}.$$

If we denote $T_1 f$ as the translation of given any function $f \in L^{r(\cdot)}(\mathbb{R})$, then we obtain

$$T_1 f(x) = \begin{cases} f(x+1) = f_2(x), & x \in A_k \\ f(x+1) = f_1(x), & x \in B_k \end{cases}.$$

It is easy to see that $T_1 f \notin L^{r(\cdot)}(\mathbb{R})$ .That means the space $L^{r(\cdot)}(\mathbb{R})$ is not translation invariant. Now, we quote this idea to the amalgam space. To show this we take same function $r(.)$ and same space $L^{r(\cdot)}(\mathbb{R})$. Let $p > 1$. Let us define a function f as

$$f(x) = \begin{cases} 0, & x \in A_k \\ \frac{1}{(k+1)^p}, & x \in B_k \\ 0, & x < 0\ (x \notin A_k \cup B_k) \end{cases}$$

Then, we obtain

$$\|f\|_{\left(L^{r(\cdot)},\ell^1\right)} = \begin{cases} \sum\limits_{k\in Z} \left\{\int_k^{k+1} |f(x)|^2 dx\right\}^{\frac{1}{2}} = \sum\limits_{k=1}^{\infty} \frac{1}{(k+1)^p} < \infty, & x \in B_k \\ 0, & x \in A_k \end{cases}.$$

Therefore we have $f \in \left(L^{r(\cdot)}, \ell^1\right)$. By the definition of $T_1 f$, we get

$$T_1 f(x) = \begin{cases} f(x+1) = \frac{1}{(k+1)^p}, & x \in A_k \\ 0, & x \in B_k \end{cases}.$$

This follows that

$$\|T_1 f\|_{\left(L^{r(\cdot)},\ell^1\right)} = \begin{cases} \sum\limits_{k\in Z} \left\{\int_k^{k+1} |f(x+1)| dx\right\} = \sum\limits_{k=1}^{\infty} \frac{1}{(k+1)^p} < \infty, & x \in A_k \\ 0, & x \in B_k \end{cases}.$$

Therefore, we have $T_1 f \in \left(L^{r(\cdot)}, \ell^1\right)$. This follows that the space $\left(L^{r(\cdot)}, \ell^1\right)$ is translation invariant. As an alternative method, it is easy to see that $\left(L^1, \ell^1\right) = L^1$ or $\left(L^2, \ell^1\right) \subset L^1$ and the space L^1 is translation invariant. Therefore, the same result is satisfied.

Remark 1. *If we consider the Theorem 3.3 in [8], then $L^{r(\cdot)} = \left(L^{r(\cdot)}, \ell^s\right)$ holds for some special cases. Therefore, the amalgam space $\left(L^{r(\cdot)}, \ell^s\right)$ is not translation invariant in general.*

Definition 3. *$L_c^{r(\cdot)}(\mathbb{R})$ denotes the functions f in $L^{r(\cdot)}(\mathbb{R})$ such that supp$f \subset \mathbb{R}$ is compact, that is,*

$$L_c^{r(\cdot)}(\mathbb{R}) = \left\{f \in L^{r(\cdot)}(\mathbb{R}) : \textit{supp} f \textit{ compact}\right\}.$$

Now, let $K \subset \mathbb{R}$ be given. The cardinality of the set

$$S(K) = \{J_k : J_k \cap K \neq \varnothing\}$$

is denoted by $|S(K)|$ where $\{J_k\}_{k\in\mathbb{Z}}$ is a collection of intervals $J_k = [k, k+1] = k + [0, 1]$, and also cover $\mathbb{R}$.

The following proposition was proved by Aydin [2].

Proposition 1. *If $g \in L_c^{r(.)}(\mathbb{R})$ and K is the compact support of g, then we have*

(i) $\|g\|_{\left(L^{r(.)},\ell^s\right)} \leq |S(K)|^{\frac{1}{s}} \|g\|_{r(.)}$ *for* $1 \leq s < \infty$.
(ii) $\|g\|_{\left(L^{r(.)},\ell^\infty\right)} \leq |S(K)| \|g\|_{r(.)}$.

Moreover, we have $L_c^{r(.)}(\mathbb{R}) \subset \left(L^{r(.)}(\mathbb{R}), \ell^s\right)$ *for* $1 \leq s \leq \infty$.

The main result of this study is to show that the space $\left(L^{r(.)}, \ell^s\right)$ be a special case of Banach function space, in other words, the norm of $\left(L^{r(.)}, \ell^s\right)$ satisfies the following properties, where f, g, f_n in $\left(L^{r(.)}, \ell^s\right)$ for all $n \in \mathbb{N}$, $\lambda \geq 0$ and E is any measurable subset of $\mathbb{R}$ ($|E| < \infty$):

1. $\|f\|_{\left(L^{r(.)},\ell^s\right)} \geq 0$
2. $\|f\|_{\left(L^{r(.)},\ell^s\right)} = 0$ if and only if $f = 0$ a.e. in $\mathbb{R}$
3. $\|\lambda f\|_{\left(L^{r(.)},\ell^s\right)} = \lambda \|f\|_{\left(L^{r(.)},\ell^s\right)}$
4. $\|f + g\|_{\left(L^{r(.)},\ell^s\right)} \leq \|f\|_{\left(L^{r(.)},\ell^s\right)} + \|g\|_{\left(L^{r(.)},\ell^s\right)}$
5. If $|g| \leq |f|$ a.e. in $\mathbb{R}$, then $\|g\|_{\left(L^{r(.)},\ell^s\right)} \leq \|f\|_{\left(L^{r(.)},\ell^s\right)}$
6. If $0 \leq f_n \uparrow f$ a.e. in $\mathbb{R}$, then $\|f_n\|_{\left(L^{r(.)},\ell^s\right)} \uparrow \|f\|_{\left(L^{r(.)},\ell^s\right)}$
7. $\|\chi_E\|_{\left(L^{r(.)},\ell^s\right)} < \infty$
8. $\int_E |f|\, dx \leq C(r(.), E) \|f\|_{\left(L^{r(.)},\ell^s\right)}$ for some $C > 0$.

Theorem 2. *The space $\left(L^{r(.)}, \ell^s\right)$ is a Banach Function space with respect to the norm $\|.\|_{\left(L^{r(.)},\ell^s\right)}$.*

Proof: We have to prove the properties (1)-(8). The first three properties follow directly from the definition of the norm $\|.\|_{\left(L^{r(.)},\ell^s\right)}$.

Proof of Property 4. Let $f, g \in \left(L^{r(.)}, \ell^s\right)$ be given. It is well known that $f, g \in \left(L^{r(.)}, \ell^s\right)$ if and only if $\left\{\|f\|_{r(.),J_k}\right\}_{k\in Z}, \left\{\|g\|_{r(.),J_k}\right\}_{k\in Z} \in \ell^s(\mathbb{Z})$. Then we have

$$\begin{aligned} \|f+g\|_{\left(L^{r(.)},\ell^s\right)} &= \left\|\|f+g\|_{r(.),J_k}\right\|_{\ell^s} \\ &\leq \left\|\|f\|_{r(.),J_k} + \|g\|_{r(.),J_k}\right\|_{\ell^s} \\ &\leq \left\|\|f\|_{r(.),J_k}\right\|_{\ell^s} + \left\|\|g\|_{r(.),J_k}\right\|_{\ell^s} \\ &= \|f\|_{\left(L^{r(.)},\ell^s\right)} + \|g\|_{\left(L^{r(.)},\ell^s\right)} . \end{aligned}$$

Proof of Property 5. Let $|g| \leq |f|$. Then we obtain

$$\begin{aligned} \|g\|_{\left(L^{r(.)},\ell^s\right)} &= \left\|\|g\|_{r(.),J_k}\right\|_{\ell^s} \\ &\leq \left\|\|f\|_{r(.),J_k}\right\|_{\ell^s} = \|f\|_{\left(L^{r(.)},\ell^s\right)} . \end{aligned}$$

Proof of Property 6. It is well known that $L^{r(.)}$ is a BF-space by Proposition 1.3 in [5]. Since $0 \leq f_n \uparrow f$ a.e. in $\mathbb{R}$, then $\|f_n\|_{r(.),J_k} \uparrow \|f\|_{r(.),J_k}$. If we consider this property for ℓ^s, we have

$$\|f_n\|_{\left(L^{r(.)},\ell^s\right)} = \left\|\|f_n\|_{r(.),J_k}\right\|_{\ell^s} \uparrow \left\|\|f\|_{r(.),J_k}\right\|_{\ell^s} = \|f\|_{\left(L^{r(.)},\ell^s\right)} .$$

Proof of Property 7. Since $|E| < \infty$ and supp$\chi_E = \overline{E} \subset \mathbb{R}$ is compact, then $\chi_E \in L_c^{r(.)}(\mathbb{R})$ and

$$\|\chi_E\|_{\left(L^{r(.)},\ell^s\right)} \leq |S(E)|^{\frac{1}{s}} \|\chi_E\|_{r(.),E} < \infty$$

by Proposition 1.
Proof of Property 8. By Hölder's inequality for variable exponent amalgam spaces (see, Corollary 2.4, [3]), we get

$$\begin{aligned} \int_E |f|\, dx &= \int_{\mathbb{R}} |f\chi_E|\, dx \leq c \|f\|_{\left(L^{r(.)},\ell^s\right)} \|\chi_E\|_{\left(L^{r'(.)},\ell^{s'}\right)} \\ &\leq C(r(.), E) \|f\|_{\left(L^{r(.)},\ell^s\right)} \end{aligned}$$

for some $C > 0$ where $\frac{1}{r(.)} + \frac{1}{r'(.)} = \frac{1}{s} + \frac{1}{s'} = 1$ and $C = C(r(.), E) = c \|\chi_E\|_{\left(L^{r'(.)},\ell^{s'}\right)}$. □

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Mathematical Behavior of Solutions of Fourth-Order Hyperbolic Equation with Logarithmic Source Term

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Abstract: The main goal of this paper is to study for a fourth-order hyperbolic equation with logarithmic nonlinearity. We obtain several results: Firstly, by using Feado-Galerkin method and a logaritmic Sobolev inequality, we proved local existence of solutions. Later, we proved global existence of solutions by potential well method. Finally, we showed the decay estimates result of the solutions.

Keywords: Decay of solution, Existence, Logarithmic nonlinearity.

1 Introduction

In this paper, we study the following fourth order hyperbolic equation with logarithmic nonlinearity

$$\begin{cases} u_{tt} + \triangle^2 u - \triangle u + u_t = u \ln |u|^k, & x \in \Omega,\ t > 0 \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = \frac{\partial}{\partial \nu} u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0 \end{cases} \tag{1}$$

where $\Omega \subset R^3$ is a bounded domain with smooth boundary $\partial\Omega$, and k is a costant to be chosen later.

This type of problems has many applications in many branches physics, such as quantum mechanics, nuclear physics, supersymmetric field theories, optics and geophysics [2–4, 6, 11].

In [5], Cazenave and Haraux studied the existence of the solution following equation

$$u_{tt} - \triangle u + u = u \ln |u|^k \tag{2}$$

in R^3. Later, Gorka [6] studied the global existence of the solution of Eq. (2) in the one dimensional case. Furthermore, existence of the solutions were studied in [1–3].

Hiramatsu et al. [9] is introduced the following equation

$$u_{tt} - \Delta u + u + u_t + u|u|^2 = u \ln |u|^2 . \tag{3}$$

In [8], Han showed the global existence of weak solutions to the initial boundary value problem (3) in R^3.

Recently, Hu et al. [14] studied exponential growth and decay estimates of the solutions for Eq. (1), without the fourth-order term ($\triangle^2 u$). Al-Gharabli and Messaoudi [12, 13] proved existence and decay of the solutions for Eq. (1), without the $\triangle u$ term.

Motivated by the above studies, we established the local and global existence, growth and decay estimates of the solution for problem (1).

The rest of our work is organized as follows. In section 2, we gave some notations and lemmas which will be used throughout this paper. In section 3, we established the local existence of the solutions of the problem. In section 4, we established the global existence of the solutions of the problem. The decay estimates result were presented in section 5.

2 Preliminaries

In this section we will give some notations and lemmas which will be used throughout this paper. We denote $\|.\|$ and $\|.\|_p$ the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively. We denote by C and C_i $(i = 1, 2, ...)$ varius positive constants.

We define energy function as follows

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{1}{2}\int_\Omega \ln |u|^k u^2 dx + \frac{k}{4}\|u\|^2 . \tag{4}$$

Lemma 1. *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$E'(t) = -\|u_t\|^2 \leq 0. \tag{5}$$

Proof: Multiplying the equation (1) by u_t and integrating on Ω, we have

$$\int_\Omega u_{tt}u_t dx + \int_\Omega \triangle^2 u u_t dx - \int_\Omega \triangle u u_t dx + \int_\Omega u_t u_t dx = \int_\Omega \ln|u|^k u u_t dx,$$

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{1}{2}\int_\Omega \ln|u|^k u^2 dx + \frac{k}{4}\|u\|^2\right) = -\|u_t\|^2,$$

$$E'(t) = -\|u_t\|^2.$$

□

Lemma 2. *[7] (Logarithmic Sobolev Inequality). Let u be any function $u \in H_0^1(\Omega)$ and $a > 0$ be any number. Then,*

$$\int_\Omega \ln|u|\, u^2 dx < \frac{1}{2}\|u\|^2 \ln\|u\|^2 + \frac{\alpha^2}{2\pi}\|\nabla u\|^2 - (1 + \ln\alpha)\|u\|^2.$$

Lemma 3. *[5] (Logarithmic Gronwall Inequality).*

Let $c > 0,\ \gamma \in L^1(0.T, R^+)$ and assume that the function $w : [0,T] \to [1,\infty]$ satisfies

$$w(t) \leq c\left(1 + \int_0^t \gamma(s)\, w(s) \ln w(s)\, ds\right),\ 0 \leq t \leq T,$$

where

$$w(t) \leq c e^{\int_0^t c\gamma(s)ds},\ 0 \leq t \leq T.$$

3 Local existence

In this section we state and prove the local existence result for problem (1). The proof is based on Faedo-Galerkin method.

Definition 4. *A function u defined on $[0,T]$ is called a weak solution of (1) if*

$$u \in C\left([0,T); H_0^2(\Omega)\right),\ u_t \in C\left([0,T); L^2(\Omega)\right)$$

and u satisfies

$$\begin{cases} \int_\Omega u_{tt}(x,t)\, w(x)\, dx + \int_\Omega \triangle u(x,t)\, \Delta w(x)\, dx \\ + \int_\Omega \nabla u(x,t)\, \nabla w(x)\, dx + \int_\Omega u_t(x,t)\, w(x)\, dx \\ \qquad = \int_\Omega \ln|u(x,t)|^k u(x,t)\, w(x)\, dx, \end{cases}$$

for $w \in H_0^2(\Omega)$.

Theorem 5. *Let $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$, then the problem (1) has a weak solution on $[0,T]$.*

Proof: We will use the Faedo-Galerkin method to construct approximate solutions. Let $\{w_j\}_{j=1}^\infty$ be an orthogonal basis of the "separable" space $H_0^2(\Omega)$ which is orthonormal in $L^2(\Omega)$. Let

$$V_m = \text{span}\{w_1, w_2, ..., w_m\}$$

and let the projections of the initial data on the finite dimensional subspace V_m be given by

$$\begin{aligned} u_0^m(x) &= \sum_{j=1}^m a_j w_j(x) \to u_0 \text{ in } H_0^2(\Omega), \\ u_1^m(x) &= \sum_{j=1}^m b_j w_j(x) \to u_1 \text{ in } L^2(\Omega), \end{aligned}$$

for $j = 1, 2, ..., m$.

We look for the approximate solutions

$$u^m(x,t)=\sum_{j=1}^{m}h_j^m(t)\,w_j(x)$$

of the approximate problem in V_m

$$\left\{\begin{array}{l}\int\limits_{\Omega}(u_{tt}^m w dx+\triangle u^m\Delta w+\nabla u^m\nabla w+u_t^m w)\,dx=\int\limits_{\Omega}\ln|u^m|^k u^m w dx,\ w\in V_m,\\ \quad u^m(0)=u_0^m=\sum\limits_{j=1}^{m}(u_0,w_j)\,w_j,\\ \quad u_t^m(0)=u_1^m=\sum\limits_{j=1}^{m}(u_1,w_j)\,w_j.\end{array}\right. \tag{6}$$

This lead to a system of ordinary differantial equations for unknown functions $h_j^m(t)$. Based on standard existence theory for ordinary differantial equation, one can obtain functions

$$h_j:[0,t_m)\to R,\ \ j=1,2,...,m,$$

which satisfy (6) in a maximal interval $[0,t_m)$, $0<t_m\le T$. Next, we show that $t_m=T$ and that the local solution is uniformly bounded independent of m and t. For this purpose, we replace w by u_t^m in (6) and integrate by parts we obtain

$$\frac{d}{dt}E^m(t)=-\left\|u_t^m\right\|^2\le 0 \tag{7}$$

where

$$E^m(t)=\frac{1}{2}\left(\left\|u_t^m\right\|^2+\left\|\Delta u^m\right\|^2+\left\|\nabla u^m\right\|^2+\frac{k}{2}\left\|u^m\right\|^2-\int\limits_{\Omega}\ln|u^m|^k|u^m|^2dx\right). \tag{8}$$

Integrating (7) with respect to t from 0 to t, we obtain

$$E^m(t)\le E^m(0). \tag{9}$$

The last inequality and the Logarithmic Sobolev Inequality lead to

$$\begin{aligned}E^m(t)\ &\ge\ \frac{1}{2}\left[\left\|u_t^m\right\|^2+\left\|\Delta u^m\right\|^2+\left\|\nabla u^m\right\|^2+\frac{k}{2}\left\|u^m\right\|^2\right]\\ &\quad-\frac{k}{2}\left[\frac{1}{2}\left\|u^m\right\|^2\ln\left\|u^m\right\|^2+\frac{\alpha^2}{2\pi}\left\|\nabla u^m\right\|^2-(1+\ln\alpha)\left\|u^m\right\|^2\right],\\ &=\ \frac{1}{2}\left(\left\|u_t^m\right\|^2+\left\|\Delta u^m\right\|^2+\left(1-\frac{k\alpha^2}{2\pi}\right)\left\|\nabla u^m\right\|^2\right.\\ &\quad\left.+\left(\frac{k}{2}+k(1+\ln\alpha)\right)\left\|u^m\right\|^2-\frac{k}{2}\left\|u^m\right\|^2\ln\left\|u^m\right\|^2\right),\end{aligned}$$

$$\begin{aligned}&\left\|u_t^m\right\|^2+\left\|\Delta u^m\right\|^2+\left(1-\frac{k\alpha^2}{2\pi}\right)\left\|\nabla u^m\right\|^2\\ &+\left(\frac{k}{2}+k(1+\ln\alpha)\right)\left\|u^m\right\|^2\\ \le\ &C+\frac{k}{2}\left\|u^m\right\|^2\ln\left\|u^m\right\|^2.\end{aligned} \tag{10}$$

where $C=2E^m(0)$.

Choosing

$$e^{-\frac{3}{2}}<\alpha<\sqrt{\frac{2\pi}{k}} \tag{11}$$

will make

$$1-\frac{k\alpha^2}{2\pi}>0,$$

$$\sqrt{\frac{2\pi}{k}}>\alpha$$

and

$$\frac{k+2}{2}+k(1+\ln\alpha)>0,$$

$$\alpha>e^{-\frac{3}{2}}.$$

This selection is possible thanks to (A). So, we have

$$\left\|u_t^m\right\|^2+\left\|\Delta u^m\right\|^2+\left\|\nabla u^m\right\|^2+\left\|u^m\right\|^2 \leq C\left(1+\left\|u^m\right\|^2 \ln \left\|u^m\right\|^2\right) \tag{12}$$

We know that

$$u^m(., t)=u^m(., 0)+\int_0^t \frac{\partial u^m}{\partial \tau}(., \tau)\, d\tau.$$

We make use of the following Cauchy-Schwarz inequality

$$(a+b)^2 \leq 2\left(a^2+b^2\right),$$

we obtain

$$\begin{aligned}
\left\|u^m(t)\right\|^2 &= \left\|u^m(., 0)+\int_0^t \frac{\partial u^m}{\partial \tau}(., \tau)\, d\tau\right\|^2 \\
&\leq 2\left\|u^m(0)\right\|^2+2\left\|\int_0^t \frac{\partial u^m}{\partial \tau}(., \tau)\, d\tau\right\|^2 \\
&\leq 2\left\|u^m(0)\right\|^2+2T \int_0^t\left\|u_t^m(\tau)\right\|^2 d\tau
\end{aligned} \tag{13}$$

So if we write inequality (12) instead of inequality (13), we get

$$\left\|u^m\right\|^2 \leq 2\left\|u^m(0)\right\|^2+2TC\left(1+\left\|u^m\right\|^2 \ln \left\|u^m\right\|^2\right) \tag{14}$$

If we put $C_1=\max \left\{2TC,\ 2\left\|u^m(0)\right\|^2\right\}$, (14) leads to

$$\left\|u^m\right\|^2 \leq 2C_1\left(1+\int_0^t\left\|u^m\right\|^2 \ln \left\|u^m\right\|^2 d\tau\right).$$

Without loss of generality, we take $C_1 \geq 1$, we have

$$\left\|u^m\right\|^2 \leq 2C_1\left(1+\int_0^t\left(C_1+\left\|u^m\right\|^2\right) \ln \left(C_1+\left\|u^m\right\|^2\right) d\tau\right).$$

Thanks to Logarithmic Gronwall inequality, we obtain

$$\left\|u^m\right\|^2 \leq 2C_1 e^{2C_1 T}=C_2.$$

Hence, from inequality (12), it follows that

$$\left\|u_t^m\right\|^2+\left\|\Delta u^m\right\|^2+\left\|\nabla u^m\right\|^2+\left\|u^m\right\|^2 \leq C_3=C\left(1+C_2 \ln C_2\right)$$

where C_3 is a positive constant independent of m and t. If these operations (12) are applied to each term of inequality, this implies

$$\sup_{t \in(0, t_m)}\left\|u_t^m\right\|^2+\sup_{t \in(0, t_m)}\left\|\Delta u^m\right\|^2+\sup_{t \in(0, t_m)}\left\|\nabla u^m\right\|^2+\sup_{t \in(0, t_m)}\left\|u^m\right\|^2 \leq 4C_3. \tag{15}$$

So, the approximate solution is uniformly bounded independent of m and t. Therefore, we can extend t_m to T. Moreover, we obtain

$$\left\{\begin{array}{l}
u^m, \text{ is uniformly bounded in } L^{\infty}\left(0, T; H_0^2(\Omega)\right), \\
u_t^m, \text{ is uniformly bounded in } L^{\infty}\left(0, T; L^2(\Omega)\right).
\end{array}\right. \tag{16}$$

Hence we can infer from (15) and (16) that there exists a subsequence of (u^m) (still denoted by (u^m)), such that

$$\left\{\begin{array}{l}
u^m \rightarrow u, \text{ weakly* in } L^{\infty}\left(0, T; H_0^2(\Omega)\right), \\
u_t^m \rightarrow u_t, \text{ weakly* in } L^{\infty}\left(0, T; L^2(\Omega)\right), \\
u^m \rightarrow u, \text{ weakly in } L^2\left(0, T; H_0^2(\Omega)\right), \\
u_t^m \rightarrow u_t, \text{ weakly in } L^2\left(0, T; L^2(\Omega)\right).
\end{array}\right. \tag{17}$$

Then using (17) and Aubin–Lions' lemma, we have

$$u^m \to u, \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right)$$

which implies

$$u^m \to u, \ \ \Omega\times(0,T).$$

Since the map $s\to s\ln|s|^k$ is continuous, we have the convergence

$$u^m\ln|u^m|^k \to u\ln|u|^k, \ \ \Omega\times(0,T). \tag{18}$$

By the Sobolev embedding theorem ($H_0^2(\Omega)\hookrightarrow L^\infty(\Omega)$), it is clear that $\left|u^m\ln|u^m|^k - u\ln|u|^k\right|$ is bounded in $L^\infty(\Omega\times(0,T))$. Next, taking into account the Lebesgue bounded convergence theorem, we have

$$u^m\ln|u^m|^k \to u\ln|u|^k, \text{strongly in } L^2\left(0,T;L^2(\Omega)\right). \tag{19}$$

We integrate (6) over $(0,t)$ to obtain, $\forall w\in V_m$

$$\begin{aligned}\int_0^t\int_\Omega \ln|u^m|^k u^m w\,dx\,d\tau &= \int_\Omega u_t^m w\,dx\,d\tau - \int_\Omega u_1^m w\,dx\,d\tau + \int_0^t\int_\Omega \triangle u^m\Delta w\,dx\,d\tau \\ &\quad + \int_0^t\int_\Omega \nabla u^m\nabla w\,dx\,d\tau + \int_0^t\int_\Omega u_t^m w\,dx\,d\tau. \end{aligned} \tag{20}$$

Convergences (17), (19) are sufficient to pass to the limit in (20)

$$\begin{aligned}\int_\Omega u_t w\,dx &= \int_\Omega u_1 w\,dx - \int_0^t\int_\Omega \triangle u\Delta w\,dx\,d\tau - \int_0^t\int_\Omega \nabla u\nabla w\,dx\,d\tau \\ &\quad - \int_0^t\int_\Omega u_t w\,dx\,d\tau + \int_0^t\int_\Omega \ln|u|^k u w\,dx\,d\tau \end{aligned} \tag{21}$$

which implies that (20) is valid $\forall w\in H_0^2(\Omega)$ .Using the fact that the terms in the right-hand side of (21) are absolutely continuous since they are functions of t defined by integrals over $(0,t)$, hence it is differentiable for a.e. $t\in R^+$. Thus, differentiating (21), we obtain, for a.e. $t\in(0,T)$ and any $w\in H_0^2(\Omega)$,

$$\begin{aligned} & \int_0^t\int_\Omega \ln|u(x,t)|^k u(x,t)\,w(x)\,dx\,d\tau \\ = & \int_\Omega u_{tt}(x,t)\,w(x)\,dx + \int_\Omega \triangle u(x,t)\,\Delta w(x)\,dx\,d\tau \\ & - \int_0^t\int_\Omega \nabla u(x,t)\,\nabla w(x)\,dx\,d\tau - \int_0^t\int_\Omega u_t(x,t)\,w(x)\,dx\,d\tau. \end{aligned}$$

If we take initial data, we note that

$$\begin{aligned} u^m &\to u, \text{ weakly in } L^2\left(0,T;H_0^2(\Omega)\right) \\ u_t^m &\to u_t, \text{ weakly in } L^2\left(0,T;L^2(\Omega)\right) \end{aligned}$$

Thus, using Lion's Lemma [10], we have

$$u^m \to u, C\left(\left([0,T];L^2(\Omega)\right)\right).$$

Therefore, $u^m(x,0)$ makes sense and

$$u^m(x,0)\to u(x,0), \ \ L^2(\Omega).$$

We have

$$u^m(x,0) \to u_0(x,0), \ \ H_0^2(\Omega)$$

hence

$$u(x) = u_0(x).$$

Now, multiply (1) by $\phi \in C_0^\infty(0,T)$ and integrate over $(0,T)$, we obtain for $\forall w \in V_m$, and because of

$$\left(u_t^m \phi(t)\right)' = u_{tt}^m \phi(t) + u_t^m \phi'(t)$$

we get

$$\begin{aligned} -\int_0^T\int_\Omega u_t^m w\phi'(t)\,dxdt &= -\int_0^T\int_\Omega \triangle u^m \Delta w\phi(t)\,dxdt - \int_0^T\int_\Omega \nabla u^m \nabla w\phi(t)\,dxdt \\ &\quad -\int_0^T\int_\Omega u_t^m w\phi(t)\,dxdt + \int_0^T\int_\Omega \ln\left|u^m\right|^k u^m w\phi(t)\,dxdt. \end{aligned}$$

As $m \to \infty$, we have for $\forall w \in H_0^2(\Omega)$ and $\phi \in C_0^\infty(0,T)$

$$\begin{aligned} -\int_0^T\int_\Omega u_t w\phi'(t)\,dxdt &= -\int_0^T\int_\Omega \triangle u \Delta w\phi(t)\,dxdt - \int_0^T\int_\Omega \nabla u \nabla w\phi(t)\,dxdt \\ &\quad -\int_0^T\int_\Omega u_t w\phi(t)\,dxdt + \int_0^T\int_\Omega \ln|u|^k uw\phi(t)\,dxdt. \end{aligned}$$

This means

$$u_{tt} \in L^2[0,T), H^{-2}(\Omega),$$

on the other hand, because of

$$u_t \in \left(L^2[0,T), L^2(\Omega)\right),$$

we obtain

$$u_t \in C\left([0,T), H^{-2}(\Omega)\right).$$

So that

$$u_t^m(x,0) \to u_t(x,0), \ \ H^{-2}(\Omega),$$

but

$$u_t^m(x,0) = u_1^m(x) \to u_1(x), \ \ L^2(\Omega).$$

Hence

$$u_t(x,0) = u_1(x).$$

This finishes the proof of the theorem. □

4 Global existence

In this section we study global existence of problem (1). We prove a global existence result using the potential wells corresponding to the logarithmic nonlinearity.

Now, we define the following functionals

$$J(t) = \frac{1}{2}\left(\|\Delta u\|^2 + \|\nabla u\|^2 + \frac{k}{2}\|u\|^2 - \int_\Omega \ln|u|^k u^2 dx\right), \tag{22}$$

$$I(t) = \|\Delta u\|^2 + \|\nabla u\|^2 - \int_\Omega \ln|u|^k u^2 dx. \tag{23}$$

Then, it is obvious that

$$J(t) = \frac{1}{2}I(u) + \frac{k}{4}\|u\|^2 \tag{24}$$

and

$$E(t) = \frac{1}{2}\|u_t\|^2 + J(u). \tag{25}$$

According to the Logarithmic Sobolev inequality, $J(u)$ and $I(u)$ are well defined. The potential well depth is defined as

$$0 < d = \inf_{u} \left\{ \sup_{\lambda \geq 0} J\left(\lambda u\right) : u \in H_0^2\left(\Omega\right), \left\|\Delta u\right\| \neq 0 \right\} \tag{26}$$

and the well-known Nehari manifold

$$N = \left\{ u : u \in H_0^2\left(\Omega\right) / I\left(u\right) = 0, \left\|\Delta u\right\| \neq 0 \right\}, \tag{27}$$

$$0 < d = \inf_{u \in N} J\left(u\right). \tag{28}$$

Then, we introduce

$$W = \left\{ u : u \in H_0^2\left(\Omega\right) / I\left(u\right) > 0,\ J\left(u\right) < d \right\} \cup \left\{0\right\}.$$

Lemma 6. *For any* $u \in H_0^2\left(\Omega\right),\ \left\|u\right\| \neq 0$ *and let* $g\left(\lambda\right) = J\left(\lambda u\right)$*. Then we have*

$$I\left(\lambda u\right) = \lambda g'\left(\lambda\right) \begin{cases} > 0, & 0 \leq \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda < \lambda^* < \infty \end{cases}$$

where

$$\lambda^* = \exp\left(\frac{\left\|\Delta u\right\|^2 + \left\|\nabla u\right\|^2 - \int\limits_{\Omega} \ln\left|u\right|^k u^2 dx}{k\left\|u\right\|^2} \right).$$

Proof: By the defination of $J\left(u\right)$, we obtain

$$\begin{aligned} g\left(\lambda\right) &= J\left(\lambda u\right) \\ &= \frac{1}{2}\left(\left\|\lambda \Delta u\right\|^2 + \left\|\lambda \nabla u\right\|^2 + - \int\limits_{\Omega} \ln\left|\lambda u\right|^k \left(\lambda u\right)^2 dx \right) + \frac{k}{4}\left\|\lambda u\right\|^2 \\ &= \frac{\lambda^2}{2}\left(\left\|\Delta u\right\|^2 + \left\|\nabla u\right\|^2 \right) + \frac{\lambda^2}{2}\left(\frac{k}{2} - k \ln\left|\lambda\right| \right)\left\|u\right\|^2 - \frac{k\lambda^2}{2}\int\limits_{\Omega} \ln\left|u\right|\left|u\right|^2 dx. \end{aligned}$$

Since $\left\|u\right\| \neq 0,\ \lim\limits_{\lambda \to 0} g\left(\lambda\right) = 0,\ \lim\limits_{\lambda \to \infty} g\left(\lambda\right) = -\infty$. Now, differentiating $g\left(\lambda\right)$ with respect to λ, we have

$$g'\left(\lambda\right) = \lambda\left(\left\|\Delta u\right\|^2 + \left\|\nabla u\right\|^2 - k \ln \lambda \left\|u\right\|^2 - k \int\limits_{\Omega} \ln\left|u\right| \left(u\right)^2 dx \right).$$

We can see clearly that

$$\lambda \frac{dJ\left(\lambda u\right)}{d\lambda} = \lambda g'\left(\lambda\right) = I\left(\lambda u\right).$$

We can derive $I\left(\lambda u\right) = 0$ when

$$\lambda^* = \exp\left(\frac{\left\|\Delta u\right\|^2 + \left\|\nabla u\right\|^2 - \int\limits_{\Omega} \ln\left|u\right|^k u^2 dx}{k\left\|u\right\|^2} \right).$$

Thus, we have

$$I\left(\lambda u\right) = \lambda g'\left(\lambda\right) \begin{cases} > 0, & 0 \leq \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda < \lambda^* < \infty. \end{cases}$$

□

Lemma 7. *Let* $u \in H_0^2\left(\Omega\right)$ *and* $l = e^{\frac{2}{k} + 2\frac{2\pi}{k}}$.

i) If $0 < \left\|u\right\|^2 < l$, then $I\left(u\right) > 0$;
ii) If $I\left(u\right) = 0$ and $\left\|u\right\| \neq 0$, then $\left\|u\right\|^2 > l$;
iii)The constant d in (26) satisfies

$$d \geq \frac{\pi}{k} e^{\frac{2}{k} + 2}.$$

Proof: Thanks to Logarithmic Sobolev Inequality to the last term of the $I(u)$ function, we have

$$\begin{aligned} I(u) &= \|\Delta u\|^2 + \|\nabla u\|^2 - \int_\Omega \ln|u|^k u^2 dx, \\ &\geq \|\Delta u\|^2 + \|\nabla u\|^2 - \frac{k}{2}\|u\|^2 \ln\|u\|^2 \\ &\quad - \frac{k\alpha^2}{2\pi}\|\nabla u\|^2 + k(1+\ln\alpha)\|u\|^2, \\ &\geq \left(1 - \frac{k\alpha^2}{2\pi}\right)\|\nabla u\|^2 + k\left((1+\ln\alpha) - \frac{1}{2}\ln\|u\|^2\right)\|u\|^2 \end{aligned} \tag{29}$$

Taking any α satisfying $0 < \alpha \leq \sqrt{\frac{2\pi}{k}}$ in (29), we have

$$I(u) \geq \left(k(1 + \ln\sqrt{\frac{2\pi}{k}}) - \frac{k}{2}\ln\|u\|^2\right)\|u\|^2. \tag{30}$$

i) If $0 < \|u\|^2 < l$, then $I(u) > 0$ from the above inequality.
ii) If $I(u) = 0$ and $\|u\| \neq 0$, then

$$\|u\|^2 \geq e^2 \frac{2\pi}{k} = l.$$

iii) Because of (26), we write

$$\sup_{\lambda\geq 0} J(\lambda u) = J(\lambda^* u) = \frac{1}{2} I(\lambda^* u) + \frac{k}{4}(\lambda^*)^2\|u\|^2 \tag{31}$$

By the Lemma 7 and (30), we obtain

$$0 = I(\lambda^* u) \geq \left(k(1 + \ln\sqrt{\frac{2\pi}{k}}) - \frac{k}{2}\ln\|\lambda^* u\|^2\right)\|\lambda^* u\|^2.$$

Therefore; we have

$$0 \geq k(1 + \ln\sqrt{\frac{2\pi}{k}}) - \frac{k}{2}\ln\|\lambda^* u\|^2,$$

$$\ln\|\lambda^* u\|^2 \geq 2 + 2\ln\sqrt{\frac{2\pi}{k}}),$$

$$\|\lambda^* u\|^2 \geq e^2\frac{2\pi}{k} = l. \tag{32}$$

Thus, by using of (26), (31) and (32) , we obtain

$$d \geq \frac{\pi}{2k}e^2.$$

□

Lemma 8. *Let* $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ *and* $l = e^2\frac{2\pi}{k}$ *such that* $0 < E(0) < \frac{k}{4}l < d$ *and* $I(u_0) > 0$*. Then any solution of (1),* $u \in W$*.*

Proof: Let T be maximal existence time of weak solution of u. From (25) and (9), we have

$$\frac{1}{2}\|u_t\|^2 + J(u) \leq \frac{1}{2}\|u_1\|^2 + J(u_0) < d, \forall t \in [0, T). \tag{33}$$

Then we claim that $u(t) \in W$ for all $t \in [0, T)$. If it is false, then there is a $t_0 \in [0, T)$ such that $u(t_0) \in \partial W$, so we have
(a) either $I(u(t_0)) = 0$ and $\|\Delta u(t_0)\| \neq 0$, or (b) $J(u(t_0)) = d$.
By (33), (b) is impossible, thus we have $I(u(t_0)) = 0$ and $\|\Delta u(t_0)\| \neq 0$. However, at least one $J(u(t_0)) \geq d$ exists if $0 < d = \inf_{u\in N} J(u)$. Because of this contradiction, $u(t) \in W$ is found for $\forall t \in [0, T)$. □

5 Decay of solution

In this section, we will prove decay of solutions to problem (1).
For this purpose, we use the Lyapunov functional

$$L(t) = E(t) + \varepsilon\int_\Omega u u_t dx + \frac{\varepsilon}{2}\int_\Omega u^2 dx \tag{34}$$

where ϵ is a positive constant. We will show the $L(t)$ and $E(t)$ are equivalent:

Lemma 9. *For $\epsilon > 0$ small enough, the relation*

$$\beta_1 L(t) \leq E(t) \leq \beta_2 L(t) \tag{35}$$

holds for two positive constants β_1 and β_2.

We can choose ε small enough such that $L \sim E$.

Theorem 10. *Let $u_0 \in W$, $u_1 \in L^2(\Omega)$. Assume further $0 < E(0) < \alpha l < d$, where*

$$l = \frac{2\pi}{k} e^2 \quad \text{and } 0 < \alpha \leq \frac{\sqrt{2\pi}}{k} \alpha^{\frac{1}{2}} e$$

then there exist two positive constants c_1 and c_2 such that

$$0 < E(t) \leq c_1 e^{-c_2 t},\ t \geq 0.$$

Proof: By taking the time derivative of the $L(t)$ and using Eq. (1), we obtain

$$\begin{aligned} L'(t) &= E'(t) + \varepsilon \int_\Omega \left(u_{tt}u + u_t^2\right) dx + \varepsilon \int_\Omega u u_t dx \\ &= (\varepsilon - 1)\|u_t\|^2 - \varepsilon\left(\|\triangle u\|^2 + \|\nabla u\|^2\right) + \varepsilon \int_\Omega \ln|u|^k u^2. \end{aligned} \tag{36}$$

Adding and subtracting $\varepsilon\beta E(t)$ into (36) where β is a positive constant, we get

$$\begin{aligned} L'(t) &= \left(\varepsilon + \frac{\varepsilon}{2}\beta - 1\right)\|u_t\|^2 + \varepsilon\left(\frac{\beta}{2} - 1\right)\|\triangle u\|^2 + \varepsilon\left(\frac{\beta}{2} - 1\right)\|\nabla u\|^2 \\ &\quad + \varepsilon\left(1 - \frac{\beta}{2}\right)\int_\Omega \ln|u|^k u^2 + \frac{k}{4}\varepsilon\beta\|u\|^2 - \varepsilon\beta E(t). \end{aligned} \tag{37}$$

By the Logarithmic Sobolev inequality and embedding theorems and choosing c_p is smallest enough positive constant, we have

$$\begin{aligned} L'(t) &\leq \left(\varepsilon + \frac{\varepsilon}{2}\beta - 1\right)\|u_t\|^2 + \varepsilon\left(\frac{\beta}{2} - 1\right)\|\triangle u\|^2 \\ &\quad + \varepsilon\left(\frac{\beta}{2} - 1\right)\|\nabla u\|^2 + \frac{k}{4}\varepsilon\beta\|u\|^2 - \varepsilon\beta E(t) \\ &\quad + \varepsilon\left(1 - \frac{\beta}{2}\right)k\left(\frac{1}{2}\|u\|^2 \ln\|u\|^2 + \frac{\alpha^2}{2\pi}\|\nabla u\|^2 - (1 + \ln\alpha)\|u\|^2\right), \\ &= -\varepsilon\beta E(t) + \left(\varepsilon + \frac{\varepsilon}{2}\beta - 1\right)\|u_t\|^2 \\ &\quad + \varepsilon\left(\frac{\beta}{2} - 1\right).\left(1 - k\frac{\alpha^2}{2\pi}\right)\|\nabla u\|^2 + \varepsilon\left(\frac{\beta k c_p}{4} + \frac{\beta}{2} - 1\right)\|\triangle u\|^2 \\ &\quad + \varepsilon k\left[\left(1 - \frac{\beta}{2}\right).\left(\frac{1}{2}\ln\|u\|^2 - (1 + \ln\alpha)\right)\right]\|u\|^2. \end{aligned}$$

Noting that since $\beta = \min\left\{2, \frac{4}{2 + kc_p}\right\}$, and $\varepsilon > 0$ sufficiently small so that

$$\varepsilon + \frac{\varepsilon}{2}\beta - 1 < 0,$$

we get

$$\begin{aligned} L'(t) &\leq -\varepsilon\beta E(t) - \varepsilon\left(1 - \frac{\beta}{2}\right)\left(1 - k\frac{\alpha^2}{2\pi}\right)\|\nabla u\|^2 - \varepsilon\left(1 - \frac{\beta}{2} - \frac{\beta k c_p}{4}\right)\|\triangle u\|^2 \\ &\quad + \varepsilon k\left[\left(1 - \frac{\beta}{2}\right)\left(\frac{1}{2}\ln\|u\|^2 - (1 + \ln\alpha)\right)\right]\|u\|^2. \end{aligned} \tag{38}$$

Using (4), (5), (22) and assumption in the Theorem 10, we have

$$
\begin{aligned}
\ln\|u\|^2 &\leq \ln\left(\frac{4}{k}E(t)\right) \\
&\leq \ln\left(\frac{4}{k}E(t)\right) \\
&\leq \ln\left(\frac{4}{k}E(0)\right) \\
&\leq \ln\left(\frac{4}{k}\alpha l\right) \\
&= \ln\left(\frac{2\pi}{k^2}\alpha e^2\right).
\end{aligned}
$$

Taking α satisfying

$$\frac{\sqrt{2\pi}}{k}\alpha^{\frac{1}{2}}e < \alpha \leq \sqrt{\frac{2\pi}{k}}$$

we guarantee

$$\frac{1}{2}\ln\|u\|^2 - (1+\ln\alpha) < 0.$$

Consequently, inequlatiy (38) becomes

$$L'(t) \leq -\varepsilon\beta E(t).$$

By (35), we have

$$L'(t) \leq -\varepsilon\beta\beta_2 L(t) \tag{39}$$

setting $c_2 = \varepsilon\beta\beta_2 > 0$ and integrating (39) between $(0,t)$ gives the following estimate

$$L(t) \leq c_1 e^{-c_2 t}$$

Consequently, by using (35) once again. This completes the proof. □

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Weak Type Estimates Of Hardy Integral Operators On Morrey Spaces With Variable Exponent Lebesgue spaces

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Abstract: We show that when the infimum of the exponent function, Hardy integral operator is a bounded operator from the Morrey space with variable exponent to the weak Morrey space with variable exponent.

Keywords: Hardy integral operator, Morrey spaces, Weak Morrey spaces, Variable exponent.

1 Introduction

We show that when $\inf_{x\in R^n} p(x) = 1$ Hardy integral operator is a bounded linear operator from the Morrey space with variable exponent to the weak Morrey space with variable exponent. In this work, we obtain the weak type estimates for Hardy integral operators on Morrey spaces with variable exponents. We introduce the weak(w) Morrey spaces with variable exponent $M^{p(.)}_{u,w}$ (see Definition 2.2) and show that H is a bounded linear operator that maps $M^{p(.)}_u$ to $M^{p(.)}_{u,w}$. The weak Morrey spaces has applications on the study of Navier-Stokes equations, see [7, 10]. The duality of weak Morrey space is investigated in [11]. Furthermore, we also have the atomic decompositions of weak-Hardy Morrey spaces in [4].

2 Definitions and Auxillary Statements

For any $p(.) : R^n \longrightarrow [1,\infty]$, we define $p^+ = \sup_{x\in R^n} p(x)$ and $p^- = \inf_{x\in R^n} p(x)$ and also

$$R^{p(.)}_\infty = \{x \in R^n : p(x) = \infty\}.$$

And also any $x \in R^n$ and $r > 0$, write $B(x,r) = \{z : |z - x| < r\}$.
Define $\Psi = \{B(x,r) : x \in R^n, r > 0\}$.Furthermore we define

$$\Gamma_{log} = \{p(.) : R^n \longrightarrow [1,\infty] : \frac{1}{p(.)} \quad is \quad globally \quad log-Holder \quad continuous\}.$$

Definition 2.1. The weak Lebesgue space with variable exponent $L^{p(.)}_w$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L^{p(.)}_w} = \sup_{\lambda>0} \lambda \|\chi_{\{x:|f(x)|>\lambda\}}\|_{L^{p(.)}}$$

We call $p(.)$ the exponent function of $L^{p(.)}_w$.

Lemma 1. *(See [5]) If $p(.) : R^n \longrightarrow [1,\infty]$, then $\|.\|_{L^{p(.)}_w}$ is a quasi-norm. We now recall some basic results for $L^{p(.)}$. For some details on the study of $L^{p(.)}$. the reader is referred to [2, 8]. For any exponent function $p(.) : R^n \longrightarrow [1,\infty]$, define $p^{'}(.)$ by*

$$\frac{1}{p(.)} + \frac{1}{p^{'}(.)} = 1$$

with the convention that $\frac{1}{\infty} = 0$.

Lemma 2. *(See [5]) Let $p(.) : R^n \longrightarrow [1,\infty]$. For any Lebesgue measurable set E with $|E| < \infty$, we have*

$$\|\chi_E\|_{L^{p(.)}} = \|\chi_E\|_{L^{p(.)}_w}.$$

Theorem 1. *(See $[8, Theorem 4.3.8]$) Let $p(.) : R^n \longrightarrow [1,\infty]$. If $p(.) \in \Gamma_{log}$ with $p^- > 1$, then the Hardy-Littlewood maximal operator M is bounded on $L^{p(.)}$.*

Lemma 3. *(See $[5]$) Let $p(.) : R^n \longrightarrow [1,\infty]$ be a globally log-Holder continuous with $1 \le p^- \le p^+ < \infty$. Then, there exists a constant $C > 0$ such that for any $B \in \psi$ we have*

$$|B| \le \|\chi_B\|_{L^{p(.)}} \|\chi_B\|_{L^{p'(.)}} \le C|B|.$$

Lemma 4. *(See $[8, Corollary 4.5.9]$) Let $p(.) \in \Gamma_{log}$. There exist constants $K, C > 0$ such that for any $B \in \psi$, we have*

$$K|B|^{\frac{1}{p_B}} \le \|\chi_B\|_{L^{p(.)}} \le C|B|^{\frac{1}{p_B}}.$$

Theorem 2. *(See $[1, Theorem 1.8(for \quad \alpha = 1)]$).Let $p(.) : R^n \longrightarrow [1,\infty]$. Suppose that $p(.)$ is globally log-Holder continuous and satisfies $1 < p^- \le p^+ < n$. Define $q(.)$ by*

$$\frac{1}{p(.)} - \frac{1}{q(.)} = \frac{1}{n} \tag{1}$$

We have a constant $C > 0$ such that for any $f \in L^{p(.)}$,

$$\|Hf\|_{L^{q(.)}} \le C\|f\|_{L^{p(.)}}$$

We see that whenever $p(.)$ and $q(.)$ satisfy (1), we have

$$\frac{1}{p_B} - \frac{1}{q_B} = \frac{1}{n}, \qquad \forall B \in \psi \tag{2}$$

Theorem 3. *(See $[1, Theorem 1.8(for \quad \alpha = 1)]$).Let $p(.) : R^n \longrightarrow [1,\infty]$. Suppose that $p(.)$ is globally log-Holder continuous and satisfies $1 \le p^- \le p^+ < n$. Let $q(.)$ be defined by (1). We have a constant $C > 0$ such that for any $f \in L^{p(.)}$,*

$$\|Hf\|_{L^{q(.)}_w} \le C\|f\|_{L^{p(.)}}.$$

Definition 2.2. Let $p(.) : R^n \longrightarrow [1,\infty)$ and $u : R^n \times (0,\infty) \longrightarrow (0,\infty)$. The Morrey space with variable exponent $M_k^{p(.)}$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{M_k^{p(.)}} = \sup_{B(x,r)\in\psi} \frac{1}{k(x,r)} \|f\chi_{B(x,r)}\|_{L^{p(.)}} < \infty$$

The weak Morrey space with variable exponent $M_{k,w}^{p(.)}$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{M_{k,w}^{p(.)}} = \sup_{B(x,r)\in\psi} \frac{1}{k(x,r)} \|f\chi_{B(x,r)}\|_{L_w^{p(.)}} < \infty.$$

3 Main Result

Theorem 4. *Let $p(.) : R^n \longrightarrow [1,\infty)$ and $k : R^n \times (0,\infty) \longrightarrow (0,\infty)$. Suppose that $p(.)$ is globally log-Holder continuous and satisfies $1 \le p^- \le p^+ < n$. Let $q(.)$ be defined by (1).If there exists a constant $C > 0$ such that for any $x \in R^n$ and $r > 0$, k satisfies*

$$\sum_{j=0}^{\infty} = \frac{\|\chi_{B(x,r)}\|_{L^{q(.)}}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{q(.)}}} k(x, 2^{j+1}r) \le Ck(x,r) \tag{3}$$

then we have a constant $C > 0$ such that for any $f \in M_k^{p(.)}$,

$$\|Hf\|_{M_{k,w}^{q(.)}} \le C\|f\|_{M_k^{p(.)}}.$$

Proof: Let $f \in M_k^{p(.)}$. For any $z \in R^n$ and $r > 0$, write $f_0 = \chi_{B(z,2r)}f$ and $f_j = \chi_{B(z,2^{j+1}r)/B(z,2^j r)}f$, $j \in N/\{0\}$. We have $f = \sum_{j=0}^{\infty} f_j$. In view of Theorem 2.7, we find that

$$\|\chi_{B(z,r)}Hf_0\|_{L_w^{q(.)}} \le C\|f_0\|_{L^{p(.)}} = C\|f\chi_{B(z,2r)}\|_{L^{p(.)}} \tag{4}$$

Notice that there exists a constant $C > 0$ such that for any $z \in R^n$ and $r > 0$,

$$\chi_{B(z,2r)} \le CM_{\chi_{B(x,r)}}$$

Moreover, whenever $p(.)$ is globally log-Holder continuous with $1 \le p^- \le p^+ < \infty$, then $q(.)$ is globally log-Holder continuous with $1 < p^- \le p^+ < \infty$. Therefore, Theorem 2.3 asserts that

$$\|\chi_{B(z,2r)}\|_{L^{q(.)}} \le C\|M_{\chi_{B(z,r)}}\| \le C\|\chi_{B(z,r)}\|_{L^{q(.)}}$$

for some $C > 0$. Consequently, (3) gives $k(z, 2r) < Ck(z,r)$ for some $C > 0$ independent of z and r. As a result of the above inequality, (4) yields

$$\frac{1}{k(z,r)}\|\chi_{B(z,r)}Hf_0\|_{L_w^{q(.)}} \leq C\frac{1}{k(z,r)}\|\chi_{B(z,2r)}f\|_{L^{p(.)}}$$

$$\leq C\frac{1}{k(z,2r)}\|\chi_{B(z,2r)}f\|_{L^{p(.)}} \leq C\|f\|_{M_k^{p(.)}} \tag{5}$$

Next, for any $j \geq 1$, we have that for any $x \in B(z,r)$

$$|Hf_j(x)| \leq C2^{-j(n-1)}r^{-n+1}\int_{B(z,2^{j+1}r)}|f(y)|dy.$$

The Holder inequality for $L^{p(.)}$ gives

$$\chi_{B(z,r)}(x)|Hf_j(x)|$$

$$\leq C2^{-j(n-1)}r^{-n+1}\chi_{B(z,r)}(x) \times \|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(.)}}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(.)}} \tag{6}$$

Since $p(.)$ is globally log-Holder continuous with $1 \leq p^- \leq p^+ < \infty$. Lemma 2.4 ensures that

$$D_j \leq C2^{-j(n-1)}r^{-n+1}2^{n(j+1)}r^n\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(.)}}} \leq C2^jr\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(.)}}}$$

Lemma 2.5 and (2) show that

$$K\frac{|B(z,2^{j+1}r)|^{\frac{1}{n}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(.)}}} \leq \frac{1}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(.)}}} \leq C\frac{|B(z,2^{j+1}r)|^{\frac{1}{n}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(.)}}}$$

for some $C, K > 0$ independent of z and r.
Since $|B(z,2^{j+1}r)|^{\frac{1}{n}} = C2^jr$, where $C > 0$ is a constant independent of z and $r > 0$, we obtain

$$D_j \leq C\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(.)}}}.$$

Consequently,

$$\chi_{B(z,r)}(x)\sum_{j=1}^{\infty}|Hf_j(x)| \leq C\chi_{B(z,r)}(x)\sum_{j=1}^{\infty}\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(.)}}}.$$

By applying the quasi-norm $\|.\|_{L_w^{q(.)}}$ on both sides of the above inequality, we get

$$\|\chi_{B(z,r)}(x)\sum_{j=1}^{\infty}|Hf_j(x)|\|_{L_w^{q(.)}} \leq C\|\chi_{B(z,r)}\|_{L_{w,k}^{q(.)}}\sum_{j=1}^{\infty}\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(.)}}}$$

$$\leq \sum_{j=1}^{\infty}\frac{\|\chi_{B(z,r)}f\|_{L_w^{q(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(.)}}}k(z,2^{j+1}r)\|f\|_{M_k^{p(.)}}.$$

Lemma 2.2 gives

$$\frac{1}{k(y,r)}\|\chi_{B(z,r)}(x)\sum_{j=1}^{\infty}|Hf_j(x)|\|_{L_w^{q(.)}} \leq \sum_{j=1}^{\infty}\frac{k(z,2^{j+1}r)}{k(z,r)}\frac{\|\chi_{B(z,r)}f\|_{L_w^{q(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(.)}}}\|f\|_{M_k^{p(.)}}$$

$$\leq \sum_{j=1}^{\infty}\frac{k(z,2^{j+1}r)}{k(z,r)}\frac{\|\chi_{B(z,r)}f\|_{L^{q(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(.)}}}\|f\|_{M_k^{p(.)}}.$$

Therefore, (3) and (5) yield

$$\frac{1}{k(z,r)}\|\chi_{B(z,r)}(x)Hf\|_{L_w^{q(.)}}$$

$$C(\frac{1}{k(z,r)}\|\chi_{B(z,r)}(x)Hf_0\|_{L_w^{q(.)}} + \frac{1}{k(z,r)}\|\chi_{B(z,r)}(x)\sum_{j=1}^{\infty}|Hf_j|\|_{L_w^{q(.)}}) \leq C\|f\|_{M_k^{p(.)}}.$$

for some $C > 0$ independent of $B(z,r) \in \psi$. By taking the supremum over $z \in R^n$ and $r > 0$, we obtain

$$\|Hf\|_{M_{k,w}^{q(.)}} \leq C\|f\|_{M_k^{p(.)}}.$$

Thus the proof of Theorem 3.1 is completed.

□

The reader is referred to $([6], pp.366-367)$ for some examples of k that satisfies (3) and the relation between (3) with the conditions imposed on k for the results obtained in $[3, 9]$.

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A Characterization of Homogeneous Fractional Hardy-Type Integrals on Variable Exponent Spaces

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Abstract: In this study, we establish boundedness of homogeneous fractional Hardy-type integral on variable exponent spaces.

Keywords: Boundedness, Fractional Hardy-Type integral, Variable exponent Lebesgue space.

1 Introduction

The theory of variable exponent Lebesgue spaces are started by Orlicz in 1931 and by Nakano in 1950 and 1951. However, the variable exponent function space, due to the failure of translation invariance and related properties, is very difficult to analysis. Nowadays there is an evident increase of investigations related to both the theory of the spaces $L^{q(.)}(R^n)$ themselves and the operator theory in these spaces (See[1-8]). This is caused by possible applications to models with non-standard local growth in elasticity theory, fluid mechanics, differential equations and is based on recent breakthrough result on boundedness of the Hardy-Littlewood maximal operator in these spaces.

Let S^{n-1} denote the unit sphere in Euclidean space R^n and $\Phi \in L^r(S^{n-1})$
$(r \geq 1)$ be homogeneous of degree zero on R^n. For $0 < \beta < n$, the homogeneous fractional integral is defined by

$$T^{\beta}_{\Phi} f(x) = \int_{R^n} \frac{\Phi(x-y)}{|x-y|^{n-\beta}} f(y)dy.$$

It is obvious that T^{β}_{Φ} just be the Riesz potential I^{β} when $\Phi \equiv 1$. Let E be a measurable set in R^n. We denote $p^-_E = inf_{x\in E} p(x)$ and $p^+_E = sup_{x\in E} p(x)$. Especially, we denote $p^- = p^-(R^n)$ and $p^+ = p^+(R^n)$. Let $p(.) : R^n \longrightarrow (0,\infty)$ be a measurable function with $0 < p^- \leq p^+ < \infty$ and $\Delta^0(R^n)$ be the set of all these $p(.)$. Let $\Delta(R^n)$ be the set of all measurable functions $p(.) : R^n \longrightarrow [1,\infty)$ such that $1 < p^- \leq p^+ < \infty$.

The variable Lebesgue space $L^{p(.)}(R^n)$ is defined as the set of all measurable function f for which the quantity $\int_{R^n} |\delta f(x)|^{p(x)} dx$ is finite for some $\delta > 0$ and

$$\|f\|_{L^{p(.)}(R^n)} = inf\{\lambda > 0 : \int_{R^n} (\frac{|f(x)|}{\lambda})^{p(x)} dx \leq 1\}.$$

As a special case of the theory of Nakano and Luxemburg, we see that $L^{p(.)}(R^n)$ is a quasi-normed space. Especially, when $p^- \geq 1$, $L^{p(.)}(R^n)$ is a Banach space. We say that(Log-Holder condition) $p(.) \in LH(R^n)$ if $p(.)$ satisfies

$$|p(x) - p(y)| \leq \frac{C}{-log(|x-y|)}, \quad |x-y| \leq \frac{1}{2}$$

and

$$|p(x) - p(y)| \leq \frac{C}{log|x| + e}, \quad |y| \leq |x|$$

Let $B_k = \{x \in R^n : |x| \leq 2^k\}, \quad A_k = B_k \backslash B_{k-1}, \quad k \in Z$ Let f be a locally integrable function on R^n. The n-dimensional Hardy operator is defined by

$$Hf(x) = \frac{1}{|x|^n} \int_{|t|<|x|} f(t)dt, \quad x \in R^n \backslash \{0\}.$$

In 1995, Christ and Grafakos [2] obtained the result for the boundedness of H on $L^p(R^n)$, $(1 < p < \infty)$ spaces, and they also found the exact operator norms of H on this space. In 2007, Fu et al. [8] gave the central BMO estimates for commutators of n-dimensional fractional and Hardy operators.

Now, we define the n-dimensional fractional Hardy-type operators of variable order $\beta(x)$ as follows.

Definition 1.1. Let f be a locally integrable function on R^n, $0 \leq \beta(x) < n$. The n-dimensional fractional Hardy-type operators of variable order $\beta(x)$ are defined by

$$H_{\beta(.)}f(x) = \frac{1}{|x|^{n-\beta(x)}} \int_{|t|<|x|} f(t)dt$$

and

$$H^*_{\beta(.)}f(x) = \int_{|t|\geq|x|} \frac{f(t)}{|t|^{n-\beta(x)}} dt$$

where $x \in R^n \backslash \{0\}$. Obviously, when $\beta(x) = 0$, $H_{\beta(.)}$ is just H and denote by $H^* = H^*_{\beta(.)} = H^*_0$. And when $\beta(x)$ is constant, $H_{\beta(.)}$ and $H^*_{\beta(.)}$ will become, H_β and H^*_β respectively.
We say that $\omega \in A(p, q)$ with $1 < p, q < \infty$, if there exists a constant $C > 0$, such that for any cube $Q \in R^n$,

$$(\frac{1}{|Q|} \int_Q \omega(x)^q dx)^{\frac{1}{q}} \ (\frac{1}{|Q|} \int_Q \omega(x)^{-p'} dx)^{\frac{1}{p'}} \leq C < \infty.$$

Let $\omega_r(\epsilon)$ be the integral modulus of continuity of order r of Φ defined by

$$\omega_r(\epsilon) = \sup_{|\rho|<\epsilon} (\int_{S^{n-1}} |\Phi(px') - \Phi(x')|^r d\sigma(x'))^{\frac{1}{r}},$$

where ρ is rotation in R^n and $|\rho| = \|\rho - I\|$.

Lemma 1.2. [1] Υ denote a family of ordered pairs of non-negative measurable functions (f, g). Assume that for some p_0 and q_0 with $0 < p_0 \leq q_0 < \infty$ and every weight $\omega \in A_1$,

$$(\int_{R^n} f(x)^{q_0} \omega(x) dx)^{\frac{1}{q_0}} \leq C_0 (\int_{R^n} g(x)^{p_0} \omega(x)^{\frac{p_0}{q_0}} dx)^{\frac{1}{p_0}}, \ \ (f, g) \in \Upsilon$$

Given $p(.) \in \Delta^0(R^n)$ such that $p_0 < p^- \leq p^+ < p_0 q_0 \backslash (q_0 - p_0)$, the function $q(.)$ is defined by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}$ for any $x \in R^n$. If $p(.) \in LH(R^n)$, then for any $(f, g) \in \Upsilon$ and $f \in L^{q(.)}(R^n)$, we have

$$\|f\|_{L^{q(.)}(R^n)} \leq C\|g\|_{L^{p(.)}(R^n)}.$$

Lemma1.3.[7] Suppose that $0 < \beta < n$, $1 \leq r' < p < n \backslash \beta$ and $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$. If $\Phi \in L^r(S^{n-1})$ and $\omega^{r'} \in A(\frac{p}{r'}, \frac{q}{r'})$, then there exists a constant C independent of f such that

$$(\int_{R^n} |T^\beta_\Phi f(x)\omega(x)|^q dx)^{\frac{1}{q}} \leq C (\int_{R^n} |f(x)\omega(x)|^p dx)^{\frac{1}{p}}.$$

2 Result and Discussion

Now let us declare and prove the theorem that gives boundedness of the fractional Hardy-Type integral.

Theorem 2.1. Let $p(.), q(.) \in \Delta(R^n)$, $0 < \beta < n$, $1 < p^- \leq p^+ < \frac{n}{\beta}$ and $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\beta}{n}$ for any $x \in R^n$. If $p(.) \in LH(R^n)$, $\Phi \in L^r(S^{n-1})$ and $1 \leq r' < p^-$, then

$$\|H^\beta_\Phi\|_{L^{q(.)}(R^n)} \leq C\|f\|_{L^{p(.)}(R^n)}$$

Proof. Choose $0 < p_0 \leq q_0 < \infty$ such that $r' < p_0 < p^-$ and $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\beta}{n}$. For any weight function $W(x) = \omega(x)^{q_0} \in A_1$ and any cube $Q \in R^n$ we have

$$\frac{1}{|Q|} \int_Q \omega(x)^{q_0} dx \leq C \inf_{x \in Q} \omega(x)^{q_0}$$

and

$$(\frac{1}{|Q|} \int_Q \omega(x)^{-r'(\frac{p_0}{r'})'} dx)^{\frac{1}{(\frac{p_0}{r'})'}} \leq \sup_{x \in Q} \omega(x)^{-r'} = (\inf_{x \in Q} \omega(x))^{-r'}.$$

These follow that

$$(\frac{1}{|Q|} \int_Q \omega(x)^(r'(\frac{q_0}{r'})) dx)^{\frac{r'}{q_0}} \ (\frac{1}{|Q|} \int_Q \omega(x)^{-r'(\frac{p_0}{r'})'} dx)^{\frac{1}{(\frac{p_0}{r'})'}} \leq C.$$

Thus we see that $\omega^{r'} \in A(\frac{p_0}{r'}, \frac{q_0}{r'})$. By Lemma 1.3, we obtain that

$$(\int_{R^n} |H^\beta_\Phi f(x)\omega(x|^{q_0} dx)^{\frac{1}{q_0}} \leq C (\int_{R^n} |f(x)\omega(x|^{p_0} dx)^{\frac{1}{p_0}}.$$

Finally, we choose the exponent function $p(.)$ and $q(.)$ such that $p_0 < p^- \leq p^+ < \frac{p_0 q_0}{(q_0 - p_0)}$, $p(.) \in LH(R^n)$ and for any $x \in R^n$

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}$$

By Lemma 1.2, we have

$$\|H_\Phi^\beta\|_{L^{q(\cdot)}(R^n)} \leq C\|f\|_{L^{p(\cdot)}(R^n)}.$$

This completes the proof of Theorem 2.1.

3 Conclusion

Under the given conditions, we obtained the boundary of the homogeneous fractional Hardy-type integral in variable exponential spaces. This method can also be applied to different operators and integrals.

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On Spectral Properties of Discontinuous Differential Operator with Second Order

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Abstract: In this work, we consider the spectral problem for a second-order discontinuous differential operator with a spectral parameter in the boundary condition in $L_p, 1 < p < \infty$. We study a method for establishing the basicity of eigenfunctions for such a problem. Such spectral problems arise while one solves the problem of a loaded string fixed at both ends with a load placed in the between ends of the string by the Fourier method.

Keywords: Completeness, Eigenfunctions, Minimality, Spectral problem.

1 Introduction

The spectral problems with discontinuity conditions inside the interval play an important role in mathematics, mechanics, physics and other fields of science. The applications of boundary value problems are related to discontinuous material properties.

The study of spectral properties of many discrete differential operators requires new methods for constructing a basis. This was the motivation for many mathematicians to study intensively the basis properties (such as completeness, minimality, basicity) of the systems of special functions mostly eigen and associated functions of differential operators. For this purpose, various methods were developed for these properties [1]-[8]. However, in the case of a discontinuous differential operator, a system of eigenfunctions emerges, which cannot be demonstrated the basicity properties by standard methods. An example of this situation has been the subject of our study.

In this paper, we consider the following spectral problem with a point of discontinuity

$$y''(x) + \lambda y(x) = 0, \qquad x \in (-1,0) \cup (0,1), \tag{1}$$

$$\begin{aligned} y(-1) = y(1) = 0 \\ y(-0) = y(+0) \\ y'(-0) - y'(+0) = \lambda m y(0) \end{aligned} \tag{2}$$

where λ is the spectral parameter,m is a non-zero complex number. This problem comes from the problem of vibrations of a loaded string with the fixed ends with a load placed in the middle of a string when the problem was solved by applying Fourier methods [[9]-[11]]. For these methods, basis properties of the eigenfunctions system should be studied suitable spaces of functions (generally Lebesgue spaces or Sobolev space).

Grand Lebesgue Spaces introduced by Iwaniec and Sbordone come from integrability properties of the Jacobian determinant ([12]), and the spaces play an important role in PDEs theory (see e.g. [13]) and in Functions Spaces Theory (see e.g. [14]). There are many applications in analysis, see [12]-[19]. These spaces attracted the interest of many researchers, either in Harmonic Analysis (see [20],[21]) and Interpolation-Extrapolation Theory ([22]) or in P.D.Es ([23],[24]).

In subsequent years, quite a number of problems in Harmonic Analysis and the theory of non- linear differential equations were studied in these spaces (see, e.g., the papers [25]-[29]). So, in this work, we study the basicity properties of the eigenfunctions system of the problem (1),(2) in grand Lebesgue spaces. For this purpose, at first, we find corresponding spaces dense in grand Lebesgue spaces. Then we denote that the eigenfunctions system of (1),(2) form a basis on these spaces.

2 Auxiliary informations

Let $\Omega \subset R^n, n \geq 2$ be a measurable set of Lebesgue measure $|\Omega| < +\infty$. In 1992, grand Lebesgue space $L^{n}(\Omega)$ was established by Iwaniec and Sbordone [12] as space such that

$$|Df| \in L^{n)}(\Omega) \Rightarrow |Jf| \in L^1_{loc}(\Omega)$$

for all Sobolev mappings $f : \Omega \to \mathbb{R}^n$, $f = (f_1, \dots, f_n)$. After that we will use letter p instead of n, supposing $1 < p < +\infty$. Grand Lebesgue spaces are defined by

$$L^{p)}(\Omega) = \left\{ f \in M_0 : f_{p)} = \varrho(|f|) = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \left(\frac{1}{|\,\Omega\,|} \int_\Omega |f|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} < +\infty \right\},$$

where M_0 is the set of all real-valued measurable functions on Ω, M_0^+ show the subset nonnegative functions of M_0 and $\varrho : M_0^+ \to [0, +\infty]$ satisfy the following conditions for all $f, g, f_n (n = 1, 2, 3, \dots), \lambda \leq 0$ constants, measurable subsets $E \subset \Omega$

- $\varrho(f) = 0 \Leftrightarrow f = 0$ a.e. in Ω,
- $\varrho(\lambda f) = \lambda \varrho(f)$,
- $\varrho(f + g) \leq \varrho(f) + \varrho(g)$
- $0 \leq g \leq f$ a.e. in $\Omega \Rightarrow \varrho(g) \leq \varrho(f)$,
- $0 \leq f_n \uparrow f$ a.e. in $\Omega \Rightarrow \varrho(f_n) \uparrow \varrho(f)$,
- $E \subset \Omega \Rightarrow \varrho(\chi_E) < +\infty$,
- $E \subset \Omega \Rightarrow \int_E f d\chi \leq C_E \varrho(f)$,

where $C_E, 0 < C_E < \infty$ depend on E and ϱ but not to f. Grand Lebesgue spaces are a special category of Banach Function spaces: The spaces are rearrangement-invariant:

$$\mu_f(\lambda) = |\{\chi \in \Omega : |f(\chi)| > \lambda\}|\,, \qquad for \quad all \quad \lambda \geq 0$$

it is $\varrho(f) = \varrho(g)$ if $\mu_f = \mu_g$. $L_{p)}$ is nonseparable spaces. Because for $\alpha \in \mathbb{R}$

$$f_\alpha(x) = \begin{cases} x^{-1/p}; & x \in [0, \alpha) \\ 0; & x \in [\alpha, 1] \end{cases}$$

functions satisfy the following inequality. For all $\alpha, \beta \in \mathbb{R}, (\alpha \neq \beta)$ there exists $\varepsilon_0 > 0$:

$$\|f_\alpha - f_\beta\|_{p)} \geq \varepsilon_0 > 0,$$

so $L_{p)}(0, 1)$ is nonseparable spaces. But these spaces must be separable so that we can talk about basicity properties. That's why we should study on separable subspaces of $L_{p)}$. Thus, for $\delta > 0$ we give shift operatör in $L_{p)}$,

$$(T_\delta f)(x) = \begin{cases} f(x + \delta); & x + \delta \in [0, 1] \\ 0; & x + \delta \notin [0, 1] \end{cases},$$

where $f \in L_{p)}(0, 1)$. Let us define the following set

$$\widetilde{G}^{p)}(0, 1) = \{f \in L_{p)}(0, 1) : \|T_\delta f - f\|_{p)} \to 0, \delta \to 0\}$$

then it is evident that

$$\overline{(\widetilde{G}^{p)}, \|.\|_{p)})} = G_{p)} \subset L_{p)}.$$

Hence we can express the following lemma.

Lemma 1. *For $1 < p < \infty$, the following expressions are true.*

1. $\overline{(C_0^\infty, \|.\|_{p)})} = G_{p)}$;
2. $\overline{(C_0^\infty, \|.\|_p)} = L_p$

The proof of Lemma 1 can be easily shown.

Let us mention the continuous embedding and we can give the following inclusions from [30]

$$L_p \subsetneq L_{p)} \subsetneq L_{p-\varepsilon}, \qquad 0 < \varepsilon < p - 1.$$

Then we conclude that

$$L_p \subsetneq G_{p)} \subsetneq L_{p)} \subsetneq L_{p-\varepsilon}, \qquad 0 < \varepsilon < p - 1.$$

Because we have the following example. Let us consider the series

$$f(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{n^2},$$

where $f_n(x) = \begin{cases} x^{-1/p}, & x \in (e^{-n^{2p}}; 1] \\ 0, & x \notin (e^{-n^{2p}}; 1] \end{cases}$ .Here $f \notin L_p(0, 1)$ since $\|f_n\|_p^p = n^{2p}$.

Let us denote $f \in G_{p)}(0,1)$.

$$\|f_n\|_{p)} \leq \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^1 x^{\frac{-1}{p}(p-1)} dx \right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon<p-1} p^{\frac{1}{p-\varepsilon}} = p,$$

and we use the partial sum of the series,

$$S_m(x) = \sum_{n=1}^{m} \frac{f_n(x)}{n^2}.$$

From here

$$\left\| \sum_{n=m}^{m+p} \frac{f_n(x)}{n^2} \right\|_{p)} \leq \sum_{n=m}^{m+p} \frac{\|f_n(x)\|_{p)}}{n^2} < p \sum_{n=m}^{m+p} \frac{1}{n^2} < +\infty,$$

then $f \in G_{p)}(0,1)$. Thus

$$L_p(0,1) \subsetneq G_{p)}(0,1)$$

and from definitions

$$G_{p)}(0,1) \subsetneq L_{p)}(0,1)$$

We conclude that $\overline{G_{p)}} = L_{p)}$ and from Lemma 1, $G_{p)}$ is separable for $1 < p < \infty$.

Let us recall the definition of completeness, minimality, basicity and theirs criterions from [31] in any Banach space. Let X be a Banach space.

"A system $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called **complete** in X if $\overline{L[\{x_n\}_{n\in\mathbb{N}}]} = X$."

Completeness Criterion. Let X be a normed space. A system $\{x_n\}_{n\in\mathbb{N}} \subset X$ is complete in X if and only if for all $f \in X^* :< x_n, f >= 0$ for each $n \in \mathbb{N}$ implies $f = 0$.

"A system $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called **minimal** in X if $x_k \notin \overline{L[\{x_n\}_{n\in\mathbb{N}_k}]}$ for all $k \in \mathbb{N}$, where $\mathbb{N}_k = \mathbb{N}\backslash\{k\}$."

"Systems $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{x_n^*\}_{n\in\mathbb{N}} \subset X^*$ are called **biorthogonal** if $< x_m, x_n^* >= \delta_{nm}$ for all $n, m \in \mathbb{N}$."

Minimality Criterion. *A system in a Banach space is minimal if and only if it has a biorthogonal system.*

Minimality Criterion. *A system $\{x_n\}_{n\in\mathbb{N}} \subset X$ form a basis for X if and only if the following conditions are satisfied:*

1. *$\{x_n\}_{n\in\mathbb{N}}$ is complete in X;*
2. *$\{x_n\}_{n\in\mathbb{N}}$ is minimal in X;*
3. *The projectors $P_m(.) = \sum_{n=1}^{m} < ., x_n^* > x_n$ are uniformly bounded, i.e., there exists $M > 0$ such that*

$$\|P_m x\|_X \leq M\|x\|_X, \qquad \forall x \in X,$$

where $\{x_n^*\}_{n\in\mathbb{N}} \subset X^*$ is a system biorthogonal to $\{x_n\}_{n\in\mathbb{N}}$.

Let's give the Dirac delta functional that can find from many sources.

$$\delta(x) = \begin{cases} +\infty; & x = 0 \\ 0; & x \neq 0 \end{cases}$$

imposing that

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1$$

For δ to satisfy the above property, we define δ_ε as

$$\int_{-\infty}^{+\infty} \delta(x)dx = \lim_{\varepsilon\to 0^+} \int_{-\infty}^{+\infty} \delta_\varepsilon(x)dx,$$

where δ_ε is a generic function of both x and ε such that

$$\lim_{\varepsilon\to 0^+} \delta_\varepsilon(x) = \begin{cases} +\infty; & x = 0 \\ 0; & x \neq 0 \end{cases},$$

and

$$\int_{-\infty}^{+\infty} \delta_\varepsilon(x)dx = 1.$$

From here

$$\int_{-\infty}^{+\infty} \delta(x) f(x) dx = \lim_{\varepsilon \to 0^+} \int_{-\infty}^{+\infty} \delta_\varepsilon(x) f(x) dx,$$

for any function $f(x)$ and

$$\int_{-\infty}^{+\infty} \delta(x-c) f(x) dx = f(c).$$

Also, we will use the Muckenhoupt condition [32] in this work. So we mention Hardy's inequality for $1 \le p \le \infty$ and $bp < -1$,

$$\left[\int_0^\infty \left| x^b \int_0^x f(t)dt \right|^p dx\right]^{\frac{1}{p}} \le \frac{-p}{bp+1} \left[\int_0^\infty |x^{b+1} f(x)|^p dx\right]^{\frac{1}{p}}.$$

Later several authors such as Tomaselli, Talenti and Artola investigated the problem of for what functions, $U(x)$ and $V(x)$, there is a finite constant C such that

$$\left[\int_0^\infty \left| U(x) \int_0^x f(t)dt \right|^p dx\right]^{\frac{1}{p}} \le C \left[\int_0^\infty |V(x) f(x)|^p dx\right]^{\frac{1}{p}}. \tag{3}$$

where $U(x)$ and $V(x)$ are weight functions. In 1972, Muckenhoupt gives a condition for the inequality (3):

Theorem 1. *[32] If $1 \le p \le \infty$, there is a finite C for which* (3) *is true if and only if*

$$\sup_{r>0} \left[\int_r^\infty |U(x)|^p dx\right]^{\frac{1}{p}} \left[\int_0^r |V(x)|^{-p'} dx\right]^{\frac{1}{p'}} < \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Now we need to give some notation and results from [33] that will use throughout the paper.

Let us take $\lambda = \rho^2$ and denote the following designation for boundary forms of (2)

$$U_v(y) = U_{v1}(y) + U_{v2}(y), \qquad v = \overline{1,4}$$

where

$$U_{11} = y(-1) \qquad U_{12} \equiv 0,$$
$$U_{21} \equiv 0 \qquad U_{22} = y(1),$$
$$U_{31} = y(0-) \qquad U_{32} = y(0+),$$
$$U_{41} = y'(0-) \qquad U_{42} = -y'(0+) - \lambda m y(0).$$

Lemma 2. *[33] Spectral problem* (1),(2) *has two series of simple eigenvalues:*
$\lambda_{1,n} = (\pi n)^2, \quad n = 1, 2, ...$ *and* $\lambda_{2,n} = (\rho_{2,n})^2, \quad n = 0, 1, 2, ...$ *where* $\rho_{2,n}$*has asymptotic form*

$$\rho_{2,n} = \pi n + \frac{2}{\pi m n} + o\left(\frac{1}{n^2}\right).$$

The eigenfunctions $u_n(x), \quad n = 0, 1, 2, \ldots$ *prescribed by formula*

$$u_{2n-1}(x) = \sin \pi n x, \qquad n = 1, 2, ...,$$

$$u_{2n}(x) = \begin{cases} \sin \rho_{2,n}(1+x) & at x \in [-1, 0] \\ \sin \rho_{2,n}(1-x) & at x \in [0, 1] \end{cases}$$

correspond to them.

Lemma 3. *[33] For Green function components $G_{kj}(x, \xi, \rho)$ the following expressions*

$$G_{11}(x,\xi,\rho) = \begin{cases} -\frac{1}{\rho}\sin\rho(x-\xi) + \frac{1}{\Delta(\rho)}\sin\rho(1+x)\sin\rho(1+\xi) - \frac{1}{\rho\sin\rho}\sin\rho(1+x)\sin\rho\xi, & -1 \le \xi < x \le 0 \\ \frac{1}{\rho}\sin\rho(x-\xi) + \frac{1}{\Delta(\rho)}\sin\rho(1+x)\sin\rho(1+\xi) - \frac{1}{\rho\sin\rho}\sin\rho x\sin\rho(1+\xi), & -1 \le x \le \xi \le 0 \end{cases};$$

$$G_{22}(x,\xi,\rho) = \begin{cases} -\frac{1}{\rho}\sin\rho(x-\xi) + \frac{1}{\Delta(\rho)}\sin\rho(1-x)\sin\rho(1-\xi) + \frac{1}{\rho\sin\rho}\sin\rho x\sin\rho(1-\xi), & 0 \le \xi < x \le 1 \\ \frac{1}{\rho}\sin\rho(x-\xi) + \frac{1}{\Delta(\rho)}\sin\rho(1-x)\sin\rho(1-\xi) - \frac{1}{\rho\sin\rho}\sin\rho(1-x)\sin\rho\xi, & 0 \le x \le \xi \le 1 \end{cases};$$

$$G_{12}(x,\xi,\rho) = \frac{1}{\Delta(\rho)}\sin\rho(1+x)\sin\rho(1-\xi), \quad x \in [-1,0], \xi \in [0,1];$$

$$G_{21}(x,\xi,\rho) = \frac{1}{\Delta(\rho)}\sin\rho(1-x)\sin\rho(1+\xi), \quad x \in [0,1], \xi \in [-1,0].$$

Let (a, b) be an interval on $\mathbb{R}$ and let us define G-Sobolev spaces

$$GW^{p)}(a,b) = \left\{ f, f' \in G^{p)}(a,b) : \|f\|_{W_{p)}} = \|f\|_{p)} + \|f'\|_{p)} \right\}.$$

$GW^{p)}(-1,0) \times GW^{p)}(0,1)$ denotes a space functions whose shrinkages on intervals $[-1,0]$ and $[0,1]$ belong respectively to G-Sobolev Spaces $GW^{p)}(-1,0)$ and $GW^{p)}(0,1)$. We define the operator L in $G_{p)}(-1,1)$ spaces as

$$D(L) = \left\{ \hat{u} \in G_{p)}(-1,1) \oplus \mathbb{C} : \hat{u} = (u, mu(0)); u \in W_G^{p)}; u(-1) = u(1) = 0; u(0-) = u(0+) \right\} \tag{4}$$

where $W_G^{p)} = GW^{p)}(-1,0) \times GW^{p)}(0,1)$ and for $\hat{u} \in D(L)$

$$L\hat{u} = \left(-u''; u'(0-) - u'(0+) \right). \tag{5}$$

Let us take the following equation to construct the resolvent of L.

$$L\hat{u} - \lambda\hat{u} = \hat{f}, \tag{6}$$

where $\hat{u} \in D(L)$, $\hat{f} = (f;\beta) \in G_{p)}(-1,1) \oplus \mathbb{C}$. This equation can be expressed as follows.

$$\begin{cases} -u'' = \lambda u + f, \\ u'(0-) - u'(0+) - \lambda m u(0) = \beta, \\ U_v(u) = 0, \quad v = 1,2,3 \end{cases} \tag{7}$$

We shall use the following Lemma to prove basicity in grand Lebesgue spaces.

Lemma 4. *[33] For solution $\hat{u} = (u; mu(0))$ of the equation (6) it holds the following representations*

$$\begin{aligned} u(x,\rho) = {} & \frac{\beta \sin\rho(1+x)}{\rho(2\cos\rho - \rho m \sin\rho)} - \frac{1}{\rho}\int_{-1}^{x} f(\xi)\sin\rho(x-\xi)d\xi + \frac{1}{\rho}\int_{x}^{0} f(\xi)\sin\rho(x-\xi)d\xi + \\ & + \frac{1}{\Delta(\rho)}\int_{-1}^{0} f(\xi)\sin\rho(1+x)\sin\rho(1+\xi)d\xi - \frac{1}{\rho\sin\rho}\int_{-1}^{x} f(\xi)\sin\rho(1+x)\sin\rho\xi d\xi - \\ & - \frac{1}{\rho\sin\rho}\int_{x}^{0} f(\xi)\sin\rho x \sin\rho(1+\xi)d\xi + \frac{1}{\Delta(\rho)}\int_{0}^{1} f(\xi)\sin\rho(1+x)\sin\rho(1-\xi)d\xi, \end{aligned} \tag{8}$$

if $x \in [-1,0]$;

$$\begin{aligned} u(x,\rho) = {} & \frac{\beta \sin\rho(1-x)}{\rho(2\cos\rho - \rho m \sin\rho)} - \frac{1}{\rho}\int_{0}^{x} f(\xi)\sin\rho(x-\xi)d\xi + \frac{1}{\rho}\int_{x}^{1} f(\xi)\sin\rho(x-\xi)d\xi + \\ & + \frac{1}{\Delta(\rho)}\int_{0}^{1} f(\xi)\sin\rho(1-x)\sin\rho(1-\xi)d\xi + \frac{1}{\rho\sin\rho}\int_{0}^{x} f(\xi)\sin\rho x\sin\rho(1-\xi)d\xi + \\ & + \frac{1}{\rho\sin\rho}\int_{x}^{1} f(\xi)\sin\rho(1-x)\sin\rho\xi d\xi + \frac{1}{\Delta(\rho)}\int_{-1}^{0} f(\xi)\sin\rho(1-x)\sin\rho(1+\xi)d\xi, \end{aligned} \tag{9}$$

if $x \in [0,1]$;

$$u(0,\rho) = \frac{1}{\rho(2\cos\rho - \rho m \sin\rho)}\left[\beta\sin\rho + \int_{-1}^{0} f(\xi)\sin\rho(1+\xi)d\xi + \int_{0}^{1} f(\xi)\sin\rho(1-\xi)d\xi\right]. \tag{10}$$

Finally, let us give the Riesz theorem, which we will apply to the Hilbert transformation. This theorem can be reached from many sources.

Theorem 2. *(Riesz Theorem) Let $\Gamma \in L^p(X,\mu)^*$, where $1 \le p < \infty$ and μ is $\sigma-$finite. Then if $\frac{1}{p} + \frac{1}{q} = 1$, there exists a unique $g \in L^q(X,\mu)^*$ such that*

$$\Gamma(f) = \int_X fg d\mu = \Phi_g(f).$$

Moreover $\|\gamma\| = \|g\|_q$.

3 Main results

Lemma 5. *The operator defined by* (4),(5) *is a linear closed operator with dense definitional domain in* $G_{p)}(-1,1) \oplus \mathbb{C}$*. Eigenfunctions of the operator* L *and problem* (1),(2) *overlap, and* $\hat{u}_k$ *are eigenvectors of the operator* L*, where* $\hat{u}_{2n-1} = (u_{2n-1}(x); 0)\hat{u}_{2n} = (u_{2n}(x); m \sin \rho_{2,n})$.

Proof: For the proof of dense, we take $\hat{u} = (u; \alpha) \in G_{p)}(-1,1) \oplus \mathbb{C}$ and define functional $F(\hat{u})$ as follows

$$F(\hat{u}) = mu(0) - \alpha.$$

Assume that

$$U_v(\hat{u}) = U_v(u), \quad v = 1,2,3.$$

Let us show that F and U_v are bounded linear functionals on $W_G^{p)} \oplus \mathbb{C}$, but unbounded on $G_{p)}(-1,1) \oplus \mathbb{C}$. For boundedness of F and $U_v, v = 1,2,3$ it is sufficient to prove that $\delta_{x_0}(f) = f(x_0)$ Dirac functional is bounded on $W_G^{p)}$ where $x_0 \in (-1,1)$ is any fixed point. For any $f \in W_G^{p)}$,

$$|f(x_0)| = \left| \int_{x_0}^{x} f'(t)dt - f(x) \right| \le \int_{x_0}^{x} |f'(t)|dt + |f(x)|$$

$$2|f(x_0)| \le \int_{-1}^{1} \int_{x_0}^{x} |f'(t)|dtdx + \int_{-1}^{1} |f(x)|dx \le 2 \int_{-1}^{1} |f'(t)|dt + \int_{-1}^{1} |f(x)|dx \le (2\|f'\|_{p-\varepsilon_0} +$$

$$\|f\|_{p-\varepsilon_0} 2^{1-\frac{1}{p-\varepsilon_0}} \le 2^{2-\frac{1}{p-\varepsilon_0}} \varepsilon_0^{\frac{1}{p-\varepsilon_0}} (\|f'\|_{p)} + \|f\|_{p)}),$$

then

$$|\delta_{x_0}(f)| \le 2^{2-\frac{1}{p-\varepsilon_0}} \varepsilon_0^{\frac{1}{p-\varepsilon_0}} \|f\|_{W_G^{p)}}.$$

So δ_{x_0} is bounded on $W_G^{p)}$ but unbounded on $G_{p)}(-1,1)$ because for $f \in G_{p)}(-1,1)$,

$$\|f\|_{p)} \le (p-1) 2^{\frac{p-1}{p}} \|f\|_p = C_p \|f\|_p,$$

then for $g \in L_p(-1,1)$

$$\sup_{\|g\|_p \le 1} |\delta_{x_0}(g)| = \sup_{C_p \|f\|_p \le 1} |\delta_{x_0}(C_p f)| = C_p \sup_{C_p \|f\|_p \le 1} |\delta_{x_0}(f)| \le C_p \sup_{\|f\|_{p)} \le 1} |\delta_{x_0}(f)|.$$

We conclude that δ_{x_0} is unbounded on $W_G^{p)}(-1,1)$ since it is unbounded on $L_p(-1,1)$ [10]. It is evident that $F, U_v, v = 1,2,3$ are bounded on $W_G^{p)} \oplus \mathbb{C}$ and unbounded on $G^{p)}(-1,1) \oplus \mathbb{C}$. Therefore the set

$$D(L) = \left\{ \hat{u} = (u, \alpha) : u \in W_G^{p)}; F(\hat{u}) = U_v(\hat{u}) = 0, v = 1,2,3 \right\}$$

is everywhere dense in $G^{p)}(-1,1) \oplus \mathbb{C}$ and L is a closed operator as a contraction of the corresponding closed maximal operator. The second part of the lemma is certified directly. □

Theorem 3. *Eigenvectors of operator* L *form a basis in spaces* $G^{p)}(-1,1) \oplus \mathbb{C}, 1 < p < \infty$.

4 Conclusion

In this study, the problem (1),(2) is discussed in grand spaces and basic properties are examined. It is foreseen that these properties can be examined in more general cases of this problem as arbitrary point for discontinuity.

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On a New Metric Space

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Abstract: In this presentations, the definition of new metric space with neutrosophic numbers is given. Several topological and structural properties have been investigated. The analogue of Baire Category Theorem is given for Neutrosophic metric spaces.

Keywords: Baire category theorem, Completeness, Hausdorffness, Neutrosophic metric space, Nowhere dense.

1 Introduction

Fuzzy Sets (FSs) put forward by Zadeh [1] has influenced deeply all the scientific fields since the publication of the paper. It is seen that this concept, which is very important for real-life situations, had not enough solution to some problems in time. New quests for such problems have been coming up. Atanassov [2] initiated Intuitionistic fuzzy sets (IFSs) for such cases. Neutrosophic set (NS) is a new version of the idea of the classical set which is defined by Smarandache [3]. Examples of other generalizations are FS [1] interval-valued FS [4], IFS [2], interval-valued IFS [5], the sets paraconsistent, dialetheist, paradoxist, and tautological [6], Pythagorean fuzzy sets [7] .

Using the concepts Probabilistic metric space and fuzzy, fuzzy metric space (FMS) is introduced in [8]. Kaleva and Seikkala [9] have defined the FMS as a distance between two points to be a non-negative fuzzy number. In [10] some basic properties of FMS studied and the Baire Category Theorem for FMS proved. Further, some properties such as separability, countability are given and Uniform Limit Theorem is proved in [11]. Afterward, FMS has used in the applied sciences such as fixed point theory, image and signal processing, medical imaging, decision-making et al. After defined of the intuitionistic fuzzy set (IFS), it was used in all areas where FS theory was studied. Park [12] defined IF metric space (IFMS), which is a generalization of FMSs. Park used George and Veeramani's [10] idea of applying t-norm and t-conorm to the FMS meanwhile defining IFMS and studying its basic features.

Bera and Mahapatra defined the neutrosophic soft linear spaces (NSLSs) [13]. Later, neutrosophic soft normed linear spaces(NSNLS) has been defined by Bera and Mahapatra [14]. In [14], neutrosophic norm, Cauchy sequence in NSNLS, convexity of NSNLS, metric in NSNLS were studied.

In present study, from the idea of neutrosophic sets, new metric space was defined which is called Neutrosophic metric Spaces (NMS). We investigate some properties of NMS such as open set, Hausdorff, neutrosophic bounded, compactness, completeness, nowhere dense. Also we give Baire Category Theorem and Uniform Convergence Theorem for NMSs.

2 Preliminaries

Let's consider that K is a space of points(objects). Denote the $T_U(a)$ is a truth-MF, $I_U(a)$ is an indeterminacy-MF and $F_U(a)$ is a falsity-MF, where U is a set in K with $a \in K$. Then, if we take $J =]0^-, 1^+[$

$$T_U(a) : K \to J,$$
$$I_U(a) : K \to J,$$
$$F_U(a) : K \to J.$$

There is no restriction on the sum of $T_U(a)$, $I_U(a)$ and $F_U(a)$. Therefore,

$$0^- \leq \sup T_U(a) + \sup I_U(a) + \sup F_U(a) \leq 3^+.$$

The set U which consist of with $T_U(a)$, $I_U(a)$ and $F_U(a)$ in K is called a neutrosophic sets(NS) and can be denoted by

$$U = \{< a, (T_U(a), I_U(a), F_U(a)) >: a \in K, T_U(a), I_U(a), F_U(a) \in J\} \tag{1}$$

Clearly, NS is an enhancement of $[0, 1]$ of IFSs.

An NS U is included in another NS V, ($U \subseteq V$), if and only if,

$$\begin{aligned}
&\inf T_U(a) \leq \inf T_V(a), \quad \sup T_U(a) \leq \sup T_V(a),\\
&\inf I_U(a) \geq \inf I_V(a), \quad \sup I_U(a) \geq \sup I_V(a),\\
&\inf F_U(a) \geq \inf F_V(a), \quad \sup F_U(a) \geq \sup F_V(a).
\end{aligned}$$

for any $a \in K$. However, NSs are inconvenient to practice in real problems. To cope with this inconvenient situation, Wang et al [15] customized NS's definition and single-valued NSs (SVNSs) suggested. Ye [16], described the notion of simplified NSs, which may be characterized by three real numbers in the $[0,1]$. At the same time, the simplified NSs' operations may be impractical, in some cases [16]. Hence, the operations and comparison way between SNSs and the aggregation operators for simplified NSs are redefined in [17].

According to the Ye [16], a simplification of an NS U, in (1), is

$$U = \{< a, (T_U(a), I_U(a), F_U(a)) >: a \in K\},$$

which called an simplified NS. Especially, if K has only one element $< G_U(a), B_U(a), Y_U(a) >$ is said to be an simplified NN. Expressly, we may see simplified NSs as a subclass of NSs.

An simplified NS U is comprised in another simplified NS V ($U \subseteq V$), iff $G_U(a) \leq G_V(a)$, $B_U(a) \geq B_V(a)$ and $Y_U(a) \geq Y_V(a)$ for any $a \in K$. Then, the following operations are given by Ye[16]:

$$\begin{aligned}
U + V &= \langle G_U(a) + G_V(a) - G_U(a).G_V(a), B_U(a) + B_V(a) - B_U(a).B_V(a), Y_U(a) + Y_V(a) - Y_U(a).Y_V(a)\rangle,\\
U.V &= \langle G_U(a).G_V(a), B_U(a).B_V(a), Y_U(a).Y_V(a)\rangle,\\
\alpha.U &= \langle 1 - (1 - G_U(a))^\alpha, 1 - (1 - B_U(a))^\alpha, 1 - (1 - Y_U(a))^\alpha\rangle \qquad for \quad \alpha > 0,\\
U^\alpha &= \langle G_U^\alpha(a), B_U^\alpha(a), Y_U^\alpha(a)\rangle \qquad for \quad \alpha > 0.
\end{aligned}$$

Definition 1. *Give an operation $\circ : [0,1] \times [0,1] \to [0,1]$. If the operation $\circ$ is satisfying the following conditions, then it is called that the operation $\circ$ is* continuous TN*: For $s, t, u, v \in [0,1]$,*

i. $s \circ 1 = s$,
ii. If $s \leq u$ and $t \leq v$, then $s \circ t \leq u \circ v$,
iii. $\circ$ is continuous,
iv. $\circ$ is commutative and associative.

Definition 2. *Give an operation $\bullet : [0,1] \times [0,1] \to [0,1]$. If the operation $\bullet$ is satisfying the following conditions, then it is called that the operation $\bullet$ is* continuous TC*:*

i. $s \bullet 0 = s$,
ii. If $s \leq u$ and $t \leq v$, then $s \bullet t \leq u \bullet v$,
iii. $\bullet$ is continuous,
iv. $\bullet$ is commutative and associative.

Form above definitions, we note that if we choose $0 < \varepsilon_1, \varepsilon_2 < 1$ for $\varepsilon_1 > \varepsilon_2$, then there exist $0 < \varepsilon_3, \varepsilon_4 < 0, 1$ such that $\varepsilon_1 \circ \varepsilon_3 \geq \varepsilon_2$, $\varepsilon_1 \geq \varepsilon_4 \bullet \varepsilon_2$. Further, if we choose $\varepsilon_5 \in (0,1)$, then there exist $\varepsilon_6, \varepsilon_7 \in (0,1)$ such that $\varepsilon_6 \circ \varepsilon_6 \geq \varepsilon_5$ and $\varepsilon_7 \bullet \varepsilon_7 \leq \varepsilon_5$.

3 New metric spaces

Definition 3. *Take K be an arbitrary set, $\mathcal{N} = \{< a, T(a), I(a), F(a) >: a \in K\}$ be a NS such that $\mathcal{N} : K \times K \times \mathbb{R}^+ \to [0,1]$. Let $\circ$ and $\bullet$ show the continuous TN and continuous TC, respectively. The four-tuple $(K, \mathcal{N}, \circ, \bullet)$ is called neutrosophic metric space(NMS) when the following conditions are satisfied. $\forall a, b, c \in K$,*

i. $0 \leq T(a,b,\lambda) \leq 1, \quad 0 \leq I(a,b,\lambda) \leq 1, \quad 0 \leq F(a,b,\lambda) \leq 1 \quad \forall \lambda \in \mathbb{R}^+$,
ii. $T(a,b,\lambda) + I(a,b,\lambda) + F(a,b,\lambda) \leq 3$, (for $\lambda \in \mathbb{R}^+$),
iii. $T(a,b,\lambda) = 1 \quad$ (for $\lambda > 0$) if and only if $a = b$,
iv. $T(a,b,\lambda) = T(b,a,\lambda) \quad$ (for $\lambda > 0$),
v. $T(a,b,\lambda) \circ T(b,c,\mu) \leq T(a,c,\lambda + \mu) \quad (\forall \lambda, \mu > 0)$,
vi. $T(a,b,.) : [0,\infty) \to [0,1]$ is continuous,
vii. $lim_{\lambda \to \infty} T(a,b,\lambda) = 1 \quad (\forall \lambda > 0)$,
viii. $I(a,b,\lambda) = 0 \quad$ (for $\lambda > 0$) if and only if $a = b$,
ix. $I(a,b,\lambda) = I(b,a,\lambda) \quad$ (for $\lambda > 0$),
x. $I(a,b,\lambda) \bullet I(b,c,\mu) \geq I(a,c,\lambda + \mu) \quad (\forall \lambda, \mu > 0)$,
xi. $I(a,b,.) : [0,\infty) \to [0,1]$ is continuous,
xii. $lim_{\lambda \to \infty} I(a,b,\lambda) = 0 \quad (\forall \lambda > 0)$,
xiii. $F(a,b,\lambda) = 0 \quad$ (for $\lambda > 0$) if and only if $a = b$,
xiv. $F(a,b,\lambda) = F(b,a,\lambda) \quad (\forall \lambda > 0)$,
xv. $F(a,b,\lambda) \bullet F(b,c,\mu) \geq F(a,c,\lambda + \mu) \quad (\forall \lambda, \mu > 0)$,

xvi. $F(a,b,.):[0,\infty)\to[0,1]$ *is continuous,*
xvii. $lim_{\lambda\to\infty}F(a,b,\lambda)=0$ *(for* $\lambda>0$*),*
xviii. If $\lambda\leq 0$*, then* $T(a,b,\lambda)=0$*,* $I(a,b,\lambda)=1$ *and* $F(a,b,\lambda)=1$.

Then $\mathcal{N}=(T,I,F)$ *is called Neutrosophic metric(NM) on* K.

The functions $T(a,b,\lambda), I(a,b,\lambda), F(a,b,\lambda)$ denote the degree of nearness, the degree of neutralness and the degree of non-nearness between a and b with respect to λ, respectively.

Example 1. *Let* $(K,\boldsymbol{d})$ *be a MS. Give the operations* $\circ$ *and* $\bullet$ *as default (min) TN* $a\circ b=\min\{a,b\}$ *and default(max) TC* $a\bullet b=\max\{a,b\}$.

$$T(a,b,\lambda)=\frac{\lambda}{\lambda+d(a,b)},\quad I(a,b,\lambda)=\frac{d(a,b)}{\lambda+d(a,b)},\quad F(a,b,\lambda)=\frac{d(a,b)}{\lambda},$$

$\forall a,b\in K$ *and* $\lambda>0$*. Then,* $(K,\mathcal{N},\circ,\bullet)$ *is NMS such that* $\mathcal{N}:K\times K\times\mathbb{R}^+\to[0,1]$*. This NMS is expressed as produced by a metric* ***d*** *the NM.*

Example 2. *Choose* K *as natural numbers set. Give the operations* $\circ$ *and* $\bullet$ *as TN* $a\circ b=\max\{0,a+b-1\}$ *and TC* $a\bullet b=a+b-ab$*.* $\forall a,b\in F,\quad \lambda>0$

$$T(a,b,\lambda)=\begin{cases}\frac{a}{b} & , \quad (a\leq b),\\ \frac{b}{a} & , \quad (b\leq a),\end{cases}$$

$$I(a,b,\lambda)=\begin{cases}\frac{b-a}{y} & , \quad (a\leq b),\\ \frac{a-b}{x} & , \quad (b\leq a),\end{cases}$$

$$F(a,b,\lambda)=\begin{cases}b-a & , \quad (a\leq b)\\ a-b & , \quad (b\leq a).\end{cases}$$

Then, $(K,\mathcal{N},\circ,\bullet)$ *is NMS such that* $\mathcal{N}:K\times K\times\mathbb{R}^+\to[0,1]$.

Example 3. $\mathcal{N}=\{<a,G(a),B(a),Y(a)>: a\in K\}$ *defined in Example* 1 *is not a NM with TN* $a\circ b=\max\{0,a+b-1\}$ *and TC* $a\bullet b=a+b-ab$.

Example 4. $\mathcal{N}=\{<a,G(a),B(a),Y(a)>: a\in K\}$ *defined in Example* 2 *is not a NM with TN* $a\circ b=\min\{a,b\}$ *and TC* $a\bullet b=\max\{a,b\}$.

Definition 4. *Give* $(K,\mathcal{N},\circ,\bullet)$ *be a NMS,* $0<\varepsilon<1$*,* $\lambda>0$ *and* $a\in K$*. The set* $O(a,\varepsilon,\lambda)=\{b\in K: T(a,b,\lambda)>1-\varepsilon,\quad I(a,b,\lambda)<\varepsilon,\quad F(a,b,\lambda)<\varepsilon\}$ *is said to be the open ball (OB) (center* a *and radius* ε *with respect to* λ*).*

Theorem 1. *Every OB* $O(a,\varepsilon,\lambda)$ *is an open set (OS).*

Theorem 2. *Every NMS is Hausdorff.*

Definition 5. *Let* $(K,\mathcal{N},\circ,\bullet)$ *be a NMS. A subset* A *of* K *is called Neutrosophic-bounded (NB), if there exist* $\lambda>0$ *and* $\varepsilon\in(0,1)$ *such that* $T(a,b,\lambda)>1-\varepsilon$*,* $I(a,b,\lambda)<\varepsilon$ *and* $F(a,b,\lambda)<\varepsilon\quad(\forall a,b\in A)$*.*

Theorem 3. *Every compact subset* A *of a NMS is NB.*

If $(K,\mathcal{N},\circ,\bullet)$ is NMS produces by a metric **d** on K and $A\subset K$, then A is NB if and only if it is bounded. Consequently, with Theorems 2 and 3, we can write:

Corollary 1. *In a NMS, every compact set is closed and bounded.*

Definition 6. *Take* $(K,\mathcal{N},\circ,\bullet)$ *to be a NMS. A sequence* (a_n) *in* K *is called* ***Cauchy*** *if for each* $\varepsilon>0$ *and each* $\lambda>0$*, there exist* $N\in\mathbb{N}$ *such that* $T(a_n,a_m,\lambda)>1-\varepsilon$*,* $I(a_n,a_m,\lambda)<\varepsilon$*,* $F(a_n,a_m,\lambda)<\varepsilon\quad\forall n,m\geq N$*.* $(K,\mathcal{N},\circ,\bullet)$ *is called* ***complete*** *if every Cauchy sequence is convergent with respect to* $\tau_{\mathcal{N}}$*.*

Theorem 4. *Take* $(K,\mathcal{N},\circ,\bullet)$ *to be a NMS. Let's every Cauchy sequence in* K *has a convergent subsequences. Then the NMS* $(K,\mathcal{N},\circ,\bullet)$ *is complete.*

Theorem 5. *Let* $(K,\mathcal{N},\circ,\bullet)$ *is NMS and let* A *be a subset of* K *with the subspace NM* $(T_A,I_A,F_A)=(T|_{A^2\times\mathbb{R}^+},I|_{A^2\times\mathbb{R}^+},F|_{A^2\times\mathbb{R}^+})$*. Then* $(A,\mathcal{N}_A,\circ,\bullet)$ *is complete if and only if* A *is closed subset of* F*.*

Theorem 6. *(Baire Category Theorem) Let* $\{\gamma_n: n\in\mathbb{N}\}$ *be a sequence of dense open subsets of a complete NMS* $(K,\mathcal{N},\circ,\bullet)$*. Then* $\cap_{n\in\mathbb{N}}\gamma_n$ *is also dense in* K*.*

4 Conclusion

The aim of this presentations is to define a neutrosophic metric spaces and examine some properties. The structural characteristic properties of NMSs such as open ball, open set, Hausdorffness, compactness, completeness, nowhere dense in NMS have been established. Analogue of Baire Category Theorem is given for NMS.

This new concept can also be studied to the fixed point theory, as in metric fixed metric theory and so it can constructed the NMS fixed point theory. As is well known, in recent years, the study of metric fixed point theory has been widely researched because of the this theory has a fundamental role in various areas of mathematics, science and economic studies.

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Exact Solutions for Generalized (3+1)-Dimensional Shallow Water-Like (SWL) Equation

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Abstract: In this paper, we demonstrate the exact solutions of the generalized (3+1)-dimensional shallow water-like (SWL) equation by using Bernoulli sub-equation function method. Some new solutions are successfully constructed. We carried out all the computations by Wolfram Mathematica.

Keywords: Generalized (3+1) shallow water-like (SWL) equation , Bernoulli sub-equation function method, Exact solution.

1 Introduction

In this paper, generalized (3 + 1)-dimensional shallow water-like (SWL) equation [1, 2] which is one of nonlinear evolution equations will be discussed and new solutions will be examined.

$$u_{xxxy} + 3u_{xx} * u_y + 3u_x * u_{xy} - u_{yt} - u_{xz} = 0 \quad (1)$$

There are several studies on this equation. Rational solutions and lump solutions are obtained for equation(1) by Zhang et.al.[1] and Grammian and Pfaffian solutions are obtained by Tang et.al.[2]. Also, this equation solved by Tian and Gao[3] via the tanh method,by Zayed[4] via the (G'/G)-expansion method. Lump-type solutions and their interaction solutions are generated by Sadat[5].

2 Material and method

In this part, we use the Bernoulli sub-equation function method[6, 7]. for solutions equation(1).
Step 1. Let's consider the following partial differential equation;

$$P(u, u_x, u_y, u_z, u_t, u_{xx}, u_{xy}, ...) = 0 \quad (2)$$

and take the wave transformation;

$$u(x, y, z, t) = U(\eta), \eta = x + ky + mz - wt \quad (3)$$

where k, m and w are nonzero constants.Substituting equation(3) into equation(2), gives the following nonlinear ordinary differential equation:

$$N = (U, U^{'}, U^{''}, U^{'''}, ...) = 0 \quad (4)$$

Step 2. Considering trial equation of solution in equation(4), it can be written as following;

$$U(\eta) = \sum_{i=0}^{n} a_i F^i(\eta) = a_0 + a_1 * F + a_2 * F^2 + ... + a_n * F^n \quad (5)$$

According to the Bernoulli theory, we can consider the general form of Bernoulli differential equation for as following;

$$F^{'} = \alpha F + \beta * F^M, \alpha \neq 0, \beta \neq 0, M \in R - 0, 1, 2 \quad (6)$$

where $F = F(\eta)$ is Bernoulli differential polynomial. Substituting equation(5,6) into equation(4), it converts an equations of polynomial (F) as following;

$$\Omega(F) = \rho_s F^s + ... + \rho_1 F + \rho_0 = 0 \tag{7}$$

According to the balance principle, we can determine the relationship between n and M.
Step 3. The coefficients of (F) all be zero will yield us an algebraic system of equations;

$$\rho_i = 0, i = 0, ..., s \tag{8}$$

Solving this system , we will specify the values of $a_0, ..., a_n$. Step 4. When we solve nonlinear Bernoulli differential equation equation (6), we obtain the following two situations according to and ;

$$F(\eta) = [\frac{}{} + \frac{E}{e^{(M-1)\eta}}]^{\frac{1}{1-M}}, \neq \tag{9}$$

$$F(\eta) = [\frac{(E-1) + (E+1)tanh(\frac{(1-M)\eta}{2}}{1 - tanh(\frac{(1-M)\eta}{2}}]^{\frac{1}{1-M}}, =, E \in R \tag{10}$$

3 Implementation of proposed method

In this section, application of the Bernoulli sub-equation function method to SWL equation is presented. Using the wave transformation on equation(1)

$$(u(x, y, z, t) = U(\eta), \eta = x + ky + mz - wt, \tag{11}$$

we get the following nonlinear ordinary differential equation:

$$(kU^4 + 6kU^{'}U^{''} + (kw - m)U^{''} = 0, \tag{12}$$

Integrating the equation in equation (12), we get

$$(kU^{'''} + 3k(U^{'})^2 + (kw - m)U^{'} = 0, \tag{13}$$

Finally, If we write V instead of U', the equation (13) becomes a second order nonlinear ordinary differential equation:

$$(kV^{''} + 3kV^2 + (kw - m)V = 0, \tag{14}$$

Balancing equation(14) by considering the highest derivative $(V^{''})$ and the highest power (V^2) , we obtain $n + 2 = 2M$. When determining the value of M, we pay attention to the fact that it is greater than two and take the smallest M value for easier calculation.Choosing $M = 3$ gives $n = 4$. Thus, the trial solution to equation(1) takes the following form:
Choosing $M = 3$,$m = 1$, gives $n = 3$. Thus, the trial solution to Eq.(1) takes the following form:

$$U(\eta) = a_0 + a_1 * F + a_2 * F^2 + a_3 * F^3 + a_4 * F^4 \tag{15}$$

where $F^{'} = wF + dF^3$, $\neq 0$, $\neq 0$. Substituting Eq.(15), its second derivative and power along with $F^{'} = wF + dF^3$, $\neq 0$, $\neq 0$ into equation(14), yields a polynomial in F. Solving the system of the algebraic equations, yields the values of the parameter involved. Substituting the obtained values of the parameters into equation (15), yields the solutions to equation (1). We can find following coefficients: Case 1

$$a_0 = -(1 + k)w/3k; a_1 = 0; a_2 = (4w^(32))/(k/(1 + k)); a_3 = 0; a_4 = -8w^2; = -w/(2(k/(1 + k))) \tag{16}$$

Case 2

$$a_0 = 0; a_1 = 0; a_2 = (4w(-(1 + k)w))/k; a_3 = 0; a_4 = -8w^2; = -(-(1 + k)w)/(2k) \tag{17}$$

Case 3

$$a_0 = -(4^2)/3; a_1 = 0; a_2 = -(32k^3)/(1 + k); a_3 = 0; a_4 = -(128k^{24})/(1 + k)^2; w = (4k^2)/(1 + k). \tag{18}$$

Substituting equation(16) into Eq.(15), gives

$$
\begin{aligned}
u_1(x,y,z,t) = & \frac{4(1+k)w^{\frac{5}{2}}\left(2d\sqrt{\frac{k}{1+k}}-e^{\frac{\sqrt{w}(x+ky+mz-wt)}{\sqrt{\frac{k}{1+k}}}}\sqrt{w}\in\right)}{d\left(-4d^2k+e^{\frac{2\sqrt{w}(x+ky+mz-wt)}{\sqrt{\frac{k}{1+k}}}}w\in^2+e^{\frac{2\sqrt{w}(x+ky+mz-wt)}{\sqrt{\frac{k}{1+k}}}}kw\in^2\right)} \\
& -\frac{(1+k)w(x+ky+mz-wt)}{3k}+\frac{2(1+k)w^2(x+ky+mz-wt)}{dk} \\
& -\frac{2(1+k)w^3(x+ky+mz-wt)}{d^2k}-\frac{2w^{\frac{3}{2}}Arc\tanh[\frac{e^{\frac{2\sqrt{w}(x+ky+mz-wt)}{\sqrt{\frac{k}{1+k}}}}\sqrt{1+k}\sqrt{w\in}}{2d\sqrt{k}}]}{d\sqrt{\frac{k}{1+k}}} \\
& +\frac{2\sqrt{1+k}w^{\frac{5}{2}}Arc\tanh[\frac{e^{\frac{2\sqrt{w}(x+ky+mz-wt)}{\sqrt{\frac{k}{1+k}}}}\sqrt{1+k}\sqrt{w\in}}{2d\sqrt{k}}]}{d^2\sqrt{k}} \\
& -\frac{w^{\frac{3}{2}}Log[4d^2k-e^{\frac{2\sqrt{w}(x+ky+mz-wt)}{\sqrt{\frac{k}{1+k}}}}w\in^2-e^{\frac{2\sqrt{w}(x+ky+mz-wt)}{\sqrt{\frac{k}{1+k}}}}kw\in^2]}{d\sqrt{\frac{k}{1+k}}} \\
& +\frac{w^{\frac{5}{2}}Log[4d^2k-e^{\frac{2\sqrt{w}(x+ky+mz-wt)}{\sqrt{\frac{k}{1+k}}}}w\in^2-e^{\frac{2\sqrt{w}(x+ky+mz-wt)}{\sqrt{\frac{k}{1+k}}}}kw\in^2]}{d^2\sqrt{\frac{k}{1+k}}}
\end{aligned}
\tag{19}
$$

Substituting Eq.(17) into Eq.(15), gives

$$
u_2(x,y,z,t)=\frac{(1+k)w^2\left(\begin{array}{l}\frac{8d^2kw-4de^{\frac{\sqrt{-(1+k)w}(x+ky+mz-wt)}{\sqrt{k}}}\sqrt{k}w\sqrt{-(1+k)w}\in}{\sqrt{-(1+k)w}\left(4d^2k+e^{\frac{\sqrt{-(1+k)w}(x+ky+mz-wt)}{\sqrt{k}}}(1+k)w\in^2\right)} \\ +\frac{2(-d+w)(x+ky+mz-wt)}{\sqrt{k}} \\ +\frac{2(d-w)Arc\tan\left[e^{\frac{\sqrt{-(1+k)w}(x+ky+mz-wt)}{\sqrt{k}}}\sqrt{w}\in\right]}{\sqrt{1+k}\sqrt{w}} \\ +\frac{(d-w)Log\left[4d^2k+e^{\frac{\sqrt{-(1+k)w}(x+ky+mz-wt)}{\sqrt{k}}}(1+k)w\in^2\right]}{\sqrt{-(1+k)w}}\end{array}\right)}{d^2\sqrt{k}}
\tag{20}
$$

Substituting Eq.(18) into Eq.(15), gives

$$
u_3(x,y,z,t)=\frac{4}{3}\sigma^2\left(\begin{array}{l}-(x+ky+mz-wt)+\frac{48k^2\in\sigma^4}{d^2(1+k)^2\left(de^{2(x+ky+mz-wt)\sigma}-\in\sigma\right)} \\ +\frac{12k\sigma\left(d+dk-4k\sigma^2\right)Log\left[de^{2(x+ky+mz-wt)\sigma}-\in\sigma\right]}{d^2(1+k)^2}\end{array}\right)
\tag{21}
$$

4 Result and discussion

New solutions are obtained for the SWL equation using the Bernoulli sub-equation function method. We have seen that the results we obtained are new solutions when we compare them with previous ones.

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On the Conversion of Convex Functions to Certain within the Unit Disk

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Abstract: A function $g(z)$ is said to be univalent in a domain D if it provides a one-to-one mapping onto its image, $g(D)$. Geometrically , this means that the representation of the image domain can be visualized as a suitable set of points in the complex plane. We are mainly interested in univalent functions that are also regular (analytic, holomorphik) in U . Without lost of generality we assume D to be unit disk $U = \{z : |z| < 1\}$. One of the most important events in the history of complex analysis is Riemann's mapping theorem, that any simply connected domain in the complex plane $\mathbb{C}$ which is not the whole complex plane, can be mapped by any analytic function univalently on the unit disk U. The investigation of analytic functions which are univalent in a simply connected region with more than one boundary point can be confined to the investigation of analytic functions which are univalent in U. The theory of univalent functions owes the modern development the amazing Riemann mapping theorem. In 1916, Bieberbach proved that for every $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in class S , $|a_2| \leq 2$ with equality only for the rotation of Koebe function $k(z) = \frac{z}{(1-z)^2}$. We give an example of this univalent function with negative coefficients of order $\frac{1}{4}$ and we try to explain $B_{\frac{1}{4}}\left(1, \frac{\pi}{3}, -1\right)$ with convex functions.

Keywords: Class s, Convex functions, Univalent functions.

1 Introduction

A indicates the class of the functions of form $g(z)$

$$g(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + a_6z^6 + a_7z^7 + a_8z^8 + ...$$

that are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$.
Let $A(n)$ show the A subclass of form's functions

$$g(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + a_6z^6 + a_7z^7 + a_8z^8 + ... \;\; (a_k \geq 0).$$

Let $T(n)$ denote the subclass of $A(n)$ consisting of functions which are univalent in U. Further a function in $T(n)$ is said to be starlike of order $\frac{1}{4}$ if and only if satisfies

$$\Re\left(\frac{zg^{'}(z)}{g(z)}\right) > \frac{1}{4} \qquad (z \in U),$$

and such a subclass of $A(n)$ consisting of all the starlike functions of order $\frac{1}{4}$ is denote by $T_{\frac{1}{4}}(n)$. Also, $g(z) \in T(n)$ is said to be convex of order $\frac{1}{4}$ if and only if satisfies

$$\Re\left\{1 + \frac{zg^{''}(z)}{g^{'}(z)}\right\} > \frac{1}{4} \qquad (z \in U),$$

and the subclass by $C_{\frac{1}{4}}(n)$[1][2][3][6].

For $n = 1$, these notations are usually used as $T_{\frac{1}{4}}(1) = T^*\left(\frac{1}{4}\right)$ this form with starlike function and show us we have this form for convex functions with $C_{\frac{1}{4}}(1) = C^*(\frac{1}{4})$[5].

Theorem 1. *[1] A function $g(z)$ in $A(n)$ is in $T_{\frac{1}{4}}(n)$ if and only if*

$$\sum_{k=n+1}^{\infty}\left(k-\tfrac{1}{4}\right)a_k \le 1-\tfrac{1}{4}=\tfrac{3}{4}.$$

Theorem 2. *[1] A function $g(z)$ in $A(n)$ is in $C_{\frac{1}{4}}(n)$ if and only if*

$$\sum_{k=n+1}^{\infty} k\left(k-\tfrac{1}{4}\right)a_k \le 1-\tfrac{1}{4}=\tfrac{3}{4}.$$

We introduced subclass $A(n,\theta)$ of A, and the subclass $T^*_{\frac{1}{4}}(n,\theta)$ and $C_{\frac{1}{4}}(n,\theta)$ of $A(n,\theta)$ in the we define the subclass with this way. Let $A(n,\theta)$ denote the subclass of A consisting of function of the form

$$g(z)=z-\sum_{k=n+1}^{\infty} e^{i(k-1)\theta}a_k z^n \qquad (a_k\ge 0,\ \ n\in\mathbb{N})\ [4].$$

We note that $A(n,\theta)=A(n)$, that is $A(n,\theta)$ is the subclass of analytic functions with negative coefficients. We denote by $T^*_{\frac{1}{4}}(n,\theta)$ starlike functions and $C_{\frac{1}{4}}(n,\theta)$ the subclass of $A(n,\theta)$ of convex functions of order $\frac{1}{4}$ in U.

Theorem 3. *A function $g(z)$ in $A(n,\theta)$ is in $T^*_{\frac{1}{4}}(n,\theta)$ if and only if*

$$\sum_{k=n+1}^{\infty}\left(k-\tfrac{1}{4}\right)a_k \le 1-\tfrac{1}{4}=\tfrac{3}{4}[4].$$

Theorem 4. *A function $g(z)$ in $A(n,\theta)$ is in $C_{\frac{1}{4}}(n,\theta)$ if and only if*

$$\sum_{k=n+1}^{\infty} k\left(k-\tfrac{1}{4}\right)a_k \le 1-\tfrac{1}{4}=\tfrac{3}{4}[4].$$

Theorem 5. *If g(z) is in $C_{\frac{1}{4}}(n,\theta)$, then*

$$|z|-\tfrac{3}{(n+1)(4n+3)}|z|^{n+1}\le |g(z)|\le |z|+\tfrac{3}{(n+1)(4n+3)}|z|^{n+1}.$$

The right hand equality holds for the function

$$g(z)=z-e^{in\theta}\tfrac{3}{(n+1)(4n+3)}z^{n+1} \qquad \left(z=re^{-i\left(\theta+\frac{\pi}{n}\right)},\quad r<1\right)$$

and the left hand equality holds for the function

$$g(z)=z-e^{in\theta}\tfrac{3}{(n+1)(4n+3)}z^{n+1} \qquad \left(z=re^{-i\theta},\quad r<1\right)[4].$$

Theorem 6. *(Main theorem) If $g(z)\in B_{\frac{1}{4}}(1,\frac{\pi}{3},-1)$, then we have*

$$g(z)=z-\tfrac{3+3i\sqrt{3}}{49}z^2+\tfrac{3-3i\sqrt{3}}{49}z^3+\tfrac{6}{49}z^4-\tfrac{3+3i\sqrt{3}}{49}z^5-\tfrac{3-3i\sqrt{3}}{49}z^6-\ldots \qquad .$$

Proof:
Let $B_{\frac{1}{4}}(n,\theta,h)$ denote the subclass of $A(n,\theta)$ consisting of functions of the form

$$g(z)=z-\sum_{k=n+1}^{\infty} e^{i(k-1)\theta}a_k z^n \qquad (h\ge -n),$$

where

$$b_{k,h}=\frac{(1-\frac{1}{4})^2}{(2-1-\frac{1}{4})(2+1+1-\frac{1}{4})(2-\frac{1}{4})2}=\frac{\frac{9}{16}}{\frac{3}{4}.\frac{7}{4}.\frac{7}{4}.2}=\frac{6}{49}.$$

If we put in place at that $b_{k,h}=\frac{6}{49}$

$$\begin{aligned}
g(z)&=z-\sum_{k=n+1}^{\infty} e^{i(k-1)\theta}\tfrac{6}{49}z^n \qquad (h\ge n,\ \ n\in\mathbb{N},\ \ n\ge 1)\,.\\
g(z)&=z-\tfrac{6e^{i\frac{\pi}{3}}}{49}z^2-\tfrac{6e^{2i\frac{\pi}{3}}}{49}z^3-\tfrac{6e^{3i\frac{\pi}{3}}}{49}z^4-\tfrac{6e^{4i\frac{\pi}{3}}}{49}z^5-\tfrac{6e^{5i\frac{\pi}{3}}}{49}z^6-\ldots\\
&=z-\tfrac{6cis\frac{\pi}{3}}{49}z^2-\tfrac{6cis\frac{2\pi}{3}}{49}z^3-\tfrac{6cis\pi}{49}z^4-\tfrac{6cis\frac{4\pi}{3}}{49}z^5-\tfrac{6cis\frac{5\pi}{3}}{49}z^6-\ldots\\
&=z-\tfrac{2\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}{49}z^2-\tfrac{2\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}{49}z^3-\tfrac{-2}{49}z^4-\tfrac{2\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}{49}z^5-\ldots\\
&=z-\tfrac{3+3i\sqrt{3}}{49}z^2+\tfrac{3-3i\sqrt{3}}{49}z^3+\tfrac{6}{49}z^4+\tfrac{3+3i\sqrt{3}}{49}z^5-\tfrac{3-3i\sqrt{3}}{49}z^6+\ldots
\end{aligned}$$

We have proved the desired answer and we show that $g(z)\in B_{\frac{1}{4}}(1,\frac{\pi}{3},-1)$ and $B_{\frac{1}{4}}(1,\frac{\pi}{3},-1)\in C_{\frac{1}{4}}(1,\frac{\pi}{3})$ so $g(z)$ is convex function.

□

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Negative Coefficient of Starlike Functions of Order 1/2

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Abstract: A function $g(z)$ is said to be univalent in a domain D if it provides a one-to-one mapping onto its image, $g(D)$. Geometrically , this means that the representation of the image domain can be visualized as a suitable set of points in the complex plane. We are mainly interested in univalent functions that are also regular (analytic, holomorphik) in U . Without lost of generality we assume D to be unit disk $U = \{z : |z| < 1\}$. One of the most important events in the history of complex analysis is Riemann's mapping theorem, that any simply connected domain in the complex plane $\mathbb{C}$ which is not the whole complex plane, can be mapped by any analytic function univalently on the unit disk U. The investigation of analytic functions which are univalent in a simply connected region with more than one boundary point can be confined to the investigation of analytic functions which are univalent in U. The theory of univalent functions owes the modern development the amazing Riemann mapping theorem. In 1916, Bieberbach proved that for every $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in class S , $|a_2| \le 2$ with equality only for the rotation of Koebe function $k(z) = \frac{z}{(1-z)^2}$. We give an example of this univalent function with negative coefficients of order $\frac{1}{4}$ and we try to explain $B_{\frac{1}{4}}\left(1, \frac{\pi}{3}, -1\right)$ with convex functions.

Keywords: Class s, Convex functions, Univalent functions.

1 Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (n \in \mathbb{N} = \{1, 2, 3, ...\}) .$$

$f(z)$ is a function in unit disk $U = \{z : |z| < 1\}$and analytic.
Let $A(n)$ denote the subclass of A consisting of functions of form

$$f(z) = z - \sum k = n + 1^{\infty} a_k z^k \quad (a_k \ge 0,\ n \in \mathbb{N} = \{1, 2, 3, ...\}) .$$

Let $T(n)$ denote the subclass of $A(n)$ consisting of functions which are univalent in U. Further a function in $T(n)$ is said to be starlike of order $\dfrac{1}{2}$ if and only if

$$\frac{zf'(z)}{f(z)} > \frac{1}{2} \,(z \in U)$$

and such a subclass of $A(n)$ consisting of all the starlike functions of order $\frac{1}{2}$ is denote by $T_{\frac{1}{2}}(n)$. Also, $f(z) \in T(n)$ is said to be convex of order $\dfrac{1}{2}$ if and only if satisfies

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1}{2} \,(z \in U)$$

and the subclass by $C_{\frac{1}{2}}(n)$ [1][2][3][6].

For $n = 1$, these notations are usually used as $T_{\frac{1}{2}}(1) = T^* \left(\dfrac{1}{2}\right)$ and $C_{\frac{1}{2}}(n) = C^* \left(\dfrac{1}{2}\right)$ [5].

Theorem 1. *A function $f(z)$ in $A(n)$ is in $T_{\frac{1}{2}}(n)$ if and only if*

$$\sum_{k=n+1}^{\infty} \left(k - \tfrac{1}{2}\right) a_k \le 1 - \frac{1}{2} = \frac{1}{2}[1].$$

Theorem 2. *A function $f(z)$ in $A(n)$ is in $C_{\frac{1}{2}}(n)$ if and only if*

$$\sum_{k=n+1}^{\infty}\left(k-\frac{1}{2}\right)a_k \leq 1-\frac{1}{2}=\frac{1}{2}[1].$$

We introduced the subclass A(n,θ) of A, and the subclass $T^*_{\frac{1}{2}}(n,\theta)$ and $C_{\frac{1}{2}}(n,\theta)$of $A(n,\theta)$ in the following manner. Let $A(n,\theta)$ denote the subclass of A consisting of function of the form

$$f(z)=z-\sum_{k=n+1}^{\infty}e^{i(k-1)\theta}a_k z^k \quad (a_k \geq 0,\ n\in\mathbb{N})\ [4].$$

We note that $A(n,\theta)=A(n)$, that is $A(n,0)$ is the subclass of analytic functions with negative coefficients. We denote by $T^*_{\frac{1}{2}}(n,\theta)$ and $C_{\frac{1}{2}}(n,\theta)$ the subclass of $A(n,\theta)$ of starlike and convex functions of order $\dfrac{1}{2}$ in U.

Theorem 3. *A function $f(z)$ in $A(n,\theta)$ is in $T^*_{\frac{1}{2}}(n,\theta)$ if and only if*

$$\sum_{k=n+1}^{\infty}\left(k-\frac{1}{2}\right)a_k \leq 1-\frac{1}{2}=\frac{1}{2}\textit{[4].}$$

Theorem 4. *A function $f(z)$ in $A(n,\theta)$ is in $C_{\frac{1}{2}}(n,\theta)$ if and only if*

$$\sum_{k=n+1}^{\infty}k\left(k-\frac{1}{2}\right)a_k \leq 1-\frac{1}{2}=\frac{1}{2}\textit{[4].}$$

Theorem 5. $f(z)\in A_{\frac{1}{2}}(n,\theta,h)$ *,then* $f(z)\in T^*_{\frac{1}{2}}(n,\theta)$.

Proof:

$$\begin{aligned}\sum_{k=n+1}^{\infty}\left(k-\frac{1}{2}\right)a_{k,h} &= \sum_{k=n+1}^{\infty}\left(k-\frac{1}{2}\right)\frac{\frac{1}{4}}{\left(k+h-\frac{1}{2}\right)\left(k+h+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)}\\ &= \frac{1}{4}\sum_{k=n+1}^{\infty}\left(\frac{2}{2k+2h-1}-\frac{2}{2k+2h+1}\right)=\frac{1}{4}\frac{1}{n+h+\frac{1}{2}}=\frac{1}{4n+4h+2}\end{aligned}$$

$$=\begin{cases}\dfrac{(1/2)^2}{1/2}=\frac{1}{2}, & h=-n,\\ \dfrac{(1/2)^2}{n+h+1/2}<\dfrac{(1/2)^2}{1/2}=\frac{1}{2}, & h>-n.\end{cases}$$

Hence we know that $f(z)$ is an element of $T^*_{\frac{1}{2}}(n,\theta)$. □

Theorem 6. *(Main theorem) If $f(z)\in A_{\frac{1}{2}}\left(1,\frac{\pi}{3},0\right)$, then we have starlike function and $A_{\frac{1}{2}}\left(1,\frac{\pi}{3},0\right)\in S^*$.*

$$f(z)=z-\frac{1+i\sqrt{3}}{45}z^2+\frac{1-i\sqrt{3}}{45}z^3-\frac{2}{45}z^4-\frac{1+i\sqrt{3}}{45}z^5-...$$

Proof: Let $f(z)\in A_{\frac{1}{2}}\left(1,\frac{\pi}{3},0\right)$ denote the subclass of $A\left(1,\frac{\pi}{3}\right)$ consisting of functions of the form

$$f(z)=z-\sum_{k=n+1}^{\infty}e^{i(k-1)\theta}a_k z^k \quad (h\geq -n,\ n\in\mathbb{N}=\{1,2,3,...\})$$

where

$$a_{k,h}=a_{2,0}=\frac{\left(1-\frac{1}{2}\right)}{2\left(2+0-\frac{1}{2}\right)\left(2+0+1-\frac{1}{2}\right)\left(2-\frac{1}{2}\right)}=\frac{\frac{1}{4}}{\frac{45}{8}}=\frac{2}{45}.$$

$$\begin{aligned}f(z)&=z-\sum_{k=n+1}^{\infty}e^{i(k-1)\theta}\frac{2}{45}\quad (h\geq -n,\ n\in\mathbb{N}, n\geq 1)\\ &=z-\frac{2\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}{45}z^2+\frac{2\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}{45}z^3-\frac{-2}{45}z^4-...\end{aligned}$$

$$= z - \frac{2e^{i\frac{\pi}{3}}}{45}z^2 + \frac{2e^{i\frac{\pi}{3}}}{45}z^3 - \frac{2e^{i\frac{\pi}{3}}}{45}z^4 - \frac{2e^{i\frac{\pi}{3}}}{45}z^5 - ...$$
$$= z - \frac{1+i\sqrt{3}}{45}z^2 + \frac{1-i\sqrt{3}}{45}z^3 - \frac{2}{45}z^4 - \frac{1+i\sqrt{3}}{45}z^5 - ...$$

We show the results we've achieved our proof. □

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Neutrosophic Metric Spaces and Fixed Point Results

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Abstract: In this paper, we define the neutrosophic contraction mapping and give a fixed point theorem in neutrsophic metric spaces.

Keywords: Fixed point theorem, Neutrosophic contraction, Neutrosophic metric spaces.

1 Introduction

Fuzzy Sets (FSs) put forward by Zadeh [23] has influenced deeply all the scientific fields since the publication of the paper. It is seen that this concept, which is very important for real-life situations, had not enough solution to some problems in time. New quests for such problems have been coming up. Atanassov [1] initiated Intuitionistic fuzzy sets (IFSs) for such cases. Neutrosophic set (NS) is a new version of the idea of the classical set which is defined by Smarandache [17]. Examples of other generalizations are FS [23] interval-valued FS [19], IFS [1], interval-valued IFS [2], the sets paraconsistent, dialetheist, paradoxist, and tautological [18], Pythagorean fuzzy sets [21] .

Using the concepts Probabilistic metric space and fuzzy, fuzzy metric space (FMS) is introduced in [12]. Kaleva and Seikkala [8] have defined the FMS as a distance between two points to be a non-negative fuzzy number. In [5] some basic properties of FMS studied and the Baire Category Theorem for FMS proved. Further, some properties such as separability, countability are given and Uniform Limit Theorem is proved in [6]. Afterward, FMS has used in the applied sciences such as fixed point theory, image and signal processing, medical imaging, decision-making et al. After defined of the intuitionistic fuzzy set (IFS), it was used in all areas where FS theory was studied. Park [14] defined IF metric space (IFMS), which is a generalization of FMSs. Park used George and Veeramani's [5] idea of applying t-norm and t-conorm to the FMS meanwhile defining IFMS and studying its basic features.

Bera and Mahapatra defined the neutrosophic soft linear spaces (NSLSs) [3]. Later, neutrosophic soft normed linear spaces(NSNLS) has been defined by Bera and Mahapatra [4]. In [4], neutrosophic norm, Cauchy sequence in NSNLS, convexity of NSNLS, metric in NSNLS were studied.

New metric space was defined which is called Neutrosophic metric Spaces (NMS) from the idea of neutrosophic sets [11]. In [11], some properties of NMS such as open set, Hausdorff, neutrosophic bounded, compactness, completeness, nowhere dense are investigated. Also we give Baire Category Theorem and Uniform Convergence Theorem for NMSs.

In this paper, fixed point results for NMSs are given.

2 Preliminaries

Some definitions related to the fuzziness, intuitionistic fuzziness and neutrosophy are given as follows:

The fuzzy subset F of $\mathbb{R}$ is said to be a fuzzy number(FN). The FN is a mapping $F : \mathbb{R} \to [0, 1]$ that corresponds to each real number a to the degree of membership $F(a)$.

Let F is a FN. Then, it is known that [9]

- If $F(a_0) = 1$, for $a_0 \in \mathbb{R}$, F is said to be normal,
- If for each $\mu > 0$, $F^{-1}\{[0, \tau + \mu)\}$ is open in the usual topology $\forall \tau \in [0, 1)$, F is said to be upper semi continuous, ,
- The set $[F]^\tau = \{a \in \mathbb{R} : F(a) \geq \tau\}$, $\tau \in [0, 1]$ is called $\tau-$cuts of F.

Choose non-empty set F. An IFS in F is an object U defined by

$$U = \{< a, G_U(a), Y_U(a) >: a \in F\}$$

where $G_U(a) : F \to [0,1]$ and $Y_U(a) : F \to [0,1]$ are functions for all $a \in F$ such that $0 \leq G_U(a) + Y_U(a) \leq 1$ [1]. Let U be an IFN. Then,

- an IF subset of the $\mathbb{R}$,
- If $G_U(a_0) = 1$ and, $Y_U(a_0) = 0$ for $a_0 \in \mathbb{R}$, normal,
- If $G_U(\lambda a_1 + (1-\lambda)a_2) \geq \min(G_U(a_1), G_U(a_2))$, $\forall a_1, a_2 \in \mathbb{R}$ and $\lambda \in [0,1]$, then the membership function(MF) $G_U(a)$ is called convex,
- If $Y_U(\lambda a_1 + (1-\lambda)a_2) \geq \min(Y_U(a_1), Y_U(a_2))$, $\forall a_1, a_2 \in \mathbb{R}$ and $\lambda \in [0,1]$, then the nonmembership function(NMF)$Y_U(a)$ is concav,
- G_U is upper semi continuous and Y_U is lower semi continuous
- $suppU = cl(\{a \in F : Y_U(a) < 1\})$ is bounded.

An IFS $U = \{< a, G_U(a), Y_U(a) >: a \in F\}$ such that $G_U(a)$ and $1 - Y_U(a)$ are FNs, where $(1 - Y_U)(a) = 1 - Y_U(a)$, and $G_U(a) + Y_U(a) \leq 1$ is called an IFN.

Let's consider that F is a space of points(objects). Denote the $G_U(a)$ is a truth-MF, $B_U(a)$ is an indeterminacy-MF and $Y_U(a)$ is a falsity-MF, where U is a set in F with $a \in F$. Then, if we take $I =]0^-, 1^+[$

$$\begin{aligned} G_U(a) &: F \to I, \\ B_U(a) &: F \to I, \\ Y_U(a) &: F \to I, \end{aligned}$$

There is no restriction on the sum of $G_U(a)$, $B_U(a)$ and $Y_U(a)$. Therefore,

$$0^- \leq \sup G_U(a) + \sup B_U(a) + \sup Y_U(a) \leq 3^+.$$

The set U which consist of with $G_U(a)$, $B_U(a)$ and $Y_U(a)$ in F is called a neutrosophic sets(NS) and can be denoted by

$$U = \{< a, (G_U(a), B_U(a), Y_U(a)) >: a \in F, G_U(a), B_U(a), Y_U(a) \in I\} \tag{1}$$

Clearly, NS is an enhancement of $[0,1]$ of IFSs.

An NS U is included in another NS V, $(U \subseteq V)$, if and only if,

$$\begin{aligned} &\inf G_U(a) \leq \inf G_V(a), \quad \sup G_U(a) \leq \sup G_V(a), \\ &\inf B_U(a) \geq \inf B_V(a), \quad \sup B_U(a) \geq \sup B_V(a), \\ &\inf Y_U(a) \geq \inf Y_V(a), \quad \sup Y_U(a) \geq \sup Y_V(a). \end{aligned}$$

for any $a \in F$. However, NSs are inconvenient to practice in real problems. To cope with this inconvenient situation, Wang et al [20] customized NS's definition and single-valued NSs (SVNSs) suggested.

To cope with this inconvenient situation, Wang et al [20] customized NS's definition and single-valued NSs suggested. Ye [22], described the notion of simplified NSs, which may be characterized by three real numbers in the $[0,1]$. At the same time, the simplified NSs' operations may be impractical, in some cases [22]. Hence, the operations and comparison way between SNSs and the aggregation operators for simplified NSs are redefined in [15].

According to the Ye [22], a simplification of an NS U, in (1), is

$$U = \{< a, (G_U(a), B_U(a), Y_U(a)) >: a \in F\},$$

which called an simplified NS. Especially, if F has only one element $< G_U(a), B_U(a), Y_U(a) >$ is said to be an simplified NN. Expressly, we may see simplified NSs as a subclass of NSs.

An simplified NS U is comprised in another simplified NS V $(U \subseteq V)$, iff $G_U(a) \leq G_V(a)$, $B_U(a) \geq B_V(a)$ and $Y_U(a) \geq Y_V(a)$ for any $a \in F$. Then, the following operations are given by Ye[22]:

$$\begin{aligned} U + V &= \langle G_U(a) + G_V(a) - G_U(a).G_V(a), B_U(a) + B_V(a) - B_U(a).B_V(a), Y_U(a) + Y_V(a) - Y_U(a).Y_V(a) \rangle, \\ U.V &= \langle G_U(a).G_V(a), B_U(a).B_V(a), Y_U(a).Y_V(a) \rangle, \\ \alpha.U &= \langle 1 - (1 - G_U(a))^\alpha, 1 - (1 - B_U(a))^\alpha, 1 - (1 - Y_U(a))^\alpha \rangle \qquad for \quad \alpha > 0, \\ U^\alpha &= \langle G_U^\alpha(a), B_U^\alpha(a), Y_U^\alpha(a) \rangle \qquad for \quad \alpha > 0. \end{aligned}$$

Triangular norms (t-norms) (TN) were initiated by Menger [13]. In the problem of computing the distance between two elements in space, Menger offered using probability distributions instead of using numbers for distance. TNs are used to generalize with the probability distribution of triangle inequality in metric space conditions. Triangular conorms (t-conorms) (TC) know as dual operations of TNs. TNs and TCs are very significant for fuzzy operations(intersections and unions).

Definition 1. *Give an operation* $\circ : [0,1] \times [0,1] \to [0,1]$. *If the operation* $\circ$ *is satisfying the following conditions, then it is called that the operation* $\circ$ *is* continuous TN*: For* $s, t, u, v \in [0,1]$,

i. $s \circ 1 = s$
ii. If $s \le u$ *and* $t \le v$*, then* $s \circ t \le u \circ v$*,*
iii. $\circ$ *is continuous,*
iv. $\circ$ *is commutative and associative.*

Definition 2. *Give an operation* $\bullet : [0,1] \times [0,1] \to [0,1]$*. If the operation* $\bullet$ *is satisfying the following conditions, then it is called that the operation* $\bullet$ *is* continuous TC*:*

i. $s \bullet 0 = s$*,*
ii. If $s \le u$ *and* $t \le v$*, then* $s \bullet t \le u \bullet v$*,*
iii. $\bullet$ *is continuous,*
iv. $\bullet$ *is commutative and associative.*

Form above definitions, we note that if we choose $0 < \varepsilon_1, \varepsilon_2 < 1$ for $\varepsilon_1 > \varepsilon_2$, then there exist $0 < \varepsilon_3, \varepsilon_4 < 0, 1$ such that $\varepsilon_1 \circ \varepsilon_3 \ge \varepsilon_2$, $\varepsilon_1 \ge \varepsilon_4 \bullet \varepsilon_2$. Further, if we choose $\varepsilon_5 \in (0,1)$, then there exist $\varepsilon_6, \varepsilon_7 \in (0,1)$ such that $\varepsilon_6 \circ \varepsilon_6 \ge \varepsilon_5$ and $\varepsilon_7 \bullet \varepsilon_7 \le \varepsilon_5$.

Definition 3. *[11] Take* F *be an arbitrary set,* $V = \mathcal{N} = \{< a, G(a), B(a), Y(a) >: a \in F\}$ *be a NS such that* $\mathcal{N} : F \times F \times \mathbb{R}^+ \to [0,1]$*. Let* $\circ$ *and* $\bullet$ *show the continuous TN and continuous TC, respectively. The four-tuple* $(F, \mathcal{N}, \circ, \bullet)$ *is called neutrosophic metric space(NMS) when the following conditions are satisfied.* $\forall a, b, c \in F$*,*

i. $0 \le G(a,b,\lambda) \le 1, \quad 0 \le B(a,b,\lambda) \le 1, \quad 0 \le Y(a,b,\lambda) \le 1 \quad \forall \lambda \in \mathbb{R}^+$*,*
ii. $G(a,b,\lambda) + B(a,b,\lambda) + Y(a,b,\lambda) \le 3$*, (for* $\lambda \in \mathbb{R}^+$*),*
iii. $G(a,b,\lambda) = 1$ *(for* $\lambda > 0$*) if and only if* $a = b$*,*
iv. $G(a,b,\lambda) = G(b,a,\lambda)$ *(for* $\lambda > 0$*),*
v. $G(a,b,\lambda) \circ G(b,c,\mu) \le G(a,c,\lambda+\mu) \quad (\forall \lambda, \mu > 0)$*,*
vi. $G(a,b,.) : [0,\infty) \to [0,1]$ *is continuous,*
vii. $lim_{\lambda \to \infty} G(a,b,\lambda) = 1 \quad (\forall \lambda > 0)$*,*
viii. $B(a,b,\lambda) = 0$ *(for* $\lambda > 0$*) if and only if* $a = b$*,*
ix. $B(a,b,\lambda) = B(b,a,\lambda)$ *(for* $\lambda > 0$*),*
x. $B(a,b,\lambda) \bullet B(b,c,\mu) \ge B(a,c,\lambda+\mu) \quad (\forall \lambda, \mu > 0)$*,*
xi. $B(a,b,.) : [0,\infty) \to [0,1]$ *is continuous,*
xii. $lim_{\lambda \to \infty} B(a,b,\lambda) = 0 \quad (\forall \lambda > 0)$*,*
xiii. $Y(a,b,\lambda) = 0$ *(for* $\lambda > 0$*) if and only if* $a = b$*,*
xiv. $Y(a,b,\lambda) = Y(b,a,\lambda) \quad (\forall \lambda > 0)$*,*
xv. $Y(a,b,\lambda) \bullet Y(b,c,\mu) \ge Y(a,c,\lambda+\mu) \quad (\forall \lambda, \mu > 0)$*,*
xvi. $Y(a,b,.) : [0,\infty) \to [0,1]$ *is continuous,*
xvii. $lim_{\lambda \to \infty} Y(a,b,\lambda) = 0$ *(for* $\lambda > 0$*),*
xviii. If $\lambda \le 0$*, then* $G(a,b,\lambda) = 0$*,* $B(a,b,\lambda) = 1$ *and* $Y(a,b,\lambda) = 1$*.*

Then $\mathcal{N} = (G, B, Y)$ *is called Neutrosophic metric(NM) on* F*.*

The functions $G(a,b,\lambda), B(a,b,\lambda), Y(a,b,\lambda)$ denote the degree of nearness, the degree of neutralness and the degree of non-nearness between a and b with respect to λ, respectively.

Definition 4. *[11] Give* V *be a NMS,* $0 < \varepsilon < 1$*,* $\lambda > 0$ *and* $a \in F$*. The set* $O(a,\varepsilon,\lambda) = \{b \in F : G(a,b,\lambda) > 1-\varepsilon, \quad B(a,b,\lambda) < \varepsilon, \quad Y(a,b,\lambda) < \varepsilon\}$ *is said to be the open ball (OB) (center* a *and radius* ε *with respect to* λ*).*

Lemma 1. *[11] Every OB* $O(a,\varepsilon,\lambda)$ *is an open set (OS).*

3 Fixed point results

Definition 5. *[7] Let* F *be a set. A non-negative real-valued function* f *on* $F \times F$ *is called as a quasi-metric on* F *if it satisfies the following axioms:*

i. $f(a,b) = f(b,a) = 0$ *if and only if* $a = b$*,*
ii. $f(a,b) \le f(a,c) + f(c,b)$*,*

for all $a, b, c \in F$*.*

From this definition we can understand: It is possible $f(a,b) \ne f(b,z)$ for some $a, b \in F$.

A quasi-metric is a distance function which satisfies the triangle inequality but is not symmetric in general. Quasi-metrics are a subject of comprehensive investigation both in pure and applied mathematics in areas such as in functional analysis, topology and computer science.

Proposition 1. *Let* V *be the NMS. For any* $\varepsilon \in (0,1]$*, define* $h : F \times F \to R^+$ *as follows:*

$$h_\varepsilon(a,b) = \inf\{\lambda > 0 : G(a,b,\lambda) > 1-\varepsilon, \quad B(a,b,\lambda) < \varepsilon, \quad Y(a,b,\lambda) < \varepsilon\}$$

Then,

i. $(F, h_\varepsilon : \varepsilon \in (0,1])$ *is a generating space of quasi-metric family.*
ii. The topology $\tau_{\mathcal{N}}$ *on* $(F, h_\varepsilon : \varepsilon \in (0,1])$ *coincides with the* $\mathcal{N}-$*topology on* V*, that is,* h_ε *is a compatible symmetric for* $\tau_{\mathcal{N}}$*.*

Definition 6. *Let* V *be a NMS. The mapping* $f : F \to F$ *is called neutrosophic contraction(NC) if there exists* $k \in (0,1)$ *such that*

$$\frac{1}{G(f(a), f(b), \lambda)} - 1 \leq k(\frac{1}{G(a,b,\lambda) - 1}), \quad B(f(a), f(b), \lambda) \leq kB(a,b,\lambda), \quad Y(f(a), f(b), \lambda) \leq kY(a,b,\lambda)$$

for each $a, b \in F$ *and* $\lambda > 0$*.*

Definition 7. *Let* V *be a NMS and let* $f : F \to F$ *be a NC mapping. Then there exists* $c \in F$ *such that* $c = f(c)$*. That is,* c *is called neutrosophic fixed point (NFP) of* f*.*

Generally, we claim that the contractions have fixed point. If all contractions(including NC) have fixed points, then we can easily say that f^2 should have a fixed point. In below proposition, we will show that if f^n is a NC then, f^n has fixed point.

Proposition 2. *Suppose that* f *is a NC. Then* f^n *is also a NC. Furthermore, if* k *is the constant for* f*, then* k^n *is the constant for* f^n*.*

Remark 1. *From Proposition* 2*, we can say that each* f^n *has the same fixed point. Because, if we take* $f(a) = a$*, then* $f^2 = f(f(a)) = f(a) = a$ *and by induction,* $f^n(a) = a$*.*

Proposition 3. *Let* f *be a NC and* $a \in F$*.* $f[O(a, \varepsilon, \lambda)] \subset O(a, \varepsilon, \lambda)$ *for large enough values of* ε*.*

Remark 2. *From Proposition* 3 *and the definitions neutrosophic open ball and neutrosophic closed ball, if the inclusion* $f[O(a, \varepsilon, \lambda)] \subset O(a, \varepsilon, \lambda)$ *is hold, then the inclusion also* $\overline{f[O(a, \varepsilon, \lambda)]} \subset \overline{O(a, \varepsilon, \lambda)}$ *is hold.*

Proposition 4. *The inclusion* $f^n[O(a, \varepsilon, \lambda)] \subset O(f^n(a), \epsilon, \lambda)$ *is hold for all* n*, where* $\epsilon = k^n \times \varepsilon$*.*

Remark 3. *It is fact that if the inclusion* $f^n[O(a, \varepsilon, \lambda)] \subset O(f^n(a), \epsilon, \lambda)$ *is hold, then the inclusion also* $\overline{f^n[O(a, \varepsilon, \lambda)]} \subset \overline{O(f^n(a), \epsilon, \lambda)}$ *is hold.*

Propositions 2-4 are proved as similar in [10].

Theorem 1. *Let* V *be a complete NMS. Let* $f : F \to F$ *be a NC mapping. Then,* f *has a unique NFP.*

Theorem 1 is a consequence of Theorem 3.6 in [16]. Hence, using the consept of neutrosophy, Theorem 1 is proved as similar Theorem 3.6 in [16].

4 Conclusion

The purpose of this paper is to apply the NMS which defined by Kirisci and Simsek [11]. NC mapping is defined. After the properties related to NC are proved, fixed point theorem is given.

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On Generalized Sister Celine's Polynomials

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Abstract: In this research, we establish some properties for the generalized Sister Celine's polynomials. We derive various families of multilinear and multilateral generating functions for a family of generalized Sister Celine's polynomials.

Keywords: Generalized Sister Celine's polynomials, Multilinear and multilateral generating functions, Recurrence relations.

1 Introduction

Sister Celine [1] has introduced the polynomial $f_n(x)$

$$f_n\left[\begin{array}{cc} a_1,...,a_p; & \\ & x \\ b_1,...,b_q; & \end{array}\right] = {}_{p+2}F_{q+2}\left[\begin{array}{cc} -n,n+1,a_1,...,a_p; & \\ & x \\ 1,\frac{1}{2},b_1,...,b_q; & \end{array}\right], \tag{1}$$

which is defined by the following generating function (see [2], p.290)

$$\sum_{n=0}^{\infty} f_n\left[\begin{array}{cc} a_1,...,a_p; & \\ & x \\ b_1,...,b_q; & \end{array}\right] t^n = (1-t)^{-1}\,{}_{p+2}F_{q+2}\left[\begin{array}{cc} a_1,...,a_p; & \\ & \frac{-4xt}{(1-t)^2} \\ b_1,...,b_q; & \end{array}\right] |t|<1, \tag{2}$$

where ${}_pF_q$ denotes the generalized hypergeometric function [2].
For $p=1,\ q=1,\ a_1=\frac{1}{2},\ b_1=1$ the following integral representation of Sister Celine polynomials is given by

$$f_n(\frac{1}{2};1;x) = \frac{1}{\sqrt{\pi}}\int_0^{\infty} y^{-1/2}e^{-y}f_n(-;1;xy)dy.$$

Equation (1) with no a's and no b's denotes simply

$$f_n(x) = {}_2F_2\left[-n,n+1;1;\frac{1}{2};x\right] = \sum_{r=0}^{n}\frac{(-1)^n(n)!x^r}{(r!)^2(\frac{1}{2})_r(n-r)!}.$$

For the $f_n(x)$ the generating function (2) becomes

$$\sum_{n=0}^{\infty} f_n(x)t^n = (1-t)^{-1}\exp\left(\frac{-4xt}{(1-t)^2}\right),\ |t|<1. \tag{3}$$

In the view of above results, we define the generalized Sister Celine polynomial in following manner [3]

$$f_n^{(\alpha,\beta)}\left[\begin{array}{cc} a_1,...,a_p; & \\ & x \\ b_1,...,b_q; & \end{array}\right] \tag{4}$$

$$= \frac{(1+\alpha+\beta)_n}{n!}\,{}_{p+2}F_{q+2}\left[\begin{array}{cc} -n,n+\alpha+\beta+1,a_1,...,a_p; & \\ & x \\ 1+\alpha,\frac{1}{2},b_1,...,b_q; & \end{array}\right].$$

Equation (4) with no a's and no b's denotes simply [3]

$$f_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha+\beta)_n}{n!}\ {}_2F_2\left[\begin{array}{cc} -n, n+\alpha+\beta+1; & \\ & x \\ 1+\alpha, \frac{1}{2}; & \end{array}\right] \tag{5}$$
$$= \frac{(1+\alpha+\beta)_n}{n!}\sum_{r=0}^{n}\frac{(-n)_r(n+\alpha+\beta+1)_r x^r}{(1+\alpha)_r(\frac{1}{2})_r r!}.$$

Indeed [3]

$$f_n^{(0,0)}(x) = f_n(x).$$

The following generating function can be easily obtained [3]

$$\sum_{n=0}^{\infty} f_n^{(\alpha,\beta)}(x)t^n = (1-t)^{-1-\alpha-\beta}\ {}_2F_2\left[\begin{array}{cc} \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; & \\ & \frac{-4xt}{(1-t)^2} \\ 1+\alpha, \frac{1}{2}; & \end{array}\right]$$

$$\sum_{n=0}^{\infty}(C)_n f_n^{(\alpha,\beta)}(x)t^n = (1-t)^{-C-\alpha-\beta}\ {}_3F_3\left[\begin{array}{cc} C, \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; & \\ & \frac{-4xt}{(1-t)^2} \\ 1+\alpha, 1+\alpha+\beta, \frac{1}{2}; & \end{array}\right]. \tag{6}$$

Obviously for $C = 1$, $\alpha = \beta = 0$, equation (6) reduces to the generating function (3), and (see, [3])

$$\sum_{n=0}^{\infty} f_n^{(\alpha,\beta)}\left[\begin{array}{cc} a_1, ..., a_p; & \\ & x \\ b_1, ..., b_q; & \end{array}\right] t^n = (1-t)^{-1-\alpha-\beta}$$
$$\times\ {}_{p+2}F_{q+2}\left[\begin{array}{cc} a_1, ..., a_p, \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; & \\ & \frac{-4xt}{(1-t)^2} \\ b_1, ..., b_q, 1+\alpha, \frac{1}{2}, & \end{array}\right].$$

The main object of this paper to study several properties of the Sister Celine polynomials $f_n(x)$ and the generalized Sister Celine's polynomials $f_n^{(\alpha,\beta)}(x)$. Various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials are given.

2 Generating and Special Functions

In this section, we derive several families of bilinear and bilateral generating functions for the Sister Celine polynomials $f_n(x)$ and the generalized Sister Celine's polynomials $f_n^{(\alpha,\beta)}(x)$ generated by using the similar method considered in (see, [4] - [10]).

We begin by stating the following theorem.

Theorem 1. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, ..., y_r)$ of r complex variables $y_1, ..., y_r$ $(r \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(y_1, ..., y_r; \zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r)\zeta^k,\ (a_k \neq 0,\ \mu, \psi \in \mathbb{C})$$

and

$$\Theta_{n,p}^{\mu,\psi}(x; y_1, ..., y_r; \xi) := \sum_{k=0}^{[n/p]} a_k f_{n-pk}(x)\Omega_{\mu+\psi k}(y_1, ..., y_r)\xi^k.$$

Then, for $p \in \mathbb{N}$, we have

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}(x; y_1, ..., y_r; \eta)\,t^n = (1-t)^{-1}\exp\left(\frac{-4xt}{(1-t)^2}\right)\Lambda_{\mu,\psi}(y_1, ..., y_r; \eta t^p), \tag{7}$$

provided that each member of (7) exists.

Proof: For convenience, let H denote the first member of the assertion (7). Then,

$$H = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k f_{n-pk}(x) \Omega_{\mu+\psi k}(y_1, ..., y_r) \eta^k t^n.$$

Replacing n by $n + pk$, we may write that

$$\begin{aligned} H &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k \, f_n(x) \Omega_{\mu+\psi k}(y_1, ..., y_r) \eta^k t^{n+pk} \\ &= \sum_{n=0}^{\infty} f_n(x) t^n \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r)(\eta t^p)^k \\ &= (1-t)^{-1} \exp\left(\frac{-4xt}{(1-t)^2}\right) \Lambda_{\mu,\psi}(y_1, ..., y_r; \eta t^p), \end{aligned}$$

which completes the proof. □

If we set $r = 1$ and

$$\Omega_{\mu+\psi k}(y_1) = f_{\mu+\psi k}(y_1)$$

in Theorem 1 , where the Sister Celine's polynomials $f_n(x)$, generated by [3]

$$\sum_{n=0}^{\infty} f_n(x) t^n = (1-t)^{-1} \exp\left(\tfrac{-4xt}{(1-t)^2}\right), \ |t| < 1.$$

Thus, we have the following result which provides a class of bilinear generating functions for the Sister Celine's polynomials $f_n(x)$, as follows:

Corollary 1. *If*

$$\Lambda_{\mu,\psi}(y_1; w) := \sum_{k=0}^{\infty} a_k f_{\mu+\psi k}(y_1) w^k \quad (a_k \neq 0\,, \ \ \mu, \psi \in \mathbb{C})\,,$$

then, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k f_n(x) f_{\mu+\psi k}(y_1) w^k t^n \\ = \ & (1-t)^{-1} \exp\left(\frac{-4xt}{(1-t)^2}\right) \Lambda_{\mu,\psi}(y_1, ..., y_r; w), \end{aligned} \tag{8}$$

*provided that each member of (*8*) exists.*

Theorem 2. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, ..., y_r)$ of r complex variables $y_1, ..., y_r$ $(r \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(y_1, ..., y_r; \zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r) \zeta^k \ \ (a_k \ \ \neq \ \ 0 \ \ \mu, \psi \in \mathbb{C}),$$

and

$$\Theta_{n,p}^{\mu,\psi}(x; y_1, ..., y_r; \xi) := \sum_{k=0}^{[n/p]} a_k f_{n-pk}^{(\alpha,\beta)}(x) \Omega_{\mu+\psi k}(y_1, ..., y_r) \xi^k.$$

Then, for $p \in \mathbb{N}$, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}(x; y_1, ..., y_r; \eta)\, t^n \\ = \ & (1-t)^{-1-\alpha-\beta}_{\ 2} F_2 \left[\begin{array}{cc} \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; & \\ & \frac{-4xt}{(1-t)^2} \\ 1+\alpha, \frac{1}{2}; & \end{array} \right] \Lambda_{\mu,\psi}(y_1, ..., y_r; \eta t^p), \end{aligned} \tag{9}$$

*provided that each member of (*9*) exists.*

Proof: For convenience, let S denote the first member of the assertion (9). Then,

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k f_{n-pk}^{(\alpha,\beta)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^n.$$

Replacing n by $n+pk$, we may write that

$$\begin{aligned}
S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\, f_n^{(\alpha,\beta)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^{n+pk}\\
&= \sum_{n=0}^{\infty}f_n^{(\alpha,\beta)}(x)t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)(\eta t^p)^k\\
&= (1-t)^{-1-\alpha-\beta}\ {}_2F_2\left[\begin{array}{c}\frac{1+\alpha+\beta}{2},\frac{2+\alpha+\beta}{2};\\ \\ 1+\alpha,\frac{1}{2};\end{array}\ \frac{-4xt}{(1-t)^2}\right]\Lambda_{\mu,\psi}(y_1,...,y_r;\eta t^p),
\end{aligned}$$

which completes the proof. □

If we set

$$\Omega_{\mu+\psi k}(y_1,...,y_r)=\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)$$

in Theorem 2 , where the multivariable polynomials $\Phi_{\mu+\psi k}^{(\alpha)}(x_1,...,x_r)$, generated by [9]

$$\sum_{n=0}^{\infty}\Phi_n^{(\alpha)}(x_1,...,x_r)\,t^n=(1-x_1t)^{-\alpha}e^{(x_2+...+x_r)\,t},\quad \left(\alpha\in\mathbb{C}\,;\ \ |t|<\left\{|x_1|^{-1}\right\}\right). \tag{10}$$

Thus, we have the following result which provides a class of bilateral generating functions for the multivariable polynomials $\Phi_{\mu+\psi k}^{(\alpha)}(x_1,...,x_r)$ and the generalized Sister Celine's polynomials as follows:

Corollary 2. *If*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;w):=\sum_{k=0}^{\infty}a_k\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)w^k\ \ (a_k\neq 0\,,\ \ \mu,\psi\in\mathbb{C}),$$

then, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k f_n^{(\alpha,\beta)}(x)\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)w^k t^n\\
&= (1-t)^{-1-\alpha-\beta}\ {}_2F_2\left[\begin{array}{c}\frac{1+\alpha+\beta}{2},\frac{2+\alpha+\beta}{2};\\ \\ 1+\alpha,\frac{1}{2};\end{array}\ \frac{-4xt}{(1-t)^2}\right]\Lambda_{\mu,\psi}(y_1,...,y_r;w),
\end{aligned}\tag{11}$$

provided that each member of (11) *exists.*

Remark 1. *Using the generating relation* (10) *for the multivariable polynomials* $\Phi_n^{(\alpha)}(x_1,...,x_r)$ *and getting* $a_k=1,\ \mu=0,\ \psi=1$ *in Corollary 2, we find that*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}f_n^{(\alpha,\beta)}(x)\Phi_k^{(\alpha)}(x_1,...,x_r)w^k t^n\\
&= (1-t)^{-1-\alpha-\beta}\ {}_2F_2\left[\begin{array}{c}\frac{1+\alpha+\beta}{2},\frac{2+\alpha+\beta}{2};\\ \\ 1+\alpha,\frac{1}{2};\end{array}\ \frac{-4xt}{(1-t)^2}\right](1-x_1w)^{-\alpha}e^{(x_2+...+x_r)w},\\
&\quad\left(\alpha_j\in\mathbb{C},\ \ |w|<\left\{|x_1|^{-1}\right\},\ |t|<1\right).
\end{aligned}$$

Theorem 3. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,...,y_r)$ of r complex variables $y_1,...,y_r$ $(r\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1,...,y_r)\zeta^k \; (a_k \neq 0, \; \mu,\psi\in\mathbb{C}),$$

and

$$\Theta_{n,p}^{\mu,\psi}(x;y_1,...,y_r;\xi) := \sum_{k=0}^{[n/p]} a_k (C)_{n-pk} f_{n-pk}^{(\alpha,\beta)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k.$$

Then, for $p\in\mathbb{N}$, we have

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}(x;y_1,...,y_r;\eta)\,t^n \tag{12}$$
$$= (1-t)^{-C-\alpha-\beta}\; {}_3F_3\left[\begin{array}{c} C, \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; \\ 1+\alpha, 1+\alpha+\beta, \frac{1}{2}; \end{array} \frac{-4xt}{(1-t)^2}\right] \Lambda_{\mu,\psi}(y_1,...,y_r;\eta t^p),$$

provided that each member of (12) exists.

Proof: For convenience, let K denote the first member of the assertion (12). Then,

$$K = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k (C)_{n-pk} f_{n-pk}^{(\alpha,\beta)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^n.$$

Replacing n by $n+pk$, we may write that

$$\begin{aligned} K &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k (C)_n f_n^{(\alpha,\beta)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^{n+pk} \\ &= \sum_{n=0}^{\infty}(C)_n f_n^{(\alpha,\beta)}(x)t^n \sum_{k=0}^{\infty} a_k\Omega_{\mu+\psi k}(y_1,...,y_r)(\eta t^p)^k \\ &= (1-t)^{-C-\alpha-\beta}\; {}_3F_3\left[\begin{array}{c} C, \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; \\ 1+\alpha, 1+\alpha+\beta, \frac{1}{2}; \end{array} \frac{-4xt}{(1-t)^2}\right] \Lambda_{\mu,\psi}(y_1,...,y_r;\eta t^p), \end{aligned}$$

which completes the proof. □

Furthermore, for every suitable choice of the coefficients a_k $(k\in\mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $r\in\mathbb{N}$, are expressed as an appropriate product of several simpler functions, the assertions of Theorem 1, Theorem 2 and Theorem 3 can be applied in order to derive various families of multilinear and multilateral generating functions for the family of the Sister Celine's polynomials and the generalized Sister Celine's polynomials given explicitly by (1) and (5).

3 Conclusion

In this paper, we establish some properties for the generalized Sister Celine's polynomials. Various families of multilinear and multilateral generating functions and their miscellaneous properties are obtained. With the method used here, it is possible to obtain bilinear and bilateral generating functions for other polynomials.

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A note on Shively's Pseudo-Laguerre Polynomials

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Abstract: In this research, we establish some properties for the Shively's Pseudo-Laguerre polynomials. We derive various families of multilinear and multilateral generating functions for a family of Shively's Pseudo-Laguerre polynomials.

Keywords: Multilinear and multilateral generating functions, Recurrence relations, Shively's Pseudo-Laguerre polynomials.

1 Introduction

Shivley (see, for example, [1]; see also [[2], p. 298, Eq. 152 (1)];[[3], p. 127, Eq. (47)] and [[4], p. 1758, Eq. (3)]) has defined the polynomial $R_n(a,x)$ by

$$R_n(a,x) := \frac{(a+n)_n}{n!}\ {}_1F_1(-n; a+n; x) \tag{1}$$

in which n is any non-negative integer, and a is independent of n.

The pseudo-Laguerre polynomial $R_n(a,x)$ may also be written as

$$R_n(a,x) = \frac{(a)_{2n}}{n!(a)_n}\ {}_1F_1(-n; a+n; x)$$

which are related to the proper simple Laguerre polynomial

$$L_n(x) = {}_1F_1(-n; 1; x)$$

by

$$R_n(a,x) = \frac{1}{(a-1)_n}\sum_{k=0}^{n}\frac{(a-1)_{n-k}}{k!}L_{n-k}(x).$$

Toscano [5] had already shown that

$$\sum_{n=0}^{\infty} R_n(a,x)t^n = (1-4t)^{-1/2}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1}\exp\left(\frac{-4xt}{\left(1+\sqrt{1-4t}\right)^2}\right). \tag{2}$$

Shively obtained Toscano's other generating relation

$$\sum_{n=0}^{\infty}\frac{R_n(a,x)}{\left(\frac{1}{2}+\frac{1}{2}a\right)_n}t^n = e^{2t}\ {}_0F_1(-; \frac{1}{2}+\frac{1}{2}a; t^2-xt),$$

and extended Toscano's (2) to

$$\sum_{n=0}^{\infty} S_n(x)t^n = (1-4t)^{-1/2}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1}_{p} F_q\left[\begin{array}{ll}\alpha_1,...,\alpha_p; & \frac{-4xt}{\left(1+\sqrt{1-4t}\right)^2}\\ \beta_1,...,\beta_q; & \end{array}\right],$$

in which

$$S_n(x) = \frac{(a)_{2n}}{n!(a)_n}\ {}_{p+1}F_{q+1}\left[\begin{array}{ll}-n,\alpha_1,...,\alpha_p; & x\\ a+n,\beta_1,...,\beta_q; & \end{array}\right].$$

For the particular choice $p = 0, \;\; q = 1, \;\; b_1 = 1, \;\; a = 1$ the $S_n(x)$ becomes

$$\sigma_n(x) = \frac{(2n)!}{(n!)^2} \, {}_1F_2(-n; 1+n, 1; x)$$

for which Shively has the additional generating relation [6]

$$\sum_{n=0}^{\infty} \frac{\sigma_n(x)}{(2n)!} t^n = \, {}_0F_1\left(-; 1; \frac{t - \sqrt{4xt + t^2}}{2}\right) \, {}_0F_1\left(-; 1; \frac{t + \sqrt{4xt - t^2}}{2}\right).$$

The $R_n(a,x)$ of (1) is of Sheffer A-type zero, as pointedout by Shively. He obtains many other properties of $R_n(a,x)$. Here ${}_pF_q$ denotes, as usual, a generalized hypergeometric function with p numerator and q denominator parameters and as usual, $(\lambda)_\nu$ denotes the Pochhammer symbol or the *shifted factorial*, since

$$(1)_n = n! \;\; (n \in \mathbb{N}_0),$$

which is defined (for λ, $\nu \in \mathbb{C}$ and in terms of Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1, & (\nu = 0; \; \lambda \in \mathbb{C}\backslash\{0\}) \\ \lambda(\lambda+1)...(\lambda+n-1), & (\nu = n \in \mathbb{N}; \; \lambda \in \mathbb{C}) \end{cases},$$

it getting understood conventionally that $(0)_0 := 1$.

The main object of this paper is to study several properties of the pseudo-Laguerre polynomial $R_n(a,x)$. Various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials are given.

2 Generating functions

In this section, we derive several families of bilinear and bilateral generating functions for the pseudo-Laguerre polynomial $R_n(a,x)$ generated by using the similar method considered in (see, [7] - [12]).

We begin by stating the following theorem.

Theorem 1. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, ..., y_r)$ of r complex variables $y_1, ..., y_r$ $(r \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(y_1, ..., y_r; \zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r) \zeta^k \;\; (a_k \;\; \neq \;\; 0\,, \; \mu, \psi \in \mathbb{C}),$$

and

$$\Theta_{n,p}^{\mu,\psi}\left(a, x; y_1, ..., y_r; \xi\right) := \sum_{k=0}^{[n/p]} a_k R_{n-pk}(a,x) \Omega_{\mu+\psi k}(y_1, ..., y_r) \xi^k.$$

Then, for $p \in \mathbb{N}$, we have

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}\left(a, x; y_1, ..., y_r; \eta\right) t^n = (1-4t)^{-1/2} \left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1} \exp\left(\frac{-4xt}{\left(1+\sqrt{1-4t}\right)^2}\right) \Lambda_{\mu,\psi}(y_1, ..., y_r; \eta) \tag{3}$$

provided that each member of (3) exists.

Proof: For convenience, let S denote the first member of the assertion (3). Then,

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k R_{n-pk}(a,x) \Omega_{\mu+\psi k}(y_1, ..., y_r) \eta^k t^n.$$

Replacing n by $n + pk$, we may write that

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k \, R_n(a,x) y_n(x, \alpha - n, \beta) \Omega_{\mu+\psi k}(y_1, ..., y_r) \eta^k t^n \\ &= \sum_{n=0}^{\infty} R_n(a,x) t^n \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r) \eta^k \\ &= (1-4t)^{-1/2} \left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1} \exp\left(\frac{-4xt}{\left(1+\sqrt{1-4t}\right)^2}\right) \Lambda_{\mu,\psi}(y_1, ..., y_r; \eta), \end{aligned}$$

which completes the proof. □

If we set

$$\Omega_{\mu+\psi k}(y_1,...,y_r) = \Phi^{(\alpha)}_{\mu+\psi k}(y_1,...,y_r)$$

in Theorem 1 , where the multivariable polynomials $\Phi^{(\alpha)}_{\mu+\psi k}(x_1,...,x_r)$, generated by [10]

$$\sum_{n=0}^{\infty} \Phi^{(\alpha)}_n (x_1,...,x_r)\, t^n = (1-x_1 t)^{-\alpha} e^{(x_2+...+x_r)\,t}, \quad \left(\alpha \in \mathbb{C}\,;\ |t| < \left\{|x_1|^{-1}\right\}\right). \tag{4}$$

Thus, we have the following result which provides a class of bilateral generating functions for the multivariable polynomials $\Phi^{(\alpha)}_{\mu+\psi k}(x_1,...,x_r)$ and the pseudo-Laguerre polynomial $R_n(a,x)$ as follows:

Corollary 1. *If*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;w) := \sum_{k=0}^{\infty} a_k \Phi^{(\alpha)}_{\mu+\psi k}(y_1,...,y_r) w^k \ \ (a_k \neq 0\, \mu,\psi \in \mathbb{C}),$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k R_{n-pk}\,(a,x)\,\Phi^{(\alpha)}_{\mu+\psi k}(y_1,...,y_r) w^k t^n = (1-4t)^{-1/2}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1}\exp\left(\frac{-4xt}{\left(1+\sqrt{1-4t}\right)^2}\right)\Lambda_{\mu,\psi}(y_1,...,y_r;\eta), \tag{5}$$

provided that each member of (5) *exists.*

Remark 1. *Using the generating relation* (4) *for the multivariable polynomials* $\Phi^{(\alpha)}_n(x_1,...,x_r)$ *and getting* $a_k = 1,\ \mu = 0,\ \psi = 1$ *in Corollary 1 , we find that*

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} R_{n-pk}\,(a,x)\,\Phi^{(\alpha)}_k(x_1,...,x_r) w^k t^n &= (1-4t)^{-1/2}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1}\exp\left(\frac{-4xt}{\left(1+\sqrt{1-4t}\right)^2}\right)\\ &\quad\times (1-x_1 w)^{-\alpha} e^{(x_2+...+x_r)w},\\ &\quad\left(\alpha_j \in \mathbb{C},\ \ |w| < \left\{|x_1|^{-1}\right\}, |t| < \frac{1}{4}\right).\end{aligned}$$

3 Conclusion

In this paper, we esteblish some properties for the Shively's Pseudo-Laguerre polynomials. Various families of multilinear and multilateral generating functions and their miscellaneous properties are obtained. With the method used here, it is possible to obtain bilinear and bilateral generating functions for other polynomials.

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On the Bitsadze-Samarskii Type Nonlocal Boundary Value Problem with the Integral Condition for an Elliptic Equation

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Abstract: In the present paper, the Bitsadze-Samarskii type nonlocal boundary value problem with the integral condition for an abstract elliptic differential equation in a Hilbert space is studied. Theorem on well-posedness of this problem in Hölder spaces with a weight is established. The nonlocal boundary value problem for multidimensional elliptic equations with the Dirichlet condition is studied. The first order of accuracy difference scheme for the approximate solution of the Bitsadze-Samarskii type nonlocal boundary value problem is investigated. Theorem on well-posedness of this difference scheme in difference analogue of Hölder spaces with a weight is established.

Keywords: Bitsadze-Samarskii type nonlocal boundary value problem, Difference scheme, Elliptic equation, Well-posedness.

1 Introduction

The simply nonlocal boundary value problem was presented and investigated for the first time by A.V. Bitsadze and A.A. Samarskii in the paper [1] . Further in papers [2–13], the Bitsadze-Samarskii type nonlocal boundary value problem and its generalizations for various differential and difference equations of elliptic equations were investigated by many scientists. Coercivity inequalities in Hölder norms with a weight for the solutions of an abstract differential equation of elliptic type were established for the first Sobolevskii in the paper [12]. Further, in papers [14–25] coercive inequalities in Hölder norms with a weight were obtained for the solutions of various local and nonlocal boundary-value problems for differential and difference equations of elliptic type. In the present paper, we consider the Bitsadze-Samarskii type nonlocal boundary value problem with the integral condition

$$\begin{cases} -\frac{d^2u(t)}{dt^2} + Au(t) = f(t),\ 0 < t < 1, \\ u(0) = \varphi,\ u(1) = \int\limits_0^1 \rho(\lambda)u(\lambda)d\lambda + \psi \end{cases} \tag{1}$$

for the differential equation of elliptic type in a Hilbert space H with the self-adjoint positive definite operator A with a closed domain $D(A) \subset H$. Here, let $f(t)$ be a given abstract continuous function defined on $[0,1]$ with values in H, φ, and ψ are elements of $D(A)$ and $\rho(t)$ is a scalar continous function. A function $u(t)$ is called a solution of problem (1) if the following conditions are satisfied:

i. $u(t)$ is twice continuously differentiable on the segment $[0,1]$.

ii. The element $u(t)$ belongs to $D(A)$ for all $t \in [0,1]$, and the function $Au(t)$ is continuous on the segment $[0,1]$.

iii. $u(t)$ satisfies the equation and nonlocal boundary conditions (1).

A solution of problem (1) defined in this manner will from now on be referred to as a solution of problem (1) in the space $C([0,1], H)$. Here, $C([0,1],\ H)$ stands for the Banach space of all continuous functions $\varphi(t)$ defined on $[0,1]$ with values in H with the norm

$$\|\varphi\|_{C([0,1],H)} = \max_{0\le t\le 1} \|\varphi(t)\|_H .$$

We say that the problem (1) is well-posed in $C([0,1], H)$, if there exists the unique solution $u(t)$ in $C([0,1], H)$ of problem (1) for any $f(t) \in C([0,1], H)$ and the following coercivity inequality is satisfied:

$$\|u''\|_{C([0,1],H)} + \|Au\|_{C([0,1],H)} \le M_c \left[\|f\|_{C([0,1],H)} + \|A\varphi\|_H + \|A\psi\|_H\right],$$

where M_c does not depend on $f(t)$ and φ, ψ. Unfortunately, the problem (1) is ill-posed in the space $C([0,1], H)$.

In this paper, positive constants, which can differ in time (hence: not a subject of precision), will be indicated with M. On the other hand $M(\alpha; \beta; ...)$ is used to focus on the fact that the constant depends only on α;β;....

Let us denote by $C_{01}^{\alpha}([0,1], H)$, $0<\alpha<1$, the Banach spaces obtained by completion of the set of all smooth $H-$values functions $\varphi(t)$ on $[0,1]$ in the norms

$$\|\varphi\|_{C_{01}^{\alpha}([0,1],H)} = \|\varphi\|_{C([0,1],H)} + \sup_{0\le t<t+\tau\le 1} \frac{(1-t)^{\alpha}(t+\tau)^{\alpha}\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\alpha}}.$$

We say that the problem (1) is well-posed in $C_{01}^{\alpha}([0,1], H)$, if there exists a unique solution $u(t)$ in $C_{01}^{\alpha}([0,1], H)$ of problem (1) for any $f(t)\in C_{01}^{\alpha}([0,1], H)$ and the following coercivity inequality is satisfied:

$$\|u''\|_{C_{01}^{\alpha}([0,1],H)} + \|Au\|_{C_{01}^{\alpha}([0,1],H)} \le M(\delta,\alpha)\left[\|A\varphi\|_H + \|A\psi\|_H + \| f \|_{C_{01}^{\alpha}([0,1],H)}\right].$$

We will study the problem (1) under the assumption:

$$\int_0^1 |\rho(\lambda)|\, d\lambda < 1. \tag{2}$$

In the present paper, the well-posedness of the nonlocal boundary value problem (1) in $C_{01}^{\alpha}([0,1], H)$ spaces is established. The first order of accuracy difference scheme for the approximate solution of this problem (1) is presented. The coercive inequalities for the solution of this difference scheme in difference analogue of $C_{01}^{\alpha}([0,1], H)$ spaces are established. In applications, difference scheme for approximate nonlocal boundary value problem for elliptic equation is investigated.

2 The Bitsadze-Samarskii type nonlocal boundary value problem

In this section, let $B = A^{\frac{1}{2}}$. Then, it is clear that B is a self-adjoint positive definite operator and $B\ge \delta I$. The following lemmas will be needed below.

Lemma 1. *[8] The following estimates hold:*

$$\left\|B^{\alpha}\exp(-tB)\right\|_{H\longrightarrow H} \le t^{-\alpha},\ 0\le\alpha\le 1, \tag{3}$$

$$\left\|(I-e^{-2B})^{-1}\right\|_{H\longrightarrow H} \le M. \tag{4}$$

Lemma 2. *[17] For any $0\le t<t+\tau\le 1$ and $0\le\alpha\le 1$ one has the inequality*

$$\|\exp(-tB)-\exp(-(t+\tau)B)\|_{H\to H} \le M\frac{\tau^{\alpha}}{(\tau+t)^{\alpha}}. \tag{5}$$

Lemma 3. *Let*

$$D = \int_0^1 \rho(\lambda)(I-e^{-2B})^{-1}(e^{-(1-\lambda)B}-e^{-(1+\lambda)B})d\lambda.$$

Then, under the assumption (1), the operator $I-D$ has an inverse

$$P = (I-D)^{-1}$$

and the following estimate is satisfied:

$$\|P\|_{H\to H} \le M(\delta). \tag{6}$$

It is clear that (see [17]) the boundary value problem for elliptic equation

$$-\frac{d^2u(t)}{dt^2} + Au(t) = f(t),\ 0<t<1, u(0)=u_0, u(1)=u_1 \tag{7}$$

has a unique solution

$$u(t) = (I-e^{-2B})^{-1}\left\{(e^{-tB}-e^{-(2-t)B})\varphi + (e^{-(1-t)B}-e^{-(1+t)B})u(1) - (e^{-(1-t)B}-e^{-(1+t)B})\right. \tag{8}$$

$$\left.\times(2B)^{-1}\int_0^1(e^{-(1-s)B}-e^{-(1+s)B})f(s)ds\right\} + (2B)^{-1}\int_0^1(e^{-|t-s|B}-e^{-(t+s)B})f(s)ds,$$

$$u(1) = P\left[\psi + \int_0^1 \rho(\lambda)(I-e^{-2B})^{-1}\left\{\left(e^{-\lambda B}-e^{-(2-\lambda)B}\right)\varphi\right.\right. \tag{9}$$

$$-(e^{-(1-\lambda)B}-e^{-(1+\lambda)B})\,(2B)^{-1}\int_0^1(e^{-(1-s)B}-e^{-(1+s)B})f(s)ds\Bigg\}d\lambda$$

$$+\int_0^1\rho(\lambda)\,(2B)^{-1}\left(\int_0^\lambda e^{-(\lambda-s)B}f(s)ds+\int_\lambda^1 e^{-(s-\lambda)B}f(s)ds-\int_0^1 e^{-(\lambda+s)B}f(s)ds\right)d\lambda\Bigg],$$

where

$$P=\left(I-\int_0^1\rho(\lambda)\left(I-e^{-2B}\right)^{-1}\left(e^{-(1-\lambda)B}-e^{-(1+\lambda)B}\right)d\lambda\right)^{-1}.$$

Theorem 1. *Suppose $\varphi,\psi\in D(A)$, $f\left(t\right)\in C_{01}^{\alpha}\left(\left[0,1\right],H\right)\,(0<\alpha<1)$. Then, for the solution $u\left(t\right)$ of the boundary value problem (1) the coercivity inequality*

$$\|u''\|_{C_{01}^{\alpha}([0,1],H)}+\|Au\|_{C_{01}^{\alpha}([0,1],H)}\leq M(\delta)\left[\|A\varphi\|_H+\|A\psi\|_H+\frac{1}{\alpha\left(1-\alpha\right)}\parallel f\parallel_{C_{01}^{\alpha}([0,1],H)}\right]$$

holds.

Proof: By [17], we had the following coercivity inequality

$$\|u''\|_{C_{01}^{\alpha}([0,1],H)}+\|Au\|_{C_{01}^{\alpha}([0,1],H)}\leq\frac{M(\delta)}{\alpha(1-\alpha)}\|f\|_{C_{01}^{\alpha}([0,1],H)}+M(\delta)\left\{\|Au(0)\|_H+\|Au(1)\|_H\right\}\tag{10}$$

for the solution of boundary value problem (7). Then the proof of Theorem 1 is based on coercivity inequality (10) and on the following estimate

$$\|Au\left(1\right)\|_H\leq\frac{M(\delta)}{\alpha(1-\alpha)}\|f\|_{C_{01}^{\alpha}([0,1],H)}+M(\delta)\left\{\|A\varphi\|_H+\|A\psi\|_H\right\}.\tag{11}$$

Therefore, we will prove (11). First, applying formula (9), we can write

$$Au(1)=P\left(\int_0^1\rho(\lambda)(I-e^{-2B})^{-1}\left\{\left(e^{-\lambda B}-e^{-(2-\lambda)B}\right)A\varphi+\left(I-e^{-\lambda B}\right)\left(I-e^{-(1-\lambda)B}\right)\right.\right.$$

$$\times\left(I-e^{-B}\right)f(\lambda)+\frac{B}{2}\left(I-e^{-2(1-\lambda)B}\right)\int_0^\lambda\left(e^{-(\lambda-s)B}\left(I-e^{-2sB}\right)\right)(f(s)-f(\lambda))ds$$

$$\left.\left.+\frac{B}{2}\left(I-e^{-2\lambda B}\right)\int_\lambda^1\left(e^{-(s-\lambda)B}\left(I-e^{-2(1-s)B}\right)\right)(f(s)-f(\lambda))ds\right\}d\lambda+A\psi\right)$$

$$=J_1+J_2+J_3+J_4,$$

where

$$J_1=P\left(\int_0^1\rho(\lambda)(I-e^{-2B})^{-1}(e^{-\lambda B}-e^{-(2-\lambda)B})A\varphi d\lambda+A\psi\right),$$

$$J_2=P\left(\int_0^1\rho(\lambda)(I-e^{-2B})^{-1}\left(I-e^{-\lambda B}\right)\left(I-e^{-(1-\lambda)B}\right)\left(I-e^{-B}\right)f(\lambda)d\lambda\right),$$

$$J_3=\frac{1}{2}P\int_0^1\rho(\lambda)(I-e^{-2B})^{-1}\int_0^\lambda\left(Be^{-(\lambda-s)B}\left(I-e^{-2(1-\lambda)B}\right)\left(I-e^{-2sB}\right)\right)(f(s)-f(\lambda))dsd\lambda,$$

$$J_4=\frac{1}{2}P\int_0^1\rho(\lambda)B\left(I-e^{-2B}\right)^{-1}\int_\lambda^1\left(e^{-(s-\lambda)B}\left(I-e^{-2(1-s)B}\right)\left(I-e^{-2\lambda B}\right)\right)(f(s)-f(\lambda))dsd\lambda.$$

Let us estimate J_k for $k=1,\cdots,4$, separately. First, we estimate J_1. Using estimates (4), (5) and (6), we obtain

$$\|J_1\|_H\leq\|P\|_{H\to H}\left(\int_0^1|\rho(\lambda)|\left\|\left(I-e^{-2B}\right)^{-1}\right\|_{H\to H}\left\|e^{-\lambda B}-e^{-(2-\lambda)B}\right\|_{H\to H}\|A\varphi\|_H\,d\lambda+\|A\psi\|_H\right)$$

$$\leq M(\delta)\left[\int_0^1 |\rho(\lambda)|\, d\lambda \,\|A\varphi\|_H + \|A\psi\|_H\right].$$

Thus, from condition (2) it follows that

$$\|J_1\|_H \leq M_1(\delta)\left[\|A\varphi\|_H + \|A\psi\|_H\right].$$

Let us estimate J_2.

$$\|J_2\|_H \leq \|P\|_{H\to H} \int_0^1 |\rho(\lambda)| \left\|\left(I - e^{-2B}\right)^{-1}\right\|_{H\to H} \left\|I - e^{-\lambda B}\right\|_{H\to H}$$

$$\times \left\|I - e^{-(1-\lambda)B}\right\|_{H\to H} \left\|I - e^{-B}\right\|_{H\to H} \|f(\lambda)\|_H \, d\lambda.$$

Further, using estimates (3), (5), (6) and the definition of the norm of the space $C_{01}^{\alpha}([0,1], H)$, we get

$$\|J_2\|_H \leq M_2(\delta)\int_0^1 |\rho(\lambda)|\, d\lambda \max_{0\leq t\leq 1} \|f(t)\|_H.$$

Thus, from (2) it follows that

$$\|J_2\|_H \leq M_2(\delta)\,\|f(t)\|_{C([0,1],H)} \leq M_2(\delta)\,\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

To estimate J_3, we will put $J_3 = J_{3,1} + J_{3,2}$, where

$$J_{3,1} = P\int_0^{\frac{1}{2}} \rho(\lambda)(I - e^{-2B})^{-1}\frac{1}{2}\int_0^{\lambda} \left(Be^{-(\lambda-s)B}\left(I - e^{-2(1-\lambda)B}\right)\left(I - e^{-2sB}\right)\right)(f(s) - f(\lambda))ds d\lambda,$$

$$J_{3,2} = P\int_{\frac{1}{2}}^{1} \rho(\lambda)(I - e^{-2B})^{-1}\frac{1}{2}\int_0^{\lambda} \left(Be^{-(\lambda-s)B}\left(I - e^{-2(1-\lambda)B}\right)\left(I - e^{-2sB}\right)\right)(f(s) - f(\lambda))ds d\lambda.$$

First, we will estimate $J_{3,1}$. Applying estimates (3), (5), (6) and the definition of the norm of the space $C_{01}^{\alpha}([0,1], H)$, we obtain

$$\|J_{3,1}\|_H \leq M(\delta)\int_0^{\frac{1}{2}} |\rho(\lambda)| \int_0^{\lambda} \frac{ds}{(\lambda - s)^{1-\alpha}\lambda^{\alpha}(1-s)^{\alpha}} d\lambda \,\|f\|_{C_{01}^{\alpha}([0,1],H)} \leq \frac{M(\delta)}{\alpha(1-\alpha)}\int_0^{\frac{1}{2}} |\rho(\lambda)|\, d\lambda \,\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

Second, we will estimate $J_{3,2}$. For $J_{3,2}$, using estimates (3), (5), (6) and the definition of the norm of the space $C_{01}^{\alpha}([0,1], H)$, we get

$$\|J_{3,2}\|_H \leq M(\delta)\int_{\frac{1}{2}}^{1} |\rho(\lambda)| \int_0^{\lambda} \frac{(2(1-\lambda))^{\alpha}\, ds}{(\lambda - s)(2 - \lambda - s)^{\alpha}\lambda^{\alpha}(1-s)^{\alpha}} d\lambda \,\|f\|_{C_{01}^{\alpha}([0,1],H)}$$

$$\leq \frac{M(\delta)2^{\alpha-1}}{\alpha}\int_{\frac{1}{2}}^{1} \frac{|\rho(\lambda)|}{(1-\lambda)^{\alpha}} d\lambda \,\|f\|_{C_{01}^{\alpha}([0,1],H)} \leq \frac{M(\delta)2^{2\alpha-2}}{\alpha(1-\alpha)}\int_{\frac{1}{2}}^{1} |\rho(\lambda)|\, d\lambda \,\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

Applying estimates for $\|J_{3,1}\|_H$ and $\|J_{3,2}\|_H$, we get

$$\|J_3\|_H \leq \frac{M_3(\delta)}{\alpha(1-\alpha)}\int_0^1 |\rho(\lambda)|\, d\lambda \,\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

Using condition (2), we get

$$\|J_3\|_H \leq \frac{M_4(\delta)}{\alpha(1-\alpha)}\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

Let us estimate J_4. We will put $J_4 = J_{4,1} + J_{4,2}$, where

$$J_{4,1} = P\int_0^{\frac{1}{2}} \rho(\lambda)\frac{1}{2}\left(I - e^{-2B}\right)^{-1}\int_{\lambda}^{1} \left(Be^{-(s-\lambda)B}\left(I - e^{-2(1-s)B}\right)\left(I - e^{-2\lambda B}\right)\right)(f(s) - f(\lambda))ds d\lambda,$$

$$J_{4,2} = P\int_{\frac{1}{2}}^{1} \rho(\lambda)\frac{1}{2}\left(I - e^{-2B}\right)^{-1}\int_{\lambda}^{1} \left(Be^{-(s-\lambda)B}\left(I - e^{-2(1-s)B}\right)\left(I - e^{-2\lambda B}\right)\right)(f(s) - f(\lambda))ds d\lambda.$$

The estimates (3), (5), (6) and the definition of the norm of the space $C_{01}^{\alpha}([0,1], H)$ give

$$\|J_{4,1}\|_H \le M(\delta)\int_0^{\frac{1}{2}}|\rho(\lambda)|\int_{\lambda}^{1}\frac{(1-s)^{\alpha}\,ds}{(1-\lambda)^{\alpha}s^{\alpha}(s-\lambda)^{1-\alpha}(2-s-\lambda)^{\alpha}}d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)}$$

$$\le M(\delta)\int_0^{\frac{1}{2}}\frac{|\rho(\lambda)|}{\lambda^{\alpha}(1-\lambda)^{\alpha}}\int_{\lambda}^{1}\frac{ds}{(s-\lambda)^{1-\alpha}}d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)} \le \frac{M_5(\delta)}{\alpha(1-\alpha)}\int_0^{\frac{1}{2}}|\rho(\lambda)|\,d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

Finally, we estimate $J_{4,2}$. For $J_{4,2}$, applying estimates (3), (5), (6) and the definition of the norm of the space $C_{01}^{\alpha}([0,1], H)$, we obtain

$$\|J_{4,2}\|_H \le M(\delta)\int_{\frac{1}{2}}^{1}\frac{|\rho(\lambda)|}{\lambda^{\alpha}\alpha}d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)} \le \frac{M(\delta)2^{\alpha-2}}{\alpha(1-\alpha)}\int_{\frac{1}{2}}^{1}|\rho(\lambda)|\,d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

Applying estimates for $\|J_{4,1}\|_H$ and $\|J_{4,2}\|_H$, we get

$$\|J_4\|_H \le \frac{M_6(\delta)}{\alpha(1-\alpha)}\int_0^1|\rho(\lambda)|\,d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

So, from (2) it follows that

$$\|J_4\|_H \le \frac{M_7(\delta)}{\alpha(1-\alpha)}\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

Combining estimates for $\|J_k\|_H$, $k = 1, \cdots, 4$, we obtain estimate (11). Theorem 1 is proved. □

Now, we consider the application of Theorem 1.

Let Ω is the unit open cube in $\mathbb{R}^n$ $\{x = (x_1, \cdots, x_n) : 0 < x_k < 1, 1 \le k \le n\}$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1]\times\Omega$, the Dirichlet-Bitsadze-Samarskii type mixed boundary value problem for the multidimensional elliptic equation

$$\begin{cases} -u_{tt} - \sum\limits_{r=1}^{n}(a_r(x)u_{x_r})_{x_r} = f(t,x), 0 < t < 1,\ x = (x_1,\ldots,x_n) \in \Omega, \\ u(0,x) = \varphi(x),\ u(1,x) = \int\limits_0^1 \rho(\lambda)u(\lambda,x)d\lambda + \psi(x),\ x \in \overline{\Omega}, \\ u(t,x)\mid_{x\epsilon S} = 0, x \in \overline{\Omega},\ 0 \le t \le 1 \end{cases} \tag{12}$$

is considered. We will study the problem (12) under the assumption (2). The problem has an unique smooth solution $u(t,x)$ for the smooth $f(t,x)$ $(t \in (0,1), x \in \overline{\Omega})$, $\varphi(x)$ and $\psi(x)$ functions, and $a_r(x) \ge a > 0$ $(x \in \Omega)$. We introduce the Hilbert space $L_2(\overline{\Omega})$ of all square-integrable functions f defined on $\overline{\Omega}$, equipped with the norm

$$\| f \|_{L_2(\overline{\Omega})} = \left\{\int\cdots\int_{x\in\overline{\Omega}}|f(x)|^2\,dx_1\cdots dx_n\right\}^{\frac{1}{2}}.$$

We can reduce the Dirichlet-Bitsadze-Samarskii type mixed boundary value problem (12) to the nonlocal boundary problem (1) in Hilbert space $H = L_2(\overline{\Omega})$ with a self -adjoint positive definite operator A defined by (12) .

Theorem 2. *The solution of the nonlocal boundary value problem (12) satisfies the coercivity inequality*

$$\|u_{tt}\|_{C_{01}^{\alpha}([0,1],L_2(\overline{\Omega}))} + \|u\|_{C_{01}^{\alpha}([0,1],W_2^2(\overline{\Omega}))}$$

$$\le \frac{M(\delta)}{\alpha(1-\alpha)}\| f \|_{C_{01}^{\alpha}([0,1],L_2(\overline{\Omega}))} + M(\delta)\left[\| \varphi \|_{W_2^2(\overline{\Omega})} + \| \psi \|_{W_2^2(\overline{\Omega})}\right].$$

Here, the Sobolev space $W_2^2(\overline{\Omega})$ is defined as the set of all functions f defined on $\bar{\Omega}$ such that f and all second order partial derivative functions f_{x_r,x_r}, $r = 1, ...n$ is both locally integrable in $L_2(\overline{\Omega})$, equipped with the norm

$$\| f \|_{W_2^2(\overline{\Omega})} = \| f \|_{L_2(\overline{\Omega})} + \left(\int\cdots\int_{x\in\overline{\Omega}}\sum_{r=1}^{n}|f_{x_r,x_r}|^2\,dx_1\cdots dx_n\right)^{1/2}.$$

The proof of Theorem 2 is based on Theorem 1, on the symmetry properties of the space operator A generated by the problem (12), and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 3. *For the solution of the elliptic differential problem*

$$\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r} = \omega(x), x \in \Omega,$$

$$u(x) = 0,\ x \in S,$$

the following coercivity inequality holds [22]:

$$\|u\|_{W_2^2(\overline{\Omega})} \leq M\|\omega\|_{L_2(\overline{\Omega})}.$$

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3 The first order of accuracy difference scheme

The nonlocal boundary value problem (1) is associated with the corresponding first order of accuracy difference scheme

$$\begin{cases} -\frac{1}{\tau^2}[u_{k+1} - 2u_k + u_{k-1}] + Au_k = \varphi_k, \\ \varphi_k = f(t_k),\ t_k = k\tau,\ 1 \leq k \leq N-1,\ N\tau = 1, \\ u_0 = \varphi,\ u_N = \sum\limits_{j=1}^{N} \rho\left(t_j\right) u_j \tau + \psi. \end{cases} \tag{13}$$

A study of discreatization over time of the nonlocal boundary value problem also permits one to include general difference schemes in applications, if the differential operator in space variables, A is replaced by the difference operators A_h that act in the Hilbert spaces H_h and are uniformly self-adjoint positive definite in h for $0 \leq h \leq h_0$. It is known that for a self-adjoint positive definite operator A it follows that $B = \frac{1}{2}(\tau A + \sqrt{4A + \tau^2 A^2})$ is self-adjoint positive definite and $R = (I + \tau B)^{-1}$ which defined on the whole space H is a bounded operator. Here, I is the identity operator. We will study the problem (13) under the assumption:

$$\sum_{j=1}^{N} \left|\rho\left(t_j\right)\right| \tau < 1. \tag{14}$$

Now, let us give some lemmas and theorem that will be needed below.

Lemma 4. *The estimates hold [17]*

$$\begin{cases} \left\|\left(I - R^{2N}\right)^{-1}\right\|_{H\to H} \leq M(\delta), \\ \|R^k\|_{H\to H} \leq M(\delta)(1+\delta\tau)^{-k},\ k\tau\|BR^k\|_{H\to H} \leq M(\delta), k \geq 1, \delta > 0, \\ \|B^\beta\left(R^{k+r} - R^k\right)\|_{H\to H} \leq M(\delta)\frac{(r\tau)^\alpha}{(k\tau)^{\alpha+\beta}}, 1 \leq k < k+r \leq N, 0 \leq \alpha, \beta \leq 1. \end{cases} \tag{15}$$

Lemma 5. *Suppose A is the positive operator in Hilbert space H. Then, the following estimate holds [17]:*

$$\sum_{j=1}^{N-1} \left\|(I-R)R^{j-1}\right\|_{H\to H} \leq M \min\left(\ln(\frac{1}{\tau}), 1 + \tau\left|\ln\|B\|_{H\to H}\right|\right). \tag{16}$$

Lemma 3.3. *The operator*

$$I - \sum_{j=1}^{N} \rho\left(t_j\right)\tau\left(I - R^{2N}\right)^{-1}\left(R^{N-j} - R^{N+j}\right)$$

has an inverse

$$K_\tau = \left(I - \sum_{j=1}^{N} \rho\left(t_j\right)\tau\left(I - R^{2N}\right)^{-1}\left(R^{N-j} - R^{N+j}\right)\right)^{-1}$$

and the following estimate is satisfied under the assumption (14)

$$\|K_\tau\|_{H\longrightarrow H} \leq M(\delta)\tau. \tag{17}$$

Theorem 4. *For any φ_k, $1 \le k \le N-1$, the solution of the problem (13) exists and the following formula holds for $k = 1, \cdots, N-1$,*

$$u_k = (I - R^{2N})^{-1} \left\{ \left(R^k - R^{2N-k}\right) \varphi + \left(R^{N-k} - R^{N+k}\right) u_N \right.$$

$$\left. - \left(R^{N-k} - R^{N+k}\right) (I+\tau B)(2I+\tau B)^{-1} B^{-1} \sum_{i=1}^{N-1} \left(R^{N-1-i} - R^{N-1+i}\right) \varphi_i \tau \right\} \tag{18}$$

$$+ (I+\tau B)(2I+\tau B)^{-1} B^{-1} \sum_{i=1}^{N-1} \left(R^{|k-i|-1} - R^{k+i-1}\right) \varphi_i \tau \,,$$

$$u_N = K_\tau \left(\sum_{j=1}^{N} \rho\left(t_j\right) \tau \left(I - R^{2N}\right)^{-1} \left\{ \left(R^j - R^{2N-j}\right) \varphi - \left(R^{N-j} - R^{N+j}\right) \right.\right.$$

$$\left. \times (I+\tau B)(2I+\tau B)^{-1} \sum_{i=1}^{N-1} B^{-1} \left(R^{N-1-i} - R^{N-1+i}\right) \varphi_i \tau \right\} + (2I+\tau B)^{-1} B^{-1}$$

$$\left. \times (I+\tau B) \left(\sum_{i=1}^{j} R^{j-i-1} \varphi_i \tau + \sum_{i=j+1}^{N-1} R^{i-j-1} \varphi_i \tau - \sum_{i=1}^{N-1} R^{j+i-1} \varphi_i \tau \right) + \psi \right)$$

for $k = N$.
Let $F([0,1]_\tau, H)$ be the linear space of the mesh functions $\varphi^\tau = \{\varphi_k\}_1^{N-1}$ with values in the Hilbert space H. We denote by $C([0,1]_\tau, H)$ and $C_{01}^\alpha([0,1]_\tau, H)$, $0 < \alpha < 1$, Banach spaces with the norms

$$\|\varphi^\tau\|_{C([0,1]_\tau, H)} = \max_{1 \le k \le N-1} \|\varphi_k\|_H \,,$$

$$\|\varphi^\tau\|_{C_{01}^\alpha([0,1]_\tau, H)} = \|\varphi^\tau\|_{C([0,1]_\tau, H)} + \sup_{1 \le k \le k+r \le N-1} \frac{((N-k)\tau)^\alpha ((k+r)\tau)^\alpha}{(r\tau)^\alpha} \|\varphi_{k+r} - \varphi_k\|_H \,.$$

Theorem 5. *The solution of the difference problem (13) in $C([0,1]_\tau, H)$ under the assumption (14) obeys the almost coercive inequality*

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C([0,1]_\tau, H)} + \| \{Au_k\}_1^N \|_{C([0,1]_\tau, H)} \tag{19}$$

$$\le M(\delta) \left[\min\left\{ \ln\frac{1}{\tau}, 1 + \left|\ln \| B \|_{H \to H}\right| \right\} \| \varphi^\tau \|_{C([0,1]_\tau, H)} + \|A\varphi\|_H + \|A\psi\|_H \right].$$

Proof: By [17],

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C([0,1]_\tau, H)} + \| \{Au_k\}_1^N \|_{C([0,1]_\tau, H)} \tag{20}$$

$$\le M(\delta) \left[\min\left\{ \ln\frac{1}{\tau}, 1 + \left|\ln \| B \|_{H \to H}\right| \right\} \| \varphi^\tau \|_{C([0,1]_\tau, H)} + \|A\varphi\|_H + \|Au_N\|_H \right]$$

was proved for the solution of the boundary value problem

$$\begin{cases} -\frac{1}{\tau^2}[u_{k+1} - 2u_k + u_{k-1}] + Au_k = \varphi_k, 1 \le k \le N-1, N\tau = 1, \\ u_0 = \varphi,\ u_N \text{ are given.} \end{cases} \tag{21}$$

Using the estimates (15), (17), and the formula (18), we obtain

$$\| Au_N \|_H \le M(\delta) \left(\min\left\{ \ln\frac{1}{\tau}, 1 + \left|\ln \| B \|_{H \to H}\right| \right\} \| \varphi^\tau \|_{C([0,1]_\tau, H)} + \|A\varphi\|_H + ||Au_N||_H \right) \tag{22}$$

for the solution of difference scheme (13). Applying formula (18) and $A = B^2 R$, we get

$$Au_N = J_1 + J_2,$$

where

$$J_1 = K_\tau \left(\sum_{j=1}^{N} \rho\left(t_j\right) \tau (I - R^{2N})^{-1} \left(R^j - R^{2N-j}\right) A\varphi + A\psi \right), \tag{23}$$

$$J_2 = K_\tau \sum_{j=1}^{N} \rho\left(t_j\right)\tau\left\{(I-R^{2N})^{-1}\left(\left(-R^{N-j}+R^{N+j}\right)(I+\tau B)\right.\right. \tag{24}$$

$$\times(2I+\tau B)^{-1}B\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\varphi_i\tau\right)+(I+\tau B)(2I+\tau B)^{-1}B$$

$$\left.\times\left(\sum_{i=1}^{j-1}R^{j-i}\varphi_i\tau+\sum_{i=j}^{N-1}R^{i-j}\varphi_i\tau-\sum_{i=1}^{N-1}R^{j+i}\varphi_i\tau\right)\right\}.$$

To this end, it suffices to show that

$$\|J_1\|_H \le M(\delta)[\|A\varphi\|_H + \|A\psi\|_H] \tag{25}$$

and

$$\|J_2\|_H \le M(\delta)\min\left\{\ln\frac{1}{\tau}, 1+\left|\ln\| B\|_{H\to H}\right|\right\}\|\varphi^\tau\|_{C([0,1]_\tau,H)}. \tag{26}$$

The estimate (25) follows from formula (23) and estimates (15), (17). Using formula (24) and estimates (15), (16), and (17), we obtain

$$\|J_2\|_H \le \|K_\tau\|_{H\to H}\left(\sum_{j=1}^{N}\left|\rho\left(t_j\right)\right|\tau\left(\left\|(I-R^{2N})^{-1}\right\|_{H\to H}\left\{\left(\left\|R^{N-j}\right\|_{H\to H}+\left\|R^{N+j}\right\|_{H\to H}\right)\right.\right.\right.$$

$$\left.\times\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\sum_{i=1}^{N-1}\left(\left\|(I-R)R^{N-i-1}\right\|_{H\to H}+\left\|(I-R)R^{N+i-1}\right\|_{H\to H}\right)\|\varphi_i\|_H\right\}$$

$$+\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\left(\sum_{i=1}^{j}\left\|(I-R)R^{j-i-1}\right\|_{H\to H}\|\varphi_i\|_H\right.$$

$$\left.\left.\left.+\sum_{i=j+1}^{N-1}\left\|(I-R)R^{i-j-1}\right\|_{H\to H}\|\varphi_i\|_H+\sum_{i=1}^{N-1}\left\|(I-R)R^{j+i-1}\right\|_{H\to H}\|\varphi_i\|_H\right)\right)\right)$$

$$\le M(\delta)\min\left\{\ln\frac{1}{\tau}, 1+\left|\ln\| B\|_{H\to H}\right|\right\}\|\varphi^\tau\|_{C([0,1]_\tau,H)}.$$

So, from the last estimate and the estimate (16) it follows the estimate (26). Theorem 5 is proved. □

Theorem 6. *The difference problem* (13) *is well posed in the Hölder spaces* $C_{01}^{\alpha}([0,1]_\tau, H)$ *under the assumption* (14) *and the following coercivity inequality holds:*

$$\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\|_{C_{01}^{\alpha}([0,1]_\tau,H)}+\|\{Au_k\}_1^{N}\|_{C_{01}^{\alpha}([0,1]_\tau,H)} \tag{27}$$

$$\le M(\delta)\left[\frac{1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C_{01}^{\alpha}([0,1]_\tau,H)}+||A\varphi||_H+||A\psi||_H\right].$$

Proof: By [17],

$$\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\|_{C_{01}^{\alpha}([0,1]_\tau,H)}+\|\{Au_k\}_1^{N-1}\|_{C_{01}^{\alpha}([0,1]_\tau,H)} \tag{28}$$

$$\le M(\delta)\frac{1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C_{01}^{\alpha}([0,1]_\tau,H)}+M(\delta)[||A\varphi||_H+||Au_N||_H]$$

was proved for the solution of difference scheme (21). Then the proof of (27) is based on (28) and on the estimate

$$\|Au_N\|_H \le M(\delta)\frac{1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C_{01}^{\alpha}([0,1]_\tau,H)}+M(\delta)[||A\varphi||_H+||A\psi||_H].$$

Applying the triangle inequality, formulas (23), (24), and estimate (25), we get

$$\|Au_N\|_H \le \|J_1\|_H + \|J_2\|_H \le \|J_2\|_H + M(\delta)[||A\varphi||_H+||A\psi||_H].$$

To this end, it suffices to show that

$$\|J_2\|_H \le M(\delta)\frac{1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C_{01}^{\alpha}([0,1]_\tau,H)}. \tag{29}$$

Applying formula (24), we get

$$J_2=K_\tau\sum_{j=1}^{N}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}\Bigg\{-\left(R^{N-j}-R^{N+j}\right)\tau^{-2}(I-R)^2\sum_{i=1}^{j-1}\tau^2\left(R^{N-i}-R^{N+i}\right)$$

$$\times\left(I-R^2\right)^{-1}(\varphi_i-\varphi_j)+\left(-\left(R^{N-j}-R^{N+j}\right)\right)\tau^{-2}(I-R)^2\sum_{i=j+1}^{N-1}\tau^2\left(R^{N-i}-R^{N+i}\right)$$

$$\times\left(I-R^2\right)^{-1}(\varphi_i-\varphi_j)+(I-R^{2N})\tau^{-2}(I-R)^2\sum_{i=1}^{j-1}\tau^2\left(R^{j-i}-R^{j+i}\right)\left(I-R^2\right)^{-1}(\varphi_i-\varphi_j)$$

$$+(I-R^{2N})\tau^{-2}(I-R)^2\sum_{i=j+1}^{N-1}\tau^2\left(R^{i-j}-R^{j+i}\right)\left(I-R^2\right)^{-1}(\varphi_i-\varphi_j)-\left(R^{N-j}-R^{N+j}\right)\tau^{-2}(I-R)^2$$

$$\times\sum_{i=1}^{j-1}\tau^2\left(R^{N-i}-R^{N+i}\right)\left(I-R^2\right)^{-1}\varphi_j-\left(R^{N-j}-R^{N+j}\right)\tau^{-2}(I-R)^2$$

$$\times\sum_{i=j+1}^{N-1}\tau^2\left(R^{N-i}-R^{N+i}\right)\left(I-R^2\right)^{-1}\varphi_j+(I-R^{2N})\tau^{-2}(I-R)^2\sum_{i=1}^{j-1}\tau^2\left(R^{j-i}-R^{j+i}\right)\left(I-R^2\right)^{-1}\varphi_j$$

$$+(I-R^{2N})\tau^{-2}(I-R)^2\sum_{i=j+1}^{N-1}\tau^2\left(R^{i-j}-R^{j+i}\right)\left(I-R^2\right)^{-1}\varphi_j\Bigg\}=\sum_{z=2}^{4}J_2^z,$$

where

$$J_2^2=K_\tau\sum_{j=1}^{N}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}\left(R^{2N-j-1}\left(I-R-R^2+R^3\right)+R^{2N+j}\left(I+R-R^j-R^{-1}\right)\right)\varphi_j,$$

$$J_2^3=K_\tau\sum_{j=1}^{N}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}(I-R)\left(I-R^{2N-2j}\right)$$

$$\times\sum_{i=1}^{j-1}R^{j-i}\left(I-R^{2i}\right)(I+R)^{-1}(\varphi_i-\varphi_j)=J_2^{3,1}+J_2^{3,2},$$

$$J_2^{3,1}=K_\tau\sum_{j=1}^{\left[\frac{N}{2}\right]}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}(I-R)\left(I-R^{2N-2j}\right)\sum_{i=1}^{j-1}R^{j-i}\left(I-R^{2i}\right)(I+R)^{-1}(\varphi_i-\varphi_j),$$

$$J_2^{3,2}=K_\tau\sum_{j=\left[\frac{N}{2}\right]+1}^{N}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}(I-R)\left(I-R^{2N-2j}\right)$$

$$\times\sum_{i=1}^{j-1}R^{j-i}\left(I-R^{2i}\right)(I+R)^{-1}(\varphi_i-\varphi_j),$$

$$J_2^4=K_\tau\sum_{j=1}^{N}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}(I-R)\left(I-R^{2j}\right)\sum_{i=j+1}^{N-1}R^{i-j}\left(I-R^{2N-2i}\right)$$

$$\times(I+R)^{-1}(\varphi_i-\varphi_j)=J_2^{4,1}+J_2^{4,2},$$

$$J_2^{4,1}=K_\tau\sum_{j=1}^{\left[\frac{N}{2}\right]}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}(I-R)\left(I-R^{2j}\right)\sum_{i=j+1}^{N-1}R^{i-j}\left(I-R^{2N-2i}\right)(I+R)^{-1}(\varphi_i-\varphi_j),$$

$$J_2^{4,2}=K_\tau\sum_{j=\left[\frac{N}{2}\right]+1}^{N}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}(I-R)\left(I-R^{2j}\right)\sum_{i=j+1}^{N-1}R^{i-j}\left(I-R^{2N-2i}\right)(I+R)^{-1}(\varphi_i-\varphi_j).$$

Second, let us estimate J_2^m for any $m=2,\cdots,4$, separately. We start with J_2^2, using estimates (15), (17), and the definition of the norm of the space $C_{01}^{\alpha}([0,1]_\tau,H)$, we obtain

$$\left\|J_2^2\right\|_H \leq M_1(\delta)\sum_{j=1}^{N}\left|\rho\left(t_j\right)\right|\tau\max_{1\leq j\leq N}\left\|\varphi_j\right\|_H \leq M_1(\delta)\sum_{j=1}^{N}\left|\rho\left(t_j\right)\right|\tau\left\|\varphi^\tau\right\|_{C_{01}^\alpha([0,1]_\tau,H)}.$$

From (14) it follows that

$$\left\|J_2^2\right\|_H \leq M_2(\delta)\left\|\varphi^\tau\right\|_{C_{01}^\alpha([0,1]_\tau,H)}.$$

Now, let us estimate $J_2^{3,1}$. Using the estimates (15), (17), and the definition of the norm of the space $C_{01}^\alpha([0,1]_\tau,H)$, we obtain

$$\left\|J_2^{3,1}\right\|_H \leq \left\|K_\tau\right\|_{H\to H}\sum_{j=1}^{\left[\frac{N}{2}\right]}\left|\rho\left(t_j\right)\right|\tau\left\|(I-R^{2N})^{-1}\right\|_{H\to H}$$

$$\times\sum_{i=1}^{j-1}\left\|R^{j-i}\left(I-R^{2N-2j}\right)(I-R)\right\|_{H\to H}\left\|I-R^{2i}\right\|_{H\to H}\left\|(I+R)^{-1}\right\|_{H\to H}\left\|\varphi_i-\varphi_j\right\|_H$$

$$\leq M(\delta)\sum_{j=1}^{\left[\frac{N}{2}\right]}\frac{\left|\rho\left(t_j\right)\right|}{(j\tau)^\alpha\left((N-j)\tau\right)^\alpha}\sum_{i=1}^{j-1}\frac{\tau}{((j-i)\tau)^{1-\alpha}}\left\|\varphi^\tau\right\|_{C_{01}^\alpha([0,1]_\tau,H)}.$$

The sum

$$\sum_{i=1}^{j-1}\frac{\tau}{((j-i)\tau)^{1-\alpha}}$$

is the lower Darboux integral sum for the integral

$$\int_0^{j\tau}\frac{ds}{(j\tau-s)^{1-\alpha}}.$$

It follows that

$$\left\|J_2^{3,1}\right\|_H \leq M(\delta)\sum_{j=1}^{\left[\frac{N}{2}\right]}\frac{\left|\rho\left(t_j\right)\right|\tau}{\alpha\left((N-j)\tau\right)^\alpha}\left\|\varphi^\tau\right\|_{C_{01}^\alpha([0,1]_\tau,H)}.$$

By the lower Darboux integral sum for the integral, it concludes that

$$\left\|J_2^{3,1}\right\|_H \leq M(\delta)\frac{2^{\alpha-1}}{\alpha(1-\alpha)}\sum_{j=1}^{\left[\frac{N}{2}\right]}\left|\rho\left(t_j\right)\right|\tau\left\|\varphi^\tau\right\|_{C_{01}^\alpha([0,1]_\tau,H)}.$$

For $J_2^{3,2}$, applying (15), (17), and the definition of the norm of the space $C_{01}^\alpha([0,1]_\tau,H)$, we get

$$\left\|J_2^{3,2}\right\|_H \leq M(\delta)\sum_{j=\left[\frac{N}{2}\right]+1}^{N}\frac{\left|\rho\left(t_j\right)\right|2^\alpha((N-j)\tau)^\alpha}{((N-j)\tau)^\alpha(j\tau)^\alpha}$$

$$\times\sum_{i=1}^{j-1}\frac{\tau}{((j-i)\tau)^{1-\alpha}((N-j-i+N)\tau)^\alpha}\left\|\varphi^\tau\right\|_{C_{01}^\alpha([0,1]_\tau,H)}.$$

The sum

$$\sum_{i=1}^{j-1}\frac{\tau}{((j-i)\tau)^{1-\alpha}((N-j-i+N)\tau)^\alpha}$$

is the lower Darboux integral sum for the integral

$$\int_0^{j\tau}\frac{ds}{(j\tau-s)^{1-\alpha}(1-j\tau-s+1)^\alpha}.$$

Since

$$\int_0^{j\tau}\frac{ds}{(j\tau-s)^{1-\alpha}(N\tau-j\tau-s+N\tau)^\alpha} \leq \frac{1}{(1-j\tau)^\alpha}\int_0^{j\tau}\frac{ds}{(j\tau-s)^{1-\alpha}} \leq \frac{M}{\alpha(j\tau)^\alpha},$$

it follows that

$$\left\|J_2^{3,2}\right\|_H \leq M(\delta) \sum_{j=\left[\frac{N}{2}\right]+1}^{N} \left|\rho\left(t_j\right)\right| \tau \frac{2^{\alpha}}{(j\tau)^{\alpha}\left(N\tau - j\tau\right)^{\alpha} \alpha (j\tau)^{-\alpha}} \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

By the lower Darboux integral sum for the integral it follows that

$$\left\|J_2^{3,2}\right\|_H \leq \frac{M(\delta)2^{2\alpha-1}}{\alpha\left(1-\alpha\right)} \sum_{j=\left[\frac{N}{2}\right]+1}^{N} \left|\rho\left(t_j\right)\right| \tau \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

Applying estimates for $\left\|J_2^{3,1}\right\|_H$ and $\left\|J_2^{3,2}\right\|_H$, we get

$$\left\|J_2^{3}\right\|_H \leq \frac{M_3(\delta)}{\alpha\left(1-\alpha\right)} \sum_{j=1}^{N} \left|\rho\left(t_j\right)\right| \tau \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

From (14) it follows that

$$\left\|J_2^{3}\right\|_H \leq \frac{M_4(\delta)}{\alpha\left(1-\alpha\right)} \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

Next, let us estimate $J_2^{4,1}$. Using estimates (15), (17), and the definition of the norm space $C_{01}^{\alpha}([0,1]_{\tau}, H)$, we obtain

$$\left\|J_2^{4,1}\right\|_H \leq M(\delta) \sum_{j=1}^{\left[\frac{N}{2}\right]} \frac{\left|\rho\left(t_j\right)\right|(N-j)^{\alpha}}{\left((N-j)\,\tau\right)^{\alpha}(j\tau)^{\alpha}} \sum_{i=j+1}^{N-1} \frac{\tau}{\left((i-j)\tau\right)^{1-\alpha}\left((N-j-i+N)\,\tau\right)^{\alpha}} \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

The sum

$$\sum_{i=j+1}^{N-1} \frac{\tau}{\left((i-j)\tau\right)^{1-\alpha}}$$

is the lower Darboux integral sum for the integral

$$\int_{j\tau}^{1} \frac{ds}{\left(s-j\tau\right)^{1-\alpha}}.$$

Since

$$\int_{j\tau}^{1} \frac{ds}{\left(s-j\tau\right)^{1-\alpha}\left(2N-j\tau-s\right)^{\alpha}} \leq \frac{1}{\left(N\tau-j\tau\right)^{\alpha}} \int_{j\tau}^{1} \frac{ds}{\left(s-j\tau\right)^{1-\alpha}}$$

$$\leq \frac{\left(N\tau-j\tau\right)^{\alpha}}{\alpha\left(N\tau-j\tau\right)^{\alpha}},$$

we have that

$$\left\|J_2^{4,1}\right\|_H \leq M(\delta) \sum_{j=1}^{\left[\frac{N}{2}\right]} \frac{\left|\rho\left(t_j\right)\right|\tau}{(j\tau)^{\alpha}\alpha} \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

By the lower Darboux integral sum for the integral, it follows that

$$\left\|J_2^{4,1}\right\|_H \leq \frac{M(\delta)2^{2\alpha}}{\alpha\left(1-\alpha\right)} \sum_{j=1}^{\left[\frac{N}{2}\right]} \left|\rho\left(t_j\right)\right| \tau \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

Finally, let us estimate $J_2^{4,2}$. Using estimates (15), (17), and the definition of the norm space $C_{01}^{\alpha}([0,1]_{\tau}, H)$, we get

$$\left\|J_2^{4,2}\right\|_H \leq M(\delta) \sum_{j=\left[\frac{N}{2}\right]+1}^{N} \frac{\left|\rho\left(t_j\right)\right|\tau}{\left((N-j)\,\tau\right)^{\alpha}(j\tau)^{\alpha}} \sum_{i=j+1}^{N-1} \frac{\tau}{\left((i-j)\tau\right)^{1-\alpha}} \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

The sum

$$\sum_{i=j+1}^{N-1}\frac{\tau}{((i-j)\tau)^{1-\alpha}}$$

is the lower Darboux integral sum for the integral

$$\int_{j\tau}^{1}\frac{ds}{(s-j\tau)^{1-\alpha}}.$$

Thus, we show that

$$\left\|J_2^{4,2}\right\|_H\le M(\delta)\sum_{j=\left[\frac{N}{2}\right]+1}^{N}\frac{\left|\rho\left(t_j\right)\right|\tau}{(j\tau)^{\alpha}\alpha}\left\|\varphi^\tau\right\|_{C_{01}^{\alpha}([0,1]_\tau,H)}.$$

By the lower Darboux integral sum for the integral, it follows that

$$\left\|J_2^{4,2}\right\|_H\le M(\delta)\frac{2^{\alpha-1}}{\alpha(1-\alpha)}\sum_{j=\left[\frac{N}{2}\right]+1}^{N}\left|\rho\left(t_j\right)\right|\tau\left\|\varphi^\tau\right\|_{C_{01}^{\alpha}([0,1]_\tau,H)}.$$

Applying estimates for $\left\|J_2^{4,1}\right\|_H$ and $\left\|J_2^{4,2}\right\|_H$, we get

$$\left\|J_2^{4}\right\|_H\le M(\delta)\left(\frac{2^{\alpha-1}}{\alpha(1-\alpha)}+\frac{2^{2\alpha}}{\alpha(1-\alpha)}\right)\sum_{j=1}^{N}\left|\rho\left(t_j\right)\right|\tau\left\|\varphi^\tau\right\|_{C_{01}^{\alpha}([0,1]_\tau,H)}.$$

From (14) it follows that

$$\left\|J_2^{4}\right\|_H\le\frac{M_5(\delta)}{\alpha(1-\alpha)}\left\|\varphi^\tau\right\|_{C_{01}^{\alpha}([0,1]_\tau,H)}.$$

Combining the estimates for $\|J_2^m\|_H, m=2,\cdots,4$ we get the estimate (29). Theorem 6 is proved. □

Now, we consider the applications of Theorems 3.2- 3.3.

The Bitsadze-Samarskii type nonlocal boundary value problem for the multidimensional elliptic equation (12) is considered. The discretization of problem (12) is carried out in two steps. In the first step, let us define the grid sets

$$\overline{\Omega}_h=\{x=x_m=(h_1m_1,\cdots,h_nm_n),m=(m_1,\cdots,m_n),0\le m_r\le N_r,$$

$$h_rN_r=1,\ r=1,\cdots,n\},\Omega_h=\widetilde{\Omega}_h\cap\Omega,S_h=\widetilde{\Omega}_h\cap S.$$

We introduce the Hilbert spaces $L_{2h}=L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2(\widetilde{\Omega}_h)$of the grid functions $\varphi^h(x)=\{\varphi(h_1m_1,\cdots,h_nm_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}(\widetilde{\Omega}_h)}=(\sum_{x\in\overline{\Omega}_h}\left|\varphi^h(x)\right|^2h_1\cdots h_n)^{1/2},$$

$$\left\|\varphi^h\right\|_{W_{2h}^2(\widetilde{\Omega}_h)}=\left\|\varphi^h\right\|_{L_{2h}(\widetilde{\Omega}_h)}+\left(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^{n}\left|\varphi^h_{x_r\overline{x_r},\,m_r}\right|^2h_1\cdots h_n\right)^{1/2}.$$

To the differential operator A generated by the problem (12), we assign the difference operator A_h^x by the formula

$$A_h^xu^h=-\sum_{r=1}^{n}(a_r(x)u^h_{\overline{x_r}})_{x_r,m_r},\tag{30}$$

acting in the space of the grid functions $u^h(x)$, satisfying the conditions $u^h=0$ for all $x\in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in $L_{2h}(\widetilde{\Omega}_h)$. With the help of A_h^x, we arrive at the nonlocal boundary value problem for an infinite system of ordinary differential equations

$$\begin{cases}-\frac{d^2u^h(t,x)}{dt^2}+A_h^xu^h(t,x)=f^h(t,x),\ 0<t<1,\ x\in\Omega_h,\\ u^h(0,x)=\varphi^h(x);\ u^h(1,x)=\int_0^1\rho(t)u^h(t,x)dt+\psi^h(x),\ x\in\widetilde{\Omega}_h.\end{cases}\tag{31}$$

In the second step, (31) is replaced by the difference scheme (13), and we get the following difference scheme:

$$\begin{cases}-\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^xu_k^h(x)=\varphi_k^h(x),\\ \varphi_k^h(x)=f^h(t_k,x),\ x\in\Omega_h,t_k=k\tau,1\le k\le N-1,\ N\tau=1,\\ u_0^h(x)=\varphi^h(x),x\in\widetilde{\Omega}_h,\\ u_N^h(x)=\sum_{j=1}^{N}\rho\left(t_j\right)\tau u_j^h(x)+\psi^h(x),\ x\in\widetilde{\Omega}_h.\end{cases}\tag{32}$$

Theorem 7. *Let τ and $|h|$ be sufficiently small positive numbers. Under the assumption (*14*), the solution of the difference scheme (*32*) satisfies the following almost coercivity estimate:*

$$\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^{h}-2u_{k}^{h}+u_{k-1}^{h}\right)\right\|_{L_{2h}}+\max_{1\le k\le N-1}\left\|u_{k}^{h}\right\|_{W_{2h}^{2}}$$

$$\le M(\delta)\left[\ln\frac{1}{\tau+|h|}\max_{1\le k\le N-1}\left\|\varphi_{k}^{h}\right\|_{L_{2h}}+\left\|\varphi^{h}\right\|_{W_{2h}^{2}}+\left\|\psi^{h}\right\|_{W_{2h}^{2}}\right].$$

The proof of Theorem 3.4 is based on Theorem 3.2 on the estimate

$$\min\left\{\ln\frac{1}{\tau},1+\left|\ln\|\,B_{h}^{x}\|_{L_{2h}\to L_{2h}}\right|\right\}\le M\ln\,\frac{1}{\tau+|h|},$$

on the symmetry properties of the difference operator A_h^x defined by (30) in L_{2h}, and on the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 8. *For the solution of the elliptic difference problem*

$$A_h^x u^h(x)=\omega^h(x),x\in\Omega_h, \tag{33}$$

$$u^h(x)=0,x\in S_h$$

the following coercivity inequality holds [22]:

$$\left\|u^{h}\right\|_{W_{2h}^{2}}\le M\left(\delta\right)||\omega^{h}||_{L_{2h}}.$$

Theorem 9. *τ and $|h|$ be sufficiently small positive numbers. Then under the assumption (*14*) the solution of the difference scheme (*32*) satisfies the following coercivity stability estimate*

$$\|\left\{\tau^{-2}\left(u_{k+1}^{h}-2u_{k}^{h}+u_{k-1}^{h}\right)\right\}_{1}^{N-1}\|_{C_{01}^{\alpha}([0,1]_{\tau},L_{2h})}+\|\left\{u_{k}^{h}\right\}_{1}^{N-1}\|_{C_{01}^{\alpha}([0,1]_{\tau},W_{2h}^{2})}$$

$$\le M(\delta)\left[\left\|\varphi^{h}\right\|_{W_{2h}^{2}}+\left\|\psi^{h}\right\|_{W_{2h}^{2}}+\frac{1}{\alpha(1-\alpha)}\left\|\left\{\varphi_{k}^{h}\right\}_{1}^{N-1}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}L_{2h}\right)}\right].$$

The proof of Theorem 9 is based on Theorem 6, on the symmetry properties of the difference operator A_h^x defined by the formula (30), and on Theorem 8 on the coercivity inequality for the solution of the elliptic difference equation (13) in L_{2h}.

4 Conclusion

In this paper, the well-posedness of problem (1) in Hölder spaces with a weight is established. The coercivity inequality for the solution of the nonlocal boundary value problem for elliptic equation is obtained. The first order of accuracy difference scheme for the approximate solution of the Bitsadze-Samarskii type nonlocal boundary value problem with integral condition for elliptic equation is studied. Theorems on the almost coercive stability estimates and coercive stability estimates for the solution of difference scheme for elliptic equations are proved.

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A New Modular Space Derived by Euler Totient Function

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Abstract: In this study, we introduce the Euler Totient sequence spaces in generalized Orlicz space and we examine some topological properties of these spaces by using the Luxemburg norm.

Keywords: Euler Totient function, Modular space, Orlicz sequence space, Luxemburg norm

1 Introduction and background

Lindenstrauss and Tzafriri [1] used the idea of Orlicz function M to construct the sequence space ℓ_M of all sequences of scalars (x_k) such that $\sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty$ for some $\rho > 0$. The space ℓ_M with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

is a Banach space and it is called as Orlicz sequence space. The space ℓ_M is closely related to the space $\ell_p = \{(x_k) : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$ which is an Orlicz space with $M(x) = x^p$, for $1 \leq p < \infty$.

Definition 1. *[2] A function $M : [0, \infty) \to [0, \infty)$ is called an Orlicz function if it is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for all $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.*

An Orlicz function M is said to satisfy the Δ_2-condition if there exists a constant $K > 0$ such that $M(2x) \leq KM(x)$ for all $x \geq 0$. It is easy to see that always $K > 2$.

Equivalently, an Orlicz function M is said to satisfy the Δ_2-condition if $M(lx) \leq K(l)M(x)$ for all $x \geq 0$, where $l > 1$.

A simple example of an Orlicz function which satisfies the Δ_2-condition is given by $M(x) = \alpha|x|^\alpha$ $(\alpha > 1)$, since we have $M(2x) = \alpha 2^\alpha |x|^\alpha = 2^\alpha M(x)$.

Definition 2. *[2] Let X be a linear space over $\mathbb{R}$. A function $\rho : X \to [0, \infty]$ is called a modular if the following conditions hold:*
(1) $\rho(x) = 0 \Leftrightarrow x = \theta$ (zero vector of X),
(2)$\rho(x) = \rho(-x)$ for all $x \in X$,
(3)$\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.
If the condition
(3')$\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$
holds instead of (3), then ρ is called a convex modular.
If ρ is a modular on X, then the linear space

$$X_\rho = \{x \in X : \lim_{\delta \to 0} \rho(\delta x) = 0\}$$

is called a modular space.

Definition 3. *[2] A sequence (x_n) in X_ρ is called ρ-convergent to $x_0 \in X_\rho$ if $\rho(\delta(x_n - x_0)) \to 0$ as $n \to \infty$ for some $\delta > 0$.*
A sequence (x_n) in X_ρ is called ρ-Cauchy if $\rho(\delta(x_n - x_m)) \to 0$ as $n, m \to \infty$ for some $\delta > 0$.
The space X_ρ is called ρ-complete if every ρ-Cauchy sequence in this space is ρ-convergent.

Definition 4. *Let E be a Lebesgue measurable subset of $\mathbb{R}$. The generalized Orlicz space is defined as follows:*

$$L_M = \{f : E \to \mathbb{R} : f \text{ is Lebesgue measurable and } \int_E M(\delta\,|f(x)|)dx < \infty \text{ for some } \delta > 0\}.$$

The function $\rho_M : L_M \to [0, \infty)$ defined by

$$\rho_M(f) = \int_E M(|f(x)|)dx$$

is a modular on L_M and the space L_M is ρ_M-complete.

The generalized Orlicz space L_M is a Banach space with the Luxemburg norm given by

$$\|f\|_M = \inf\{\gamma > 0 : \rho_M\left(\frac{f}{\gamma}\right) \le 1\}.$$

Throughout the study, by $\omega(L_M)$, we denote the space of all sequences in L_M.

Let φ denote the Euler function. For every $m \in \mathbb{N}$ with $m > 1$, $\varphi(m)$ is the number of positive integers less than m which are coprime with m and $\varphi(1) = 1$. If $p_1^{a_1} p_2^{a_2} ... p_r^{a_r}$ is the prime factorization of a natural number $m > 1$, then

$$\varphi(m) = m(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})...(1 - \frac{1}{p_r}).$$

Also, the equality

$$m = \sum_{k|m} \varphi(k)$$

holds for every $m \in \mathbb{N}$ and $\varphi(m_1 m_2) = \varphi(m_1)\varphi(m_2)$, where $m_1, m_2 \in \mathbb{N}$ are coprime [4]. One can consult to [5] for more details related to these functions.

The Φ-summability was introduced by Schoenberg [3] for the purpose of studying the Riemann integrability of a generalized Dirichlet function in the range $[0, 1]$. This method is called φ-convergence which is a weaker form of usual convergence. The infinite matrix $\Phi = (\phi_{ij})$ is defined as

$$\phi_{ij} = \begin{cases} \frac{\varphi(j)}{i} & , \quad \text{if } j \mid i \\ 0 & , \quad \text{if } j \nmid i \end{cases}$$

The matrix Φ satisfies the following conditions:

1. $\sup_{i\in\mathbb{N}}(\sum_{j=1}^{\infty} |\phi_{ij}|) < \infty$,
2. $\lim_{i\to\infty} \phi_{ij} = 0$ for each fixed $j \in \mathbb{N}$,
3. $\lim_{i\to\infty} \sum_{j=1}^{\infty} \phi_{ij} = 1$

and so it is a regular matrix.

By using this matrix, İlkhan and Kara [6] have introduced the sequence spaces $\ell_p(\Phi)$ and $\ell_\infty(\Phi)$ as

$$\ell_p(\Phi) = \left\{ u = (u_n) \in \omega : \sum_n \left| \frac{1}{n} \sum_{k|n} \varphi(k) u_k \right|^p < \infty \right\} \quad (1 \le p < \infty)$$

and

$$\ell_\infty(\Phi) = \left\{ u = (u_n) \in \omega : \sup_n \left| \frac{1}{n} \sum_{k|n} \varphi(k) u_k \right| < \infty \right\}.$$

In the literature, there are many papers on sequence spaces using Orlicz function. Later these spaces are generalized by using the Lebesgue integral with Orlicz function. In [7], the authors have generalized the Cesàro sequence spaces in the classical Banach space L_p to the generalized Orlicz space L_M. In this paper, we generalize Euler sequence spaces to the generalized Orlicz space and obtain a modular space. Also, we examine some topological properties of these spaces by using the Luxemburg norm.

2 Main results

Now, we introduce the Euler Totient sequence spaces in generalized Orlicz space as follows:

$$W(M, \Phi) = \{(f_k) \in \omega(L_M) : \lim_{n\to\infty} \frac{1}{n} \sum_{k|n} \varphi(k)\rho_M(\lambda|f_k - f_0|) = 0 \text{ for some } \lambda > 0, f_0 \in L_M\},$$

$$W^\infty(M, \Phi) = \{(f_k) \in \omega(L_M) : \sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{k|n} \varphi(k)\rho_M(\lambda|f_k|) = 0 \text{ for some } \lambda > 0\}.$$

Theorem 1. *If the Orlicz function M satisfies the Δ_2-condition, then the following equalities hold:*

$$W(M,\Phi) = \{(f_k) \in \omega(L_M) : \lim_{n\to\infty} \frac{1}{n} \sum_{k|n} \varphi(k)\rho_M(|f_k - f_0|) = 0, f_0 \in L_M\},$$

$$W^{\infty}(M,\Phi) = \{(f_k) \in \omega(L_M) : \sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{k|n} \varphi(k)\rho_M(|f_k|) = 0\}.$$

Proof: Denote the right hand side of the first equality by $W_0(\rho_M,\Phi)$. It is clear that $W_0(\rho_M,\Phi) \subset W(M,\Phi)$.
Now, choose $(f_k) \in W(M,\Phi)$. If $\lambda \geq 1$, we have $(f_k) \in W_0(\rho_M,\Phi)$ since M is a non-decreasing function. If $\lambda < 1$, there exists $K(\lambda) > 0$ such that $M(\frac{x}{\lambda}) \leq K(\lambda)M(x)$ for all $x \geq 0$ since M satisfies Δ_2-condition. Hence, we deduce that

$$\begin{aligned}
\frac{1}{n}\sum_{k|n}\varphi(k)\rho_M(|f_k - f_0|) &= \frac{1}{n}\sum_{k|n}\varphi(k)\int_E M\left(\frac{\lambda}{\lambda}|f_k(x) - f_0(x)|\right)dx \\
&\leq \frac{K(\lambda)}{n}\sum_{k|n}\varphi(k)\int_E M\left(\lambda|f_k(x) - f_0(x)|\right)dx \to 0
\end{aligned}$$

as $n \to \infty$. This proves that $(f_k) \in W_0(\rho_M,\Phi)$. Hence, we conclude that $W(M,\Phi) = W_0(\rho_M,\Phi)$. □

Remark 1. *Note that if the Orlicz function M is defined by $M(x) = |x|^p$ for $1 < p < \infty$, then the space is reduced to the following space*

$$W(p,\Phi) = \{(f_k) \in \omega(L_p) : \lim_{n\to\infty} \frac{1}{n} \sum_{k|n} \varphi(k) \int_E |f_k(x) - f_0(x)|^p dx = 0, f_0 \in L_p\},$$

where $L_p = \{f : E \to \mathbb{R} : \int_E |f(x)|^p dx < \infty\}$.

Using the fact that ρ_M is a convex modular on L_M, we obtain the following results.

Theorem 2. *The function $\rho : \omega(L_M) \to [0,\infty)$ given by*

$$\rho(f) = \sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{k|n}\rho_M(f_k)$$

is a convex modular, where $f = (f_k) \in \omega(L_M)$.

Theorem 3. *The space*

$$W^{\infty}(M,\Phi) = \{f \in \omega(L_M) : \rho(\lambda f) < \infty, \lambda > 0\}$$

is a modular space.

Proof: Clearly, the space $W^{\infty}(M,\Phi)$ is linear. Also, $(\omega(L_M))_\rho = \{f \in \omega(L_M) : \lim_{\lambda\to 0}\rho(\lambda f) = 0\} \subset W^{\infty}(M,\Phi)$ holds. To prove the inverse inclusion, choose $f \in W^{\infty}(M,\Phi)$ which means $\rho(\lambda f) < \infty$ for some $\lambda > 0$. By convexity of ρ, for $|\frac{\alpha}{\lambda}| < 1$, we have

$$\lim_{\alpha\to 0}\rho(\alpha f) = \lim_{\alpha\to 0}\frac{\alpha}{\lambda}\rho(\lambda f) = 0.$$

This implies that $f \in (\omega(L_M))_\rho$. □

Since ρ is a modular, we can define the Luxemburg norm $\|.\|_\rho$ on $W^{\infty}(M,\Phi)$ as

$$\|f\|_\rho = \inf\{\gamma > 0 : \rho\left(\frac{f}{\gamma}\right) \leq 1, f \in W^{\infty}(M,\Phi)\}.$$

Definition 5. *Let (f^n) be a sequence in $W^{\infty}(M,\Phi)$.*
It is said to be ρ-convergent or modular convergent to $f \in W^{\infty}(M,\Phi)$ if there exists $\lambda > 0$ such that $\lim_{n\to\infty}\rho(\lambda(f^n - f)) = 0$.
It is said to be ρ-Cauchy if there exists $\lambda > 0$ such that $\lim_{n,m\to\infty}\rho(\lambda(f^n - f^m)) = 0$.

Theorem 4. *The space $W^{\infty}(M,\Phi)$ is ρ-complete.*

Theorem 5. *The space $W^{\infty}(M,\Phi)$ is complete with the Luxemburg norm $\|.\|_\rho$.*

Theorem 6. *If the Orlicz function M satisfies Δ_2-condition, then the norm convergence and modular convergence are equivalent.*

Theorem 7. *The space $W(M,\Phi)$ is a closed subspace of $W^{\infty}(M,\Phi)$.*

Proof: It is clear that $W(M,\Phi)$ is a linear subspace of $W^{\infty}(M,\Phi)$. Now, let (f^m) be a convergent sequence in $W(M,\Phi)$. Since $f^m \in W(M,\Phi)$ for each $m \in \mathbb{N}$, then there exists $f_0^m \in L_M$ and $\lambda > 0$ such that $\lim_n \frac{1}{n}\sum_{k|n}\varphi(k)\rho_M(\lambda(f_k^m - f_0^m)) = 0$. Also, since (f^m) is convergent, then $\rho(\lambda(f^m - f)) \to 0$ as $m \to \infty$ for some $f = (f_k) \in W^{\infty}(M,\Phi)$. Hence, we have

$$\frac{1}{n}\sum_{k|n}\varphi(k)\rho_M(\lambda(f_k^m - f_k)) \to 0$$

. It follows that

$$\frac{1}{n}\sum_{k|n}\varphi(k)\rho_M(\lambda(f_k - f_0^m)) \to 0$$

as $n \to \infty$ which implies that $f = (f_k) \in W(M,\Phi)$. Thus the space $W(M,\Phi)$ is closed. □

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Numerical Solution of Riesz Fractional Differential Equation via Meshless Method

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Abstract: In this study, we present the numerical solution of Riesz fractional differential equation with the help of meshless method. In accordance with this purpose, we benefit the radial basis functions (RBFs) interpolation method and conformable fractional calculus. We finally present the results of numerical experimentation to show that presented algorithm provide successful consequences.

Keywords: Numerical solution, Fractional calculus, RBF interpolation

1 Introduction

In this study, we provide a meshfree algorithm for the numerical solution of Riesz fractional differential equation by taking advantageous of radial basis function interpolation [5], [6]. The aim of this scheme is to acquire approximate solution of Riesz fractional differential equation with RBF collocation method. Of course this approach would provide an insight the solution of more complex cases.

The remainder of this work is organized as follows: In Section 2, the conformable derivatives are summarised, along with the conformable fractional calculus. In Section 3, the RBF interpolation method is reviewed while in Section 4 the numerical scheme of solving conformable ordinary differential equation using meshfree method is introduced and we also review the RBF collocation technique. Numerical examples are given in Section 5, while some conclusions are discussed in Section 6.

2 Conformable fractional calculus

In this manuscript, meshfree solution of Riesz fractional differential equation will be presented and tested via conformable fractional calculus. More detail, conformable derivatives for $\boldsymbol{\alpha} \in (0, 1]$ and $t \in [0, \infty)$ given by

$$\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f})(t) = \lim_{\varepsilon \to 0} \frac{\mathfrak{f}\left(te^{\varepsilon t^{-\alpha}}\right) - \mathfrak{f}(t)}{\varepsilon}, \qquad \mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f})(0) = \lim_{t \to 0} \mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f})(t), \tag{1}$$

provided the limits exist (for more detail see, [1]). If $\mathfrak{f}$ is fully differentiable at t, then

$$\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f})(t) = t^{1-\alpha} \frac{d\mathfrak{f}}{dt}(t). \tag{2}$$

A function $\mathfrak{f}$ is $\boldsymbol{\alpha}-$differentiable at a point $t \geq 0$ if the limit in (1) exists and is finite. This yields the following consequences.

Theorem 1. *Let $\boldsymbol{\alpha} \in (0, 1]$ and $\mathfrak{f}, \mathfrak{g}$ be $\boldsymbol{\alpha}-$differentiable at a point $t > 0$. Then*

1. $\mathcal{D}^{\boldsymbol{\alpha}}(a\mathfrak{f} + b\mathfrak{g}) = a\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f}) + bD^{\alpha}(\mathfrak{g})$, *for all* $a, b \in \mathbb{R}$,
2. $\mathcal{D}^{\boldsymbol{\alpha}}(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$,
3. $\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f}\mathfrak{g}) = \mathfrak{f}\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{g}) + vD^{\alpha}(\mathfrak{f})$,
4. $\mathcal{D}^{\boldsymbol{\alpha}}\left(\frac{\mathfrak{f}}{\mathfrak{g}}\right) = \frac{\mathfrak{g}\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f}) - \mathfrak{f}\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{g})}{\mathfrak{g}^2}$,
5. $\mathcal{D}^{\boldsymbol{\alpha}}(t^n) = nt^{n-\alpha}$ *for all* $n \in \mathbb{R}$,
6. $\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f} \circ \mathfrak{g})(t) = \mathfrak{f}'(\mathfrak{g}(t))\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{g})(t)$ *for* $\mathfrak{f}$ *is differentiable at* $\mathfrak{g}(t)$.

3 Meshless method

In this part, the basic facts of the meshless radial basis function interpolation are explained. Consider a function $\mathfrak{f} : \mathbb{R}^d \to \mathbb{R}$ a real valued function with d variables, which is to be approximated by $\mathcal{I}_{\mathcal{X}} : \mathbb{R}^d \to \mathbb{R}$, for given values $\mathfrak{f}(\mathbf{x}_i) : i = 1, 2, \cdots, n$, where $\mathbf{x}_i : i = 1, 2, \cdots, n$ is a set of distinct points in $\mathbb{R}^d$, named the center set $\mathcal{X}$.

Then the approximation to the function $\mathfrak{f}$ is of the form

$$\mathfrak{I}_{\mathfrak{X}}(\mathbf{x}) = \sum_{k=1}^{N} \mathfrak{a}_k \tau_k(\|\mathbf{x} - \mathbf{x}_k\|), \tag{3}$$

where $\tau_k : \mathbb{R}^d \to \mathbb{R}$ is a univariate radial basis function. Then the interpolation condition can be constructed as $\mathfrak{I}_{\mathfrak{X}}(\mathbf{x}_m) = \mathfrak{f}(\mathbf{x}_m)$, $m = 1, 2, \cdots, N$. Namely, the interpolation condition is

$$\sum_{k=1}^{N} \mathfrak{a}_k \tau_k(\|\mathbf{x}_m - \mathbf{x}_k\|) = \mathfrak{f}(\mathbf{x}_m), \qquad m = 1, 2, \cdots, N \tag{4}$$

In other words the system of matrix for interpolation condition can be written as

$$[A]\{\mathfrak{a}\} = \{\mathfrak{f}\} \tag{5}$$

where the entries of the matrix A are $A_{m,k} = \tau_k(\|\mathbf{x}_m - \mathbf{x}_k\|)$ such that $m, k = 1, 2, \cdots, N$, $\mathfrak{a} = \{\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_N\}^T$ and $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \cdots, \mathfrak{f}_N\}^T$. The interpolant of $\mathfrak{f}(\mathbf{x})$ is unique if and only if the matrix X is non-singular. It has been discussed about sufficient conditions for $\tau(r)$ to guarantee non-singularity of the a matrix [2], [3]

Well known radial basis functions are given below:

RBFs	$\tau(r)$
Piecewise Polynomial (R_n)	$\lvert r\rvert^n$, n odd
Thin Plate Spline (TPS_n)	$\lvert r\rvert^n ln\lvert r\rvert$, n even
Multiquadric (MQ)	$\sqrt{1+r^2}$
Inverse Multiquadric (IMQ)	$\frac{1}{\sqrt{1+r^2}}$
Inverse Quadratic (IQ)	$\frac{1}{1+r^2}$
Gaussian (GA)	e^{-r^2}
Bessel (BE)	$J_0(2r)$

4 Numerical approach

In this section, we provide a numerical scheme to solve Riesz fractional differential equation via meshless method. To begin with, consider the following Riesz fractional differential equation [4],

$$\frac{\partial \mathfrak{f}(x,t)}{\partial t} = -\kappa_{\boldsymbol{\alpha}} \frac{\partial^{\boldsymbol{\alpha}}}{\partial |x|^{\boldsymbol{\alpha}}} \mathfrak{f}(x,t), \qquad x \in [0, \pi], \qquad t \in (0, T],$$

$$\mathfrak{f}(x, 0) = \mathfrak{f}_0(x),$$

$$\mathfrak{f}(0, t) = \mathfrak{f}(\pi, t) = 0,$$

where $\kappa_{\boldsymbol{\alpha}}$ represents the dispersion coefficient and $\mathfrak{f}(x,t)$ is a solute concentration. Let x_i be equally spaced grid points in the interval $0 \leq x_i \leq \pi$ such that $1 \leq i \leq N$, $x_1 = 0$ and $x_N = \pi$. Then by solving the equation $\omega(x) = \tau(x)\lambda^{-1}$ for $\omega(x) = (\omega_k(x))_{1 \leq k \leq N}$, $\tau(x) = (\tau(|x - x_k|))_{1 \leq k \leq N}$ and $\lambda = (\tau(|x_i - x_k|))_{1 \leq i \leq N, 1 \leq k \leq N}$ one can construct the Lagrange basis $\omega_1(x), \omega_1(x), \cdots, \omega_N(x)$ of the span of the functions $(\tau(|x - x_k|))_{1 \leq k \leq N}$. If $\mathfrak{D}$ is a differential operator, and if the radial basis function τ is sufficiently smooth to to allow implementation of $\mathfrak{D}$, the desired derivatives $\mathfrak{D}\omega_k$ of the Lagrange basis ω_k come via solving $(\mathfrak{D}\omega)(x) = (\mathfrak{D}\tau)(x)\lambda^{-1}$. So one can write the approximation solution as

$$\mathfrak{f}(x,t)=\sum_{k=2}^{N-1}\varphi_k(t)\omega_k(x)$$

where $\varphi(t)=\varphi_k(t), 2\le k\le N-1$, which yields

$$\sum_{k=2}^{N-1}\varphi_k'(t)\omega_k(x_i)=-\kappa_{\boldsymbol{\alpha}}\sum_{k=2}^{N-1}\varphi_k(t)\frac{\partial^{\boldsymbol{\alpha}}}{\partial|x|^{\boldsymbol{\alpha}}}\omega_k(x_i),\qquad \varphi_k(0)=\mathfrak{f}_0(x_k),\quad 2\le k\le N-1.$$

Hence one obtain the following system of ordinary differential equation:

$$\varphi'(t)=-\kappa_{\boldsymbol{\alpha}}\left[\frac{\partial^{\boldsymbol{\alpha}}}{\partial|x|^{\boldsymbol{\alpha}}}\omega*\varphi(t)\right],\varphi(0)=\Delta_0.$$

where

$$\frac{\partial^{\boldsymbol{\alpha}}}{\partial|x|^{\boldsymbol{\alpha}}}\omega=\left[\frac{\partial^{\boldsymbol{\alpha}}}{\partial|x|^{\boldsymbol{\alpha}}}\omega_k(x_i)\right]_{2\le i\le N-1,2\le k\le N-1},$$

and

$$\Delta_0=\begin{pmatrix}\mathfrak{f}_0(x_2)\\ \vdots\\ \vdots\\ \mathfrak{f}_0(x_{N-1})\end{pmatrix}.$$

5 Numerical experiments

Now, in order to validate our approach, we present a numerical experiment which performed by MATLAB. Here, let consider the parameter $\alpha=0.5, \kappa_{\boldsymbol{\alpha}}=0.25, T=0.4$ and $\mathfrak{f}_0(x)=\sin(\pi-3x)$. In Figure, we present the numerical solutions of given fractional differential equations by using the Gaussian RBF with $\varepsilon=1$, and taking 101 discretization grids.

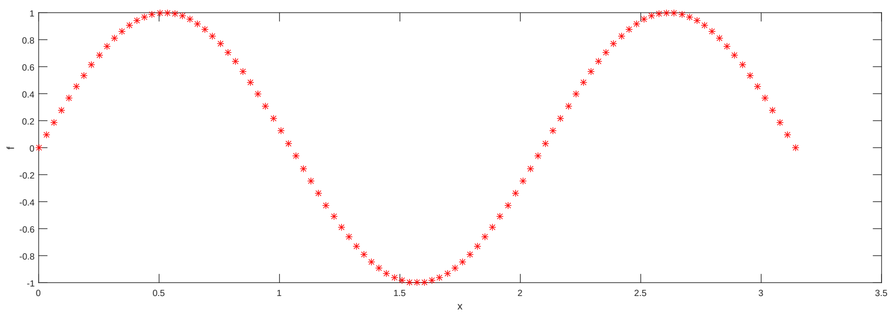

Fig. 1: $\mathfrak{f}$ versus x using Gaussain RBF with $\varepsilon=1$ taking 101 discretization grids

6 Conclusion

In this article, we presented and tested the numerical solution of Riesz fractional differential equation with the help of meshless method. Here, we benefit the radial basis functions (RBFs) interpolation method and conformable fractional calculus. We finally present the results of numerical experimentation to show that presented algorithm provide successful consequences.

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Some Pascal Spaces of Difference Sequences Spaces of Order m

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Abstract: The main purpose of this article is to introduce new sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ which are consisted by sequences whose m^{th} order differences are in the Pascal sequence spaces p_∞, p_c and p_0, respectively. Furthermore, the bases of the new difference sequence spaces $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$, and the α-, β- and γ-duals of the new difference sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ have been determined. Finally, necessary and sufficient conditions on an infinite matrix belonging to the classes $(p_c\left(\Delta^{(m)}\right) : l_\infty)$ and $(p_c\left(\Delta^{(m)}\right) : c)$ are obtained.

Keywords: Difference operator of order m, Matrix mappings, Pascal difference sequence spaces, α-, β- and γ-duals.

1 Introduction

By w, we shall denote the space of all real or complex valued sequences. Any vector subspace of w is called as a sequence space. We shall write l_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs and l_1 we denote the spaces of all bounded, convergent and absolutely convergent series, respectively.

Let X, Y be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,\, k \in N$. Then, the matrix A defines a transformation from X into Y and we denote it by $A : X \to Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in Y, where

$$(Ax)_n = \sum_k a_{nk} x_k \tag{1}$$

for each $n \in N$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(X : Y)$, we denote the class of all matrices A such that $A : X \to Y$. Thus $A \in (X : Y)$ if and only if the series on the right side of (1) converges for each $n \in N$ and every $x \in X$, and we have $Ax = \{(Ax)_n\} \in Y$ for all $x \in X$.

In the study on the sequence spaces, there are some basic approaches which are determination of topologies, matrix mapping and inclusions of sequence spaces [2]. These methods are applied to study the matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\},$$

which is a sequence space. Although in the most cases the new sequence space X_A generated by the limitation matrix A from a sequence space X is the expansion or the contraction of the original space X, in some cases it may be observed that those spaces overlap. Indeed, one can easily see that the inclusions $X_S \subset X$ and $X \subset X_\Delta$ strictly hold for $X \in \{l_\infty, c, c_0\}$ [1]. Especially, the sequence spaces and the difference operator which are special cases for the matrix A have been studied extensively via the methods mentioned above.

Define the difference matrices $\Delta^1 = (\delta_{nk})$ by

$$\delta_{nk} = \begin{cases} \{(-1)^{n-k}, & (n-1 \le k \le n), \\ 0, & (0 < n-1 \; or \; n > k), \end{cases}$$

for each $k, n \in N$.

In the literature, the difference sequence spaces $l_\infty(\Delta) = \{x = (x_k) \in w : \Delta x \in l_\infty\}$, $c(\Delta) = \{x = (x_k) \in w : \Delta x \in c\}$ and $c_0(\Delta) = \{x = (x_k) \in w : \Delta x \in c_0\}$ are first defined by Kızmaz [3]. Difference sequence spaces have been defined and studied by various authors [9]-[20]. The idea of difference sequences was generalized by Et and Çolak [9] and Murseelan [10]. Let λ denotes one of the sequence spaces l_∞, c, and c_0. They defined the sequence spaces $\lambda\left(\Delta^{(m)}\right) = \{x = (x_k) \in w : \Delta^{(m)} x \in \lambda\}$, where $m \in N$ and $\left(\Delta^{(m)} x\right)_n = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k-i}$. The operator $\Delta^{(m)} : w \to w$ is defined by $\left(\Delta^{(1)} x\right)_k = (x_k - x_{k+1})$ and $\Delta^{(m)} x = \left(\Delta^{(1)} x\right)_k o \left(\Delta^{(m-1)} x\right)_k$ $(m \ge$

2). Throughout the article, we shall use the convention that a term with a negative subscript is equal to naught. Also throughout this work, by F and K, respectively, we shall denote the collection of all finite subsets of N.

Let P denote the Pascal means defined by the Pascal matrix [4] is defined by

$$P = [p_{nk}] = \begin{cases} \binom{n}{n-k}, \ 0 \le k \le n) \\ 0, \ (k > n) \end{cases}, (n, k \in N)$$

and the inverse of Pascal's matrix $P_n = (p_{nk})$ is given by

$$P^{-1} = [p_{nk}]^{-1} = \begin{cases} (-1)^{n-k}\binom{n}{n-k}, (0 \le k \le n) \\ 0, (k > n) \end{cases}, (n, k \in N).$$

There is some interesting properties of Pascal matrix. For example; we can form three types of matrices: symmetric, lower triangular, and upper triangular, for any integer $n > 0$. The symmetric Pascal matrix of order n is defined by

$$S_n = (s_{ij}) = \binom{i+j-2}{j-1}, \text{for } i, j = 1, 2, ..., n, \tag{2}$$

we can define the lower triangular Pascal matrix of order n by

$$L_n = (l_{ij}) = \begin{cases} \binom{i-1}{j-1}, (0 \le j \le i) \\ 0, \ (j > i) \end{cases}, \tag{3}$$

and the upper triangular Pascal matrix of order n is defined by

$$U_n = (u_{ij}) = \begin{cases} \binom{j-1}{i-1}, (0 \le i \le j) \\ 0, \ (j > i) \end{cases}. \tag{4}$$

We notice that $U_n = (L_n)^T$, for any positive integer n.

i. Let S_n be the symmetric Pascal matrix of order n defined by (2), L_n be the lower triangular Pascal matrix of order n defined by (3), and U_n be the upper triangular Pascal matrix of order n defined by (4), then $S_n = L_n U_n$ and $det(S_n) = 1$ [5].

ii. Let A and B be $n \times n$ matrices. We say that A is similar to B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$ [6].

iii. Let S_n be the symmetric Pascal matrix of order n defined by (2), then S_n is similar to its inverse S_n^{-1} [5].

iv. Let L_n be the lower triangular Pascal matrix of order n defined by (3), then $L_n^{-1} = ((-1)^{i-j} l_{ij})$ [7].

Recently Polat [8] has defined the Pascal sequence spaces p_∞, p_c and p_0 like as follows:

$$p_\infty = \left\{ x = (x_k) \in w : \sup_n \left| \sum_{k=0}^{n} \binom{n}{n-k} x_k \right| < \infty \right\},$$

$$p_c = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{n-k} x_k \text{ exists} \right\},$$

and

$$p_0 = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{n-k} x_k = 0 \right\}.$$

In the present paper, we define Pascal difference sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ which consist of all sequences whose m^{th} order differences are in the Pascal sequence spaces p_∞, p_c and p_0, respectively. Furthermore, the Schauder bases of the sequence spaces $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$, and the α-, β- and γ- duals of the sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ have been determined. The last section of the article is devoted to the characterization of some matrix mappings on the sequence space $p_c\left(\Delta^{(m)}\right)$.

2 New Pascal difference sequence spaces of order m

The triangle matrix $\Delta^{(m)} = (\delta_{nk}^{(m)})$ is defined by

$$\delta_{nk}^{(m)} = \begin{cases} (-1)^{n-k}\left(\binom{m}{n-k}\right), (\max\{0, n-m\} \le k \le n), \\ 0, \ (0 \le k < \max\{0, n-m\} \ or \ n > k), \end{cases}$$

for all $k, n \in N$ and for any fixed $m \in N$. Using this matrix, we introduce the sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ as the set of all sequences such that $\Delta^{(m)}$-transforms of them are in the Pascal sequence spaces p_∞, p_c and p_0, respectively, that is,

$$\begin{aligned} p_\infty\left(\Delta^{(m)}\right) &= \left\{x = (x_k) \in w : \Delta^{(m)} x \in p_\infty\right\}, \\ p_c\left(\Delta^{(m)}\right) &= \left\{x = (x_k) \in w : \ \Delta^{(m)} x \in p_c\right\}, \end{aligned}$$

and

$$p_0\left(\Delta^{(m)}\right) = \left\{x = (x_k) \in w : \ \Delta^{(m)} x \in p_0\right\}.$$

Define the sequence $y = \{y_k\}$, which is frequently used, as the H- transform of a sequence $x = (x_k)$, i.e.,

$$\begin{aligned} y_n = (Hx)_n &= \sum_{k=0}^{n} \binom{n}{n-k} \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k-i} \\ &= \sum_{k=0}^{n} \left[\sum_{i=k}^{n} \binom{i}{i-k} (-1)^{i-k} \binom{m}{i-k}\right] x_k \end{aligned} \tag{5}$$

for each $n, m \in N$. Here by H, we denote the matrix $H = P\Delta^{(m)} = (h_{nk})$ defined by

$$h_{nk} = \begin{cases} \sum_{i=k}^{n} \binom{i}{i-k} (-1)^{i-k} \binom{m}{i-k}, \ (0 \le k \le n) \\ 0, \ (k > n) \end{cases}, (n, k \in N).$$

It can be easily shown that $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ are normed linear spaces by the following norm:

$$\|x\|_\Delta = \|Hx\|_\infty = \sup_n \left|\sum_{k=0}^{n} \left[\sum_{i=k}^{n} \binom{i}{i-k} (-1)^{i-k} \binom{m}{i-k}\right] x_k\right|. \tag{6}$$

Theorem 1. *The sequence spaces* $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ *and* $p_0\left(\Delta^{(m)}\right)$ *are Banach spaces with the norm* (6).

Proof: Let $\left\{x^i\right\}$ be any Cauchy sequence in the space $p_\infty\left(\Delta^{(m)}\right)$, where $\left\{x^i\right\} = \left\{x_k^i\right\} = \left\{x_0^i, x_1^i, ...\right\} \in p_\infty\left(\Delta^{(m)}\right)$ for each $i \in N$. Then, for a given $\varepsilon > 0$ there exists a positive integer $N_0(\varepsilon)$ such that $\left\|x_i^k - x_i^n\right\|_\Delta < \varepsilon$ for all $k, n > N_0(\varepsilon)$. Hence

$$\left|H(x_i^k - x_i^n)\right| < \varepsilon$$

for all $k, n > N_0(\varepsilon)$ and for each $i \in N$. Therefore, $\left\{(Hx)_i^k\right\} = \left\{(Hx)_i^0, (Hx)_i^1, (Hx)_i^2, ...\right\}$ is a Cauchy sequence of real numbers for every fixed $i \in N$. Since the set of real numbers R is complete, it converges, say

$$\lim_{i \to \infty} (Hx^i)_k \to (Hx)_k$$

for each $k \in N$. So, we have

$$\lim_{n \to \infty} \left|H(x_i^k - x_i^n)\right| = \left|H(x_i^k - x_i)\right| \le \varepsilon$$

for each $k \ge N_0(\varepsilon)$. This implies that $\left\|x^k - x\right\|_\Delta < \varepsilon$ for $k \ge N_0(\varepsilon)$, that is, $x^i \to x$ as $i \to \infty$.

Now, we must show that $x \in p_\infty\left(\Delta^{(m)}\right)$. We have

$$\begin{aligned} \|x\|_\Delta &= \|Hx\|_\infty = \sup_n \left|\sum_{k=0}^{n} \left[\sum_{i=k}^{n} \binom{i}{i-k} (-1)^{i-k} \binom{m}{i-k}\right] x_k\right| \\ &\le \sup_n \left|H(x_k^i - x_k)\right| + \sup_n \left|Hx_k^i\right| \\ &\le \left\|x^i - x\right\|_\Delta + \left|P\Delta^{(m)} x_k^i\right| < \infty \end{aligned}$$

for all $i \in N$. This implies that $x = (x_i) \in p_\infty\left(\Delta^{(m)}\right)$. Therefore $p_\infty\left(\Delta^{(m)}\right)$ is a Banach space. It can be shown that $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ are closed subspaces of $p_\infty\left(\Delta^{(m)}\right)$, which leads us to the consequence that the spaces $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ are also Banach spaces with the norm (6). Furthermore, since $p_\infty\left(\Delta^{(m)}\right)$ is a Banach space with continuous coordinates, i.e., $\left\|P(x^k - x)\right\|_\Delta \to 0$ implies $\left|H(x_i^k - x_i)\right| \to 0$ for all, it is a BK-space. □

3 The bases of sequence spaces $p_c(\Delta^{(m)})$ and $p_0(\Delta^{(m)})$

In this section, we shall give the Schauder bases for the spaces $p_c(\Delta^{(m)})$ and $p_0(\Delta^{(m)})$. First we define the Schauder bases. A sequence $\left\{b^{(k)}\right\}_{k\in\mathbb{N}}$ in a normed sequence space X is called a Schauder bases (or briefly bases), if for every $x\in X$ there is a unique sequence (λ_k) of scalars such that

$$\lim_{n\to\infty}\|x-(\lambda_0x_0+\lambda_1x_1+...+\lambda_nx_n)\|=0.$$

Theorem 2. *Define the sequence* $b^{(k)}=\left\{b_n^{(k)}\right\}_{n\in N}$ *of the elements of the space* $p_0(\Delta^{(m)})$ *for every fixed* $k\in\mathbb{N}$ *by*

$$b_n^{(k)}=\begin{cases} 0,\ (n<k) \\ \sum_{i=k}^{n}\binom{m+n-i-1}{n-i}(-1)^{i-k}\binom{i}{i-k},\ (n\geq k)\end{cases}. \tag{7}$$

Then, the following assertions are true:

i. *The sequence* $\left\{b^{(k)}\right\}_{k\in\mathbb{N}}$ *is bases for the space* $p_0\left(\Delta^{(m)}\right)$ *and for any* $x\in p_0\left(\Delta^{(m)}\right)$ *has a unique representation of the form*

$$x=\sum_k (Hx)\,b^{(k)}.$$

ii. *The set* $\left\{t,b^{(1)},b^{(2)},...\right\}$ *is a basis for the space* $p_c\left(\Delta^{(m)}\right)$ *and for any* $x\in p_c\left(\Delta^{(m)}\right)$ *has a unique representation of form*

$$x=lt+\sum_k\left[(Hx)_k-l\right]b^{(k)},$$

where $t=\{t_n\}=\sum_{k=0}^{n}\sum_{i=k}^{n}\binom{i}{i-k}(-1)^{i-k}\binom{m}{i-k}$, $(m,n\in\mathbb{N})$, $l=\lim_{k\to\infty}(Hx)_k$ *and* $H=P\Delta^{(m)}$.

Theorem 3. *The sequence spaces* $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ *and* $p_0\left(\Delta^{(m)}\right)$ *are linearly isomorphic to the spaces* l_∞, c *and* c_0 *respectively, i.e.,* $p_\infty\left(\Delta^{(m)}\right)\cong l_\infty$, $p_c\left(\Delta^{(m)}\right)\cong c$ *and* $p_0\left(\Delta^{(m)}\right)\cong c_0$.

Proof: To prove the fact $p_0\left(\Delta^{(m)}\right)\cong c_0$, we should show the existence of a linear bijection between the spaces $p_0(\Delta)$ and c_0. Consider the transformation T defined, with the notation (5), from $p_0\left(\Delta^{(m)}\right)$ to c_0 by $x\to y=Tx$. The linearity of T is clear. Further, it is trivial that $x=0$ whenever $Tx=0$ and hence T is injective. Let $y\in c_0$ and define the sequence $x=\{x_n\}$ by

$$x_n=\sum_{k=0}^{n}\left[\sum_{i=k}^{n}\binom{m+n-i-1}{n-i}(-1)^{i-k}\binom{i}{i-k}\right]y_k \tag{8}$$

for each m, $n\in\mathbb{N}$. Then,

$$\lim_{n\to\infty}(Hx)_k=\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{n-k}\Delta^{(m)}x_k$$

$$=\lim_{\substack{n\to\infty\\k=0}}\sum^{n}\binom{n}{n-k}\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k-i}$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n}\left[\sum_{i=k}^{n}\binom{i}{i-k}(-1)^{i-k}\binom{m}{i-k}\right]x_k=\lim_{n\to\infty}y_n=0$$

Thus, we have that $x\in p_0\left(\Delta^{(m)}\right)$. Consequently, T is surjective and is norm preserving. Hence, T is a linear bijection which implies that the spaces $p_0\left(\Delta^{(m)}\right)$ and c_0 are linearly isomorphic. In the same way, it can be shown that $p_\infty\left(\Delta^{(m)}\right)$ and $p_c\left(\Delta^{(m)}\right)$ are linearly isomorphic to l_∞ and c, respectively, and so we omit the detail. □

4 The α-, β- and γ- duals of the sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$

In this section, we state and prove the theorems determining the α-, β- and γ- duals of Pascal difference sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$. For the sequence spaces λ and μ , define the set $S(\lambda,\mu)$ by

$$S(\lambda,\mu)=\{z=(z_k)\in w: xz=(x_kz_k)\in\mu\ \text{ for all } x\in\lambda\}. \tag{9}$$

The α-, β- and γ- duals of a sequence space λ, which are respectively denoted by λ^{α}, λ^{β} and λ^{γ} are defined

$$\lambda^{\alpha}=S\left(\lambda,l_1\right),\lambda^{\beta}=S\left(\lambda,cs\right)\text{ and }\lambda^{\gamma}=S\left(\lambda,bs\right).$$

We shall begin with some lemmas due to Stieglitz and Tietz [21] that are needed in proving (4)-(6).

Lemma 1. $A\in(c_0:l_1)$ if and only if

$$\sup_{K\in F}\sum_{n}\left|\sum_{k\in K}a_{nk}\right|<\infty\,. \tag{10}$$

Lemma 2. $A\in(c_0:c)$ if and only if

$$\sup_{n}\sum_{k}|a_{nk}|<\infty, \tag{11}$$

$$\lim_{n\to\infty}a_{nk}-\alpha_k=0. \tag{12}$$

Lemma 3. $A\in(c_0:l_\infty)$ if and only if

$$\sup_{n}\sum_{k}|a_{nk}|<\infty. \tag{13}$$

Theorem 4. *Let $a=(a_k)\in w$ and the matrix $B=(b_{nk})$ by*

$$b_{nk}=\begin{cases}\sum_{i=k}^{n}\binom{m+n-i-1}{n-i}(-1)^{i-k}\binom{i}{i-k}a_n, & (0\le k\le n)\\ 0\,, & (k>n)\end{cases}$$

for all m, $n\in\mathbb{N}$. Then the α- dual of the spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ is the set

$$D=\left\{a=(a_n)\in w:\sup_{K\in F}\sum_{n}\left|\sum_{k\in K}b_{nk}\right|<\infty\right\}.$$

Proof: Let $a=(a_n)\in w$ and consider the matrix B whose rows are the products of the rows of the matrix $H^{-1}=\left(P\Delta^{(m)}\right)^{-1}=(\Delta^{(m)})^{-1}P^{-1}$ and sequence $a=(a_n)$. Bearing in mind the relation (5), we immediately derive that

$$a_nx_n=\sum_{k=0}^{n}\left[\sum_{i=k}^{n}\binom{m+n-i-1}{n-i}(-1)^{i-k}\binom{i}{i-k}a_n\right]y_k=\sum_{k=1}^{n}b_{nk}y_k=(By)_n \tag{14}$$

$m,n\in\mathbb{N}$, we therefore observe Lemma 1 and by (14) that $ax=(a_nx_n)\in l_1$ whenever $x\in p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ if and only if $By\in l_1$ whenever $y=(y_k)\in l_\infty$, c and c_0. This means that $a=(a_n)\in\left[p_\infty\left(\Delta^{(m)}\right)\right]^{\alpha}$, $\left[p_c\left(\Delta^{(m)}\right)\right]^{\alpha}$ and $\left[p_0\left(\Delta^{(m)}\right)\right]^{\alpha}$ if and only if $By\in(\left[p_\infty\left(\Delta^{(m)}\right)\right]^{\alpha}:l_1)$, $(\left[p_\infty\left(\Delta^{(m)}\right)\right]^{\alpha}:l_1)$ and $(\left[p_\infty\left(\Delta^{(m)}\right)\right]^{\alpha}:l_1)$ which yields the consequence that $\left[p_\infty\left(\Delta^{(m)}\right)\right]^{\alpha}=\left[p_c\left(\Delta^{(m)}\right)\right]^{\alpha}=\left[p_0\left(\Delta^{(m)}\right)\right]^{\alpha}=D.$ □

Theorem 5. *Let $a=(a_k)\in w$ and the matrix $C=(c_{nk})$ by*

$$c_{nk}=\begin{cases}\sum_{i=k}^{n}\left[\sum_{j=k}^{n}\binom{m+i-j-1}{i-j}(-1)^{j-k}\binom{j}{j-k}\right]a_i, & (0\le k\le n)\\ 0, & (k>n)\end{cases}$$

and define the sets c_1, c_2, c_3 and c_4 by

$$\begin{aligned}c_1&=\left\{a=(a_k)\in w:\sup_{n}\sum_{k}|c_{nk}|<\infty\right\},\\ c_2&=a=(a_k)\in w:\lim_{n\to\infty}c_{nk}\text{ exists for each }k\in N,\\ c_3&=\left\{a=(a_k)\in w:\lim_{n\to\infty}\sum_{k}|c_{nk}|=\sum_{k}\left|\lim_{n\to\infty}c_{nk}\right|\right\},\end{aligned}$$

and

$$c_4 = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_k c_{nk} \text{ exists} \right\}.$$

Then $\left[p_\infty\left(\Delta^{(m)}\right)\right]^\beta$, $\left[p_c\left(\Delta^{(m)}\right)\right]^\beta$ *and* $\left[p_0\left(\Delta^{(m)}\right)\right]^\beta$ *is* $c_2 \cap c_3$, $c_1 \cap c_2 \cap c_4$ *and* $c_1 \cap c_2$, *respectively.*

Proof: We only give the proof the space $p_0\left(\Delta^{(m)}\right)$. Since the rest of the proof can be obtained by the same way for the spaces $p_c\left(\Delta^{(m)}\right)$ and $p_\infty\left(\Delta^{(m)}\right)$. Consider the equation

$$\begin{aligned}
\sum_{k=1}^{n} a_k x_k &= \sum_{k=0}^{n}\sum_{i=0}^{k}\left[\sum_{j=i}^{k}\binom{m+k-j-1}{k-j}(-1)^{j-i}\binom{j}{j-i}\right] a_k y_i \\
&= \sum_{k=0}^{n}\left[\sum_{i=k}^{n}\left[\sum_{j=k}^{i}\binom{m+i-j-1}{i-j}(-1)^{j-k}\binom{j}{j-k}\right] a_i\right] y_k \\
&= (Cy)_n .
\end{aligned} \tag{15}$$

Thus , we deduce from Lemma 2 and (15) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in h_0\left(\Delta^{(m)}\right)$ if and only if $Cy \in c$ whenever $y = (y_k) \in c_0$. That is to say that $a = (a_k) \in \left[p_0\left(\Delta^{(m)}\right)\right]^\beta$ if and only if $C \in (c_0 : c)$ which yields us $[p_0\left(\Delta^{(m)}\right)]^\beta = c_1 \cap c_2$. The β- dual of the sequence spaces $[p_c\left(\Delta^{(m)}\right)]$ and $[p_\infty\left(\Delta^{(m)}\right)]$ may be obtained in the similiar way, we omit their proofs. ☐

Theorem 6. *The* γ*- dual of the spaces* $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ *and* $p_0\left(\Delta^{(m)}\right)$ *is the set* c_1.

Proof: This may be obtained in the similiar way used in the prof of Theorem (5) together with Lemma 3 instead of Lemma 2. So, we omit the detail. ☐

5 Matrix transformations on the sequence space $p_c\left(\Delta^{(m)}\right)$

We shall write throughout for brevity that

$$\tilde{a}_{nk} = \sum_{j=k}^{\infty}\binom{m+n-j-1}{n-j}(-1)^{j-k}\binom{j}{j-k} a_{nj},$$

and

$$\hat{g}_{nk} = \sum_{j=k}^{s}\binom{m+n-j-1}{n-j}(-1)^{j-k}\binom{j}{j-k} a_{nj}$$

for all $m, n, s \in \mathbb{N}$.

In this section, we give the characterization of the classes $\left(p_c\left(\Delta^{(m)}\right) : l_\infty\right)$ and $\left(h_c\left(\Delta^{(m)}\right) : c\right)$. Following theorems can be proved using standart methods, we omit the detail.

Theorem 7. $A \in \left(p_c\left(\Delta^{(m)}\right) : l_\infty\right)$ *if and only if*

$$\sup_n \sum_k |\hat{g}_{nk}| < \infty, \tag{16}$$

$$\lim_{n\to\infty} \sum_k \hat{g}_{nk} \text{ exists for all } m \in N, \tag{17}$$

$$\sup_{n\in N} \sum_k |\tilde{a}_{nk}| < \infty, \ (n \in N), \tag{18}$$

and

$$\lim_{n\to\infty} \tilde{a}_{nk} \text{ exists for all } n \in N. \tag{19}$$

Theorem 8. $A \in \left(p_c\left(\Delta^{(m)}\right):c\right)$ *if and only if* (16)-(19) *hold, and*

$$\lim_{n\to\infty}\sum_k \tilde{a}_{nk}=\alpha, \tag{20}$$

$$\lim_{n\to\infty}(\tilde{a}_{nk})=\alpha_k,\ (k\in N) \tag{21}$$

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