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CONSTRUCTIVE MATHEMATICAL ANALYSIS

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Editorial Staff

Special issue dedicated to Prof. Dr. Francesco Altomare: on the occasion of his 70th birthday

Prof. Dr. Francesco Altomare
University of Bari Aldo Moro

Contents

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Research Article

Generalized Cesàro summability of Fourier series and its applications

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ABSTRACT. In this paper, by using generalized Cesàro means based on q-integers, we study on approximating continuous and periodic functions by their Fourier series. We also discuss its connection with the concept of statistical convergence. At the end of the paper, some applications and graphical illustrations are also provided.

Keywords: Fourier analysis, Cesàro summability, Fejér's kernel, q-integers, statistical convergence.

2020 Mathematics Subject Classification: 42A24, 40G15.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

The question of whether the Fourier series of a periodic function converges to the given function is researched by a field known as classical harmonic analysis. It is well-known that convergence is not necessarily given in the general case. However, by using some summability methods, such as Cesàro means and Riesz means, the convergence is possible in some sense (see, for instance, [21]). In the present paper, by using generalized Cesàro means based on q-integers (see the next section for details), we study on approximating continuous and periodic functions by their Fourier series.

Let $S_n(f)$ denote the partial sums of an integrable and 2π-periodic function f, that is

$$S_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx),$$

where

$$a_k := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos ktdt,\ k = 0, 1, ...$$

and

$$b_k := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin ktdt,\ k = 1, 2, ...$$

Then, we may write that

$$S_n(f;x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_n(t) dt,$$

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where $D_n(t)$ denotes Dirichlet's kernel given by

$$D_n(t) = \frac{\sin((n+1/2)\,t)}{2\sin(t/2)}.$$

Furthermore, the classical Cesàro means of $S_n(f)$ can be written as follows:

$$\begin{aligned}\sigma_n(f;x) =& \frac{S_0(f;x) + S_1(f;x) + \ldots + S_n(f;x)}{n+1} \\ =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left(\frac{1}{n+1}\sum_{k=0}^{n} D_k(t)\right)dt \\ =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K_n(t)dt,\end{aligned}$$

where $K_n(t)$ denotes Fejér's kernel given by

$$K_n(t) = \frac{\sin^2((n+1/2)t)}{2(n+1)\sin^2(t/2)}. \tag{1.1}$$

Now, let $C^{2\pi}$ denote the space of all continuous and 2π periodic functions. Then, it is well-known that, for any $f \in C^{2\pi}$, the sequence $(\sigma_n(f))$ is uniformly convergent to f, i.e., $(S_n(f))$ is uniformly Cesàro summable to f.

In order to generalize this summability, we will consider the generalized Cesàro means based on q-integers introduced in [1, 4].

2. q-Cesàro summability of Fourier series

We first recall some concepts and notation from the q-calculus (see [11] for details). For a given $q > 0$, the q-integer $[n]_q$ is given by

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1} \text{ with } [0]_q = 0.$$

Then, for each $n = 1, 2, \ldots,$ we may write that

$$[n]_q = \frac{1-q^n}{1-q} \text{ for } q \neq 1.$$

Now, for a given $q > 0$, consider the q-Cesàro matrix $C(q) = [c_{nk}(q)]$ $(k, n = 0, 1, 2, \ldots)$ defined by (see [1, 4])

$$c_{nk}(q) = \begin{cases} \frac{q^k}{[n+1]_q}, & k = 0, 1, \ldots, n \\ 0, & \text{otherwise.}\end{cases} \tag{2.2}$$

Then, we can write the matrix $C(q)$ as follows:

$$C(q) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \frac{1}{[2]_q} & \frac{q}{[2]_q} & 0 & \cdots & 0 & 0 & \cdots \\ \frac{1}{[3]_q} & \frac{q}{[3]_q} & \frac{q^2}{[3]_q} & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \frac{1}{[n+1]_q} & \frac{q}{[n+1]_q} & \frac{q^2}{[n+1]_q} & \cdots & \frac{q^n}{[n+1]_q} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Observe that the case of $q = 1$ reduces to the classical Cesàro matrix. About regularity of q-Cesàro matrix, we can say the following:

- For any fixed $q \geq 1$, the matrix $C(q)$ is regular (see [1]).
- For a given $0 < q < 1$, the corresponding matrix $C(q)$ cannot be regular due to the fact that $[n+1]_q \to \frac{1}{1-q}$ as $n \to \infty$.
- Instead of a fixed q, take a sequence $q = (q_n)$ such that the following conditions hold:

(2.3) $$0 < q_n < 1 \text{ for all } n \in \mathbb{N}_0 = \{0, 1, ...\}$$

and

(2.4) $$\lim_{n\to\infty} q_n = 1.$$

Then, $C(q)$ is still regular. Indeed, from (2.3) and (2.4) , we may write that $[n+1]_{q_n} \to \infty$ as $n \to \infty$ (see, for instance, [17, 18, 19]). Hence, using the well-known Silverman-Toeplitz conditions, we immediately get the regularity of $C(q)$ for $q = (q_n)$ (see Example 2.1 for such a sequence).

Because the sequence of partial sums $(S_n(f))$ need not converge to f, we may try looking at their q-Cesàro means as follows:

$$\begin{aligned}\sigma_n(f;q;x) &= \frac{1}{[n+1]_q}\sum_{k=0}^{n} q^k S_k(f;x) \\ &= \frac{S_0(f;x) + qS_1(f;x) + ... + q^n S_n(f;x)}{[n+1]_q},\end{aligned}$$

which implies

$$\sigma_n(f;q;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left(\frac{1}{[n+1]_q}\sum_{k=0}^{n} q^k D_k(t)\right) dt.$$

Hence, we may write that

(2.5) $$\sigma_n(f;q;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K_n(q;t)dt,$$

where

(2.6) $$K_n(q;t) := \frac{1}{2[n+1]_q}\sum_{k=0}^{n} q^k \frac{\sin\left((k+1/2)\,t\right)}{\sin\left(t/2\right)},$$

say q-Fejér's kernel.

We should note that the q-Cesàro means in (2.5) may be regarded as a special case of Nörlund or Riesz means of the partial sums of Fourier series. However, it is more convenient to examine the behavior of the corresponding q-Fejér's kernel in (2.6) by taking into account the known properties of the q-integers. Another important reason for using q-integers in this process is that it is possible to weaken the classical limit condition needed in the approximation (see Section 3 for details).

We now start with the fundamental properties of q-Fejér's kernel.

Lemma 2.1. *Let $q > 0$ and $n \in \mathbb{N}_0$. Then, we get the followings:*

(a) $$K_n(q;t) = \frac{(1+q)\sin(t/2) + q^{n+2}\sin((n+1/2)t) - q^{n+1}\sin((n+3/2)t)}{2[n+1]_q \sin(t/2)\left\{(1-q)^2\cos^2(t/2) + (1+q)^2\sin^2(t/2)\right\}}.$$

(b) $$K_n(q;t) = \frac{(n+1)}{[n+1]_q}\left\{q^n K_n(t) + \sum_{k=0}^{n-1}\left(q^k - q^{k+1}\right)K_k(t)\right\},$$

where $K_n(t)$ is the classical Fejér's kernel given by (1.1).

(c) $\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(q;t)dt = \frac{n+1}{[n+1]_q}$.
(d) *If* $0 < q \leq 1$, *then* $K_n(q;\cdot) \geq 0$.

Proof. (a) From the definition of $K_n(q;t)$ in (2.6), we may write that

$$\begin{aligned} K_n(q;t) =& \frac{1}{2[n+1]_q \sin(t/2)} \operatorname{Im}\left\{\sum_{k=0}^{n} q^k e^{i(k+1/2)t}\right\} \\ =& \frac{1}{2[n+1]_q \sin(t/2)} \operatorname{Im}\left\{e^{it/2}\sum_{k=0}^{n} \left(qe^{it}\right)^k\right\} \\ =& \frac{1}{2[n+1]_q \sin(t/2)} \operatorname{Im}\left\{e^{it/2}\frac{1-q^{n+1}e^{i(n+1)t}}{1-qe^{it}}\right\} \\ =& \frac{1}{2[n+1]_q \sin(t/2)} \operatorname{Im}\left\{\frac{1-q^{n+1}e^{i(n+1)t}}{e^{-it/2}-qe^{it/2}}\right\}. \end{aligned}$$

Observe that

$$\operatorname{Im}\left\{\frac{1-q^{n+1}e^{i(n+1)t}}{e^{-it/2}-qe^{it/2}}\right\} = \frac{A_n(q,t)-B_n(q,t)}{(1-q)^2\cos^2(t/2)+(1+q)^2\sin^2(t/2)},$$

where

$$\begin{aligned} A_n(q,t) &= (1+q)\left(1-q^{n+1}\cos((n+1)t)\right)\sin(t/2), \\ B_n(q,t) &= (1-q)\,q^{n+1}\sin((n+1)t)\cos(t/2). \end{aligned}$$

Hence, using some appropriate trigonometric identities, we obtain that

$$\begin{aligned} A_n(q,t)-B_n(q,t) =& (1+q)\sin(t/2) + q^{n+2}\sin((n+1/2)t) \\ & - q^{n+1}\sin((n+3/2)t), \end{aligned}$$

which immediately gives the equality in (a).

(b) If we use Abel's partial sums identity in (2.6), then we observe from (1.1) that

$$\begin{aligned} K_n(q;t) =& \frac{(n+1)}{[n+1]_q}\sum_{k=0}^{n} q^k \frac{\sin((k+1/2)t)}{2(n+1)\sin(t/2)} \\ =& \frac{(n+1)}{[n+1]_q}\left\{q^n K_n(t) + \sum_{k=0}^{n-1}\left(q^k - q^{k+1}\right)K_k(t)\right\}, \end{aligned}$$

which completes the proof of (b).

(c) We may write from (b) that

$$\begin{aligned} &\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(q;t)dt \\ =& \frac{n+1}{[n+1]_q}\left\{q^n\left(\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)dt\right) + (1-q)\sum_{k=0}^{n-1} q^k\left(\frac{1}{\pi}\int_{-\pi}^{\pi} K_k(t)dt\right)\right\}. \end{aligned}$$

Since $\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)dt = 1$, we see that

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(q;t)dt =& \frac{n+1}{[n+1]_q}\left\{q^n + (1-q)\sum_{k=0}^{n-1} q^k\right\} \\ =& \frac{n+1}{[n+1]_q}\left\{q^n + (1-q)[n]_q\right\} \\ =& \frac{n+1}{[n+1]_q},\end{aligned}$$

which gives (c).

(d) It is clear from (b), since $0 < q \leq 1$. □

Remark 2.1. *If one takes $q = 1$ in Lemma 2.1 then (a) implies*

$$\begin{aligned}K_n(1;t) =& \frac{2\sin(t/2) + \sin\left((n+1/2)t\right) - \sin\left((n+3/2)t\right)}{8(n+1)\sin^3(t/2)} \\ =& \frac{1-\cos\left((n+1)t\right)}{4(n+1)\sin^2(t/2)} \\ =& \frac{\sin^2\left((n+1)t/2\right)}{2(n+1)\sin^2(t/2)} \\ =& K_n(t),\end{aligned}$$

and (b) implies the same equality $K_n(1;t) = K_n(t)$, and also (c) and (d) implies the classical results $\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)dt = 1$ and $K_n(\cdot) \geq 0$, respectively.

Theorem 2.1. *Assume that the sequence $q = (q_n)$ satisfies the conditions (2.3) and (2.4). Then, for the operators in (2.5), we get*

$$\lim_{n\to\infty} \sigma_n(f;q_n;x) = f(x) \text{ uniformly with respect to } x$$

for every $f \in C^{2\pi}$.

Proof. Since the operators in (2.5) are positive and linear, from the well-known Korovkin theorem for 2π-periodic continuous functions (see [2, 12]), it is enough to show that

(2.7) $$\sigma_n\left(f_i;q_n;x\right) \rightrightarrows f_i(x) \text{ for } i = 0,1,2,$$

where $f_0(x) = 1$, $f_1(x) = \sin x$ and $f_2(x) = \cos x$. As usual, the symbol $\rightrightarrows$ denotes the uniform convergence. Now, it is easy to check that

(2.8) $$\sigma_n(f_0;q_n;x) = f_0(x) = 1.$$

From the definition of the operators, we observe that

$$\begin{aligned}\sigma_n\left(f_1;q_n;x\right) =& \frac{S_0(f_1;x) + q_nS_1(f_1;x) + q_n^2S_2(f_1;x) + \cdots + q_n^nS_n(f_1;x)}{[n+1]_{q_n}} \\ =& \frac{0 + q_n\sin x + q_n^2\sin x + \cdots + q_n^n\sin x}{[n+1]_{q_n}} \\ =& \frac{q_n + q_n^2 + \cdots + q_n^n}{[n+1]_{q_n}}\sin x,\end{aligned}$$

which implies

(2.9) $$\sigma_n\left(f_1;q_n;x\right) = \left(1 - \frac{1}{[n+1]_{q_n}}\right)\sin x.$$

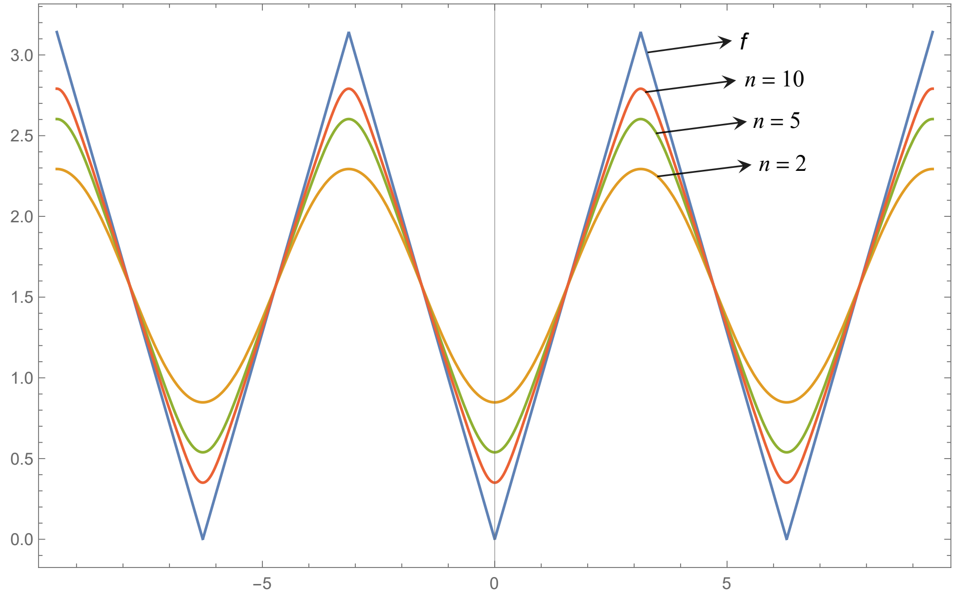

FIGURE 1. q-Cesàro approximation to the function $f(x) = |x|$ by the operators $\sigma_n(f; q_n; x)$ associated with the sequence $q = (q_n)$ given by (2.11)

Similarly, we also get

$$\sigma_n\left(f_2; q_n; x\right) = \left(1 - \frac{1}{[n+1]_{q_n}}\right) \cos x. \tag{2.10}$$

Taking limit as $n \to \infty$ in (2.8), (2.9) and (2.10) and also considering the assumptions (2.3) and (2.4), we obtain (2.7) , which completes the proof. □

Example 2.1. *Define the function f, for $x \in [-\pi, \pi]$, by $f(x) = |x|$ and extend its domain periodically to the whole real line which coincides on $[-\pi, \pi]$. Consider the sequence $q = (q_n)$ given by*

$$q_n = 1 - \frac{1}{n+2}. \tag{2.11}$$

Then, all conditions of Theorem 2.1 holds, which implies

$$\lim_{n\to\infty} \sigma_n(f; q_n; x) = f(x)$$

uniformly with respect to x. This (uniform) q-Cesàro summability is indicated in Figure 1 with the parameter values $n = 2, 5, 10$.

Example 2.2. *Consider the 2π-periodic and even function f defined on $[0, \pi]$ by*

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sin\left(\left(2^{k^3} + 1\right) \frac{x}{2}\right). \tag{2.12}$$

Then, according to Weierstrass M-test, we get the continuity of f on $\mathbb{R}$. Hence, Theorem 2.1 implies that, for any sequence $q = (q_n)$ satisfying (2.3) and (2.4), $\lim_{n\to\infty} \sigma_n(f; q_n; x) = f(x)$ uniformly with respect to x. However, one can observe that the classical partial sums of the function f in (2.12) cannot converge to f at the origin. More precisely, the sequence $(S_n(f; 0))$ diverges to the infinity as $n \to \infty$ (see, for instance, [20]).

3. Extension to the statistical convergence and concluding remarks

In this section, we will work on the situation where the limit condition in (2.4) is weakened. For example, we can consider the concept of statistical convergence (see [8] by Fast). We note that this type of convergence has been introduced a few years earlier by Zygmund with the name "almost convergence" (see [21, Vol. II, Chap. XIII]). Later on, the statistical convergence has been frequently used not only in the summability theory, but also in the approximation theory (see [3, 6, 7, 10, 13, 14, 15, 16]). We recall that the (asymptotic) density, $\delta(K)$, of a set $K \subset \mathbb{N}$ is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n+1} \#\{0 \le k \le n : k \in K\}$$

provided that the limit exists, where the symbol $\#$ denotes the cardinal number of a set. Using this density, a sequence (x_n) is said to be statistically convergent to a number L, denoted by $st - \lim_{n\to\infty} x_n = L$, if for every $\varepsilon > 0$,

$$\delta\left(\{0 \le k \le n : |x_k - L| \ge \varepsilon\}\right) = 0,$$

that is

$$\lim_{n\to\infty} \frac{1}{n+1} \#\{0 \le k \le n : |x_k - L| \ge \varepsilon\} = 0.$$

It is well-known that every convergent sequence is statistically convergent to the same value, but the converse is not always true. Furthermore, for a given sequence (x_n), $st - \lim_{n\to\infty} x_n = L$ if and only if there exists an index set $K = \{k_n : n \in \mathbb{N}_0\}$ of density 1 such that the subsequence (x_{k_n}) converges to L (in the usual sense) as $n \to \infty$ (see [5, 9] for further properties of statistical convergence).

For a given sequence $q = (q_n)$ with $0 < q_n < 1$, we replace the limit condition (2.4) with the following weaker condition:

$$st - \lim_{n\to\infty} q_n = 1. \tag{3.13}$$

In this case, the corresponding q-Cesàro matrix in (2.2) does not need to be regular. For example, consider the sequence $q = (q_n)$ defined by

$$q_n = \begin{cases} \dfrac{1}{2}\left(1 - \dfrac{1}{n+2}\right), & \text{if } n = m^2 \ (m = 0, 1, \ldots) \\ 1 - \dfrac{1}{n+1}, & \text{otherwise.} \end{cases} \tag{3.14}$$

Observe that, in this case, $[m^2 + 1]_{q_{m^2}} \to 2$ as $m \to \infty$. Despite this negative situation, we obtain the following statistical approximation theorem.

Theorem 3.2. *Assume that the sequence $q = (q_n)$ satisfy the conditions (2.3) and (3.13). Then, for the operators in (2.5), we get*

$$st - \lim_{n\to\infty} \sigma_n(f; q_n; x) = f(x) \text{ uniformly with respect to } x \tag{3.15}$$

for every $f \in C^{2\pi}$.

Proof. This immediately follows from the statistical Korovkin theorem for periodic functions (see [6]) since, for each $i = 0, 1, 2,$

$$st - \lim_{n\to\infty} \sigma_n(f_i; q_n; x) = f_i(x) \text{ uniformly with respect to } x,$$

where f_i are the test functions stated before. □

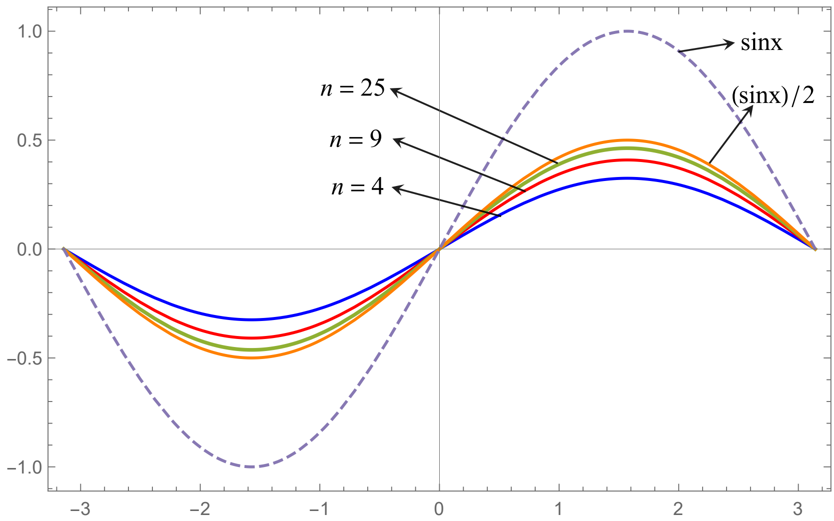

FIGURE 2. q-Cesàro approximation to the function $f_1(x) = \sin x$ by $(\sigma_n(f_1; q_n; x))$ associated with the sequence $q = (q_n)$ given by (3.14) fails for the values $n = m^2$

Remark 3.2. *We know from Example 2.2 that there exists a function f in $C^{2\pi}$ such that approximation to f by the partial sums $(S_n(f))$ fails. Now consider the sequence $q = (q_n)$ given by (3.14). Then, using the test function $f_1(x) = \sin x$, we get*

$$\sigma_{m^2}(f_1; q_{m^2}; x) = \left(1 - \frac{1}{[m^2+1]_{q_{m^2}}}\right)\sin x \to \frac{\sin x}{2} \ (\text{as } m \to \infty),$$

which is indicated in Figure 2 for some values $n = m^2$. Hence, for the sequence $q = (q_n)$ in (3.14), q-Cesàro approximation in Theorem 2.1 fails either. However, one can obtain from Theorem 3.2 that (3.15) holds for any function in $C^{2\pi}$. This situation is indicated in Figure 3 for some values $n \neq m^2$, where the set of all nonnegative integers n satisfying $n \neq m^2$ has density one.

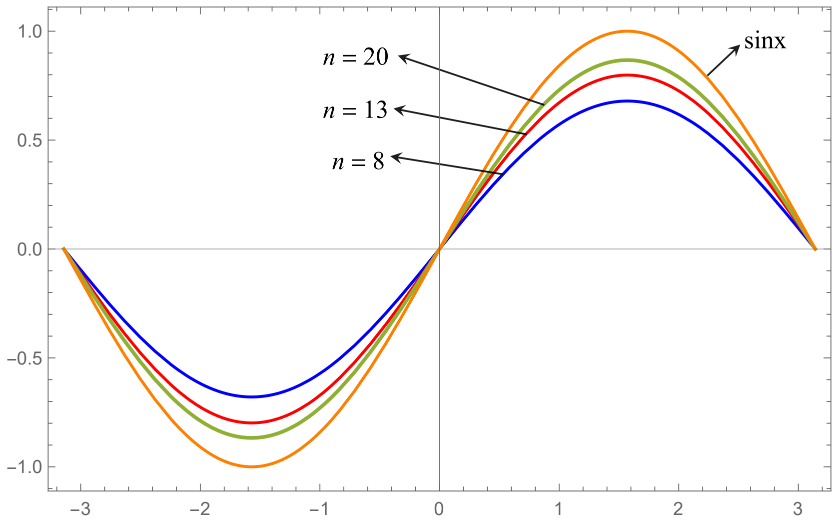

FIGURE 3. Statistical q-Cesàro approximation to the function $f_1(x) = \sin x$ by $(\sigma_n(f_1; q_n; x))$ associated with the sequence $q = (q_n)$ given by (3.14) for the values $n \neq m^2$

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Research Article

Modulars from Nakano onwards

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ABSTRACT. We discuss and compare a number of notions of modulars appeared in literature, among which there is a selection of the well known ones. We highlight the connections between the various definitions and provide several examples, taken from existing literature, recalling known results and completing the picture with some original considerations.

Keywords: Modulars, norms, function norm, function space, modular function spaces, Banach function spaces, Riesz spaces, lattices, Nakano modulars.

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This paper is dedicated to Professor Francesco Altomare.

1. INTRODUCTION

The Functional Analysis theory is typically built on spaces having at least the structures of vector spaces (essentially infinite-dimensional) endowed with a topology. The notions of Topological Vector space, Metric space, Normed space, Banach space, Hilbert space are a kind of chain of structures, each of them beginning with a set of properties which allows to build an entire theory. The overall result is a branch of Mathematical Analysis of great interest in its own and in applications to several other fields like PDEs. The central notion of norm raises naturally the question of how it can be *generated:* important questions are to establish whether a topology is induced by a norm (see the classical result by Kolmogorov, for instance Swartz [139, Theorem 1 p.182]), and whether a given metric can be derived from some norm (see e.g. Singh, Narang [138]), and, on the other hand, the check that inner products induce a norm ensures that properties defining inner products allow to gain results known for normed spaces. Norms of the most familiar infinite dimensional spaces (such as sequence spaces or Lebesgue spaces over sets in Euclidean spaces) characterize the elements of the spaces they generate in the sense that an element belongs to the space if and only if its norm is finite.

The relevance of the notion we are going to treat in this paper is that it allows to characterize easily certain sets of functions, namely, the sets of functions on which a certain functional, built through a so-called *modular,* is finite. Several norms are modulars (in Preface of the book Kozlowski [95] we read: *roughly speaking, modulars are the functionals that generalize norms*), but the heart of the matter is that modulars are not necessarily norms, and this may happen even for functionals characterizing – through its finiteness – vector spaces: in such case, the crucial role of the modular is that it allows to define a new functional (on the set where it is defined if it is a vector space, otherwise in its linear hull) which is a norm. Hence again, modulars represent

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a way to answer to the question of how generate a norm. The most popular (we could also say: *historical*) example is that one of Orlicz spaces: if $\Omega \subset \mathbb{R}^n$ is a Lebesgue measurable set with positive measure and if Φ is a non-negative, convex, increasing function over $[0, \infty[$, vanishing at zero, then the condition

$$\rho(f) = \int_\Omega \Phi(|f(x)|)dx < \infty$$

defines a set of measurable functions on Ω; but, in general, ρ does not enjoy the properties of a norm. In the case of functions Φ satisfying the Δ_2 condition (see e.g. Rao, Ren [134, Theorem 2 p.46] for details), the set $\{f : \rho(f) < \infty\}$, endowed with usual addition and scalar multiplication, is a vector space; but ρ is never a norm, unless $\Phi(t) = ct$ for some $c > 0$. The well known norm of classical Orlicz spaces is maybe the most standard example of norm built from a modular (in this case from ρ).

Overall, a natural question is to establish which properties should define a modular ρ so that one can build, starting from ρ, a structure of normed space.

While the classical structures (topological spaces, metric spaces, normed and Banach spaces, etc.) studied in Functional Analysis are defined in standard ways, the literature containing the notion of modular starts frequently from sets of axioms which may have differences among authors. This is not a serious problem, because each treatise must be built from all their necessary prerequisites. However, a comparison between the various notions seems missing at our knowledge, because authors are mostly interested in deriving their particular results. The goal of this paper is to give a contribution to this apparent lack of investigation, which may be of help for researchers which try to extend results previously known in classical structures to the context of modular spaces and need to choose a suitable set of properties from which to derive their results.

The plan of the paper is the following: in Section 2, we recall the definitions of normed and Riesz spaces: this will be of help when we will need to specify the domains of the modulars we will consider; *some* of the modulars introduced by Nakano are recalled and studied in Section 3, which contains Theorem 3.1 about how to build, starting from a modular, a structure of normed space; we will devote Section 4 to modulars introduced after Nakano by various authors; some of these notions – besides, of course, those ones introduced by Nakano – are among the most quoted in literature. They will be not presented in chronological order, because it will be privileged someway the mathematical sequence of the structures, even if the chain of modulars we will consider cannot be ordered rigorously: from the logical point of view the various notions of modular spaces are someway pairwise slanting. We hope that this paper will raise new questions, even from old literature, never studied in a systematic way; few ideas will be listed in the final Section 5.

We will see that the notion of modular is strongly linked, not only historically, to ordered vector spaces. In Subsection 2.2, we will recall that this topic has been within the interests of Professor Francesco Altomare, to which this paper is dedicated.

We close this section pointing out that the term *modular* is used also in contexts different from that one considered in this paper: for instance, it is used in abstract set function theory (see the comment on the property (P.3.1.11) below) and in the framework of finite lattices (see e.g. Section 4 in Kohonen [84]).

2. Normed spaces and Riesz spaces

In next subsection, we recall the standard notion of norm (see e.g. Dunford, Schwartz [46, p.59] or Megginson [118, p.ix or Definition 1.2.1 p.9]), which represents in fact the most important category of one of the next sets of modulars.

2.1. **Normed spaces.** Let $\mathcal{R}$ be a real or complex vector space. A functional

(2.1) $$\|\cdot\| : \mathcal{R} \to [0, +\infty[$$

is said to be a *norm* if it satisfies the following properties ($f, g \in \mathcal{R}$, $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C}$):
(P.2.1.1) $\|f\| = 0 \Leftrightarrow f = 0$,
(P.2.1.2) $\|\lambda f\| = |\lambda|\|f\|$,
(P.2.1.3) $\|f + g\| \leqslant \|f\| + \|g\|$.
Norms appear in every book dealing with Functional Analysis and/or Function spaces and their applications; normed spaces are not a topic of interest of this paper. However, when we think to the norm as functional, it may be of interest to highlight some interesting properties: for instance, the triangle inequality can be strengthened. In this framework, we mention just two papers about inequalities for the norm, see Maligranda [111, 112].

2.2. **Riesz spaces.** The set of the real numbers $\mathbb{R}$ may be enriched by the structure of complete ordered field, but it is also the substructure for an ordered *vector* field over $\mathbb{R}$ itself. In other familiar vector spaces order relations can be defined, but the compatibility with operations must be lost: for instance, it is well known from undergraduate Calculus that in $\mathbb{R}^2$ one cannot define at all an order compatible with the structure of field. However, a partial order compatible with the structure of vector space can be defined (see e.g. Example 2.1 below). One gets this way an important structure, which can be built also starting from other well known sets: for instance, the real vector space $\mathcal{M}$ of the Lebesgue measurable real-valued functions defined in $(0,1) \subset \mathbb{R}$. We are going to recall the definition of this structure.

The *Riesz spaces* (see Nakano [126, p.9], who called them "semi-ordered linear spaces"; for a modern exposition see e.g. Luxemburg, Zaanen [107, p.48]) are real vector spaces $\mathcal{R}$, where a binary relation (= subset of $\mathcal{R} \times \mathcal{R}$), denoted by $\geqslant$, is defined (for semiorders see also Luce [104, Section 2 p.181]; roughly speaking, we stress that elements f, g such that both $f \leqslant g$, $g \leqslant f$ do not hold, are allowed), satisfying a set of properties, which in turn is divided in other sets of properties with appropriate terminology. The various definitions are collected in the following scheme, where $\alpha \in \mathbb{R}$:

$$\left.\begin{array}{l}
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\left.\begin{array}{l}
f \geqslant g\,,\ g \geqslant h \ \Rightarrow f \geqslant h \\
f \geqslant f \\
f \geqslant g\,,\ g \geqslant f \ \Rightarrow f = g
\end{array}\right\}\text{partial ordering} \\
f \geqslant g\,,\ h \in \mathcal{R} \ \Rightarrow f + h \geqslant g + h \\
f \geqslant g\,,\ \alpha \geqslant 0 \ \Rightarrow \alpha f \geqslant \alpha g
\end{array}\right\}\text{(partially) ordered vector space} \\
\left.\begin{array}{l}
\forall f, g\ \exists f \vee g \in \mathcal{R} : f \vee g \geqslant f, f \vee g \geqslant g \\
h \geqslant f \vee g \ \ \forall h : h \geqslant f, h \geqslant g
\end{array}\right\}\text{lattice property}
\end{array}\right\}\begin{array}{c}\text{(semiordered) vector lattice}\\ \textit{or}\\ \text{lattice ordered vector space}\\ \textit{or}\\ \textbf{Riesz space}\\ \textit{or}\\ \text{semiordered linear space}\end{array} \\
\begin{array}{l}
\text{There exists a norm } \|\cdot\| \text{ such that, setting} \\
\text{"absolute value of } f\text{"} = |f| := f \vee (-f) \in \mathcal{R}\,, \\
|f| \geqslant |g| \ \Rightarrow \|f\| \geqslant \|g\| \quad \text{(lattice norm property)} \\
\text{(note that this implies } \|\,f\,\| = \|\,|f|\,\|\text{, see (P.3.1.9) below)}
\end{array}
\end{array}\right\}\text{normed Riesz space} \\
\text{The norm } \|\cdot\| \text{ is complete}
\end{array}\right\}\text{Banach lattice}$$

The element $f \vee g$ is, by definition, the *least upper bound* of the set $\{f, g\}$; its existence for all sets of two elements is equivalent to the existence of the greatest lower bound $f \wedge g$ for all sets of two elements, because it can be easily shown that $(-f) \wedge (-g) = -f \vee g$ (see e.g. Aliprantis, Border [11, Theorem 8.6 (1.) p.318]). For a treatise on lattices and order, see Davey, Priestley [40].

Example 2.1. *Examples of Banach lattices.* Banach lattices are common in Analysis. $\mathbb{R}$ with usual operations, as real vector space, is a Banach lattice, the standard $\vee$ being the maximum of two real numbers. Also, the Euclidean space $\mathbb{R}^n$ under the componentwise ordering (see e.g. Zaanen [149, Example 1.2 (i) p.2])

$$x = (x_1, \dots, x_n) \geqslant y = (y_1, \dots, y_n) \quad \Leftrightarrow \quad x_i \geqslant y_i \;\; \forall i = 1, \dots, n$$

is a Riesz space under

$$x \vee y = (\max\{x_1, y_1\}, \dots, \max\{x_n, y_n\})$$

(see e.g. Aliprantis, Border [11, Example 8.1, n.1, p.313]), and with the Euclidean norm it becomes a Banach lattice (see e.g. Aliprantis, Border [11, Example 9.1 p.348]).

Also, classical Lebesgue spaces are examples of Banach lattices (see e.g. Meyer-Nieberg [121, Example (v) p.9]), under the ordering defined by $f \geqslant g$ whenever $f(x) \geqslant g(x)$ almost everywhere; this is stated also in Altomare, Campiti [13, (1.2.39) p.30].

Moreover, the Banach space of all real valued continuous functions on a compact Hausdorff space X, endowed with the pointwise order and the supremum norm, is a Banach lattice (see e.g. Meyer-Nieberg [121, Example (ii) p.8]). For other examples, see Chill, F., Król [32, Remark 4.4 p.522] and Szankowski [140]. □

The reader can find several examples of Riesz spaces in Luxemburg, Zaanen [107, Example 11.2 p.48], Zaanen [149, Example 4.2 p.13 and Example 7.3 p.28]. Other interesting examples are in Aliprantis, Border [11, Examples 8.1 p.313].

We remark that properties of ordered vector spaces play a role in Korovkin-type approximation theory, a subject in which Professor Francesco Altomare gave important contributions (see the treatise Altomare, Campiti [13]).

We close this section showing, through next examples, that structures having the properties above are not only abstract, but they really exist. In particular, we are going to see that any additional property required in the chain of notions above is necessary, because it is not a consequence of the previous ones.

Example 2.2. *There exist ordered sets (i.e., sets with partial ordering) without a real vector space structure.* $\mathbb{N}$, the natural numbers with the usual order, constitute an ordered set. Obviously with the usual notion of addition $\mathbb{N}$ cannot have a real vector spaces structure (because, for instance, the opposite of *1* is missing), but we can assert that it cannot exist any notion of sum and scalar multiplication which gives to $\mathbb{N}$ a real vector space structure. In fact, $\alpha 1$ should belong to $\mathbb{N}$ for every $\alpha \in \mathbb{R}$, but this cannot happen because $\mathbb{N}$ has not the cardinality of the continuum. □

Example 2.3. *There exist (real) ordered sets without the lattice property.* Consider for instance $C^1([0,2])$ as the real vector space of all continuously differentiable real-valued functions on the closed interval $[0,2]$: it is a real vector space with the usual pointwise sum and pointwise scalar multiplication. It has a partial ordering with $\geqslant$ defined pointwise. However, there exist couples of functions in $C^1([0,2])$ without a least upper bound (in $C^1([0,2])$). Take for instance $f(x) = x$, $g(x) = 2 - x$. Then, a $C^1(0,2)$ function $f \vee g$ satisfying the properties $f \vee g \geqslant f$, $f \vee g \geqslant g$, $h \geqslant f \vee g \; \forall h : h \geqslant f, h \geqslant g$ does not exist. In fact, on the contrary, from $f \vee g \geqslant f$, $f \vee g \geqslant g$ it would be $(f \vee g)(x) \geqslant \max\{x, 2 - x\}$ and any function h of the type

$$h(x) = \alpha(x)(f \vee g)(x) + (1 - \alpha(x)) \max\{x, 2 - x\},$$

where $\alpha \in C^1(0,2)$, $0 \leqslant \alpha(x) \leqslant 1$, $\alpha \not\equiv 1$, $\alpha(1) = 1$ (hence $\alpha'(1) = 0$; say, $\alpha(x) = \exp(-(x-1)^2)$) would be such that

$$h \in C^1(0,2), \quad \max\{x, 2 - x\} \leqslant h(x) \leqslant (f \vee g)(x), \quad h \neq f \vee g$$

which is absurd. $\square$

Example 2.4. *There exist a Riesz space without the lattice norm property.* An example of Riesz space is the Sobolev space $W^{1,p}(\Omega)$, that we consider here for $1 \leqslant p \leqslant \infty$ and $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$) open bounded set. It is the real vector space of the real valued functions in $L^p(\Omega)$ whose weak derivatives of first order exist in $L^p(\Omega)$; endowed with the norm (here, as usual, $f = D^0 f$)

$$\|f\|_{W^{1,p}(\Omega)} = \sum_{0 \leqslant |\beta| \leqslant 1} \|D^\beta f\|_{L^p(\Omega)} \quad \text{if} \quad 1 \leqslant p \leqslant \infty,$$

it becomes a Banach space (see e.g. Brezis [28, Proposition 9.1 p.264], Gilbarg, Trudinger [66, Section 7.5 p.153], Adams, Fournier [2, Theorem 3.3 p.60]). Moreover, under the ordering defined by $f \geqslant g$ whenever $f(x) \geqslant g(x)$ almost everywhere, it becomes a Riesz space: in fact, it is well known that the positive part of a weakly differentiable function is again a weakly differentiable function, and $Df^+ = \chi_{\{f>0\}} Df$ (see e.g. Gilbarg, Trudinger [66, Lemma 7.6 p.152]), from which $f \in W^{1,p}(\Omega)$ entrains $f^+, f^- \in W^{1,p}(\Omega)$, and therefore from

$$f \vee g = [(f-g) \vee 0] + g = (f-g)^+ + g,$$

it is immediate to realize that the lattice property holds: if $f, g \in W^{1,p}(\Omega)$, then also $f \vee g \in W^{1,p}(\Omega)$. On the other hand, the lattice norm property does not hold. We can verify this statement on a particular case. Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, $f(x,y) \equiv 1$, $g(x,y) \equiv x$. Then, $f, g \in W^{1,p}(\Omega)$, $|f| \geqslant |g|$; but

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \left\|\frac{\partial f}{\partial x}\right\|_{L^p(\Omega)} + \left\|\frac{\partial f}{\partial y}\right\|_{L^p(\Omega)} = \|1\|_{L^p(\Omega)}$$

and

$$\|g\|_{W^{1,p}(\Omega)} = \|g\|_{L^p(\Omega)} + \left\|\frac{\partial g}{\partial x}\right\|_{L^p(\Omega)} + \left\|\frac{\partial g}{\partial y}\right\|_{L^p(\Omega)} = \|g\|_{L^p(\Omega)} + \left\|\frac{\partial g}{\partial x}\right\|_{L^p(\Omega)} = \|x\|_{L^p(\Omega)} + \|1\|_{L^p(\Omega)},$$

hence $\|f\|_{W^{1,p}(\Omega)} < \|g\|_{W^{1,p}(\Omega)}$. The conclusion is that $W^{1,p}(\Omega)$ is not a Banach lattice (for a delicate result in this direction see Pełczyński, Wojciechowski [131]). We stress, however, that there are situations where the lattice norm property is not needed for the whole Sobolev space, but just for the space to which f and $|Df|$ belong (see the recent study in Jain, Molchanova, Singh, Vodopyanov [74], where a characterization in terms of the boundedness of the maximal operator is proved).

We recall here also another example, which is in a finite dimensional vector space. Consider the following example from Chill, F., Król [32, Remark2.3(a) p.513]. If we equip the Riesz space $\mathbb{R}^2$ with either of the norms

$$N_1(x_1, x_2) := \begin{cases} |x_1| + |x_2| & \text{if } x_1 x_2 \geq 0 \\ \sup\{|x_1|, |x_2|\} & \text{if } x_1 x_2 < 0 \end{cases}$$

or

$$N_2(x_1, x_2) := \begin{cases} \sup\{|x_1|, |x_2|\} & \text{if } x_1 x_2 \geq 0 \\ |x_1| + |x_2| & \text{if } x_1 x_2 < 0 \end{cases},$$

since $N_i(1,-1) \neq N_i(1,1) = N_i(|(1,-1)|)$, then $(\mathbb{R}^2, N_i)$, $i = 1,2$, do not enjoy the lattice norm property. $\square$

Example 2.5. *There exist a normed Riesz space whose norm does not satisfy the completeness property.* A first, immediate, example is $\mathbb{Q}$, the set of rational numbers endowed with usual operations. A second example is $C([0,1])$, the real vector space of all continuous real-valued functions on the closed interval $[0,1]$: it is a real vector space with the usual pointwise sum and pointwise scalar multiplication, and it has a partial ordering with $\geqslant$ defined pointwise. It is a normed space under the L^1 lattice norm $\|f\| := \int_0^1 |f(x)|dx$, but such norm is not complete (see e.g. Aliprantis, Border [11, p.348]): setting for instance $f_n(x) = \min\{x^{-1/2}, n\}$, $f_n(0) = 0$, the sequence (f_n) is a Cauchy sequence not converging to any element in $C([0,1])$. However, as written above, endowed with the standard supremum norm, it is a Banach lattice.

Another example is the following. Let $\mathcal{R}$ be the space of sequences $f = (a_1, a_2, \dots, a_n, \dots)$, where all the a_i's are real numbers, and $a_i \neq 0$ for only finitely many values of i. It is a real vector space with the usual pointwise sum and pointwise scalar multiplication, and it has a partial ordering with $\geqslant$ defined pointwise. If $f = (a_1, a_2, \dots, a_n, \dots)$ and $g = (b_1, b_2, \dots, b_n, \dots)$, then clearly

$$f \vee g = (\max\{a_1, b_1\}, \max\{a_2, b_2\}, \dots, \max\{a_n, b_n\}, \dots).$$

Setting $\|f\| = \max_i\{|a_i|\}$, we obtain a Riesz space, however, the norm is not complete because, for instance, setting

$$f_n = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, 0, \dots\right) \quad n \in \mathbb{N},$$

the sequence $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence not converging to any element in $\mathcal{R}$. □

3. Some Nakano modulars

Modulars were historically introduced by Hidegorô Nakano. The reader interested in the life and in the scientific activity of Hidegorô Nakano should consult the beautiful exposition by Maligranda [113]. Nakano introduced and studied modulars in various different frameworks: he introduced modulars on *universally continuous semi-ordered linear spaces* (actually called *Dedekind complete Riesz spaces*), and gave a generalized version for *general semi-ordered linear spaces* (actually called *Riesz spaces*, see Nakano [126]). In a subsequent paper, he introduced a still more general definition for real vector spaces without assuming the existence of an order (Nakano [127]). Here, we will limit ourselves to these three notions (so that we can deserve attention to notions introduced later) which do not cover all variants, appeared already in the pioneering book Nakano [126] (which are, for instance, complete modulars, monotone complete modulars, simple modulars, semi-simple modulars, singular modulars, linear modulars, etc.) or in other papers (see e.g. Nakano [129, 130]).

For our goals, it is convenient to fix a unique notation for the discussion about the various notions of modulars. It is therefore natural that our symbols may differ from the original references.

Modulars introduced by Nakano are particular functionals defined on real vector spaces or on richer structures (as Riesz spaces), which will be denoted always by $\mathcal{R}$. In the sequel, the same symbol will be used also to denote complex vector spaces (in which case *complex* will be specified). Elements of $\mathcal{R}$ will usually be denoted by $f, g, h, \dots$, and the zero vector in $\mathcal{R}$ will be denoted by 0; greek letters $\alpha, \beta, \dots$ will be used to denote constants in $\mathbb{R}$ which act as scalar multipliers, and the real number zero will be denoted by 0 (so that it can be easily distinguished by $0 \in \mathcal{R}$). Modulars will be denoted by ρ. Natural numbers are denoted by $\mathbb{N}$ $(= \{1, 2, \dots\})$.

3.1. **Nakano modulars on Dedekind complete Riesz spaces.** A Riesz space $\mathcal{R}$ is said to be *Dedekind complete* (Nakano used to say *universally continuous,* as recalled in Luxemburg, Zaanen [107, p.124]) if every non-empty subset of $\mathcal{R}$ with an upper bound admits in $\mathcal{R}$ the supremum, i.e., the least upper bound (equivalently, if every non-empty subset of $\mathcal{R}$ with a lower bound admits in $\mathcal{R}$ the infimum). The definition makes sense because the notion of Riesz space requires the existence of a supremum (or of an infimum) only for sets of two elements (which can be shown to be equivalent to the same requirement for finite sets, see Luxemburg, Zaanen [107, (vii) p.56]), and the extension to every non-empty subset is not automatic, as we can see from Example 3.8 below.

Example 3.6. *Some standard examples of Dedekind complete Riesz spaces.* $\mathbb{R}^n$, as Riesz space with the usual coordinatewise ordering, is Dedekind complete (see Zaanen [149, p.65]). More generally, the Riesz space of all real (pointwise) functions on any non-empty set X (if X is finite, consisting of n points, we get back $\mathbb{R}^n$), endowed with pointwise ordering, is Dedekind complete (see e.g. Luxemburg, Zaanen [107, p.466]).

The affine functions on a closed interval $[a,b]$ in $\mathbb{R}$ are a Dedekind complete Banach lattice with the Euclidean norm (see Zaanen [149, p.67]).

Moreover, the classical Lebesgue spaces (on totally $\sigma-$finite measures, i.e., the elements of the Lebesgue spaces are defined on a measure space which is union of a countable collection of sets of finite positive measure) are examples of Dedekind complete Banach lattices, and therefore Dedekind complete Riesz spaces (see e.g. Meyer-Nieberg [121, Example (v) p.9]). Another example is $L^0(X)$, the Riesz space – under the a.e. ordering – of all measurable real-valued functions over a measure space (X, Λ, μ) where μ is non-negative, $\sigma-$additive (equivalently, countably additive, i.e., the measure of the union of a countable collection of pairwise disjoint sets coincides with the sum of the measures) and $\sigma-$finite, with identification of the functions which are equal almost everywhere on X, which is Dedekind complete (see e.g. Boccuto, Riečan, Vrábelová [26, p.37], Luxemburg, Zaanen [107, p.459]). □

In order to understand how abstract Dedekind complete Riesz spaces can be, we exhibit the details of an example of such a structure (we will mainly follow Luxemburg [106, p.110]), whose elements are, in general, not functions but measures.

Example 3.7. *A Dedekind complete Riesz space of measures.* Let X be a non-empty set and let Λ be an algebra (i.e., a non-empty family of subsets closed under finite unions and complementation, that is, if $A, B \in \Lambda$, then $A \cup B \in \Lambda$ and $X \setminus A \in \Lambda$, see e.g. Aliprantis, Border [11, p.129]; for instance, if X is a topological space, one can consider the Borel sets in X) in X whose elements will be called *measurable sets.* Let $C(X, \Lambda)$ be the completion of the real-valued $\Lambda-$step functions (i.e., linear combinations of characteristic functions of measurable sets) with respect to the topology of the uniform convergence, i.e., with respect to the supremum norm. Let $\mathcal{R}$ be the (norm) dual space of $C(X, \Lambda)$, usually denoted by $ba(X, \Lambda)$, i.e., the vector space of the linear and bounded ($\Leftrightarrow$ continuous) operators $X \to \mathbb{R}$ (see e.g. Aliprantis, Border [11, Sect.6.3 p.230]; this norm dual space is not to be confused with the *order* dual of Riesz spaces, made of linear functionals which are bounded in the sense of the order, see e.g. Aliprantis, Border [11, p.327] – the two spaces may be different!). This dual can be represented by the space of the finitely additive set functions μ on Λ (*finitely additive* means that for each finite family of pairwise disjoint sets $\{A_i\}$ whose union belongs to Λ, μ computed in the union equals the sum of the $\mu(A_i)$'s, see Aliprantis, Border [11, p.374]) of finite total variation $\|\mu\|_1$. Here, $\|\mu\|_1$ denotes the total variation of μ over X, defined by

$$\|\mu\|_1 = \lim_{\pi} V(\mu; \pi) = \lim_{\pi} \sum_{k=1}^{n} |\mu(E_k)|,$$

where $\pi = \pi(E_1, \ldots, E_n)$ is a $\Lambda-$partition (note that the limit exists since $V(\mu;\pi)$ is increasing in π). If $f \in C(X, \Lambda)$ and $\mu \in ba(X, \Lambda)$, then the bilinear form determining the duality between $C(X, \Lambda)$ and $ba(X, \Lambda)$ is denoted by

$$< f, \mu >= \int_X f d\mu.$$

It can be proved (see Aliprantis, Border [11, p.374], see also Meyer-Nieberg [121, Example (vi) p.9]) that $\mathcal{R} = ba(X, \Lambda)$ is a Dedekind complete Banach lattice (in particular, a Riesz space) under the ordering defined by $\mu \geqslant \nu$ whenever $\mu(A) \geqslant \nu(A)$ for all $A \in \Lambda$ (see Aliprantis, Border [11, p.314]). The induced supremum is given by

$$\mu \vee \nu(E) = \mu(E) + \sup\{(\nu - \mu)(A) : A \in \Lambda, A \subset E\} = \sup\{\mu(A) + \nu(E \setminus A) : A \in \Lambda, A \subset E\}. \quad \square$$

Example 3.8. *There exists a Riesz space which is not Dedekind complete.* Consider an infinite set X, and let $\mathcal{R}$ be the Riesz space of all real-valued functions defined on X whose range is finite, with the pointwise ordering. Then, $\mathcal{R}$ is not a Dedekind complete Riesz space (see Aliprantis, Burkinshaw [12, Example 2.13(3) p.15] or Luxemburg, Zaanen [107, (iii) p.139]). We give here the full details of the proof. Fix a sequence $(x_i)_{i\in\mathbb{N}}$ of pairwise disjoint elements in X, and consider the subset $\{f_i\}_{i\in\mathbb{N}} \subset \mathcal{R}$, where

$$f_i(x) = \begin{cases} j^{-1} & \text{if} \quad x = x_j \text{ and } j \leqslant i \\ 0 & \text{otherwise} \end{cases}.$$

Clearly, $\{f_i\}_{i\in\mathbb{N}}$ admits an upper bound, because every f_i is smaller than the function identically 1 (which has finite range); however, we can show that the supremum does not exist. In fact, let $g \in \mathcal{R}$ be an upper bound for $\{f_i\}_{i\in\mathbb{N}}$; we show that there exists $h \in \mathcal{R}$ such that $h < g$, h being an upper bound for $\{f_i\}_{i\in\mathbb{N}}$ as well. Since g has a finite range, there exists $k \in \mathbb{N}$ such that k^{-1} does not belong to the range of g. Set

$$h(x) = \begin{cases} g(x) & \text{if} \quad x \neq x_k \\ k^{-1} & \text{if} \quad x = x_k \end{cases}.$$

Since $f_k(x_k) \leqslant g(x_k)$, we have $k^{-1} \leqslant g(x_k)$, and since k^{-1} does not belong to the range of g, we can assert that in fact $k^{-1} < g(x_k)$. It follows that $h < g$. On the other hand, $f_i \leqslant h$ for all $i \in \mathbb{N}$: if $x \neq x_k$, then $f_i(x) \leqslant h(x)$ because it is equivalent to $f_i(x) \leqslant g(x)$; if $x = x_k$ and $i < k$, then $f_i(x_k) = 0 < k^{-1} = h(x_k)$; finally, if $x = x_k$ and $i \geqslant k$, then $f_i(x_k) = k^{-1} = h(x_k)$.

Another example of Riesz space which is not Dedekind complete is $C([0,1])$ (see e.g. Meyer-Nieberg [121, p.7]), which has been considered in Example 2.5: the set $\{x^\alpha\}_{0<\alpha<1} \subset C([0,1])$ is bounded by the constant function 1, but it does not admit a supremum in $C([0,1])$.

Finally, the Riesz space of all real bounded functions f over the interval $[0,1] \subset \mathbb{R}$ such that $f(x) \neq f(0)$ holds for at most countably many x, with pointwise ordering, is not Dedekind complete (see e.g. Luxemburg, Zaanen [107, (ii) p.139] for details). □

In Dedekind complete Riesz spaces, it is defined a monotone convergence (see Nakano [126, p.17]) as follows (here Λ denotes a set of indices):

$$f_\lambda \uparrow_{\lambda\in\Lambda} f \Leftrightarrow \begin{cases} \forall\, \lambda_1, \lambda_2 \in \Lambda\ \exists\, \lambda \in \Lambda \text{ such that } f_\lambda \geqslant f_{\lambda_1} \vee f_{\lambda_2} \\ f = \text{least upper bound of } \{f_\lambda\}_{\lambda\in\Lambda} \end{cases}.$$

A functional ρ defined on a Dedekind complete Riesz space $\mathcal{R}$ is said to be a *(Nakano) modular on a Dedekind complete Riesz space* (see Nakano [126, p.153]) if it satisfies the following properties ($f, f_\lambda, g \in \mathcal{R}$, $\alpha \in \mathbb{R}$):

(P.3.1.1) $0 \le \rho(f) \le \infty, \quad \forall f \in \mathcal{R}$,
(P.3.1.2) $\rho(\alpha f) = 0, \quad \forall \alpha \ge 0 \quad \Rightarrow \quad f = 0$,
(P.3.1.3) $\forall\, f \in \mathcal{R}, \quad \exists\, \alpha > 0$ such that $\rho(\alpha f) < \infty$,
(P.3.1.4) $\forall\, f \in \mathcal{R},\ \alpha \to \rho(\alpha f)$ is a convex function,
(P.3.1.5) $|f| \le |g| \quad \Rightarrow \quad \rho(f) \le \rho(g)$,
(P.3.1.6) $f \wedge g = 0 \quad \Rightarrow \quad \rho(f+g) = \rho(f) + \rho(g)$,
(P.3.1.7) $0 \le f_\lambda \uparrow_{\lambda\in\Lambda} f \quad \Rightarrow \quad \rho(f) = \sup_{\lambda\in\Lambda} \rho(f_\lambda)$.

When on $\mathcal{R}$ a modular is defined, $\mathcal{R}$ is said to be a *modulared space*.
Let us record few consequences of this definition of modular:
(P.3.1.8) $\rho(0) = 0$.

Proof. By (P.3.1.3), for some $\alpha > 0$ it is $\rho(\alpha 0) < \infty$, but, since $\alpha 0 = 0$, we have $\rho(0) < \infty$; on the other hand, since $0 \wedge 0 = 0$ and $0 + 0 = 0$, by (P.3.1.6)

$$\rho(0) = \rho(0+0) = \rho(0) + \rho(0) = 2\rho(0),$$

from which $\rho(0) = 0$. □

(P.3.1.9) $\rho(f) = \rho(|f|)$.

Proof. Since $|f| \leqslant |f| \vee (-|f|) = ||f||$, by (P.3.1.5), with g replaced by $|f|$, we have $\rho(f) \leqslant \rho(|f|)$; on the other hand, setting $h = 0$ in the triangle inequality $||f| - |h|| \leqslant |f + h|$ (whose simple proof coincides with the standard one for real numbers, however, for Riesz spaces see e.g. Luxemburg, Zaanen [107, Theorem 12.1 p.62]), we get $||f|| \leqslant |f|$, and again by (P.3.1.5), with f replaced by $|f|$ and g replaced by f, we get the reversed inequality. □

(P.3.1.10) $\lim\limits_{\alpha \to 0} \rho(\alpha f) = 0, \quad \forall f \in \mathcal{R}$.

Proof. For every $f \in \mathcal{R}$, using the compatibility of $\geqslant$ with respect with the addition (included in the definition of order given in the scheme at the beginning of subsection 2.2), we have

$$\begin{aligned} |f| &= f \vee (-f) = f + f \vee (-f) - f = (2f) \vee 0 - f = (2f) \vee 0 + [f \vee 0 - f] - f \vee 0 \\ &= 2(f \vee 0) + [0 \vee (-f)] - (f \vee 0) = (f \vee 0) + (-f) \vee 0 \geqslant 0 + (-f) \vee 0 \\ &= (-f) \vee 0 \geqslant 0. \end{aligned} \tag{3.2}$$

On the other hand, for $\alpha \in \mathbb{R}$,

$$|\alpha f| = (\alpha f) \vee (-(\alpha f)) = (\alpha f) \vee ((-\alpha) f) = |\alpha| (f \vee (-f)) = |\alpha||f|. \tag{3.3}$$

Now, let $0 \leqslant \alpha_1 < \alpha_2 \leqslant 1$, so that $\alpha_2 - \alpha_1 > 0$. Using the compatibility of $\geqslant$ with respect with the scalar multiplication, by (3.2), we have $(\alpha_2 - \alpha_1)|f| \geqslant 0$. Then, by (3.3), using again the compatibility of $\geqslant$ with respect with the addition,

$$|\alpha_1 f| = \alpha_1 |f| \leqslant \alpha_1 |f| + (\alpha_2 - \alpha_1)|f| = \alpha_2 |f| = |\alpha_2 f|$$

from which, by (P.3.1.5), we get that $\alpha \in [0, 1] \to \rho(\alpha f)$ is an increasing function. Now, by (P.3.1.8), we have $\rho(0) = 0$, by (P.3.1.3) the same function is finite around $\alpha = 0$, and by (P.3.1.4) the same function is also continuous. Property (P.3.1.10) is therefore proved. □

Remark 3.1. *The proof above uses implicitly some properties which are true also in the more general framework of ordered vector spaces. Readers interested in a systematic exposition containing a huge sequence of simple propositions may consult, for instance, Luxemburg, Zaanen* [107, Chapter 2, Section 11 p.48].

A less immediate consequence is the identity (see Nakano [126, (12) p.154]).
(P.3.1.11) $\rho(f \vee g) + \rho(f \wedge g) = \rho(f) + \rho(g) \quad$ for $f, g \geqslant 0$.
It should be remarked, here, that in the abstract set function theory (which is not within the topic of this paper), the property (P.3.1.11) alone (where f, g are sets, $\vee$ is the union and $\wedge$ the intersection) defines ρ as $modular$ (for details see Konig [89, p.11], see also Weber [146]).
By using a very technical result (one of the tools being Zorn's Lemma), Nakano obtains the convexity of ρ in $\mathcal{R}$ (see Nakano [126, Theorem 36.8 p.163]):
(P.3.1.12) $\rho(\alpha f + \beta g) \leqslant \alpha \rho(f) + \beta \rho(g) \quad$ for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$
and also, in the same statement,
(P.3.1.13) $\rho(f + g) \geqslant \rho(f) + \rho(g) \geqslant \rho(f - g) \quad$ for $f, g \geqslant 0$.
A description of the whole theory built by Nakano and the several contributions concerning vector lattices are out of the goal of this paper; the reader interested in this topic is warmly invited to read the already mentioned nice exposition by Lech Maligranda [113], full of historical details. We mention here also Koshi, Shimogaki [93], where the authors introduced the $quasi - modular$ spaces, weakened the theory of modular spaces and summarized the work in

Nakano [126]. The same authors then built a theory developed in Koshi, Shimogaki [92], Koshi [91]. See also Yamamuro [147].

For our purposes, we just highlight that essentially the model example of modular satisfying properties (P.3.1.1)-(P.3.1.7) is the following, highlighted by Nakano himself in the introduction in [126, p.4]. If $\Phi(\xi, t)$ is measurable as a function of t for $0 \leqslant t \leqslant 1$, and non-decreasing, convex as a function of $\xi \geqslant 0$, $\Phi(0,t) = 0$, $\Phi(\xi, t) = \lim_{\varepsilon \to 0+} \Phi(\xi - \varepsilon, t)$ for $\xi > 0$, and $\Phi(\alpha_t, t) < \infty$ for some $\alpha_t > 0$, then the class $\mathcal{R}$ of all real a. e. measurable functions φ on $[0,1]$ such that

$$\int_0^1 \Phi(\alpha|\varphi(t)|, t) d\mu_t < \infty$$

for some $\alpha > 0$, is a Dedekind complete Riesz space, and

$$\rho(\varphi) := \int_0^1 \Phi(|\varphi(t)|, t) d\mu_t < \infty \tag{3.4}$$

is a modular on $\mathcal{R}$. In particular, $\mathcal{R}$ can be the Lebesgue space $L^1(\Omega, \mu)$. For another, less standard, example, see Albrycht, Orlicz [10].

Example 3.9. *There exist norms which are (Nakano) modulars (on Dedekind complete Riesz space).* The model example discussed above, in the case $\Phi(\xi, t) = \xi$, reduces $\mathcal{R}$ to the Lebesgue space $L^1(\Omega, \mu)$, and the corresponding modular ρ is the standard norm in $L^1(\Omega, \mu)$. A still more particular, nevertheless relevant, example is that one of μ equal to the sum of a finite number n of Dirac masses: in this case the set of the a.e. measurable functions can be identified with $\mathbb{R}^n$. The identification holds also as Riesz spaces: the a.e. ordering of measurable functions corresponds to the componentwise ordering, and the modular ρ, which is the L^1 norm, corresponds to the so-called *Taxicab norm* or *Manhattan norm.* □

Example 3.10. *There exist norms which are not (Nakano) modulars (on Dedekind complete Riesz space).* Norms of well known spaces whose elements are real measurable functions, in general, are *not* modulars. The key point is property (P.3.1.6), which is a fundamental tool to get the convexity property (P.3.1.12), but it throws away some important norms, for instance, the norm in $L^p(\Omega, \mu)$, $1 < p < \infty$: in fact, if for instance $A, B \subset \Omega$ are disjoint and $\mu(A) = \mu(B) = 1$, then $\chi_A \wedge \chi_B = 0$ and

$$\begin{aligned} \rho(\chi_A + \chi_B) &= \|\chi_A + \chi_B\|_{L^p(\Omega,\mu)} = (\mu(A) + \mu(B))^{1/p} = 2^{1/p} \\ &\neq 2 = \mu(A) + \mu(B) = \|\chi_A\|_{L^p(\Omega,\mu)} + \|\chi_B\|_{L^p(\Omega,\mu)} = \rho(\chi_A) + \rho(\chi_B), \end{aligned}$$

and therefore (P.3.1.6) is not satisfied. As in the previous example, the consideration of the case of measures which are sums of Dirac masses tells that when $n \geq 2$ the Euclidean norm is not a modular on $\mathbb{R}^n$, in the sense introduced in Nakano [126]. □

3.2. **Nakano modulars on Riesz spaces.** Still in the same Nakano [126], there is a notion of modular over Riesz spaces which does not satisfy necessarily the Dedekind completeness assumption, which clearly plays its role in property (P.3.1.7). Since such property had a role in the proof of convexity, then the one-dimensional convexity (P.3.1.4) is changed definitively in the whole convexity of the modular, and this had some consequences also on other properties. We are going to list the resulting set of properties, which define modulars that Nakano called *general modulars.* A functional ρ defined on a Riesz space $\mathcal{R}$ is said to be a *(Nakano) modular on Riesz space* (see Nakano [126, p.271]) if it satisfies the following properties ($f, f_\lambda, g \in \mathcal{R}$, $\alpha \in \mathbb{R}$):

(P.3.2.1) $0 \leq \rho(f) \leq \infty, \quad \forall f \in \mathcal{R}$,

(P.3.2.2) $\rho(\alpha f) = 0, \quad \forall \alpha \geq 0 \quad \Rightarrow \quad f = 0$,

(P.3.2.3) $\forall\, f \in \mathcal{R}, \quad \exists\, \alpha > 0$ such that $\rho(\alpha f) < \infty$,

(P.3.2.4) $\rho(\alpha f + \beta g) \leqslant \alpha\rho(f) + \beta\rho(g)$ for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$,

(P.3.2.5) $|f| \leq |g| \quad \Rightarrow \quad \rho(f) \leq \rho(g)$,

(P.3.2.6) $\rho(f + g) \geqslant \rho(f) + \rho(g) \geqslant \rho(f \vee g)$ for $f, g \geqslant 0$, (P.3.2.7) $\sup_{0\leqslant\alpha<1} \rho(\alpha f) = \rho(f)$.

The consequences (P.3.1.8), (P.3.1.9), (P.3.1.10) are still true with almost the same proofs. It is worth to make a comment on (P.3.2.7). Given $f \in \mathcal{R}$, if for some $0 \leqslant \alpha < 1$ we have $\rho(\alpha f) = \infty$, then (P.3.2.7) is not a condition on ρ, because (P.3.2.7) is satisfied: in fact, in all Riesz spaces it is known that $|\alpha f| = |\alpha||f|$ (the proof in (3.3) does not use the Dedekind completeness), and therefore, by (P.3.1.9), since $\alpha \geqslant 0$,

$$\rho(\alpha f) = \rho(|\alpha f|) = \rho(|\alpha||f|) = \rho(\alpha|f|) \leqslant \rho(|f|) = \rho(f).$$

On the other hand, if for all $0 \leqslant \alpha < 1$ we have $\rho(\alpha f) < \infty$, then

$$\sup_{0\leqslant\alpha<1} \rho(\alpha f) \leqslant \sup_{\substack{|f|\geqslant|g| \\ \rho(g)<\infty}} \rho(g) \leqslant \rho(f),$$

hence, (P.3.2.7) is not weaker than

$$\sup_{\substack{|f|\geqslant|g| \\ \rho(g)<\infty}} \rho(g) = \rho(f)$$

which is a condition on ρ called by Nakano *modular continuity* (see Nakano [126, p.182] and Nakano [126, p.192]).

When considering Nakano modulars on Riesz spaces, the gain is, for instance, the possibility to consider the modular (3.4) restricted to the Riesz space of all real-valued functions whose range is finite, with the pointwise ordering, considered in Example 3.8. However, the gain of new structures still leaves out, in general, several norms. In fact, a norm $\|\cdot\|$ satisfying (P.3.2.6) and, at the same time, the triangle inequality, necessarily must satisfy

$$\|f + g\| = \|f\| + \|g\| \quad \text{for } f, g \geqslant 0,$$

which is not satisfied, for instance, by the norm in $L^p(\Omega, \mu)$ when $p > 1$ (as we saw in Example 3.10). Moreover, it would be good to have a notion of modular which admits norms on structures not enjoying the requirements imposed by Riesz spaces, for instance, the real vector space $C^1([0,1])$ considered in Example 2.3 (as ordered vector space which is not a Riesz space) which is a normed space (see e.g. Kufner, John, Fučik [101, (1) p.25]) when endowed with the (standard) norm

$$\|f\|_{C^1([0,1])} = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|.$$

The hole is filled again by Nakano, in a 1951 paper, as we are going to see.

3.3. **Nakano modulars on real vector spaces.** A functional ρ defined on a real vector space $\mathcal{R}$ is said to be a *(Nakano) modular on real vector space* (see Nakano [127]) if it satisfies the following properties ($f, g \in \mathcal{R}$):

(P.3.3.1) $0 \leq \rho(f) \leq \infty$,
(P.3.3.2) $\rho(f) = \rho(-f)$,
(P.3.3.3) $\exists f \in \mathcal{R},\ \exists \alpha > 0$ such that $\rho(\alpha f) < \infty$,
(P.3.3.4) $\rho(\alpha f) = 0, \quad \forall \alpha > 0 \quad \Leftrightarrow \quad f = 0$,
(P.3.3.5) $\sup_{0\leqslant\alpha<1} \rho(\alpha f) = \rho(f)$,
(P.3.3.6) ρ is convex: $0 \leqslant \alpha \leqslant 1 \Rightarrow \rho(\alpha f + (1-\alpha)g) \leqslant \alpha\rho(f) + (1-\alpha)\rho(g) \qquad (0 \cdot \infty = 0)$.

Remark 3.2. *Unfortunately reference is not easily accessible, and properties (P.3.3.1)-(P.3.3.6) are taken from the review MR44048 (13,362a) in MathSciNet, where the list of properties is given in a style raising doubts. This means that our translation into a precise style could be not faithful with the original source. The sentence "$m(\lambda x) = \infty$ for all positive λ does not occur for any x" appearing in the review could have different interpretations; maybe the original source intends, "For any x, $m(\lambda x) = \infty$ for all positive λ does not occur", because the existence of x in (P.3.3.3) is trivial (hence, there would be no reason to have* *this extra property) in the case $\rho(0) = 0$, hence implicit in (P.3.3.4). However, the whole question is not really important, because we will never use it and we stated it just to try to respect the historical value of this notion. We observe that in Nakano's terminology (see e.g. [129]), an element $x \in \mathcal{R}$ is said to be finite whenever $\rho(\lambda x) < \infty$ for all $\lambda > 0$, is said to be a null element if $\rho(\lambda x) = 0$ for all $\lambda > 0$, and a modular ρ is said to be pure if 0 is the only null element: when applying results from a set of axioms, it may be worth to choose properties which avoid unpleasant situations (for instance: all elements must be finite and/or the modular must be pure, etc.).*

Proposition 3.1. *Norms $\|\cdot\|$ on a real vector space $\mathcal{R}$ are (Nakano) modulars (on real vector space).*

Proof. Property (P.3.3.1) follows directly from (2.1). Property (P.3.3.2) follows applying (P.2.1.2) with $\alpha = -1$: $\|f\| = |-1|\|f\| = \|-f\|$. Property (P.3.3.3) holds choosing any $f \in \mathcal{R}$, any $\alpha \in \mathbb{R}$, because by (2.1) the norm is always finite. About property (P.3.3.4), if $\|\alpha f\| = 0 \quad \forall \alpha \geq 0$, then by (P.2.1.2) we have $\alpha \|f\| = 0 \quad \forall \alpha \geq 0$, which means that $\|f\| = 0$. By (P.2.1.1), we get $f = 0$. The viceversa comes directly again from (P.2.1.1). Property (P.3.3.5) is consequence of (P.2.1.2):

$$\sup\{\|\lambda f\| \ : 0 \leqslant \lambda < 1\} = \sup\{\lambda \|f\| \ : 0 \leqslant \lambda < 1\} = \|f\|.$$

Finally, convexity follows by (P.2.1.3) and (P.2.1.2): for $0 \leqslant \alpha \leqslant 1$, $f, g \in \mathcal{M}$,

$$\|\alpha f + (1-\alpha) g\| \leqslant \|\alpha f\| + \|(1-\alpha) g\| = \alpha \|f\| + (1-\alpha)\|g\|.$$

□

As a consequence of Proposition 3.1, the norms of several well known spaces highly used in Analysis and applications are modulars, for instance, Musielak-Orlicz spaces (and therefore Orlicz spaces and variable Lebesgue spaces with their weighted versions), Lorentz spaces, grand and small Lebesgue spaces, spaces of continuous differentiable functions, Hölder-continuous differentiable functions, Morrey and Campanato spaces, Sobolev spaces. Such spaces are treated in many books, an incomplete list being Adams, Fournier [2], Bennett, Sharpley [24], Brezis [28], Castillo, Rafeiro [31], Cruz-Uribe, F. [38], Cruz-Uribe, Martell, Pérez [39], Demengel, Demengel [41], Diening, Harjulehto, Hästö, Růžička [43], Edmunds, Evans [47], Edmunds, Triebel [49], Fiorenza [64], Genebashvili, Gogatishvili, Kokilashvili, Krbec [65], Harjulehto, Hästö [69], Haroske [70], Haroske, Triebel [71], Kokilashvili, Krbec [85], Kokilashvili, Meskhi, Rafeiro, Samko [86, 87], Kufner [100], Kufner, John, Fučik [101], Lindenstrauss, Tzafriri [102, 103], Maligranda [110], Maz'ja [116], Mendez, Lang [119], Meskhi [120], Musielak [124], Pick, Kufner, John, Fučik [132], Rakotoson [133], Rao, Ren [134], Schmeisser, Triebel [137], Triebel [142], Triebel [143], Triebel [141], Turett [144]. We mention here also other nonstandard norms, which are obtained as roots of polynomials (Anatriello, F., Vincenzi [15]) or as fixed points (F., Talponen [63]).

Example 3.11. *There exist (Nakano) modulars (on real vector space) which are not norms.* If $\mathcal{R}$ is a normed space with norm $\|\cdot\|$, setting $\rho(f) = \|f\|^2$, we have a modular which is not a norm. The fact that it is not a norm is a consequence of (P.2.1.2): since

$$\rho(\alpha f) = \|\alpha f\|^2 = \alpha^2 \|f\|^2 = \alpha^2 \rho(f),$$

then ρ cannot satisfy (P.2.1.2) if $\alpha \neq 0, 1, -1$. On the other hand, ρ is a modular: the proof of properties (P.3.3.1)-(P.3.3.6) using (2.1), (P.2.1.1)-(P.2.1.3) is immediate. The reader may check that, in general, the square can be replaced by any nondecreasing, convex function on $[0, +\infty[$ assuming value 0 in the origin and not identically 0. □

Norms are also special *quasinorms*, however, in next example, we will see that quasinorms are not necessarily modulars. Incidentally, we recall that the notion of modular in Nakano [129] includes quasinorms, but we will not deal with it in this paper.

Example 3.12. *There exist quasinorms, i.e., functionals $\rho : \mathcal{R} \to [0, \infty[$ such that for some $C > 0$, for every $f, g \in \mathcal{R}$: (j) $\rho(f) = 0$ if and only if $f = 0$, (jj) $\rho(\lambda f) = |\lambda| \rho(f)$ for all $\lambda \in \mathbb{R}$, (jjj) $\rho(f+g) \leqslant$*

$C(\rho(f)+\rho(g))$ which are not (Nakano) modulars (on real vector space). We consider the following example, borrowed from Anatriello, F., Vincenzi [15, Example 2.1 p.4]: set $\mathcal{R}=L^1(0,1)$ and

$$\rho(f)=(2-\sin(\pi|\mathrm{supp}f|))\int_0^1 |f(x)|dx,$$

where $|\mathrm{supp}f|$ denotes the measure of the support of f. It is easy to check that it is a quasinorm, using the fact that the factor of the integral on the right hand side is in the interval $[1,2]$. On the other hand, ρ is not a (Nakano) modular because it is not convex: in fact,

$$\rho\left(\frac{1}{2}\chi_{(0,\frac{1}{2})}+\frac{1}{2}(-\chi_{(\frac{1}{2},1)})\right)=2\int_0^1 \frac{1}{2}\chi_{(0,\frac{1}{2})}(x)+\frac{1}{2}\chi_{(\frac{1}{2},1)}(x)dx=1;$$

on the other hand,

$$\frac{1}{2}\rho\left(\chi_{(0,\frac{1}{2})}\right)+\frac{1}{2}\rho\left(-\chi_{(\frac{1}{2},1)}\right)=\frac{1}{2}\int_0^1 \chi_{(0,\frac{1}{2})}(x)dx+\frac{1}{2}\int_0^1 \chi_{(\frac{1}{2},1)}(x)dx=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}.$$

□

Example 3.13. *There exist (Nakano) modulars (on real vector space) for which $\rho(f)=0$ does not imply $f=0$.* Let $\mathcal{M}$ be the vector space of the Lebesgue measurable functions defined in the real interval $(0,1)$, with values in the set of the real numbers $\mathbb{R}$ (i.e., almost everywhere finite), and let us set

$$\rho(f)=\begin{cases}0 & \text{if} \quad \operatorname{ess\,sup}|f|\leqslant 1\\ \infty & \text{if} \quad \operatorname{ess\,sup}|f|>1\end{cases}.$$

It is easy to check that ρ is a modular. Moreover, ρ vanishes on any function whose modulus is bounded by 1. We stress that this example is standard, and it appears in literature also with minor variations (see e.g. Koshi [90], Bachar, Mendez, Bounkhel [17, (2)]). □

The importance of Nakano modulars relies upon the following result, appeared in a primitive form in Nakano [126, Theorem 43.6 p.192]. The heart of the matter is that properties (P.3.3.1)-(P.3.3.6) guarantee the existence of a vector subspace on which a certain functional, whose expression is given explicitly, is a norm. The expression in (3.6) below is usually said to be *Luxemburg norm* (because it is known from the celebrated Luxemburg's thesis, see [105, Definition 1 p.43], given in the restricted framework of Orlicz spaces), however, following Maligranda (see [113] and [108, Comment 2]), we call it Luxemburg-Nakano norm. However, it should be stressed also what it is remarked in Diening, Harjulehto, Hästö, Růžička [43, p.25] (see also Maligranda [73]), namely, that the Luxemburg norm has the structure of the Minkowski functional introduced in Kolmogoroff [88] long before; in Diening, Harjulehto, Hästö, Růžička [43, Remark 2.1.16] it is shown that the proof of the fact that it is a norm comes from a more general (and nowadays classical) statement of Functional Analysis (see e.g. Schechter [136, 12.29.g p.317]).

Theorem 3.1. *Let ρ be a (Nakano) modular on a real vector space $\mathcal{R}$, and set*

$$\widetilde{\mathcal{R}}:=\left\{f\in\mathcal{R}: \textit{the set } \left\{\lambda>0:\rho\left(\frac{f}{\lambda}\right)<\infty\right\} \textit{ is non-empty}\right\}.$$

The following statements hold:

(i) *$\widetilde{\mathcal{R}}$ is a vector subspace of $\mathcal{R}$ (in particular, $\widetilde{\mathcal{R}}$ is non-empty).*

(ii) $\widetilde{\mathcal{R}}=\left\{f\in\mathcal{R}:\lim\limits_{\lambda\to 0+}\rho(\lambda f)=0\right\}$.

(iii) *For every $\alpha>0$,*

$$\widetilde{\mathcal{R}}=\left\{f\in\mathcal{R}: \textit{the set } \left\{\lambda>0:\rho\left(\frac{f}{\lambda}\right)\leqslant\alpha\right\} \textit{ is non-empty}\right\}. \tag{3.5}$$

(iv) *The functional* $\|\cdot\|_\alpha : \widetilde{\mathcal{R}} \to [0, +\infty[$ *defined by*

$$\|f\|_\alpha := \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \alpha\right\} \quad (\textit{Luxemburg-Nakano norm}) \tag{3.6}$$

is a norm on $\widetilde{\mathcal{R}}$.

(v) *For all* $f \in \widetilde{\mathcal{R}}$, *we have* $\|f\|_\alpha = \inf_{\lambda>0} \max\left\{\frac{1}{\lambda}, \frac{\rho(\lambda f)}{\alpha\lambda}\right\}$.

(vi) *The norms* $\|f\|_\alpha$ *are pairwise equivalent, and if* $\alpha > \beta > 0$, *then for all* $f \in \widetilde{\mathcal{R}}$

$$\|f\|_\alpha \leqslant \|f\|_\beta \leqslant \frac{\alpha}{\beta}\|f\|_\alpha. \tag{3.7}$$

(vii) *The functional* $|||\cdot|||_\alpha : \widetilde{\mathcal{R}} \to [0, +\infty[$ *defined by*

$$|||f|||_\alpha := \inf_{\lambda>0} \frac{\alpha + \rho(\lambda f)}{\alpha\lambda} \quad (\textit{Amemiya norm}) \tag{3.8}$$

is a norm on $\widetilde{\mathcal{R}}$.

(viii) *The norms* $\|\cdot\|_\alpha$ *and* $|||\cdot|||_\alpha$ *are equivalent, and for all* $f \in \widetilde{\mathcal{R}}$

$$\|f\|_\alpha \leqslant |||f|||_\alpha \leqslant 2\|f\|_\alpha.$$

Proof of **(i)**. By (P.3.3.4), if $f = 0$, then $\rho(0) = \rho(1f) = 0$, hence $\widetilde{\mathcal{R}}$ is non-empty and we have also that $0 \in \widetilde{\mathcal{R}}$. Now, we show that $\widetilde{\mathcal{R}}$ is a subspace of $\mathcal{R}$. If $f \in \widetilde{\mathcal{R}}$ is such that for some $\lambda > 0$ we have $\rho(f/\lambda) < \infty$, then for every $\alpha > 0$ also $\alpha f \in \widetilde{\mathcal{R}}$, because we have $\rho(\alpha f/(\alpha\lambda)) < \infty$. On the other hand, let $f, g \in \widetilde{\mathcal{R}}$ be such that $\rho(f/\lambda_1) < \infty$, $\rho(g/\lambda_2) < \infty$, where $\lambda_1, \lambda_2 > 0$. By the convexity property (P.3.3.6), we have

$$\rho\left(\frac{f+g}{\lambda_1+\lambda_2}\right) = \rho\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{f}{\lambda_1} + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{g}{\lambda_2}\right) \leqslant \frac{\lambda_1}{\lambda_1+\lambda_2}\rho\left(\frac{f}{\lambda_1}\right) + \frac{\lambda_2}{\lambda_1+\lambda_2}\rho\left(\frac{g}{\lambda_2}\right) < \infty,$$

hence also the sum of elements of $\widetilde{\mathcal{R}}$ belongs to $\widetilde{\mathcal{R}}$. □

Proof of **(ii)**. Let $f \in \widetilde{\mathcal{R}}$, hence $\rho(f/\lambda_0) < \infty$ for some $\lambda_0 > 0$. Recalling again that by (P.3.3.4) $\rho(0) = \rho(10) = 0$, by the convexity property (P.3.3.6), for every $0 \leqslant \lambda \leqslant 1/\lambda_0$, we have

$$0 \leqslant \rho(\lambda f) = \rho\left(\lambda\lambda_0\frac{f}{\lambda_0} + (1-\lambda\lambda_0)0\right) \leqslant \lambda\lambda_0\rho\left(\frac{f}{\lambda_0}\right) + (1-\lambda\lambda_0)0,$$

from which, letting $\lambda \to 0$, we get

$$\widetilde{\mathcal{R}} \subset \left\{f \in \mathcal{R} : \lim_{\lambda\to 0+} \rho(\lambda f) = 0\right\}.$$

Viceversa, if $f \in \mathcal{R}$ is such that

$$\lim_{\lambda\to 0+} \rho(\lambda f) = 0,$$

then for λ small we have

$$\rho(\lambda f) < 1 < \infty,$$

hence, we get also

$$\widetilde{\mathcal{R}} \supset \left\{f \in \mathcal{R} : \lim_{\lambda\to 0+} \rho(\lambda f) = 0\right\}.$$

Part **(ii)** is therefore proved. □

Remark 3.3. *Equality* **(ii)** *is currently used in literature. For instance, recently, it has been stated in Costarelli, Vinti* [36, p.9, after (3)].

Remark 3.4. *In principle, the same proof of Part* **(ii)** *could have been used to prove (P.3.1.10), because we just used convexity (which is still true, see (P.3.1.12)). However, we observed that the proof of (P.3.1.12) is very technical; in the case of Part* **(ii)**, *convexity is directly in the assumption (P.3.3.6).*

Proof of **(iii)**. Equality (3.5) follows from **(ii)** and the definition of $\widetilde{\mathcal{R}}$: in fact, if for some $f \in \mathcal{R}$ the set $\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \alpha\right\}$ is non-empty, then f is such that $\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) < \infty\right\}$ is non-empty, therefore in (3.5) the $\supset$ holds; on the other hand, if $f \in \widetilde{\mathcal{R}}$, by **(ii)**, we have

$$\lim_{\lambda \to 0+} \rho(\lambda f) = 0 \tag{3.9}$$

and therefore the definition of limit gives that for every $\alpha > 0$ there exists $\lambda_0 > 0$ such that $\rho(\lambda_0 f) < \alpha$, hence

$$\frac{1}{\lambda_0} \in \left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \alpha\right\},$$

from which (3.5) follows. □

Proof of **(iv)**. We begin by showing that $\|\cdot\|_\alpha$ satisfies (P.2.1.1). We have

$$\|0\|_\alpha = \inf\left\{\lambda > 0 : \rho\left(\frac{0}{\lambda}\right) \leqslant \alpha\right\},$$

and since for every $\lambda > 0$

$$\rho\left(\frac{0}{\lambda}\right) = 0 < \alpha,$$

we have $\|0\|_\alpha = 0$. On the other hand, let $f \in \widetilde{\mathcal{R}}$ be such that $\|f\|_\alpha = 0$, so that

$$\rho\left(\frac{f}{\lambda}\right) \leqslant \alpha \quad \forall \lambda > 0.$$

Fix $\beta > 0$, and let $\lambda > 0$ be such that $0 < \beta\lambda < 1$. By the convexity property (P.3.3.6), we have

$$\rho(\beta f) = \rho\left(\beta\lambda\frac{f}{\lambda} + (1-\beta\lambda)0\right) \leqslant \beta\lambda\rho\left(\frac{f}{\lambda}\right) + (1-\beta\lambda)0 \leqslant \beta\lambda \cdot \alpha, \tag{3.10}$$

from which, letting $\lambda \to 0$, we get $\rho(\beta f) = 0$ for all $\beta > 0$. By property (P.3.3.4), we conclude that $f = 0$.

We now show (P.2.1.2). Fix $f \in \widetilde{\mathcal{R}}$, so that $\|f\|_\alpha$ is well defined and finite. If $\lambda = 0$, we have to show that $\|0f\|_\alpha = 0\|f\|_\alpha$, i.e., $\|0\|_\alpha = 0$, but this is already known from (P.2.1.1). If $\lambda \neq 0$, by property (P.3.3.2),

$$\begin{aligned}
\|\lambda f\|_\alpha := \inf&\left\{\mu > 0 : \rho\left(\frac{\lambda f}{\mu}\right) \leqslant \alpha\right\} \\
= \inf&\left\{\mu > 0 : \rho\left(\frac{|\lambda| f}{\mu}\right) \leqslant \alpha\right\} \\
= \inf&\left\{\mu|\lambda| > 0 : \rho\left(\frac{f}{\mu}\right) \leqslant \alpha\right\} \\
= |\lambda| \inf&\left\{\mu > 0 : \rho\left(\frac{f}{\mu}\right) \leqslant \alpha\right\} \\
= |\lambda|&\|f\|_\alpha.
\end{aligned}$$

Finally, we show that $\|\cdot\|_\alpha$ satisfies (P.2.1.3). At first, we observe that for every $f \in \widetilde{\mathcal{R}}$, $f \neq 0$,

$$\rho\left(\frac{f}{\|f\|_\alpha}\right) \leqslant \alpha : \tag{3.11}$$

in fact, on the contrary, by property (P.3.3.5), it would exist $0 < \lambda < 1$ such that

$$\rho\left(\frac{\lambda f}{\|f\|_\alpha}\right) > \alpha,$$

hence $\|f\|_\alpha/\lambda > \|f\|_\alpha$ does not belong to the set defining $\|f\|_\alpha$, which is absurd. Now let $f, g \in \widetilde{\mathcal{R}}$, $f, g \neq 0$ (otherwise (P.2.1.3) is trivially true). We have

$$\begin{aligned}
\rho\left(\frac{f+g}{\|f\|_\alpha + \|g\|_\alpha}\right) &= \rho\left(\frac{\|f\|_\alpha}{\|f\|_\alpha + \|g\|_\alpha}\frac{f}{\|f\|_\alpha} + \frac{\|g\|_\alpha}{\|f\|_\alpha + \|g\|_\alpha}\frac{g}{\|g\|_\alpha}\right)\\
&\leqslant \frac{\|f\|_\alpha}{\|f\|_\alpha + \|g\|_\alpha}\rho\left(\frac{f}{\|f\|_\alpha}\right) + \frac{\|g\|_\alpha}{\|f\|_\alpha + \|g\|_\alpha}\rho\left(\frac{g}{\|g\|_\alpha}\right)\\
&\leqslant \frac{\|f\|_\alpha}{\|f\|_\alpha + \|g\|_\alpha}\alpha + \frac{\|g\|_\alpha}{\|f\|_\alpha + \|g\|_\alpha}\alpha = \alpha
\end{aligned}$$

from which $\|f+g\|_\alpha \leqslant \|f\|_\alpha + \|g\|_\alpha$. □

Remark 3.5. *Recently, it has been shown that* (3.11) *holds for every convex pseudomodulars on real vector spaces which are left lower semicontinuous (see F., Talponen* [63, Proposition 1.2] *for details); for Musielak-Orlicz spaces see Harjulehto, Hästö* [69, Lemma 3.2.3 p.53], *for variable Lebesgue spaces see Cruz-Uribe, F.* [38, Proposition 2.21 p.24]*).*

Proof of **(v)**. Let us set temporarily (it is the symbol of norm without α)

$$\|f\| = \inf_{\lambda>0} \max\left\{\frac{1}{\lambda}, \frac{\rho(\lambda f)}{\alpha\lambda}\right\}.$$

Fix $f \in \mathcal{R}$, and let us split the positive λ's into two sets. If $\lambda > 0$ is such that $\rho(\lambda f) \leqslant \alpha$, then

$$\rho\left(\frac{f}{1/\lambda}\right) \leqslant \alpha,$$

from which we get $\|f\|_\alpha \leqslant 1/\lambda$. On the other hand, if $\lambda > 0$ is such that $\alpha < \rho(\lambda f) < \infty$, then by the convexity property (P.3.3.6), we have

$$\rho\left(\frac{f}{\rho(\lambda f)/(\alpha\lambda)}\right) = \rho\left(\frac{\alpha\lambda f}{\rho(\lambda f)}\right) \leqslant \frac{\alpha}{\rho(\lambda f)}\rho(\lambda f) = \alpha,$$

and therefore in this case $\|f\|_\alpha \leqslant \rho(\lambda f)/(\alpha\lambda)$. Note that the same inequality is obviously true if $\rho(\lambda f) = \infty$. Overall, in any case, for every $\lambda > 0$, we get $\|f\|_\alpha \leqslant \|f\|$.

Viceversa, it will be sufficient to show that $\|f\|$ is smaller than any positive number which is in the set defining $\|f\|_\alpha$. Let us denote such generic positive number by $1/\mu$, so that

$$\rho\left(\frac{f}{1/\mu}\right) \leqslant \alpha. \tag{3.12}$$

We have

$$\|f\| = \inf_{\lambda>0} \max\left\{\frac{1}{\lambda}, \frac{\rho(\lambda f)}{\alpha\lambda}\right\} \leqslant \max\left\{\frac{1}{\mu}, \frac{\rho(\mu f)}{\alpha\mu}\right\} \leqslant \frac{1}{\mu}.$$

Since the infimum of all $1/\mu$ satisfying (3.12) is $\|f\|_\alpha$, we get also the inequality $\|f\| \leqslant \|f\|_\alpha$. □

Proof of **(vi)**. Fix $f \in \widetilde{\mathcal{R}}$. If $\alpha > \beta > 0$, then clearly

$$\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \beta\right\} \subset \left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \alpha\right\},$$

hence

$$\inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \beta\right\} \geqslant \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \alpha\right\},$$

i.e.,

$$\|f\|_\alpha \leqslant \|f\|_\beta.$$

On the other hand, if

$$\lambda \in \left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \alpha\right\},$$

then

$$\rho\left(\frac{\beta}{\alpha\lambda}f\right) = \rho\left(\frac{\beta}{\alpha}\frac{f}{\lambda} + \left(1 - \frac{\beta}{\alpha}\right)0\right) \leqslant \frac{\beta}{\alpha}\rho\left(\frac{f}{\lambda}\right) + \left(1 - \frac{\beta}{\alpha}\right)0 \leqslant \frac{\beta}{\alpha}\alpha = \beta,$$

hence

$$\frac{\alpha\lambda}{\beta} \in \left\{\mu > 0 : \rho\left(\frac{f}{\mu}\right) \leqslant \beta\right\},$$

from which

$$\|f\|_\beta \leqslant \frac{\alpha\lambda}{\beta}.$$

Passing to the infimum over λ, we get the right wing inequality in (3.7). □

Proof of **(vii)**. We begin by showing that $|||\cdot|||_\alpha$ satisfies (P.2.1.1). If $f = 0$, then $|||f|||_\alpha = 0$: in fact,

$$|||0|||_\alpha = \inf_{\lambda>0} \frac{\alpha + \rho(\lambda 0)}{\alpha\lambda} = \inf_{\lambda>0} \frac{1}{\lambda} = 0.$$

On the other hand, if $f \in \widetilde{\mathcal{R}}$ is such that $|||f|||_\alpha = 0$, we have

$$0 = |||f|||_\alpha = \inf_{\lambda>0} \frac{\alpha + \rho(\lambda f)}{\alpha\lambda} \geqslant \inf_{\lambda>0} \max\left\{\frac{1}{\lambda}, \frac{\rho(\lambda f)}{\alpha\lambda}\right\} = \|f\|_\alpha \geqslant 0, \tag{3.13}$$

hence $\|f\|_\alpha = 0$, from which we already showed in **(iv)** that $f = 0$. Property (P.2.1.2) follows observing that

$$\begin{aligned} |||\lambda f|||_\alpha &= \inf_{\mu>0} \frac{\alpha + \rho(\mu\lambda f)}{\alpha\mu} = \inf_{\mu>0} \frac{\alpha + \rho(\mu|\lambda|f)}{\alpha\mu} = \inf_{\mu>0} \frac{\alpha + \rho(\mu f)}{\alpha(\mu/|\lambda|)} \\ &= |\lambda| \inf_{\mu>0} \frac{\alpha + \rho(\mu f)}{\alpha\mu} = |\lambda||||f|||_\alpha. \end{aligned}$$

Finally, we show that $|||\cdot|||_\alpha$ satisfies (P.2.1.3). At first, we observe that for every $f, g \in \widetilde{\mathcal{R}}$, for arbitrary $\varepsilon > 0$ there exist $\lambda, \mu > 0$ such that

$$\frac{\alpha + \rho(\lambda f)}{\alpha\lambda} < |||f|||_\alpha + \varepsilon, \quad \frac{\alpha + \rho(\mu g)}{\alpha\mu} < |||g|||_\alpha + \varepsilon,$$

and therefore, by the convexity property (P.3.3.6), we have

$$\begin{aligned}|||f+g|||_\alpha &\leqslant \frac{\alpha+\rho\left(\frac{\lambda\mu}{\lambda+\mu}(f+g)\right)}{\alpha\frac{\lambda\mu}{\lambda+\mu}}\\ &= \frac{\lambda+\mu}{\alpha\lambda\mu}\left[\alpha+\rho\left(\frac{\mu}{\lambda+\mu}\lambda f+\frac{\lambda}{\lambda+\mu}\mu g\right)\right]\\ &\leqslant \frac{\lambda+\mu}{\alpha\lambda\mu}\left[\alpha+\frac{\mu}{\lambda+\mu}\rho\left(\lambda f\right)+\frac{\lambda}{\lambda+\mu}\rho\left(\mu g\right)\right]\\ &= \frac{1}{\mu}+\frac{1}{\lambda}+\frac{\rho\left(\lambda f\right)}{\alpha\lambda}+\frac{\rho\left(\mu g\right)}{\alpha\mu} < |||f|||_\alpha+|||g|||_\alpha+2\varepsilon,\end{aligned}$$

from which $|||f+g|||_\alpha \leqslant |||f|||_\alpha+|||g|||_\alpha$. □

Proof of **(viii)**. Estimates in the chain (3.13) show already that not only for $f=0$, but for all $f\in\widetilde{\mathcal{R}}$, we have

$$\|f\|_\alpha \leqslant |||f|||_\alpha.$$

On the other hand, for all $f\in\widetilde{\mathcal{R}}$, by **(v)**, we have

$$|||f|||_\alpha = \inf_{\lambda>0}\frac{\alpha+\rho(\lambda f)}{\alpha\lambda} \leqslant \inf_{\lambda>0} 2\max\left\{\frac{1}{\lambda},\frac{\rho(\lambda f)}{\alpha\lambda}\right\} = 2\|f\|_\alpha.$$

□

Remark 3.6. *It can happen that the vector subspace $\widetilde{\mathcal{R}}$ is strictly contained in $\mathcal{R}$: for instance, in the case of the modular in Example 3.13, we have $\widetilde{\mathcal{R}}=L^\infty(0,1)\subsetneq\mathcal{M}=\mathcal{R}$. Of course, if one introduces modulars imposing the further condition that* (3.9) *must hold on the whole vector space, then $\widetilde{\mathcal{R}}=\mathcal{R}$ (see e.g. Biegert* [25, (M4) p.295]*).* □

Theorem 3.1, especially in the case $\alpha=1$, is well known and repetitively quoted and proved in literature, often with some variants in the assumptions and/or with only some of the implications included in our exposition. For instance, it appears in Diening, Harjulehto, Hästö, Růžička [43, Theorem 2.1.7 p.24], where essentially **(i)-(iv)** are proved. In Harjulehto, Hästö [69, Lemma 3.1.3. p.48] the result is proved in the framework of Generalized Orlicz spaces: in this case the interest is in the assumptions on the modular, written in terms of properties of $\Phi-$functions. For the case of variable Lebesgue spaces, see e.g. Cruz-Uribe, F. [38, Theorem 2.17]. In Maligranda [110, Theorem 1.2 p.5] (see also Maligranda [113, Theorem 4 p.125], Musielak [124, Theorem 1.5 p.3], Bardaro, Musielak, Vinti [23, Theorem 1.1(b) p. 4]) the assertions in Theorem 3.1 are analyzed in the case of a weaker assumption of convexity (introduced in Musielak, Orlicz [125]; we will consider it later, see property (P.4.1.4)): in such case **(ii)** is not necessarily true and the two vector subspaces of $\mathcal{R}$ are denoted with different symbols. Equality **(ii)** appears in Mendez, Lang [119, Lemma 1.3.1 p.28], and in the same reference Proposition 1.3.2 contains the proof of **(iv)**, which appears also in Edmunds, Mendez, Lang [48, Proposition 1.3 p.11]. The idea to introduce the parameter α in the statement of Theorem 3.1 is inspired by Miranda [122, (49.7) p.265, proof in p. 266], stated for Orlicz spaces; the same trick has been used also more recently, see e.g. Greco, Iwaniec, Moscariello [68, Lemma 4.2]: the advantage is to get a "clean" Hölder inequality (this is explicitly remarked in Miranda [122, (49.IV) p.270]).

Theorem 3.1 opens the way to define the norm of some of the familiar function spaces. If it is applied to a modular which is already a norm, one gets again the same norm (multiplied by a constant if $\alpha\neq 1$): setting, say, $\rho(f)=\|f\|$ in

$$\|f\|_\alpha = \inf\left\{\lambda>0:\rho\left(\frac{f}{\lambda}\right)\leqslant\alpha\right\},$$

one gets

$$\|f\|_\alpha = \inf\left\{\lambda > 0 : \left\|\frac{f}{\lambda}\right\| \leqslant \alpha\right\} = \inf\left\{\lambda > 0 : \left\|\frac{f}{\alpha}\right\| \leqslant \lambda\right\} = \frac{1}{\alpha}\|f\|,$$

and the same happens substituting ρ with $\|\cdot\|$ in

$$|||f|||_\alpha = \inf_{\lambda>0} \frac{\alpha + \rho(\lambda f)}{\alpha\lambda} :$$

in fact,

$$|||f|||_\alpha = \inf_{\lambda>0} \frac{\alpha + \rho(\lambda f)}{\alpha\lambda} = \inf_{\lambda>0} \frac{\alpha + \|\lambda f\|}{\alpha\lambda} = \inf_{\lambda>0} \frac{1}{\lambda} + \frac{\|f\|}{\alpha} = \frac{1}{\alpha}\|f\|.$$

However, the value of Theorem 3.1 is that it is the key to define function spaces from modulars which are not norms. The most "popular" example is that one known as Musielak-Orlicz space (as stressed by Maligranda in [108, Comment 1], they should be called *variable Orlicz spaces* or *Orlicz-Nakano spaces*, because – as highlighted also in Bardaro, Musielak, Vinti [23, Section 1.5] – they were introduced in Nakano [126]), whose standard norm is the Luxemburg-Nakano norm built from a modular of the type

$$\rho(f) = \int_\Omega \Phi(x, f)d\mu.$$

Musielak-Orlicz spaces never lost their interest among researchers: we mention for instance the recent research Youssfi, Ahmida [148] on approximation results, and applications in Ahmida, Chlebicka, Gwiazda, Youssfi [4], Ahmida, F., Youssfi [5]. The example of Musielak-Orlicz spaces is not only popular, but in some sense is *the* example of modular, because under suitable assumptions, modulars have an integral representations of this type (see Drewnowski, Orlicz [45] for details; see also Kranz, Wnuk [98]).

Note that also other norms are defined using the same machinery, for instance, the Orlicz-Lorentz spaces, which are a common generalization of the Orlicz spaces and the Lorentz spaces (see part 4 of Mastyło [115], Maligranda [109], Kaminska [75, 76, 77], Montgomery-Smith [123], Kamińska, Leśnik, Raynaud [78]).

Moreover, we observe that one could introduce the grand Lebesgue spaces over a (Lebesgue) measurable set $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, $0 < |\Omega| < \infty$ (see F., Formica, Gogatishvili [53] and the more recent papers Farroni, F., Giova [50], Di Fratta, F., Slastikov [42], F., Formica [52]), as the set of the real valued, measurable functions such that

$$\rho(f) = \sup_{0<\varepsilon<p-1} \frac{\varepsilon}{|\Omega|} \int_\Omega |f(x)|^{p-\varepsilon} dx < \infty;$$

the Luxemburg-Nakano norm build from this modular gives back the usual norm (see F., Giannetti [55, Remark 4.3])

$$\|f\|_{L^{p)}(\Omega)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|\Omega|} \int_\Omega |f(x)|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}}.$$

We recall that still in the same paper by Nakano (Nakano [127]), other norms appear ("first norm", "second norm"), whose expressions involve the so-called *modular bounded linear functionals over $\mathcal{R}$.*

A comparison between the (Nakano) modulars on Riesz spaces and the (Nakano) modulars on real vector spaces is in order. In the former notion, property (P.3.2.5) requires the structure of Riesz space, because the absolute value of some $f \in \mathcal{R}$ is defined in the terms of the order, while the latter requires just the general structure of vector space, which is a minimum requirement to state the convexity property (P.3.3.6).

After a look at the properties defining the two notions and the consequences of the first one, it is immediate to realize that:

Proposition 3.2. *If $\mathcal{R}$ is a Riesz space and if ρ satisfies the properties defining the (Nakano) modulars on Riesz spaces, then ρ satisfies also the properties defining the (Nakano) modulars on real vector spaces.*

We close this section quoting the existence of another notion of modular, again by Nakano, again on real vector spaces (besides the already mentioned Nakano [129]): according to Musielak [124, p.164], Nakano in his second book [128, Sect.78 p.204] gave a notion of modular, which is more restrictive with respect to that one analyzed here. In fact, it is assumed that for every $f \in \mathcal{R}$ there exists a $\lambda > 0$ such that $\rho(\lambda f) < \infty$; therefore, for instance, the Example 3.13 would be excluded (because, for instance, $f(x) = 1/x$ would be such that $\rho(\lambda f) = \infty$ for all $\lambda > 0$) and this restriction is not necessary to get Theorem 3.1.

4. Some post Nakano modulars

4.1. **Musielak-Orlicz modulars: a way to weaken convexity.** A functional ρ defined on a real vector space $\mathcal{R}$ is said to be a *(Musielak-Orlicz) modular* (see Musielak, Orlicz [125]) if it satisfies the following properties ($f, g \in \mathcal{R}$):

(P.4.1.1) $0 \leqslant \rho(f) \leqslant \infty$,
(P.4.1.2) $\rho(f) = \rho(-f)$,
(P.4.1.3) $\rho(f) = 0 \quad \Leftrightarrow \quad f = 0$,
(P.4.1.4) $0 \leqslant \alpha \leqslant 1 \;\Rightarrow\; \rho(\alpha f + (1-\alpha)g) \leqslant \rho(f) + \rho(g)$.

It must be noted that the original definition is with (P.4.1.1) replaced by $-\infty \leqslant \rho(f) \leqslant \infty$, but the authors proved (see 1.02(a) therein) that from (P.4.1.2) and (P.4.1.4) one gets that $\rho(f) \geqslant 0$, therefore the notion given in this paper is equivalent to the original one.

Proposition 4.3. *Norms $\|\cdot\|$ on a real vector space $\mathcal{R}$ are (Musielak-Orlicz) modulars.*

Proof. Property (P.4.1.1) follows directly from (2.1). Property (P.4.1.2) follows applying (P.2.1.2) with $\alpha = -1$: $\|f\| = |-1|\|f\| = \|-f\|$. Property (P.4.1.3) coincides with (P.2.1.1). Finally, property (P.4.1.4) follows by (P.2.1.3) and (P.2.1.2): for $0 \leqslant \alpha \leqslant 1$, $f, g \in \mathcal{M}$,

$$\|\alpha f + (1-\alpha)g\| \leqslant \|\alpha f\| + \|(1-\alpha)g\| = \alpha\|f\| + (1-\alpha)\|g\| \leqslant \|f\| + \|g\|.$$

□

Example 4.14. *There exist (Musielak-Orlicz) modulars which are not norms.* If $\mathcal{R}$ is a normed space with norm $\|\cdot\|$, setting $\rho(f) = \|f\|^2$, we have a modular which is not a norm. The fact that it is not a norm has been shown in Example 3.11. On the other hand, ρ is a modular: the proof of properties (P.4.1.1)-(P.4.1.4) is immediate. The reader may check that, in general, the square can be replaced by any increasing, convex function on $[0, +\infty[$ assuming value 0 in the origin. □

The original paper Musielak, Orlicz [125] contains a list of examples of modulars. More examples are e.g. in Maligranda [110] (where the second chapter is entirely dedicated to examples of modulars) and the list in Bardaro, Musielak, Vinti [23, Example 1.5 p.5] (we highlight the interesting (e) in p.7, due to the same authors, mentioned again in the final Section 5).

Next two examples show that the class of Musielak-Orlicz modulars and the class of Nakano modulars (on real vector space) are not comparable with respect to inclusion.

Example 4.15. *There exist Nakano modulars (on real vector space) which are not Musielak-Orlicz modulars.* In Example 3.13, we have seen that there exist Nakano modulars for which $\rho(f) = 0$ does not imply $f = 0$, i.e., such that (P.4.1.3) does not hold. □

Example 4.16. *There exist Musielak-Orlicz modulars which are not Nakano modulars (on real vector space).* Set $\mathcal{R} = \mathbb{R}$, $\rho(x) := \sqrt{|x|}$ for every $f \in \mathbb{R}$. Clearly ρ is a Musielak-Orlicz modular; in particular, (P.4.1.4) holds because for $0 \leqslant \alpha \leqslant 1$, $x, y \in \mathcal{R}$,

$$\rho(\alpha x + (1-\alpha)y) = \sqrt{|\alpha x + (1-\alpha)y|} \leqslant \sqrt{\alpha|x| + (1-\alpha)|y|} \leqslant \sqrt{|x|} + \sqrt{|y|} = \rho(x) + \rho(y).$$

On the other hand, ρ is not a Nakano modular, because the convexity property (P.3.3.6) is lost. □

The previous example shows that Musielak-Orlicz modulars may loose convexity, which has been used in Theorem 3.1 to prove that the Luxemburg-Nakano norm satisfies the properties of the norm. And in fact, we can consider the following:

Example 4.17. *There exist Musielak-Orlicz modulars ρ such that*

$$[f] := \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant 1\right\} \tag{4.14}$$

is not a norm. Let $\mathcal{M}$ be the real vector space of the Lebesgue measurable functions defined in the real interval $(0,1)$, with values in $\mathbb{R}$ (i.e., almost everywhere finite), and let us set

$$\rho(f) = \int_0^1 \sqrt{|f(x)|}dx, \qquad f \in \mathcal{M}.$$

After the chain of inequalities in the previous example, clearly ρ is a modular. However, in this case

$$[f] = \left(\int_0^1 \sqrt{|f(x)|}dx\right)^2$$

which is not a norm, because the triangle inequality (P.2.1.3) fails (see e.g. Castillo, Rafeiro [31, p.51 and Theorem 3.79 p.124], where the authors proved also that $\{[f] < \infty\}$ is not normalizable). More generally, one can consider modulars of the type $\rho(f) = \|f^{1/q}\|_{L^p(0,1)}$, with $q > p$. □

On the other hand, convexity is not necessary, for a modular ρ, to get that (4.14) is a norm: the Musielak-Orlicz modular in Example 4.16 is not convex, nevertheless, in this case (4.14) gives $[f] = |f|$ (see also the example in Maligranda [110, Remark 5 p.8]).

Musielak-Orlicz modulars are the starting point of a rich theory developed in the book by Musielak [124] (see also Maligranda [110]), where the definition has been extended to complex vector spaces, replacing property (P.4.1.2) $\rho(f) = \rho(-f)$ with $\rho(e^{it}f) = \rho(f)$ for all $t \in \mathbb{R}$. They owe its success from a result, proved in the original paper Musielak, Orlicz [125], analogous to Theorem 3.1: from (P.4.1.1)-(P.4.1.4), hence even without the convexity property (P.3.3.6) (replaced by the weaker property (P.4.1.4)), it is possible to build, on the vector subspace of $\mathcal{R}$

$$\widetilde{\mathcal{R}} = \left\{f \in \mathcal{R} : \lim_{\lambda \to 0+} \rho(\lambda f) = 0\right\}, \tag{4.15}$$

a kind of "norm", called *$F-$norm* (originated by Mazur, Orlicz [117, 1.82 p.105] in the framework of Orlicz spaces; in Maligranda [113, p.128] it is called *Mazur-Orlicz $F-$norm*), defined by

$$\|f\|_\rho := \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \lambda\right\}. \tag{4.16}$$

This functional, which is a modified version of the Luxemburg-Nakano, satisfies *almost* all the properties of a norm (see e.g. Maligranda [110, Theorem 1.1 p.2], Bardaro, Musielak, Vinti [23, Theorem 1.1(a) p.4], Rolewicz [135, Theorem 1.2.4 p.8]; a version for quasi-modular spaces is in Koshi, Shimogaki [92]). For this reason, quite frequently, in literature, the notion of modular is given in the Musielak-Orlicz sense, and therefore, in particular, with the weaker version of convexity (P.4.1.4) (see e.g. Maligranda [110, p.1], Rolewicz [135, p.6], Abdou, Khamsi [1, Definition 2.1 p.4047], Mantellini, Vinti [114], etc.). The missing property is the homogeneity property (P.2.1.2): in fact, at first, in Maligranda [110, Example 1 p.4] it is observed that setting $\rho(f) = \|f\|$, where $\|\cdot\|$ is some norm, then $\|f\|_\rho = \|f\|^{1/2}$ (which clearly does not satisfy (P.2.1.2)). Moreover, it must be noted also that in general **(ii)** of Theorem 3.1 does not hold for Musielak-Orlicz modulars: a careful look at the proof tells that, while the inclusion

$$\left\{f \in \mathcal{R} : \lim_{\lambda \to 0+} \rho(\lambda f) = 0\right\} \subset \left\{f \in \mathcal{R} : \text{the set } \left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) < \infty\right\} \text{ is non-empty}\right\} \tag{4.17}$$

is immediate (see also Maligranda [110, Property 2 p.2]), the opposite inclusion has been proved in **(ii)** using convexity, which is now missing. The fact that the inclusion (4.17) can be proper is shown by the following:

Example 4.18. *There exist Musielak-Orlicz modulars ρ such that*

$$\left\{f \in \mathcal{R} : \lim_{\lambda \to 0+} \rho(\lambda f) = 0\right\} \subsetneq \left\{f \in \mathcal{R} : \textit{the set } \left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) < \infty\right\} \textit{ is non-empty}\right\}.$$

Set $\mathcal{R} = \mathbb{R}$ and

$$\rho(x) := \begin{cases} 0 & \text{if} \quad x = 0 \\ 1 + x^2 & \text{if} \quad x \neq 0 \end{cases}, \qquad x \in \mathbb{R}.$$

Note that ρ is not convex and that properties (P.4.1.1)-(P.4.1.4) are satisfied: the first three are immediate; about (P.4.1.4) it suffices to consider the three cases

⋆ $xy = 0$: if, say, $y = 0$, since $\rho(0) = 0$, the property is reduced to $0 \leqslant \alpha \leqslant 1 \ \Rightarrow \ \rho(\alpha x) \leqslant \rho(x)$, which is true because, if $x \neq 0$, the inequality is equivalent to $1 + (\alpha x)^2 \leqslant 1 + x^2$; if $x = 0$, it is reduced to $0 \leqslant 0$;

⋆ $xy > 0$: it suffices to recall that $x \to 1 + x^2$ is convex;

⋆ $xy < 0$: in this case there are two possibilities: if $\alpha x + (1 - \alpha)y \neq 0$, then again it suffices to recall that $x \to 1 + x^2$ is convex; otherwise, the inequality to be proved is reduced, taking into account that $\rho(0) = 0$, to $0 \leqslant \rho(x) + \rho(y)$.

The two sets to be analyzed are different, because clearly

$$\left\{x \in \mathcal{R} : \lim_{\lambda \to 0+} \rho(\lambda x) = 0\right\} = \{0\}$$

and

$$\left\{x \in \mathcal{R} : \text{the set } \left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) < \infty\right\} \text{ is non-empty}\right\} = \mathbb{R}.$$

□

The existence of examples like Example 4.18 motivates the fact that in the literature concerning Musielak-Orlicz modulars the vector space on which norms are considered is that one given in (4.15).

We close this section recalling that Musielak-Orlicz modulars are not only a tool to build norms, but there are contexts where modulars are of interest in their own. For instance, we mention the modular inequalities studied in Cruz-Uribe, Di Fratta, F. [37]: the replacement of norms with modulars, in Harmonic Analysis, has often the effect to restrict the validity of certain inequalities to a smaller set of functions.

4.2. **Q-quasi convex Musielak-Orlicz modulars: the role of Q-quasi convexity.** The notion of Q-quasi convexity in the framework of modular spaces goes back to Bardaro, Musielak, Vinti [22], where the authors considered Musielak-Orlicz modulars ρ on the vector space of the $\mu-$measurable complex-valued functions over a measure space (X, Λ, μ) with $\sigma-$finite measure, with equality $\mu-$a.e. . They imposed on ρ the condition

$$\rho\left(\int_X p(t)h(t)d\mu(t)\right) \leqslant Q \int_X p(t)\rho(Qh(t))d\mu(t)$$

satisfied for some $Q \geqslant 1$, for every $p(\cdot) \in L^1(X)$, $p(\cdot) \geqslant 0$, $\int_X p(t)d\mu(t) = 1$, and every $h(\cdot)$ $\mu-$measurable complex-valued functions over (X, Λ, μ) (note that in Bardaro, Musielak, Vinti [22] the function h is written with a second variable, because the paper concerns double integrals – in fact, in such paper the authors extend the Fubini-Tonelli identity for double integral to the more general context of modulars, obtaining inequalities which are then applied to linear and nonlinear integral operators).

The generalization to the abstract setting is the notion of Q-*quasi convex Musielak-Orlicz modular* (see Bardaro, Mantellini [21]), where Q$\geqslant 1$ is the parameter involved in the property (additional to (P.4.1.1)-(P.4.1.4))

(Q) $0 \leqslant \alpha \leqslant 1 \;\Rightarrow\; \rho(\alpha f + (1-\alpha)g) \leqslant Q\alpha\rho(Qf) + Q(1-\alpha)\rho(Qg)$.

It is clear that 1-quasi convex modulars are convex and that the greater is Q, the weaker is the condition (for a detailed study of properties quasiconvex functions on $[0,\infty[$ see Gogatishvili, Kokilashvili [67, Section 1 p.646]). However, whatever Q$\geqslant 1$ is given, Q-quasi convex modulars are an important selection of Musielak-Orlicz modulars: in fact, while in general **(ii)** of Theorem 3.1 does not hold for Musielak-Orlicz modulars, the Q-quasi convexity ensures that **(ii)** of Theorem 3.1 is still true, namely,

$$\left\{ f \in \mathcal{R} : \lim_{\lambda \to 0+} \rho(\lambda f) = 0 \right\} = \left\{ f \in \mathcal{R} : \text{the set } \left\{ \lambda > 0 : \rho\left(\frac{f}{\lambda}\right) < \infty \right\} \text{ is non-empty} \right\}$$

(see e.g. Bardaro, Mantellini [21] and references therein). Moreover, it must be stressed that 1-quasi convex modulars, i.e., the class of the convex Musielak-Orlicz modulars, is not comparable with the class of the Nakano modulars on real vector spaces (note that modulars in both classes enjoy the standard convexity property): in fact in Example 4.15, (which goes back to Example 3.13), we saw that there exist Nakano modulars which are not Musielak-Orlicz modulars; on the other hand, we can consider the following:

Example 4.19. *There exist 1-quasi (hence Q-quasi, for any given Q$\geqslant 1$) convex modulars, i.e., convex Musielak-Orlicz modulars which are not Nakano modulars on real vector spaces.* Let $\mathcal{M}$ be the real vector space of the Lebesgue measurable functions defined in the real interval $(0,1)$, with values in $\mathbb{R}$ (i.e., almost everywhere finite), and let us set

$$\rho(f) = \begin{cases} \operatorname{ess\,sup}|f| & \text{if} \quad \operatorname{ess\,sup}|f| < 1 \\ \infty & \text{if} \quad \operatorname{ess\,sup}|f| \geqslant 1 \end{cases}.$$

Then, ρ is a convex Musielak-Orlicz modular, but it is not a Nakano modular on real vector spaces, because the property (P.3.3.5) $\sup\limits_{0\leqslant\alpha<1} \rho(\alpha f) = \rho(f)$ is missing (it suffices to consider $f \equiv 1$). □

For applications of Q-quasi convex modulars, see e.g. Bardaro, Musielak, Vinti [23], Bardaro, Mantellini [20] (see also Bardaro, Boccuto, Dimitriou, Mantellini [18] for *Q-quasi semiconvex modulars*).

We mention also that Bardaro and Mantellini introduced also another class of abstract modular spaces, which we will not treat in this exposition: they are generated by modulars defined on the vector space of measurable real functions defined on a locally compact Hausdorff topological space (see Bardaro, Mantellini [19] for details).

4.3. **Luxemburg Banach function spaces: a selection of norms of spaces of functions.** A functional ρ defined on $L^0_+(X)$, the cone of the non-negative elements of $L^0(X)$ (which in turn is the vector space of the $\mu-$measurable real-valued functions over a complete measure space (X,Λ,μ) – *complete* means that $\mu(E) = 0$ implies $F \in \Lambda$ for any set $F \subset E$), $\sigma-$additive and $\sigma-$finite, with identification of the functions which are equal almost everywhere on X) is said to be a *Banach function norm* (see Luxemburg [105]) if it satisfies the following properties ($f, g \in L^0_+(X)$):

(P.4.3.1) $0 \leqslant \rho(f) \leqslant \infty$,

(P.4.3.2) $\rho(f) = 0 \quad \Leftrightarrow \quad f = 0\,$,

(P.4.3.3) $\rho(f+g) \leqslant \rho(f) + \rho(g)\,$,

(P.4.3.4) $\rho(\alpha f) = \alpha\rho(f), \quad \forall \alpha \geq 0 \quad (0\cdot\infty = 0)$,

(P.4.3.5) $f_n \in L^0_+(X)\ (n \in \mathbb{N})\,,\ f_n \uparrow f$ a.e. $\quad\Rightarrow\quad \rho(f_n) \uparrow \rho(f)$, (P.4.3.6) $E \subset X$ "bounded" $\Rightarrow$ $\rho(\chi_E) < \infty$,

(P.4.3.7) $E \subset X$ "bounded" $\Rightarrow$ $\exists\, c_E \geqslant 0$ such that $\int_E f d\mu \leqslant c_E\, \rho(f), \quad \forall f \in L^0_+(X)$.

Here, as usual, χ_E denotes the characteristic function of E. Moreover, E "bounded"set in a measure space (which cannot mean that E is contained in a ball, since we don't assume that X is a metric space and therefore balls are not defined) means that if X is the union of a fixed, once for all, increasing sequence of sets X_n, $(n \in \mathbb{N})$ of finite measure μ, then there exists $m \in \mathbb{N}$ such that $E \subset X_m$.

Now, let $\mathcal{R}$ be the complex vector space of the $\mu-$measurable complex-valued functions on X such that $\rho(|f|) < \infty$ (here, of course, $|\cdot|$ denotes the modulus in $\mathbb{C}$). While ρ is not defined on a vector space, the functional $\rho(|\cdot|)$ is defined on the complex (and therefore also real) vector space $\mathcal{R}$ (for complex vector spaces which can be considered also real vector spaces the reader may consult Brezis [28, Section 11.4 p.361]), and using (P.4.3.1)-(P.4.3.4) it is immediate to realize that it is a norm on $\mathcal{R}$ (this is stated also, for instance, in Bennett, Sharpley [24, Theorem 1.4 p.3]). Properties (P.4.3.5)-(P.4.3.7) impose further conditions on the norm, which allow to build the theory begun with the Luxemburg's thesis [105] and described in several treatises (for instance, Bennett, Sharpley [24]).

We already noticed, in Proposition 3.1, that all norms are Nakano modulars on real vector spaces. Therefore, it is legitimate to include the functional $\rho(|\cdot|)$ among special Nakano modulars. Analogously, by Proposition 4.3, the functional $\rho(|\cdot|)$ is also a special Musielak-Orlicz modular. The theory goes on setting $\|f\| := \rho(|f|)$ for all $f \in \mathcal{R}$ and the resulting normed spaces, widely known as *Banach function spaces,* include several classical Banach spaces of functions, some of them listed in next example.

Example 4.20. *Examples of Banach function spaces.* Several function spaces treated in the books listed after Proposition 3.1 are Banach function spaces: many are classical, such as Lebesgue, Lorentz, Orlicz spaces (which include the Zygmund spaces and the space denoted by EXP, which is the Orlicz space generated by $\Phi(t) = e^t - 1$) and the Musielak-Orlicz spaces, which include the weighted Lebesgue spaces and the variable exponent Lebesgue spaces (see e.g. Cruz-Uribe, F. [38]). We mention here also the Orlicz-Lorentz spaces, already considered after Theorem 3.1. Of still high interest we mention the grand Lebesgue spaces (see the survey F., Formica, Gogatishvili [53]; a detailed proof of the properties of Banach function spaces is in Anatriello [14]) and small Lebesgue spaces (see e.g. F. [51], F., Rakotoson [61], Capone, F. [30], F., Krbec, Schmeisser [60]). Grand and small Lebesgue spaces stimulated the introduction of several variants and generalizations, such as the weighted grand Lebesgue spaces (see e.g. F., Gupta, Jain [56], F., Kokilashvili [58]), the weighted grand variable Lebesgue spaces (see e.g. F., Kokilashvili, Meskhi [59] and references therein), the $G\Gamma$ spaces introduced in F., Rakotoson [62], which are special cases of the $G\Gamma$ spaces with double weights (see F., Formica, Gogatishvili, Kopaliani, Rakotoson [54], Ahmed, F., Formica, Gogatishvili, Rakotoson [3]). We close the (obviously incomplete) list mentioning the (maybe most) important spaces $L^1 + L^\infty$ and $L^1 \cap L^\infty$: all rearrangement-invariant Banach function spaces X over resonant measure spaces satisfy

$$L^1 \cap L^\infty \;\subset\; X \;\subset\; L^1 + L^\infty$$

(see Bennett Sharpley [24, Theorem 6.6] for details; see also Chill, F., Król [32]). □

It must be stressed that Banach function spaces are, in some sense, the "nicest" Banach spaces of functions, because properties (P.4.3.5)-(P.4.3.7) allow to prove in a unified theory several results (concerning, for instance, separability, duality, reflexivity); however, they do not cover all possible Banach spaces of functions (in spite of the standard name "Banach function spaces"), as we are going to see in next

Example 4.21. *Examples of Banach spaces of functions which are not "Banach function spaces".* An important class of Banach spaces of functions is that of the Sobolev spaces. They are not Banach function spaces: in fact, property (P.4.3.5) applied to the sequence $f, g, \ldots, g, \ldots$ entrains that $|f| \leqslant |g| \;\Rightarrow \|f\| \leqslant \|g\|$, but this implication is generally not true (see Example 2.4, when we showed that the lattice norm property does not hold). More generally, other examples are all the proper, closed Banach subspaces of Banach function spaces: for instance, if $\Omega \subset \mathbb{R}^n$, the Banach space exp, defined as the closure in $L^\infty(\Omega)$ in EXP, is not a Banach function space.

We quote also the John-Nirenberg BMO space and other BMO-like spaces like the recent ones introduced in Bourgain, Brezis, Mironescu [27] (see also D'Onofrio, Greco, Perfekt, Sbordone, Schiattarella [44]), which are Banach spaces whose elements are measurable functions modulo constants, and only representatives from each equivalence class belong to L^0. □

4.4. **Kozlowski modular function spaces: modulars for applications of function space theory.** In order to treat problems linked to nonlinear operators (e.g., to find a maximal domain of continuity, or to establish the existence of fixed points, or to find conditions for the extension of functions of several complex variables to holomorphic functions, etc.), in 1988 Kozlowski introduced the modular function spaces, i.e., a class of function spaces defined through modulars (hence definitively something more concrete with respect to the abstract theory built from modulars on fairly general structures) having the properties necessary to develop both a general theory and tools for several applications. Starting from integrals, which can be seen as functionals depending both on functions and sets, now modulars are defined on the pair $(\mathcal{E}, \Sigma)$. Here, $\mathcal{E}$ denotes the vector space of all $\mathcal{P}-$simple functions, i.e., (finite) linear combinations of characteristic functions of pairwise disjoint sets in a non-trivial $\delta-$ring (a ring closed with respect to countable intersections) $\mathcal{P}$ of subsets of a non-empty set X, with values in a Banach space $(S, |\cdot|)$. On the other hand, $\Sigma \supset \mathcal{P}$ denotes the smallest $\sigma-$algebra of subsets of X having the properties:

$$(*)_1 \qquad E \cap A \in \mathcal{P}; \quad \forall E \in \mathcal{P},\ A \in \Sigma,$$

$$(*)_2 \qquad \text{there exists a non-decreasing sequence } X_1 \subset X_2 \subset \cdots,\ X_i \in \mathcal{P};\ X = \bigcup_{i=1}^{\infty} X_i.$$

A functional $\rho : (\mathcal{E}, \Sigma) \to [0, \infty]$ is said to be a *(Kozlowski) function modular* (see Kozlowski [94]) if it satisfies the following properties ($f, g \in \mathcal{E}$, $E, E_1, \dots, E_n, \dots, F \in \Sigma$):

(P.4.4.1) $0 \leqslant \rho(f, E) \leqslant \infty$,
(P.4.4.2) $\rho(0, E) = 0$,
(P.4.4.3) $|f(x)| \leqslant |g(x)|,\ \forall x \in E \quad \Rightarrow \rho(f, E) \leqslant \rho(g, E)$,
(P.4.4.4) $\rho(f, \cdot)$ is a $\sigma-$submeasure:
(i) $\rho(f, \emptyset) = 0$, (ii) $\rho(f, E) \leqslant \rho(f, F)$ if $E \subset F$, (iii) $\rho(f, \cup E_n) \leqslant \sum \rho(f, E_n)$,
(P.4.4.5) $\lim_{\alpha \to 0+} \overline{\rho}_\alpha(E) := \lim_{\alpha \to 0+} \sup\{\rho(g, E) \ :\ g \in \mathcal{E},\ |g(x)| \leqslant \alpha\ \forall x \in E\} = 0$,
(P.4.4.6) $(\ \exists\, \alpha > 0\ :\ \overline{\rho}_\alpha(E) = 0\ ) \quad \Rightarrow \quad (\ \overline{\rho}_\beta(E) = 0,\ \forall\, \beta > 0\ )$,
(P.4.4.7) For every $\alpha > 0$, $\overline{\rho}_\alpha$ is order continuous on $\mathcal{P}$:
for each sequence $(E_n) \subset \Sigma$ such that $E_n \searrow \emptyset$, $\lim_{n \to \infty} \overline{\rho}_\alpha(E_n) = 0$.

Next result, in line with the statements given for the previous modulars, is the essence of Kozlowski [94, Theorem 2.5 p.91]:

Theorem 4.2. *Let $M(X, S)$ be the vector space consisting of all measurable functions $f : X \to S$, i.e., of the functions f for which there exists a sequence of $\mathcal{P}-$simple functions (f_n) such that $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in X$, with identification of functions which differ only on $\rho-$null sets (i.e., sets on which $\overline{\rho}_\alpha$ is zero for all $\alpha > 0$).*

The functional

$$f \in M(X, S) \quad \to \quad \rho(f, X) := \sup\{\rho(g, X) : g \in \mathcal{E}, |g| \leqslant |f| \text{ in } E\} \in [0, \infty] \tag{4.18}$$

is a Musielak-Orlicz modular.

It must be noticed that by Musielak-Orlicz modular it must be intended the aforementioned extension, made in the Musielak's book [124], of the original Musielak-Orlicz modulars to complex vector spaces.

Theorem 4.2 inserts Kozlowski modulars into the Musielak-Orlicz theory, so that now it is automatically defined the *modular function space* as $\widetilde{\mathcal{R}}$ defined in (4.15) (setting $\mathcal{R} = M(X, S)$ therein), endowed with the F-norm (4.16). According to one of the main features of the Musielak-Orlicz theory, in general the theory is not affected by convexity, even if in some occasions this assumption allows to rephrase/improve the results. As a consequence, from a general perspective, without the explicit addition of assumptions, the modular (4.18) is not a Nakano modular.

After the definition of (Kozlowski) function modular, a natural question arises, namely, how to build such kind of modulars. Roughly speaking, $\rho(f, E)$ mimics the integral of $|f|$ over E, but in general, if ρ is a Nakano modular on $M(X, S)$, the functional

$$\widehat{\rho}\ : (f, E) \ni (\mathcal{E}, \Sigma) \ \to \ \widehat{\rho}(f, E) := \rho(f\chi_E) \in [0, \infty] \tag{4.19}$$

is not necessarily a Kozlowski modular: it suffices to consider, for instance, Sobolev spaces norms (see Example (2.4)): since they are norms, they fit into the category of Nakano modulars on real vector spaces (see Proposition 3.1), however, as shown in Example 2.4, the corresponding $\widehat{\rho}$ does not satisfy property (P.4.4.3). In Kozlowski [96] the author uses the trick (4.19) to build modulars, starting from the following notion.

A nontrivial functional ρ defined on $\mathcal{M}_\infty$, the space of all extended measurable functions, i.e., all functions $f : X \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leqslant |f|$ and $g_n(x) \to f(x)$ for all $x \in X$, is said to be a *regular convex function modular* if it satisfies the following properties ($f, g \in \mathcal{M}_\infty$):

(P.4.4.**i**) $0 \leqslant \rho(f) \leqslant \infty$,
(P.4.4.**ii**) $\rho(f) = \rho(-f)$,
(P.4.4.**iii**) $\rho(f) = 0 \quad \Leftrightarrow \quad f = 0$,
(P.4.4.**iv**) ρ is convex: $0 \leqslant \alpha \leqslant 1 \ \Rightarrow \ \rho(\alpha f + (1-\alpha)g) \leqslant \alpha\rho(f) + (1-\alpha)\rho(g) \qquad (0 \cdot \infty = 0)$,
(P.4.4.**v**) ρ is monotone: $|f(x)| \leq |g(x)|,\ \forall x \in X \quad \Rightarrow \quad \rho(f) \leq \rho(g)$,
(P.4.4.**vi**) ρ is orthogonally subadditive: $A, B \in \Sigma,\ A \cap B \neq \emptyset \Rightarrow \rho(f\chi_{A\cup B}) \leq \rho(f\chi_A) + \rho(f\chi_B)$,
(P.4.4.**vii**) ρ has the Fatou property: $f_n \in \mathcal{M}_\infty\ (n \in \mathbb{N}),\ f_n(x) \uparrow f(x),\ \forall x \in X \Rightarrow \rho(f_n) \uparrow \rho(f)$,
(P.4.4.**viii**) ρ is order continuous in $\mathcal{E}$: $f_n \in \mathcal{M}_\infty\ (n \in \mathbb{N}),\ |f_n(x)| \downarrow 0,\ \forall x \in X \ \Rightarrow \ \rho(f_n) \downarrow 0$.

The following result (see Kozlowski [96, p.479, after Definition 2.2]) is a method to build modulars:

Proposition 4.4. *Let $\mathcal{M}$ be the vector space consisting of all functions $f \in \mathcal{M}_\infty$ which are $\rho-$a.e. finite, i.e., finite up to a $\rho-$null set ($A \in \Sigma$ is $\rho-$null if $\rho(f\chi_A) = 0$ for every $f \in \mathcal{E}$), with identification of functions which differ only on $\rho-$null sets. If ρ is a regular convex function modular, then the functional in* (4.19)*:*

$$\widehat{\rho}\ : (f, E) \ni (\mathcal{M}, \Sigma) \ \to \ \widehat{\rho}(f, E) := \rho(f\chi_E) \in [0, \infty]$$

is a Kozlowski modular.

Regular convex function modulars are particular Nakano modulars on real vector spaces, hence Kozlowski modulars built from Proposition 4.4 are an important category of modulars, which benefit either the theory shown in Kozlowski [95], either, for instance, Theorem 3.1. It is worth to mention, here, that regular convex function modulars generate in a natural way a quite general structure called *modulated topological vector space*, introduced very recently in Kozlowski [97].

Example 4.22. *Examples of Modular function spaces.* Musielak-Orlicz spaces (see Kozlowski [95, Sect. 4.1p.86]), defined through the modular

$$\rho(f, E) = \int_E \phi(x, |f(x)|) d\mu,$$

where μ is a $\sigma-$additive measure on (X, Σ) and $\phi = \phi(x, u)$ is measurable and locally integrable in $x \in X$, continuous in $u \geqslant 0$, and such that $\phi(x, 0) = 0$ for every $x \in X$, $\phi(x, \infty) = \infty$, and, finally, $\phi(\cdot, u) > 0$ $\rho-$a.e. for every $u > 0$ are Modular function spaces. Moreover, one can consider generalizations of Musielak-Orlicz spaces (see Kozlowski [95, Sect. 4.2.1 p.92]) defined through

$$\rho(f, E) = \sup_{\mu} \int_E \phi(x, |f(x)|) d\mu,$$

where μ varies in a family of $\sigma-$additive measures on (X, Σ); the theory includes Lorentz type L^p-spaces (see Kozlowski [95, Sect. 4.2.2 p.93]) defined through

$$\rho(f, E) = \sup_{z} \int_E |f(x)|^p z(x) d\mu,$$

where μ is a fixed measure and z varies in a family of non-negative $\mu-$measurable functions; moreover, the theory includes also countably modulared spaces (see Kozlowski [95, Sect. 4.2.3 p.93]), whose modulars are defined, for instance, as suprema of modulars, and, for instance, also a class of Fenchel-Orlicz spaces (see Kozlowski [95, Sect. 4.2.4 p.94]), which are Orlicz-like spaces constituted by functions Banach-space valued.

Modular function spaces have an extensive applications to Fixed Point Theory: see the pioneering paper Khamsi, Kozlowski, Reich [82] and e.g. Khamsi, Kozlowski [81], Al-Mezel, Al-Solamy, Ansari [9], Alfuraidan, Khamsi, Manav [8], Alfuraidan, Bachar, Khamsi [7] and references therein.

4.5. **Chistyakov modular metric spaces: structures born from modulars on arbitrary sets.** In the opposite direction with respect to the previous class of modulars, there exist questions where the notion of modular over vector spaces (or, in fact, *any* algebraic structure) is restrictive and therefore it may be of help an abstract notion of modular acting on arbitrary sets, which leads to an extension of the theories built by Nakano and Musielak-Orlicz.

Let X be a non-empty set. A functional

$$w : (0, \infty) \times X \times X \to [0, \infty]$$

is said to be a *metric modular on* X (see Chistyakov [33, 34, 35]) if it satisfies the following properties:

(P.4.5.1) Given $x, y \in X$,

$$w(\lambda, x, y) = 0 \text{ for all } \lambda > 0 \Leftrightarrow x = y,$$

(P.4.5.2) $w(\lambda, x, y) = w(\lambda, y, x)$ for all $\lambda > 0, x, y \in X$,

(P.4.5.3) $w(\lambda + \mu, x, y) \leqslant w(\lambda, x, z) + w(\mu, y, z)$ for all $\lambda, \mu > 0, x, y, z \in X$.

Now fix an element $x_0 \in X$ arbitrarily. The subsets of X of the type

$$X_w := \{y \in X : \lim_{\lambda \to \infty} w(\lambda, x, y) = 0\}$$

are said to be *modular sets*. Endowed with the metric given by

$$d_w^0(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \leqslant \lambda\},$$

X_w becomes a metric space. Moreover, setting

$$d_w^1(x, y) = \inf_{\lambda > 0} (\lambda + w(\lambda, x, y)),$$

also d_w^1 is a metric and $d_w^0 \leqslant d_w^1 \leqslant 2d_w^0$ on $X_w \times X_w$. The analogy with Theorem 3.1 is evident; for examples, variants and applications see Chistyakov [33, 34], Ansari, Demma, Guran, Lee, Park [16], Aksoy, Karapinar, Erhan, Rakočević [6]. For a survey on Generalized metric spaces, see Khamsi [80].

We close this subsection mentioning the paper Turkoglu, Manav [145], where a new type of modular metric space has been introduced.

4.6. **A recent class of Banach-function-norm-like modulars.** A classical result in Sobolev space theory states that in general, if $\Omega \subset \mathbb{R}^n$ is an open set and $N \subset \Omega$ is a closed set of zero Lebesgue measure,

$$W_0^{1,p}(\Omega) \neq W_0^{1,p}(\Omega \setminus N).$$

At a first look this is surprising, because functions in Sobolev spaces are defined a.e. and the class of measurable functions on Ω coincides with the class of measurable functions on $\Omega \setminus N$. But recalling that functions in Sobolev spaces with zero boundary values are defined as approximations of regular functions which attain value zero on the boundary, then it is clear that even a very *small* set N – even a single point – forces a decay of regular functions on N and the approximation recognizes, or does not recognize, such decay depending on the topology. The interplay between the smallness of N and the topology has its heart in the notion of capacity, and in fact the precise result is the following (see e.g. Heinonen, Kilpeläinen, Martio [72, Theorem 2.43 p.51], Kilpeläinen, Kinnunen, Martio [83, Theorems 4.6, 4.8, Remark 4.2(4)]):

$$W_0^{1,p}(\Omega) = W_0^{1,p}(\Omega \setminus N)$$

(i.e., the closure in $W^{1,p}(\Omega)$ of the space of C^1 functions in Ω coincides with the closure of the space of C^1 functions having compact support in $\Omega \setminus N$) if and only if $cap_p(N) = 0$, where

$$cap_p(N) = \inf\left\{ \int_{\mathbb{R}^n} |u|^p dx + \int_{\mathbb{R}^n} |\nabla u|^p dx : u \in W^{1,p}(\mathbb{R}^n),\ u = 1 \text{ in an open set containing } N \right\}.$$

This statement can be generalized in several different ways: since Lebesgue spaces are particular Orlicz, Lorentz or variable exponent Lebesgue spaces, similarly one can consider the corresponding Sobolev spaces with zero boundary values and prove analogous *removability* results for sets with zero capacity. The whole question has been investigated in a quite general framework in F., Giannetti [55], where the Lebesgue norm (which appears either in the norm in Sobolev spaces and in the definition of capacity) has been replaced by a functional more general than a norm of a Luxemburg Banach function space, namely, a modular. Unfortunately the minimal requirements to impose to modulars, requirements needed for the extension of such classical result, do not match in any notion of modular considered before: all of them have some extra and/or missing property. For instance, the regular convex function modulars considered in Section 4.4 are convex, while for the removability result it is needed just the orthogonal subadditivity. The notion of modular introduced in F., Giannetti [55, Section 2] – which looks very close to that one of Banach function norm, but which allows, for instance, suitable powers of norms (see F., Giannetti [55, Example 2.7]) – is the following: let $\Omega \subset \mathbb{R}^n$ be an open set and let $\mathcal{M}(\Omega)$ be the set of all measurable, real valued functions with respect to the Lebesgue measure, defined on Ω. Given a mapping $\rho_X(\cdot) : \mathcal{M}(\Omega) \to [0, \infty]$, the set

$$X(\Omega) = \{u \in \mathcal{M}(\Omega) : \rho_X(u) < \infty\}$$

is a modular function space over Ω if the pair $(X(\Omega), \rho_X)$ satisfies the following properties for all $u,\ v \in \mathcal{M}(\Omega)$:

i $\rho_X(u) = \rho_X(|u|)$ and $\rho_X(u) = 0$ if and only if $u \equiv 0$,
ii $|u| \leqslant |v|$ a.e. $\Rightarrow \rho_X(u) \leqslant \rho_X(v)$,
iii $\rho_X(u + v) \leqslant \rho_X(u) + \rho_X(v) \quad \forall u, v : uv \equiv 0$,
iv if $E \subset \Omega$ is measurable set and $|E| < \infty$, then $\rho_X(\chi_E) < \infty$,
v $|u_j| \uparrow |u|$ a.e. $\Rightarrow \ \rho_X(u_j) \uparrow \rho_X(u)$,
vi $\forall k > 1,\ \exists\, c_k > 1 : \rho_X(ku) \leqslant c_k \rho_X(u)$.

A link with the Banach function spaces in the sense of Bennett, Sharpley is the following: adding the assumption of the convexity of ρ_X and the imbedding in L^1, one gets a full set of axioms which, analogously to Theorem 3.1, allows to build a Banach function space (see F., Giannetti [55, Proposition 4.1] for details).

We close this subsection recalling that property ***vi*** is well known in Orlicz spaces theory, and that in the framework of modular spaces it has been considered also in Krbec [99], where an interpolation method in modular spaces has been built, generalizing the well known K-method (see e.g. Bennett, Sharpley [24], Maligranda [110], Triebel [142]). We note that modulars in Krbec [99] must be convex and do not satisfy necessarily the Fatou property ***v***.

5. Hints for further research

1. For every set of properties defining a modular, prove independence. Namely, for each property, find an example of functional which is *not* a modular and which satisfies *all* the other properties.

2. For each couple of distinct modulars, are there nontrivial additional properties to impose to a modular, so that one notion fits into the other? Results of this type seem missing, even for the popular Musielak-Orlicz notion of modular. Answers could be different in the case of finite/infinite dimensional vector spaces.

3. Recently, in F., Jain [57], it has been shown that if $\rho(f) = \|f\|_{L^1(0,\ell)}$ and $\psi : [0,\ell] \to [0,\infty[$ is absolutely continuous, nondecreasing, and such that $\psi(\ell) > \psi(0)$, $\psi(t) > 0$ for $t > 0$, then

$$\rho\left(\frac{\psi'(\cdot)}{\psi(\cdot)^2}\int_0^{\cdot} f^*(s)\psi(s)ds\right) \approx \rho(f),$$

where by f^* we denote the decreasing rearrangement of f. It would be interesting to extend the validity of these two inequalities to some class of modulars.

4. In Example 4.17, we showed Musielak-Orlicz modulars for which the Luxemburg-Nakano norm is not a norm; however, after Example 4.17 we observed that convexity is not necessary for having that the Luxemburg-Nakano norm is a norm. Find a necessary and sufficient condition for a Musielak-Orlicz modular so that the Luxemburg-Nakano norm is a norm.

5. Different modulars may generate the same Luxemburg-Nakano norm: are there criteria to characterize the whole class of modulars, for a given norm?

6. Is the sum of modulars a modular? Are suprema of modulars, modulars? Is the multiple of a modular, a modular? Is a functional equivalent to a modular, a modular? These questions could be posed by each category of modulars, and in case of negative answers it would be interesting to know conditions on modulars so that the answers become positive. In particular, such kind of questions can be posed for the *grand modulars* introduced in Farroni, F., Giova [50, 3.11 p.762].

7. In literature, several particular Musielak-Orlicz modulars are well known (e.g., those ones generating Orlicz spaces, variable Lebesgue spaces, weighted Lebesgue spaces, double-phase functionals, etc.), frequently applied in several contexts (say, Harmonic Analysis, PDEs, etc.). A maybe less explored functional, in applications, is a functional of the following type, which combines Orlicz and variable exponents:

$$f \to \int_\Omega \Phi(|f|^{p(x)})dx.$$

This functional is suggested by a look at Bardaro, Musielak, Vinti [23, Example 1.5(e) p.7] and it appears e.g. in an estimate for the local maximal operator (see Capone, Cruz-Uribe, F. [29]).

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Research Article

Unrestricted Cesàro summability of d-dimensional Fourier series and Lebesgue points

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ABSTRACT. We generalize the classical Lebesgue's theorem to multi-dimensional functions. We prove that the Cesàro means of the Fourier series of the multi-dimensional function $f \in L_1(\log L)^{d-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d)$ $(1 < p \leq \infty)$ converge to f at each strong Lebesgue point.

Keywords: Cesàro summability, strong Hardy-Littlewood maximal function, strong Lebesgue points.

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Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

In 1904, Fejér [3] investigated the arithmetic means of the partial sums of the trigonometric Fourier series of a one-dimensional function f, the so called Fejér means and proved that if the left and right limits $f(x-0)$ and $f(x+0)$ exist at a point x, then the Fejér means

$$\sigma_n f(x) := \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) \widehat{f}(k) e^{\imath kx}$$

converge to $(f(x-0)+f(x+0))/2$. Here, $\widehat{f}(k)$ denotes the k-th Fourier coefficient. One year later, Lebesgue [11] extended this theorem and obtained that every one-dimensional integrable function is Fejér summable at each Lebesgue point, thus almost everywhere. Some years later, M. Riesz [15] generalized this theorem for the Cesàro means of one-dimensional integrable functions (the definition can be found later).

The Cesàro summability is investigated in a great number of papers (see e.g. Gát [4, 5, 6], Goginava [7, 8, 9], Simon [17, 18], Nagy, Persson, Tephnadze and Wall [13, 14] and Weisz [19, 20]). In this short note, we generalize the result of Lebesgue and Riesz to this summability of multi-dimensional functions. We generalize the Lebesgue points and introduce the so called strong Lebesgue points. It is known that almost every point is a strong Lebesgue point of $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$. We introduce the strong Hardy-Littlewood maximal function $M_s f$ and show that the Cesàro means of $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$ can be estimated by $M_s f$ pointwise. Our main result is the following. If $M_s f(x)$ is finite and x is a strong Lebesgue point of $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$, then

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x),$$

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where $\sigma_n^\alpha f$ denotes the n-th Cesàro means of the Fourier series of f. This implies the convergence of the Cesàro means almost everywhere as well as covers the one-dimensional results mentioned above. Note that $L_1(\log L)^{d-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d)$ with $1 < p \leq \infty$. The results are not true for $L_1(\mathbb{T}^d)$ if $d > 1$. Similar theorems are known for the θ-means generated by a single function θ (see Feichtinger and Weisz [2] and the references therein). However, those results and proofs do not contain the results for Cesàro means. For the multi-dimensional Cesàro means, we need new ideas.

2. Strong maximal function and strong Lebesgue points

Let us fix $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$, let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \ldots \times \mathbb{Y}$ taken with itself d times. We briefly write $L_p(\mathbb{T}^d)$ instead of the $L_p(\mathbb{T}^d, \lambda)$ space equipped with the norm

$$\|f\|_p := \left(\int_{\mathbb{T}^d} |f|^p \, d\lambda \right)^{1/p} \qquad (1 \leq p < \infty),$$

with the usual modification for $p = \infty$, where λ is the Lebesgue measure. We identify the torus $\mathbb{T}$ with $[-\pi, \pi]$. Set $\log^+ u := \max(0, \log u)$. For $k \in \mathbb{N}$ and $1 \leq p < \infty$, a measurable function f is in the set $L_p(\log L)^k(\mathbb{T}^d)$ if

$$\|f\|_{L_p(\log L)^k} := \left(\int_{\mathbb{T}^d} |f|^p (\log^+ |f|)^k \, d\lambda \right)^{1/p} < \infty.$$

For $k = 0$, we get back the $L_p(\mathbb{T}^d)$ spaces. We have for all $k \in \mathbb{P}$ and $1 \leq p < r \leq \infty$ that

$$L_p(\mathbb{T}^d) \supset L_p(\log L)^{k-1}(\mathbb{T}^d) \supset L_p(\log L)^k(\mathbb{T}^d) \supset L_r(\mathbb{T}^d).$$

For $f \in L_1(\mathbb{T}^d)$, the strong Hardy-Littlewood maximal function is defined by

$$M_s f(x) := \sup_{h \in \mathbb{R}_+^d} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t)| \, dt.$$

For $d > 1$, it is known that there is a function $f \in L_1(\mathbb{T}^d)$ such that $M_s f = \infty$ almost everywhere (see Jessen, Marcinkiewicz and Zygmund [10] and Saks [16]). Thus, in contrary to the one-dimensional case, M_s cannot be of weak type $(1,1)$ if $d > 1$. However, we know the following weak type inequality. If $f \in L(\log L)^{d-1}(\mathbb{T}^d)$, then

$$\sup_{\rho > 0} \rho \lambda(M_s f > \rho) \leq C + C \left\| |f| \left(\log^+ |f| \right)^{d-1} \right\|_1. \tag{2.1}$$

Moreover, for $1 < p \leq \infty$, we have

$$\|M_s f\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)). \tag{2.2}$$

In this paper, the constants C and C_p may vary from line to line. If $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$, then

$$\lim_{h \to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} f(x-t) \, dt = f(x)$$

for almost every $x \in \mathbb{T}^d$. Here $h \to 0$ means that $h_j \to 0$ for all $j = 1, \ldots, d$. Note that this result does not hold for all $f \in L_1(\mathbb{T}^d)$ if $d > 1$ (see Jessen, Marcinkiewicz and Zygmund [10] and Saks [16]).

Motivated by this convergence result, a point $x \in \mathbb{T}^d$ is called a strong Lebesgue point of $f \in L_p(\mathbb{T}^d)$ if

$$\lim_{h \to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t) - f(x)| \, dt = 0.$$

Theorem 2.1. *Almost every point $x \in \mathbb{T}^d$ is a strong Lebesgue point of $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$.*

This is not true for $f \in L_1(\mathbb{T}^d)$ if $d > 1$. Note that $L_1(\log L)^{d-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d)$ for all $1 < p \leq \infty$. For the results of this section, see Chang and Fefferman [1], Zygmund [21] or Weisz [19, 20].

3. Rectangular Cesàro summability

For $\alpha \neq -1, -2, \ldots$ and $n \in \mathbb{N}$, let

$$A_n^\alpha := \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}.$$

Then $A_0^\alpha = 1$, $A_n^0 = 1$ and $A_n^1 = n+1$ ($n \in \mathbb{N}$). The k-th Fourier coefficient of a d-dimensional integrable function $f \in L_1(\mathbb{T}^d)$ is defined by

$$\widehat{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-\imath k \cdot x}\, dx \qquad (k \in \mathbb{Z}^d),$$

where $u \cdot x := \sum_{k=1}^d u_k x_k$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$. Since the Fourier series of f has bad convergence properties (see e.g. Weisz [20]), we consider the Cesàro summability.

Let $f \in L_1(\mathbb{T}^d)$, $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}_+^d$. The n-th rectangular Cesàro means $\sigma_n^\alpha f$ of the Fourier series of f and the Cesàro kernel K_n^α are introduced by

$$\sigma_n^\alpha f(x) := \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{|k_1| \leq n_1} \cdots \sum_{|k_d| \leq n_d} \prod_{i=1}^d A_{n_i-1-|k_i|}^\alpha \widehat{f}(k) e^{\imath k \cdot x}$$

and

$$K_n^\alpha(t) := \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{|k_1| \leq n_1} \cdots \sum_{|k_d| \leq n_d} \prod_{i=1}^d A_{n_i-1-|k_i|}^\alpha e^{\imath k \cdot t},$$

respectively. It is easy to see that

$$\sigma_n^\alpha f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n^\alpha(t)\, dt$$

and

$$K_n^\alpha = K_{n_1}^{\alpha_1} \otimes \cdots \otimes K_{n_d}^{\alpha_d},$$

where the functions $K_{n_i}^{\alpha_i}$ are the one-dimensional Cesàro kernels. The Cesàro means are also called (C,α)-means. If all $\alpha_i = 1$, then we get back the rectangular Fejér means. For the one-dimensional Cesàro kernels, it is known (see Zygmund [21]) that

$$K_n^\alpha(t) \leq C \min\left(n, \frac{1}{n^\alpha |t|^{\alpha+1}}\right) \tag{3.3}$$

and

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{T}} |K_n^\alpha|\, d\lambda \leq C,$$

where $n \in \mathbb{N}$, $0 < \alpha \leq 1$ and $t \in (-\pi, \pi)$.

4. Unrestricted convergence at Lebesgue points

Before proving the main results of this paper, we introduce the Herz space $E_\infty(\mathbb{R}^d)$ with the norm

$$\|f\|_{E_\infty} := \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} 2^{k_1+\ldots+k_d} \|f 1_{P_k}\|_\infty < \infty,$$

where

$$P_k := P_{k_1} \times \cdots \times P_{k_d} \qquad (k \in \mathbb{Z}^d)$$

and

$$P_i = \{x \in \mathbb{R} : 2^{i-1}\pi \le |x| < 2^i \pi\} \qquad (i \in \mathbb{Z}).$$

Obviously, $L_1(\mathbb{R}^d) \supset E_\infty(\mathbb{R}^d)$. First, we will estimate pointwise the maximal operator

$$\sigma_*^\alpha f := \sup_{n \in \mathbb{N}^d} |\sigma_n^\alpha f|$$

by the strong Hardy-Littlewood maximal function. To this end, we introduce the functions

$$h^{\alpha_j}(t) := \min\left\{1, |t|^{-\alpha_j - 1}\right\} \qquad (t \in \mathbb{R})$$

and

$$h^\alpha := h^{\alpha_1} \otimes \cdots \otimes h^{\alpha_d}.$$

We get from (3.3) that

$$\frac{1}{n_j} \left| \left(1_{(-\pi,\pi)} K_{n_j}^{\alpha_j}\right) \left(\frac{t}{n_j}\right) \right| \le \frac{C}{n_j} \min\left\{n_j, \frac{n_j}{|t|^{\alpha_j+1}}\right\} = C h^{\alpha_j}(t) \qquad (t \in \mathbb{R}). \tag{4.4}$$

It is easy to see that

$$\|h^\alpha\|_{E_\infty(\mathbb{R}^d)} = \prod_{j=1}^{d} \|h^{\alpha_j}\|_{E_\infty(\mathbb{R})} \le C_\alpha. \tag{4.5}$$

Theorem 4.2. *Suppose that $0 < \alpha_j \le 1$ for all $j = 1, \ldots, d$. If $f \in L_1(\mathbb{T}^d)$ and $x \in \mathbb{T}^d$, then*

$$\sigma_*^\alpha f(x) \le C M_s f(x).$$

Proof. Observe that

$$\begin{aligned} |\sigma_n^\alpha f(x)| &= \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} f(x-t) \left(1_{(-\pi,\pi)^d} K_n^\alpha\right)(t)\, dt \right| \\ &= \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t)| \left| \left(1_{(-\pi,\pi)^d} K_n^\alpha\right)(t) \right| dt, \end{aligned}$$

where

$$P_{k_j}(n_j) := \{x \in \mathbb{R} : 2^{k_j-1}\pi/n_j \le |x| < 2^{k_j}\pi/n_j\} \qquad (j = 1, \ldots, d).$$

Then,

$$\begin{aligned}|\sigma_n^\alpha f(x)| &\leq \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t)|\, dt \\ &\quad \times \sup_{t \in P_{k_1}(n_1)\times\cdots\times P_{k_d}(n_d)} \left|\left(1_{(-\pi,\pi)^d} K_n^\alpha\right)(t)\right| \\ &= \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t)|\, dt \\ &\quad \times \sup_{t \in P_{k_1}\times\cdots\times P_{k_d}} \left|\left(1_{(-\pi,\pi)^d} K_n^\alpha\right)\left(\frac{t_1}{n_1},\ldots,\frac{t_d}{n_d}\right)\right|. \end{aligned} \tag{4.6}$$

Consequently, by (4.4),

$$\begin{aligned}|\sigma_n^\alpha f(x)| &\leq \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} 2^{k_1+\ldots+k_d} M_s f(x) \sup_{t\in P_k} |h^\alpha(t)| \\ &= C \left\|h^\alpha\right\|_{E_\infty(\mathbb{R}^d)} M_s f(x).\end{aligned}$$

Inequality (4.5) finishes the proof. □

Inequalities (2.1) and (2.2) imply:

Corollary 4.1. *Suppose that $0 < \alpha_j \leq 1$ for all $j = 1, \ldots, d$. If $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$, then*

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\alpha f > \rho) \leq C + C \left\|f\right\|_{L_1(\log L)^{d-1}}.$$

If $1 < p \leq \infty$ and $f \in L_p(\mathbb{T}^d)$, then

$$\left\|\sigma_*^\alpha f\right\|_p \leq C_p \|f\|_p.$$

The usual density argument due to Marcinkiewicz and Zygmund [12] implies:

Corollary 4.2. *Suppose that $0 < \alpha_j \leq 1$ for all $j = 1, \ldots, d$. If $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \quad \textit{a.e.}\ .$$

In this paper, $n \to \infty$ means that $n_j \to \infty$ for all $j = 1, \ldots, d$. Now, we prove that the convergence in Corollary 4.2 holds at each strong Lebesgue point, whenever the corresponding strong Hardy-Littlewood maximal function is finite.

Theorem 4.3. *Suppose that $0 < \alpha_j \leq 1$ for all $j = 1, \ldots, d$. If $M_s f(x)$ is finite and x is a strong Lebesgue point of $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x).$$

Proof. Let

$$G(u) := \int_{-u_1}^{u_1} \cdots \int_{-u_d}^{u_d} |f(x-t) - f(x)|\, dt \qquad (u \in \mathbb{R}_+^d).$$

Since x is a strong Lebesgue point of f, for all $\epsilon > 0$, we can find an integer $m \leq 0$ such that

$$\frac{G(u)}{\prod_{j=1}^d (2u_j)} \leq \epsilon \quad \text{if} \quad 0 < u_j \leq 2^m\pi, j = 1, \ldots, d. \tag{4.7}$$

Since

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n^\alpha(t)\, dt = 1,$$

we have

$$|\sigma_n^\alpha f(x) - f(x)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n^\alpha\right)(t)\right| \, dt := A_1(x) + A_2(x),$$

where

$$\begin{aligned} A_1(x) := & \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{m+\lfloor \log_2 n_1 \rfloor} \cdots \sum_{k_d=-\infty}^{m+\lfloor \log_2 n_d \rfloor} \\ & \times \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n^\alpha\right)(t)\right| \, dt \end{aligned}$$

and

$$\begin{aligned} A_2(x) := & \frac{1}{(2\pi)^d} \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=m+\lfloor \log_2 n_{\pi_1} \rfloor+1}^{\infty} \cdots \sum_{k_{\pi_j}=m+\lfloor \log_2 n_{\pi_j} \rfloor+1}^{\infty} \sum_{k_{\pi_{j+1}}=-\infty}^{\infty} \cdots \sum_{k_{\pi_d}=-\infty}^{\infty} \\ & \times \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n^\alpha\right)(t)\right| \, dt. \end{aligned}$$

Here $\{\pi_1, \ldots, \pi_d\}$ is a permutation of $\{1, \ldots, d\}$ and $1 \leq j \leq d$. As in (4.6),

$$\begin{aligned} A_1(x) \leq & C \sum_{k_1=-\infty}^{m+\lfloor \log_2 n_1 \rfloor} \cdots \sum_{k_d=-\infty}^{m+\lfloor \log_2 n_d \rfloor} \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t) - f(x)| \, dt \\ & \times \sup_{t \in P_{k_1} \times \cdots \times P_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n^\alpha\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right) \right| \\ \leq & C \sum_{k_1=-\infty}^{m+\lfloor \log_2 n_1 \rfloor} \cdots \sum_{k_d=-\infty}^{m+\lfloor \log_2 n_d \rfloor} G\left(\frac{2^{k_1}\pi}{n_1}, \ldots, \frac{2^{k_d}\pi}{n_d}\right) \left(\prod_{j=1}^d n_j\right) \sup_{t \in P_k} |h^\alpha(t)|. \end{aligned}$$

Inequalities (4.7), (4.5) and $2^{k_j}/n_j \leq 2^m$ imply

$$A_1(x) \leq C\epsilon \sum_{k_1=-\infty}^{m+\lfloor \log_2 n_1 \rfloor} \cdots \sum_{k_d=-\infty}^{m+\lfloor \log_2 n_d \rfloor} 2^{k_1+\ldots+k_d} \sup_{t \in P_k} |h^\alpha(t)| \leq C\epsilon \, \|h^\alpha\|_{E_\infty(\mathbb{R}^d)} \leq C_\alpha \epsilon.$$

Similarly,

$$\begin{aligned} A_2(x) \leq & C \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=m+\lfloor \log_2 n_{\pi_1} \rfloor+1}^{\infty} \cdots \sum_{k_{\pi_j}=m+\lfloor \log_2 n_{\pi_j} \rfloor+1}^{\infty} \sum_{k_{\pi_{j+1}}=-\infty}^{\infty} \cdots \sum_{k_{\pi_d}=-\infty}^{\infty} \\ & \times \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t) - f(x)| \, dt \left(\prod_{j=1}^d n_j\right) \sup_{t \in P_k} |h^\alpha(t)| \\ \leq & C_p \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=m+\lfloor \log_2 n_{\pi_1} \rfloor+1}^{\infty} \cdots \sum_{k_{\pi_j}=m+\lfloor \log_2 n_{\pi_j} \rfloor+1}^{\infty} \sum_{k_{\pi_{j+1}}=-\infty}^{\infty} \cdots \sum_{k_{\pi_d}=-\infty}^{\infty} \\ & \times 2^{k_1+\ldots+k_d} \sup_{t \in P_k} |h^\alpha(t)| \Big(M_s f(x) + |f(x)| \Big). \end{aligned}$$

Since $M_s f(x)$ and $f(x)$ are finite, the fact $\lfloor \log_2 n_{\pi_j} \rfloor \to \infty$ as $T \to \infty$ imply that $A_2(x) \to 0$ as $n \to \infty$. □

In the one-dimensional case, if x is a strong Lebesgue point, then $M_s f(x)$ is finite and $L_1(\log L)^{d-1}(\mathbb{T}^d) = L_1(\mathbb{T}^d)$, hence we get back the results due to Lebesgue [11] and Riesz [15] mentioned in the introduction. Recall that $L_1(\log L)^{d-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d)$ for $1 < p \leq \infty$ and $d > 1$. Since by Theorem 2.1 and (2.1) almost every point is a strong Lebesgue point and the strong maximal operator $M_s f$ is almost everywhere finite for $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$, Theorem 4.3 implies Corollary 4.2. If f is continuous at a point x, then x is also a strong Lebesgue point. So we obtain:

Corollary 4.3. *Suppose that $0 < \alpha_j \leq 1$ for all $j = 1, \ldots, d$. If $M_s f(x)$ is finite and $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$ is continuous at a point x, then*

$$\lim_{n \to \infty} \sigma_n^\alpha f(x) = f(x).$$

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Research Article

Some numerical applications of generalized Bernstein operators

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ABSTRACT. In this paper, some recent applications of the so-called Generalized Bernstein polynomials are collected. This polynomial sequence is constructed by means of the samples of a continuous function f on equispaced points of $[0, 1]$ and depends on an additional parameter which can be suitable chosen in order to improve the rate of convergence to the function f, as the smoothness of f increases, overcoming the well-known low degree of approximation achieved by the classical Bernstein polynomials or by the piecewise polynomial approximation. The applications considered here deal with the numerical integration and the simultaneous approximation. Quadrature rules on equidistant nodes of $[0, 1]$ are studied for the numerical computation of ordinary integrals in one or two dimensions, and usefully employed in Nyström methods for solving Fredholm integral equations. Moreover, the simultaneous approximation of the Hilbert transform and its derivative (the Hadamard transform) is illustrated. For all the applications, some numerical details are given in addition to the error estimates, and the proposed approximation methods have been implemented providing numerical tests which confirm the theoretical estimates. Some open problems are also introduced.

Keywords: Bernstein polynomials, approximation by polynomials, numerical integration on uniform grids, Fredholm integral equations on uniform grids.

2020 Mathematics Subject Classification: 41A10, 65D32, 65R10, 65R20.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

Bernstein polynomials $B_m f$ constitute a classical approximation of a continuous function f based on the samples of f at equidistant nodes of $[0, 1]$. They have been widely studied in literature (see for instance [23], [3]) and provide a constructive proof of the Weierstrass theorem, since the positive Bernstein operators $B_m : f \to B_m f$ fits the assumptions of the Korovkin theorem (see e.g. [2]).

On the other hand, in many applications, the available data are often the values of the target function at equally spaced point sets, which would make suitable to apply the Bernstein polynomials. However, such polynomials are rarely used in the numerical approximation because a rate of convergence faster than $\frac{1}{m}$ cannot be obtained for more regular functions than absolutely continuous functions f s.t. $\|f''\varphi^2\| < \infty$, where $\varphi(x) = \sqrt{x(1-x)}$ (see e.g. [13]).

In order to get an higher rate of approximation, independently Micchelli [30], Felbeker [17] and Mastroianni-Occorsio [24] introduced and studied the following combinations of iterates

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of the Bernstein operator B_m

$$B_{m,s} = I - (I - B_m)^s = \sum_{i=1}^{s} \binom{s}{i} (-1)^{i-1} B_m^i, \qquad s \in \mathbb{N}, \tag{1.1}$$

where $B_m^i = B_m(B_m^{i-1}),\ i \geq 1,\ B_m^0 = I$ and I is the identity operator.

Similarly to B_m, for all $s \in \mathbb{N}$, the operators $B_{m,s}$ map continuous functions f into polynomials of degree m. The polynomials $B_{m,s}f$ are known in the literature as Generalized Bernstein polynomials of parameter s (shortly GBs polynomials). Like $B_m f$, they require the samples of f at the $m+1$ equispaced points of $[0,1]$ and interpolate f at the extremes. However, differently from the "originating" Bernstein operator, GBs operators are not always positive, as it can be clearly expected, since as $m \in \mathbb{N}$ is fixed and $s \to \infty$, we have that $B_{m,s}f \to L_m f$ uniformly in $[0,1]$, where $L_m f$ is the Lagrange polynomial interpolating f at the same equispaced nodes [24].

Nonetheless, for any fixed $s \in \mathbb{N}$ and $m \to \infty$, we have that $B_{m,s}f \to f$ uniformly in $[0,1]$ and suitable choices of the additional parameter $s > 1$, allow to increase the approximation rate achieved by the classical Bernstein polynomials, being m^{-s} the saturation order of $B_{m,s}$ [30]. In addition, as main property and, in some sense, cornerstone of the study, in [30, 24] the authors independently stated how the rate of convergence in approximating f improves as the smoothness of f increases. To be more precise, they proved that any function $f \in C^{2s}([0,1])$ can be uniformly approximated by the sequence $\{B_{m,s}f\}_{m\in\mathbb{N}}$ with the rate of convergence $\mathcal{O}(m^{-s})$. A more refined error estimate was proven in [19] by using the Ditzian–Totik $\varphi-$modulus of f having order $2s$. GBs polynomials were further investigated from other many authors and from many different points of view (see e.g. [4], [36], [38], [15], [8], [10], [34], [35]). In particular, a short history of GBs polynomials can be found in [19], with a wide bibliography on the topic.

The aim of the present paper is to "promote" GBs approximation in the applications by collecting some numerical methods based on GBs polynomials, which show how these polynomials may be useful from the applicative point of view.

It is in fact known that in many applications the samples of f are sometimes obtained by devices, or by measures detected at equidistant times. Such an experimental nature of the data precludes the use of those global techniques of approximation that have optimal performance, but are based on specified (non uniform) distribution of nodes such as the zeros of orthogonal polynomials.

On the contrary, GBs polynomials require data at equally spaced points and, differently from piecewise polynomials or ordinary Bernstein approximations, as the smoothness of f increases, suitable choices of s allow to improve the rate of convergence. For this reason, GBs polynomials might be fruitfully used in the applications, where the data are taken at equidistant nodes.

In particular, they have been successfully employed in some applications that we will summarize in this paper, where we provide some improvements of the already known results.

The numerical quadrature of ordinary integrals on the interval $[0,1]$ and on the square $[0,1]^2$, the numerical solution of Fredholm Integral Equations of the second kind (FIEs) on such domains, and, finally, the numerical computation of the finite Hilbert and Hadamard transforms on $[0,1]$ are the applications we will deal with.

FIEs play an important role in various fields of the applied sciences, since they model many problems in elasticity, fluid-dynamics, etc.. Also, the Hilbert transform is widely used for applications in several fields. Among them, there are partial differential equations, optics (X-ray crystallography, electron-atom scattering), electrodynamics and quantum mechanics

(Kramers-Kronig relation), signal processing (phase retrieval, transfer functions of linear systems, spectral factorization) (see e.g. [22]). Moreover, both the Hilbert and Hadamard transforms, the latter regarded as the first derivative of the Hilbert one, are widely used to formulate boundary–value problems in many areas of mathematical physics (potential theory, fracture mechanics, aerodynamics, elasticity, etc.) in terms of singular integral equations in $(0, 1)$ (see e.g. [21, 27, 28, 29, 37] and the references therein). Hence, numerical methods based on GBs for the above applications can be applied in different contexts whenever the discrete data are available at equally spaced nodes.

We recall that in [32, 33] stable and convergent quadrature and cubature rules have been obtained by replacing the integrand by GBs polynomials in one and in two variables. Based on these rules, in the same papers, Nyström methods have been proposed for the numerical solution of one and two-dimensional FIEs. Studying such equations in Sobolev type spaces,the authors proved that in both the dimensions the methods are numerically stable, convergent and the involved linear systems are well-conditioned. Here, we extend the results given in [32] by providing error estimates in the wider class of Hölder–Zygmund spaces. Moreover, in the bivariate case, we consider the tensor product of GBs operators using different values for both the degrees and the parameters. In this way, we get a more flexible approximation tool than that proposed in [33].

The approximation of the Hilbert transform in $(0, 1)$, by means of GBs polynomials, was firstly investigated in [25, 26]. Such idea has been recently revised in [18], where the simultaneous approximation of the Hilbert transform and its first derivative was proposed in $(-1, 1)$.

In this paper, following the ideas in [18], we construct quadrature rules for both the Hilbert and Hadamard transforms in $(0, 1)$, by means of a shrewd use of the simultaneous approximation by GBs polynomials. Such approach allows to approximate both the integral transforms, by using the same samples of f at a grid of equally spaced nodes. Moreover, as in [18], some improvements from both the theoretical and computational point of view, are achieved w.r.t. those shown in [25, 26]. We determine weighted pointwise estimates of the quadrature errors, in the general case of density functions satisfying the Dini-type conditions involving the Ditzian–Totik moduli of smoothness [13], as well as in the case of smoother functions in Sobolev and Hölder–Zygmund spaces. Concerning the numerical computation, recurrence relations for the quadrature coefficients in $[0, 1]$ are given either for the Hilbert and the Hadamard transforms. Moreover, such new recurrence relations preserve the more stable Bernstein polynomial basis and do not require the transformation into the basis $\{1, x, \dots, x^m\}$ as done in [25, 26].

For all the applications, we give some numerical tests and graphs in order to confirm the theoretical results and to show some numerical evidences on the role of the involved parameters m and s and the interaction between them.

Finally along the paper, the reader can find some open problems that could be interesting for further investigations.

The outline of the paper is as follows. Section 2 contains some notation and preliminary results about the approximation tools and the functional spaces. Section 3 is devoted to the GBs polynomials, their properties and the convergence results also for the simultaneous approximation. In Section 4, the quadrature formula based on GBs approximation is proposed. Section 5 is devoted to the Nyström method based on the quadrature rule of Section 4. Section 6 contains the results on the simultaneous approximation of the Hilbert and Hadamard transforms. Section 7 shows some recent results in the bivariate case. Finally, Section 8 includes some computational details, that describe the practical implementation of the formulae based on GBs polynomials, and used in the paper. The tests given for each application were performed in double precision arithmetic.

2. Notation and preliminary results

In the sequel, $\mathcal{C}$ will denote a generic positive constant which may differ at different occurrences and $\mathcal{C} \neq \mathcal{C}(a, b, ..)$ indicates that $\mathcal{C}$ is independent of $a, b, \ldots$. Moreover, if $A, B > 0$ depend on some parameters the notation $A \sim B$ means that there are fixed constants $\mathcal{C}_1, \mathcal{C}_2 > 0$ (independent of the parameters in A, B) such that $\mathcal{C}_1 A \leq B \leq \mathcal{C}_2 A$.

For any integer $m \geq 0$, we set $N_0^m := \{0, 1, 2, \ldots, m\}$ and denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m. In the Banach space $C^0([0,1])$ of the continuous functions on $[0,1]$ endowed with the uniform norm $\|f\| := \max_{x\in[0,1]} |f(x)|$, the error of best approximation of $f \in C^0([0,1])$ in $\mathbb{P}_m$ is defined as

$$E_m(f) = \min_{P\in\mathbb{P}_m} \|f - P\|$$

and the Weierstrass theorem ensures that

$$f \in C^0([0,1]) \Longleftrightarrow \lim_{m\to\infty} E_m(f) = 0.$$

A constructive proof of this result is given by the well-known Bernstein polynomials

$$B_m f(x) = \sum_{k=0}^{m} p_{m,k}(x) f(t_k), \qquad t_k := \frac{k}{m}, \quad x \in [0,1], \quad m \geq 1, \tag{2.2}$$

where

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \qquad k \in N_0^m, \qquad x \in [0,1], \tag{2.3}$$

are the fundamental Bernstein polynomials of degree m, which satisfy the following recurrence relation

$$p_{m,k}(x) = (1-x)p_{m-1,k}(x) + x p_{m-1,k-1}(x), \qquad k \in N_0^m, \quad m \geq 1, \tag{2.4}$$

being $p_{m,k}(x) \equiv 0$, for $k \notin N_0^m$.

It is well–known that, for all $f \in C^2([0,1])$ and $m \in \mathbb{N}$ sufficiently large, Bernstein polynomials satisfy

$$\|f - B_m f\| \leq \frac{\mathcal{C}}{m}, \qquad \mathcal{C} \neq \mathcal{C}(m),$$

and that the convergence rate does not improve by increasing the smoothness of f as, instead, it happens for $E_m(f)$.

A useful tool to measure the smoothness of $f \in C^0([0,1])$ is the following Ditzian–Totik modulus of smoothness [13, (2.1.2)]

$$\omega_\varphi^r(f, t) = \sup_{0<h\leq t} \|\Delta_{h\varphi}^r f\|, \qquad r \in \mathbb{N},$$

defined by means of the following finite differences with variable step–size

$$\Delta_{h\varphi(x)}^r f(x) = \sum_{k=0}^{r} (-1)^k \left(\begin{array}{c} r \\ k \end{array} \right) f\left(x + (r-2k)\frac{h}{2}\varphi(x)\right),$$

where throughout the paper, it is $\varphi(x) := \sqrt{x(1-x)}$ and $x \in [0,1]$.

Denoting by AC_{Loc} the space of all locally absolutely continuous functions on $[0,1]$ (i.e. which are absolutely continuous in every closed subinterval $[a,b]$ in $(0,1)$), such modulus can be estimated by means of the following equivalent K–functional [13, Th. 2.1.1]

$$\omega_\varphi^r(f,t) \sim K_{r,\varphi}(f, t^r) := \inf\{\|f - g\| + t^r \|g^{(r)}\varphi^r\| : \ g^{(r-1)} \in AC_{Loc}\}. \tag{2.5}$$

Similarly to the classical modulus of smoothness given by

$$\omega^r(f,t) = \sup_{0<h\leq t} \|\Delta_h^r f\|, \qquad r \in \mathbb{N},$$

$$\omega^r(f,t) \sim K_r(f,t^r) := \inf\{\|f-g\| + t^r\|g^{(r)}\| : \; g^{(r-1)} \in AC_{Loc}\},$$

we have

$$\lim_{t\to 0} \omega_\varphi^r(f,t) = 0, \qquad \forall f \in C^0([0,1]).$$

Nevertheless, by taking the variable step–size $h\varphi(x)$, that decreases more and more as x approaches to the extremes of $[0,1]$, the Ditzian–Totik modulus better describes the behaviour of the polynomial approximation close to the endpoints. In fact, the following Jackson and Stechkin-type inequalities hold true [13, Th. 7.2.1 and Th. 7.2.4]

$$E_m(f) \leq \mathcal{C}\omega_\varphi^r\left(f, \frac{1}{m}\right), \quad \forall r < m, \qquad \mathcal{C} \neq \mathcal{C}(m,f), \tag{2.6}$$

$$\omega_\varphi^r(f,t) \leq \mathcal{C}t^r \sum_{0\leq k\leq 1/t} (1+k)^{r-1}E_k(f), \quad \mathcal{C} \neq \mathcal{C}(t,f) \tag{2.7}$$

and these direct and converse results yield [13, Corollary 7.2.5]

$$E_m(f) = \mathcal{O}(m^{-r}) \Longleftrightarrow \omega_\varphi^k(f,t) = \mathcal{O}(t^r), \quad k > r > 0. \tag{2.8}$$

We point out that the implication "$\Longrightarrow$" does not hold for the classical moduli, that are related to the Ditzian–Totik ones as follows

$$\omega_\varphi^r(f,t) \leq \mathcal{C}\omega^r(f,t), \qquad \mathcal{C} \neq \mathcal{C}(f,t), \qquad r \in \mathbb{N}.$$

Now, let us consider the following Sobolev–type spaces

$$W_r = \left\{ f \in C^0([0,1]) : f^{(r-1)} \in AC_{Loc}, \|f^{(r)}\varphi^r\| < \infty \right\}, \qquad r \in \mathbb{N},$$

equipped with the norm $\|f\|_{W_r} := \|f\| + \|f^{(r)}\varphi^r\|$. By virtue of the previous results, the following properties hold for all $f \in W_r$ and $r \in \mathbb{N}$,

$$E_m(f) \leq \frac{\mathcal{C}}{m^r}, \qquad \mathcal{C} \neq \mathcal{C}(m), \tag{2.9}$$

$$\omega_\varphi^k(f,t) \leq \mathcal{C}t^r\|f^{(r)}\varphi^r\|, \qquad \forall k \geq r, \qquad \mathcal{C} \neq \mathcal{C}(t,f), \tag{2.10}$$

$$\omega_\varphi^r(f,t) = o(t^r) \implies f \in \mathbb{P}_{r-1}. \tag{2.11}$$

Denoting by $C^k([0,1])$ the space of all continuously differentiable functions till the order $k \in \mathbb{N}$, we also recall that classical moduli satisfy

$$\omega^k(f,t) \leq \mathcal{C}t^r\|f^{(r)}\|, \qquad \forall f \in C^r([0,1]), \quad \forall k \geq r, \qquad \mathcal{C} \neq \mathcal{C}(t,f). \tag{2.12}$$

The Hölder–Zygmund type spaces based on Ditzian–Totik modulus of smoothness are defined as follows

$$Z_\lambda = \left\{ f \in C^0 : \quad \sup_{t>0} \frac{\omega_\varphi^r(f,t)}{t^\lambda} < \infty, \quad r > \lambda \right\}, \qquad \forall \lambda > 0 \tag{2.13}$$

and equipped with the norm

$$\|f\|_{Z_\lambda} = \|f\| + \sup_{t>0} \frac{\omega_\varphi^r(f,t)}{t^\lambda}, \qquad r > \lambda. \tag{2.14}$$

For any $\lambda > 0$, the space Z_λ constitutes a particular case of the Besov–type spaces studied in [14] and in the case $\lambda = r \in \mathbb{N}$ and the previously introduced Sobolev space W_r in continuously imbedded in Z_r. More generally, it has been proved that [14, Theorem 2.1]

$$\|f\|_{Z_\lambda} \sim \|f\| + \sup_{n>0}(n+1)^\lambda E_n(f), \qquad \forall\lambda > 0. \tag{2.15}$$

Such equivalence ensures that the definitions (2.13) and (2.14) are indeed independent of the integer $r > \lambda$ we choose. Moreover, (2.15) yields the following characterization of the continuous functions $f \in Z_\lambda$

$$f \in Z_\lambda \iff E_n(f) = \mathcal{O}(n^{-\lambda}), \qquad \forall\lambda > 0. \tag{2.16}$$

In particular, for all $f \in Z_\lambda$ and any $r > \lambda > 0$, we get

$$\omega_\varphi^r(f,t) \le \mathcal{C}t^\lambda \|f\|_{Z_\lambda}, \qquad \mathcal{C} \ne \mathcal{C}(f,t). \tag{2.17}$$

3. The generalized Bernstein polynomials

Let $f \in C^0([0,1])$ and

$$B_m^i f(x) := B_m(B_m^{i-1}f)(x), \qquad B_m^0 f := f, \qquad m,i \in \mathbb{N} \tag{3.18}$$

be the i-th iterate of the Bernstein polynomial (2.2). Fixed an integer parameter $s \ge 1$, the Generalized Bernstein polynomial of parameter s and degree $m \in \mathbb{N}$ is defined as follows

$$B_{m,s}f(x) = \sum_{i=1}^{s}(-1)^{i+1}\binom{s}{i}B_m^i f(x). \tag{3.19}$$

Such GBs polynomials have been independently introduced and studied in [30], [17], [24].

By (3.19) and (3.18), for any $m, s \in \mathbb{N}$, the polynomial $B_{m,s}f$ takes the form

$$B_{m,s}f(x) = \sum_{j=0}^{m} p_{m,j}^{(s)}(x) f\left(\frac{j}{m}\right), \quad 0 \le x \le 1, \tag{3.20}$$

where

$$p_{m,j}^{(s)}(x) = \sum_{i=1}^{s}\binom{s}{i}(-1)^{i-1}B_m^{i-1}p_{m,j}(x), \qquad j = 0,\dots,m \tag{3.21}$$

are the so–called fundamental Generalized Bernstein polynomials of degree m.

Note that the map $B_{m,s} : f \in C^0([0,1]) \to B_{m,s}f \in \mathbb{P}_m$ is a linear map, not always positive, and for $\forall m, s \in \mathbb{N}$, we have

$$B_{m,s}f(0) = f(0); \quad B_{m,s}f(1) = f(1),$$
$$B_{m,s}e_i(x) = x^i, \ i = 1,2, \quad e_i(x) := x^i.$$

For all degrees $m \in \mathbb{N}$, if we fix $s = 1$, then we get the classical Bernstein polynomial, i.e., $B_{m,1} = B_m$. For increasingly $s \in \mathbb{N}$, the sequence $\{B_{m,s}f\}_s$, continuously links Bernstein polynomials $B_m f$ to the Lagrange polynomials $L_m f \in \mathbb{P}_m$ interpolating f at the nodes $\{t_i\}_{i=0}^m$, i.e.,

$$L_m f(x) = \sum_{k=0}^{m} f(t_k) l_{m,k}(x), \quad l_{m,k}(x) = \prod_{k \ne i=0}^{m} \frac{x - t_i}{t_k - t_i}. \tag{3.22}$$

Such property, advisable in different contexts (see [38], [34], [5]), was given in the following.

Theorem 3.1. [24] *For any $f \in C^0([0,1])$, we have*

$$\lim_{s\to\infty} B_{m,s} f(x) = L_m f(x) \tag{3.23}$$

uniformly w.r.t. $x \in [0,1]$.

Figure 1 displays the behaviour of the fundamental GBs polynomials $p_{m,k}^{(s)}$, with fixed m,k and increasing values of the parameter s. The plots confirm the continuous relation between Bernstein and Lagrange polynomials. In fact, as $s \to \infty$, in Figure 1 we see that the fundamental GBs polynomial $p_{m,k}^{(s)}(x)$ uniformly tends to the $k-$th fundamental Lagrange polynomial $l_{m,k}(x)$, according with Theorem 3.1.

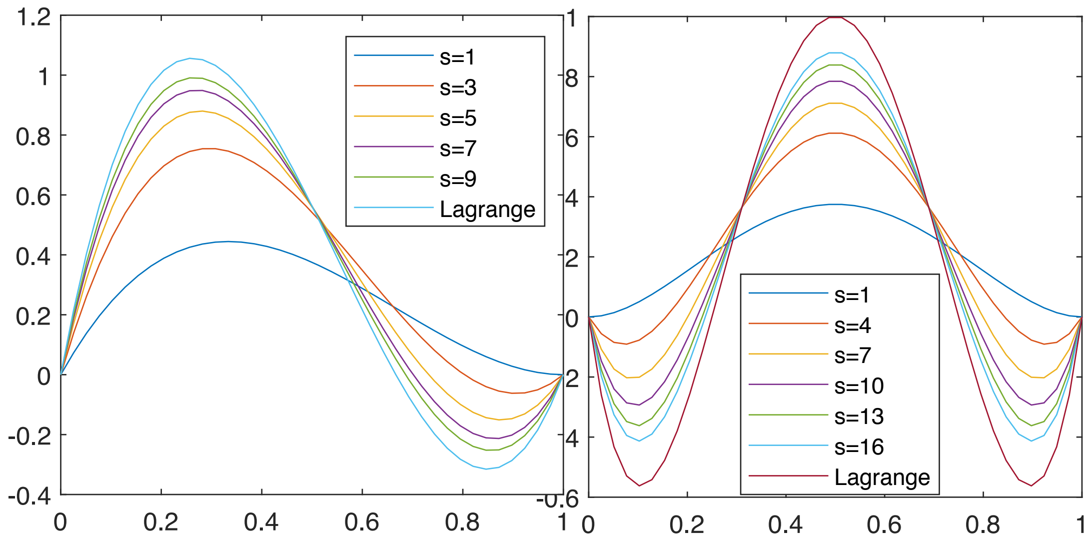

FIGURE 1. Plots of the fundamental GBs and Lagrange polynomials $p_{m,k}^{(s)}(x)$ and $l_{m,k}(x)$ for $m = 3$, $k = 1$ on the left and $m = 2$, $k = 2$ on the right

For all $m, s \in \mathbb{N}$, the fundamental GBs polynomials $\{p_{m,j}^{(s)}(x)\}_{j=0}^m$ form a partition of the unity, i.e.,

$$\sum_{j=0}^{m} p_{m,j}^{(s)}(x) = 1, \qquad \forall x \in [0,1].$$

A handle vectorial form of the basis $\{p_{m,k}^{(s)}\}_{k=0}^m$ is given by the following theorem proved in [34].

Theorem 3.2. *For all $m, s \in \mathbb{N}$ and for any $x \in [0,1]$, let $\mathbf{p}_m^{(s)}(x) = [p_{m,0}^{(s)}(x), p_{m,1}^{(s)}(x), \dots, p_{m,m}^{(s)}(x)]$ be the row–vector of fundamental GBs polynomials that, for $s = 1$, reduces to the vector of fundamental Bernstein polynomials $\mathbf{p}_m(x) = [p_{m,0}(x), \dots, p_{m,m}(x)]$. Moreover, let $C_{m,s}$ be the following square matrix of order $(m+1)$*

$$\begin{aligned} C_{m,s} :=& A^{-1}[I-(I-A)^s] = [I-(I-A)^s]A^{-1} \\ =& [I+(I-A)+(I-A)^2+\dots+(I-A)^{s-1}], \end{aligned} \tag{3.24}$$

where I is the identity matrix and $A \in \mathbb{R}^{(m+1)\times(m+1)}$ is defined by

$$A_{i,j} = p_{m,j}(t_i), \qquad t_i = \frac{i}{m}, \qquad (i,j) \in N_0^m \times N_0^m. \tag{3.25}$$

We have

$$\mathbf{p}_m^{(s)}(x) = \mathbf{p}_m(x) \cdot C_{m,s}, \qquad \forall m,s \in \mathbb{N}, \qquad \forall x \in [0,1]. \tag{3.26}$$

Moreover, for any $f \in C^0([0,1])$, the polynomial $B_{m,s}f$ can be represented in the following form

$$B_{m,s}f(x) = \mathbf{p}_m(x) \cdot C_{m,s} \cdot \mathbf{f}_m, \qquad \forall m,s \in \mathbb{N}, \qquad \forall x \in [0,1], \tag{3.27}$$

where $\mathbf{f}_m \in \mathbb{R}^{m+1}$ is the sampling (column) vector of the function f evaluated at the nodes t_i, i.e.,

$$\mathbf{f}_m := \left[f(0), \ldots, f\left(\frac{i}{m}\right), \ldots, f(1) \right]^T. \tag{3.28}$$

Remark 3.1. *Theorem 3.2 provides a useful tool for computing GBs polynomial. Indeed from (3.27), it follows that the GBs polynomial $B_{m,s}f$ can be considered as the $m-$th classical Bernstein polynomial of a function g having a suitable sampling vector, i.e.,*

$$B_{m,s}f(x) = B_m g(x) = \mathbf{p}_m(x) \cdot \mathbf{g}_m,$$

where the sampling vector of g is given by

$$\mathbf{g}_m := C_{m,s} \cdot \mathbf{f}_m.$$

As a consequence, we can compute the polynomial $B_{m,s}f$ by using the de Casteljau recursive scheme, which, as it is well-known, is a fast and stable algorithm [16].

Additional details on the fast computation of GBs polynomials are given in Section 7. In the sequel, we are going to analyze the approximation provided by GBs polynomials of fixed parameter $s \in \mathbb{N}$ and increasing degrees. About the estimate of the remainder term $f - B_{m,s}f$, the following error bound in $C^0([0,1])$ was proved in [19].

Theorem 3.3. [19] *Let $s \in \mathbb{N}$ be fixed. Then, for all $m \in \mathbb{N}$ and any $f \in C^0([0,1])$, we have*

$$\|f - B_{m,s}f\| \le \mathcal{C} \left\{ \omega_\varphi^{2s}\left(f, \frac{1}{\sqrt{m}}\right) + \frac{\|f\|}{m^s} \right\}, \qquad C \neq \mathcal{C}(m,f). \tag{3.29}$$

Moreover, for any $0 < \mu \le 2s$, we obtain

$$\|f - B_{m,s}f\| = \mathcal{O}(m^{-\frac{\mu}{2}}),\ m \to \infty \iff \omega_\varphi^{2s}(f,t) = \mathcal{O}(t^\mu)$$

and the o–saturation class is characterized by the equivalence

$$\|f - B_{m,s}f\| = o(m^{-s}) \iff f \text{ is a linear function.}$$

Applying the properties of the moduli of smoothness given in Section 2, several error estimates can be deduced from (3.29). In particular, for all $m,s \in \mathbb{N}$, by (2.11) and by (2.17), we have

$$\|f - B_{m,s}f\| \le \frac{\mathcal{C}}{\sqrt{m^r}}, \qquad \mathcal{C} \neq \mathcal{C}(m), \qquad \forall f \in W_r, \quad r \le 2s, \tag{3.30}$$

$$\|f - B_{m,s}f\| \le \frac{\mathcal{C}}{\sqrt{m^\lambda}}, \qquad \mathcal{C} \neq \mathcal{C}(m), \qquad \forall f \in Z_\lambda, \quad \lambda < 2s. \tag{3.31}$$

Hence, we remark that by introducing the additional parameter $s \in \mathbb{N}$, the saturation order m^{-1} occurring with classical Bernstein polynomials is enlarged to m^{-s} and, using the same function samples at the $(m+1)$ equidistant nodes of $[0,1]$, the GBs polynomial may provide the square root of the order of the best uniform polynomial approximation of f in $\mathbb{P}_m$ (see (2.16)).

Nevertheless, s cannot be chosen arbitrarily high and the reason is given by Theorem 3.1, stating that for $s \to \infty$ the operator $B_{m,s}$ tends to the Lagrange interpolating polynomial on equispaced points, which is a well known unstable operator.

In order to show the real degree in approximating a given function f, for increasing values of m and s, in Table 1, we report the maximum errors

$$\mathcal{E}_{m,s}f = \max_{x\in\mathcal{X}} |f(x) - B_{m,s}f(x)|,$$

attained in a sufficiently large set $\mathcal{X} \subset [0,1]$, for the test function $f(x) = |x-0.6|^{\frac{7}{2}} \in Z_{\frac{7}{2}}$, whose theoretical error goes like $\mathcal{O}\left(m^{-\frac{7}{4}}\right)$. In each column of Table 1, the errors for m fixed and s varying, starting from 2^6 on, until the errors decrease, are reported. The empty boxes mean that for the corresponding s, the error does not decrease anymore. We note that for any m there

s	$m=16$	$m=32$	$m=64$	$m=128$	$m=256$	$m=512$	$m=1024$
2^6	$2.50e-5$	$6.90e-6$	$1.95e-6$	$5.47e-7$	$1.49e-7$	$3.7e-8$	$9.86e-9$
2^7	$1.93e-5$	$5.20e-6$	$1.46e-6$	$4.10e-7$	$1.09e-7$	$2.76e-8$	$6.81e-9$
2^8	$3.78e-5$	$4.08e-6$	$1.15e-6$	$3.20e-7$	$8.45e-8$	$2.16e-8$	$4.58e-9$
2^9		$3.31e-6$	$9.34e-7$	$2.57e-7$	$6.69e-8$	$1.73e-8$	$2.98e-9$
2^{10}			$7.34e-7$	$2.11e-7$	$5.42e-8$	$1.46e-8$	$2.07e-9$
2^{11}			$6.52e-7$	$1.77e-7$	$4.47e-8$	$1.22e-8$	$2.07e-9$
2^{12}				$1.50e-7$	$3.74e-8$	$1.02e-8$	$2.04e-9$
2^{13}				$1.29e-7$	$3.31e-8$	$8.44e-9$	$1.97e-9$
2^{14}				$1.12e-7$	$2.93e-8$	$6.98e-9$	$1.89e-9$
2^{15}				$9.89e-8$	$2.59e-8$	$5.74e-9$	$1.79e-9$
2^{16}					$2.29e-8$	$4.70e-9$	$1.67e-9$
2^{17}					$2.02e-8$	$3.81e-9$	$1.60e-9$
2^{18}					$1.79e-8$	$3.24e-9$	$1.50e-9$

TABLE 1. Convergence behaviour w.r.t. m and s

exist a threshold $\widetilde{s} = \widetilde{s}(m)$, until which the errors decrease, while for $s > \widetilde{s}$ the situation is quite reversed. In these reverse cases, m has to be increased for speeding up again the convergence. To highlight this behaviour, in Fig. 2, we plotted the error curves w.r.t. the same function $f(x) = |x-0.6|^{\frac{7}{2}}$, for fixed m and s varying from 2^6 to 2^{18}.

The investigation on the mutually relation between m and s for obtaining the optimal s for each m is still an open problem.

In Figure 3, the plots of the polynomials $B_{m,s}f$, for m fixed and s varying, are given. Since in the whole interval the curves seem to coincide, on the right a magnification is given in the restricted interval $[0.5, 0.7]$.

In conclusion of the section, let us consider the case that f is a continuously differentiable function up to a certain order $k \in \mathbb{N}$. In this case, GBs polynomials can be used for the simultaneous approximation of f and its first k derivatives. In fact, we have

$$\lim_{m\to\infty} \|f^{(k)} - (B_{m,s}f)^{(k)}\| = 0, \qquad \forall f \in C^k,$$

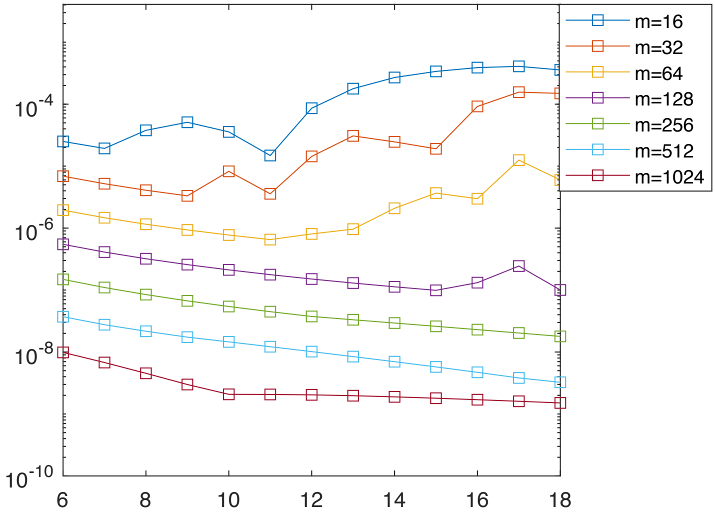

FIGURE 2. Plots of $\mathcal{E}_{m,s}f$ for $s = 2^n$, $n = 6, 7, \ldots, 18$. On the abscissas the values of n are reported.

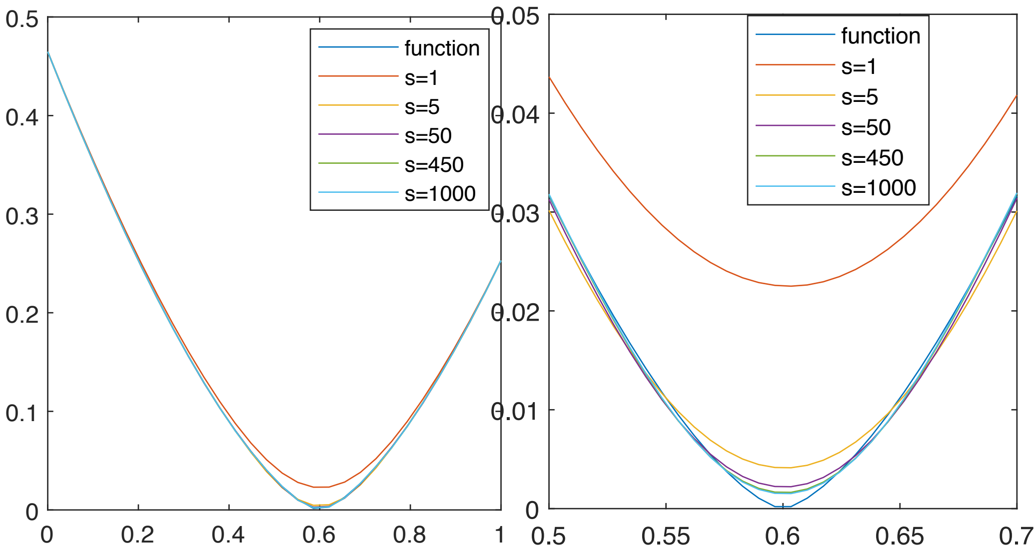

FIGURE 3. On the left the plots of the functions $f(x) = |x - 0.6|^{\frac{7}{2}}$ and $B_{m,s}f$ for $m = 31$ and different values of s. On the right the same plots are zoomed in $[0.5, 0.7]$.

where the derivatives of GBs polynomials are all based on the same sampling vector of f. Some computational details on the derivatives of the GBs polynomials can be found in Section 7. Here, we recall the following error estimate.

Theorem 3.4. [15, Corollary 1.6] *Let $s \geq 1$ be fixed. Then, for all $m, k \in \mathbb{N}$ and any $f \in C^k$, we have*

$$\|(f - B_{m,s}f)^{(k)}\| \leq \mathcal{C} \begin{cases} \omega_\varphi^{2s}\left(f', \frac{1}{\sqrt{m}}\right) + \omega^s\left(f', \frac{1}{m}\right) + \omega\left(f', \frac{1}{m^s}\right), & k = 1 \\ \omega_\varphi^{2s}\left(f^{(k)}, \frac{1}{\sqrt{m}}\right) + \omega^s\left(f^{(k)}, \frac{1}{m}\right) + \frac{\|f^{(k)}\|}{m^s}, & k \geq 2 \end{cases},$$

where $\omega := \omega^1$ and $\mathcal{C} \neq \mathcal{C}(m, f)$.

4. A QUADRATURE RULE ON EQUALLY SPACED KNOTS

Based on GBs polynomials, the following quadrature rule was introduced in [24]

$$\int_0^1 f(x)dx = \Sigma_m^{(s)} f + R_m^{(s)} f, \tag{4.32}$$

where the quadrature sum is defined by

$$\Sigma_m^{(s)} f := \int_0^1 B_{m,s}f(x)dx = \sum_{j=0}^m Q_j^{(s)} f\left(\frac{j}{m}\right), \qquad Q_j^{(s)} := \int_0^1 p_{m,j}^{(s)}(x)\, dx \tag{4.33}$$

and the quadrature error is given by

$$R_m^{(s)} f = \int_0^1 [f(x) - B_{m,s}f(x)]\, dx.$$

Such rule is easy to construct since the quadrature weights are representable in the following form [35]

$$Q_j^{(s)} = \frac{1}{m+1} \sum_{i=0}^m (C_{m,s})_{i,j}, \qquad j \in N_0^m,$$

where $(C_{m,s})_{i,j}$ denotes the (i, j)–entry of the matrix $C_{m,s}$ in (3.24).

We point out that these quadrature weights are not always positive. Nevertheless, the quadrature formula is always stable. Indeed, by Theorem 3.3 and by the Uniform Boundedness Principle, it is possible to deduce the following theorem of convergence and stability.

Theorem 4.5. [32] *For all $f \in C^0([0, 1])$ and any $s, m \in \mathbb{N}$, there holds*

$$|R_m^{(s)} f| \leq \mathcal{C}\left(\omega_\varphi^{2s}\left(f, \frac{1}{\sqrt{m}}\right) + \frac{\|f\|}{m^s}\right), \qquad \mathcal{C} \neq \mathcal{C}(f, m), \quad \mathcal{C} = \mathcal{C}(s). \tag{4.34}$$

Moreover, the quadrature formula is stable, i.e.,

$$\sup_m \sum_{j=0}^m |Q_j^{(s)}| < \infty.$$

By estimate (4.34), it is possible to deduce the order of convergence for functions belonging to several functional spaces. For instance, if $f \in W_r$ with $r \in \mathbb{N}$, then for sufficiently large m and for any integer $s \geq \frac{r}{2}$, we have by (2.10)

$$|R_m^{(s)} f| \leq \mathcal{C}\left(\frac{\|f^{(r)}\varphi^r\|}{\sqrt{m^r}} + \frac{\|f\|}{m^s}\right), \qquad \mathcal{C} \neq \mathcal{C}(f, m), \quad \mathcal{C} = \mathcal{C}(s). \tag{4.35}$$

Another example is given for $f \in Z_\lambda$. Indeed for any $s > \frac{\lambda}{2}$, by (2.17), we get

$$|R_m^{(s)} f| \leq \mathcal{C}\frac{\|f\|_{Z_\lambda}}{\sqrt{m^\lambda}}, \qquad \mathcal{C} \neq \mathcal{C}(f, m), \quad \mathcal{C} = \mathcal{C}(s). \tag{4.36}$$

Now, we propose two tests, comparing the performance of formula (4.33) with the classical Romberg integration scheme

$$\int_0^1 f(t)dt = T_{N,N}f + e_N f,$$

where $T_{N,N}$ is the Romberg rule and $e_N f$ denotes the quadrature error. This rule besides the well-known triangular scheme (see e.g. [11]), can be also represented as a linear combination of the samples of f

$$T_{N,N}f = \sum_{i=0}^{2^N} \sigma_i f\left(\frac{i}{2^N}\right), \tag{4.37}$$

that is a more convenient form in the implementation of other procedure, for instance in Nyström methods for integral equations. Details on the coefficients σ_i are given in [32]. Here, we only recall the following result about the convergence which holds true for functions having a continuous derivative of fixed order $r \geq 1$ in $[0,1]$ [7]:

$$|e_N f| \leq \mathcal{C}\frac{2^{(r-1)^2/4}}{(2\pi)^r}\frac{\|f^{(r)}\|}{(2^N)^r}, \qquad \forall N \geq \frac{r-1}{2}, \qquad f \in C^r([0,1]), \tag{4.38}$$

where $\mathcal{C}$ is a positive constant independent of f and depending on N and r and such that $1.5 \leq \mathcal{C} \leq 3.1$.

Example 4.1.

$$I(f_1) = \int_0^1 \cos(x)(1-x)^{\frac{5}{2}}dx, \quad I(f_2) = \int_0^1 \sqrt{1 - \frac{\sin^2(x)}{3}}dx.$$

In order to compare rules (4.33) and (4.37), we choose $m = 2^N$ in (4.33), reporting in Tables 2 and 3 the absolute errors. The empty boxes mean that no improvement is attained w.r.t. the errors obtained for the same values of m.

The values of the integrals $I(f_1) \sim 0.2744041660389273$ and $I(f_2) \sim 0.9526594143223039835$ were computed with 16 and 19 exact digits respectively by means of the software Mathematica.

m	*Romberg*	$s = 32$	$s = 64$	$s = 2048$
8	$1.41e-06$	$1.35e-06$	$1.47e-06$	$9.04e-07$
16	$1.47e-07$	$1.70e-07$	$1.39e-07$	$7.98e-08$
32	$1.39e-08$	$1.53e-08$	$1.22e-08$	$6.51e-09$
64	$1.28e-09$	$1.35e-09$	$1.07e-09$	$5.54e-10$
128	$1.15e-10$	$1.19e-10$	$9.43e-11$	$4.81e-11$
256	$1.03e-11$	$1.05e-11$	$8.31e-12$	$4.30e-12$
512	$9.10e-13$	$9.31e-13$	$7.35e-13$	$3.91e-13$
1024	$8.11e-14$	$8.26e-14$	$6.54e-14$	$1.67e-14$

TABLE 2. Errors for $I(f_1)$

Regarding $I(f_1)$, according to (4.35) and (4.38), since $f_1 \in W_5 \cap C^2([0,1])$ the error $R_m^{(s)}f$ will go as $\mathcal{O}(m^{-\frac{5}{2}})$, while the error e_N of the Romberg rule will behave like $\mathcal{O}(m^{-2})$, where $m = 2^N$.

About the second (elliptic) integral, we note that $f_2 \in W_r$ for any r and the convergence is very fast by both the quadrature rules.

m	*Romb*	$s = 64$	$s = 256$	$s = 1024$
8	$5.46e-09$	$5.33e-09$	$1.72e-09$	$4.44e-11$
16	$2.29e-11$	$1.43e-11$	$6.26e-13$	$2.18e-13$
32	$5.66e-15$	$1.33e-15$	$2.22e-15$	
64	$1.11e-16$	$1.11e-16$		
128	$4.44e-16$	$1.11e-15$		
256	$2.22e-16$	$3.33e-16$		

TABLE 3. Errors for $I(f_2)$

So, the tables show that the two quadrature rules are comparable for small values of s. However, in both the cases, by using the same number of samples m the free parameter s can be enhanced, allowing to gain better results.

We remark that in both the previous examples the speed of convergence is faster than the theoretical estimate of this speed. For instance, in the case of $I(f_1)$ with $m = 1024$, the errors would be around 9.5×10^{-7} for the Romberg formula and 2.8×10^{-8} for the GBs rule. This means in particular that estimate (4.34) is not sharp.

So an open problem is to estimate $R_m^{(s)} f$ "directly" or, which is the same, to have an L^1 estimate of $f - B_{m,s}f$. In addition, it would be useful to understand how the constant $\mathcal{C}$ in (4.34) depends on s.

5. A NYSTRÖM METHOD FOR SOLVING FREDHOLM INTEGRAL EQUATIONS

Based on the quadrature rule (4.33), a Nyström method has been introduced in [32] for solving the following Fredholm Integral Equation (FIE)

$$f(x) - \mu \int_0^1 f(t)k(x,t)dt = g(x), \qquad \mu \in \mathbb{R}, \qquad x \in [0,1]. \tag{5.39}$$

Such equation can be rewritten in operator form as follows

$$(I - K)f = g, \tag{5.40}$$

where I denotes the identity operator and

$$Kf(x) := \mu \int_0^1 k(x,t)f(t)\,dt, \qquad x \in [0,1]. \tag{5.41}$$

It is known [6] that if the kernel $k(x,t)$ is continuous, then $K : C^0([0,1]) \to C^0([0,1])$ is a compact operator.

In order to consider the case of more regular kernels, here and in the sequel, we will use the notation k_t (respectively k_x) for the bivariate function $k(x,t)$ considered as a function of the single variable x (respectively t). Using this notation, it is also known [20, Proposition 4.12] that if k is continuous w.r.t. both the variables and we have

$$\sup_{t\in[0,1]} \|k_t\|_{Z_\lambda} < +\infty, \qquad \lambda > 0, \tag{5.42}$$

then $K : f \in C^0([0,1]) \to Kf \in Z_\lambda$ is a countinuous map and hence, due to the compact embedding $Z_\lambda \subset C^0([0,1])$ ([20, Lemma3.2]), we have that $K : Z_\lambda \to Z_\lambda$ is a compact operator.

The previous mapping properties and the Fredholm Alternative yield the following theorem concerning the existence, the uniqueness and the degree of smoothness of the solution of the FIE (5.39).

Theorem 5.6. *Suppose that the homogeneous equation associated with (5.40) has only the trivial solution. If the kernel k is continuous w.r.t. both the variables in $[0,1]$, then there exists a unique solution of (5.40), that is $f^* \in C^0([0,1])$, for any continuous function g.*
If, in addition, for some $\lambda > 0$ we have $k_t \in Z_\lambda$ uniformly w.r.t. $t \in [0,1]$, i.e., (5.42) holds, then (5.40) is uniquely solvable in the Hölder–Zygmund spaces Z_ρ with $0 < \rho \le \lambda$, that is $f^* \in Z_\rho$, for all $g \in Z_\rho$.

In order to numerically solve the FIE (5.39), we recall that several fast convergent methods can be found in the literature concerning projection and Nyström methods based on Gauss-Jacobi quadrature rules (see for instance [6] and [12] and the references therein).

Nevertheless, many problems in engineering and mathematical physics are often modelled by (5.39), where the only available data are discrete values of the kernel k and the right–hand side g at a uniform grid of nodes. In such cases, the implementation of all the methods based on Jacobi zeros needs a further approximation step in order to derive, from the available data, the sampling vectors at the involved Jacobi grid. On the other hand, classical methods based on piecewise polynomial approximation are also available (see [6]) but they offer lower degree of approximation.

On the contrary, the Nyström method based on the quadrature rule (4.33) can be directly applied to numerically solve the equation in the case that the values of the kernel k and the term g are known at equidistant nodes of $[0,1]$. To be more precise, in order to find a numerical approximation of the solution f^*, for all m, we consider the following approximation K_m of the operator K

$$K_m f(x) = \mu \sum_{i=0}^{m} Q_i^{(s)} k(x,t_i) f(t_i), \qquad t_i = \frac{i}{m}, \qquad s \in \mathbb{N}, \tag{5.43}$$

where $Q_i^{(s)}$ are the weights of the quadrature rule (4.33) that has been applied to $Kf(x)$ given by (5.41). The operator K_m defines the following approximate equation

$$(I - K_m) f_m = g \tag{5.44}$$

whose solution f_m, if existing, has to satisfy the following identity that is a consequence of (5.44) and (5.43)

$$f_m(x) = \mu \sum_{i=0}^{m} Q_i^{(s)} k(x,t_i) \alpha_i^* + g(x), \qquad \alpha_i^* := f_m(t_i), \qquad x \in [0,1]. \tag{5.45}$$

This function is known as the Nyström interpolant of the solution f^*. In order to compute the unknowns α_i^*, $i \in N_0^m$, we collocate the approximate equation (5.44) on the knots t_h, with $h \in N_0^m$, obtaining the following linear system of $(m+1)$ equations and $(m+1)$ unknowns $\{\alpha_i\}_{i \in N_0^m}$

$$\alpha_h - \mu \sum_{i=0}^{m} Q_i^{(s)} k(t_h, t_i) \alpha_i = g(t_h), \qquad t_i = \frac{i}{m}, \qquad h \in N_0^m. \tag{5.46}$$

The solution of this system, if existing, provides the values $\{\alpha_i^*\}_{i=0}^m$ that we need in (5.45) and vice versa, the values $f_m(t_i)$, $i \in N_0^m$, are solutions of system (5.46). In other words, (5.44) and (5.46) are equivalent.

Denoting by $\mathbf{V}_m$ the coefficient matrix of the system (5.46) and by $\mathrm{cond}(\mathbf{V}_m) = \|\mathbf{V}_m\|_\infty \|\mathbf{V}_m^{-1}\|_\infty$ its condition number w.r.t. the matrix infinity norm, we have the following result which extends a previous one stated for Sobolev spaces in [32].

Theorem 5.7. *If for some $\lambda > 0$, the kernel k satisfies (5.42) and*

$$\sup_{x\in[0,1]} \|k_x\|_{Z_\lambda} < +\infty, \tag{5.47}$$

then for all integers $m \in \mathbb{N}$ and $s > \lambda/2$, the system (5.46) is uniquely solvable and well-conditioned, i.e.,

$$cond(\mathbf{V}_m) \le \mathcal{C}, \qquad \mathcal{C} \ne \mathcal{C}(m).$$

Moreover, for all $g \in Z_\lambda$ the unique solution $f^* \in Z_\lambda$ of the FIE (5.39) can be approximated by its Nyström interpolant f_m in (5.45) according with the following error estimate

$$\|f^* - f_m\| \le \mathcal{C}\frac{\|f\|_{Z_\lambda}}{\sqrt{m^\lambda}}, \tag{5.48}$$

where $\mathcal{C} \ne \mathcal{C}(m, f^*)$ and $\mathcal{C} = \mathcal{C}(s)$.

Proof. The proof can be led using classical arguments (see for instance [6, Th.4.1.2]). Indeed, it is well known that if the Nyström method is based on a quadrature formula converging for continuous functions, then the linear system (5.46) in uniquely solvable and the condition number of the matrix of coefficients is bounded by $\|I - K_m\|\|(I - K_m)^{-1}\|$ that is uniformly bounded for the collectively compactness of the sequence $\{K_m\}_m$. Moreover, it is also known that

$$\|f^* - f_m\| \sim \|Kf^* - K_m f^*\|.$$

Therefore, the Theorem is proved if we estimate the quadrature error for the function $k_x f^*$. Taking into account (4.36), we have just to estimate $\|f^* k_x\|_{Z_\lambda}$, being $f^*, k_x \in Z_\lambda$, uniformly w.r.t. x. Fix $x \in [0, 1]$. Using (2.15), we have

$$\|f^* k_x\|_\lambda \sim \|f^* k_x\| + \sup_n (n + 1)^\lambda E_n(f^* k_x).$$

In the case $n = 2m$ is not hard to prove that

$$E_{2m}(f^* k_x) \le 2\|k_x\|E_m(f^*) + \|f^*\|E_m(k_x).$$

Analogously if $n = 2m + 1$, then

$$E_{2m+1}(f^* k_x) \le 2\|k_x\|E_{m+1}(f^*) + \|f^*\|E_m(k_x).$$

Therefore,

$$\|f^* k_x\| + \sup_n (n + 1)^\lambda E_n(f^* k_x) \le \mathcal{C} \left(\|k_x\|\|f^*\|_{Z_\lambda} + \|f^*\|\|k_x\|_{Z_\lambda}\right).$$

Hence, assuming the sup on $x \in [0, 1]$ and holding (5.47), we finally get

$$\sup_x \|f^* k_x\|_{Z_\lambda} \le \mathcal{C}\|f^*\|_{Z_\lambda}, \qquad \mathcal{C} \ne \mathcal{C}(m, f^*)$$

and (5.48) follows. □

Remark 5.2. *The convergence estimate (5.48) says that if the known functions in equation (5.39) are in Z_λ, then the order of convergence is $\mathcal{O}(m^{-\frac{\lambda}{2}})$. This means that in the Hölder–Zygmund spaces the method converges with an order that is the half of the order of the best polynomial approximation in Z_λ (see (2.16)).*

In the sequel, we propose a numerical test in order to check the previous theoretical estimate.

Example 5.2. *We consider the following equation*

$$f(x) - 0.2 \int_0^1 f(t)|x-t|^{7.5}\, dt = |\arctan(x-0.5)|^{10.4}.$$

Here, $\mu = 0.2$*, the kernel* $k(x,t) = |x-t|^{7.5} \in Z_{7.5}$ *w.r.t. both the variables and* $g(x) = |\arctan(x - 0.5)|^{10.4} \in Z_{10.4}$*. In Table 4, we report the maximum errors attained in a discrete sufficiently large set of point in* $[0,1]$*, for increasing values of* m *and* s*.*

m	$s=16$	$s=32$	$s=64$
16	$0.71e-02$	$0.34e-02$	$0.16e-02$
32	$0.58e-04$	$0.43e-05$	$0.37e-06$
64	$0.41e-07$	$0.15e-09$	$0.15e-10$
128	$0.18e-11$	$0.27e-14$	$0.24e-14$
256	$0.12e-14$	$\mathbf{0.33e-15}$	$\mathbf{0.69e-15}$

TABLE 4. Errors by means of the Nyström interpolant

According with (5.48) the theoretical error behaves like $\mathcal{O}(m^{-15/4})$ *for* $s > 3$ *and as shown in Table 4, the Nyström method goes faster than the attended speed of convergence. For instance, the machine precision is attained for* $m = 256$*,* $s = 32$*.*

6. SIMULTANEOUS APPROXIMATION OF THE HILBERT TRANSFORM AND ITS FIRST DERIVATIVE

For any $f \in C^0([0,1])$, let

$$\mathcal{H}f(t) = \int\!\!\!\!\!\!- _0^1 \frac{f(x)}{x-t}dx \qquad \text{and} \qquad \mathcal{H}^1 f(t) = \int\!\!\!\!\!\!= _0^1 \frac{f(x)}{(x-t)^2}dx, \qquad 0<t<1, \tag{6.49}$$

be the (finite) Hilbert and Hadamard transforms of f, respectively, where we used the single and double bar-integral notation to indicate that the integrals have to be understood as the Cauchy principal value integral and the Hadamard finite–part integral, respectively, namely (see for instance [31], [37, (1.3)])

$$\mathcal{H}f(t) = \lim_{\epsilon\to 0}\left[\int_0^{t-\epsilon} \frac{f(x)}{x-t}dx + \int_{t+\epsilon}^1 \frac{f(x)}{x-t}dx\right], \qquad 0<t<1,$$

$$\mathcal{H}^1 f(t) = \lim_{\varepsilon\to 0}\left[\int_0^{t-\varepsilon} \frac{f(x)}{(x-t)^2}dx + \int_{t+\varepsilon}^1 \frac{f(x)}{(x-t)^2}dx - \frac{2f(t)}{\varepsilon}\right], \quad 0<t<1. \tag{6.50}$$

An alternative definition interprets the Hadamard transform as the first derivative of the Hilbert transform, i.e.,

$$\mathcal{H}^1 f(t) = \frac{d}{dt}\int\!\!\!\!\!\!- _0^1 \frac{f(x)}{x-t}dx, \quad 0<t<1 \tag{6.51}$$

and in that case that f' is Hölder continuous on $[0,1]$, the definitions (6.51) and (6.50) are indeed equivalent (see [37]).

Both the previous transforms are widely used in many areas of mathematical physics (potential theory, fracture mechanics, aerodynamics, elasticity, etc.), where several boundary–value problems can be formulated as singular integral equations in $[0,1]$ involving such integrals (see e.g. [21, 27, 28, 29, 37] and the references therein).

The following theorem provide some upper bounds of $|\mathcal{H}f(t)|$ and $|\mathcal{H}^1 f(t)|$, respectively, in the case that a Dini-type condition is satisfied by f and f', respectively. We omit the proof since it can be deduced mutatis mutandis by the analogous results in [9, Th.2.1],[18], concerning the case $[-1,1]$.

Theorem 6.8. *Let $0<t<1$. For all functions f such that*

$$\int_0^1 \frac{\omega_\varphi(f,u)}{u}du<\infty, \tag{6.52}$$

we have

$$\log^{-1}\left(\frac{e}{t(1-t)}\right)|\mathcal{H}f(t)|\le \mathcal{C}\left(\|f\|+\int_0^1\frac{\omega_\varphi(f,u)}{u}du\right),\qquad \mathcal{C}\ne\mathcal{C}(f,t).$$

Moreover, if

$$\int_0^1 \frac{\omega_\varphi(f',u)}{u}du<\infty, \tag{6.53}$$

then we have

$$\varphi^2(t)|\mathcal{H}^1f(t)|\le \mathcal{C}\left(\|f\|+\int_0^1\frac{\omega_\varphi(f',\tau)}{\tau}d\tau\right),\qquad \mathcal{C}\ne\mathcal{C}(f,t). \tag{6.54}$$

In many applications, the data are discrete and often consist of only the values of f at equidistant points of $[0,1]$. In this case, the simultaneous approximation properties of GBs polynomials turn out to be useful in constructing quadrature rules for the simultaneous approximation of the Hilbert and Hadamard transforms.

As regards the Hilbert transform, a first numerical approach based on GBs polynomials can be found in [25, 26]. Such method has been recently improved in [18], where efficient quadrature rules provide the simultaneous approximation of $\mathcal{H}f(t)$ and $\mathcal{H}^1f(t)$ by using the same samples of f, taken at equidistant nodes of $[-1,1]$. Here, we consider similar quadrature rules on the interval $[0,1]$.

Such formulas have been constructed starting from the following standard decompositions

$$\begin{aligned}\mathcal{H}f(t)&=\int_0^1\frac{f(x)-f(t)}{x-t}dx+f(t)\log\left(\frac{1-t}{t}\right)\\ \mathcal{H}^1f(t)&=\int_0^1\frac{f(x)-f(t)-f'(t)(x-t)}{(x-t)^2}dx+f'(t)\log\left(\frac{1-t}{t}\right)-\frac{f(t)}{t(1-t)}\end{aligned}$$

that are based on the ordinary integrals

$$\mathcal{F}f(t):=\int_0^1\frac{f(x)-f(t)}{x-t}dx, \tag{6.55}$$

$$\mathcal{F}^1f(t):=\int_0^1\frac{f(x)-f(t)-f'(t)(x-t)}{(x-t)^2}dx. \tag{6.56}$$

These integrals are approximated by the following quadrature rules based on the GBs polynomials

$$\begin{aligned}\mathcal{F}f(t)&=\mathcal{F}_{m,s}f(t)+\Phi_{m,s}f(t)\\ \mathcal{F}^1f(t)&=\mathcal{F}^1_{m,s}f(t)+\Phi^1_{m,s}f(t),\end{aligned}$$

where $\Phi_{m,s}f$ and $\Phi^1_{m,s}f$ denotes the quadrature errors and $\mathcal{F}_{m,s}f$, $\mathcal{F}^1_{m,s}f$ are quadrature sums defined as follows

$$\mathcal{F}_{m,s}f(t) := \int_0^1 \frac{B_{m,s}f(x) - B_{m,s}f(t)}{x-t}dx, \tag{6.57}$$

$$\mathcal{F}^1_{m,s}f(t) := \int_0^1 \frac{B_{m,s}f(x) - B_{m,s}f(t) - \left(B_{m,s}f\right)'(t)(x-t)}{(x-t)^2}dx. \tag{6.58}$$

Recalling (3.20), the above integrals can be written as follows

$$\mathcal{F}_{m,s}f(t) = \sum_{j=0}^m f\left(\frac{j}{m}\right) D^{(s)}_{m,j}(t), \qquad D^{(s)}_{m,j}(t) := \int_0^1 \frac{p^{(s)}_{m,j}(x) - p^{(s)}_{m,j}(t)}{x-t}dx \tag{6.59}$$

$$\mathcal{F}^1_{m,s}f(t) = \sum_{j=0}^m f\left(\frac{j}{m}\right) \overline{D}^{(s)}_{m,j}(t), \qquad \overline{D}^{(s)}_{m,j}(t) := \frac{d}{dt}D^{(s)}_{m,j}(t). \tag{6.60}$$

Thus, assuming at first instance that the values $f(t), f'(t)$ are available, we get the following quadrature rules for the Hilbert and Hadamard transforms at the point $t \in (0,1)$

$$\begin{aligned}\mathcal{H}f(t) &= \mathcal{F}_{m,s}f(t) + f(t)\log\left(\frac{1-t}{t}\right) + \Phi_{m,s}f(t) \\ &=: H_{m,s}f(t) + \Phi_{m,s}f(t),\end{aligned} \tag{6.61}$$

$$\begin{aligned}\mathcal{H}^1 f(t) &= \mathcal{F}^1_{m,s}f(t) + f'(t)\log\left(\frac{1-t}{t}\right) - \frac{f(t)}{t(1-t)} + \Phi^1_{m,s}f(t) \\ &=: H^1_{m,s}f(t) + \Phi^1_{m,s}f(t).\end{aligned} \tag{6.62}$$

In the case that the values of $f(t)$ and $f'(t)$ are unknown, we approximate them by $B_{m,s}f(t)$ and its derivative, respectively, obtaining

$$\begin{aligned}\mathcal{H}f(t) &= \mathcal{F}_{m,s}f(t) + \log\left(\frac{1-t}{t}\right)B_{m,s}f(t) + \mathcal{E}_{m,s}f(t) \\ &=: \mathcal{H}_{m,s}f(t) + \mathcal{E}_{m,s}f(t)\end{aligned} \tag{6.63}$$

and

$$\begin{aligned}\mathcal{H}^1 f(t) &= \mathcal{F}^1_{m,s}f(t) + \log\left(\frac{1-t}{t}\right)\left(B_{m,s}f\right)'(t) - \frac{1}{t(1-t)}B_{m,s}f(t) + \mathcal{E}^1_{m,s}f(t) \\ &=: \mathcal{H}^1_{m,s}f(t) + \mathcal{E}^1_{m,s}f(t),\end{aligned} \tag{6.64}$$

where $\mathcal{E}_{m,s}f(t), \mathcal{E}^1_{m,s}f(t)$ denote the errors and the remaining part at the right–hand side of (6.63), (6.64) reduces to a quadrature sum based on the same samples of f.

Further numerical details on the computation of the previous quadrature rules will be given in Section 7. In the sequel, we are going to discuss the convergence rate of such formulas.

The following result has been stated in [18] for the interval $[-1,1]$.

Theorem 6.9. *Let $0 < t < 1$. For all $f \in C^0([0,1])$ satisfying (6.52), we get*

$$|\mathcal{E}_{m,s}f(t)| \leq \mathcal{C}\log\left(\frac{e}{t(1-t)}\right)\left[\log m\,\|f - B_{m,s}f\| + \int_0^{\frac{1}{m}} \frac{\omega^r_\varphi(f,u)}{u}du\right], \qquad \mathcal{C} \neq \mathcal{C}(m,f,t).$$

Moreover, in the case that (6.53) holds, we have

$$\varphi^2(t)|\mathcal{E}^1_{m,s}f(t)| \leq \mathcal{C}\left[\|f - B_{m,s}f\| + \log m\|(f - B_{m,s}f)'\| + \int_0^{\frac{1}{m}} \frac{\omega^r_\varphi(f',u)}{u}du\right] \tag{6.65}$$

with $r < m$ and $\mathcal{C} \neq \mathcal{C}(m, f, t)$.
Finally, the errors $\Phi_{m,s}f(t)$ and $\Phi^1_{m,s}f(t)$ satisfy the same estimates of $\mathcal{E}_{m,s}f(t)$ and $\mathcal{E}^1_{m,s}f(t)$, and in addition we have

$$|\Phi_{m,s}f(t)| \leq \mathcal{C}\|(f - B_{m,s}f)'\|, \qquad \forall f \in C^1([0,1]), \qquad \mathcal{C} \neq \mathcal{C}(m, f, t), \tag{6.66}$$

$$|\Phi^1_{m,s}f(t)| \leq \mathcal{C}\|(f - B_{m,s}f)''\|, \qquad \forall f \in C^2([0,1]), \qquad \mathcal{C} \neq \mathcal{C}(m, f, t). \tag{6.67}$$

From this theorem, several error estimates of the quadrature errors can be obtained by the error estimates of GBs polynomials recalled in Section 3 and based on several moduli of smoothness of f and f'. For instance, as regards the approximation of the Hilbert transform, by (3.29) and Theorem 3.4, we get that

$$\log^{-1}\left(\frac{e}{t(1-t)}\right) |\, \mathcal{E}_{m,s}f(t)| \leq \mathcal{C} \log m \left[\omega_\varphi^{2s}\left(f, \frac{1}{\sqrt{m}}\right) + \frac{\|f\|}{m^s}\right] + \mathcal{C}\int_0^{\frac{1}{m}} \frac{\omega_\varphi^r(f,u)}{u} du$$

holds with $r < m$ and $\mathcal{C} \neq \mathcal{C}(m, f, t)$, for any f satisfying (6.52). Moreover,

$$|\Phi_{m,s}f(t)| \leq \mathcal{C}\left[\omega_\varphi^{2s}\left(f', \frac{1}{\sqrt{m}}\right) + \omega^s\left(f', \frac{1}{m}\right) + \omega\left(f', \frac{1}{m^s}\right)\right], \qquad \forall f \in C^1([0,1]).$$

For brevity, we omit the details and only state the following corollary to Theorem 6.9, which easily follows by using (2.10) and (2.17) in the estimates of Theorem 3.3 and Theorem 3.4.

Corollary 6.1. *Let $0 < t < 1$. For any given $s \in \mathbb{N}$ and sufficiently large $m \in \mathbb{N}$, as regards the quadrature errors for the Hilbert transform, we have*

$$|\mathcal{E}_{m,s}f(t)| \leq \mathcal{C}\log\left(\frac{e}{t(1-t)}\right)\frac{\|f\|_{W_r}}{\sqrt{m^r}}\log m, \qquad \forall f \in W_r, \quad r \leq 2s,$$
$$|\mathcal{E}_{m,s}f(t)| \leq \mathcal{C}\log\left(\frac{e}{t(1-t)}\right)\frac{\|f\|_{Z_\lambda}}{\sqrt{m^\lambda}}\log m, \qquad \forall f \in Z_\lambda, \quad 0 < \lambda < 2s,$$

with $\mathcal{C} \neq \mathcal{C}(m, f, t)$. The same estimate holds for $|\Phi_{m,s}f(t)|$, which also satisfies

$$|\Phi_{m,s}f(t)| \leq \frac{\mathcal{C}}{\sqrt{m^k}}, \qquad \forall f \in C^{k+1}([0,1]),\ 1 \leq k \leq 2s, \qquad \mathcal{C} \neq \mathcal{C}(m, t).$$

Moreover, concerning the approximation of the Hadamard transform, we have

$$\varphi^2(t)|\mathcal{E}^1_{m,s}f(t)| \leq \mathcal{C}\frac{\log m}{\sqrt{m^k}}, \qquad \forall f \in C^{k+1}([0,1]),\ 1 \leq k \leq 2s, \qquad \mathcal{C} \neq \mathcal{C}(m, t)$$

and the error $\Phi^1_{m,s}f(t)$ satisfies the same estimate and

$$|\Phi^1_{m,s}f(t)| \leq \frac{\mathcal{C}}{\sqrt{m^k}}, \qquad \forall f \in C^{k+2}([0,1]),\ 1 \leq k \leq 2s, \qquad \mathcal{C} \neq \mathcal{C}(m, t).$$

We conclude by proposing the following test.

Example 6.3. *Consider $\mathcal{H}f(t)$, $\mathcal{H}^1 f(t)$ with $f(x) = \frac{e^x}{1+x^2}$. By the previous algorithms, the reconstruction of $\mathcal{H}f(t)$ and $\mathcal{H}^1 f(t)$, has been performed in high precision (more than 10 exact digits) by using only 200 samples of f for both of them. In the graph are shown the approximation $H_{m,s}f(t)$ and $H^1_{m,s}f(t)$ of the functions $\mathcal{H}f(t)$ and $\mathcal{H}^1 f(t)$, respectively.*

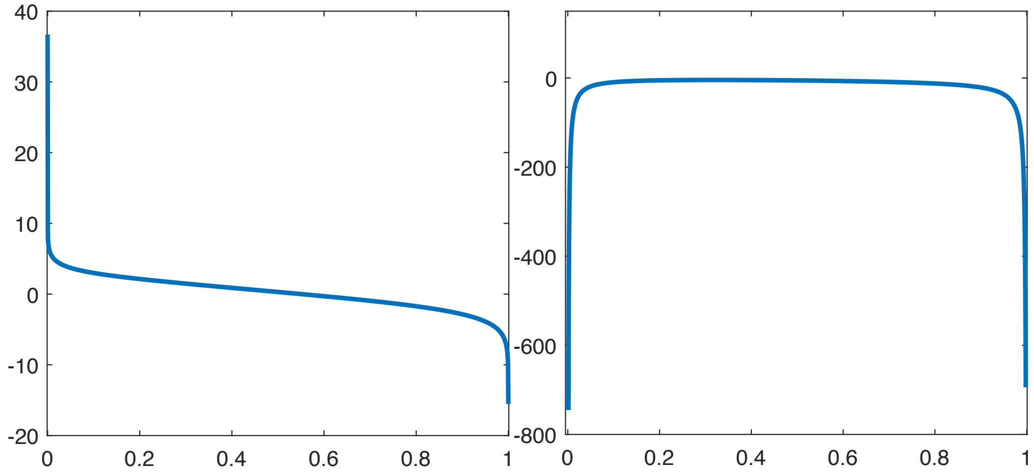

FIGURE 4. $H_{m,s}f$ and $H^1_{m,s}f$, $m = 200, s = 20$

7. BIVARIATE GBS POLYNOMIALS AND APPLICATIONS

7.1. **The bivariate generalized Bernstein operator.** Let $\mathrm{S} = [0,1]^2$ and $C(\mathrm{S})$ indicate the space of continuous functions f in two variables, equipped with the uniform norm on the square S

$$\|f\|_{\mathrm{S}} = \max_{(x,y)\in \mathrm{S}} |f(x,y)|.$$

Denote by $\mathbf{m} = (m_1, m_2)$, $m_1, m_2 \in \mathbb{N}$ and by $\mathbf{s} = (s_1, s_2)$, $s_1, s_2 \in \mathbb{N}$. From now on, let $\mathbf{P}_{ij} = \left(\frac{i}{m_1}, \frac{j}{m_2}\right)$, $(i,j) \in N_0^{m_1} \times N_0^{m_2}$ and $\mathbb{P}_{m_1,m_2}$ denote the space of the bivariate algebraic polynomials of degree m_1 w.r.t. the variable x and m_2 w.r.t. the variable y.

With these notation and by $B_{m,s}$ given in (3.20), we can introduce the bivariate Generalized Bernstein operator $\mathbf{B}_{\mathbf{m},\mathbf{s}}$ on S as the tensor product

$$\mathbf{B}_{\mathbf{m},\mathbf{s}} := B_{m_1,s_1} \otimes B_{m_2,s_2} : C(\mathrm{S}) \to \mathbb{P}_{m_1,m_2}, \quad \mathbf{m} = (m_1, m_2), \quad \mathbf{s} = (s_1, s_2).$$

This operator with $m_1 = m_2$ and $s_1 = s_2$ was introduced in [33]. Here, we are proposing a more general definition in order to get a more flexible approximation tool, according to the different smoothness properties of the approximating bivariate function, with respect to the single variables. In other words, we want to make the most of the advantages of the definition of $\mathbf{B}_{\mathbf{m},\mathbf{s}}$ as a tensor product.

By definition and taking into account (3.20), the polynomial $\mathbf{B}_{\mathbf{m},\mathbf{s}}f(x,y)$ can be expressed as

$$\mathbf{B}_{\mathbf{m},\mathbf{s}}f(x,y) = \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} p^{(s_1)}_{m_1,i}(x) p^{(s_2)}_{m_2,j}(y) f(\mathbf{P}_{ij}) \tag{7.68}$$

with $\left\{p^{(s)}_{m,k}\right\}_{k\in N_0^m}$ defined in (3.21).

Let $f \in C(\mathrm{S})$. For $\mathbf{s} = (1,1)$, $\mathbf{B}_{\mathbf{m},\mathbf{s}} f$ reduces to the bivariate Bernstein polynomial (see for instance [10] for the case $m_1 \equiv m_2$),

$$\mathbf{B}_{\mathbf{m}} f(x,y) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} p_{m_1,i}(x) p_{m_2,j}(y) f(\mathbf{P}_{ij}).$$

Using the vector representation for the Bernstein basis, the bivariate Bernstein polynomial can be rewritten as

$$\mathbf{B}_{\mathbf{m}} f(x,y) = \mathbf{p}_{m_1}(x)^T \mathbf{F}_{\mathbf{m}} \mathbf{p}_{m_2}(y), \tag{7.69}$$

where the entries of the matrix $\mathbf{F}_{\mathbf{m}} \in \mathbb{R}^{(m_1+1)\times(m_2+1)}$ are

$$(\mathbf{F}_{\mathbf{m}})_{i,j} = f(\mathbf{P}_{ij}), \quad (i,j) \in N_0^{m_1} \times N_0^{m_2}. \tag{7.70}$$

Extending some properties holding true in the univariate case, it is not hard to prove that $\mathbf{B}_{\mathbf{m},\mathbf{s}} f$ interpolates f at the corners of the square S and preserves bivariate polynomials of degree 1 in each variable separately.

By (3.26), $\mathbf{B}_{\mathbf{m},\mathbf{s}} f$ can be also represented in the Bernstein basis. Indeed, using the definiton of $C_{m,s}$ given in (3.24), it results

$$\mathbf{B}_{\mathbf{m},\mathbf{s}} f(x,y) = \mathbf{p}_{m_1}(x)^T C_{m_1,s_1} \mathbf{F}_{\mathbf{m}} C_{m_2,s_2}^T \mathbf{p}_{m_2}(y). \tag{7.71}$$

Setting

$$G_{\mathbf{m},\mathbf{s}} = C_{m_1,s_1} \mathbf{F}_{\mathbf{m}} C_{m_2,s_2}^T, \tag{7.72}$$

by (7.71), it follows

$$\mathbf{B}_{\mathbf{m},\mathbf{s}} f(x,y) = \mathbf{p}_{m_1}(x)^T G_{\mathbf{m},\mathbf{s}} \mathbf{p}_{m_2}(y), \tag{7.73}$$

i.e., according to (7.69) the polynomial $\mathbf{B}_{\mathbf{m},\mathbf{s}} f$ can be seen as the bivariate Bernstein polynomial of a continuous function g such that $g(\mathbf{P}_{ij}) = (G_{\mathbf{m},\mathbf{s}})_{i,j}$, $(i,j) \in N_0^{m_1} \times N_0^{m_2}$.

From now on, let f_x and f_y denote the function $f(x,y)$ when considered as a function of the only variable y and x, respectively.

We give a convergence result of the proposed approximation which is a generalization of that given in [33].

Theorem 7.10. *Let $f \in C(\mathrm{S})$. For any fixed $\mathbf{s} = (s_1, s_2)$, it results*

$$\|\mathbf{B}_{\mathbf{m},\mathbf{s}} f\|_{\mathrm{S}} \le 2^{s_1+s_2} \|f\|_{\mathrm{S}}, \quad \forall \mathbf{m}. \tag{7.74}$$

Moreover, for m_1 and m_2 sufficiently large (say $m_1, m_2 > m_0$ fixed)

$$\|f - \mathbf{B}_{\mathbf{m},\mathbf{s}} f\|_{\mathrm{S}} \le \mathcal{C} \left\{ \sup_{y\in[0,1]} \left[\omega_\varphi^{2s_1}\left(f_y, \frac{1}{\sqrt{m_1}}\right) + \frac{\|f_y\|}{m_1^{s_1}} \right] + \sup_{x\in[0,1]} \left[\omega_\varphi^{2s_2}\left(f_x, \frac{1}{\sqrt{m_2}}\right) + \frac{\|f_x\|}{m_2^{s_2}} \right] \right\}, \tag{7.75}$$

where $\mathcal{C}$ is a positive constant depending on s_1, s_2 and independent of f, m_1 and m_2.

Proof. The proof can be led repeating word by word that of Theorem 3.1 in [33]. Indeed, denoting by $\|A\|_\infty$, the infinity norm of a matrix A, then it results $\|C_{m_h,s_h}\|_\infty \le 2^{s_h} - 1$ and hence (7.74) immediately follows by (7.72) and (7.73). Moreover, (7.75) can be deduced with the same arguments in [33], taking into account estimate (3.29). □

Remark 7.3. *From the previous estimate, it is possible to deduce the rate of convergence of the approximation according to the smoothness properties of $f \in C(\mathrm{S})$. For instance, if $f_y \in W_{r_1}$ and $f_x \in W_{r_2}$, uniformly w.r.t. y and x respectively, then choosing $s_1 \geq \frac{r_1}{2}$ and $s_2 \geq \frac{r_2}{2}$, by (2.10), we immediately deduce that*

$$\|f - \mathbf{B}_{\mathbf{m},\mathbf{s}} f\|_{\mathrm{S}} \leq \mathcal{C} \left\{ \sup_{y\in[0,1]} \left[\frac{\|f_y\|_{W_{r_1}}}{\sqrt{m_1^{r_1}}} + \frac{\|f_y\|}{m_1^{s_1}} \right] + \sup_{x\in[0,1]} \left[\frac{\|f_x\|_{W_{r_2}}}{\sqrt{m_2^{r_2}}} + \frac{\|f_x\|}{m_2^{s_2}} \right] \right\}. \tag{7.76}$$

Estimate (7.76) suggests that when the smoothness of the function f is different w.r.t the two variables, then it is possible to obtain a significant reduction in the computational cost, both in terms of function samples and in the construction of matrices C_{m_h,s_h}, $h = 1, 2$, with respect to the case $m_1 = m_2$.

7.2. **A cubature rule.** The above introduced $\mathbf{B}_{\mathbf{m},\mathbf{s}}$ operator can be usefully employed in the numerical cubature. Indeed for integrals of the type $\int_{\mathrm{S}} f(x,y)dxdy$, it is possible to deduce the following cubature rule

$$\begin{aligned} \int_{\mathrm{S}} f(x,y)dxdy &= \int_{\mathrm{S}} \mathbf{B}_{\mathbf{m},\mathbf{s}} f(x,y)dxdy + \mathbf{R}_{\mathbf{m},\mathbf{s}} f \\ &=: \mathbf{\Sigma}_{\mathbf{m},\mathbf{s}} f + \mathbf{R}_{\mathbf{m},\mathbf{s}} f. \end{aligned} \tag{7.77}$$

By (7.73) and taking into account that

$$\int_0^1 p_{m,k}(t)dt = \frac{1}{m+1}, \qquad k \in N_0^m,$$

it is not hard to prove that

$$\mathbf{\Sigma}_{\mathbf{m},\mathbf{s}} f = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \mathbf{T}_{i,j}^{(\mathbf{s})} f(t_i, t_j), \tag{7.78}$$

where for any $(i,j) \in N_0^{m_1} \times N_0^{m_2}$,

$$\mathbf{T}_{i,j}^{(\mathbf{s})} = \frac{1}{(m_1+1)(m_2+1)} \left(\sum_{r=0}^{m_1} (C_{m_1,s_1})_{r,i} \right) \left(\sum_{k=0}^{m_2} (C_{m_2,s_2})_{k,j} \right).$$

The previous rule for $m_1 = m_2$ and $s_1 = s_2$ was introduced in [33]. The stability and convergence of the cubature rule are stated in the next theorem which can be obtained as a direct application of (7.75).

Theorem 7.11. *With the notation used in (7.77)-(7.78) and for any $f \in C(\mathrm{S})$, the cubature formula is convergent , holding*

$$\begin{aligned} |\mathbf{R}_{\mathbf{m},\mathbf{s}} f| \leq \mathcal{C} &\left\{ \sup_{y\in[0,1]} \left[\omega_\varphi^{2s_1} \left(f_y, \frac{1}{\sqrt{m_1}} \right) + \frac{\|f_y\|}{m_1^{s_1}} \right] \right. \\ &\left. + \sup_{x\in[0,1]} \left[\omega_\varphi^{2s_2} \left(f_x, \frac{1}{\sqrt{m_2}} \right) + \frac{\|f_x\|}{m_2^{s_2}} \right] \right\}, \quad \mathcal{C} \neq \mathcal{C}(\mathbf{m}, f), \quad \mathcal{C} = \mathcal{C}(\mathbf{s}) \end{aligned} \tag{7.79}$$

and numerically stable, i.e.,

$$\sup_{m_1} \sup_{m_2} \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} |\mathbf{T}_{i,j}^{(\mathbf{s})}| < \infty. \tag{7.80}$$

Remark 7.4. *As for the generalized Bernstein approximation, also in this case from estimate (7.79), it is possible to deduce suitable convergence order for specific classes of functions. For instance for functions that are in Sobolev spaces w.r.t the single variable and estimate like (7.76) can be obtained also for $\mathbf{R}_{\mathbf{m},\mathbf{s}}f$. In addition taking into account the different estimates w.r.t the single variables of f, it is possible to construct, with the same attained error, a cubature rule with a computational saving respect both the computation of the function samples and the construction of the weights of the rule.*

Now, we want to give just two examples of the proposed cubature formula. In both cases, the value of the integral is computed in machine precision by means of a Gaussian cubature formula obtained as a tensor product of two Gauss-Legendre rules.

Example 7.4. *Consider the double integral*

$$\int_0^1 \int_0^1 \cos(xy)e^{|y-0.2|^{\frac{17}{3}}}\, dx\, dy \sim 9.7795428911044441e-01.$$

The integrating function is in W_{r_1} for any integer r_1 with respect to the variable x, while is in W_5 with respect to the variable y. Therefore from (7.79), we get that the order of convergence of the formula is $\mathcal{O}(m^{-5/2})$. Nevertheless, it is clear that m_1 and s_1 could be taken reasonably small while m_2 has to increase in order to get an high number of correct digits. Therefore we fixed $m_1 = 64$, $s_1 = 1024$, while for any value of m_2 we consider different values for s_2. The results are shown in the following Table 5

m_2	$s_2 = 4$	$s_2 = 8$	$s_2 = 16$	$s_2 = 32$	$s_2 = 64$	$s_2 = 128$
64	$6.31e-07$	$7.04e-08$	$1.86e-08$	$6.69e-09$	$2.86e-09$	$1.38e-09$
128	$6.04e-08$	$2.79e-09$	$4.18e-10$	$9.29e-11$	$2.57e-11$	$8.39e-12$
256	$6.22e-09$	$1.05e-10$	$8.00e-12$	$1.01e-12$	$1.63e-13$	$3.13e-14$
512	$6.89e-10$	$4.01e-12$	$1.38e-13$	$8.10e-15$	$7.77e-16$	$1.32e-15$
1024	$8.05e-11$	$1.63e-13$	$7.77e-16$	$9.99e-16$	$1.33e-15$	$1.44e-15$

TABLE 5. Absolute errors for Example 7.4

As the table shows, the numerical behavior in surely better than the theoretical estimate predicts. Moreover, it is possible to show that increasing m_1 does not lead to any relevant improvement in the exact digits in the results.

Example 7.5. *Consider the double integral*

$$\int_0^1 \int_0^1 e^{(x+y)^2}|\sin(y)-0.5|^{\frac{7}{2}}\, dx\, dy \sim 4.8794503105779e-02.$$

The integrating function is in W_{r_1} for any integer r_1 with respect to the variable x while is in W_3 with respect to the variable y. Therefore from (7.79), we get that the order of convergence of the formula is $\mathcal{O}(m^{-3/2})$. As in the previous example, we fix m_1 and s_1, while m_2 is taken increasing in order to get an high number of correct digits. Therefore we fixed $m_1 = 256$, $s_1 = 4096$, while for any value of m_2 we consider different values for s_2. The results are shown in the following Table 6.

Also in this case, if we take higher values for m_1, no benefits can be found on the obtained results. Moreover also here it is evident that for a fixed value of m_2, the choice of higher values for s_2 leads to gain more exact digits, till some threshold $\widetilde{s}_2$ depending on m_2 (see for instance the results for $m_2 = 256$ and $s_2 = 32, 64, 128$ or $m_2 = 512$ and $s_2 = 16, 32, 64, 128$).

m_2	$s_2 = 4$	$s_2 = 8$	$s_2 = 16$	$s_2 = 32$	$s_2 = 64$	$s_2 = 128$
64	$1.14e-06$	$6.53e-08$	$8.69e-09$	$1.48e-09$	$1.90e-10$	$6.92e-11$
128	$1.18e-07$	$2.57e-09$	$1.69e-10$	$2.19e-11$	$8.05e-12$	$6.62e-12$
256	$1.31e-08$	$1.05e-10$	$3.08e-12$	$4.11e-13$	$2.85e-13$	$2.79e-13$
512	$1.53e-09$	$4.72e-12$	$5.92e-14$	$1.22e-14$	$1.11e-14$	$1.11e-14$
1024	$1.84e-10$	$2.35e-13$	$1.41e-15$	$5.55e-16$	$5.55e-16$	$5.55e-16$

TABLE 6. Absolute errors for Example 7.5

7.3. **Fredholm integral equations on the square.** In this section, we want to show a possible application of the cubature rule introduced above to the numerical approximation of the solution of a Fredholm integral equation (FIE) defined on S . For the sake of simplicity, we will limit ourselves to the case $m_1 = m_2$ and $s_1 = s_2$, but all the results, mutatis mutandis can be obtained in the more general case.

First of all, we introduce the following bivariate Sobolev–type space

$$\mathbf{W}_r = \left\{ f \in C(\mathrm{S}) : \mathcal{M}_r f := \max\left\{ \max_{(x,y)\in \mathrm{S}} |f_y^{(r)}(x)\varphi^r(x)|, \max_{(x,y)\in \mathrm{S}} |f_x^{(r)}(y)\varphi^r(y)| \right\} < \infty \right\}, \tag{7.81}$$

where the superscript (r) denotes the r-th derivative of the one-dimensional function f_y or f_x and the function $\varphi(z) = \sqrt{z(1-z)}$. $\mathbf{W}_r$ will be equipped with the norm $\|f\|_{\mathbf{W}_r} = \|f\|_{\mathrm{S}} + \mathcal{M}_r f$.

Let us consider the following bivariate FIE on the square S

$$f(x,y) - \mu \int_{\mathrm{S}} f(z,t)k(x,y,z,t)dz\, dt = g(x,y), \ (x,y) \in \mathrm{S}, \tag{7.82}$$

where $\mu \in \mathbb{R}$, k defined on $\mathrm{S}\times\mathrm{S}$ and g defined on S are given functions, while f is the unknown function. Denoting by

$$\mathbf{K}f(x,y) = \mu \int_{\mathrm{S}} k(x,y,z,t)f(z,t)\, dz\, dt$$

(7.82) can be rewritten in operatorial form as

$$(\mathbf{I} - \mathbf{K})f = g, \tag{7.83}$$

where $\mathbf{I}$ is the identity operator on $C(\mathrm{S})$. Here and in the sequel, we will denote $k_{(z,t)}$ for meaning that the function of four variables k is considered as a function of the only pair (x,y).

Using standard arguments, it is possible to prove that if $k(x,y,z,t)$ is continuous, then $\mathbf{K} : C(\mathrm{S}) \to C(\mathrm{S})$ is compact and consequently the Fredholm Alternative holds true for (7.83) in $C(\mathrm{S})$ (see for instance [6]). Moreover if for some $r \in \mathbb{N}$,

$$\sup_{(z,t)\in\mathrm{S}} \|k_{(z,t)}\|_{W_r} < +\infty, \tag{7.84}$$

then $\mathbf{K}f \in W_r$ for any $f \in C(\mathrm{S})$.

Starting with the cubature rule (7.78) written with $m_1 = m_2 =: m$ and $s_1 = s_2 =: s$, we can define the following discrete operator

$$\mathbf{K}_m f(x,y) = \mu \sum_{i=0}^{m}\sum_{j=0}^{m} \mathbf{T}_{i,j}^{(\mathbf{s})} k(x,y,t_i,t_j) f(t_i,t_j)$$

and consider the operator equation

$$(\mathbf{I} - \mathbf{K}_m)f_m = g, \tag{7.85}$$

where f_m is unknown. Collocating on the pairs (t_h, t_ℓ), $(h, \ell) \in N_0^m \times N_0^m$, the quantities $\beta_{ij} = f(\mathbf{P}_{i,j})$, $(i, j) \in N_0^m \times N_0^m$, come out to be the unknowns of the linear system

$$\beta_{h\ell} - \mu \sum_{i=0}^{m} \sum_{j=0}^{m} \mathbf{T}_{i,j}^{(s)} k(t_h, t_\ell, t_i, t_j)\beta_{ij} = g(t_h, t_\ell), \quad (h, \ell) \in N_0^m \times N_0^m. \tag{7.86}$$

The matrix solution $(\beta_{ij}^*)_{i,j=0,1...,m}$ of this system, if it exists, allows us to construct the Nyström interpolant in two variables

$$f_m(x, y) = \mu \sum_{i=0}^{m} \sum_{j=0}^{m} \mathbf{T}_{i,j}^{(s)} k(x, y, t_i, t_j)\beta_{ij}^* + g(x, y) \tag{7.87}$$

which will approximate the unknown f. Now, denote by $\mathbf{\Gamma}_{m,s}$ the coefficient matrix of system (7.86), which is a $(m + 1)$ block matrix, the entries of which are matrices of order $m + 1$.

Denoting by $\mathrm{cond}(\mathbf{\Gamma}_{m,s})$ the condition number in infinity norm of $\mathbf{\Gamma}_{m,s}$, the following theorem holds true (see [33]).

Theorem 7.12. *Assume that k is continuous w.r.t. its four variables and that $Ker\{\mathbf{I} - \mathbf{K}\} = \{0\}$ in $C(\mathrm{S})$. Denote by f the unique solution of (7.85) in $C(\mathrm{S})$ for a given $g \in C(\mathrm{S})$. If in addition, for some $r \in \mathbb{N}$, k satisfies (7.84), $\mathbf{g} \in \mathbf{W}_r$, and*

$$\sup_{(x,y)\in \mathrm{S}} \|k_{(x,y)}\|_{\mathbf{W}_r} < +\infty, \tag{7.88}$$

then, for m sufficiently large, the system (7.86) is uniquely solvable and well-conditioned too, since

$$\mathit{cond}(\mathbf{\Gamma}_{m,s}) \le \mathcal{C}, \qquad \mathcal{C} \ne \mathcal{C}(m), \quad \mathcal{C} = \mathcal{C}(s).$$

Moreover, for any $s \ge \frac{r}{2}$, there results

$$\|f - f_m\|_{\mathrm{S}} \le \mathcal{C} \frac{\|f\|_{\mathbf{W}_r}}{\sqrt{m^r}}, \tag{7.89}$$

where $\mathcal{C} \ne \mathcal{C}(m, f)$ and $\mathcal{C} = \mathcal{C}(s)$.

Remark 7.5. *Several computational details and numerical tests about the proposed Nyström method can be found in [33]. In particular, in that paper the case in which the function kernel $k(x, y, z, t)$ shows some symmetry was discussed. Indeed, the symmetry properties of the kernel are inherited by the matrix $\mathbf{\Gamma}_{m,s}$. And this fact could be useful for reducing the computational effort in solving the linear system (7.86).*

8. Computational details

8.1. **On the computation of $B_{m,s}f$.** In order to construct $B_{m,s}f$ as defined in (3.27), we give some computational details about the matrix $C_{m,s}$ [33]. Such matrix is defined by the matrix A in (3.25) which can be constructed by rows by making use of the triangular scheme in (2.4). In this way, for each row, m^2 long operations are required. On the other hand, since A is centrosymmetric, i.e., $A = JAJ$, where J is the counter-identity matrix of order $m + 1$ (i.e., $J_{i,j} = \delta_{i,m-j}$, $\forall i, j \in N_0^m$, being $\delta_{h,k}$ the Kronecker delta), it will be enough to compute only the first $\left(\frac{m+1}{2}\right)$ rows for odd m, or $\left(\frac{m+2}{2}\right)$ rows, when m is even. Therefore, the construction of A requires about $\frac{m^3}{2}$ long operations. Furthermore, since the product of two centrosymmetric matrices can be performed in almost $\frac{m^3}{4}$ long operations [1], the matrix $C_{m,s}$ in (3.24) can be constructed in almost $(s - 2)m^3/4$ long operations, instead of $(s - 2)m^3$ ones, i.e., with a saving of about the 75%. A more significant reduction is achieved when the parameter $s = 2^p,\ p \in N$. Indeed, by using [32, (14)]

$$C_{m,2^p} = C_{m,2^{p-1}} + (I - A)^{2^{p-1}} C_{m,2^{p-1}}, \tag{8.90}$$

that allow to the following relation among the polynomials

$$B_{m,2^p} f(x) = 2B_{m,2^{p-1}} f(x) - B^2_{m,2^{p-1}} f(x), \tag{8.91}$$

the matrix $C_{m,s}$ can be determined by $2(\log_2 s - 1)$ products of centrosymmetric matrices and therefore requiring almost $\frac{m^3}{2}(\log_2 s - 1)$ long operations. For instance, for $s = 256$, if we use definition (3.24), then we have 255 products of centrosymmetric matrices that require about $255\frac{m^3}{4} \sim 63.7m^3$ long operations. On the contrary, if we use (8.90), then approximatively only $3.5m^3$ long operations are needed.

8.2. **Computation of the derivatives of $B_{m,s}f$.** As regards the first derivative of the Bernstein polynomials $B_{m,s}f$, by (3.27), we obtain the following useful representation

$$\left(B_{m,s} f\right)'(x) = \mathbf{p}^1_m(x) C_{m,s} \mathbf{f}_m, \tag{8.92}$$

where $\mathbf{f}_m$ was defined in (3.28) and we set

$$\mathbf{p}^1_m(x) := [p'_{m,0}(x), \ldots, p'_{m,m}(x)],$$

where

$$p'_{m,k}(x) = m\left(p_{m-1,k-1}(x) - p_{m-1,k}(x)\right), \qquad k \in \mathbb{N}^m_0, \tag{8.93}$$

with the usual convention $p_{m,j}(x) = 0$ if $j \notin \mathbb{N}^m_0$.

8.3. **Coefficients of the quadrature rules for Hilbert and Hadamard transforms.** The coefficients of the rule (6.59) take the following expression

$$D^{(s)}_{m,j}(t) = \sum_{i=0}^{m} (C_{m,s})_{i,j} \int_0^1 \frac{p_{m,i}(x) - p_{m,i}(t)}{x - t} dx =: \sum_{i=0}^{m} (C_{m,s})_{i,j} q_{m,i}(t), \tag{8.94}$$

where the polynomials $q_{m,i}(t)$ can be computed via recurrence relation as stated in the following proposition.

Proposition 8.1. *For all $m \in \mathbb{N}$, $m > 1$, the polynomials $q_{m,k}(t)$, with $k \in \mathbb{N}^m_0$, satisfy the following recurrence relation*

$$\begin{aligned}
q_{0,0}(t) &= 0, \quad q_{1,0}(t) = -1, \quad q_{1,1}(t) = 1, \\
q_{m,0}(t) &= (1-t) q_{m-1,0}(t) - \frac{1}{m}, \\
q_{m,k}(t) &= (1-t) q_{m-1,k}(t) + t q_{m-1,k-1}(t), \quad 1 \le k \le m-1, \\
q_{m,m}(t) &= t q_{m-1,m-1}(t) + \frac{1}{m}.
\end{aligned}$$

Proof. For $1 \le k \le m-1$, by using recurrence relation (2.4) and taking into account that $\int_0^1 p_{m,k}(x)dx = \frac{1}{m+1}$ for all $k \in \mathbb{N}_0^m$, we get

$$\begin{aligned}
q_{m,k}(t) &= \int_0^1 \frac{(1-x)p_{m-1,k}(x) - (1-t)p_{m-1,k}(t)}{x-t}dx \\
&+ \int_0^1 \frac{xp_{m-1,k-1}(x) - tp_{m-1,k-1}(t)}{x-t}dx \\
&= q_{m-1,k}(t) - \int_0^1 \frac{xp_{m-1,k}(x) - tp_{m-1,k}(t)}{x-t}dx \\
&+ \int_0^1 \frac{xp_{m-1,k-1}(x) - tp_{m-1,k-1}(t)}{x-t}dx \\
&= q_{m-1,k}(t) - \frac{1}{m} - tq_{m-1,k}(t) + \frac{1}{m} + tq_{m-1,k-1}(t) \\
&= (1-t)q_{m-1,k}(t) + tq_{m-1,k-1}(t).
\end{aligned}$$

For $k = 0$,

$$\begin{aligned}
q_{m,0}(t) &= \int_0^1 \frac{(1-x)p_{m-1,0}(x) - (1-t)p_{m-1,0}(t)}{x-t}dx = q_{m-1,0}(t) - \frac{1}{m} - tq_{m-1,0}(t) \\
&= (1-t)q_{m-1,0}(t) - \frac{1}{m}.
\end{aligned}$$

For $k = m$, we proceed in the same way. □

Setting

(8.95) $$\mathbf{q}_m(t) = [q_{m,0}(t), q_{m,1}(t), \dots, q_{m,m}(t)],$$

the quadrature rule (6.59) can be rewritten as

(8.96) $$\mathcal{F}_{m,s}f(t) = \mathbf{q}_m(t)C_{m,s}\mathbf{f}_m.$$

Moreover, the quadrature rule $\mathcal{H}_{m,s}$ in (6.63) tales the form

(8.97) $$\mathcal{H}_{m,s}f(t) = \left[\mathbf{q}_m(t) + \log\left(\frac{1-t}{t}\right)\mathbf{p}_m(t)\right]C_{m,s}\mathbf{f}_m.$$

About the coefficients of the formula (6.60), by (6.59) and (8.94), we get $\forall j \in \mathbb{N}_0^m$,

(8.98) $$\overline{D}_{m,j}^{(s)}(t) = \sum_{i=0}^{m}(C_{m,s})_{i,j}d_{m,i}(t), \qquad d_{m,i}(t) := q'_{m,i}(t),$$

where the polynomials $d_{m,i}(t), i = 0, \dots, m$ can be computed recursively according to next proposition which easily follows by Proposition 8.1.

Proposition 8.2. *For all $m \in \mathbb{N}$, $m > 1$, the polynomials $d_{m,k}(t), k \in \mathbb{N}_0^m$, satisfy the following recurrence relation*

$$\begin{aligned}
d_{1,0}(t) &= 0, \quad d_{1,1}(t) = 0, \\
d_{m,0}(t) &= (1-t)d_{m-1,0}(t) - q_{m-1,0}(t), \\
d_{m,k}(t) &= (1-t)d_{m-1,k}(t) - q_{m-1,k}(t) + td_{m-1,k-1}(t) + q_{m-1,k-1}(t), \quad 0 < k < m, \\
d_{m,m}(t) &= td_{m-1,m-1}(t) + q_{m-1,m-1}(t).
\end{aligned}$$

Setting

$$\mathbf{d}_m(t) = [d_{m,0}(t), d_{m,1}(t), \ldots, d_{m,m}(t)]\,,$$

the quadrature rule (6.60) takes the following form

$$\mathcal{F}^1_{m,s} f(t) = \mathbf{d}_m(t) C_{m,s} \mathbf{f}_m. \tag{8.99}$$

Finally by (8.99), (3.27) and (8.92), the rule $\mathcal{H}^1_{m,s}$ defined in (6.64) takes the following vectorial form

$$\mathcal{H}^1_{m,s} f(t) = \left[\mathbf{d}_m(t) + \log\left(\frac{1-t}{t}\right) \mathbf{p}^1_m(t) - \frac{2}{t(1-t)}\, \mathbf{p}_m(t)\right] C_{m,s} \mathbf{f}_m. \tag{8.100}$$

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Research Article

Durrmeyer type operators on a simplex

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ABSTRACT. The paper contains the definition and certain approximation properties of a sequence of Durrmeyer type operators on a simplex, which preserve affine functions and make a link between the multidimensional "genuine" Durrmeyer operators and the multidimensional Bernstein operators.

Keywords: Multidimensional linear positive operators, Durrmeyer type operators, Bernstein operators on a simplex, limit operators, estimates with moduli of continuity.

2020 Mathematics Subject Classification: 41A36, 41A25.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and consideration.

1. INTRODUCTION

Durrmeyer operators introduced in [10] and independently by Lupaş [17], were one of the most fecund source of inspiration in approximation by positive linear operators. They were be known especially after the paper by Derriennic [7]. In References, we give only a very partial review of contributions in this field. The extension to Jacobi weight was considered by the author in [20], see also [21], [5]. The limit of Durrmeyer operators with Jacobi weight yields the so named "genuine" Durrmeyer operators considered firstly by Chen [6], Goodman and Sharma [14], see also [19], [26], [11]. The eigen-structure of this operators was studied in [22]. For other modifications of Bernstein-Durrmeyer operators mention [18], [27], [16], [3], [1], [15].

In this paper, we are especially interested in the following modification. In [24], there was constructed a family of operators depending on a parameter ρ, with property that they preserve linear functions, which make a link between the genuine-Durrmeyer operators and the Bernstein operators in the following mode:

$$\mathbb{U}_n^{\rho}(f)(x) = (1-x)^n f(0) + x^n f(1) + \sum_{k=0}^{n} \frac{\int_0^1 f(t) t^{k\rho-1}(1-t)^{(n-k)\rho-1} dt}{B(k\rho,(n-k)\rho)} p_{n,k}(x),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ (0 \le k \le n),\ f \in C[0,1],\ x \in [0,1]$. For $\rho = 1$, these operators coincide with genuine–Durrmeyer operators and on the other hand $\lim_{\rho\to\infty} U_n^{\rho} = B_n$, where B_n are the Bernstein operators. These operators are studied more completely in Gonska and the author in [12]. The eigen-structure of operators U_n^{ρ} was given in Gonska, Raşa, Stănilă [13].

The extension of Durrmeyer operators on a simplex is very natural. Mention that the first Durrmeyer operators on a simplex were considered by Derriennic [8]. The multidimensional Durrmeyer operators with Jacobi weight were considered by Ditzian [9] and the equivalent of

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the "genuine"-Durrmeyer operators on a simplex are given by Waldron [28]. The genuine Durrmeyer operators on a simplex preserves affine functions. The generalization of the Durrmeyer operators on a simplex with regard to a arbitrary measure was made by Berdysheva and Jetter [4], see also [25].

The aim of this paper is to extend operators U_n^ρ on a simplex, obtaining the family of operators $\mathbb{U}_n^\rho$, which preserve affine functions.

We also construct an additional class of operators $\mathbb{M}_n^{\rho,\mathbf{a}}$, depending on a scalar parameter ρ and on a vector parameter $\mathbf{a}$ and we prove that operators $\mathbb{U}_n^\rho$ are the limit of operators $\mathbb{M}_n^{\rho,\mathbf{a}}$ when $\mathbf{a} \to (-1,\ldots,-1)$. This class of operators $\mathbb{M}_n^{\rho,\mathbf{a}}$ allows to obtain more simply certain properties of operators $\mathbb{U}_n^\rho$.

2. Preliminaries and Definitions

Let $p \in \mathbb{N}$. For any vector $\mathbf{x} = (x_1,\ldots,x_p)$, denote $|\mathbf{x}| = x_1+\ldots+x_p$. For any $p \in \mathbb{R}$, consider the standard simplex in $\mathbb{R}^p$.

$$\Delta_p = \{(x_1,\ldots,x_p)|\ x_i \geq 0,\ \ |\mathbf{x}| \leq 1\}.$$

If $g \in C(\Delta_p)$, $p \in \mathbb{N}$, denote by $\int_{\Delta_p} g$ the volume integral of g on Δ_p.

Fix $m \in \mathbb{N}$. Denote $\mathbf{e}_k = (0,\ldots,0,1,0,\ldots,0) \in \mathbb{R}^m$,$(1 \leq k \leq m)$, where the digit 1 appears at the k-th place. Denote also $\mathbf{e}_0 = (0,\ldots,0) \in \mathbb{R}^m$.

Let $\mathbf{x} = (x_1,\ldots,x_m) \in \mathbb{R}^m$. Denote the Euclidean norm of $\mathbf{x}$ by $\|\mathbf{x}\| = \sqrt{x_1^2+\ldots+x_m^2}$, and the L_1 norm of $\mathbf{x}$ by $\|\mathbf{x}\|_1 = |x_1|+\ldots+|x_m|$. If $f \in C(\Delta_m)$, denote $\|f\| = \max_{\mathbf{x}\in\Delta_m} |f(\mathbf{x})|$.

For vectors $\mathbf{v}_0,\ldots,\mathbf{v}_p \in \mathbb{R}^m$, denote

$$\Delta_{[\mathbf{v}_0,\ldots,\mathbf{v}_p]} = \left\{ \sum_{i=0}^p t_i\mathbf{v}_i \middle| t_0,\ldots,t_p \geq 0, t_0+\ldots t_p = 1 \right\},$$

the simplex with vertices $\mathbf{v}_0,\ldots,\mathbf{v}_p$. Numbers $t_0,\ldots,t_p$ are the barycenter coordinates of a point in $\Delta_{[\mathbf{v}_0,\ldots,\mathbf{v}_r]}$. Note that $\Delta_m = \Delta_{[\mathbf{e}_0,\ldots,\mathbf{e}_m]}$.

Fix also a number $n \in \mathbb{N}$. Put

$$\Lambda = \{\mathbf{k} = (k_0,\ldots,k_m)|\mathbf{k} \geq 0,\ |\mathbf{k}| = n\}.$$

For $\mathbf{k} \in \Lambda$, denote $\operatorname{supp} \mathbf{k} := \{i \in \{0,1,\ldots,m\}|k_i > 0\}$. If $\operatorname{supp} \mathbf{k} = \{i_0 < \ldots < i_p\}$, define $D_{\mathbf{k}} = \Delta_{[\mathbf{e}_{i_0},\ldots,\mathbf{e}_{i_p}]}$.

If $g \in C(D_{\mathbf{k}})$, denote by $\int_{D_k} g\,d\sigma$ the integral of g on D_k. In the case when $D_{\mathbf{k}} = \Delta_m$, $\int_{D_k} g\,d\sigma = \int_{\Delta_m} g$. If $g \in C(\Delta_m)$, then the restriction of g to $D_{\mathbf{k}}$ is denoted also by g.

For $\mathbf{k} \in \Lambda$, with $\operatorname{supp} \mathbf{k} = \{i_0 < \ldots < i_p\}$ consider function $\theta_{\mathbf{k}} : \Delta_p \to D_{\mathbf{k}}$ defined by

$$\theta_{\mathbf{k}}(x_{i_1},\ldots,x_{i_p}) = \sum_{s=1}^p x_{i_s}\mathbf{e}_{i_s} + \left(1 - \sum_{s=1}^p x_{i_s}\right)\mathbf{e}_{i_0},\ (x_{i_1},\ldots,x_{i_p}) \in \Delta_p. \tag{2.1}$$

Lemma 2.1. *Let* $\mathbf{k} \in \Lambda$, *with* $\operatorname{supp} \mathbf{k} = \{i_0 < \ldots < i_p\}$.

i) If $i_0 = 0$, *then*

$$\int_{D_{\mathbf{k}}} g d\sigma = \int_{\Delta_p} g \circ \theta_{\mathbf{k}},\ g \in C(D_{\mathbf{k}}); \tag{2.2}$$

ii) If $i_0 > 0$, *then*

$$\int_{D_{\mathbf{k}}} g d\sigma = \sqrt{p+1}\int_{\Delta_p} g \circ \theta_{\mathbf{k}},\ g \in C(D_{\mathbf{k}}). \tag{2.3}$$

Proof. Let prove ii). We have $\theta(\Delta_p) = D_{\mathbf{k}}$. We can write $\theta_{\mathbf{k}}(x_{i_1}, \dots, x_{i_p}) = \mathbf{e}_{i_0} + \sum_{s=1}^{p} x_{i_s}(\mathbf{e}_{is} - \mathbf{e}_{i_0})$. Then $\frac{\partial \theta_{\mathbf{k}}}{\partial x_{i_s}} = \mathbf{e}_{i_s} - \mathbf{e}_{i_0}$. Hence

$$\det[\partial\theta_{\mathbf{k}} \cdot (\partial\theta_{\mathbf{k}})^T] := \det\left[\left\langle \frac{\partial\theta_{\mathbf{k}}}{\partial x_{i_s}}, \frac{\partial\theta_{\mathbf{k}}}{\partial x_{i_t}} \right\rangle\right]_{1\le s,t\le p} = \det\begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2 \end{pmatrix} = p+1.$$

Then,

$$\int_{D_{\mathbf{k}}} g\, d\sigma = \int_{\Delta_p} (g \circ \theta_{\mathbf{k}}) \sqrt{\det[\partial\theta_{\mathbf{k}} \cdot (\partial\theta_{\mathbf{k}})^T]} = \sqrt{p+1} \int_{\Delta_p} g \circ \theta_{\mathbf{k}}.$$

Using the same method, point i) is immediate. □

Let $\mathbf{k} = (k_0, \dots, k_m) \in \Lambda$. For $\mathbf{x} = (x_1, \dots, x_m) \in \Delta_m$, denote

$$p_{n,\mathbf{k}}(\mathbf{x}) = \binom{n}{k_0\ k_1\ \dots k_m} (1 - |\mathbf{x}|)^{k_0} (x_1)^{k_1} \dots (x_m)^{k_m},$$

where

$$\binom{n}{k_0\ k_1\ \dots k_m} = \frac{n!}{k_0! k_1! \dots k_m!}.$$

The Bernstein operators on the simplex Δ_m are given by

$$\mathbb{B}_n(f)(\mathbf{x}) = \sum_{\mathbf{k}\in\Lambda} f\left(\frac{\mathbf{k}}{n}\right) p_{n,\mathbf{k}}(\mathbf{x}),\ f \in C(\Delta_m),\ \mathbf{x} \in \Delta_m. \tag{2.4}$$

Fix a number $\rho > 0$. For $\mathbf{k} \in \Lambda$ such that $\operatorname{supp} \mathbf{k} = \{i_0, \dots, i_p\}$ consider function $Q_{\mathbf{k}}^{\rho} : D_{\mathbf{k}} \to \mathbb{R}$ defined by

$$Q_{\mathbf{k}}^{\rho}\left(\sum_{s=0}^{p} t_s \mathbf{e}_{i_s}\right) = \prod_{s=0}^{p} t_s^{k_{i_s}\rho - 1},\ \sum_{s=0}^{p} t_s \mathbf{e}_{i_s} \in D_{\mathbf{k}}. \tag{2.5}$$

For $\beta = (\beta_0, \dots, \beta_p)$, $b_0, \dots, b_p > 0$, consider multidimensional beta function

$$B(\beta) = \frac{\Gamma(\beta_0) \dots \Gamma(\beta_p)}{\Gamma(|\beta|)},$$

where Γ is gamma function. If $p = 0$, then $B(\beta) = 1$.
Let $\mathbf{k} \in \Lambda$, $\operatorname{supp} \mathbf{k} = \{i_0 < \dots < i_p\}$. From relation (2.5) and relations (2.2) and (2.3), it follows that

$$\int_{D_{\mathbf{k}}} Q_{\mathbf{k}}^{\rho} d\sigma = B(k_{i_0}\rho, \dots, k_{i_p}\rho),\ \text{if } i_0 = 0; \tag{2.6}$$

$$\int_{D_{\mathbf{k}}} Q_{\mathbf{k}}^{\rho} d\sigma = \sqrt{p+1} B(k_{i_0}\rho, \dots, k_{i_p}\rho),\ \text{if } i_0 > 0. \tag{2.7}$$

Definition 2.1. Operators $\mathbb{U}_n^{\rho} : C(\Delta_m) \to C(\Delta_m)$ are defined by

$$\mathbb{U}_n^{\rho}(f)(\mathbf{x}) = \sum_{\mathbf{k}\in\Lambda} F_{n,\mathbf{k}}^{\rho}(f) p_{n,\mathbf{k}}(\mathbf{x}),\ f \in C(\Delta_m),\ \mathbf{x} \in \Delta_m, \tag{2.8}$$

where

$$F_{n,\mathbf{k}}^{\rho}(f) = \frac{\int_{D_{\mathbf{k}}} f Q_{\mathbf{k}}^{\rho} d\sigma}{\int_{D_{\mathbf{k}}} Q_{\mathbf{k}}^{\rho} d\sigma},\ \mathbf{k} \in \Lambda,\ f \in C(\Delta_m). \tag{2.9}$$

Remark 2.1. For $\rho = 1$, operators $\mathbb{U}_n^{\rho}$ coincide with operators constructed by Waldron [28].

Definition 2.2. For a vector $\mathbf{a} = (a_0, \ldots, a_m)$, with $a_i > -1$, $(0 \le i \le m)$, $\rho \ge 1$ and $n \in \mathbb{N}$ define

$$\mathbb{M}_n^{\rho,\mathbf{a}}(f, \mathbf{x}) = \sum_{\mathbf{k}\in\Lambda} F_{n,\mathbf{k}}^{\rho,\mathbf{a}}(f) p_{n,\mathbf{k}}(\mathbf{x}),\ f \in C(\Delta_m),\ \mathbf{x} \in \Delta_m, \tag{2.10}$$

where

$$F_{n,\mathbf{k}}^{\rho,\mathbf{a}}(f) = \frac{\int_{\Delta_m} f P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}},\ f \in C(\Delta_m),\ \mathbf{k} \in \Lambda \tag{2.11}$$

and

$$P_{\mathbf{k}}^{\rho,\mathbf{a}}(\mathbf{x}) = \prod_{s=0}^{m} x_s^{k_s\rho + a_s},\ \mathbf{x} = (x_1, \ldots, x_m) \in \Delta_m,\ x_0 = 1 - |\mathbf{x}|.$$

3. Link properties

Theorem 3.1. *For any $\rho \ge 1$, $n \in \mathbb{N}$ and $f \in C(\Delta_m)$, we have*

$$\lim_{\mathbf{a}\to-\mathbf{1}} \mathbb{M}_n^{\rho,\mathbf{a}}(f)(\mathbf{x}) = \mathbb{U}_n^{\rho}(f)(\mathbf{x}),\ \textit{uniformly for } \mathbf{x} \in \Delta_m, \tag{3.12}$$

where $-\mathbf{1} = (-1, \ldots, -1) \in \mathbb{N}^{m+1}$.

Proof. It is sufficient to show that

$$\lim_{\mathbf{a}\to-\mathbf{1}} F_{n,\mathbf{k}}^{\rho,\mathbf{a}}(f) = F_{n,\mathbf{k}}^{\rho}(f),\ \mathbf{k} \in \Lambda,\ f \in C(\Delta_m). \tag{3.13}$$

If supp $\mathbf{k} = \{0, 1, \ldots, m\}$, then it is possible to pass to limit $\mathbf{a} \to -\mathbf{1}$ by simple replacement $\rho = -1$, because

$$\lim_{\mathbf{a}\to-\mathbf{1}} F_{n,\mathbf{k}}^{\rho,\mathbf{a}}(f) = \frac{\int_{\Delta_m} (f Q_{\mathbf{k}}^{\rho}) \circ \theta_{\mathbf{k}}}{\int_{\Delta_m} Q_{\mathbf{k}}^{\rho} \circ \theta_{\mathbf{k}}} = \frac{\int_{D_{\mathbf{k}}} f Q_{\mathbf{k}}^{\rho} d\sigma}{\int_{D_{\mathbf{k}}} Q_{\mathbf{k}}^{\rho} d\sigma}$$

and these integrals exist.
In the sequel, we consider that supp $\mathbf{k} = \{i_0 < \ldots < i_p\} \subset \{0, 1, \ldots, m\}$, with $p < m$. Also, we denote $\{i_{p+1}, \ldots, i_m\} := \{0, 1, \ldots, m\} \setminus$ supp $\mathbf{k}$.
If $p = 0$, then $D_{\mathbf{k}} = \{\mathbf{e}_{i_0}\}$ and $\pi_{\mathbf{k}}(\mathbf{x}) = \mathbf{e}_{i_0}$, $\mathbf{x} \in \Delta_m$. Then $\frac{\int_{\Delta_m} (f\circ\pi_{\mathbf{k}}) P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} = f(\mathbf{e}_{i_0})$ and on the other hand it follows $F_{n,\mathbf{k}}^{\rho}(f) = \frac{\int_{D_{\mathbf{k}}} (f Q_{\mathbf{k}}^{\rho})\circ\theta_{\mathbf{k}}}{\int_{D_{\mathbf{k}}} Q_{\mathbf{k}}^{\rho}\circ\theta_{\mathbf{k}}} = f(\mathbf{e}_{i_0})$ and (3.13) is clear. We consider now that $p \ge 1$. We have to consider two cases.
<u>Case 1.</u> $0 \notin$ supp $\mathbf{k}$. Then, $0 \in \{i_{p+1}, \ldots, i_m\}$. Consider function $\pi_{\mathbf{k}} : \Delta_m \to D_{\mathbf{k}}$, given by

$$\pi_{\mathbf{k}}(\mathbf{x}) = \sum_{s=1}^{p} x_{i_s} \mathbf{e}_{i_s} + \left(1 - \sum_{s=1}^{p} x_{i_s}\right) \mathbf{e}_{i_0},\ \mathbf{x} \in \Delta_m.$$

Hence $D_{\mathbf{k}} = \Delta_{[\mathbf{e}_{i_0}, \ldots, \mathbf{e}_{i_p}]} = \pi(\Delta_m)$. We decompose

$$F_{n,\mathbf{k}}^{\rho,\mathbf{a}}(f) = \frac{\int_{\Delta_m} (f \circ \pi_{\mathbf{k}}) P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} + \frac{\int_{\Delta_m} (f - f \circ \pi_{\mathbf{k}}) P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}}. \tag{3.14}$$

We show that

$$\lim_{\mathbf{a}\to-\mathbf{1}} \frac{\int_{\Delta_m} (f \circ \pi_{\mathbf{k}}) P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} = F_{n,\mathbf{k}}^{\rho}(f). \tag{3.15}$$

We can write

$$\int_{\Delta_m} (f \circ \pi_{\mathbf{k}}) P_{\mathbf{k}}^{\rho,\mathbf{a}}$$
$$= \int_0^1 dx_{i_1} \int_0^{1-x_{i_1}} dx_{i_2} \dots \int_0^{1-\sum_{s=1}^{p-1} x_{i_s}} f(\pi_{\mathbf{k}}(\mathbf{x})) \prod_{s=1}^{p} x_{i_s}^{k_{i_s}\rho + a_{i_s}} V_{\mathbf{k}}(x_{i_1}, \dots, x_{i_p}) dx_{i_p},$$

where

$$V_{\mathbf{k}}(x_{i_1}, \dots, x_{i_p}) = \int_0^{1-\sum_{s=1}^{p} x_{i_s}} dx_{i_{p+1}} \dots \int_0^{1-\sum_{s=1}^{m-1} x_{i_s}} x_{i_0}^{k_{i_0}\rho + a_{i_0}} \prod_{s=p+1}^{m} x_{i_s}^{a_{i_s}} dx_{i_m},$$

where $x_{i_0} = 1 - \sum_{s=1}^{m} x_{i_s}$. Denote $u = 1 - \sum_{s=1}^{p} x_{i_s}$. Using the change of variables $x_{i_s} = u y_{i_s}$, $p+1 \le s \le m$ one obtains $x_{i_0} = u\left(1 - \sum_{s=p+1}^{m} y_{i_s}\right)$ and then

$$\begin{aligned} V_{\mathbf{k}}(x_{i_1}, \dots, x_{i_p}) &= u^{m-p+\sum_{s=p+1}^{m} a_{i_s} + a_{i_0} + \rho k_{i_0}} B(\rho k_{i_0} + a_{i_0} + 1, a_{i_{p+1}} + 1, \dots a_{i_m} + 1) \\ &= u^{m-p+\sum_{s=p+1}^{m} a_{i_s} + a_{i_0} + \rho k_{i_0}} \frac{\Gamma(k_{i_0}\rho + a_{i_0} + 1) \prod_{s=p+1}^{m} \Gamma(a_{i_s} + 1)}{\Gamma\left(a_{i_0} + \sum_{s=p+1}^{m} a_{i_s} + \rho k_{i_0} + m - p + 1\right)}. \end{aligned}$$

We have

$$\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}} = \frac{\prod_{s=0}^{p} \Gamma(k_{i_s}\rho + a_{i_s} + 1) \prod_{s=p+1}^{m} \Gamma(a_{i_s} + 1)}{\Gamma\left(|a| + n\rho + m + 1\right)}. \tag{3.16}$$

By combining the relations above, we get

$$\frac{\int_{\Delta_m} (f \circ \pi_{\mathbf{k}}) P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} = \int_0^1 dx_{i_1} \int_0^{1-x_{i_1}} dx_{i_2} \dots \int_0^{1-\sum_{s=1}^{p-1} x_{i_s}} f(\pi_{\mathbf{k}}(\mathbf{x})) T_{\mathbf{k}}^{\mathbf{a}}(x_{i_1}, \dots, x_{i_p}) dx_{i_p}, \tag{3.17}$$

where

$$\begin{aligned} T_{\mathbf{k}}^{\mathbf{a}}(x_{i_1}, \dots, x_{i_p}) &= \frac{\Gamma(|\mathbf{a}| + n\rho + m + 1)}{\Gamma\left(a_{i_0} + \sum_{s=p+1}^{m} a_{i_s} + \rho k_{i_0} + m - p + 1\right) \prod_{s=1}^{p} \Gamma(k_{i_s}\rho + a_{i_s} + 1)} \\ &\times \prod_{s=1}^{p} x_{i_s}^{k_{i_s}\rho + a_{i_s}} \left(1 - \sum_{s=1}^{p} x_{i_s}\right)^{m-p+\sum_{s=p+1}^{m} a_{i_s} + k_{i_0}\rho + a_{i_0}}. \end{aligned} \tag{3.18}$$

It is possible to pass to limit $\mathbf{a} \to -\mathbf{1}$ in (3.18) and it follows

$$\lim_{\mathbf{a} \to -\mathbf{1}} T_{\mathbf{k}}^{\mathbf{a}}(x_{i_1}, \dots, x_{i_p}) = \frac{\Gamma(n\rho)}{\prod_{s=0}^{p} \Gamma(k_{i_s}\rho)} \prod_{s=1}^{p} x_{i_s}^{k_{i_s}\rho - 1} \left(1 - \sum_{s=1}^{p} x_{i_s}\right)^{\rho k_{i_0} - 1}. \tag{3.19}$$

By taking into account relations (3.17), (3.19), (2.5), (2.7), (2.3) and then (2.9), we have successively

$$\begin{aligned}\lim_{\mathbf{a}\to-\mathbf{1}}\frac{\int_{\Delta_m}(f\circ\pi_{\mathbf{k}})P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m}P_{\mathbf{k}}^{\rho,\mathbf{a}}}&=\frac{\int_{\Delta_p}(f\cdot Q_{\mathbf{k}}^{\rho})\circ\theta_{\mathbf{k}}}{B(k_{i_0}\rho,\ldots,k_{i_p}\rho)}\\&=\frac{\sqrt{p+1}\int_{\Delta_p}(f\cdot Q_{\mathbf{k}}^{\rho})\circ\theta_{\mathbf{k}}}{\sqrt{p+1}B(k_{i_0}\rho,\ldots,k_{i_p}\rho)}\\&=\frac{\int_{D_{\mathbf{k}}}f\cdot Q_{\mathbf{k}}^{\rho}\,d\sigma}{\int_{D_{\mathbf{k}}}Q_{\mathbf{k}}^{\rho}\,d\sigma}\\&=F_{n,\mathbf{k}}^{\rho}(f).\end{aligned}$$

So that relation (3.15) was proved. Now, we show that

$$\lim_{\mathbf{a}\to-\mathbf{1}}\frac{\int_{\Delta_m}(f-f\circ\pi_{\mathbf{k}})P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m}P_{\mathbf{k}}^{\rho,\mathbf{a}}}=0. \tag{3.20}$$

Consider on $\mathbb{R}^m$ the norm $\|\mathbf{x}\|_1$, defined in the beginning. Let $\varepsilon>0$. There exist $0<\delta<1$, such that if $\mathbf{x}\in\Delta_m$, $\|\mathbf{x}-\pi_{\mathbf{k}}(\mathbf{x})\|<\delta$, then $|f(\mathbf{x})-f(\pi_{\mathbf{k}}(\mathbf{x}))|<\varepsilon$. Decompose $\Delta_m=A\cup B$, where $A=\{\mathbf{x}\in\Delta_m|\ \|\mathbf{x}-\pi_{\mathbf{k}}(x)\|_1<\delta\}$ and $B=\Delta_m\setminus A$. Then

$$\left|\frac{\int_A(f-f\circ\pi_{\mathbf{k}})P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m}P_{\mathbf{k}}^{\rho,\mathbf{a}}}\right|\le\varepsilon. \tag{3.21}$$

Let $\mathbf{x}\in B$. We have

$$\mathbf{x}-\pi_{\mathbf{k}}(\mathbf{x})=\sum_{j=1}^m x_j\mathbf{e}_j-\sum_{s=1}^p x_{i_s}\mathbf{e}_{i_s}-\left(1-\sum_{s=1}^p x_{i_s}\right)\mathbf{e}_{i_0}=\sum_{s=p+1}^m x_{i_s}\mathbf{e}_{i_s}+\left(-1+\sum_{s=1}^p x_{i_s}\right)\mathbf{e}_{i_0}.$$

Therefore

$$\|\mathbf{x}-\pi_{\mathbf{k}}(\mathbf{x})\|_1=\sum_{s=p+1}^m x_{i_s}\mathbf{e}_{i_s}+1-\sum_{s=1}^p x_{i_s}=x_0+2\sum_{s=p+1}^m x_{i_s}.$$

Since $x_0\in\{i_{p+1},\ldots,i_m\}$, it results $\|\mathbf{x}-\pi_{\mathbf{k}}(\mathbf{x})\|_1\le 3\sum_{s=p+1}^m x_{i_s}$. It follows that there is at least an index $j\in\{i_{p+1},\ldots,i_m\}$, such that $x_i\ge\frac{\delta}{3m}$. Define

$$B_j=\left\{\mathbf{x}\in\Delta_m|x_j\ge\frac{\delta}{3m}\right\},\ j\in\{i_{p+1},\ldots,i_m\}.$$

From above, it follows that $B\subset\bigcup_{j\in\{0,i_{p+1},\ldots,i_m\}}B_j$. Therefore

$$\left|\frac{\int_B(f-f\circ\pi_{\mathbf{k}})P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m}P_{\mathbf{k}}^{\rho,\mathbf{a}}}\right|\le\frac{2\|f\|\int_B P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m}P_{\mathbf{k}}^{\rho,\mathbf{a}}}\le 2\|f\|\sum_{j\in\{i_{p+1},\ldots,i_m\}}\frac{\int_{B_j}P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m}P_{\mathbf{k}}^{\rho,\mathbf{a}}}.$$

We show that

$$\lim_{\mathbf{a}\to-\mathbf{1}}\frac{\int_{B_j}P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m}P_{\mathbf{k}}^{\rho,\mathbf{a}}}=0,\ j\in\{i_{p+1},\ldots,i_m\}. \tag{3.22}$$

We can consider that $a_j<0$, $(0\le j\le m)$. Let $j=i_r$, with $p+1\le r\le m$. In integral $\int_{B_j}P_{\mathbf{k}}^{\rho,\mathbf{a}}$ if we make the change of variables: $x_j=\frac{\delta}{3m}+\left(1-\frac{\delta}{3m}\right)y_j$ and $x_\ell=\left(1-\frac{\delta}{3m}\right)y_\ell$, for

$\ell \in \{1,\dots,m\} \setminus \{j\}$. Also $x_0 = \left(1 - \frac{\delta}{3m}\right) y_0$, where $y_0 = 1 - (y_1 + \dots + y_m)$. Then, we have the equivalence $(x_1,\dots,x_m) \in B_j \iff (y_1,\dots,y_m) \in \Delta_m$. We get

$$\begin{aligned}&\int_{B_j} P_{\mathbf{k}}^{\rho,\mathbf{a}}\\ &= \left(1 - \frac{\delta}{3m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_r}} \int_{\Delta_m} \prod_{s=0}^{p} y_{i_s}^{k_{i_s}\rho+a_{i_s}} \prod_{s=p+1,\, s\neq r}^{m} y_{i_s}^{a_{i_s}} \left(\frac{\delta}{3m} + \left(1 - \frac{\delta}{3m}\right) y_{i_r}\right)^{a_{i_r}}.\end{aligned}$$

Since $a_{i_r} < 0$ and $\frac{\delta}{3m} < 1$, we obtain $\left(\frac{\delta}{3m} + \left(1 - \frac{\delta}{3m}\right) y_{i_r}\right)^{a_{i_r}} \le \left(\frac{\delta}{3m}\right)^{a_{i_r}} \le \left(\frac{\delta}{3m}\right)^{-1} = \frac{3m}{\delta}$. Consequently

$$\begin{aligned}&\int_{B_j} P_{\mathbf{k}}^{\rho,\mathbf{a}}\\ &\le \frac{3m}{\delta}\left(1 - \frac{\delta}{3m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_r}} \frac{\prod_{s=0}^{p} \Gamma(k_{i_s}\rho + a_{i_s} + 1) \prod_{s=p+1,\, s\neq r}^{m} \Gamma(a_{i_s} + 1)\Gamma(1)}{\Gamma(n\rho + |a| - a_{i_r} + m + 1)}.\end{aligned}$$

By taking into account relation (3.16), it results:

$$\frac{\int_{B_j} P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} \le \frac{3m}{\delta}\left(1 - \frac{\delta}{3m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_r}} \frac{\Gamma(n\rho + |\mathbf{a}| + m + 1)}{\Gamma(n\rho + |\mathbf{a}| - a_{i_r} + m + 1)\Gamma(a_{i_r} + 1)}.$$

But

$$\lim_{\mathbf{a}\to-\mathbf{1}} \left(1 - \frac{\delta}{3m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_r}} \frac{\Gamma(n\rho + |\mathbf{a}| + m + 1)}{\Gamma(n\rho + |\mathbf{a}| - a_{i_r} + m + 1)} = \left(1 - \frac{\delta}{3m}\right)^{n\rho} \frac{\Gamma(n\rho)}{\Gamma(n\rho + 1)}$$

and $\lim_{\mathbf{a}\to-\mathbf{1}} \Gamma(a_{i_r} + 1) = \infty$. Then, one obtains relation (3.22).
Case 2. $i_0 = 0$. Then $\operatorname{supp} \mathbf{k} = \{0 = i_0 < i_1 < \dots < i_p\}$, where $0 \le p \le m-1$. Define the function $\pi_{\mathbf{k}} : \Delta_m \to D_{\mathbf{k}}$, by

$$\pi_{\mathbf{k}}(\mathbf{x}) = \sum_{s=1}^{p} x_{i_s}\mathbf{e}_{i_s}, \quad \mathbf{x} = (x_1,\dots,x_m) \in \Delta_m.$$

The method of the proof is similar as in Case 1. Consider the decomposition of the form given in (3.14). First, we show the corresponding relation (3.15). For $(x_{i_1},\dots,x_{i_p}) \in \Delta_p$, we denote

$$U(x_{i_1},\dots,x_{i_p}) = \left\{ (x_{i_{p+1}},\dots,x_{i_m}) \middle| x_{i_s} \ge 0,\ (p+1 \le s \le m),\ \sum_{s=p+1}^{m} x_{i_s} \le 1 - \sum_{s=1}^{p} x_{i_s} \right\}.$$

Then, we can write

$$\begin{aligned}&\int_{\Delta_m} (f \circ \pi_{\mathbf{k}}) \cdot P_{\mathbf{k}}^{\rho,\mathbf{a}}\\ &= \int_{\Delta_p} f\left(\sum_{s=1}^{p} x_{i_s}\mathbf{e}_{i_s}\right) \prod_{s=1}^{p} x_{i_s}^{k_{i_s}\rho+a_{i_s}} dx_{i_1}\dots dx_{i_p}\\ &\times \int_{U(x_{i_1},\dots,x_{i_p})} \prod_{s=p+1}^{m} x_{i_s}^{a_{i_s}} x_0^{k_0\rho+a_0} dx_{i_{p+1}}\dots dx_{i_m}.\end{aligned}$$

Denote $u = 1 - x_{i_1} - \ldots - x_{i_p}$. Using the change of variables $x_\ell = u y_{i_\ell}$, $p+1 \le \ell \le m$ in the interior integral, we obtain

$$\begin{aligned}&\int_{U(x_{i_1},\ldots,x_{i_p})} \prod_{s=p+1}^{m} x_{i_s}^{a_{i_s}} x_0^{k_0\rho+a_0} dx_{i_{p+1}}\ldots dx_{i_m}\\ =&u^{m-p+a_0+a_{i_{p+1}}+\ldots+a_{i_m}+k_0\rho} B(k_0\rho+a_0+1, a_{i_{p+1}}+1,\ldots,a_{i_m}+1).\end{aligned}$$

By taking also into account relation (3.16), we obtain

$$\begin{aligned}&\frac{\int_{\Delta_m}(f\circ\pi)\cdot P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}}\\ =&\int_{\Delta_p} f\left(\sum_{s=1}^{p} x_{i_s}\mathbf{e}_{i_s}\right)\prod_{s=1}^{p} x_{i_s}^{k_{i_s}\rho+a_{i_s}}\left(1-\sum_{s=1}^{p} x_{i_s}\right)^{m-p+a_0+\sum_{s=p+1}^{m} a_{i_s}+k_0\rho} dx_{i_1}\ldots dx_{i_p}\\ \times&\frac{\Gamma(n\rho+|\mathbf{a}|+m+1)}{\Gamma(k_0\rho+a_0+a_{i_{p+1}}+\ldots+a_{i_m}+m-p+1)\prod_{s=1}^{p}\Gamma(k_{i_s}\rho+a_{i_s}+1)}.\end{aligned}$$

Using (2.1), (2.6), (2.2) and (2.9), it follows

$$\lim_{\mathbf{a}\to\mathbf{1}} \frac{\int_{\Delta_m}(f\circ\pi_{\mathbf{k}})\cdot P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} = \frac{\int_{\Delta_p}(f\cdot Q_{\mathbf{k}}^{\rho})\circ\theta_{\mathbf{k}}}{B(k_{i_0}\rho,\ldots,k_{i_p}\rho)} = \frac{\int_{D_{\mathbf{k}}} f\cdot Q_{\mathbf{k}}^{\rho}}{\int_{D_{\mathbf{k}}} Q_{\mathbf{k}}^{\rho}} = F_{n,\mathbf{k}}^{\rho}(f).$$

So that relation (3.15) is proved.
In order to prove the corresponding relation (3.20), let $\varepsilon > 0$ arbitrarily chosen. There is $0 < \delta < 1$, such that inequality $\|\mathbf{x}-\pi_{\mathbf{k}}(\mathbf{x})\|_1 < \delta$, $\mathbf{x}\in\Delta_m$ implies $|f(\mathbf{x}) - f(\pi_{\mathbf{k}}(x))| < \varepsilon$. Consider the sets $A = \{\mathbf{x}\in\Delta_m |\ \|\mathbf{x}-\pi_{\mathbf{k}}(\mathbf{x})\|_1 < \delta\}$ and $B = \Delta_m \setminus A$. We have

$$\left|\frac{\int_{\Delta_m}(f-f\circ\pi_{\mathbf{k}})P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}}\right| \le \varepsilon + 2\|f\|\frac{\int_B P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}}.$$

If $\mathbf{x}\in B$, there is $j\in\{p+1,\ldots,m\}$ such that $x_j \ge \frac{\delta}{m}$. Indeed, otherwise we have $\|\mathbf{x}-\pi_{\mathbf{k}}(\mathbf{x})\|_1 = x_{i_{p+1}} + \ldots x_{i_m} < (m-p)\frac{\delta}{m} \le \delta$, which is a contradiction. Define

$$B_j := \{\mathbf{x} = (x_1,\ldots,x_m)\in\Delta_m | x_j \ge \frac{\delta}{m}\},\ j\in\{i_{p+1},\ldots,i_m\}.$$

Therefore $B \subset \bigcup_{j=p+1}^{m} B_j$, which implies

$$\frac{\int_B P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} \le \sum_{j=p+1}^{m} \frac{\int_{B_j} P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}}.$$

Fix $r\in\{p+1,\ldots,m\}$ and $j = i_r$. With the change of variables $x_{i_r} = \frac{\delta}{m} + \left(1-\frac{\delta}{m}\right) y_{i_r}$ and $x_\ell = \left(1-\frac{\delta}{m}\right) y_\ell$, $\ell\in\{1,\ldots,m\}\setminus\{r\}$; the condition $\mathbf{x}\in B_j$ is equivalent to $(y_1,\ldots,y_m)\in\Delta_m$. We obtain $x_0 = \left(1-\frac{\delta}{m}\right) y_0$ and then

$$\begin{aligned}&\int_{B_j} P_{\mathbf{k}}^{\rho,\mathbf{a}}\\ =&\left(1-\frac{\delta}{m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_s}} \int_{\Delta_m} \prod_{\ell=0,\ \ell\ne i_r}^{m} y^{k_\ell\rho+a_\ell}\left(\frac{\delta}{m}+\left(1-\frac{\delta}{m}\right)y_{i_r}\right)^{a_{i_r}} dy_1\ldots dy_m.\end{aligned}$$

We have $\left(\frac{\delta}{m}+\left(1-\frac{\delta}{m}\right)y_{i_r}\right)^{a_{i_r}} \le \left(\frac{\delta}{m}\right)^{a_{i_r}} < \left(\frac{\delta}{m}\right)^{-1} = \frac{m}{\delta}$. Then

$$\int_{B_j} P_{\mathbf{k}}^{\rho,\mathbf{a}} \leq \frac{m}{\delta}\left(1-\frac{\delta}{m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_r}}$$
$$\times \frac{\prod_{s=0}^{p}\Gamma(k_{i_s}\rho+a_{i_s}+1)\prod_{s=p+1,\ s\neq r}^{m}\Gamma(a_{i_s}+1)\Gamma(1)}{\Gamma(n\rho+|\mathbf{a}|-a_{i_r}+m+1)}.$$

Using also relation (3.16), it follows

$$\frac{\int_{B_j} P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} \leq \frac{m}{\delta}\left(1-\frac{\delta}{m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_r}} \frac{\Gamma(n\rho+|\mathbf{a}|+m+1)}{\Gamma(n\rho+|\mathbf{a}|-a_j+m+1)\Gamma(a_j+1)}.$$

But

$$\lim_{\mathbf{a}\to\mathbf{1}}\left(1-\frac{\delta}{m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_r}} \frac{\Gamma(n\rho+|\mathbf{a}|+m+1)}{\Gamma(n\rho+|\mathbf{a}|-a_j+m+1)} = \left(1-\frac{\delta}{m}\right)^{n\rho}\frac{\Gamma(n\rho)}{\Gamma(n\rho+1)}$$

and $\lim_{\mathbf{a}\to\mathbf{1}}\Gamma(a_j+1)=\infty$. Then, the corresponding relations (3.22) are true. Now, it is simple to deduce that (3.20) is valid. □

Remark 3.2. In the case $\rho = 1$, Theorem 3.1 was proved in [28] but using a method which is not applicable here. In unidimensional case, Theorem 3.1 was proved in [12].

Theorem 3.2. *For any $f \in C(\Delta_m)$, we have*

(3.23) $$\lim_{\rho\to\infty}\mathbb{U}_n^{\rho}(f)(\mathbf{x}) = \mathbb{B}_n(f)(\mathbf{x}),\ \textit{uniformly for } \mathbf{x}\in\Delta_m.$$

Proof. It is sufficient to show that

(3.24) $$\lim_{\rho\to\infty} F_{n,\mathbf{k}}^{\rho}(f) = f\left(\frac{k_1}{n},\ldots,\frac{k_m}{n}\right),\ \mathbf{k}=(k_0,k_1,\ldots,k_m)\in\Lambda,\ f\in C(\Delta_m).$$

Let supp $\mathbf{k} = \{i_0,\ldots,i_p\}\subset\{0,1,\ldots,m\}$, $p\geq 0$. If $p=0$, then relation (3.23) is immediate. Now consider that $p\geq 1$. We introduce simplified notations as follows. Denote $\mu_j = k_{i_j}$, $(0\leq j\leq p)$. Recall that $D_{\mathbf{k}} = \left\{\sum_{j=0}^{p} y_j\mathbf{e}_{i_j} | y_j\geq 0,\ (0\leq j\leq p),\ \sum_{j=0}^{p} y_j = 1\right\}$. Define $\varphi:\Delta_p\to\mathbb{R}$ by

$$\varphi(\mathbf{y}) = \prod_{j=1}^{p} y_j^{\mu_j}(1-|\mathbf{y}|)^{\mu_0},\ \mathbf{y}=(y_1,\ldots,y_p)\in\Delta_p.$$

We have $\varphi\geq 0$ on Δ_p. Since $\mu_j\geq 1$, $0\leq j\leq p$, it follows that $\varphi = 0$ on the frontier of Δ_p. Consequently, the maximum of φ is reached in the interior of domain Δ_p. It is simple to show that the unique interior critical point of φ is $\mathbf{y}^* = \left(\frac{\mu_1}{n},\ldots,\frac{\mu_p}{n}\right)\in\Delta_p$. Then, $\mathbf{y}^*$ is the unique maximum point of φ.
Define $g\in C(\Delta_p)$, $g = f\circ\theta_{\mathbf{k}}$, where $\theta_{\mathbf{k}}$ was defined in (2.1).
Let $\varepsilon>0$ arbitrarily chose. We can choose a number $r>0$, such that $B_r(\mathbf{y}^*) = \{\mathbf{y}\in\mathbb{R}^p | \ \|\mathbf{y}-\mathbf{y}^*\|<r\}\subset \mathrm{Int}\Delta_p$ and $|g(\mathbf{y})-g(\mathbf{y}^*)|<\frac{\varepsilon}{2}$, for all $\mathbf{y}\in B_r(\mathbf{y}^*)$. Define $M = \max\{\varphi(\mathbf{y})|\ \mathbf{y}\in\overline{\Delta_p\setminus B_r(\mathbf{y}^*)}\}$. Then $M<\varphi(\mathbf{y}^*)$. Choose $M<M_1<\varphi(\mathbf{y}^*)$. There is $\delta>0$, such that $0<\delta<r$ and $\varphi(\mathbf{y})\geq M_1$, for all $\mathbf{y}\in B_\delta(\mathbf{y}^*)$.
For $\rho>1$, define $\Psi=\Psi_{\rho,\mathbf{k}}\in C(\Delta_p)$, $\Psi = Q_{\mathbf{k}}^{\rho}\circ\theta_{\mathbf{k}}$, where $Q_{\mathbf{k}}^{\rho}$ and $\theta_{\mathbf{k}}$ are defined in (2.5) and (2.1), respectively. Then we can write $\Psi = \varphi^{\rho-1}\cdot\eta$, where

$$\eta(\mathbf{y}) = \prod_{j=1}^{p} y_j^{\mu_j-1}(1-|\mathbf{y}|)^{\mu_0-1},\ \mathbf{y}=(y_1,\ldots,y)\in\Delta_p.$$

We have
$$\int_{\Delta_p\setminus B_r(y^*)}\Psi=\int_{\Delta_p\setminus B_r(y^*)}(\varphi)^{\rho-1}\eta\leq\|\eta\|M^{\rho-1}\mathrm{vol}(\Delta_p)$$
and
$$\int_{B_r(y^*)}\Psi\geq\int_{B_\delta(y^*)}(\varphi)^{\rho-1}\eta\geq h\cdot M_1^{\rho-1}\mathrm{vol}(B_\delta(\mathbf{y}^*)),$$
where $h=\min\{\eta(\mathbf{y})|\ \mathbf{y}\in\overline{B_r(\mathbf{y}^*)}\}>0$. Then
$$\frac{\int_{\Delta_p\setminus B_r(\mathbf{y}^*)}\Psi}{\int_{B_r(y^*)}\Psi}\leq\frac{\|\eta\|\cdot\mathrm{vol}(\Delta_p)}{h\cdot\mathrm{vol}(B_\delta(\mathbf{y}^*))}\left(\frac{M}{M_1}\right)^{\rho-1}.$$
It is possible to choose $\rho_0>1$, such that
$$2\|f\|\cdot\frac{\int_{\Delta_p\setminus B_r(\mathbf{y}^*)}\Psi}{\int_{B_r(y^*)}\Psi}<\frac{\varepsilon}{2},\ \forall\rho>\rho_0.$$
Using formula (2.2) or formula (2.3) depending on the condition $0\in\mathrm{supp}\ \mathbf{k}$ or $0\notin\mathrm{supp}\ \mathbf{k}$, in both cases it results
$$F^\rho_{n,\mathbf{k}}(f)=\frac{\int_{D_\mathbf{k}}f\cdot Q^\rho_\mathbf{k}\,d\sigma}{\int_{D_\mathbf{k}}Q^\rho_\mathbf{k}\,d\sigma}=\frac{\int_{\Delta_p}(f\cdot Q^\rho_\mathbf{k})\circ\theta_\mathbf{k}}{\int_{\Delta_p}Q^\rho_\mathbf{k}\circ\theta_\mathbf{k}}=\frac{\int_{\Delta_p}g\cdot\Psi}{\int_{\Delta_p}\Psi}.$$
Then, for $\rho>\rho_0$:
$$\begin{aligned}|F^\rho_{n,\mathbf{k}}(f)-g(\mathbf{y}^*)|&=\left|\frac{\int_{\Delta_p}g\cdot\Psi}{\int_{\Delta_p}\Psi}-g(\mathbf{y}^*)\right|\\&\leq\frac{\int_{\Delta_p}|(g-g(y^*)|\cdot\Psi}{\int_{\Delta_p}\Psi}\\&\leq\frac{\int_{\Delta_p\setminus B_r(\mathbf{y}^*)}|g-g(y^*)|\cdot\Psi}{\int_{\Delta_p}\Psi}+\frac{\int_{B_r(\mathbf{y}^*)}|(g-g(y^*)|\cdot\Psi}{\int_{\Delta_p}\Psi}\\&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$
Finally note that $g(\mathbf{y}^*)=f\left(\frac{k_1}{n},\dots,\frac{k_m}{n}\right)$. □

4. Convergence properties

The moments of operators play a crucial role in the study of the convergence properties of a sequence of linear positive operators. The computation of moments of operators $\mathbb{M}^{\rho,\mathbf{a}}_n$ and $\mathbb{U}^\rho_n$ can be reduced to the moments of the Bernstein operators $\mathbb{B}_n$.

Define the functions $\mathbf{1}_{\Delta_m}\in C(\Delta_m)$, $\mathbf{1}_{\Delta_m}(\mathbf{x})=1$ and $\mathrm{pr}_i\in C(\Delta_m)$, $(1\leq i\leq m)$, $\mathrm{pr}_i(\mathbf{x})=x_i$, $\mathbf{x}=(x_1,\dots,x_m)\in\Delta_m$.

Define
$$\|\bullet-\overline{x}\|(t_1,\dots,t_m)=\sqrt{\sum_{i=1}^m(t_i-x_i)^2}.$$

Lemma 4.2. *For $m\in\mathbb{N}$, $\mathbf{a}=(a_0,\dots,a_m)\in\mathbb{R}^{m+1}$, $\mathbf{a}>-\mathbf{1}$, $\rho\geq1$, $n\in\mathbb{N}$ and $\mathbf{x}=(x_1,\dots,x_m)\in\Delta_m$:*

i) $\mathbb{M}^{\rho,\mathbf{a}}_n(\mathbf{1}_{\Delta_m})(\mathbf{x})=1$,
ii) $\mathbb{M}^{\rho,\mathbf{a}}_n(\mathrm{pr}_i)(\mathbf{x})=\frac{n\rho x_i+a_i+1}{\rho n+|\mathbf{a}|+m+1}$, $(1\leq i\leq m)$,

iii) $\mathbb{M}_n^{\rho,\mathbf{a}}(\|\bullet-\mathbf{x}\|^2)(\mathbf{x}) = \sum_{i=1}^m \frac{n\rho(\rho+1)x_i(1-x_i)+\lambda_i(\mathbf{a},m,\mathbf{x})}{(\rho n+|\mathbf{a}|+m+1)(\rho n+|\mathbf{a}|+m+2)}$, *where*

$$\lambda_i(\mathbf{a},m,\mathbf{x}) := (|\mathbf{a}|+m+1)(|\mathbf{a}|+m+2)x_i^2 - 2(|\mathbf{a}|+m+2)(a_i+1)x_i + (a_i+1)(a_i+2).$$

Proof. For any $\mathbf{k} \in \Lambda$, $\mathbf{k} = (k_0, \dots, k_m)$, we have

a) $F_{n,\mathbf{k}}^{\rho}(\mathbf{1}_{\Delta_m}) = 1$;

b) $F_{n,\mathbf{k}}^{\rho}(\mathrm{pr}_i) = \frac{\rho k_i + a_i + 1}{\rho n+|\mathbf{a}|+m+1}$, $(1 \le i \le m)$;

c) $F_{n,\mathbf{k}}^{\rho}(\mathrm{pr}_i^2) = \frac{(\rho k_i+a_i+1)(\rho k_i+a_i+2)}{(\rho n+|\mathbf{a}|+m+1)(\rho n+|\mathbf{a}|+m+2)}$, $(1 \le i \le m)$.

Then, we can apply the known results for Bernstein operator on a simplex. □

By passing to limit $\mathbf{a} \to -\mathbf{1}$ and using Lemma 4.2 and Theorem 3.1, we obtain:

Corollary 4.1. *For $m \in \mathbb{N}$, $\rho \ge 1$, $n \in \mathbb{N}$ and $\mathbf{x} = (x_1, \dots, x_m) \in \Delta_m$:*

i) $\mathbb{U}_n^{\rho}(\ell) = \ell$, *for any affine function,*

ii) $\mathbb{U}_n^{\rho}(\|\bullet-\mathbf{x}\|^2)(\mathbf{x}) = \frac{\rho+1}{n\rho+1}\sum_{i=1}^m x_i(1-x_i)$.

Lemma 4.3. *For $m \ge 2$, we have*

$$\max\left\{\sum_{i=1}^m x_i(1-x_i)\Big|(x_1,\dots,x_m) \in \Delta_m\right\} = \frac{m-1}{m}.$$

Proof. We can apply for instance the Kuhn-Tucker conditions for this maximization problem and the optimum is obtained for $x_i = \frac{1}{m}$, $(1 \le i \le m)$. □

For $f \in C(\Delta_m)$, $h > 0$, define

$$\omega_1(f,h) = \sup\{|f(\mathbf{x}) - f(\mathbf{y})|,\ \mathbf{x},\mathbf{y} \in \Delta_m,\ \|\overline{x}-\overline{y}\| \le h\}.$$

Theorem 4.3. *For $m \in \mathbb{N}$, $\mathbf{a} = (a_0, \dots, a_m) \in \mathbb{R}^{m+1}$, $\mathbf{a} > -\mathbf{1}$, and $\rho \ge 1$, we have*

$$\|\mathbb{M}_n^{\rho,\mathbf{a}}(f) - f\| \le 2\omega_1(f, \sqrt{\mu_n}),\ f \in C(\Delta_m),\ n \in \mathbb{N}, \tag{4.25}$$

where

$$\mu_n = \sup_{\mathbf{x}\in\Delta_m} \mathbb{M}_n^{\rho,\mathbf{a}}(\|\bullet-\mathbf{x}\|^2)(\mathbf{x})$$

and

$$\mu_n = \mathrm{O}\left(\frac{1}{n}\right),\ \textit{uniformly with regard to } \rho \in [1,\infty).$$

Proof. For $m \ge 2$, from Lemma 4.2, Lemma 4.3, since $|\mathbf{a}|+m+1 > 0$, it follows:

$$\begin{aligned}\mu_n &\le \sum_{i=1}^m \frac{n\rho(\rho+1)x_i(1-x_i) + \|\lambda_i(\mathbf{a},m,\bullet)\|}{(n\rho)^2}\\ &\le \frac{1}{n}\left[\frac{\rho+1}{\rho}\cdot\frac{m-1}{m} + \frac{1}{n\rho^2}\sum_{i=1}^m \|\lambda_i(\mathbf{a},m,\bullet)\|\right] \le \frac{1}{n}\left(2 + \sum_{i=1}^m \|\lambda_i(\mathbf{a},m,\bullet)\|\right).\end{aligned}$$

This final estimate exists also in the case $m = 1$. Then, we apply the generalized theorem of Shisha and Mond, given in Altomare and Campiti [2]- Proposition 5.1.5. in the following form:

$$|L(f)(\mathbf{x}) - f(\mathbf{x})| \le \left(1 + \frac{1}{\delta^2}(L(e)(\mathbf{x}) - e(\mathbf{x}))\right)\omega_1(f,\delta),$$

where $L : C(K) \to B(K)$ is a positive linear operator which preserves affine functions, K is a compact set in an inner product space, $e(\mathbf{x}) = \|\mathbf{x}\|^2$, $\mathbf{x} \in K$, $f \in C(K)$ and $\delta > 0$. Here, we take $K = \Delta_m$, $L = \mathbb{M}_n^{\rho,\mathbf{a}}$ and $\delta = \sqrt{\mu_n}$. □

Theorem 4.4. *For $m \in \mathbb{N}$ and $\rho \geq 1$, we have*

$$|\mathbb{U}_n^\rho(f)(\mathbf{x}) - f(\mathbf{x})| \leq 2\omega_1\left(f, \sqrt{\frac{\rho+1}{n\rho+1}\sum_{i=1}^m x_i(1-x_i)}\right),\ f \in C(\Delta_m),\ n \in \mathbb{N},\ \mathbf{x} \in \Delta_m,$$

$$\|\mathbb{U}_n^\rho(f) - f\| \leq 2\omega_1\left(f, \sqrt{\frac{\rho+1}{n\rho+1}\cdot \max\left\{\frac{1}{4}, \frac{m-1}{m}\right\}}\right),\ f \in C(\Delta_m),\ n \in \mathbb{N}.$$

Proof. We apply Corollary 4.1, Lemma 4.3 and the generalized theorem of Shisha and Mond as in the proof of Theorem 4.3. □

Corollary 4.2. *For any $f \in C(\Delta_m)$, we have*

i) $\lim_{n\to\infty} \|\mathbb{M}_n^{\rho,\mathbf{a}}(f) - f\| = 0$, *where* $\mathbf{a} > -\mathbf{1},\ \rho \geq 0$,
ii) $\lim_{n\to\infty} \|\mathbb{U}_n^\rho(f) - f\| = 0$, *where* $\rho \geq 0$, $m \geq 2$.

Corollary 4.3. *For any $m \in \mathbb{N}$, $\rho \geq 1$ and $n \in \mathbb{N}$, operator $\mathbb{U}_n^\rho$ interpolates each function $f \in C(\Delta_m)$ in the vertices of the simplex Δ_m, i.e.,*

$$\mathbb{U}_n^\rho(f)(\mathbf{e}_i) = f(\mathbf{e}_i),\ (0 \leq i \leq m).$$

More refined estimates with second order moduli can be given for operators $\mathbb{U}_n^\rho$ because they reproduce the affine functions.

For $f \in C(\Delta_m, Y)$, $h > 0$, define

$$\omega_2(f,h) = \sup\left\{\left|f(\mathbf{x}) - 2f\left(\frac{1}{2}(\mathbf{x}+\mathbf{y})\right) + f(\mathbf{y})\right|,\ \mathbf{x},\mathbf{y} \in \Delta_m,\ \|\mathbf{x}-\mathbf{y}\| < h\right\}.$$

We apply the following scalar version of a theorem given in [23, Th. 7.2.4].

Theorem A. *Let $D \subset \mathbb{R}^m$ be a compact convex set. Let $F : C(D) \to \mathbb{R}$ be a functional given by a positive Borel measure μ. Suppose $\mu(D) = 1$. Let $\mathbf{x} \in D$ be the barycenter of μ. Then*

$$|F(f) - f(\mathbf{x})| \leq \left[m + \frac{1}{2}h^{-2}F(\|\bullet - \mathbf{x}\|^2)\right]\omega_2(f,h)$$

for $f \in C(D)$, $h > 0$.

Theorem 4.5. *For $n \in \mathbb{N}$, $\rho > 0$, $f \in C(\Delta_m)$, $m \geq 2$ and $h > 0$*

$$\|\mathbb{U}_n^\rho(f) - f\| \leq \left(m + \frac{1}{2h^2}\frac{\rho+1}{\rho n+1}\cdot\frac{m-1}{m}\right)\omega_2(f,h).$$

Proof. For any fixed $\mathbf{x} \in \Delta_m$, define the functional on $C(\Delta_m)$, $F(f) = \mathbb{U}_n^\rho(f,\mathbf{x})$. This is a functional defined by a positive Borel measure, say μ. From Corollary 4.1 - i), it follows that $\mathbf{x}$ is the barycenter of μ. Then, we can apply Theorem A. □

An other second modulus can be defined as follows. For $f \in C(\Delta_m)$ and $h > 0$, define

$$\tilde{\omega}_2(f,h) = \sup\left\{\left|\sum_{i=1}^p \lambda_i f(\mathbf{y}_i) - f(\mathbf{x})\right|,\ p \in \mathbb{N},\ \mathbf{x},\mathbf{y}_i \in \Delta_m,\right.$$
$$\left.\mathbf{x} = \sum_{i=1}^p \lambda_i \mathbf{y}_i,\ \lambda_i \in (0,1),\ \sum_{i=1}^p \lambda_i = 1,\ \|\mathbf{x}-\mathbf{y}_i\| \leq h\right\}.$$

The theorem below is a scalar version of a result given in [23, Th. 6.2.9].

Theorem B. *Let $D \subset \mathbb{R}^m$ be a compact convex set. Let $F : C(D) \to \mathbb{R}$ be a functional given by a positive Borel measure μ. Suppose $\mu(D) = 1$. Let $\mathbf{x} \in D$ be the barycenter of μ. Then*

$$|F(f) - f(\mathbf{x})| \le \left[1 + h^{-2} F(\| \bullet - \overline{x}\|^2)\right] \tilde{\omega}_2(f, h)$$

for $f \in C(D)$ and $h > 0$.

In a similar mode as in the proof of Theorem 4.5, we obtain

Theorem 4.6. *For $n \in \mathbb{N}$, $\rho > 0$, $f \in C(\Delta_m)$, $m \in \mathbb{N}$ and $h > 0$,*

$$\|\mathbb{U}_n^\rho(f) - f\| \le \left(1 + h^{-2} \frac{\rho+1}{\rho n + 1} \cdot \max\left\{\frac{1}{4}, \frac{m-1}{m}\right\}\right) \tilde{\omega}_2(f, h).$$

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Research Article

Multivariate sampling Kantorovich operators: quantitative estimates in Orlicz spaces

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ABSTRACT. In this paper, we establish a quantitative estimate for multivariate sampling Kantorovich operators by means of the modulus of smoothness in the general setting of Orlicz spaces. As a consequence, the qualitative order of convergence can be obtained, in case of functions belonging to suitable Lipschitz classes. In the particular instance of L^p-spaces, using a direct approach, we obtain a sharper estimate than that one that can be deduced from the general case.

Keywords: Multivariate sampling Kantorovich operators, Orlicz spaces, modulus of smoothness, quantitative estimates, Lipschitz classes.

2020 Mathematics Subject Classification: 41A25, 41A35.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

The theory of generalized sampling-type operators is known since the 80's, when it has been introduced by P. L. Butzer and his school with the aim to study approximate version of the celebrated Whittaker-Kotel'nikov-Shannon sampling theorem. It is closely related to positive linear operators and therefore this work fits into a field of Approximation Theory in which Prof. Francesco Altomare, to whom this paper is dedicated, has given basic and fundamental results (see, e.g., [2–5]).

Considering the long relationship of esteem, sharing of scientific interests and friendship by many of the authors of this paper with Prof. Francesco Altomare, it was an honor for us to have been invited to write this contribution dedicated to him. He was and will certainly continue to be a leading exponent of the Approximation Theory in Italy and abroad.

The Kantorovich version of the sampling-type operators has been introduced in [13], in one-dimensional setting, with the aim to provide a family of linear operators suitable in order to reconstruct not necessarily continuous signals. Indeed, approximation results have been established in the very general context of Orlicz spaces ([12, 32–34]), in which are included the L^p-spaces and several other cases of well-known function spaces. Later on, a very complete theoretical study on all the above operators have been given, see [1, 6–8, 10, 15, 22, 29, 30, 35]. For instance, saturation results and inverse theorems of approximation have been established in [14, 26–28].

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The multivariate version of the above operators has been treated in [24] (see also [11]). This revealed to be very useful in order to face the problem of image reconstruction in several applied fields, since digital images are typical examples of multivariate discontinuous signals. For more details and, as concerns some applications of the above theory to concrete problems, the readers can see [9,16,20,21].

In the present paper, a quantitative estimate for the multivariate sampling Kantorovich operators has been proved for functions belonging to Orlicz spaces. In order to get the above task, a suitable definition of the modulus of smoothness, based on the modular of the space, has been employed. We recall that the concept of modular arises from the theory of the modular spaces (see, e.g., [12,32]), which represent a further generalization of the Orlicz spaces. Moreover, let us remark that the results proved in the present paper contain, as particular cases, some of the results established in [25] in the one-dimensional setting.

Formulating the prefixed results in the setting of Orlicz spaces allows a unifying approach that naturally includes results in L^p-spaces, Zygmund spaces, exponential spaces, and others; see, e.g., [12,25,32–34].

In particular, if we face the above problem in the L^p-case (that coincides with the Orlicz space generated by the φ-function $\varphi(u) = u^p$) by a direct approach, the modulus of smoothness of the Orlicz space reduces to the usual L^p-modulus of smoothness $\omega(f, \delta)_p$. This allows to exploit the well-known properties of $\omega(f, \delta)_p$ in order to establish an estimate that turns out to be sharper than that one achieved in the general (Orlicz) case.

Finally, the qualitative versions of the above results have been obtained assuming the involved functions in suitable Lipschitz classes.

2. Preliminary Notions

We begin this section recalling the general setting of Orlicz spaces, in which we will work. First, we recall the notion of φ-function. A function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be a $\varphi-$function if it satisfies the following conditions:

(Φ_1) φ is continuous and non-decreasing on $\mathbb{R}_0^+$;
(Φ_2) $\varphi(0) = 0$, $\varphi(u) > 0$ for every $u > 0$;
(Φ_3) $\lim\limits_{u\to+\infty} \varphi(u) = +\infty$.

For any fixed $\varphi-$function φ, we introduce the following modular functional $I^\varphi : M(\mathbb{R}^n) \to [0, +\infty]$ defined by

$$I^\varphi[f] := \int_{\mathbb{R}^n} \varphi(|f(\underline{x})|)\, d\underline{x}$$

for every $f \in M(\mathbb{R}^n)$, where here $M(\mathbb{R}^n)$ denotes the set of all Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{R}$. Then, the Orlicz space generated by a φ function φ is defined by

$$L^\varphi(\mathbb{R}^n) := \{f \in M(\mathbb{R}^n) : I^\varphi[\lambda f] < +\infty \text{ for some } \lambda > 0\}.$$

Now, we can recall a well-known concept of convergence in Orlicz spaces, i.e., the *modular convergence* (see, e.g., [12,31,32]). We say that a net of functions $(f_w)_{w>0} \subset L^\varphi(\mathbb{R}^n)$ is modularly convergent to a function $f \in L^\varphi(\mathbb{R}^n)$, if

$$\lim_{w\to+\infty} I^\varphi[\lambda(f_w - f)] = \lim_{w\to+\infty} \int_{\mathbb{R}^n} \varphi(\lambda|f_w(\underline{x}) - f(\underline{x})|)\, d\underline{x} = 0 \tag{2.1}$$

for some $\lambda > 0$. Now, in order to establish a quantitative estimate for the order of approximation of a family of linear multivariate operators, we recall the definition of the modulus of smoothness in Orlicz spaces $L^{\varphi}(\mathbb{R}^n)$. For any fixed $f \in M(\mathbb{R}^n)$ and $\delta > 0$, we define the Orlicz-type modulus of smoothness by

$$\omega(f,\delta)_{\varphi} := \sup_{\|\underline{t}\|_2 \le \delta} I^{\varphi}\left[f(\cdot + \underline{t}) - f(\cdot)\right], \tag{2.2}$$

where $\|\cdot\|_2$ denotes the usual Euclidean norm of $\mathbb{R}^n$. It is well-known (see [12, Theorem 2.4]) that if $f \in L^{\varphi}(\mathbb{R}^n)$, then there exists $\lambda > 0$ such that

$$\lim_{\delta\to 0} \omega(\lambda f,\delta)_{\varphi} = 0.$$

Now, we recall the definition of the class of operators we work with. Let $\Pi^n = \left(t_{\underline{k}}\right)_{\underline{k}\in\mathbb{Z}^n} \subset \mathbb{R}^n$ be a sequence defined by $t_{\underline{k}} = (t_{k_1}, ..., t_{k_n})$, where each $(t_{k_i})_{k_i\in\mathbb{Z}}$, $i = 1, ..., n$ is a sequence of real numbers such that $-\infty < t_{k_i} < t_{k_i+1} < +\infty$, $\lim_{k_i\to\pm\infty} t_{k_i} = \pm\infty$ for every $i = 1, ..., n$ and there are two positive constants Δ, δ for which $\delta \le \Delta_{k_i} := t_{k_i+1} - t_{k_i} \le \Delta$ for every $i = 1, ..., n$. Moreover, we denote by

$$R_{w,t_{\underline{k}}} := \left[\frac{t_{k_1}}{w}, \frac{t_{k_1+1}}{w}\right] \times ... \times \left[\frac{t_{k_n}}{w}, \frac{t_{k_n+1}}{w}\right], \qquad (w > 0),$$

the $n-$dimensional interval associated to the sequence Π^n. For the sake of simplicity, we denote the Lebesgue measure of any $R_{w,t_{\underline{k}}}$ by $A_{\underline{k}}/w^n$, where $A_{\underline{k}} := \Delta_{k_1} \cdot \Delta_{k_2} \cdot \ldots \cdot \Delta_{k_n}$. In general, sequences of the form Π^n are not necessarily equally distributed on $\mathbb{R}^n$, in fact producing a non-uniform sampling scheme. Clearly, if we consider the sequence $t_{\underline{k}} = \underline{k}$, $\underline{k} \in \mathbb{Z}^n$, we obtain an equally spaced grid of nodes, with $\delta = \Delta = \Delta_{k_i} = 1$ and $A_{\underline{k}} = 1$ for every $\underline{k} \in \mathbb{Z}^n$. From now on, a function $\chi : \mathbb{R}^n \to \mathbb{R}$ will be called a kernel if it satisfies the following conditions:

(χ_1) $\chi \in L^1(\mathbb{R}^n)$ and is bounded in a neighborhood of $\underline{0} \in \mathbb{R}^n$;

(χ_2) for every $\underline{u} \in \mathbb{R}^n$, with $\underline{u} = (u_1, ..., u_n)$, we have

$$\sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(\underline{u} - t_{\underline{k}}\right) = 1;$$

(χ_3) for some $\beta > 0$,

$$m_{\beta,\Pi^n}(\chi) := \sup_{\underline{u}\in\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} \left|\chi\left(\underline{u} - t_{\underline{k}}\right)\right| \left\|\underline{u} - t_{\underline{k}}\right\|_2^{\beta} < +\infty,$$

i.e., the discrete absolute moment of order β of χ is finite.

For examples of multivariate kernels see, e.g., [24]. We recall that, several (but not all) examples of multivariate kernels are defined as the product of n one-dimensional kernels, such as the Fejér kernel, the central B-splines, and many others ([17–19]). Now, we recall the following lemma that will be useful in the proof of the results of the next section.

Lemma 2.1. (see [24]) *Let χ be a kernel satisfying conditions (χ_1) and (χ_3). Then, we have*

$$m_{0,\Pi^n}(\chi) := \sup_{\underline{u}\in\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} \left|\chi\left(\underline{u} - t_{\underline{k}}\right)\right| < +\infty,$$

where the convergence of the series $\sum_{\underline{k}\in\mathbb{Z}^n} \left|\chi\left(\underline{u} - t_{\underline{k}}\right)\right|$ is uniform on the compact subsets of $\mathbb{R}^n$.

Thus, for any given kernel χ, the corresponding family of multivariate sampling Kantorovich operators is defined by

$$\left(S_w^\chi f\right)(\underline{x}) := \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x}-t_{\underline{k}}\right)\left[\frac{w^n}{A_{\underline{k}}}\int_{R_{w,t_{\underline{k}}}} f\left(\underline{u}\right)d\underline{u}\right], \qquad \underline{x}\in\mathbb{R}^n,$$

where $f:\mathbb{R}^n\to\mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x}\in\mathbb{R}^n$. It is well-known that the above operators are well-defined, for instance, if $f\in L^\infty(\mathbb{R})$, or if $f\in L^\varphi(\mathbb{R})$, where φ is any convex φ-function (see [24]). In particular, in the setting of Orlicz spaces, the modular convergence of the family $S_w^\chi f$ to f has been established in Theorem 4.5 of [24].

3. Main Results

In this section, we establish a quantitative estimate for the multivariate sampling Kantorovich operators using the modulus of smoothness in Orlicz spaces $L^\varphi\left(\mathbb{R}^n\right)$, recalled in Section 2.

Theorem 3.1. *Let φ be a convex φ-function. Suppose that for any fixed $0<\alpha<1$, we have*

$$w^n\int_{\|\underline{y}\|_2>1/w^\alpha}\left|\chi\left(w\underline{y}\right)\right|d\underline{y}\leq Mw^{-\gamma}, \text{ as } w\to+\infty \tag{3.3}$$

for suitable positive constants M,γ depending on α and χ. Then, for every $f\in L^\varphi\left(\mathbb{R}^n\right)$ and a suitable $\lambda_f=\lambda>0$, the following estimate holds:

$$\begin{aligned} I^\varphi\left[\lambda\left(S_w^\chi f-f\right)\right] \leq{}& \frac{\|\chi\|_1}{2\delta^n m_{0,\Pi^n}(\chi)}\omega\left(2m_{0,\Pi^n}(\chi)\,f,\frac{1}{w^\alpha}\right)_\varphi \\ &+\frac{MI^\varphi\left[4\lambda m_{0,\Pi^n}(\chi)\,f\right]}{2\delta^n m_{0,\Pi^n}(\chi)}w^{-\gamma} \\ &+\frac{\Delta^n}{2\delta^n}\,\omega\left(2m_{0,\Pi^n}(\chi)\,f,\sqrt{n}\,\frac{\Delta}{w}\right)_\varphi \end{aligned}$$

for every sufficiently large $w>0$, where $m_{0,\Pi^n}(\chi)<+\infty$ in view of Lemma 2.1. In particular, if $\lambda>0$ is sufficiently small, this inequality implies the modular convergence of multivariate sampling Kantorovich operators $S_w^\chi f$ to f.

Proof. Let $\lambda>0$ be fixed. Taking into account that φ is convex and non-decreasing, we have

$$\begin{aligned} &I^\varphi\left[\lambda\left(S_w^\chi f-f\right)\right] \\ \leq&\frac{1}{2}\left\{\int_{\mathbb{R}^n}\varphi\left(2\lambda\left|\left(S_w^\chi f\right)(\underline{x})-\sum_{\underline{k}\in\mathbb{Z}^n}\chi\left(w\underline{x}-t_{\underline{k}}\right)\frac{w^n}{A_{\underline{k}}}\int_{R_{w,t_{\underline{k}}}}f\left(\underline{u}+\underline{x}-\frac{t_{\underline{k}}}{w}\right)d\underline{u}\right|\right)d\underline{x}\right. \\ &\left.+\int_{\mathbb{R}^n}\varphi\left(2\lambda\left|\sum_{\underline{k}\in\mathbb{Z}^n}\chi\left(w\underline{x}-t_{\underline{k}}\right)\frac{w^n}{A_{\underline{k}}}\int_{R_{w,t_{\underline{k}}}}f\left(\underline{u}+\underline{x}-\frac{t_{\underline{k}}}{w}\right)d\underline{u}-f(\underline{x})\right|\right)d\underline{x}\right\} \\ =&:J_1+J_2. \end{aligned}$$

Now, we estimate J_1. Using Jensen inequality twice (see, e.g., [23]), the change of variable $\underline{y} = \underline{x} - \dfrac{t_{\underline{k}}}{w}$, and Fubini-Tonelli theorem, we obtain

$$
\begin{aligned}
2J_1 \leq & \int_{\mathbb{R}^n} \varphi\left(2\lambda \sum_{\underline{k}\in\mathbb{Z}^n} \left|\chi\left(w\underline{x} - t_{\underline{k}}\right)\right| \frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} \left|f\left(\underline{u}\right) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right| d\underline{u}\right) d\underline{x} \\
\leq & \frac{1}{m_{0,\Pi^n}\left(\chi\right)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} \left|\chi\left(w\underline{x} - t_{\underline{k}}\right)\right| \varphi\left(2\lambda m_{0,\Pi^n}\left(\chi\right) \frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} \left|f\left(\underline{u}\right) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right| d\underline{u}\right) d\underline{x} \\
\leq & \frac{1}{m_{0,\Pi^n}\left(\chi\right)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} \left|\chi\left(w\underline{x} - t_{\underline{k}}\right)\right| \frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} \varphi\left(2\lambda m_{0,\Pi^n}\left(\chi\right) \left|f\left(\underline{u}\right) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right|\right) d\underline{u} d\underline{x} \\
\leq & \frac{1}{m_{0,\Pi^n}\left(\chi\right) \delta^n} \int_{\mathbb{R}^n} \left|\chi\left(w\underline{y}\right)\right| w^n \sum_{\underline{k}\in\mathbb{Z}^n} \int_{R_{w,t_{\underline{k}}}} \varphi\left(2\lambda m_{0,\Pi^n}\left(\chi\right) \left|f\left(\underline{u}\right) - f\left(\underline{u} + \underline{y}\right)\right|\right) d\underline{u} d\underline{y} \\
= & \frac{1}{m_{0,\Pi^n}\left(\chi\right) \delta^n} \int_{\mathbb{R}^n} \left|\chi\left(w\underline{y}\right)\right| w^n \int_{\mathbb{R}^n} \varphi\left(2\lambda m_{0,\Pi^n}\left(\chi\right) \left|f\left(\underline{u}\right) - f\left(\underline{u} + \underline{y}\right)\right|\right) d\underline{u} d\underline{y} \\
= & \frac{1}{m_{0,\Pi^n}\left(\chi\right) \delta^n} \int_{\mathbb{R}^n} w^n \left|\chi\left(w\underline{y}\right)\right| I^{\varphi}\left[2\lambda m_{0,\Pi^n}\left(\chi\right)\left(f\left(\cdot\right) - f\left(\cdot + \underline{y}\right)\right)\right] d\underline{y}.
\end{aligned}
$$

Now, let $0 < \alpha < 1$ be fixed. We now split the above integral as follows:

$$
\begin{aligned}
& \frac{1}{m_{0,\Pi^n}\left(\chi\right) \delta^n} \left\{ \int_{\left\|\underline{y}\right\|_2 \leq 1/w^{\alpha}} + \int_{\left\|\underline{y}\right\|_2 > 1/w^{\alpha}} \right\} w^n \left|\chi\left(w\underline{y}\right)\right| I^{\varphi}\left[2\lambda m_{0,\Pi^n}\left(\chi\right)\left(f\left(\cdot\right) - f\left(\cdot + \underline{y}\right)\right)\right] d\underline{y} \\
& =: J_{1,1} + J_{1,2}.
\end{aligned}
$$

For $J_{1,1}$, one has

$$
\begin{aligned}
J_{1,1} \leq & \frac{1}{m_{0,\Pi^n}\left(\chi\right) \delta^n} \int_{\left\|\underline{y}\right\|_2 \leq 1/w^{\alpha}} w^n \left|\chi\left(w\underline{y}\right)\right| \omega\left(2m_{0,\Pi^n}\left(\chi\right) f, \left\|\underline{y}\right\|_2\right)_{\varphi} d\underline{y} \\
\leq & \, \omega\left(2m_{0,\Pi^n}\left(\chi\right) f, \frac{1}{w^{\alpha}}\right)_{\varphi} \frac{w^n}{m_{0,\Pi^n}\left(\chi\right) \delta^n} \int_{\left\|\underline{y}\right\|_2 \leq 1/w^{\alpha}} \left|\chi\left(w\underline{y}\right)\right| d\underline{y} \\
\leq & \, \omega\left(2m_{0,\Pi^n}\left(\chi\right) f, \frac{1}{w^{\alpha}}\right)_{\varphi} \frac{\left\|\chi\right\|_1}{m_{0,\Pi^n}\left(\chi\right) \delta^n}.
\end{aligned}
$$

On the other hand, taking into account that φ is convex, for $J_{1,2}$, we can write

$$
\begin{aligned}
J_{1,2} \leq & \frac{1}{m_{0,\Pi^n}\left(\chi\right) \delta^n} \int_{\left\|\underline{y}\right\|_2 > 1/w^{\alpha}} w^n \left|\chi\left(w\underline{y}\right)\right| \frac{1}{2} \left\{ I^{\varphi}\left[4\lambda m_{0,\Pi^n}\left(\chi\right) f\left(\cdot\right)\right] \right. \\
& \left. + I^{\varphi}\left[4\lambda m_{0,\Pi^n}\left(\chi\right) f\left(\cdot + \underline{y}\right)\right] \right\} d\underline{y}.
\end{aligned}
$$

Moreover, it can be easily seen that

$$I^{\varphi}\left[4\lambda m_{0,\Pi^n}(\chi) f(\cdot)\right] = I^{\varphi}\left[4\lambda m_{0,\Pi^n}(\chi) f\left(\cdot + \underline{y}\right)\right]$$

for every $\underline{y}$. Therefore, by assumption (3.3), we finally have

$$\begin{aligned} J_{1,2} &\le \frac{1}{m_{0,\Pi^n}(\chi)\,\delta^n} \int\limits_{\|\underline{y}\|_2 > 1/w^{\alpha}} w^n \left|\chi\left(w\underline{y}\right)\right| I^{\varphi}\left[4\lambda m_{0,\Pi^n}(\chi) f(\cdot)\right] d\underline{y} \\ &\le \frac{I^{\varphi}\left[4\lambda m_{0,\Pi^n}(\chi) f\right]}{m_{0,\Pi^n}(\chi)\,\delta^n} M w^{-\gamma} \end{aligned}$$

for $w > 0$ sufficiently large. Now, we can estimate J_2. Using the singularity assumption (χ_2), we immediately have

$$2J_2 \le \int\limits_{\mathbb{R}^n} \varphi\left(2\lambda \sum_{\underline{k}\in\mathbb{Z}^n} \left|\chi\left(w\underline{x} - t_{\underline{k}}\right)\right| \frac{w^n}{A_{\underline{k}}} \int\limits_{R_{w,t_{\underline{k}}}} \left| f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) - f(\underline{x})\right| d\underline{u}\right) d\underline{x}.$$

Now, using twice Jensen inequality as above and the change of variable $\underline{y} = \underline{u} - \dfrac{t_{\underline{k}}}{w}$, we obtain

$$\begin{aligned} 2J_2 \le& \frac{1}{m_{0,\Pi^n}(\chi)} \int\limits_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} \left|\chi\left(w\underline{x} - t_{\underline{k}}\right)\right| \frac{w^n}{A_{\underline{k}}} \int\limits_{R_{w,t_{\underline{k}}}} \varphi\left(2\lambda m_{0,\Pi^n}(\chi) \left| f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) - f(\underline{x})\right|\right) d\underline{u}\, d\underline{x} \\ \le& \frac{1}{m_{0,\Pi^n}(\chi)\,\delta^n} \int\limits_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} \left|\chi\left(w\underline{x} - t_{\underline{k}}\right)\right| w^n \int\limits_{\left(R_{w,t_{\underline{k}}} - \frac{t_{\underline{k}}}{w}\right)} \varphi\left(2\lambda m_{0,\Pi^n}(\chi) \left| f\left(\underline{x} + \underline{y}\right) - f(\underline{x})\right|\right) d\underline{y}\, d\underline{x}, \end{aligned}$$

where the symbol $\left(R_{w,t_{\underline{k}}} - \dfrac{t_{\underline{k}}}{w}\right) := \left[0, \dfrac{\Delta_{k_1}}{w}\right] \times \ldots \times \left[0, \dfrac{\Delta_{k_n}}{w}\right]$ for every $\underline{k} \in \mathbb{Z}^n$ and $w > 0$. Hence, by the Fubini-Tonelli theorem,

$$\begin{aligned} 2J_2 &\le \frac{1}{\delta^n} \int\limits_{\mathbb{R}^n} w^n \int\limits_{\left(R_{w,t_{\underline{k}}} - \frac{t_{\underline{k}}}{w}\right)} \varphi\left(2\lambda m_{0,\Pi^n}(\chi) \left| f\left(\underline{x} + \underline{y}\right) - f(\underline{x})\right|\right) d\underline{y}\, d\underline{x} \\ &\le \frac{w^n}{\delta^n} \int\limits_{\left(R_{w,t_{\underline{k}}} - \frac{t_{\underline{k}}}{w}\right)} I^{\varphi}\left[2\lambda m_{0,\Pi^n}(\chi) \left(f\left(\cdot + \underline{y}\right) - f(\cdot)\right)\right] d\underline{y} \\ &\le \frac{w^n}{\delta^n} \int\limits_{(\Delta_w)} I^{\varphi}\left[2\lambda m_{0,\Pi^n}(\chi) \left(f\left(\cdot + \underline{y}\right) - f(\cdot)\right)\right] d\underline{y}, \end{aligned}$$

where $(\Delta_w) := \left[0, \frac{\Delta}{w}\right] \times \ldots \times \left[0, \frac{\Delta}{w}\right]$. Then, we get

$$2J_2 \leq \frac{w^n}{\delta^n} \int\limits_{(\Delta_w)} \omega \left(2m_{0,\Pi^n}(\chi) f, \sqrt{n} \frac{\Delta}{w} \right)_\varphi d\underline{y}$$

$$\leq \frac{\Delta^n}{\delta^n} \omega \left(2m_{0,\Pi^n}(\chi) f, \sqrt{n} \frac{\Delta}{w} \right)_\varphi ,$$

which completes the proof. □

Remark 3.1.

(1) *Note that, it is easy to show that for any kernels such that $\chi(\underline{u}) = \mathcal{O}(\|\underline{u}\|_2^{-\theta})$, as $\|\underline{u}\|_2 \to +\infty$, for $\theta > 1$, we have that assumption (3.3) is satisfied for some constant $M > 0$ and $\gamma = (1-\alpha)(\theta-1) > 0$ for every fixed $0 < \alpha < 1$ (see, e.g., [25]).*

(2) *For further examples of kernels satisfying assumption (3.3), with and without compact support, the reader can refer to [15].*

Now, we recall the definition of Lipschitz classes in Orlicz spaces $L^\varphi(\mathbb{R}^n)$. We define by $Lip_\varphi(\nu)$, $0 < \nu \leq 1$, the set of all functions $f \in M(\mathbb{R}^n)$ such that there exists $\lambda > 0$ with

$$I^\varphi[\lambda(f(\cdot) - f(\cdot + \underline{t}))] = \int\limits_{\mathbb{R}^n} \varphi(\lambda |f(\underline{x}) - f(\underline{x} + \underline{t})|)\, d\underline{x} = O(\|\underline{t}\|_2^\nu)$$

as $\|\underline{t}\|_2 \to 0$. From Theorem 3.1, we immediately obtain the following corollary.

Corollary 3.1. *Under the assumptions of Theorem 3.1 with $0 < \alpha < 1$ and for any $f \in Lip_\varphi(\nu)$, $0 < \nu \leq 1$, there exist $K > 0$ and $\lambda > 0$ such that*

$$I^\varphi[\lambda(S_w^\chi f - f)] \leq K w^{-\theta}$$

for sufficiently large $w > 0$, where $\theta := \min\{\alpha\nu, \gamma\}$.

Note that, the results established in Theorem 3.1 and Corollary 3.1 are valid in case of functions belonging to $L^\varphi(\mathbb{R}^n)$, with φ convex. Hence, applications can be easily obtained in some well-known cases of Orlicz spaces, such as the L^p-spaces, the Zygmund (or interpolation) spaces, and the exponential spaces. For more details concerning the above instances of Orlicz spaces, see, e.g., [13,24].

Actually, in the particular case of L^p-spaces (i.e., when $\varphi(u) = u^p$, $u \in \mathbb{R}_0^+$, $p \geq 1$), thanks to the well-known properties of the first order modulus of smoothness in L^p, we can also establish the following direct quantitative estimate, which turns out to be sharper than that one established in the general case considered in Theorem 3.1 (and consequently also in Corollary 3.1).

In order to obtain the above mentioned result for the multivariate sampling Kantorovich operators, we recall, for $f \in L^p(\mathbb{R}^n)$, the definition of the L^p-first order modulus of smoothness of f, given by

$$\omega(f, \delta)_p = \sup_{\|\underline{h}\|_2 \leq \delta} \left(\int\limits_{\mathbb{R}^n} |f(\underline{t} + \underline{h}) - f(\underline{t})|^p \, d\underline{t} \right)^{1/p},$$

with $\delta > 0$, $1 \leq p < +\infty$. We can prove the following estimate.

Theorem 3.2. *Suppose that*

$$M_p(\chi) := \int_{\mathbb{R}^n} |\chi(\underline{t})| \, \|\underline{t}\|_2^p \, d\underline{t} \; < \; +\infty \tag{3.4}$$

for some $1 \leq p < +\infty$. *Then, for every* $f \in L^p(\mathbb{R}^n)$, *the following quantitative estimate holds*

$$\|S_w^\chi f - f\|_p \leq \frac{[2m_{0,\Pi^n}(\chi)]^{(p-1)/p}}{\delta^{n/p}} \left[\|\chi\|_1 + M_p(\chi)\right]^{1/p} \omega(f, 1/w)_p + \left(\frac{\Delta}{\delta}\right)^{n/p} m_{0,\Pi^n}(\chi)\, \omega\left(f, \sqrt{n}\,\frac{\Delta}{w}\right)_p$$

for every sufficiently large $w > 0$.

Proof. Proceeding as in the first part of the proof of Theorem 3.1, and using the Minkowsky inequality, the concavity and hence the subadditivity of the function $|\cdot|^{1/p}$, we have

$$\|S_w^\chi f - f\|_p \leq \left(\int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} |\chi(w\underline{x} - t_{\underline{k}})| \frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} \left| f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right| d\underline{u} \right]^p d\underline{x} \right)^{1/p}$$
$$+ \left(\int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} |\chi(w\underline{x} - t_{\underline{k}})| \frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} \left| f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) - f(\underline{x})\right| d\underline{u} \right]^p d\underline{x} \right)^{1/p}$$
$$=: J_1 + J_2.$$

We now estimate J_1. Proceeding as in the proof of Theorem 3.1, i.e., applying Jensen inequality twice, Fubini-Tonelli theorem and the change of variable $\underline{y} = \underline{x} - \dfrac{t_{\underline{k}}}{w}$, we get

$$J_1^p = \int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} |\chi(w\underline{x} - t_{\underline{k}})| \frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} \left| f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right| d\underline{u} \right]^p d\underline{x}$$
$$\leq m_{0,\Pi^n}(\chi)^{p-1} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} |\chi(w\underline{x} - t_{\underline{k}})| \left[\frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} \left| f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right| d\underline{u} \right]^p d\underline{x}$$
$$\leq m_{0,\Pi^n}(\chi)^{p-1} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} |\chi(w\underline{x} - t_{\underline{k}})| \left[\frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} \left| f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right|^p d\underline{u} \right] d\underline{x}$$
$$\leq m_{0,\Pi^n}(\chi)^{p-1} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} |\chi(w\underline{y})| \left[\frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} |f(\underline{u}) - f(\underline{u} + \underline{y})|^p \, d\underline{u} \right] d\underline{y}$$
$$\leq \frac{m_{0,\Pi^n}(\chi)^{p-1}}{\delta^n} \int_{\mathbb{R}^n} w^n |\chi(w\underline{y})| \left[\sum_{\underline{k}\in\mathbb{Z}^n} \int_{R_{w,t_{\underline{k}}}} |f(\underline{u}) - f(\underline{u} + \underline{y})|^p \, d\underline{u} \right] d\underline{y}$$

$$= \frac{m_{0,\Pi^n}(\chi)^{p-1}}{\delta^n} \int_{\mathbb{R}^n} w^n |\chi(w\underline{y})| \left[\int_{\mathbb{R}^n} |f(\underline{u}) - f(\underline{u}+\underline{y})|^p \, d\underline{u} \right] d\underline{y}$$
$$\le \frac{m_{0,\Pi^n}(\chi)^{p-1}}{\delta^n} \int_{\mathbb{R}^n} w^n |\chi(w\underline{y})| \, \omega\left(f, \|\underline{y}\|_2\right)_p^p \, d\underline{y}$$
$$\le \frac{m_{0,\Pi^n}(\chi)^{p-1}}{\delta^n} \int_{\mathbb{R}^n} w^n |\chi(w\underline{y})| \left(1 + w\|\underline{y}\|_2\right)^p \omega\left(f, \frac{1}{w}\right)_p^p d\underline{y}$$
$$\le \frac{m_{0,\Pi^n}(\chi)^{p-1}}{\delta^n} 2^{p-1} \omega\left(f, \frac{1}{w}\right)_p^p \int_{\mathbb{R}^n} w^n |\chi(w\underline{y})| \left[1 + \left(w\|\underline{y}\|_2\right)^p\right] d\underline{y}$$
$$= \frac{m_{0,\Pi^n}(\chi)^{p-1}}{\delta^n} 2^{p-1} \omega\left(f, \frac{1}{w}\right)_p^p \left\{ \int_{\mathbb{R}^n} w^n |\chi(w\underline{y})| \, d\underline{y} + \int_{\mathbb{R}^n} w^n |\chi(w\underline{y})| \left(w\|\underline{y}\|_2\right)^p d\underline{y} \right\}$$
$$= \frac{m_{0,\Pi^n}(\chi)^{p-1}}{\delta^n} 2^{p-1} \omega\left(f, \frac{1}{w}\right)_p^p \left(\|\chi\|_1 + M_p(\chi)\right) < +\infty$$

for every $w > 0$, where $\|\chi\|_1$ and $M_p(\chi)$ are both finite, in view of (χ_1) and (3.4). Note that, in the above estimates, we used the well-known inequality:

$$\omega(f, \lambda\delta)_p \le (1+\lambda)\omega(f,\delta)_p, \qquad \lambda,\, \delta > 0.^1$$

Now, we estimate J_2. Using Jensen inequality twice, the change of variable $\underline{y} = \underline{u} - \frac{t_{\underline{k}}}{w}$ and Fubini-Tonelli theorem, we have

$$J_2^p = \int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} |\chi(w\underline{x} - t_{\underline{k}})| \frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} \left| f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) - f(\underline{x}) \right| d\underline{u} \right]^p d\underline{x}$$
$$\le \int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} |\chi(w\underline{x} - t_{\underline{k}})| \frac{w^n}{A_{\underline{k}}} \int_{\left(R_{w,t_{\underline{k}}} - \frac{t_{\underline{k}}}{w}\right)} |f(\underline{x}+\underline{y}) - f(\underline{x})| \, d\underline{y} \right]^p d\underline{x}$$
$$\le m_{0,\Pi^n}(\chi)^{p-1} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} |\chi(w\underline{x} - t_{\underline{k}})| \left[\frac{w^n}{A_{\underline{k}}} \int_{\left(R_{w,t_{\underline{k}}} - \frac{t_{\underline{k}}}{w}\right)} |f(\underline{x}+\underline{y}) - f(\underline{x})| \, d\underline{y} \right]^p d\underline{x}$$

[1] In general, this inequality does not hold in the case of $\omega(f,\delta)_\varphi$ (i.e., in Orlicz spaces).

$$\leq \frac{m_{0,\Pi^n}(\chi)^{p-1}}{\delta^n} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} \left|\chi\left(w\underline{x}-t_{\underline{k}}\right)\right| w^n \left[\int_{\left(R_{w,t_{\underline{k}}}-\frac{t_{\underline{k}}}{w}\right)} \left|f\left(\underline{x}+\underline{y}\right)-f\left(\underline{x}\right)\right|^p d\underline{y}\right] d\underline{x}$$

$$\leq \frac{m_{0,\Pi^n}(\chi)^{p}}{\delta^n} \int_{\mathbb{R}^n} w^n \left[\int_{(\Delta_w)} \left|f\left(\underline{x}+\underline{y}\right)-f\left(\underline{x}\right)\right|^p d\underline{y}\right] d\underline{x},$$

where $(\Delta_w) := \left[0, \frac{\Delta}{w}\right] \times \ldots \times \left[0, \frac{\Delta}{w}\right]$. Then, we obtain

$$\begin{aligned} J_2^p &\leq \frac{m_{0,\Pi^n}(\chi)^{p}}{\delta^n} \int_{(\Delta_w)} w^n \left[\int_{\mathbb{R}^n} \left|f\left(\underline{x}+\underline{y}\right)-f\left(\underline{x}\right)\right|^p d\underline{x}\right] d\underline{y} \\ &\leq \frac{m_{0,\Pi^n}(\chi)^{p}}{\delta^n} \int_{(\Delta_w)} w^n \left[\omega\left(f, \sqrt{n}\,\frac{\Delta}{w}\right)_p\right]^p d\underline{y} \\ &= \frac{m_{0,\Pi^n}(\chi)^{p}}{\delta^n} \Delta^n \left[\omega\left(f, \sqrt{n}\,\frac{\Delta}{w}\right)_p\right]^p. \end{aligned}$$

This proves the theorem. □

Note that also assumption (3.4) is quite standard and it is satisfied for kernels χ having sufficiently rapid decay, as for instance χ with compact support. Moreover, we remark that assumption (3.4) implies (3.3) but, in this context, (3.4) appears more natural to assume rather than (3.3). Also, note that what it allows us to achieve a sharper quantitative estimate depends only on the properties of the L^p-modulus of smoothness.

As made in the general context of Orlicz spaces, from the above quantitative estimate, we can directly deduce the qualitative order of approximation, assuming f in suitable Lipschitz spaces.

Firstly, we recall that the Lipschitz class of Zygmund-type in L^p-spaces, with $0 < \alpha \leq 1$, are defined as follows:

(3.5) $$Lip(\alpha, p) := \left\{f \in L^p(\mathbb{R}^n) : \|f(\cdot+\underline{t}) - f(\cdot)\|_p = O\left(\|\underline{t}\|_2^\alpha\right), \text{ as } \|\underline{t}\|_2 \to 0\right\}.$$

Now, we can state the following result.

Corollary 3.2. *Suppose that*

$$M_p(\chi) := \int_{\mathbb{R}^n} |\chi(\underline{t})|\, \|\underline{t}\|_2^p\, d\underline{t} < +\infty$$

for some $1 \leq p < +\infty$. *Then, for every* $f \in Lip\,(\alpha, p)\,,\, 0 < \alpha \leq 1$, *we have*

$$\|S_w^\chi f - f\|_p \leq \frac{[2m_{0,\Pi^n}(\chi)]^{(p-1)/p}}{\delta^{n/p}} \left[\|\chi\|_1 + M_p(\chi)\right]^{1/p} C_1 \frac{1}{w^\alpha} + \left(\frac{\Delta}{\delta}\right)^{n/p} m_{0,\Pi^n}(\chi)\, C_1 \left(\sqrt{n}\,\frac{\Delta}{w}\right)^\alpha$$

for every sufficiently large $w > 0$, *where* $C_1 > 0$ *is the constant coming from definition (3.5).*

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Research Article

Approximation in weighted spaces of vector functions

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ABSTRACT. In this paper, we present the duality theory for general weighted space of vector functions. We mention that a characterization of the dual of a weighted space of vector functions in the particular case $V \subset C^+(X)$ is mentioned by J. B. Prolla in [6]. Also, we extend de Branges lemma in this new setting for convex cones of a weighted spaces of vector functions (Theorem 4.2). Using this theorem, we find various approximations results for weighted spaces of vector functions: Theorems 4.2-4.6 as well as Corollary 4.3. We mention also that a brief version of this paper, in the particular case $V \subset C^+(X)$, is presented in [3], Chapter 2, subparagraph 2.5.

Keywords: Nachbin family, weighted space of vector functions, $p-$Radon measure, polar set, extreme point, convex cone, antialgebraic set with respect to a pair (M, C).

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Dedicated to Professor Francesco Altomare on the occasion of his 70th birthday.

1. INTRODUCTION

The weighted spaces of scalar functions was introduced and studied by L. Nachbin in [4] (see also [5]). We recall that if V is a Nachbin family of upper semi-continuous functions on the locally compact spaces X, then the weighted space associated to V, denoted by $CV_0(X)$, is the set of all continuous functions f on X such that the function $f \cdot v$ vanishes at infinity. Any weight $v \in V$ generate a seminorm $p_v : CV_0(X) \to \mathbb{R}_+$ defined by $p_v(f) = \sup\{v(x) \cdot |f(x)| : \ x \in X\}$. The locally convex topology defined by this family of seminorms is denoted by ω_V and it will be called the weighted topology on $CV_0(X)$. For some specific families of weights V, some different classes of continuous functions on a locally compact space are obtained, namely the functions with compact support, bounded functions, the functions vanishing at infinity, the rapidly decreasing functions at infinity and so on. A characterization of the dual space of the locally convex spaces $(CV_0(X), \omega_V)$ was obtained by W. H. Summers in [7]. More precisely, he showed that if $V \leq C^+(X)$ then, the dual space $[CV_0(X)]^*$ is isomorphic with the space $V \cdot M_b(X)$, where $M_b(X)$ is the space of all bounded Radon measure on X. A similar result for weighted spaces of **vector functions**, in the particular case $V \subset C^+(X)$, is mentioned by J. B. Prolla in [6]. In Theorem 3.1 of this paper, we obtain a characterization of the dual of a weighted space of vector functions in the general case of the upper semi-continuous weights. The key to getting this result is a new result of Measure Theory, namely Proposition 2.1, in which it is proved that if $U : \mathsf{K}(X, E) \to \mathbb{R}$ is a $p-$Radon measure, then there exists a smallest

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positive Radon measure on X, denoted by $|U|$, such that

$$|U(f)| \leq \int p \circ f d|U|, \forall f \in \mathsf{K}(X, E).$$

Using two fundamental tools in functional analysis: Hahn-Banach and Krein –Milman theorems, in 1959, Louis de Branges [1] give a nice proof of Stone-Weierstrass theorem on algebras of real continuous functions on a compact Hausdorff space. Some generalizations of de Branges lemma for weighted space of scalar functions was obtained in [2]. In the last part of this paper, we present a generalization of de Branges lemma for a convex cone in a weighted spaces of **vector functions** (Theorem 4.2). Using this theorem, we obtain various approximations results for weighted spaces of vector functions: Theorems 4.2-4.6 as well as Corollary 4.3.

2. Weighted spaces of vector functions

Let X be a locally compact Hausdorff space, let E be a locally convex complete space endowed with a family P of seminorms of E. We denote by $C(X, E)$ the set of all continuous functions $f : X \to E$ and by $C_0(X, E)$ respectively $\mathsf{K}(X, E)$, the set of continuous functions vanishing at infinity, respectively having compact support. We recall that a function $f : X \to E$ ***vanishes at infinity*** if $\lim\limits_{x\to\infty} f(x) = 0$, i.e., for any $p \in \mathsf{P}$ and any $\varepsilon > 0$, there exists a compact subset $K_{\varepsilon,p}$ of X such that

$$p[f(x)] < \varepsilon, \ \forall x \in X \backslash K_{\varepsilon,p}.$$

Further, we shall denote by $\mathcal{F}_0(X, E)$ the set of all functions $f : X \to E$ vanishing at infinity.

Definition 2.1. *A family V of upper semi-continuous, non-negative functions on X such that for any $v_1, v_2 \in V$ and any $\lambda \in \mathbb{R}$, $\lambda > 0$ there exists $w \in V$ such that*

$$v_i(x) \leq \lambda \cdot w(x), \forall x \in X, i = 1, 2$$

*will be called a **Nachbin family** on X. Any element of V will be called **a weight.***

If V is a Nachbin family of weights on X, we denote by

$$CV_0(X, E) = \{f \in C(X, E); \ v \cdot f \in C_0(X, E), \ \forall v \in V\}.$$

We endow this linear space with so called **the weighted topology** $\omega_{V,\mathsf{P}}$, given by the family of seminorms $\|\cdot\|_{v,p}$ or $\|\cdot\|_{p_v}$ defined by

$$\|f\|_{p_v} = \|f\|_{v,p} = \sup\{v(x) \cdot p[f(x)], \ \forall x \in X\}, \ \forall f \in CV_0(X, E).$$

A base of neighborhoods of the origin in $CV_0(X, E)$ is the family $(B_{v,p})_{v\in V, p\in\mathsf{P}}$ given by

$$B_{v,p} = \left\{f \in CV_0(X, E); \ \|f\|_{v,p} \leq 1\right\}.$$

Further, the space $CV_0(X, E)$ endowed with the weighted topology $\omega_{V,\mathsf{P}}$ will be called **the weighted space of vector functions.** As in the scalar case, one can see that $\mathsf{K}(X, E)$ is a dense subset of $CV_0(X, E)$ with respect to the weighted topology $\omega_{V,\mathsf{P}}$. For any $p \in \mathsf{P}$ and any $f \in \mathsf{K}(X, E)$, we denote

$$\|f\|_p = \sup_{x\in X} p[f(x)].$$

Obviously, $\|f\|_p < \infty$ since $p : E \to \mathbb{R}_+$ is a continuous function on the locally compact space E and $f(X) = f(K_f) \cup \{0\}$ is a compact subset of E, where K_f denotes the support of f. If we endow $\mathsf{K}(X, E)$ with the family of seminorms $\left(\|\cdot\|_p\right)_{p\in\mathsf{P}}$, then $\mathsf{K}(X, E)$ becomes a locally convex space and we shall denote by τ_{P} the topology given by these seminorms $\left(\|\cdot\|_p\right)_{p\in\mathsf{P}}$.

Definition 2.2. *A linear map $U : \mathsf{K}(X,E) \to \mathbb{R}$ is called a* $p-$***Radon measure,*** *where $p \in \mathsf{P}$, if for any compact subset $K \subset X$ there exists a positive number α_K such that for any $f \in \mathsf{K}(X,E)$, $f = 0$ on $X \backslash K$, we have*

$$|U(f)| \leq \alpha_K \cdot \|f\|_p \,.$$

If α_K does not depend of the compact K, then U is called a $p-$ ***bounded Radon measure.*** *The smallest $\alpha \in \mathbb{R}_+$, such that $|U(f)| \leq \alpha \cdot \|f\|_p$ will be denoted by $\|U\|_p$.*

Proposition 2.1. *If $U : \mathsf{K}(X,E) \to \mathbb{R}$ is a $p-$Radon measure, then there exists a smallest positive Radon measure on X, denoted by $|U|$, such that*

$$|U(f)| \leq \int p \circ f d\,|U|\,,\ \forall f \in \mathsf{K}(X,E).$$

Moreover, for any function $\varphi \in \mathsf{K}(X,\mathbb{R})$, the map $\varphi U : \mathsf{K}(X,E) \to \mathbb{R}$ given by

$$\varphi U(\psi) = U(\varphi \cdot \psi),\ \forall \psi \in \mathsf{K}(X,E)$$

is a $p-$ bounded Radon measure and we have
a) $\|\varphi U\|_p = |\varphi U|\,(1)$ and generally $\|U\|_p = |U|\,(1)$ if U is $p-$ bounded,
b) $|\varphi U| = |\varphi| \cdot |U|\,,\ \|\varphi U\|_p = |\varphi U|\,(1) = (|\varphi| \cdot |U|)\,(1) = \int |\varphi|\,d\,|U|$.

Proof. Passing to a factorization, we may suppose that p is a norm on X. We consider a relatively compact open subset D of the locally compact space X and for any $\varphi \in \mathsf{K}(X,\mathbb{R})$, $\varphi \geq 0$ and $supp\varphi \subset D$, we put by definition

$$|U|\,(\varphi) = \sup\{U(\psi);\ \psi \in \mathsf{K}(X,E),\ p \circ \psi \leq \varphi\} = \sup\{|U(\psi)|\,; \psi \in \mathsf{K}(X,E),\ p \circ \psi \leq \varphi\}\,.$$

Since $\overline{D}$ is compact and $\psi(x) = 0$, if $\varphi(x) = 0$, we deduce that $\psi = 0$ outside $\overline{D}$ and therefore there exists $\alpha \in \mathbb{R}_+$ such that $|U(\psi)| \leq \alpha \cdot \|\psi\|_p \leq \alpha \cdot \|\varphi\|$, where $\|\varphi\|$ is the uniform norm of φ on X. Hence $|U|\,(\varphi) \leq \alpha \cdot \|\varphi\|$ for all $\varphi \in \mathsf{K}(X,\mathbb{R})$, $\varphi \geq 0$ and supp$\varphi \subset D$. We show now that for any $\varphi_i \in \mathsf{K}(X,\mathbb{R}),\ \varphi_i \geq 0,\ \text{supp}\varphi_i \subset D,\ i = 1,2$, we have

$$|U|\,(\varphi_1 + \varphi_2) = |U|\,(\varphi_1) + |U|\,(\varphi_2).$$

The inequality $|U|\,(\varphi_1 + \varphi_2) \geq |U|\,(\varphi_1) + |U|\,(\varphi_2)$ follows just from the definition. Let $\psi \in \mathsf{K}(X,E)$, $p(\psi) \leq \varphi_1 + \varphi_2$. For any $n \in \mathbb{N}^*$, we consider the functions $\psi_i \in \mathsf{K}(X,E)$ given by

$$\psi_i = \frac{\varphi_i}{\varphi_1 + \varphi_2 + \frac{1}{n}} \cdot \psi,\ i = 1,2.$$

Obviously, we have successively

$$p(\psi_i) = \varphi_i \cdot \frac{p(\psi)}{\varphi_1 + \varphi_2 + \frac{1}{n}} \leq \varphi_i,\ i = 1,2,$$

$$\psi - (\psi_1 + \psi_2) = \frac{1}{n} \cdot \frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}},$$

$$p\left(\psi - (\psi_1 + \psi_2)\right) \leq \frac{1}{n} \cdot p\left(\frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}}\right),$$

$$\text{supp}\left(\frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}}\right) \subset D,\ p\left(\frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}}\right) \leq 1,\ \left|U\left(\frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}}\right)\right| \leq \alpha,$$

$$|U(\psi) - U(\psi_1) - U(\psi_2)| \leq \frac{\alpha}{n},\ U(\psi) \leq U(\psi_1) + U(\psi_2) + \frac{\alpha}{n},$$

$$U(\psi) \leq |U|\,(\varphi_1) + |U|\,(\varphi_2) + \frac{\alpha}{n},\ \forall n \in \mathbb{N}^*,$$

$$U(\psi) \leq |U|\,(\varphi_1) + |U|\,(\varphi_2),\ |U|\,(\varphi_1 + \varphi_2) = \sup\{U(\psi);\ \psi \in \mathsf{K}(X,E),\ p(\psi) \leq \varphi_1 + \varphi_2\}\,,$$

$$|U|(\varphi_1+\varphi_2) \leq |U|(\varphi_1)+|U|(\varphi_2),\ |U|(\varphi_1+\varphi_2) = |U|(\varphi_1)+|U|(\varphi_2).$$

Obviously, we have

$$|U|(\lambda\cdot\varphi) = \lambda\cdot|U|(\varphi),\ \forall\lambda\in\mathbb{R}_+$$

and the map $|U| : \mathsf{K}^+(X,\mathbb{R}) \to \mathbb{R}_+$ is a positive Radon measure on X. Just from the definition, we have

$$|U(\psi)| \leq |U|(p(\psi)),\ \forall\psi\in\mathsf{K}(X,E).$$

On the other hand, taking a positive Radon measure μ on X such that $|U(\psi)| \leq \int p(\psi)d\mu$ then for any $\varphi\in\mathsf{K}(X,\mathbb{R}),\ \varphi\geq 0$, we have

$$\int\varphi d\mu \geq \int p(\psi)d\mu,\ \forall\psi\in\mathsf{K}(X,E),\ p(\psi)\leq\varphi,$$

$$\int\varphi d\mu \geq |U(\psi)|,\ \forall\psi\in\mathsf{K}(X,E),\ p(\psi)\leq\varphi,$$

$$\int\varphi d\mu \geq |U|(\varphi),\ |U|\leq\mu \text{ on } \mathsf{K}^+(X,\mathbb{R}).$$

a) For any $\varphi\in\mathsf{K}(X,\mathbb{R})$, the map $\varphi U : \mathsf{K}(X,E)\to\mathbb{R}$ defined by $\varphi U(\psi) = U(\varphi\cdot\psi)$ is linear and we have

$$|\varphi U(\psi)| \leq \alpha_K\cdot\|\varphi\cdot\psi\|_p \leq \alpha_K\cdot\|\varphi\|\cdot\|\psi\|_p\,,$$

where $K = \mathrm{supp}\varphi$ and therefore φU is a $p-$ bounded Radon measure on $\mathsf{K}(X,E)$. Further, we have

$$\begin{aligned}|\varphi U|(1) &= \int 1d\,|\varphi U|\\ &= \sup\left\{\int hd\,|\varphi U|\,;\ 0\leq h\leq 1,\ h\in\mathsf{K}(X,\mathbb{R})\right\}\\ &= \sup\{(\varphi U)(\psi);\ \psi\in\mathsf{K}(X,\mathbb{R}),\ p(\psi)\leq 1\}\\ &= \|\varphi U\|_p\end{aligned}$$

(In fact, for any $p-$ bounded Radon measure $U' : \mathsf{K}(X,E)\to\mathbb{R}$ we have, using the definition of $|U'|$:

$$\|U'\|_p = |U'|(1) = \int_X d\,|U'|),$$

b) The inequality $|\varphi U| \leq |\varphi|\cdot|U|$ follows immediately. Indeed, if $h\in\mathsf{K}(X,\mathbb{R}),\ h\geq 0$ then,

$$\begin{aligned}|\varphi U|(h) &= \sup\{U(\varphi\cdot\psi);\ p(\psi)\leq h\}\\ &\leq \sup\{|U|(p(\varphi\cdot\psi)\,;\ p(\psi)\leq h\}\\ &= \sup\{(|\varphi|\cdot|U|)(p(\psi))\,;\ p(\psi)\leq h\}\\ &= (|\varphi|\cdot|U|)(h).\end{aligned}$$

Hence $|\varphi U|(h) \leq |\varphi|\cdot|U|(h)$ for any $h\in\mathsf{K}(X,\mathbb{R}),\ h\geq 0$. For the converse inequality, we restrict ourself to the case $\varphi\geq 0$. Let us consider $\psi\in\mathsf{K}(X,E)$ such that $p(\psi)\leq h\cdot\varphi$ and for any $n\in\mathbb{N}^*$, we consider the function $f_n\in\mathsf{K}(X,E)$ defined by

$$f_n = \frac{\psi}{\varphi+\frac{1}{n}}.$$

Obviously, $p(f_n)\leq h$ and therefore

$$|\varphi U|(h) \geq U(\varphi\cdot f_n),\ p(\varphi\cdot f_n)\leq h\cdot\varphi,\ p(\psi-\varphi\cdot f_n)\leq\frac{1}{n}\cdot p(h).$$

Since $\psi = 0$ outside $K = \mathrm{supp}\varphi$, we have

$$\psi - \varphi \cdot f_n = 0 \text{ on } X \backslash K,\, p\left(\psi - \varphi \cdot f_n\right) \leq \frac{1}{n} \cdot \|h\|\,,\ \left|U\left(\psi - \varphi \cdot f_n\right)\right| \leq \alpha_K \cdot \frac{1}{n} \cdot \|h\|$$

and therefore

$$|\varphi U|\,(h) \geq U(\varphi \cdot f_n) \geq U(\psi) - \alpha_K \cdot \|h\| \cdot \frac{1}{n},\ |\varphi U|\,(h) \geq U(\psi).$$

But

$$\left(\varphi\,|U|\right)(h) = |U|\,(\varphi \cdot h) = \sup\left\{U(\psi);\ \psi \in \mathsf{K}(X, E),\ p(\psi) \leq h \cdot \varphi\right\}.$$

From the preceding two lines, we get $|\varphi U|\,(h) \geq \left(\varphi\,|U|\right)(h)$ and finally $|\varphi U| = |\varphi| \cdot |U|$. □

Proposition 2.2. *Let $U : \mathsf{K}(X, E) \to E$ be a $p-$Radonn measure, $f : X \to \overline{\mathbb{R}}$ be an integrable function with respect to the positive Radon measure $|U|$ (i.e., $f \in \mathsf{L}^1\left(|U|\right)$) and let$\left(\varphi_n\right)_n$ be a sequence in $\mathsf{K}(X, \mathbb{R})$ such that $\lim\limits_{n\to\infty} \varphi_n(x) = f(x)$,$|U|$ $-$a.e. on X and such that*

$$\lim_{n\to\infty} \int |f - \varphi_n|\, d\,|U| = 0.$$

Then, the sequence of $p-$ bounded Radon measures $\left(\varphi_n U\right)_n$ is convergent to a $p-$ bounded Radon measure (depending of f only), denoted by fU, i.e., $\lim\limits_{n\to\infty} \|fU - \varphi_n U\|_p = 0$. Moreover, we have

$$|fU| = |f| \cdot |U|\,.$$

Proof. Since $\lim\limits_{n\to\infty} \int |f - \varphi_n|\, d\,|U| = 0$, we deduce that $\lim\limits_{n,m\to\infty} \int |\varphi_n - \varphi_m|\, d\,|U| = 0$ and therefore, using Proposition 2.1, we have

$$\lim_{n,m\to\infty} \|\varphi_n U - \varphi_m U\|_p = \lim_{n,m\to\infty} \int |\varphi_n - \varphi_m|\, d\,|U| = 0.$$

Hence for any $\psi \in \mathsf{K}(X, E)$, the sequence $\left(\varphi_n U(\psi)\right)_n$ of real numbers is convergent to a number denoted $fU(\psi)$ and for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}^*$ such that

$$|\varphi_n U(\psi) - \varphi_m U(\psi)| \leq \|\varphi_n U - \varphi_m U\|_p \cdot \|\psi\|_p \leq \varepsilon \cdot \|\psi\|_p\,,\ \forall n, m \geq n_\varepsilon,$$

$$|fU(\psi) - \varphi_m U(\psi)| \leq \varepsilon \cdot \|\psi\|_p\,,\ \forall m \geq n_\varepsilon,$$

$$|fU(\psi)| \leq |\varphi_m U(\psi)| + \varepsilon \cdot \|\psi\|_p \leq \left(\|\varphi_m U\|_p + \varepsilon\right) \cdot \|\psi\|_p\,.$$

Hence fU is a $p-$ bounded Radon measure on $\mathsf{K}(X, E)$, $\lim\limits_{m\to\infty} \|fU - \varphi_m U\|_p = 0$ (Particularly if $f = 0\,|U|$ a.e., from the relation $\lim\limits_{n\to\infty} \int |f - \varphi_n|\, d\,|U| = 0$, we deduce $\lim\limits_{n\to\infty} \int |\varphi_n|\, d\,|U| = 0$ and therefore $\lim\limits_{n\to\infty} \|\varphi_n U\|_p = \lim\limits_{n\to\infty} \int |\varphi_n|\, d\,|U| = 0$, $\lim\limits_{n\to\infty} \left(\varphi_n U\right)(\psi) = 0$, $\forall \psi \in \mathsf{K}(X, E)$. This shows that the element fU, previously defined, depends only on f, does not depend on the choice of the sequence $\left(\varphi_n\right)_n$ tending to f). Let now $h \in \mathsf{K}(X, \mathbb{R})$, $0 \leq h \leq 1$ and let $\psi \in \mathsf{K}(X, E)$ be such that $p(\psi) \leq h$. We have

$$|fU(\psi) - \varphi_n U(\psi)| \leq \|fU - \varphi_n U\|_p \cdot \|\psi\|_p \leq \|fU - \varphi_n U\|\,,\ \forall n \in \mathrm{N},$$

$$\left(\varphi_n U\right)(\psi) - \|fU - \varphi_n U\|_p \leq fU(\psi) \leq \varphi_n U(\psi) + \|fU - \varphi_n U\|_p\,,$$

$$|\varphi_n U|\,(h) - \|fU - \varphi_n U\|_p \leq |fU|\,(h) \leq |\varphi_n U|\,(h) + \|fU - \varphi_n U\|_p\,.$$

Using Proposition 2.1 b), we deduce that

$$|\varphi_n| \cdot |U|\,(h) - \|fU - \varphi_n U\|_p \leq |fU|\,(h) \leq |\varphi_n| \cdot |U|\,(h) + \|fU - \varphi_n U\|_p$$

$$\int |\varphi_n| \cdot hd\,|U| - \|fU - \varphi_n U\|_p \leq |fU|\,(h) \leq \int |\varphi_n| \cdot hd\,|U| + \|fU - \varphi_n U\|_p \,.$$

Passing to the limit on n, we get

$$\int |f| \cdot hd\,|U| \leq |fU|\,(h) \leq \int |f| \cdot hd\,|U| \,,$$

$$|fU|\,(h) = \int |f| \cdot hd\,|U| = |f| \cdot |U|\,(h).$$

The last equality holds for $0 \leq h \leq 1$ and therefore for all $h \in \mathsf{K}(X, \mathbb{R}),\ h \geq 0$, i.e.,

$$|fU| = |f| \cdot |U| \,.$$

□

3. On the dual of weighted spaces of vector functions

Let E, P, X and V as in the preceding section. For any $p \in \mathsf{P}$ and $v \in V$, let

$$B_{v,p} = \{f \in CV_0(X, E);\ p_v(f) \leq 1\} \,,$$

where $p_v(f) = \sup\{v(x) \cdot p[f(x)];\ \forall x \in X\} = \|f\|_{v,p}\,,\ \ \forall f \in CV_0(X, E)$. The linear vector space $CV_0(X, E)$ endowed with the family $(p_v)_{p\in\mathsf{P},\ v\in V}$ of seminorms is a locally convex space whose fundamental system of neighborhoods of the origin is just the family $(B_{v,p})_{v\in V,\ p\in\mathsf{P}}$. We recall that we have denoted by $\omega_{V,\mathsf{P}}$ the weighted topology on $CV_0(X, E)$ given by the family of seminorms $(p_v)_{p\in\mathsf{P},\ v\in V}$. It is no lost of generality if we suppose that for any real number $\alpha,\ \alpha > 0$, we have $\alpha \cdot p \in \mathsf{P},\ \ \alpha \cdot v \in V$ for any $p \in \mathsf{P}$ and any $v \in V$. So the dual of the locally convex space $(CV_0(X, E), \omega_{V,\mathsf{P}})$ is the set $\bigcup_{v\in V, p\in\mathsf{P}} B^0_{v,p}$, where

$$B^0_{v,p} = \{T : CV_0(X, E) \to \mathbb{R};\ T \text{ linear},\ T(f) \leq 1,\ \forall f \in B_{v,p}\} \,.$$

If we denote by $[CV_0(X, E)]^*$ this dual, then for any subset M of $CV_0(X, E)$ (respectively of $[CV_0(X, E)]^*$), we denote by M^0 the polar of M i.e.,

$$M^0 = \{T \in [CV_0(X, E)]^*;\ T(m) \leq 1,\ \forall m \in M\}$$

respectively

$$M^0 = \{f \in CV_0(X, E);\ m(f) \leq 1,\ \forall m \in M\} \,.$$

The map on $CV_0(X, E) \times [CV_0(X, E)]^* \to \mathbb{R}$, $(f, T) \to \langle f, T\rangle = T(f)$ is a natural duality between the linear space $CV_0(X, E)$ and $[CV_0(X, E)]^*$. The smallest topology on $[CV_0(X, E)]^*$ making continuous the maps

$$T \to \langle f, T\rangle : [CV_0(X, E)]^* \to \mathbb{R},\ \forall f \in CV_0(X, \mathbb{R})$$

is the ***weak topology*** on $[CV_0(X, E)]^*$. It is known (Alaoglu's Theorem) that for any $(p, v) \in \mathsf{P} \times V$, the set $B^0_{p,v}$ is a weakly compact subset of $[CV_0(X, E)]^*$. We know also that the topological space $[CV_0(X, E)]^*$ is a Hausdorff one with respect to this weak topology. Moreover, since $\mathsf{K}(X, E)$ is a dense subset of $CV_0(X, E)$ with respect to the weighted topology $\omega_{V,\mathsf{P}}$, we deduce that

1) any continuous linear functional $L : CV_0(X, E) \to \mathbb{R}$ is completely determined by its restriction to $\mathsf{K}(X, E)$,

2) the smallest topology on $[CV_0(X, E)]^*$ making continuous all linear functionals

$$T \to \langle f, T\rangle : [CV_0(X, E)]^* \to \mathbb{R},\ \forall f \in \mathsf{K}(X, \mathbb{R})$$

is also a Hausdorff one and therefore its restriction to $B^0_{p,v}$ coincides with the restriction to $B^0_{p,v}$ of the weak topology on $[CV_0(X, E)]^*$.

We conclude that any element of the dual of the locally convex space $(\mathsf{K}(X,E),\ \omega_{V,\mathsf{P}}\,|\mathsf{K}(X,E))$ may be uniquely extended to an element of $[CV_0(X,E)]^*$. The following assertion characterizes the elements of $[CV_0(X,E)]^*$ in terms of Radon measures on $\mathsf{K}(X,E)$. With the above notations, we have

Theorem 3.1. *For any $(p,v)\in\mathsf{P}\times V$, we have*
a) The restriction of any element $T\in B^0_{p,v}$ to $\mathsf{K}(X,E)$ is a $p-$Radon measure on $\mathsf{K}(X,E)$ such that the function $\frac{1}{v}$ is integrable with respect to the positive Radon measure $|T|$ on X.
Moreover, the following relation holds:

$$\int\frac{1}{v}d\,|T| = \|T\|_{p,v} = \sup\left\{T(f);\ f\in B_{p,v}\right\},$$

b) For any $p-$Radon measure U on $\mathsf{K}(X,E)$ such that the function $\frac{1}{v}$ is $|U|-$ integrable, there exists $T\in B^0_{p,v}$ such that U is the restriction of T to $\mathsf{K}(X,E)$.

Proof. *a)* Let $T\in B^0_{p,v}$ and let K be a compact subset of X. Since $v:X\to[0,\infty)$ is an upper semi-continuous function, its upper bound α_K on K is finite. Let $\varphi\in\mathsf{K}(X,E)$ such that $\varphi=0$ on $X\backslash K$. We have

$$\sup\left\{v(x)\cdot p(\varphi(x)):\ x\in X\right\}\le\alpha_K\cdot\sup\left\{p(\varphi(x)):\ x\in X\right\}=\alpha_K\cdot\|\varphi\|_p\,,$$

$$\frac{\varphi}{\alpha_K\cdot\|\varphi\|_p}\in B_{p,v},\ \left|T\left(\frac{\varphi}{\alpha_K\cdot\|\varphi\|_p}\right)\right|\le 1,\ |T(\varphi)|\le\alpha_K\cdot\|\varphi\|_p\,,$$

i.e., the restriction of T to $\mathsf{K}(X,E)$, denoted also by T, is a $p-$Radon measure. We have

$$\begin{aligned}\|T\|_{p,v}&=\sup\left\{T(f),\ f\in CV_0(X,E),\ p_v(f)\le 1\right\}\\&=\sup\left\{T(f),\ f\in\mathsf{K}(X,E),\ p_v(f)\le 1\right\}\\&=\sup\left\{T(f),\ f\in\mathsf{K}(X,E),\ p(f)\le\frac{1}{v}\right\}\\&=\int\frac{1}{v}d\,|T|\,.\end{aligned}$$

b) Let U be a $p-$Radon measure on $\mathsf{K}(X,E)$ such that the function $\frac{1}{v}$ is $|U|-$ integrable. Then, we have

$$\begin{aligned}\infty>\int\frac{1}{v}d\,|U|&=\sup\left\{\int\varphi d\,|U|\,;\ \varphi\in\mathsf{K}(X,\mathbb{R}),\ 0\le\varphi\le\frac{1}{v}\right\}\\&=\sup_{\varphi\le\frac{1}{v}}\left\{U(\psi);\ \psi\in\mathsf{K}(X,E),\ p(\psi)\le\varphi\right\}\\&=\sup\left\{U(\psi);\ \psi\in\mathsf{K}(X,E),\ p(\psi)\le\frac{1}{v}\right\}\\&=\sup\left\{U(\psi);\ \psi\in\mathsf{K}(X,E),\ v(x)\cdot p(\varphi(x))\le 1\right\}\\&=\|U\|_{p,v}\,.\end{aligned}$$

□

Remark 3.1. *From the above considerations, we deduce that:*
The elements $T\in B^0_{p,v}$ are $p-$Radon measure on $\mathsf{K}(X,E)$ such that the function $\frac{1}{v}$ is $|T|-$ integrable and $\|T\|_{p,v}=\int\frac{1}{v}d\,|T|\le 1$.

Proposition 3.3. *Let T be a $p-$ Radon measure, $T \in B^0_{p,v}$. If $f \in CV_0(X,E)$, then*

$$|T(f)| \leq \int p(f)d\,|T|\,.$$

Proof. Let $(\psi_n)_n$ be a sequence in $\mathsf{K}(X,E)$ such that $\lim\limits_{n\to\infty} \|f-\psi_n\|_{p,v} = 0$. We know that $|T(\psi_n)| \leq \int p(\psi_n)d\,|T|$ and $T(f) = \lim\limits_{n\to\infty} T(\psi_n)$. On the other hand

$$p(f-\psi_n) \leq \frac{\|f-\psi_n\|_{p,v}}{v} \text{ on } X,$$

$$\int p(f-\psi_n)d\,|T| \leq \|f-\psi_n\|_{p,v} \cdot \int \frac{1}{v} d\,|T| \leq \|f-\psi_n\|_{p,v}$$

$$\int |p(f)-p(\psi_n)|\,d\,|T| \leq \int p(f-\psi_n)d\,|T| \leq \|f-\psi_n\|_{p,v}\,,$$

$$\int p(f)d\,|T| = \lim_{n\to\infty} \int p(\psi_n)d\,|T|\,.$$

Hence

$$|T(f)| = \lim_{n\to\infty} |T(\psi_n)| \leq \lim_{n\to\infty} \int p(\psi_n)d\,|T| = \int p(f)d\,|T|\,.$$

$\square$

Corollary 3.1. *If $T \in B^0_{p,v}$ and $f \in CV_0(X,E)$ is such that $f = 0$ on $\operatorname{supp}|T|$, then $T(f) = 0$.*

4. Lemma de Branges and approximation results

In this section, we preserve all notations used in the preceding paragraphs. For any subset $A \subset CV_0(X,E)$, we denote by A^0 the polar of A, i.e.,

$$A^0 = \{T \in [CV_0(X,E)]^*;\ T(a) \leq 1,\ \forall a \in A\}\,.$$

If C is a convex cone of the real vector space $CV_0(X,E)$ then, one can see that

$$\mathsf{C}^0 = \{T \in [CV_0(X,E)]^*;\ T(c) \leq 0,\ \forall c \in \mathsf{C}\}\,.$$

Theorem 4.2. *Let C be a convex cone in $CV_0(X,E)$, $p \in \mathsf{P}$, $v \in V$ and let $L \in B^0_{p,v} \cap \mathsf{C}^0$, $L \neq 0$ be an extreme point of the convex and compact subset $B^0_{p,v} \cap \mathsf{C}^0$. If $h \in C(X,[0,1])$ is such for any $c \in \mathsf{C}$, we have $h \cdot c\,|\sigma(|L|) \in \mathsf{C}\,|\sigma(|L|)$ and $(1-h)\cdot c\,|\sigma(|L|) \in \mathsf{C}\,|\sigma(|L|)$, then h is constant on $\sigma(|L|)-$ the support of the positive Radon measure $|L|$ on X.*

Proof. Since $L \neq 0$ and L is an extreme point of the subset $B^0_{p,v} \cap \mathsf{C}^0$, we have $\|L\|_{p,v} = \int \frac{1}{v} d\,|L|\,.$ If h is an arbitrary element in$C(X,[0,1])$, then the map $hL : \mathsf{K}(X,E) \to \mathbb{R}$, given by $hL(\psi) = L(h\cdot\psi)$, is a $p-$Radon measure on $\mathsf{K}(X,E)$. It is not so difficult to show, using the definition, that $|hL| = |h|\cdot|L|$. Obviously, the function $\frac{1}{v}$ is $|h|\cdot|L|-$ integrable and using Remark 3.1 and the relations

$$\|hL\|_{p,v} = \int \frac{1}{v} d\,|hL| = \int \frac{h}{v} d\,|L| \leq \int \frac{1}{v} d\,|L| \leq 1,$$

we get $hL \in B^0_{p,v}$. Analogously, the map $(1-h)L : \mathsf{K}(X,E) \to \mathbb{R}$ given by $(1-h)L(\psi) = L((1-h)\cdot(\psi))$ is a $p-$Radon measure and

$$\|(1-h)L\|_{p,v} = \int \frac{1-h}{v} d\,|L| \leq \int \frac{1}{v} d\,|L| = 1,\ (1-h)L \in B^0_{p,v}.$$

If we denote $\alpha = \|hL\|_{p,v} = \int \frac{h}{v} d\,|L|\,$, $\beta = \|(1-h)L\|_{p,v} = \int \frac{1-h}{v} d\,|L|$, we have $\alpha + \beta = \int \frac{1}{v} d\,|L| = 1$. We remark also that the function $\frac{1}{v}$ is strictly positive on X. If $\alpha = 0$, then

$h = 0$ $|L|$ a.e. on $\sigma(|L|)$. Since the function h is continuous, it results that $h = 0$ on $\sigma(|L|)$, i.e., h is constant on $\sigma(|L|)$. Analogously, if $\beta = 0$, we obtain $h = 1$ on $\sigma(|L|)$, i.e., h is constant on $\sigma(|L|)$. We suppose further $\alpha \neq 0,\ \beta \neq 0$ and we denote

$$L_1 = \frac{1}{\alpha} \cdot hL,\ L_2 = \frac{1}{\beta} \cdot (1-h)L.$$

Obviously, $\|L_i\|_{p,v} = 1,\ i = 1,2$ and $\alpha \cdot L_1 + \beta \cdot L_2 = L$. We show now that $L_i \in \mathsf{C}^0,\ i = 1,2$, if for any $c \in \mathsf{C}$ there exist $c_1, c_2 \in \mathsf{C}$ such that $h \cdot c = c_1,\ (1-h) \cdot c = c_2$ on $\sigma(|L|)$. Since the functions $h \cdot c,\ (1-h) \cdot c,\ c_1,\ c_2$ belong to $CV_0(X,E)$ and $h \cdot c = c_1$ on $\sigma(|L|)$, respectively $(1-h) \cdot c = c_2$ on $\sigma(|L|)$, using Corollary 3.1, we get

$$L(h \cdot c) = L(c_1) \leq 0,\ L((1-h) \cdot c) = L(c_2) \leq 0,$$

$$L_1(c) = \frac{1}{\alpha} \cdot L(h \cdot c) = \frac{1}{\alpha} \cdot L(c_1) \leq 0,\ L_2(c) = \frac{1}{\beta} \cdot L((1-h) \cdot c) = \frac{1}{\beta} \cdot L(c_2) \leq 0.$$

Hence L_1, L_2 belong to the set $B^0_{p,v} \cap \mathsf{C}^0$ and since $L = \alpha \cdot L_1 + \beta \cdot L_2$, we get $L_1 = L_2 = L$. Hence $|L_1| = |L|$, i.e., the measures $\frac{h}{\alpha} \cdot |L|$ and $|L|$ coincide and therefore $\frac{h}{\alpha} = 1$ almost everywhere on $\sigma(|L|)$. But h is continuous and hence $h = \alpha$ on $\sigma(|L|)$. $\square$

Definition 4.3. *A subset $\mathsf{M} \subset C(X,[0,1])$ is called **complemented**, if for any $h \in \mathsf{M}$, the function $1-h$ belongs to M. If $\mathsf{C} \subset CV_0(X,E)$ is a convex cone and $\mathsf{M} \subset C(X,[0,1])$ is a complemented family, then a subset $S \subset X$ is called **antialgebraic with respect to the pair** (M,C) (or simpler $(\mathsf{M},\mathsf{C})-$antialgebraic), if any $h \in \mathsf{M}$ such that the restriction to S of the functions $h \cdot c$ and $(1-h) \cdot c$ belong to the restriction of C to S (i.e., $h \cdot c\,|S \in \mathsf{C}\,|S$, $(1-h) \cdot c\,|S \in \mathsf{C}\,|S$) for any $c \in \mathsf{C}$, is a constant function on S.*

We can reformulate de Branges lemma (Theorem 4.2) as follows:

Corollary 4.2. *For any extreme point L of $B^0_{p,v} \cap \mathsf{C}^0$, the support $\sigma(|L|)$ of the positive Radon measure $|L|$ on X is an antialgebraic subset with respect to the pair $(C(X,[0,1]),\mathsf{C})$. Further, we denote by S the family of all subsets of X antialgebraic with respect to the pair (M,C).*

The following assertions are almost obvious.
i) $\{x\} \in \mathsf{S},\ \forall x \in X$,
ii) $S_1, S_2 \in \mathsf{S},\ S_1 \cap S_2 \neq \phi \Rightarrow S_1 \cup S_2 \in \mathsf{S}$,
iii) $S \in \mathsf{S} \Rightarrow \bar{S} \in \mathsf{S}$,
iv) For any upper directed family $(S_\alpha)_{\alpha \in I}$ from S, we have $\bigcup_{\alpha \in I} S_\alpha \in \mathsf{S}$.

If for any $x \in X$, we denote by $S_x = \cup\{S;\ S \in \mathsf{S}, x \in S\}$, then we have

$$S_x = \overline{S_x} \in \mathsf{S},\ S_x \cap S_y = \phi \ \text{ if } \ S_x \neq S_y.$$

The family $(S_x)_{x \in X}$ is a partition of X and for any $S \in \mathsf{S}$ there exists $x \in X$ such that $S \subset S_x$. For the general theory of duality, we have for any convex cone C, $\mathsf{C} \subset CV_0(X,E)$, the closure $\bar{\mathsf{C}}$ in $CV_0(X,E)$ with respect to the weighted topology $\omega_{\mathsf{P},V}$ coincides with the bipolar of C i.e., $\bar{\mathsf{C}} = \mathsf{C}^{00}$. In the our special case, we have the following general approximation theorem.

Theorem 4.3. *If $\mathsf{C} \subset CV_0(X,E)$ is a convex cone, then the closure of C in $(CV_0(X,E),\ \omega_{\mathsf{P},V})$ is given by*

$$\overline{\mathsf{C}} = \left\{ f \in CV_0(X,E);\ f\,|\sigma(|L|) \in \overline{\mathsf{C}\,|\sigma(|L|)},\ \forall L \in Ext\left(B^0_{p,v} \cap \mathsf{C}^0\right),\ \forall v \in V,\ \forall p \in \mathsf{P} \right\}.$$

Proof. We show only that for any function $g \in CV_0(X,E)\backslash\bar{\mathsf{C}}$ there exist $p \in \mathsf{P}$, $v \in V$ and $L \in Ext\left(B^0_{p,v} \cap \mathsf{C}^0\right)$ such that $g\,|\sigma(|L|) \notin \overline{\mathsf{C}\,|\sigma(\mu)}$. Indeed, using Hahn-Banach separation theorem, there exists $T \in [CV_0(X,E)]^*$ such that $T \in \mathsf{C}^0$ and $T(g) > 0$. Let $p \in \mathsf{P}$ and $v \in V$ be such that $|T(f)| \leq \|f\|_{p,v}$, $\forall f \in CV_0(X,E)$ i.e., $|T|\left(\frac{1}{v}\right) \leq 1$. Hence $T \in B^0_{p,v} \cap \mathsf{C}^0$. Since $B^0_{p,v} \cap \mathsf{C}^0$ is a compact convex subset of $[CV_0(X,E)]^*$ with respect to the weak topology and $T(g) > 0$, it follows from Krein-Milman theorem that there exists $L \in Ext\left(B^0_{p,v} \cap \mathsf{C}^0\right)$ such that $L(g) > 0$. Since $L \in \mathsf{C}^0$, we deduce that $\int \varphi d\,|L| \leq 0$ for any $\varphi \in \overline{\mathsf{C}\,|\sigma\,(|L|)}$. Hence $g\left|\sigma(|L|) \notin \overline{\mathsf{C}\,|\sigma(|L|)}\right.$. $\square$

Let now $\mathsf{M} \subset C(X,[0,1])$ be a complemented family and for any $x \in X$ let S_x be the greatest $(\mathsf{M},\mathsf{C})-$ antialgebraic subset of X containing x.

Theorem 4.4. *If $\mathsf{C} \subset CV_0(X,E)$ is a convex cone, then the closure of C in $(CV_0(X,E),\ \omega_{\mathsf{P},V})$ is given by*

$$\bar{\mathsf{C}} = \left\{ f \in CV_0(X,E);\ f\,|S_x \in \overline{\mathsf{C}\,|S_x}\,,\ \forall x \in X \right\}.$$

Proof. For any $p \in \mathsf{P}$,$v \in V$ and any extreme point L of the compact convex subset $B^0_{p,v} \cap \mathsf{C}^0$, the support $\sigma\,(|L|)$ is a $(\mathsf{M},\mathsf{C})-$ antialgebraic subset of X. If we choose a point $x \in \sigma\,(|L|)$, then $\sigma\,(|L|) \subset S_x$, and therefore if $f\,|S_x \in \overline{\mathsf{C}\,|S_x}$, we have also $f\,|\sigma(L) \in \overline{\mathsf{C}\,|\sigma(L)}$. Further, we may use Theorem 4.3. $\square$

Theorem 4.5. *If $\mathsf{M} \subset C(X,[0,1])$ is a complemented family and the convex cone $\mathsf{C} \subset CV_0(X,E)$ is stable with respect to the multiplication of M (i.e., $c \cdot m \in \mathsf{C}, \forall c \in \mathsf{C}, m \in \mathsf{M}$), then we have*

$$\bar{\mathsf{C}} = \left\{ f \in CV_0(X,E);\ f\,|\ [x]_{\mathsf{M}} \in \overline{\mathsf{C}\,|\ [x]_{\mathsf{M}}}\,,\ \forall x \in X \right\},$$

where for any $x \in X$ we denote $[x]_{\mathsf{M}} = \{y \in X;\ m(y) = m(x),\ \forall m \in \mathsf{M}\}$.

Proof. Using just the definitions and previous notations, we deduce that for any $x \in X$ we have $[x]_{\mathsf{M}} = S_x$. Further, we use Theorem 4.4. $\square$

The following assertion needs to define so called **"section in C" by the points of X**, namely to consider the following convex cone $\mathsf{C}(x)$ in E given by

$$\mathsf{C}(x) = \{c(x);\ c \in \mathsf{C}\}$$

and also its closure $\overline{\mathsf{C}(x)}$ in E. Certainly the starting convex cone C in $CV_0(X,E)$ may be a linear subspace and in this case $\mathsf{C}(x)$ is a linear subspace in E.

Theorem 4.6. *If $\mathsf{M} \subset C(X,[0,1])$ is a complemented family and the convex cone $\mathsf{C} \subset CV_0(X,E)$ is stable with respect to the multiplication with elements of M and M separates the points of X, i.e., for any $x, y \in X$ there exists $m \in \mathsf{M}$ such that $m(x) \neq m(y)$, then we have*

$$\bar{\mathsf{C}} = \left\{ f \in CV_0(X,E);\ f(x) \in \overline{\mathsf{C}(x)},\ \forall x \in X \right\}.$$

Indeed, in this case, for any $x \in X$, we have $[x]_{\mathsf{M}} = \{x\}$ and we close the proof applying Theorem 4.5.

Corollary 4.3. *If $\mathsf{M} \subset C(X,[0,1])$ is a complemented family, separating the points of X and $\mathsf{W} \subset CV_0(X,E)$ is a linear subspace which is stable with respect to the multiplication with elements of M and for any $x \in X$ the section $\mathsf{W}(x)$ is a dense subset of the locally convex space (E,P), then*

$$\overline{\mathsf{W}} = CV_0(X,E).$$

Remark 4.2. *For the scalar case $E = \mathbb{R}$, the density of $\mathsf{W}(x)$ in $\mathbb{R}$ is automatically fulfilled unless the case where $\mathsf{W}(x) = \{0\}$ for the points x of a closed subset $F \subset X$. In this case, we have*

$$\overline{\mathsf{W}} = \{f \in CV_0(X);\ f = 0 \text{ on } F\}\,.$$

Even this assertion may be drown from Theorem 4.6 as a particular case where there exists $F \subset X$ such that the section of C by x is trivial for all $x \in F$ i.e., $\mathsf{C}(x) = \{0_E\}$, $\forall x \in F$. Anyway Theorem 4.6 may be used in different manners to obtain density results.

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Research Article

Approximation properties related to the Bell polynomials

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ABSTRACT. The authors provide a complete asymptotic expansion for a class of functions in terms of the complete Bell polynomials. In particular, they obtain known asymptotic expansions of some Keller type sequences.

Keywords: Asymptotic expansions, Bell polynomials.

2020 Mathematics Subject Classification: 41A60.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

The first references to the number e were published in 1618 in the table of an appendix of a work on logarithms by John Napier [1, p. xiii]. The discovery of the constant itself is credited to Jacob Bernoulli in 1690 who considered the problem of continuous compounding of interest,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n .$$

Leonhard Euler introduced the letter e as the base for natural logarithms, writing in a letter to Christian Goldbach on 25 November 1731. In 1665, Newton [1, p. 151] discovered

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots .$$

Let us consider the sequence

$$(n+1)\left(1+\frac{1}{n+1}\right)^{n+1} - n\left(1+\frac{1}{n}\right)^n, \qquad n = 1, 2, \ldots . \tag{1.1}$$

The sequence (1.1) is attributed to Felix A. Keller (see, e.g., [2], [3, p. 14], but its origin surely lay in the Euler age). In 1998, H. J. Brothers and J. A. Knox [4, Eq. (8)] gave the following approximation to e,

$$\begin{aligned}
&\frac{(x+1)^{x+1}}{x^x} - \frac{x^x}{(x-1)^{x-1}}\\
= \;& (1+x)\left(1+\frac{1}{x}\right)^x + (1-x)\left(1-\frac{1}{x}\right)^{-x}\\
= \;& e\left(1+\frac{1}{24x^2}+\frac{11}{640x^4}+\frac{5525}{580608x^6}+\mathcal{O}\left(\frac{1}{x^8}\right)\right), \qquad x\to\infty.
\end{aligned}$$

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We recall an excellent result of Alzer and Berg [5, (2.2)]:

$$(x+1)\left(e-\left(1+\frac{1}{x}\right)^{x}\right)=\frac{e}{2}+\frac{1}{\pi}\int_0^1\frac{s^s(1-s)^{1-s}\,\sin(\pi\,s)}{x+s}\mathrm{d}s,\qquad x>0. \tag{1.2}$$

Next, using the identity

$$\frac{1}{x+s}=\frac{1}{x+a}\sum_{n=0}^{\infty}\left(\frac{a-s}{x+a}\right)^n,\qquad a\ge 0,\ |a-s|<|x+a|,\ \text{for } s\in[0,1],$$

from (1.2), we deduce that

$$\begin{aligned}(x+1)\left(e-\left(1+\frac{1}{x}\right)^{x}\right)\\ =\frac{e}{2}+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{(x+a)^{n+1}}\int_0^1 s^s(1-s)^{1-s}\,\sin(\pi\,s)\,(a-s)^n\,\mathrm{d}s.\end{aligned} \tag{1.3}$$

For $a=\frac{11}{12}$, (1.3) yields the result of Mortici and Hu [6, (3.1)];
for $a=1$, (1.3) gives an expansion in [7];
for $a=0$, $x>1$, (1.3) becomes

$$(x+1)\left(e-\left(1+\frac{1}{x}\right)^{x}\right)=\frac{e}{2}+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{x^{n+1}}\int_0^1 s^s(1-s)^{n+1-s}\,\sin(\pi\,s)\mathrm{d}s. \tag{1.4}$$

We will review the integrals involved in (1.4) in subsection 3.1. The main result of the paper is the series expansion (2.9) of the function $x\mapsto\left(1+\dfrac{1}{x+a}\right)^{x+b}$ in terms of Bell polynomials. This extends many known results.

1.1. **Complete asymptotic expansion.** Let $(b_n)_{n\ge 0}$ be a sequence of real numbers and $f\colon(0,\infty)\to\mathbb{R}$. Use the symbol $\mathcal{O}$ for Landau's big "O" notation. We recall that $\sum\limits_{n=0}^{\infty}\frac{b_n}{x^n}$ is said to be a complete asymptotic expansion of f as $x\to\infty$, and use the notation

$$f(x)\sim\sum_{n=0}^{\infty}\frac{b_n}{x^n},\qquad\text{as } x\to\infty,$$

if

$$f(x)=\sum_{n=0}^{p}\frac{b_n}{x^n}+\mathcal{O}(x^{-p-1}),\qquad\text{as } x\to\infty$$

for all integers $p\ge 0$.

1.2. **The Bell polynomials.** Let $(x_n)_{n\ge 1}$ be a sequence of numbers. The *complete exponential Bell polynomials* $B_n(x_1,\dots,x_n)$ (see, e.g., [8, Chapter 2, Section 8], [9, p. 134]) denoted in the sequel by $\mathbf{Bell}_n[x_i]$, are given by the *formal* series identity

$$\exp\left(\sum_{i=1}^{\infty}x_i\frac{t^i}{i!}\right)=\sum_{n=0}^{\infty}\mathbf{Bell}_n[x_i]\,\frac{t^n}{n!}. \tag{1.5}$$

They may be recursively defined as

$$\mathbf{Bell}_0[x_i]:=1,\quad \mathbf{Bell}_{n+1}[x_i]=\sum_{j=0}^{n}\binom{n}{j}\mathbf{Bell}_{n-j}[x_i]\,x_{j+1},\qquad n=0,1,\dots. \tag{1.6}$$

The following table can be obtained immediately from (1.6).

$$\begin{array}{rcl}
\mathbf{Bell}_0[x_i] & = & 1, \\
\mathbf{Bell}_1[x_i] & = & x_1, \\
\mathbf{Bell}_2[x_i] & = & x_1^2 + x_2, \\
\mathbf{Bell}_3[x_i] & = & x_1^3 + 3x_2x_1 + x_3, \\
\mathbf{Bell}_4[x_i] & = & x_1^4 + 6x_2x_1^2 + 4x_3x_1 + 3x_2^2 + x_4, \\
\mathbf{Bell}_5[x_i] & = & x_1^5 + 10x_2x_1^3 + 10x_3x_1^2 + 15x_2^2x_1 + 5x_4x_1 + 10x_2, x_3 + x_5.
\end{array}$$

2. Main Results

It is well known that if $z \mapsto g(z)$ is holomorphic in the disk $|z| < R$, then $z \mapsto \exp(g(z))$ is holomorphic in the disk $|z| < R$. In consequence, the power series expansion of $\exp(g(z))$ has a radius of convergence at least R. So, if the power series $\sum_{i=1}^{\infty} x_i \frac{t^i}{i!}$ has the radius of convergence $R > 0$, then the formal equality (1.5) becomes an equality

$$\exp\left(\sum_{i=1}^{\infty} x_i \frac{t^i}{i!}\right) = \sum_{n=0}^{\infty} \mathbf{Bell}_n[x_i] \frac{t^n}{n!}, \quad |t| < R. \tag{2.7}$$

The following is the main result of the paper.

Theorem 2.1. *Let $a, b \in \mathbb{R}$. For*

$$x_{i,a,b} \quad := \quad (-1)^i i! \left(\frac{(a+1)^{i+1} - a^{i+1}}{i+1} - \frac{b\left((a+1)^i - a^i\right)}{i} \right), \quad i = 1, 2, \ldots, \tag{2.8}$$

we obtain the following equality

$$\left(1 + \frac{1}{x+a}\right)^{x+b} = e \sum_{n=0}^{\infty} \frac{\mathbf{Bell}_n[x_{i,a,b}]}{n!} \frac{1}{x^n}, \quad |x| > \max(|a|, |a+1|). \tag{2.9}$$

Proof. The following expansion can be obtained by simple calculation,

$$\begin{aligned}
& (x+b)\log\left(1 + \frac{1}{x+a}\right) \\
=& (x+b)\log\frac{1+\frac{a+1}{x}}{1+\frac{a}{x}} \\
=& 1 + \sum_{i=1}^{\infty} (-1)^i i! \left(\frac{(a+1)^{i+1} - a^{i+1}}{i+1} - \frac{b\left((a+1)^i - a^i\right)}{i} \right) \frac{1}{i! x^i},
\end{aligned} \tag{2.10}$$

$|x| > \max(|a|, |a+1|)$. Using (2.7), the proof is complete. □

In particular, we obtain:

Example 2.1. *The following asymptotic expansions hold true:*

$$\begin{aligned}
&\left(1+\frac{1}{x+a}\right)^{x+b} \\
= \;& e \\
&-\frac{e(2a-2b+1)}{2}\frac{1}{x} \\
&+\frac{e\left(36a^2-48ab+36a+12b^2-24b+11\right)}{24}\frac{1}{x^2} \\
&-\frac{e\left(104a^3-168a^2b+156a^2+72ab^2-168ab+94a-8b^3+36b^2-50b+21\right)}{48}\frac{1}{x^3} \\
&+\mathcal{O}(x^{-4}), \qquad x\to\infty,
\end{aligned}$$

$$\left(1+\frac{1}{x+a-\frac{1}{4}}\right)^{x+a+\frac{1}{4}} = e+\mathcal{O}(x^{-2}), \qquad x\to\infty.$$

3. Applications

All known or new results in this section stem from Theorem 2.1.

Corollary 3.1. *From (2.9), we deduce*

$$(x+c)\left(1+\frac{1}{x+a}\right)^{x+b}-ex = e\sum_{k=0}^{\infty}\left(\frac{\mathbf{Bell}_{k+1}[x_{i,a,b}]}{(k+1)!}+c\frac{\mathbf{Bell}_{k}[x_{i,a,b}]}{k!}\right)\cdot\frac{1}{x^k} \tag{3.11}$$

and, in particular,

$$\begin{aligned}
&(x+c)\left(\frac{1}{x+a}+1\right)^{x+b}-ex \\
&=-\frac{1}{2}e(2a-2b-2c+1) \\
&+e\left(36a^2-48ab-24ac+36a+12b^2+24bc-24b-12c+11\right)\frac{1}{24x} \\
&-e\Big(104a^3-168a^2b-72a^2c+156a^2+72ab^2+96abc-168ab \\
&-72ac+94a-8b^3-24b^2c+36b^2+48bc-50b-22c+21\Big)\frac{1}{48x^2} \\
&+\mathcal{O}(x^{-3}).
\end{aligned}$$

We note that particular cases of (2.9) can be found, e.g., in papers of H. J. Brothers and J. A. Knox [4, 10], C. Mortici and X.-J. Jang [11], C. Mortici and Y. Hu [6].

3.1. **Evaluating the integrals in (1.4).** In this subsection, we obtain the following evaluation of the integrals involved in (1.4).

Proposition 3.1.

$$J_k := \int_0^1 s^s(1-s)^{k-s}\sin(\pi s)\,\mathrm{d}s = (-1)^k\pi\, e\left(\frac{\mathbf{Bell}_{k+1}[x_{i,0,0}]}{(k+1)!}+\frac{\mathbf{Bell}_{k}[x_{i,0,0}]}{k!}\right), \tag{3.12}$$

$k=1,2,\dots$.

Proof. Taking $a = 0$, $b = 0$, and $c = 1$ in (3.11), we obtain

$$(3.13)\quad (x+1)\left(1+\frac{1}{x}\right)^x - ex = e\sum_{k=0}^{\infty}\left(\frac{\mathbf{Bell}_{k+1}[x_{i,0,0}]}{(k+1)!} + \frac{\mathbf{Bell}_k[x_{i,0,0}]}{k!}\right)\cdot\frac{1}{x^k}, \quad x > 1,$$

where

$$(3.14)\quad x_{0,0,i} = \frac{(-1)^i i!}{i+1}, \qquad i = 1, 2, \ldots.$$

Comparing (1.4) with (3.13), we succeeded in calculating the integrals (3.12). □

For example,

$$\begin{aligned}\int_0^1 s^s(1-s)^{1-s}\sin(\pi s)\,\mathrm{d}s = J_1 &= \frac{e\pi}{24},\\ J_2 &= \frac{e\pi}{48},\\ J_3 &= \frac{73e\pi}{5760},\\ J_4 &= \frac{11e\pi}{1280}.\end{aligned}$$

Note that MATHEMATICA and other assistant software failed to evaluate the integrals (3.12).

3.2. **A generalized Keller function.** Extend now the Keller sequence (1.1) to the function

$$(3.15)\quad K(a,b,c;x) := (x+c)\left(1+\frac{1}{x+a}\right)^{x+b} - (x+c-1)\left(1+\frac{1}{x+a-1}\right)^{x+b-1} - e,$$

$|x| > \max(|a-1|, |a|, |a+1|) = |a| + 1$.
From (3.11), we obtain

$$\begin{aligned}(3.16)\quad K(a,b,c;x) &= e\sum_{k=2}^{\infty}\frac{1}{x^k}\left(\frac{c\,\mathbf{Bell}_k[x_{i,a,b}] - (c-1)\,\mathbf{Bell}_k[x_{i,a-1,b-1}]}{k!}\right.\\ &\left.\qquad + \frac{\mathbf{Bell}_{k+1}[x_{i,a,b}] - \mathbf{Bell}_{k+1}[x_{i,a-1,b-1}]}{(k+1)!}\right),\end{aligned}$$

$|x| > |a| + 1$. We note that, for any parameters a, b, $c \in \mathbb{R}$, the function $K(a,b,c;x)$ is a $\mathcal{O}(x^{-2})$, as $x \to \infty$. For example,

$$\begin{aligned}&K(a,b,c;x)\\ =&\frac{e}{24x^2}\cdot\left(-36a^2+48ab+24ac-36a-12b^2-24bc+24b+12c-11\right)\\ +&\frac{e}{24x^3}\cdot\Big(104a^3-168a^2b-72a^2c+84a^2+72ab^2+96abc-72ab\\ &-24ac+22a-8b^3-24b^2c+12b^2-2b+2c-1\Big)\\ &+\mathcal{O}(x^{-4}), \quad x\to\infty.\end{aligned}$$

In particular, we obtain

$$K(a,a,1;x) = \frac{e-12ea}{24x^2} + \frac{e(4a(6a-1)+1)}{24x^3} + \mathcal{O}(x^{-4}), \qquad x \to \infty,$$

$$K\left(\frac{1}{12}, \frac{1}{12}, 1; x\right) = \frac{5e}{144x^3} + \mathcal{O}(x^{-4}), \qquad x \to \infty,$$

which are cases considered in [7] and [11]. Taking benefit of three free parameters a, b, c, we obtain

$$K\left(-\frac{1}{2}, \sqrt{\frac{1}{2}+\frac{1}{\sqrt{6}}}, -\frac{1}{6}\sqrt{9+\sqrt{6}}; x\right) = \frac{(3+5\sqrt{6})\,e}{720x^4} + \mathcal{O}(x^{-5}).$$

3.3. **On an expansion of Yang.** In [12, Theorem 1], X. Yang obtained the following expansion

$$\left(1+\frac{1}{x}\right)^x = e\left(1 - \sum_{k=1}^{\infty} \frac{b_k}{(1+x)^k}\right), \quad x > 0, \tag{3.17}$$

where

$$b_1 = \frac{1}{2}, \quad b_{k+1} = \frac{1}{k+1}\left(\frac{1}{k+2} - \sum_{i=1}^{k} \frac{b_i}{k+2-i}\right), \quad k = 1, 2, \ldots.$$

We prove that Yang's formula (3.17) is a particular case of the general Bell-type expansion (2.7) for

$$t = \frac{1}{1+x} \quad \text{and} \quad x_i = -\frac{(i-1)!}{i+1}, \quad i = 1, 2, \ldots.$$

Indeed, we have

$$\begin{aligned}
\left(1+\frac{1}{x}\right)^x e^{-1} &= (1-t)^{1-\frac{1}{t}}\, e^{-1} = \exp\left(\frac{(t-1)\log(1-t)-t}{t}\right) \\
&= \exp\left(-\sum_{i=1}^{\infty} \frac{t^i}{i(i+1)}\right) \\
&= \exp\left(-\sum_{i=1}^{\infty} \frac{(i-1)!}{i+1} \cdot \frac{t^i}{i!}\right) = \sum_{k=0}^{\infty} \frac{\mathbf{Bell}_k[x_i]}{k!} \cdot t^k,
\end{aligned}$$

hence

$$\left(1+\frac{1}{x}\right)^x = e \sum_{k=0}^{\infty} \frac{\mathbf{Bell}_k[x_i]}{k!} \cdot \frac{1}{(1+x)^k}, \quad x > 0. \tag{3.18}$$

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