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Praface

International Geometry Symposium has been held regularly every year for 17 years and many distinguished participants from Turkey and all over the world have been included.

We have successfully completed the 17th International Geometry Symposium at Erzincan Binali Yildirim University, between June 19th-22nd, 2019.

In total, 8 invited speakers, 130 oral and 4 poster presentations were accepted by the scientific committee of 17th International Geometry Symposium. Having accepted these speaks, posters and invited speaks, oral and presentation posters were contributed in parallel sessions.

The main aim of symposium is to share current studies in the geometry field.

This volume contains the proceeding the selected contributions of the participants for the 17th International Geometry Symposium scheduled during June 19th-22nd, 2019 in Erzincan, Turkey.

The selection of papers in this volume has been carried out coptiously. Experts in committee from different disciplines review papers. Having been submitted, papers were sent for peer review in the related area. To be able accepted, papers need to be recommended by at least two reviewers.

We deeply thank all of the contributors for their great effort during 17th International Geometry Symposium.

We are also grateful to the dear members of committee for their continuous effort and guidance which led to the selection of the contributed talks and the papers published in this volume

See you in future symposium!,

Assist. Prof. Dr. Tülay ERİŞİR

Guest Editor

17th International Geometry Symposium (2019)

Guest Editors

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On Quasi-Einstein Manifolds Admitting Space-Matter Tensor

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Abstract: The subject matter of this paper lies in the interesting domain of Differential Geometry and the Theory of General Relativity. Although the space has its motivation in Relativity, we study the geometric properties of the space, inspired by the papers on the geometry related to curvature restrictions. Such a study was joined by A. Z. Petrov to Einstein spaces. We extend the study on quasi-Einstein spaces which can be considered as a generalization of Einstein spaces. This study is supported by an example.

Keywords: Einstein's field equation, Quasi-Einstein manifold, Scalar curvature, Space-Matter tensor.

1 Introduction

In 1949, the celebrated theorem [8] showing the existence of three types of Einstein space with signature (-, -, -, +) and the corresponding three canonical forms were established. During this study, the gravitation fields are classified on the basis of the algebraic structures of the space-matter tensor. A. Z. Petrov introduced a tensor field P of type (0, 4) and defined as follows:

$$P = R + \frac{k}{2}\, g \wedge T - \sigma G, \tag{1}$$

where R is the Riemannian curvature tensor of type (0, 4), T is the energy-momentum tensor of type (0, 2), k is a cosmological constant, σ is the energy density (scalar), G is a tensor of type $(0,\ 4)$ given by

$$G(X,\ Y,\ Z,\ U) = g(X,\ U)g(Y,\ Z) - g(X,\ Z)g(Y,\ U) \tag{2}$$

for all $X,\ Y,\ Z,\ U \in \chi(M)$, $\chi(M)$ being the Lie algebra of smooth vector fields on M and the Kulkarni-Nomizu product $E \wedge F$ of two $(0,\ 2)$ tensors E and F is defined by

$$\begin{aligned}(E \wedge F)(X_1,\ X_2,\ X_3,\ X_4) &= E(X_1,\ X_4)F(X_2,\ X_3) + E(X_2,\ X_3)F(X_1,\ X_4) \\ &- E(X_1,\ X_3)F(X_2,\ X_4) - E(X_2,\ X_4)F(X_1,\ X_3),\end{aligned}$$

$X_i \in \chi(M)$, $i = 1,\ 2,\ 3,\ 4$. The tensor P is known as the space-matter tensor of type (0, 4) of the manifold M.
Einstein's field equation with cosmological constant is given by

$$kT = S + \left(\lambda - \frac{r}{2}\right) g, \tag{3}$$

where λ is a cosmological constant, r is the scalar curvature and S is the Ricci tensor of type (0, 2) and r is the scalar curvature defined by the following equations:

$$S(X,Y) = g(QX,Y) = \sum_{i=1}^{n} R(e_i,\ X,\ Y,\ e_i) \quad , \quad r = \sum_{i=1}^{n} S(e_i,\ e_i) = \sum_{i=1}^{n} g((QE)_i,\ e_i),$$

where $\{e_i : i = 1,\ 2,\ \ldots,\ n\}$ be an orthonormal basis of the tangent space at any point of the manifold and Q denotes the symmetric endomorphism corresponding to the Ricci tensor S.

(1) takes the form

$$P = R + \frac{1}{2} g \wedge S - \left(\sigma - \lambda + \frac{r}{2}\right) G, \tag{4}$$

by virtue of (3).

If the energy momentum tensor is of Codazzi type and the energy density is constant in our manifold, then the space-matter tensor satisfies the second Bianchi identity [7] i.e.,

$$(\nabla_X P)(Y,\ Z,\ U,\ V) + (\nabla_Y P)(Z,\ X,\ U,\ V) + (\nabla_Z P)(X,\ Y,\ U,\ V) = 0. \tag{5}$$

We know that the space matter tensor comprises of two parts. First part deals with the curvature of the space and the remaining on the motion of the matter. Our investigation focused on attributing different constraints on space-matter tensor-their admissibility, and if admissible the after effects. As our intention is purely geometric, we investigate the corresponding change in its scalar curvature.

Section 2 deals with several properties of space-matter tensor P satisfying certain curvature conditions. In the last section an example proving the existence of a quasi-Einstein manifold with space-matter tensor is given.

2 Quasi-Einstein manifold with space-matter tensor

This section is concerned with a quasi-Einstein manifold with space-matter tensor satisfying certain curvature conditions. A Riemannian manifold $(M^n,\ g)$ is said to be quasi-Einstein ([1], [2]-[6]) if its Ricci tensor S is not identically zero and if there exists two real valued functions α_1 and $\alpha_2(\neq 0)$ and a smooth unit 1-form π on M, such that the Ricci tensor S satisfies

$$S = \alpha_1 g + \alpha_2 \pi \otimes \pi. \tag{6}$$

The vector field ς, which is metrically equivalent to the unit 1-form π, is called the generator of the manifold. Such a manifold of dimension n is usually denoted by $(QE)_n$. The scalars α_1, α_2 are known as the associated scalars.

Firstly let us consider a Riemannian manifold $(M^n,\ g)$ $(n > 3)$ admitting Einstein's field equation in which the space-matter tensor P of type $(0,\ 4)$ is recurrent i.e.,it satisfies the relation

$$(\nabla_X P)(Y,\ Z,\ U,\ V) = L(X) P(Y,\ Z,\ U,\ V), \tag{7}$$

where L is the non-zero 1-form of recurrence and ρ be the vector field associated with L. In view of the above relation, (4) is reduced to the following equation

$$\begin{aligned} & 2(\nabla_X R)(Y,\ Z,\ U,\ V) + (\nabla_X S)(Y,\ V) g(Z,\ U) + (\nabla_X S)(Z,\ U) g(Y,\ V) \\ & -(\nabla_X S)(Y,\ U) g(Z,\ V) - (\nabla_X S)(Z,\ V) g(Y,\ U) - [2d\sigma(X) + dr(X)] G(Y,\ Z,\ U,\ V) \\ = \ & L(X)[2R(Y,\ Z,\ U,\ V) + (g \wedge S)(Y,\ Z,\ U,\ V) - 2(\sigma + \frac{r}{2} - \lambda) G(Y,\ Z,\ U,\ V)]. \end{aligned} \tag{8}$$

Taking contraction of (8) with respect to Y and V, we obtain

$$\begin{aligned} & n(\nabla_X S)(Z,\ U) - \{(n-2) dr(X) + 2(n-1) d\sigma(X)\} g(Z,\ U) \\ = \ & L(X)[nS(Z,\ U) - \{(n-2) r + 2(n-1)(\sigma - \lambda)\} g(Z,\ U)]. \end{aligned} \tag{9}$$

Setting $Z = U = e_i$ in (9) and taking summation over i, $1 \le i \le n$, we find

$$dq(X) = q L(X), \tag{10}$$

where $q = (n-3) r + 2(n-1)(\sigma - \lambda)$. If q is a constant then the only possible value of q is zero, since $L(X) \neq 0$. Hence we have

$$r = \frac{2(n-1)}{(n-3)} (\lambda - \sigma), \ \text{ since } \ L(X) \neq 0. \tag{11}$$

Putting $X = \rho$ in (10) and by the virtue of (6), we get

$$(n-3)[n d\alpha_1(\rho) + d\alpha_2(\rho)] + 2(n-1) d\sigma(\rho) = 2(n-1)(\sigma - \lambda) + (n-3)(n\alpha_1 + \alpha_2). \tag{12}$$

Hence we get the following:

Theorem 1. *The scalars ρ, σ, λ, α_1, α_2 are connected by the relation (12) in a $(QE)_n$ $(n > 3)$ admitting Einstein's field equation and recurrent space-matter tensor.*

In the view of (6), (11) is reduced to the following form

$$(n-3)(n\alpha_1 + \alpha_2) + 2(n-1)(\sigma - \lambda) = 0. \tag{13}$$

This leads to the following:

Theorem 2. *If $\alpha_1,\ \alpha_2$ and σ are constants in a $(QE)_n$ $(n > 3)$ admitting Einstein's field equation and recurrent space-matter tensor, then they are connected by the relation* (13).

Again in the view of (5), the relation (7) reduces to

$$L(X)P(Y,\ Z,\ U,\ V) + L(Y)P(Z,\ X,\ U,\ V) + L(Z)P(X,\ Y,\ U,\ V) = 0.$$

Setting $Y = V = e_i$ in the above relation and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned}&\{2(n-1)(\lambda-\sigma)-(n-2)r\}[L(X)g(Z,\ U)-L(Z)g(X,\ U)]\\&+n[L(X)S(Z,\ U)-L(Z)S(X,\ U)]+2L(P(Z,\ X)U)=0,\end{aligned}$$

by the virtue of (4). Now contracting the above relation with respect to X and U we obtain

$$L(QZ) = \frac{r_0}{2n}L(Z),\ \text{ which yields }\ S(Z,\ \rho) = \frac{r_0}{2n}g(Z,\ \rho), \tag{14}$$

where $r_0 = (n-1)[2(n-2)(\lambda-\sigma)-(n-4)r]$ and $g(X,\rho) = L(X)$.

By the virtue of (6) and (14), we get

$$L(QZ) = \frac{n-1}{2n}[2(n-2)(\lambda-\sigma)-(n-4)(n\alpha_1+\alpha_2)]L(Z),$$

which yields

$$S(Z,\ \rho) = \frac{n-1}{2n}[2(n-2)(\lambda-\sigma)-(n-4)(n\alpha_1+\alpha_2)]g(Z,\ \rho).$$

This gives the following:

Theorem 3. *If the energy-momentum tensor is of Codazzi type in a $(QE)_n$ $(n > 3)$ admitting Einstein's field equation and recurrent space-matter tensor, then $\frac{n-1}{2n}[2(n-2)(\lambda-\sigma)-(n-4)(n\alpha_1+\alpha_2)]$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ, defined by $g(X,\ \rho) = L(X)$, provided that the energy density is constant.*

Now we assume that the space-matter tensor P of type (0, 4) in a Riemannian manifold $(M^n,\ g)$ $(n > 3)$ admitting Einstein's field equation, is weakly symmetric [9] in nature. Then there exist 1-forms A, B and E (not simultaneously zero) such that the following relation holds:

$$\begin{aligned}(\nabla_X P)(Y,\ Z,\ U,\ V) \ &= \ A(X)P(Y,\ Z,\ U,\ V) + B(Y)P(X,\ Z,\ U,\ V)\\&\quad +B(Z)P(Y,\ X,\ U,\ V) + E(U)P(Y,\ Z,\ X,\ V)\\&\quad +E(V)P(Y,\ Z,\ U,\ X).\end{aligned} \tag{15}$$

Here ρ_1, ρ_2, ρ_3 be the vector fields metrically equivalent to A, B, E respectively.
Contracting Y and V and using (4), the equation (15) yields

$$\begin{aligned}&\frac{n}{2}(\nabla_X S)(Z,\ U) - \left[\frac{n-2}{2}dr(X) + (n-1)d\sigma(X)\right]g(Z,\ U)\\=\ &A(X)\left[\frac{n}{2}S(Z,\ U) - \left\{\frac{n-2}{2}r + (n-1)(\sigma-\lambda)\right\}g(Z,\ U)\right]\\&+B(Z)\left[\frac{n}{2}S(X,\ U) - \left\{\frac{n-2}{2}r + (n-1)(\sigma-\lambda)\right\}g(X,\ U)\right]\\&+E(U)\left[\frac{n}{2}S(Z,X) - \left\{\frac{n-2}{2}r + (n-1)(\sigma-\lambda)\right\}g(Z,X)\right]\\&+P(X,Z,U,\rho_2) + P(X,U,Z,\rho_3).\end{aligned} \tag{16}$$

Setting $Z = U = e_i$ in (16) and taking summation over i, $1 \leq i \leq n$, we have

$$\begin{aligned}&n(n-3)dr(X) + 2n(n-1)d\sigma(X) + 2nH_1(QX)\\=\ &n[(n-3)r + 2(n-1)(\sigma-\lambda)]A(X) + 2[(n-2)r + 2(n-1)(\sigma-\lambda)]H_1(X)\end{aligned} \tag{17}$$

by the virtue of (4), where $H_1(X) = B(X) + E(X)$ for all vector fields X.
Contracting (16) with respect to $X,\ Z$ and replacing U by X and by the virtue of (4), we find

$$\begin{aligned}\frac{n-4}{4}dr(X)\ &+\ (n-1)d\sigma(X) + \frac{n}{2}[A(QX) + B(QX) - E(QX)]\\&=\ \left\{\frac{n-2}{2}r + (n-1)(\sigma-\lambda)\right\}[A(X) + B(X)]\\&+\ \left\{\frac{n^2-4n+2}{2}r + (n-1)^2(\sigma-\lambda)\right\}E(X).\end{aligned} \tag{18}$$

Further contracting (16) with respect to $X,\ U$ and replacing Z by X, we find

$$\begin{aligned} & \frac{n-4}{4} dr(X) + (n-1) d\sigma(X) + \frac{n}{2}[A(QX) - B(QX) + E(QX)] \\ = & \left\{\frac{n-2}{2} r + (n-1)(\sigma-\lambda)\right\}[A(X)+E(X)] + \left\{\frac{n^2-4n+2}{2} r + (n-1)^2(\sigma-\lambda)\right\} B(X) \end{aligned} \tag{19}$$

by the virtue of (4). Now (18) and (19) both yield

$$H_2(QX) = -\frac{n-1}{2n}[(n-4)r + 2(n-2)(\sigma-\lambda)]H_2(X), \tag{20}$$

which gives

$$S(X,\ \tau_2) = \frac{n-1}{2n}[2(n-2)(\lambda-\sigma) - (n-4)r]g(X,\ \tau_2),$$

where $g(X,\ \tau_2) = H_2(X) = B(X) - E(X)$ for all vector fields X.

Now using (18) and (20), we obtain

$$\begin{aligned} & (n-4)dr(X) + 4(n-1)d\sigma(X) + 2nA(QX) \\ = & 2[(n-2)r + 2(n-1)(\sigma-\lambda)]A(X) + n[(n-3)r + 2(n-1)(\sigma-\lambda)]H_1(X). \end{aligned} \tag{21}$$

In the view of (17), (6) and (21), we get

$$\begin{aligned} & (n^2-2n-4)[nd\alpha_1(X) + d\alpha_2(X)] + 2(n-1)(n+2)d\sigma(X) \\ = & [(n^2-n-4)(n\alpha_1+\alpha_2) + 2(n-1)(n+2)(\sigma-\lambda)]H_3(X) - 2nH_3(QX), \end{aligned}$$

where $H_3(X) = g(X,\tau_3) = A(X) + B(X) + E(X)$ for all vector fields X. If $\alpha_1,\ \alpha_2,\ \sigma$ are constants then we get from the above relation

$$H_3(QX) = \frac{1}{2n}[(n^2-n-4)(n\alpha_1+\alpha_2) + 2(n-1)(n+2)(\sigma-\lambda)]H_3(X),$$

which yields

$$S(X,\tau_3) = \frac{1}{2n}[(n^2-n-4)(n\alpha_1+\alpha_2) + 2(n-1)(n+2)(\sigma-\lambda)]g(X,\tau_3).$$

Again by (17), (6) and (21), we also get

$$\begin{aligned} & (n^2-4n+4)[nd\alpha_1(X) + d\alpha_2(X)] + 2(n-1)(n-2)d\sigma(X) \\ = & 2nH_4(QX) + (n-1)[(n-4)(n\alpha_1+\alpha_2) + 2(n-2)(\sigma-\lambda)]H_4(X), \end{aligned}$$

where $H_4(X) = g(X,\tau_4) = A(X) - B(X) - E(X)$ for all vector fields X. If $\alpha_1,\ \alpha_2,\ \sigma$ are constants, then the above relation gives us

$$H_4(QX) = \frac{1}{2n}(n-1)[2(n-2)(\lambda-\sigma) - (n-4)(n\alpha_1+\alpha_2)]H_4(X),$$

which yields

$$S(X,\tau_4) = \frac{n-1}{2n}[2(n-2)(\lambda-\sigma) - (n-4)(n\alpha_1+\alpha_2)]g(X,\tau_4).$$

Thus we have the following:

Theorem 4. *$\frac{1}{2n}[(n^2-n-4)(n\alpha_1+\alpha_2) + 2(n-1)(n+2)(\sigma-\lambda)]$ and $\frac{n-1}{2n}[2(n-2)(\lambda-\sigma) - (n-4)(n\alpha_1+\alpha_2)]$ are the eigenvalues of the Ricci tensor S corresponding to the eigenvectors τ_3 and τ_4, respectively, in a $(QE)_n$ $(n>3)$ admitting Einstein's field equation and weakly symmetric space-matter tensor, provided $\alpha_1,\ \alpha_2,\ \sigma$ are constants.*

3 An example proving the existence of a Quasi-Einstein Manifold with Space-Matter Tensor

This section deals with an example of a quasi-Einstein manifold admitting Einstein's field equation and space-matter tensor satisfying certain curvature restrictions.

Example 1: Let $M^4 = \{(x^1,\ x^2,\ x^3,\ x^4) \in \mathbb{R}^4\}$ be an open subset of $\mathbb{R}^4$ endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1+1}(dx^1)^2 + e^{x^1}[(dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{22}$$

where i, j run from 1 to 4. Then the only non-vanishing components of the curvature tensor, the Ricci tensor and the scalar curvature are given by

$$R_{2323} = -\frac{1}{4}e^{x^1-1} = R_{2424} = R_{3434};\ S_{22} = \frac{1}{2e} = S_{33} = S_{44};\ r = \frac{3}{2}e^{-(x^1+1)}. \tag{23}$$

We shall now verify that our M^4 is a quasi-Einstein manifold. To verify that the manifold M^4 is a $(QE)_4$ let us consider the 1-form π and associated scalars α_1, α_2 as follows:

$$\pi(\frac{\partial}{\partial x^i}) \quad = \quad \pi_i = \begin{cases} e^{\frac{x^1+1}{2}} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{24}$$

$$\alpha_1 = \frac{1}{2}e^{-(x^1+1)};\ \alpha_2 = -\frac{1}{2}e^{-(x^1+1)}. \tag{25}$$

According to our M^4, (6) is reduced to following equations

$$S_{ii} = \alpha_1 g_{ii} + \alpha_2 \pi_i \pi_i, \qquad \text{for } i = 1,\ 2,\ 3,\ 4, \tag{26}$$

since the components of (6) vanishes identically for the cases other than (26) and the relation (6) holds trivially. By the virtue of (22), (23), (24), (25), it follows that
r. h. s. of (26) $= \frac{1}{2}e^{-(x^1+1)}g_{ii} - \frac{1}{2}e^{-(x^1+1)}\pi_i\pi_i = 0 =$ l. h. s. of (26) for $i = 1$. By a similar argument, it can be easily shown that the relation (26) holds for $i = 2,\ 3,\ 4$. Therefore our $(M^4,\ g)$ is a $(QE)_4$. Now, considering σ as a constant, we calculate the non-vanishing components of the space-matter tensor and its covariant derivatives as follows:

$$\begin{cases} P_{1212} = -\frac{1}{2}e^{x^1}[-1 + 2e^{x^1+1}(\lambda - \sigma)] = P_{1313} = P_{1414}, \\ P_{2323} = -e^{2x^1}(\lambda - \sigma) = P_{2424} = P_{3434}; \\ P_{1212,1} = e^{x^1}[-1 + 2e^{x^1+1}(\lambda - \sigma)] = P_{1313,1} = P_{1414,1}, \\ P_{2323,1} = 2e^{2x^1}(\lambda - \sigma) = P_{2424,1} = P_{3434,1}, \\ P_{1323,2} = -\frac{1}{4}e^{x^1-1} = P_{1424,2} = P_{1232,3} = P_{1434,3} = P_{1242,4} = P_{1343,4}; \end{cases} \tag{27}$$

where ',' denotes the covariant differentiation with respect to the coordinates. Let us consider the 1-form L as follows:

$$L(\frac{\partial}{\partial x^i}) = L_i = \begin{cases} -2 & \text{for } i = 1 \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

at any point of M. One can easily check the validity of the following relations with these 1-forms defined above:

$$P_{1j1j,1} = L_1 P_{1j1j}, \tag{29}$$
$$P_{jkjk,1} = L_1 P_{jkjk}, \tag{30}$$
$$P_{1jkj,k} = L_k P_{1jkj}, \tag{31}$$

where j and k run from 2 to 4 and $j \neq k$. We simply show one of the above here:

$$\text{r. h. s. of (29)} = -2P_{1j1j} = e^{x^1}[-1 + 2e^{x^1+1}(\lambda - \sigma)] = \text{l. h. s. of (29) for } j = 2.$$

In other possible cases either the result is trivial or both sides vanish identically. Here it can also be easily shown that our considered manifold is a recurrent manifold. Hence we can state the following theorem:

Theorem 5. *Let (M^4, g) be a Riemannian manifold endowed with the metric*

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1+1}(dx^1)^2 + e^{x^1}[(dx^2)^2 + (dx^3)^2 + (dx^4)^2] \text{ where } (i,\ j = 1,\ 2,\ 3,\ 4).$$

Then $(M^4,\ g)$ is a $(QE)_4$ admitting Einstein's field equation and recurrent space-matter tensor with non-vanishing scalar curvature such that σ is constant. It is also a recurrent manifold.

4 Conclusion

In this article we provide several properties of space-matter tensor P on quasi-Einstein manifolds. Our future research will be focused on the study of space-matter tensor P on some other kind of manifolds.

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Smarandache Curves According to q-Frame in Minkowski 3-Space

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Abstract: In this study, we investigate special Smarandache curves according to q-frame in Minkowski 3-space and we give some differential geometric properties of Smarandache curves.

Keywords: Frenet frame, Natural curvatures, q-frame, Smarandache curves.

1 Introduction

The most well-known adapted frame is the Frenet frame. The Frenet frame plays an important role in classical differential geometry [30], where it is useful to investigate Bertrand curves [27] and tube surfaces [17, 21].

Let $\alpha(t)$ be a regular space curve [6, 8], then the Frenet frame is defined as follows

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{b} = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}, \mathbf{n} = \mathbf{b} \wedge \mathbf{t}. \tag{1}$$

The curvature κ and the torsion τ are given by

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2}. \tag{2}$$

The well-known Frenet formulas are given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \|\alpha'\| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{3}$$

A regular curve in Euclidean 3-space, whose position vector is composed of Frenet frame vectors on another regular curve, is called a Smarandache curve [3]. Recently, Smarandache curves have been stutied in various ambient spaces [1, 10, 16, 26, 28]. Moreover, some special Smarandache curves with reference to Darboux frame in Euclidean 3-space is studied in [4]. A. T. Ali has introduced some special Smarandache curves in the Euclidean space [1]. Taking $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be its moving Serret-Frenet frame, $\mathbf{tn}-$Smarandache curve, $\mathbf{nb}-$Smarandache curve and $\mathbf{tnb}-$Smarandache curve are defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{t} + \mathbf{n}), \beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{n} + \mathbf{b}) \text{ and } \beta(s^*) = \frac{1}{\sqrt{3}}(\mathbf{t} + \mathbf{n} + \mathbf{b}) \tag{4}$$

respectively [1].

However, Frenet frame has several disadvantages in applications. For instance, Frenet frame is undefined wherever the curvature vanishes. Moreover, the main drawback of the Frenet frame is that it has undesirable rotation about tangent vector [6, 18]. Therefore, Bishop [5] introduced a new frame along a space curve which is more suitable for applications. But, it is well known that Bishop frame calculations is not a easy task [29]. In order to construct the 3D curve offset, Coquillart [9] introduced the quasi-normal vector of a space curve. The quasi-normal vector is defined for each point of the curve, and lies in the plane perpendicular to the tangent of the curve at this point [24]. Then using the quasi-normal vector Dede et al. in [11] introduced the q-frame along a space curve. Given a space curve $\alpha(t)$ the q-frame consists of three orthonormal vectors, these being the unit tangent vector $\mathbf{t}$, the quasi-normal $\mathbf{n}_q$ and the quasi-binormal vector $\mathbf{b}_q$. The q-frame $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}\}$ is given by

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q \tag{5}$$

where $\mathbf{k}$ is the projection vector [11]. The q-frame has many advantages versus other frames (Frenet, Bishop). For instance the q-frame can be defined even along a line($\kappa = 0$) and the construction of the q-frame doesn't change if the space curve has unit speed or not. Moreover the q-frame can be calculated easily [12].

For simplicity, we have chosen the projection vector $\mathbf{k} = (0, 0, 1)$ in this paper. However, the q-frame is singular in all cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel. Thus, in those cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k} = (0, 1, 0)$ or $\mathbf{k} = (1, 0, 0)$.

A q-frame along a space curve is shown in Figure 1.

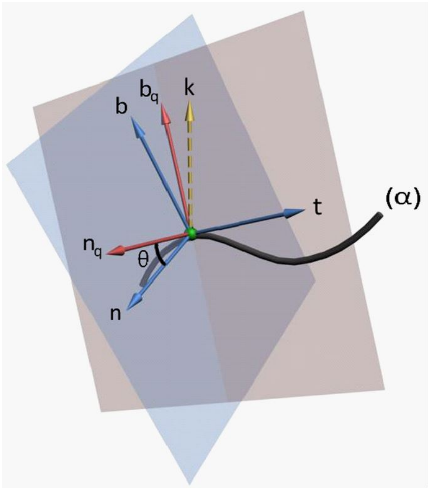

Fig. 1: The q-frame and Frenet frame.

The variation equations of the directional q-frame is given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \|\alpha'\| \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}, \tag{6}$$

where the q-curvatures are expressed as follows

$$k_1 = -\frac{\langle \mathbf{t}, \mathbf{n}_q' \rangle}{\|\alpha'\|}, k_2 = -\frac{\langle \mathbf{t}, \mathbf{b}_q' \rangle}{\|\alpha'\|}, k_3 = \frac{\langle \mathbf{n}_q', \mathbf{b}_q \rangle}{\|\alpha'\|}. \tag{7}$$

In the three dimensional Minkowski space $\mathbb{R}_1^3$, the inner product and the cross product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are defined as

$$< \mathbf{u}, \mathbf{v} >= u_1 v_1 + u_2 v_2 - u_3 v_3$$

and

$$\mathbf{u} \wedge \mathbf{v} = (u_3 v_2 - u_2 v_3, u_1 v_3 - u_3 v_1, u_1 v_2 - u_2 v_1)$$

where $e_1 \wedge e_2 = e_3$, $e_2 \wedge e_3 = -e_1$, $e_3 \wedge e_1 = -e_2$, respectively [2]. If $\mathbf{u}$ and $\mathbf{v}$ are timelike vectors then $\mathbf{u} \wedge \mathbf{v}$ is a spacelike vector [22].

The norm of the vector $\mathbf{u}$ is given by

$$\|\mathbf{u}\| = \sqrt{|\langle u, u \rangle|} \tag{8}$$

We say that a Lorentzian vector $\mathbf{u}$ is spacelike, lightlike or timelike if $\langle \mathbf{u}, \mathbf{u} \rangle > 0$, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ and $\mathbf{u} \neq 0$, $\langle \mathbf{u}, \mathbf{u} \rangle < 0$, respectively. In particular, the vector $\mathbf{u} = 0$ is spacelike.

An arbitrary curve $\alpha(s)$ in $\mathbb{R}_1^3$, can locally be spacelike, timelike or null(lightlike), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null [20]. A null curve α is parameterized by pseudo-arc s if $\langle \alpha''(s), \alpha''(s) \rangle = 1$. On the other hand, a non-null curve α is parameterized by arc-lenght parameter s if $\langle \alpha'(s), \alpha'(s) \rangle = \pm 1$ [7, 23].

Then Frenet formulas of timelike curve may be written as

$$\frac{d}{dt} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = v \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{9}$$

where $v = \|\mathbf{t}\|$. The Minkowski curvature and torsion of timelike curve $\alpha(t)$ are obtained by

$$\kappa = \|\mathbf{t}'\|, \tau =< \mathbf{n}', \mathbf{b} > \tag{10}$$

respectively [2, 22].

As an alternative to the Frenet frame we define a new adapted frame along a timelike space curve, called as the q-frame. Given a regular timelike space curve $\alpha(t)$ the q-frame consists of three orthonormal vectors, these being the unit tangent vector $\mathbf{t}$(timelike), the quasi-normal $\mathbf{n}_q$(spacelike) and the quasi-binormal vector $\mathbf{b}_q$(spacelike). The q-frame $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}\}$ along $\alpha(t)$ is given by

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q. \tag{11}$$

For simplicity, we have chosen the projection vector $\mathbf{k} = (0, 1, 0)$(spacelike) in this paper. However, the q-frame is singular in all cases where $\mathbf{t} \wedge \mathbf{k}$ vanishes. Therefore, in those cases where $\mathbf{t} \wedge \mathbf{k}$ vanishes the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k} = (1, 0, 0)$(spacelike). Interestingly if we chose $\mathbf{k} = (0, 0, 1)$(timelike) we get the same results in this paper [14].

The variation equations of the directional q-frame is given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & k_3 \\ k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{12}$$

where the q-curvatures are expressed as follows

$$k_1 = \langle \mathbf{t}', \mathbf{n}_q \rangle, k_2 = \langle \mathbf{t}', \mathbf{b}_q \rangle, k_3 = \langle \mathbf{n}_q', \mathbf{b}_q \rangle. \tag{13}$$

Theorem 1.1. Let $\alpha(s)$ be a timelike curve that is parameterized by arc length s. The relation between the q-curvatures and the Frenet curvatures (the torsion τ and the curvature κ) may be expressed as,

$$\begin{aligned} k_1 &= \kappa \cos\theta, \\ k_2 &= -\kappa \sin\theta, \\ k_3 &= d\theta + \tau, \end{aligned} \tag{14}$$

where θ is the angle between the vectors, the principal normal vector $\mathbf{n}$ and the quasi-normal vector $\mathbf{n}_q$ [13, 14].

We can define the Euclidean angle θ between the principal normal $\mathbf{n}$ and quasi-normal $\mathbf{n}_q$ spacelike vectors. Then, as one can see immediately, the relation matrix may be expressed as

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{15}$$

Thus,

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{16}$$

[11, 15]. Furthermore, from (15) and (11) we have

$$\cos\theta = \langle \mathbf{n}_q, \mathbf{n} \rangle = \frac{\langle \mathbf{t} \wedge \mathbf{k}, \alpha'' \rangle}{\|\mathbf{t} \wedge \mathbf{k}\| \, \|\alpha''\|} = \frac{\det(\alpha'', \alpha', \mathbf{k})}{\|\alpha' \wedge \mathbf{k}\| \, \|\alpha''\|}. \tag{17}$$

In [10], the authors investigated the Smarandache curves with respect to the Bishop frame. Smarandache curves have also been studied in other ambient frames [4, 19, 25]**.** Taking $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q\}$ be its moving q-frame, $\mathbf{t}\mathbf{n}_q-$Smarandache curve, $\mathbf{t}\mathbf{b}_q-$Smarandache curve and $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve are defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{t} + \mathbf{n}_q), \beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{t} + \mathbf{b}_q) \text{ and } \beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{n}_q + \mathbf{b}_q) \tag{18}$$

respectively [16].

Now, in this paper we investigate special Smarandache curves according to q-frame in Minkowski 3-space and we give some differential geometric properties of Smarandache curves.

2 Smarandache curves of Timelike Space Curve according to q-frame

In this section we will investigate the Smarandache curves of timelike space curve according to q-frame in Minkowski space.

2.1 $\mathbf{t}\mathbf{n}_q$- *Smarandache curves of a timelike curve in* $\mathbb{R}^3_1$

Let $\alpha = \alpha(s)$ be a unit speed regular curve in $\mathbb{R}^3_1$ and $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q\}$ be its moving q-frame. $\mathbf{t}\mathbf{n}_q-$Smarandache curve can be defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{t} + \mathbf{n}_q) \tag{19}$$

We can get q-invariants of $\mathbf{t}\mathbf{n}_q-$Smarandache curves according to $\alpha = \alpha(s)$. From (19) we obtain

$$\beta' = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(\mathbf{t}' + \mathbf{n}_q') \tag{20}$$

and

$$\mathbf{t}_\beta \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(k_1\mathbf{t} + k_1\mathbf{n}_q + (k_2 + k_3)\mathbf{b}_q) \tag{21}$$

where

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}} |k_2 + k_3| \tag{22}$$

We can write tangent vector of curve β as follow

$$\mathbf{t}_\beta = \frac{1}{|k_2 + k_3|}(k_1\mathbf{t} + k_1\mathbf{n}_q + (k_2 + k_3)\mathbf{b}_q) \tag{23}$$

Let's assume that $k_2 \neq -k_3$. The causal character of $\mathbf{tn}_q-$Smarandache curve is obtained as

$$\langle \mathbf{t}_\beta, \mathbf{t}_\beta \rangle = \frac{1}{|k_2 + k_3|^2}(k_1^2 \langle \mathbf{t}, \mathbf{t}\rangle + k_1^2 \langle \mathbf{n}_q, \mathbf{n}_q \rangle + (k_2 + k_3)^2 \langle \mathbf{b}_q, \mathbf{b}_q \rangle) = 1$$

this shows that $\mathbf{tn}_q-$Smarandache curve is a spacelike curve. Then differentiating (23) with respect to s, we obtain

$$\frac{d\mathbf{t}_\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{|k_2 + k_3|^2}(\zeta_1\mathbf{t} + \zeta_2\mathbf{n}_q + \zeta_3\mathbf{b}_q) \tag{24}$$

where

$$\begin{aligned} \zeta_1 &= -\left(\frac{|k_2 + k_3|}{k_1}\right)' k_1^2 + |k_2 + k_3| (k_1^2 + k_2^2 + k_2k_3) \\ \zeta_2 &= -\left(\frac{|k_2 + k_3|}{k_1}\right)' k_1^2 + |k_2 + k_3| (k_1^2 + k_3^2 - k_2k_3) \\ \zeta_3 &= k_1 |k_2 + k_3| (k_2 + k_{3.}) \end{aligned}$$

Substituting (22) into (24), we get

$$\mathbf{t}'_\beta = \frac{\sqrt{2}}{|k_2 + k_3|^3}(\zeta_1\mathbf{t} + \zeta_2\mathbf{n}_q + \zeta_3\mathbf{b}_q) \tag{25}$$

Then, the first curvature and the principal normal vector field of curve β are calculated

$$\kappa_\beta = \sqrt{\langle \mathbf{t}'_\beta, \mathbf{t}'_\beta \rangle} = \frac{\sqrt{2}}{|k_2 + k_3|^3}\sqrt{|-\zeta_1^2 + \zeta_2^2 + \zeta_3^2|} \tag{26}$$

and

$$\mathbf{n}_\beta = \frac{1}{\sqrt{\xi}} (\zeta_1\mathbf{t} + \zeta_2\mathbf{n}_q + \zeta_3\mathbf{b}_q) \tag{27}$$

where $\xi \neq 0$ and $\xi = \left|-\zeta_1^2 + \zeta_2^2 + \zeta_3^2\right|$. Then, the binormal vector of curve β is

$$\mathbf{b}_\beta = \frac{1}{|k_2 + k_3| \sqrt{\xi}} (\varsigma_1\mathbf{t} + \varsigma_2\mathbf{n}_q + \varsigma_3\mathbf{b}_q) \tag{28}$$

where

$$\varsigma_1 = k_1\zeta_3 - |k_2 + k_3| \zeta_2, \quad \varsigma_2 = k_1\zeta_3 - |k_2 + k_3| \zeta_1, \quad \varsigma_3 = -k_1 \zeta_2 + k_1\zeta_1.$$

In order to calculate the torsion, differentiating (20) with respect to s gives

$$\beta'' = \frac{1}{\sqrt{2}}[(k_1' + k_1^2 + k_2(k_2 + k_3))\mathbf{t} + (k_1' + k_1^2 - k_3(k_2 + k_3))\mathbf{n}_q + (k_2' + k_3' + k_1(k_2 + k_3))\mathbf{b}_q] \tag{29}$$

and

$$\beta''' = \frac{1}{\sqrt{2}} (\nu_1\mathbf{t} + \nu_2\mathbf{n}_q + \nu_3\mathbf{b}_q) \tag{30}$$

where

$$\begin{aligned} \nu_1 &= k_1'' + 3k_1'k_1 + (k_2(k_2 + k_3))' + k_2(k_2 + k_3)' + k_1(\kappa - k_3^2) \\ \nu_2 &= k_1'' - 3k_1k_1' + k_3'(k_2 + k_3) - 2k_3(k_2 + k_3)' - k_1(\kappa - k_3^2) \\ \nu_3 &= k_2'' + k_3'' + (k_1(k_2 + k_3))' + (k_2 + k_3)(\kappa - k_3^2) \end{aligned}$$

The torsion of the curve β is

$$\tau_\beta = \left[\frac{\sqrt{2}(\psi(\nu_1 - \nu_2) - (k_2 + k_3)^2(\nu_1k_3 + \nu_2k_2 - \nu_3k_1))}{(\psi - k_3(k_2 + k_3)^2)^2 + (\psi + k_2(k_2 + k_3)^2)^2 - (k_1(k_2 + k_3)^2)^2}\right] \tag{31}$$

where $\psi = -\left(\frac{|k_2+k_3|}{k_1}\right)' k_1^2$. Using (15) and (16) to compute the quasi-normal and quasi-binormal vectors of the curve β, we obtain

$$\mathbf{n}_q^\beta = \frac{1}{|k_2+k_3|\sqrt{\xi}}\begin{bmatrix} (|k_2+k_3|\,\zeta_1\cos\theta_\beta + \varsigma_1\sin\theta_\beta)\mathbf{t} \\ +(|k_2+k_3|\,\zeta_2\cos\theta_\beta + \varsigma_2\sin\theta_\beta)\mathbf{n}_q \\ +(|k_2+k_3|\,\zeta_3\cos\theta_\beta + \varsigma_3\sin\theta_\beta)\mathbf{b}_q \end{bmatrix} \tag{32}$$

and

$$\mathbf{b}_q^\beta = \frac{-1}{|k_2+k_3|\sqrt{\xi}}\begin{bmatrix} (|k_2+k_3|\,\zeta_1\sin\theta_\beta - \varsigma_1\cos\theta_\beta)\mathbf{t} \\ +(|k_2+k_3|\,\zeta_2\sin\theta_\beta - \varsigma_2\cos\theta_\beta)\mathbf{n}_q \\ +(|k_2+k_3|\,\zeta_3\sin\theta_\beta - \varsigma_3\cos\theta_\beta)\mathbf{b}_q \end{bmatrix} \tag{33}$$

Using (13) to calculate q-curvatures of the curve β, we get

$$k_1^\beta = \frac{\sqrt{2}}{|k_2+k_3|\sqrt{\xi}}\left(|k_2+k_3|\,\xi\cos\theta_\beta + \sin\theta_\beta(-\varsigma_1\zeta_1 + \varsigma_2\,\zeta_2 + \varsigma_3\zeta_3)\right) \tag{34}$$

$$k_2^\beta = \frac{-\sqrt{2}}{|k_2+k_3|\sqrt{\xi}}\left(|k_2+k_3|\,\xi\sin\theta_\beta - \cos\theta_\beta(-\varsigma_1\zeta_1 + \varsigma_2\zeta_2 + \varsigma_3\zeta_3)\right) \tag{35}$$

and

$$k_3^\beta = \frac{1}{|k_2+k_3|\sqrt{\xi}}\begin{bmatrix} \cos\theta_\beta\sin\theta_\beta\begin{pmatrix} \sqrt{|k_2+k_3|}(\zeta_1'\varsigma_1 + \zeta_2'\varsigma_2 + \zeta_3'\varsigma_3) \\ +\theta_\beta'(\varsigma_1^2 + \varsigma_2^2 + \varsigma_3^2 + |k_2+k_3|\,\xi) \end{pmatrix} \\ +\sqrt{|k_2+k_3|}\theta_\beta'(\zeta_1\varsigma_1 + \zeta_2\varsigma_2 + \ \zeta_3\varsigma_3) \\ +|k_2+k_3|\cos^2\theta_\beta(\zeta_1\zeta_1' + \zeta_2\zeta_2' + \zeta_3\zeta_3') \end{bmatrix}. \tag{36}$$

EXAMPLE: In this example, we derived the Smarandache curve of a timelike curve parametrized by

$$\alpha(s) = (2\cosh s, \sqrt{3}s, 2\sinh s)$$

for $\mathbf{k} = (0,1,0)$(spacelike), the q-frame of the curve is obtained by

$$\mathbf{t} = (2\sinh s, \sqrt{3}, 2\cosh s) \tag{37}$$

$$\mathbf{n}_q = (\cosh s, 0, \sinh s) \tag{38}$$

and

$$\mathbf{b}_q = (\sqrt{3}\sinh s, 2, \sqrt{3}\cosh s). \tag{39}$$

Thus, $\mathbf{tn}_q-$Smarandache curve is obtained by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(2\sinh s + \cosh s, \sqrt{3}, 2\cosh s + \sinh s) \tag{40}$$

and its Frenet curvatures as follows

$$\kappa_\beta = \frac{\sqrt{6}}{3} \text{ and } \tau_\beta = 0$$

The q-frame and q-curvatures of the $\mathbf{tn}_q-$Smarandache curve are calculated by

$$\begin{aligned} \mathbf{n}_q^\beta &= \frac{-1}{3}\left[\sqrt{2}\cos\theta_\beta(2\sinh s + \cosh s), \sin\theta_\beta, \sqrt{2}\cos\theta_\beta(2\cosh s + \sinh s)\right] \\ \mathbf{b}_q^\beta &= \frac{1}{3}\left[\sqrt{2}\sin\theta_\beta(2\sinh s + \cosh s), -\cos\theta_\beta, \sqrt{2}\sin\theta_\beta(2\cosh s + \sinh s)\right] \end{aligned}$$

and

$$\begin{aligned} k_1^\beta &= \cos\theta_\beta = -\frac{\sqrt{6}}{3} \\ k_2^\beta &= -\sin\theta_\beta = 0 \\ k_3^\beta &= \frac{5}{3}\sin^2\theta_\beta\theta_\beta' = 0 \end{aligned}$$

respectively. Finally the curve (black) and the $\mathbf{tn}_q-$Smarandache curve (red) are shown in Figure 2.

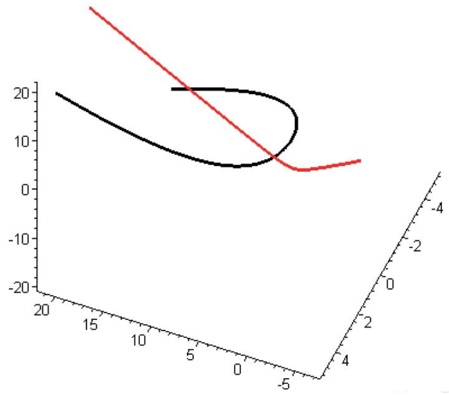

Fig. 2: The curve and $\mathbf{tn}_q$-Smarandache curve.

2.2 $\mathbf{n}_q\mathbf{b}_q$- *Smarandache curves of a timelike curve in* $\mathbb{R}^3_1$

Let $\alpha = \alpha(s)$ be a unit speed regular curve in $\mathbb{R}^3_1$ and $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q\}$ be its moving q-frame. $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve can be defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{n}_q + \mathbf{b}_q) \tag{41}$$

We can get q-invariants of $\mathbf{n}_q\mathbf{b}_q-$Smarandache curves according to $\alpha = \alpha(s)$. From (41) we have

$$\beta' = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(\mathbf{n}_q' + \mathbf{b}_q') \tag{42}$$

It follows that

$$\mathbf{t}_\beta \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}((k_1 + k_2)\mathbf{t} - k_3\mathbf{n}_q + k_3\mathbf{b}_q) \tag{43}$$

where

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\sqrt{\left|2k_3^2 - (k_1 + k_2)^2\right|} \tag{44}$$

We can write tangent vector of curve β as follow

$$\mathbf{t}_\beta = \frac{1}{\sqrt{\left|2k_3^2 - (k_1 + k_2)^2\right|}}((k_1 + k_2)\mathbf{t} - k_3\mathbf{n}_q + k_3\mathbf{b}_q) \tag{45}$$

Let's assume that $2k_3^2 \neq (k_1 + k_2)^2$ and investigate the causal character of $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve, we get

$$\langle \mathbf{t}_\beta, \mathbf{t}_\beta \rangle = \frac{2k_3^2 - (k_1 + k_2)^2}{\left|2k_3^2 - (k_1 + k_2)^2\right|}$$

Therefore, there are two possibilities for the causal character of $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve; the β curve is spacelike if

$$2k_3^2 > (k_1 + k_2)^2 \Rightarrow \langle \mathbf{t}_\beta, \mathbf{t}_\beta \rangle > 0$$

and the β curve is timelike if

$$2k_3^2 < (k_1 + k_2)^2 \Rightarrow \langle \mathbf{t}_\beta, \mathbf{t}_\beta \rangle < 0$$

Let's assume that $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve is a spacelike curve. Then differentiating (45) with respect to s, we obtain

$$\frac{d\mathbf{t}_\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{|\upsilon|^{\frac{3}{2}}}\left(\widetilde{\zeta}_1\mathbf{t} + \widetilde{\zeta}_2\mathbf{n}_q + \widetilde{\zeta}_3\mathbf{b}_q\right) \tag{46}$$

where $\upsilon = 2k_3^2 - (k_1 + k_2)^2$ and

$$\begin{aligned}
\widetilde{\zeta}_1 &= 2k_3^3\left(\frac{k_1 + k_2}{k_3}\right)' - \upsilon k_3(k_1 - k_2) \\
\widetilde{\zeta}_2 &= -(k_1 + k_2)k_3^2\left(\frac{k_1 + k_2}{k_3}\right)' + \upsilon(k_1^2 - k_3^2 + k_1k_2) \\
\widetilde{\zeta}_3 &= (k_1 + k_2)k_3^2\left(\frac{k_1 + k_2}{k_3}\right)' + \upsilon(k_2^2 - k_3^2 + k_1k_2)
\end{aligned}$$

Substituting (44) into (46), we get

$$\mathbf{t}'_\beta = \frac{\sqrt{2}}{\upsilon^2}\left(\widetilde{\zeta}_1\mathbf{t} + \widetilde{\zeta}_2\mathbf{n}_q + \widetilde{\zeta}_3\mathbf{b}_q\right) \tag{47}$$

Then, the first curvature and the principal normal vector field of curve β are respectively

$$\kappa_\beta = \|\mathbf{t}'_\beta\| = \sqrt{\left\langle \mathbf{t}'_\beta, \mathbf{t}'_\beta \right\rangle} = \frac{1}{\upsilon^2}\sqrt{2\left|-\widetilde{\zeta}_1^2 + \widetilde{\zeta}_2^2 + \widetilde{\zeta}_3^2\right|} \tag{48}$$

and

$$\mathbf{n}_\beta = \frac{1}{\sqrt{\widetilde{\xi}}}\left(\widetilde{\zeta}_1\mathbf{t} + \widetilde{\zeta}_2\mathbf{n}_q + \widetilde{\zeta}_3\mathbf{b}_q\right) \tag{49}$$

where $\xi \neq 0$ and $\xi = \left|-\widetilde{\zeta}_1^2 + \widetilde{\zeta}_2^2 + \widetilde{\zeta}_3^2\right|$. Then, the binormal vector of curve β is

$$\mathbf{b}_\beta = \frac{1}{\sqrt{\widetilde{\xi}}\,|\upsilon|}\left(\widetilde{\varsigma}_1\mathbf{t} + \widetilde{\varsigma}_2\mathbf{n}_q + \widetilde{\varsigma}_3\mathbf{b}_q\right) \tag{50}$$

where

$$\widetilde{\varsigma}_1 = -k_3\widetilde{\zeta}_3 - k_3\widetilde{\zeta}_2, \quad \widetilde{\varsigma}_2 = (k_1+k_2)\widetilde{\zeta}_3 - k_3\widetilde{\zeta}_1, \quad \widetilde{\varsigma}_3 = -(k_1+k_2)\widetilde{\zeta}_2 - k_3\widetilde{\zeta}_1.$$

In order to calculate the torsion, differentiating (20) with respect to s gives

$$\beta'' = \frac{1}{\sqrt{2}}[(k'_1 + k'_2 + k_3(k_2 - k_1))\mathbf{t} + (-k'_3 - k_3^2 + k_1(k_1+k_2))\mathbf{n}_q + (k'_3 - k_3^2 + k_2(k_1+k_2))\mathbf{b}_q] \tag{51}$$

and

$$\beta''' = \frac{1}{\sqrt{2}}\begin{bmatrix} (k''_1 + k''_2 + (k_3(k_2-k_1))' + k'_3(k_2-k_1) + (k_1+k_2)(\kappa - k_3^2))\mathbf{t} \\ +(-k''_3 - 3k_3k'_3 + (k_1(k_1+k_2))' + k_1(k_1+k_2)' - k_3(\kappa - k_3^2))\mathbf{n}_q \\ +(k''_3 - 3k_3k'_3 + (k_2(k_1+k_2))' + k_2(k_1+k_2)' + k_3(\kappa - k_3^2))\mathbf{b}_q \end{bmatrix} \tag{52}$$

where

$$\begin{aligned} \widetilde{\nu}_1 &= k''_1 + k''_2 + (k_3(k_2-k_1))' + k'_3(k_2-k_1) + (k_1+k_2)(\kappa - k_3^2) \\ \widetilde{\nu}_2 &= -k''_3 - 3k_3k'_3 + (k_1(k_1+k_2))' + k_1(k_1+k_2)' - k_3(\kappa - k_3^2) \\ \widetilde{\nu}_3 &= k''_3 - 3k_3k'_3 + (k_2(k_1+k_2))' + k_2(k_1+k_2)' + k_3(\kappa - k_3^2) \end{aligned}$$

The torsion of the curve β is

$$\tau_\beta = \left[\frac{-\sqrt{2}(\delta(\widetilde{\nu}_2 + \widetilde{\nu}_3) + \upsilon(k_3\widetilde{\nu}_1 + k_2\widetilde{\nu}_2 + k_1\widetilde{\nu}_3))}{(k_3\upsilon)^2 + (\delta + k_2\upsilon)^2 + (\delta + k_1\upsilon)^2}\right] \tag{53}$$

where $\delta = \left(\frac{k_1+k_2}{k_3}\right)' k_3^2$. The quasi-normal and quasi-binormal vectors of curve β are as follow.

$$\mathbf{n}_q^\beta = \frac{1}{\sqrt{\widetilde{\xi}}\,|\upsilon|}\begin{bmatrix} (|\upsilon|\,\widetilde{\zeta}_1\cos\theta_\beta + \widetilde{\varsigma}_1\sin\theta_\beta)\mathbf{t} \\ +(|\upsilon|\,\widetilde{\zeta}_2\cos\theta_\beta + \widetilde{\varsigma}_2\sin\theta_\beta)\mathbf{n}_q \\ +(|\upsilon|\,\widetilde{\zeta}_3\cos\theta_\beta + \widetilde{\varsigma}_3\sin\theta_\beta)\mathbf{b}_q \end{bmatrix} \tag{54}$$

and

$$\mathbf{b}_q^\beta = \frac{-1}{\sqrt{\widetilde{\xi}}\,|\upsilon|}\begin{bmatrix} (|\upsilon|\,\widetilde{\zeta}_1\sin\theta_\beta - \widetilde{\varsigma}_1\cos\theta_\beta)\mathbf{t} \\ +(|\upsilon|_2\sin\theta_\beta - \widetilde{\varsigma}_2\cos\theta_\beta)\mathbf{n}_q \\ +(|\upsilon|\,\widetilde{\zeta}_3\sin\theta_\beta - \widetilde{\varsigma}_3\cos\theta_\beta)\mathbf{b}_q \end{bmatrix} \tag{55}$$

We can calculate q-curvatures of curve β, so from (13) we get

$$k_1^\beta = \frac{\sqrt{2}}{\sqrt{\widetilde{\xi}}\,|\upsilon|}\left(|\upsilon|\,\widetilde{\xi}\cos\theta_\beta + \sin\theta_\beta(-\widetilde{\varsigma}_1\widetilde{\zeta}_1 + \widetilde{\varsigma}_2\widetilde{\zeta}_2 + \widetilde{\varsigma}_3\widetilde{\zeta}_3)\right) \tag{56}$$

and

$$k_2^{\beta} = \frac{-\sqrt{2}}{\sqrt{\widetilde{\xi}}\,|\upsilon|}\left(|\upsilon|\,\widetilde{\xi}\sin\theta_{\beta} - \cos\theta_{\beta}(-\widetilde{\varsigma}_1\widetilde{\zeta}_1 + \widetilde{\varsigma}_2\widetilde{\zeta}_2 + \widetilde{\varsigma}_3\widetilde{\zeta}_3)\right) \tag{57}$$

and

$$k_3^{\beta} = \frac{1}{|\upsilon|\,\sqrt{\widetilde{\xi}}}\left[\begin{array}{l} \cos\theta_{\beta}\sin\theta_{\beta}\left(\begin{array}{c} \sqrt{|\upsilon|}(\widetilde{\zeta}_1'\widetilde{\varsigma}_1 + \widetilde{\zeta}_2'\widetilde{\varsigma}_2 + \widetilde{\zeta}_3'\widetilde{\varsigma}_3) \\ +\theta_{\beta}'(\widetilde{\varsigma}_1^2 + \widetilde{\varsigma}_2^2 + \widetilde{\varsigma}_3^2 + |\upsilon|\,\widetilde{\xi}) \end{array}\right) \\ \\ +\sqrt{|\upsilon|}\theta_{\beta}'(\widetilde{\zeta}_1\widetilde{\varsigma}_1 + \ \widetilde{\zeta}_2\widetilde{\varsigma}_2 + \widetilde{\zeta}_3\widetilde{\varsigma}_3) \\ \\ +|\upsilon|\cos^2\theta_{\beta}(\widetilde{\zeta}_1\widetilde{\zeta}_1' + \widetilde{\zeta}_2\widetilde{\zeta}_2' + \widetilde{\zeta}_3\widetilde{\zeta}_3') \end{array}\right]. \tag{58}$$

EXAMPLE: In this example, we derived the Smarandache curve of a timelike curve parametrized by

$$\alpha(s) = (\cosh s, 1, \sinh s)$$

for $\mathbf{k} = (0, 1, 0)$(spacelike), the q-frame of the curve is obtained by

$$\mathbf{t} = (\sinh s, 0, \cosh s) \tag{59}$$

$$\mathbf{n}_q = (\cosh s, 0, \sinh s) \tag{60}$$

and

$$\mathbf{b}_q = (0, 1, 0). \tag{61}$$

Thus, $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve is obtained by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\cosh s, 1, \sinh s) \tag{62}$$

and its Frenet curvatures as follows

$$\kappa_{\beta} = \sqrt{2} \text{ and } \tau_{\beta} = 0$$

The q-frame and q-curvatures of the $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve is obtained by

$$\begin{aligned} \mathbf{n}_q^{\beta} &= (\cosh s, 0, \sinh s) \\ \mathbf{b}_q^{\beta} &= (0, 1, 0) \end{aligned}$$

$$\begin{aligned} k_1^{\beta} &= \cos\theta_{\beta} = \sqrt{2} \\ k_2^{\beta} &= -\sin\theta_{\beta} = 0 \\ k_3^{\beta} &= \theta_{\beta}' = 0 \end{aligned}$$

respectively. Finally the curve (blue) and the $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve (red) are shown in Figure 3.

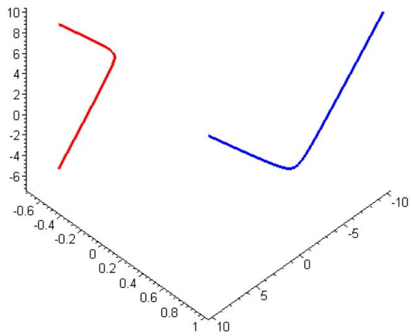

Fig. 3: The curve and $\mathbf{n}_q\mathbf{b}_q$- Smarandache curve.

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On Special Curves of General Hyperboloid in E_1^3

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Abstract:
In this work, we give the Darboux vectors $\{\gamma(s), T(s), Y(s)\}$ of a given curve using the hyperbolically motion and hyperbolically inner product defined by Simsek and Özdemir in [9]. Then, we present the variations of the geodesic curvature function $\kappa_g(s,w)$ and the speed function $v(s,w)$ of the curve γ at $w=0$. Also, we define the new type curves whose Darboux frame vectors of a given curve makes a constant angle with the constant Killing vector field and also we obtain the parametric characterizations of these curves. At the end of this article, we exemplify these curves on the general hyperboloid with their figures using the program Mathematica.

Keywords: General hyperboloid, Special curves, Lorentzian space, Darboux frame.

1 Introduction

The helix is known as a curve in DNA double, carbon nanotubes, form of plants. In the geometry, the definition of helix is a curve that its tangent vector field makes a constant angle with a fixed straight line called the axis of the helix. The study of these curves in the 3-dimensional Lorentzian space forms in [4]. On the other hand, the slant helices are studied by Izumiya and Takeuchi. If the curve is called as a slant helix, then its principal normal vector make a constant angle with a fixed direction [6]. In [8], the authors show that the path of a charged particle moves in a static magnetic field in 3D Riemannian space is the circular helix or slant helix path. Another of the important curves in geometry is the spherical curves. Thus, so many authors study in this field (see for details in [2, 3, 7]). In [10, 11], the authors give a characterization for a curve to be on a sphere. Breuer et al. obtain an explicit characterization of the spherical curve [5].

In this study, we summarize the some basic notations of general hyperbolical space which are defined by Simsek and Özdemir in [9]. In addition, we give the features of variation vector field along a curve as well as the variational formulas for its Darboux curvatures in the third section. The connection between the geometric variational formulas for curvatures and the Killing equations along a space curve according to Darboux frame. Then, we define the special curves whose Darboux frame vectors of a given curve makes a constant angle with the constant Killing vector field. Also, we generate the parametric representations of all kind of helices on the general hyperboloid $\mathcal{GH}$ and illustrate these curves on the general hyperboloid with their figures using the Mathematica program language.

2 Preliminaries

In [9], the $g-$hyperbolic $2-$space and the $g-$de Sitter $2-$space are defined as

$$H^{2,1}_{a_1,a_2,a_3} = \left\{ u=(x,y,z)\in\mathbb{R}^3 \,\middle|\, -a_1x^2+a_2y^2+a_3z^2=-1 \right\},$$
$$S^{2,1}_{a_1,a_2,a_3} = \left\{ u=(x,y,z)\in\mathbb{R}^3 \,\middle|\, -a_1x^2+a_2y^2+a_3z^2=1 \right\},$$

respectively. The scalar product on the $g-$hyperbolic $2-$space and the $g-$de Sitter $2-$space of $\mathbb{R}^3$ with the cartesian equation $-a_1x^2+a_2y^2+a_3z^2=\pm1$, $a_1,a_2,a_3\in\mathbb{R}^+$ is defined as

$$g:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R};\; g(u,v)=-a_1x_1y_1+a_2x_2y_2+a_3x_3y_3$$

where $u=(u_1,u_2,u_3)$, $v=(v_1,v_2,v_3)\in\mathbb{R}^3$. The real vector space $\mathbb{R}^3$ equipped with the hyperbolical $g-$inner product will be represented by $\mathbb{R}^{2,1}_{a_1,a_2,a_3}$. The norm of a vector associated with the scalar product g is defined as

$$\|u\|_g=\sqrt{|g(u,u)|}.$$

Two vectors u and v are called hyperbolically orthogonal vectors if $g(u,v)=0$. If u is a hyperbolically orthonormal vector then $g(u,u)=1$. The hyperbolic angle of the between two timelike vectors u and v on the same timecone is given by

$$\cosh\theta = \frac{-g(u,v)}{\|u\|_g \|v\|_g}$$

where θ is compatible with the parameters of the angular parametric equations of pseudo-spheres.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a standard unit vectors of $\mathbb{R}^3$. The $g-$vector product of the vector fields $u, v \in \mathbb{R}^3$ is described as

$$\mathcal{V}_g(u\times v) = \Delta^* \begin{vmatrix} -\mathbf{e}_1/a_1 & \mathbf{e}_2/a_2 & \mathbf{e}_3/a_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \tag{1}$$

where $\Delta^* = \sqrt{a_1 a_2 a_3}$, $a_1, a_2, a_3 \in \mathbb{R}^+$ (*see for details* in [9]).

Let's take the general hyperboloid $-a_1x^2 + a_2y^2 + a_3z^2 = \pm 1$. The sectional curvature of the hyperboloid generated by the non-degenerated plane $\{u, v\}$ is defined as

$$K(u,v) = \frac{g(R(u,v)u,v)}{g(u,u)g(v,v) - g(u,v)^2} \tag{2}$$

where R is the Riemannian curvature tensor given by

$$R(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z \tag{3}$$

where $X, Y, Z \in \chi(H^{2,1}_{a_1,a_2,a_3})$ or $\chi(S^{2,1}_{a_1,a_2,a_3})$. The general hyperboloid has the constant sectional curvature. Therefore, the curvature tensor R written as follows

$$R(X,Y)Z = \mathcal{C}\{g(Z,X)Y - g(Z,Y)X\} \tag{4}$$

where $\mathcal{C}$ is the constant sectional curvature.

Let γ be a curve with arc length parameter s on the general hyperboloid ($\mathcal{GH}$). To calculate the Darboux frame apparatus $\{\gamma(s), T(s), Y(s), \kappa_g(s)\}$ along the curve γ on the hyperboloid surface, firstly we will calculate the unit normal vector field of the general hyperboloid. The surfaces $H^{2,1}_{a_1,a_2,a_3}$ and $S^{2,1}_{a_1,a_2,a_3}$ are Lorentzian spheres according to the hyperbolical inner product, the unit normal vector field along the general hyperboloid equal to the position vector of the curve γ. Then we found an orthonormal frame $\{\gamma(s), T(s) = \gamma'(s), Y(s) = \mathcal{V}_g(\gamma(s)\times T(s))\}$ which is called the hyperbolical Darboux frame along the curve γ. The corresponding Darboux formulas of the curve γ is written as

$$\begin{aligned} \gamma'(s) &= T(s), \\ T'(s) &= -\varepsilon_\gamma\varepsilon_T\gamma(s) + \varepsilon_Y\kappa_g(s)Y(s), \\ Y'(s) &= \varepsilon_\gamma\varepsilon_Y\kappa_g(s)T(s), \end{aligned} \tag{5}$$

$$\mathcal{V}_g(\gamma\times T) = Y,\ \ \mathcal{V}_g(\gamma\times Y) = -\varepsilon_T\varepsilon_Y T\ ,\ \mathcal{V}_g(T\times Y) = \varepsilon_\gamma\varepsilon_Y\gamma$$

where $\kappa_g(s) = g(T'(s), Y(s))$ is an *geodesic curvature function* of the curve γ on the Lorentzian spheres $H^{2,1}_{a_1,a_2,a_3}$ and $S^{2,1}_{a_1,a_2,a_3}$. Also, $\varepsilon_\gamma = g(\gamma,\gamma)$, $\varepsilon_T = g(T,T)$ and $\varepsilon_Y = g(Y,Y)$ are shown the signature functions of the hyperbolical Darboux vectors. Namely, $\varepsilon_k = 1$ $(k = \gamma, T, Y)$ when the vector k is spacelike vector and $\varepsilon_k = -1$ $(k = \gamma, T, Y)$ when the vector k is timelike vector [1].

Lemma 1. *Let $\varphi: U \subset \mathbb{E}^2 \to \mathbb{E}^3$, $\varphi(U) = H^{2,1}_{a_1,a_2,a_3}$ or $S^{2,1}_{a_1,a_2,a_3}$ be a general hyperboloid and $\gamma: I \subset \mathbb{R} \to U$ be a regular curve on semi-Riemannian manifold M. Provided that V be a vector field along the curve γ then the variation of γ defined by $\Gamma: I\times(-\varepsilon,\varepsilon) \to \mathcal{GH}(C)$ with $\Gamma(s,0) = \gamma(s)$ where $\gamma(s,0)$ is the initial curve. The variations of the geodesic curvature function $\kappa_g(s,w)$ and the speed function $v(s,w)$ at $w=0$ are calculated as follows:*

$$V(v) = \left(\frac{\partial v}{\partial t}(s,w)\right)\Big|_{w=0} = -v\wp, \tag{6}$$

$$V(\kappa_g) = \left(\frac{\partial \kappa_g}{\partial w}(s,w)\right)\Big|_{w=0} = g(Y, -R(V,T)T + \nabla^2_T V) + \frac{1}{\kappa_g}g(\gamma, -R(V,T)T + \nabla^2_T V) + 2\wp(\kappa_g - \frac{\varepsilon_T}{\kappa_g}).$$

where $\wp = g(\nabla_T V, T)$ and R stands for the curvature tensor of general hyperboloid [1].

Proposition 1. *[1] If $V(s)$ is the restriction to $\gamma(s)$ of a Killing vector field V of $\mathcal{GH}$ then the variations of the hyperbolical Darboux curvature functions and speed function of γ satisfy the following condition*

$$V(v) = 0 = V(\kappa_g). \tag{7}$$

3 Special helices on the general hyperboloid $\mathcal{GH}$

In this section, we define a new kind of slant helices called as type-1, type-2 and type-3 special helices in 3-dimensional Lorentzian space. Moreover, we obtain some characterizations using the Killing vector field and give examples of these curves. We plot the figures of these curves on the general hyperboloid by using the Mathematica.

Definition 1. *Let $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}$, $\varphi(U) = \mathcal{GH}$ be a general hyperboloid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on the $\mathcal{GH}$. Then we say that γ is a type-1 special helix, type-2 special helix, or type-3 special helix if $g(V,T) = const.$, $g(V,\gamma) = const.$, and $g(V,Y) = const.$, respectively.*

Theorem 1. *[1] Let $\varphi : U \subset \mathbb{E}^2 \to \mathbb{E}^3_1$, $\varphi(U) = \mathcal{GH}$ be a general hyperboloid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on $\mathcal{GH}$ and V be a Killing vector field along the curve γ. Then γ is a type-1 special helix with the axis V if and only if the geodesic curvature of the curve γ satisfy the following equations:*
(i) If the curve γ has the spacelike tangent vector, then the geodesic curvature is

$$\kappa_g(s) = -\varepsilon_\gamma \coth\theta(s)$$

where $\theta''(s)\sinh^2\theta(s) + \varepsilon_\gamma \omega\theta'(s)\cosh\theta(s) = 0$ and $\varepsilon_\gamma = \pm 1$,
(ii) If the curve γ has the timelike tangent vector, then the geodesic curvature is

$$\kappa_g(s) = -\cot\theta(s)$$

where $\theta''(s)\sin^2\theta(s) + \omega\theta'(s)\cos\theta(s) = 0$.

Theorem 2. *Let $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}$, $\varphi(U) = \mathcal{GH}$ be a general hyperboloid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on the $\mathcal{GH}$. Then γ is a type-2 special helix with the axis V if and only if the geodesic curvature of the curve γ satisfies the following conditions which are given according to the casual character of the position vector field of the curve γ:*
(i) If $g(\gamma,\gamma) = 1$ is satisfied, then the geodesic curvature is

$$k_g(s) = -\left(\frac{c_1}{\sinh\theta(s)} + \theta'(s)\right) \tag{8}$$

with $\theta = constant$ or $\varepsilon_T(C+1)\sinh^4\theta(s) + c_1\theta'(s)\sinh\theta(s) + c_1^2 = 0$ and C is a constant.

(ii) If $g(\gamma,\gamma) = -1$ is satisfied, then the geodesic curvature is

$$k_g(s) = -\theta'(s) + \frac{c_1}{\sin\theta(s)} \tag{9}$$

with $\theta = constant$ or $(1-C)\sin^4\theta(s) + c_1\theta'(s)\sin\theta(s) - c_1^2 = 0$ and C is a constant.

Proof: (i) If $g(\gamma,\gamma) = 1$ is satisfied and γ is a type-2 special helix, then the Killing axis is written as

$$V = \cosh\theta(s)T(s) + c_1\gamma(s) + \varepsilon_Y \sinh\theta(s)Y(s), c_1 = const. \tag{10}$$

Differentiating eq.(10) with respect to s, we obtain the following equation

$$\begin{aligned} \nabla_T V &= ((\theta'(s)\sinh\theta(s) + k_g(s)\sinh\theta(s) + c_1)T(s) + (-\varepsilon_T\cosh\theta(s))\gamma(s) \\ &\quad +(\varepsilon_Y\cosh\theta(s)k_g(s) + \varepsilon_Y\theta'(s)\cosh\theta(s))Y(s). \end{aligned} \tag{11}$$

Using $V(v) = 0$ in Lemma 1, we calculate

$$k_g(s) = -\left(\frac{c_1}{\sinh\theta(s)} + \theta'(s)\right). \tag{12}$$

The differentiation of eq.(11) is given by

$$\begin{aligned} \nabla_T^2 V &= (-\varepsilon_T cosh\theta(s) + (\theta'(s) + k_g(s))k_g(s)cosh\theta(s))T(s) - \varepsilon_T\theta'(s)sinh\theta(s)\gamma(s) \\ &\quad +\varepsilon_Y(\theta'(s) + k_g(s))cosh\theta(s)Y(s). \end{aligned} \tag{13}$$

Also, we have the following equation

$$R(V,T(s))T(s) = C(g(T(s),V)T(s) - g(T(s),T(s))V).$$

Using the Darboux frame equations and eq.(10) we deduce

$$R(V,T(s))T(s) = -\varepsilon_T C(c_1\gamma(s) + \varepsilon_Y\sinh\theta(s)Y(s)). \tag{14}$$

Considering the eqs.(13) and (20) with the second equation in Lemma 1 and the Proposition 1, we obtain the desired differential equation for θ.

(ii) If $g(\gamma, \gamma) = -1$ is satisfied and γ is a type-2 special helix, then the Killing axis is written as

$$V = \cos\theta(s)T(s) + c_1\gamma(s) + \sin\theta(s)Y(s), c_1 = const.$$

If we make similar calculations in (i), then we obtain

$$k_g(s) = -\theta'(s) + \frac{c_1}{\sin\theta(s)},$$

$$\begin{aligned}\nabla_T^2 V &= (cos\theta(s) - \frac{c_1 k_g(s) cos\theta(s)}{\sin\theta(s)})T(s) - \theta'(s)sin\theta(s)\gamma(s) \\ &\quad - \left(\frac{c_1\theta'(s)}{\sin^2\theta(s)}\right)Y(s)\end{aligned}$$

and

$$R(V, T(s))T(s) = -C(c_1\gamma(s) + \sin\theta(s)Y(s)).$$

Considering the last two equalities with the second equation in Lemma 1 and the Proposition 1, the desired result is obtained. □

Corollary 1. *Let γ be a type-2 special helix on the general hyperboloid.*
(i) If the axis $V = \cosh\theta T(s) + c_1\gamma(s) + \varepsilon_Y \sinh\theta Y(s)$; $\theta = const.$ of the type-2 special helix is taken, then the curve γ has the following parametric representation

$$\gamma(s) = A_1 + \frac{A_2}{\sqrt{\frac{c_1^2}{\sinh^2\theta} - \varepsilon_T}} \exp\left(\left(\sqrt{\frac{c_1^2}{\sinh^2\theta} - \varepsilon_T}\right)s\right) + \frac{A_3}{\sqrt{\frac{c_1^2}{\sinh^2\theta} - \varepsilon_T}} \exp\left(-\left(\sqrt{\frac{c_1^2}{\sinh^2\theta} - \varepsilon_T}\right)s\right)$$

where $A_1, A_2, A_3 \in \mathbb{R}^3_{a_1,a_2,a_3}$ and $c_1 \in \mathbb{R}$.
(ii) If the axis of the type-2 special helix is $V = \cos\theta T(s) + c_2\gamma(s) + \sin\theta Y(s)$; $\theta = const.$, then the curve γ has the following parametric representation

$$\gamma(s) = B_1 + \frac{B_2}{\sqrt{\frac{c_2^2}{\sin^2\theta} + 1}} \exp\left(\left(\sqrt{\frac{c_2^2}{\sin^2\theta} + 1}\right)s\right) + \frac{B_3}{\sqrt{\frac{c_2^2}{\sin^2\theta} + 1}} \exp\left(-\left(\sqrt{\frac{c_2^2}{\sin^2\theta} + 1}\right)s\right)$$

where $B_1, B_2, B_3 \in \mathbb{R}^3_{a_1,a_2,a_3}$ and $c_2 \in \mathbb{R}$.

Proof: (i) Let γ be a type-2 special helix on the general hyperboloid with the axis

$$V = \cosh\theta T(s) + c_1\gamma(s) + \varepsilon_Y \sinh\theta Y(s); \quad \theta = const.$$

then the hyperbolical curvature of γ calculated as

$$k_g = -\frac{c_1}{\sinh\theta}. \tag{15}$$

On the other hand, from the Darboux frame equations γ satisfy the following third order differential equation

$$k_g\gamma''' - k_g'\gamma'' + (\varepsilon_T k_g - k_g^3)\gamma' - \varepsilon_T k_g'\gamma = 0. \tag{16}$$

If k_g is written in the eq.(16) and the differential equation is solved then it is obtained that γ has the following parametric representation

$$\gamma(s) = A_1 + \frac{A_2}{\sqrt{\frac{c_1^2}{\sinh^2\theta} - \varepsilon_T}} \exp\left(\left(\sqrt{\frac{c_1^2}{\sinh^2\theta} - \varepsilon_T}\right)s\right) + \frac{A_3}{\sqrt{\frac{c_1^2}{\sinh^2\theta} - \varepsilon_T}} \exp\left(-\left(\sqrt{\frac{c_1^2}{\sinh^2\theta} - \varepsilon_T}\right)s\right) \tag{17}$$

here $A_1, A_2, A_3 \in \mathbb{R}^3_{a_1,a_2,a_3}$ and $c_1 \in \mathbb{R}$.
(ii) Let γ be a type-2 special helix on the general hyperboloid with the axis

$$V = \cos\theta T(s) + c_2\gamma(s) + \sin\theta Y(s); \quad \theta = const.$$

then the hyperbolical curvature of γ calculated as

$$k_g = \frac{c_2}{\sin\theta}. \tag{18}$$

From the Darboux frame equations, the γ satisfy the following differential equation

$$k_g\gamma''' - k_g'\gamma'' + (k_g^3 - k_g)\gamma' + k_g'\gamma = 0. \tag{19}$$

If k_g is written in the eq.(19) and the differential equation is solved then it is obtained parametric representation of the curve γ. □

Theorem 3. *Let $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}$, $\varphi(U) = \mathcal{GH}$ be a general hyperboloid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on the $\mathcal{GH}$. Then γ is type-3 special helix with the axis V if and only if the geodesic curvature of the curve γ satisfy the following conditions:*
(i) If $g(Y(s), Y(s)) = 1$ is satisfied, then the geodesic curvature is

$$k_g(s) = -\frac{\varepsilon_\gamma \left(1 + \varepsilon_T \theta'(s)\right) \cosh\theta(s)}{c_3} \tag{20}$$

with $\theta'' \sinh\theta \left(-\varepsilon_\gamma \left(1 + \varepsilon_T \theta'\right) \cosh^2\theta - c_3^2\right) + \theta' \left((C + \varepsilon_\gamma) c_3^2 + \left(1 + \varepsilon_T \theta'\right)^2 (\sinh^2\theta + \cosh^2\theta)\right) \cosh\theta - c_3^2 (\theta')^2 \cosh\theta = 0$ *or $\theta = constant$ and C is a constant.*
(ii) If $g(Y(s), Y(s)) = -1$ is satisfied, then the geodesic curvature is

$$k_g(s) = \frac{(\theta'(s) + 1)\cos\theta(s)}{c_4} \tag{21}$$

with $\theta = constant$ or $\theta'' \sin\theta \left(\left(1 + \theta'\right)\cos^2\theta - c_4^2\right) - c_4^2 \theta' \cos\theta \left(C + \theta' + 1\right) + (\theta' + 1)^2 \theta' \cos\theta \cos 2\theta = 0$ *and C is a constant.*

Proof: (i) If γ is a type-3 special helix with the Killing axis V then V is written as

$$V = \varepsilon_T \sinh\theta T(s) + \cosh\theta \gamma(s) + c_3 Y(s). \tag{22}$$

By differentiating eq.(22) we get

$$\nabla_T V = (\cosh\theta + \varepsilon_T \theta' \cosh\theta + \varepsilon_\gamma c_3 k_g) T(s) + (\theta' \sinh\theta - \varepsilon_\gamma \sinh\theta)\gamma(s) + \varepsilon_T k_g \sinh\theta Y(s). \tag{23}$$

By using the equation $V(v) = 0$ in Lemma 1 we reach

$$k_g = -\frac{\varepsilon_\gamma (1 + \varepsilon_T \theta') \cosh\theta}{c_3}. \tag{24}$$

If we take the differentiation of eq.(23) we obtain

$$\begin{aligned} \nabla_T^2 V = & ((\theta' - \varepsilon_\gamma)\sinh\theta - k_g^2 \sinh\theta) T(s) + (\theta'' \sinh\theta + (\theta' - \varepsilon_\gamma)\theta' \cosh\theta)\gamma(s) \\ & + (\varepsilon_T k_g' \sinh\theta + \varepsilon_T k_g \theta' \cosh\theta) Y(s). \end{aligned} \tag{25}$$

Furthermore, we have the following equation

$$R(V, T(s))T(s) = C(g(T(s), V)T(s) - g(T(s), T(s))V). \tag{26}$$

By using the Darboux frame equations and eq.(22) we obtain

$$R(V, T(s))T(s) = -C\varepsilon_T (\cosh\theta \gamma(s) - c_3 Y(s)). \tag{27}$$

If we consider the eq.(25) and eq.(27) with the second equation in Lemma 1 and the Proposition 1, we deduce

$$\theta = const. \tag{28}$$

or satisfy the following equation

$$\theta'' \sinh\theta \left(-\varepsilon_\gamma \left(1 + \varepsilon_T \theta'\right)\cosh^2\theta - c_3^2\right) + \theta' \left((C + \varepsilon_\gamma) c_3^2 + \left(1 + \varepsilon_T \theta'\right)^2 (\sinh^2\theta + \cosh^2\theta)\right)\cosh\theta - c_3^2 (\theta')^2 \cosh\theta = 0. \tag{29}$$

(ii) If γ is a type-3 special helix with the Killing axis V then V is written as

$$V = \sin\theta T(s) + \cos\theta \gamma(s) + c_4 Y(s). \tag{30}$$

If we make similar calculations in (i), then we obtain

$$k_g(s) = \frac{(\theta'(s) + 1)\cos\theta(s)}{c_4},$$

$$\begin{aligned} \nabla_T^2 V \;=\; & (-(\theta'(s) + 1)\sin\theta(s) + k_g^2(s)\sin\theta(s))T(s) - (\theta''(s)\sin\theta(s) + (\theta'(s) + 1)\theta'(s)\cos\theta(s))\gamma(s) \\ & - \left(k_g'(s)\sin\theta(s) + k_g(s)\theta'(s)\cos\theta(s)\right) Y(s) \end{aligned}$$

and

$$R(V, T(s))T(s) = -C(\cos\theta(s)\gamma(s) + c_4 Y(s)).$$

Considering the last two equalities with the second equation in Lemma 1 and the Proposition 1, the desired result is obtained. □

Corollary 2. *Let γ be a type-3 special helix on the general hyperboloid.*
(i) If the axis $V = \varepsilon_T \sinh\theta T(s) + \cosh\theta\gamma(s) + c_3 Y(s)$; $\theta = const.$ of the type-3 special helix is taken, then the curve γ has the following parametric representation

$$\gamma(s) = C_1 + \frac{C_2}{\sqrt{\frac{\varepsilon_\gamma \cosh^2\theta}{c_3^2}+1}} \exp\left(\left(\sqrt{\frac{\varepsilon_\gamma \cosh^2\theta}{c_3^2}}+1\right)s\right) + \frac{C_3}{\sqrt{\frac{\varepsilon_\gamma \cosh^2\theta}{c_3^2}+1}} \exp\left(-\left(\sqrt{\frac{\varepsilon_\gamma \cosh^2\theta}{c_3^2}}+1\right)s\right) \tag{31}$$

where $C_1, C_2, C_3 \in \mathbb{R}^3_{a_1,a_2,a_3}$ and $c_3 \in \mathbb{R}$.
(ii) If the axis of the type-3 special helix is $V = \sin\theta T(s) + \cos\theta\gamma(s) + c_4 Y(s)$; $\theta = const.$, then the curve γ has the following parametric representation

$$\gamma(s) = D_1 + \frac{D_2}{\sqrt{\frac{\cos^2\theta}{c_4^2}-1}} \exp\left(\left(\sqrt{\frac{\cos^2\theta}{c_4^2}}-1\right)s\right) + \frac{D_3}{\sqrt{\frac{\cos^2\theta}{c_4^2}-1}} \exp\left(-\left(\sqrt{\frac{\cos^2\theta}{c_4^2}}-1\right)s\right) \tag{32}$$

where $D_1, D_2, D_3 \in \mathbb{R}^3_{a_1,a_2,a_3}$ and $c_4 \in \mathbb{R}$.

Proof: (i) Let γ be a type-3 special helix on the general hyperboloid and the hyperbolical curvature of γ is calculated as

$$k_g = -\frac{\varepsilon_\gamma \cosh\theta}{c_3} \tag{33}$$

where θ is constant. From the Darboux frame equations, it is calculated

$$k_g\gamma''' - k_g'\gamma'' - (\varepsilon_\gamma k_g^3 + k_g)\gamma' + k_g'\gamma = 0. \tag{34}$$

If k_g is written in the eq.(34) and solution of the differential equation is eq. (31).
(ii) Let γ be a type-3 special helix on the general hyperboloid with the axis

$$V = \sin\theta T(s) + \cos\theta\gamma(s) + c_4 Y(s); \quad \theta = const.$$

then the hyperbolical curvature of γ calculated as

$$k_g = \frac{\cos\theta}{c_4}. \tag{35}$$

From the Darboux frame equations, the γ satisfy the following differential equation

$$k_g\gamma''' - k_g'\gamma'' - (k_g^3 - k_g)\gamma' - k_g'\gamma = 0. \tag{36}$$

If k_g is written in the eq.(36) and the differential equation is solved then it is obtained parametric representation of the curve γ. □

Example 1. *Let us take the timelike curve parameterized as*

$$\gamma(s) = \left(\frac{15}{16}\cos 17s, \frac{9}{64}\cos 25s + \frac{25}{64}\cos 9s, \frac{1}{16}\sin 25s - \frac{25}{144}\sin 9s\right) \tag{37}$$

on the hyperboloid $-4x^2 + 16y^2 + 81z^2 = 1$. The hyperbolical curvature of the curve γ calculated as

$$k_g(s) = -\cot(17s). \tag{38}$$

Thus we can easily see that γ is a type-1 special helix. It is illustrated in Figure 1.

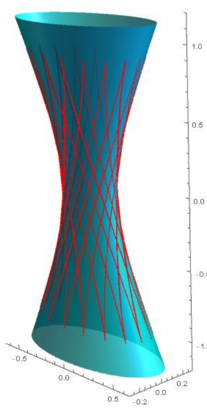

Figure 1. Type-1 special helix on the one sheet hyperboloid $S^{2,1}_{4,16,81}$.

Example 2. *Type-2 (type-3) special helices corresponding to different values of the A_i, C_i, $i = 1, 2, 3$. are illustrated in Figure 2.*

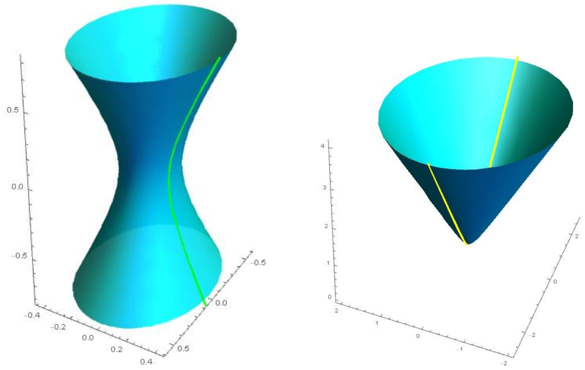

Figure 2. Type-2 (type-3) special helices on the hyperboloid $S^{2,1}_{2\sqrt{2},3\sqrt{2},4\sqrt{2}}$ and $H^{2,1}_{2\sqrt{2},3\sqrt{2},4\sqrt{2}}$.

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A Study on Generalized Tubes

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Abstract: In this paper, we consider generalized tubes, which we refer to in the paper as hereafter GTs, according to q-frame in Euclidean space E^3. First, we give a parametric representation of directional generalized tubes (DGTs). Since GT class is divided by two important subclasses, we investigate geometric properties of these two classes with respect to the q-frame.

Keywords: Adapted frame, Frenet frame, Generalized tubes.

1 Introduction

A canal surface is introduced by the French mathematician Gaspard Monge in 1850. Canal surface can be defined as envelope of a nonparameter set of spheres, centered at a spine curve $\alpha(s)$ with radius $r(s)$. As a special case of such a surface is called a pipe surface or tubular surface at a constant function radius $r(s)$with spine curve $\alpha(s)$ [12]. Moreover, tubular surfaces are used in many applications such as CAGD, shape reconstruction, transition surfaces between pipes,robotic path planning etc [16]. A generalized tube (or GT) is the surface constructed by sweeping some planar closed curve along an arbitrary 3D space curve. One good reason to understand GTs are used in many man-made objects. It has a spine that is generally not straight, and includes some bending and twisting and maybe even some knots. Many cables, garden hoses,wires, poles, objects fit into this category of objects [8]. Horaud and Brady present a method for recovering the underlying GC such that a cross-section curve is found that extremizes compactness subject to an orthogonality constraint between the cross-section plane and the GC axis [10]. Recovery of surfaces of revolution, an important subclass of SHGCs, is considered by Richetin et al. in [13] and LaVest [11]. In Ulupinar and Nevatia [14], general definitions are given for both parallel and mirror symmetry. In Ulupinar [15], SHGC contours are combined with various heuristic constraints to recover 3D shape from contour.

Let $\alpha : (a, b) \to E^3$ be a curve that is parametrized by arc lenght s. Denote by $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ the moving Frenet frame along the unit speed curve $\alpha(s)$. A parameterization of the GT is written as in [8]

$$\psi(s, \theta) = \alpha(s) + r(\theta) (\mathbf{n}(s) \cos\theta + \mathbf{b}(s) \sin\theta) \tag{1}$$

where $\theta \in (0, 2\pi)$. For the GT to be a regular, well-defined surface, we impose the additional restrictions that r is twice differentiable, $r(\theta) > 0$ and $r(0) = r(2\pi)$.

Theorem 1.1. The directions of the parameter curves at a non-umbilical point on a patch are in the direction of the principal directions if and only if $F = M = 0$ at the point, where F and M are the respective first and second fundamental coefficients. At an umbilical point, every direction is a principal direction.

Theorem 1.2. All straight lines on a surface are geodesics. A curve not a straight line is a geodesic if and only if the osculating plane of the curve is perpendicular to the tangent plane to the surface at each point [6].

The tube surface can be parameterized using Frenet frame. However, this frame is undefined wherever the curvature vanishes, such as at points of inflection or along straight sections of the curve [3]. Therefore, new frames have been investigated as an alternative to Frenet frame such as Bishop (parallel transport), q-frames etc [1, 5]. The directional q-frame offers two key advantages over the Frenet frame The first, it is well defined even if the curve has vanishing second derivative and the second, it avoids the unnecessary twist around the tangent. Also, the directional q-frame is easier than the rotation minimizing frames, one of them is Bishop frame to calculate [5]. The q-frame of a regular curve $\alpha(t)$ as follows

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q \tag{2}$$

where $\mathbf{t}$ is the unit tangent vector, $\mathbf{n}_q$ is quasi-normal and $\mathbf{b}_q$ is quasi-binormal vector. Also, k is the projection vector and is usually chosen as $k = (0, 0, 1)$ [5]. The q-frame and Frenet frame along a space curve are shown in Figure 1.

The variation equations of the directional q-frame are given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \|\alpha'\| \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{3}$$

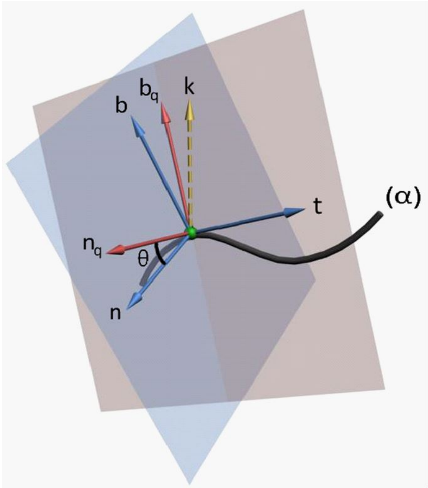

Fig. 1: The q-frame and Frenet frame.

where the q-curvatures are expressed as follows

$$k_1 = \frac{\langle \mathbf{t}', \mathbf{n}_q \rangle}{\|\alpha'\|}, k_2 = \frac{\langle \mathbf{t}', \mathbf{b}_q \rangle}{\|\alpha'\|}, k_3 = -\frac{\langle \mathbf{n}_q, \mathbf{b}_q' \rangle}{\|\alpha'\|}. \tag{4}$$

Dede et al. have been defined the q-frame for tubular surface modeling, called D-tubular surface and gave a parametric representation of these surface in Euclidean and Minkowski spaces[4, 7] The D-tubular surface, at a distance r from the spine curve $\alpha(s)$, may be represented as

$$\psi^r(s,v) = \alpha(s) + r(s)(\cos v \mathbf{n}_q + \sin v \mathbf{b}_q). \tag{5}$$

2 A study on generalized tubes

In this study, we consider generalized tubes, which we refer to in the paper as hereafter GTs, according to q-frame in E^3. we give a parametric representation of directional generalized tubes (DGTs).

The directional generalized tube (DGT) centered at a spine curve $\alpha(s)$ with radius $r(v) = r$ may be represented as

$$\psi^r(s,v) = \alpha(s) + r(v)\left(\cos(v)\,\mathbf{n_q} + \sin(v)\,\mathbf{b_q}\right) \tag{6}$$

where $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, k\}$ is the q-frame of the spine curve $\alpha(s)$. Since the DGT should be a regular, well-defined surface, we take the additional restrictions that r is twice differentiable, $r(v) > 0$ and $r(0) = r(2\pi)$. The partial derivatives of $\psi^r(s,v)$, with respect to v and s, respectively, are determined by

$$\psi_v^r = (r'\cos v - r\sin v)\mathbf{n}_q + (r'\sin v - r\cos v)\mathbf{b}_q \tag{7}$$

and

$$\psi_s^r(s,v) = \begin{array}{c} (1 - r(k_1\cos(v) + k_2\sin(v)))\mathbf{t} \\ -rk_3\left(\sin(v)\,\mathbf{n_q} - \cos(v)\,\mathbf{b_q}\right) \end{array}. \tag{8}$$

It follows that the unit normal vector of the DGT is

$$U = \frac{1}{A}\left(\begin{array}{c} rr'k_3 t + (1 - r(k_1\cos(v) + k_2\sin(v)))\cdot \\ \left[(r'\sin v + r\cos v)\mathbf{n}_q - (r'\cos v - r\sin v)\mathbf{b}_q\right] \end{array}\right) \tag{9}$$

where $A = \sqrt{(1 - r(k_1\cos(v) + k_2\sin(v)))^2(r^2 + r'^2) + r^2r'^2k_3^2}$.

From (7) and (8), the coefficients $E = \langle \psi_v^r, \psi_v^r \rangle$, $F = \langle \psi_s^r, \psi_v^r \rangle$ and $G = \langle \psi_s^r, \psi_s^r \rangle$ of the first fundamental form are calculated by

$$E = r'^2 + r^2,\ F = r^2k_3, \tag{10}$$

and

$$G = (1 - r(k_1\cos(v) + k_2\sin(v)))^2 + r^2k_3^2. \tag{11}$$

By using $L = \langle \psi_{vv}^r, U \rangle$, $M = \langle \psi_{vs}^r, U \rangle$ and $N = \langle \psi_{ss}^r, U \rangle$, the coefficients of second fundamental form are obtained as

$$L = \frac{1}{A}\left[(1 - r(k_1\cos(v) + k_2\sin(v)))\left(rr'' - r^2 - 2r'^2\right)\right] \tag{12}$$

$$M = -\frac{1}{A}k_3\left(\begin{array}{l} r^2(1 - r(k_1\cos(v) + k_2\sin(v))) \\ +r'\left(r^2(1 - r(k_1\sin(v) - k_2\cos(v))) + r'\right) \end{array}\right), \tag{13}$$

$$N=\frac{1}{A}\left[\begin{array}{c} k_3r^2r'(k_1k_3\sin v-k_1\cos v)- \\ (1-r(k_1\cos(v)+k_2\sin(v)))(r^2k_3^2+rr'k_3') \\ +k_2(1-r(k_1\cos(v)+k_2\sin(v)))^2\left(r'\sin v+r\cos v\right) \end{array}\right] \tag{14}$$

In this study, we consider according to the lines of curvature in terms of natural parametrization of DGTs. Thus, we check whether s and v parameter curves of a DGT are its lines of curvature.

Theorem 2.1. The parameter curves of a DGT are lines of curvature if and only if k_3 vanishes.

Proof: If the k_3 vanishes, then by substituting $k_3=0$ into the above expressions for M and F , we obtain $M=F=0$. To show the "only if" part of the theorem, assume the DGT parameter curves are lines of curvature. Then from the Theorem 1.1., we have $M=F=0$. But since $F=r^2k_3=0$ and by definition $u>0$, it follows that $k_3=0$.

Parameter curve is also a geodesic. Thus, we are interested in when DGT parameter curves are either lines of curvature or geodesics. We only determine for the DGT v- parameter curves.

Theorem 2.2. The non-linear cross-sections of a DGT are geodesics if and only if the DGT is either a ZDGT(zero gaussian directional generalized tube) or a CDGT(constant cross section directional generalized tubes).

Proof: The osculating plane of a DGT cross-section at some point $\alpha(s)$ on the DGT axis is always perpendicular to $t(s)$, the tangent at $\alpha(s)$, by construction. But then, for a point $\psi^r(s,v)$ on the DGT surface, the osculating plane of the cross-section curve is perpendicular to the tangent plane exactly when $t(s)$ is perpendicular to $U(s,v)$. Since $U_t=\langle U,t\rangle=rr'k_3=0$, $r>0$, it follows from the geodesic theorem that $U_t=0$ exactly when either $r'=0$ or $k_3=0$.

Theorem 2.3. The non-linear cross-sections of a DGT are lines of curvature if the DGT is either a ZDGT or a CDGT.

Proof: This follows directly from the fact that a geodesic is planar if and only if it is a line of curvature.

Corollary 2.1. There seems to be a relationship between the parametric curves and the set of intrinsic directions as defined for both ZDGTs and CDGTs. This relationship implies that, in some sense, they are "natural subclasses" of DGTs, and should be studied further with respect to their surface.

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Detecting Similarities of Bézier Curves for the Groups $LSim(E_2), LSim^+(E_2)$

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Abstract: In this paper, for linear similarity groups, global invariants of plane Bézier curves (plane polynomial curves) in E_2 are introduced. Using complex numbers and the global G-invariants of a plane Bézier curve(a plane polynomial curve), for given two plane Bézier curves (plane polynomial curves) $x(t)$ and $y(t)$, evident forms of all transformations $g \in G$, carrying $x(t)$ to $y(t)$, are obtained.

Keywords: Polynomial curve, Bézier curve, Invariant, Linear similarity group.

1 Introduction

Let E_2 be the 2-dimensional Euclidean space, $G = LSim(E_2)$ be the group of all linear similarities of E_2 and $G = LSim^+(E_2)$ be the group of all orientation-preserving linear similarities of E_2.

In [1], using local differential invariants and Frenet frames of two curves, uniqueness and existence theorems for a curve determined up to a direct similarity of E_2.

For the group $Sim^+(n)$, this theorem shows that a necessary and sufficient conditions for two curves in E_n to be equivalent is that they have same shape curvatures and the other specially conditions.

The complete systems of global G-invariants of a path and a curve in E_2 are obtained. For the groups G, existence and uniqueness theorems for a curve and a path are given in the terms of global G-invariants of a path and a curve in [2].

$LSim(2)$-equivalence of two Bézier curves without using differential invariants of Bézier curves in the terms of control invariants of Bézier curves is proved in [3, 4].

In this work, starting from the ideas in [2–4, 8–11], we address how to compute explicitly an linear similarity transformation which carrying a Bézier curve into another Bézier curve in the terms of control invariants of a Bézier curve for the groups $LSim(E_2)$ and $LSim^+(E_2)$ without using differential invariants of Bézier curves.

2 Preliminaries

The following definitions and propositions are known in [2].

Let R be the field of real numbers and $\mathbb{C}$ be the field of complex numbers. The multiplication in $\mathbb{C}$ has the form $(a_1 + ia_2)(b_1 + ib_2) = (a_1b_1 - a_2b_2) + i(a_1b_2 + a_2b_1)$. We will consider element $a = a_1 + ia_2$ also in the form $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. For $a = a_1 + ia_2$, denote by P_a the matrix $\begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$ and consider P_a also as the transformation $P_a : \mathbb{C} \to \mathbb{C}$, where $P_a b = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 - a_2b_2 \\ a_1b_2 + a_2b_1 \end{pmatrix}$ for all $b = b_1 + ib_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{C}$. Then we have the equality

$$ab = P_a b. \tag{1}$$

for all $a, b \in \mathbb{C}$. Let $P(\mathbb{C})$ denote the set of all matrices P_a, where $a \in \mathbb{C}$. We consider on $P(\mathbb{C})$ the following standard matrix operations: the component-wise addition, a scalar multiplication and the multiplication of matrices. Then $P(\mathbb{C})$ is a field, where the unit element is the unit matrix. The following Propositions are known.

Proposition 1. *The mapping $P : \mathbb{C} \to P(\mathbb{C})$, where $P : a \to P_a$ for all $a \in \mathbb{C}$, is an isomorphism of fields.*

For vectors $a = a_1 + ia_2, b = b_1 + ib_2 \in \mathbb{C}$, we put $< a, b >= a_1b_1 + a_2b_2$. Then $< a, b >$ is a bilinear form on E_2 and $< a, a >= a_1^2 + a_2^2$ is a quadratic form on E_2. Put $Q(a) =< a, a >$. We consider the field $\mathbb{C}$ also as the two-dimensional Euclidean space E_2 with the scalar product $< a, b >$. Then $\|a\| = |a| = \sqrt{Q(a)}, \forall a \in \mathbb{C}$.

Proposition 2. *(i) Equalities $Q(a) = det(P_a)$, $Q(ab) = Q(a)Q(b)$, $|ab| = |a|\,|b|$, $Q(a) = det(P_a) =$ hold for all $a, b \in \mathbb{C}$.*
(ii) Let $a = a_1 + ia_2 \in \mathbb{C}^*$. Then $det(P_a) = Q(a) > 0$.

An endomorphism ψ of a vector space $\mathbb{C}$ is called an involution of the field $\mathbb{C}$ if $\psi(\psi(a)) = a$ and $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in \mathbb{C}$. For an element $a = a_1 + ia_2 \in \mathbb{C}$, we set $\overline{a} = a_1 - ia_2$.

Proposition 3. *The mapping $a \to \overline{a}$ is an involution of the field $\mathbb{C}$. In addition, for an arbitrary element $a = a_1 + ia_2 \in \mathbb{C}$, equalities $a + \overline{a} = 2a_1, < a, a >= a\overline{a} = a_1^2 + a_2^2 \in R$ hold.*

Proposition 4. *Let $x \in \mathbb{C}$. Then the element x^{-1} exists if and only if $Q(x) \neq 0$. In the case $Q(x) \neq 0$, equalities $x^{-1} = \frac{\overline{x}}{Q(x)}$ and $Q(x^{-1}) = \frac{1}{Q(x)}$ hold.*

Let $W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We will use W also for the writing of the element $\overline{z}$ in the form $\overline{z} = Wz$.

Proposition 5. *$Q(Wx) = Q(x)$ for all $x \in \mathbb{C}$ and $< Wx, Wy >=< x, y >$ for all $x, y \in \mathbb{C}$.*

Put $\mathbb{C}^* = \{z \in \mathbb{C} \mid Q(z) \neq 0\}$. $\mathbb{C}^*$ is a group with respect to the multiplication operation in the field $\mathbb{C}$. Let $a = a_1 + ia_2 \in \mathbb{C}^*$ that is $|a| \neq 0$. Put

$$P_a^+ = \begin{pmatrix} \frac{a_1}{|a|} & \frac{-a_2}{|a|} \\ \frac{a_2}{|a|} & \frac{a_1}{|a|} \end{pmatrix}.$$

Proposition 6. *Let $a = a_1 + ia_2 \in \mathbb{C}^*$. Then the equality $P_a = |a|\, P_a^+$ holds, where $P_a^+ \in SO(2)$.*

Put $S(\mathbb{C}^*) = \{z \in \mathbb{C} \mid Q(z) = 1\}$, $P(\mathbb{C}^*) = \{P_z \mid z \in \mathbb{C}^*\}$ and $P(S(\mathbb{C}^*)) = \{P_z \mid z \in S(\mathbb{C}^*)\}$. $S(\mathbb{C}^*)$ is a subgroup of the group $\mathbb{C}^*$ and $S(\mathbb{C}^*) = \left\{e^{i\varphi} \mid \varphi \in R\right\}$. Denote the set of all matrices $\{gW \mid g \in P(\mathbb{C}^*)\}$ by $P(\mathbb{C}^*)W$, where gW is the multiplication of matrices g and W.

Theorem 1. *(see [7, p.172]) The following equalities are hold:*

(i) $LSim^+(E_2) = \{P_a : E_2 \to E_2 | a \in \mathbb{C}^*\} = P(\mathbb{C}^*)$.
(ii) $LSim^-(E_2) = \{P_aW : E_2 \to E_2 | a \in \mathbb{C}^*\} = P(\mathbb{C}^*)W$.
(iii) $LSim(E_2) = LSim^+(E_2) \cup LSim^-(E_2)$.

Proposition 7. *(i) Let $u, v \in \mathbb{C}$. Assume that $Q(u) \neq 0$. Then the element vu^{-1} exists, the following equalities hold:*

$$vu^{-1} = \frac{< u, v >}{Q(u)} + i\frac{[uv]}{Q(u)}$$

and

$$P_{vu^{-1}} = \begin{pmatrix} \frac{<u,v>}{Q(u)} & -\frac{[uv]}{Q(u)} \\ \frac{[uv]}{Q(u)} & \frac{<u,v>}{Q(u)} \end{pmatrix}. \tag{2}$$

(ii) Assume that $Q(u) \neq 0$. Then $det(P_{vu^{-1}}) = (\frac{<u,v>}{Q(u)})^2 + (\frac{[uv]}{Q(u)})^2 \neq 0$ if and only if $Q(v) \neq 0$.

3 Control invariants of planar Bézier curve

A planar Bézier curve is a parametric curve(or a I-path, where $I = [0, 1]$) whose points $x(t)$ are defined by $x(t) = \sum_{i=0}^{m} p_i B_{i,m}(t)$, where the $p_i \in E_2$ are control points and $B_{i,m}\,(t)$ are Bernstein basis polynomials.(for more details, see [6].)

A planar polynomial curve is a parametric curve whose points $x(t)$ are defined by $x(t) = \sum_{i=0}^{m} a_i t^i$, where the $a_i \in E_2$ are monomial control points.(for more details, see [6, p.181].)
All polynomial curves can be represented in Bézier form. The following lemma is given in [6, p.181].

Lemma 1. *The following equalities*

$$a_i \quad = \textstyle\sum_{j=0}^{i} (-1)^{i-j} \frac{m!}{i!(m-i)!} \frac{i!}{j!(i-j)!} b_j \tag{3}$$

hold for all $i = 1, 2, \ldots, m$ and $i \geq j$.

Let $G = LSim(E_2), LSim^+(E_2)$.

Definition 1. *(see [5]) A function $f(z_0, z_1, \ldots, z_m)$ of points $z_0, z_1, \ldots, z_m$ in E_2 will be called G-invariant if $f(Fz_0, Fz_1, \ldots, Fz_m) = f(z_0, z_1, \ldots, z_m)$ for all $F \in G$.*
A G-invariant function $f(b_0, b_1, \ldots, b_m)$ of control points $b_0, b_1, \ldots, b_m$ of a Bézier curve $x(t) = \sum_{j=0}^{m} b_j B_{j,m}(t)$ will be called a control G-invariant of $x(t)$, where $B_{j,m}(t)$ are Bernstein basis polynomials. A G-invariant function $f(a_0, a_1, \ldots, a_m)$ of monomial control points $a_0, a_1, \ldots, a_m$ of a polynomial curve $x(t) = \sum_{j=0}^{m} a_j t^j$ will be called a monomial G-invariant of $x(t)$.

Definition 2. *(see [5]) Bézier curves $x(t)$ and $y(t)$ in E_2 will be called G -similar and written $x \overset{G}{\sim} y$ if there exists $F \in G$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$.*

Since Bézier curves can be introduced by control points, we will define the problem of G-similarity of points in E_2.

Definition 3. *(see [5]) m-uples $\{z_1, z_2, \ldots, z_m\}$ and $\{w_1, w_2, \ldots, w_m\}$ of points in E_2 will be called G-similar and written by $\{z_1, z_2, \ldots, z_m\} \overset{G}{\sim} \{w_1, w_2, \ldots, w_m\}$ if there exists $F \in G$ such that $w_j = Fz_j$ for all $j = 1, 2, \ldots, m$.*

Let u, v be points in E_2. We denote the the matrix of column-vectors u, v by $\|u\ v\|$ and its determinant by $[u\ v]$.

Example 1. *Since $\frac{<g(u),g(v)>}{<g(u),g(u)>} = \frac{<u,v>}{<u,u>}$ for all $g \in LSim(E_2)$, we obtain that the function $\frac{<u,v)>}{<u,u>}$ of points $u, v \in E_2$ is $LSim(E_2)$-invariant. Similarly, the function $\frac{[u\ v]}{<u,u>}$ is $LSim^+(E_2)$-invariant.*

Example 2. *Let $x(t)$ and $y(t)$ be Bézier curves of degrees of m and k, respectively. Assume that $x \overset{LSim(E_2)}{\sim} y$. Then $m = k$ that is the degree of a Bézier curve $x(t)$ is $LSim(E_2)$-invariant.*

4 Similarity of planar Bézier curves

Theorem 2. *Let $x(t) = \sum_{j=0}^{m} a_j t^j = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} c_j t^j = \sum_{j=0}^{m} q_j B_{j,m}(t)$ be Bézier curves in E_2 of degree m, where $m \geq 1$. Then following conditions are equivalent:*

(i) $x(t) \overset{LSim(E_2)}{\sim} y(t)$
(ii) $\{p_0, p_1, \ldots, p_m\} \overset{LSim(E_2)}{\sim} \{q_0, q_1, \ldots, q_m\}$
(iv) $\{a_0, a_1, \ldots, a_m\} \overset{LSim(E_2)}{\sim} \{c_0, c_1, \ldots, c_m\}$

Theorem 3. *Let $x(t) = \sum_{j=0}^{m} a_j t^j = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} c_j t^j = \sum_{j=0}^{m} q_j B_{j,m}(t)$ be Bézier curves in E_2 of degree m, where $m \geq 1$. Then following conditions are equivalent:*

(i) $x(t) \overset{LSim^+(E_2)}{\sim} y(t)$
(ii) $\{p_0, p_1, \ldots, p_m\} \overset{LSim^+(E_2)}{\sim} \{q_0, q_1, \ldots, q_m\}$
(iv) $\{a_0, a_1, \ldots, a_m\} \overset{LSim^+(E_2)}{\sim} \{c_0, c_1, \ldots, c_m\}$

Remark 1. *In Theorems 2 and 3, we have considered the problem of G-similarity of polynomial curves in the case $m \geq 1$. For the case $m = 0$, the problem of G-similarity of polynomial curves $x(t) = a_0$ and $y(t) = c_0$ reduces to the problem of G-similarity of points a_0 and c_0 in E_2. For the groups $G = LSim(E_2), LSim^+(E_2)$, it is obvious that $a_0 \overset{G}{\sim} c_0$ for all a_0 and c_0 in E_2. In what follows, $m \geq 1$. The case $m = 0$ is easily considered.*

Theorem 4. *Let $A = \{a_0, \ldots, a_m\}$ and $C = \{c_0, \ldots, c_m\}$ be two systems in E_2 such that $a_k \neq 0$, $c_k \neq 0$, where $k \in \{0, 1, \ldots, m\}$. Then, A and C are $LSim^+(E_2)$-similar if and only if*

$$\begin{cases} \dfrac{< a_i, a_k >}{< a_k, a_k >} = \dfrac{< c_i, c_k >}{< c_k, c_k >}, \\ \dfrac{[a_i\ a_k]}{< a_k, a_k >} = \dfrac{[c_i\ c_k]}{< c_k, c_k >} \end{cases} \tag{4}$$

for all $i = 0, 1, \ldots k-1, k+1, k+2, \ldots, m$. Moreover, there exists the unique element $F \in LSim^+(E_2)$ such that $c_j = Fa_j$ for all $j = 0, 1, \ldots, m$, where the matrix F can be written as

$$F = \begin{pmatrix} \frac{<a_k,c_k>}{Q(a_k)} & -\frac{[a_k c_k]}{Q(a_k)} \\ \frac{[a_k c_k]}{Q(a_k)} & \frac{<a_k,c_k>}{Q(a_k)} \end{pmatrix}. \tag{5}$$

Theorem 5. *Let $A = \{a_0, \ldots, a_m\}$ and $C = \{c_0, \ldots, c_m\}$ be two systems in E_2 such that $a_k \neq 0$, $c_k \neq 0$ for $k \in \{0, 1, \ldots, m\}$ and $rankA = rankC = 1$. Then, A and C are $LSim(E_2)$-similar if and only if*

$$\frac{< a_i, a_k >}{< a_k, a_k >} = \frac{< c_i, c_k >}{< c_k, c_k >} \tag{6}$$

for all $i = 0, 1, \ldots k-1, k+1, k+2, \ldots, m$. Moreover, there exists the unique element $H \in LSim(E_2)$ such that $c_j = Ha_j$ for all $j = 0, 1, \ldots, m$, where the matrix H has the form (5).

Remark 2. *Let* $A = \{a_0, \ldots, a_m\}$*. In the case* $rankA = 2$*, denote by* $indexA$ *smallest of* s*,* $0 \le s \le m$*, such that* $a_s \neq \lambda a_k$ *for all* $\lambda \in E_2$ *and* $a_k \neq 0$*. The number* $indexA$ *is* $LSim(E_2)$*-invariant.*

Theorem 6. *Let* $A = \{a_0, \ldots, a_m\}$ *and* $C = \{c_0, \ldots, c_m\}$ *be two systems in* E_2 *such that* $a_k \neq 0$*,* $c_k \neq 0$ *,* $rankA = rankC = 2$ *and* $indexA = indexC = l$ *for* $k, l \in \{0, 1, \ldots, m\}$*,* $l \neq k$*. Then,* A *and* C *are* $LSim(E_2)$*-similar if and only if*

$$\begin{cases} \dfrac{< a_i, a_k >}{< a_k, a_k >} = \dfrac{< c_i, c_k >}{< c_k, c_k >} \\ \left(\dfrac{[a_l\ a_k]}{< a_k, a_k >}\right)^2 = \left(\dfrac{[c_l\ c_k]}{< c_k, c_k >}\right)^2 \\ \dfrac{[a_i\ a_k]}{[a_l\ a_k]} = \dfrac{[c_i\ c_k]}{[c_l\ c_k]} \end{cases} \tag{7}$$

for all $i = 0, 1, \ldots, \ldots, m$*,* $i \neq k$ *and* $i \neq l$*. Moreover, there exists the unique element* $M \in LSim(E_2)$ *such that* $c_j = Ma_j$ *for all* $j = 1, \ldots, m$*. Then there exist following cases:*

(i) *In the case* $\frac{[a_l\ a_k]}{<a_k, a_k>} = \frac{[c_l\ c_k]}{<c_k, c_k>}$*, the matrix* $M \in LSim^+(E_2)$ *and it has the form* (5)*.*
(ii) *In the case* $\frac{[a_l\ a_k]}{<a_k, a_k>} = -\frac{[c_l\ c_k]}{<c_k, c_k>}$*, the matrix* $MW \in LSim(E_2)$ *and it can be represented by*

$$M = \begin{pmatrix} \frac{<Wa_k, c_k>}{Q(a_k)} & -\frac{[Wa_k c_k]}{Q(a_k)} \\ \frac{[Wa_k c_k]}{Q(a_k)} & \frac{<Wa_k, c_k>}{Q(a_k)} \end{pmatrix}. \tag{8}$$

Theorem 7. (i) *Let* $x(t) = \sum_{j=0}^{m} a_j t^j$ *and* $y(t) = \sum_{j=0}^{m} c_j t^j$ *be two polynomial curves in* E_2 *of degree* m*,where* $m \ge 1$ *such that* $x(t) \overset{LSim^+(E_2)}{\sim} y(t)$*. Then, the equalities* (4) *in Theorem* 4 *hold.*
(ii) *Conversely, if* $x(t) = \sum_{j=0}^{m} a_j t^j$ *and* $y(t) = \sum_{j=0}^{m} c_j t^j$ *are two polynomial curves in* E_2 *of degree* m*, where* $m \ge 1$ *such that the equalities* (4) *in Theorem* 4 *hold, then* $x(t) \overset{LSim^+(E_2)}{\sim} y(t)$*. Moreover, there exists the unique* $F \in LSim^+(E_2)$ *such that* $y(t) = Fx(t)$ *for all* $t \in [0, 1]$ *and* F *has the form* (5)*.*

Theorem 8. (i) *Let* $x(t) = \sum_{j=0}^{m} a_j t^j$ *and* $y(t) = \sum_{j=0}^{m} c_j t^j$ *be two polynomial curves in* E_2 *of degree* m*,where* $m \ge 1$ *such that* $x(t) \overset{LSim(E_2)}{\sim} y(t)$*. Then, the equalities* (7) *in Theorem* 6 *hold.*
(ii) *Conversely, if* $x(t) = \sum_{j=0}^{m} a_j t^j$ *and* $y(t) = \sum_{j=0}^{m} c_j t^j$ *are two polynomial curves in* E_2 *of degree* m*, where* $m \ge 1$ *such that the equalities* (7) *in Theorem* 6 *hold, then* $x(t) \overset{LSim(E_2)}{\sim} y(t)$*. Moreover, there exists the unique* $F \in LSim(E_2)$ *such that* $y(t) = Fx(t)$ *for all* $t \in [0, 1]$ *. Then,*
(a) *In the case* $\frac{[a_l\ a_k]}{<a_k, a_k>} = \frac{[c_l\ c_k]}{<c_k, c_k>}$*,* F *has the form* (5)*.*
(b) *In the case* $\frac{[a_l\ a_k]}{<a_k, a_k>} = -\frac{[c_l\ c_k]}{<c_k, c_k>}$*,* F *has the form* (8)*.*

Theorem 9. (i) *Let* $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ *and* $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ *be two Bézier curves in* E_2 *of degree* m*, where* $m \ge 1$ *such that* $x(t) \overset{LSim^+(E_2)}{\sim} y(t)$*. Then by Lemma* 1*, the equalities* (4) *in Theorem* 4 *hold.*
(ii) *Conversely, if Let* $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ *and* $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ *be two Bézier curves in* E_2 *of degree* m*, where* $m \ge 1$ *such that the equalities* (4) *in Theorem* 4 *and Lemma* 1 *hold, then* $x(t) \overset{LSim^+(E_2)}{\sim} y(t)$*. Moreover, there exists the unique* $F \in LSim^+(E_2)$ *such that* $y(t) = Fx(t)$ *for all* $t \in [0, 1]$ *and* F *in the terms of the equalities given in Lemma* 1 *has the form* (5) *.*

Theorem 10. (i) *Let* $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ *and* $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ *be two Bézier curves in* E_2 *of degree* m*, where* $m \ge 1$ *such that* $x(t) \overset{LSim(E_2)}{\sim} y(t)$*. Then by Lemma* 1*, the equalities* (7) *in Theorem* 6 *hold.*
(ii) *Conversely, if* $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ *and* $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ *be two Bézier curves in* E_2 *of degree* m*, where* $m \ge 1$ *such that the equalities* (7) *in Theorem* 6 *and Lemma*1 *hold, then* $x(t) \overset{LSim(E_2)}{\sim} y(t)$*. Moreover, there exists the unique* $F \in LSim(E_2)$ *such that* $y(t) = Fx(t)$ *for all* $t \in [0, 1]$ *. Then,*
(a) *In the case* $\frac{[p_l\ p_k]}{<p_k, p_k>} = \frac{[q_l\ q_k]}{<q_k, q_k>}$*,* F *in the terms of the equalities given in Lemma* 1 *has the form* (5)*.*
(b) *In the case* $\frac{[p_l\ p_k]}{<p_k, p_k>} = -\frac{[q_l\ q_k]}{<q_k, q_k>}$*,* F *in the terms of the equalities given in Lemma* 1 *has the form* (8)*.*

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5 References

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Involute Curves in 4-Dimensional Galilean Space G4

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Abstract: In this paper, we define the (0,2)-involute of a given curve in 4-dimensional Galilean space, and for the curve with a generalized involute, the necessary and sufficient condition is obtained.

Keywords: Frenet formula, Galilean space, Involute curve. (Please, alphabetical order and at lease one keyword)

1 Introduction

Galilean geometry is one of the nine projective space geometries which was discussed by Cayley-Klein at the beginning of 20th century. After that, the curvature-related studies were maintained and the curve properties in Galilean space were studied in $[1, 2]$. The involute of a given curve in Euclidean space is a famous concept, whereas the idea of an involute string is due to C. Huygens, who is well known for his job in optics and who found involutes while attempting to construct a more accurate clock in 1968 $[3, 4]$. The theories of the Involute and Evolute Curves in Minkowski Space are extensively studied in $[5, 6, 7]$.

In classical differential geometry, an evolute of a curve is defined as the locus of the centers of curvatures of the curve, which is the envelope of the normal of the given curve. While an Involute of a specified curve is a curve in which all tangents of a specified curve are normal $[3, 8, 9, 10]$.

In $[11]$, the author created Frenet-Serret curve frame in the Galilean 4-space and acquired constant ratio curves in Galilean 4-space. Aydin and Ergüt constructed equiform differential geometry of curves and obtained the angle between the equiform Frenet vectors and their derivatives in G_4 $[12]$.

In $[13, 14]$, the authors studied some curves of Galilean geometry in both plane and space, they obtained the characterization of slant helices in 3- dimensional Galilean space G_3.

2 Preliminaries

The Galilean space can be described as a three dimentional complex projective space with absolute figures $\{m, l, p_1, p_2\}$ which consists of a real plane m, a real line $l \subset m$ and two complex conjugate points $p_1, p_2 \in l$.

The study of plane-parallel motion mechanics decreases the study of a 3-space geometry with $\{x, y, t\}$ coordinates by the motion formula $[2]$. This geometry can be described as geometry of Galilean 3-space. It is clarified in $[2]$ that four dimensional Galilean space, which studies all invariant features under object movements in space is even more complicated.

Moreover, it is indicated that this geometry can be more accurately defined as studying those four dimensional space characteristics with co-ordinates that are invariant under the general Galilean transformations as follows:

$$\begin{aligned} x^{'} &= (\cos\theta\cos\phi - \cos\gamma\sin\theta\sin\phi)\,x + (\sin\theta\cos\phi - \cos\gamma\cos\theta\sin\phi)\,y \\ &\quad + (\sin\gamma\sin\phi)\,z + (v\cos\beta_1)\,t + a \\ y^{'} &= -(\cos\theta\sin\phi + \cos\gamma\sin\theta\cos\phi)\,x + (-\sin\theta\sin\phi + \cos\gamma\cos\theta\cos\phi)\,y \\ &\quad + (\sin\gamma\cos\phi)\,z + (v\cos\beta_2)\,t + b \\ z^{'} &= (\sin\gamma\sin\theta)\,x - (\sin\gamma\cos\theta)\,y + (\cos\gamma)\,z + (v\cos\beta_3)\,t + c \\ t^{'} &= t + d \end{aligned}$$

with $\cos^2\beta_1 + \cos^2\beta_2 + \cos^2\beta_3 = 1$

The following chapter provides some basic characteristics of curves in Galilean 4-space for the uses of the conditions.

A curve $\alpha : I \to G_4$, $I \subset R$ can be given as

$$\alpha(t) = (x_1(t), x_2(t), x_3(t), x_4(t)),$$

where $x_i(t) \in C^4$ i=1,2,3,4 and $t \in I$. Let α be a curve in G_4, which is parameterized by arclength $t = s$, and its coordinate form can be written as

$$\alpha(s) = (s, x_2(s), x_3(s), x_4(s)).$$

In affine coordinates the Galilean inner product between two points $P_i = (x_{i1}, x_{i2}, x_{i3}, x_{i4})$, $i = 1, 2$, is defined by

$$\begin{aligned} g(P_1, P_2) &= |x_{21} - x_{11}|, \ if \ \ x_{21} \neq x_{11} \\ g(P_1, P_2) &= \sqrt{(x_{22} - x_{12})^2 + (x_{23} - x_{13})^2 + (x_{24} - x_{14})^2}, \ if \ \ x_{21} = x_{11} \end{aligned}$$

For the vectors $p = (p_1, p_2, p_3, p_4)$, $q = (q_1, q_2, q_3, q_4)$ and $r = (r_1, r_2, r_3, r_4)$, Galilean cross product in G_4 is defined as follows:

$$p \wedge q \wedge r = \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \\ r_1 & r_2 & r_3 & r_4 \end{vmatrix}$$

where e_i are the standard basis vectors.

The notation $< x, y >_G$ we use in this paper denotes the inner product of the vectors x, y in Galilean space.

Let $\alpha(s) = (s, x_2(s), x_3(s), x_4(s))$ be a curve parameterized by arclength s in G_4, the Frenet formulas can written as

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{2.1}$$

where T, N, B_1, B_2 are mutually orthogonal vector fields which the following equations hold

$$\begin{aligned} &< T, T >_G = < N, N >_G = < B_1, B_1 >_G = < B_2, B_2 >_G = 1 \\ &< T, N >_G = < T, B_1 >_G = < T, B_2 >_G = < N, B_1 >_G = < N, B_2 >_G = < B_1, B_2 >_G = 0. \end{aligned}$$

We use some terms in this journal. The plane spanned by { T, B_1 } is called (0,2)-tangent plane at any point of the curve ϕ. The plane spanning { N, B_2 } is called the (1,3)-normal plane of ϕ.

Let $\phi : I \to G_4$ and $\phi^* : I \to G_4$, $I \subset R$ be two regular parameterized curves in Galilean 4-space G_4. Let $s^* = f(s)$ be an arc-length parameter of ϕ^*. $\forall$ $s \in I$, if the $(0, 2)$-tangent plane at $\phi(s)$ of ϕ overlaps with the $(1, 3)$-normal plane of ϕ^*at $\phi^*(s)$, then ϕ^* is said to be $(0, 2)$-involute curve of ϕ in G_4 while ϕ is called $(1, 3)$-evolute curve of ϕ^* in G_4.

3 The (0,2)-involute curve in a Galilean 4-space G_4

In this chapter, we investigate the existence and representation of the (0,2)-involute curve in Galilean 4-space.

Let $\phi : I \subset R \to G_4$ be a regular parameterized curve, and k_1, k_2 and k_3 to be its curvatures $k_i \neq 0$, and let $\phi^* : I \subset R \to G_4$ be a $(0, 2)$-involute curve of ϕ. Donate $\{T^*, N^*, B_1^*, B_2^*\}$ to be the Frenet Frame along ϕ^* and k_1^*, k_2^* and k_3^* to be the curvatures of ϕ^*. Then

$$\begin{aligned} span\{T, B_1\} &= span\{N^*, B_2^*\} \\ span\{N, B_2\} &= span\{T^*, B_1^*\} \end{aligned} \tag{3.1}$$

and

$$\left\langle T^*, T \right\rangle = 0.$$

Moreover, α^* can be expressed as

$$\phi^*(s) = \phi(s) + a(s)T(s) + b(s)B_1(s) \tag{3.2}$$

where $a, b \in C^\infty$ functions on I.

By differentiating (3.2) with respect to s and using (2.1)

$$\phi^{*'}(s) = \phi'(s) + a'(s)T(s) + a(s)T'(s) + b'(s)B_1 + b(s)B_1'(s) \tag{3.3}$$

$$f'T^* = (1 + a')\,T + (ak_1 - bk_2)\,N + b'B_1 + bk_3B_2.$$

So by taking dot product on both-sides of (3.3) with T and B_1

$$
\begin{aligned}
\left\langle f^{'}T^{^{*}},T\right\rangle &= \left\langle\left(1+a'\right)T+\left(ak_1-bk_2\right)N+b^{'}B_1+bk_3B_2,T\right\rangle\\
0 &= 1+a'\\
a' &= -1
\end{aligned}
$$

integrate both sides of the above equation

$$
\begin{aligned}
\int\frac{da}{ds}ds &= -\int ds\\
a &= a_0-s,\quad (a_0 \text{ is a constant})
\end{aligned}
$$

and

$$
\begin{aligned}
\left\langle f^{'}T^{^{*}},B_1\right\rangle &= \left\langle\left(1+a'\right)T+\left(ak_1-bk_2\right)N+b^{'}B_1+bk_3B_2,B_1\right\rangle\\
0 &= b^{'},
\end{aligned}
$$

which implies that b is a constant, thus (3.3) turns to

$$
f^{'}T^{^{*}}=\left(ak_1-bk_2\right)N+bk_3B_2, \tag{3.4}
$$

let

$$
\delta=\frac{\left(ak_1-bk_2\right)}{f'}\quad\text{and}\quad\gamma=\frac{bk_3}{f'}, \tag{3.5}
$$

therefore

$$
\begin{aligned}
T^{^{*}} &= \delta N+\gamma B_2, \qquad (3.6)\\
\delta^2+\gamma^2 &= 1.
\end{aligned}
$$

Case 1
$b\neq 0$, in this case $\gamma=\frac{bk_3}{f'}\neq 0$. Denote $\frac{\delta}{\gamma}=t_1$, then $\delta=\gamma t_1$ and

$$
f^{'}=\frac{bk_3}{\gamma}=b\gamma^{-1}k_3 \tag{3.7}
$$

From (3.5) and (3.7)

$$
\begin{aligned}
\delta &= \frac{\left(ak_1-bk_2\right)}{f'}\\
bt_1k_3 &= ak_1-bk_2. \qquad (3.8)
\end{aligned}
$$

From (3.6)

$$
\begin{aligned}
\delta^2+\gamma^2 &= 1\\
\gamma^2 &= \frac{1}{t_1^2+1}. \qquad (3.9)
\end{aligned}
$$

Differentiate (3.6) with respect to s and using (2.1)

$$
\begin{aligned}
T^{^{*'}} &= \delta^{'}N+\delta N^{'}+\gamma^{'}B_2+\gamma B_2^{'}\\
f^{'}k_1^{^{*}}N^{^{*}} &= \delta^{'}N+\left(\delta k_2-\gamma k_3\right)B_1+\gamma^{'}B_2 \qquad (3.10)
\end{aligned}
$$

So by taking dot product on both-sides of (3.10) with N and B_2

$$
\begin{aligned}
\left\langle f^{'}k_1^{^{*}}N^{^{*}},N\right\rangle &= \left\langle\delta^{'}N+\left(\delta k_2-\gamma k_3\right)B_1+\gamma^{'}B_2,N\right\rangle\\
0 &= \delta^{'}\\
\left\langle f^{'}k_1^{^{*}}N^{^{*}},B_2\right\rangle &= \left\langle\delta^{'}N+\left(\delta k_2-\gamma k_3\right)B_1+\gamma^{'}B_2,B_2\right\rangle\\
0 &= \gamma^{'}
\end{aligned}
$$

which implies that γ and δ are constants, thus (3.10) turns to

$$f^{'}\overset{*}{k_1}\overset{*}{N} = (\delta k_2 - \gamma k_3)\, B_1. \tag{3.11}$$

We suppose that

$$\begin{aligned} f^{'}\overset{*}{k_1} &= \delta k_2 - \gamma k_3, \\ \overset{*}{N} &= B_1. \end{aligned} \tag{3.12}$$

Differentiate (3.12) with respect to s

$$\begin{aligned} \overset{*'}{N} &= B_1^{'} \\ f^{'}\overset{*}{k_2}\overset{*}{B_1} &= -k_2 N + k_3 B_2. \end{aligned} \tag{3.13}$$

Let

$$c = \frac{-k_2}{f'\overset{*}{k_2}}, \quad e = \frac{k_3}{f'\overset{*}{k_2}}, \tag{3.14}$$

then (3.13) turns into

$$\begin{aligned} \overset{*}{B_1} &= cN + eB_2, \\ c^2 + e^2 &= 1. \end{aligned}$$

Let $\frac{c}{e} = t_2$, then $c = et_2$, from (3.14)

$$\begin{aligned} c &= \frac{-k_2}{f'\overset{*}{k_2}} \\ et_2 &= \frac{-k_2}{f'\overset{*}{k_2}} \\ \frac{k_3}{f'\overset{*}{k_2}} t_2 &= \frac{-k_2}{f'\overset{*}{k_2}} \end{aligned}$$

$$k_3 = \frac{-k_2}{t_2}, \tag{3.16}$$

from (3.15)

$$\begin{aligned} c^2 + e^2 &= 1 \\ e^2 &= \frac{1}{t_2^2 + 1}, \end{aligned} \tag{3.17}$$

from (3.8) and (3.16)

$$\begin{aligned} bt_1 k_3 &= ak_1 - bk_2 \\ bt_1\left(\frac{-k_2}{t_2}\right) &= ak_1 - bk_2 \\ \tau &= \frac{k_2}{k_1} = \frac{at_2}{b\,(t_2 - t_1)} \end{aligned}$$

$$\tau = \frac{\frac{a}{b} t_2}{(t_2 - t_1)}. \tag{3.18}$$

From (3.16)

$$k_2 = -k_3 t_2, \tag{3.19}$$

substitute (3.19) in (3.8)

$$bt_1 k_3 = ak_1 - bk_2$$

$$\frac{k_3}{k_1} = \frac{a}{(bt_1 - bt_2)} = -\frac{1}{t_2}\tau. \tag{3.20}$$

Let $\frac{\gamma}{e} = t_3$, then $\gamma = et_3$, from (3.14)

$$e = \frac{k_3}{f' k_2^*}$$

$$f' k_2^* = \frac{k_3 t_3}{\gamma} = e^{-1} k_3 \tag{3.21}$$

but

$$\begin{aligned} t_3 &= \frac{\gamma}{e} \\ t_3^2 &= \frac{\gamma^2}{e^2}, \end{aligned}$$

substitute (3.9) and (3.17) in the above equation

$$t_3^2 = \frac{\gamma^2}{e^2}$$

$$t_3^2 = \frac{1 + t_2^2}{1 + t_1^2}. \tag{3.22}$$

Differentiate (3.15) with respect to s and using (2.1)

$$B_1^{*\prime} = c' N + c N' + e' B_2 + e B_2'$$

$$f' k_3^* B_2^* = f' k_2^* N^* + c' N + (c k_2 - e k_3) B_1 + e' B_2. \tag{3.23}$$

So by taking inner product on both-sides of (3.23) with N and B_2

$$\begin{aligned} \left\langle f' k_3^* B_2^*, N \right\rangle &= \left\langle f' k_2^* N^* + c' N + (c k_2 - e k_3) B_1 + e' B_2, N \right\rangle \\ 0 &= c' \\ \left\langle f' k_3^* B_2^*, B_2 \right\rangle &= \left\langle f' k_2^* N^* + c' N + (c k_2 - e k_3) B_1 + e' B_2, B_2 \right\rangle \\ 0 &= e', \end{aligned}$$

which implies that c and e are constants, thus (3.23) turns to

$$f' k_3^* B_2^* = f' k_2^* N^* + (c k_2 - e k_3) B_1, \tag{3.24}$$

substitute (3.12) and (3.21) in (3.24)

$$f' k_3^* B_2^* = e^{-1} k_3 B_1 + (c k_2 - e k_3) B_1,$$

$$f' k_3^* B_2^* = c (t_2 k_3 + k_2) B_1, \tag{3.25}$$

we may choose that

$$B_2^* = c B_1 \tag{3.26}$$

$$f' k_3^* = (t_2 k_3 + k_2).$$

Summarising the above discussion, we obtain the following

Theorem 1. *Let $\phi : I \subset R \to G_4$ be a regular parameterized curve and k_1, k_2 and k_3 are its curvatures $k_i \neq 0$. If ϕ has the $(0,2)$-involute mate curve $\phi^*(s) = \phi(s) + (a_0 - s) T(s) + b(s) B_1(s)$ with $b \neq 0$, then k_1, k_2 and k_3 satisfy*

$$\frac{k_2}{k_1} = \tau, \quad \frac{k_3}{k_1} = -\frac{1}{t_2}\tau \ \text{ and } \ \tau = \frac{(a_0 - s)\, t_2}{b\,(t_2 - t_1)},$$

where a_0, b and t_2 are constants, moreover, the three curvatures of $\phi^*(s)$ are given by

$$k_1^* = -\frac{(t_1 t_2 + 1)}{b\,(1 + t_2^2)}, \quad k_2^* = \frac{t_3}{b} \ \text{ and } \ k_3^* = 0,$$

its frenet frame can be written as

$$\begin{aligned} T^{*} &= et_3\left(t_1 N + B_2\right), \\ N^{*} &= B_1 \\ B_1^{*} &= e\left(t_2 N + B_2\right) \\ B_2^{*} &= et_2 B_1. \end{aligned}$$

Case 2
$b = 0$, in this case (3.2) turns to

$$\phi^*(s) = \phi(s) + (a_0 - s)T(s). \tag{3.27}$$

Differentiate (3.27) with respect to s and using (2.1)

$$f^{'} T^{*} = (a_0 - s)k_1 N, \tag{3.28}$$

we suppose that

$$f^{'} = (s - a_0)k_1$$

$$T^{*} = -N. \tag{3.29}$$

Differentiate (3.29) with respect to s and using (2.1)

$$\begin{aligned} T^{*'} &= -N^{'} \\ f^{'} k_1^{*} N^{*} &= -k_2 B_1. \end{aligned}$$

Let

$$N^{*} = eB_1 \tag{3.30}$$

$$e = \frac{-k_2}{f' k_1^{*}}.$$

By differentiating (3.30) with respect to s we get

$$f^{'} k_2^{*} B_1^{*} = -ek_2 N + e^{'} B_1 + ek_3 B_2, \tag{3.31}$$

so by taking dot product on both-sides of (3.31) with B_1

$$\begin{aligned} \left\langle f^{'} k_2^{*} B_1^{*}, B_1 \right\rangle &= \left\langle -ek_2 N + e^{'} B_1 + ek_3 B_2, B_1 \right\rangle \\ 0 &= e^{'} \end{aligned}$$

which implies that e is a constant, therefore (3.31) turns to

$$f^{'} k_2^{*} B_1^{*} = -ek_2 N + ek_3 B_2, \tag{3.32}$$

let

$$p = \frac{ek_3}{f' k_2^{*}}, \quad q = \frac{-ek_2}{f' k_2^{*}} \tag{3.33}$$

$$\begin{aligned} B_1^{*} &= pB_2 + qN \\ p^2 + q^2 &= 1 \end{aligned}$$

from (3.33) we get

$$pk_2 + qk_3 = 0. \tag{3.34}$$

From (3.34) we get

$$\frac{k_3}{k_1} = -\frac{p}{q}\tau. \tag{3.35}$$

Let $\frac{e}{p}=t_1$, $e=pt_1$, from (3.33)

$$\begin{aligned} p &= \frac{ek_3}{f'k_2^{*}} \\ \frac{e}{t_1} &= \frac{ek_3}{f'k_2^{*}} \end{aligned}$$

$$f^{'}k_2^{*}=t_1k_3, \tag{3.36}$$

By differentiating (3.33) we get

$$f^{'}k_3^{*}B_2^{*}=f^{'}k_2^{*}N^{*}+q^{'}N+(qk_2-pk_3)B_1+p^{'}B_2, \tag{3.37}$$

so by taking dot product on both-sides of (3.37) with N and B_2

$$\begin{aligned} \left\langle f^{'}k_3^{*}B_2^{*},N\right\rangle &= \left\langle f^{'}k_2^{*}N^{*}+q^{'}N+(qk_2-pk_3)B_1+p^{'}B_2,N\right\rangle \\ 0 &= q^{'} \\ \left\langle f^{'}k_3^{*}B_2^{*},B_2\right\rangle &= \left\langle f^{'}k_2^{*}N^{*}+q^{'}N+(qk_2-pk_3)B_1+p^{'}B_2,B_2\right\rangle \\ 0 &= p^{'} \end{aligned}$$

which implies that p and q are constants, thus (3.37) turns to

$$f^{'}k_3^{*}B_2^{*}=f^{'}k_2^{*}N^{*}+(qk_2-pk_3)B_1, \tag{3.38}$$

by substituting (3.33) and (3.36) in (3.38) we have

$$f^{'}k_3^{*}B_2^{*}=k_1\left\{p\frac{k_3}{k_1}\left(t_1^2-1\right)+q\frac{k_2}{k_1}\right\}B_1=\frac{k_1\tau}{q}\left(1-e^2\right)B_1. \tag{3.39}$$

We suppose that

$$f^{'}k_3^{*}=k_1\tau\left(e^2-1\right) \tag{3.40}$$

$$B_2^{*}=-q^{-1}B_1$$

Summarising the above discussion, we obtain the following.

Theorem 2. *Let $\phi: I\subset R\to G_4$ be a regular parameterized curve and k_1, k_2 and k_3 are its curvatures $k_i\neq 0$. If ϕ has the $(0,2)$-involute mate curve $\phi^*(s)=\phi(s)+(a_{0-}s)T(s)$, then k_2 and k_3 satisfy*

$$pk_2+qk_3=0, \tag{3.41}$$

$$\frac{k_2}{k_1}=\tau,\quad \frac{k_3}{k_1}=-\frac{p}{q}\tau,$$

where a_0, p and q are given constants, moreover, the three curvatures of $\alpha^*(s)$ are given by

$$k_1^{*}=-\frac{k_2}{e(s-a_0)k_1},\quad k_2^{*}=\frac{-pt_1\tau}{q(s-a_0)}\ \text{ and }\ k_3^{*}=\frac{\tau\left(e^2-1\right)}{(s-a_0)},$$

its frenet frame can be written as

$$\begin{aligned} T^{*} &= -N \\ N^{*\prime} &= pt_1B_1 \\ B_1^{*} &= qN+pB_2 \\ B_2^{*} &= -q^{-1}B_1. \end{aligned}$$

Remark 1. *From theorems 1 and 2 we can see that the above two cases are quite different with each other.*

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On Minimal Surfaces in Galilean Space

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Abstract: In this paper, we investigated the minimal surfaces in three dimensional Galilean space $\mathbb{G}^3$. We showed that the condition of minimality of a surface area is locally equivalent to the mean curvature vector H vanishes identically. Then, we derived the necessary and sufficient conditions that the minimal surfaces have to satisfy in Galilean space.

Keywords: Minimal surfaces, Area of a surface, Galilean space.

1 Introduction

Minimal surfaces are one of the most interesting subject in mathematics. The study and computation of minimal surfaces has a long history [12]. Lagrange made the first investigation of the minimal surfaces by asking a simple question named as Plateau's problem which concerns with finding a surface of least area that spans a given fixed one-dimensional contour in three-dimensional Euclidean space [15]. Later G. Monge discovered that the condition for minimality of a surface leads to the condition that vanishing mean curvature, and therefore surfaces with $H = 0$ are called "minimal" [6].

The study of minimal surface has arised many interesting applications in other fields in science, such as the interface between crystals and organic matter in the skeletal element of sea urchins can be described as a minimal surface [11]. Moreover, the surfaces with vanishing mean curvature are also studied in other ambient spaces [16]. In Lorentz–Minkowski Space, a spacelike surface has maximal area if its mean curvature vanishes [10]. Moreover, the timelike minimal surfaces are investigated in [7, 8].

Simultaneity gives the characterization of the Galilean space. The simultaneous events (points), which are events that occur at the same time [9]. The Galilean space $\mathbb{G}^3$ has three-dimensions (x, y, z): two dimensions of Euclidean plane (y, z) and one dimension of time (x) [17]. The distance between non-simultaneous events $p = (x, y, z)$ and $q = (x_1, y_1, z_1)$ in $\mathbb{G}^3$ is defined by

$$d_1(p, q) = |x - x_1|\,.$$

On the other hand, the distance between two simultaneous events $p = (x, y, z)$ and $q_1 = (x, y_1, z_1)$ is defined by

$$d_2(p, q_1) = \sqrt{(y - y_1)^2 + (z - z_1)^2}.$$

Moreover, the Galilean space $\mathbb{G}^3$ can be considered as a Cayley-Klein space equipped with the projective metric of signature $(0, 0, +, +)$[5]. There are two types of plane in the Galilean space. Euclidean planes are in the following form $x = k$ $(y = z = 0)$ $k \in \mathbb{R}$ [4]. The other planes are isotropic [2, 3]. The non-isotropic vectors are in the following form $\mathbf{u} = (u_1, u_2, u_3)$, $u_1 \neq 0$. For isotropic vectors $u_1 = 0$ holds. More information about Galilean space can be found in [1, 13, 18]

Let $\mathbf{a} = (x, y, z)$ and $\mathbf{b} = (x_1, y_1, z_1)$ be vectors in the Galilean space. The scalar product $<, >$ is defined by

$$< \mathbf{a}, \mathbf{b} >= x_1 x. \tag{1}$$

In addition, when both of the vectors $\mathbf{p} = (0, y, z)$ and $\mathbf{q} = (0, y_1, z_1)$ are isotropic, the scalar product $<, >_1$ is given by

$$< \mathbf{p}, \mathbf{q} >_1= y y_1 + z z_1. \tag{2}$$

Let M be a surface in $\mathbb{G}^3$ given by parametrization

$$\varphi(v^1, v^2) = (x(v^1, v^2), y(v^1, v^2), z(v^1, v^2)).$$

The isotropic unit normal vector N is defined by

$$N = \frac{\varphi_1 \wedge \varphi_2}{w} \tag{3}$$

where the partial derivatives of the surface M with respect to v^1 and v^2 is denoted by φ_1 and φ_2, respectively and $w = \|\varphi_1 \wedge \varphi_2\|_1$ [14]. The first fundamental form of the surface is given by

$$I = I_1 + \epsilon I_2 \tag{4}$$

where $I_1 = g_{ij} dv^i dv^j$ and $I_2 = h_{ij} dv^i dv^j$. If $I_1 = 0$ then $\epsilon = 1$, in the other cases $\epsilon = 0$. The induced metrics h_{ij} and g_{ij} $(i, j = 1, 2)$ on the surface are given by

$$h_{ij} = \langle \varphi_i, \varphi_j \rangle_1 , \quad g_{ij} = \langle \varphi_i, \varphi_j \rangle . \tag{5}$$

The components of the inverse metric are given by

$$g^1 = \frac{x_2}{w}, \quad g^2 = -\frac{x_1}{w} \quad g^{ij} = g^i g^j \tag{6}$$

where the partial derivatives of the first component $x(v^1, v^2)$ of the surface with respect to v^1 and v^2 is denoted by x_1 and x_2, respectively.

In [14], the coefficients L_{ij} of the second fundamental form, the Gauss curvature K and mean curvature H of the surface are given by

$$L_{ij} = < \frac{\varphi_{,ij} x_{,1} - x_{,ij} \varphi_{,1}}{x_{,1}}, N >_1, K = \frac{\det L_{ij}}{w^2}, 2H = g^{ij} L_{ij}. \tag{7}$$

The partial derivatives of the normal vector is obtained by

$$N_i = -g^{jm} L_{ij} \varphi_m. \tag{8}$$

2 Minimal surfaces in galilean space

In this section, we will give a mathematical definition of the minimal surface in Galilean space $\mathbb{G}^3$. Since the minimal surfaces locally minimizes area, firstly we need to show that it is also meaningful in Galilean space. Similar to the three dimensional Euclidean space, the norm of the cross product measures the area spanned by two vectors in the three dimensional Galilean space [17]. Therefore we state the following definition.

Let M be the surface parametrized by $\varphi(u, v) = (x(u, v), y(u, v), z(u, v))$. We can see that $\|\varphi_u \wedge \varphi_v\|_1$ is the area of the parallelogram determined by φ_u and φ_v. Therefore, in Galilean space, the area of the surface can be obtained by

$$A(\varphi) = \int\int \|\varphi_u \wedge \varphi_v\|_1 \, dudv. \tag{9}$$

The minimal surface is the problem of minimizing $A(\varphi)$. To do this, let us consider a normal variation of the surface M in Galilean space. Let $t(u, v)$ be any smooth function on such that it vanishes on the boundary of the surface and N be the unit surface normal. For some small λ a normal variation of the surface M^σ can be parametrized by

$$\omega^\sigma(u, v) = \varphi(u, v) + \sigma t(u, v) N(u, v) \tag{10}$$

where $-\lambda < \sigma < \lambda$. This motivates the following theorem:

Theorem 2.1 Let $A(\sigma)$ be the area of the normal variation of the surface M in Galilean space. The critical point of the area of the normal variation M^σ is given by

$$A'(0) = -2 \int\int t(u, v) H \|\varphi_u \wedge \varphi_v\|_1 \, dudv \tag{11}$$

where prime denotes differentiation respect to σ.

Proof: The area $A(\sigma)$ of the surface M^σ is given by

$$A(\sigma) = \int\int \|\omega_u^\sigma \wedge \omega_v^\sigma\|_1 \, dudv. \tag{12}$$

On the other hand from (10), the partial derivatives of the normal variation M^σ are obtained as

$$\omega_u^\sigma = \varphi_u + \sigma t_u \mathbf{N} + \sigma t \mathbf{N}_v, \quad \omega_v^\sigma = \varphi_v + \sigma t_v \mathbf{N} + \sigma t \mathbf{N}_u.$$

By using $\mathbf{N}_u \wedge \mathbf{N}_v = 0$ we arrive at

$$\omega_u^\sigma \wedge \omega_v^\sigma = \varphi_u \wedge \varphi_v + t\sigma(\varphi_u \wedge \mathbf{N}_v + \mathbf{N}_u \wedge \varphi_v) + \sigma^2 t(t_u \mathbf{N} \wedge \mathbf{N}_v + t_v \mathbf{N}_u \wedge \mathbf{N}).$$

Taking norms on both sides of above equation, then differentiating with respect to σ, finally putting $\sigma = 0$ into the result gives

$$A'(0) = \int\int \frac{t \langle \varphi_u \wedge \mathbf{N}_v + \mathbf{N}_u \wedge \varphi_v, \varphi_u \wedge \varphi_v \rangle_1}{\|\varphi_{,1} \wedge \varphi_{,2}\|_1} dudv \tag{13}$$

where $\left.\frac{dA(\sigma)}{d\sigma}\right|_{\sigma=0} = A'(0).$

On the other hand, from (8) we have

$$\mathbf{N}_u \wedge \varphi_v = (-g^{11}L_{11} - g^{12}L_{12})\varphi_u \wedge \varphi_v \tag{14}$$

and

$$\varphi_u \wedge \mathbf{N}_v = \varphi_u \wedge \varphi_v(-g^{12}L_{12} - g^{22}L_{22}). \tag{15}$$

Substituting (14) and (15) into (13) gives

$$A'(0) = \int\int t(-2g^{12}L_{12} - g^{22}L_{22} - g^{11}L_{11}) \left\|\varphi_{,1} \wedge \varphi_{,2}\right\|_1 dudv.$$

From (7), we get

$$A'(0) = -2\int\int t(u,v)H \left\|\varphi_{,1} \wedge \varphi_{,2}\right\|_1 dudv$$

which completes the proof.

As a corollary, one deduces that a surface is a critical point for area under all smooth compactly supported variations if and only if the mean curvature vanishes identically.

Definition 2.1 In Galilean space, a regular surface M is called a area minimizing surface (minimal surface) if and only if its mean curvature is zero at each point.

Theorem 2.2 Suppose a surface M is the graph of a function of two variables. Then, the surface M can be parametrized by

$$\varphi(x,y) = (x, y, f(x,y)).$$

The surface M in $\mathbb{G}^3$ is minimal if and only if it can be locally expressed as the graph of a solution of

$$f_{yy} = 0.$$

Proof: The partial derivatives of the surface are obtained as

$$\varphi_x = (1, 0, f_x), \varphi_y = (0, 1, f_y).$$

Thus, we have the unit normal vector as follows

$$N = \frac{(0, -f_y, 1)}{\sqrt{1+f_y^2}}.$$

The components of the second fundamental form are given by

$$L_{11} = \frac{f_{xx}}{\sqrt{1+f_y^2}}, L_{12} = \frac{f_{xy}}{\sqrt{1+f_y^2}}, L_{22} = \frac{f_{yy}}{\sqrt{1+f_y^2}}. \tag{16}$$

On the other hand, using (6) gives

$$g^{11} = g^{12} = 0, g^{22} = \frac{1}{1+f_y^2}. \tag{17}$$

From (7), (16) and (17), we have the Gauss and mean curvatures as follows

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1+f_y^2)^2}$$

and

$$2H = \frac{f_{yy}}{(1+f_y^2)^{\frac{3}{2}}}.$$

Consequently, this surface is minimal if and only if f_{yy} vanishes.

The geometric interpretation of the above expression is that let us consider the $y-$parameter curves $\varphi(x_0, y) = (x_0, y, f(x_0, y))$ of the surface, hence $f_{yy} = 0$ the $y-$parameter curves are isotropic lines. Thus we state the following corollary.

Corollary 2.1 The minimal surfaces in the Galilean space given with a Monge patch are ruled surfaces type C parametrized by

$$\varphi(x,y) = \alpha(x) + y\beta(x)$$

where $\alpha(x) = (x, 0, z(x))$ is the non-isotropic plane curve, $\beta(x) = (0, 1, w(x))$ is the isotropic plane line.

Example 2.1 Let us consider the surface given by

$$\varphi(x,y) = (x, y, x^3y + xy).$$

It is easy to see that hence we have $f_{yy} = 0$, this surface is a minimal surface shown in Fig.1.

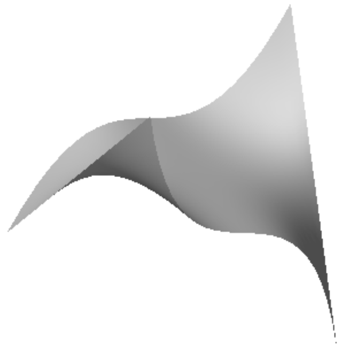

Fig. 1:

This surface is also a ruled surface of type C parametrized by

$$\varphi(x,y)=(x,0,0)+y(0,1,x^3+x).$$

There is not isothermal coordinates in Galilean space. In order to have a similar view of the minimal surface in Galilean space. We give following definition.

Definition 3.2 In Galilean space, a surface $\varphi(u,v)=(x(u,v),y(u,v),z(u,v))$ can be parameterized by using a isothermal-like parameterization as follows

$$g_{11}=g_{22}=g_{12}=\lambda^2 \tag{18}$$

and Euclid isothermal-like parameterization as follows

$$h_{11}=h_{22}=\lambda_1^2,\ \ h_{12}=0 \tag{19}$$

where g_{11},g_{22},g_{12} and h_{11},h_{22},h_{12} are the coefficients of the first fundamental forms I_1 and I_2, respectively.

Theorem 2.3 Let M be a surface described by an isothermal-like and Euclid isothermal-like patch parameterization in Galilean space. Then we have

$$w^2=\|\varphi_u\wedge\varphi_v\|_1^2=2\lambda^2\lambda_1^2. \tag{20}$$

Proof: From (3) we have

$$\|\varphi_u\wedge\varphi_v\|_1^2=x_u^2(z_v^2+y_v^2)+x_v^2(y_u^2+z_u^2)-2x_ux_v(z_uz_v+y_uy_v). \tag{21}$$

It is easy to see that

$$g_{11}=x_u^2, g_{12}=x_ux_v, g_{22}=x_v^2 \tag{22}$$

and

$$h_{11}=y_u^2+z_u^2, h_{12}=z_uz_v+y_uy_v, h_{22}=z_v^2+y_v^2. \tag{23}$$

Substituting (22) and (23) into (21) gives

$$w^2=g_{11}h_{22}+g_{22}h_{11}-2g_{12}h_{12}.$$

From (18) and (19) we have

$$w^2=2\lambda^2\lambda_1^2.$$

Theorem 2.4 Let M be a surface described by an isothermal-like patch in Galilean space. Then the first fundamental form I of the surface is given by

$$I=I_1+\epsilon I_2 \tag{24}$$

where if $du\neq -dv$ then $\epsilon=1$, in the other cases $\epsilon=0$ and I_1, I_2 are obtained as

$$I_1=\lambda^2(du+dv)^2$$

and

$$I_2=2\lambda_1^2 dv^2.$$

Proof: The proof is straightforward.

Theorem 2.5 Let M be a surface described by an isothermal-like patch given by parameterization $\varphi(u,v)=(x(u,v),y(u,v),z(u,v))$ in Galilean space. The surface M is minimal if and only if the partial derivatives of the surface satisfy the following equation

$$\varphi_{uu}-2\varphi_{uv}+\varphi_{vv}=0.$$

Proof: From (18) and (22), it follows that

$$x_u(u,v)=\lambda,\ \ x_v(u,v)=\lambda \tag{25}$$

where the partial derivatives of the first component $x(u,v)$ of the surface with respect to u and v is denoted by x_u and x_v, respectively.

Combining (6), (25) and (20) we arrive at

$$g^{11} = \frac{\lambda^2}{w^2}, g^{22} = \frac{\lambda^2}{w^2}, g^{12} = -\frac{\lambda^2}{w^2}. \tag{26}$$

Substituting (26) into (7) we get

$$\frac{2Hw^2}{\lambda^2} = L_{11} - 2L_{12} + L_{22}.$$

Which implies that

$$\frac{2Hw^2}{\lambda^2} = \langle \varphi_{uu} - 2\varphi_{uv} + \varphi_{vv}, N \rangle_1 .$$

It follows that

$$\frac{2Hw^2 N}{\lambda^2} = \varphi_{uu} - 2\varphi_{uv} + \varphi_{vv}.$$

Thus, the surface is minimal if and only if

$$\varphi_{uu} - 2\varphi_{uv} + \varphi_{vv} = 0$$

holds.

Example 2.2 Let us consider the surface given by

$$\varphi(u, v) = (u + v, u^2 - v^2, u - v).$$

From (5) we get

$$g_{11} = g_{12} = g_{22} = 1.$$

In addition, it is easy to see that

$$\varphi_{uu} = (0, 2, 0), \varphi_{uv} = (0, 0, 0), \varphi_{vv} = (0, -2, 0).$$

Hence we have $\varphi_{uu} - 2\varphi_{uv} + \varphi_{vv} = 0$, the surface is a minimal surface shown in Fig.2.

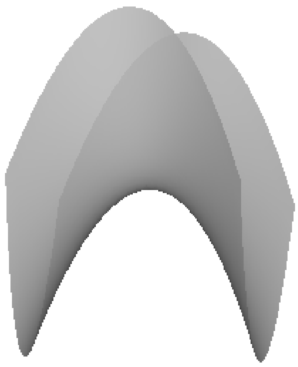

Fig. 2:

Special cases:

- If $x_u = \lambda$ and $x_v = 0$ then, from (6) we have

$$g^{11} = g^{12} = 0, g^{22} = \frac{1}{w^2}. \tag{27}$$

It is easy to see that

$$L_{22} = \frac{1}{w} \langle (0, z_v x_u, -y_v x_u), (0, y_{vv}, z_{vv}) \rangle_1 . \tag{28}$$

Substituting (27) and (28) into (7) we get

$$2Hw^3 = \langle \varphi_{vv}, N \rangle_1 .$$

Thus, the surface is minimal if and only if φ_{vv} vanishes. One of the interesting example of this case is that the ruled surface of type *C*.

Example 2.3 Assume that the ruled surface of type *C* is parametrized by

$$\Phi(u, v) = (u, u^2 + v \cos u, v \sin u)$$

where $r(u) = (u, u^2, 0)$ is the directrix and $a(u) = (0, \cos u, \sin u)$ is the generator.

Observe that $x_u = 1$, $x_v = 0$ and $\Phi_{vv} = 0$. Thus this surface is a minimal surface shown in Fig.3.

- If $x_u = 0$ and $x_v = \lambda$ then, similar to the previous case, the surface is minimal if and only if $\varphi_{uu} = 0$ holds. One of the exciting example of this case is the helicoid parametrized by

$$\varphi(u, v) = (v, u \cos v, u \sin v).$$

- If $x_u = x_v = 0$ then, the surface is a part of a plane with vanishing mean curvature.

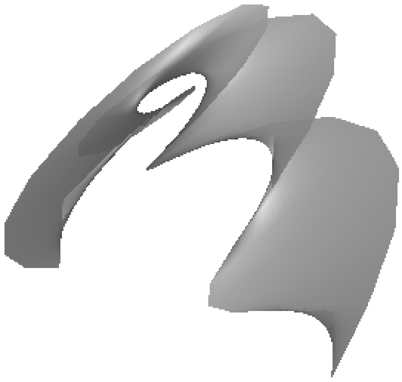

Fig. 3:

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On Developable Ruled Surfaces in Pseudo-Galilean Space

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Abstract: In this paper, we investigated the ruled surfaces in the three-dimensional pseudo-Galilean space. We obtained the distribution parameter of the ruled surface by using the Frenet frame of directrix curve. Moreover, we derived the necessary conditions to construct a developable ruled surface in the pseudo-Galilean space.

Keyword: Developable, Distribution parameter, Pseudo-Galilean space, Ruled surface.

1 Introduction

A surface formed by a one-parameter family of straight lines is called a ruled surface. The ruled surfaces are one of the most important topics of the differential geometry since, the ruled surfaces appear in many different areas such as geometric design and manufacturing [6, 7, 9, 11], tool path planning and robot motion planning [13]. So, many geometers studied on the ruled surfaces in different spaces such as Minkowski and Galilean. Turgut and Hacısalihoglu [14, 15] defined timelike ruled surfaces and obtained some properties in the Minkowski space. Yaylı [17] obtained the distribution parameter of a spacelike ruled surface with respect to Frenet frame. O. Röschel [12] introduced the ruled surfaces in the Galilean space. He classified ruled surfaces into three types in the Galilean space. After that, Kamenarovic [8] obtained the defining equations for the ruled surfaces of types *A, B* and *C* in the Galilean space. Divjak and Milin-Sipus [4, 5] calculated the special curves on the ruled surfaces in the Galilean and pseudo-Galilean spaces. Recently, Milin-Sipus [10] investigated the ruled Weingarten surfaces in the Galilean space.

The pseudo-Galilean space is a Cayley-Klein space equipped with the projective metric of signature $(0, 0, +, -)$. The absolute figure of the pseudo-Galilean geometry consists of an ordered triple $\{\omega, f, I\}$, where ω is the real (absolute) plane, f is the real line (absolute line) in ω. I is the fixed hyperbolic involution of points of f. More information can be found in [1–3, 6, 16].

Definition 1.1. Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be vectors in the pseudo-Galilean space. The scalar product is given by

$$< \mathbf{a}, \mathbf{b} >= a_1 b_1. \tag{1}$$

The scalar product of two isotropic vectors $\mathbf{p} = (0, p_2, p_3)$ and $\mathbf{q} = (0, q_2, q_3)$ is given by

$$< \mathbf{p}, \mathbf{q} >_1= p_2 q_2 - p_3 q_3. \tag{2}$$

Definition 1.2. The isotropic angle measure ϑ between two vectors $\mathbf{u} = (1, u_2, u_3)$ and $\mathbf{v} = (1, v_2, v_3)$ is defined by

$$\vartheta = \|\mathbf{u} - \mathbf{v}\|_1 = \sqrt{|(u_2 - v_2)^2 - (u_3 - v_3)^2|}. \tag{3}$$

On the otherhand, the angle measure between two isotropic vectors $\mathbf{p} = (0, p_2, p_3)$ and $\mathbf{q} = (0, q_2, q_3)$ is given by

$$\cosh\theta = \frac{< \mathbf{p}, \mathbf{q} >_1}{\|\mathbf{p}\|_1 \, \|\mathbf{q}\|_1}. \tag{4}$$

Definition 1.3. An admissible curve c is given by the parametrization

$$r(u) = (u, y(u), z(u)). \tag{5}$$

The Frenet frame is given by

$$\begin{aligned}
\mathbf{t} &= (1, y'(u), z'(u)) \\
\mathbf{n} &= \frac{1}{\kappa}(0, y''(u), z''(u)) \\
\mathbf{b} &= \frac{1}{\kappa}(0, z''(u), y''(u))
\end{aligned}$$

where $\kappa = \sqrt{|y''^2 - z''^2|}$ is the curvature.

Frenet formulas are given by

$$\frac{d}{dx}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{6}$$

where $\tau = \dfrac{1}{\kappa^2}\det[r', r'', r''']$ is the torsion.

There are there types of the ruled surfaces in the pseudo-Galilean space.

Definition 1.4. A ruled surface of type *A* is parametrized by

$$\Phi_A(x, u) = m(x) + u\mathbf{a}(x) \tag{7}$$

where the directrix curve $m(x) = (x, y(x), z(x))$ does not lie in an pseudo-Euclidean plane and the generator $\mathbf{a}(x) = (1, a_2(x), a_3(x))$ is non-isotropic.

Definition 1.5. A ruled surface of type *B* can be parametrized by

$$\Phi_B(x, u) = r(x) + u\mathbf{a}(x) \tag{8}$$

where the directrix curve $r(x) = (0, y(x), z(x))$ lies in an pseudo-Euclidean plane and $\mathbf{a}(x) = (1, a_2(x), a_3(x))$ is the generator.

Definition 1.6. A ruled surface of type *C* is given by the parametrization

$$\Phi_C(x, u) = n(x) + u\mathbf{k}(x) \tag{9}$$

where the directrix $n(x) = (x, y(x), 0)$ lies in an isotropic plane and the generator $\mathbf{k}(x) = (0, a_2(x), a_3(x))$ is isotropic.

The ruled surfaces of type B and C are the special case of the ruled surfaces of type A. Moreover, the distribution parameters of the ruled surfaces of type B and C never vanish since, there are not developable ruled surfaces of type B and C. Hence, we omit them in this study.

2 Ruled surface of type *A*

The natural frame of the ruled surface of type *A* is defined by

$$\begin{aligned} \mathbf{e}_1(x) &= (1, a_2(x), a_3(x)) \\ \mathbf{e}_2(x) &= \frac{1}{\kappa}(0, a_2'(x), a_3'(x)) \\ \mathbf{e}_3(x) &= \frac{1}{\kappa}(0, a_3'(x), a_2'(x)) \end{aligned} \tag{10}$$

where $\kappa = \sqrt{\left|a_2'^2 - a_3'^2\right|}$.

Frenet formulas are given as follows

$$\frac{d}{dx}\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \tag{11}$$

where $\tau = \dfrac{\det(\mathbf{a}, \mathbf{a}', \mathbf{a}'')}{\kappa^2}$ is called the torsion.

The parameter of distribution d is given by

$$d = \frac{\det(m', \mathbf{a}, \mathbf{a}')}{\langle \mathbf{a}', \mathbf{a}' \rangle_1}. \tag{12}$$

The striction curve of the ruled surface of type *A* can be given by

$$s(x) = m(x) + u(x)\mathbf{a}(x)$$

where

$$u(x) = \frac{\langle \mathbf{a} - m', \mathbf{a}' \rangle_1}{\langle \mathbf{a}', \mathbf{a}' \rangle_1}. \tag{13}$$

Using the directrix curve $m(x)$ of the ruled surface of type *A*, we can define an another moving frame which is called the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$. The Frenet frame can be given in the following form

$$\begin{aligned} \mathbf{t} &= (1, y'(x), z'(x)) \\ \mathbf{n} &= \frac{1}{\kappa^*}(0, y''(x), z''(x)) \\ \mathbf{b} &= \frac{1}{\kappa^*}(0, z''(x), y''(x)) \end{aligned} \tag{14}$$

where $\kappa^* = \sqrt{|y''^2 - z''^2|}$.

Frenet formulas are obtained by

$$\frac{d}{dx}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}=\begin{bmatrix}0 & \kappa^* & 0\\ 0 & 0 & \tau^*\\ 0 & \tau^* & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} \tag{15}$$

where $\tau^*=\frac{1}{\kappa^{*2}}\det[m'(x),m''(x),m'''(x)]$.

The two moving coordinate systems $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ and $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ are orthogonal coordinate systems in the pseudo-Galilean space which represent the moving space H and the fixed space H', respectively.

CASE 1 Assume that spine curve $\gamma(u)$ is an admissible spacelike curve with a timelike normal vector $\mathbf{n}$. Therefore, the binormal $\mathbf{b}$ is a spacelike vector.

Theorem 2.1. Let $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ and $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be two orthogonal coordinate systems along the ruled surface of type *A*. The distribution parameter can be given by

$$d=\frac{\tau^*(x_3^2-x_2^2)+\kappa^* x_3}{x_2^2\tau^{*2}-(\kappa^*-x_3\tau^*)^2}. \tag{16}$$

Proof: The generator vector $\mathbf{a}(x)$ of the ruled surface of type *A* can be written in terms of Frenet frame base vectors $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ as follows

$$\mathbf{a}=x_1\mathbf{t}+x_2\mathbf{n}+x_3\mathbf{b} \tag{17}$$

where $x_1,x_2,x_3\in\mathbb{R}$.

Using (1) and (17) it is easy to see that

$$\langle\mathbf{a},\mathbf{t}\rangle=x_1=1. \tag{18}$$

Since $\mathbf{n}$ and $\mathbf{b}$ are isotropic vectors, it is convenient to rewrite (17) in the following form

$$\mathbf{a}-\mathbf{t}=x_2\mathbf{n}+x_3\mathbf{b}.$$

Then, using (2) gives

$$x_2=-\langle\mathbf{a}-\mathbf{t},\mathbf{n}\rangle_1 \tag{19}$$

and

$$x_3=\langle\mathbf{a}-\mathbf{t},\mathbf{b}\rangle_1\,. \tag{20}$$

In addition, differentiating (17) then, substituting (15) into the result gives

$$\mathbf{a}'=(\kappa^*+x_3\tau^*)\mathbf{n}+x_2\tau^*\mathbf{b}. \tag{21}$$

From (12), (17) and (21), we have the distribution parameter of the ruled surface of type *A* in the following form

$$d=\frac{\tau^*(x_3^2-x_2^2)+\kappa^* x_3}{x_2^2\tau^{*2}-(\kappa^*-x_3\tau^*)^2}. \tag{22}$$

A ruled surface is said to be developable if and only if the parameter of distribution *d* is zero. Hence, we state the following corollary:

Corollary 2.1. The ruled surface of type *A* is developable, if and only if the directrix curve is a helix with the following Frenet curvatures

$$\frac{\kappa^*}{\tau^*}=\frac{x_2^2-x_3^2}{x_3}. \tag{23}$$

Theorem 2.2. Let $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ and $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be two orthogonal coordinate systems along the ruled surface of type *A*. The relation matrix between the two moving coordinate systems $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ and $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ are given by

$$\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{bmatrix}=\begin{bmatrix}1 & \frac{\kappa\sin\phi}{\tau^*} & \frac{\kappa^*-\kappa\cos\phi}{\tau^*}\\ 0 & \cos\phi & \sin\phi\\ 0 & -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} \tag{24}$$

where ϕ is the pseudo-Euclidean angle between the vectors $\mathbf{e}_2$ and $\mathbf{n}$.

Proof: It is easy to see that combining (21) and $\mathbf{a}'=\kappa\mathbf{e}_2$ gives

$$\mathbf{e}_2=(\frac{\kappa^*+x_3\tau^*}{\kappa})\mathbf{n}+\frac{x_2\tau^*}{\kappa}\mathbf{b}. \tag{25}$$

This last equation implies that $\mathbf{e}_2$ is on the normal plane spanned by $\{\mathbf{n},\mathbf{b}\}$. Since the isotropic vectors are on the pseudo-Euclidean planes in the pseudo-Galilean space. we can define the hyperbolic angle between the isotropic vectors. In order to obtain the relationship between the natural and Frenet frames. Let ϕ be the hyperbolic angle between the timelike isotropic vectors $\mathbf{e}_2$ and $\mathbf{n}$, we have

$$\mathbf{e}_2 = \cosh\phi\mathbf{n} + \sinh\phi\mathbf{b}. \tag{26}$$

It is easy to see that

$$\mathbf{e}_3 = \sinh\phi\mathbf{n} + \cosh\phi\mathbf{b}. \tag{27}$$

From (25), (26) and (27), we have

$$x_2 = \frac{\kappa\sinh\phi}{\tau^*} \tag{28}$$

and

$$x_3 = \frac{-\kappa^* + \kappa\cosh\phi}{\tau^*}. \tag{29}$$

Substituting (28) and (29) into (17) gives

$$\mathbf{a} = \mathbf{t} + \frac{\kappa\sinh\phi}{\tau^*}\mathbf{n} + \frac{-\kappa^* + \kappa\cosh\phi}{\tau^*}\mathbf{b}. \tag{30}$$

We may express the results in the following matrix form as

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{\kappa\sinh\phi}{\tau^*} & \frac{-\kappa^* + \kappa\cosh\phi}{\tau^*} \\ 0 & \cosh\phi & \sinh\phi \\ 0 & \sinh\phi & \cosh\phi \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{31}$$

Combining (26) and $\mathbf{a}' = \kappa\mathbf{e}_2$, we obtain

$$\mathbf{a}' = \kappa\cosh\phi\mathbf{n} + \kappa\sinh\phi\mathbf{b}. \tag{32}$$

Corollary 2.2. From (12), (30) and (32), the distribution parameter of the ruled surface of type *A* is obtained by

$$d = \frac{\kappa - \kappa^*\cosh\phi}{\tau^*\kappa}. \tag{33}$$

Corollary 2.3. The ruled surface of type *A* is developable, if and only if there is a relation between κ and κ^* in the following form

$$\kappa = \kappa^*\cosh\phi. \tag{34}$$

Theorem 2.3. The striction curve of the ruled surface of type *A* can be given by

$$s(x) = m(x) + \frac{\kappa^*\sin\phi}{\kappa\tau^*}\mathbf{a}(x). \tag{35}$$

Proof: Substituting (28) and (29) into (17) gives

$$u(x) = \frac{\kappa^*\sinh\phi}{\kappa\tau^*} \tag{36}$$

and we obtain the equation (35).

3 Examples

Example 3.1. Let the ruled surface of type *A* be given by the parametrization

$$\Phi_A(x,u) = (x, 2\sinh x, 2\cosh x) + u((1, 3\cosh x + \sqrt{3}\sinh x, 3\sinh x + \sqrt{3}\cosh x)). \tag{37}$$

The natural frame is obtained by

$$\begin{aligned} \mathbf{e}_1 = &\quad (1, 3\cosh x + \sqrt{3}\sinh x, 3\sinh x + \sqrt{3}\cosh x) \\ \mathbf{e}_2 = &\quad (0, 3\sinh x + \sqrt{3}\cosh x, 3\cosh x + \sqrt{3}\sinh x) \\ \mathbf{e}_3 = &\quad (0, 3\cosh x + \sqrt{3}\sinh x, 3\sinh x + \sqrt{3}\cosh x) \end{aligned}. \tag{38}$$

The Frenet frame is

$$\begin{aligned} \mathbf{t} = &\quad (1, 2\cosh x, 2\sinh x) \\ \mathbf{n} = &\quad (0, \sinh x, \cosh x) \\ \mathbf{b} = &\quad (0, \cosh x, \sinh x) \end{aligned} \tag{39}$$

where $\kappa^* = 2, \tau^* = 1$.

From (19) and (20), we have

$$x_1 = 1, x_2 = \sqrt{3}, x_3 = 1$$

It follows that

$$\frac{\kappa^*}{\tau^*} = \frac{x_2^2 - x_3^2}{x_3} = 2.$$

Thus this is a developable ruled surface of type A shown in Fig. 1.

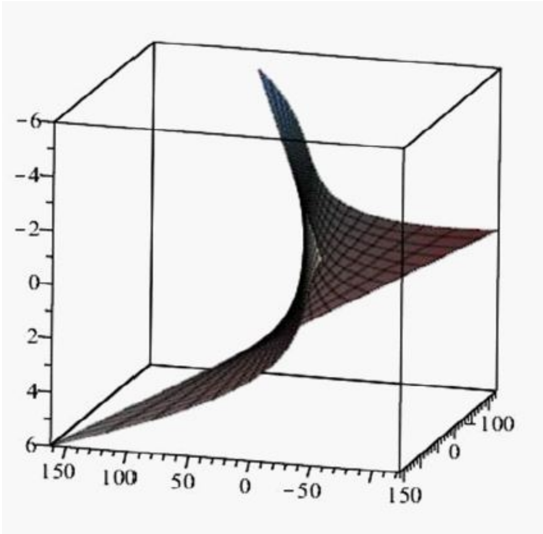

Fig. 1: A developable ruled surface of type A

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Special Helices on the Ellipsoid

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Abstract: In this study, we investigate three types of special helices whose axis is a fixed constant Killing vector field on the Ellipsoid $\mathbb{S}^2_{a_1,a_2,a_3}$ in $\mathbb{R}^3_{a_1,a_2,a_3}$. Then, we obtain the curvatures of all special helices on the ellipsoid $\mathbb{S}^2_{a_1,a_2,a_3}$ and give some characterizations of these curves. Moreover, we present various examples and visualize their images using the Mathematica program.

Keywords: Frame fields, Killing vector field, Special curves and surfaces.

1 Introduction

The spherical curves are the special space curves that lie on the sphere. If the sphere is constructed by using the elliptical inner product, then the elliptical 2-sphere is obtained. This sphere is an ellipsoid according to the Euclidean sense. We summarize some studies about spherical curves: Firstly, Wong proved the condition for a curve to be on a sphere and gave some characterizations for this curve [10, 11]. In [3], Breuer et al. gave an explicit characterization of the spherical curve. In [6], the author investigated the characterization of the dual spherical curve. Then, in [2], the author obtained a differential equation for characterizing of the dual spherical curves. Besides, in [4], İlarslan presented the spherical curve characterization for non-null regular curves in Lorentzian 3-space. Ayyıldız introduced the dual Lorentzian spherical curves [1]. Moreover, Izumiya and Takeuchi defined the slant helices and conical geodesic curve and gave a classification of special developable surfaces under the condition of the existence of such a special helix as a geodesic [5]. Scofield derived a curve of constant precession and proved that this curve is tangent indicatrix of a spherical helix [9].

In the present work, we give some characterizations for the special helices whose axis is the fixed constant Killing vector field on the elliptical 2-sphere. Furthermore, we give various examples and draw their images by using the Mathematica program.

2 Preliminaries

Let we take $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ and $a_1, a_2, a_3 \in \mathbb{R}^+$ then the elliptical inner product defined as

$$B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R};\ B(u, v) = a_1x_1y_1 + a_2x_2y_2 + a_3x_3y_3. \tag{1}$$

The 3-dimensional real vector space $\mathbb{R}^3$ equipped with the elliptical inner product will be represented by $\mathbb{R}^n_{a_1,a_2,a_3}$. The norm of a vector associated with the scalar product B is defined as

$$\|u\|_B = \sqrt{B(u, u)}. \tag{2}$$

Two vectors u and v are called elliptically orthogonal vectors if $B(u, v) = 0$. In addition, if u is an elliptically orthonormal vector then $B(u, u) = 1$. The cosine of the angle between two vectors u and v is defined as

$$\cos\theta = \frac{B(u, v)}{\|u\|_B \|v\|_B}, \tag{3}$$

where θ is compatible with the parameters of the angular parametric equations of ellipse or ellipsoid. The cross product of two vector fields $X, Y \in \mathbb{R}^3_{a_1,a_2,a_3}$ is given by

$$X \times_E Y = \Delta \begin{vmatrix} \frac{e_1}{a_1} & \frac{e_2}{a_2} & \frac{e_3}{a_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}, \tag{4}$$

where $\Delta = \sqrt{a_1a_2a_3}$, $a_1, a_2, a_3 \in \mathbb{R}^+$ [7].

Let us take the ellipsoid denoted by $\mathbb{S}^2_{a_1,a_2,a_3}$ in $\mathbb{R}^3_{a_1,a_2,a_3}$. Then, the sectional curvature of the ellipsoid generated by the non-degenerated plane $\{u, v\}$ is defined as

$$K(u,v) = \frac{B(R(u,v)u,v)}{B(u,u)B(v,v) - B(u,v)^2}, \tag{5}$$

where R is the Riemannian curvature tensor given by

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z. \tag{6}$$

The ellipsoid has the constant sectional curvature. Therefore, the curvature tensor R is written as follows

$$R(X,Y)Z = C\{B(Z,X)Y - B(Z,Y)X\}, \tag{7}$$

where C is the constant sectional curvature.

A curve γ on the ellipsoid $\mathbb{S}^2_{a_1,a_2,a_3}$ defined by $\gamma(s) = \varphi(\alpha(s))$ and a unit normal vector field Z along the surface $\mathbb{S}^2_{a_1,a_2,a_3}$ defined

$$Z = \frac{\varphi_u \times_E \varphi_v}{\|\varphi_u \times \varphi_v\|}. \tag{8}$$

Since $\mathbb{S}^2_{a_1,a_2,a_3}$ is sphere according to the elliptical inner product, the unit normal vector field Z along the surface $\mathbb{S}^2_{a_1,a_2,a_3}$ equal to the position vector of the curve γ. Then, we found an orthonormal frame $\{t = \gamma', y = \gamma \times_E \gamma', \gamma\}$ which is called the elliptical Darboux frame along the curve γ. The corresponding Darboux formulae of γ is written as

$$\begin{aligned} t' &= -\gamma + k_{g_E} y, \\ \gamma' &= t, \\ y' &= -k_{g_E} t, \end{aligned} \tag{9}$$

where $k_{n_E} = -1$, $k_{g_E} = B(\gamma'', y)$ and $\tau_r = 0$ are *geodesic curvature*, *asymptotic curvature*, and *principal curvature* of γ on the surface $\mathbb{S}^2_{a_1,a_2,a_3}$, respectively. Moreover, it is found as the following relation

$$y \times_E t = \gamma, \; z \times_E y = t, \; z \times_E t = -y, \tag{10}$$

[8].

Lemma 1. *Let $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}$, $\varphi(U) = \mathbb{S}^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on the $\mathbb{S}^2_{a_1,a_2,a_3}$. Provided that V be a vector field along the curve γ then the variation of γ defined by $\Gamma : I \times (-\varepsilon, \varepsilon) \to \mathbb{S}^2_{a_1,a_2,a_3}(C)$ with $\gamma(s,0)$ the initial curve satisfy $\Gamma(s,0) = \gamma(s)$. The variations of the geodesic curvature function $k_{g_E}(s,w)$ and the speed function $v(s,w)$ at $w = 0$ are calculated as follows:*

$$\begin{aligned} V(v) &= \left(\frac{\partial v}{\partial w}(s,w)\right)\Big|_{w=0} = -v\rho, \\ V(k_{g_E}) &= \left(\frac{\partial k_{g_E}}{\partial w}(s,w)\right)\Big|_{w=0} = B(-R(V,t)t + \nabla_t^2 V, y) - \frac{1}{k_{g_E}} B(-R(V,t)t + \nabla_t^2 V, \gamma), \end{aligned} \tag{11}$$

where $\rho = B(\nabla_t V, t)$ and R stands for the curvature tensor of $\mathbb{S}^2_{a_1,a_2,a_3}$ [8].

Proposition 1. *If $V(s)$ is the restriction to $\gamma(s)$ of a Killing vector field V of $\mathbb{S}^2_{a_1,a_2,a_3}$ then the variations of the elliptical Darboux curvature functions and speed function of γ satisfy:*

$$V(v) = V(k_{g_E}) = 0, \tag{12}$$

[8].

3 Special helices on the ellipsoid $\mathbb{S}^2_{a_1,a_2,a_3}$

Definition 1. *Let $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}$, $\varphi(U) = \mathbb{S}^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on the $\mathbb{S}^2_{a_1,a_2,a_3}$. Then we say that γ is a type-1 special helix, type-2 special helix, or type-3 special helix if $B(V,t) = const.$, $B(V,\gamma) = const.$, and $B(V,y) = const.$, respectively.*

Theorem 1. *Let $\varphi : U \subset \mathbb{E}^2 \to \mathbb{E}^3$, $\varphi(U) = \mathbb{S}^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on $\mathbb{S}^2_{a_1,a_2,a_3}$ and V be a Killing vector field along the curve γ. Then γ is a type-1 special helix with the axis V if and only if the geodesic curvature of the curve γ satisfy the following equation:*

$$k_{g_E} = \cot\theta,$$

where θ satisfy

$$\theta'' \sin^2\theta - \omega\theta' \cos\theta = 0,$$

[8].

Now, we can give the following corollary without proof. The proof of the corollary similar to Scofield's work [9].

Corollary 1. *Let $\varphi : U \subset \mathbb{E}^2 \to \mathbb{E}^3$, $\varphi(U) = \mathbb{S}^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \to U$ be a type-1 special helix with the Killing axis V on $\mathbb{S}^2_{a_1,a_2,a_3}$. Then, the integral curve of γ is an elliptical constant procession curve on the elliptical hyperboloid.*

Theorem 2. *Let $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}$, $\varphi(U) = \mathbb{S}^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on the $\mathbb{S}^2_{a_1,a_2,a_3}$. Then γ is a type-2 special helix with the axis V if and only if the geodesic curvature of the curve γ satisfy the following equation:*

$$k_{g_E} = \frac{c_1}{\sin\theta} - \theta', \tag{13}$$

here θ satisfies

$$\theta = const. \text{ or } (C+1)\sin^4\theta - c_1\theta'\sin\theta + c_1 = 0,$$

where c is a constant.

Proof: If γ is a type-2 special helix with the Killing axis V then V is written as

$$V = \cos\theta t + c_1\gamma + \sin\theta y, c_1 = const. \tag{14}$$

Differentiating eq.(14) with respect to s, we found the following equation

$$\begin{aligned} \nabla_T V &= ((-\theta' - k_{g_E})\sin\theta + c_1)t + (-\cos\theta)\gamma \\ &\quad +(\cos\theta k_{g_E} + \theta'\cos\theta)y. \end{aligned} \tag{15}$$

Using the equation $V(v) = 0$ in Lemma 1, we found

$$k_{g_E} = \frac{c_1}{\sin\theta} - \theta' \tag{16}$$

The differentiation of eq.(15) is obtained as

$$\nabla^2_T V = (-1 - k^2_{g_E} + k_{g_E}\theta')\cos\theta t + \theta'\sin\theta\gamma + ((k_{g_E} + \theta')\cos\theta)'y. \tag{17}$$

Moreover, we have the following equation

$$R(V,t)t = C(B(t,V)t - B(t,t)V). \tag{18}$$

Using the Darboux frame equations and eq.(14), we deduce

$$R(V,t)t = -C(c_1\gamma + \sin\theta y). \tag{19}$$

Considering the eq.(17) and eq.(19) with the second equation in Lemma 1 and the Proposition 1, we reach the following equations

$$\theta = const. \text{ or } (C+1)\sin^4\theta - c_1\theta'\sin\theta + c_1 = 0.$$

□

Corollary 2. *Let γ be a type-2 special helix on the ellipsoid with the axis*

$$V = \cos\theta t + c_1\gamma + \sin\theta y, \quad \theta = const.,$$

then γ has the following parametric representation

$$\gamma(s) = A_1 + \frac{A_2}{\sqrt{1+\frac{c_1^2}{\sin^2\theta}}}\cos(\sqrt{1+\frac{c_1^2}{\sin^2\theta}}) + \frac{A_3}{\sqrt{1+\frac{c_1^2}{\sin^2\theta}}}\sin(\sqrt{1+\frac{c_1^2}{\sin^2\theta}}s),$$

where $A_1, A_2, A_3 \in \mathbb{R}^3_{a_1,a_2,a_3}$ and $c_1 \in \mathbb{R}$.

Proof: Let γ be a type-2 special helix on the ellipsoid with the axis

$$V = \cos\theta t + c_1\gamma + \sin\theta y, \quad \theta = const., \tag{20}$$

then the elliptical curvature of γ calculated as

$$k_{g_E} = \frac{c_1}{\sin\theta}. \tag{21}$$

On the other hand, from the Darboux frame equations γ satisfy the following third order differential equation

$$k_{g_E}\gamma''' - k'_{g_E}\gamma'' + (k^3_{g_E} + k_{g_E})\gamma' - k'_{g_E}\gamma = 0. \tag{22}$$

If k_{g_E} is written in the eq.(22) and the differential equation is solved then it is obtained that γ has the following parametric representation

$$\gamma(s) = A_1 + \frac{A_2}{\sqrt{1+\frac{c_1^2}{\sin^2\theta}}}\cos(\sqrt{1+\frac{c_1^2}{\sin^2\theta}}) + \frac{A_3}{\sqrt{1+\frac{c_1^2}{\sin^2\theta}}}\sin(\sqrt{1+\frac{c_1^2}{\sin^2\theta}}s), \tag{23}$$

where $A_1, A_2, A_3 \in \mathbb{R}^3_{a_1,a_2,a_3}$ and $c_1 \in \mathbb{R}$. □

Theorem 3. *Let $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}$, $\varphi(U) = \mathbb{S}^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on the $\mathbb{S}^2_{a_1,a_2,a_3}$. Then γ is type-3 special helix with the axis V if and only if the geodesic curvature of the curve γ satisfy the following equation:*

$$k_{g_E} = \frac{(1-\theta')\sin\theta}{c_2}, \tag{24}$$

here θ satisfies

$$(1-\theta')\sin\theta(-c_2^2\theta' - \theta''\sin\theta\cos\theta + (1-\theta')\theta'\cos 2\theta) - \theta'\sin\theta - c_2^2\theta''\cos\theta = 0,$$

where c_2 is a constant.

Proof: If γ is a type-3 special helix with the Killing axis V then V is written as

$$V = \cos\theta t + \sin\theta\gamma + c_2 y. \tag{25}$$

By differentiating eq.(25), we get

$$\nabla_T V = ((1-\theta')\sin\theta - c_2 k_{g_E})t + (1-\theta')\cos\theta\gamma + k_{g_E}\cos\theta y. \tag{26}$$

By using the equation $V(v) = 0$ in Lemma 1, we reach

$$k_{g_E} = \frac{(1-\theta')\sin\theta}{c_2}. \tag{27}$$

If we take the differentiation of eq.(26), we obtain

$$\begin{aligned}\nabla_T^2 V = &((1-\theta')\cos\theta - k_{g_E}^2\cos\theta)t + (-\theta''\cos - (1-\theta')\theta'\sin\theta)\gamma \\ &+(k_{g_E}^2\cos\theta - k_{g_E}\theta'\sin\theta)y.\end{aligned} \tag{28}$$

Furthermore, we have the following equation

$$R(V,t)t = C(B(t,V)t - B(t,t)V). \tag{29}$$

By using the Darboux frame equations and eq.(25), we obtain

$$R(V,T)T = C(-\sin\theta\gamma - c_2 y). \tag{30}$$

If we consider the eq.(28) and eq.(30) with the second equation in Lemma 1 and the Proposition 1, we deduce

$$\theta = const. \tag{31}$$

or satisfy the following equation

$$(1-\theta')\sin\theta(-c_2^2\theta' - \theta''\sin\theta\cos\theta + (1-\theta')\theta'\cos 2\theta) - \theta'\sin\theta - c_2^2\theta''\cos\theta = 0. \tag{32}$$

□

Corollary 3. *Let γ be a type-3 special helix on the ellipsoid with the axis*

$$V = \cos\theta t + \sin\theta\gamma + c_2 y, \quad \theta = const., \tag{33}$$

then γ has the following parametric representation

$$\gamma(s) = B_1 + \frac{B_2}{\sqrt{1+\frac{\sin^2\theta}{c_2^2}}}\cos(\sqrt{1+\frac{\sin^2\theta}{c_2^2}}) + \frac{B_3}{\sqrt{1+\frac{\sin^2\theta}{c_2^2}}}\sin(\sqrt{1+\frac{\sin^2\theta}{c_2^2}}s),$$

where $B_1, B_2, B_3 \in \mathbb{R}^3_{a_1,a_2,a_3}$ and $c_2 \in \mathbb{R}$.

Proof: Let γ be a type-3 special helix on the ellipsoid with the axis

$$V = \cos\theta t + \sin\theta\gamma + c_2 y, \quad \theta = const., \tag{34}$$

then the elliptical curvature of γ calculated as

$$k_{g_E} = \frac{\sin\theta}{c_2}. \tag{35}$$

On the other hand, from the Darboux frame equations, γ satisfy the following third order differential equation

$$k_{g_E}\gamma''' - k'_{g_E}\gamma'' + (k_{g_E}^3 + k_{g_E})\gamma' - k'_{g_E}\gamma = 0. \tag{36}$$

If k_{g_E} is written in the eq.(34) and the differential equation is solved then it is obtained that γ has the following parametric representation

$$\gamma(s) = B_1 + \frac{B_2}{\sqrt{1+\frac{\sin^2\theta}{c_2^2}}}\cos(\sqrt{1+\frac{\sin^2\theta}{c_2^2}}) + \frac{B_3}{\sqrt{1+\frac{\sin^2\theta}{c_2^2}}}\sin(\sqrt{1+\frac{\sin^2\theta}{c_2^2}}s), \tag{37}$$

where $B_1, B_2, B_3 \in \mathbb{R}^3_{a_1,a_2,a_3}$ and $c_2 \in \mathbb{R}$. □

In the following examples we give various special helices on the ellipsoid.

Examle 1. *Let us take the curve parameterized as*

$$\gamma(s) = \frac{1}{2}\frac{(1+k)\cos(1-k)t-(1-k)\cos(1+k)t}{2}\frac{1}{2}\frac{(1+k)\sin(1-k)t-(1-k)\sin(1+k)t}{4}\frac{\sqrt{1-k^2}\cos kt}{9}. \quad (38)$$

The elliptical curvature of the helix calculated as

$$k_{g_E}(s) = \cot(ks). \quad (39)$$

Thus, we can easily see that γ is a type-1 special helix. It is illustrated in Figure 1.

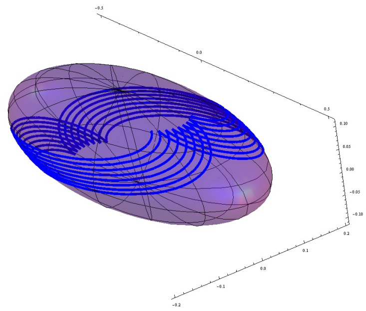

Figure1. Type-1 special Helices on the Ellipsoid $\mathbb{S}^2_{2,4,9}, k = 0.505$.

Examle 2. *Type-2 (type-3) special helices corresponding to different values of the A_i, B_i, $i = 1, 2, 3$. are illustrated in Figure 2.*

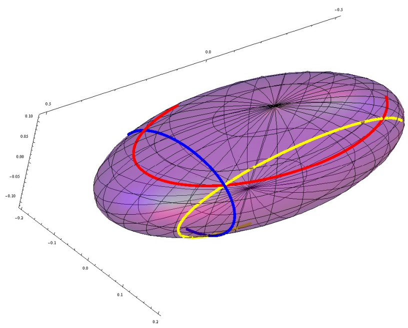

Figure2. Type-2 (type-3) Special Helices on the Ellipsoid $\mathbb{S}^2_{2,4,9}$.

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Spherical Curves in Finsler 3-Space

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Abstract: In this work, we investigate the general characteristics of the Finslerian spherical curves in Finsler 3-space. We obtain some characterizations for these curves. Moreover, we give various examples and visualized their images on Randers sphere.

Keywords: Finsler space, Special curves and surfaces.

1 Introduction

Finsler geometry is introduced with the doctoral thesis of Finsler. This geometry has more general metric and includes the Riemannian metric. Therefore, it has numerous applications in thermodynamics, optics, ecology, evolution, biology, geometry, physics, engineering, and computer sciences, etc. [1, 3–5, 7, 14]. Remizov [13] researched the singularities of geodesics flows in two-dimensional Finsler space. In [15, 16], Yıldırım, et al. investigated the helices in Finsler space. Ergüt, et al. gave the characterizations of AW(k)-type curves in three-dimensional Finsler manifold [9]. Furthermore, there exist various studies in literature examining methodology to use spherical curves to construct some specialized curves (see [2, 8, 10, 11, 17]). Deshmukh et al. study rectifying curves via the dilation of unit speed curves on the unit sphere $\mathbb{S}^2$ [8] in the Euclidean space $\mathbb{E}^3$. In [10], Izuyama and Takeuchi, created Bertrand curves using the spherical curve. In [11], the authors examined the spherical images of the tangent vector and binormal vector of a slant helix and gave various characterizations of these curve. The authors, in [6] gave the spherical curve characterization in the Sasakian 3-space. The spherical spirals are studied by the author, [12].

The purpose of this study is to introduce the Finslerian spherical curves and give some characterizations of these curves on the Finslerian 3 sphere.

2 Preliminaries

Let $c : I \to M;\ s \to x^i(s)$ be a smooth curve in M and $c' : I \to TM;\ s \to (x^i(s), \dot{x}^i(s))$ be the tangent bundle of the curve c in M' then Finslerian 3-manifold can be denoted by $\mathbb{F}^3 = \left(M, M', F\right)$. Also, the notion of the one dimensional Finsler submanifold of $\mathbb{F}^3$has the notion $\mathbb{F}^1 = (c, c', F_1)$. c has unit Finslerian speed on the condition that

$$F(x, \dot{x}) = 1.$$

The unit speed curve c has the following representation

$$x^i = x^i(s) : \ i \in \{1, 2, ..., m+1\}, \quad s \in (a, b) \tag{1}$$

Let (s, v) denoted by the coordinates on c' and s be an arc length parameter of the curve c. Then, we have the following equalities

$$\frac{\partial}{\partial s} = \frac{dx^i}{ds}\frac{\partial}{\partial x^i} + v\frac{d^2x^i}{ds^2}\frac{\partial}{\partial y^i} \text{ and } \frac{\partial}{\partial v} = \frac{dx^i}{ds}\frac{\partial}{\partial y^i}. \tag{2}$$

Here, $\dfrac{\partial}{\partial v}$ is a unit Finslerian vector field. These imply following equation

$$y^i(s, v) = v\frac{dx^i}{ds}, \ \ i \in \{1, 2, ..., m+1\} \tag{3}$$

where $\left\{\dfrac{\partial}{\partial s}, \dfrac{\partial}{\partial v}\right\}$ is the natural frame on c'.

A Finsler vector field X on $\mathbb{F}^{m+1}$ along c' is projectable on c. Then, it can be expressed as follows:

$$X(x(s), vx'(s)) = X^i(s)\frac{\partial}{\partial y^i}\left(x(s), vx'(s)\right) \tag{4}$$

at any point $\left(x(s), vx'(s)\right) \in c'$. From here, a vector field X^* on c denoted by the following formula

$$X^*(x(s)) = X^i(s)\frac{\partial}{\partial x^i}\left(x(s)\right). \tag{5}$$

Thus, the vector field $X^*(x(s))$ considered as the projection of the Finsler vector $X(x(s), vx'(s))$ on the tangent space TM of M at $x(s) \in c$ *(see for details* [4]).

The covariant derivatives according to Cartan connection of any projectable Finsler vector field X in the direction of $\frac{\partial}{\partial v}$ vanish identically on c'.

$$(\nabla^*_{\frac{\partial}{\partial v}} X)\left(x(s), vx'(s)\right) = 0,\ s \in (-\varepsilon, \varepsilon) \tag{6}$$

and

$$\nabla^*_{\frac{\partial}{\partial v}}\frac{\partial}{\partial v} = 0 \tag{7}$$

which enable us to express that the vertical covariant derivatives along c with respect to Cartan connection do not give any Frenet frame for c [4].

Let $c = c(s)$ be a smooth curve in $\mathbb{F}^3$ and s be an arc length parameter of the curve c. Suppose that the moving Frenet frame along the curve c denoted by $\{T := \frac{\partial}{\partial v}, N, B\}$ in the Finsler space $\mathbb{F}^3$. Then, the Frenet formulas of the curve c are given by

$$\begin{aligned} \nabla^*_{\frac{\partial}{\partial s}} T(s) &= \varkappa(s)N(s), \\ &= -\varkappa(s)T(s) + \tau(s)B(s), \\ \nabla^*_{\frac{\partial}{\partial s}} B(s) &= -\tau(s)N(s), \end{aligned} \tag{8}$$

where the vector fields $N(s)$ and $B(s)$ are Finslerian principal normal and binormal vector fields of c, respectively. The Finslerian curvature and torsion of the curve c are defined by

$$\varkappa(s) = \left\{g_{ij}(s)(c''^i(s) + 2G^i(s))(c''^j(s) + 2G^j(s))\right\}^{\frac{1}{2}}, \tag{9}$$

$$\tau(s) = -g\left(\nabla^*_{\frac{\partial}{\partial s}} N, B\right)(s) = -g_{ij}(s)B^i(s)\left\{\frac{\partial N^j}{\partial s} + N^k(s)S^j_k(s)\right\}$$

[4].

3 Spherical curves in Finslerian 3-space $\mathbb{F}^3$

In this section, we investigate the Finslerian spherical curves. The similar characterizations of those curves are found in Euclidean and Lorentzian spaces. However, since the Finsler metric is derived from the Minkowski norm, the results are quite different in practice. To explain this difference, we give various examples with the help of the Randers metric that is described in the Remark 1.

Theorem 1. *Let c be a smooth curve in* $3-$*dimensional Finsler space* $\mathbb{F}^3$ *with an arc length parameter* s*. Then the center of the osculating sphere of the curve c is written as*

$$A(s) = c(s) + A_2 N + A_3 B \tag{10}$$

where $A_2 = \frac{1}{\varkappa}$, $A_3 = \frac{1}{\tau(s)}\nabla^*_{\frac{\partial}{\partial s}} A_2$.

Proof: Suppose that c is a smooth curve in $3-$dimensional Finsler space $\mathbb{F}^3$ and $A(s)$ is the center of a Finslerian sphere having four common neighbor points with c. Let us define the following function

$$f : I \to R;\ s \to f(s) = g_F(A(s) - c(s), A(s) - c(s)) - r^2. \tag{11}$$

If the Finslerian sphere has four common neighbor points with c then we have

$$f(s) = \nabla^*_{\frac{\partial}{\partial s}} f(s) = \nabla^{2*}_{\frac{\partial}{\partial s}} f(s) = \nabla^{3*}_{\frac{\partial}{\partial s}} f(s) = 0. \tag{12}$$

If $f(s) = 0$ then we obtain

$$g_F(A(s) - c(s), A(s) - c(s)) = r^2. \tag{13}$$

Using the equation $\nabla^*_{\frac{\partial}{\partial s}} f(s) = 0$ we reach

$$g_F(A(s) - c(s), T(s)) = 0. \tag{14}$$

If $\nabla^{2*}_{\frac{\partial}{\partial s}} f(s) = 0$ then we get

$$g_F(A(s) - c(s), N) = \frac{1}{\varkappa}. \tag{15}$$

Then the equation $\nabla^{3*}_{\frac{\partial}{\partial s}} f(s) = 0$ gives us

$$g_F(A(s) - c(s), B) = \frac{1}{\tau(s)} \nabla^*_{\frac{\partial}{\partial s}} \frac{1}{\varkappa(s)}. \tag{16}$$

Since $A(s) - c(s) \in sp\{T, N, B\}$, we can write

$$A(s) - c(s) = A_1 T + A_2 N + A_3 B. \tag{17}$$

This gives $A_1 = g_F(A(s) - c(s), T(s)) = 0$, $A_2 = g_F(A(s) - c(s), N) = \frac{1}{\varkappa}$, and $A_3 = g_F(A(s) - c(s), B) = \frac{1}{\tau(s)} \nabla^*_{\frac{\partial}{\partial s}} \frac{1}{\varkappa(s)}$. This completes the proof. □

Corollary 1. *Let $\mathbb{FS}^2$ be a contact Finslerian sphere whose center is at the origin. Then the contact osculating Finslerian sphere of the curve γ is $\mathbb{FS}^2$, for all $s \in (-\varepsilon, \varepsilon)$.*

Proof: From the Theorem 1, we have

$$A(s) = c(s) + A_2 N + A_3 B, \tag{18}$$

here $A_2 = -g_F(c(s), N)$ and $A_3 = -g_F(c(s), B)$. On the other hand, we can write

$$c(s) = \zeta_1 T + \zeta_2 N + \zeta_3 B. \tag{19}$$

Then, we calculate that $\zeta_1 = 0$, $\zeta_2 = g_F(c(s), N)$,and $\zeta_3 = g_F(c(s), B)$. Therefore, using the eq.(18) and eq.(19), we obtain

$$A(s) = c(s) - c(s) = 0 \tag{20}$$

which completes the proof. □

Theorem 2. *Let $\mathbb{FS}^2$ be a contact Finslerian sphere whose center is denoted by $A(s)$. The radius of contact osculating sphere is constant, for all $s \in (-\varepsilon, \varepsilon)$ if and only if the centers of contact osculating spheres are the same constants.*

Proof: From the Theorem 1, we have

$$g_F(A(s) - c(s), A(s) - c(s)) = r^2, \tag{21}$$

and

$$A(s) - c(s) = A_2 N + A_3 B. \tag{22}$$

Therefore, we reach the following equation

$$A_2^2 + A_3^2 = r^2. \tag{23}$$

If we differentiate the eq.(23), we deduce

$$A_2 \nabla^*_{\frac{\partial}{\partial s}} A_2 + A_3 \nabla^*_{\frac{\partial}{\partial s}} A_3 = 0. \tag{24}$$

Besides, we have

$$\nabla^*_{\frac{\partial}{\partial s}} A_2 = \tau(s) A_3. \tag{25}$$

If we consider with the eq.(24) and eq.(25) we compute

$$A_2 \tau(s) + \nabla^*_{\frac{\partial}{\partial s}} A_3 = 0. \tag{26}$$

On the other hand, differentiating eq.(22) we obtain

$$\nabla^*_{\frac{\partial}{\partial s}} A(s) = (A_2 \tau(s) + \nabla^*_{\frac{\partial}{\partial s}} A_3) B = 0. \tag{27}$$

Thus, $A(s)$ is a constant for all $s \in (-\varepsilon, \varepsilon)$.

Conversely, if $A(s)$ is a constant then we have $A_2 \tau(s) + \nabla^*_{\frac{\partial}{\partial s}} A_3 = 0$. If we differentiate the eq.(23) we get

$$r \nabla^*_{\frac{\partial}{\partial s}} r = A_2 \nabla^*_{\frac{\partial}{\partial s}} A_2 + A_3 \nabla^*_{\frac{\partial}{\partial s}} A_3 = 0. \tag{28}$$

Therefore r is a constant. □

Theorem 3. *Let c be a smooth curve in the 3−dimensional Finsler space $\mathbb{F}^3$ with an arc length parameter s. Then c is a spherical curve if and only if the following equation holds*

$$A_2\tau(s) + \nabla^*_{\frac{\partial}{\partial s}} A_3 = 0 \tag{29}$$

for all $s \in I$.

Proof: From the Theorem 1 we have

$$A(s) = c(s) + A_2 N + A_3 B, \tag{30}$$

then using the Theorem 2 we get

$$\nabla^*_{\frac{\partial}{\partial s}} A(s) = (A_2\tau(s) + \nabla^*_{\frac{\partial}{\partial s}} A_3)B = 0. \tag{31}$$

These imply

$$A_2\tau(s) + \nabla^*_{\frac{\partial}{\partial s}} A_3 = 0. \tag{32}$$

Conversely, if we have $A_2\tau(s) + \nabla^*_{\frac{\partial}{\partial s}} A_3 = 0$ then it gives $\nabla^*_{\frac{\partial}{\partial s}} A(s) = 0$. Therefore we can say that c is a spherical curve. □

Remark 1. *The curves in the examples throughout the article are the unit speed curves according to the Randers metric. The Randers metric defines as*

$$F(x, y) := \alpha(x, y) + \beta(x, y) \tag{33}$$

here, $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ be Riemannian metric and $\beta(x, y) = b_i(x)y^i$ be a 1-form on a manifold M. The authors, in [7], obtained the coefficients of the Randers metric as follows:

$$g_{ij} = \frac{F}{\alpha}\{a_{ij} - \frac{y^i}{\alpha}\frac{y^j}{\alpha} + \frac{\alpha}{F}(b_i + \frac{y_i}{\alpha})(b_j + \frac{y_j}{\alpha})\}$$

where $y_i := a_{ij}y^j$. Since the bilinear form (g_{ij}) is positive definite, then the length of β is less then 1, i.e., $\|\beta\|_\gamma := \sqrt{a^{ij}b_i b_j} < 1$ where $\left(a^{ij}\right) := \left(a_{ij}\right)^{-1}$. A Minkowski norm in the form eq.(33) is called as the Randers norm [7].

4 Examples of the Finslerian spherical curves

The Finslerian sphere with center origin O and radius r according to the Randers metric can be parameterized as follows

$$\mathbb{FS}^2(u, v) = (\frac{r\cos u \sin v - b}{1 - b^2}, \frac{r\cos u\cos v}{\sqrt{1 - b^2}}, \frac{r\sin u}{\sqrt{1 - b^2}}).$$

In this subsection we obtain various spherical curves and visulated their images on the Finslerian sphere $\mathbb{FS}^2$.

Examle 1. *We consider an arc length parameterized curve γ in Finsler space $\mathbb{F}^3$ is defined by*

$$\gamma(s) = \begin{pmatrix} \frac{R}{2}\frac{(1+k)\cos(1-k)s-(1-k)\cos(1+k)s-2b}{1-b^2}, \\ \frac{R}{2}\frac{(1+k)\sin(1-k)s-(1-k)\sin(1+k)s}{\sqrt{1-b^2}}, \\ \frac{R\sqrt{1-k^2}\cos(ks)}{\sqrt{1-b^2}} \end{pmatrix}$$

where $b \in (0, 1)$ is a constant real number. From the Theorem 3, the curves γ are spherical curve illustrated in Figure 1.

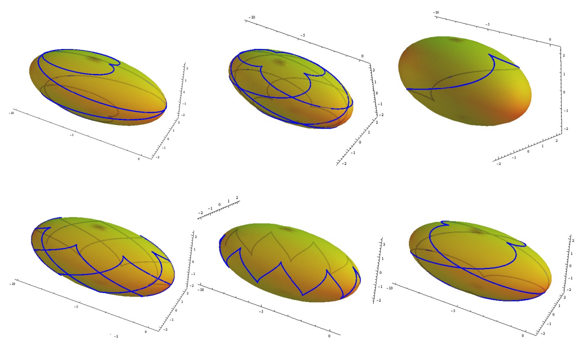

Figure 1. Curves on the Finslerian sphere $\mathbb{FS}^2$ for $b = 0.9$.

4.1 *Finslerian spherical spirals*

The spiral curve γ defined as a curve for which the tangent vector of γ makes a constant angle with fixed line. First we obtain Finslerian spherical spirals that obtained the intersection of the Finslerian sphere and the spherical helicoid.

Let f be a positive continuous function then the surface of revolution generated by rotating the curve $y = f(z), z \in [a, b]$ around the $oz-$axis has the parametric equation

$$\begin{cases} x = \frac{f(t)\cos s - b}{1-b^2}, \\ y = \frac{f(t)\sin s}{\sqrt{1-b^2}}, \\ z = \frac{t}{\sqrt{1-b^2}} \end{cases}, \quad (t, s) \in [a, b] \times [0, 2\pi).$$

From here, the helicoid is obtained as

$$\begin{cases} x = \frac{t\cos s - b}{1-b^2}, \\ y = t\frac{\sin s}{\sqrt{1-b^2}}, \\ z = \frac{ct}{\sqrt{1-b^2}} \end{cases}, \quad (t, s) \in [a, b] \times [0, 2\pi),$$

the sphere is obtained as

$$\begin{cases} x = \frac{\sqrt{r^2-t^2}\cos s - b}{1-b^2}, \\ y = \frac{\sqrt{r^2-t^2}\sin s}{\sqrt{1-b^2}}, \\ z = \frac{t}{\sqrt{1-b^2}} \end{cases}, \quad (t, s) \in [a, b] \times [0, 2\pi).$$

As the intersection of the circular helicoid and Finslerian sphere gives rice to a three-dimensional spiral obtained as follows:

$$\begin{cases} x = \frac{\sqrt{r^2-t^2}\cos\frac{t}{c} - b}{1-b^2}, \\ y = \frac{\sqrt{r^2-t^2}\sin\frac{t}{c}}{\sqrt{1-b^2}}, \\ z = \frac{t}{\sqrt{1-b^2}} \end{cases}, \quad t \in [a, b].$$

From the above calculation we can give the following parametrization of the Finslerian spherical helicoid.

Finslerian Spherical helicoid: The sphere of radius r is generated by the function $f : [-r, r] \to [0, r]$, $f(z) = \sqrt{r^2 - z^2}$. Then the Finslerian spherical helicoid are given in the following parametric representation:

$$\begin{cases} x = \frac{u\sqrt{r^2-v^2}\cos\frac{v}{c} - b}{1-b^2}, \\ y = \frac{u\sqrt{r^2-v^2}\sin\frac{v}{c}}{\sqrt{1-b^2}}, \\ z = \frac{v}{\sqrt{1-b^2}} \end{cases}, \quad (u, v) \in [0, 1] \times [-r, r].$$

Now we give various examples of the Finslerian spherical helical curves and the Finslerian spherical helicoid. The images of the spirals and Finslerian spherical helicoid are plotted in Figure 2.

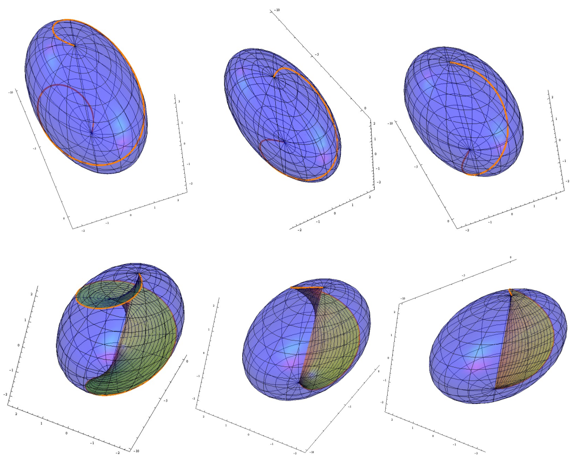

Figure 2. Spirals and helicoid on the $\mathbb{FS}^2$ for $b = 0.9$.

Let us observe that if we take different values of c, then we obtain different shapes of the spherical helical curves. Moreover, the $x - y$ projections of these curves are different. The family of these curves, which reminds us the Limacon of Pascal with a different view, is given by the following parametric representation:

$$\begin{cases} x = \frac{\sqrt{r^2-t^2}\cos\frac{t}{c}-b}{1-b^2}, \\ y = \frac{\sqrt{r^2-t^2}\sin\frac{t}{c}}{\sqrt{1-b^2}}, \end{cases}, \quad t \in [-r, r].$$

The images of the curve are illustrated in Figure 3. The first image demonstrate the projections of the values $c = 1\ (green)$, $c = \frac{1}{2}\ (orange)$ and $c = \frac{1}{4}\ (blue)$. The second images show projection of the value$c = \frac{1}{6}$ and $c = \frac{1}{8}$, respectively, in Figure 3.

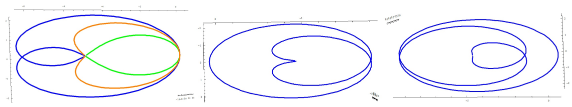

Figure 3. projections of the spirals.

Now we obtain the Viviani's curve obtained the intersection of a circular cylinder and a sphere in Euclidean 3 space Gray (1997, p. 201). This curve can be given the intersection of the Finslerian cylinder of radius a and center $(a, 0)$

$$F_S(u, v) = (\frac{\cos u + a - b}{1 - b^2}, \frac{\sin u}{\sqrt{1 - b^2}} v)$$

and the Finslerian sphere with center $(0, 0, 0)$ and radius $2a$.

$$\mathbb{FS}^2(u, v) = (\frac{2a\cos u\sin v - b}{1 - b^2}, \frac{2a\cos u\cos v}{\sqrt{1 - b^2}}, \frac{2a\sin u}{\sqrt{1 - b^2}})$$

Then the Viviani's curve on the Finslerian sphere has the following parametric representation

$$\gamma(t) = (\frac{a(1 + \cos t) - b}{1 - b^2}, \frac{a\sin t}{\sqrt{1 - b^2}}, \frac{2a\sin\frac{t}{2}}{\sqrt{1 - b^2}}).$$

The images of this curve is shown in Figure 4.

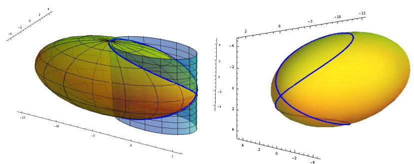

Figure 4. Viviani's curve on the $\mathbb{FS}^2$ for $b = 0.9$.

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