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OPTIMAL STEP STRESS ACCELERATED LIFE TESTING FOR THE LENGTH-BIASED EXPONENTIAL CUMULATIVE EXPOSURE MODEL

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Abstract: This paper considers a simple step stress accelerated life test for units modeled by a length-biased exponential distribution. The cumulative exposure model of time to failure holds in this accelerated life test model. The optimal test plan is constructed by determining the optimal stress change time. Parameters of the model are estimated by using the maximum likelihood estimation method. The corresponding approximate confidence intervals are obtained by using the asymptotic normality features of the maximum likelihood estimators. Theoretical outcomes are illustrated with simulation studies and a real data example.

Key words: Accelerated life test, cumulative exposure model,length-biased exponential distribution, maximum likelihood estimation, optimal test plan, step-stress model

1. Introduction

Studies based on the accelerated life tests (ALT) have been popular since they let the experimenters control the higher stress levels to be used for components or units in the life tests. It is clear that long lifetimes of highly reliable products make observing the experiment difficult. In such cases, ALT approaches provide higher than usual stress conditions for units/components. These tests are used for estimation of the lifetime of highly reliable components within an acceptable period (for more details Nelson [14] is recommended to the readers). Various types of ALT plans take part in reliability theory such as constant stress, step stress or progressive stress ALTs. These plans differ from each other depending on how to apply stresses to components. In a constant stress plan, the stress applied to a component does not vary with time. In contrast, there is a time point in step stress and more time point to increase stress levels in progressively stress. Studies on these different cases of ALTs take part in various studies by various authors.

Nelson [13] introduced the step-stress ALTs that allows test conditions to change during testing. Among step stress experiments, the cumulative exposure model (CEM) is one of the most useful and used models. A simple step stress model starts with initial low stress and if it does not fail in a predetermined time point, τ, the stress level is increased. Simple step stress models contain only one stress change point. The CEM defined by Nelson [13] for simple step-stress testing with stresses and is given as

$$F_0(t) = \begin{cases} F_1(t) & , t \leq \tau \\ F_2(t-\tau+\tau') & , t \geq \tau \end{cases} \tag{1.1}$$

where τ' (the equivalent start time) is the solution of $F_1(\tau) = F_2(\tau')$.

The ALT plans are considered for many different probability distributions by various authors. For instance; Miller and Nelson [12] considered optimum ALT plan under exponentially distributed lifetimes. Chung and Bai [4] studied ALT for log-normal lifetime distributions, Ebrahem and Al-Masri

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[6] considered ALT for log-logistic distribution, Ma and Meeker [11] explored ALT for log-location-scale distributions, Saxena et al. [16] considered ALT for Rayleigh distribution, Haghighi [10] considered ALT for an extension of the exponential distribution, Abbas and Firdos [1] studied ALT for Fréchet distribution. For more probability distribution, the literature list can be extended. On the other hand, ALT plans mostly are also considered for censored cases. Based on many different censoring schemes, ALT plans are considered for different probability models.
It is known that several probability models have been extensively used over the past decade in describing lifetime data. The exponential distribution is one of the major distributions for modeling lifetime datasets. For this purpose, many generalizations and modifications of exponential distribution take part in the literature. For example, generalized exponential distribution by Gupta and Kundu [9] and exponential-geometric distribution by Adamidis and Loukas [2] and etc. Length-biased distributions have great importance in reliability, biomedicine, and ecology among other distributions due to their greater flexibility in modeling data in different areas such as lifetime analysis, engineering, economics, finance, demography, actuarial and medical sciences (Akhter et al. [3]). Recently, Dara and Ahmed [5] proposed a new extension of exponential distribution denoted by "moment exponential distribution". Then, it is called as length-biased exponential (LBE) distribution by some authors.
The probability density (pdf) and distribution (cdf) function of the LBE distribution are given as

$$f(t) = \frac{t}{\theta^2} exp\left\{-\frac{t}{\theta}\right\}, \, x > 0, \, \theta > 0 \tag{1.2}$$

$$F(t) = 1 - \left[1 + \frac{t}{\theta}\right] exp\left\{-\frac{t}{\theta}\right\} \tag{1.3}$$

where θ is the scale parameter.

The LBE distribution has never been studied for any ALT plans before. In this study, we aimed to obtain parameter estimation of LBE distribution under simple step- stress cumulative exposure model. Maximum likelihood estimation (MLE) method is used to obtain point estimates and their credible intervals. For this purpose, in Section 2, we presented the model description. Following, MLE and approximate confidence intervals are obtained in Section 3. An optimization criterion to obtain optimum stress change time and its applications are given in Section 4. The simulation studies and a real data example are given to illustrate the theoretical outcomes in Sections 5 and 6, respectively.

2. Model Description

The lifetime of a test unit or component follows a LBE distribution under any constant stress. We assume that the scale parameter of the distribution is a log-linear function of stress. The following assumptions are provided for a LBE distributed lifetime units.

- Test procedure is done at stresses S_1 and $S_2 (S_1 < S_2)$ levels.
- Under any level of stress, the lifetime of a test unit follows a LBE distribution with the given cdf as

$$F_i(t) = 1 - \left[1 + \frac{t}{\theta_i}\right] exp\left\{-\frac{t}{\theta_i}\right\}$$

- The scale parameter θ_i is the log-linear function of stresses as $\log\theta_i = \beta_0 + \beta_1 S_i$ where $i = 1, 2$, β_0 and $\beta_1 (< 0)$ are unknown parameters depending on the nature of the product and the method of the test.
- All test units are independently and identically distributed variables from the LBE distribution.

• In this test, the cumulative exposure model which is defined by Nelson [14] for the simple step-stress testing with stresses S_1 and S_2 is used.

• n_i failure times $t_{ij}, j=1,2,\cdots,n_i$ of test units are observed under test operation at stress level $S_i, i=1,2$.

Based on the given assumptions above, length-biased exponential cumulative exposure (LBECE) model is given as follows. Firstly, the equivalent start time $\tau^{'}$ for the LBECE model which is the solution of $F_1(\tau)=F_2(\tau^{'})$ is equal to

$$\tau^{'}=\left(\frac{\theta_2}{\theta_1}\right)\tau$$

Then, by replacing $\tau^{'}$ in (1) the cdf of a test unit is obtained as

$$F_0(t)=\begin{cases} 1-\left[1+\frac{t}{\theta_1}\right]exp\left\{-\frac{t}{\theta_1}\right\} & ,t\leq\tau \\ 1-\left[1+\frac{t-\tau}{\theta_2}+\frac{\tau}{\theta_1}\right]exp\left\{-\frac{t-\tau}{\theta_2}-\frac{\tau}{\theta_1}\right\} & ,t\geq\tau \end{cases} \tag{2.1}$$

and the pdf is given as

$$f_0(t)=\begin{cases} \frac{t}{\theta_1^2}exp\left\{-\frac{t}{\theta_1}\right\} & ,t\leq\tau \\ \frac{\theta_1 t+(\theta_2-\theta_1)\tau}{\theta_1\theta_2^2}exp\left\{-\frac{t-\tau}{\theta_2}-\frac{\tau}{\theta_1}\right\} & ,t\geq\tau \end{cases} \tag{2.2}$$

3. Maximum Likelihood Estimation

This section considers obtaining MLEs of model parameters. Let t_{ij} denotes the observed failure time of a test component j under $i-$th stress level. Also, n_1 be the number of components failed at stress S_1 and n_2 at stress S_2. Corresponding likelihood function on the observed sample is given as

$$L(\theta)=\prod_{j=1}^{n_1} f_1(t_{1j},\theta)\prod_{j=1}^{n_2} f_2(t_{2j},\theta) \tag{3.1}$$

where $j=1,2,\cdots,n_i$ and $i=1,2$, $f_1(.)$ and $f_2(.)$ denotes the cases of the pdf due to the stress levels. By replacing Equation (2.2) in Equation (3.1) we obtain the likelihood function as

$$\begin{aligned} L(\theta_1,\theta_2)=\frac{1}{\theta_1^{2n_1+n_2}}\frac{1}{\theta_2^{2n_2}}\prod_{j=1}^{n_1} t_{1j}exp\left\{-\frac{t_{1j}}{\theta_1}\right\}\prod_{j=1}^{n_2}\left(\theta_1 t_{2j}+(\theta_2-\theta_1)\tau\right) \\ \times exp\left\{-\frac{t_{2j}-\tau}{\theta_2}-\frac{\tau}{\theta_1}\right\} \end{aligned} \tag{3.2}$$

and the log-likelihood function is obtained as

$$\begin{aligned} \ell(\theta_1,\theta_2)=&-(2n_1+n_2)\log\theta_1-2n_2\log\theta_2+\sum_{j=1}^{n_1}\log(t_{1j})-\sum_{j=1}^{n_1}\frac{t_{1j}}{\theta_1} \\ &+\sum_{j=1}^{n_2}\log\left(\theta_1 t_{2j}+(\theta_2-\theta_1)\tau\right)-\sum_{j=1}^{n_2}\frac{t_{2j}-\tau}{\theta_2}-\frac{\tau n_2}{\theta_1} \end{aligned} \tag{3.3}$$

By replacing the relation $\log\theta_i=\beta_0+\beta_1 S_i$ in log-likelihood function, we obtain

$$\begin{aligned} \ell(\beta_0,\beta_1)=&-2n\beta_0-2\beta_1(n_1S_1+n_2S_2)-n_2(\beta_0+\beta_1S_1)+\sum_{j=1}^{n_1}\log(t_{1j}) \\ &-e^{-(\beta_0+\beta_1S_1)}\sum_{j=1}^{n_1}t_{1j}+\sum_{j=1}^{n_2}\log\left[e^{(\beta_0+\beta_1S_1)}(t_{2i}-\tau)+e^{(\beta_0+\beta_1S_2)}\tau\right] \\ &-e^{-(\beta_0+\beta_1S_2)}\sum_{j=1}^{n_2}(t_{2j}-\tau)-n_2\tau e^{-(\beta_0+\beta_1S_1)} \end{aligned} \tag{3.4}$$

where $n = n_1 + n_2$. To obtain the MLEs of the parameters, denoted by $\hat{\beta}_0$ and $\hat{\beta}_1$ we should equate the partial derivates of $\ell(\beta_0, \beta_1)$ to zero with respect to β_0 and β_1 respectively as given in the following

$$\frac{\partial \ell}{\partial \beta_0} = -2n - n_2 + e^{-(\beta_0+\beta_1 S_1)} \sum_{j=1}^{n_1} t_{1j} + e^{-(\beta_0+\beta_1 S_2)} \sum_{j=1}^{n_2} (t_{2j} - \tau) + n_2 \tau e^{-(\beta_0+\beta_1 S_1)} \tag{3.5}$$

and

$$\begin{aligned}\frac{\partial \ell}{\partial \beta_1} =& -2(n_1 S_1 + n_2 S_2) - n_2 S_1 + S_1 e^{-(\beta_0+\beta_1 S_1)} \sum_{j=1}^{n_1} t_{1j} + S_2 e^{-(\beta_0+\beta_1 S_2)} \sum_{j=1}^{n_2} (t_{2j} - \tau) \\ &+ S_1 n_2 \tau e^{-(\beta_0+\beta_1 S_1)} + \sum_{j=1}^{n_2} \frac{S_1 e^{(\beta_0+\beta_1 S_1)} (t_{2j} - \tau) + S_2 e^{(\beta_0+\beta_1 S_2)} \tau}{e^{(\beta_0+\beta_1 S_1)} (t_{2j} - \tau) + e^{(\beta_0+\beta_1 S_2)} \tau}\end{aligned} \tag{3.6}$$

These non-linear equations can not be solved analytically and some iterative methods are needed. Thus, approximate solutions of the system of these non-linear equations are the MLEs of the β_0 and β_1.

Approximate confidence intervals for MLEs of the parameters can be obtained by using the inverse of the asymptotic Fisher information matrix. The inverse Fisher information matrix is given as follows

$$F^{-1} = \begin{bmatrix} -E\left[\frac{\partial^2 \ell}{\partial \beta_0^2}\right] & -E\left[\frac{\partial^2 \ell}{\partial \beta_0 \partial \beta_1}\right] \\ -E\left[\frac{\partial^2 \ell}{\partial \beta_1 \partial \beta_0}\right] & -E\left[\frac{\partial^2 \ell}{\partial \beta_1^2}\right] \end{bmatrix}^{-1}_{(\beta_0,\beta_1)=(\hat{\beta}_0,\hat{\beta}_1)} = \begin{bmatrix} Var(\hat{\beta}_0) & Cov(\hat{\beta}_0 \hat{\beta}_1) \\ Cov(\hat{\beta}_1 \hat{\beta}_0) & Var(\hat{\beta}_1) \end{bmatrix}$$

where

$$\frac{\partial^2 \ell}{\partial \beta_0^2} = -e^{-(\beta_0+\beta_1 S_1)} \sum_{j=1}^{n_1} t_{1j} - e^{-(\beta_0+\beta_1 S_2)} \sum_{j=1}^{n_2} (t_{2j} - \tau) - n_2 \tau e^{-(\beta_0+\beta_1 S_1)}$$

$$\frac{\partial^2 \ell}{\partial \beta_0 \partial \beta_1} = -S_1 e^{-(\beta_0+\beta_1 S_1)} \sum_{j=1}^{n_1} t_{1j} - S_2 e^{-(\beta_0+\beta_1 S_2)} \sum_{j=1}^{n_2} (t_{2j} - \tau) - n_2 S_1 \tau e^{-(\beta_0+\beta_1 S_1)}$$

$$\begin{aligned}\frac{\partial^2 \ell}{\partial \beta_1^2} =& -S_1^2 e^{-(\beta_0+\beta_1 S_1)} \sum_{j=1}^{n_1} t_{1j} - S_2^2 e^{-(\beta_0+\beta_1 S_2)} \sum_{j=1}^{n_2} (t_{2j} - \tau) - n_2 S_1^2 \tau e^{-(\beta_0+\beta_1 S_1)} \\ &+ (S_1 - S_2)^2 e^{(\beta_0+\beta_1 S_1)} e^{(\beta_0+\beta_1 S_2)} \tau \sum_{j=1}^{n_2} \frac{t_{2j} - \tau}{\left[e^{(\beta_0+\beta_1 S_1)} (t_{2j} - \tau) + e^{(\beta_0+\beta_1 S_2)} \tau\right]^2}\end{aligned}$$

Thus,

$$-E\left[\frac{\partial^2 \ell}{\partial \beta_0^2}\right] = n_1 \gamma\big(3, \tau/\theta_1\big) + n_2 \big[\Gamma(3, \tau/\theta_1) - (\tau/\theta_1)\Gamma(2, \tau/\theta_1)\big] + n_2 \tau/\theta_1$$

$$-E\left[\frac{\partial^2 \ell}{\partial \beta_0 \partial \beta_1}\right] = S_1 n_1 \gamma\big(3, \tau/\theta_1\big) + S_2 n_2 \big[\Gamma(3, \tau/\theta_1) - (\tau/\theta_1)\Gamma(2, \tau/\theta_1)\big] + n_2 S_1 \tau/\theta_1$$

$$\begin{aligned}-E\left[\frac{\partial^2 \ell}{\partial \beta_1^2}\right] =& S_1^2 n_1 \gamma\big(3, \tau/\theta_1\big) + S_2^2 n_2 \big[\Gamma(3, \tau/\theta_1) - (\tau/\theta_1)\Gamma(2, \tau/\theta_1)\big] + n_2 S_1^2 \tau/\theta_1 \\ &- \frac{(S_1 - S_2)^2 \tau n_2}{\theta_1} \left\{ e^{-\tau/\theta_1} \left(1 - \sqrt{\frac{\tau}{\theta_1}} W_{-\frac{1}{2},0}\big(\theta_1/\tau\big) \right) \right\}\end{aligned}$$

where $\theta_1 = e^{\beta_0+\beta_1 S_1}$ and $\theta_2 = e^{\beta_0+\beta_1 S_2}$. Also, $\gamma(a,b)$ denotes the lower incomplete gamma function and $\Gamma(a,b)$ denotes the upper incomplete gamma function as given in the following

$$\gamma(a,b) = \int_0^b t^{a-1}e^{-t}dt \quad \text{and} \quad \Gamma(a,b) = \int_b^\infty t^{a-1}e^{-t}dt$$

Further, the expression $W_{\lambda,\mu}(z)$ in $-E\left[\frac{\partial^2 \ell}{\partial \beta_1^2}\right]$ denotes the Whittaker functions and used for the solution of the integral $\int_{\tau/\theta_1}^\infty u^{-1}e^{-u}du$ (Gradshteyn and Ryzhik [8], Eq. 3.381.6, pg. 346).
It is knows that the MLEs under some regularity conditions are consistent and normally distributed (Godambe [7]). Thus, the a $100(1-\delta)\%$ asymptotic confidence intervals of β_0 and β_1 can be constructed by

$$\hat{\beta}_0 \mp Z_{\frac{\delta}{2}}\sqrt{Var(\hat{\beta}_0)} \text{ and } \hat{\beta}_1 \mp Z_{\frac{\delta}{2}}\sqrt{Var(\hat{\beta}_1)}$$

where Z_δ is 100 δth percentile of standard normal distribution $N(0,1)$.

4. Optimal Test Plan

The optimal test plan emphasizes an optimum stress change timepoint τ which determines the lifetime of lower stress level. In simple step stress accelerated life test plan, optimal stress change time is determined by minimizing the asymptotic variance of MLEs of a given log $100p$-th percentile at the design stress level S_0 (Ebrahem and Masri [6]). The $100p$-th percentile of the length-biased exponential distribution, denoted by $Q_p(S_0)$ at the design stress level S_0 is obtained as

$$Q_p(S_0) = -e^{\beta_0+\beta_1 S_0}\left[1 + W_{-1}\left(\frac{p-1}{e}\right)\right]$$

where $W_{-1}(.)$ is the negative branch of the Lambert W function (i.e., the solution of the equation $W(z)e^{W(z)} = z$). The asymptotic variance (AV) of the MLEs of the log $100p$-th percentile at the design stress level can be obtained by using

$$\begin{aligned} AV\left\{\log\left[\hat{Q}_p(S_0)\right]\right\} &= AV\left\{-\hat{\beta}_0 - \hat{\beta}_1 S_0 - \log\left[1 + W_{-1}\left(\frac{p-1}{e}\right)\right]\right\} \\ &= H\Sigma H^T \end{aligned}$$

where

$$H = \left[\frac{\partial \log\left[\hat{Q}_p(S_0)\right]}{\partial \hat{\beta}_0} \quad \frac{\partial \log\left[\hat{Q}_p(S_0)\right]}{\partial \hat{\beta}_1}\right] = \left[-1 \quad -S_0\right]$$

and Σ is the variance-covariance matrix which is obtained by using the inverse of the Fisher information matrix. Thus, the asymptotic variance of the MLEs of the log $100p$-th percentile at the design stress level can be obtained as follows

$$AV\left\{\log\left[\hat{Q}_p(S_0)\right\} = \Sigma_{11} + 2S_0\Sigma_{12} + S_0^2\Sigma_{22}$$

The Σ_{ij} values are already given in Section 3. Consequently, the optimal stress change time, denoted by τ^* is the τ value that minimizing the $AV\left\{\log\left[\hat{Q}_p(S_0)\right\}$. The NMinimize option of Mathematica 11 is a very useful tool to obtain the optimal τ value that minimizing the asymptotic variance.

We performed a small numerical study to observe the existence and evaluate the optimal stress change time with minimizing $AV\left\{\log\left[\hat{Q}_p(S_0)\right\}$. For given values of design stress level S_0 and parameters β_0 and β_1, different combinations of two levels of stress S_1 and S_2 as $(S_1 < S_2)$, we obtained the optimal stress-change times which provide variance optimality and reported in Table 1. We used the NMinimize option of Mathematica 11 for calculations.

TABLE 1. Optimal time, τ^*, changing stress for $s_0 = 0.25, \beta_0 = 2.5$ and $\beta_1 = -1.5$

			s_2		
s_1	1.25	1.50	1.75	2.00	2.50
0.30	5.55	6.46	7.32	8.15	9.74
0.40	4.38	5.17	5.92	6.63	7.99
0.50	3.44	4.12	4.77	5.38	6.54
0.75	1.78	2.28	2.74	3.16	3.96
1.00	0.80	1.19	1.52	1.82	2.37

It is observed that the optimal stress change time increases as parallel to increasing S_2 stress level when S_1 is fixed. On the other hand, decreases are observed on optimal stress change time in parallel to increasing on S_1 stress level when S_2 is fixed. These results are reasonable and acceptable.

5. Simulation Study

In this section, we provide a simulation study to illustrate the theoretical outcomes. We performed simulations and obtained the MLEs of the parameters and their corresponding confidence intervals.

We take the parameter values as $\beta_0 = 2.5$ and $\beta_1 = -1.5$, different stress levels as $(S_1, S_2) = (0.50, 1.50)$, $(S_1, S_2) = (0.50, 2.00)$, $(S_1, S_2) = (0.75, 1.50)$ and $(S_1, S_2) = (0.75, 2.00)$ and the stress change times τ as $4.12, 5.38, 2.28$ and 3.16, respectively. We consider different sample sizes as $n = 25, 50, 100, 250, 500$.

We first generate random samples of the LBECE model from the cdf in Eq. (2.1) with size n. Then, we generate 10 000 samples with each size n. We use R software (Team R.C. [15]) to perform this simulation. We obtained the maximum likelihood estimates of β_0 and β_1 with their mean squared errors (MSE), relative errors (RE), the %95 approximate confidence intervals (CI) and coverage probabilities (CP). We presented simulation results in Tables 2,3,4 and 5. The MSEs and REs for an arbitrary parameter can be obtained as follows

$$MSE_\xi = E_\xi[(\hat{\xi} - \xi)^2] \quad \text{and} \quad RE_\xi = \frac{|\xi - \hat{\xi}|}{\xi} 100\%$$

We observed that both estimates are obtained quite close to their actual values. In parallel to increase sample sizes, estimations are almost same with the actual values. As expected, MSEs and REs are getting smaller at the same time. Lengths of the approximate confidence intervals also decrease with increasing sample sizes. Coverage probabilities of the CIs have quite close to their actual value 0.95. In all cases of stress levels, consistent results are obtained. It is known that using the optimal stress change time makes estimations better than using arbitrary stress change times. However, it is clearly seen that differences between estimates for our examples are not very large. Of course, various combinations can be worth trying according to the needs of many engineering problems.

TABLE 2. The MLEs, MSEs, REs and approximate CIs of β_0 and β_1 based on 3000 replications.($\beta_0 = 2.5, \beta_1 = -1.5, \tau = 4.12$ and $S_1 = 0.50, S_2 = 1.50$)

n	Parameter	MLE	MSE	RE	Lower CI	Upper CI	CP
25	β_0	2.55967	0.00356	2.38671	1.58066	3.53867	96.33
	β_1	-1.55907	0.00349	3.93796	-2.36209	-0.75605	96.65
50	β_0	2.55195	0.00270	2.07806	1.87081	3.23310	96.48
	β_1	-1.54341	0.00188	2.89384	-2.10417	-0.98265	96.43
100	β_0	2.52732	0.00075	1.09283	2.05321	3.00143	96.29
	β_1	-1.52281	0.00052	1.52069	-1.91407	-1.13155	96.10
250	β_0	2.50917	0.00008	0.36662	2.21222	2.80611	95.54
	β_1	-1.50768	0.00006	0.51176	-1.75317	-1.26218	95.93
500	β_0	2.50381	0.00001	0.15248	2.2945	2.71312	95.88
	β_1	-1.50337	0.00001	0.22498	-1.67651	-1.33024	96.04

TABLE 3. The MLEs, MSEs, REs and approximate CIs of β_0 and β_1 based on 3000 replications.($\beta_0 = 2.5, \beta_1 = -1.5, \tau = 5.38$ and $S_1 = 0.50, S_2 = 2$).

n	Parameter	MLE	MSE	RE	Lower CI	Upper CI	CP
25	β_0	2.55861	0.00343	2.34421	1.81832	3.29889	97.01
	β_1	-1.54283	0.00183	2.85562	-2.04983	-1.03584	97.17
50	β_0	2.53178	0.00101	1.27110	2.01902	3.04453	96.73
	β_1	-1.52261	0.00051	1.50708	-1.87412	-1.17109	96.63
100	β_0	2.51489	0.00022	0.59551	2.15599	2.87379	96.25
	β_1	-1.51008	0.00010	0.67184	-1.75649	-1.26366	96.65
250	β_0	2.50750	0.00006	0.30019	2.28171	2.73330	96.11
	β_1	-1.50571	0.00003	0.38041	-1.66080	-1.35061	96.54
500	β_0	2.50312	0.00001	0.12484	2.3438	2.66244	95.87
	β_1	-1.50195	0.0000*	0.12979	-1.61142	-1.39247	96.33

(* denotes smaller values than $\times 10^{-5}$)

TABLE 4. The MLEs, MSEs, REs and approximate CIs of β_0 and β_1 based on 3000 replications.($\beta_0 = 2.5, \beta_1 = 1.5, \tau = 2.28$ and $S_1 = 0.75, S_2 = 1.5$)

n	Parameter	MLE	MSE	RE	Lower CI	Upper CI	CP
25	β_0	2.61059	0.01223	4.42372	1.06294	4.15825	97.85
	β_1	-1.58820	0.00778	5.88030	-2.76213	-0.41428	98.51
50	β_0	2.61029	0.01216	4.41170	1.52793	3.69266	96.98
	β_1	-1.58200	0.00672	5.46663	-2.40209	-0.76191	96.96
100	β_0	2.54484	0.00201	1.79351	1.80052	3.28915	95.86
	β_1	-1.53358	0.00113	2.23876	-2.09971	-0.96746	95.96
250	β_0	2.51936	0.00037	0.77450	2.05444	2.98428	95.88
	β_1	-1.51444	0.00021	0.96248	-1.86853	-1.16034	96.11
500	β_0	2.50866	0.00007	0.34635	2.18138	2.83594	95.71
	β_1	-1.50673	0.00005	0.44882	-1.75615	-1.25732	95.85

TABLE 5. The MLEs, MSEs, REs and approximate CIs of β_0 and β_1 based on 3000 replications.($\beta_0 = 2.5, \beta_1 = 1.5, \tau = 3.16$ and $S_1 = 0.75, S_2 = 2$)

n	Parameter	MLE	MSE	RE	Lower CI	Upper CI	CP
25	β_0	2.59675	0.00936	3.87000	1.59086	3.60264	98.14
	β_1	-1.56215	0.00386	4.14333	-2.19076	-0.93354	97.94
50	β_0	2.55287	0.00280	2.11473	1.85966	3.24607	96.58
	β_1	-1.53325	0.00111	2.21657	-1.96787	-1.09863	96.40
100	β_0	2.52397	0.00057	0.95869	2.04077	3.00716	96.14
	β_1	-1.51480	0.00022	0.98642	-1.81849	-1.21110	9627
250	β_0	2.51157	0.00013	0.46271	2.20816	2.81497	95.96
	β_1	-1.50817	0.00007	0.54448	-1.69906	-1.31727	96.27
500	β_0	2.50405	0.00002	0.16189	2.29010	2.71800	95.56
	β_1	-1.50243	0.00001	0.16186	-1.63711	-1.36775	96.02

6. Real Data Example

In this section, a real data set is presented to illustrate the theoretical outcomes. We used the data set of the amount of annual rainfall (in inches, from 1984 to 2008) recorded at the Los Angeles Civic Center that is available on the website of Los Angeles Almanac: www.laalmanac.com. Recently, Tarvirdizade and Ahmadpour [17] were used this data set in the reliability context. The data set is given as in the following;

$$12.82, 17.86, 7.66, 2.48, 8.08, 7.35, 11.99, 21.00, 7.36, 8.11, 24.35, 12.44, 12.40$$
$$31.01, 9.09, 11.57, 17.94, 4.42, 16.42, 9.25, 37.96, 13.19, 3.21, 13.53, 9.08$$

We fit this data set to LBE distribution and we obtain MLE of the scale parameter as $\hat{\theta} = 6.6114$. The corresponding Kolmogorov-Smirnov test statistics and associated p-values are obtained 0.16 and 0.915. Therefore, we can reject the null hypothesis that this dataset comes from the LBE distribution. Also, the estimated density and the empirical cdf plots support these observations (Figure 1). Then, we considered different stress levels S_1, S_2 and stress change times τ to exemplify our findings.

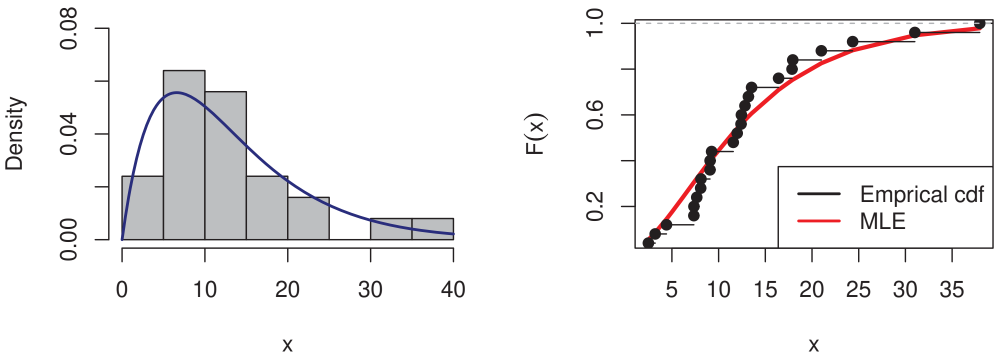

FIGURE 1. Estimated density and empirical cdf for real-data example fitted by the LBE distribution.

TABLE 6. Parameter estimates and their approximate confidence intervals for the real dataset.

S_1	S_2	τ	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\theta}_1$	$\hat{\theta}_2$
0.5	1.5	7.5	2.4429 (1.7592;3.1266)	-0.5342 (-1.1007;0.0324)	8.8096	5.1639
		12.5	1.9753 (1.4205;2.5302)	-0.1141 (-0.7407;0.5125)	6.8092	6.0749
		15	1.8768 (1.3144;2.4392)	-0.0172 (-0.6987;0.7331)	6.5890	6.7034
0.5	2	7.5	2.3543 (1.7556;2.9529)	-0.3564 (-0.7341; 0.0213)	8.8118	5.1629
		12.5	1.9564 (1.4891;2.4236)	-0.0764 (-0.4941;0.3412)	6.8085	6.0711
		15	1.8796 (1.4172;2.3419)	-0.0118 (-0.4656;0.4892)	6.5895	6.7076
0.75	1.5	7.5	2.7112 (1.7614;3.6610)	-0.7131 (-1.4686;0.0423)	8.8141	5.1629
		12.5	2.0313 (1.1905;2.8722)	-0.1513 (-0.9868;0.6842)	6.8063	6.0761
		15	1.8681 (0.9765;2.7597)	-0.0233 (-0.9314;0.9780)	6.5900	6.7061
0.75	1.5	7.5	2.4974 (1.7614;3.2334)	-0.4279 (-0.8812;0.0254)	8.8147	5.1631
		12.5	1.9866 (1.3768;2.5964)	-0.0916 (-0.5927;0.4096)	6.8071	6.0710
		15	1.8758 (1.2502;2.5014)	-0.0129 (-0.5596;0.5854)	6.5897	6.6969

It is observed that estimates are getting closer to its MLE value under the normal conditions when stress change time equal to 15 for all stress levels. Similarly, estimations are worsening with decreasing stress change time. The mean of this real data set is 13.22. We may conclude that close values of stress change time to the mean of the sample help to obtain better estimates for these determined stress levels.

7. Conclusions

The importance of length-biased distributions in especially reliability studies and their greater flexibility in modeling data in different areas such as lifetime data analysis, engineering, actuarial etc. inspired to consider accelerated life test plans under this distribution. Therefore, the length-biased exponential distribution is used which is one of the most used length-biased distributions in the literature. As a first attempt based on LBE distribution, we considered a cumulative exposure model under simple step stress accelerated life test.
We see that maximum likelihood estimations are obtained using some iterative methods as in many inference problems. Therefore, approximate confidence intervals are used in place of exact ones. Nevertheless, performances of the estimator and its confidence intervals are quite well performed. In addition, different combinations of the stress levels are compared with simulations and real data examples. We see that for the high stress with fixed level, results for lower stress level gives better result than higher ones. Similarly, results for higher stress level gives better result in the case of fixed lower stress level. Even so, there are not very important differences between estimates for our combinations. We also obtained optimal stress change times to construct the best ALT plans for this cumulative exposure model. All simulations and real data studies were applied according to this optimal plan.
As open problems, this ALT plan can be extended for cumulative exposure models under multiple stress levels. Also, censored cases can be considered for these plans.

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EXPONENTIATED GOMPERTZ EXPONENTIAL (EGoE) DISTRIBUTION: DERIVATION, PROPERTIES AND APPLICATIONS

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Abstract: In this paper, a new probability distribution called Exponentiated Gompertz Exponential distribution was introduced which can help researchers to model different types of data sets. In proposed distribution we introduce a new shape parameter to Gompertz Exponential distribution, varied its tail weight such that it enhances its flexibility and performance. Furthermore, the maximum likelihood method was used in estimating the model's parameters. Simulation method was used to investigate the behaviours of the parameters of the proposed distribution; the results showed that the mean square error and standard error for the chosen parameter values decrease as the sample size increases. The proposed distribution was tested on real life data, the results showed that EGoE performed better than the existing distribution in the literature and a strong competitor to other distributions of the same class. The results also showed that the distribution can be used as an alternative model in modelling lifetime processes.

Key words: Exponentiated gompertz exponential distribution, maximum Likelihood, means square error, quantile function, parameters

1. Introduction

The Gompertz distribution is a continuous probability distribution, named after Benjamen Gompertz. It is often used by demographers and actuaries to describe the distribution of adult life spans. It is a two parameter distribution that lies on support $[0, \infty]$. In the fields of Science and Biology, Gompertz distribution was used for survival analysis. This paper proposed a new continuous distribution called Exponentiated Gompertz Exponential distribution with increasing hazard rate. The proposed distribution added a shape parameter to the existing Gompertz Exponential distribution using the gompertz generalized family of distribution to enhance flexibility and better performance. There are so many generalized forms of Gompertz distribution in the literature. For instance, [1] extended Lomax distribution obtained by using Gompertz generalized family of distribution proposed by [3]. Also, [2] studied Gompertz Exponential distribution

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by extending exponential distribution using Gompertz family. [4] studied generalized Gompertz distribution by generalizing exponential and Gompertz distribution. The main advantage of [4] is that the shape of the hazard function could be increasing, decreasing, constant or bathtub depending on the value of the shape parameters thereby making it a suitable tool for reliability analysis. [5] introduced Beta Gompertz distribution which is quite flexible and can be used effectively in modelling survival data and reliability problems. Beta Gompertz distribution can have a decreasing, increasing, and bathtub-shaped failure rate function depending on its parameters.

In recent time, [8] proposed Gompertz flexible weibull distribution by extending the flexible weibull distribution using the gompertz generalized family of distributions proposed by [3] and used by [1]. The superiority of Gompertz flexible weibull distribution over Gompertz Weibull, Gompertz Burr type XII, Gompertz Lomax, exponentiated flexible weibull, exponentiated flexible weibull extension and Kumaraswamy flexible weibull distributions was demonstrated through its application to real data sets.

The remaining part of this paper is organized as follows: In section 2, the densities of Exponentiated Gompertz Exponential distribution which will henceforth be referred to as EGoE distribution are derived, its statistical properties like reliability function, distribution of order statistics, quantile function, mode, mean and variance (in integral form) are obtained ; including the estimation of the unknown parameters. In section 3, a simulation study was carried out to assess the performance of the unknown parameters of EGoE distribution. Applications to real data sets are provided in section 4 while concluding remark is provided in section 5.

2. The Exponentiated Gompertz Exponential (EGoE) distribution

The cdf of a random variable X from Exponentiated Gompertz Exponential distribution is derived by raising the cdf of Gompertz Exponential distribution to a shape parameter α. The associated expression is given as

$$F(x) = \left\{1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x \gamma}\right]}\right\}^{\alpha} x > 0,\ \ \theta > 0\ ,\ \gamma > 0\ ,\ \lambda > 0,\ \ \alpha > 0. \tag{2.1}$$

where θ, λ and γ are the shape parameters. The resulting plot is shown in figure 1.

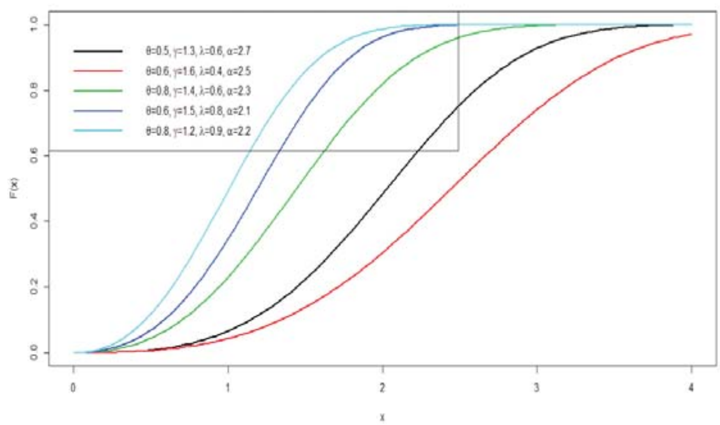

FIGURE 1. The plot of cumulative distribution function of EGoE distribution

Its corresponding pdf is obtained by differentiating equation (2.1) with respect to x

$$f(x) = \alpha\theta\lambda e^{\lambda x\gamma} e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]} \left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]^{\alpha-1} x > 0,\ \ \theta > 0\,,\ \gamma > 0\,,\ \lambda > 0,\ \ \alpha > 0 \tag{2.2}$$

The expansion of the pdf will be of the form

$$f(x) = \alpha\theta\lambda \sum_{i,j,s=0}^{\infty} \frac{(-1)^{i+s}}{j!} \binom{\alpha-1}{i} \binom{j}{s} \left(\frac{i\theta}{\gamma} + \frac{\theta}{\gamma}\right)^{j} e^{(s+1)\lambda x\gamma}\,. \tag{2.3}$$

and its probability density function shown in figure 2.

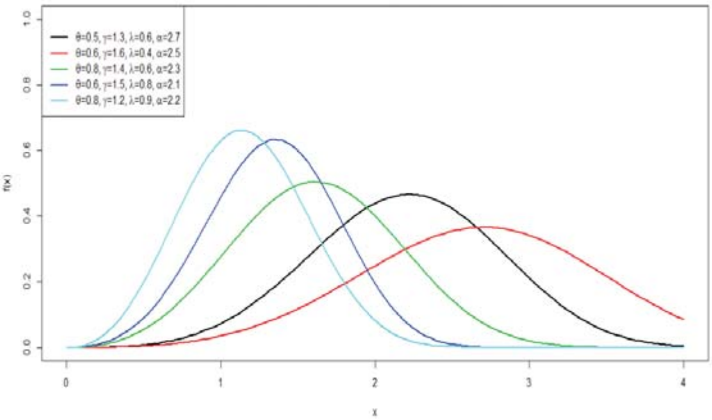

FIGURE 2. Plot of the probability density function of EGoE distribution.

Reliability analysis

The reliability analysis of EGoE distribution discussed in this sub – section are survival function and hazard function. Survival function is the probability that a system will survive beyond a specified time while hazard function also known as failure rate can be interpreted as the conditional probability of failure, given it has survived to time t. Survival and hazard functions are very important in Biological sciences for survival analysis and engineering for reliability analysis.

Reliability or survival function can be obtained mathematically as the complement of the cumulative density function (cdf) as follows:

$$S(x) = 1 - F(x) \tag{2.4}$$

Therefore, the reliability function of EGoE distribution is given by

$$S(x) = 1 - \left\{1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right\}^{\alpha}; x > 0,\ \ \theta > 0\,,\ \gamma > 0\,,\ \lambda > 0,\ \alpha > 0 \tag{2.5}$$

Hazard function can be obtained from

$$h(x) = \frac{f(x)}{S(x)}. \tag{2.6}$$

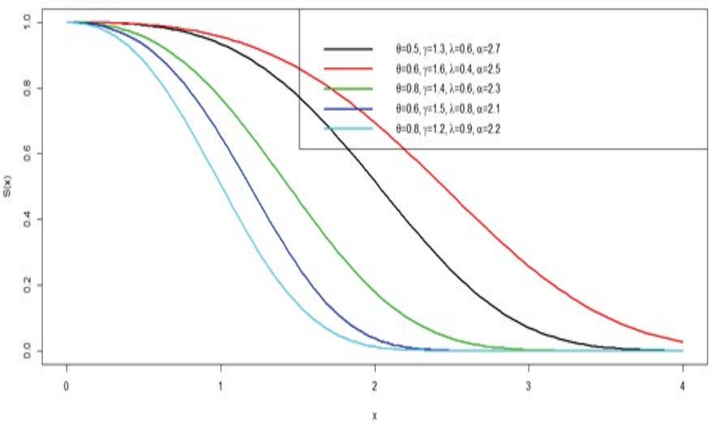

FIGURE 3. Plot of the survival function of EGoE distribution.

Therefore the hazard function of EGoE distribution is given by

$$h(x) = \frac{\alpha\theta\lambda e^{\lambda x\gamma} e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\left[1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]^{\alpha-1}}{1-\left\{1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right\}^{\alpha}}\;, x>0,\;\; \theta>0\,,\; \gamma>0\,,\; \lambda>0,\;\; \alpha>0. \tag{2.7}$$

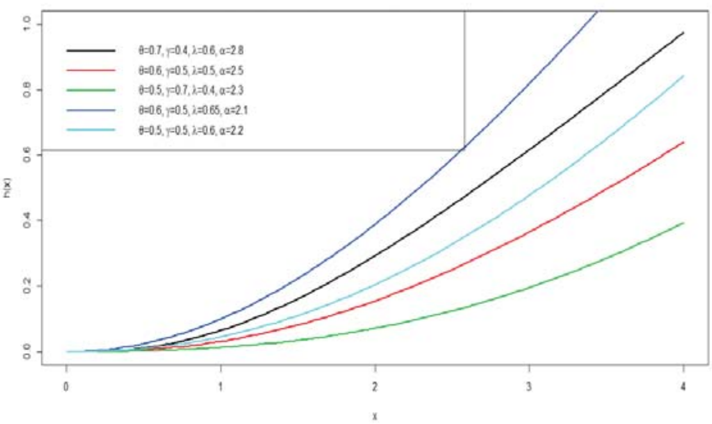

FIGURE 4. Plot of the hazard function of EGoE distribution.

Quantile function and median of EGoE distribution

In this sub-section, quantile and median of EGoE distribution are derived as follows

The quantile X_q of a random variable from EGoE distribution (θ , γ , λ, α) random variable X is given by

$$(X_q)_{EGoE} = \frac{1}{\lambda\gamma}\ln\left[1 - \frac{\gamma}{\theta}ln\left(1 - q^{\frac{1}{\alpha}}\right)\right]$$

The final form of the quantile function of Exponentiated Gompertz Exponential distribution can be written as

$$(X_q)_{EGoE} = \frac{1}{\lambda\gamma}\ln\left[1 - \frac{\gamma}{\theta}ln\left(1 - q^{\frac{1}{\alpha}}\right)\right] \tag{2.8}$$

The median of EGoE can be derived from equation (2.8) by setting $q = 0.5$
Therefore,

$$median = \frac{1}{\lambda\gamma}\ln\left[1 - \frac{\gamma}{\theta}ln\left(1 - (0.5)^{\frac{1}{\alpha}}\right)\right] \tag{2.9}$$

The mode
The mode of EGoE distribution can be derived by first differentiating its probability density function with respect to x and then equating the resulting derivative to zero.

$$\left[\left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]^{\alpha-1}\left[\lambda\left(\gamma - \theta e^{2\lambda x\gamma}e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right)\right] + \left[\theta\lambda e^{2\lambda x\gamma}e^{\frac{2\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]\right] = 0. \tag{2.10}$$

gives the mode of EGoE distribution.
It should be noted that the non -linear equation (2.10) does not have analytical solution in x but can be solved numerically when data sets are available with the use of statistical packages.

Order statistics
The pdf of the j_{th} order statistics of the EGoE distribution is

$$\begin{gathered} f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!}\alpha\theta\lambda e^{\lambda x\gamma}e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]^{\alpha-1} \\ \left[\left\{1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right\}^{\alpha}\right]^{j-1}\left[1 - \left\{1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right\}^{\alpha}\right]^{n-j} \end{gathered} \tag{2.11}$$

The distributions of minimum and maximum order statistics for the Exponentiated Gompertz Exponential distribution are given below.

$$f_{1:n}(x) = n\alpha\theta\lambda e^{\lambda x\gamma}e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]^{\alpha-1}\left[1 - \left\{1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right\}^{\alpha}\right]^{n-1}. \tag{2.12}$$

$$f_{n:n}(x) = n\alpha\theta\lambda e^{\lambda x\gamma}e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]^{\alpha-1}\left[\left\{1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right\}^{\alpha}\right]^{n-1}. \tag{2.13}$$

Moments of EGoE distribution
Let X be a random variable that has the EGoE distribution, then, the r^{th} non-central moments is given by

$$E(X^r) = \int_0^\infty x^r f(x)\,dx \tag{2.14}$$

$$E(X^r) = \int_0^\infty x^r\alpha\theta\lambda\sum_{i,j,s=0}^{\infty}\frac{(-1)^{i+s}}{j!}\binom{\alpha-1}{i}\binom{j}{s}\left(\frac{i\theta}{\gamma} + \frac{\theta}{\gamma}\right)^j e^{(s+1)\lambda x\gamma}\,dx$$

$$= \alpha\theta\lambda\sum_{i,j,s=0}^{\infty}\frac{(-1)^{i+s}}{j!}\binom{\alpha-1}{i}\binom{j}{s}\left(\frac{i\theta}{\gamma} + \frac{\theta}{\gamma}\right)^j\int_0^\infty x^r e^{(s+1)\lambda x\gamma}\,dx.$$

The first moment ($r = 1$) which is the mean of the distribution is given by

$$E(X) = \alpha\theta\lambda \sum_{i,j,s=0}^{\infty} \frac{(-1)^{i+s}}{j!} \binom{\alpha-1}{i} \binom{j}{s} \left(\frac{i\theta}{\gamma} + \frac{\theta}{\gamma}\right)^j \int_0^\infty x e^{(s+1)\lambda x\gamma}\, dx. \tag{2.15}$$

The second moment ($r = 2$)is given by

$$E\left(X^2\right) = \alpha\theta\lambda \sum_{i,j,s=0}^{\infty} \frac{(-1)^{i+s}}{j!} \binom{\alpha-1}{i} \binom{j}{s} \left(\frac{i\theta}{\gamma} + \frac{\theta}{\gamma}\right)^j \int_0^\infty x^2 e^{(s+1)\lambda x\gamma}\, dx. \tag{2.16}$$

Thus, the variance of EGoE distribution using Equations (2.15) and (2.16) is given by

$$Var(X) = E(X^2) - [E(X)]^2$$

$$\begin{aligned} Var(X) = {} & \alpha\theta\lambda \sum_{i,j,s=0}^{\infty} \frac{(-1)^{i+s}}{j!} \binom{\alpha-1}{i} \binom{j}{s} \left(\frac{i\theta}{\gamma} + \frac{\theta}{\gamma}\right)^j \int_0^\infty x^2 e^{(s+1)\lambda x\gamma}\, dx \\ & - \left[\alpha\theta\lambda \sum_{i,j,s=0}^{\infty} \frac{(-1)^{i+s}}{j!} \binom{\alpha-1}{i} \binom{j}{s} \left(\frac{i\theta}{\gamma} + \frac{\theta}{\gamma}\right)^j \int_0^\infty x e^{(s+1)\lambda x\gamma}\, dx\right]^2. \end{aligned} \tag{2.17}$$

Maximum likelihood estimators

The parameters of the EGoE distribution can be estimated using the method of Maximum Likelihood Estimation (MLE) as follows:
Let $x_1, x_2, \ldots, x_n$ be a random sample from the Exponentiated Gompertz Exponential (EGoE) distribution. Then, the likelihood function is given by

Thus,

$$f(x_1, x_2, \ldots, x_n; \alpha, \theta, \gamma, \lambda) = \alpha^n \theta^n \lambda^n e^{\lambda\gamma \sum_{i=1}^n x} e^{\frac{\theta}{\gamma} \sum_{i=1}^n \left[\, 1 - e^{\lambda x\gamma}\right]} \sum_{i=1}^n \left[1 - e^{\frac{\theta}{\gamma}\left[1 - e^{\lambda x\gamma}\right]}\right]^{\alpha-1}$$

Let $l = log f(x_1, x_2, \ldots, x_n; \alpha, \theta, \gamma, \lambda)$,

$$l = nlog\alpha + nlog\theta + nlog\lambda + \lambda\gamma \sum_{i=1}^n x + \frac{\theta}{\gamma} \sum_{i=1}^n \left[\, 1 - e^{\lambda x\gamma}\right] + (\alpha - 1) \sum_{i=1}^n log\left[1 - e^{\frac{\theta}{\gamma}\left[1 - e^{\lambda x\gamma}\right]}\right].$$

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n log\left[1 - e^{\frac{\theta}{\gamma}\left[1 - e^{\lambda x\gamma}\right]}\right].$$

Therefore,

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \frac{1}{\gamma} \sum_{i=1}^n \left[\, 1 - e^{\lambda x\gamma}\right] + \frac{(1-\alpha)(1 - e^{\lambda x\gamma})}{\gamma} \sum_{i=1}^n \frac{e^{\frac{\theta}{\gamma}\left[1 - e^{\lambda x\gamma}\right]}}{1 - e^{\frac{\theta}{\gamma}\left[1 - e^{\lambda x\gamma}\right]}}. \tag{2.18}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + \gamma \sum_{i=1}^n x - \theta \sum_{i=1}^n x e^{\lambda x\gamma} + \frac{\theta(1-\alpha) \sum_{i=1}^n x e^{\frac{\theta}{\gamma}\left[1 - e^{\lambda x\gamma}\right]}}{\left[1 - e^{\frac{\theta}{\gamma}\left[1 - e^{\lambda x\gamma}\right]}\right]}. \tag{2.19}$$

$$\begin{aligned} \frac{\partial l}{\partial \gamma} = {} & \lambda \sum_{i=1}^n x - \frac{\theta\lambda}{\gamma} \sum_{i=1}^n x e^{\lambda x\gamma} - \frac{\theta}{\gamma^2} \sum_{i=1}^n \left[\, 1 - e^{\lambda x\gamma}\right] \\ & + \frac{\theta(\alpha-1) \sum_{i=1}^n \left[\left(\lambda x e^{\lambda x\gamma} + \frac{1}{\gamma}\left(\, 1 - e^{\lambda x\gamma}\right)\right) e^{\frac{\theta}{\gamma}\left[1 - e^{\lambda x\gamma}\right]}\right]}{\gamma\left[1 - e^{\frac{\theta}{\gamma}\left[1 - e^{\lambda x\gamma}\right]}\right]}. \end{aligned} \tag{2.20}$$

Equating $\frac{\partial l}{\partial \alpha}=0$, $\frac{\partial l}{\partial \theta}=0$, $\frac{\partial l}{\partial \lambda}=0\ and\ \frac{\partial l}{\partial \gamma}=0$, which become

$$\frac{n}{\alpha}+\sum_{i=1}^{n} log\left[1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]=0. \tag{2.21}$$

$$\frac{n}{\theta}+\frac{1}{\gamma}\sum_{i=1}^{n}\left[\,1-e^{\lambda x\gamma}\right]+\frac{(1-\alpha)(1-e^{\lambda x\gamma})}{\gamma}\sum_{i=1}^{n}\frac{e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}}{1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}}=0. \tag{2.22}$$

$$\frac{n}{\lambda}+\gamma\sum_{i=1}^{n}x-\theta\sum_{i=1}^{n}xe^{\lambda x\gamma}+\frac{\theta\,(1-\alpha)\sum_{i=1}^{n}xe^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}}{\left[1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]}=0. \tag{2.23}$$

and

$$\begin{aligned}&\lambda\sum_{i=1}^{n}x-\frac{\theta\lambda}{\gamma}\sum_{i=1}^{n}xe^{\lambda x\gamma}-\frac{\theta}{\gamma^{2}}\sum_{i=1}^{n}\left[\,1-e^{\lambda x\gamma}\right]\\&+\frac{\theta(\alpha-1)\sum_{i=1}^{n}\left[\left(\lambda xe^{\lambda x\gamma}+\frac{1}{\gamma}\left(\,1-e^{\lambda x\gamma}\right)\right)e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]}{\gamma\left[1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]}=0.\end{aligned} \tag{2.24}$$

The MLE of α can then be obtained from equation (2.21) for a given $\theta, \lambda\ and\ \gamma$ in the form below.

$$\alpha=\frac{-n}{\sum_{i=1}^{n} log\left[1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]}. \tag{2.25}$$

Substituting equation (2.25) into equations (2.22), (2.23) and (2.24) and by solving the resulting systems of three non-linear equations numerically, we get the MLE of $\theta, \lambda\ and\ \gamma$.

3. Simulation study

Simulation study was conducted using R statistical software. Data sets were generated from EGoE distribution. Random samples of sizes n=10, 100 and 1000 were used. The simulation was conducted for three different cases using varying true parameters in each case. The selected true parameter values are $\theta = 0.5, \quad \lambda = 0.5, \quad \gamma = 0.5, and\ \alpha = 0.5$; $\theta = 1.0, \quad \lambda = 1.0, \ \gamma = 1.0, and\ \alpha = 1.0$; $\theta = 0.5, \lambda = 0.5, \gamma = 0.5, and\ \alpha = 0.5$ for the first, second and third cases respectively. The maximum likelihood estimates of the true parameters including the Bias, Standard Error and Root Mean Square Error (RMSE) were obtained with the result of the simulation studies shown in Tables 3.1, 3.2, and 3.3.

Table 3.1: Simulation study at $\theta\ = 0.5$, $\lambda = 0.5$, $\gamma = 0.5$ and $\alpha = 0.5$

n	Parameters	Means	Bias	Std. Error	RMSE
10	θ =0.5	2.3028	-1.8028	2.3183	0.4815
	$\lambda = 0.5$	0.7712	-0.2712	7.5652	0.8698
	$\gamma = 0.5$	1.3686	-0.8686	13.2869	1.1527
	$\alpha = 0.5$	0.2217	0.2783	2.2168	0.4708
100	θ=0.5	0.4251	0.0749	0.0681	0.0261
	$\lambda = 0.5$	0.2363	0.2637	0.2333	0.0483
	$\gamma = 0.5$	0.5676	- 0.0676	0.3534	0.0594
	$\alpha = 0.5$	0.6930	- 0.1930	0.3114	0.0558
1000	θ=0.5	0.4972	0.0028	0.0230	0.0048
	$\lambda = 0.5$	0.3693	0.1307	0.0432	0.0066
	$\gamma = 0.5$	0.6537	- 0.1537	0.0357	0.0060
	$\alpha = 0.5$	0.3632	0.1368	0.0483	0.0069

Table 3.2: Simulation study at$\theta = 1.0$,$\lambda = 1.0$, $\gamma = 1.0$ and $\alpha = 1.0$

n	Parameters	Means	Bias	Std. Error	RMSE
10	$\theta = 1.0$	0.9102	0.0898	0.6518	0.2553
	$\lambda = 1.0$	0.6371	0.3629	15.4529	1.2431
	$\gamma = 1.0$	1.1214	- 0.1214	27.0708	1.6453
	$\alpha = 1.0$	1.2441	0.1368	29.9894	1.7317
100	$\theta = 1.0$	0.7499	0.2501	0.1390	0.0373
	$\lambda = 1.0$	0.3780	0.6220	1.9413	0.1393
	$\gamma = 1.0$	1.4201	- 0.4201	7.2655	0.2695
	$\alpha = 1.0$	1.0126	- 0.0126	5.1842	0.2277
1000	$\theta = 1.0$	1.0628	- 0.0628	0.0648	0.0080
	$\lambda = 1.0$	1.1703	-0.1703	4.0535	0.0637
	$\gamma = 1.0$	0.9296	- 0.0704	3.2057	0.0566
	$\alpha = 1.0$	0.9498	0.0502	3.2610	0.0571

Table 3.3: Simulation study at $\theta = 1.5$, $\lambda = 1.5$, $\gamma = 1.5$ and $\alpha = 1.5$

n	Parameters	Means	Bias	Std. Error	RMSE
10	$\theta = 1.5$	1.0776	0.4224	0.9760	0.3124
	$\lambda = 1.5$	0.4382	1.0618	4.7415	0.6886
	$\gamma = 1.5$	1.9782	- 0.4782	21.7080	1.4734
	$\alpha = 1.5$	2.3560	- 0.8560	26.3076	1.6220
100	$\theta = 1.5$	2.0174	- 0.5174	0.4849	0.0696
	$\lambda = 1.5$	1.5719	-0.0719	4.6175	0.2149
	$\gamma = 1.5$	1.5852	- 0.0852	4.5838	0.2141
	$\alpha = 1.5$	1.6677	- 0.1677	4.8031	0.2192
1000	$\theta = 1.5$	1.5388	- 0.0388	0.1102	0.0105
	$\lambda = 1.5$	1.7839	-0.2839	1.3853	0.0372
	$\gamma = 1.5$	1.2857	0.2143	0.9888	0.0314
	$\alpha = 1.5$	1.6850	- 0.1850	1.3079	0.0361

Application to real-life data sets

In this section, Exponentiated Gompertz Exponential distribution was compared with four other four – parameter compound distributions – Gompertz Weibull distribution (GOWE), Gompertz Burrxii distribution (GOBXII), Gompertz Lomax (GOLOM) distribution and Gompertz Flexible Weibull distribution (GOFLWE). The distributions were fitted to three real data sets presented below:

Dataset I: The first data set represents the reproducibility of median – time – to – failure (t 50) measurements. It has been previously used by [1].
6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.12, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.7, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.64, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923.

Dataset II: This data represents the waiting times (in minutes) before service of 100 Bank customers. It has been used previously by [2], [3] and [4].
0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5 .

Dataset III: This data set represents the strength of carbon fibers tested under tension at gauge lengths of 10mm. It has been used previously by [5] and [6]. The observations are as follows:
1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

Table 3.4: The descriptive statistics for the three data sets above are provided in the table below.

Parameters	N	Min.	Q_1	Median	Q_3	Mean	Max.	Skewness	Kurtosis	Variance
Dataset 1	59	2.997	6.052	6.923	7.941	6.980	11.040	0.1932	3.0874	2.6051
Dataset II	100	0.800	4.675	8.100	13.02	9.877	38.500	1.4728	5.5403	52.3741
Dataset III	63	1.901	2.554	2.996	3.422	3.059	5.020	0.6328	3.2863	0.3855

The goodness-of-fit statistics including Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC),Bayesian Information Criterion (BIC), Log-likelihood (LL), Hannan-Quinn Information Criterion (HQIC), Shapiro – Wilk test (W), Anderson – Darling test (A) and Kolmogorov-Smirnov (K-S) statistics are computed to compare the fitted models.

Table 3.5: Performance rating of EGoE distribution for data set 1

Distribution	Parameter Estimates	- LL	AIC	CAIC	BIC	HQIC
EGoE	$\hat{\alpha}$=1.7868	111.2993	230.5985	231.3395	238.9087	233.8425
	$\hat{\theta}$=2.4673					
	$\hat{\lambda}$=7.7670					
	$\hat{\gamma}$=0.0866					
GOWE	$\hat{\alpha}$=0.3838	111.7834	231.5668	232.3075	239.877	234.8107
	$\hat{\theta}$=-0.0589					
	$\hat{\lambda}$=0.1617					
	$\hat{\gamma}$=5.3304					
GOFLWE	$\hat{\alpha}$ =0.0507	111.8013	231.6027	232.3434	239.9128	234.8466
	$\hat{\theta}$=15.4192					
	$\hat{\lambda}$=-0.0115					
	$\hat{\gamma}$=6.8100					
GOLOM	$\hat{\alpha}$=0.0046	114.5715	237.0230	237.7638	245.3332	240.2670
	$\hat{\theta}$=3.3722					
	$\hat{\lambda}$=0.1747					
	$\hat{\gamma}$=2.3061					
GOBUXII	$\hat{\alpha}$ =0.0027	114.5667	237.1335	237.8742	245.4436	240.3774
	$\hat{\theta}$=7.5043					
	$\hat{\lambda}$=0.2720					
	$\hat{\gamma}$=1.9276					

Table 3.6: Test statistic of EGOE and the competing distributions using data set 1

Distribution	W	A	KS	p - value
EGoE	0.0380	0.2132	0.0664	0.9419
GOWE	0.0456	0.2567	0.0735	0.8842
GOFLWE	0.0511	0.2860	0.0760	0.8589
GOLOM	0.1232	0.7036	0.0979	0.5888
GOBUXII	0.0644	0.3602	0.1339	0.2198

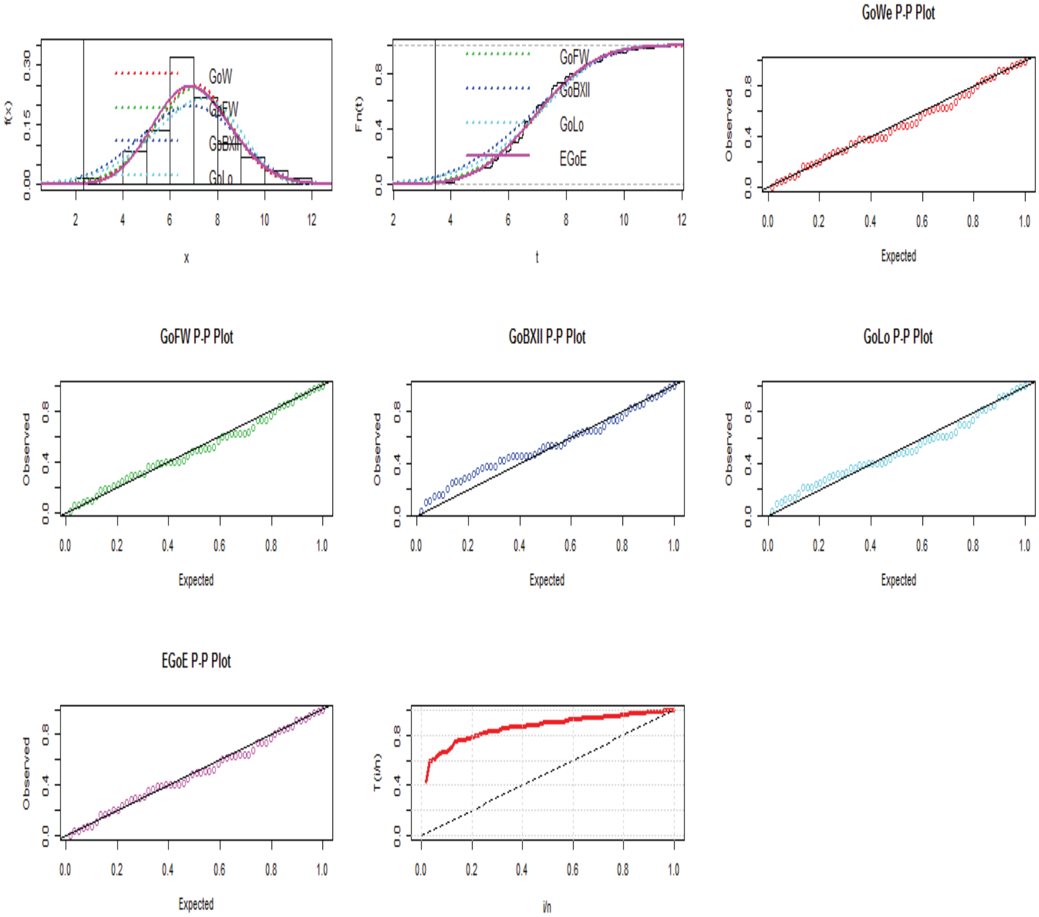

FIGURE 5. Graphical displays of EGoE and the competing distributions with respect to data set 1

Table 3.7: Performance rating of EGoE distribution for data set II

Distribution	Parameter Estimates	- LL	AIC	CAIC	BIC	HQIC
EGoE	$\hat{\alpha}$=0.6558	317.0708	642.1415	642.5626	652.5622	646.3590
	$\hat{\theta}$=-0.0146					
	$\hat{\lambda}$ =2.2605					
	$\hat{\gamma}$=0.2538					
GOWE	$\hat{\alpha}$=1.1742	317.9028	643.8055	644.2266	654.2262	648.0230
	$\hat{\theta}$ =-0.1589					
	$\hat{\lambda}$ =0.0893					
	$\hat{\gamma}$=1.6221					
GOFLWE	$\hat{\alpha}$=0.1224	317.1068	642.2136	642.6347	652.6343	646.4311
	$\hat{\theta}$=4.5968					
	$\hat{\lambda}$=0.0061					
	$\hat{\gamma}$=2.9838					
GOLOM	$\hat{\alpha}$ =0.0159	319.3325	646.6649	647.0860	657.0856	650.8823
	$\hat{\theta}$ =3.0670					
	$\hat{\lambda}$=3.0723					
	$\hat{\gamma}$=0.4841					
GOBUXII	$\hat{\alpha}$ =0.1057	317.3427	642.6854	643.1065	653.1061	646.9028
	$\hat{\theta}$=2.4126					
	$\hat{\lambda}$ =0.1618					
	$\hat{\gamma}$=3.4293					

Considering the values of the AIC, CAIC, BIC and HQIC; Exponentiated Gompertz Exponential distribution seems to perform better than the competing distributions since it has the lowest value. The graphical representations of comparative analysis in table 3.7 are shown in figure 5.

Table 3.8: Test statistic of EGoE and the competing distributions using data set II

Distribution	W	A	KS	p - value
EGoE	0.0187	0.1320	0.0387	0.9983
GOWE	0.0373	0.2346	0.0447	0.9882
GOFLWE	0.0264	0.2074	0.0389	0.9981
GOLOM	0.0728	0.4578	0.0577	0.8926
GOBUXII	0.0321	0.2207	0.0573	0.9551

With EGoE having the lowest value of W, A and KS, it shows that it is the best among the competing distributions.

Table 3.9: Performance rating of EGoE distribution for data set III

Distribution	Parameter Estimates	- LL	AIC	CAIC	BIC	HQIC
EGoE	$\hat{\alpha}$ =1.2366	56.2729	120.5458	121.2354	129.1183	123.9174
	$\hat{\theta}$ =0.2069					
	$\hat{\lambda}$=58.5181					
	$\hat{\gamma}$=0.8998					
GOWE	$\hat{\alpha}$ =0.0081	67.1454	142.2908	142.9805	150.8634	145.6624
	$\hat{\theta}$ =5.6238					
	$\hat{\lambda}$=0.3697					
	$\hat{\gamma}$=0.7116					
GOFLWE	$\hat{\alpha}$ =0.0076	58.3618	124.7236	125.4132	133.2961	128.0952
	$\hat{\theta}$ =27.5730					
	$\hat{\lambda}$=-0.0845					
	$\hat{\gamma}$=3.0426					
GOLOM	$\hat{\alpha}$ =0.0044	64.9569	137.9139	138.6035	146.4864	141.2855
	$\hat{\theta}$ =5.3157					
	$\hat{\lambda}$=0.4533					
	$\hat{\gamma}$=1.4547					
GOBUXII	$\hat{\alpha}$ =0.0074	62.4951	132.9903	133.6799	141.5628	136.3619
	$\hat{\theta}$ =3.6613					
	$\hat{\lambda}$=0.4632					
	$\hat{\gamma}$=3.0294					

Considering the values of the AIC, CAIC, BIC and HQIC; Exponentiated Gompertz Exponential distribution is having the lowest value, therefore, performs better than the competing distributions.

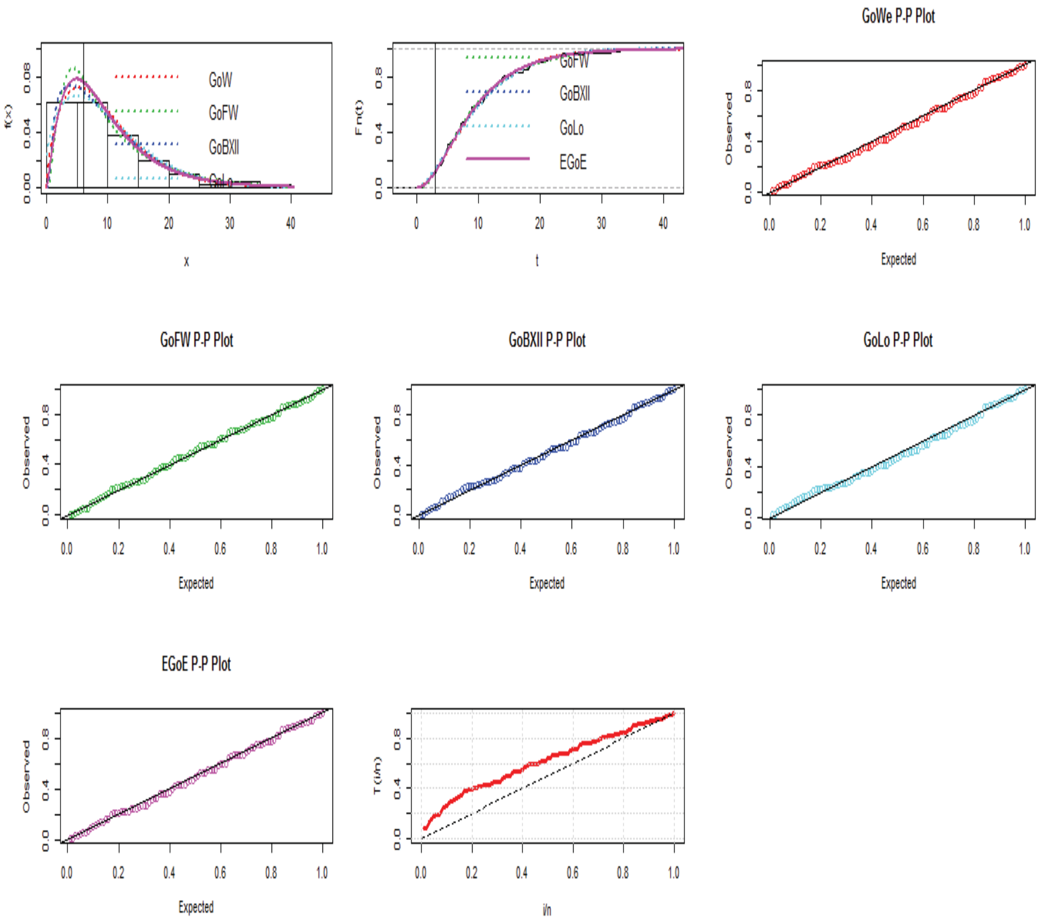

FIGURE 6. Graphical displays of EGoE and the competing distributions with respect to data set II

Table 3.10: Test statistic of EGoE and the competing distributions using data set III

Distribution	W	A	KS	p - value
EGoE	0.0583	0.3166	0.0798	0.8178
GOWE	0.2082	1.3967	0.1370	0.1877
GOFLWE	0.0801	0.5138	0.0914	0.6688
GOLOM	0.1624	1.1128	0.1282	0.2515
GOBUXII	0.1361	0.9429	0.0934	0.6411

With EGoE having the lowest value of W, A and KS, it shows that it is the best among the competing distributions.

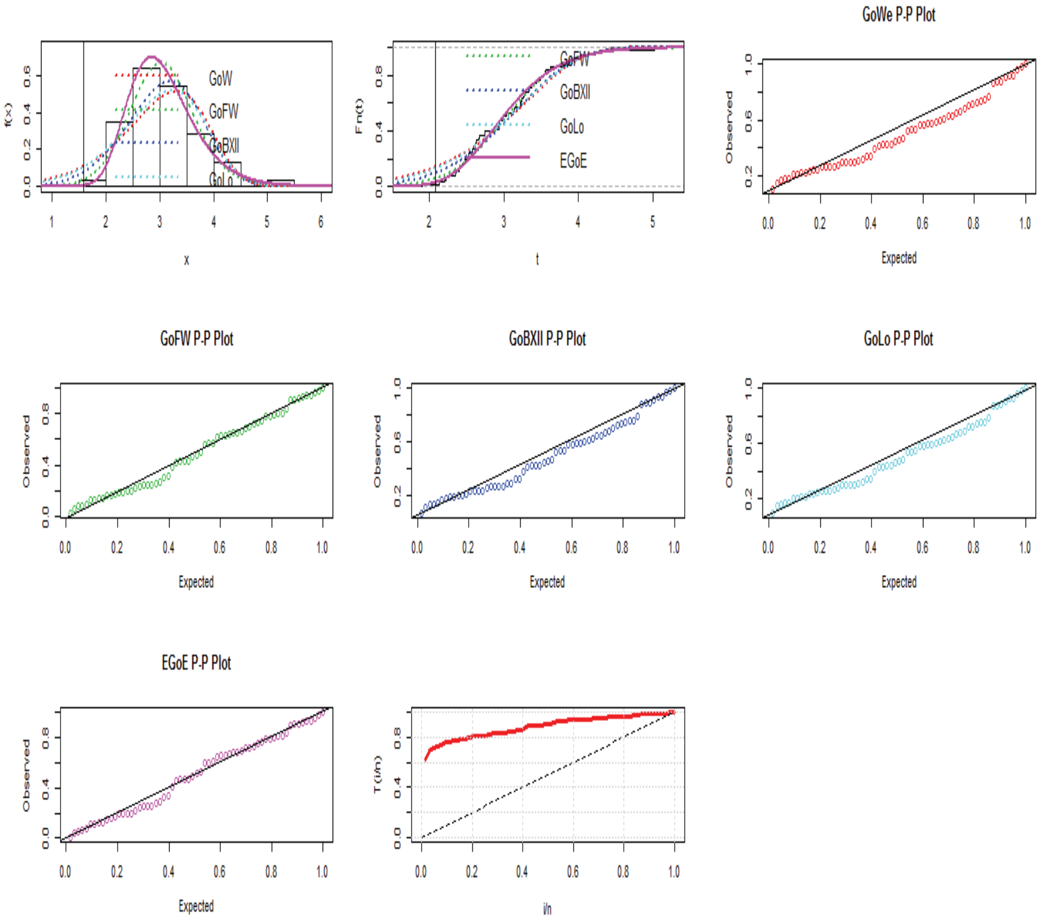

FIGURE 7. Graphical displays of EGoE and the competing distributions with respect to data set III.

4. Discussion of results

Tables 3.1, 3.2 and 3.3 present simulation results performed using R statistical software. The results showed that as parameter $\theta, \lambda, \gamma, and \alpha$; values increases and sample size n also increase then the root mean square error (RMSE) and the standard error of the results decreases. Three different real data sets were used for this study. The results are as shown in the Tables 3.5, 3.7, and 3.9 and Tables 3.6, 3.8 and 3.10.
Tables 3.5, 3.7 and 3.9 the descriptive statistics result showed that life data is positively skewed distribution. Furthermore the test of performance rating was done as shown in the above table the result showed that the proposed distribution Exponentiated Gompertz Exponential distribution has the lowest value of AIC,

CAIC, BIC and HQIC. This implies that the distribution is better fit and performed better than the compared distribution when compared with Gompertz Weibull distribution, Gompertz Flexible Weibull distribution, Gompertz Lomax distribution and Gompertz Burr XII distribution. The goodness of fit for Tables 3.6, 3.8 and 3.10 showed that the performance of EGoE distribution over other distributions is with the lowest values of W, A and KS. This implies that the proposed distribution is a strong competitor to other distribution of the same class and can also be used as alternative model in modeling lifetime processes.

5. Conclusion

In this paper, a new continuous probability distribution with increasing hazard rate is introduced and discussed. Its statistical properties were investigated. The mean and variance of the proposed distribution were obtained in integral form. The maximum likelihood method was used in estimating the parameters of the distribution. Simulation study was carried out to assess the performance of the distribution and its stability. Real life data was carried out on the proposed distribution. The result revealed that EGoE distribution performed better with lower AIC and BIC than the existing distribution when compared with Gompertz Weibull, Gompertz flexible Weibull, Gompertz Lomax and Gompertz Burr XII distributions.

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ON THE DUS-KUMARASWAMY DISTRIBUTION

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Abstract: Kumaraswamy distribution is introduced by [7] and it is particularly useful for many natural phenomena whose outcomes have lower and upper bounds or bounded outcomes in biomedical and epidemiological research (see [12]). In this paper, a new statistical distribution called DUS-Kumaraswamy is introduced by using DUS transformation (which is recently introduced by [6]) on Kumaraswamy distribution. The proposed distribution has the same domain as Kumaraswamy and it can be used as an alternative model to describe the natural phenomena mentioned above. Several distributional properties such as mean, variance, skewness, kurtosis, Lorenz and Bonferroni curves are studied. The statistical inference on the parameters of Dus-Kumaraswamy is discussed by maximum likelihood methodology. A simulation study is conducted to observe the behaviors of maximum likelihood estimates under different conditions. A numerical example is also presented.

Key words: Data analysis, Kumaraswamy distribution, maximum likelihood estimator, monte carlo simulation

1. Introduction

In this study, the DUS transformation of [7] is used to introduce a new distribution bounded within (0,1). The Kumaraswamy distribution is considered as a baseline distribution in their DUS transformation. Let $F(x)$ and $f(x)$ denote respectively the cumulative distribution function (cdf) and probability density function (pdf) of baseline distribution. Then the pdf and cdf of the DUS family are given, respectively, by

$$f_{DUS}(x) = \frac{1}{e-1} f(x) e^{F(x)}, \ x \in D \tag{1.1}$$

and

$$F_{DUS}(x) = \frac{1}{e-1}\left(e^{F(x)} - 1\right), \tag{1.2}$$

where D is a domain of the baseline distribution with cdf F.

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The DUS transformation with a exponential cdf is considered by [7]. Using DUS transformation, the DUS-Lomax distribution is proposed by [4]. [5] introduced a new lifetime distribution called DUS-Weibull distribution by using same mechanism. Recently, [10] generalized the DUS transformation and they studied the exponential baseline in their generalized DUS transformation.

It was reported in these studies that the DUS transformation increases the distribution flexibility. It can also be said that the DUS transformation has become a center of attraction in recent years. In this paper, the DUS transformation is applied to the Kumaraswamy cdf to get new distribution. The paper is organized as follows. In Section 2, moments, hazard rate, survival and quantile functions are obtained. The maximum likelihood method is discussed in Section 3. In Section 4, a simulation study is also performed to observe the performance of the maximum likelihood estimate. A numerical example is given to illustrate the capability of the proposed distribution for modeling a real data in Section 5. In Section 6, concluding remarks are provided.

2. DUS-Kumaraswamy distribution

In this section, DUS transformation is applied to Kumaraswamy cdf. The pdf and cdf of the Kumaraswamy distribution are given, respectively, by

$$f_K(x) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}, \; 0<x<1, \tag{2.1}$$

and

$$F_K(x) = 1-(1-x^{\alpha})^{\beta}, \tag{2.2}$$

where $\alpha, \beta > 0$ are the parameters.

Using Eqs. (2.1) and (2.2) in Eqs. (1.1) and (1.2), respectively, the pdf and cdf of the new distribution are given, respectively, by

$$f_{DUS-K}(x) = \tfrac{1}{e-1}\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1} e^{\left(1-(1-x^{\alpha})^{\beta}\right)}, \; 0<x<1 \tag{2.3}$$

and

$$F_{DUS-K}(x) = \frac{1}{e-1}\left(e^{\left(1-(1-x^{\alpha})^{\beta}\right)}-1\right), \tag{2.4}$$

with parameters $\alpha > 0$ and $\beta > 0$. The random variable X with cdf (2.2) is said to have two-parameter DUS-Kumaraswamy distribution and it is denoted by $DUS-K(\alpha, \beta)$.

Fig. 1 presents the plots of the $DUS-K(\alpha, \beta)$ probability density function for some choices of α and β. From Fig. 1, it is concluded that the pdf of $DUS-K(\alpha, \beta)$ can be unimodal as well as have decreasing and increasing forms.

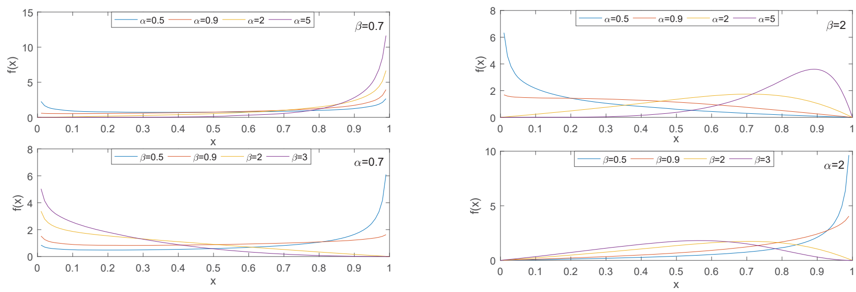

FIGURE 1. Pdf plots of the $DUS-K$ distribution for selected parameters values

The survival function, $S(x)$, and the hazard rate function, $h(x)$, for $DUS-K(\alpha,\beta)$ distribution are in the following forms:

$$S_{DUS-K(\alpha,\beta)}(x) = \frac{e - e^{\left(1-(1-x^{\alpha})^{\beta}\right)}}{e-1} \tag{2.5}$$

and

$$h_{DUS-K(\alpha,\beta)}(x) = \frac{\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1} e^{\left(1-(1-x^{\alpha})^{\beta}\right)}}{e - e^{\left(1-(1-x^{\alpha})^{\beta}\right)}}. \tag{2.6}$$

Fig. 2 presents plots of the hazard rate function of $DUS-K(\alpha,\beta)$ for some selected values of α and β. From Fig. 2, it is observed that the hazard function of introduced distribution is increasing.

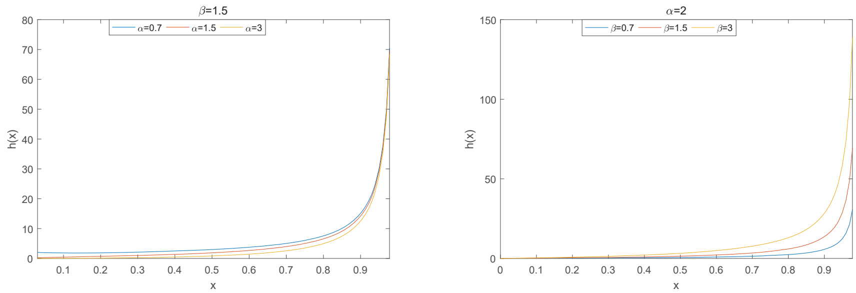

FIGURE 2. Hazard rate function plots of the $DUS-K$ distribution for selected parameters values

The quantile function of the $DUS-K(\alpha,\beta)$ distribution is given by

$$Q(u;\alpha,\beta) = \left\{1 - \left[1 - \log\left(1 + u(e-1)\right)\right]^{\frac{1}{\beta}}\right\}^{\frac{1}{\alpha}}, \ u \in (0,1). \tag{2.7}$$

Using Eq. (2.7), the median is obtained as

$$Q(0.5;\alpha,\beta) = \left\{1 - \left[1 + \log(2) - \log(1+e)\right]^{\frac{1}{\beta}}\right\}^{\frac{1}{\alpha}}.$$

Let X be an absolutely continuous random variable with distribution function F. Then, by using probability integral transformation, we can write

$$E(X_{i:n}) \simeq F^{-1}\left(\frac{i}{n+1}\right), \ i = 1, 2, \ldots, m, \tag{2.8}$$

where $X_{i:n}$ is the ith order statistic of the sample of size n and F^{-1} is the inverse of F.

Let X be a random variable from DUS family with a baseline cdf F in (1.2). Using Taylor series expansion, the distribution function of X can be written as

$$\begin{aligned} G(x) &= \frac{1}{e-1}\left(e^{F(x)} - 1\right) \\ &= \frac{1}{e-1}\sum_{i=0}^{\infty}\frac{(F(x))^{i}}{i!} - 1 \\ &= \frac{1}{e-1}\sum_{i=0}^{\infty}\frac{F_{Y_{i:i}}(x)}{i!} - 1, \end{aligned} \tag{2.9}$$

where $Y_{i:i}$ is the ith order statistic of sample of size i from $K(\alpha,\beta)$ with cdf $F_{Y_{i:i}}(x)$. Using (2.9), the expected value of $DUS-K(\alpha,\beta)$ can be represented by

$$E(X)=\frac{1}{e-1}\sum_{i=0}^{\infty}\frac{1}{i!}E(Y_{i:i}), \tag{2.10}$$

where $f_{i:i}(y)$ is pdf of $Y_{i:i}$. Using (2.8) in (2.10) and quantile function of Kumaraswamy distribution, the approximate expected value of $DUS-K(\alpha,\beta)$ is obtained as

$$E(X)\simeq\frac{1}{e-1}\sum_{i=0}^{\infty}\frac{1}{i!}\left(1-\left(\frac{i}{n+1}\right)^{1/\beta}\right)^{1/\alpha}. \tag{2.11}$$

Let X be the $DUS-K(\alpha,\beta)$ random variable with pdf (2.3). Then, for $r=1,2,...,$ the approximate r-th moment of X is given by

$$\begin{aligned}\mu_r &= \int_0^1 x^r f(x)\,dx\\ &= \int_{-1}^{1}\frac{1}{2}\left(\frac{y_\ell+1}{2}\right)^r f\left(\frac{y_\ell+1}{2}\right)dy\\ &\simeq \sum_{l=1}^{N}\varpi_\ell\frac{1}{2}\left(\frac{y_\ell+1}{2}\right)^r f\left(\frac{y_\ell+1}{2}\right),\end{aligned} \tag{2.12}$$

where $f(\cdot)$ is the pdf given in Eq. (2.3), y_ℓ and ϖ_ℓ are the zeros and the corresponding Christoffel numbers of the Legendre-Gauss quadrature formula on the interval $(-1,1)$, respectively, see [2]. It is also noticed here, ϖ_ℓ is given by

$$\varpi_\ell=\frac{2}{(1-y_\ell)^2\left[L'_{N+1}(y_\ell)\right]^2}, \tag{2.13}$$

where

$$L'_{N+1}(y_\ell)=\frac{dL_{N+1}(y)}{dy} \tag{2.14}$$

at $y=y_\ell$ and $L_{N+1}(\cdot)$ is the Legendre polynomial of degree N.

The relation between the approximation of the mean and degree (N) of Legendre polynomial is presented in Fig. 3. It can be observed that $N=30$ is enough to reach the acceptable approximation to the true mean.

For some selected parameters, the mean, variance, skewness and kurtosis of $DUS-K$ distribution are presented in Table 1. The values of N has been taken to be $N=30$ in the numerical calculations.

The Bonferroni and Lorenz curves are introduced by [1]. They have applications in economics and insurance. In the following, we give the Bonferroni and Lorenz curves of $DUS-K(\alpha,\beta)$ distribution.

Let the random variable X have $DUS-K(\alpha,\beta)$ distribution with pdf (2.3). Then, the Bonferroni and Lorenz curves are given, respectively, by

$$BC(\xi)=\frac{q^2}{4\xi\mu_1}\sum_{\ell=0}^{N}\varpi_\ell\left(y_\ell+1\right)f\left(\frac{q}{2}\left(y_\ell+1\right)\right), \tag{2.15}$$

and

$$LC(\xi)=\frac{q^2}{4\mu_1}\sum_{\ell=0}^{N}\varpi_\ell\left(y_\ell+1\right)f\left(\frac{q}{2}\left(y_\ell+1\right)\right), \tag{2.16}$$

where ϖ_ℓ is given by (2.13) and $q=F^{-1}(\xi)$.

In the following, we compute the well-known stress-strength reliability $R=P(Y<X)$ for the model under concern.

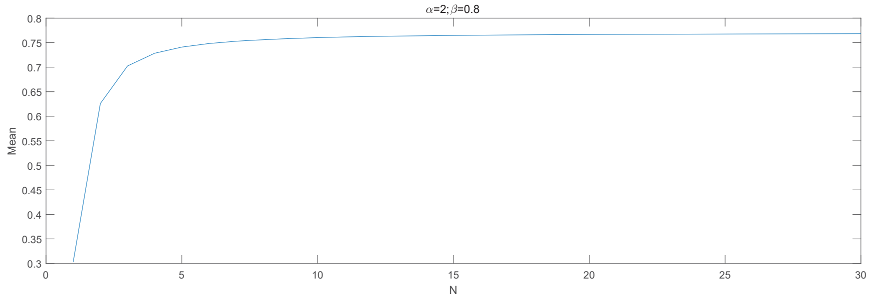

FIGURE 3. The relationship between approximated mean and degree (N) of Legendre polynomial

TABLE 1. The mean, variance, skewness and kurtosis of $DUS-K(\alpha,\beta)$ distribution for different values of α and β

α	β	Mean	Variance	Skewness	Kurtosis
0.5	0.5	0.6251	0.1134	-0.5366	1.8555
	1.5	0.2951	0.0680	0.7450	2.4860
	3	0.1341	0.0237	1.6482	5.6597
1.5	0.5	0.7948	0.0592	-1.4293	4.2872
	1.5	0.5822	0.0548	-0.3473	2.1865
	3	0.4268	0.0432	0.1011	2.2450
3	0.5	0.8698	0.0341	-2.4082	9.7298
	1.5	0.7411	0.0328	-0.9021	3.3317
	3	0.6294	0.0307	-0.5212	2.7864

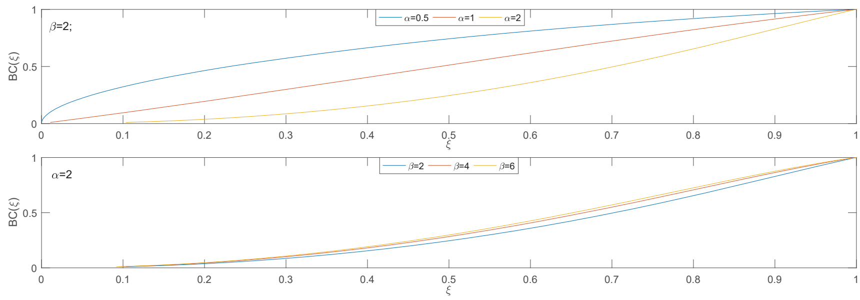

FIGURE 4. The Bonferroni curves of $DUS-K(\alpha,\beta)$ distribution

PROPOSITION 1. *Let Y and X be independent stress and strength random variables that follow Dus-K distribution with parameters (α,β_1) and (α,β_2), respectively. Then, the stress–strength reliability R is*

$$R = \frac{1}{(e-1)^2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\frac{(-1)^{j+i}}{n_1!n_2!} \times \binom{n_1}{i}\binom{n_2}{j}\frac{\beta_1}{i\beta_1+j\beta_2+\beta_1} - \frac{1}{e-1},$$

where β_1 and β_2 are positive integers.

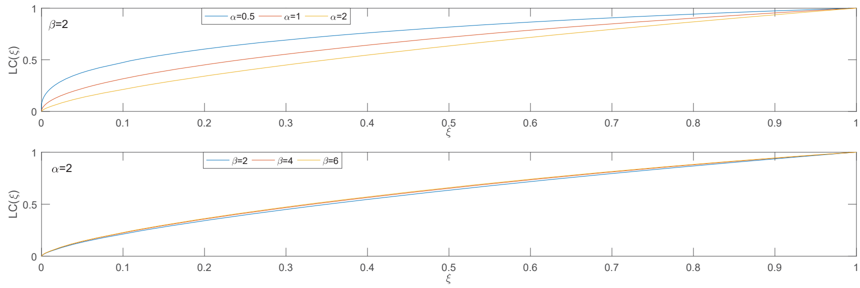

FIGURE 5. The Lorenz curves of $DUS - K(\alpha, \beta)$ distribution

PROOF. The stress-strength reliability can be written as

$$\begin{aligned} R &= \int_0^1 P(Y < X \mid X = x) f_X(x)\, dx \\ &= \int_0^1 F_Y(x) f_X(x)\, dx \\ &= \frac{1}{(e-1)^2} \int_0^1 \left(\exp\left(1 - (1 - x^\alpha)^{\beta_1}\right) - 1 \right) \\ &\quad \times \alpha \beta_2 x^{\alpha - 1} (1 - x^\alpha)^{\beta_2 - 1} \exp\left(1 - (1 - x^\alpha)^{\beta_2}\right) dx. \end{aligned} \tag{2.17}$$

Hence the proof follows by using Taylor and binomial expansions on terms in (2.17).
It is said that X is smaller than Y according to likelihood ratio ordering if

$$\frac{f_X(x)}{f_Y(x)} \text{ is nondecreasing in } x,$$

where $f_X(\cdot)$ and $f_Y(\cdot)$ are the pdfs of X and Y random variables, respectively. We write $X \leq_{lr} Y$ to represent that the random variable X is smaller than Y in the likelihood ratio ordering. The following proposition gives likelihood ratio order properties for the random variables with Dus-K distribution.

PROPOSITION 2. *Let $X \sim$DUS-K(α, β_1) and $Y \sim$DUS-K(α, β_2). If $\beta_1 > \beta_2$ then $X \leq_{lr} Y$.*

PROOF. For any $x \in (0, 1)$ the ratio of the densities of X and Y is given by

$$g(x) = \frac{\beta_1 (1 - x^\alpha)^{\beta_1 - 1} \exp\left(1 - (1 - x^\alpha)^{\beta_1}\right)}{\beta_2 (1 - x^\alpha)^{\beta_2 - 1} \exp\left(1 - (1 - x^\alpha)^{\beta_2}\right)}.$$

$X \leq_{lr} Y$ is equivalent to $g(x)$ is decreasing in x. Let us consider

$$\frac{d \log (g(x))}{dx} = r(x)\, h(x),$$

where

$$r(x) = \frac{\alpha x^{\alpha}}{x(1-x^{\alpha})}$$

and

$$h(x) = \beta_1\left[(1-x^{\alpha})^{\beta_1}-1\right] - \beta_2\left[(1-x^{\alpha})^{\beta_2}-1\right].$$

It is pointed out that we can easily write $r(x) > 0$ for all $\alpha > 0$ and $x \in (0,1)$. It is also clearly that $(1-x^{\alpha})^{\beta}$ is decreasing function in β for $x \in (0,1)$. Then we can write

$$\begin{aligned}(\beta_1 > \beta_2) &\Longrightarrow (1-x^{\alpha})^{\beta_2} > (1-x^{\alpha})^{\beta_1}\\ &\Longrightarrow (1-x^{\alpha})^{\beta_2} - 1 > (1-x^{\alpha})^{\beta_1} - 1\\ &\Longrightarrow \beta_2\left[(1-x^{\alpha})^{\beta_2}-1\right] > \beta_1\left[(1-x^{\alpha})^{\beta_1}-1\right]\\ &\Longrightarrow \beta_1\left[(1-x^{\alpha})^{\beta_1}-1\right] - \beta_2\left[(1-x^{\alpha})^{\beta_2}-1\right] < 0\\ &\Longrightarrow h(x) < 0\end{aligned}$$

Therefore, we have

$$\frac{d\log(g(x))}{dx} = \underbrace{r(x)}_{>0}\underbrace{h(x)}_{<0} < 0$$

for $\beta_1 > \beta_2$. The last inequality shows that $g(x)$ is decreasing in x and it implies $X \leq_{lr} Y$ for $\beta_1 > \beta_2$.

Corollary 1. *It follows from [11] that X is also smaller than Y in the hazard rate, mean residual life and stochastic orders under the conditions given in Proposition 2.*

3. Maximum likelihood estimation

Let $X_1, X_2, \ldots, X_n$ be the i.i.d sample from $DUS-K(a,\beta)$, then the likelihood and log-likelihood functions can be written as

$$L(\alpha,\beta) = \prod_{i=1}^{n}\left(\frac{1}{e-1}\alpha\beta x_i^{\alpha-1}(1-x_i^{\alpha})^{\beta-1}e^{\left(1-(1-x_i^{\alpha})^{\beta}\right)}\right) \tag{3.1}$$

and

$$\begin{aligned}\ell(\alpha,\beta) = & -n\log(e-1) + n\log(\alpha) + n\log(\beta) + (\alpha-1)\sum_{i=1}^{n}\log(x_i) \\ & +(\beta-1)\sum_{i=1}^{n}\log(1-x_i^{\alpha}) + \sum_{i=1}^{n}\left(1-(1-x_i^{\alpha})^{\beta}\right),\end{aligned} \tag{3.2}$$

respectively. The corresponding likelihood equations are found to be

$$\begin{aligned}\frac{\partial\ell(\alpha,\beta)}{\partial\alpha} = & \sum_{i=1}^{n}\log(x_i) + (\beta-1)\sum_{i=1}^{n}\left(-\frac{x_i^{\alpha}\log(x_i)}{1-x_i^{\alpha}}\right)\\ & +\sum_{i=1}^{n}\frac{(1-x_i^{\alpha})^{\beta}\beta x_i^{\alpha}\log(x_i)}{1-x_i^{\alpha}} + \frac{n}{\alpha} = 0,\end{aligned} \tag{3.3}$$

$$\frac{\partial\ell(\alpha,\beta)}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log(1-x_i^{\alpha}) + \sum_{i=1}^{n}\left(-(1-x_i^{\alpha})^{\beta}\log((1-x_i^{\alpha}))\right) = 0. \tag{3.4}$$

The Eqs. (3.3) and (3.4) cannot be solved explicitly. It can be solved by some iterative methods. In the next section, the fminsearch (MATLAB function) command is used for this purpose. fminsearch function uses the simplex search method of [8].

4. Simulation study

In this section, a simulation study is conducted to observe the properties of MLE discussed in the Section 3. In Table 2, for different choices of (n, α, β), we present the biases and mean squares errors (MSEs) of the estimates with 5000 replications. From Tables 1, it is observed that the MLEs are biased but asymptotically unbiased. Also, when the sample size n increases, the bias and MSEs of the MLEs decrease to zero as desired.

TABLE 2. Bias and MSEs of MLE estimators for selected parameters

			Bias		MSE	
α	β	n	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\beta}$
2	2	50	0.0870	0.1212	0.0075	0.0147
		100	0.0466	0.0644	0.0021	0.0041
		200	0.0256	0.0330	0.0006	0.0010
		300	0.0162	0.0210	0.0002	0.0004
		500	0.0098	0.0131	0.0001	0.0001
3	0.5	50	0.2787	0.0185	0.0787	0.0003
		100	0.1429	0.0101	0.0204	0.0001
		200	0.0749	0.0050	0.0056	0.0000
		300	0.0467	0.0031	0.0021	0.0000
		500	0.0283	0.0020	0.0008	0.0000
0.7	1.5	50	0.0348	0.0885	0.0012	0.0078
		100	0.0179	0.0424	0.0003	0.0018
		200	0.0099	0.0221	0.0001	0.0004
		300	0.0062	0.0140	0.0000	0.0001
		500	0.0038	0.0087	0.0000	0.0000

5. Real data application

In this section, we provide an application with real data to illustrate the flexibility of the $DUS-K(\alpha,\beta)$ model. For illustrative purposes, we consider a real data set and compare with some statistical distributions. The data set represents the total milk production in the first birth of 107 cows from the SINDI race. This data can be found in [3]. We consider the Kumaraswamy (Kw) ([7]), exponentiated Kumararaswamy (EKw) ([9]), Weibull (W) ([13]) and beta (B) distributions to compare the fitting ability of the $DUS-K(\alpha,\beta)$ distribution.

The pdf of the distributions used in comparison study are given as follows:

Kw distribution:

$$f_{Kw}(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1},\ \alpha,\beta>0$$

EKw distribution:

$$f_{EKw}(x;\theta,\alpha,\beta) = \alpha\beta\theta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\left(1-(1-x^{\alpha})^{\beta}\right)^{\theta-1},\ \theta,\alpha,\beta>0$$

W distribution

$$f_W(x;\alpha,\theta) = \frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{\alpha-1}\exp\left(-\left(\frac{x}{\theta}\right)\right)^{\alpha},\ \alpha,\theta>0$$

B distribution

$$f_B(x;a,b) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \ a,b > 0$$

The unknown parameters are estimated for each distribution by the maximum likelihood method. The goodness-of-fit statistics including the values of the Akaike information criterion (AIC), Bayesian information criterion (BIC) where the lower values of AIC,BIC and -2ℓ values are presented in Table 3. From the Table 3, we observed that the $DUS-K$ model is the best model to fit the milk data.

Table 3. Results of AIC, BIC and log-likelihood for $DUS-K$ and other distributions for the data set summaries

Model	*Parameters*	*AIC*	*BIC*	-2ℓ
Dus-K	$\hat{\alpha}=1.9198, \hat{\beta}=3.6421$	-50.360	-45.015	-54.361
Kw	$\hat{\alpha}=2.1949, \widehat{\beta}=3.4363$	-46.789	-41.443	-50.789
EKw	$\hat{\theta}=0.3361, \hat{\alpha}=5.315, \widehat{\beta}=7.141$	-27.557	-49.114	-41.443
W	$\hat{\alpha}=2.6012, \hat{\theta}=0.5236$	-38.695	-33.349	-42.695
B	$\hat{a}=2.4125, \widehat{b}=2.8296$	-43.554	-38.208	-23.777

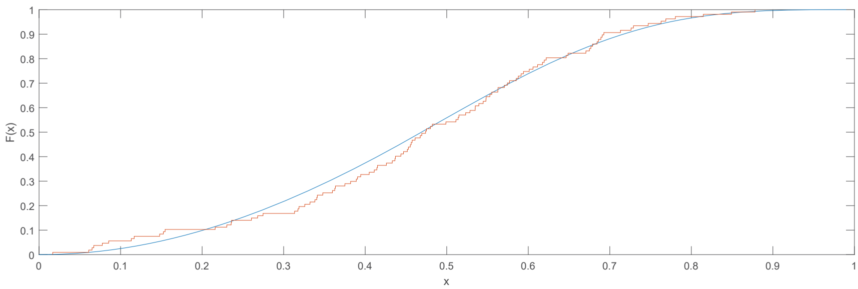

Figure 6. Empirical and fitted distribution function based on milk data

6. Conclusions

In this paper, we introduce a new lifetime distribution by using Dus transformation on Kumaraswamy cdf. Several characteristics have been calculated for the new distribution. Based on our example with the real data, we observed that Dus transformation increases the modelling capability of Kumaraswamy distribution. The proposed distribution has been found to be better than the other well-known distributions in terms of AIC.

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