SOME RESULTS ON SEVERAL NUMBERICAL P_{+53} SETS

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Abstract:

Diophantine set theory has an importence role in Mathematics.In this paper,we consider prime number p=+53 and give some Diophantine P_{+53} triples. Some of the such sets are extended but others not. We give several of them with proofs. Also, some types of elements of the Diophantine P_{+53} m-tubles are determined. One can be work on other Diophantine P_{+53} - m tubles and discover extendibility of them.

Keywords: Diophantine P_s 3-Tuple, Number Theory, Pell Equations, Elements of Diophantine P_s m-tuples, Quadratic Reciprocity Law.

Introduction and Preliminaries

To obtain proofs of our main results we need following definitions, lemmas, theorems and so on... All of the following informations are found in the references [1-25].

1. Let n be an non-zero integer. A set of m positive integers

$$\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$$

such that $\alpha_i\alpha_j + n$ *is a* perfect square for all $1 \leq i < j \leq m$ is called α Diophantine m-tuple with the property $D(n)$.

2. Let p be an odd prime and let a be an integer. The Legendre symbol of a with respect to p is defined by

$$\left(\frac{\alpha}{p}\right) = \begin{cases} 1 & if\ \alpha\ is\ \text{a}\ quadratic\ residue\ modulo\ p\ and\ \alpha\ \not\equiv 0\ (mod\ p\ ) \\ -1 & if\ \alpha\ is\ a\ quadratic\ non-residue\ modulo\ p \\ 0 & if\ \alpha \equiv 0\ (mod\ p). \end{cases}$$

(a) $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$, so it is 1 if and only if $p \equiv 1\ mod 4$.

(b) $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$ for an odd prime p , so it is 1 if and only if $p \equiv \pm 1\ mod\ 8$.

3. **Law of Quadratic Reciprocity** is given by

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}},$$

where p and q are odd prime numbers, and $\left(\frac{p}{q}\right)$ denotes the Legendre symbol.

Note: (Extension of the law of quadratic reciprocity) If m and n are coprime positive odd integers,

$$\left(\frac{m}{n}\right)\left(\frac{n}{m}\right) = (-1)^{\frac{m-1}{2}\frac{n-1}{2}}.$$

Main Results

Theorem 1. $P_{+53} = \{11, 13, 52\}$ Diophantine triple can not be extended to Diophantine P_{+53} quadruple.

<u>Proof .</u>

Assume that d is in the set of Diophantine P_{+53} set. So, we obtain following result from the definition of Diophantine P_{+53} set.

$\{11, 13, 52, d\} \rightarrow$ (1) $11d + 53 = x^2$

(2) $13d + 53 = y^2$

(3) $52d + 53 = z^2$.

These (2) and (3) equations imply that $z^2 - 4y^2 = -159$. Table 1 gives us integer solutions of the equation as follow:

Table 1. $z^2 - 4y^2 = -159$

(z, y)	(z, y)
$(\pm 40, \pm 79)$	$(\pm 14, \pm 25)$

From the (1) and (2), we get,

$$13x^2 - 11y^2 = 2.53 \implies 13x^2 - 11y^2 = 106$$

And also integers solutions of the $13x^2 - 11y^2 = 106$ can be given as Table 2.

Table 2. $13x^2 - 11y^2 = 106$

(x, y)	(x, y)	(x, y)	(x, y)	(x, y)
$(\pm 1125, \pm 1223)$	$(\pm 597, \pm 649)$	$(\pm 47, \pm 51)$	$(\pm 25, \pm 27)$	$(\pm 3,1)$

If we compare Table 1 and Table 2, we obtain that there is no common integer solution for the system of pell equations. So, $\{11,13,52\}$ can not be extended to Diophantine P_{+53} Quadruple.

Theorem 2. Diophantine $P_{+53} = \{4,119,169\}$ Triple can not be extended to P_{+53} Quadruple.

Proof: Let us consider Diophantine $P_{+53} = \{4,119,169\}$. If d is an element of the such property set, then it is written by Diophantine $\{4,119,169, d\}$ 4- tuples. Then we obtain following results

$$\left.\begin{array}{r}(1)\ 4d + 53 = A^2 \\ (2)\ 119 + 53 = B^2 \\ (3)\ 169d + 53 = C^2\end{array}\right\}$$

From (1) and (3), it is obtained that

$$\begin{array}{l}169\ \ /\ 4d + 53 = A^2 \\ \underline{-4\ \ /\ 169d + 53 = C^2}\end{array}$$

$$\Rightarrow 169A^2 - 4C^2 = 165.53$$

$$\Longrightarrow 169A^2 - 4C^2 = 8745 \qquad (4)$$

Also, from (1) and (2), we get;

$$119A^2 - 4B^2 = 115.53 \quad \Longrightarrow 119A^2 - 4B^2 = 6095 \qquad (5)$$

For (4) and (5) , we have Table 3 and Table 4 include integer solutions.

Tablo 3. $169A^2 - 4C^2 = 8745$

(A, C)	(A, C)
$(\pm 31, \pm 196)$	$(\pm 23, \pm 142)$

Tablo 4. $119A^2 - 4B^2 = 6095$

(A, B)	(A, B)	(A, B)	(A, B)	(A, B)
$(\pm 1389, \pm 7576)$	$(\pm 531, \pm 2896)$	$(\pm 37, \pm 198)$	$(\pm 27, \pm 142)$	$(\pm 19, \pm 96)$

From Tablo 3 and Tablo 4, we can not get common integer solutions for (3) and (4). So, {4,119,169} can not be extended.

Theorem 3. $P_{+53} = \{4,169,227\}$ can not be extendable to Diophantine P_{+53} quadruple.

Proof. It is proven like previous proofs of the theorems.

Theorem 4. There is no elements in the set of Diophantine P_{+53} m- tuples if they are written by three fold or five fold or thirtyone fold or fortyone fold or thirtynine fold, so on…

Proof.

(a) Assume that 3k $(k \in Z^+)$ is in the set of Diophantine P_{+53} m- tuples.So, following equation have solution ;

$$3k.s + 53 = x^2$$

for $s \in P_{+53}$ m- tuples. It implies that

$$x^2 = 2 \ (mod\ 3).$$

This congruents can solvable if $\left(\frac{2}{3}\right) = +1$ but $\left(\frac{2}{3}\right) = (-1)^{\frac{9-1}{8}} = (-1).$

This implies that 3 ∉ Diophantine P_{+53} m- tuples.

(b) Suppose that $31r$ $(r \in Z^{+})$ is an element of the Diophantine P_{+53} m- tuples.Then, we obtain following equation from the definition of the Diophantine P_{+53} m- tuples.

$$31r.u + 53 = A^2 \quad \ni \quad u \in \text{ Diophantine } P_{+53} \text{ m} - \text{ tuples.}$$

It implies that

$$A^2 \equiv 22 \ (mod\ 31) \ \text{ solvable } \Leftrightarrow \left(\frac{22}{31}\right) = 1 \quad (?)$$

$\left(\frac{22}{31}\right) = \left(\frac{2}{31}\right).\left(\frac{11}{31}\right)$ and from Quadratic reciprocity;

$$\left(\frac{11}{31}\right).\left(\frac{31}{11}\right) = (-1)^{\left(\frac{11-1}{2}\right).\left(\frac{31-1}{2}\right)} \Rightarrow \quad \left(\frac{11}{31}\right) = -1$$

$\left(\frac{2}{31}\right) = (-1)^{\frac{31^2-1}{8}} = (-1)^{120} = +1$ then $31r \notin$ Diophantine P_{+53} m- tuples.

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