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Research Article

The A-integral and Restricted Riesz Transform

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ABSTRACT. It is known that the restricted Riesz transform of a Lebesgue integrable function is not Lebesgue integrable. In this paper, we prove that the restricted Riesz transform of a Lebesgue integrable function is A-integrable and the analogue of Riesz's equality holds.

Keywords: Riesz transform, A-integral, Riesz's equality, covering theorem, singular integrals.

2010 Mathematics Subject Classification: 44A15, 42B20, 26A39, 26B10.

1. INTRODUCTION

The j-th Riesz transform of a function $f \in L_p(R^d)$, $1 \le p < +\infty$ is defined as the following singular integral:

$$R_j(f)(x) = C_{(d)} \lim_{\varepsilon \to 0} \int_{\{y \in R^d:\, |x-y|>\varepsilon\}} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)dy,\ j = \overline{1,d},$$

where $C_{(d)} = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}$, $\Gamma(z) = \int\limits_0^{+\infty} t^{z-1}e^{-t}dt$ - Euler Gamma function and d - dimension of the space R^d.

Let $\Omega \subset R^d$ be a bounded domain and $f \in L_1(\Omega)$. In the present paper, we consider the corresponding modification of R_j. Namely, the restricted Riesz transform $R_{j,\Omega}$ is defined as

$$(R_{j,\Omega}f)(z) = R_j(\chi_\Omega f)(z)$$
$$= C_{(d)} \lim_{\varepsilon \to 0} \int_{\{y \in \Omega:\, |x-y|>\varepsilon\}} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)dy, j = \overline{1,d}, z \in \Omega,$$

where $\chi_\Omega(z) = 1$ for $z \in \Omega$ and $\chi_\Omega(z) = 0$ for $z \in R^d \backslash \Omega$ is the characteristic function of the set Ω.

From the theory of singular integrals (see [15]) it is known that the Riesz transform is a bounded operator in the space $L_p(\Omega)$, $p > 1$, that is, if $f \in L_p(\Omega)$, then $R_{j,\Omega}(f) \in L_p(\Omega)$ and the inequality

$$\|R_{j,\Omega}f\|_{L_p} \le C^{(p)}\|f\|_{L_p} \tag{1.1}$$

holds. In the case $f \in L_1(\Omega)$ only the weak inequality holds:

$$m\{x \in \Omega:\ |(R_{j,\Omega}f)(x)| > \lambda\} \le \frac{C_1}{\lambda}\|f\|_{L_1}, \tag{1.2}$$

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where m stands for the Lebesgue measure, $C^{(p)}$, C_1 are constants independent of f. From the inequalities (1.1), (1.2) it follows that the Riesz transform of the function $f \in L_1(\Omega)$ satisfies the condition

$$m\{x \in \Omega : \; |(R_{j,\Omega} f)(x)| > \lambda\} = o\left(\frac{1}{\lambda}\right), \lambda \to +\infty.$$

Note that the Riesz transform of a function $f \in L_1(\Omega)$ is not Lebesgue integrable. In this paper, we prove that the Riesz transform of a function $f \in L_1(\Omega)$ is A-integrable on Ω and the analogue of Riesz's equality holds.

2. A - INTEGRAL

For a measurable complex function $f(x)$ on domain Ω we set

$$[f(x)]_n = [f(x)]^n = f(x) \text{ for } |f(x)| \le n$$

$$[f(x)]_n = n \operatorname{sgn} f(x), \; [f(x)]^n = 0 \text{ for } |f(x)| > n, \; n \in N,$$

where $\operatorname{sgn} z = z/|z|$ for $z \neq 0$ and $\operatorname{sgn} 0 = 0$.

In 1929, E.Titchmarsh [16] introduced the notions of Q- and Q'-integrals of function, measurable on Ω.

Definition 2.1. If a finite limit $\lim\limits_{n\to\infty} \int_\Omega [f(x)]_n dx$ ($\lim\limits_{n\to\infty} \int_\Omega [f(x)]^n dx$, respectively) exists, then f is said to be *Q-integrable* (*Q'-integrable*, respectively) on Ω, that is $f \in Q(\Omega)$ ($f \in Q'(\Omega)$), and the value of this limit is referred to as the *Q-integral* (*Q'-integral*) of this function and is denoted by

$$(Q) \int_\Omega f(x) dx \; \left((Q') \int_\Omega f(x) dx\right).$$

In the same paper, E.Titchmarsh established that, when studying the properties of trigonometric series conjugate to Fourier series of Lebesgue integrable functions, Q-integration leads to a series of natural results. A very uncomfortable fact impeding the application of Q-integrals and Q'-integrals when studying diverse problems of function theory is the absence of the additivity property, that is, the Q-integrability (Q'-integrability) of two functions does not imply the Q-integrability (Q'-integrability) of their sum. If one adds the condition

$$m\{\, z \in \Omega : \; |f(x)| > \lambda \,\} = o\left(\frac{1}{\lambda}\right), \lambda \to +\infty \tag{2.3}$$

to the definition of Q-integrability (Q'-integrability) of a function f, then the Q-integral and Q'-integral coincide ($Q(\Omega) = Q'(\Omega)$), and these integrals become additive.

Definition 2.2. If $f \in Q'(\Omega)$ (or $f \in Q(\Omega)$) and condition (2.3) holds, then f is said to be *A-integrable* on Ω, $f \in A(\Omega)$, and the limit $\lim\limits_{n\to\infty} \int_\Omega [f(x)]_n dx$ (or the limit $\lim\limits_{n\to\infty} \int_\Omega [f(x)]^n dx$) is denoted in this case by

$$(A) \int_\Omega f(x) dx.$$

The properties of Q- and Q'-integrals were investigated in [2, 8, 9, 16, 17], and for the applications of A-, Q- and Q'-integrals in the theory of functions of real and complex variables, covering and flattering arguments, we refer the reader to [1, 2, 3, 4, 5, 6, 11, 12, 13, 14, 17, 18] and references therein.

3. A - Integrability and Riesz's equality for the Riesz transform of Lebesgue integrable functions

From the properties of singular integrals it follows that (see [15]) if $f \in L_p(\Omega)$, $p > 1$ and $g \in L_q(\Omega)$, $q > 1$, $1/p + 1/q = 1$, then

$$\int_\Omega g(x)(R_{j,\Omega} f)(x)dx$$

$$= C_d \lim_{\varepsilon\to 0} \iint_{\{x,y\in\Omega:\,|x-y|>\varepsilon\}} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)g(x)dydx$$

$$= -\int_\Omega f(x)(R_{j,\Omega} g)(x)dx. \tag{3.4}$$

In this section, we prove that the Riesz transform of the function $f \in L_1(\Omega)$ is A-integrable on Ω and the analogue of (3.1) holds.

Theorem 3.1. Let $f \in L_1(\Omega)$ and $g(x)$ is a bounded function on Ω such that the $(R_{j,\Omega} g)(x)$ is also bounded on Ω. Then the function $g(x) \cdot (R_{j,\Omega} f)(x)$ is A-integrable on Ω and the equation

$$(A)\int_\Omega g(x)(R_{j,\Omega} f)(x)dx = -\int_\Omega f(x)(R_{j,\Omega} g)(x)dx. \tag{3.5}$$

holds.

Proof. Since the A-integral satisfies the additivity property, it can be assumed that the function f is real, $f(x) \geq 0$ for any $x \in \Omega$, and

$$\max_{x\in\Omega}\{|g(x)|,\ |(R_{j,\Omega} g)(x)|\} \leq 1.$$

For $x \notin \Omega$, we assume that $f(x) = 0$.

Our proof will depend on a certain refinement of Besicovitch's method [7] for a direct proof of the existence of conjugate function (this method employs only the machinery of the theory of sets of points). This method was improved by Titchmarsh [16] and Ul'yanov [17] for the study of properties of the conjugate function. It is worth noting that Besicovitch–Titchmarsh–Ul'yanov's method is applicable only to functions of one real variable (because this method relies on some facts that are valid only in one-dimensional case). For example, it depends on the fact that any open set is a union of at most a countable number of intervals (to overcome this difficulty, we used Vitali's covering theorem). To make this method to work in the setting of functions of several variables, we have slightly improved the construction.

Denote $\Phi_n(x) = f(x) - [f(x)]^n$. Then $\alpha_n = \int_\Omega \Phi_n(x)dx \to 0$ at $n \to \infty$. Take $n \in N$ such that $\alpha_n < 1$. Let $E_n = \{x \in \Omega:\ f(x) > n\}$. For any $x \in E_n$, we set

$$r_x = \sup\{r > 0 : \int_{B(x;r)} \Phi_n(y)dy = \frac{1}{2} \cdot \frac{\pi^{d/2}}{\Gamma(1+d/2)} r^d \cdot n\}$$

if $\{r > 0 :\ \int_{B(x;r)} \Phi_n(y)dy = \frac{1}{2} \cdot \frac{\pi^{d/2}}{\Gamma(1+d/2)} r^d \cdot n\} \neq \emptyset$, and $r_x = 0$ otherwise, where $B(x;r)$ - open ball with center at x and with radius $r > 0$. Note that if $x \in E_n$ is a Lebesgue point of the function $\Phi_n(x)$, then $r_x > 0$ and, therefore, the set $E_n \backslash E'_n$ has a zero measure, where $E'_n = \{x \in E_n :\ r_x > 0\}$.

Consider the system of sets $\{B(x; r_x)\}_{x\in E'_n}$. It follows from the covering theorem (see [10]) that there exists at most a countable points $x_k \in E'_n$, $k \in I \subset N$ such that the balls $B(x_k; r_{x_k})$,

$k \in I$ are pairwise disjoint and

$$\bigcup_{x \in E_n'} B(x;\, r_x) \subset \bigcup_{k \in I} B(x_k;\, 5r_{x_k}).$$

Denote

$$G_1 = B(x_1; 5\, r_{x_1}) \setminus \bigcup_{k>1} B(x_k;\, r_{x_k}),$$

$$G_p = B(x_p; 5r_{x_p}) \setminus \left[\bigcup_{j=1}^{p-1} G_j \bigcup \left(\bigcup_{k>p} B(x_k;\, r_{x_k}) \right) \right], p \geq 2, p \in I.$$

Then, the measurable sets G_p, $p \in I$ are pairwise disjoint, and

$$B(x_p;\, r_{x_p}) \subset G_p \subset B(x_p;\, 5r_{x_p}), p \in I,$$

$$E_n' \subset \bigcup_{x \in E_n} B(x;\, r_x) \subset \bigcup_{p \in I} G_p = \bigcup_{p \in I} B(x_p;\, 5r_{x_p}).$$

Denote $\Phi_n^*(x) = \frac{1}{m(G_p)} \int_{G_p} \Phi_n(y)dy$ for $z \in G_p$, $p \in I$ and $\Phi_n^*(x) = 0$ for $x \in R^d \setminus \bigcup_{p \in I} G_p$. Then for any $p \in I$, we have

$$\int_{G_p} \Phi_n(x)dx = \int_{G_p} \Phi_n^*(x)dx. \tag{3.6}$$

Note that for any $x \in G_p$, $p \in I$, the inequalities

$$0 \leq \Phi_n^*(x) \leq \frac{1}{m(B(x_p;\, r_{x_p}))} \int_{B(x_p;\, 5r_{x_p})} \Phi_n(y)dy$$

$$\leq \frac{\Gamma(1+d/2)}{\pi^{d/2} r_{x_p}^d} \cdot \frac{1}{2} \frac{\pi^{d/2} \cdot 5^d r_{x_p}^d}{\Gamma(1+d/2)} \cdot n = \frac{5^d n}{2}$$

hold. Denote $L_n = \bigcup_{p \in I} G_p$, $L_n' = \bigcup_{p \in I} B(x_p;\, 10r_{x_p})$. Then

$$m(L_n) \leq \sum_{p \in I} \frac{\pi^{d/2} \cdot 5^d r_{x_p}^d}{\Gamma(1+d/2)} \leq 5^d \cdot \frac{2}{n} \sum_{p \in I} \int_{B(x_p;\, r_{x_p})} \Phi_n(y)dy$$

$$\leq 5^d \cdot \frac{2}{n} \sum_{p \in I} \int_{B(x_p;\, r_{x_p})} \Phi_n(y)dy \leq \frac{2 \cdot 5^d}{n} \int_{\Omega} \Phi_n(y)dy = \frac{2 \cdot 5^d \alpha_n}{n},$$

$$m(L_n') \leq \sum_{p \in I} \frac{\pi^{d/2} \cdot 10^d r_{x_p}^d}{\Gamma(1+d/2)} \leq \frac{2 \cdot 10^d \alpha_n}{n}. \tag{3.7}$$

Let $T_n = \Omega \setminus L_n'$. First we prove that the inequality

$$\int_{T_n} |R_{j,\Omega}(\Phi_n - \Phi_n^*)(x)|dx < C_2 \cdot \alpha_n \tag{3.8}$$

holds, where C_2 is a constant, independent of n. Denote

$$h_n(x) = R_{j,\Omega}(\Phi_n - \Phi_n^*)(x).$$

For any $x \in T_n$, we have

$$|h_n(x)| = C_{(d)} \left| \int_{\Omega} \frac{x_j - y_j}{|x-y|^{d+1}} [\Phi_n(y) - \Phi_n^*(y)]dy \right|$$

$$= C_{(d)} \left| \sum_{p \in I} \int_{G_p} \frac{x_j - y_j}{|x-y|^{d+1}} [\Phi_n(y) - \Phi_n^*(y)] dy \right|$$

$$\leq C_{(d)} \sum_{p \in I} \left| \int_{G_p} \frac{x_j - y_j}{|x-y|^{d+1}} \Phi_n(y) dy - \int_{G_p} \frac{x_j - y_j}{|x-y|^{d+1}} \Phi_n^*(y) dy \right|. \tag{3.9}$$

It follows from the integral mean value theorem that for any $p \in I$ there are points $y^{(p)}, \tilde{y}^{(p)} \in B(x_p; 5r_{x_p})$, such that

$$\int_{G_p} \frac{x_j - y_j}{|x-y|^{d+1}} \Phi_n(y) dy = \frac{x_j - y_j^{(p)}}{|x - y^{(p)}|^{d+1}} \cdot \int_{G_p} \Phi_n(y) dy,$$

$$\int_{G_p} \frac{x_j - y_j}{|x-y|^{d+1}} \Phi_n^*(y) dy = \frac{x_j - \tilde{y}_j^{(p)}}{|x - \tilde{y}^{(p)}|^{d+1}} \cdot \int_{G_p} \Phi_n^*(y) dy.$$

Then from (3.6) and (3.9), we obtain that

$$|h_n(x)| \leq C_{(d)} \sum_{p \in I} \left| \frac{x_j - y_j^{(p)}}{|x - y^{(p)}|^{d+1}} - \frac{x_j - \tilde{y}_j^{(p)}}{|x - \tilde{y}^{(p)}|^{d+1}} \right| \cdot \int_{G_p} \Phi_n(y) dy. \tag{3.10}$$

Since for any $y, \tilde{y} \in B(x_p; 5r_{x_p})$ and $x \in T_n$, the inequality

$$\left| \frac{x_j - y_j}{|x-y|^{d+1}} - \frac{x_j - \tilde{y}_j}{|x-\tilde{y}|^{d+1}} \right|$$

$$\leq \frac{|(x_j - y_j) \cdot [|x-\tilde{y}|^{d+1} - |x-y|^{d+1}] - (y_j - \tilde{y}_j) \cdot |x-y|^{d+1}|}{|x-y|^{d+1} \cdot |x-\tilde{y}|^{d+1}}$$

$$\leq \frac{|x_j - y_j| \cdot ||x-\tilde{y}| - |x-y|| \cdot \sum_{k=0}^{d} |x-\tilde{y}|^k \cdot |x-y|^{d-k}}{|x-y|^{d+1} \cdot |x-\tilde{y}|^{d+1}} + \frac{|y_j - \tilde{y}_j|}{|x-\tilde{y}|^{d+1}}$$

$$\leq \frac{(d+1) \cdot 10 r_{x_p} \cdot 2^{d+1}}{|x-x_p|^{d+1}} + \frac{10 r_{x_p} \cdot 2^{d+1}}{|x-x_p|^{d+1}} = \frac{10(d+2) \cdot 2^{d+1} \cdot r_{x_p}}{|x-x_p|^{d+1}}$$

holds, then it follows from (3.10) that

$$|h_n(x)| \leq C_{(d)} \sum_{p \in I} \frac{10(d+2) \cdot 2^{d+1} \cdot r_{x_p}}{|x-x_p|^{d+1}} \cdot \int_{G_p} \Phi_n(y) dy$$

$$\leq C_{(d)} \sum_{p \in I} \frac{10(d+2) \cdot 2^{d+1} \cdot r_{x_p}}{|x-x_p|^{d+1}} \cdot \left(\frac{1}{2} \cdot \frac{\pi^{d/2}}{\Gamma(1+d/2)} r_{x_p}^d \cdot n\right)$$

$$= C_3 \cdot n \cdot \sum_{p \in I} \frac{r_{x_p}^{d+1}}{|x-x_p|^{d+1}},$$

where $C_3 = \frac{\Gamma((d+1)/2) \cdot 10(d+2) \cdot 2^d}{\pi^{1/2} \cdot \Gamma(1+d/2)}$. From this, we get that

$$\int_{T_n} |h_n(x)| dx \leq C_3 \cdot n \cdot \sum_{p \in I} r_{x_p}^{d+1} \int_{T_n} \frac{dx}{|x-x_p|^{d+1}}$$

$$\leq C_3 \cdot n \cdot \sum_{p \in I} r_{x_p}^{d+1} \int_{\{x \in R^d : |x-x_p| \geq 10 r_{x_p}\}} \frac{dx}{|x-x_p|^{d+1}}$$

$$= C_3 \cdot n \cdot \sum_{p\in I} r_{x_p}^{d+1} \cdot \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{10r_{x_p}}^{+\infty} \frac{dr}{r^2} = \frac{C_3\pi^{d/2}}{5\Gamma(d/2)} \cdot n \cdot \sum_{p\in I} r_{x_p}^d$$
$$= \frac{C_3\pi^{d/2}}{5\Gamma(d/2)} \cdot n \cdot \frac{2\alpha_n}{n} \cdot \frac{\Gamma(1+d/2)}{\pi^{d/2}} = \frac{C_3 d}{5}\alpha_n.$$

That is, inequality (3.8) is satisfied.

We represent the function $f(x)$ in the form

$$f(x) = [f(x)]^n + \Phi_n^*(x) + [\Phi_n - \Phi_n^*](x). \tag{3.11}$$

Let us prove that the equality

$$\lim_{n\to\infty} \int_{T_n} g(x)(R_{j,\Omega}f)(x)dx = -\int_\Omega f(x)(R_{j,\Omega}g)(x)dx. \tag{3.12}$$

holds. Consider the integral

$$\int_{T_n} g(x)(R_{j,\Omega}f)(x)dx$$
$$= \int_{T_n} g(x)\{(R_{j,\Omega}[f]^n)(x) + (R_{j,\Omega}\Phi_n^*)(x) + R_{j,\Omega}(\Phi_n - \Phi_n^*)(x)\}dx$$
$$= \int_{T_n} g(x)(R_{j,\Omega}[f]^n)(x)dx + \int_{T_n} g(x)(R_{j,\Omega}\Phi_n^*)(x)dx$$
$$+ \int_{T_n} g(x)R_{j,\Omega}(\Phi_n - \Phi_n^*)(x)dx = S_1 + S_2 + S_3. \tag{3.13}$$

By (3.4) for square integrable functions, we obtain

$$S_1 = \int_{T_n} g(x)(R_{j,\Omega}[f]^n)(x)dx$$
$$= \int_\Omega g(x)(R_{j,\Omega}[f]^n)(x)dx - \int_{L_n'} g(x)(R_{j,\Omega}[f]^n)(x)dx$$
$$= -\int_\Omega [f(x)]^n(R_{j,\Omega}g)(x)dx - \int_{L_n'} g(x)(R_{j,\Omega}[f]^n)(x)dx = S_1^{(1)} + S_1^{(2)}.$$

Since

$$|S_1^{(2)}| = \left|\int_{L_n'} g(x)(R_{j,\Omega}[f]^n)(x)dx\right| \le \int_{L_n'} |(R_{j,\Omega}[f]^n)(x)|dx$$
$$\le \left[m(L_n') \cdot \int_\Omega (R_{j,\Omega}[f]^n)^2(x)dx\right]^{1/2}$$
$$\le C^{(2)}\left[m(L_n') \cdot \int_\Omega ([f(x)]^n)^2 dx\right]^{1/2}$$
$$\le C^{(2)}\left[n \cdot m(L_n') \cdot \int_\Omega f(x)dx\right]^{1/2},$$

then it follows from (3.7) that

$$\lim_{n\to\infty} S_1 = -\lim_{n\to\infty} \int_\Omega [f(x)]^n(R_{j,\Omega}g)(x)dx$$
$$= -\int_\Omega f(x)(R_{j,\Omega}g)(x)dx. \tag{3.14}$$

For the integral S_2, we also have

$$S_2 = \int_{T_n} g(x)(R_{j,\Omega}\Phi_n^*)(x)dx$$
$$= \int_{\Omega} g(x)(R_{j,\Omega}\Phi_n^*)(x)dx - \int_{L'_n} g(x)(R_{j,\Omega}\Phi_n^*)(x)dx$$
$$= -\int_{\Omega} \Phi_n^*(x)(R_{j,\Omega}g)(x)dx - \int_{L'_n} g(x)(R_{j,\Omega}\Phi_n^*)(x)dx = S_2^{(1)} + S_2^{(2)}.$$

The following estimations are valid.

$$|S_2^{(1)}| = \left|\int_{\Omega} \Phi_n^*(x)(R_{j,\Omega}g)(x)dx\right| \le \int_{\Omega} |\Phi_n^*(x)(R_{j,\Omega}g)(x)|dx$$
$$\le \int_{\Omega} \Phi_n^*(x)dx = \int_{\Omega} \Phi_n(x)dx = \alpha_n,$$
$$|S_2^{(2)}| = \left|\int_{L'_n} g(x)(R_{j,\Omega}\Phi_n^*)(x)dx\right| \le \int_{L'_n} |(R_{j,\Omega}\Phi_n^*)(x)|dx$$
$$\le \left[m(L'_n)\cdot\int_{\Omega}(R_{j,\Omega}\Phi_n^*)^2(x)dx\right]^{1/2} \le C^{(2)}\left[m(L'_n)\cdot\int_{\Omega}(\Phi_n^*(x))^2dx\right]^{1/2}$$
$$\le C^{(2)}\left[\frac{5^d n}{2}\cdot m(L'_n)\cdot\int_{\Omega}\Phi_n^*(x)dx\right]^{1/2} = C^{(2)}\left[\frac{5^d n}{2}\cdot m(L'_n)\cdot\alpha_n\right]^{1/2}.$$

Then it follows from (3.7) that

$$\lim_{n\to\infty} S_2 = 0. \tag{3.15}$$

To estimate the integral S_3, we need to apply the inequality (3.8):

$$|S_3| = \left|\int_{T_n} g\left(x\right) R_{j,\Omega}\left(\Phi_n - \Phi_n^*\right)(x)\, dx\right|$$
$$\le \int_{T_n} \left|g\left(x\right) R_{j,\Omega}\left(\Phi_n - \Phi_n^*\right)(x)\right| dx$$
$$\le \int_{T_n} \left|R_{j,\Omega}\left(\Phi_n - \Phi_n^*\right)(x)\right| dx < C_2\cdot\alpha_n.$$

This implies the equality

$$\lim_{n\to\infty} S_3 = 0. \tag{3.16}$$

From the equalities (3.13), (3.14), (3.15) and (3.16), we obtain the equality (3.12). It remains to prove the equality

$$(A)\int_{\Omega} g(x)(R_{j,\Omega}f)(x)dx = \lim_{n\to\infty}\int_{T_n} g(x)(R_{j,\Omega}f)(x)dx. \tag{3.17}$$

Consider the difference of integrals

$$\int_{T_n} g(x)(R_{j,\Omega}f)(x)dx - \int_{\Omega}[g(x)(R_{j,\Omega}f)(x)]^n dx$$
$$= -\int_{L'_n}[g(x)(R_{j,\Omega}f)(x)]^n dx+$$

$$+\int_{T_n}\{g(x)(R_{j,\Omega}f)(x)-[g(x)(R_{j,\Omega}f)(x)]^n\}dx=S^{(1)}+S^{(2)}. \tag{3.18}$$

From the inequality $|S^{(1)}|\le n\cdot m(L_n')$, it follows that

$$\lim_{n\to\infty}S^{(1)}=0. \tag{3.19}$$

Denote $\sigma_n=\{x\in\Omega:\ |g(x)(R_{j,\Omega}f)(x)|>n\}$. Since

$$m\{x\in\Omega:\ |(R_{j,\Omega}f)(x)|>n\}=o\left(\frac{1}{n}\right),n\to\infty,$$

then $m(\sigma_n)=o\left(\frac{1}{n}\right)$, $n\to\infty$. Using (3.8) and (3.11), we obtain

$$|S^{(2)}|\le\int_{T_n\bigcap\sigma_n}|g(x)(R_{j,\Omega}f)(x)|dx$$

$$\le\int_{T_n\bigcap\sigma_n}|(R_{j,\Omega}f)(x)|dx\le\int_{\sigma_n}|(R_{j,\Omega}[f]^n)(x)|dx$$

$$+\int_{\sigma_n}|(R_{j,\Omega}\Phi_n^*)(x)|dx+\int_{T_n}|R_{j,\Omega}(\Phi_n-\Phi_n^*)(x)|dx$$

$$\le\left[m(\sigma_n)\cdot\int_{\Omega}(R_{j,\Omega}[f]^n)^2(x)dx\right]^{1/2}$$

$$+\left[m(\sigma_n)\cdot\int_{\Omega}(R_{j,\Omega}\Phi_n^*)^2(x)dx\right]^{1/2}+C\cdot\alpha_n$$

$$\le c_2\left[m(\sigma_n)\cdot\int_{\Omega}([f(x)]^n)^2dx\right]^{1/2}$$

$$+c_2\left[m(\sigma_n)\cdot\int_{\Omega}(\Phi_n^*(x))^2dx\right]^{1/2}+C\cdot\alpha_n$$

$$\le c_2\left[n\cdot m(\sigma_n)\cdot\int_{\Omega}f(x)dx\right]^{1/2}$$

$$+c_2\left[\frac{5^dn}{2}\cdot m(\sigma_n)\cdot\int_{\Omega}\Phi_n^*(x)dx\right]^{1/2}+C\cdot\alpha_n.$$

It follows that

$$\lim_{n\to\infty}S^{(2)}=0. \tag{3.20}$$

From the equalities (3.18), (3.19) and (3.20), we obtain the equality (3.17). Theorem 3.1 is proved.

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Research Article

Ulam Stability in Real Inner-Product Spaces

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ABSTRACT. Roughly speaking an equation is called Ulam stable if near each approximate solution of the equation there exists an exact solution. In this paper, we prove that Cauchy-Schwarz equation, Orthogonality equation and Gram equation are Ulam stable.

Keywords: Ulam stability, inner-product space.

2010 Mathematics Subject Classification: 39B72, 39B82.

1. INTRODUCTION

This paper is concerned with the Ulam stability of some classical equations arising in the context of inner-product spaces. For the general notion of Ulam stability see, e.q., [1]. Roughly speaking an equation is called Ulam stable if near every approximate solution there exists an exact solution; the precise meaning in each case presented in this paper is described in three theorems. Related results can be found in [2, 3, 4]. See also [5] for some inequalities in inner product spaces.

2. THE CAUCHY-SCHWARZ EQUATION

Let $(V, (\cdot|\cdot))$ be a real inner-product space. Consider the Cauchy-Schwarz equation, i.e.,

$$\|x\|^2\|y\|^2 - (x|y)^2 = 0. \tag{2.1}$$

The set of its solutions is

$$S = \{(x, y) \in V^2 : x, y \ \ are \ \ linearly \ \ dependent \ \ vectors\}. \tag{2.2}$$

Theorem 2.1. *Let $\varepsilon > 0$ and $(u, v) \in V^2$ an approximate solution of (2.1), i.e.,*

$$\|u\|^2\|v\|^2 - (u|v)^2 \leq \varepsilon. \tag{2.3}$$

Then there exists an exact solution $(x, y) \in S$ such that

$$\|u - x\|^2 + \|v - y\|^2 \leq \sqrt{\varepsilon}. \tag{2.4}$$

Proof. If u and v are linearly dependent, then it suffices to take $x = u, y = v$. So, let u and v be linearly independent. Then $u \neq 0$, $v \neq 0$; suppose that $\|v\| \leq \|u\|$ and let $w_t := u + tv, t \in \mathbb{R}$. Then $w_t \neq 0, t \in \mathbb{R}$; let $z_t := w_t/\|w_t\|$ and $W_t := Span\{z_t\}$. Let $x_t := pr_{W_t}u$ and $y_t := pr_{W_t}v$ be the orthogonal projections of u, respectively v, on W_t. Then x_t and y_t are linearly dependent

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vectors, i.e., $(x_t, y_t) \in V^2$ is a solution of (2.1). Moreover, $\|u-x_t\|^2+\|v-y_t\|^2 = \|u\|^2-(u|z_t)^2+\|v\|^2-(v|z_t)^2$. Let $a := (v|v), b := (u|v), c := (u|u)$. Then

$$ac - b^2 > 0; 0 < a \leq c. \tag{2.5}$$

We have

$$\|u - x_t\|^2 + \|v - y_t\|^2 = c + a - (u|z_t)^2 - (v|z_t)^2 \tag{2.6}$$

and

$$s := s(t) := (u|z_t)^2 + (v|z_t)^2 = \frac{(u|w_t)^2 + (v|w_t)^2}{\|w_t\|^2} = \frac{(c+tb)^2 + (b+ta)^2}{\|w_t\|^2},$$

so that

$$s = \frac{(a^2+b^2)t^2 + 2b(a+c)t + b^2 + c^2}{at^2 + 2bt + c}.$$

It follows that

$$(a^2 + b^2 - as)t^2 + 2b(a + c - s)t + b^2 + c^2 - cs = 0,$$

which entails

$$b^2(a + c - s)^2 - (a^2 + b^2 - as)(b^2 + c^2 - cs) \geq 0. \tag{2.7}$$

From (2.5) and (2.7), we deduce

$$s^2 - (a + c)s + ac - b^2 \leq 0.$$

Let s_1 and s_2 be the roots of the corresponding equation, i.e.,

$$s_1 = \frac{a + c - \sqrt{(a-c)^2 + 4b^2}}{2}, s_2 = \frac{a + c + \sqrt{(a-c)^2 + 4b^2}}{2}.$$

Then $s_1 \leq s(t) \leq s_2$ for all $t \in \mathbb{R}$. By using (2.5), it is easy to prove that there exists $\tau \in \mathbb{R}$ such that $s(\tau) = s_2$. Now,

$$\|u - x_\tau\|^2 + \|v - x_\tau\|^2 = a + c - s(\tau) = a + c - s_2 = a + c - \frac{a + c + \sqrt{(a-c)^2 + 4b^2}}{2} =$$

$$= \frac{a + c - \sqrt{(a-c)^2 + 4b^2}}{2} \leq \sqrt{ac - b^2} = \sqrt{\|u\|^2\|v\|^2 - (u|v)^2}.$$

Combined with (2.3), this gives (2.4) and the proof is finished. □

3. The orthogonality equation

Consider the orthogonality equation $(x|y) = 0$.

Theorem 3.2. *Let $\varepsilon > 0$ and $(u, v) \in V^2$ such that $\|u\| = \|v\| = 1$ and $|(u|v)| \leq \varepsilon$. Then, there exists $(x, y) \in V^2$ such that $\|x\| = \|y\| = 1, (x|y) = 0$ and*

$$\|u - x\|^2 + \|v - y\|^2 \leq (4 - 2\sqrt{2})\varepsilon. \tag{3.8}$$

Proof. (i) Let $(u|v) > 0$. Choose $w \in Span\{u, v\}, \|w\| = 1, (w|u) = 0$. Then $v = u\cos\alpha + w\sin\alpha$, for a suitable $\alpha \in [0, \frac{\Pi}{2})$. Define $x_t := u\cos t - w\sin t, y_t := u\sin t + w\cos t, t \in \mathbb{R}$. Then $\|x_t\| = 1, \|y_t\| = 1, (x_t|y_t) = 0$, and $\|u-x_t\|^2+\|v-y_t\|^2 = \|(1-\cos t)u+w\sin t\|^2+\|(\cos\alpha-\sin t)u+(\sin\alpha-\cos t)w\|^2 = (1-\cos t)^2+\sin^2 t+(\cos\alpha-\sin t)^2+(\sin\alpha-\cos t)^2 = 4-2((1+\sin\alpha)\cos t+\cos\alpha\sin t)$. Clearly $(1+\sin\alpha)\cos t+\cos\alpha\sin t \leq \sqrt{2+2\sin\alpha}, t \in \mathbb{R}$. Choose $\tau \in \mathbb{R}$ such that

$$(1 + \sin\alpha)\cos\tau + \cos\alpha\sin\tau = \sqrt{2 + 2\sin\alpha}.$$

Then

$$\|u - x_\tau\|^2 + \|v - y_\tau\|^2 = 4 - 2\sqrt{2}\sqrt{1 + \sin\alpha}. \tag{3.9}$$

Now consider the function $f(\alpha) = (4 - 2\sqrt{2})\cos\alpha - 4 + 2\sqrt{2}\sqrt{1 + \sin\alpha}$, $\alpha \in [0, \frac{\Pi}{2}]$. It is easy to verify that $f(0) = f(\frac{\Pi}{2}) = 0$ and there exists $0 < \beta < \frac{\Pi}{2}$ such that f is increasing on $[0, \beta]$ and decreasing on $[\beta, \frac{\Pi}{2}]$. It follows that $f(\alpha) \geq 0, \alpha \in [0, \frac{\Pi}{2}]$; combined with (3.9), this yields

$$\|u - x_\tau\|^2 + \|v - y_\tau\|^2 \leq (4 - 2\sqrt{2})\cos\alpha. \tag{3.10}$$

On the other hand, $\cos\alpha = (u|v) \leq \varepsilon$, and so (3.8) is a consequence of (3.10).

(ii) If $(u|v) < 0$, it suffices to use the proof of (i) with v replaced by $-v$. Thus, the theorem is proved.

□

4. The Gram equation

Denote by $G(u_1, ..., u_m)$ the Gram determinant of the vectors $u_1, ..., u_m \in V$. Let $v_1, ..., v_n \in V$ be linearly independent vectors. Consider the equation

$$G(x, v_1, ..., v_n) = 0. \tag{4.11}$$

Theorem 4.3. *Let $\varepsilon > 0$ and $u \in V$ such that*

$$G(u, v_1, ..., v_n) \leq \varepsilon.$$

Then, there exists $x \in V$ which satisfy (4.11) and

$$\|u - x\| \leq \frac{1}{\sqrt{G(v_1, ..., v_n)}}\sqrt{\varepsilon}.$$

Proof. Let $W = Span\{v_1, ..., v_n\}$ and $x := pr_W u$. Then $x \in W$ and therefore it satisfies (4.11). Moreover,

$$\|u - x\| = \sqrt{\frac{G(u, v_1, ..., v_n)}{G(v_1, ..., v_n)}} \leq \frac{1}{\sqrt{G(v_1, ..., v_n)}}\sqrt{\varepsilon}.$$

□

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Research Article

A New Class of Kantorovich-Type Operators

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ABSTRACT. The purpose of the paper called "A new class of Kantorovich-type operators", as the title says, is to introduce a new class of Kantorovich-type operators with the property that the test functions e_1 and e_2 are reproduced. Furthermore, in our approach, an asymptotic type convergence theorem, a Voronovskaja type theorem and two error approximation theorems are given. As a conclusion, we make a comparison between the classical Kantorovich operators and the new class of Kantorovich - type operators.

Keywords: Bernstein polynomials, Kantorovich operators, King operators, fixed points.

2010 Mathematics Subject Classification: 41A36, 41A60.

1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We denote by e_j the monomial of j degree, $j \in \mathbb{N}_0$, $L_1([0,1]) = \{f | f : [0,1] \longrightarrow \mathbb{R}$ and f integrable Lebesgue on $[0,1]\}$.

In 1930, L. Kantorovich [7] constructed and studied the linear positive operators $K_m : L_1([0,1]) \longrightarrow C([0,1])$, defined for any $f \in L_1([0,1])$, $x \in [0,1]$ and $m \in \mathbb{N}$ by

$$(K_m f)(x) = (m+1) \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt. \tag{1.1}$$

The operators (1.1) are known as Kantorovich operators and they preserve the test function e_0. Following the ideas from [3]-[6], in this paper we introduce a general class which preserves the test functions e_1 and e_2. For our operators a convergence theorem, a Voronovskaja-type theorem and two error approximation theorems are obtained.

The paper is organized as follows: in Section 2 we introduce some preliminary notions which we will use in the construction of the new type of Kantorovich operators, in Section 3 we will construct the new operators and in Section 4 we give an asymptotic type convergence theorem, a Voronovskaja type theorem, two error approximation theorems and a comparison between the classical Kantorovich operators and the new one.

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2. PRELIMINARIES

In this section, we recall some notions and results which we will use in what follows. We consider I, J real intervals with the property $I \cap J \neq \emptyset$, let $E(I), F(J)$ be certain subsets of the space of all real functions defined on I, respectively J,

$$B(I) = \{f | f : I \to \mathbb{R}, f \text{ bounded on } I\},$$
$$C(I) = \{f | f : I \to \mathbb{R}, f \text{ continuous on } I\}$$

and

$$C_B(I) = B(I) \cap C(I).$$

For $x \in I$, we consider the function $\psi_x : I \to \mathbb{R}, \psi_x(t) = t - x, t \in I$. For any $m \in \mathbb{N}$, we consider the functions $\varphi_{m,k} : J \to \mathbb{R}$, with the property $\varphi_{m,k}(x) \geq 0$, for any $x \in J, k \in \{0, 1, ..., m\}$ and the linear positive functionals $A_{m,k} : E(I) \to \mathbb{R}$, $k \in \{0, 1, ..., m\}$. For $m \in \mathbb{N}$, we define the operators $L_m : E(I) \to F(J)$ by

$$(L_m f)(x) = \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(f). \tag{2.1}$$

Remark 2.1. *The operators $(L_m)_{m\in\mathbb{N}}$ are linear and positive on $E(I \cap J)$.*

For any $f \in E(I), x \in I \cap J$ and for $i \in \mathbb{N}_0$, we define $T_{m,i}$ by

$$(T_{m,i} L_m)(x) = m^i (L_m \psi_x^i)(x) = m^i \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(\psi_x^i). \tag{2.2}$$

In the following, let s be a fixed even natural number and we suppose that the operators $(L_m)_{m\in\mathbb{N}}$ verifies the following conditions:
there exists the smallest $\alpha_s, \alpha_{s+2} \in [0, \infty)$ such that

$$\lim_{m \longrightarrow \infty} \frac{(T_{m,j} L_m)(x)}{m^{\alpha_j}} = B_j(x) \in \mathbb{R}, \tag{2.3}$$

for any $x \in I \cap J, j \in \{s, s+2\}$ and

$$\alpha_{s+2} < \alpha_s + 2. \tag{2.4}$$

If $I \subset \mathbb{R}$ is a given interval and $f \in C_B(I)$, then the first order modulus of smoothness of f is the function $\omega_1(f; \cdot) : [0, +\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by $\omega_1(f, \delta) = sup\{|f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta\}$.

Theorem 2.1. ([8]) *Let $f : I \longrightarrow \mathbb{R}$ be a function. If $x \in I \cap J$ and f is s times derivable function on I, the function $f^{(s)}$ is continuous on I, then*

$$\lim_{m \longrightarrow \infty} m^{s-\alpha_s} \Big((L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} (T_{m,i} L_m)(x) \Big) = 0. \tag{2.5}$$

If f is a s times differentiable function on I, the function $f^{(s)}$ is continuous on I and there exists $m(s) \in \mathbb{N}$ and $k_j \in \mathbb{R}$ such that for any natural number $m \geq m(s)$ and for any $x \in I \cap J$ we have

$$\frac{(T_{m,j} L_m)(x)}{m^{\alpha_j}} \leq k_j, \tag{2.6}$$

where $j \in \{s, s+2\}$, then the convergence given in (2.5) is uniformly on $I \cap J$ and

$$m^{s-\alpha_s} \Big| (L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} (T_{m,i} L_m)(x) \Big| \tag{2.7}$$

$$\leq \frac{1}{s!}(k_s + k_{s+2})\omega_1\Big(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\Big)$$

for any $x \in I \cap J$ and $m \geq m(s)$.

Let φ_x be defined by

$$\varphi_x(t) = |t - x|, t \in I, x \in I. \tag{2.8}$$

Theorem 2.2. [9] *Let $L : C(I) \longrightarrow B(I)$ be a linear positive operator. Let φ_x be defined by (2.8).*
(i) If $f \in C_B(I)$, then for every $x \in I$ and $\delta > 0$, one has

$$\begin{aligned}|(Lf)(x) - f(x)| &\leq |f(x)|\,|(Le_0)(x) - 1| \\ &+ \Big((Le_0)(x) + \delta^{-1}\sqrt{(Le_0)(x)\cdot(L\varphi_x^2)(x)}\Big)\,\omega_1(f;\delta).\end{aligned}$$

(ii) If f is differentiable on I and $f' \in C_B(I)$, then for every $x \in I$ and $\delta > 0$, one has

$$\begin{aligned}|(Lf)(x) - f(x)| &\leq |f(x)|\,|(Le_0)(x) - 1| + |f'(x)||(Le_1)(x) - x(Le_0)(x)| \\ &+ \sqrt{(L\varphi_x^2)(x)}\Big(\sqrt{(Le_0)(x)} + \delta^{-1}\cdot\sqrt{(L\varphi_x^2)(x)}\Big)\,\omega_1(f';\delta).\end{aligned}$$

3. A new class of Kantorovich-type operators

Let $a_m, b_m : J \longrightarrow \mathbb{R}$ be functions such that $a_m(x) \geq 0$, $b_m(x) \geq 0$ for any $x \in J$ and $m \in \mathbb{N}_1$, where J and $\mathbb{N}_1 \subset \mathbb{N}$ will be determined later. We define the operators of the following form

$$(K_m^* f)(x) = (m+1)\sum_{k=0}^{m}\binom{m}{k}(a_m(x))^k(b_m(x))^{m-k}\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt \tag{3.1}$$

for any $x \in J$, $m \in \mathbb{N}_1$ and $f \in L_1([0,1])$. Then, we get

$$(K_m^* e_0)(x) = \big(a_m(x) + b_m(x)\big)^m, \tag{3.2}$$

$$(K_m^* e_1)(x) = \frac{m}{m+1}a_m(x)\big(a_m(x) + b_m(x)\big)^{m-1} + \frac{1}{2(m+1)}\big(a_m(x) + b_m(x)\big)^m \tag{3.3}$$

and

$$\begin{aligned}(K_m^* e_2)(x) &= \frac{m(m-1)}{(m+1)^2}a_m^2(x)\big(a_m(x) + b_m(x)\big)^{m-2} \\ &+ \frac{2m}{(m+1)^2}a_m(x)\big(a_m(x) + b_m(x)\big)^{m-1} \\ &+ \frac{1}{3(m+1)^2}\big(a_m(x) + b_m(x)\big)^m\end{aligned} \tag{3.4}$$

for any $x \in J$ and $m \in \mathbb{N}_1$.

In what follows, we impose the additional condition to be fulfilled by our operators

$$(K_m^* e_0)(x) = 1 + u_m(x), \tag{3.5}$$

where $x \in J$, $m \in \mathbb{N}_1$ and $u_m : J \longrightarrow \mathbb{R}$.

Remark 3.1. *We want that $K_m^*, m \in \mathbb{N}_1$ be positive operators, then from $(K_m^* e_0) \geq 0$ and (3.5), we have*

$$1 + u_m(x) \geq 0, x \in J, m \in \mathbb{N}_1. \tag{3.6}$$

We will show in Lemma 3.3 that $1+u_m(x)>0, x\in J, m\in\mathbb{N}_1$. From (3.2), we get

$$(a_m(x)+b_m(x))^m=1+u_m(x), x\in J, m\in\mathbb{N}_1, \tag{3.7}$$

from where

$$a_m(x)+b_m(x)=(1+u_m(x))^{\frac{1}{m}}, x\in J, m\in\mathbb{N}_1. \tag{3.8}$$

The next conditions will be read as follows

$$(K_m^*e_1)(x)=x \tag{3.9}$$

and

$$(K_m^*e_2)(x)=x^2 \tag{3.10}$$

for any $x\in J$ and $m\in\mathbb{N}_1$.

From (3.3), (3.8) and (3.9), we get

$$a_m(x)=\frac{m+1}{m}(1+u_m(x))^{\frac{1-m}{m}}\left(x-\frac{1}{2(m+1)}(1+u_m(x))\right), \tag{3.11}$$

$x\in J, m\in\mathbb{N}_1$.

From (3.8) and (3.11) we obtain

$$\begin{aligned} b_m(x)=&(1+u_m(x))^{\frac{1}{m}}\left(1-\frac{m+1}{m}\cdot\frac{1}{1+u_m(x)}\cdot\right.\\ &\left.\left(x-\frac{1}{2(m+1)}(1+u_m(x))\right)\right), \end{aligned} \tag{3.12}$$

$x\in J, m\in\mathbb{N}_1$.

Because $a_m(x)\geq 0, b_m(x)\geq 0, x\in J, m\in\mathbb{N}_1$, from (3.7), (3.11) and (3.12) we get

$$x-\frac{1}{2(m+1)}(1+u_m(x))\geq 0$$

and

$$1-\frac{m+1}{m}\cdot\frac{1}{1+u_m(x)}\left(x-\frac{1}{2(m+1)}(1+u_m(x))\right)\geq 0,$$

$x\in J, m\in\mathbb{N}_1$, from where we obtain

$$\frac{2(m+1)}{2m+1}x-1\leq u_m(x)\leq 2(m+1)x-1, \tag{3.13}$$

$x\in J, m\in\mathbb{N}_1$.

From (3.4), (3.8), (3.10) and (3.11) it follows

$$\begin{aligned} &(-5m-3)u_m^2(x)+\\ &(-12m(m+1)^2x^2+12(m+1)^2x-2(5m+3))u_m(x)+\\ &(-12(m+1)^2x^2+12(m+1)^2x-(5m+3))=0. \end{aligned} \tag{3.14}$$

The relation (3.14) is an equation in $u_m(x)$ with the discriminant

$$\Delta_m(x)=48(m+1)^2x^2\Big(3(m+1)^2(mx-1)^2+(5m+3)(m-1)\Big). \tag{3.15}$$

The discriminant $\Delta_m(x)$, after some calculation, has the following form

$$\Delta_m(x)=\Big(12m(m+1)^2x^2-12(m+1)^2x\Big)^2+4(5m+3)12(m+1)^2x^2(m-1), \tag{3.16}$$

so for $x\neq 0$ and $m\in\mathbb{N}$ we obtain that $\Delta_m(x)>0$.

Then, in the above conditions, we have the solutions of the equation (3.14)

$$u_{m,1}(x) = \frac{-6m(m+1)^2x^2 + 6(m+1)^2x - (5m+3)}{5m+3} - \frac{2(m+1)x\sqrt{9(m+1)^2(mx-1)^2 + 3(5m+3)(m-1)}}{5m+3} \tag{3.17}$$

and

$$u_{m,2}(x) = \frac{-6m(m+1)^2x^2 + 6(m+1)^2x - (5m+3)}{5m+3} + \frac{2(m+1)x\sqrt{9(m+1)^2(mx-1)^2 + 3(5m+3)(m-1)}}{5m+3} \tag{3.18}$$

for any $x \in J, m \in \mathbb{N}_1$.

For $u_{m,1}(x)$, we have $\lim\limits_{m\longrightarrow\infty} u_{m,1}(x) = -\infty$ then $u_{m,1}(x)$ does not satisfy the relation (3.6), so from the relation (3.18) follows that $u_m(x) = u_{m,2}(x)$.

Lemma 3.1. *The relation (3.13) happens for any $x \in J, m \in \mathbb{N}_1$ if and only if*

$$\frac{2}{3(m+1)} \le x \le \frac{2(3m^2+3m+1)}{3(m+1)(2m+1)}. \tag{3.19}$$

Proof. After some calculation, it follows from the relations (3.13) and (3.18). □

Remark 3.2. *(i) The following inequalities state*

$$\frac{2}{3(m+1)} > 0$$

and

$$\frac{2(3m^2+3m+1)}{3(m+1)(2m+1)} < 1$$

for $m \in \mathbb{N}$.

(ii) The sequence $\left(\frac{2}{3(m+1)}\right)_{m\in\mathbb{N}}$ is decreasing and the sequence $\left(\frac{2(3m^2+3m+1)}{3(m+1)(2m+1)}\right)_{m\in\mathbb{N}}$ is increasing.

(iii) From (ii), the following relations state

$$\frac{2}{3(m+1)} \le \frac{1}{3}$$

and

$$\frac{7}{9} \le \frac{2(3m^2+3m+1)}{3(m+1)(2m+1)}, m \in \mathbb{N}.$$

(iv) From (3.19) and (iii) follows $\frac{1}{3} \le x \le \frac{7}{9}$, so the operators K_m^* are positive for $m \in \mathbb{N}$.

(v) If $c \in \left(0, \frac{1}{3}\right)$, because $\lim\limits_{m\longrightarrow\infty} \dfrac{2}{3(m+1)} = 0$ it follows that there exists $m(c) \in \mathbb{N}$ such that $\frac{2}{3(m+1)} \le c$, for any $m \in \mathbb{N}$ and $m \ge m(c)$.

(vi) If $d \in \left(\frac{7}{9}, 1\right)$, because $\lim\limits_{m\longrightarrow\infty} \dfrac{2(3m^2+3m+1)}{3(m+1)(2m+1)} = 1$ follows that there exists $m(d) \in \mathbb{N}$ such that $d \le \frac{2(3m^2+3m+1)}{3(m+1)(2m+1)}$, for any $m \in \mathbb{N}$ and $m \ge m(d)$.

(vii) Let $\mathbb{N}_1 = \{m \in \mathbb{N} \mid m \ge \max(m(c), m(d)) = m(c,d)\}$.

Lemma 3.2. *If $0 < c < d < 1$, then exists $m(c,d) \in \mathbb{N}$ such that the operators K_m^* are positive on $[c,d]$, for $m \in \mathbb{N}, m \ge m(c,d)$.*

Proof. It follows from Lemma 3.1 and Remark 3.2. □

Lemma 3.3. *The inequality*

$$1 + u_m(x) > 0 \tag{3.20}$$

holds for any $x \in [c, d]$ *and* $m \in \mathbb{N}_1$.

Proof. We take into account the relation (3.18). □

Let c and d be real numbers with $0 < c < d < 1$, then $I = [0, 1]$, $J = [c, d]$,

$$\begin{aligned}\varphi_{m,k}(x) = (m+1)(1+u_m(x))^{1-k} \\ \times \left(\frac{m+1}{m}\left(x - \frac{1}{2(m+1)}(1+u_m(x))\right)\right)^k \\ \times \left(1 - \frac{m+1}{m(1+u_m(x))}\left(x - \frac{1}{2(m+1)}(1+u_m(x))\right)\right)^{m-k}\end{aligned}$$

and

$$A_{m,k}(f) = \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt,$$

$f \in L_1([0, 1])$, $x \in [c, d]$, $m \in \mathbb{N}_1$.

Then the operators (3.1) become

$$\begin{aligned}(K_m^* f)(x) = (m+1)\sum_{k=0}^{m}\binom{m}{k}(1+u_m(x))^{1-k} \\ \times \left(\frac{m+1}{m}\left(x - \frac{1}{2(m+1)}(1+u_m(x))\right)\right)^k \\ \times \left(1 - \frac{m+1}{m(1+u_m(x))}\left(x - \frac{1}{2(m+1)}(1+u_m(x))\right)\right)^{m-k} \\ \times \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt,\end{aligned} \tag{3.21}$$

$x \in [c, d]$, $m \in \mathbb{N}_1$.

Lemma 3.4. *For* $x \in [c, d]$ *and* $m \in \mathbb{N}_1$, *the following identities*

$$(T_{m,0}K_m^*)(x) = 1 + u_m(x), \tag{3.22}$$

$$(T_{m,1}K_m^*)(x) = -mxu_m(x), \tag{3.23}$$

$$(T_{m,2}K_m^*)(x) = m^2x^2u_m(x) \tag{3.24}$$

hold.

Proof. We take (2.2), (3.9) and (3.10) into account. □

Lemma 3.5. *For* $x \in [c, d]$, $m \in \mathbb{N}_1, m \geq m_*, m_* = \max\big(m(0), m(2)\big)$, *we have*

$$\begin{aligned}\alpha_0 &= 0, & \alpha_2 &= 1,\\ B_0(x) &= 1, & B_2(x) &= x(1-x),\\ k_0 &= 1, & k_2 &= \frac{1}{4}.\end{aligned}$$

Proof. We have that

$$(T_{m,0}K_m^*)(x) = 1 + u_m(x),$$

then

$$\lim_{m\longrightarrow\infty} \frac{(T_{m,0}K_m^*)(x)}{m^0} = 1,$$

so from relations (2.3), (2.4) and (2.6) we get $\alpha_0 = 0, B_0(x) = 1$ and $k_0 = 1$ for $x \in [c,d]$, $m \in \mathbb{N}_1, m \geq m(0)$.

We have that

$$(T_{m,2}K_m^*)(x) = m^2x^2u_m(x).$$

Because

$$\lim_{m\longrightarrow\infty} mu_m(x) = \frac{1-x}{x},$$

we obtain

$$\lim_{m\longrightarrow\infty} \frac{(T_{m,2}K_m^*)(x)}{m^1} = x(1-x).$$

Then from relations (2.3), (2.4) and (2.6), we get $\alpha_2 = 1, B_2(x) = x(1-x)$ and $k_2 = \frac{1}{4}$ for $x \in [c,d], m \in \mathbb{N}_1, m \geq m(2)$. □

4. Properties for the new class of Kantorovich type operators

In this section, we present some properties of the new class of Kantorovich type operators, where c and d are real fixed numbers, $0 < c < d < 1$.

Theorem 4.3. *If $f \in C([0,1])$, then*

$$\lim_{m\longrightarrow\infty} (K_m^* f)(x) = f(x) \tag{4.1}$$

uniformly on $[c,d]$ and

$$|(K_m^* f)(x) - f(x)| \leq |f(x)| \cdot |u_m(x)| + \frac{5}{4} \cdot \omega_1\left(f; \frac{1}{\sqrt{m}}\right), \tag{4.2}$$

for any $x \in [c,d]$ and $m \in \mathbb{N}_1$.

Proof. From (2.7), for $\alpha_0 = 0$, $\alpha_2 = 2$, $k_0 = 0$ and $k_2 = \frac{1}{4}$, we get

$$|(K_m^* f)(x) - f(x)(1 + u_m(x))| \leq \frac{5}{4} \cdot \omega_1\left(f; \frac{1}{\sqrt{m}}\right), \tag{4.3}$$

for any $x \in [c,d]$, $m \in \mathbb{N}_1$, $m \geq m_*$ which is equivalent with (4.2).

□

Theorem 4.4. *Let $f : [0,1] \longrightarrow \mathbb{R}$ be a function. If f is two times differentiable on $[0,1]$, the function $f^{(2)}$ is continuous on $[0,1]$ and $x \in [c,d]$, then*

$$\begin{aligned} \lim_{m\longrightarrow\infty} m((K_m^* f)(x) - f(x)) &= \frac{1-x}{x} f(x) + (x-1)f^{(1)}(x) \\ &+ \frac{x(1-x)}{2} f^{(2)}(x), \end{aligned} \tag{4.4}$$

for any $x \in [c,d]$, $m \in \mathbb{N}_1$.

Proof. Using the relation (2.5) and Lemma (3.1), the relation (4.4) follows. □

The relation (4.4) is a Voronovskaja type theorem.

Theorem 4.5. *If* $f \in C([0,1])$, *then*

$$|(K_m^* f)(x) - f(x)| \leq |f(x)| \cdot |u_m(x)| + 3 \cdot \omega_1(f; \delta_1) \tag{4.5}$$

for any $x \in [c,d]$, $m \in \mathbb{N}_1$, *where* $\delta_1 = \sqrt{\frac{mx+1}{m^2}}$.

Proof. Using Theorem 2.2 (i), from relation (3.2) for $\delta = \sqrt{(K_m^* e_0)(x) \cdot (K_m^* \varphi_x^2)(x)}$, we have

$$|(K_m^* f)(x) - f(x)| \leq |f(x)| \cdot |u_m(x)| + 3 \cdot \omega_1(f; \delta_1) \tag{4.6}$$

for any $x \in [c,d], m \in \mathbb{N}_1$.

After some calculus, we get $\delta = \sqrt{(1+u_m(x)) \cdot x^2 \cdot u_m(x)}$. Because $\lim\limits_{m \longrightarrow \infty} m u_m(x) = \dfrac{1-x}{x}$, we have that there exists $m(1) \in \mathbb{N}_1$ such that $u_m(x) < \frac{1}{mx}$ for any $x \in [c,d], m \geq m(1), m(1) \in \mathbb{N}_1$. Then $\delta < \sqrt{\frac{mx+1}{m^2}} = \delta_1$ and from (4.6) we obtain (4.5).

We observe that for the genuine Kantorovich operators we have the relation $|(K_m f)(x) - f(x)| \leq 2 \cdot \omega_1 \left(f; \frac{1}{2\sqrt{m+1}}\right)$ and for our operators we have the relation (4.5) and if we make a comparison between this two results, we remark that $\delta_1 < \frac{1}{2\sqrt{m+1}}$, for any $x \in [c,h]$, $m \geq m_1$, $m_1 \in \mathbb{N}_1$, where h is a real number that has the following properties:
(i) $0 < c < h < d$ and $h < \frac{1}{4}$;
(ii) there exists $m(h) \in \mathbb{N}$ such that for any $m \geq m(h)$, the inequality $h < \frac{m^2-4m-4}{4m^2+4m}$ holds, where $\delta_1 < \frac{1}{2\sqrt{m+1}}$ is equivalent with $x < \frac{m^2-4m-4}{4m^2+4m}$;
(iii) $m_1 = max\Big(m(c), m(h), m(d)\Big), m_1 \in \mathbb{N}_1$.

□

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