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Research Article

Hermite–Hadamard Type Inclusions for m-Polynomial Harmonically Convex Interval-Valued Functions

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ABSTRACT. We introduce the notion of m-polynomial harmonically convex interval-valued function. A relationship between a given interval-valued function and its component real-valued functions is pointed out. Moreover, some new Hermite–Hadamard type results are established for this class of functions. In particular, we show that if a non-negative interval-valued function F, defined on a harmonically convex set $\mathbf{S}$, is m-polynomial harmonically convex with $\alpha < \beta$ and $\alpha, \beta \in \mathbf{S}$, then

$$\frac{2^{-1}\, m}{m + 2^{-m} - 1} F\left(\frac{2\alpha\beta}{\alpha + \beta}\right) \supseteq \frac{\alpha\beta}{\beta - \alpha} \int_{\alpha}^{\beta} \frac{F(r)}{r^2}\, dr \supseteq \frac{F(\alpha) + F(\beta)}{m} \sum_{p=1}^{m} \frac{p}{p+1},$$

where F is Lebesgue integrable on $[\alpha, \beta]$. Our results complement and extend existing results in the literature. By taking $m \geq 2$, we derive loads of new and interesting inclusions. We anticipate that the idea outlined herein will trigger further investigations in this direction.

Keywords: Hermite–Hadamard, m-polynomial harmonically convex, interval-valued function.

2010 Mathematics Subject Classification: 26D15, 26E25, 28B20.

1. INTRODUCTION

The Hermite–Hadamard inequality (HHI) stipulates that the average value of a convex function on an interval is bounded below by the value of the function at the midpoint of the interval and above by the average value of the function at the endpoints of the interval. Whenever a new class of function is introduced, researchers want to know if the analogue of the HHI can be established for such class. Loads of articles, in this direction, are bound in the literature. See for example, [3, 4, 9, 10, 12, 13, 14, 15, 22, 23, 26, 24] and the references cited therein. One of such is the harmonically convex function: a set $\mathbf{S} \subset \mathbb{R} \setminus \{0\}$ is called a harmonically convex set if

$$\frac{xy}{\tau x + (1-\tau) y} \in \mathbf{S}$$

for all $x, y \in \mathbf{S}$ and $\tau \in [0, 1]$. In 2014, İşcan [11] proposed and defined a harmonically convex function as follows: a real valued function $f : \mathbf{S} \to \mathbb{R}^+ := (0, \infty)$ is harmonically convex if

$$f\left(\frac{xy}{\tau x + (1-\tau) y}\right) \leq \tau f(y) + (1-\tau) f(x)$$

for all $x, y \in \mathbf{S}$ and $\tau \in [0, 1]$. In the same paper, the author established the following Hermite–Hadamard type inequality for this class of functions:

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Theorem 1.1 ([11]). *Let $f : \mathbf{S} \to \mathbb{R}$ be a harmonically convex function. If $\alpha, \beta \in \mathbf{S}$ with $\alpha < \beta$, and f is Lebesgue integrable on $[\alpha, \beta]$, then the following Hermite–Hadamard type inequality holds:*

$$f\left(\frac{2\alpha\beta}{\alpha+\beta}\right) \leq \frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{f(r)}{r^2}\,dr \leq \frac{f(\alpha)+f(\beta)}{2}.$$

Recently, Awan et al. [1] introduced the notion of m-polynomial harmonically convex functions as a generalization of the harmonically convex functions, and then proved, among other things, the result that follows:

Definition 1.1 ([1]). *Let $m \in \mathbb{N}$. Then, a real-valued function $f : \mathbf{S} \to \mathbb{R}^+$ is said to be m-polynomial harmonically convex (concave) if*

$$f\left(\frac{xy}{\tau x + (1-\tau)y}\right) \leq (\geq) \frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\tau)^p\right] f(x) + \frac{1}{m}\sum_{p=1}^{m}\left[1-\tau^p\right] f(y)$$

for all $x, y \in \mathbf{S}$ and $\tau \in [0,1]$. The sets of all m-polynomial harmonically convex and m-polynomial harmonically concave functions from $\mathbf{S}$ into $\mathbb{R}^+$ is denoted by $\mathbf{HXP}_m(\mathbf{S}, \mathbb{R}^+)$ and $\mathbf{HVP}_m(\mathbf{S}, \mathbb{R}^+)$, respectively.

Theorem 1.2 ([1]). *Let $f : [\alpha, \beta] \to \mathbb{R}^+$ be an m-polynomial harmonically convex function. If f is Riemann integrable on $[\alpha, \beta]$, then*

$$\frac{2^{-1}\,m}{m+2^{-m}-1} f\left(\frac{2\alpha\beta}{\alpha+\beta}\right) \leq \frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{f(r)}{r^2}\,dr \leq \frac{f(\alpha)+f(\beta)}{m}\sum_{p=1}^{m}\frac{p}{p+1}.$$

In 1966, the late American Mathematician Ramon E. Moore initiated the theory of interval analysis [18]: simply put, the analysis of interval-valued functions. Ever since, this field has received great deal of attention from researchers in various areas of the mathematical sciences (like experts in global optimization and constraint solution algorithms) and has grown steadily in popularity over the past four decades. Interval analysis has been found to be valuable to engineers and scientists interested in scientific computation, especially in reliability, effects of roundoff error, and automatic verification of results, see [7, 8, 6, 5]. With the advent of interval analysis, mathematicians, those who work in the field of mathematical inequalities, want to know if the inequalities in the above mentioned results can be replaced with inclusions. In some cases, the answer to the question is in the affirmative. In lieu of this, E. Sadowska (see also [17]) established the following result for a given interval-valued function:

Theorem 1.3 ([25]). *Let F be a nonnegative continuous convex set-valued function on $[\alpha, \beta]$. Then,*

$$F\left(\frac{\alpha+\beta}{2}\right) \supset \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} F(r)\,dr \supset \frac{F(\alpha)+F(\beta)}{2}. \tag{1.1}$$

Results akin to (1.1), for different classes of set-valued convex functions, have been established. For example, see the papers [28, 27, 21, 16, 7, 8, 6, 5, 2, 29]. Motivated by the above mentioned articles, it is our goal in this article to introduce a new class of interval-valued function called the m-polynomial harmonically convex function and then obtain the interval-valued counterpart of Theorem 1.2. Thereafter, we will establish four more results in this direction. Our results complement and extend known results in the literature. The paper is organized in the following manner: Section 2 contains some brief background information in the theory of interval analysis. In Section 3, we state and prove our main results; followed by an open problem in Section 4.

2. Preliminaries

In this section, we give a brief overview of the theory of interval analysis. For an indepth study of this subject, we invite the interested reader to see the books [18, 19, 20]. We shall call $\mathbb{K}_c$ the class of all bounded closed nonempty intervals in $\mathbb{R}$, i.e.,

$$\mathbb{K}_c := \left\{ \left[\alpha^-, \alpha^+\right] \mid \alpha^-, \alpha^+ \in \mathbb{R} \quad \text{and} \quad \alpha^- \leq \alpha^+ \right\}.$$

The numbers α^- and α^+ are called the left and right endpoints of $[\alpha^-, \alpha^+]$, respectively. The interval $[\alpha^-, \alpha^+]$ is called degenerated if $\alpha^- = \alpha^+$; positive if $\alpha^- > 0$ and negative if $\alpha^+ < 0$. We denote the sets of all negative intervals and positive intervals in $\mathbb{R}$ by $\mathbb{K}_c^-$ and $\mathbb{K}_c^+$, respectively. That is;

$$\mathbb{K}_c^- := \left\{ \left[\alpha^-, \alpha^+\right] \in \mathbb{K}_c \mid \alpha^+ < 0 \right\}$$

and

$$\mathbb{K}_c^+ := \left\{ \left[\alpha^-, \alpha^+\right] \in \mathbb{K}_c \mid \alpha^- > 0 \right\}.$$

Let $A = [\alpha^-, \alpha^+]$, $B = [\beta^-, \beta^+] \in \mathbb{K}_c$ and $\gamma \in \mathbb{R}$. We say $A \subseteq B$ (or $B \supseteq A$) if and only if $\beta^- \leq \alpha^-$ and $\alpha^+ \leq \beta^+$. The following arithmetic operations are defined thus

$$\gamma A = \begin{cases} [\gamma\alpha^-, \gamma\alpha^+] & \text{if} \quad \gamma > 0 \\ \{0\} & \text{if} \quad \gamma = 0 \\ [\gamma\alpha^+, \gamma\alpha^-] & \text{if} \quad \gamma < 0; \end{cases}$$

$$A + B = \left[\alpha^-, \alpha^+\right] + \left[\beta^-, \beta^+\right] := \left[\alpha^- + \beta^-, \alpha^+ + \beta^+\right];$$

$$A - B = \left[\alpha^-, \alpha^+\right] - \left[\beta^-, \beta^+\right] := \left[\alpha^- - \beta^+, \alpha^+ - \beta^-\right];$$

$$A \cdot B := \left[\min\left\{\alpha^-\beta^-, \alpha^-\beta^+, \alpha^+\beta^-, \alpha^+\beta^+\right\}, \max\left\{\alpha^-\beta^-, \alpha^-\beta^+, \alpha^+\beta^-, \alpha^+\beta^+\right\}\right];$$

$$\frac{A}{B} := \left[\min\left\{\frac{\alpha^-}{\beta^-}, \frac{\alpha^-}{\beta^+}, \frac{\alpha^+}{\beta^-}, \frac{\alpha^+}{\beta^+}\right\}, \max\left\{\frac{\alpha^-}{\beta^-}, \frac{\alpha^-}{\beta^+}, \frac{\alpha^+}{\beta^-}, \frac{\alpha^+}{\beta^+}\right\}\right], \quad 0 \notin B.$$

The Pompieu–Hausdorff distance $d_H : \mathbb{K}_c \times \mathbb{K}_c \to \mathbb{R}_+ \cup \{0\}$ is defined by

$$d_H := \max\left\{\max_{\alpha \in A} d(\alpha, B), \ \max_{\beta \in B} d(\beta, A)\right\} \quad \text{with} \quad d(\beta, A) = \min_{\alpha \in A} |\beta - \alpha|.$$

It is generally known that $(\mathbb{K}_c, d_H)$ is a complete metric space. The concept of a convergent sequence of intervals $(A_n)_{n\in\mathbb{N}}$, $A_n \in \mathbb{K}_c$ is considered in the complete metric space $\mathbb{K}_c$, endowed with the d_H distance. We say that $\lim\limits_{n\to\infty} A_n = A$ if and only if for any real number $\epsilon > 0$ there exists an $N_\epsilon \in \mathbb{N}$ such that

$$d_H(A_n, A) < \epsilon \quad \text{for all} \quad n > N_\epsilon.$$

Next, we turn our attention to interval-valued functions.

Definition 2.2. *An interval-valued function is defined to be any* $F : [\alpha, \beta] \to \mathbb{K}_c$ *with* $F(x) = [f^-(x), f^+(x)] \in \mathbb{K}_c$ *and* $f^-(x) \leq f^+(x)$ *for all* $x \in [\alpha, \beta]$. *We say that* F *is Lebesgue integrable on* $[\alpha, \beta]$ *if the real-valued functions* f^- *and* f^+ *are Lebesgue integrable on* $[\alpha, \beta]$, *and then write*

$$\int_\alpha^\beta F(r)\,dr = \left[\int_\alpha^\beta f^-(r)\,dr, \int_\alpha^\beta f^+(r)\,dr\right].$$

3. Main Results

We start by introducing the concept of m-polynomial harmonically convex interval-valued function in the following definition.

Definition 3.3. *Let $\mathbf{S}$ be a harmonically convex set, $F : \mathbf{S} \to \mathbb{K}_c^+$ an interval-valued function and $m \in \mathbb{N}$. We say that F is m-polynomial harmonically convex (concave) if and only if*

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\tau)^p\right]F(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\tau^p\right]F(y) \subseteq (\supseteq) F\left(\frac{xy}{\tau x+(1-\tau)y}\right) \tag{3.2}$$

for all $x, y \in \mathbf{S}$ and $\tau \in [0,1]$. In what follows, we shall denote the sets of all m-polynomial harmonically convex and m-polynomial harmonically concave interval-valued functions from $\mathbf{S}$ into $\mathbb{K}_c^+$ by $\mathbf{HXP}_m(\mathbf{S}, \mathbb{K}_c^+)$ and $\mathbf{HVP}_m(\mathbf{S}, \mathbb{K}_c^+)$, respectively.

Remark 3.1. For a specific value of m, we get a corresponding set inclusion. For instance,

(1) If $m = 1$, then we get the definition of harmonically convex interval-valued function:

$$F\left(\frac{xy}{\tau x+(1-\tau)y}\right) \supseteq \tau F(x) + (1-\tau)F(y)$$

for all $x, y \in \mathbf{S}$ and $\tau \in [0,1]$.

(2) For $m = 2$, we get the following inclusion for a 2-polynomial harmonically convex interval-valued function:

$$F\left(\frac{xy}{\tau x+(1-\tau)y}\right) \supseteq \frac{3\tau-\tau^2}{2}F(x) + \frac{2-\tau-\tau^2}{2}F(y)$$

for all $x, y \in \mathbf{S}$ and $\tau \in [0,1]$.

(3) For $m = 3$, we deduce the succeeding relation for a 3-polynomial harmonically convex interval-valued function:

$$F\left(\frac{xy}{\tau x+(1-\tau)y}\right) \supseteq \frac{6\tau-4\tau^2+\tau^3}{3}F(x) + \frac{3-\tau-\tau^2-\tau^3}{3}F(y)$$

for all $x, y \in \mathbf{S}$ and $\tau \in [0,1]$.

The following theorem gives a relationship between a given interval-valued function F and its component real-valued functions f^- and f^+.

Theorem 3.4. *Let $F : \mathbf{S} \to \mathbb{K}_c^+$ be an interval-valued function such that $F(x) = [f^-(x), f^+(x)] \in \mathbb{K}_c$ and $f^-(x) \leq f^+(x)$ for all $x \in [\alpha, \beta]$. Then, $F \in \mathbf{HXP}_m(\mathbf{S}, \mathbb{K}_c^+)$ if and only if $f^- \in \mathbf{HXP}_m(\mathbf{S}, \mathbb{R}^+)$ and $f^+ \in \mathbf{HVP}_m(\mathbf{S}, \mathbb{R}^+)$.*

Proof. Let $x, y \in \mathbf{S}$ and $\tau \in [0,1]$. Then,

$$F \in \mathbf{HXP}_m\left(\mathbf{S}, \mathbb{K}_c^+\right)$$

if and only if

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\tau)^p\right]F(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\tau^p\right]F(y) \subseteq F\left(\frac{xy}{\tau x+(1-\tau)y}\right)$$

if and only if

$$\left[\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\tau)^p\right]f^-(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\tau^p\right]f^-(y),\right.$$
$$\left.\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\tau)^p\right]f^+(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\tau^p\right]f^+(y)\right]$$
$$\subseteq\left[f^-\left(\frac{xy}{\tau x+(1-\tau)y}\right),f^+\left(\frac{xy}{\tau x+(1-\tau)y}\right)\right]$$

if and only if

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\tau)^p\right]f^-(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\tau^p\right]f^-(y)\geq f^-\left(\frac{xy}{\tau x+(1-\tau)y}\right)$$

and

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\tau)^p\right]f^+(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\tau^p\right]f^+(y)\leq f^+\left(\frac{xy}{\tau x+(1-\tau)y}\right)$$

if and only if

$$f^-\in\mathbf{HXP}_m\left(\mathbf{S},\mathbb{R}^+\right)\quad\text{and}\quad f^+\in\mathbf{HVP}_m\left(\mathbf{S},\mathbb{R}^+\right).$$

That completes the proof in both directions. □

Following a similar line of argument, one can easily prove the following result.

Theorem 3.5. *Let $F:\mathbf{S}\to\mathbb{K}_c^+$ be an interval-valued function such that $F(x)=[f^-(x),f^+(x)]\in\mathbb{K}_c$ and $f^-(x)\leq f^+(x)$ for all $x\in[\alpha,\beta]$. Then, $F\in\mathbf{HVP}_m(\mathbf{S},\mathbb{K}_c^+)$ if and only if $f^-\in\mathbf{HVP}_m(\mathbf{S},\mathbb{R}^+)$ and $f^+\in\mathbf{HXP}_m(\mathbf{S},\mathbb{R}^+)$.*

For the remaining part of this article, we shall assume that $F:\mathbf{S}\to\mathbb{K}_c^+$ is always of the form $F(x)=[f^-(x),f^+(x)]\in\mathbb{K}_c$ and $f^-(x)\leq f^+(x)$ for all $x\in[\alpha,\beta]$. We are now ready to formulate and prove some Hermite–Hadamard type results for m-polynomial harmonically convex (concave) interval-valued functions.

Theorem 3.6. *Let $F:\mathbf{S}\to\mathbb{K}_c^+$ be an interval-valued function with $\alpha<\beta$ and $\alpha,\beta\in\mathbf{S}$, and Lebesgue integrable on $[\alpha,\beta]$. If $F\in\mathbf{HXP}_m(\mathbf{S},\mathbb{K}_c^+)$, then*

$$\frac{2^{-1}m}{m+2^{-m}-1}F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\supseteq\frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta\frac{F(r)}{r^2}\,dr\supseteq\frac{F(\alpha)+F(\beta)}{m}\sum_{p=1}^{m}\frac{p}{p+1}. \tag{3.3}$$

The inclusions are reversed if $F\in\mathbf{HVP}_m(\mathbf{S},\mathbb{K}_c^+)$.

Proof. Assuming $F\in\mathbf{HXP}_m(\mathbf{S},\mathbb{K}_c^+)$, we get from (3.2) the following relation:

$$F\left(\frac{xy}{\frac{1}{2}x+\frac{1}{2}y}\right)\supseteq\frac{1}{m}\sum_{p=1}^{m}\left[1-\frac{1}{2^p}\right]F(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\frac{1}{2^p}\right]F(y).$$

This implies that for all $x,y\in\mathbf{S}$

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-\frac{1}{2^p}\right]\Big(F(x)+F(y)\Big)\subseteq F\left(\frac{2xy}{x+y}\right). \tag{3.4}$$

Now, let $x = \frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}$ and $y = \frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}$. Then, (3.4) becomes:

$$\frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left\{F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)+F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right\}\subseteq F\left(\frac{2\alpha\beta}{\alpha+\beta}\right). \tag{3.5}$$

Integrating both sides of (3.5) with respect to τ over $[0,1]$, we get

(3.6)

$$\begin{aligned}
\int_0^1 F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)d\tau &\supseteq \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\int_0^1\left\{F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)+F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right\}d\tau\\
&=\frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left[\int_0^1\left\{f^-\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)+f^-\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right\}d\tau,\right.\\
&\qquad\left.\int_0^1\left\{f^+\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)+f^+\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right\}d\tau\right]\\
&=\frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left[\frac{2\alpha\beta}{\beta-\alpha}\int_\alpha^\beta\frac{f^-(r)}{r^2}\,dr,\frac{2\alpha\beta}{\beta-\alpha}\int_\alpha^\beta\frac{f^+(r)}{r^2}\,dr\right]\\
&=\frac{2\alpha\beta}{\beta-\alpha}\frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left[\int_\alpha^\beta\frac{f^-(r)}{r^2}\,dr,\int_\alpha^\beta\frac{f^+(r)}{r^2}\,dr\right]\\
&=\frac{2\alpha\beta}{\beta-\alpha}\frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\int_\alpha^\beta\frac{F(r)}{r^2}\,dr.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_0^1 F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)d\tau &= \left[\int_0^1 f^-\left(\frac{2\alpha\beta}{\alpha+\beta}\right)d\tau,\int_0^1 f^+\left(\frac{2\alpha\beta}{\alpha+\beta}\right)d\tau\right]\\
&=\left[f^-\left(\frac{2\alpha\beta}{\alpha+\beta}\right),f^+\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\right]\\
&=F\left(\frac{2\alpha\beta}{\alpha+\beta}\right).
\end{aligned}\tag{3.7}$$

Using (3.7) in (3.6), one gets

$$\frac{m}{m+2^{-m}-1}F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\supseteq\frac{2\alpha\beta}{\alpha+\beta}\int_\alpha^\beta\frac{F(r)}{r^2}\,dr. \tag{3.8}$$

Next, we substitute $x=\alpha$ and $y=\beta$ into (3.2) and integrate the resulting inclusion with respect to τ over $[0,1]$, to obtain

$$\begin{aligned}\frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta \frac{F(r)}{r^2}\,dr &= \int_0^1 F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)d\tau\\ &\supseteq \int_0^1\left\{\frac{1}{m}\sum_{p=1}^m\left[1-(1-\tau)^p\right]F(\alpha)+\frac{1}{m}\sum_{p=1}^m\left[1-\tau^p\right]F(\beta)\right\}d\tau\\ &= \frac{1}{m}\sum_{p=1}^m\int_0^1\left[1-(1-\tau)^p\right]F(\alpha)\,d\tau+\frac{1}{m}\sum_{p=1}^m\int_0^1\left[1-\tau^p\right]F(\beta)\,d\tau\\ &= \frac{F(\alpha)+F(\beta)}{m}\sum_{p=1}^m\frac{p}{p+1}.\end{aligned}$$

This gives

$$\frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta\frac{F(r)}{r^2}\,dr\supseteq\frac{F(\alpha)+F(\beta)}{m}\sum_{p=1}^m\frac{p}{p+1}. \tag{3.9}$$

Combining (3.8) and (3.9), we get the desired result (3.3). If $F\in \mathbf{HVP}_m(\mathbf{S},\mathbb{K}_c^+)$, then we establish the reverse inclusions in a similar manner. □

Remark 3.2. Using Theorem 3.6, we obtain the following corollaries:

(1) For $m=1$, we deduce the result for 1-polynomial harmonically convex interval-valued functions:
$$F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\supseteq\frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta\frac{F(r)}{r^2}\,dr\supseteq\frac{F(\alpha)+F(\beta)}{2}.$$

(2) If $m=2$, then we obtain the result for 2-polynomial harmonically convex interval-valued functions:
$$\frac{4}{35}F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\supseteq\frac{\alpha\beta}{7(\beta-\alpha)}\int_\alpha^\beta\frac{F(r)}{r^2}\,dr\supseteq\frac{F(\alpha)+F(\beta)}{12}.$$

Theorem 3.7. *Let $F:\mathbf{S}\to\mathbb{K}_c^+$ be an interval-valued function with $\alpha<\beta$ and $\alpha,\beta\in\mathbf{S}$, and Lebesgue integrable on $[\alpha,\beta]$. If $F\in\mathbf{HXP}_m(\mathbf{S},\mathbb{K}_c^+)$, then*

$$\begin{aligned}\frac{1}{4}\left(\frac{m}{m+2^{-m}-1}\right)^2 F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)&\supseteq\Omega_1\supseteq\frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta\frac{F(r)}{r^2}\,dr\supseteq\Omega_2\\ &\supseteq\Big(F(\alpha)+F(\beta)\Big)\frac{m^2+2m+2^{1-m}-2}{2m^2}\sum_{p=1}^m\frac{p}{p+1},\end{aligned}\tag{3.10}$$

where

$$\Omega_1:=\frac{1}{4}\frac{m}{m+2^{-m}-1}\left\{F\left(\frac{4\alpha\beta}{\alpha+3\beta}\right)+F\left(\frac{4\alpha\beta}{\beta+3\alpha}\right)\right\};$$

$$\Omega_2:=\frac{1}{2}\left[\frac{F(\alpha)+F(\beta)+2F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)}{m}\right]\sum_{p=1}^m\frac{p}{p+1}.$$

The inclusions are reversed if $F\in\mathbf{HVP}_m(\mathbf{S},\mathbb{K}_c^+)$.

Proof. Using the fact that $F \in \mathbf{HXP}_m(\mathbf{S}, \mathbb{K}_c^+)$ and recalling (3.4)

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-\frac{1}{2^p}\right]\Big(F(x)+F(y)\Big) \subseteq F\left(\frac{2xy}{x+y}\right) \tag{3.11}$$

for all $x, y \in \mathbf{S}$. So, in particular for

$$x = \frac{\alpha\lambda}{\tau\alpha+(1-\tau)\lambda} \quad \text{and} \quad y = \frac{\alpha\lambda}{\tau\lambda+(1-\tau)\alpha}, \quad \text{where} \quad \lambda = \frac{2\alpha\beta}{\alpha+\beta},$$

the inclusion in (3.11) becomes:

$$F\left(\frac{4\alpha\beta}{\alpha+3\beta}\right) \supseteq \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left\{F\left(\frac{\alpha\lambda}{\tau\alpha+(1-\tau)\lambda}\right)+F\left(\frac{\alpha\lambda}{\tau\lambda+(1-\tau)\alpha}\right)\right\}.$$

Integrating both sides of the above relation with respect to τ over $[0,1]$, one gets

$$\begin{aligned}
F\left(\frac{4\alpha\beta}{\alpha+3\beta}\right) &\supseteq \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\int_0^1\left\{F\left(\frac{\alpha\lambda}{\tau\alpha+(1-\tau)\lambda}\right)+F\left(\frac{\alpha\lambda}{\tau\lambda+(1-\tau)\alpha}\right)\right\}d\tau \\
&= \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left[\int_0^1\left\{f^-\left(\frac{\alpha\lambda}{\tau\alpha+(1-\tau)\lambda}\right)+f^-\left(\frac{\alpha\lambda}{\tau\lambda+(1-\tau)\alpha}\right)\right\}d\tau,\right. \\
&\qquad \left.\int_0^1\left\{f^+\left(\frac{\alpha\lambda}{\tau\alpha+(1-\tau)\lambda}\right)+f^+\left(\frac{\alpha\lambda}{\tau\lambda+(1-\tau)\alpha}\right)\right\}d\tau\right] \\
&= \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\frac{4\alpha\beta}{\beta-\alpha}\left[\int_\alpha^\lambda\frac{f^-(r)}{r^2}\,dr, \int_\alpha^\lambda\frac{f^+(r)}{r^2}\,dr\right] \\
&= \frac{m+2^{-m}-1}{m}\,\frac{4\alpha\beta}{\beta-\alpha}\int_\alpha^\lambda\frac{F(r)}{r^2}\,dr.
\end{aligned}$$

Thus, we have

$$F\left(\frac{4\alpha\beta}{\alpha+3\beta}\right) \supseteq \frac{m+2^{-m}-1}{m}\,\frac{4\alpha\beta}{\beta-\alpha}\int_\alpha^\lambda\frac{F(r)}{r^2}\,dr. \tag{3.12}$$

If we also let

$$x = \frac{\beta\lambda}{\tau\lambda+(1-\tau)\beta} \quad \text{and} \quad y = \frac{\beta\lambda}{\tau\beta+(1-\tau)\lambda}$$

and then proceed as outlined above, we obtain

$$F\left(\frac{4\alpha\beta}{\beta+3\alpha}\right) \supseteq \frac{m+2^{-m}-1}{m}\,\frac{4\alpha\beta}{\beta-\alpha}\int_\lambda^\beta\frac{F(r)}{r^2}\,dr. \tag{3.13}$$

Also, by setting $x = \frac{4\alpha\beta}{\alpha+3\beta}$ and $y = \frac{4\alpha\beta}{\beta+3\alpha}$ into (3.11) and then using (3.12) and (3.13), we obtain

$$
\begin{aligned}
F\left(\frac{2\alpha\beta}{\alpha+\beta}\right) &\supseteq \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left\{F\left(\frac{4\alpha\beta}{\alpha+3\beta}\right)+F\left(\frac{4\alpha\beta}{\beta+3\alpha}\right)\right\} \\
&\supseteq \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left\{\frac{m+2^{-m}-1}{m}\frac{4\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\lambda}\frac{F(r)}{r^2}\,dr\right. \\
&\left. +\frac{m+2^{-m}-1}{m}\frac{4\alpha\beta}{\beta-\alpha}\int_{\lambda}^{\beta}\frac{F(r)}{r^2}\,dr\right\} \\
&= \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\frac{m+2^{-m}-1}{m}\frac{4\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{F(r)}{r^2}\,dr \\
&= 4\left(\frac{m+2^{-m}-1}{m}\right)^2\frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{F(r)}{r^2}\,dr.
\end{aligned} \tag{3.14}
$$

From (3.14), we get the following chain of inclusions:

$$
\begin{aligned}
\frac{1}{4}\left(\frac{m}{m+2^{-m}-1}\right)^2 F\left(\frac{2\alpha\beta}{\alpha+\beta}\right) &\supseteq \frac{1}{4}\frac{m}{m+2^{-m}-1}\left\{F\left(\frac{4\alpha\beta}{\alpha+3\beta}\right)+F\left(\frac{4\alpha\beta}{\beta+3\alpha}\right)\right\} \\
&\supseteq \frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{F(r)}{r^2}\,dr.
\end{aligned} \tag{3.15}
$$

Employing the second inclusion of (3.3) from Theorem 3.6 and (3.11), we get

$$
\begin{aligned}
\frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{F(r)}{r^2}\,dr &= \frac{1}{2}\left[\frac{2\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\frac{2\alpha\beta}{\alpha+\beta}}\frac{F(r)}{r^2}\,dr+\frac{2\alpha\beta}{\beta-\alpha}\int_{\frac{2\alpha\beta}{\alpha+\beta}}^{\beta}\frac{F(r)}{r^2}\,dr\right] \\
&\supseteq \frac{1}{2}\left[\frac{F(\alpha)+F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)}{m}+\frac{F(\beta)+F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)}{m}\right]\sum_{p=1}^{m}\frac{p}{p+1} \\
&= \frac{1}{2}\left[\frac{F(\alpha)+F(\beta)+2F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)}{m}\right]\sum_{p=1}^{m}\frac{p}{p+1} \\
&\supseteq \left[\frac{F(\alpha)+F(\beta)}{2}+\frac{m+2^{-m}-1}{m^2}\Big(F(\alpha)+F(\beta)\Big)\right]\sum_{p=1}^{m}\frac{p}{p+1} \\
&= \Big(F(\alpha)+F(\beta)\Big)\frac{m^2+2m+2^{1-m}-2}{2m^2}\sum_{p=1}^{m}\frac{p}{p+1}.
\end{aligned} \tag{3.16}
$$

We get the intended result by putting together (3.15) and (3.16). □

Theorem 3.8. *Let $F, G : \mathbf{S} \to \mathbb{K}_c^+$ be two interval-valued functions with $\alpha < \beta$ and $\alpha, \beta \in \mathbf{S}$, and suppose FG is Lebesgue integrable on $[\alpha, \beta]$. If $F \in \mathbf{HXP}_{m_1}(\mathbf{S}, \mathbb{K}_c^+)$ and $G \in \mathbf{HXP}_{m_2}(\mathbf{S}, \mathbb{K}_c^+)$, then*

$$\begin{aligned}&\frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{F(r)\,G(r)}{r^2}\,dr\\ &\supseteq F(\alpha)G(\alpha)\int_0^1\Delta_1(\tau)\,d\tau+F(\alpha)G(\beta)\int_0^1\Delta_2(\tau)\,d\tau\\ &+F(\beta)G(\alpha)\int_0^1\Delta_3(\tau)\,d\tau+F(\beta)G(\beta)\int_0^1\Delta_4(\tau)\,d\tau,\end{aligned}\tag{3.17}$$

where

$$\Delta_1(\tau):=\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-(1-\tau)^p]\sum_{p=1}^{m_2}[1-(1-\tau)^p];$$

$$\Delta_2(\tau):=\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-(1-\tau)^p]\sum_{p=1}^{m_2}[1-\tau^p];$$

$$\Delta_3(\tau):=\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-\tau^p]\sum_{p=1}^{m_2}[1-(1-\tau)^p];$$

$$\Delta_4(\tau):=\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-\tau^p]\sum_{p=1}^{m_2}[1-\tau^p].$$

The inclusions are reversed if $F\in\mathbf{HVP}_{m_1}(\mathbf{S},\mathbb{K}_c^+)$ *and* $G\in\mathbf{HVP}_{m_2}(\mathbf{S},\mathbb{K}_c^+)$.

Proof. Given that $F\in\mathbf{HXP}_{m_1}(\mathbf{S},\mathbb{K}_c^+)$ and $G\in\mathbf{HXP}_{m_2}(\mathbf{S},\mathbb{K}_c^+)$, we get

$$\frac{1}{m_1}\sum_{p=1}^{m_1}[1-(1-\tau)^p]\,F(\alpha)+\frac{1}{m_1}\sum_{p=1}^{m_1}[1-\tau^p]\,F(\beta)\subseteq F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\tag{3.18}$$

and

$$\frac{1}{m_2}\sum_{p=1}^{m_2}[1-(1-\tau)^p]\,G(\alpha)+\frac{1}{m_2}\sum_{p=1}^{m_2}[1-\tau^p]\,G(\beta)\subseteq G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right).\tag{3.19}$$

This implies

$$\begin{aligned}&F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\\ &\supseteq\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-(1-\tau)^p]\sum_{p=1}^{m_2}[1-(1-\tau)^p]\,F(\alpha)G(\alpha)\\ &+\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-(1-\tau)^p]\sum_{p=1}^{m_2}[1-\tau^p]\,F(\alpha)G(\beta)\\ &+\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-\tau^p]\sum_{p=1}^{m_2}[1-(1-\tau)^p]\,F(\beta)G(\alpha)\\ &+\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-\tau^p]\sum_{p=1}^{m_2}[1-\tau^p]\,F(\beta)G(\beta)\\ &:=\Delta_1(\tau)F(\alpha)G(\alpha)+\Delta_2(\tau)F(\alpha)G(\beta)+\Delta_3(\tau)F(\beta)G(\alpha)+\Delta_4(\tau)F(\beta)G(\beta).\end{aligned}\tag{3.20}$$

Now, integrating both sides of (3.20) with respect to τ over $[0,1]$, gives

$$\begin{aligned}
&\int_0^1 F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) d\tau \\
&=\int_0^1 \left[f^-\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) g^-\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right),\right. \\
&\left. f^+\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) g^+\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\right] d\tau \\
&=\left[\int_0^1 f^-\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) g^-\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) d\tau,\right. \\
&\left.\int_0^1 f^+\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) g^+\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) d\tau\right] \\
&=\left[\frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta \frac{f^-(r)g^-(r)}{r^2}\,dr,\ \frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta \frac{f^+(r)g^+(r)}{r^2}\,dr\right] \\
&=\frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta \frac{F(r)\,G(r)}{r^2}\,dr \\
&\supseteq F(\alpha)G(\alpha)\int_0^1 \Delta_1(\tau)\,d\tau + F(\alpha)G(\beta)\int_0^1 \Delta_2(\tau)\,d\tau \\
&+F(\beta)G(\alpha)\int_0^1 \Delta_3(\tau)\,d\tau + F(\beta)G(\beta)\int_0^1 \Delta_4(\tau)\,d\tau.
\end{aligned}$$

Hence that completes the proof. □

Theorem 3.9. *Let $F, G : \mathbf{S} \to \mathbb{K}_c^+$ be two interval-valued functions with $\alpha < \beta$ and $\alpha, \beta \in \mathbf{S}$, and suppose FG is Lebesgue integrable on $[\alpha, \beta]$. If $F \in \mathbf{HXP}_{m_1}(\mathbf{S}, \mathbb{K}_c^+)$, $G \in \mathbf{HXP}_{m_2}(\mathbf{S}, \mathbb{K}_c^+)$, $R(\alpha,\beta) = F(\alpha)G(\alpha) + F(\beta)G(\beta)$ and $Q(\alpha,\beta) = F(\alpha)G(\beta) + F(\beta)G(\alpha)$, then*

$$\begin{aligned}
&\frac{m_1 m_2}{(m_1+2^{-m_1}-1)(m_2+2^{-m_2}-1)} F\left(\frac{2\alpha\beta}{\alpha+\beta}\right) G\left(\frac{2\alpha\beta}{\alpha+\beta}\right) \\
&\supseteq \frac{2\alpha\beta}{\alpha+\beta}\int_\alpha^\beta \frac{F(r)\,G(r)}{r^2}\,dr + R(\alpha,\beta)\int_0^1 \left[\Lambda_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau) + \tilde{\Lambda}_{m_1}(\tau)\Lambda_{m_2}(\tau)\right] d\tau \\
&+Q(\alpha,\beta)\int_0^1 \left[\Lambda_{m_1}(\tau)\Lambda_{m_2}(\tau) + \tilde{\Lambda}_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau)\right] d\tau,
\end{aligned}$$

where $\Lambda_m(\tau) = \frac{1}{m}\sum_{p=1}^m [1-(1-\tau)^p]$ and $\tilde{\Lambda}_m(\tau) = \frac{1}{m}\sum_{p=1}^m [1-\tau^p]$. The inclusions are reversed if $F \in \mathbf{HVP}_{m_1}(\mathbf{S}, \mathbb{K}_c^+)$ and $G \in \mathbf{HVP}_{m_2}(\mathbf{S}, \mathbb{K}_c^+)$.

Proof. Let $\tau \in [0,1]$. From the definition of $\tilde{\Lambda}_m$ and Λ_m above, one observes that

$$\tilde{\Lambda}_m\left(\frac{1}{2}\right) = \Lambda_m\left(\frac{1}{2}\right) := P_m := \frac{m+2^{-m}-1}{m}.$$

Hence, from (3.5), one gets

$$P_{m_1}\left\{F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) + F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right\} \subseteq F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)$$

and

$$P_{m_2}\left\{G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) + G\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right\} \subseteq G\left(\frac{2\alpha\beta}{\alpha+\beta}\right).$$

Now,

$$\begin{aligned}
&F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)G\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\\
\supseteq& P_{m_1}P_{m_2}\left[F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\right.\\
&\left.+F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)G\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right]\\
&+P_{m_1}P_{m_2}\left[F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)G\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right.\\
&\left.+F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\right]\\
\supseteq& P_{m_1}P_{m_2}\left[F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\right.\\
&\left.+F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)G\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right]\\
&+P_{m_1}P_{m_2}\left\{\left[\Lambda_{m_1}(\tau)F(\alpha)+\tilde{\Lambda}_{m_1}(\tau)F(\beta)\right]\left[\Lambda_{m_2}(\tau)G(\beta)+\tilde{\Lambda}_{m_2}(\tau)G(\alpha)\right]\right.\\
&\left.+\left[\Lambda_{m_1}(\tau)F(\beta)+\tilde{\Lambda}_{m_1}(\tau)F(\alpha)\right]\left[\Lambda_{m_2}(\tau)G(\alpha)+\tilde{\Lambda}_{m_2}(\tau)G(\beta)\right]\right\}\\
=& P_{m_1}P_{m_2}\left[F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\right.\\
&\left.+F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)G\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right]\\
&+P_{m_1}P_{m_2}\left\{\left[\Lambda_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau)+\tilde{\Lambda}_{m_1}(\tau)\Lambda_{m_2}(\tau)\right][F(\alpha)G(\alpha)+F(\beta)G(\beta)]\right.\\
&\left.+\left[\Lambda_{m_1}(\tau)\Lambda_{m_2}(\tau)+\tilde{\Lambda}_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau)\right][F(\alpha)G(\beta)+F(\beta)G(\alpha)]\right\}\\
=& P_{m_1}P_{m_2}\left[F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\right.\\
&\left.+F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)G\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right]\\
&+P_{m_1}P_{m_2}\left\{\left[\Lambda_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau)+\tilde{\Lambda}_{m_1}(\tau)\Lambda_{m_2}(\tau)\right]R(\alpha,\beta)\right.\\
&\left.+\left[\Lambda_{m_1}(\tau)\Lambda_{m_2}(\tau)+\tilde{\Lambda}_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau)\right]Q(\alpha,\beta)\right\}.
\end{aligned} \tag{3.21}$$

Integrating with respect to τ over $[0,1]$, we get from (3.21) the following inclusion:

$$\begin{aligned}
&\frac{1}{P_{m_1}P_{m_2}}F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)G\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\\
\supseteq&\frac{2\alpha\beta}{\alpha+\beta}\int_\alpha^\beta\frac{F(r)\,G(r)}{r^2}\,dr+R(\alpha,\beta)\int_0^1\left[\Lambda_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau)+\tilde{\Lambda}_{m_1}(\tau)\Lambda_{m_2}(\tau)\right]d\tau\\
&+Q(\alpha,\beta)\int_0^1\left[\Lambda_{m_1}(\tau)\Lambda_{m_2}(\tau)+\tilde{\Lambda}_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau)\right]d\tau.
\end{aligned}$$

That completes the proof. □

4. Conclusion

A new class of interval-valued function has been proposed. We show that an interval-valued function $F(x) = [f^-(x), f^+(x)]$ is m-polynomial harmonically convex if and only if its component real-valued functions f^- and f^+ are m-polynomial harmonically convex and m-polynomial harmonically concave, respectively. Furthermore, some new set-inclusions of the Hermite–Hadamard type are hereby established. We therefore pose the following open question:

Open question 1. *Let $m_1, m_2 \in \mathbb{N}$. Is it possible to compare $\mathbf{HXP}_{m_1}(\mathbf{S}, \mathbb{K}_c^+)$ and $\mathbf{HXP}_{m_2}(\mathbf{S}, \mathbb{K}_c^+)$?*

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Research Article

Sequential Abstract Generalized Right Side Fractional Landau Inequalities

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ABSTRACT. We give uniform and L_p Caputo-Bochner abstract sequential generalized right fractional Landau inequalities over $\mathbb{R}_-$. These estimates the size of second and third sequential abstract generalized right fractional derivatives of a Banach space valued function over $\mathbb{R}_-$. We give an application when the basic fractional order is $\frac{1}{2}$.

Keywords: Sequential abstract generalized right fractional Landau inequality, sequential Caputo abstract generalized right fractional derivative.

2020 Mathematics Subject Classification: 26A33, 26D10, 26D15.

1. INTRODUCTION

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of f, such that

$$\|f'\|_{p,I} \leq C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \|f''\|_{p,I}^{\frac{1}{2}}, \tag{1}$$

where $\|\cdot\|_{p,I}$ is the p-norm on the interval I, see [1], [5]. The research on these inequalities started by E. Landau [10] in 1913. For the case of $p = \infty$, he proved that

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2} \tag{2}$$

are the best constants in (1). In 1932, G. H. Hardy and J. E. Littlewood [7] proved (1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \text{ and } C_2(\mathbb{R}) = 1. \tag{3}$$

In 1935, G. H. Hardy, E. Landau and J. E. Littlewood [8] showed that the best constants $C_p(\mathbb{R}_+)$ in (1) satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2 \text{ for } p \in [1, \infty), \tag{4}$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$. In fact, in [6] and [9] was shown that $C_p(\mathbb{R}) \leq \sqrt{2}$. We need the following concept from abstract fractional calculus. Our integral next is of Bochner type [11]. We need

Definition 1.1. *([4], p. 105) Let $[a, b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ a Banach space, $g \in C^1([a, b])$ and increasing, $f \in C([a, b], X)$, $\nu > 0$.*

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We define the right Riemann-Liouville generalized fractional Bochner integral operator

$$\left(J_{b-;g}^{\nu}f\right)(x) := \frac{1}{\Gamma(\nu)}\int_x^b (g(z)-g(x))^{\nu-1} g'(z) f(z)\,dz, \tag{5}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function. The last integral is of Bochner type. Since $f \in C([a,b],X)$, then $f \in L_\infty([a,b],X)$. By Theorem 4.11, p. 101, [4], we get that $\left(J_{b-;g}^{\nu}f\right) \in C([a,b],X)$. Above we set $J_{b-;g}^{0}f := f$ and see that $\left(J_{b-;g}^{\nu}f\right)(b) = 0$.

We also need

Definition 1.2. *([4], p. 107) Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b],X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a),g(b)])$. We define the right generalized g-fractional derivative X-valued of f of order α as follows:*

$$\left(D_{b-;g}^{\alpha}f\right)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b (g(t)-g(x))^{n-\alpha-1} g'(t)\left(f\circ g^{-1}\right)^{(n)}(g(t))\,dt, \tag{6}$$

$\forall\, x \in [a,b]$. The last integral is of Bochner type. Ordinary vector valued derivative is as in [12], similar to numerical one. If $\alpha \notin \mathbb{N}$, by Theorem 4.11, p. 101, [4], we have that $\left(D_{b-;g}^{\alpha}f\right) \in C([a,b],X)$. We see that

$$\left(J_{b;g}^{n-\alpha}\left((-1)^n\left(f\circ g^{-1}\right)^{(n)}\circ g\right)\right)(x) = \left(D_{b-;g}^{\alpha}f\right)(x),\ \ \forall\, x\in[a,b]. \tag{7}$$

We set

$$D_{b-;g}^{n}f(x) := (-1)^n\left(\left(f\circ g^{-1}\right)^{n}\circ g\right)(x) \in C([a,b],X),\ \ n\in\mathbb{N}, \tag{8}$$

$$D_{b-;g}^{0}f(x) = f(x),\ \ \forall\, x\in[a,b].$$

When $g = id$, then

$$D_{b-;g}^{\alpha}f = D_{b-;id}^{\alpha}f = D_{b-}^{\alpha}f, \tag{9}$$

the usual left X-valued Caputo fractional derivative, see [4], Chapter 2.

By convention, we suppose that

$$\left(D_{x_0-;g}^{\alpha}f\right)(x) = 0,\ \text{for } x > x_0 \tag{10}$$

for any $x, x_0 \in [a,b]$.

Denote the sequential (also called iterated) generalized left fractional derivative by

$$D_{b-;g}^{n\alpha} := D_{b-;g}^{\alpha}D_{b-;g}^{\alpha}...D_{b-;g}^{\alpha}\ \ (n \text{ times}),\ n\in\mathbb{N}. \tag{11}$$

We need the following g-right generalized modified X-valued Taylor's formula.

Theorem 1.1 **([4, p. 120]).** *Let $0 < \alpha \le 1$, $n \in \mathbb{N}$, $f \in C^1([a,b],X)$, $g \in C^1([a,b])$ strictly increasing, such that $g^{-1} \in C^1([g(a),g(b)])$. Let $F_k := D_{b-;g}^{k\alpha}f$, $k = 1,...,n$, that fulfill $F_k \in C^1([a,b],X)$, and $F_{n+1} \in C([a,b],X)$. Then,*

$$f(x) = \sum_{i=0}^{n}\frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{b-;g}^{i\alpha}f\right)(b) + \frac{1}{\Gamma((n+1)\alpha)}\int_x^b (g(t)-g(x))^{(n+1)\alpha-1} g'(t)\left(D_{b-;g}^{(n+1)\alpha}f\right)(t)\,dt, \tag{12}$$

$\forall\, x\in[a,b]$.

We make

Remark 1.1 (to Theorem 1.1). *When* $0 < \alpha < 1$, *by* (6), *we get*

$$\left(D_{b-;g}^{\alpha} f\right)(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^b (g(t) - g(x))^{-\alpha} g'(t) \left(f \circ g^{-1}\right)'(g(t))\, dt, \tag{13}$$

$\forall\, x \in [a, b]$.
Hence,

$$\begin{aligned} \left\| \left(D_{b-;g}^{\alpha} f\right)(x) \right\| &\leq \frac{1}{\Gamma(1-\alpha)} \int_x^b (g(t) - g(x))^{-\alpha} g'(t) \left\| \left(f \circ g^{-1}\right)'(g(t)) \right\| dt \\ &\leq \frac{\left\| \left\| \left(f \circ g^{-1}\right)' \circ g \right\| \right\|_{\infty,[a,b]}}{\Gamma(1-\alpha)} \left(\int_x^b (g(t) - g(x))^{-\alpha} g'(t)\, dt \right) \\ &= \frac{\left\| \left\| \left(f \circ g^{-1}\right)' \circ g \right\| \right\|_{\infty,[a,b]}}{\Gamma(2-\alpha)} (g(b) - g(x))^{1-\alpha}. \end{aligned} \tag{14}$$

That is

$$\left\| \left(D_{b-;g}^{\alpha} f\right)(x) \right\| \leq \frac{\left\| \left\| \left(f \circ g^{-1}\right)' \circ g \right\| \right\|_{\infty,[a,b]}}{\Gamma(2-\alpha)} (g(b) - g(x))^{1-\alpha} < \infty, \tag{15}$$

$\forall\, x \in [a, b]$, $0 < \alpha < 1$. *Hence, it holds*

$$\left\| \left(D_{b-;g}^{\alpha} f\right)(b) \right\| = 0,$$

i.e.

$$\left(D_{b-;g}^{\alpha} f\right)(b) = 0, \tag{16}$$

when $0 < \alpha < 1$.

The author has already done an extensive amount of work on fractional Landau inequalities, see [3], and on abstract fractional Landau inequalities, see [4]. However, there the proving methods came out of applications of fractional Ostrowski inequalities ([2], [4]). Usually there the domains, where $[A, +\infty)$ or $(-\infty, B]$, with $A, B \in \mathbb{R}$ and in one mixed case the domain was all of $\mathbb{R}$.
In this work with less assumptions, we establish uniform and L_p type right Caputo-Bochner abstract sequential generalized fractional Landau inequalities over $\mathbb{R}_-$. The method of proving is based on right Caputo-Bochner sequential generalized fractional Taylor's formula with integral remainder, see Theorem 1.1.
We give also an application for $\alpha = \frac{1}{2}$. Clearly we are also inspired by [3], [4].

2. Main Results

We present the following abstract sequential generalized fractional Landau inequalities over $\mathbb{R}_-$.

Theorem 2.2. *Let* $g \in C^1(\mathbb{R}_-)$ *strictly increasing, with* $g^{-1} \in C^1(g(\mathbb{R}_-))$. *Let* $0 < \alpha < 1$, $f \in C^1(\mathbb{R}_-, X)$ *with* $\|\|f\|\|_{\infty,\mathbb{R}_-}$, $\left\| \left\| \left(f \circ g^{-1}\right)' \circ g \right\| \right\|_{\infty,\mathbb{R}_-} < \infty$. *For* $k = 1, 2, 3$, *we assume that* $D_{b-;g}^{k\alpha} f \in C^1((-\infty, b], X)$ *and* $D_{b-;g}^{4\alpha} f \in C((-\infty, b], X)$, $\forall\, b \in \mathbb{R}_-$. *We further assume that*

$$\overline{K_g} := \left\| \left\| D_{b-;g}^{4\alpha} f(t) \right\| \right\|_{\infty,\mathbb{R}_-^2} < \infty, \tag{17}$$

where $(b,t) \in \mathbb{R}_-^2$. *Then*

$$\sup_{b\in\mathbb{R}_-} \left\| \left(D_{b-;g}^{2\alpha} f\right)(b) \right\| \leq \frac{\Gamma(2\alpha+1)}{2^{2\alpha-1}(2^\alpha-1)} \sqrt{\frac{2^{3\alpha+1}(2^{3\alpha}+1)(2^\alpha+1)}{\Gamma(4\alpha+1)} \left\| \|f\| \right\|_{\infty,\mathbb{R}_-} \overline{K_g}} \tag{18}$$

and

$$\sup_{b\in\mathbb{R}_-} \left\| \left(D_{b-;g}^{3\alpha} f\right)(b) \right\| \leq \frac{4\sqrt[4]{2}\,\Gamma(3\alpha+1)(\Gamma(4\alpha+1))^{-\frac{3}{4}}(2^{2\alpha}+1)}{\left(\sqrt[4]{3}\right)^3 \left(\sqrt{2}\right)^\alpha (2^\alpha-1)} \left\| \|f\| \right\|_{\infty,\mathbb{R}_-}^{\frac{1}{4}} \overline{K_g}^{\frac{3}{4}}. \tag{19}$$

That is $\sup_{b\in\mathbb{R}_-} \left\| \left(D_{b-;g}^{2\alpha} f\right)(b) \right\|, \sup_{b\in\mathbb{R}_-} \left\| \left(D_{b-;g}^{3\alpha} f\right)(b) \right\| < \infty.$

Proof. We notice easily again here that $\left(D_{b-;g}^{\alpha} f\right)(b) = 0$, $\forall\, b \in \mathbb{R}_-$. We make use of Theorem 1.1 for $0 < \alpha < 1$ and $n = 3$, applied for any $b \in \mathbb{R}_-$ and $a = -\infty$. Momentarily, we fix $b \in \mathbb{R}_-$. Let $x_2 < x_1 < b$, then $g(x_2) < g(x_1) < g(b)$, and

$$\begin{aligned} f(x_1) - f(b) = {} & \frac{(g(b)-g(x_1))^{2\alpha}}{\Gamma(2\alpha+1)} \left(D_{b-;g}^{2\alpha} f\right)(b) + \frac{(g(b)-g(x_1))^{3\alpha}}{\Gamma(3\alpha+1)} \left(D_{b-;g}^{3\alpha} f\right)(b) \\ & + \frac{1}{\Gamma(4\alpha)} \int_{x_1}^{b} (g(t)-g(x_1))^{4\alpha-1} g'(t) \left(D_{b-;g}^{4\alpha} f\right)(t)\, dt, \end{aligned} \tag{20}$$

and

$$\begin{aligned} f(x_2) - f(b) = {} & \frac{(g(b)-g(x_2))^{2\alpha}}{\Gamma(2\alpha+1)} \left(D_{b-;g}^{2\alpha} f\right)(b) + \frac{(g(b)-g(x_2))^{3\alpha}}{\Gamma(3\alpha+1)} \left(D_{b-;g}^{3\alpha} f\right)(b) \\ & + \frac{1}{\Gamma(4\alpha)} \int_{x_2}^{b} (g(t)-g(x_2))^{4\alpha-1} g'(t) \left(D_{b-;g}^{4\alpha} f\right)(t)\, dt. \end{aligned} \tag{21}$$

That is

$$\begin{aligned} & \frac{(g(b)-g(x_1))^{2\alpha}}{\Gamma(2\alpha+1)} \left(D_{b-;g}^{2\alpha} f\right)(b) + \frac{(g(b)-g(x_1))^{3\alpha}}{\Gamma(3\alpha+1)} \left(D_{b-;g}^{3\alpha} f\right)(b) \\ = & f(x_1) - f(b) - \frac{1}{\Gamma(4\alpha)} \int_{x_1}^{b} (g(t)-g(x_1))^{4\alpha-1} g'(t) \left(D_{b-;g}^{4\alpha} f\right)(t)\, dt =: A, \end{aligned} \tag{22}$$

and

$$\begin{aligned} & \frac{(g(b)-g(x_2))^{2\alpha}}{\Gamma(2\alpha+1)} \left(D_{b-;g}^{2\alpha} f\right)(b) + \frac{(g(b)-g(x_2))^{3\alpha}}{\Gamma(3\alpha+1)} \left(D_{b-;g}^{3\alpha} f\right)(b) \\ = & f(x_2) - f(b) - \frac{1}{\Gamma(4\alpha)} \int_{x_2}^{b} (g(t)-g(x_2))^{4\alpha-1} g'(t) \left(D_{b-;g}^{4\alpha} f\right)(t)\, dt =: B. \end{aligned} \tag{23}$$

We are solving the above system of two equations with two unknowns $\left(D_{b-;g}^{2\alpha}f\right)(b)$, $\left(D_{b-;g}^{3\alpha}f\right)(b)$. The main determinant of system is

$$D := \begin{vmatrix} \frac{(g(b)-g(x_1))^{2\alpha}}{\Gamma(2\alpha+1)} & \frac{(g(b)-g(x_1))^{3\alpha}}{\Gamma(3\alpha+1)} \\ \\ \frac{(g(b)-g(x_2))^{2\alpha}}{\Gamma(2\alpha+1)} & \frac{(g(b)-g(x_2))^{3\alpha}}{\Gamma(3\alpha+1)} \end{vmatrix}$$

$$= \frac{1}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)}$$

$$\times\left[(g(b)-g(x_1))^{2\alpha}(g(b)-g(x_2))^{3\alpha}-(g(b)-g(x_1))^{3\alpha}(g(b)-g(x_2))^{2\alpha}\right]$$

$$= \frac{(g(b)-g(x_1))^{2\alpha}(g(b)-g(x_2))^{2\alpha}}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)}\left[(g(b)-g(x_2))^{\alpha}-(g(b)-g(x_1))^{\alpha}\right]>0,$$

i.e.

$$D = \frac{(g(b)-g(x_1))^{2\alpha}(g(b)-g(x_2))^{2\alpha}}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)}\left[(g(b)-g(x_2))^{\alpha}-(g(b)-g(x_1))^{\alpha}\right]>0. \quad (24)$$

We obtain the unique solution

$$\left(D_{b-;g}^{2\alpha}f\right)(b) = \frac{\begin{vmatrix} A & \frac{(g(b)-g(x_1))^{3\alpha}}{\Gamma(3\alpha+1)} \\ \\ B & \frac{(g(b)-g(x_2))^{3\alpha}}{\Gamma(3\alpha+1)} \end{vmatrix}}{D},$$

(25)

$$\left(D_{b-;g}^{3\alpha}f\right)(b) = \frac{\begin{vmatrix} \frac{(g(b)-g(x_1))^{2\alpha}}{\Gamma(2\alpha+1)} & A \\ \\ \frac{(g(b)-g(x_2))^{2\alpha}}{\Gamma(2\alpha+1)} & B \end{vmatrix}}{D}.$$

Therefore, we have

$$\left(D_{b-;g}^{2\alpha}f\right)(b) = \frac{\frac{(g(b)-g(x_2))^{3\alpha}}{\Gamma(3\alpha+1)}A-\frac{(g(b)-g(x_1))^{3\alpha}}{\Gamma(3\alpha+1)}B}{D},$$

(26) and

$$\left(D_{b-;g}^{3\alpha}f\right)(b) = \frac{\frac{(g(b)-g(x_1))^{2\alpha}}{\Gamma(2\alpha+1)}B-\frac{(g(b)-g(x_2))^{2\alpha}}{\Gamma(2\alpha+1)}A}{D}.$$

We have the following

$$\|A\| = \left\|f(x_1)-f(b)-\frac{1}{\Gamma(4\alpha)}\int_{x_1}^{b}(g(t)-g(x_1))^{4\alpha-1}g'(t)\left(D_{b-;g}^{4\alpha}f\right)(t)\,dt\right\|$$

$$\leq 2\,\||f|\|_{\infty,\mathbb{R}_-}+\frac{\left\|\left\|D_{b-;g}^{4\alpha}f(t)\right\|\right\|_{\infty,\mathbb{R}_-^2}}{\Gamma(4\alpha+1)}(g(b)-g(x_1))^{4\alpha} \quad (27)$$

under the assumption $\||f|\|_{\infty,\mathbb{R}_-}<\infty$. That is

$$\|A\|\leq 2\,\||f|\|_{\infty,\mathbb{R}_-}+\frac{\overline{K_g}}{\Gamma(4\alpha+1)}(g(b)-g(x_1))^{4\alpha}, \quad (28)$$

and similarly,

$$\|B\| \leq 2\,\||f\||_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)}\left(g(b)-g(x_2)\right)^{4\alpha}, \tag{29}$$

where by assumption

$$\overline{K_g} := \left\|\left\|D^{4\alpha}_{b-;g} f(t)\right\|\right\|_{\infty,\mathbb{R}^2_-} < \infty, \tag{30}$$

with $(b,t) \in \mathbb{R}^2_-$. Consequently, we have

$$\begin{aligned}\left\|\left(D^{2\alpha}_{b-;g} f\right)(b)\right\| \leq \frac{1}{\Gamma(3\alpha+1)\,D}&\left[\left(g(b)-g(x_2)\right)^{3\alpha}\left(2\,\||f\||_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)}\left(g(b)-g(x_1)\right)^{4\alpha}\right)\right. \\ &\left.+\left(g(b)-g(x_1)\right)^{3\alpha}\left(2\,\||f\||_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)}\left(g(b)-g(x_2)\right)^{4\alpha}\right)\right], \end{aligned}\tag{31}$$

and

$$\begin{aligned}\left\|\left(D^{3\alpha}_{b-;g} f\right)(b)\right\| \leq \frac{1}{\Gamma(2\alpha+1)\,D}&\left[\left(g(b)-g(x_1)\right)^{2\alpha}\left(2\,\||f\||_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)}\left(g(b)-g(x_2)\right)^{4\alpha}\right)\right. \\ &\left.+\left(g(b)-g(x_2)\right)^{2\alpha}\left(2\,\||f\||_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)}\left(g(b)-g(x_1)\right)^{4\alpha}\right)\right]. \end{aligned}\tag{32}$$

Set now $g(x_1) := g(b) - h$, $g(x_2) := g(b) - 2h$, where $h > 0$, so that $g(b) - g(x_1) = h$, $g(b) - g(x_2) = 2h$. Hence, we get

$$D = \frac{2^{2\alpha} h^{5\alpha}\left(2^{\alpha}-1\right)}{\Gamma(2\alpha+1)\,\Gamma(3\alpha+1)} > 0. \tag{33}$$

Therefore, we derive (from (26))

$$\begin{aligned}\left\|\left(D^{2\alpha}_{b-;g} f\right)(b)\right\| &\leq \frac{\Gamma(2\alpha+1)}{2^{2\alpha}h^{5\alpha}\left(2^{\alpha}-1\right)}\left[2^{3\alpha}h^{3\alpha}\left(2\,\||f\||_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)}h^{4\alpha}\right)\right. \\ &\left.+h^{3\alpha}\left(2\,\||f\||_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)}2^{4\alpha}h^{4\alpha}\right)\right] \end{aligned}\tag{34}$$

$$\begin{aligned}&= \frac{\Gamma(2\alpha+1)}{2^{2\alpha}h^{5\alpha}\left(2^{\alpha}-1\right)}\left[2\,\||f\||_{\infty,\mathbb{R}_-}\left(2^{3\alpha}+1\right)h^{3\alpha} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)}\left(2^{3\alpha}+2^{4\alpha}\right)h^{7\alpha}\right] \\ &= \left(\frac{\Gamma(2\alpha+1)}{2^{2\alpha}\left(2^{\alpha}-1\right)}\right)\left[\frac{2\left(2^{3\alpha}+1\right)\||f\||_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{2^{3\alpha}\left(2^{\alpha}+1\right)}{\Gamma(4\alpha+1)}\overline{K_g}h^{2\alpha}\right]. \end{aligned}\tag{35}$$

That is

$$\begin{aligned}\left\|\left(D^{2\alpha}_{b-;g} f\right)(b)\right\| &\leq \left(\frac{\Gamma(2\alpha+1)}{2^{2\alpha}\left(2^{\alpha}-1\right)}\right) \\ &\times\left[\frac{2\left(2^{3\alpha}+1\right)\||f\||_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{2^{3\alpha}\left(2^{\alpha}+1\right)}{\Gamma(4\alpha+1)}\overline{K_g}h^{2\alpha}\right], \end{aligned}\tag{36}$$

$\forall\, b \in \mathbb{R}_-$, $\forall\, h > 0$. I.e., it holds

$$\begin{aligned}\sup_{b\in\mathbb{R}_-}\left\|\left(D^{2\alpha}_{b-;g} f\right)(b)\right\| &\leq \left(\frac{\Gamma(2\alpha+1)}{2^{2\alpha}\left(2^{\alpha}-1\right)}\right) \\ &\times\left[\frac{2\left(2^{3\alpha}+1\right)\||f\||_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{2^{3\alpha}\left(2^{\alpha}+1\right)}{\Gamma(4\alpha+1)}\overline{K_g}h^{2\alpha}\right] < \infty, \end{aligned}\tag{37}$$

$\forall\, h > 0$, $0 < \alpha < 1$. By (26), we derive

$$\left\|\left(D_{b-;g}^{3\alpha}f\right)(b)\right\| \leq \frac{\Gamma(3\alpha+1)}{2^{2\alpha}h^{5\alpha}(2^{\alpha}-1)}\left[h^{2\alpha}\left(2\,\||f\||_{\infty,\mathbb{R}_-}+\frac{\overline{K_g}}{\Gamma(4\alpha+1)}2^{4\alpha}h^{4\alpha}\right)\right.$$
$$\left.+2^{2\alpha}h^{2\alpha}\left(2\,\||f\||_{\infty,\mathbb{R}_-}+\frac{\overline{K_g}}{\Gamma(4\alpha+1)}h^{4\alpha}\right)\right]$$
$$(38)\qquad = \frac{\Gamma(3\alpha+1)}{2^{2\alpha}h^{5\alpha}(2^{\alpha}-1)}\left[2\,\||f\||_{\infty,\mathbb{R}_-}\left(2^{2\alpha}+1\right)h^{2\alpha}+\frac{\overline{K_g}}{\Gamma(4\alpha+1)}\left(2^{4\alpha}+2^{2\alpha}\right)h^{6\alpha}\right]$$
$$= \left(\frac{\Gamma(3\alpha+1)}{2^{2\alpha}(2^{\alpha}-1)}\right)\left[\frac{2\left(2^{2\alpha}+1\right)\||f\||_{\infty,\mathbb{R}_-}}{h^{3\alpha}}+\frac{2^{2\alpha}\left(2^{2\alpha}+1\right)}{\Gamma(4\alpha+1)}\overline{K_g}h^{\alpha}\right]$$
$$= \frac{\Gamma(3\alpha+1)\left(2^{2\alpha}+1\right)}{2^{2\alpha}(2^{\alpha}-1)}\left[\frac{2\,\||f\||_{\infty,\mathbb{R}_-}}{h^{3\alpha}}+\frac{2^{2\alpha}\overline{K_g}}{\Gamma(4\alpha+1)}h^{\alpha}\right].$$

That is

$$\left\|\left(D_{b-;g}^{3\alpha}f\right)(b)\right\| \leq \frac{\Gamma(3\alpha+1)\left(2^{2\alpha}+1\right)}{2^{2\alpha}(2^{\alpha}-1)}$$
$$(39)\qquad \times\left[\frac{2\,\||f\||_{\infty,\mathbb{R}_-}}{h^{3\alpha}}+\frac{2^{2\alpha}\overline{K_g}}{\Gamma(4\alpha+1)}h^{\alpha}\right],$$

$\forall\, b \in \mathbb{R}_-$, $\forall\, h > 0$. I.e., it holds

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-;g}^{3\alpha}f\right)(b)\right\| \leq \frac{\Gamma(3\alpha+1)\left(2^{2\alpha}+1\right)}{2^{2\alpha}(2^{\alpha}-1)}$$
$$(40)\qquad \times\left[\frac{2\,\||f\||_{\infty,\mathbb{R}_-}}{h^{3\alpha}}+\frac{2^{2\alpha}\overline{K_g}}{\Gamma(4\alpha+1)}h^{\alpha}\right] < \infty,$$

$\forall\, h > 0$, $0 < \alpha < 1$. Call

$$(41)\qquad \mu := 2\left(2^{3\alpha}+1\right)\||f\||_{\infty,\mathbb{R}_-},$$

$$(42)\qquad \theta = \frac{2^{3\alpha}\left(2^{\alpha}+1\right)\overline{K_g}}{\Gamma(4\alpha+1)},$$

both are greater than zero. Set also $\rho := 2\alpha$; $0 < \rho < 2$. We consider the function

$$(43)\qquad y(h) := \mu h^{-\rho}+\theta h^{\rho},\ \ \forall\, h > 0.$$

We have

$$(44)\qquad y'(h) = -\rho\mu h^{-\rho-1}+\rho\theta h^{\rho-1} = 0,$$

then

$$\theta h^{2\rho} = \mu,$$

with a unique solution

$$(45)\qquad h_0 := h_{crit.no.} = \left(\frac{\mu}{\theta}\right)^{\frac{1}{2\rho}}.$$

We have that

$$(46)\qquad y''(h) = \rho(\rho+1)\mu h^{-\rho-2}+\rho(\rho-1)\theta h^{\rho-2}.$$

We see that

$$y''(h_0) = y''\left(\left(\frac{\mu}{\theta}\right)^{\frac{1}{2\rho}}\right) = \rho(\rho+1)\mu\left(\frac{\mu}{\theta}\right)^{\frac{-\rho-2}{2\rho}} + \rho(\rho-1)\theta\left(\frac{\mu}{\theta}\right)^{\frac{\rho-2}{2\rho}}$$

$$= \rho\left(\frac{\theta}{\mu}\right)^{\frac{1}{\rho}}\left[(\rho+1)\sqrt{\mu\theta} + (\rho-1)\sqrt{\mu\theta}\right]$$

$$= \rho\left(\frac{\theta}{\mu}\right)^{\frac{1}{\rho}}\left(2\rho\sqrt{\mu\theta}\right) = 2\rho^2\sqrt{\mu\theta}\left(\frac{\theta}{\mu}\right)^{\frac{1}{\rho}} > 0.$$

Therefore, y has a global minimum at $h_0 = \left(\frac{\mu}{\theta}\right)^{\frac{1}{2\rho}}$, which is

$$y(h_0) = \mu\left(\frac{\mu}{\theta}\right)^{-\frac{1}{2}} + \theta\left(\frac{\mu}{\theta}\right)^{\frac{1}{2}} = \mu\left(\frac{\theta}{\mu}\right)^{\frac{1}{2}} + \sqrt{\theta\mu} = 2\sqrt{\theta\mu}.$$

We have proved that (see (37))

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D^{2\alpha}_{b-;g}f\right)(b)\right\| \leq \frac{\Gamma(2\alpha+1)}{2^{2\alpha-1}(2^\alpha-1)}$$

(47) $$\times\sqrt{\frac{2^{3\alpha+1}(2^{3\alpha}+1)(2^\alpha+1)}{\Gamma(4\alpha+1)}\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}\overline{K_g}}.$$

Call

(48) $$\xi := 2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}, \qquad \psi := \frac{2^{2\alpha}\overline{K_g}}{\Gamma(4\alpha+1)},$$

both are greater than zero. We consider the function

(49) $$\gamma(h) := \xi h^{-3\alpha} + \psi h^\alpha, \quad \forall\, h > 0.$$

We have

$$\gamma'(h) = -3\alpha\xi h^{-3\alpha-1} + \alpha\psi h^{\alpha-1} = 0,$$

then

$$\psi h^{4\alpha} = 3\xi,$$

with unique solution

(50) $$h_0 := h_{crit.no.} = \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4\alpha}}.$$

We have that

(51) $$\gamma''(h) = 3\alpha(3\alpha+1)\xi h^{-3\alpha-2} + \alpha(\alpha-1)\psi h^{\alpha-2}.$$

We see

$$\gamma''(h_0) = 3\alpha(3\alpha+1)\xi\left(\frac{3\xi}{\psi}\right)^{\frac{-3\alpha-2}{4\alpha}} + \alpha(\alpha-1)\psi\left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}}$$

$$= \alpha\left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}}\left[3(3\alpha+1)\xi\frac{\psi}{3\xi} + (\alpha-1)\psi\right]$$

(52) $$= \alpha\left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}}(4\alpha\psi) = 4\alpha^2\psi\left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}} > 0.$$

Therefore, γ has a global minimum at $h_0 = \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4\alpha}}$, which is

$$\gamma(h_0) = \xi\left(\frac{3\xi}{\psi}\right)^{-\frac{3}{4}} + \psi\left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}}$$

$$= \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}}\left(\xi\frac{\psi}{3\xi} + \psi\right) = \frac{4}{3}\psi\left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}}. \tag{53}$$

Consequently,

$$\gamma(h_0) = \frac{4}{3}\psi\left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}} = \frac{4}{\left(\sqrt[4]{3}\right)^3}\psi^{\frac{3}{4}}\xi^{\frac{1}{4}}. \tag{54}$$

We have proved that (see (40))

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D^{3\alpha}_{b-;g}f\right)(b)\right\| \leq \frac{4\Gamma(3\alpha+1)\left(2^{2\alpha}+1\right)}{\left(\sqrt[4]{3}\right)^3 2^{2\alpha}\left(2^{\alpha}-1\right)}$$

$$\times\left(2\,\|\|f\|\|_{\infty,\mathbb{R}_-}\right)^{\frac{1}{4}}\left(\frac{2^{2\alpha}\overline{K_g}}{\Gamma(4\alpha+1)}\right)^{\frac{3}{4}}$$

$$= \frac{4\sqrt[4]{2}\Gamma(3\alpha+1)\Gamma(4\alpha+1)^{-\frac{3}{4}}\left(2^{2\alpha}+1\right)}{\left(\sqrt[4]{3}\right)^3 2^{\frac{\alpha}{2}}\left(2^{\alpha}-1\right)}\|\|f\|\|^{\frac{1}{4}}_{\infty,\mathbb{R}_-}\overline{K_g}^{\frac{3}{4}}. \tag{55}$$

The theorem is proved. □

We continue with abstract L_p right sequential generalized fractional Landau inequalities over $\mathbb{R}_-$.

Theorem 2.3. *Let $g \in C^1(\mathbb{R}_-)$ strictly increasing, with $g^{-1} \in C^1(g(\mathbb{R}_-))$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $0 < \alpha < 1$. Let $f \in C^1(\mathbb{R}_-, X)$ with $\|\|f\|\|_{\infty,\mathbb{R}_-}$, $\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,\mathbb{R}_-} < \infty$. For $k = 1, 2, 3$, we assume that $D^{k\alpha}_{b-;g}f \in C^1((-\infty, b], X)$ and $D^{4\alpha}_{b-;g}f \in C((-\infty, b], X)$, $\forall\, b \in \mathbb{R}_-$. We further assume that*

$$\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D^{4\alpha}_{b-;g}f\right\|\right\|_{p,\mathbb{R}_-}\right) < \infty. \tag{56}$$

Then
1) under $\frac{1}{2p} < \alpha < 1$, we get

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D^{2\alpha}_{b-;g}f\right)(b)\right\| \leq \left[\left(\frac{2^{\alpha}\Gamma(2\alpha)\left(4\alpha-\frac{1}{p}\right)}{2^{\alpha}-1}\right)\left(\frac{4\alpha\left(1+2^{-3\alpha}\right)}{2\alpha-\frac{1}{p}}\right)^{\left(\frac{2\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\right.$$

$$\left.\times\left(\frac{1+2^{\alpha-\frac{1}{p}}}{\Gamma(4\alpha)\left(q(4\alpha-1)+1\right)^{\frac{1}{q}}}\right)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)}\right]\|\|f\|\|^{\left(\frac{2\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}_{\infty,\mathbb{R}_-}$$

$$\times\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D^{4\alpha}_{b-;g}f\right\|\right\|_{p,\mathbb{R}_-}\right)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)} < \infty. \tag{57}$$

2) under $\frac{1}{p} < \alpha < 1$*, we get*

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-;g}^{3\alpha}f\right)(b)\right\| \leq \left[\left(\frac{\Gamma(3\alpha)\left(4\alpha-\frac{1}{p}\right)}{2^\alpha-1}\right)\left(\frac{6\alpha\left(1+2^{-2\alpha}\right)}{\alpha-\frac{1}{p}}\right)^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\right.$$

$$\left.\times\left(\frac{1+2^{2\alpha-\frac{1}{p}}}{\Gamma(4\alpha)\left(q(4\alpha-1)+1\right)^{\frac{1}{q}}}\right)^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}\right]\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}$$

$$\times\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D_{b-;g}^{4\alpha}f\right\|\right\|_{p,\mathbb{R}_-}\right)^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)} < \infty. \tag{58}$$

That is $\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-;g}^{2\alpha}f\right)(b)\right\|$, $\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-;g}^{3\alpha}f\right)(b)\right\| < \infty$.

Proof. As in the proof of Theorem 2.2, we have that

$$\|A\| \overset{(22)}{=} \left\|f(x_1)-f(b)-\frac{1}{\Gamma(4\alpha)}\int_{x_1}^{b}\left(g(t)-g(x_1)\right)^{4\alpha-1}g'(t)\left(D_{b-;g}^{4\alpha}f\right)(t)\,dt\right\|$$

$$\leq 2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{1}{\Gamma(4\alpha)}\int_{x_1}^{b}\left(g(t)-g(x_1)\right)^{4\alpha-1}g'(t)\left\|\left(D_{b-;g}^{4\alpha}f\right)(t)\right\|dt$$

$$\leq 2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{1}{\Gamma(4\alpha)}\frac{\left(g(b)-g(x_1)\right)^{\frac{(q(4\alpha-1)+1)}{q}}}{\left(q(4\alpha-1)+1\right)^{\frac{1}{q}}}\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D_{b-;g}^{4\alpha}f\right\|\right\|_{p,\mathbb{R}_-}\right) \tag{59}$$

$$\overset{(g(b)-g(x_1)=:h>0)}{=} 2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{1}{\Gamma(4\alpha)}\frac{h^{\left(4\alpha-\frac{1}{p}\right)}}{\left(q(4\alpha-1)+1\right)^{\frac{1}{q}}}\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D_{b-;g}^{4\alpha}f\right\|\right\|_{p,\mathbb{R}_-}\right),$$

with $\frac{1}{4p} < \alpha < 1$. That is

$$\|A\| \leq 2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{h^{4\alpha-\frac{1}{p}}}{\Gamma(4\alpha)\left(q(4\alpha-1)+1\right)^{\frac{1}{q}}}\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D_{b-;g}^{4\alpha}f\right\|\right\|_{p,\mathbb{R}_-}\right), \tag{60}$$

where $\frac{1}{4p} < \alpha < 1$. We also have

$$\|B\| \overset{(23)}{=} \left\|f(x_2)-f(b)-\frac{1}{\Gamma(4\alpha)}\int_{x_2}^{b}\left(g(t)-g(x_2)\right)^{4\alpha-1}g'(t)\left(D_{b-;g}^{4\alpha}f\right)(t)\,dt\right\| \tag{2.1}$$

$$\leq 2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{\left(g(b)-g(x_2)\right)^{\frac{(q(4\alpha-1)+1)}{q}}}{\Gamma(4\alpha)\left(q(4\alpha-1)+1\right)^{\frac{1}{q}}}\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D_{b-;g}^{4\alpha}f\right\|\right\|_{p,\mathbb{R}_-}\right) \tag{61}$$

$$\overset{(g(b)-g(x_2)=:2h)}{=} 2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{2^{4\alpha-\frac{1}{p}}h^{4\alpha-\frac{1}{p}}}{\Gamma(4\alpha)\left(q(4\alpha-1)+1\right)^{\frac{1}{q}}}\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D_{b-;g}^{4\alpha}f\right\|\right\|_{p,\mathbb{R}_-}\right).$$

That is

$$\|B\| \leq 2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{2^{4\alpha-\frac{1}{p}}h^{4\alpha-\frac{1}{p}}}{\Gamma(4\alpha)\left(q(4\alpha-1)+1\right)^{\frac{1}{q}}}\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D_{b-;g}^{4\alpha}f\right\|\right\|_{p,\mathbb{R}_-}\right), \tag{62}$$

where $\frac{1}{4p} < \alpha < 1$. We have assumed that

$$\overline{M_g} := \left(\sup_{b \in \mathbb{R}_-} \left\| \left\| D^{4\alpha}_{b-;g} f \right\| \right\|_{p,\mathbb{R}_-} \right) < \infty. \tag{63}$$

For convenience, we call

$$c := \Gamma(4\alpha)\left(q(4\alpha-1)+1\right)^{\frac{1}{q}} > 0. \tag{64}$$

So, we have

$$\|A\| \leq 2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-} + \frac{h^{4\alpha-\frac{1}{p}}}{c}\overline{M_g} \quad \text{and} \quad \|B\| \leq 2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-} + \frac{2^{4\alpha-\frac{1}{p}}h^{4\alpha-\frac{1}{p}}}{c}\overline{M_g}, \tag{65}$$

where $\frac{1}{4p} < \alpha < 1$. Next, we estimate the (26)-quantities and we have

$$\begin{aligned} \left\|\left(D^{2\alpha}_{b-;g}f\right)(b)\right\| &\leq \frac{1}{D\Gamma(3\alpha+1)}\left[2^{3\alpha}h^{3\alpha}\|A\| + h^{3\alpha}\|B\|\right] \\ &\overset{(33)}{=} \frac{h^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha}h^{5\alpha}(2^\alpha-1)}\left[2^{3\alpha}\|A\|+\|B\|\right] \\ &\overset{(65)}{\leq} \frac{\Gamma(2\alpha+1)}{2^{2\alpha}(2^\alpha-1)h^{2\alpha}}\left[2^{3\alpha+1}\left\|\|f\|\right\|_{\infty,\mathbb{R}_-} + \frac{2^{3\alpha}h^{4\alpha-\frac{1}{p}}}{c}\overline{M_g}\right. \\ &\left. + 2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-} + \frac{2^{4\alpha-\frac{1}{p}}h^{4\alpha-\frac{1}{p}}}{c}\overline{M_g}\right] \\ &= \frac{\Gamma(2\alpha+1)}{2^{2\alpha}(2^\alpha-1)}\left[\frac{\left(2^{3\alpha+1}+2\right)\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{\left(2^{3\alpha}+2^{4\alpha-\frac{1}{p}}\right)}{c}\overline{M_g}h^{2\alpha-\frac{1}{p}}\right] \end{aligned} \tag{66}$$

$$= \frac{2^\alpha\Gamma(2\alpha+1)}{(2^\alpha-1)}\left[\frac{2\left(1+2^{-3\alpha}\right)\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{\left(1+2^{\alpha-\frac{1}{p}}\right)\overline{M_g}}{c}h^{2\alpha-\frac{1}{p}}\right]. \tag{67}$$

That is

$$\left\|\left(D^{2\alpha}_{b-;g}f\right)(b)\right\| \leq \left(\frac{2^\alpha\Gamma(2\alpha+1)}{2^\alpha-1}\right)\left[\frac{2\left(1+2^{-3\alpha}\right)\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{\left(1+2^{\alpha-\frac{1}{p}}\right)\overline{M_g}}{c}h^{2\alpha-\frac{1}{p}}\right], \tag{68}$$

$\forall\, b \in \mathbb{R}_-$, $\forall\, h > 0$. I.e., it holds

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D^{2\alpha}_{b-;g}f\right)(b)\right\| \leq \left(\frac{2^\alpha\Gamma(2\alpha+1)}{2^\alpha-1}\right) \times \left[\frac{2\left(1+2^{-3\alpha}\right)\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{\left(1+2^{\alpha-\frac{1}{p}}\right)\overline{M_g}}{c}h^{2\alpha-\frac{1}{p}}\right], \tag{69}$$

$\forall\, h > 0$, under $\frac{1}{4p} < \alpha < 1$. Again from (26), we get

$$\left\| \left(D_{b-;g}^{3\alpha} f \right)(b) \right\| \leq \frac{1}{\Gamma(2\alpha+1)\, D} \left[h^{2\alpha} \|B\| + 2^{2\alpha} h^{2\alpha} \|A\| \right]$$

(70)
$$= \frac{h^{2\alpha}\Gamma(3\alpha+1)}{2^{2\alpha} h^{5\alpha} (2^{\alpha}-1)} \left[\|B\| + 2^{2\alpha} \|A\| \right]$$

$$\leq \left(\frac{h^{2\alpha}\Gamma(3\alpha+1)}{2^{2\alpha} h^{5\alpha} (2^{\alpha}-1)} \right) \left[2 \left\|\|f\|\right\|_{\infty,\mathbb{R}_-} + \frac{2^{4\alpha-\frac{1}{p}} h^{4\alpha-\frac{1}{p}}}{c} \overline{M_g} \right.$$
$$\left. + 2^{2\alpha+1} \left\|\|f\|\right\|_{\infty,\mathbb{R}_-} + \frac{2^{2\alpha} h^{4\alpha-\frac{1}{p}}}{c} \overline{M_g} \right]$$

(71)
$$= \frac{\Gamma(3\alpha+1)}{2^{2\alpha}(2^{\alpha}-1)} \left[\frac{\left(2+2^{2\alpha+1}\right) \left\|\|f\|\right\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}} + \frac{\left(2^{2\alpha}+2^{4\alpha-\frac{1}{p}}\right)}{c} \overline{M_g} h^{\alpha-\frac{1}{p}} \right]$$

$$= \frac{\Gamma(3\alpha+1)}{(2^{\alpha}-1)} \left[\frac{2\left(1+2^{-2\alpha}\right) \left\|\|f\|\right\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}} + \frac{\left(1+2^{2\alpha-\frac{1}{p}}\right)}{c} \overline{M_g} h^{\alpha-\frac{1}{p}} \right].$$

That is

(72)
$$\left\| \left(D_{b-;g}^{3\alpha} f \right)(b) \right\| \leq \left(\frac{\Gamma(3\alpha+1)}{2^{\alpha}-1} \right) \left[\frac{2\left(1+2^{-2\alpha}\right) \left\|\|f\|\right\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}} + \frac{\left(1+2^{2\alpha-\frac{1}{p}}\right)}{c} \overline{M_g} h^{\alpha-\frac{1}{p}} \right],$$

$\forall\, b \in \mathbb{R}_-$, $\forall\, h > 0$. I.e., it holds

$$\sup_{b\in\mathbb{R}_-} \left\| \left(D_{b-;g}^{3\alpha} f \right)(b) \right\| \leq \left(\frac{\Gamma(3\alpha+1)}{2^{\alpha}-1} \right)$$

(73)
$$\times \left[\frac{2\left(1+2^{-2\alpha}\right) \left\|\|f\|\right\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}} + \frac{\left(1+2^{2\alpha-\frac{1}{p}}\right)}{c} \overline{M_g} h^{\alpha-\frac{1}{p}} \right],$$

$\forall\, h > 0$, $\frac{1}{4p} < \alpha < 1$. Call

(74)
$$\mu := 2\left(1+2^{-3\alpha}\right) \left\|\|f\|\right\|_{\infty,\mathbb{R}_-},$$
$$\theta := \frac{\left(1+2^{\alpha-\frac{1}{p}}\right)\overline{M_g}}{c},$$

both are greater than zero. We consider the function

(75)
$$y(h) = \mu h^{-2\alpha} + \theta h^{2\alpha-\frac{1}{p}}, \quad \forall\, h > 0.$$

We have

(76)
$$y'(h) = -2\alpha\mu h^{-2\alpha-1} + \left(2\alpha - \frac{1}{p}\right) \theta h^{2\alpha-\frac{1}{p}-1} = 0,$$

then

$$\left(2\alpha - \frac{1}{p}\right) \theta h^{2\alpha-\frac{1}{p}-1} = 2\alpha\mu h^{-2\alpha-1},$$

i.e.,

$$\left(2\alpha-\frac{1}{p}\right)\theta h^{4\alpha-\frac{1}{p}}=2\alpha\mu,$$

with a unique solution

(77) $$h_0:=h_{crit.no.}=\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{1}{4\alpha-\frac{1}{p}}}$$

(assuming $\frac{1}{2p}<\alpha<1$). We have that

(78) $$y''(h)=2\alpha(2\alpha+1)\mu h^{-2\alpha-2}+\left(2\alpha-\frac{1}{p}\right)\left(2\alpha-\frac{1}{p}-1\right)\theta h^{2\alpha-\frac{1}{p}-2}.$$

We see that

$$y''(h_0)=2\alpha(2\alpha+1)\mu\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{-2\alpha-2}{4\alpha-\frac{1}{p}}}$$

$$+\left(2\alpha-\frac{1}{p}\right)\left(2\alpha-\frac{1}{p}-1\right)\theta\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{2\alpha-\frac{1}{p}-2}{4\alpha-\frac{1}{p}}}$$

(79) $$=\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{-2\alpha-2}{4\alpha-\frac{1}{p}}}\left[2\alpha(2\alpha+1)\mu+2\alpha\mu\left(2\alpha-\frac{1}{p}-1\right)\right]$$

$$=2\alpha\mu\left(\frac{\left(2\alpha-\frac{1}{p}\right)\theta}{2\alpha\mu}\right)^{\left(\frac{2(\alpha+1)}{4\alpha-\frac{1}{p}}\right)}\left(4\alpha-\frac{1}{p}\right)>0.$$

Therefore, y has a global minimum at

$$h_0=\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{1}{4\alpha-\frac{1}{p}}},$$

which is

(80) $$y(h_0)=\mu\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{-2\alpha}{4\alpha-\frac{1}{p}}}+\theta\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{2\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}}$$

$$=\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{-2\alpha}{4\alpha-\frac{1}{p}}}\left[\mu+\theta\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right]$$

$$=\mu\left(\frac{\left(2\alpha-\frac{1}{p}\right)\theta}{2\alpha\mu}\right)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)}\left(1+\frac{2\alpha}{2\alpha-\frac{1}{p}}\right)$$

$$= \frac{\left(4\alpha - \frac{1}{p}\right)}{(2\alpha)^{\left(\frac{2\alpha}{4\alpha - \frac{1}{p}}\right)}} \left(2\alpha - \frac{1}{p}\right)^{\frac{-2\alpha + \frac{1}{p}}{4\alpha - \frac{1}{p}}} \mu^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \theta^{\left(\frac{2\alpha}{4\alpha - \frac{1}{p}}\right)}. \tag{81}$$

That is

$$y(h_0) = \frac{\left(4\alpha - \frac{1}{p}\right)\left(2\alpha - \frac{1}{p}\right)^{\left(\frac{-2\alpha + \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)}}{(2\alpha)^{\left(\frac{2\alpha}{4\alpha - \frac{1}{p}}\right)}} \mu^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \theta^{\left(\frac{2\alpha}{4\alpha - \frac{1}{p}}\right)}. \tag{82}$$

Therefore, we derive (see (69))

$$\sup_{b \in \mathbb{R}_-} \left\| \left(D_{b-;g}^{2\alpha} f\right)(b) \right\| \leq \left(\frac{2^\alpha \Gamma(2\alpha)}{2^\alpha - 1}\right) \left(\frac{2\alpha}{2\alpha - \frac{1}{p}}\right)^{\frac{\left(2\alpha - \frac{1}{p}\right)}{\left(4\alpha - \frac{1}{p}\right)}} \left(4\alpha - \frac{1}{p}\right)$$

$$\times \left(2\left(1 + 2^{-3\alpha}\right)\right)^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \left(\frac{\left(1 + 2^{\alpha - \frac{1}{p}}\right)}{\Gamma(4\alpha)\left(q(4\alpha - 1) + 1\right)^{\frac{1}{q}}}\right)^{\left(\frac{2\alpha}{4\alpha - \frac{1}{p}}\right)} \left\|\left\| f \right\|\right\|_{\infty, \mathbb{R}_-}^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)}$$

$$\times \left(\sup_{b \in \mathbb{R}_-} \left\|\left\| D_{b-;g}^{4\alpha} f \right\|\right\|_{p, \mathbb{R}_-}\right)^{\left(\frac{2\alpha}{4\alpha - \frac{1}{p}}\right)} < \infty, \tag{83}$$

where $\frac{1}{2p} < \alpha < 1$. Call

$$\xi := 2\left(1 + 2^{-2\alpha}\right) \left\|\left\| f \right\|\right\|_{\infty, \mathbb{R}_-},$$

$$\psi := \frac{\left(1 + 2^{2\alpha - \frac{1}{p}}\right)}{c} \overline{M_g}, \tag{84}$$

both are greater than zero. We consider the function

$$\gamma(h) := \xi h^{-3\alpha} + \psi h^{\alpha - \frac{1}{p}}, \quad \forall\, h > 0. \tag{85}$$

We have

$$\gamma'(h) = -3\alpha \xi h^{-3\alpha - 1} + \left(\alpha - \frac{1}{p}\right) \psi h^{\alpha - \frac{1}{p} - 1} = 0,$$

then

$$\left(\alpha - \frac{1}{p}\right) \psi h^{\alpha - \frac{1}{p} - 1} = 3\alpha \xi h^{-3\alpha - 1}$$

and

$$\left(\alpha - \frac{1}{p}\right) \psi h^{4\alpha - \frac{1}{p}} = 3\alpha \xi,$$

with unique solution

$$h_0 := h_{crit.no.} = \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi}\right)^{\frac{1}{4\alpha - \frac{1}{p}}} \tag{86}$$

(assuming $\frac{1}{p} < \alpha < 1$). We have that

(87) $$\gamma''(h) = 3\alpha(3\alpha+1)\xi h^{-3\alpha-2} + \left(\alpha - \frac{1}{p}\right)\left(\alpha - \frac{1}{p} - 1\right)\psi h^{\alpha-\frac{1}{p}-2}.$$

We observe

$$\gamma''(h_0) = 3\alpha(3\alpha+1)\xi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha-2}{4\alpha-\frac{1}{p}}\right)}$$

$$+\left(\alpha-\frac{1}{p}\right)\left(\alpha-\frac{1}{p}-1\right)\psi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{\alpha-\frac{1}{p}-2}{4\alpha-\frac{1}{p}}\right)}$$

(88) $$=\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha-2}{4\alpha-\frac{1}{p}}\right)}\left[3\alpha(3\alpha+1)\xi+\left(\alpha-\frac{1}{p}-1\right)3\alpha\xi\right]$$

$$=3\alpha\xi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha-2}{4\alpha-\frac{1}{p}}\right)}\left(4\alpha-\frac{1}{p}\right)>0.$$

Therefore, y has a global minimum at

$$h_0=\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\frac{1}{4\alpha-\frac{1}{p}}},$$

which is

(89) $$\gamma(h_0)=\xi h_0^{-3\alpha}+\psi h_0^{\alpha-\frac{1}{p}}=h_0^{-3\alpha}\left(\xi+\psi h_0^{4\alpha-\frac{1}{p}}\right)$$

$$=\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha}{4\alpha-\frac{1}{p}}\right)}\left(\xi+\psi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)\right)$$

$$=\xi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha}{4\alpha-\frac{1}{p}}\right)}\left(\frac{4\alpha-\frac{1}{p}}{\alpha-\frac{1}{p}}\right).$$

That is

(90) $$\gamma(h_0)=\xi\left(\frac{4\alpha-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha}{4\alpha-\frac{1}{p}}\right)}$$

$$=\left(\frac{4\alpha-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)\xi^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\left(\frac{\left(\alpha-\frac{1}{p}\right)}{3\alpha}\right)^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}\psi^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}.$$

I.e., we have found

(91) $$\gamma(h_0) = \frac{\left(4\alpha - \frac{1}{p}\right)}{\left(\alpha - \frac{1}{p}\right)^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} (3\alpha)^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}}\right)}} \xi^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \psi^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}}\right)}.$$

We have proved that (see (73))

(92)
$$\sup_{b \in \mathbb{R}_-} \left\| \left(D_{b-;g}^{3\alpha} f\right)(b)\right\| \leq \left(\frac{\left(4\alpha - \frac{1}{p}\right)\Gamma(3\alpha)}{2^{\alpha} - 1}\right)\left(\frac{3\alpha}{\alpha - \frac{1}{p}}\right)^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)}$$
$$\times \left(2\left(1 + 2^{-2\alpha}\right)\right)^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \left(\frac{\left(1 + 2^{2\alpha - \frac{1}{p}}\right)}{\Gamma(4\alpha)\left(q(4\alpha - 1) + 1\right)^{\frac{1}{q}}}\right)^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}}\right)} \||f\||_{\infty,\mathbb{R}_-}^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)}$$
$$\times \left(\sup_{b \in \mathbb{R}_-} \left\|\left\|D_{b-;g}^{4\alpha} f\right\|\right\|_{p,\mathbb{R}_-}\right)^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}}\right)} < \infty,$$

where $\frac{1}{p} < \alpha < 1$. The theorem is proved. □

We give an application when $\alpha = \frac{1}{2}$ and $g(t) = e^t|_{\mathbb{R}_-}$.

Corollary 2.1. *Let $f \in C^1(\mathbb{R}_-, X)$ with $\||f\||_{\infty,\mathbb{R}_-}$, $\left\|\left\|(f \circ \ln)' \circ e^t\right\|\right\|_{\infty,\mathbb{R}_-} < \infty$, where $(X, \|\cdot\|)$ is a Banach space. For $k = 1, 2, 3$, we assume that $D_{b-;e^t}^{k\frac{1}{2}} f \in C^1((-\infty, b], X)$ and $D_{b-;e^t}^{4\frac{1}{2}} f \in C((-\infty, b], X)$, $\forall\, b \in \mathbb{R}_-$. We further assume that*

(93) $$\left\|\left\|D_{b-;e^t}^{4\frac{1}{2}} f(t)\right\|\right\|_{\infty,\mathbb{R}_-^2} < \infty,$$

where $(b, t) \in \mathbb{R}_-^2$. Then,

(94) $$\sup_{b \in \mathbb{R}_-} \left\|\left(D_{b-;e^t}^{2\frac{1}{2}} f\right)(b)\right\| \leq \left(\frac{\sqrt{12 + 6\sqrt{2}}}{\sqrt{2} - 1}\right) \||f\||_{\infty,\mathbb{R}_-}^{\frac{1}{2}} \left(\left\|\left\|D_{b-;e^t}^{4\frac{1}{2}} f(t)\right\|\right\|_{\infty,\mathbb{R}_-^2}\right)^{\frac{1}{2}} < \infty$$

and

(95) $$\sup_{b \in \mathbb{R}_-} \left\|\left(D_{b-;e^t}^{3\frac{1}{2}} f\right)(b)\right\| \leq \left(\frac{9\sqrt{\pi}}{\left(2 - \sqrt{2}\right)\sqrt[4]{2}\left(\sqrt[4]{3}\right)^3}\right) \||f\||_{\infty,\mathbb{R}_-}^{\frac{1}{4}} \left(\left\|\left\|D_{b-;e^t}^{4\frac{1}{2}} f(t)\right\|\right\|_{\infty,\mathbb{R}_-^2}\right)^{\frac{3}{4}} < \infty.$$

That is $\sup_{b \in \mathbb{R}_-} \left\|\left(D_{b-;e^t}^{2\frac{1}{2}} f\right)(b)\right\|, \sup_{b \in \mathbb{R}_-} \left\|\left(D_{b-;e^t}^{3\frac{1}{2}} f\right)(b)\right\| < \infty.$

Proof. By Theorem 2.2. □

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Research Article

Some New Fourier and Jackson–Nikol'skii Type Inequalities in Unbounded Orthonormal Systems

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ABSTRACT. We consider the generalized Lorentz space $L_{\psi,q}$ defined via a continuous and concave function ψ and the Fourier series of a function with respect to an unbounded orthonormal system. Some new Fourier and Jackson-Nikol'skii type inequalities in this frame are stated, proved and discussed. In particular, the derived results generalize and unify several well-known results but also some new applications are pointed out.

Keywords: Inequalities, generalized Lorentz spaces, unbounded orthonormal system, Fourier inequalities, Jackson–Nikol'skii inequality.

2020 Mathematics Subject Classification: 42A16, 42B05, 26D15, 26D20, 46E30.

1. INTRODUCTION

Let the function ψ be continuous and concave by $[0,1]$, $\psi(0)=0$ and $0<q\leqslant\infty$. Such functions are called Φ functions. The generalized Lorentz space $L_{\psi,q}$ is the set of measurable functions f on $[0,1]$ for which

$$\|f\|_{\psi,q}:=\left(\int_0^1 f^{*^q}(t)\psi^q(t)\frac{dt}{t}\right)^{1/q}<\infty,$$

where f^* is the non-increasing rearrangement of the function $|f|$ (see e.g. [36]).
For a given function $\psi(t), t\in[0,1]$, we define

$$\alpha_\psi:=\underline{\lim}_{t\to 0}\frac{\psi(2t)}{\psi(t)},\quad \beta_\psi:=\overline{\lim}_{t\to 0}\frac{\psi(2t)}{\psi(t)}.$$

It is known that $1\leqslant\alpha_\psi\leqslant\beta_\psi\leqslant 2$ (see e.g. [35]) .
Note that for $\psi(t)=t^{1/p}$, the space $L_{\psi,q}$ coincides with the Lorentz space $L_{p,q}$, $0<q,p<\infty$, which consists of all functions f such that (see e.g. [38, p. 228])

$$\|f\|_{p,q}:=\left(\int_0^1 f^{*^q}(t)t^{\frac{q}{p}-1}dt\right)^{1/q}.$$

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In particular, for the case $p = q$, we have the usual Lebesgue space with the norm (quasi-norm if $0 < q < 1$)

$$\|f\|_q := \left(\int_0^1 |f(x)|^q dx \right)^{1/q}, \quad 0 < q < \infty.$$

Let $q, p \in (0, +\infty)$ and $\alpha \in R = (-\infty, +\infty)$. The Lorentz-Zygmund space $L_{p,q}(\log L)^\alpha$ is the set of all functions f measurable on $[0, 1]$ for which (see e.g. [37])

$$\|f\|_{p,q,\alpha} := \left\{ \int_0^1 (f^*(t))^q (1 + |\log t|)^{\alpha q} t^{\frac{q}{p}-1} dt \right\}^{\frac{1}{q}} < +\infty.$$

For A, B the notation $A \asymp B$ means that there exits positive constants C_1, C_2 such that $C_1 A \leqslant B \leqslant C_2 A$.

We consider the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}} \subset L_2[0, 1]$ (see [22, p. 58]) satisfying the condition

$$\|\varphi_n\|_r := \left(\int_0^1 |\varphi_n(x)|^r dx \right)^{\frac{1}{r}} \leqslant M_n, \quad n \in \mathbb{N} \tag{1}$$

for some $r \in (2, +\infty]$. Here, we assume that $\{M_n\}$ is a non-decreasing sequence.

Let $\hat{f}(n)$ be the Fourier coefficients of the function f with respect to the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$.

J. Marcinkiewicz and A. Zygmund [22] proved some inequalities for the sums of the Fourier coefficients of the orthogonal system $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfying condition (1) and norms of the function $f \in L_p$, $1 < p < \infty$. Later, many authors investigated this problem in other functional spaces (for example, see [3], [6], [7], [8], [11], [13], [21], [30], [32], [33], [42] and bibliographic references in them).

In particular, the following statement is known (see S.V. Bochkarev [11]):

Theorem 1.1. *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal system of complex-valued functions*

$$\|\varphi_n\|_\infty \leqslant M, \ n = 1, 2, ... \tag{2}$$

for some $M < \infty$. Then, for any $2 < q \leqslant \infty$ and $n = 2, 3, ...$, the following inequality holds:

$$\left[\sum_{k=1}^{n} (\hat{f}^*(k))^2 \right]^{\frac{1}{2}} \leqslant CM \|f\|_{2,q} (\log n)^{\frac{1}{2}-\frac{1}{q}}.$$

In the case $q = \infty$, Theorem 1.1 was previously proved by V.I. Ovchinnikov, V.D. Raspopova and V.A. Rodin [32].

In the case when $\{\varphi_n\}_{n\in\mathbb{N}}$ is a trigonometric system, in the Lorentz-Zygmund space $L_{2,q,}(\log L)^\alpha$ H. Oba , E. Sato and Y. Sato [30] stated and proved the following:

Theorem 1.2. *Let $2 < q \leqslant \infty$, $n \geqslant 3$ and $\alpha \in \mathbb{R}$. Then the following inequality holds:*

$$\left[\sum_{k=1}^{n} (\hat{f}^*(k))^2 \right]^{\frac{1}{2}} \leqslant C A_n \|f\|_{2,q,\alpha}$$

for some constant C which is independent of n and f, and A_n is as follows:

$$A_n = \begin{cases} (\log n)^{\frac{1}{2}-\frac{1}{q}-\alpha}, & \text{if } \alpha < \frac{1}{2} - \frac{1}{q}, \\ (\log(\log n))^\alpha, & \text{if } \alpha = \frac{1}{2} - \frac{1}{q}, \\ 1, & \text{if } \alpha > \frac{1}{2} - \frac{1}{q}. \end{cases}$$

A generalization of this theorem for the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfying condition (2) was proved by L.R.Ya. Doktorski (see [13]). Moreover, N. Tleukhanova and G. Mussabaeva [42] for the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfying condition (2) proved the inequality

$$\sup_{n\in\mathbb{N}} \frac{1}{n^{1/2}(\log(n+1))^{\frac{1}{2}-\frac{1}{q}}} \sum_{k=1}^{n} \hat{f}^*(k) \leqslant C\|f\|_{2,q} \tag{3}$$

for any function $f \in L_{2,q}, 2 < q \leqslant \infty$.

Most results concerning Fourier inequalities are derived for bounded orthonormal systems. However, for several applications it is also important to derive such results for unbounded orthonormal systems like those described in our final Remark 4.11. One aim of this paper is to further complement our recent research in this direction (see [6], [7] and [8]) and also prove and discuss some new related Nikol'skii type inequalities of this type. Let us first mention that in [3] for an unbounded orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$, the following statement was proved (for the case $\alpha = 0$, see [2]).

Theorem 1.3. *Let the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2, +\infty]$ satisfy the condition (1). Then, for any function $L_{2,q}(\log L)^{\alpha}, 2 < q \leqslant \infty, \alpha < \frac{1}{2} - \frac{1}{q}, n \in \mathbb{N}$, the following inequality holds:*

$$\left[\sum_{k=1}^{n} |\hat{f}(k)|^2\right]^{\frac{1}{2}} \leqslant C\|f\|_{2,q,\alpha}\left[\ln(1+\sum_{j=1}^{n} M_j^2)\right]^{\frac{1}{2}-\frac{1}{q}-\alpha}.$$

For a trigonometric polynomial

$$T_n(x) = \sum_{k=-n}^{n} a_k e^{ikx}, n \in \mathbb{N}$$

the following Jackson–Nikol'skii inequality is well known (see [17], [27])

$$\|T_n\|_q \leqslant 2n^{1/p}\|T_n\|_p \tag{4}$$

for $1 \leqslant p < q \leqslant \infty$. This inequality is also called the inequality of different metrics for a trigonometric polynomial.

For case $0 < p < q \leqslant \infty$, inequality (4) was proved in [16] and [10]. Moreover, for $p = 0 < q < \infty$, it was proved by V.V. Arestov [10].

Nowadays, there are various generalizations of the Jackson-Nikol'skii inequality (see [5], [12], [29] and the bibliography therein). One of the generalizations is its extension to polynomials in orthonormal systems of functions. In particular, M.F. Timan [40] proved the following statement:

Theorem 1.4. *Let $1 \leqslant p \leqslant 2, p < q \leqslant \infty$ and $\{\varphi_n\}_{n=1}^{\infty}$ be a uniformly bounded sequence of orthonormal systems of functions. Then for the polynomial*

$$f_n(x) = \sum_{k=1}^{n} c_k \varphi_k(x), n \in \mathbb{N},$$

holds the following inequality:

$$\|f_n\|_q \leqslant Cn^{1/p-1/q}\|f_n\|_p. \tag{5}$$

A multidimensional version of inequality (5) in the spaces L_p was established by R.J. Nessel and G. Wilmes [25], [26]. The Jackson-Nikol'skii inequality for polynomials in a uniformly bounded system of functions in some symmetric spaces was proved by V.A. Rodin [34]. Moreover, L.R.Ya. Doktorski and D.Gendler [14] proved the inequality of different metrics for polynomials in a uniformly bounded orthonormal system of functions in the Lorentz–Zygmund

space. Jackson–Nikol'skii inequality is also known for polynomials in an unbounded orthonormal system of functions (see, for example, [19], [20], [23], [24]).
In this paper, we complement the results above by proving some new Fourier and Jackson-Nikol'skii type inequalities in the generalized Lorentz space $L_{\psi,q}$ and in unbounded systems satisfying (1).
In Section 2, we present and discuss our main results. The announced generalizations and unifications of Fourier type inequalities can be found in Theorem 2.1 while the corresponding results concerning Jackson-Nikol'skii type inequalities are given in Theorem 2.2. These detailed proofs are presented; in Section 3 and Section 4 is reserved for some concluding remarks and result (see Proposition 4.1).

2. The main results

We denote by SVL (slowly varing) the set of all non-negative functions on $[0,1]$ of $\psi(t)$ for which $(\log 2/t)^{\varepsilon}\psi(t) \uparrow +\infty$ and $(\log 2/t)^{-\varepsilon}\psi(t) \downarrow 0$ for $t \downarrow 0$ (see e.g. [8]).
First, we formulate the following generalization and unification of Theorem 1.1, Theorem 1.2 for the case $\alpha < \frac{1}{2} - \frac{1}{q}$, assertion 1) of Theorem 1.3 and inequality (3):

Theorem 2.1. *Let ψ a function satisfying the conditions $1 < \alpha_\psi = \beta_\psi = 2^{1/2}$, $\frac{t^{1/2}}{\psi(t)} \in SVL$,*

$$\sup_{t\in(0,1]} \frac{\psi(t)}{t^{1/2}} < \infty,$$

and assume that the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2,+\infty]$ satisfies the condition (1). Then, for any function $f \in L_{\psi,q}$, $2 < q \leqslant \infty$, the following inequality holds:

$$\Big[\sum_{k\in A} |\hat{f}(k)|^2\Big]^{\frac{1}{2}} \leqslant C\|f\|_{\psi,q}\Big[\ln(1+\sum_{j\in A} M_j^2)\Big]^{\frac{1}{2}-\frac{1}{q}} \frac{\sqrt{(1+\sum_{j\in A} M_j^2)^{-1}}}{\psi((1+\sum_{j\in A} M_j^2)^{-1})},$$

where A is a non-empty set in $\mathbb{N}$ and C is positive constant which depends only on q and r.

Corollary 2.1. *Let ψ be a function satisfying the conditions of Theorem 2.1 and the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2,+\infty]$ satisfying the condition (2). Then, for any function $f \in L_{\psi,q}, 2 < q \leqslant \infty$, we have the inequality*

$$\Big[\sum_{k=1}^{|A|}(\hat{f}^*(k))^2\Big]^{\frac{1}{2}} \leqslant C\|f\|_{\psi,q}\Big[\log(1+|A|M^2)\Big]^{\frac{1}{2}-\frac{1}{q}} \frac{\sqrt{(1+|A|M^2)^{-1}}}{\psi((1+|A|M^2)^{-1})},$$

where $|A|$ is the number of elements in the set $A \subset \mathbb{N}$.

Corollary 2.2. *Let ψ be a function satisfying the conditions of Theorem 2.1 and let the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2,+\infty]$ satisfying the condition (2). Then, for any function $f \in L_{\psi,q}, 2 < q \leqslant \infty$, the following inequality holds:*

$$\sup_{n\in\mathbb{N}} n^{-1/2}\Big[\log(1+nM^2)\Big]^{\frac{1}{q}-\frac{1}{2}} \left(\frac{\sqrt{(1+nM^2)^{-1}}}{\psi((1+nM^2)^{-1})}\right)^{-1} \sum_{k=1}^{n} \hat{f}^*(k) \leqslant C\|f\|_{\psi,q}.$$

Remark 2.1. *In the case $\psi(t) = t^{1/2}$ from Corollary 2.1 and Corollary 2.2, we accordingly obtain the statement of Theorem 1.1 and inequality (3).*

Remark 2.2. *In the case $\psi(t) = t^{1/2}(1+|\log t|)^{\alpha}$ and $\{\varphi_n\}$ the trigonometric system from Corollary 2.2, we obtain the statement in Theorem 1.2 for $\alpha < \frac{1}{2} - \frac{1}{q}$.*

Remark 2.3. *If $\psi(t) = t^{1/2}(1+|\log t|)^\alpha$ and the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2,+\infty]$ satisfies condition (2), then from Corollary 2.2, we obtain assertion 1) of Theorem 1.3.*

Remark 2.4. *In the case $\psi(t) = t^{1/2}$ and $A = \{1, ..., n\}$, it was proved in [11] that the inequality in Corollary 2.1 is exact for the multiplicative Crestenson–Levy system . This fact for a trigonometric system in the Lorentz–Zygmund space $L_{2,q,}(\log L)^\alpha$ was proved in [30]. By also using Theorem 2 in [5], we obtain the following statement:*

Corollary 2.3. *Let ψ be a function satisfying the conditions of Theorem 2.1, $2 < q < \infty$ and $\{e^{inx}\}_{n\in\mathbb{Z}}$ be the trigonometric system. Then*

$$\sup_{f\neq 0} \frac{\left(\sum_{k=1}^{2n+1}(\hat{f}^*(k))^2\right)^{1/2}}{\|f\|_{\psi,q}} \asymp \frac{\sqrt{(1+n)^{-1}}}{\psi((1+n)^{-1})}\left[\log(1+n)\right]^{\frac{1}{2}-\frac{1}{q}}.$$

Next, we state a Jackson–Nikol'skii type inequality which generalizes some results for the trigonometric system in [17] and [27], [28] (for a complementary bibliography see also [4], [5]).

Theorem 2.2. *Let the function ψ satisfy the conditions $1 < \alpha_\psi = \beta_\psi = 2^{1/2}$, $\frac{\psi(t)}{t^{1/2}} \in SVL$,*

$$\sup_{t\in(0,1]} \frac{t^{1/2}}{\psi(t)} < \infty, \tag{6}$$

let the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2,+\infty]$ satisfy the condition (1) and $f_n(x) = \sum_{k=1}^n c_k\varphi_k(x)$.
1) If $1 < q < 2$, then

$$\|f_n\|_{\psi,q} \leqslant C\left(\frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\psi((1+\sum_{j=1}^n M_j^2)^{-1})}\right)^{-1}\left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{q}-\frac{1}{2}}\|f_n\|_2$$

for some constant C depending only on q.
2) If $1 < p < 2 < q < +\infty$, then

$$\|f_n\|_{\psi,p} \leqslant C(p,q)\|f_n\|_{\psi,q}\left(\log(1+\sum_{k=1}^n M_k^2)\right)^{\frac{1}{p}-\frac{1}{q}}$$

for some constant C depending only on p and q.
3) If $2 < p < q < +\infty$, then

$$\|f_n\|_{\psi,p} \leqslant C(p,q)\|f_n\|_{\psi,q}\left(\log(1+\sum_{k=1}^n M_k^2)\right)^{\frac{1}{p}-\frac{1}{q}}$$

for some constant C depending only on p and q.

3. Proofs

Proof of Theorem 2.1. Let $f \in L_{\psi,q}$. This function can be represented as $f(x) = f_1(x) + f_2(x)$, where

$$f_1(x) = \begin{cases} f(x), & \text{when } |f(x)| \leqslant f^*(\tau), \\ 0, & \text{when } |f(x)| > f^*(\tau), \end{cases}$$
$$f_2(x) = f(x) - f_1(x), \quad 0 < \tau < 1.$$

Then, by the Minkowski inequality, we have that

$$\Big[\sum_{k\in A}|\hat{f}(k)|^2\Big]^{1/2}\leqslant\Big[\sum_{k\in A}|\hat{f}_1(k)|^2\Big]^{1/2}+\Big[\sum_{k\in A}|\hat{f}_2(k)|^2\Big]^{1/2}. \tag{7}$$

Now, we prove that each of the functions $f_i, i=1,2$, satisfies the inequality

$$\Big[\sum_{k\in A}|\hat{f}_i(k)|^2\Big]^{1/2}\leqslant C(q,r)\Big(\ln(1+\sum_{k\in A}M_k^2)\Big)^{\frac{1}{2}-\frac{1}{q}}\frac{\sqrt{(1+\sum_{j\in A}M_j^2)^{-\frac{r}{2(r-2)}}}}{\psi((1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}})}\|f\|_{\psi,q}. \tag{8}$$

According to the Parseval equality for an orthonormal system and Hölder's inequality for $\theta=\frac{q}{2}>1$, $\frac{1}{\theta}+\frac{1}{\theta'}=1$ for the function f_1, we find that

$$\sum_{k\in A}|\hat{f}_1(k)|^2\leqslant\|f_1\|_2^2\leqslant\int_\tau^1 {f^*}^2(t)dt\leqslant\|f\|_{\psi,q}^2\Big[\int_\tau^1\Big(\frac{t^{1/2}}{\psi(t)}\Big)^{2\theta'}t^{-1}dt\Big]^{\frac{1}{\theta'}}. \tag{9}$$

Since $\frac{t^{1/2}}{\psi(t)}\in SVL$, then $\frac{t^{1/2}}{\psi(t)}\log^\varepsilon 2/t\leqslant\frac{\tau^{1/2}}{\psi(\tau)}\log^\varepsilon 2/\tau$ for $t\in[\tau,1]$, $\forall\varepsilon>0$. Therefore

$$\Big[\int_\tau^1\Big(\frac{t^{1/2}}{\psi(t)}\Big)^{2\theta'}t^{-1}dt\Big]^{\frac{1}{\theta'}}\leqslant\Big(\frac{\tau^{1/2}}{\psi(\tau)}\Big)^2\log^{2\varepsilon}2/\tau\Big[\int_\tau^1(\log 2/t)^{-2\varepsilon\theta'}t^{-1}dt\Big]^{\frac{1}{\theta'}}. \tag{10}$$

Choose the number $\varepsilon\in(0,\frac{1}{2}-\frac{1}{q})$. Then, $1-2\varepsilon\theta'>0$ so that

$$\int_\tau^1(\log 2/t)^{-2\varepsilon\theta'}t^{-1}dt=\frac{1}{1-2\varepsilon\theta'}\Big[(\log 2/t)^{1-2\varepsilon\theta'}-1\Big].$$

Therefore, from inequality (10), it follows that

$$\Big[\int_\tau^1\Big(\frac{t^{1/2}}{\psi(t)}\Big)^{2\theta'}t^{-1}dt\Big]^{\frac{1}{\theta'}}\leqslant\frac{1}{1-2\varepsilon\theta'}\Big(\frac{\tau^{1/2}}{\psi(\tau)}\Big)^2(\log 2/t)^{\frac{1}{\theta'}}. \tag{11}$$

Now by using inequalities (9) and (11), we obtain that

$$\Big(\sum_{k\in A}|\hat{f}_1(k)|^2\Big)^{\frac{1}{2}}\leqslant\frac{1}{1-2\varepsilon\theta'}\frac{\tau^{1/2}}{\psi(\tau)}(\log 2/\tau)^{\frac{1}{2}-\frac{1}{q}}\|f\|_{\psi,q}. \tag{12}$$

In this formula, we put $\tau=(1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}}$. Then, for the function f_1 from (12), we can conclude that

$$\Big(\sum_{k\in A}|\hat{f}_1(k)|^2\Big)^{\frac{1}{2}}$$
$$\leqslant C\Big(\ln(1+\sum_{k\in A}M_k^2)\Big)^{\frac{1}{2}-\frac{1}{q}}\frac{\sqrt{(1+\sum_{j\in A}M_j^2)^{-\frac{r}{2(r-2)}}}}{\psi((1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}})}\Big(\ln(1+\sum_{k\in A}M_k^2)\Big)^{\frac{1}{2}-\frac{1}{q}}\|f\|_{\psi,q},$$

so (8) holds with $i=1$. For the function $f_2\in L_{r'}$ by the definition of the coefficient expansions and Hölder's inequality ($2<r<+\infty, r'=\frac{r}{r-1}$), we have that

$$|\hat{f}_2(k)|=\Big|\int_0^1 f_2(x)\varphi_k(x)dx\Big|\leqslant\|f_2\|_{r'}\|\varphi_k\|_r\leqslant M_k\|f\|_{r'}.$$

Hence,

$$\sum_{k\in A}|\hat{f}_2(k)|^2\leqslant\|f_2\|_{r'}^2\sum_{k\in A}M_k^2=\Big(\int_0^\tau {f^*}^{r'}(t)dt\Big)^{2/r'}\sum_{k\in A}M_k^2. \tag{13}$$

Since the function f^* is non-increasing and ψ is non-decreasing, then

$$\|f\|_{\psi,q} \geqslant \Big(\int\limits_{x/2}^{x} {f^*}^q(t)\psi^q(t)\frac{dt}{t}\Big)^{1/q}$$
$$\geqslant f^*(x)\psi(x/2)\Big(\int\limits_{x/2}^{x} \frac{dt}{t}\Big)^{1/q} = f^*(x)\psi(x/2)(\ln 2)^{1/q},\ x \in (0,1].$$

Therefore, from inequality (13), it follows that

$$\sum_{k\in A} |\hat{f}_2(k)|^2 \leqslant \|f\|^2_{\psi,q}\Big(\int\limits_0^{\tau} \psi^{-r'}(t/2)dt\Big)^{2/r'} \sum_{k\in A} M_k^2. \tag{14}$$

Since $\frac{t^{1/2}}{\psi(t)} \in SVL$, then

$$\Big(\int\limits_0^{\tau} \psi^{-r'}(t/2)dt\Big)^{2/r'} = \Big(\int\limits_0^{\tau}\Big(\frac{\sqrt{t/2}}{\psi(t/2)}\Big)^{r'}(t/2)^{-r'/2}dt\Big)^{2/r'}$$
$$\leqslant \Big(\frac{\sqrt{\tau/2}}{\psi(\tau/2)}\log^{\varepsilon}\frac{2}{\tau/2}\Big)^2\Big(\int\limits_0^{\tau}(\log\frac{2}{t/2})^{-\varepsilon r'}(t/2)^{-r'/2}dt\Big)^{2/r'}. \tag{15}$$

If $0<t<\tau$, then $(\log\frac{2}{t/2})^{-\varepsilon} < (\log\frac{2}{\tau/2})^{-\varepsilon}$, for $\varepsilon > 0$. Therefore, by using (15), we obtain that

$$\Big(\int\limits_0^{\tau} \psi^{-r'}(t/2)dt\Big)^{2/r'} \leqslant \Big(\frac{\sqrt{\tau/2}}{\psi(\tau/2)}\Big)^2\Big(\int\limits_0^{\tau}(t/2)^{-r'/2}dt\Big)^{2/r'}$$
$$= (\frac{2}{2-r'})^{2/r'}\Big(\frac{\sqrt{\tau/2}}{\psi(\tau/2)}\Big)^2(\tau/2)^{\frac{2}{r'}-1} = (\frac{2}{2-r'})^{2/r'}\Big(\frac{1}{\psi(\tau/2)}\Big)^2 2^{-\frac{2}{r'}}\tau^{\frac{2}{r'}}. \tag{16}$$

Now, it follows from inequalities (14) and (16) that

$$\Big(\sum_{k\in A} |\hat{f}_2(k)|^2\Big)^{1/2} \leqslant C\|f\|_{\psi,q}\frac{1}{\psi(\tau)}\tau^{\frac{1}{r'}}\Big(\sum_{k\in A} M_k^2\Big)^{1/2}.$$

In this formula, we put $\tau = (1+\sum_{j\in A} M_j^2)^{-\frac{r}{r-2}}$. Then

$$\Big(\sum_{k=1}^{n} |\hat{f}_2(k)|^2\Big)^{1`/2} \leqslant C\|f\|_{\psi,q}\frac{1}{\psi((1+\sum_{j=1}^n M_j^2)^{-\frac{r}{r-2}})}\Big(1+\sum_{j=1}^{n} M_j^2\Big)^{-\frac{r}{r'(r-2)}}\Big(\sum_{k=1}^{n} M_k^2\Big)^{1/2}$$
$$= C\frac{1}{\psi((1+\sum_{j=1}^n M_j^2)^{-\frac{r}{r-2}})}\Big(1+\sum_{j=1}^{n} M_j^2\Big)^{-\frac{r}{2(r-2)}}\|f\|_{\psi,q}.$$

Now, taking into account that $1/2-1/q>0$, we get from here that

$$\Big(\sum_{k\in A} |\hat{f}_2(k)|^2\Big)^{1`/2}$$
$$\leqslant C\frac{1}{\psi((1+\sum_{j\in A} M_j^2)^{-\frac{r}{r-2}})}\Big(1+\sum_{j\in A} M_j^2\Big)^{-\frac{r}{2(r-2)}}\Big(\log\Big(1+\sum_{j\in A} M_j^2\Big)\Big)^{1/2-1/q}\|f\|_{\psi,q},$$

so (8) holds also for $i = 2$. From inequalities (7) and (8), it follows that

$$\Big(\sum_{k\in A} |\hat{f}(k)|^2\Big)^{1`/2}$$
$$\leqslant C\frac{1}{\psi((1+\sum_{j\in A} M_j^2)^{-\frac{r}{r-2}})}\Big(1+\sum_{j\in A} M_j^2\Big)^{-\frac{r}{2(r-2)}}\Big(\log\Big(1+\sum_{j\in A} M_j^2\Big)\Big)^{1/2-1/q}\|f\|_{\psi,q}. \qquad (17)$$

Since $\frac{t^{1/2}}{\psi(t)} \in SVL$ and $\Big(1+\sum_{j\in A} M_j^2\Big)^{-\frac{r}{2(r-2)}} < \Big(1+\sum_{j\in A} M_j^2\Big)^{-1}$, then

$$\frac{\sqrt{\Big(1+\sum_{j\in A} M_j^2\Big)^{-\frac{r}{(r-2)}}}}{\psi((1+\sum_{j\in A} M_j^2)^{-\frac{r}{r-2}})}$$
$$\leqslant\frac{\sqrt{\Big(1+\sum_{j\in A} M_j^2\Big)^{-1}}}{\psi((1+\sum_{j\in A} M_j^2)^{-1})}\Big(\log\frac{2}{\Big(1+\sum_{j\in A} M_j^2\Big)^{-1}}\Big)^{-\varepsilon}\Big(\log\frac{2}{\Big(1+\sum_{j\in A} M_j^2\Big)^{-\frac{r}{(r-2)}}}\Big)^{\varepsilon}$$
$$\leqslant\frac{\sqrt{\Big(1+\sum_{j\in A} M_j^2\Big)^{-1}}}{\psi((1+\sum_{j\in A} M_j^2)^{-1})}\Big(\log 2\Big(1+\sum_{j\in A} M_j^2\Big)\Big)^{-\varepsilon}\Big(\frac{r}{r-2}\log 2\Big(1+\sum_{j\in A} M_j^2\Big)\Big)^{\varepsilon}$$
$$=\frac{r}{r-2}\frac{\sqrt{\Big(1+\sum_{j\in A} M_j^2\Big)^{-1}}}{\psi((1+\sum_{j\in A} M_j^2)^{-1})}. \qquad (18)$$

It follows from inequalities (17) and (18) that

$$\Big(\sum_{k\in A} |\hat{f}(k)|^2\Big)^{1`/2} \leqslant \frac{r}{r-2}\frac{\sqrt{\Big(1+\sum_{j\in A} M_j^2\Big)^{-1}}}{\psi((1+\sum_{j\in A} M_j^2)^{-1})}\Big(\log\Big(1+\sum_{j\in A} M_j^2\Big)\Big)^{1/2-1/q}\|f\|_{\psi,q}.$$

The proof is complete. □

Proof of Corollary 2.1. In view of the fact that $M_j = M$, $j = 1, 2, \ldots$ and the property of non-increasing rearrangement of numbers, it yields that

$$\sum_{k\in A} |\hat{f}(k)|^2 = \sum_{k=1}^{|A|} (\hat{f}^*(k))^2,$$

so the proof follows by just applying Theorem 2.1. □

Proof of Corollary 2.2. According to Hölder's inequality, we have that

$$\sum_{k=1}^{n} \hat{f}^*(k) \leqslant n^{1/2}\Big(\sum_{k=1}^{n} (\hat{f}^*(k))^2\Big)^{1/2}.$$

Therefore, the assertion of Corollary 2.2 follows by applying Corollary 2.1 with $A = \{1, 2, \ldots, n\}$. □

Proof of Corollary 2.3. For the set $A = \{-n, ..., -1, 0, 1, ..., n\}$ from Corollary 2.1, we get

$$\sup_{f\neq 0} \frac{\left(\sum_{k=1}^{2n+1}(\hat{f}^*(k))^2\right)^{1/2}}{\|f\|_{\psi,q}} \leqslant C\frac{\sqrt{(1+n)^{-1}}}{\psi((1+n)^{-1})}\Big[\log(1+n)\Big]^{\frac{1}{2}-\frac{1}{q}}.$$

To prove the reversed inequality, we consider the trigonometric polynomial

$$f_n(x) = \sum_{k=-n}^{n} a_k e^{ikx}.$$

Then, by using Theorem 2 in [5] for $\psi_1(t) = t^{1/2}$, $\tau_1 = 2$, $\psi_2(t) = \psi(t)$, $\tau_2 = q$, we have that

$$\sup_{f_n\neq 0} \frac{\|f_n\|_2}{\|f_n\|_{\psi,q}} \geqslant C\frac{\sqrt{(1+n)^{-1}}}{\psi((1+n)^{-1})}\Big[\log(1+n)\Big]^{\frac{1}{2}-\frac{1}{q}}.$$

Therefore

$$\sup_{f\neq 0} \frac{\left(\sum_{k=1}^{2n+1}(\hat{f}^*(k))^2\right)^{1/2}}{\|f\|_{\psi,q}} \geqslant \sup_{f_n\neq 0} \frac{\|f_n\|_2}{\|f_n\|_{\psi,q}} \geqslant C\frac{\sqrt{(1+n)^{-1}}}{\psi((1+n)^{-1})}\Big[\log(1+n)\Big]^{\frac{1}{2}-\frac{1}{q}}.$$

The proof is complete. □

Proof of Theorem 2.2. For the generalized Lorentz space $L_{\psi,q}$, we have the relation (see [2])

$$\|f\|_{\psi,q} \asymp \sup_{\|f\|_{\bar{\psi},q'}\leqslant 1} \left|\int_0^1 f(x)g(x)dx\right|, \tag{19}$$

where $\bar{\psi}(t) = \frac{t}{\psi(t)}$, $t \in (0,1]$, $1 < q < \infty$, $q' = \frac{q}{q-1}$. Since the system $\{\varphi_n\}$ is orthonormal, then

$$\int_0^1 f_n(x)g(x)dx = \sum_{k=1}^{n} c_k\hat{g}(k), \;\; g \in L_{\bar{\psi},q'}$$

for any $n \in \mathbb{N}$.
Note that condition (6) implies that

$$\sup_{t\in(0,1]} \frac{\bar{\psi}(t)}{t^{1/2}} < \infty.$$

By applying Hölder's inequality, Theorem 2.1, and Parseval's equality, we obtain that

$$\left|\int_0^1 f_n(x)g(x)dx\right| \leqslant \left(\sum_{k=1}^{n}|c_k|^2\right)^{1/2}\left(\sum_{k=1}^{n}|\hat{g}(k)|^2\right)^{1/2}$$
$$\leqslant C\frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\bar{\psi}((1+\sum_{j=1}^n M_j^2)^{-1})}\left(\log\Big(1+\sum_{j=1}^{n} M_j^2\Big)\right)^{1/2-1/q'}\|g\|_{\bar{\psi},q'}\|f_n\|_2.$$

Therefore, in virtue of relation (19), we have that

$$\|f_n\|_{\psi,q} \leqslant C\frac{\psi((1+\sum_{j=1}^n M_j^2)^{-1})}{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}\left(\log\Big(1+\sum_{j=1}^{n} M_j^2\Big)\right)^{1/q-1/2}\|f_n\|_2$$

and 1) is proved.

We will now prove the second statement. Since $1 < p < 2$, according to item 1), it yields that

$$\|f_n\|_{\psi,p} \leqslant C \left(\frac{\sqrt{\left(1+\sum_{j=1}^{n} M_j^2\right)^{-1}}}{\psi((1+\sum_{j=1}^{n} M_j^2)^{-1})} \right)^{-1} \left(\log\left(1+\sum_{k=1}^{n} M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{2}} \|f_n\|_2. \tag{20}$$

Moreover, since $2 < q < \infty$, by Theorem 2.1 and Parseval's equality, we find that

$$\|f_n\|_2 \leqslant \frac{\sqrt{\left(1+\sum_{j=1}^{n} M_j^2\right)^{-1}}}{\psi((1+\sum_{j=1}^{n} M_j^2)^{-1})} \left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/2-1/q} \|f\|_{\psi,q}. \tag{21}$$

Now from inequalities (20) and (21), it follows that

$$\|f_n\|_{\psi,p} \leqslant C \left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/p-1/q} \|f\|_{\psi,q}$$

and 2) is proved.
Finally, let $2 < p < q < +\infty$. In the generalized Lorentz space $L_{\psi,q}$, the following inequality hold (see [36], p. 491):

$$\|g\|_{\psi,p} \leqslant \|g\|_{\psi,q}^{\frac{\frac{1}{\tau}-\frac{1}{p}}{\frac{1}{\tau}-\frac{1}{q}}} \|g\|_{\psi,\tau}^{\frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{\tau}-\frac{1}{q}}} \tag{22}$$

for $1 < \tau < p < q < +\infty$. Choose the number $\tau \in (1,2)$. Then, according to the second statement, we have that

$$\|f_n\|_{\psi,\tau} \leqslant C \left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/\tau-1/q} \|f\|_{\psi,q}. \tag{23}$$

Now by in equality (22) setting $g = f_n$ and taking into account (23), we obtain that

$$\|f_n\|_{\psi,p} \leqslant \|f_n\|_{\psi,q}^{\frac{\frac{1}{\tau}-\frac{1}{p}}{\frac{1}{\tau}-\frac{1}{q}}} \left\{ C \left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/\tau-1/q} \|f\|_{\psi,q} \right\}^{\frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{\tau}-\frac{1}{q}}}$$
$$= C \left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/p-1/q} \|f\|_{\psi,q}$$

and also 3) is proved. The proof is complete. □

4. Concluding remarks result

Remark 4.5. *In the case $\psi(t) = t^{1/p}(1+|\log t|)^{\alpha}$, $1 < p < \infty$, Theorem 2.2 was previously proved in* [3]. *For the case $\alpha = 0$ see also* [2].

Remark 4.6. *In the case $\psi(t) = t^{1/p}(1+|\log t|)^{\alpha}$, $0 < p < 2$, Theorem 2.2 for polynomials in a uniformly bounded system was proved in* [14], *Theorem 3 i).*

Remark 4.7. *A similar statement as that in Theorem 2.1 was recently proved and discussed in* [8].

Remark 4.8. *It is well-known that each concave function $\psi = \psi(t)$ has the quasi-monotonicity properties that $\frac{\psi(t)}{t}$ is non-increasing and $\psi(t)$ is non-decreasing. Moreover, the definition of the SVL clam means that the functions satisfy two quasi-monotonicity conditions but now on a logarithmic scale.*

These facts opens the possibility that some of the results in this paper can be further generalized in this direction.

From Theorem 2.1 and Theorem 2.2, we can also derive the following generalization of a result in [5]:

Proposition 4.1. *Let the functions ψ_1 and ψ_2 satisfy the conditions $1 < \alpha_{\psi_1} = \beta_{\psi_2} = 2^{1/2}$, $\frac{t^{1/2}}{\psi_1(t)} \in SVL$, $\frac{t^{1/2}}{\psi_2(t)} \in SVL$,*

$$\sup_{t\in(0,1]} \frac{\psi_2(t)}{\psi_1(t)} < \infty \tag{24}$$

and assume that the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2, +\infty]$ satisfies condition (1). If $1 < p \leqslant 2 < q < \infty$, then for any polynomial

$$f_n(x) = \sum_{k=1}^{n} c_k \varphi_k(x),$$

the following inequality holds:

$$\|f_n\|_{\psi_1,p} \leqslant C \frac{\psi_1((1+\sum_{j=1}^n M_j^2))^{-1}}{\psi_2((1+\sum_{j=1}^n M_j^2)^{-1})} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{q}} \|f_n\|_{\psi_2,q}.$$

Proof. Since $\frac{t^{1/2}}{\psi_1(t)} \in SVL$ and $1 < p \leqslant 2$, according to the first statement of Theorem 2.2, the following inequality holds:

$$\|f_n\|_{\psi_1,p} \leqslant C \left(\frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\psi_1((1+\sum_{j=1}^n M_j^2)^{-1})} \right)^{-1} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{2}} \|f_n\|_2.$$

Taking into account that $\frac{t^{1/2}}{\psi_2(t)} \in SVL$ and $2 < q < \infty$ by Theorem 2.1, we have that

$$\|f_n\|_2 \leqslant C \left(\frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\psi_2((1+\sum_{j=1}^n M_j^2)^{-1})} \right) \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{2}-\frac{1}{q}} \|f_n\|_{\psi_2,q}.$$

From these inequalities, it follows that

$$\begin{aligned} \|f_n\|_{\psi_1,p} &\leqslant C \left(\frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\psi_1((1+\sum_{j=1}^n M_j^2)^{-1})} \right)^{-1} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{2}} \\ &\quad \times \frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\psi_2((1+\sum_{j=1}^n M_j^2)^{-1})} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{2}-\frac{1}{q}} \|f_n\|_{\psi_2,q} \\ &= \frac{\psi_1((1+\sum_{j=1}^n M_j^2)^{-1})}{\psi_2((1+\sum_{j=1}^n M_j^2)^{-1})} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{q}} \|f_n\|_{\psi_2,q} \end{aligned}$$

for $1 < p \leqslant 2 < q < \infty$. The proof is complete. □

Remark 4.9. *To investigate a statement as that Proposition 4.1 in the case of* $1 < p < q \leqslant 2$ *is an interesting open question. This case for polynomials in the trigonometric system was investigated in* [5]*. Furthermore, it seems to be possible to consider Proposition 4.1 also in the more general case* $1 \leqslant \beta_{\psi_2} < \alpha_{\psi_1} \leqslant 2$ *.*

Remark 4.10. *In* [4]*, it was proved that condition (24) implies that* $L_{\psi_1,p} \subset L_{\psi_2,q}$*,* $1 < p < q < \infty$*, in the case* $\psi_1 = \psi_2$ *see* [36]*.*

Remark 4.11 (Final Remark)**.** *Most results concerning Fourier and Jackson–Nikol'skii type inequalities are derived for the case with bounded orthonormal systems. But since there are many important unbounded orthonormal systems, it is of importance to develop the theory to cover such cases too. Examples of such unbounded systems are the following:*
(a) $\{\chi_n\}$*–orthonormal system of Haar functions (see e.g.* [9]*). The functions* $\chi_n(t)$ *are defined as follows:* $\chi_1(t) := 1$ *for* $t \in [0,1]$ *and for* $n = 2^m + k$*,* $k = 1, \ldots, m$ *and* $m = 0, 1, \ldots$ *put*

$$\chi_n(t) = \begin{cases} \sqrt{2^m}, & t \in (\frac{2k-2}{2^{m+1}}, \frac{2k-1}{2^{m+1}}), \\ -\sqrt{2^m}, & t \in (\frac{2k-1}{2^{m+1}}, \frac{2k}{2^{m+1}}), \\ 0, & t\overline{\in} \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right]. \end{cases}$$

The value of $\chi_n(t)$ *in a discontinuity point t is defined as*

$$\chi_n(t) = \frac{1}{2}\lim_{\varepsilon\to 0}[\chi_n(t+\varepsilon) + \chi_n(t-\varepsilon)].$$

(b) Let there be given an infinite sequence of integers $\{p_n\}$ *such that* $p_n \geqslant 2 \quad (n = 1, 2, ...)$*. We put* $m_n = p_1 ... p_n$*,* $n \geqslant 1$*. Then for any point* $t \in [0,1] \setminus A$*, there exists the unique expansion*

$$t = \sum_{k=1}^{\infty} \frac{\alpha_k(t)}{m_k}, \quad \alpha_k(t) = 0, 1, ..., p_k - 1,$$

where $A = \{\frac{l}{m_k}\}, l = 0, 1, \ldots, m_k$*. The generalized Haar system* $\chi\{p_k\} := \{\chi_n(t)\}$ *on* $[0,1]$ *is defined as follows (see* [15]*):*
$\chi_1(t) = 1$ *for* $t \in [0,1]$ *and if* $n \geqslant 2$*, then* $n = m_k + r(p_{k+1} - 1) + s$*, where* $m_0 = 1$ *and* $m_k = p_1 p_2 ... p_k; k = 1, ...; r = 0, 1, ..., m_k - 1; s = 1, 2, ..., p_{k+1} - 1$*.*
We put

$$\chi_n(t) := \chi_{k,r}^{(s)}(t) := \begin{cases} \sqrt{m_k} exp^{\frac{2\pi i s \alpha_{k+1}(t)}{p_{k+1}}} & , t \in \left(\frac{r}{m_k}, \frac{r+1}{m_k}\right) \cap B, \\ 0 & , t\overline{\in} \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right], \end{cases}$$

where $B := [0,1] \setminus A$*. At the remaining points of the interval* $(0,1)$*,* $\chi_n(t)$ *is equal to the half-sum of its right-hand and left-hand limits on the set* $[0,1] \setminus A$*, and at the endpoints of* $[0,1]$*, to the limits from within the interval.*
(c) Other generalizations of the Haar system were defined by A.M. Olevskii [31] *and A. Kamont* [18]*. Jackson–Nikol'skii inequalities for polynomials in the* $\chi\{p_n\}$ *system in the Lebesgue spaces* L_p *and Lorentz spaces* $L_{p,\tau}$ *were proved in* [1], [19], [39] *and* [41]*.*

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Research Article

Isomorphism problem in a special class of Banach function algebras and its application

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ABSTRACT. Given a weight function τ, we introduce a new class of Banach function algebras with respect to τ, denoted by $C_{0b}(X,\tau)$. We provide a complete solution to the isomorphism problem in this class. We further characterize the BSE-extension and the Inoue-Doss ideal associated with it. As an application of our results, we show the equivalence of the four statements: (i) $C_{0b}(X,\tau)$ is of BSE, (ii) $C_{0b}(X,\tau)$ is of BED, (iii) $C_{0b}(X,\tau)$ is Tauberian and (iv) τ is bounded.

Keywords: Banach function algebra of type I, BSE-extension, Inoue-Doss ideal, BSE-algebra, BED-algebra, Tauberian Banach algebra.

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Dedicated to the memory of Professor Wataru Takahashi.

1. INTRODUCTION AND MAIN RESULTS

Let X be a locally compact Hausdorff space and $C_b(X)$ be the Banach algebra of all bounded complex-valued continuous functions on X with supremum norm $\|\cdot\|_\infty$. Define $C_0(X) = \{f \in C_b(X) : f \text{ vanishes at infinity}\}$. Let τ be a positive continuous function on X with $\inf_{x\in X} \tau(x) \geq 1$. Define

$$C_{00}(X,\tau) = \{f \in C_0(X) : f\tau \in C_0(X)\},$$
$$C_{0b}(X,\tau) = \{f \in C_0(X) : f\tau \in C_b(X)\},$$
$$C_{bb}(X,\tau) = \{f \in C_b(X) : f\tau \in C_b(X)\}$$

and

$$\|f\|_{\infty,\tau} = \sup_{x\in X} |f(x)|\tau(x) \qquad (f \in C_b(X)).$$

Then both $C_{00}(X,\tau)$ and $C_{0b}(X,\tau)$ are subalgebras of $C_0(X)$, and $C_{bb}(X,\tau)$ is a subalgebra of $C_b(X)$. Moreover, these algebras become Banach algebras with norm $\|\cdot\|_{\infty,\tau}$, and they have the inclusion relation

$$C_c(X) \subseteq C_{00}(X,\tau) \subseteq C_{0b}(X,\tau) \subseteq C_{bb}(X,\tau) \subseteq C_b(X),$$

where $C_c(X)$ is the set of all complex-valued continuous functions on X with compact supports.

Remark 1.1. *Put $C_{b0}(X,\tau) = \{f \in C_b(X) : f\tau \in C_0(X)\}$. Then, it follows that $C_{00}(X,\tau) = C_{b0}(X,\tau)$ holds.*

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Note that τ is a $C_0(X)$-local function on X, and hence the algebra $C_0(X)_{\tau(1)}$ is defined (see [5, Definitions 5.1 and 5.3] for definition). Since $C_0(X)_{\tau(1)} = C_{00}(X, \tau)$, it follows from [5, Theorem 5.4 (ii)] that $C_{00}(X, \tau)$ is a Segal algebra in $C_0(X)$ with norm $\|\cdot\|_{\infty,\tau}$, and hence its Gelfand space can be identified with X (see [5, Theorem B']). Moreover, $C_{00}(X, \tau)$ is always a BED-algebra of type I, but it is not a BSE-algebra if τ is unbounded (see [2, Theorem 6.2]). On the other hand, $C_{0b}(X, \tau)$ is generally not a Segal algebra in $C_0(X)$. However, J. Inoue et al. have shown that $C_{0b}(X, \tau)$ is a Banach algebra of type I but is neither Tauberian nor of BSE nor of BED if $1/\tau$ vanishes at infinity (see [2, Theorem 7.3]). The aim of this paper is to investigate this algebra in greater detail. We refer the reader to [4, 5, 6, 9, 10, 11, 13] for more details on Segal algebras, BSE-algebras, BED-algebras and type I Banach algebras.

We first give a complete solution to the isomorphism problem in $C_{0b}(X, \tau)$. To state this, let Y be another locally compact Hausdorff space and σ be another positive continuous function on Y with $\inf_{y\in Y} \sigma(y) \geq 1$. Then we have:

Theorem 1.1. *The following three statements are equivalent:*

(i) *$C_{00}(X, \tau)$ is isomorphic to $C_{00}(Y, \sigma)$.*
(ii) *$C_{0b}(X, \tau)$ is isomorphic to $C_{0b}(Y, \sigma)$.*
(iii) *There exists a homeomorphism η from Y onto X such that $m\sigma \leq \tau \circ \eta \leq M\sigma$ for some positive constants m and M.*

Next we have:

Theorem 1.2. *The following two statements are equivalent:*

(i) *$C_{00}(X, \tau)$ is isomorphic to $C_{0b}(Y, \sigma)$.*
(ii) *Both X and Y are homeomorphic and both τ and σ are bounded.*

Moreover, we show that the BSE-extension and the Inoue-Doss ideal associated with $C_{0b}(X, \tau)$ are equal to $C_{bb}(X, \tau)$ and $C_{00}(X, \tau)$, respectively. The details will be described in Section 5.

As an application of the above results, we show the following result which is a generalization of [2, Theorem 7.3].

Theorem 1.3. *The following five statements are equivalent:*

(i) *$C_{0b}(X, \tau)$ is of BSE.*
(ii) *$C_{0b}(X, \tau)$ is of BED.*
(iii) *$C_{0b}(X, \tau)$ is Tauberian.*
(iv) *$C_{0b}(X, \tau)$ has a bounded X-weak approximate identity.*
(v) *τ is bounded.*

2. Preliminaries

In what follows, let X be a locally compact Hausdorff space and τ be a positive continuous function on X with

$$\inf_{x\in X} \tau(x) \geq 1.$$

Let A be a natural Banach function algebra on X. Then the natural embedding δ_X from X to the Gelfand space Φ_A of A is surjective, and hence δ_X is homeomorphic by [12, Theorem 3.2.4]. Thus we may identify Φ_A with X if it will cause no confusion. Then the multiplier algebra $M(A)$ of A is described as $\{f \in C_b(X) : fg \in A \ (g \in A)\}$. We say that A is of type I if $M(A) = C_b(X)$. We denote by $\mathrm{span}(X)$ the linear span of X in the dual space A^* of A. Therefore, an arbitrary element p in $\mathrm{span}(X)$ has the unique expression

$$p = \sum_{x\in X} \widehat{p}(x)x,$$

where $\widehat{p}$ is a complex-valued function on X with finite support. A function $f \in C_b(X)$ is said to be a *BSE-function* associated with A if there exists a constant $\beta > 0$ such that

$$\left|\sum_{x\in X} \widehat{p}(x)f(x)\right| \leq \beta\|p\|_{A^*}$$

for all $p \in \mathrm{span}(X)$. The BSE-norm of f, denoted by $\|f\|_{BSE(A)}$, is the infimum of all such β.

Let $C_{BSE(A)}(X)$ be the set of all BSE-functions on X associated with A. Then it is a semisimple commutative Banach algebra with the BSE-norm (see [13, Lemma 1]).

Definition 2.1. *We refer to $C_{BSE(A)}(X)$ as the BSE-extension associated with A.*

An algebra A is said to be a *BSE-algebra* if $M(A) = C_{BSE(A)}(X)$ (see [13, p.151, Definition]). If $\{e_\lambda\}$ is a net in A satisfying the condition

$$\lim_{\lambda} e_\lambda(x) = 1 \quad (x \in X),$$

then we call it a X-weak approximate identity of A. We note that $M(A) \subseteq C_{BSE(A)}(X)$ if and only if A has a bounded X-weak approximate identity (see [13, Corollary 5]). For the details on X-weak approximation identity, refer to [3, 8].

Let $\mathcal{K}(X)$ be the directed set consisting of all compact subsets of X with respect to the inclusion order. For $f \in C_{BSE(A)}(X)$ and $K \in \mathcal{K}(X)$, define

$$\|f\|_{BSE(A),K} = \sup\left\{\left|\sum_{x\in X\setminus K} \widehat{p}(x)f(x)\right| : p \in \mathrm{span}(X), \|p\|_{A^*} \leq 1\right\},$$

and put

$$C^0_{BSE(A)}(X) = \left\{f \in C_{BSE(A)}(X) : \lim_{K\in\mathcal{K}(X)} \|f\|_{BSE(A),K} = 0\right\}.$$

Then $C^0_{BSE(A)}(X)$ is a closed ideal of $C_{BSE(A)}(X)$ (see [4, Corollary 3.9]). This is an important ideal in our argument.

Definition 2.2. *We refer to $C^0_{BSE(A)}(X)$ as the Inoue-Doss ideal*[1] *associated with A.*

An algebra A is said to be a *BED-algebra* if $A = C^0_{BSE(A)}(X)$ (see [4, Definition 4.13]). A Banach function algebra B on X is called a Banach ideal of $C_0(X)$ if B is an ideal of $C_0(X)$ and $\|fg\|_B \leq \|f\|_\infty\|g\|_B$ holds for all $f \in C_0(X)$ and $g \in B$ (see [5, Definition 3.1]).

Lemma 2.1. *The algebra $C_{0b}(X,\tau)$ is a dense natural Banach ideal in $C_0(X)$.*

Proof. It is clear that $C_{0b}(X,\tau)$ is a dense Banach ideal in $C_0(X)$. Hence it suffices to show that $C_{0b}(X,\tau)$ is natural, that is, the natural embedding δ_X from X to $\Phi_{C_{0b}(X,\tau)}$ is surjective. To do this, let φ be an arbitrary element of $\Phi_{C_{0b}(X,\tau)}$. Take $h \in C_{0b}(X,\tau)$ with $\varphi(h) \neq 0$, and define

$$\widetilde{\varphi}(f) = \varphi(fh)/\varphi(h) \ (f \in C_0(X)).$$

This is well-defined because the right hand side of the above equation is independent of the choice of $h \in C_{0b}(X,\tau)$ with $\varphi(h) \neq 0$. By an easy calculation, we see that $\widetilde{\varphi} \in \Phi_{C_0(X)}$ with $\widetilde{\varphi}|_{C_{0b}(X,\tau)} = \varphi$, and hence there exists $x \in X$ such that $\delta_X(x) = \varphi$, namely, δ_X is surjective, as required. □

By Lemma 2.1, $\Phi_{C_{0b}(X,\tau)}$ can be identified with X under the natural embedding.

[1] The first author personally learned this important ideal from Professor Jyunji Inoue in the old days. The ideal was first introduced by him, but the underlying idea behind it had appeared in R. Doss [1].

Lemma 2.2. *The algebra $C_{0b}(X,\tau)$ is of type I.*

Proof. Since $\Phi_{C_{0b}}(X,\tau)$ can be identified with X, it follows that $M(C_{0b}(X,\tau)) \subseteq C_b(X)$. Also, since $C_{0b}(X,\tau)$ is an ideal of $C_b(X)$, it follows that $C_b(X) \subseteq M(C_{0b}(X,\tau))$. Thus we obtain $M(C_{0b}(X,\tau)) = C_b(X)$, that is, $C_{0b}(X,\tau)$ is of type I. □

3. Proof of Theorem 1.1

Let Y be another locally compact Hausdorff space and σ be another positive continuous function on Y with $\inf_{y\in Y}\sigma(y) \geq 1$.

(i)⇒(iii) Suppose that there is an isomorphism ρ from $C_{00}(X,\tau)$ onto $C_{00}(Y,\sigma)$. Then there are positive constants m, M such that

$$m\|\rho(f)\|_{\infty,\sigma} \leq \|f\|_{\infty,\tau} \leq M\|\rho(f)\|_{\infty,\sigma} \ (f \in C_{00}(X,\tau)). \tag{3.1}$$

Let ρ^* be the dual map of ρ from $C_{00}(Y,\sigma)^*$ onto $C_{00}(X,\tau)^*$. Then we have $\rho^*(\Phi_{C_{00}(Y,\sigma)}) = \Phi_{C_{00}(X,\tau)}$. Let δ_X and δ_Y be the natural embedding from X onto $\Phi_{C_{00}(X,\tau)}$ and the natural embedding from Y onto $\Phi_{C_{00}(Y,\sigma)}$, respectively. Define

$$\eta = (\delta_X)^{-1} \circ \rho^*|_{\Phi_{C_{00}(Y,\sigma)}} \circ \delta_Y.$$

Then η is a homeomorphism from Y onto X. We shall show that

$$\rho(f) = f\circ\eta \ (f \in C_{00}(X,\tau)). \tag{3.2}$$

In fact, let us take $f \in C_{00}(X,\tau)$ arbitrarily. Then we have

$$\begin{aligned}\rho(f)(y) &= \langle\rho(f), \delta_Y(y)\rangle = \langle f, \rho^*(\delta_Y(y))\rangle = \langle f, \rho^*|_{\Phi_{C_{00}(Y,\sigma)}}(\delta_Y(y))\rangle \\ &= \langle f, \delta_X\{\delta_X^{-1}(\rho^*|_{\Phi_{C_{00}(Y,\sigma)}}(\delta_Y(y)))\}\rangle \\ &= \langle f, \delta_X(\eta(y))\rangle = f(\eta(y)) \\ &= (f\circ\eta)(y)\end{aligned}$$

for all $y \in Y$, that is, (3.2) holds as required. By (3.2), $f\circ\eta \in C_{00}(Y,\sigma)$ holds for all $f \in C_{00}(X,\tau)$.

It remains to show that $m\sigma \leq \tau\circ\eta \leq M\sigma$. To show this, let us take $y \in Y$ and $\varepsilon > 0$ arbitrarily. Since τ is continuous, there exists a neighbourhood U of $\eta(y)$ such that $|\tau(x) - \tau(\eta(y))| < \varepsilon$ for all $x \in U$. Also since σ is continuous, there exists a neighbourhood V of y such that $|\sigma(y') - \sigma(y)| < \varepsilon$ for all $y' \in V$. Put $W = U \cap \eta(V)$. Then W is a neighbourhood of $\eta(y)$. Take a function $f_0 \in C_c(X)$ such that $f_0(\eta(y)) = 1$, $f_0|_{X\setminus W} = 0$ and $0 \leq f_0 \leq 1$. By (3.1) and (3.2), we have

$$m\|f_0\circ\eta\|_{\infty,\sigma} \leq \|f_0\|_{\infty,\tau} \leq M\|f_0\circ\eta\|_{\infty,\sigma}. \tag{3.3}$$

Therefore, it follows from the first half of (3.3) that

$$\begin{aligned}m\sigma(y) &= m|f_0(\eta(y))|\sigma(y) \leq m\|f_0\circ\eta\|_{\infty,\sigma} \leq \|f_0\|_{\infty,\tau} \\ &= \sup_{x\in X}|f_0(x)|\tau(x) = \sup_{x\in W}|f_0(x)|\tau(x) \leq \sup_{x\in U}\tau(x) \\ &\leq \tau(\eta(y)) + \varepsilon.\end{aligned}$$

Since $y \in Y$ and $\varepsilon > 0$ are arbitrary, we have $m\sigma \leq \tau \circ \eta$. Moreover, it follows from the latter half of (3.3) that

$$\begin{aligned}(\tau \circ \eta)(y) &= \tau(\eta(y)) = |f_0(\eta(y))|\tau(\eta(y)) \leq \|f_0\|_{\infty,\tau} \\ &\leq M\|f_0 \circ \eta\|_{\infty,\sigma} = M \sup_{y' \in Y} |f_0(\eta(y'))|\sigma(y') \\ &= M \sup_{y' \in \eta^{-1}(W)} |f_0(\eta(y'))|\sigma(y') \leq M \sup_{y' \in \eta^{-1}(W)} \sigma(y') \\ &\leq M \sup_{y' \in V} \sigma(y') \leq M(\sigma(y) + \varepsilon).\end{aligned}$$

Since $y \in Y$ and $\varepsilon > 0$ are arbitrary, we have $\tau \circ \eta \leq M\sigma$.

(iii)⇒(i) Suppose that there is a homeomorphism η from Y onto X such that $m\sigma \leq \tau \circ \eta \leq M\sigma$ for some positive constants m and M. Define

$$(\eta^* f)(y) = f(\eta(y)) \ (f \in C_{00}(X, \tau), y \in Y).$$

In this case, we see easily that η^* is a homomorphism from $C_{00}(X, \tau)$ to $C_0(Y)$. We shall show that $\eta^*(C_{00}(X, \tau)) = C_{00}(Y, \sigma)$. To do this, let us take $f \in C_{00}(X, \tau)$ arbitrarily. It is clear that $\eta^* f \in C_0(Y)$. Moreover, the inequality

$$|(\eta^* f)(y)|\sigma(y) \leq |f(\eta(y))|\tau(\eta(y))/m \ (y \in Y), \tag{3.4}$$

implies that $(\eta^* f)\sigma \in C_0(Y)$ since $f\tau \in C_0(X)$, and so $\eta^* f \in C_{00}(Y, \sigma)$. Namely, we obtain $\eta^*(C_{00}(X, \tau)) \subseteq C_{00}(Y, \sigma)$. To show the opposite inclusion, for $g \in C_{00}(Y, \sigma)$, let us define

$$f(x) = g(\eta^{-1}(x)) \ (x \in X).$$

It is clear that $f \in C_0(X)$. Moreover, the inequality

$$|f(x)|\tau(x) \leq |g(\eta^{-1}(x))|M\sigma(\eta^{-1}(x)) \ (x \in X) \tag{3.5}$$

implies that $f\tau \in C_0(X)$ since $g\sigma \in C_0(Y)$, and so $f \in C_{00}(X, \tau)$. Moreover, since

$$(\eta^* f)(y) = f(\eta(y)) = g(\eta^{-1}(\eta(y))) = g(y) \ (y \in Y)$$

holds, we have

$$\eta^* f = g, \tag{3.6}$$

namely, we obtain $C_{00}(Y, \sigma) \subseteq \eta^*(C_{00}(X, \tau))$. Therefore, we have the desired equality. By (3.4), (3.5) and (3.6), we have

$$m\|\eta^* f\|_{\infty,\sigma} \leq \|f\|_{\infty,\tau} \leq M\|\eta^* f\|_{\infty,\sigma} \ (f \in C_{00}(X, \tau)),$$

and hence η^* is an isomorphism from $C_{00}(X, \tau)$ onto $C_{00}(Y, \sigma)$.

(ii)⇒(iii) This can be shown in the same manner as the proof of (i)⇒(iii).

(iii)⇒(ii) This can be shown in the same manner as the proof of (iii)⇒(i).

This completes the proof of Theorem 1.1.

Corollary 3.1. *The following four statements are equivalent:*

(i) $C_{0b}(X, \tau)$ *is of BSE and of BED.*
(ii) τ *is bounded.*
(iii) $C_{0b}(X, \tau) = C_0(X)$.
(iv) $C_{0b}(X, \tau)$ *is isomorphic to some commutative C*-algebra.*

Proof. (i)⇔(iv) This immediately follows from Lemma 2.2 and [2, Corollary 4.2].

(ii)⇒(iii) Obvious.

(iii)⇒(iv) Obvious.

(iv)⇒(ii) Suppose that $C_{0b}(X,\tau)$ is isomorphic to $C_0(Y)$ for some locally compact Hausdorff space Y and define

$$1_Y(y) = 1 \ (y \in Y).$$

Then $C_{0b}(X,\tau)$ is isomorphic to $C_{0b}(Y,1_Y)$ since $C_{0b}(Y,1_Y) = C_0(Y)$. By Theorem 1.1, we can find a homeomorphism η from Y onto X such that $m1_Y \leq \tau \circ \eta \leq M1_Y$ for some positive constants m and M. Therefore, we have that $\tau(x) \leq M$ for all $x \in X$. □

Corollary 3.2. *The following three statements are equivalent:*

(i) τ *is unbounded.*

(ii) $C_{0b}(X,\tau)$ *has no bounded* X*-weak approximate identity.*

(iii) $C_{0b}(X,\tau)$ *is not of BSE.*

Proof. (i)⇒(ii) Suppose that τ is unbounded. If $C_{0b}(X,\tau)$ has a bounded X-weak approximate identity, say, $\{e_\lambda\}_{\lambda\in\Lambda}$ bounded by β, then we can take $x_0 \in X$ and $\lambda_0 \in \Lambda$ such that $\tau(x_0) \geq 2\beta + 1$ and $|e_{\lambda_0}(x_0) - 1| \leq 1/2$ since τ is unbounded. Then we have

$$\beta \geq \|e_{\lambda_0}\|_{\infty,\tau} = \sup_{x\in X} |e_{\lambda_0}(x)|\tau(x) \geq |e_{\lambda_0}(x_0)|\tau(x_0) \geq \frac{2\beta+1}{2} = \beta + \frac{1}{2},$$

which is a contradiction.

(ii)⇒(iii) This immediately follows from [13, Corollary 5].

(iii)⇒(i) This immediately follows from Corollary 3.1 since an arbitrary commutative C*-algebra is of BSE (see [13, Theorem 3]). □

4. Proof of Theorem 1.2

Let Y be another locally compact Hausdorff space and σ be another positive continuous function on Y with $\inf_{y\in Y}\sigma(y) \geq 1$.

(i)⇒(ii) Suppose that there is an isomorphism ρ from $C_{00}(X,\tau)$ onto $C_{0b}(Y,\sigma)$. Let ρ^* be the dual map of ρ from $C_{0b}(Y,\sigma)^*$ onto $C_{00}(X,\tau)^*$, and then we have $\rho^*(\Phi_{C_{0b}(Y,\sigma)}) = \Phi_{C_{00}(X,\tau)}$. Let δ_X and δ_Y be the natural embedding of X onto $\Phi_{C_{00}(X,\tau)}$ and the natural embedding of Y onto $\Phi_{C_{0b}(Y,\sigma)}$, respectively. Define

$$\eta = (\delta_X)^{-1} \circ \rho^*|_{\Phi_{C_{0b}(Y,\sigma)}} \circ \delta_Y.$$

Then η is a homeomorphism from Y onto X. Moreover, as observed in the proof of (i)⇒(iii) in Theorem 1.1, the equality $\rho(f) = f \circ \eta$ holds for all $f \in C_{00}(X,\tau)$ and there are positive constants m and M such that $m\sigma \leq \tau\circ\eta \leq M\sigma$. Define

$$(\eta^* f)(y) = f(\eta(y)) \ (f \in C_{00}(X,\tau), y \in Y).$$

Then, as observed in the proof of (iii)⇒(i) in Theorem 1.1, the equality $\eta^*(C_{00}(X,\tau)) = C_{00}(Y,\sigma)$ holds. However since $\rho = \eta^*$, it follows that

$$C_{0b}(Y,\sigma) = \rho(C_{00}(X,\tau)) = \eta^*(C_{00}(X,\tau)) = C_{00}(Y,\sigma),$$

and hence we have

$$C_{00}(Y,\sigma) = C_{0b}(Y,\sigma). \tag{4.1}$$

Assume that σ is unbounded. Then we can find a sequence $\{y_1, y_2, \cdots\}$ in Y and a sequence $\{n_1, n_2, \cdots\}$ in $\mathbf{N}$ such that

$$n_1 < \sigma(y_1) < n_2 < \sigma(y_2) < n_3 < \sigma(y_3) < n_4 < \cdots\cdots,$$

where $n_1 = 1$. In this case, $\lim_{n\to\infty} y_n = \omega_Y$ holds, where ω_Y is the point of Y at infinity. In fact, let K be an arbitrary compact subset of Y and put $\alpha_K = \max\{\sigma(y) : y \in K\}$. Take $i_0 \in \mathbf{N}$ with $\alpha_K < n_{i_0}$. Then we can easily see that $y_j \in Y \setminus K$ for all $j \geq i_0$, that is, $\lim_{n\to\infty} y_n = \omega_Y$.

Take $m \in \mathbf{N}$ arbitrarily. Since σ is continuous on Y, we can find a compact neighbourhood K_m of y_m such that $K_m \subseteq \{y \in Y : n_m < \sigma(y) < n_{m+1}\}$ and $|\sigma(y_m) - \sigma(y)| < 1$ $(y \in K_m)$. Then $K_i \cap K_j = \emptyset$ $(i \neq j)$. Take a continuous positive function g_m on Y such that

$$g_m(y_m) = \frac{1}{\sigma(y_m)}, 0 \leq g_m \leq \frac{1}{\sigma(y_m)} \text{ and } \operatorname{supp}(g_m) \subseteq K_m.$$

Since $|1 - \frac{\sigma(y)}{\sigma(y_m)}| < 1/\sigma(y_m) \leq 1$ for all $y \in K_m$, it follows that

$$0 \leq g_m(y)\sigma(y) \leq \frac{\sigma(y)}{\sigma(y_m)} \leq 2 \ (y \in K_m).$$

Define

$$g(y) = \sum_{m=1}^{\infty} g_m(y) \ (y \in Y).$$

Then it is clear that g is continuous on X such that $0 \leq g\sigma \leq 2$, and hence $g\sigma$ is bounded. Moreover, we shall show that $g \in C_0(Y)$. To do this, let ε be an arbitrary positive number. Take $j_0 \in \mathbf{N}$ with $1/n_{j_0} < \varepsilon$ and put

$$K_0 = K_1 \cup K_2 \cup \cdots \cup K_{n_{j_0}}.$$

Then K_0 is a compact set in Y. Take $y \in Y \setminus K_0$ arbitrarily. Then we have two cases:
(a) $y \notin \cup_{i=1}^{\infty} K_i$
and
(b) $y \in K_{k_0}$ for some $k_0 > n_{j_0}$.
In case (a), we have $g(y) = 0 < \varepsilon$. In case (b), we have

$$0 \leq g(y) = g_{k_0}(y) \leq 1/\sigma(y_{k_0}) < 1/n_{k_0} \leq 1/k_0 < 1/n_{j_0} < \varepsilon.$$

Then $g \in C_0(Y)$ as required. Thus we get $g \in C_{0b}(Y, \sigma)$. Therefore, it follows from (4.1) that $g \in C_{00}(Y, \sigma)$, and hence $\lim_{n\to\infty} g(y_n)\sigma(y_n) = 0$. But since $g(y_n)\sigma(y_n) = g_n(y_n)\sigma(y_n) = 1$ holds for all $n \in \mathbf{N}$, we arrive at a contradiction. Hence we conclude that σ is bounded. This implies that τ is also bounded because $\tau \circ \eta \leq M\sigma$.

(ii)⇒(i) Suppose that both τ and σ are bounded. Then we see that $C_{00}(X, \tau) = C_0(X)$ and $C_{0b}(Y, \sigma) = C_0(Y)$. If X is homeomorphic to Y, then $C_0(X)$ is isomorphic to $C_0(Y)$, and hence $C_{00}(X, \tau)$ must be isomorphic to $C_{0b}(Y, \sigma)$.

This completes the proof of Theorem 1.2.

Corollary 4.3. *The following two statements are equivalent:*

(i) $C_{0b}(X, \tau)$ *is Tauberian.*
(ii) τ *is bounded.*

Proof. (i)⇒(ii) Suppose that $C_{0b}(X, \tau)$ is Tauberian. Then $C_{00}(X, \tau) = C_{0b}(X, \tau)$ holds. In fact, take $f \in C_{0b}(X, \tau)$ and $\varepsilon > 0$ arbitrarily. Then $f \in C_0(X)$. Also since $C_{0b}(X, \tau)$ is Tauberian, it follows that there is $g \in C_c(X)$ with $\|f - g\|_{\infty,\tau} < \varepsilon$. Therefore, $g\tau \in C_c(X)$ and $\|f\tau - g\tau\|_\infty = \|f - g\|_{\infty,\tau} < \varepsilon$, and hence $f\tau \in C_0(X)$ because ε is arbitrary. Thus we have $f \in C_{00}(X, \tau)$, namely, $C_{00}(X, \tau) = C_{0b}(X, \tau)$ holds as required. Consequently, $C_{00}(X, \tau)$ must be isomorphic to $C_{0b}(X, \tau)$, and hence τ must be bounded by Theorem 1.2.

(ii)⇒(i) Suppose that τ is bounded. Then we have $C_{0b}(X, \tau) = C_0(X)$, and hence $C_{0b}(X, \tau)$ must be Tauberian. □

5. BSE-EXTENSION AND INOUE-DOSS IDEAL

Let A be $C_{00}(X,\tau)$ or $C_{0b}(X,\tau)$. In this section, we investigate the BSE-extension associated with A and the Inoue-Doss ideal associated with A. The obtained result is as follows:

Theorem 5.4.

(i) *The BSE-extensions associated with* $C_{00}(X,\tau)$ *and* $C_{0b}(X,\tau)$ *are both equal to* $C_{bb}(X,\tau)$*, that is,*

$$C_{BSE(C_{00}(X,\tau))}(X) = C_{BSE(C_{0b}(X,\tau))}(X) = C_{bb}(X,\tau).$$

(ii) *The Inoue-Doss ideals associated with* $C_{00}(X,\tau)$ *and* $C_{0b}(X,\tau)$ *are both equal to* $C_{00}(X,\tau)$*, that is,*

$$C^0_{BSE(C_{00}(X,\tau))}(X) = C^0_{BSE(C_{0b}(X,\tau))}(X) = C_{00}(X,\tau).$$

Proof. Denote by A any one of the algebras $C_{00}(X,\tau)$ and $C_{0b}(X,\tau)$, and then $A \subseteq C_c(X)$.

(i) We shall show $C_{BSE(A)}(X) = C_{bb}(X,\tau)$. To do this, we claim that

$$\|p\|_{A^*} = \sum_{x\in X} |\widehat{p}(x)|/\tau(x) \ (p \in \mathrm{span}(X)) \tag{5.1}$$

holds. In fact, let us take $p \in$ span(X) and $0 < \varepsilon < 1$ arbitrarily. Then we can write $p = \sum_{k=1}^n a_k x_k$, where $a_1, \cdots, a_n \in \mathbf{C} \setminus \{0\}$ and $x_1, \cdots, x_n \in X$ with $x_i \neq x_j \ (i \neq j)$. For each $1 \leq k \leq n$, we can take a compact neighbourhood K_k of x_k such that $K_i \cap K_j = \emptyset \ (i \neq j)$ and $(1-\varepsilon)/\tau(x_k) < 1/\tau(x)$ for all $x \in K_k$ because

$$\left\{x \in X : \frac{1-\varepsilon}{\tau(x_k)} < \frac{1}{\tau(x)}\right\}$$

is an open neighbourhood of x_k. Take a continuous positive function g_k on X such that

$$g_k(x_k) = \frac{1-\varepsilon}{\tau(x_k)}, 0 \leq g_k(x) \leq \frac{1-\varepsilon}{\tau(x_k)} \ (x \in K_k) \text{ and } \mathrm{supp}(g_k) \subseteq K_k,$$

and define

$$g_0(x) = \sum_{k=1}^n g_k(x) \ (x \in X).$$

Therefore, we can easily show that $g_0 \in C_c(X)$, $g_0(x_k) = (1-\varepsilon)/\tau(x_k) \ (1 \leq k \leq n)$ and $0 \leq g_0(x) \leq 1/\tau(x) \ (x \in X)$. Moreover, we can find a function $h_0 \in C_c(X)$ such that $\|h_0\|_\infty = 1$ and $h_0(x_k) = |a_k|/a_k$ for all $1 \leq k \leq n$. Put $f_0 = g_0 h_0$. Then we can see that $f_0 \in A$, $\|f_0\|_{\infty,\tau} \leq 1, |f_0(x_k)|\tau(x_k) = 1-\varepsilon$ and $a_k f_0(x_k) > 0$ for all $1 \leq k \leq n$. Therefore, we have

$$\begin{aligned}\|p\|_{A^*} &= \sup\left\{\left|\sum_{k=1}^n a_k f(x_k)\right| : f \in A, \|f\|_{\infty,\tau} \leq 1\right\}\\ &\geq \sum_{k=1}^n a_k f_0(x_k) = \sum_{k=1}^n |a_k|\tau(x_k)^{-1}|f_0(x_k)|\tau(x_k)\\ &= (1-\varepsilon)\sum_{k=1}^n |a_k|/\tau(x_k) = (1-\varepsilon)\sum_{x\in X}|\widehat{p}(x)|/\tau(x),\end{aligned}$$

and hence $\|p\|_{A^*} \geq \sum_{x\in X} |\widehat{p}(x)|/\tau(x)$ because ε is arbitrary.

On the other hand, we have

$$\begin{aligned}\|p\|_{A^*} &= \sup\left\{\left|\sum_{k=1}^{n} a_k f(x_k)\right| : f \in A, \|f\|_{\infty,\tau} \le 1\right\}\\ &\le \sup\left\{\sum_{k=1}^{n} |a_k|\tau(x_k)^{-1}|f(x_k)|\tau(x_k) : f \in A, \|f\|_{\infty,\tau} \le 1\right\}\\ &\le \sum_{k=1}^{n} |a_k|/\tau(x_k) = \sum_{x\in X} |\widehat{p}(x)|/\tau(x).\end{aligned}$$

Consequently, we have proved (5.1).

Now, by (5.1), for all $f \in C_{bb}(X,\tau)$, we have

$$\begin{aligned}\|f\|_{BSE(A)} &= \sup\left\{\left|\sum_{x\in X} \widehat{p}(x)f(x)\right| : p \in \operatorname{span}(X), \|p\|_{A^*} \le 1\right\}\\ &\le \sup\left\{\sum_{x\in X} |\widehat{p}(x)|\tau(x)^{-1}|f(x)|\tau(x) : p \in \operatorname{span}(X), \|p\|_{A^*} \le 1\right\}\\ &\le \|f\|_{\infty,\tau} \sup\left\{\sum_{x\in X} |\widehat{p}(x)|/\tau(x) : p \in \operatorname{span}(X), \|p\|_{A^*} \le 1\right\}\\ &= \|f\|_{\infty,\tau} \sup\{\|p\|_{A^*} : p \in \operatorname{span}(X), \|p\|_{A^*} \le 1\} \quad (\text{by } (5.1))\\ &= \|f\|_{\infty,\tau} < \infty,\end{aligned}$$

that is, $f \in C_{BSE(A)}(X)$. Therefore, we have $C_{bb}(X,\tau) \subseteq C_{BSE(A)}(X)$. To show the opposite inclusion, take $f \in C_{BSE(A)}(X)$ arbitrarily. Then $f \in C_b(X)$. For each $x \in X$, put $p_x = \tau(x)x$, that is, $p_x(f) = \tau(x)f(x)$ $(f \in A)$. Then we have

$$\begin{aligned}\|p_x\|_{A^*} &= \sup\{|p_x(f)| : f \in A, \|f\|_{\infty,\tau} \le 1\}\\ &= \sup\{|f(x)|\tau(x) : f \in A, \|f\|_{\infty,\tau} \le 1\}\\ &\le 1\end{aligned}$$

holds for all $x \in X$. Therefore, we have

$$|f(x)\tau(x)| = |\widehat{p}_x(x)f(x)| \le \|f\|_{BSE(A)}$$

for all $x \in X$, and hence $\|f\tau\|_\infty \le \|f\|_{BSE(A)} < \infty$, so $f\tau \in C_b(X)$, that is, $f \in C_{bb}(X,\tau)$. Therefore, we have $C_{BSE(A)}(X) \subseteq C_{bb}(X,\tau)$. This completes the proof of the equality $C_{BSE(A)}(X) = C_{bb}(X,\tau)$.

(ii) Let $f \in C_{bb}(X,\tau)$ and $K \in \mathcal{K}(X)$. Since

$$\begin{aligned}\|f\|_{BSE(A),K} &= \sup\left\{\left|\sum_{x\in X\setminus K} \widehat{p}(x)f(x)\right| : p \in \operatorname{span}(X), \|p\|_{A^*} \le 1\right\}\\ &\le \sup\left\{\sum_{x\in X\setminus K} |\widehat{p}(x)|\tau(x)^{-1}|f(x)|\tau(x) : p \in \operatorname{span}(X), \|p\|_{A^*} \le 1\right\}\\ &\le \sup_{x\in X\setminus K} |f(x)|\tau(x) \times \sup\left\{\sum_{x\in X\setminus K} |\widehat{p}(x)|/\tau(x) : p \in \operatorname{span}(X), \|p\|_{A^*} \le 1\right\}\end{aligned}$$

$$\le \sup_{x\in X\setminus K} |f(x)|\tau(x) \times \sup\left\{\sum_{x\in X} |\widehat{p}(x)|/\tau(x) : p \in \mathrm{span}\,(X), \|p\|_{A^*} \le 1\right\}$$
$$= \sup_{x\in X\setminus K} |f(x)|\tau(x) \times \sup\{\|p\|_{A^*} : p \in \mathrm{span}\,(X), \|p\|_{A^*} \le 1\} \quad (\text{by } (5.1))$$
$$= \sup_{x\in X\setminus K} |f(x)|\tau(x),$$

it follows that

$$\|f\|_{BSE(A),K} \le \sup_{x\in X\setminus K} |f(x)|\tau(x).$$

To show the reverse of the above inequality, put $p_x = \tau(x)x$ for each $x \in X$. Then we have $\|p_x\|_{A^*} \le 1$ $(x \in X)$ as observed in the proof of (i). Then we have

$$|f(x)\tau(x)| = |\widehat{p}_x(x)f(x)| \le \|f\|_{BSE(A),K} \ (x \in X \setminus K),$$

and hence

$$\sup_{x\in X\setminus K} |f(x)|\tau(x) \le \|f\|_{BSE(A),K}.$$

Therefore, we have

(5.2) $$\|f\|_{BSE(C_{00}(X,\tau)),K} = \|f\|_{BSE(C_{0b}(X,\tau)),K} \ (f \in C_{bb}(X,\tau), K \in \mathcal{K}(X)).$$

Hence it follows from (i) and (5.2) that

$$C^0_{BSE(C_{00}(X,\tau))}(X) = C^0_{BSE(C_{0b}(X,\tau))}(X).$$

Recall that $C_{00}(X,\tau)$ is of BED, and hence $C^0_{BSE(C_{00}(X,\tau))}(X) = C_{00}(X,\tau)$ holds. □

Remark 5.2.

(i) *If τ is bounded, then $C_{00}(X,\tau) = C_{0b}(X,\tau) = C_0(X)$ and $C_{bb}(X,\tau) = C_b(X)$, and hence Theorem 5.4 obviously holds.*

(ii) *As observed in the proof of Theorem 5.4 (i),*

$$\|f\|_{BSE(C_{00}(X,\tau))} = \|f\|_{BSE(C_{0b}(X,\tau))} = \|f\|_{\infty,\tau}$$

holds for all $f \in C_{bb}(X,\tau)$.

(iii) *As observed in the proof of Theorem 5.4 (ii),*

$$\|f\|_{BSE(C_{00}(X,\tau)),K} = \|f\|_{BSE(C_{0b}(X,\tau)),K} = \sup_{x\in X\setminus K} |f(x)\tau(x)$$

holds for all $f \in C_{bb}(X,\tau)$ and $K \in \mathcal{K}(X)$.

Corollary 5.4. *If τ is unbounded, then $C_{0b}(X,\tau)$ is not of BED.*

Proof. Suppose that τ is unbounded. Then $C_{00}(X,\tau) \subsetneqq C_{0b}(X,\tau)$ holds. In fact, suppose on the contrary that $C_{00}(X,\tau) = C_{0b}(X,\tau)$ holds. Then $C_{00}(X,\tau)$ is isomorphic to $C_{0b}(X,\tau)$, and hence τ must be bounded by Theorem 1.2. This is impossible because τ is unbounded by hypothesis. Now if $C_{0b}(X,\tau)$ is of BED, then we have from Theorem 5.4 (ii) that

$$C_{00}(X,\tau) = C^0_{BSE(C_{00}(X,\tau))}(X) = C^0_{BSE(C_{0b}(X,\tau))}(X) = C_{0b}(X,\tau)).$$

Thus we arrive at a contradiction. □

6. Proof of Theorem 1.3

We can see that:

(1) (v)⇔(iv)⇔(i) are derived from Corollary 3.2.
(2) (v)⇔(iii) is exactly the same as Corollary 4.3.
(3) (ii)⇔(v) is derived from Corollary 5.4.

By combining (1), (2) and (3), we have proved Theorem 1.3.

Remark 6.3. *The following four statements are equivalent:*

(i) τ *is bounded.*
(ii) $C_{00}(X,\tau)$ *is isomorphic to some commutative C*-algebra.*
(iii) $C_{0b}(X,\tau)$ *is isomorphic to some commutative C*-algebra.*
(iv) $C_{bb}(X,\tau)$ *is isomorphic to some commutative C*-algebra.*

In fact, since $C_{0b}(X,\tau)$ is a closed ideal of $C_{bb}(X,\tau)$, it follows that (iv) implies (iii). Also since $C_{00}(X,\tau)$ is a closed ideal of $C_{0b}(X,\tau)$, it follows that (iii) implies (ii). If $C_{00}(X,\tau)$ is isomorphic to some commutative C*-algebra, then it must be of BSE, and hence $M(C_{00}(X,\tau)) = C_{BSE(C_{00}(X,\tau))}(X)$. Moreover, we have from Theorem 5.4 (i) that $C_{BSE(C_{00}(X,\tau))}(X) = C_{bb}(X,\tau)$, and hence $M(C_{00}(X,\tau)) = C_{bb}(X,\tau)$. Define $1_X(x) = 1$ $(x \in X)$. Then $1_X \in M(C_{00}(X,\tau))$, and hence $1_X \in C_{bb}(X,\tau)$ by the above equality. Then τ must be bounded. Consequently, (ii) implies (i). If τ is bounded, then $C_{bb}(X,\tau) = C_b(X)$, and so $C_{bb}(X,\tau)$ is isomorphic to the C*-algebra $C_b(X)$. Then (i) implies (iv).

7. Examples

Let $\mathbf{R}$ be the space of real numbers with usual topology and $Homeo(\mathbf{R})$ be the set of all homeomorphisms from $\mathbf{R}$ onto itself. Let $\mathcal{T}(\mathbf{R})$ be the set of all positive continuous functions τ on $\mathbf{R}$ with $\inf\{\tau(x) : x \in \mathbf{R}\} \geq 1$. Let $\tau, \sigma \in \mathcal{T}(\mathbf{R})$. If there are $m, M > 0$ and $h \in Homeo(\mathbf{R})$ such that $m\tau \leq \sigma \circ h \leq M\tau$, τ and σ are said to be equivalent, and written as $\tau \cong \sigma$.

(i) Take $\tau \in \mathcal{T}(\mathbf{R})$ and $h \in Homeo(\mathbf{R})$ arbitrarily. Then we have $\tau \cong \tau \circ h$ because $\tau = (\tau \circ h) \circ h^{-1}$.

(ii) Define

$$\tau_1(x) = \begin{cases} (n+1)(|x| - 2n) + 1 & (2n \leq |x| \leq 2n+1) \\ -(n+1)(|x| - 2n - 2) + 1 & (2n+1 < |x| \leq 2n+2), \end{cases}$$

where $n = 0, 1, 2, \cdots$. Then $\tau_1 \in \mathcal{T}(\mathbf{R})$. Since τ_1 is unbounded, it follows from Corollary 4.3 that $C_{0b}(\mathbf{R}, \tau_1)$ is not Tauberian. However, we can confirm this by a concrete calculation as follows. Define

$$f(x) = \frac{1}{\tau_1(x)\tau_1(x-1)} \quad (x \in \mathbf{R}).$$

Then we can easily see $f \in C_{0b}(\mathbf{R}, \tau_1)$. Also since $\tau_1(2n) = 1$ for all $n = 0, \pm 1, \pm 2, \cdots$, it follows that

$$\|f - g\|_{\infty,\tau_1} = \sup_{x \in \mathbf{R}} \left| \frac{1}{\tau_1(x-1)} - g(x)\tau_1(x) \right| \geq 1$$

for all $g \in C_c(\mathbf{R})$. In other words, $C_{0b}(\mathbf{R}, \tau_1)$ is not Tauberian.

(iii) Define

$$\tau_0(x) = 1 + |x| \quad (x \in \mathbf{R}).$$

Then $\tau_0 \in \mathcal{T}(\mathbf{R})$ and we can easily see $\tau_0 \ncong \tau_1$. Therefore, $C_{0b}(\mathbf{R}, \tau_0)$ is not isomorphic to $C_{0b}(\mathbf{R}, \tau_1)$ by Theorem 1.1.

(iv) Let f be a strictly increasing continuous function on $[0, \infty)$ such that $f(0) = 0$ and $\lim_{x\to+\infty} f(x) = +\infty$. Let g be a strictly decreasing continuous function on $(-\infty, 0]$ such that $g(0) = 0$ and $\lim_{x\to-\infty} g(x) = +\infty$. Define

$$\tau_{f,g}(x) = \begin{cases} 1 + f(x) & (x \geq 0) \\ 1 + g(x) & (x < 0) \end{cases}.$$

Then we see $\tau_{f,g} \in \mathcal{T}(\mathbf{R})$ and $\tau_0 \cong \tau_{f,g}$. In fact, it is clear that $\tau_{f,g} \in \mathcal{T}(\mathbf{R})$. Define

$$h(x) = \begin{cases} f^{-1}(x) & (x \geq 0) \\ g^{-1}(-x) & (x < 0) \end{cases}.$$

Then we see $h \in Homeo(\mathbf{R})$ and $\tau_{f,g} \circ h = \tau_0$ by an easy calculation. Therefore, we obtain $\tau_{f,g} \cong \tau_0$ from (i), and hence $C_{0b}(\mathbf{R}, \tau_{f,g})$ is isomorphic to $C_{0b}(\mathbf{R}, \tau_0)$ by Theorem 1.1.

8. Open problems

Finally, let us list some open problems for further study.

Problem on vector-valued functions: Let X be a locally compact Hausdorff space, τ be a positive continuous function on X with $\inf_{x\in X} \tau(x) \geq 1$ and A be a unital commutative C*-algebra. Moreover, let $C_0(X, A)$ be the commutative Banach algebra of all continuous A-valued functions on X vanishing at infinity and $C_b(X, A)$ be the commutative Banach algebra of all bounded continuous A-valued functions. Define

$$C_{0b}(X; A, \tau) = \{f \in C_0(X, A) : \tau f \in C_b(X, A)\}.$$

Then, solve the isomorphism problem in the Banach algebra $C_{0b}(X; A, \tau)$.

Moreover, what are the BSE-extension and the Inoue-Doss ideal associated with $C_{0b}(X; A, \tau)$?

Problem on Lipschitz algebras: Let $\mathrm{Lip}_1(\mathbf{R})$ and $\mathrm{Lip}_1^0(\mathbf{R})$ be the Lipschitz algebras as defined in [4, Definition 5.8] and τ be a positive continuous function on $\mathbf{R}$ with $\inf_{x\in X} \tau(x) \geq 1$. Define

$$\mathrm{Lip}_{01}(\mathbf{R}, \tau) = \{f \in \mathrm{Lip}_1^0(\mathbf{R}) : \tau f \in \mathrm{Lip}_1(\mathbf{R})\}.$$

Then, what are the BSE-extension and the Inoue-Doss ideal associated with the Banach algebra $\mathrm{Lip}_{01}(\mathbf{R}, \tau)$?

Problem on differentiable functions: Let $C_b^n(\mathbf{R}^d)$ and $C_0^n(\mathbf{R}^d)$ be the differential algebras as defined in [7, §2] and τ be a positive continuous function on $\mathbf{R}$ with $\inf_{x\in X} \tau(x) \geq 1$. Define

$$C_{01}^n(\mathbf{R}^d, \tau) = \{f \in C_0^n(\mathbf{R}^d) : \tau f \in C_b^n(\mathbf{R}^d)\}.$$

Then, what are the BSE-extension and the Inoue-Doss ideal associated with the Banach algebra $C_{01}^n(\mathbf{R}^d, \tau)$?

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Research Article

Differential $\{e\}$-structures for equivalences of 2-nondegenerate Levi rank 1 hypersurfaces $M^5 \subset \mathbb{C}^3$

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ABSTRACT. The class IV_2 of 2-nondegenerate constant Levi rank 1 hypersurfaces $M^5 \subset \mathbb{C}^3$ is governed by Pocchiola's two primary invariants $\boldsymbol{W}_0$ and $\boldsymbol{J}_0$. Their vanishing characterizes equivalence of such a hypersurface M^5 to the tube M^5_{LC} over the real light cone in $\mathbb{R}^3$. When either $\boldsymbol{W}_0 \not\equiv 0$ or $\boldsymbol{J}_0 \not\equiv 0$, by normalization of certain two group parameters c and e, an invariant coframe can be built on M^5, showing that the dimension of the CR automorphism group drops from 10 to 5.

This paper constructs an explicit $\{e\}$-structure in case $\boldsymbol{W}_0$ and $\boldsymbol{J}_0$ do not necessarily vanish. Furthermore, Pocchiola's calculations hidden on a computer now appear in details, especially the determination of a secondary invariant R, expressed in terms of the first jet of $\boldsymbol{W}_0$. All other secondary invariants of the $\{e\}$-structure are also expressed explicitly in terms of $\boldsymbol{W}_0$ and $\boldsymbol{J}_0$.

Keywords: Levi degenerate CR manifolds, 2-nondegeneracy, G-structures, Cartan method of equivalence, Cartan Lemma, Pocchiola invariants.

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1. INTRODUCTION

We study the equivalence problem under biholomorphisms of real hypersurfaces $M^5 \subset \mathbb{C}^3$ — hence of CR dimension 2 — whose Levi form is degenerate of constant rank 1, and whose Freeman form is nowhere zero, or equivalently, which are 2-nondegenerate. There are previous approaches to this problem, and we refer our readers to the the articles of Medori-Spiro [12, 13], in which a *Cartan connection* was adressed.

In a recently published article [18], the authors exhibited two important primary invariants, $\boldsymbol{W}_0$ and $\boldsymbol{J}_0$, whose existence was not previously discovered prior to Pocchiola's prepublication [25], and which, in depth, required the help of a computer algebra system. These invariants have useful applications, such as in Isaev's study [9] of tube hypersurfaces in $\mathbb{C}^3$ that are 2-nondegenerate and uniformly Levi degenerate of rank 1.

Our first objective here is to reconstruct $\boldsymbol{W}_0$ and $\boldsymbol{J}_0$, by presenting fully detailed computations, only by hand, without the help of any computer. In contrast to [25, 18], the present text has the ambition of exhibiting all calculations, without requiring any extra work from the readers: '*no pen needed, no computer needed*'. Within the Cartan theory, this sounds quite like a challenge opposite to a certain tradition of hiding a lot of computations. But we believe that fully detailed articles can be read, checked and studied more rapidly.

As a second objective, we construct an *explicit* $\{e\}$-structure which characterizes equivalences under biholomorphisms of these types of hypersurfaces $M^5 \subset \mathbb{C}^3$. This way, we give a

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theoretical proof which will provide a definitive confirmation of the existence of *exactly* 2 primary invariants, $\boldsymbol{W}_0$ and $\boldsymbol{J}_0$. Unlike the approach of [25, 18] which proceeded at each step with *systematic* and *explicit* calculations of *all* torsion coefficients, we will bypass some of these steps, thereby economizing some computations. On the way, we will closely observe the evolution of the modified Maurer-Cartan 1-forms during the Cartan process.

The basic principle of Cartan's approach is to create a collection of 1-forms (a coframe), by absorbing as many as possible torsion terms, in order that the structure of this coframe be as close as possible to the structure of the Maurer-Cartan coframe on the (prolongation of the) model $M^5_{\mathsf{LC}} \subset \mathbb{C}^3$, the tube over the real light cone $\{x_1^2 + x_2^2 = x_3^2\}$ in $\mathbb{R}^3$:

$$M^5_{\mathsf{LC}} := \big\{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon\ (\mathsf{Re}\, z_1)^2 + (\mathsf{Re}\, z_2)^2 = (\mathsf{Re}\, z_3)^2\big\}$$

whose local CR automorphism group is known to be isomorphic to $\mathsf{SO}_{3,2}(\mathbb{R})$.

Recall that a Maurer-Cartan form ω valued in some Lie algebra $\mathfrak{g}$ satisfies the structure equation with no curvature:

$$d\omega + \tfrac{1}{2}\big[\omega \wedge \omega\big] \;=\; 0.$$

In practice, as in our current case, the right-hand side of the equation is not always zero, and this constitutes the *default* of ω being a Maurer-Cartan form. This happens when an invariant is written as a linear combination of torsion terms, and such a linear combination fails to follow the structure equations, thus obstructing the absorption process.

We now give a summary of our results. Recall that if J denotes the complex structure of $T\mathbb{C}^3$, then the tangent bundle TM^5 has a distribution $T^cM^5 := TM^5 \cap JTM^5 \subseteq TM^5$ of codimension 1 which is invariant under J at each point of M^5. Let ρ be a real 1-form with $\mathsf{Ker}\,\rho = T^cM^5$. The *Levi form* is a bilinear map on T^cM^5 defined as $(X, Y) \mapsto d\rho(X, JY)$ for any two sections X, Y of T^cM^5.

Letting $\mathbb{C}TM^5 := \mathbb{C} \otimes_{\mathbb{R}} TM^5$ be the complexification of the tangent bundle of M^5, by defining $T^{1,0}M^5 := \mathbb{C}TM^5 \cap T^{1,0}\mathbb{C}^3$ together with its complex conjugate $T^{0,1}M^5 := \overline{T^{1,0}M^5}$, we have the (classical) direct sum decomposition $\mathbb{C}T^cM^5 = T^{1,0}M^5 \oplus T^{0,1}M^5$. Let $\{\mathcal{L}_1, \mathcal{L}_2\}$ be two local generators of $T^{1,0}M^5$, *i.e.* a frame for $T^{1,0}M^5$.

Section 2 provides more information, while complete background may be found in [19].

By the assumption that the Levi form is uniformly of rank 1 at each point of M, there exists by [19] a uniquely determined *slant function* $\boldsymbol{k} \colon M \longrightarrow \mathbb{C}$ such that the vector field:

$$\mathcal{K} \;:=\; \boldsymbol{k}\, \mathcal{L}_1 + \mathcal{L}_2$$

generates the kernel of the Levi form, of constant rank $2 - 1 = 1$. If we let $\mathcal{T}$ denote a vector field with $\rho(\mathcal{T}) \equiv 1$, we may consider the coframe $\{\rho, \kappa_0, \zeta_0\}$ dual to $\{\mathcal{T}, \mathcal{L}_1, \mathcal{K}\}$. In fact, the conjugates $\overline{\kappa}_0$, $\overline{\zeta}_0$ and $\overline{\mathcal{L}}_1$, $\overline{\mathcal{K}}$ also come into play in order to really make up a (co)frame on $\mathbb{C}TM^5$, while $\overline{\rho} = \rho$ and $\overline{\mathcal{T}} = \mathcal{T}$ are real. A certain appropriate real 1-form ρ will be chosen, and denoted ρ_0.

Performing the Cartan process, we will make a series of changes to these 1-forms:

$$(\rho_0, \kappa_0, \zeta_0) \;\rightsquigarrow\; (\rho_0, \kappa_0', \zeta_0'')$$

and after (really a lot of) computations, we will obtain a 4-dimensional G-structure whose lifted 1-forms write up as:

$$\begin{pmatrix} \rho \\ \kappa \\ \zeta \end{pmatrix} := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 \\ -i\,\overline{\mathsf{c}}\mathsf{e} & \mathsf{c} & 0 \\ -\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \kappa_0' \\ \zeta_0'' \end{pmatrix}.$$

Also, after a long process, we will construct modified Maurer-Cartan forms:

$$
\begin{aligned}
\pi^1 &:= \alpha - \Big(\mathsf{t} - \frac{i}{2}\,\mathsf{Im}\,Z^2\Big)\,\rho - \Big(R^1 - \overline{K}^6\Big)\,\kappa - R^2\,\zeta - K^6\,\overline{\kappa} - 0,\\
\pi^2 &:= \beta - i\,Z^1\,\rho - \Big(\mathsf{t} - \frac{i}{2}\,\mathsf{Im}\,Z^2 + K^1\Big)\,\kappa - K^2\,\zeta - K^3\,\overline{\kappa} - K^4\,\overline{\zeta},
\end{aligned}
$$

with R^i, K^i, Z^i being some explicit functions on $M^5 \times G^4$, where t is a new real variable, and then, after meticulous absorption work, we will obtain as is stated below in Theorem 13.1 on p. 366, three finalized structure equations of the neat shape:

$$
\begin{aligned}
d\rho &= \big(\pi^1 + \overline{\pi}^1\big) \wedge \rho + i\,\kappa \wedge \overline{\kappa},\\
d\kappa &= \pi^2 \wedge \rho + \pi^1 \wedge \kappa + \zeta \wedge \overline{\kappa},\\
d\zeta &= \big(\pi^1 - \overline{\pi}^1\big) \wedge \zeta + i\,\pi^2 \wedge \kappa\\
&\quad + R\,\rho \wedge \zeta + J\,\rho \wedge \overline{\kappa} + W\,\kappa \wedge \zeta,
\end{aligned}
$$

in which are present Pocchiola's two primary invariants:

$$
W = \frac{1}{\mathsf{c}}\,\boldsymbol{W}_0 \qquad\qquad \text{and} \qquad\qquad J = \frac{i}{\overline{\mathsf{c}}^3}\,\overline{\boldsymbol{J}}_0,
$$

together with a single secondary (derived) invariant:

$$
R = \mathsf{Re}\left[i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\boldsymbol{W}_0 + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\bigg(-\frac{i}{2}\,\overline{\mathcal{L}}_1\big(\boldsymbol{W}_0\big) + \frac{i}{2}\,\bigg(-\frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} + \frac{1}{3}\,\overline{\boldsymbol{P}}\bigg)\,\boldsymbol{W}_0\bigg)\right].
$$

We would like to mention that the two invariants that Pocchiola denoted W and J are now denoted in our paper $\boldsymbol{W}_0$ and $\boldsymbol{J}_0$, with the subscript $(\bullet)_0$ designating functions defined on M^5 alone, independently of any extra group variable.

The expression of R was discovered by Pocchiola in [25, 18] thanks to intensive computer explorations, but no details of proof appeared in print at all. In Section 12 of this paper, a complete, *detailed*, *hand-done* proof, will be provided, thus verifying that R is indeed a function of the first jet of $\boldsymbol{W}_0$, hence a *secondary* invariant.

We will also construct a certain real 1-form $\Lambda = d\mathsf{t} + \cdots$, and in Section 14, the final $\{e\}$-structure that we obtain will take the following form (conjugate equations are unwritten):

$$
\begin{aligned}
d\rho &= \pi^1 \wedge \rho + \overline{\pi}^1 \wedge \rho + i\kappa \wedge \overline{\kappa},\\
d\kappa &= \pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa},\\
d\zeta &= i\pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi}^1 \wedge \zeta + W\kappa \wedge \zeta + R\rho \wedge \zeta + J\rho \wedge \overline{\kappa},\\
d\pi^1 &= \Lambda \wedge \rho - i\overline{\pi}^2 \wedge \kappa + \zeta \wedge \overline{\zeta} + \widehat{\Omega}_1,\\
d\pi^2 &= \Lambda \wedge \kappa + \pi^2 \wedge \overline{\pi}^1 - \overline{\pi}^2 \wedge \zeta + \widehat{\Omega}_2 + h\rho \wedge \kappa,\\
d\Lambda &= \Lambda \wedge \pi^1 + \Lambda \wedge \overline{\pi}^1 + i\pi^2 \wedge \overline{\pi}^2 + \Phi,
\end{aligned}
$$

with

$$\begin{aligned}
\widehat{\Omega}_1 &= -\tfrac{1}{4}W\pi^2 \wedge \rho + \tfrac{1}{4}\overline{W}\overline{\pi}^2 \wedge \rho - \tfrac{1}{2}(R_\kappa - \overline{J_\zeta})\rho \wedge \kappa - \tfrac{1}{2}R_\zeta \rho \wedge \zeta \\
&\quad + \tfrac{1}{2}(R_{\overline{\kappa}} - J_\zeta)\rho \wedge \overline{\kappa} + \tfrac{1}{2}R_{\overline{\zeta}}\rho \wedge \overline{\zeta} + \left(\tfrac{1}{2}W_{\overline{\kappa}} - iR\right)\kappa \wedge \overline{\kappa} - \overline{W}\kappa \wedge \overline{\zeta} - W\zeta \wedge \overline{\kappa}, \\
\widehat{\Omega}_2 &= -R\pi^2 \wedge \rho - \tfrac{1}{4}W\pi^2 \wedge \kappa + \tfrac{1}{4}\overline{W}\overline{\pi}^2 \wedge \kappa - i(W_\rho - 2R_\kappa + \overline{J_\zeta})\rho \wedge \zeta \\
&\quad - i(WJ - J_\kappa)\rho \wedge \overline{\kappa} - iJ\rho \wedge \overline{\zeta} - \tfrac{1}{2}R_\zeta \kappa \wedge \zeta + \tfrac{1}{2}(R_{\overline{\kappa}} - J_\zeta)\kappa \wedge \overline{\kappa} + \tfrac{1}{2}R_{\overline{\zeta}}\kappa \wedge \overline{\zeta} \\
&\quad - R\zeta \wedge \overline{\kappa}.
\end{aligned}$$

Furthermore, we will show that h and Φ can be expressed in terms of $\widehat{\Omega}_1$, of $\widehat{\Omega}_2$ and of their first-order derivatives. Thus, this demonstrates that there are exactly 2 primary invariants.

Clearly, when $W \equiv J \equiv 0$, the $\{e\}$-structure collapses to:

$$\begin{aligned}
d\rho &= \pi^1 \wedge \rho + \overline{\pi}^1 \wedge \rho + i\kappa \wedge \overline{\kappa}, \\
d\kappa &= \pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa}, \\
d\zeta &= i\pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi}^1 \wedge \zeta, \\
d\pi^1 &= \Lambda \wedge \rho - i\overline{\pi}^2 \wedge \kappa + \zeta \wedge \overline{\zeta}, \\
d\pi^2 &= \Lambda \wedge \kappa + \pi^2 \wedge \overline{\pi}^1 - \overline{\pi}^2 \wedge \zeta, \\
d\Lambda &= \Lambda \wedge \pi^1 + \Lambda \wedge \overline{\pi}^1 + i\pi^2 \wedge \overline{\pi}^2
\end{aligned}$$

and these constant coefficients equations correspond to the structure equations of the tube M^5_{LC} over the light cone, which is the reference model for this equivalence problem.

We would like to mention that, strictly speaking, Cartan's equivalence method of producing homogeneous models *requires* to normalize any group variable which occurs in some essential torsion term, and this is what Pocchiola did in Section 7 of [25] for $\mathsf{c} := (\boldsymbol{J}_0)^{1/3}$ and in Section 8 for $\mathsf{c} := \boldsymbol{W}_0$, showing afterwards that e can also be normalized in both cases.

For this deep reason, Pocchiola then *disregarded* the — essentially useless — task of constructing a general $\{e\}$-structure, since, when $\boldsymbol{J}_0 \equiv \boldsymbol{W}_0 \equiv 0$, the final Section 9 of [25] shows that one comes uniquely to the structure equations of the model M^5_{LC}, *without any further nonzero essential torsion appearing*. And this was really a discovery, because most of the times in CR geometry, primary invariants appear *after* a first prolongation.

However, because there is a tradition of setting up $\{e\}$-structures, even in absence of explicit computations, even without discovering invariants at all, and because the needs for *verifiable* computations has been expressed by some experts, we decided to set up the present article. While re-building this chapter [25] of Pocchiola's Ph.D. (Orsay University, September 2014), we found a few copying mistakes in some intermediate formulas of [25, 18], but no error in either statements or final formulas, *e.g.* $\boldsymbol{W}_0$ and $\boldsymbol{J}_0$ are correct.

For a more informative exposition of introductory aspects, the reader should read now the brief and complementary *Introduction* to the Addendum to [18].

This paper is organized as follows. In Section 2, we recall the local geometry of 2-nondegenerate Levi rank 1 real hypersurfaces M^5 in $\mathbb{C}^3$. In Section 3, we give a description of the G_1-structure of the biholomorphic equivalences of such real hypersurfaces. Section 4 gives a quick glimpse of a series of normalizations of parameters, which will be detailed in Sections 5 to 10, with the first appearance of $\boldsymbol{W}_0$ in Section 8. The explicit expression of the invariant $\boldsymbol{J}_0$ is given in Section 11, and a complete proof of the above formula for R is detailed in Section 12. Section 13 gives a short summary of the things that have been done in the previous sections, and finally Section 14 gives a proposed $\{e\}$-structure for the equivalence problem.

2. Local Geometry of 2-nondegenerate Levi rank 1 hypersurfaces $M^5 \subset \mathbb{C}^3$

This section only summarizes what has been presented and detailed in [19, 17, 18]. Let $M^5 \subset \mathbb{C}^3$ be a $\mathcal{C}^\omega$ (real-analytic) smooth, local or global, real hypersurface and let $p_0 \in M$. In any affine holomorphic coordinate system:

$$\big(z_1, z_2, w\big) \in \mathbb{C}^3 \qquad\qquad \text{with} \qquad\qquad w = u + i\, v,$$

centered at $p_0 = (0,0,0) = 0$ in which $\frac{\partial}{\partial u}\big|_0 \notin T_0M$, there is a local $\mathcal{C}^\omega$ graphing function $\boldsymbol{F} = \boldsymbol{F}\big(z_1, z_2, \overline{z}_1, \overline{z}_2, v\big)$ with $\boldsymbol{F}(0) = 0$ such that M is represented, in some (possibly small) open neighborhood of the origin 0 by

$$u = \boldsymbol{F}\big(z_1, z_2, \overline{z}_1, \overline{z}_2, v\big).$$

Convention 2.1. From now on, the hypersurface will be identified with its localization in some small open neighborhood of the origin, and it will always be denoted by M.

As is known (*see* [19] for detailed background), the complexified tangent bundle $\mathbb{C}TM := \mathbb{C} \otimes_{\mathbb{R}} TM$ inherits from $\mathbb{C}T\mathbb{C} := \mathbb{C} \otimes_{\mathbb{R}} T\mathbb{C}^3$ two biholomorphically invariant complex rank 2 vector subbundles

$$T^{1,0}M := T^{1,0}\mathbb{C}^3 \cap \mathbb{C}TM \qquad\qquad \text{and} \qquad\qquad T^{0,1}M := T^{0,1}\mathbb{C}^3 \cap \mathbb{C}TM = \overline{T^{1,0}M}$$

which are conjugate one to another. Then a check shows that the two vector fields written in the intrinsic coordinates $(z_1, z_2, \overline{z}_1, \overline{z}_2, v)$ on M:

$$\mathcal{L}_1 := \frac{\partial}{\partial z_1} + \boldsymbol{A}^1 \frac{\partial}{\partial v} \qquad\qquad \text{and} \qquad\qquad \mathcal{L}_2 := \frac{\partial}{\partial z_2} + \boldsymbol{A}^2 \frac{\partial}{\partial v},$$

whose coefficients are defined by:

$$\boldsymbol{A}^i := -\, i\, \frac{\boldsymbol{F}_{z_i}}{1 + i\, \boldsymbol{F}_v} \qquad (i = 1, 2),$$

generate $T^{1,0}M$, locally. Hence their two conjugates $\overline{\mathcal{L}}_1$, $\overline{\mathcal{L}}_2$ generate the bundle $T^{0,1}M$, also of complex rank 2.

Then visibly the differential 1-form

$$\varrho_0 := dv - \boldsymbol{A}^1\, dz_1 - \boldsymbol{A}^2\, dz_2 - \overline{\boldsymbol{A}}^1\, d\overline{z}_1 - \overline{\boldsymbol{A}}^2\, d\overline{z}_2$$

has kernel

$$\big\{\varrho_0 = 0\big\} = T^{1,0}M \oplus T^{0,1}M.$$

There are various (equivalent) aspects of the concept of *Levi form* of M, but they will not be recalled here, since several sources treat that. Here, the Levi form of M can be represented as a function of the points

$$p = \big(z_1, z_2, \overline{z}_1, \overline{z}_2, v\big) \in M,$$

valued in the space of Hermitian 2×2 matrices, and in terms of ϱ_0 and of the Lie brackets of the above vector fields, it writes as

$$\text{LF}_M(p) := \begin{pmatrix} \varrho_0\big(i\,[\mathcal{L}_1, \overline{\mathcal{L}}_1]\big) & \varrho_0\big(i\,[\mathcal{L}_2, \overline{\mathcal{L}}_1]\big) \\ \varrho_0\big(i\,[\mathcal{L}_1, \overline{\mathcal{L}}_2]\big) & \varrho_0\big(i\,[\mathcal{L}_2, \overline{\mathcal{L}}_2]\big) \end{pmatrix}(p).$$

As is known, the biholomorphic invariance of the Levi form legitimates our current

Hypothesis 2.2. [Uniform Levi rank 1] At all points $p \in M$, the Levi matrix (form) $\text{LF}_M(p)$ has constant rank 1.

After a linear change of coordinates in the (z_1, z_2) space, we may assume that its $(1,1)$-entry vanishes nowhere on M:

$$\varrho_0\big(i\,[\mathcal{L}_1, \overline{\mathcal{L}}_1]\big)(p) \neq 0 \quad (\forall\, p \in M).$$

This means that the *real* vector field

$$\mathcal{T} := i\,\big[\mathcal{L}_1, \overline{\mathcal{L}}_1\big] = i\,\Big(\mathcal{L}_1\big(\overline{\boldsymbol{A}}^1\big) - \overline{\mathcal{L}}_1\big(\boldsymbol{A}^1\big)\Big)\frac{\partial}{\partial v} =: \ell\,\frac{\partial}{\partial v}$$

has nowhere vanishing real coefficient that will be abbreviated as

$$\ell := i\,\Big(\overline{\boldsymbol{A}}^1_{z_1} + \boldsymbol{A}^1\,\overline{\boldsymbol{A}}^1_v - \boldsymbol{A}^1_{\overline{z}_1} - \overline{\boldsymbol{A}}^1\,\boldsymbol{A}^1_v\Big) \neq 0.$$

Furthermore, since the 2×2 Levi matrix has constant rank 1, the collection of its 1-dimensional kernels at all points $p \in M$ spans a $\mathcal{C}^\omega$ smooth subdistribution $K^{1,0}M \subset T^{1,0}M$ which satisfies ([19], pp. 72–73):

$$\begin{aligned}
\big[K^{1,0}M,\, K^{1,0}M\big] &\subset K^{1,0}M,\\
\big[K^{0,1}M,\, K^{0,1}M\big] &\subset K^{0,1}M,\\
\big[K^{1,0}M,\, K^{0,1}M\big] &\subset K^{1,0}M \oplus K^{0,1}M \quad \big(K^{0,1}M := \overline{K^{1,0}M}\big).
\end{aligned}$$

With this, a vector field generator $\mathcal{K}$ of $K^{1,0}M$ writes uniquely as

$$\mathcal{K} := \boldsymbol{k}\,\mathcal{L}_1 + \mathcal{L}_2,$$

where the function $\boldsymbol{k}$ — very important in the theory — is the negative of the quotient of two entries of the Levi matrix

$$\boldsymbol{k} := -\,\frac{\mathcal{L}_2\big(\overline{\boldsymbol{A}}^1\big) - \overline{\mathcal{L}}_1\big(\boldsymbol{A}^2\big)}{\mathcal{L}_1\big(\overline{\boldsymbol{A}}^1\big) - \overline{\mathcal{L}}_1\big(\boldsymbol{A}^1\big)}.$$

Hypothesis 2.3. **[2-nondegeneracy]** At all points $p \in M$, the *Freeman form* has constant (maximal possible) rank 1.

For a detailed presentation of this second concept of form, also biholomorphically invariant, *see* [19].

Proposition 2.4. ([19]) *In this formalism, M is 2-nondegenerate if and only if:*

$$\overline{\mathcal{L}}_1(\boldsymbol{k}) \neq 0 \quad (\text{everywhere on } M).$$

In summary, *two* functions will be assumed to be nowhere vanishing on M, corresponding to the two Hypotheses 2.2 and 2.3:

$$\ell(p) \neq 0 \qquad \text{and} \qquad \overline{\mathcal{L}}_1(\boldsymbol{k})(p) \neq 0 \quad (\forall\, p \in M).$$

Next, along with $\boldsymbol{k}$, introduce a second and last fundamental function

$$\boldsymbol{P} := \frac{\ell_{z_1} + \boldsymbol{A}^1\,\ell_v - \ell\,\boldsymbol{A}^1_v}{\ell}.$$

All invariants and semi-invariants in this paper will express in terms of $\boldsymbol{k}$ and $\boldsymbol{P}$.

Next, according to [17, 25, 18], there are 10 Lie bracket identities

$$\begin{aligned}
\big[\mathcal{T},\mathcal{L}_1\big] &= -\boldsymbol{P}\cdot\mathcal{T},\\
\big[\mathcal{T},\mathcal{K}\big] &= \mathcal{L}_1(\boldsymbol{k})\cdot\mathcal{T}+\mathcal{T}(\boldsymbol{k})\cdot\mathcal{L}_1,\\
\big[\mathcal{T},\overline{\mathcal{L}}_1\big] &= -\overline{\boldsymbol{P}}\cdot\mathcal{T},\\
\big[\mathcal{T},\overline{\mathcal{K}}\big] &= \overline{\mathcal{L}}_1\big(\overline{\boldsymbol{k}}\big)\cdot\mathcal{T}+\mathcal{T}\big(\overline{\boldsymbol{k}}\big)\cdot\overline{\mathcal{L}}_1,\\
\big[\mathcal{L}_1,\mathcal{K}\big] &= \mathcal{L}_1(\boldsymbol{k})\cdot\mathcal{L}_1,\\
\big[\mathcal{L}_1,\overline{\mathcal{L}}_1\big] &= -\,i\,\mathcal{T},\\
\big[\mathcal{L}_1,\overline{\mathcal{K}}\big] &= \mathcal{L}_1\big(\overline{\boldsymbol{k}}\big)\cdot\overline{\mathcal{L}}_1,\\
\big[\mathcal{K},\overline{\mathcal{L}}_1\big] &= -\,\overline{\mathcal{L}}_1(\boldsymbol{k})\cdot\mathcal{L}_1,\\
\big[\mathcal{K},\overline{\mathcal{K}}\big] &= 0,\\
\big[\overline{\mathcal{L}}_1,\overline{\mathcal{K}}\big] &= \overline{\mathcal{L}}_1\big(\overline{\boldsymbol{k}}\big)\cdot\overline{\mathcal{L}}_1.
\end{aligned}$$

Lemma 2.5. ([19, 17]) *The following* 3 *functional identities hold identically on* M*.*

$$\begin{aligned}
\mathcal{K}\big(\overline{\boldsymbol{k}}\big) &\equiv 0,\\
\mathcal{K}(\boldsymbol{P}) &\equiv -\boldsymbol{P}\,\mathcal{L}_1(\boldsymbol{k})-\mathcal{L}_1\big(\mathcal{L}_1(\boldsymbol{k})\big),\\
\mathcal{K}\big(\overline{\boldsymbol{P}}\big) &\equiv -\boldsymbol{P}\,\overline{\mathcal{L}}_1(\boldsymbol{k})-\overline{\mathcal{L}}_1\big(\mathcal{L}_1(\boldsymbol{k})\big)-i\,\mathcal{T}(\boldsymbol{k}).
\end{aligned}$$

□

Then the coframe

$$\big\{\rho_0,\,\kappa_0,\,\zeta_0,\,\overline{\kappa}_0,\,\overline{\zeta}_0\big\}$$

dual to the frame

$$\big\{\mathcal{T},\,\mathcal{L}_1,\,\mathcal{K},\,\overline{\mathcal{L}}_1,\,\overline{\mathcal{K}}\big\},$$

i.e. which satisfies by definition

$$\begin{array}{lllll}
\rho_0(\mathcal{T})=1, & \rho_0(\mathcal{L}_1)=0, & \rho_0(\mathcal{K})=0, & \rho_0(\overline{\mathcal{L}}_1)=0, & \rho_0(\overline{\mathcal{K}})=0,\\
\kappa_0(\mathcal{T})=0, & \kappa_0(\mathcal{L}_1)=1, & \kappa_0(\mathcal{K})=0, & \kappa_0(\overline{\mathcal{L}}_1)=0, & \kappa_0(\overline{\mathcal{K}})=0,\\
\zeta_0(\mathcal{T})=0, & \zeta_0(\mathcal{L}_1)=0, & \zeta_0(\mathcal{K})=1, & \zeta_0(\overline{\mathcal{L}}_1)=0, & \zeta_0(\overline{\mathcal{K}})=0,\\
\overline{\kappa}_0(\mathcal{T})=0, & \overline{\kappa}_0(\mathcal{L}_1)=0, & \overline{\kappa}_0(\mathcal{K})=0, & \overline{\kappa}_0(\overline{\mathcal{L}}_1)=1, & \overline{\kappa}_0(\overline{\mathcal{K}})=0,\\
\overline{\zeta}_0(\mathcal{T})=0, & \overline{\zeta}_0(\mathcal{L}_1)=0, & \overline{\zeta}_0(\mathcal{K})=0, & \overline{\zeta}_0(\overline{\mathcal{L}}_1)=0, & \overline{\zeta}_0(\overline{\mathcal{K}})=1,
\end{array}$$

has its 5 component 1-forms given explicitly by

$$\begin{aligned}
\rho_0 &= \frac{dv-\boldsymbol{A}^1dz_1-\boldsymbol{A}^2dz_2-\overline{\boldsymbol{A}}^1d\overline{z}_1-\overline{\boldsymbol{A}}^2d\overline{z}_2}{\ell},\\
\kappa_0 &= dz_1-\boldsymbol{k}\,dz_2,\\
\zeta_0 &= dz_2,\\
\overline{\kappa}_0 &= d\overline{z}_1-\overline{\boldsymbol{k}}\,d\overline{z}_2,\\
\overline{\zeta}_0 &= d\overline{z}_2.
\end{aligned}$$

Notice that a different notation $\rho_0\neq\varrho_0$ has been employed just now. Hence using a classical formula which goes back at least to Lie ([11, Chap. 5]) which holds for two arbitrary vector fields X and Y and for any differential 1-form ω

$$d\omega(X,Y)\,=\,X\,\big(\omega(Y)\big)-Y\big(\omega(X)\big)-\omega\big(\big[X,Y\big]\big),$$

by representing the 10 Lie brackets in some appropriate array

$$
\begin{array}{rclcccccccccc}
 & & \mathcal{T} & & \mathcal{L}_1 & & \mathcal{K} & & \overline{\mathcal{L}}_1 & & \overline{\mathcal{K}} & \\
 & & \boxed{d\rho_0} & & \boxed{d\kappa_0} & & \boxed{d\zeta_0} & & \boxed{d\overline{\kappa}_0} & & \boxed{d\overline{\zeta}_0} & \\
[\mathcal{T},\,\mathcal{L}_1] & = & -\boldsymbol{P}\cdot\mathcal{T} & + & 0 & + & 0 & + & 0 & + & 0 & \boxed{\rho_0\wedge\kappa_0} \\
[\mathcal{T},\,\mathcal{K}] & = & \mathcal{L}_1(\boldsymbol{k})\cdot\mathcal{T} & + & \mathcal{T}(\boldsymbol{k})\cdot\mathcal{L}_1 & + & 0 & + & 0 & + & 0 & \boxed{\rho_0\wedge\zeta_0} \\
[\mathcal{T},\,\overline{\mathcal{L}}_1] & = & -\overline{\boldsymbol{P}}\cdot\mathcal{T} & + & 0 & + & 0 & + & 0 & + & 0 & \boxed{\rho_0\wedge\overline{\kappa}_0} \\
[\mathcal{T},\,\overline{\mathcal{K}}] & = & \overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})\cdot\mathcal{T} & + & 0 & + & 0 & + & \mathcal{T}(\overline{\boldsymbol{k}})\cdot\overline{\mathcal{L}}_1 & + & 0 & \boxed{\rho_0\wedge\overline{\zeta}_0} \\
[\mathcal{L}_1,\,\mathcal{K}] & = & 0 & + & \mathcal{L}_1(\boldsymbol{k})\cdot\mathcal{L}_1 & + & 0 & + & 0 & + & 0 & \boxed{\kappa_0\wedge\zeta_0} \\
[\mathcal{L}_1,\,\overline{\mathcal{L}}_1] & = & -i\cdot\mathcal{T} & + & 0 & + & 0 & + & 0 & + & 0 & \boxed{\kappa_0\wedge\overline{\kappa}_0} \\
[\mathcal{L}_1,\,\overline{\mathcal{K}}] & = & 0 & + & 0 & + & 0 & + & \mathcal{L}_1(\overline{\boldsymbol{k}})\cdot\overline{\mathcal{L}}_1 & + & 0 & \boxed{\kappa_0\wedge\overline{\zeta}_0} \\
[\mathcal{K},\,\overline{\mathcal{L}}_1] & = & 0 & + & -\overline{\mathcal{L}}_1(\boldsymbol{k})\cdot\mathcal{L}_1 & + & 0 & + & 0 & + & 0 & \boxed{\zeta_0\wedge\overline{\kappa}_0} \\
[\mathcal{K},\,\overline{\mathcal{K}}] & = & 0 & + & 0 & + & 0 & + & 0 & + & 0 & \boxed{\zeta_0\wedge\overline{\zeta}_0} \\
[\overline{\mathcal{L}}_1,\,\overline{\mathcal{K}}] & = & 0 & + & 0\cdot\overline{\mathcal{L}}_1 & + & 0 & + & \overline{\mathcal{L}}_1(\overline{\boldsymbol{k}}) & + & 0 & \boxed{\overline{\kappa}_0\wedge\overline{\zeta}_0}
\end{array}
$$

and by reading this array *vertically*, we obtain the *initial Darboux-Cartan structure:*

$$
\begin{aligned}
d\rho_0 &= \boldsymbol{P}\cdot\rho_0\wedge\kappa_0 - \mathcal{L}_1(\boldsymbol{k})\cdot\rho_0\wedge\zeta_0 + \overline{\boldsymbol{P}}\cdot\rho_0\wedge\overline{\kappa}_0 - \overline{\mathcal{L}}_1\big(\overline{\boldsymbol{k}}\big)\cdot\rho_0\wedge\overline{\zeta}_0 + i\,\kappa_0\wedge\overline{\kappa}_0,\\
d\kappa_0 &= -\,\mathcal{T}(\boldsymbol{k})\cdot\rho_0\wedge\zeta_0 - \mathcal{L}_1(\boldsymbol{k})\cdot\kappa_0\wedge\zeta_0 + \overline{\mathcal{L}}_1(\boldsymbol{k})\cdot\zeta_0\wedge\overline{\kappa}_0,\\
d\zeta_0 &= 0,\\
d\overline{\kappa}_0 &= -\,\mathcal{T}\big(\overline{\boldsymbol{k}}\big)\cdot\rho_0\wedge\overline{\zeta}_0 - \mathcal{L}_1\big(\overline{\boldsymbol{k}}\big)\cdot\kappa_0\wedge\overline{\zeta}_0 - \overline{\mathcal{L}}_1\big(\overline{\boldsymbol{k}}\big)\cdot\overline{\kappa}_0\wedge\overline{\zeta}_0,\\
d\overline{\zeta}_0 &= 0.
\end{aligned}
$$

The fact that the frame $\big\{\mathcal{T},\mathcal{L}_1,\mathcal{K},\overline{\mathcal{L}}_1,\overline{\mathcal{K}}\big\}$ is dual to the coframe $\big\{\rho_0,\kappa_0,\zeta_0,\overline{\kappa}_0,\overline{\zeta}_0\big\}$ yields a formula that shall be used several times later.

Lemma 2.6. *The exterior differential of any function $\boldsymbol{G}=\boldsymbol{G}\big(z_1,z_2,\overline{z}_1,\overline{z}_2,v\big)$ on M expresses as*

$$
d\boldsymbol{G} \,=\, \mathcal{T}\big(\boldsymbol{G}\big)\,\rho_0 + \mathcal{L}_1\big(\boldsymbol{G}\big)\,\kappa_0 + \mathcal{K}\big(\boldsymbol{G}\big)\,\zeta_0 + \overline{\mathcal{L}}_1\big(\boldsymbol{G}\big)\,\overline{\kappa}_0 + \overline{\mathcal{K}}\big(\boldsymbol{G}\big)\,\overline{\zeta}_0.
$$

Proof. Indeed, starting from the definition

$$
d\boldsymbol{G} \,=\, \frac{\partial\boldsymbol{G}}{\partial v}\,dv + \frac{\partial\boldsymbol{G}}{\partial z_1}\,dz_1 + \frac{\partial\boldsymbol{G}}{\partial z_2}\,dz_2 + \frac{\partial\boldsymbol{G}}{\partial\overline{z}_1}\,d\overline{z}_1 + \frac{\partial\boldsymbol{G}}{\partial\overline{z}_2}\,d\overline{z}_2,
$$

and inverting the above coframe

$$
\begin{aligned}
dz_2 &= \zeta_0,\\
dz_1 &= \kappa_0 + \boldsymbol{k}\,\zeta_0,\\
dv &= \ell\,\rho_0 + \boldsymbol{A}^1\,\big(\kappa_0+\boldsymbol{k}\,\zeta_0\big) + \boldsymbol{A}^2\,\zeta_0 + \overline{\boldsymbol{A}}^1\,\big(\overline{\kappa}_0+\overline{\boldsymbol{k}}\,\overline{\zeta}_0\big) + \overline{\boldsymbol{A}}^2\,\overline{\zeta}_0\\
&= \ell\,\rho_0 + \boldsymbol{A}^1\,\kappa_0 + \big(\boldsymbol{A}^2+\boldsymbol{k}\boldsymbol{A}^1\big)\,\zeta_0 + \text{conjugates}
\end{aligned}
$$

we can replace, reorganize — unwritting the redundant conjugates — and reach the formula

$$
\begin{aligned}
d\boldsymbol{G} &\equiv \frac{\partial\boldsymbol{G}}{\partial v}\,\Big(\ell\,\rho_0 + \boldsymbol{A}^1\,\kappa_0 + (\boldsymbol{A}^2+\boldsymbol{k}\boldsymbol{A}^1)\,\zeta_0\Big) + \frac{\partial\boldsymbol{G}}{\partial z_1}\,(\kappa_0+\boldsymbol{k}\,\zeta_0) + \frac{\partial\boldsymbol{G}}{\partial z_2}\,\zeta_0\\
&\equiv \Big(\ell\,\frac{\partial}{\partial v}\Big)(\boldsymbol{G})\cdot\rho_0 + \Big(\frac{\partial}{\partial z_1} + \boldsymbol{A}^1\,\frac{\partial}{\partial v}\Big)(\boldsymbol{G})\cdot\kappa_0 + \Big(\frac{\partial}{\partial z_1} + \boldsymbol{A}^2\,\frac{\partial}{\partial v} + \boldsymbol{k}\,\frac{\partial}{\partial z_2} + \boldsymbol{k}\boldsymbol{A}^1\,\frac{\partial}{\partial v}\Big)(\boldsymbol{G})\cdot\zeta_0. \qquad \square
\end{aligned}
$$

For later much deeper computations, we need strong notational conventions. The order succession for our five 1-forms which we will constantly use

$$\{\rho_0,\,\kappa_0,\,\zeta_0,\,\overline{\kappa}_0,\overline{\zeta}_0,\},$$

induces an order succession for the ten generated 2-forms on the 5-dimensional CR manifold M

$$\begin{array}{cccc}
\underset{\mathbf{1}}{\rho_0\wedge\kappa_0} & \underset{\mathbf{2}}{\rho_0\wedge\zeta_0} & \underset{\mathbf{3}}{\rho_0\wedge\overline{\kappa}_0} & \underset{\mathbf{4}}{\rho_0\wedge\overline{\zeta}_0} \\
 & \underset{\mathbf{5}}{\kappa_0\wedge\zeta_0} & \underset{\mathbf{6}}{\kappa_0\wedge\overline{\kappa}_0} & \underset{\mathbf{7}}{\kappa_0\wedge\overline{\zeta}_0} \\
 & & \underset{\mathbf{8}}{\zeta_0\wedge\overline{\kappa}_0} & \underset{\mathbf{9}}{\zeta_0\wedge\overline{\zeta}_0} \\
 & & & \underset{\mathbf{10}}{\overline{\kappa}_0\wedge\overline{\zeta}_0.}
\end{array}$$

With such a numbering, we can abreviate the structure equations as — dropping their conjugates —

$$\begin{aligned}
d\rho_0 &= \boldsymbol{R}_0^1\,\rho_0\wedge\kappa_0+\boldsymbol{R}_0^2\,\rho_0\wedge\zeta_0+\boldsymbol{R}_0^3\,\rho_0\wedge\overline{\kappa}_0+\boldsymbol{R}_0^4\,\rho_0\wedge\overline{\zeta}_0+i\,\kappa_0\wedge\overline{\kappa}_0,\\
d\kappa_0 &= \boldsymbol{K}_0^2\,\rho_0\wedge\zeta_0+\boldsymbol{K}_0^5\,\kappa_0\wedge\zeta_0+\boldsymbol{K}_0^8\,\zeta_0\wedge\overline{\kappa}_0,\\
d\zeta_0 &= 0.
\end{aligned}$$

Convention 2.7. All functions of $p=(z_1,z_2,\overline{z}_1,\overline{z}_2,v)\in M$ will be denoted with a lower index $(\bullet)_0$, always employing the special auxiliary font characters $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\dots$.

After some transformations in the next sections, this initial coframe will change and become more complicated (unwriting the conjugates)

$$\{\rho_0,\,\kappa_0,\,\zeta_0\}\quad\rightsquigarrow\quad\{\rho_0,\,\kappa_0,\,\zeta_0'\}\quad\rightsquigarrow\quad\{\rho_0,\,\kappa_0',\,\zeta_0'\}\quad\rightsquigarrow\quad\{\rho_0,\,\kappa_0',\,\zeta_0''\},$$

and new structure function $\boldsymbol{R}_0^{i\prime},\boldsymbol{K}_0^{i\prime},\boldsymbol{Z}_0^{i\prime},\dots$ will appear.

We end up this section by stating some technical commutation relations that shall be constantly necessary to *normalize* incoming (complicated) expressions in order to avoid ambiguities. In fact, we can take advantage of $\overline{\mathcal{K}}(\boldsymbol{k})=0$ from Lemma 2.5, to make $\overline{\mathcal{K}}$ 'jump' above iterated derivatives like *e.g.* in

$$\overset{\curvearrowright}{\overline{\mathcal{K}}\;\big(\overline{\mathcal{L}}_1}(\boldsymbol{k})\big)\,,\qquad\qquad \overset{\curvearrowright}{\overline{\mathcal{K}}\;\big(\overline{\mathcal{L}}_1}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)\,.$$

Precisely, the last, 10th Lie bracket relation preceding Lemma 2.5

$$-\,\overline{\mathcal{L}}_1\big(\overline{\boldsymbol{k}}\big)\cdot\overline{\mathcal{L}}_1(\bullet)\;=\;\big[\overline{\mathcal{K}},\overline{\mathcal{L}}_1\big](\bullet),\tag{2.8}$$

when applied to the function $\bullet:=\boldsymbol{k}$ yields

$$\begin{aligned}
-\,\overline{\mathcal{L}}_1\big(\overline{\boldsymbol{k}}\big)\,\overline{\mathcal{L}}_1(\boldsymbol{k}) \;=\;\big[\overline{\mathcal{K}},\overline{\mathcal{L}}_1\big](\boldsymbol{k})\;&=\;\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)-\overline{\mathcal{L}}_1\big(\underline{\overline{\mathcal{K}}(\boldsymbol{k})}_{\circ}\big)\\
&=\;\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big).
\end{aligned}$$

Lemma 2.9. *One has the 3 relations*

$$\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\;=\;-\,\overline{\mathcal{L}}_1\big(\overline{\boldsymbol{k}}\big)\,\overline{\mathcal{L}}_1(\boldsymbol{k}),\tag{1}$$

$$\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)\;=\;-\,2\,\overline{\mathcal{L}}_1\big(\overline{\boldsymbol{k}}\big)\;\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)-\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\boldsymbol{k}}\big)\big)\;\overline{\mathcal{L}}_1(\boldsymbol{k}),\tag{2}$$

$$\begin{aligned}
\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)\big)\;&=\;-\,3\,\overline{\mathcal{L}}_1\big(\overline{\boldsymbol{k}}\big)\;\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)\\
&\quad-\,3\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\boldsymbol{k}}\big)\big)\;\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)-\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\boldsymbol{k}}\big)\big)\big)\;\overline{\mathcal{L}}_1(\boldsymbol{k}).
\end{aligned}\tag{3}$$

Proof. As **(1)** is done, we can apply $\overline{\mathcal{L}}_1(\bullet)$ to it, reversing sides

$$-\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{k})\big)\ \overline{\mathcal{L}}_1(k) - \overline{\mathcal{L}}_1(\overline{k})\ \overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big) \;=\; \overline{\mathcal{L}}_1\big(\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(k)\big)\big).$$

Similarly, we apply (2.8) to $\bullet := \overline{\mathcal{L}}_1(k)$ and we reach **(2)** after a replacement

$$-\overline{\mathcal{L}}_1(\overline{k})\ \overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big) \;=\; \big[\overline{\mathcal{K}},\overline{\mathcal{L}}_1\big]\big(\overline{\mathcal{L}}_1(k)\big) \;=\; \overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big) - \underbrace{\overline{\mathcal{L}}_1\big(\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(k)\big)\big)}_{\text{replace}}.$$

Now, as **(2)** is done, we can apply $\overline{\mathcal{L}}_1(\bullet)$ to it, and get after reorganization

$$\begin{aligned}\overline{\mathcal{L}}_1\big(\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)\big) \;=\; & -2\,\overline{\mathcal{L}}_1(\overline{k})\ \overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big) - 3\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{k})\big)\ \overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\\ & -\ \overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{k})\big)\big)\ \overline{\mathcal{L}}_1(k).\end{aligned}$$

Lastly, we apply (2.8) to $\bullet := \overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)$ and we reach **(3)** after a replacement

$$\begin{aligned}-\overline{\mathcal{L}}_1(\overline{k})\ \overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big) \;&=\; \big[\overline{\mathcal{K}},\overline{\mathcal{L}}_1\big]\Big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\Big)\\ &=\; \overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)\big) - \underbrace{\overline{\mathcal{L}}_1\big(\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)\big)}_{\text{replace}}.\end{aligned}$$

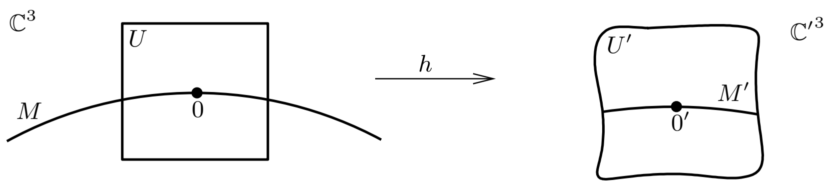
□

3. Initial G_1-structure for local biholomorphic equivalences $h\colon M \xrightarrow{\sim} M'$

Now, let $h\colon U \xrightarrow{\sim} U' \subset \mathbb{C}^3$ be a (local) biholomorphism from an open set $U \subset \mathbb{C}^3$ containing $U \ni 0$ the origin onto its image

$$h(U) \;=:\; U' \;\ni\; 0' \;=\; h(0),$$

which is also an open set $U' \subset \mathbb{C}'^3$ containing the origin $0'$ in another target complex Euclidean space $\mathbb{C}'^3$ having the same dimension.



As in Cartan's equivalence theory, assume that $h\big(M \cap U\big) \subset M'$ is contained in another real hypersurface $M' \subset \mathbb{C}'^3$, also passing through the origin $0' \in M'$, represented in holomorphic coordinates $\big(z_1', z_2', w' = u' + i\,v'\big)$ by a similar $\mathcal{C}^\omega$ graphed equation

$$u' \;=\; \boldsymbol{F'}\big(z_1', z_2', \overline{z}_1', \overline{z}_2', v'\big).$$

We now make the convention of not mentioning the open sets that must sometimes be shrunk, so that we think of $h\colon M \xrightarrow{\sim} M'$ as being a CR equivalence between hypersurfaces $M \subset \mathbb{C}^3$ and $M' \subset \mathbb{C}'^3$.

In the target space, introduce similar generators $\mathcal{L}_1'$, $\mathcal{L}_2'$ for $T^{1,0}M'$. Since h is holomorphic, its differential $h_*\colon \mathbb{C}T\mathbb{C}^3 \longrightarrow \mathbb{C}T\mathbb{C}'^3$ stabilizes holomorphic $(1,0)$ and holomorphic $(0,1)$ vector fields

$$h_*\big(T^{1,0}\mathbb{C}^3\big) \;=\; T^{1,0}\mathbb{C}'^3 \qquad\qquad \text{and} \qquad\qquad h_*\big(T^{0,1}M\big) \;=\; T^{0,1}M'.$$

Furthermore, by invariancy of the Freeman form, h respects the Levi-kernel distributions

$$h_*\big(K^{1,0}M\big) \;=\; K^{1,0}M'.$$

Consequently, there exist functions f', c', e' on M' such that

$$\begin{aligned} h_*(\mathcal{K}) &= f'\,\mathcal{K}', \\ h_*\big(\mathcal{L}_1\big) &= c'\,\mathcal{L}_1' + e'\,\mathcal{K}', \end{aligned}$$

whence by conjugation

$$\begin{aligned} h_*\big(\overline{\mathcal{K}}\big) &= \overline{f}'\,\overline{\mathcal{K}}', \\ h_*\big(\overline{\mathcal{L}}_1\big) &= \overline{c}'\,\overline{\mathcal{L}}_1' + \overline{e}'\,\overline{\mathcal{K}}'. \end{aligned}$$

On the other hand, there is *a priori* no special condition that shall be satisfied by $h_*(\mathcal{T})$, except that it be a real vector field, because $\mathcal{T}$ is real. Thus, there are a real-valued function a' and two complex-valued b' and d' on M' such that

$$h_*(\mathcal{T}) = a'\,\mathcal{T}' + b'\,\mathcal{L}_1' + d'\,\mathcal{K}' + \overline{b}'\,\overline{\mathcal{L}}_1' + \overline{d}'\,\overline{\mathcal{K}}'.$$

In fact, the function a' is determined, because

$$\begin{aligned} h_*(\mathcal{T}) = h_*\big(i\,\big[\mathcal{L}_1,\overline{\mathcal{L}}_1\big]\big) &= i\,\big[h_*\big(\mathcal{L}_1\big),\, h_*\big(\overline{\mathcal{L}}_1\big)\big] \\ &= i\,\big[c'\mathcal{L}_1' + e'\mathcal{K}',\ \overline{c}'\overline{\mathcal{L}}_1' + \overline{e}'\overline{\mathcal{K}}'\big] \\ &\equiv c'\overline{c}'\,i\,\big[\mathcal{L}_1',\,\overline{\mathcal{L}}_1'\big] \quad \mathsf{mod}\ \big(T^{1,0}M' \oplus T^{0,1}M'\big), \end{aligned}$$

whence necessarily

$$a' = c'\overline{c}'.$$

Summarizing, we have the following matrix relations

$$h_* \begin{pmatrix} \mathcal{T} \\ \mathcal{L}_1 \\ \mathcal{K} \\ \overline{\mathcal{L}}_1 \\ \overline{\mathcal{K}} \end{pmatrix} = \begin{pmatrix} c'\overline{c}' & b' & d' & \overline{b}' & \overline{d}' \\ 0 & c' & e' & 0 & 0 \\ 0 & 0 & f' & 0 & 0 \\ 0 & 0 & 0 & \overline{c}' & \overline{e}' \\ 0 & 0 & 0 & 0 & \overline{f}' \end{pmatrix} \begin{pmatrix} \mathcal{T}' \\ \mathcal{L}_1' \\ \mathcal{K}' \\ \overline{\mathcal{L}}_1' \\ \overline{\mathcal{K}}' \end{pmatrix}.$$

As h_* is invertible, the function f', and then the function c' too, must be nowhere vanishing. The relation between the coframe $\{\rho_0, \kappa_0, \zeta_0, \overline{\kappa}_0, \overline{\zeta}_0\}$ in the source space and the coframe $\{\rho_0', \kappa_0', \zeta_0', \overline{\kappa}_0', \overline{\zeta}_0'\}$ in the target space is therefore given by a plain transposition

$$h^* \begin{pmatrix} \rho_0' \\ \kappa_0' \\ \zeta_0' \\ \overline{\kappa}_0' \\ \overline{\zeta}_0' \end{pmatrix} = \begin{pmatrix} c'\overline{c}' & 0 & 0 & 0 & 0 \\ b' & c' & 0 & 0 & 0 \\ d' & e' & f' & 0 & 0 \\ \overline{b}' & 0 & 0 & \overline{c}' & 0 \\ \overline{d}' & 0 & 0 & \overline{e}' & \overline{f}' \end{pmatrix} \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \zeta_0 \\ \overline{\kappa}_0 \\ \overline{\zeta}_0 \end{pmatrix}.$$

These preliminaries, also explained in [16, 25, 18], justify that the initial G-structure for such equivalences of CR manifolds is the matrix ambiguity group G_1 is constituted of 5×5 matrices of the form

$$\begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \mathsf{f} & 0 & 0 \\ \overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ \overline{\mathsf{d}} & 0 & 0 & \overline{\mathsf{e}} & \overline{\mathsf{f}} \end{pmatrix}$$

with free variable complex entries

$$\mathsf{c},\, \mathsf{f} \in \mathbb{C}\backslash\{0\} \qquad \text{and} \qquad \mathsf{b},\, \mathsf{d},\, \mathsf{e} \in \mathbb{C},$$

namely

$$\begin{pmatrix} \rho \\ \kappa \\ \zeta \\ \overline{\kappa} \\ \overline{\zeta} \end{pmatrix} := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \mathsf{f} & 0 & 0 \\ \overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ \overline{\mathsf{d}} & 0 & 0 & \overline{\mathsf{e}} & \overline{\mathsf{f}} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \zeta_0 \\ \overline{\kappa}_0 \\ \overline{\zeta}_0 \end{pmatrix}.$$

Eliminating the conjugate 1-forms $\overline{\kappa}$, $\overline{\zeta}$ for which the structure equations are redundant, this can be abbreviated as

$$\begin{pmatrix} \rho \\ \kappa \\ \zeta \end{pmatrix} := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 \\ \mathsf{d} & \mathsf{e} & \mathsf{f} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \zeta_0 \end{pmatrix}.$$

4. A Labyrinthmap to Pocchiola's Calculations

The successive reductions of this G_1 structure will look as

$$g := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 \\ \mathsf{d} & \mathsf{e} & \mathsf{f} \end{pmatrix} \quad \rightsquigarrow \quad g := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} \end{pmatrix} \quad \rightsquigarrow \quad g := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 \\ -i\,\overline{\mathsf{c}}\mathsf{e} & \mathsf{c} & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} \end{pmatrix}$$

$$\rightsquigarrow \quad g := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 \\ -i\,\overline{\mathsf{c}}\mathsf{e} & \mathsf{c} & 0 \\ -\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{e}^2}{\mathsf{c}} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} \end{pmatrix},$$

thanks to successive normalization of some group parameters (offered by some essential torsion coefficients yielding invariants that are deeper than Levi and Freeman forms)

$$\mathsf{f} := \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\overline{\mathcal{L}}_1(\boldsymbol{k}), \qquad \mathsf{b} := -\,i\,\overline{\mathsf{c}}\,\mathsf{e} + \frac{i}{3}\,\mathsf{c}\,\boldsymbol{B}_0, \qquad \mathsf{d} := -\,\frac{i}{2}\,\frac{\overline{\mathsf{c}}\,\mathsf{e}\,\mathsf{e}}{\mathsf{c}} + i\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\boldsymbol{H}_0,$$

in terms of the following two function on M

$$\boldsymbol{B}_0 := \frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \overline{\boldsymbol{P}},$$

$$\boldsymbol{H}_0 := -\frac{1}{6}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} + \frac{2}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)^2}{\overline{\mathcal{L}}_1(\boldsymbol{k})^2} + \frac{1}{18}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\,\overline{\boldsymbol{P}}}{\overline{\mathcal{L}}_1(\boldsymbol{k})} + \frac{1}{6}\,\overline{\mathcal{L}_1}\big(\overline{\boldsymbol{P}}\big) - \frac{1}{9}\,\overline{\boldsymbol{P}}^2.$$

This function $\boldsymbol{H}_0$ coincides with Pocchiola's function H.

The next sections will present in details these successive reductions of G-structures, by these normalizations of the group parameters f, b, d. Contrary to [18, 25], all computations will be progressive, simple, detailed, readable, clear, without needing any help of either a computer or a pen. A great care will be devoted to readability.

5. First Loop: Reduction of the Group Parameter f

We recall that the initial Darboux-Cartan structure of the coframe $\{\rho_0, \kappa_0, \zeta_0, \overline{\kappa}_0, \overline{\zeta}_0\}$ is, without writing conjugate equations — remind $\overline{\rho}_0 = \rho_0$ —

$$\begin{aligned} d\rho_0 &= \boldsymbol{P}\,\rho_0\wedge\kappa_0 - \mathcal{L}_1(\boldsymbol{k})\,\rho_0\wedge\zeta_0 + \overline{\boldsymbol{P}}\,\rho_0\wedge\overline{\kappa}_0 - \overline{\mathcal{L}}_1\big(\overline{\boldsymbol{k}}\big)\,\rho_0\wedge\overline{\zeta}_0 + i\,\kappa_0\wedge\overline{\kappa}_0, \\ d\kappa_0 &= -\,\mathcal{T}(\boldsymbol{k})\,\rho_0\wedge\zeta_0 - \mathcal{L}_1(\boldsymbol{k})\,\kappa_0\wedge\zeta_0 + \overline{\mathcal{L}}_1(\boldsymbol{k})\,\zeta_0\wedge\overline{\kappa}_0, \\ d\zeta_0 &= 0. \end{aligned} \tag{5.1}$$

With the first G-structure exhibited above, introduce the *lifted differential forms*, defined by

$$\begin{pmatrix} \rho \\ \kappa \\ \zeta \end{pmatrix} := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 \\ \mathsf{d} & \mathsf{e} & \mathsf{f} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \zeta_0 \end{pmatrix},$$

id est

$$\begin{aligned} \rho &:= \mathsf{c}\overline{\mathsf{c}}\,\rho_0, \\ \kappa &:= \mathsf{b}\,\rho_0 + \mathsf{c}\,\kappa_0, \\ \zeta &:= \mathsf{d}\,\rho_0 + \mathsf{e}\,\kappa_0 + \mathsf{f}\,\zeta_0. \end{aligned}$$

Here, $\mathsf{c}, \mathsf{f} \in \mathbb{C}^*$ and $\mathsf{b}, \mathsf{e}, \mathsf{d} \in \mathbb{C}$. Mind that conjugate equations giving $\overline{\kappa}$ and $\overline{\zeta}$ are not written, but will be used.
An inversion yields

$$\begin{aligned} \rho_0 &= \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,\rho, \\ \kappa_0 &= \frac{1}{\mathsf{c}}\,\kappa - \frac{\mathsf{b}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}\,\rho, \\ \zeta_0 &= \frac{\mathsf{b}\mathsf{e} - \mathsf{c}\mathsf{d}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}\mathsf{f}}\,\rho - \frac{\mathsf{e}}{\mathsf{c}\mathsf{f}}\,\kappa + \frac{1}{\mathsf{f}}\,\zeta. \end{aligned} \tag{5.2}$$

With the above 3×3 matrix g representing the general element of a 10-dimensional (real) group $G^{10} \subset \mathsf{GL}_3(\mathbb{C})$, the Maurer-Cartan matrix is

$$\begin{aligned} dg \cdot g^{-1} &= \begin{pmatrix} \overline{\mathsf{c}}\,d\mathsf{c} + \mathsf{c}d\overline{\mathsf{c}} & 0 & 0 \\ d\mathsf{b} & d\mathsf{c} & 0 \\ d\mathsf{d} & d\mathsf{e} & d\mathsf{f} \end{pmatrix} \begin{pmatrix} \frac{1}{\mathsf{c}\overline{\mathsf{c}}} & 0 & 0 \\ -\frac{\mathsf{b}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}} & \frac{1}{\mathsf{c}} & 0 \\ \frac{\mathsf{b}\mathsf{e}-\mathsf{c}\mathsf{d}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}\mathsf{f}} & -\frac{\mathsf{e}}{\mathsf{c}\mathsf{f}} & \frac{1}{\mathsf{f}} \end{pmatrix} \\ &=: \begin{pmatrix} \alpha + \overline{\alpha} & 0 & 0 \\ \beta & \alpha & 0 \\ \gamma & \delta & \varepsilon \end{pmatrix} \end{aligned}$$

in terms of the group-invariant 1-forms

$$\begin{aligned} \alpha &:= \frac{d\mathsf{c}}{\mathsf{c}}, \\ \beta &:= \frac{d\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}\overline{\mathsf{c}}\,d\mathsf{c}}{\mathsf{c}\mathsf{c}}, \\ \gamma &:= \frac{d\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}\,d\mathsf{e}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{b}\mathsf{e} - \mathsf{c}\mathsf{d}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}\mathsf{f}}\,d\mathsf{f}, \\ \delta &:= \frac{d\mathsf{e}}{\mathsf{c}} - \frac{\mathsf{e}\,d\mathsf{f}}{\mathsf{c}\mathsf{f}}, \\ \varepsilon &:= \frac{d\mathsf{f}}{\mathsf{f}}. \end{aligned}$$

As is known, after painful computations whose outcomes are presented extensively in [25, 18], one can re-express, using (5.1) and (5.2), the exterior differentials of the 3 lifted 1-forms ρ, ζ, κ

as

$$\begin{aligned} d\rho &= \alpha \wedge \rho + \overline{\alpha} \wedge \rho \\ &\quad + R^1\, \rho \wedge \kappa + R^2\, \rho \wedge \zeta + R^3\, \rho \wedge \overline{\kappa} + R^4\, \rho \wedge \overline{\zeta} + i\, \kappa \wedge \overline{\kappa}, \\ d\kappa &= \beta \wedge \rho + \alpha \wedge \kappa \\ &\quad + K^1\, \rho \wedge \kappa + K^2\, \rho \wedge \zeta + K^3\, \rho \wedge \overline{\kappa} + K^4\, \rho \wedge \overline{\zeta} \\ &\quad + K^5\, \kappa \wedge \zeta + K^5\, \kappa \wedge \overline{\kappa} + \boxed{K^8}\, \zeta \wedge \overline{\kappa}, \\ d\zeta &= \gamma \wedge \rho + \delta \wedge \kappa + \varepsilon \wedge \zeta \\ &\quad + Z^1\, \rho \wedge \kappa + Z^2\, \rho \wedge \zeta + Z^3\, \rho \wedge \overline{\kappa} + Z^4\, \rho \wedge \overline{\zeta} \\ &\quad + Z^5\, \kappa \wedge \zeta + Z^6\, \kappa \wedge \overline{\kappa} + Z^8\, \zeta \wedge \overline{\kappa} \end{aligned}$$

in terms of certain complicated functions R^i, K^i, Z^i of the horizontal variables and of the group parameters as well

$$\big(z_1, z_2, \overline{z}_1, \overline{z}_2, v\big) \times \big(\mathsf{c}, \overline{\mathsf{c}}, \mathsf{f}, \overline{\mathsf{f}}, \mathsf{b}, \overline{\mathsf{b}}, \mathsf{d}, \overline{\mathsf{d}}, \mathsf{e}, \overline{\mathsf{e}}\big) \;\in\; M^5 \times G^{10},$$

but we shall not need the expressions of all these functions, and focus only on the boxed one, K^8, since it will bring an interesting normalization for the diagonal group parameter f.

Notation 5.3. Given a differential 2-form $\Omega \in \Gamma(M, \Lambda^2 T^*M)$ on an n-dimensional manifold M equipped with a coframe $\{\omega^1, \dots, \omega^n\}$ for its cotangent bundle T^*M, which is expanded as

$$\Omega \;=\; \sum_{1 \leqslant i < j \leqslant n} A_{i,j}\, \omega^i \wedge \omega^j,$$

with uniquely determined coefficients-functions $A_{\bullet,\bullet}$, for fixed $i < j$, the coefficient $A_{i,j}$ of $\omega^i \wedge \omega^j$ will be denoted by

$$\big[\omega^i \wedge \omega^j\big]\big\{\Omega\big\} \;:=\; A_{i,j}.$$

To capture K^8 without pain, the computation/re-expression of $d\kappa$ starts from $\kappa = \mathsf{b}\, \rho_0 + \mathsf{c}\, \kappa_0$ as follows to see how Maurer-Cartan forms enter the play

$$\begin{aligned} d\kappa &= d\mathsf{b} \wedge \rho_0 + d\mathsf{c} \wedge \kappa_0 + \mathsf{b}\, d\rho_0 + \mathsf{c}\, d\kappa_0 \\ &= d\mathsf{b} \wedge \big(\tfrac{1}{\mathsf{c}\overline{\mathsf{c}}}\, \rho\big) + d\mathsf{c} \wedge \big(\tfrac{1}{\mathsf{c}}\, \kappa - \tfrac{\mathsf{b}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}\, \rho\big) + \mathsf{Torsion} \\ &= \big(d\mathsf{b} - \tfrac{\mathsf{b}\, d\mathsf{c}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}\big) \wedge \rho + \big(\tfrac{d\mathsf{c}}{\mathsf{c}}\big) \wedge \kappa + \mathsf{Torsion} \\ &= \beta \wedge \rho + \alpha \wedge \kappa + \mathsf{Torsion}. \end{aligned}$$

Certainly, K^8 belongs to the torsion remainder, and we want to determine only

$$K^8 \;:=\; \big[\zeta \wedge \overline{\kappa}\big]\big\{d\kappa\big\} \;=\; \big[\zeta \wedge \overline{\kappa}\big]\big\{\mathsf{b}\, d\rho_0 + \mathsf{c}\, d\kappa_0\big\}.$$

For the first term $\mathsf{b}\, d\rho_0$, we look at (5.1) in which we replace *visually* $\rho_0, \zeta_0, \kappa_0$ by ρ, ζ, κ watching simultaneously (5.2) — no pen needed! computers shut down! — and we get

$$\mathsf{b}\,\big[\zeta \wedge \overline{\kappa}\big]\big\{d\rho_0\big\} \;=\; 0 + 0 + 0 + 0 + 0 \;=\; 0.$$

Proceeding similarly, just with eyes

$$\begin{aligned} \mathsf{c}\,\big[\zeta \wedge \overline{\kappa}\big]\big\{d\kappa_0\big\} &= 0 + 0 + \mathsf{c}\, \overline{\mathcal{L}}_1(\boldsymbol{k})\, \big[\zeta \wedge \overline{\kappa}\big] \Big\{ \Big(\frac{\mathsf{be} - \mathsf{bd}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}\mathsf{f}}\, \rho - \frac{\mathsf{e}}{\mathsf{c}\mathsf{f}}\, \kappa + \frac{1}{\mathsf{f}}\, \zeta \Big) \wedge \Big(- \frac{\overline{\mathsf{b}}}{\overline{\mathsf{c}\mathsf{c}}\mathsf{c}}\, \rho + \frac{1}{\overline{\mathsf{c}}}\, \overline{\kappa} \Big) \\ &= \mathsf{c}\, \overline{\mathcal{L}}_1(\boldsymbol{k})\, \big(\tfrac{1}{\mathsf{f}}\big)\, \big(\tfrac{1}{\overline{\mathsf{c}}}\big), \end{aligned}$$

whence adding

$$K^8 \;=\; \frac{\mathsf{c}}{\overline{\mathsf{c}}\mathsf{f}}\, \overline{\mathcal{L}}_1(\boldsymbol{k}).$$

Furthermore, without computation, we see that K^8 is *not* absorbable in the Maurer-Cartan part $\beta \wedge \rho + \alpha \wedge \kappa$ by means of any replacement

$$\alpha = \alpha' + a_1 \rho + a_2 \kappa + a_3 \zeta + a_4 \overline{\kappa} + a_5 \overline{\zeta},$$
$$\beta = \beta' + b_1 \rho + b_2 \kappa + b_3 \zeta + b_4 \overline{\kappa} + b_5 \overline{\zeta},$$

because the result will always be

$$\mathsf{something} \wedge \rho + \mathsf{something} \wedge \kappa,$$

whereas $K^8 \zeta \wedge \overline{\kappa}$ is not $\wedge$-divisible by either $\wedge\rho$ or $\wedge\kappa$.

Consequently, K^8 is an essential torsion coefficient, and by general Cartan theory, K^8 may bring a group parameter normalization.

In fact, since the diagonal coefficients $\mathsf{c} \neq 0 \neq \mathsf{f}$ of the invertible triangular matrix must be nonvanishing, and since $\overline{\mathcal{L}}_1(\boldsymbol{k}) \neq 0$ is nowhere vanishing by our assumption of 2-nondegeneracy, it is natural, then, to normalize K^8 to be constant nonzero, *e.g.* $K^8 := 1$, and this yields a reduction of the G^{10}-structure to an eight-dimensional G^8-structure by setting

$$\mathsf{f} := \frac{\mathsf{c}}{\overline{\mathsf{c}}} \overline{\mathcal{L}}_1(\boldsymbol{k}).$$

Inserting this in the lifted coframe

$$\begin{pmatrix} \rho \\ \kappa \\ \zeta \end{pmatrix} := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} \overline{\mathcal{L}}_1(\boldsymbol{k}) \end{pmatrix} \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \zeta_0 \end{pmatrix},$$

we are conducted to change the initial coframe by introducing the new *horizontal* — *i.e.* defined on M — 1-form

(5.4) $$\zeta_0' := \overline{\mathcal{L}}_1(\boldsymbol{k})\, \zeta_0.$$

As anticipated in a summary *supra*, we are thus changing of horizontal coframe

$$\{\rho_0, \kappa_0, \zeta_0, \overline{\kappa}_0, \overline{\zeta}_0\} \quad \rightsquigarrow \quad \{\rho_0, \kappa_0, \zeta_0', \overline{\kappa}_0, \overline{\zeta}_0'\},$$

and unavoidably, we have to set up its Darboux-Cartan structure.

Thanks to Lemma 2.6, we can compute

$$\begin{aligned} d\zeta_0' &= d\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big) \wedge \zeta_0 + \overline{\mathcal{L}}_1(\boldsymbol{k}) \wedge \underline{d\zeta_0}_{\circ} \\ &= \mathcal{T}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\, \rho_0 \wedge \zeta_0 + \mathcal{L}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\, \kappa_0 \wedge \zeta_0 + \mathcal{K}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\, \underline{\zeta_0 \wedge \zeta_0}_{\circ} + \overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\, \overline{\kappa}_0 \wedge \zeta_0 \\ &\qquad + \overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\, \overline{\zeta}_0 \wedge \zeta_0 + 0, \end{aligned}$$

and next, replacing everywhere $\zeta_0 = \frac{\zeta_0'}{\overline{\mathcal{L}}_1(\boldsymbol{k})}$, reorganizing, and transforming the last term above in application of Lemma 2.9 **(1)**, we obtain the structure equations enjoyed by this new initial base coframe

$$\begin{aligned} d\rho_0 &= \boldsymbol{P}\, \rho_0 \wedge \kappa_0 - \frac{\mathcal{L}_1(\boldsymbol{k})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\, \rho_0 \wedge \zeta_0' + \overline{\boldsymbol{P}}\, \rho_0 \wedge \overline{\kappa}_0 - \frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}\, \rho_0 \wedge \overline{\zeta}_0' + i\, \kappa_0 \wedge \overline{\kappa}_0, \\ d\kappa_0 &= -\frac{\mathcal{T}(\boldsymbol{k})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\, \rho_0 \wedge \zeta_0' - \frac{\mathcal{L}_1(\boldsymbol{k})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\, \kappa_0 \wedge \zeta_0' + \zeta_0' \wedge \overline{\kappa}_0, \\ d\zeta_0' &= \frac{\mathcal{T}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\, \rho_0 \wedge \zeta_0' + \frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\, \kappa_0 \wedge \zeta_0' - \frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\, \zeta_0' \wedge \overline{\kappa}_0 + \frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}\, \zeta_0' \wedge \overline{\zeta}_0'. \end{aligned} \tag{5.5}$$

Sometimes, it can be useful to abbreviate these formulas as

$$\begin{aligned} d\rho_0 &= \boldsymbol{R}_0^1\,\rho_0\wedge\kappa_0 + \boldsymbol{R}_0^2\,\rho_0\wedge\zeta_0' + \overline{\boldsymbol{R}}_0^1\,\rho_0\wedge\overline{\kappa}_0 + \overline{\boldsymbol{R}}_2^0\,\rho_0\wedge\overline{\zeta}_0' + i\,\kappa_0\wedge\overline{\kappa}_0,\\ d\kappa_0 &= \boldsymbol{K}_0^2\,\rho_0\wedge\zeta_0' + \boldsymbol{K}_0^5\,\kappa_0\wedge\zeta_0' + \zeta_0'\wedge\overline{\kappa}_0,\\ d\zeta_0' &= \boldsymbol{Z}_0^2\,\rho_0\wedge\zeta_0' + \boldsymbol{Z}_0^5\,\kappa_0\wedge\zeta_0' + \boldsymbol{Z}_0^8\,\zeta_0'\wedge\overline{\kappa}_0 + \boldsymbol{Z}_0^9\,\zeta_0'\wedge\overline{\zeta}_0', \end{aligned}$$

and no *primes* will be appended to these coefficients-functions, for the reason that *exactly two* further changes of initial base coframes

$$\big\{\rho_0,\,\kappa_0,\,\zeta_0',\,\overline{\kappa}_0,\,\overline{\zeta}_0'\big\} \quad\rightsquigarrow\quad \big\{\rho_0,\,\kappa_0',\,\zeta_0',\,\overline{\kappa}_0',\,\overline{\zeta}_0'\big\} \quad\rightsquigarrow\quad \big\{\rho_0,\,\kappa_0',\,\zeta_0'',\,\overline{\kappa}_0',\,\overline{\zeta}_0''\big\}$$

will force us to introduce *e.g.* $\boldsymbol{Z}_0^{i\prime}$ and $\boldsymbol{Z}_0^{i\prime\prime}$, so that we will avoid to use primes trice.

6. Second Loop: Reduction of the Group Parameter b

With this new reduced (real) eight-dimensional group G^8, the lifted coframe, in which for simplicity, we use the same letters ρ, κ, ζ as before, becomes

$$\begin{pmatrix}\rho\\ \kappa\\ \zeta\end{pmatrix} := \begin{pmatrix}\mathsf{c}\overline{\mathsf{c}} & 0 & 0\\ \mathsf{b} & \mathsf{c} & 0\\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}}\end{pmatrix}\begin{pmatrix}\rho_0\\ \kappa_0\\ \zeta_0'\end{pmatrix} \qquad\Longleftrightarrow\qquad \begin{cases}\rho := \mathsf{c}\overline{\mathsf{c}}\,\rho_0,\\ \kappa := \mathsf{b}\,\rho_0 + \mathsf{c}\,\kappa_0,\\ \zeta := \mathsf{d}\,\rho_0 + \mathsf{e}\,\kappa_0 + \dfrac{\mathsf{c}}{\overline{\mathsf{c}}}\,\zeta_0',\end{cases}$$

and inverse formulas are

$$\begin{aligned} \rho_0 &= \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,\rho,\\ \kappa_0 &= -\frac{\mathsf{b}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}\,\rho + \frac{1}{\mathsf{c}}\,\kappa,\\ \zeta_0' &= \frac{\mathsf{b}\mathsf{e}-\mathsf{c}\mathsf{d}}{\mathsf{c}\mathsf{c}\mathsf{c}}\,\rho - \frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\kappa + \frac{\overline{\mathsf{c}}}{\mathsf{c}}\,\zeta. \end{aligned} \tag{6.1}$$

The Maurer-Cartan matrix becomes

$$\begin{aligned} dg\cdot g^{-1} &= \begin{pmatrix}\overline{\mathsf{c}}\,d\mathsf{c} + \mathsf{c}\,d\overline{\mathsf{c}} & 0 & 0\\ d\mathsf{b} & d\mathsf{c} & 0\\ d\mathsf{d} & d\mathsf{e} & \frac{d\mathsf{c}}{\overline{\mathsf{c}}} - \frac{\mathsf{c}\,d\overline{\mathsf{c}}}{\overline{\mathsf{c}}\overline{\mathsf{c}}}\end{pmatrix}\begin{pmatrix}\frac{1}{\mathsf{c}\overline{\mathsf{c}}} & 0 & 0\\ -\frac{\mathsf{b}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}} & \frac{1}{\mathsf{c}} & 0\\ \frac{\mathsf{b}\mathsf{e}-\mathsf{c}\mathsf{d}}{\mathsf{c}\mathsf{c}\mathsf{c}} & -\frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}}\end{pmatrix}\\ &=: \begin{pmatrix}\alpha+\overline{\alpha} & 0 & 0\\ \beta & \alpha & 0\\ \gamma & \delta & \alpha-\overline{\alpha}\end{pmatrix}, \end{aligned}$$

in terms of the group-invariant 1-forms

$$\begin{aligned} \alpha &:= \frac{d\mathsf{c}}{\mathsf{c}},\\ \beta &:= \frac{d\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}\,d\mathsf{c}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}},\\ \gamma &:= \frac{d\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}\,d\mathsf{e}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{b}\mathsf{e}-\mathsf{c}\mathsf{d}}{\mathsf{c}\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}\,d\mathsf{c} - \frac{\mathsf{b}\mathsf{e}-\mathsf{c}\mathsf{d}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}\overline{\mathsf{c}}}\,d\overline{\mathsf{c}},\\ \delta &:= \frac{d\mathsf{e}}{\mathsf{c}} - \frac{\mathsf{e}\,d\mathsf{c}}{\mathsf{c}\mathsf{c}} + \frac{\mathsf{e}\,d\overline{\mathsf{c}}}{\mathsf{c}\overline{\mathsf{c}}}. \end{aligned}$$

Now, let us exterior-differentiate the lifted coframe on the product manifold equipped with coordinates

$$\big(z_1, z_2, \overline{z}_1, \overline{z}_2, v\big)\times\big(\mathsf{c},\overline{\mathsf{c}},\mathsf{b},\overline{\mathsf{b}},\mathsf{d},\overline{\mathsf{d}},\mathsf{e},\overline{\mathsf{e}}\big) \;\in\; M^5\times G^8.$$

The computation starts as

$$
\begin{aligned}
d\rho &= \big(\overline{\mathsf{c}}\, d\mathsf{c} + \mathsf{c}\, d\overline{\mathsf{c}}\big) \wedge \rho_0 + \mathsf{c}\overline{\mathsf{c}}\, d\rho_0, \\
d\kappa &= d\mathsf{b} \wedge \rho_0 + d\mathsf{c} \wedge \kappa_0 + \mathsf{b}\, d\rho_0 + \mathsf{c}\, d\kappa_0, \\
d\zeta &= d\mathsf{d} \wedge \rho_0 + d\mathsf{e} \wedge \kappa_0 + \bigg(\frac{d\mathsf{c}}{\overline{\mathsf{c}}} - \frac{\mathsf{c}\, d\overline{\mathsf{c}}}{\overline{\mathsf{c}}\overline{\mathsf{c}}}\bigg) \wedge \zeta_0' + \mathsf{d}\, d\rho_0 + \mathsf{e}\, d\kappa_0 + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\, d\zeta_0'.
\end{aligned}
\tag{6.2}
$$

As is known, one must replace in second lines $d\rho_0$, $d\kappa_0$, $d\zeta_0'$ by the structure equations (5.5), and after, replace everywhere ρ_0, κ_0, ζ_0', using the inversion formulas (6.1).

However, contrary to Pocchiola's systematic approach, we will *not* perform these calculations completely, but select only meaningful terms.

At least, at the level of Maurer-Cartan forms, after replacements of ρ_0, κ_0, ζ_0' in the first lines of (6.2) above using (6.1), we have as usual

$$
\begin{aligned}
d\rho &= \big(\alpha + \overline{\alpha}\big) \wedge \rho + \mathsf{Torsion}, \\
d\kappa &= \beta \wedge \rho + \alpha \wedge \kappa + \mathsf{Torsion}, \\
d\zeta &= \gamma \wedge \rho + \delta \wedge \kappa + \big(\alpha - \overline{\alpha}\big) \wedge \zeta + \mathsf{Torsion}.
\end{aligned}
$$

Question 6.3. *Without computing everything, what are the shapes of the three Torsion remainders?*

Consider for instance what happens of the last term $\frac{\mathsf{c}}{\overline{\mathsf{c}}}\, d\zeta_0'$ in $d\zeta$, when performing the required replacements, and restrict attention even to the last term of $\frac{\mathsf{c}}{\overline{\mathsf{c}}}\, d\zeta_0'$ in (5.5), which becomes

$$
\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})}\,\zeta_0' \wedge \overline{\zeta}_0' = \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})}\bigg(\frac{\mathsf{be} - \mathsf{cd}}{\mathsf{cc}\overline{\mathsf{c}}}\,\rho - \frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{cc}}\,\kappa + \frac{\overline{\mathsf{c}}}{\mathsf{c}}\,\zeta\bigg) \wedge \bigg(\frac{\overline{\mathsf{be}} - \overline{\mathsf{cd}}}{\overline{\mathsf{cc}}}\,\rho - \frac{\mathsf{c}\overline{\mathsf{e}}}{\overline{\mathsf{cc}}}\,\overline{\kappa} + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\overline{\zeta}\bigg).
$$

After expansion, we see that are present the eight 2-forms

$$
\begin{array}{cccc}
(\bullet)\, \rho \wedge \kappa, & (\bullet)\, \rho \wedge \zeta, & (\bullet)\, \rho \wedge \overline{\kappa}, & (\bullet)\, \rho \wedge \overline{\zeta}, \\
 & (\bullet)\, \kappa \wedge \overline{\kappa}, & (\bullet)\, \kappa \wedge \overline{\zeta}, & (\bullet)\, \zeta \wedge \overline{\kappa}, \quad (\bullet)\, \zeta \wedge \overline{\zeta}.
\end{array}
$$

Doing the same for all torsion terms, we may realize — although it is not necessary to check this for what follows — with almost no computation that the *nonexplicit* shape of the structure equations of the lifted coframe is

$$
\begin{aligned}
d\rho &= \big(\alpha + \overline{\alpha}\big) \wedge \rho + R^1\, \rho \wedge \kappa + R^2\, \rho \wedge \zeta + \boxed{\overline{R}^1}\, \rho \wedge \overline{\kappa} + \overline{R}^2\, \rho \wedge \overline{\zeta} + i\, \kappa \wedge \overline{\kappa}, \\
d\kappa &= \beta \wedge \rho + \alpha \wedge \kappa + K^1\, \rho \wedge \kappa + K^2\, \rho \wedge \zeta + K^3\, \rho \wedge \overline{\kappa} + K^4\, \rho \wedge \overline{\zeta} \\
&\quad + K^5\, \kappa \wedge \zeta + \boxed{K^6}\, \kappa \wedge \overline{\kappa} + \mathbf{1} \cdot \zeta \wedge \overline{\kappa}, \\
d\zeta &= \gamma \wedge \rho + \delta \wedge \kappa + \big(\alpha - \overline{\alpha}\big) \wedge \zeta \\
&\quad + Z^1\, \rho \wedge \kappa + Z^2\, \rho \wedge \zeta + Z^3\, \rho \wedge \overline{\kappa} + Z^4\, \rho \wedge \overline{\zeta} \\
&\quad + Z^5\, \kappa \wedge \zeta + Z^6\, \kappa \wedge \overline{\kappa} + Z^7\, \kappa \wedge \overline{\zeta} + \boxed{Z^8}\, \zeta \wedge \overline{\kappa} + Z^9\, \zeta \wedge \overline{\zeta}.
\end{aligned}
$$

Of course, the preceding normalization $\mathsf{f} := \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\overline{\mathcal{L}}_1(k)$ forces

$$
\mathbf{1} = \big[\zeta \wedge \overline{\kappa}\big]\big\{d\kappa\big\},
$$

a fact that can also be confirmed by a direct computation of this torsion coefficient (exercise).

So we do not compute all torsion coefficients like Pocchiola did, but we determine *before* some essential torsions, so that we may focus on just the useful torsion terms. In advance, we have boxed above the 3 useful ones, shown by Pocchiola. The subtle thing is that all three structure equations are needed.

Lemma 6.4. *Here is an essential linear combination of torsion terms*

$$\overline{R}^1 - 2\,K^6 + Z^8.$$

Proof. In order to '*absorb*' as many torsion coefficients as possible, let us substitute

$$\begin{aligned}
\alpha &=: \alpha' + a_1\,\rho + a_2\,\kappa + a_3\,\zeta + a_4\,\overline{\kappa} + a_5\,\overline{\zeta},\\
\beta &=: \beta' + b_1\,\rho + b_2\,\kappa + b_3\,\zeta + b_4\,\overline{\kappa} + b_5\,\overline{\zeta},\\
\gamma &=: \gamma' + c_1\,\rho + c_2\,\kappa + c_3\,\zeta + c_4\,\overline{\kappa} + c_5\,\overline{\zeta},\\
\delta &=: \delta' + d_1\,\rho + d_2\,\kappa + d_3\,\zeta + d_4\,\overline{\kappa} + d_5\,\overline{\zeta}.
\end{aligned}$$

At first, we have to transform the structure equations after such a substitution, the task is easy, and we write out the details so that the reader needs no pen and no computer.

Substituting, the Maurer-Cartan part of $d\rho$ becomes

$$\begin{aligned}
\big(\alpha + \overline{\alpha}\big) \wedge \rho = \big(\alpha' + \overline{\alpha}'\big) \wedge \rho &+ 0 + a_2\,\kappa \wedge \rho + a_3\,\zeta \wedge \rho + a_4\,\overline{\kappa} \wedge \rho + a_5\,\overline{\zeta} \wedge \rho\\
&+ 0 + \overline{a}_2\,\overline{\kappa} \wedge \rho + \overline{a}_3\,\overline{\zeta} \wedge \rho + \overline{a}_4\,\kappa \wedge \rho + \overline{a}_5\,\zeta \wedge \rho,
\end{aligned}$$

hence adding and reorganizing visually, we get

$$\begin{aligned}
d\rho = {} & \big(\alpha' + \overline{\alpha}'\big) \wedge \rho\\
& + \rho \wedge \kappa\Big(R^1 - a_2 - \overline{a}_4\Big) + \rho \wedge \zeta\Big(R^2 - a_3 - \overline{a}_5\Big) + \rho \wedge \overline{\kappa}\Big(\boxed{\overline{R}^1 - a_4 - \overline{a}_2}\Big)\\
& + \rho \wedge \overline{\zeta}\Big(\overline{R}^2 - a_5 - \overline{a}_3\Big) + i\,\kappa \wedge \overline{\kappa}.
\end{aligned}$$

Next

$$\begin{aligned}
\beta \wedge \rho + \alpha \wedge \kappa = {} & \beta' \wedge \rho + 0 + b_2\,\kappa \wedge \rho + b_3\,\zeta \wedge \rho + b_4\,\overline{\kappa} \wedge \rho + b_5\,\overline{\zeta} \wedge \rho\\
& + \alpha' \wedge \kappa + a_1\,\rho \wedge \kappa + 0 + a_3\,\zeta \wedge \kappa + a_4\,\overline{\kappa} \wedge \kappa + a_5\,\overline{\zeta} \wedge \kappa,
\end{aligned}$$

hence

$$\begin{aligned}
d\kappa = {} & \beta' \wedge \rho + \alpha' \wedge \kappa\\
& + \rho \wedge \kappa\,\Big(K^1 + a_1 - b_2\Big) + \rho \wedge \zeta\,\Big(K^2 - b_3\Big) + \rho \wedge \overline{\kappa}\,\Big(K^3 - b_4\Big) + \rho \wedge \overline{\zeta}\,\Big(K^4 - b_5\Big)\\
& + \kappa \wedge \zeta\,\Big(K^5 - a_3\Big) + \kappa \wedge \overline{\kappa}\,\Big(\boxed{K^6 - a_4}\Big) + \kappa \wedge \overline{\zeta}\,\big(-a_5\big) + \zeta \wedge \overline{\kappa}.
\end{aligned}$$

Lastly

$$\begin{aligned}
\gamma \wedge \rho + \delta \wedge \kappa + \big(\alpha - \overline{\alpha}\big) \wedge \zeta = {} & \gamma' \wedge \rho + 0 + c_2\,\kappa \wedge \rho + c_3\,\zeta \wedge \rho + c_4\,\overline{\kappa} \wedge \rho + c_5\,\overline{\zeta} \wedge \rho\\
& + \delta' \wedge \kappa + d_1\,\rho \wedge \kappa + 0 + d_3\,\zeta \wedge \kappa + d_4\,\overline{\kappa} \wedge \kappa + d_5\,\overline{\zeta} \wedge \kappa\\
& + \alpha' \wedge \zeta + a_1\,\rho \wedge \zeta + a_2\,\kappa \wedge \zeta + 0 + a_4\,\overline{\kappa} \wedge \zeta + a_5\,\overline{\zeta} \wedge \zeta\\
& - \overline{\alpha}' \wedge \zeta - \overline{a}_1\,\rho \wedge \zeta - \overline{a}_2\,\overline{\kappa} \wedge \zeta - \overline{a}_3\,\overline{\zeta} \wedge \zeta - \overline{a}_4\,\kappa \wedge \zeta - 0,
\end{aligned}$$

hence

$$
\begin{aligned}
d\zeta \,&=\, \gamma'\wedge\rho+\delta'\wedge\kappa+\big(\alpha'-\overline{\alpha}'\big)\wedge\zeta \\
&\quad+\rho\wedge\kappa\Big(Z^1-c_2+d_1\Big)+\rho\wedge\zeta\,\Big(Z^2-c_3+a_1-\overline{a}_1\Big)+\rho\wedge\overline{\kappa}\,\Big(Z^3-c_4\Big)+\rho\wedge\overline{\zeta}\,\Big(Z^4-c_5\Big) \\
&\quad+\kappa\wedge\zeta\,\Big(Z^5-d_3+a_2-\overline{a}_4\Big)+\kappa\wedge\overline{\kappa}\,\Big(Z^6-d_4\Big)+\kappa\wedge\overline{\zeta}\,\Big(Z^7-d_5\Big) \\
&\quad+\zeta\wedge\overline{\kappa}\,\Big(\boxed{Z^8-a_4+\overline{a}_2}\Big)+\zeta\wedge\overline{\zeta}\,\Big(Z^9-a_5+\overline{a}_3\Big).
\end{aligned}
$$

Extracting the boxed three new torsion coefficients

$$
\begin{aligned}
\overline{R}^{1\prime} \,&=\, \overline{R}^1-a_4-\overline{a}_2, \\
K^{6\prime} \,&=\, K^6-a_4, \\
Z^{8\prime} \,&=\, Z^8-a_4+\overline{a}_2,
\end{aligned}
$$

we see well the announced essentiality/invariancy of this torsion combination

$$
\overline{R}^{1\prime}-2\,K^{6\prime}+Z^{8\prime} \,=\, \overline{R}^1-2\,K^6+Z^8. \qquad \square
$$

Consequently, we may restrict ourselves to computing only these three torsion coefficients.

Lemma 6.5. *Their explicit expressions are*

$$
\begin{aligned}
\overline{R}^1 \,&=\, \frac{\overline{\boldsymbol{P}}}{\overline{\mathsf{c}}}+\frac{\mathsf{c}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}-i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}}, \\
K^6 \,&=\, i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}}-\frac{\mathsf{e}}{\mathsf{c}}, \\
Z^8 \,&=\, \frac{\mathsf{e}}{\mathsf{c}}-\frac{1}{\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}-\frac{\mathsf{c}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}.
\end{aligned}
$$

Proof. We proceed by chasing coefficients. Let us treat $\overline{R}^1$. From (6.2), replacing in (5.5) by means of (6.1), we reach its expression

$$
\begin{aligned}
\overline{R}^1 \,=\, \big[\rho\wedge\overline{\kappa}\big]\big\{\mathsf{c}\overline{\mathsf{c}}\,d\rho_0\big\} \,&=\, 0+0+\big[\rho\wedge\overline{\kappa}\big]\bigg\{\mathsf{c}\overline{\mathsf{c}}\,\overline{\boldsymbol{P}}\,\Big(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,\rho\Big)\wedge\Big(-\frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}\overline{\mathsf{c}}}\,\rho+\frac{1}{\overline{\mathsf{c}}}\,\overline{\kappa}\Big) \\
&\quad-\mathsf{c}\overline{\mathsf{c}}\,\frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}\,\Big(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,\rho\Big)\wedge\Big(\frac{\overline{\mathsf{b}}\overline{\mathsf{e}}-\overline{\mathsf{c}}\overline{\mathsf{d}}}{\mathsf{c}\overline{\mathsf{c}}\overline{\mathsf{c}}}\,\rho-\frac{\mathsf{c}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}\,\overline{\kappa}+\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\overline{\zeta}\Big) \\
&\quad+\mathsf{c}\overline{\mathsf{c}}\,i\,\Big(-\frac{\mathsf{b}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}\,\rho+\frac{1}{\mathsf{c}}\,\kappa\Big)\wedge\Big(-\frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}\overline{\mathsf{c}}}\,\rho+\frac{1}{\overline{\mathsf{c}}}\,\overline{\kappa}\Big)\bigg\} \\
&=\, \underline{\mathsf{c}\overline{\mathsf{c}}}_\circ\,\overline{\boldsymbol{P}}\,\frac{1}{\underline{\mathsf{c}\overline{\mathsf{c}}}_\circ}\,\frac{1}{\overline{\mathsf{c}}}+\underline{\mathsf{c}\overline{\mathsf{c}}}_\circ\,\frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}\,\frac{1}{\underline{\mathsf{c}\overline{\mathsf{c}}}_\circ}\,\frac{\mathsf{c}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}-i\,\underline{\mathsf{c}\overline{\mathsf{c}}}_\circ\,\frac{\mathsf{b}}{\mathsf{c}\underline{\mathsf{c}\overline{\mathsf{c}}}_\circ}\,\frac{1}{\overline{\mathsf{c}}}.
\end{aligned}
$$

Next, from (6.2), let us treat

$$
K^6 \,=\, \big[\kappa\wedge\overline{\kappa}\big]\big\{\mathsf{b}\,d\rho_0+\mathsf{c}\,d\kappa_0\big\}.
$$

In $\mathsf{b}\, d\rho_0$, the first four terms in (5.5) have zero contribution, since they are multiples of ρ_0, hence of ρ, whence

$$\begin{aligned}
\big[\kappa \wedge \overline{\kappa}\big]\big\{\mathsf{b}\, d\rho_0\big\} &= 0+0+0+0+\big[\kappa \wedge \overline{\kappa}\big]\big\{\mathsf{b}\, i\, \kappa_0 \wedge \overline{\kappa}_0\big\} \\
&= \big[\kappa \wedge \overline{\kappa}\big]\bigg\{i\,\mathsf{b}\Big(-\frac{\mathsf{b}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}\,\rho+\frac{1}{\mathsf{c}}\,\kappa\Big)\wedge\Big(-\frac{\overline{\mathsf{b}}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}\,\rho+\frac{1}{\overline{\mathsf{c}}}\,\overline{\kappa}\Big)\bigg\} \\
&= i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}}.
\end{aligned}$$

Also, in $\mathsf{c}\, d\kappa_0$, the first two terms contribute 0, and it remains

$$\begin{aligned}
\big[\kappa \wedge \overline{\kappa}\big]\big\{\mathsf{c}\, d\kappa_0\big\} &= 0+0+\big[\kappa \wedge \overline{\kappa}\big]\big\{\mathsf{c}\,\zeta_0' \wedge \overline{\kappa}_0\big\} \\
&= \big[\kappa \wedge \overline{\kappa}\big]\bigg\{\mathsf{c}\Big(-\frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\kappa\Big)\wedge\Big(\frac{1}{\overline{\mathsf{c}}}\,\overline{\kappa}\Big)\bigg\} \\
&= -\frac{\mathsf{e}}{\mathsf{c}}.
\end{aligned}$$

Lastly

$$Z^8 = \big[\zeta \wedge \overline{\kappa}\big]\Big\{\mathsf{d}\, d\rho_0 + \mathsf{e}\, d\kappa_0 + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\, d\zeta_0'\Big\}.$$

Here, $\mathsf{d}\, d\rho_0$ contributes 0. Next, the first two terms in $\mathsf{e}\, d\kappa_0$ contribute 0, and it remains

$$\begin{aligned}
\big[\zeta \wedge \overline{\kappa}\big]\big\{\mathsf{e}\, d\kappa_0\big\} &= \big[\zeta \wedge \overline{\kappa}\big]\big\{\mathsf{e}\,\zeta_0' \wedge \kappa_0\big\} \\
&= \big[\zeta \wedge \overline{\kappa}\big]\bigg\{\mathsf{e}\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\,\zeta\Big)\wedge\Big(\frac{1}{\overline{\mathsf{c}}}\,\overline{\kappa}\Big)\bigg\} \\
&= \frac{\mathsf{e}}{\mathsf{c}}.
\end{aligned}$$

Also, in $\frac{\mathsf{c}}{\overline{\mathsf{c}}}\, d\zeta_0'$, the first two terms contribute 0, and the last two terms are

$$\begin{aligned}
\big[\zeta \wedge \overline{\kappa}\big]\Big\{\frac{\mathsf{c}}{\overline{\mathsf{c}}}\, d\zeta_0'\Big\} &= -\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\,\big[\zeta \wedge \overline{\kappa}\big]\bigg\{\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\,\zeta\Big)\wedge\Big(\frac{1}{\overline{\mathsf{c}}}\,\overline{\kappa}\Big)\bigg\} \\
&\quad + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})}\,\big[\zeta \wedge \overline{\kappa}\big]\bigg\{\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\,\zeta\Big)\wedge\Big(-\frac{\mathsf{c}\overline{\mathsf{e}}}{\overline{\mathsf{c}}\mathsf{c}}\,\overline{\kappa}\Big)\bigg\} \\
&= -\frac{1}{\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)} - \frac{\mathsf{c}\overline{\mathsf{e}}}{\overline{\mathsf{c}}\mathsf{c}}\,\frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})}
\end{aligned}$$

Adding, we get Z^8. □

Observing that necessarily $-a_5 = 0$ from $\big[\kappa \wedge \overline{\zeta}\big]\big\{d\kappa\big\}$, we realize that some other invariant relations between torsion coefficients appear

$$\begin{aligned}
R^{2\prime} - K^{5\prime} &= R^2 - K^5, \\
\overline{R}^{2\prime} + Z^{9\prime} &= \overline{R}^2 + Z^9,
\end{aligned}$$

that could potentially bring normalizations of some group parameters, but will not, as it will come out that they are identically satisfied. However, knowing them will be very useful later, hence we state a supplementary:

Assertion 6.6. *Three other torsion coefficients have the common explicit expression*

$$R^2 = K^5 = -\overline{Z}^9 = -\frac{\overline{\mathsf{c}}}{\mathsf{c}}\,\frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)}.$$

Proof. Our technique gives

$$\begin{aligned} R^2 &= \big[\rho\wedge\zeta\big]\big\{\mathsf{c}\overline{\mathsf{c}}\,d\rho_0\big\} \\ &= 0-\mathsf{c}\overline{\mathsf{c}}\,\frac{\mathcal{L}_1(\boldsymbol{k})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,\frac{\overline{\mathsf{c}}}{\mathsf{c}}+0+0+0. \end{aligned}$$

Next

$$\begin{aligned} K^5 &= \big[\zeta\wedge\kappa\big]\big\{\mathsf{b}\,d\rho_0+\mathsf{c}\,d\kappa_0\big\} \\ &= 0+\big[\zeta\wedge\kappa\big]\big\{\mathsf{c}\,d\kappa_0\big\} \\ &= 0-\frac{\mathcal{L}_1(\boldsymbol{k})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\mathsf{c}\,\frac{1}{\mathsf{c}}\,\frac{\overline{\mathsf{c}}}{\mathsf{c}}+0. \end{aligned}$$

Lastly

$$\begin{aligned} Z^9 &= \big[\overline{\kappa}\wedge\overline{\zeta}\big]\Big\{\mathsf{d}\,d\rho_0+\mathsf{e}\,d\kappa_0+\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,d\zeta_0'\Big\} \\ &= 0+0+\big[\overline{\kappa}\wedge\overline{\zeta}\big]\Big\{\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,d\zeta_0'\Big\} \\ &= 0+0+0+\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}\,\frac{\overline{\mathsf{c}}}{\mathsf{c}}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}. \end{aligned}$$

□

Coming back to Lemma 6.5, we can now compute in details, emphasizing one annihilation, the expression of the interesting invariant torsion combination

$$\begin{aligned} \overline{R}^1-2\,K^6+Z^8 &= \frac{\overline{\boldsymbol{P}}}{\overline{\mathsf{c}}}+\underline{\frac{\mathsf{c}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}}_{\circ}-i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} \\ &\quad -2\,i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}}+2\,\frac{\mathsf{e}}{\mathsf{c}} \\ &\quad +\frac{\mathsf{e}}{\mathsf{c}}-\frac{1}{\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}-\underline{\frac{\mathsf{c}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}}_{\circ} \\ &= -3\,i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}}+3\,\frac{\mathsf{e}}{\mathsf{c}}-\frac{1}{\overline{\mathsf{c}}}\left(\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}-\overline{\boldsymbol{P}}\right). \end{aligned}$$

Since the group parameter $\mathsf{b}\in\mathbb{C}$ is not on the diagonal, there is no restriction for it to be nonzero, hence we can normalize it by requiring that

$$0=\overline{R}^1-2\,K^6+Z^8,$$

and this produces the announced normalization

$$\mathsf{b}:=-\,i\,\overline{\mathsf{c}}\mathsf{e}+\frac{i}{3}\,\mathsf{c}\left(\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}-\overline{\boldsymbol{P}}\right). \tag{6.7}$$

For convenience, let us abbreviate

$$\boldsymbol{B}_0:=\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}-\overline{\boldsymbol{P}},$$

which is function on M, as its lower index $_0$ points out, so that

$$\mathsf{b}:=-\,i\,\overline{\mathsf{c}}\mathsf{e}+\frac{i}{3}\,\mathsf{c}\,\boldsymbol{B}_0.$$

After this normalization, the lifted coframe becomes

$$\begin{pmatrix} \rho \\ \kappa \\ \zeta \end{pmatrix} := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 \\ -i\,\overline{\mathsf{c}}\mathsf{e} + \frac{i}{3}\,\mathsf{c}\boldsymbol{B}_0 & \mathsf{c} & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \zeta_0' \end{pmatrix}.$$

Consequently, we can transform/rewrite in a natural way

$$\begin{aligned} \kappa &= \Big(- i\,\overline{\mathsf{c}}\mathsf{e} + \frac{i}{3}\,\mathsf{c}\,\boldsymbol{B}_0 \Big)\,\rho_0 + \mathsf{c}\,\kappa_0 \\ &= \big(- i\,\overline{\mathsf{c}}\mathsf{e} \big)\,\rho_0 + \mathsf{c}\,\Big(\underbrace{\kappa_0 + \frac{i}{3}\,\boldsymbol{B}_0\,\rho_0}_{=:\,\kappa_0'} \Big), \end{aligned}$$

and this conducts us to change of initial coframe on M

$$\big\{ \rho_0,\, \kappa_0,\, \zeta_0', \overline{\kappa}_0,\, \overline{\zeta}_0' \big\} \quad \rightsquigarrow \quad \big\{ \rho_0,\, \kappa_0',\, \zeta_0',\, \overline{\kappa}_0', \overline{\zeta}_0' \big\},$$

by introducing

$$\kappa_0' := \kappa_0 + \frac{i}{3}\,\boldsymbol{B}_0\,\rho_0. \tag{6.8}$$

It follows that

$$\begin{aligned} \zeta &= \mathsf{d}\,\rho_0 + \mathsf{e}\,\kappa_0 + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\zeta_0' = \mathsf{d}\,\rho_0 + \mathsf{e}\,\Big(\kappa_0' - \frac{i}{3}\,\boldsymbol{B}_0\,\rho_0 \Big) + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\zeta_0' \\ &= \Big(\underbrace{\mathsf{d} - \frac{i}{3}\,\mathsf{e}\boldsymbol{B}_0}_{=:\,\mathsf{d}'} \Big)\,\rho_0 + \mathsf{e}\,\kappa_0' + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\zeta_0'. \end{aligned}$$

Before, $\mathsf{d} \in \mathbb{C}$ was a parameter representing some unknown function. Introducing the new unknown/parameter

$$\mathsf{d}' := \mathsf{d} - \frac{i}{3}\,\mathsf{e},$$

we come to a new G-structure of real dimension 6 parametrized by $\mathsf{c}, \mathsf{e} \in \mathbb{C}^*$ and $\mathsf{d}' \in \mathbb{C}$ whose lifted coframe writes

$$\begin{pmatrix} \rho \\ \kappa \\ \zeta \end{pmatrix} := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 \\ -i\,\overline{\mathsf{c}}\mathsf{e} & \mathsf{c} & 0 \\ \mathsf{d}' & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \kappa_0' \\ \zeta_0' \end{pmatrix}.$$

We will write again d instead of d'.

7. Darboux-Cartan Structure of the Coframe $\{\rho_0, \kappa_0', \zeta_0', \overline{\kappa}_0', \overline{\zeta}_0'\}$

Before continuing, we must compute the Darboux-Cartan structure of this new initial coframe $\{\rho_0, \kappa_0', \zeta_0', \overline{\kappa}_0', \overline{\zeta}_0'\}$, for which absolutely no details were provided in [25, 18]. Here, we offer complete explanations.

Abstractly, the structure in question will have the shape

$$\begin{aligned} d\rho_0 &= \boldsymbol{R}_0^{1'}\,\rho_0 \wedge \kappa_0' + \boldsymbol{R}_0^{2'}\,\rho_0 \wedge \zeta_0' + \overline{\boldsymbol{R}}_0^{1'}\,\rho_0 \wedge \overline{\kappa}_0' + \overline{\boldsymbol{R}}_0^{2'}\,\rho_0 \wedge \overline{\zeta}_0' + i\,\kappa_0' \wedge \overline{\kappa}_0', \\ d\kappa_0' &= \boldsymbol{K}_0^{1'}\,\rho_0 \wedge \kappa_0' + \boldsymbol{K}_0^{2'}\,\rho_0 \wedge \zeta_0' + \boldsymbol{K}_0^{3'}\,\rho_0 \wedge \overline{\kappa}_0' \\ &\quad + \boldsymbol{K}_0^{5'}\,\kappa_0' \wedge \zeta_0' + \boldsymbol{K}_0^{6'}\,\kappa_0' \wedge \overline{\kappa}_0' + \zeta_0' \wedge \overline{\kappa}_0', \\ d\zeta_0' &= \boldsymbol{Z}_0^{2'}\,\rho_0 \wedge \zeta_0' + \boldsymbol{Z}_0^{5'}\,\kappa_0' \wedge \zeta_0' + \boldsymbol{Z}_0^{8'}\,\zeta_0' \wedge \overline{\kappa}_0' + \boldsymbol{Z}_0^{9'}\,\zeta_0' \wedge \overline{\zeta}_0'. \end{aligned} \tag{7.1}$$

Our goal is to compute *explicitly* all these coefficients, and the answer is stated as follows:

Proposition 7.2. *The Darboux-Cartan structure for the initial coframe* $\{\rho_0, \kappa_0', \zeta_0', \overline{\kappa}_0', \overline{\zeta}_0'\}$ *expands as*

$$\begin{aligned}
d\rho_0 \,&=\, \left(\frac{1}{3}\,\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{k})\big)}{\mathcal{L}_1(\overline{k})} + \frac{2}{3}\,\boldsymbol{P}\right)\rho_0\wedge\kappa_0' - \frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)}\,\rho_0\wedge\zeta_0' \\
&\quad + \left(\frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)} + \frac{2}{3}\,\overline{\boldsymbol{P}}\right)\rho_0\wedge\overline{\kappa}_0' - \frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})}\,\rho_0\wedge\overline{\zeta}_0' + i\,\kappa_0'\wedge\overline{\kappa}_0',
\end{aligned}$$

$$\begin{aligned}
d\kappa_0' \,&=\, \Bigg(-\frac{i}{3}\,\frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)} + \frac{i}{9}\,\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{k})\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\mathcal{L}_1(\overline{k})\,\overline{\mathcal{L}}_1(k)} \\
&\quad + \frac{i}{3}\,\frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)^2} - \frac{i}{9}\,\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{k})\big)}{\mathcal{L}_1(\overline{k})}\,\overline{\boldsymbol{P}} \\
&\quad + \frac{2\,i}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\,\boldsymbol{P} + \frac{i}{3}\,\mathcal{L}_1\big(\overline{\boldsymbol{P}}\big) - \frac{2\,i}{9}\,\boldsymbol{P}\,\overline{\boldsymbol{P}}\Bigg)\,\rho_0\wedge\kappa_0' \\
&\quad + \Bigg(-\frac{i}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)^2} + \frac{i}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)^3} \\
&\quad - \frac{i}{3}\,\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{k})\big)}{\mathcal{L}_1(\overline{k})} - \frac{i}{3}\,\frac{\overline{\mathcal{L}}_1\big(\mathcal{L}_1(k)\big)}{\overline{\mathcal{L}}_1(k)} - \frac{2}{3}\,\frac{\mathcal{T}(k)}{\overline{\mathcal{L}}_1(k)}\Bigg)\,\rho_0\wedge\zeta_0' \\
&\quad + \Bigg(-\frac{i}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)} + \frac{4\,i}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^2}{\overline{\mathcal{L}}_1(k)^2} \\
&\quad + \frac{i}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\,\overline{\boldsymbol{P}} + \frac{i}{3}\,\overline{\mathcal{L}}_1\big(\overline{\boldsymbol{P}}\big) - \frac{2\,i}{9}\,\overline{\boldsymbol{P}}\,\overline{\boldsymbol{P}}\Bigg)\,\rho_0\wedge\overline{\kappa}_0' + 0\,\rho_0\wedge\overline{\zeta}_0' \\
&\quad - \frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)}\,\kappa_0'\wedge\zeta_0' + \left(-\frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)} + \frac{1}{3}\,\overline{\boldsymbol{P}}\right)\kappa_0'\wedge\overline{\kappa}_0' + \zeta_0'\wedge\overline{\kappa}_0',
\end{aligned}$$

$$\begin{aligned}
d\zeta_0' \,&=\, \Bigg(\frac{i}{3}\,\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{k})\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\mathcal{L}_1(\overline{k})\,\overline{\mathcal{L}}_1(k)} - \frac{i}{3}\,\frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)^2} \\
&\quad - \frac{i}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\,\boldsymbol{P} + \frac{i}{3}\,\frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\,\overline{\boldsymbol{P}} + \frac{\mathcal{T}\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\Bigg)\,\rho_0\wedge\zeta_0' \\
&\quad + \frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\,\kappa_0'\wedge\zeta_0' - \frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\,\zeta_0'\wedge\overline{\kappa}_0' + \frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})}\,\zeta_0'\wedge\overline{\zeta}_0'.
\end{aligned}$$

Observe from these explicit expressions that

$$2\,\boldsymbol{K}_0^{6\prime} \,=\, \overline{\boldsymbol{R}}_0^{1\prime} + \boldsymbol{Z}_0^{8\prime} \qquad\text{and}\qquad \boldsymbol{R}_0^{2\prime} \,=\, \boldsymbol{K}_0^{5\prime}.$$

Proof. We treat first $d\rho_0$ and $d\zeta_0'$, which are easier than $d\kappa_0'$. Observing from (6.8), that

$$\rho_0\wedge\kappa_0 \,=\, \rho_0\wedge\kappa_0' \qquad\text{and}\qquad \rho_0\wedge\overline{\kappa}_0 \,=\, \rho_0\wedge\overline{\kappa}_0',$$

it comes by replacement in (5.5)

$$\begin{aligned} d\rho_0 &= \boldsymbol{P}\,\rho_0\wedge\kappa_0' - \frac{\mathcal{L}_1(\boldsymbol{k})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\rho_0\wedge\zeta_0' + \overline{\boldsymbol{P}}\,\rho_0\wedge\overline{\kappa}_0 - \frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}\,\rho_0\wedge\overline{\zeta}_0' \\ &\quad + i\left(\kappa_0' - \frac{i}{3}\left(\frac{\overline{\mathcal{L}}_1\big(\mathcal{L}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \overline{\boldsymbol{P}}\right)\rho_0\right)\wedge\left(\overline{\kappa}_0' + \frac{i}{3}\left(\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{\boldsymbol{k}})\big)}{\mathcal{L}_1(\overline{\boldsymbol{k}})}\right)\rho_0\right)\end{aligned}$$

and a plain expansion yields the stated expression of $d\rho_0$. Next, again from (6.8), it comes by replacement in (5.5)

$$\begin{aligned} d\zeta_0' &= \frac{\mathcal{T}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\rho_0\wedge\zeta_0' + \frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\left(\kappa_0' - \frac{i}{3}\left(\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \overline{\boldsymbol{P}}\right)\rho_0\right)\wedge\zeta_0' \\ &\quad - \frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\zeta_0'\wedge\left(\overline{\kappa}_0' + \frac{i}{3}\left(\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{\boldsymbol{k}})\big)}{\mathcal{L}_1(\overline{\boldsymbol{k}})} - \boldsymbol{P}\right)\rho_0\right) + \frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}\,\zeta_0'\wedge\overline{\zeta}_0'\end{aligned}$$

and visually — no pen needed —, we obtain the stated result. To treat $d\kappa_0'$, we start from

$$\kappa_0' = \kappa_0 + \frac{i}{3}\,\boldsymbol{B}_0\,\rho_0$$

and we exterior differentiate

$$d\kappa_0' = d\kappa_0 + \frac{i}{3}\,d\boldsymbol{B}_0\wedge\rho_0 + \frac{i}{3}\,\boldsymbol{B}_0\,d\rho_0. \tag{7.3}$$

As a preliminary, we need to know $d\boldsymbol{B}_0$. Let us recall that

$$\boldsymbol{B}_0 = \frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \overline{\boldsymbol{P}} \qquad \text{whence} \qquad \overline{\boldsymbol{B}}_0 = \frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{\boldsymbol{k}})\big)}{\mathcal{L}_1(\overline{\boldsymbol{k}})} - \boldsymbol{P}.$$

A plain application of Lemma 2.6 provides this exterior differential

$$\begin{aligned} d\left(\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \overline{\boldsymbol{P}}\right) &= \left(\frac{\mathcal{T}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \frac{\mathcal{T}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})^2} - \mathcal{T}\big(\overline{\boldsymbol{P}}\big)\right)\rho_0 \\ &\quad + \left(\frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})^2} - \mathcal{L}_1\big(\overline{\boldsymbol{P}}\big)\right)\kappa_0 \\ &\quad + \left(\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \frac{\mathcal{K}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})^2} - \mathcal{K}\big(\overline{\boldsymbol{P}}\big)\right)\zeta_0 \\ &\quad + \left(\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)^2}{\overline{\mathcal{L}}_1(\boldsymbol{k})^2} - \overline{\mathcal{L}}_1\big(\overline{\boldsymbol{P}}\big)\right)\overline{\kappa}_0 \\ &\quad + \left(\frac{\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \frac{\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})^2} - \overline{\mathcal{K}}\big(\overline{\boldsymbol{P}}\big)\right)\overline{\zeta}_0,\end{aligned}$$

an expression that we will abbreviate as

$$d\boldsymbol{B}_0 = \boldsymbol{U}_0\,\rho_0 + \boldsymbol{V}_0\,\kappa_0 + \boldsymbol{W}_0\,\zeta_0 + \boldsymbol{X}_0\,\overline{\kappa}_0 + \boldsymbol{Y}_0\,\overline{\zeta}_0.$$

Assertion 7.4. *After simplifications*

$$\boldsymbol{Y}_0 = -\frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} + \overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})\,\overline{\boldsymbol{P}}.$$

Proof. In the first two terms of Y_0, we replace from Lemma 2.9

$$\begin{aligned}\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(k))\big) &= -2\,\overline{\mathcal{L}}_1(\overline{k})\;\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big) - \overline{\mathcal{L}}_1\big(\mathcal{L}_1(\overline{k})\big)\;\overline{\mathcal{L}}_1(k),\\ \overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(k)\big) &= -\overline{\mathcal{L}}_1(\overline{k})\,\overline{\mathcal{L}}_1(k)\end{aligned}$$

and in the third term of Y_0, we replace from Lemma 2.5

$$\overline{\mathcal{K}}(\overline{P}) = -\overline{P}\,\mathcal{L}_1(\overline{k}) - \overline{\mathcal{L}}_1\big(\mathcal{L}_1(\overline{k})\big),$$

which yields the result after one (underlined) pair cancellation

$$\begin{aligned}Y_0 = &-\frac{2\,\overline{\mathcal{L}}_1(\overline{k})\;\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)} - \underline{\overline{\mathcal{L}}_1\big(\mathcal{L}_1(\overline{k})\big)}_{\circ} + \frac{\overline{\mathcal{L}}_1(\overline{k})\;\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\\ &+\overline{P}\,\mathcal{L}_1(\overline{k}) + \underline{\overline{\mathcal{L}}_1\big(\mathcal{L}_1(\overline{k})\big)}_{\circ}.\end{aligned}$$

□

Temporarily, let us work with the abbreviations U_0, V_0, W_0, X_0, Y_0. So, using the previous structure formulas (5.5) in which, directly we replace

$$\zeta_0 = \frac{\zeta_0'}{\overline{\mathcal{L}}_1(k)},$$

let us add line-by-line all three terms of (7.3)

$$\begin{aligned}d\kappa_0' = &-\frac{\mathcal{T}(k)}{\overline{\mathcal{L}}_1(k)}\,\rho_0\wedge\zeta_0' - \frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)}\,\kappa_0\wedge\zeta_0' + \zeta_0'\wedge\overline{\kappa}_0'\\ &+\frac{i}{3}\,\underline{U_0\,\rho_0\wedge\rho_0}_{\circ} + \frac{i}{3}\,V_0\,\kappa_0\wedge\rho_0 + \frac{i}{3}\,W_0\,\frac{\zeta_0'}{\overline{\mathcal{L}}_1(k)}\wedge\rho_0 + \frac{i}{3}\,X_0\,\overline{\kappa}_0\wedge\rho_0 + \frac{i}{3}\,Y_0\,\frac{\overline{\zeta}_0'}{\mathcal{L}_1(\overline{k})}\wedge\rho_0\\ &+\frac{i}{3}\,B_0\,P\,\rho_0\wedge\kappa_0 - \frac{i}{3}\,B_0\,\frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)}\,\rho_0\wedge\zeta_0' + \frac{i}{3}\,B_0\,\overline{P}\,\rho_0\wedge\overline{\kappa}_0 - \frac{i}{3}\,B_0\,\frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})}\,\rho_0\wedge\overline{\zeta}_0' - \frac{1}{3}\,B_0\,\kappa_0\wedge\overline{\kappa}_0,\end{aligned}$$

hence after collecting coefficients of basic 2-forms, we get

$$\begin{aligned}d\kappa_0' = \rho_0\wedge\zeta_0'&\bigg[-\frac{\mathcal{T}(k)}{\overline{\mathcal{L}}_1(k)} - \frac{i}{3}\,\frac{W_0}{\overline{\mathcal{L}}_1(k)} - \frac{i}{3}\,B_0\,\frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)}\bigg] + \rho_0\wedge\kappa_0\bigg[-\frac{i}{3}\,V_0 + \frac{i}{3}\,B_0\,P\bigg]\\ &+\rho_0\wedge\overline{\kappa}_0\bigg[-\frac{i}{3}\,X_0 + \frac{i}{3}\,B_0\,\overline{P}\bigg] + \rho_0\wedge\overline{\zeta}_0'\bigg[-\frac{i}{3}\,\frac{Y_0}{\mathcal{L}_1(\overline{k})} - \frac{i}{3}\,B_0\,\frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})}\bigg]\\ &+\kappa_0\wedge\zeta_0'\bigg[-\frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)}\bigg] + \kappa_0\wedge\overline{\kappa}_0\bigg[-\frac{1}{3}\,B_0\bigg] + \zeta_0'\wedge\overline{\kappa}_0.\end{aligned}$$

Next, replace everywhere

$$\kappa_0 = \kappa_0' - \frac{i}{3}\,B_0\,\rho_0.$$

Then using again $\kappa_0\wedge\rho_0 = \kappa_0'\wedge\rho_0$, only the last line changes, as it becomes

$$\Big(\kappa_0' - \frac{i}{3}\,B_0\,\rho_0\Big)\wedge\zeta_0'\bigg[-\frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)}\bigg] + \Big(\kappa_0' - \frac{i}{3}\,B_0\,\rho_0\Big)\wedge\Big(\overline{\kappa}_0' + \frac{i}{3}\,\overline{B}_0\,\rho_0\Big)\bigg[-\frac{1}{3}\,B_0\bigg] + \zeta_0'\wedge\Big(\overline{\kappa}_0' + \frac{i}{3}\,\overline{B}_0\,\rho_0\Big).$$

Expanding and collecting visually — no pen needed —, we get

$$\begin{aligned} d\kappa_0' = \rho_0 \wedge \zeta_0' & \bigg[- \frac{\mathcal{T}(k)}{\overline{\mathcal{L}}_1(k)} - \frac{i}{3} \frac{W_0}{\overline{\mathcal{L}}_1(k)} - \frac{i}{3} B_0 \frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)} + \frac{i}{3} B_0 \frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)} - \frac{i}{3} \overline{B}_0 \bigg] \\ & + \rho_0 \wedge \kappa_0' \bigg[- \frac{i}{3} V_0 + \frac{i}{3} B_0 P + \frac{i}{9} B_0 \overline{B}_0 \bigg] \\ & + \rho_0 \wedge \overline{\kappa}_0' \bigg[- \frac{i}{3} X_0 + \frac{i}{3} B_0 \overline{P} + \frac{i}{9} B_0 B_0 \bigg] \\ & + \rho_0 \wedge \overline{\zeta}_0' \bigg[- \frac{i}{3} \frac{Y_0}{\mathcal{L}_1(\overline{k})} - \frac{i}{3} B_0 \frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})} \bigg] \\ & + \kappa_0' \wedge \zeta_0' \bigg[- \frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)} \bigg] + \kappa_0' \wedge \overline{\kappa}_0' \bigg[- \frac{1}{3} B_0 \bigg] + \zeta_0' \wedge \overline{\kappa}_0'. \end{aligned}$$

To finish, we must yet replace V_0, W_0, X_0, Y_0 by their complete values, and we will realize, as indicated by anticipation above, that the coefficient of $\rho_0 \wedge \overline{\zeta}_0'$ vanishes identically.

Firstly, a replacement followed by a visual expansion finalizes

$$\begin{aligned} \big[\rho_0 \wedge \kappa_0'\big]\big\{d\kappa_0'\big\} = & - \frac{i}{3} \frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)} + \frac{i}{3} \frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(k)\big)\, \overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)^2} + \frac{i}{3} \mathcal{L}_1\big(\overline{P}\big) \\ & + \frac{i}{3} \frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1} P - \frac{i}{3} P \overline{P} + \frac{i}{9} \Bigg(\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)} - \overline{P} \Bigg) \Bigg(\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{k})\big)}{\mathcal{L}_1(\overline{\overline{k}})} - P \Bigg). \end{aligned}$$

Secondly

$$\begin{aligned} \big[\rho_0 \wedge \zeta_0'\big]\big\{d\kappa_0'\big\} = & - \frac{\mathcal{T}(k)}{\overline{\mathcal{L}}_1(k)} - \frac{i}{3} \frac{\mathcal{K}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)^2} + \frac{i}{3} \frac{\mathcal{K}\big(\overline{\mathcal{L}}_1(k)\big)\, \overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)^3} \\ & + \frac{i}{3} \frac{\boxed{\mathcal{K}(\overline{P})}}{\overline{\mathcal{L}}_1(k)} - \frac{i}{3} \frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{k})\big)}{\mathcal{L}_1(\overline{k})} + \frac{i}{3} P, \end{aligned}$$

but here, we must still replace the boxed term using Lemma 2.5

$$\begin{aligned} \big[\rho_0 \wedge \zeta_0'\big]\big\{d\kappa_0'\big\} = & - \frac{\mathcal{T}(k)}{\overline{\mathcal{L}}_1(k)} - \frac{i}{3} \frac{\mathcal{K}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)^2} + \frac{i}{3} \frac{\mathcal{K}\big(\overline{\mathcal{L}}_1(k)\big)\, \overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)^3} \\ & - \frac{i}{3} P - \frac{i}{3} \frac{\overline{\mathcal{L}}_1\big(\mathcal{L}_1(k)\big)}{\overline{\mathcal{L}}_1(k)} + \frac{1}{3} \frac{\mathcal{T}(k)}{\overline{\mathcal{L}}_1(k)} - \frac{i}{3} \frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{k})\big)}{\mathcal{L}_1(\overline{k})} + \frac{i}{3} P. \end{aligned}$$

A pair cancellation makes the obtained expression match precisely with what Proposition 7.2 stated, after some permutation of terms.

The third replacement conducts directly to the stated result

$$\begin{aligned} \big[\rho_0 \wedge \overline{\kappa}_0'\big]\big\{d\kappa_0'\big\} = & - \frac{i}{3} \frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)} + \frac{i}{3} \frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^2}{\overline{\mathcal{L}}_1(k)^2} + \frac{i}{3} \overline{\mathcal{L}}_1\big(\overline{P}\big) \\ & + \frac{i}{3} \frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)} \overline{P} - \frac{i}{3} \overline{P}\, \overline{P} \\ & + \frac{i}{9} \frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^2}{\overline{\mathcal{L}}_1(k)^2} - \frac{2\, i}{9} \frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)} \overline{P} + \frac{i}{9} \overline{P}\, \overline{P}, \end{aligned}$$

while the fourth (last) brings an identically zero result

$$
\tag{7.5}
\begin{aligned}
\big[\rho_0\wedge\overline{\zeta}_0'\big]\big\{d\kappa_0'\big\} \;=\;& \underline{\frac{i}{3}\,\frac{\overline{\mathcal{L}}_1(\overline{k})\;\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\mathcal{L}_1(\overline{k})\;\overline{\mathcal{L}}_1(k)}}_{\,\circ} - \underline{\frac{i}{3}\,\frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})}\,\overline{\boldsymbol{P}}}_{\,\circ\circ} - \underline{\frac{i}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\,\frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})}}_{\,\circ} \\
&+\underline{\frac{i}{3}\,\overline{\boldsymbol{P}}\,\frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})}}_{\,\circ\circ}\,.
\end{aligned}
$$

□

8. Third Loop: Reduction of the Group Parameter d

After normalization of the group parameter b from (6.7), we have a new reduced group G^6 of real dimension 6, and the lifted coframe is

$$
\tag{8.1}
\begin{pmatrix}\rho\\ \kappa\\ \zeta\end{pmatrix} := \begin{pmatrix}\mathsf{c}\overline{\mathsf{c}} & 0 & 0\\ -i\,\overline{\mathsf{c}}\mathsf{e} & \mathsf{c} & 0\\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}}\end{pmatrix}\begin{pmatrix}\rho_0\\ \kappa_0'\\ \zeta_0'\end{pmatrix} \qquad\Longleftrightarrow\qquad \left\{\begin{aligned}\rho &:= \mathsf{c}\overline{\mathsf{c}}\,\rho_0,\\ \kappa &:= -\,i\,\overline{\mathsf{c}}\mathsf{e}\,\rho_0+\mathsf{c}\,\kappa_0',\\ \zeta &:= \mathsf{d}\,\rho_0+\mathsf{e}\,\kappa_0'+\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\zeta_0',\end{aligned}\right.
$$

with inverse formulas

$$
\tag{8.2}
\begin{aligned}
\rho_0 &= \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,\rho,\\
\kappa_0' &= i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\rho+\frac{1}{\mathsf{c}}\,\kappa,\\
\zeta_0' &= \Big(-\,i\,\frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}\mathsf{c}}-\frac{\mathsf{d}}{\mathsf{c}\mathsf{c}}\Big)\,\rho-\frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\kappa+\frac{\overline{\mathsf{c}}}{\mathsf{c}}\,\zeta.
\end{aligned}
$$

The Maurer-Cartan matrix becomes

$$
\begin{aligned}
dg\cdot g^{-1} &= \begin{pmatrix}\overline{\mathsf{c}}\,d\mathsf{c}+\mathsf{c}d\overline{\mathsf{c}} & 0 & 0\\ -i\,\mathsf{e}d\overline{\mathsf{c}}-i\,\overline{\mathsf{c}}d\mathsf{e} & d\mathsf{c} & 0\\ d\mathsf{d} & d\mathsf{e} & \frac{d\mathsf{c}}{\overline{\mathsf{c}}}-\frac{\mathsf{c}\,d\overline{\mathsf{c}}}{\overline{\mathsf{c}}\overline{\mathsf{c}}}\end{pmatrix}\begin{pmatrix}\frac{1}{\mathsf{c}\overline{\mathsf{c}}} & 0 & 0\\ i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}} & \frac{1}{\mathsf{c}} & 0\\ -i\,\frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}\mathsf{c}}-\frac{\mathsf{d}}{\mathsf{c}\mathsf{c}} & -\frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}}\end{pmatrix}\\
&=: \begin{pmatrix}\alpha+\overline{\alpha} & 0 & 0\\ \beta & \alpha & 0\\ \gamma & i\,\beta & \alpha-\overline{\alpha}\end{pmatrix},
\end{aligned}
$$

in terms of the group-invariant 1-forms

$$
\begin{aligned}
\alpha &:= \frac{d\mathsf{c}}{\mathsf{c}},\\
\beta &:= i\,\frac{\mathsf{e}\,d\mathsf{c}}{\mathsf{c}\mathsf{c}}-i\,\frac{\mathsf{e}\,d\overline{\mathsf{c}}}{\mathsf{c}\overline{\mathsf{c}}}-i\,\frac{d\mathsf{e}}{\mathsf{c}},\\
\gamma &:= \Big(\frac{\mathsf{c}\mathsf{d}+i\,\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}\Big)\Big(-\,\frac{d\mathsf{c}}{\mathsf{c}}+\frac{d\overline{\mathsf{c}}}{\overline{\mathsf{c}}}\Big)+\frac{d\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}}+i\,\frac{\mathsf{e}\,d\mathsf{e}}{\mathsf{c}\mathsf{c}}.
\end{aligned}
$$

Now, if we exterior-differentiate the lifted coframe on the product manifold equipped with coordinates

$$
\big(z_1,z_2,\overline{z}_1,\overline{z}_2,v\big)\times\big(\mathsf{c},\overline{\mathsf{c}},\mathsf{d},\overline{\mathsf{d}},\mathsf{e},\overline{\mathsf{e}}\big)\;\in\;M^5\times G^6,
$$

after some computations, we may come to structure equations of the abstract shape

$$\begin{aligned}
d\rho &= \big(\alpha + \overline{\alpha}\big) \wedge \rho \\
&\quad + R^1 \, \rho \wedge \kappa + R^2 \, \rho \wedge \zeta + \overline{R}^1 \, \rho \wedge \overline{\kappa} + \overline{R}^2 \, \rho \wedge \overline{\zeta} + i \, \kappa \wedge \overline{\kappa}, \\
d\kappa &= \beta \wedge \rho + \alpha \wedge \kappa \\
&\quad + K^1 \, \rho \wedge \kappa + K^2 \, \rho \wedge \zeta + \boxed{K^3} \rho \wedge \overline{\kappa} + K^4 \, \rho \wedge \overline{\zeta} \\
&\quad + K^5 \, \kappa \wedge \zeta + K^6 \, \kappa \wedge \overline{\kappa} + \zeta \wedge \overline{\kappa}, \\
d\zeta &= \gamma \wedge \rho + i \, \beta \wedge \kappa + \big(\alpha - \overline{\alpha}\big) \wedge \zeta \\
&\quad + Z^1 \, \rho \wedge \kappa + Z^2 \, \rho \wedge \zeta + Z^3 \, \rho \wedge \overline{\kappa} + Z^4 \, \rho \wedge \overline{\zeta} \\
&\quad + Z^5 \, \kappa \wedge \zeta + \boxed{Z^6} \kappa \wedge \overline{\kappa} + Z^7 \, \kappa \wedge \overline{\zeta} + Z^8 \, \zeta \wedge \overline{\kappa} + Z^9 \, \zeta \wedge \overline{\zeta}.
\end{aligned}$$

Before really computing explicitly some of these torsion coefficients, let us examine what are the absorption equations. For this, we replace

$$\begin{aligned}
\alpha &=: \alpha' + a_1 \, \rho + a_2 \, \kappa + a_3 \, \zeta + a_4 \, \overline{\kappa} + a_5 \, \overline{\zeta}, \\
\beta &=: \beta' + b_1 \, \rho + b_2 \, \kappa + b_3 \, \zeta + b_4 \, \overline{\kappa} + b_5 \, \overline{\zeta}, \\
\gamma &=: \gamma' + c_1 \, \rho + c_2 \, \kappa + c_3 \, \zeta + c_4 \, \overline{\kappa} + c_5 \, \overline{\zeta}.
\end{aligned}$$

A moment of reflection convinces that the result for $d\rho$ is the same as in the proof of Lemma 6.4:

$$\begin{aligned}
d\rho &= \big(\alpha' + \overline{\alpha}'\big) \wedge \rho \\
&\quad + \rho \wedge \kappa \Big(R^1 - a_2 - \overline{a}_4\Big) + \rho \wedge \zeta \Big(R^2 - a_3 - \overline{a}_5\Big) + \rho \wedge \overline{\kappa} \Big(\overline{R}^1 - a_4 - \overline{a}_2\Big) \\
&\quad + \rho \wedge \overline{\zeta} \Big(\overline{R}^2 - a_5 - \overline{a}_3\Big) + i \, \kappa \wedge \overline{\kappa}.
\end{aligned}$$

Similarly, $d\kappa$ is unchanged

$$\begin{aligned}
d\kappa &= \beta' \wedge \rho + \alpha' \wedge \kappa \\
&\quad + \rho \wedge \kappa \Big(K^1 + a_1 - b_2\Big) + \rho \wedge \zeta \Big(K^2 - b_3\Big) + \rho \wedge \overline{\kappa} \Big(K^3 - b_4\Big) + \rho \wedge \overline{\zeta} \Big(K^4 - b_5\Big) \\
&\quad + \kappa \wedge \zeta \Big(K^5 - a_3\Big) + \kappa \wedge \overline{\kappa} \Big(K^6 - a_4\Big) + \kappa \wedge \overline{\zeta} \big(- a_5\big) + \zeta \wedge \overline{\kappa}.
\end{aligned}$$

However, for $d\zeta$, we have to compute

$$\begin{aligned}
\gamma \wedge \rho + i \, \beta \wedge \kappa + \big(\alpha - \overline{\alpha}\big) \wedge \zeta &= \gamma' \wedge \rho + 0 + c_2 \, \kappa \wedge \rho + c_3 \, \zeta \wedge \rho + c_4 \, \overline{\kappa} \wedge \rho + c_5 \, \overline{\zeta} \wedge \rho \\
&\quad + i \, \beta' \wedge \kappa + i \, b_1 \, \rho \wedge \kappa + 0 + i \, b_3 \, \zeta \wedge \kappa + i \, b_4 \, \overline{\kappa} \wedge \kappa + i \, b_5 \, \overline{\zeta} \wedge \kappa \\
&\quad + \alpha' \wedge \zeta + a_1 \, \rho \wedge \zeta + a_2 \, \kappa \wedge \zeta + 0 + a_4 \, \overline{\kappa} \wedge \zeta + a_5 \, \overline{\zeta} \wedge \zeta \\
&\quad - \overline{\alpha}' \wedge \zeta - \overline{a}_1 \, \rho \wedge \zeta - \overline{a}_2 \, \overline{\kappa} \wedge \zeta - \overline{a}_3 \, \overline{\zeta} \wedge \zeta - \overline{a}_4 \, \kappa \wedge \zeta - 0
\end{aligned}$$

and we get

$$\begin{aligned}
d\zeta &= \gamma' \wedge \rho + i \, \beta' \wedge \kappa + \big(\alpha' - \overline{\alpha}'\big) \wedge \zeta \\
&\quad + \rho \wedge \kappa \Big(Z^1 + i \, b_1 - c_2\Big) + \rho \wedge \zeta \Big(Z^2 - c_3 + a_1 - \overline{a}_1\Big) + \rho \wedge \overline{\kappa} \Big(Z^3 - c_4\Big) + \rho \wedge \overline{\zeta} \Big(Z^4 - c_5\Big) \\
&\quad + \kappa \wedge \zeta \Big(Z^5 - i \, b_3 + a_2 - \overline{a}_4\Big) + \kappa \wedge \overline{\kappa} \Big(Z^6 - i \, b_4\Big) + \kappa \wedge \overline{\zeta} \Big(Z^7 - i \, b_5\Big) \\
&\quad + \zeta \wedge \overline{\kappa} \Big(Z^8 - a_4 + \overline{a}_2\Big) + \zeta \wedge \overline{\zeta} \Big(Z^9 - a_5 + \overline{a}_3\Big).
\end{aligned}$$

Lemma 8.3. *Here is an essential linear combination of torsion terms*

$$i\,K^3 - Z^6.$$

Proof. Indeed,

$$K^{3\prime} = K^3 - b_4,$$
$$Z^{6\prime} = Z^6 - i\,b_4,$$

whence

$$i\,K^{3\prime} - Z^{6\prime} = i\,K^3 - Z^6.$$ □

Proposition 8.4. *Their explicit expressions are*

$$\begin{aligned} K^3 &= -\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}}\left(-2i\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \frac{i}{3}\,\overline{\boldsymbol{P}}\right) - i\,\frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1(\boldsymbol{k})}{\mathcal{L}_1(\overline{\boldsymbol{k}})} \\ &\quad + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\left(-\frac{i}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} + \frac{4\,i}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)^2}{\overline{\mathcal{L}}_1(\boldsymbol{k})^2}\right. \\ &\quad \left. + \frac{i}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\overline{\boldsymbol{P}} + \frac{i}{3}\,\overline{\mathcal{L}}_1(\overline{\boldsymbol{P}}) - \frac{2\,i}{9}\,\overline{\boldsymbol{P}}\,\overline{\boldsymbol{P}}\right), \\ Z^6 &= i\,\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}} + \frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}}\left(\frac{1}{3}\,\overline{\boldsymbol{P}} + \frac{2}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\right) + \frac{\overline{\mathsf{e}}\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1(\boldsymbol{k})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}. \end{aligned}$$

Proof. We start by differentiating (8.1), finalizing directly the Maurer-Cartan part, thanks to the Maurer-Cartan matrix shown above, and setting aside $d\rho$ for the moment

$$\begin{aligned} d\kappa &= \beta\wedge\rho + \alpha\wedge\kappa \\ &\quad - i\,\overline{\mathsf{c}}\mathsf{e}\,d\rho_0 + \mathsf{c}\,d\kappa_0', \\ d\zeta &= \gamma\wedge\rho + i\,\beta\wedge\kappa + \big(\alpha-\overline{\alpha}\big)\wedge\zeta \\ &\quad + \mathsf{d}\,d\rho_0 + \mathsf{e}\,d\kappa_0' + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,d\zeta_0'. \end{aligned}$$

So we have to compute first

$$\begin{aligned} K^3 &= \big[\rho\wedge\overline{\kappa}\big]\big\{d\kappa\big\} \\ &= -i\,\overline{\mathsf{c}}\mathsf{e}\,\big[\rho\wedge\overline{\kappa}\big]\big\{d\rho_0\big\} + \mathsf{c}\,\big[\rho\wedge\overline{\kappa}\big]\big\{d\kappa_0'\big\}. \end{aligned}$$

The first term is, by (7.1), using the inversion formulas (8.2)

$$\begin{aligned} \big[\rho\wedge\overline{\kappa}\big]\big\{d\rho_0\big\} &= \big[\rho\wedge\overline{\kappa}\big]\Big\{0+0+\overline{\boldsymbol{R}}_0^{1\prime}\left(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\right)\wedge\left(\frac{1}{\overline{\mathsf{c}}}\,\overline{\kappa}\right) + \overline{\boldsymbol{R}}_0^{2\prime}\left(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\right)\wedge\left(-\frac{\mathsf{c}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}\,\overline{\kappa}\right) + i\left(i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\rho\right)\wedge\left(\frac{1}{\overline{\mathsf{c}}}\,\overline{\kappa}\right)\Big\} \\ &= \frac{1}{\mathsf{c}\overline{\mathsf{c}}\overline{\mathsf{c}}}\,\overline{\boldsymbol{R}}_0^{1\prime} - \frac{\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}\overline{\mathsf{c}}}\,\overline{\boldsymbol{R}}_0^{2\prime} - \frac{\mathsf{e}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}. \end{aligned}$$

Similarly

$$\begin{aligned} \big[\rho\wedge\overline{\kappa}\big]\big\{d\kappa_0'\big\} &= \big[\rho\wedge\overline{\kappa}\big]\Big\{0+0+\boldsymbol{K}_0^{3\prime}\left(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,\rho\right)\wedge\left(\frac{1}{\overline{\mathsf{c}}}\,\overline{\kappa}\right) \\ &\quad + 0 + \boldsymbol{K}_0^{6\prime}\left(i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\rho\right)\wedge\left(\frac{1}{\overline{\mathsf{c}}}\,\overline{\kappa}\right) + \left(\left(-i\,\frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}\mathsf{c}} - \frac{\mathsf{d}}{\mathsf{c}\mathsf{c}}\right)\rho\right)\wedge\left(\frac{1}{\overline{\mathsf{c}}}\,\overline{\kappa}\right)\Big\} \\ &= \frac{1}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}\,\boldsymbol{K}_0^{3\prime} + i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}\,\boldsymbol{K}_0^{6\prime} - i\,\frac{\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}\mathsf{c}} - \frac{\mathsf{d}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}. \end{aligned}$$

Hence

$$\begin{aligned}
K_0^{3\prime} &= -i\,\frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}}\,\overline{R}_0^{1\prime} + i\,\frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}\,\overline{R}_0^{2\prime} + i\,\underline{\frac{\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}}}_{\circ} + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,K_0^{3\prime} + i\,\frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}}\,K_0^{6\prime} - i\,\underline{\frac{\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}}}_{\circ} - \frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} \\
&= -\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}}\left(-\frac{i}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)} - \frac{2\,i}{3}\,\overline{P} - \frac{i}{3}\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big) + \frac{i}{3}\,\overline{p}\right) - i\,\frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})} + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,K_0^{3\prime}.
\end{aligned}$$

Replacing this last term $K_0^{3\prime}$ by its value from Proposition 7.2, we reach the stated explicit expression of K^3. Next

$$\begin{aligned}
Z^6 &= \big[\kappa\wedge\overline{\kappa}\big]\big\{d\zeta\big\} \\
&= \mathsf{d}\,\big[\kappa\wedge\overline{\kappa}\big]\big\{d\rho_0\big\} + \mathsf{e}\,\big[\kappa\wedge\overline{\kappa}\big]\big\{d\kappa_0'\big\} + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\big[\kappa\wedge\overline{\kappa}\big]\big\{d\zeta_0'\big\}.
\end{aligned}$$

Separately

$$\begin{aligned}
\big[\kappa\wedge\overline{\kappa}\big]\big\{\mathsf{d}\,d\rho_0\big\} &= 0+0+0+0+\mathsf{d}\,i\,\frac{1}{\mathsf{c}\overline{\mathsf{c}}} \;=\; i\,\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}}, \\
\big[\kappa\wedge\overline{\kappa}\big]\big\{\mathsf{e}\,d\kappa_0'\big\} &= 0+0+0+0+\mathsf{e}K_0^{6\prime}\,\frac{1}{\mathsf{c}\overline{\mathsf{c}}} + \mathsf{e}\Big(-\frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}}\Big)\,\frac{1}{\overline{\mathsf{c}}} \;=\; \frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}}\,K_0^{6\prime} - \frac{\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}}, \\
\big[\kappa\wedge\overline{\kappa}\big]\Big\{\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,d\zeta_0'\Big\} &= 0+0+\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,Z_0^{8\prime}\Big(-\frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}}\Big)\Big(\frac{1}{\overline{\mathsf{c}}}\Big) + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,Z_0^{9\prime}\Big(-\frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}}\Big)\Big(-\frac{\mathsf{c}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}\Big) \\
&= -\frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}}\,Z_0^{8\prime} + \frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}\,Z_0^{9\prime},
\end{aligned}$$

hence summing and inserting the explicit expressions from Proposition 7.2, we conclude

$$\begin{aligned}
Z^6 &= i\,\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}}\,K_0^{6\prime} - \frac{\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}} - \frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}}\,Z_0^{8\prime} + \frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}\,Z_0^{9\prime} \\
&= i\,\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}} + \frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}}\left(\frac{1}{3}\,\overline{P} + \frac{2}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\right) + \frac{\overline{\mathsf{e}}\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})}.
\end{aligned}$$

□

Once we have reached the explicit expressions of both K^3 and Z^6, when we perform the essential combination $i\,K^3 - Z^6$, we see that both the coefficients of $\frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}}$ and of $\frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}$ disappear, and it remains

$$\begin{aligned}
i\,K^3 - Z^6 &= -2i\,\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}} + i\,\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,K_0^{3\prime} \\
&= -2i\,\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}} \\
&\quad + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\underbrace{\left(\frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\Big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\Big)}{\overline{\mathcal{L}}_1(k)} - \frac{4}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^2}{\overline{\mathcal{L}}_1(k)^2} - \frac{1}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\,\overline{P} - \frac{1}{3}\,\overline{\mathcal{L}}_1\big(\overline{P}\big) + \frac{2}{9}\,\overline{P}\,\overline{P}\right)}_{=:\,-2\,H_0}.
\end{aligned}$$

We introduce, as is underbraced

$$H_0 := -\frac{1}{6}\,\frac{\overline{\mathcal{L}}_1\Big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\Big)}{\overline{\mathcal{L}}_1(k)} + \frac{2}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^2}{\overline{\mathcal{L}}_1(k)^2} + \frac{1}{18}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\,\overline{P} + \frac{1}{6}\,\overline{\mathcal{L}}_1\big(\overline{P}\big) - \frac{1}{9}\,\overline{P}\,\overline{P},$$

a function which coincides with Pocchiola's function H. Then by means of the invariant condition

$$0 = i\,K^3 - Z^6,$$

we reach a convenient normalization of the group parameter

$$\begin{aligned}\mathsf{d} \,:=\, & -\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}}+i\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\boldsymbol{H}_0\\ =\, & -\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}}+i\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\bigg(-\frac{1}{6}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}+\frac{2}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)^2}{\overline{\mathcal{L}}_1(\boldsymbol{k})^2}+\frac{1}{18}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\overline{\boldsymbol{P}}+\frac{1}{6}\,\overline{\mathcal{L}}_1\big(\overline{\boldsymbol{P}}\big)-\frac{1}{9}\,\overline{\boldsymbol{P}}\,\overline{\boldsymbol{P}}\bigg).\end{aligned}$$

Before we really perform this normalization of the group parameter d, let us point out that some other invariant relations between torsion coefficients appear. In fact, we see above that:

$$\begin{aligned} i\,K^{4\prime} &= i\,K^4 - i\,b_5,\\ Z^{7\prime} &= Z^7 - i\,b_5,\end{aligned}$$

whence

$$i\,K^{4\prime} - Z^{7\prime} = i\,K^4 - Z^7.$$

However, the next lemma shows that no group parameter can be normalized so.

Lemma 8.5. *Their explicit expressions are*

$$i\,K^4 = Z^7 = -\frac{\mathsf{e}}{\mathsf{c}}\,\frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}.$$

Proof. Indeed, by (7.1), replacing $\overline{\boldsymbol{R}}_0^{2\prime}$ from Proposition 7.2, we can compute using (8.2)

$$\begin{aligned} K^4 &= \big[\rho\wedge\overline{\zeta}\big]\big\{-i\,\overline{\mathsf{c}}\mathsf{e}\,d\rho_0+\mathsf{c}\,d\kappa_0'\big\}\\ &= -i\,\overline{\mathsf{c}}\mathsf{e}\,\big[\rho\wedge\overline{\zeta}\big]\big\{d\rho_0\big\}+\mathsf{c}\,\big[\rho\wedge\overline{\zeta}\big]\big\{d\kappa_0'\big\}\\ &= -i\,\overline{\mathsf{c}}\mathsf{e}\bigg(0+0+0+\overline{\boldsymbol{R}}_0^{2\prime}\Big(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Big)\Big(\frac{\mathsf{c}}{\overline{\mathsf{c}}}\Big)\bigg)+\mathsf{c}\cdot 0\\ &= -i\,\frac{\mathsf{e}}{\overline{\mathsf{c}}}\bigg(-\frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\bigg),\end{aligned}$$

and similarly

$$\begin{aligned} Z^7 &= \big[\kappa\wedge\overline{\zeta}\big]\Big\{\mathsf{d}\,d\rho_0+\mathsf{e}\,d\kappa_0'+\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,d\zeta_0'\Big\}\\ &= \mathsf{d}\,\big[\kappa\wedge\overline{\zeta}\big]\big\{d\rho_0\big\}+\mathsf{e}\,\big[\kappa\wedge\overline{\zeta}\big]\big\{d\kappa_0'\big\}+\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\big[\kappa\wedge\overline{\zeta}\big]\big\{d\zeta_0'\big\}\\ &= 0+0+\frac{\mathsf{c}}{\overline{\mathsf{c}}}\bigg(0+0+0+\boldsymbol{Z}_0^{9\prime}\Big(-\frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}}\Big)\Big(\frac{\mathsf{c}}{\overline{\mathsf{c}}}\Big)\bigg)\\ &= -\frac{\mathsf{e}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1(\overline{\boldsymbol{k}})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}.\end{aligned}$$

□

Another invariant torsion combination is the following.

Lemma 8.6. *Here is an essential linear combination of torsion terms*

$$-i\,K^2+Z^5-\overline{Z}^8.$$

Proof. A glance at what precedes shows

$$\begin{aligned} K^{2\prime} &= K^2-b_3,\\ Z^{5\prime} &= Z^5-i\,b_3+a_2-\overline{a}_4,\\ Z^{8\prime} &= Z^8-a_4+\overline{a}_2,\end{aligned}$$

whence indeed

$$-\,i\,K^{2\prime} + Z^{5\prime} - \overline{Z}^{8\prime} \;=\; -\,i\,K^2 + Z^5 - \overline{Z}^8. \qquad \square$$

Lemma 8.7. *Their explicit expressions are:*

$$\begin{aligned}
K^2 \;&=\; i\,\frac{\overline{\mathsf{e}}}{\mathsf{c}} + \frac{1}{\mathsf{c}}\bigg(-\frac{i}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})^2} + \frac{i}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\;\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})^3} \\
&\qquad -\frac{i}{3}\,\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{\boldsymbol{k}})\big)}{\mathcal{L}_1(\overline{\boldsymbol{k}})} - \frac{i}{3}\,\frac{\overline{\mathcal{L}}_1\big(\mathcal{L}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \frac{2}{3}\,\frac{\mathcal{T}(\boldsymbol{k})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\bigg), \\
Z^5 \;&=\; \frac{1}{\mathsf{c}}\,\frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\frac{\mathcal{L}_1(\overline{\boldsymbol{k}})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}, \\
Z^8 \;&=\; \frac{\mathsf{e}}{\mathsf{c}} - \frac{1}{\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \frac{\mathsf{c}\overline{\mathsf{e}}}{\overline{\mathsf{c}\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1(\boldsymbol{k})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}.
\end{aligned}$$

Proof. Recall

$$\begin{aligned}
d\rho \;&=\; \big(\alpha + \overline{\alpha}\big) \wedge \rho + \mathsf{c}\overline{\mathsf{c}}\,d\rho_0, \\
d\kappa \;&=\; \beta \wedge \rho + \alpha \wedge \kappa - i\,\overline{\mathsf{c}}\mathsf{e}\,d\rho_0 + \mathsf{c}\,d\kappa_0', \\
d\zeta \;&=\; \gamma \wedge \rho + i\,\beta \wedge \kappa + \big(\alpha - \overline{\alpha}\big) \wedge \rho + \mathsf{d}\,d\rho_0 + \mathsf{e}\,d\kappa_0' + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,d\zeta_0',
\end{aligned}$$

hence

$$K^2 \;=\; \big[\rho \wedge \zeta\big]\Big\{ -\; i\,\overline{\mathsf{c}}\mathsf{e}\,d\rho_0 + \mathsf{c}\,d\kappa_0' \Big\}.$$

Visually

$$\begin{aligned}
\big[\rho \wedge \zeta\big]\big\{d\rho_0\big\} \;&=\; \boldsymbol{R}_0^{2\prime}\,\Big(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Big)\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big) \;=\; \frac{1}{\mathsf{c}\mathsf{c}}\,\boldsymbol{R}_0^{2\prime}, \\
\big[\rho \wedge \zeta\big]\big\{d\kappa_0'\big\} \;&=\; \boldsymbol{K}_0^{2\prime}\,\Big(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Big)\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big) + \boldsymbol{K}_0^{5\prime}\,\Big(i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\Big)\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big) - \Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big)\Big(-\,i\,\frac{\overline{\mathsf{e}}}{\overline{\mathsf{c}\mathsf{c}}}\Big) \\
&=\; \frac{1}{\mathsf{c}\mathsf{c}}\,\boldsymbol{K}_0^{2\prime} + i\,\frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}\mathsf{c}}\,\boldsymbol{K}_0^{5\prime} + i\,\frac{\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}},
\end{aligned}$$

hence

$$\begin{aligned}
K^2 \;&=\; -\,i\,\frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\boldsymbol{R}_0^{2\prime} + \frac{1}{\mathsf{c}}\,\boldsymbol{K}_0^{2\prime} + i\,\frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\boldsymbol{K}_0^{5\prime} + i\,\frac{\overline{\mathsf{e}}}{\overline{\mathsf{c}}} \\
&=\; i\,\frac{\overline{\mathsf{e}}}{\overline{\mathsf{c}}} + \frac{1}{\mathsf{c}}\bigg(-\frac{i}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})^2} + \frac{i}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\;\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})^3} \\
&\qquad -\frac{i}{3}\,\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{\boldsymbol{k}})\big)}{\mathcal{L}_1(\overline{\boldsymbol{k}})} - \frac{i}{3}\,\frac{\overline{\mathcal{L}}_1\big(\mathcal{L}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \frac{2}{3}\,\frac{\mathcal{T}(\boldsymbol{k})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\bigg).
\end{aligned}$$

Next, treat

$$Z^5 \;=\; \big[\kappa \wedge \zeta\big]\Big\{\mathsf{d}\,d\rho_0 + \mathsf{e}\,d\kappa_0' + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,d\zeta_0'\Big\},$$

using

$$\begin{aligned}
\big[\kappa \wedge \zeta\big]\big\{d\rho_0\big\} \;&=\; 0, \\
\big[\kappa \wedge \zeta\big]\big\{d\kappa_0'\big\} \;&=\; \boldsymbol{K}_0^{5\prime}\,\Big(\frac{1}{\mathsf{c}}\Big)\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big) \;=\; \frac{\overline{\mathsf{c}}}{\mathsf{c}\mathsf{c}}\,\boldsymbol{K}_0^{5\prime}, \\
\big[\kappa \wedge \zeta\big]\big\{d\zeta_0'\big\} \;&=\; \boldsymbol{Z}_0^{5\prime}\,\Big(\frac{1}{\mathsf{c}}\Big)\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big) \;=\; \frac{\overline{\mathsf{c}}}{\mathsf{c}\mathsf{c}}\,\boldsymbol{Z}_0^{5\prime},
\end{aligned}$$

so

$$Z^5 \,=\, \frac{\overline{\mathsf{c}}\,\mathsf{e}}{\mathsf{c}\mathsf{c}}\bigg(-\,\frac{\mathcal{L}_1(\boldsymbol{k})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\bigg) + \frac{1}{\mathsf{c}}\Bigg(\frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\Bigg).$$

Lastly treat

$$Z^8 \,=\, \big[\zeta\wedge\overline{\kappa}\big]\Big\{\mathsf{d}\,d\rho_0 + \mathsf{e}\,d\kappa_0' + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,d\zeta_0'\Big\},$$

using

$$\begin{aligned}
\big[\zeta\wedge\overline{\kappa}\big]\big\{d\rho_0\big\} &\,=\, 0,\\
\big[\zeta\wedge\overline{\kappa}\big]\big\{d\kappa_0'\big\} &\,=\, \frac{\overline{\mathsf{c}}}{\mathsf{c}}\,\frac{1}{\overline{\mathsf{c}}} \,=\, \frac{1}{\mathsf{c}},\\
\big[\zeta\wedge\overline{\kappa}\big]\big\{d\zeta_0'\big\} &\,=\, \boldsymbol{Z}_0^{8\prime}\,\frac{\overline{\mathsf{c}}}{\mathsf{c}}\,\frac{1}{\overline{\mathsf{c}}} + \boldsymbol{Z}_0^{9\prime}\,\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big(-\,\frac{\mathsf{c}\,\overline{\mathsf{e}}}{\overline{\mathsf{c}\mathsf{c}}}\Big) \,=\, \frac{1}{\mathsf{c}}\,\boldsymbol{Z}_0^{8\prime} - \frac{\overline{\mathsf{e}}}{\mathsf{c}}\,\boldsymbol{Z}_0^{9\prime},
\end{aligned}$$

which concludes

$$\begin{aligned}
Z^8 &\,=\, \frac{\mathsf{e}}{\mathsf{c}} + \frac{1}{\overline{\mathsf{c}}}\,\boldsymbol{Z}_0^{8\prime} - \frac{\mathsf{c}\,\overline{\mathsf{e}}}{\overline{\mathsf{c}\mathsf{c}}}\,\boldsymbol{Z}_0^{9\prime}\\
&\,=\, \frac{\mathsf{e}}{\mathsf{c}} - \frac{1}{\overline{\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \frac{\mathsf{c}\,\overline{\mathsf{e}}}{\overline{\mathsf{c}\mathsf{c}}}\,\frac{\overline{\mathcal{L}}_1(\boldsymbol{k})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}.
\end{aligned}$$

□

Thanks to these explicit expressions, we can compute the essential linear combination of torsion terms, emphasizing two important annihilations by pairs

$$\begin{aligned}
-\,i\,K^2 + Z^5 - \overline{Z}^8 \,=\, \underline{\frac{\overline{\mathsf{e}}}{\mathsf{c}}}_{\circ} + \frac{1}{\mathsf{c}}\Bigg(&-\frac{1}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})^2} + \frac{1}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})^3}\\
&-\frac{1}{3}\,\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{\boldsymbol{k}})\big)}{\mathcal{L}_1(\overline{\boldsymbol{k}})} - \frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\mathcal{L}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} + \frac{2\,i}{3}\,\frac{\mathcal{T}(\boldsymbol{k})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\Bigg)\\
&+\frac{1}{\mathsf{c}}\,\frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \underline{\frac{\overline{\mathsf{c}}\,\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\frac{\mathcal{L}_1(\overline{\boldsymbol{k}})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}}_{\circ\circ}\\
&-\underline{\frac{\overline{\mathsf{e}}}{\overline{\mathsf{c}}}}_{\circ} + \frac{1}{\mathsf{c}}\,\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{\boldsymbol{k}})\big)}{\mathcal{L}_1(\overline{\boldsymbol{k}})} + \underline{\frac{\overline{\mathsf{c}}\,\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\frac{\mathcal{L}_1(\overline{\boldsymbol{k}})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}}_{\circ\circ}.
\end{aligned}$$

Also, in order to match exactly with Pocchiola's function W introduced in [25, 18], we decompose the last term of the second line as

$$\frac{2\,i}{3}\,\frac{\mathcal{T}(\boldsymbol{k})}{\overline{\mathcal{L}}_1(\boldsymbol{k})} \,=\, -\,\frac{1}{3}\,\frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} + \frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\mathcal{L}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} + \frac{i}{3}\,\frac{\mathcal{T}(\boldsymbol{k})}{\overline{\mathcal{L}}_1(\boldsymbol{k})},$$

so that a third pair of terms disappears, and after reorganization — no pen needed —, the result is

$$\begin{aligned}
-\,i\,K^2 + Z^5 - \overline{Z}^8 \,=\, \frac{1}{\mathsf{c}}\Bigg(&-\frac{1}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})^2} + \frac{1}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})^3}\\
&+\frac{2}{3}\,\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{\boldsymbol{k}})\big)}{\mathcal{L}_1(\overline{\boldsymbol{k}})} + \frac{2}{3}\,\frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} + \frac{i}{3}\,\frac{\mathcal{T}(\boldsymbol{k})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\Bigg)\\
=:\,&\frac{1}{\mathsf{c}}\,\boldsymbol{W}_0,
\end{aligned}$$

and this defines a new horizontal function $\boldsymbol{W}_0$, equal to Pocchiola's function W.

For now, we will not use the potential normalization $\mathsf{c} = W_0$ on the open subset of $M^5 \subset \mathbb{C}^3$ on which

$$0 \neq W_0(z_1, z_2, \overline{z}_1, \overline{z}_2, v),$$

if nonempty — a hypothesis must be set up —, but we will deal with this discussion later. In fact, before proceeding, we state a technical differential relation useful later, whose proof can be skipped in a first reading.

Lemma 8.8. *One has*

$$\overline{\mathcal{K}}(H_0) \ = \ -2\,\overline{\mathcal{L}}_1(\overline{k})\,H_0.$$

Proof. Apply the derivation $\overline{\mathcal{K}}$ to H_0

$$\begin{aligned}
\overline{\mathcal{K}}(H_0) \ = \ & -\frac{1}{6}\,\frac{\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(k)))\big)}{\overline{\mathcal{L}}_1(k)} + \frac{1}{6}\,\frac{\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(k))\big)}{\overline{\mathcal{L}}_1(k)^2} \\
& + \frac{4}{9}\,\frac{\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(k))\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)^2} - \frac{4}{9}\,\frac{\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^2}{\overline{\mathcal{L}}_1(k)^3} \\
& + \frac{1}{18}\,\frac{\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(k))\big)}{\overline{\mathcal{L}}_1(k)}\,\overline{P} + \frac{1}{18}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{K}}(\overline{P})}{\overline{\mathcal{L}}_1(k)} \\
& - \frac{1}{18}\,\frac{\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{P}}{\overline{\mathcal{L}}_1(k)^2} + \frac{1}{6}\,\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(\overline{P})\big) - \frac{2}{9}\,\overline{P}\,\overline{\mathcal{K}}(\overline{P})
\end{aligned}$$

perform replacements using Lemmas 2.9 and 2.5

$$\begin{aligned}
\overline{\mathcal{K}}(H_0) \ = \ & \frac{1}{2}\,\frac{\overline{\mathcal{L}}_1(\overline{k})\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(k))\big)}{\overline{\mathcal{L}}_1(k)} + \underline{\frac{1}{2}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{k})\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}}_{\circ} + \frac{1}{6}\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(\overline{k}))\big) - \frac{1}{6}\,\frac{\overline{\mathcal{L}}_1(\overline{k})\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(k))\big)}{\overline{\mathcal{L}}_1(k)} \\
& - \frac{8}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^2\,\overline{\mathcal{L}}_1(\overline{k})}{\overline{\mathcal{L}}_1(k)^2} - \underline{\frac{4}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{k})\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}}_{\circ} + \frac{4}{9}\,\frac{\overline{\mathcal{L}}_1(\overline{k})\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^2}{\overline{\mathcal{L}}_1(k)^2} \\
& - \frac{1}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1(\overline{k})\,\overline{P}}{\overline{\mathcal{L}}_1(k)} - \frac{1}{18}\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{k})\big)\,\overline{P} - \underline{\frac{1}{18}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1(\overline{k})\,\overline{P}}{\overline{\mathcal{L}}_1(k)}}_{\circ\circ} - \underline{\frac{1}{18}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{k})\big)}{\overline{\mathcal{L}}_1(k)}}_{\circ} \\
& + \underline{\frac{1}{18}\,\frac{\overline{\mathcal{L}}_1(\overline{k})\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{P}}{\overline{\mathcal{L}}_1(k)}}_{\circ\circ} + \frac{1}{6}\,\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(\overline{P})\big) + \frac{2}{9}\,\overline{P}\,\overline{P}\,\overline{\mathcal{L}}_1(\overline{k}) + \frac{2}{9}\,\overline{P}\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{k})\big)
\end{aligned}$$

and observe some (underlined) cancellations to get an expression in which the last three terms must yet be transformed

$$\begin{aligned}
\overline{\mathcal{K}}(H_0) \ = \ & \frac{1}{3}\,\frac{\overline{\mathcal{L}}_1(\overline{k})\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(k))\big)}{\overline{\mathcal{L}}_1(k)} - \frac{4}{9}\,\frac{\overline{\mathcal{L}}_1(\overline{k})\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^2}{\overline{\mathcal{L}}_1(k)^2} - \frac{1}{9}\,\frac{\overline{\mathcal{L}}_1(\overline{k})\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{P}}{\overline{\mathcal{L}}_1(k)} \\
& + \frac{2}{9}\,\overline{P}\,\overline{P}\,\overline{\mathcal{L}}_1(\overline{k}) + \frac{1}{6}\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(\overline{k}))\big) + \frac{1}{6}\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{P} + \frac{1}{6}\,\overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(\overline{P})\big).
\end{aligned}$$

Lemma 8.9. *One has*

$$\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(\overline{k}))\big) + \overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{P} + \overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(\overline{P})\big) \ = \ -2\,\overline{\mathcal{L}}_1(\overline{k})\,\overline{\mathcal{L}}_1(\overline{P}).$$

Proof. Apply the vector field $\overline{\mathcal{L}}_1$ to Lemma 2.5

$$\overline{\mathcal{L}}_1\big(\overline{\mathcal{K}}(\overline{P})\big) \ = \ -\overline{\mathcal{L}}_1(\overline{P})\,\overline{\mathcal{L}}_1(\overline{k}) - \overline{P}\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{k})\big) - \overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(\overline{k}))\big).$$

On the other hand, apply the Lie bracket $[\overline{\mathcal{L}}_1, \overline{\mathcal{K}}](\bullet)$ to the function $\overline{P}$, using the concerned known commutation relation shown in Section 2

$$\overline{\mathcal{L}}_1\big(\overline{\mathcal{K}}(\overline{P})\big) - \overline{\mathcal{K}}\big(\overline{\mathcal{L}}_1(\overline{P})\big) \ = \ [\overline{\mathcal{L}}_1, \overline{\mathcal{K}}](\overline{P}) \ = \ \overline{\mathcal{L}}_1(\overline{k})\,\overline{\mathcal{L}}_1(\overline{P}),$$

and replace the first term $\overline{\mathcal{L}}_1\big(\overline{\mathcal{K}}(\overline{P})\big)$ by its value above to get the result. □

Consequently, after this transformation, we see that $\overline{\mathcal{K}}(H_0)$ is a multiple of $\overline{\mathcal{L}}_1(\overline{k})$ in which we recognize $-2\,H_0$ as stated

$$\overline{\mathcal{K}}(H_0) \;=\; \overline{\mathcal{L}}_1(\overline{k})\left(\frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)} - \frac{4}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^2}{\overline{\mathcal{L}}_1(k)^2} - \frac{1}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{P}}{\overline{\mathcal{L}}_1(k)} - \frac{1}{3}\,\overline{\mathcal{L}_1}(\overline{P}) + \frac{2}{9}\,\overline{P}^2\right).$$

□

As we already observed, the essential (invariant) torsion $i\,K^3 - Z^6$ can be set 0 to normalize the group parameter d as

$$\mathsf{d} \;:=\; -\,\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}} + i\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,H_0,$$

whence inserting in (8.1)

$$\begin{pmatrix}\rho\\ \kappa\\ \zeta\end{pmatrix} \;:=\; \begin{pmatrix}\mathsf{c}\overline{\mathsf{c}} & 0 & 0\\ -i\,\overline{\mathsf{c}}\mathsf{e} & \mathsf{c} & 0\\ -\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}} + i\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,H_0 & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}}\end{pmatrix}\begin{pmatrix}\rho_0\\ \kappa_0'\\ \zeta_0'\end{pmatrix}.$$

Thus, we are naturally led to change the initial coframe on M

$$\big\{\rho_0,\,\kappa_0',\,\zeta_0',\,\overline{\kappa}_0',\,\overline{\zeta}_0'\big\} \qquad \rightsquigarrow \qquad \big\{\rho_0,\,\kappa_0',\,\zeta_0'',\,\overline{\kappa}_0',\,\overline{\zeta}_0''\big\},$$

by introducing the new 1-form

$$\zeta_0'' \;:=\; \zeta_0' + i\,H_0\,\rho_0,$$

so that a new, reduced by two real dimensions, G-structure, appears

$$\begin{pmatrix}\rho\\ \kappa\\ \zeta\end{pmatrix} \;:=\; \begin{pmatrix}\mathsf{c}\overline{\mathsf{c}} & 0 & 0\\ -i\,\overline{\mathsf{c}}\mathsf{e} & \mathsf{c} & 0\\ -\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}}\end{pmatrix}\begin{pmatrix}\rho_0\\ \kappa_0'\\ \zeta_0''\end{pmatrix},$$

which is justified by the computation/reorganization

$$\begin{aligned}\zeta \;&=\; \Big(-\,\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}} + i\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,H_0\Big)\,\rho_0 + \mathsf{e}\,\kappa_0' + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\zeta_0'\\ &=\; -\,\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}}\,\rho_0 + \mathsf{e}\,\kappa_0' + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\Big(\underbrace{\zeta_0' + i\,H_0\,\rho_0}_{=:\,\zeta_0''}\Big).\end{aligned}$$

Back to previous expressions, this last coframe writes out as

$$\begin{aligned}\rho_0 \;&:=\; \frac{1}{\ell}\,\Big(dv - A^1\,dz_1 - A^2\,dz_2 - \overline{A}^1\,d\overline{z}_1 - \overline{A}^2\,d\overline{z}_2\Big),\\ \kappa_0' \;&:=\; dz_1 - k\,dz_2 + \frac{i}{3}\,B_0\,\rho_0,\\ \zeta_0'' \;&:=\; \overline{\mathcal{L}}_1(k)\,dz_2 + i\,H_0\,\rho_0.\end{aligned}$$

9. Darboux-Cartan Structure of the Coframe $\{\rho_0, \kappa_0', \zeta_0'', \overline{\kappa}_0', \overline{\zeta}_0''\}$

The present change of initial coframe expresses as

$$\zeta_0'' \;:=\; \zeta_0' + i\,H_0\,\rho_0 \qquad\qquad \Longleftrightarrow \qquad\qquad \zeta_0' \;=\; \zeta_0'' - i\,H_0.$$

The exterior differentiation of ζ_0'' comprises 3 terms that we shall compute soon

$$d\zeta_0'' \;=\; d\zeta_0' + i\,dH_0 \wedge \rho_0 + i\,H_0\,d\rho_0.$$

Back to the previous structure equations written in the abbreviated form (7.1), we may start by replacing ζ_0' in $d\rho_0$, while observing that

$$\rho_0 \wedge \zeta_0' \,=\, \rho_0 \wedge \zeta_0'' \qquad \text{and} \qquad \rho_0 \wedge \overline{\zeta}_0' \,=\, \rho_0 \wedge \overline{\zeta}_0'',$$

we come to unchanged coefficients for

$$d\rho_0 \,=\, \boldsymbol{R}_0^{1\prime}\,\rho_0 \wedge \kappa_0' + \boldsymbol{R}_0^{2\prime}\,\rho_0 \wedge \zeta_0'' + \overline{\boldsymbol{R}}_0^{1\prime}\,\rho_0 \wedge \overline{\kappa}_0' + \overline{\boldsymbol{R}}_0^{2\prime}\,\rho_0 \wedge \overline{\zeta}_0'' + i\,\kappa_0' \wedge \overline{\kappa}_0',$$

hence without computation, the third term is

$$i\,\boldsymbol{H}_0\,d\rho_0 \,=\, i\,\boldsymbol{H}_0\,\boldsymbol{R}_0^{1\prime}\,\rho_0 \wedge \kappa_0' + i\,\boldsymbol{H}_0\,\boldsymbol{R}_0^{2\prime}\,\rho_0 \wedge \zeta_0' + i\,\boldsymbol{H}_0\,\overline{\boldsymbol{R}}_0^{1\prime}\,\rho_0 \wedge \overline{\kappa}_0' + i\,\boldsymbol{H}_0\,\overline{\boldsymbol{R}}_0^{2\prime}\,\rho_0 \wedge \overline{\zeta}_0' - \boldsymbol{H}_0\,\kappa_0 \wedge \overline{\kappa}_0'.$$

Next, we do the same replacement of ζ_0' in

$$\begin{aligned} d\kappa_0' \,=\, & \boldsymbol{K}_0^{1\prime}\,\rho_0 \wedge \kappa_0' + \boldsymbol{K}_0^{2\prime}\,\rho_0 \wedge \Big(\zeta_0'' - i\,\boldsymbol{H}_0\,\rho_0\Big) + \boldsymbol{K}_0^{3\prime}\,\rho_0 \wedge \overline{\kappa}_0' \\ & + \boldsymbol{K}_0^{5\prime}\,\kappa_0' \wedge \Big(\zeta_0'' - i\,\boldsymbol{H}_0\,\rho_0\Big) + \boldsymbol{K}_0^{6\prime}\,\kappa_0' \wedge \overline{\kappa}_0' + \Big(\zeta_0'' - i\,\boldsymbol{H}_0\,\rho_0\Big) \wedge \overline{\kappa}_0', \end{aligned}$$

hence

$$\begin{aligned} d\kappa_0' \,=\, & \Big(\underbrace{\boldsymbol{K}_0^{1\prime} + i\,\boldsymbol{K}_0^{5\prime}\,\boldsymbol{H}_0}_{=:\,\boldsymbol{K}_0^{1\prime\prime}}\Big)\,\rho_0 \wedge \kappa_0' + \boldsymbol{K}_0^{2\prime}\,\rho_0 \wedge \zeta_0'' + \Big(\underbrace{\boldsymbol{K}_0^{3\prime} - i\,\boldsymbol{H}_0}_{=:\,\boldsymbol{K}_0^{3\prime}}\Big)\,\rho_0 \wedge \overline{\kappa}_0' \\ & + \boldsymbol{K}_0^{5\prime}\,\kappa_0' \wedge \zeta_0'' + \boldsymbol{K}_0^{6\prime}\,\kappa_0' \wedge \overline{\kappa}_0' + \zeta_0'' \wedge \overline{\kappa}_0'. \end{aligned}$$

Similarly, do the same for

$$\begin{aligned} d\zeta_0' \,=\, & \boldsymbol{Z}_0^{2\prime}\,\rho_0 \wedge \Big(\zeta_0'' - i\,\boldsymbol{H}_0\,\rho_0\Big) + \boldsymbol{Z}_0^{5\prime}\,\kappa_0' \wedge \Big(\overline{\zeta}_0'' + i\,\boldsymbol{H}_0\,\rho_0\Big) \\ & + \boldsymbol{Z}_0^{8\prime}\,\Big(\zeta_0'' - i\,\boldsymbol{H}_0\,\rho_0\Big) \wedge \overline{\kappa}_0' + \boldsymbol{Z}_0^{9\prime}\,\Big(\zeta_0'' - i\,\boldsymbol{H}_0\,\rho_0\Big) \wedge \Big(\overline{\zeta}_0'' + i\,\overline{\boldsymbol{H}}_0\,\rho_0\Big), \end{aligned}$$

hence

$$\begin{aligned} d\zeta_0' \,=\, & i\,\boldsymbol{Z}_0^{5\prime}\,\boldsymbol{H}_0\,\rho_0 \wedge \kappa_0' + \Big(\boldsymbol{Z}_0^{2\prime} - i\,\boldsymbol{Z}_0^{9\prime}\,\overline{\boldsymbol{H}}_0\Big)\,\rho_0 \wedge \zeta_0'' - i\,\boldsymbol{Z}_0^{8\prime}\,\boldsymbol{H}_0\,\rho_0 \wedge \overline{\kappa}_0' \\ & -\, i\,\boldsymbol{Z}_0^{9\prime}\,\boldsymbol{H}_0\,\rho_0 \wedge \overline{\zeta}_0'' + \boldsymbol{Z}_0^{5\prime}\,\kappa_0' \wedge \zeta_0'' + \boldsymbol{Z}_0^{8\prime}\,\zeta_0'' \wedge \overline{\kappa}_0' + \boldsymbol{Z}_0^{9\prime}\,\zeta_0'' \wedge \overline{\zeta}_0''. \end{aligned}$$

Next, we have to compute the second term in $d\zeta_0''$, and using

$$d\boldsymbol{H}_0 \,=\, \mathcal{T}(\boldsymbol{H}_0)\,\rho_0 + \mathcal{L}_1(\boldsymbol{H}_0)\,\kappa_0 + \mathcal{K}(\boldsymbol{H}_0)\,\zeta_0 + \overline{\mathcal{L}}_1(\boldsymbol{H}_0)\,\overline{\kappa}_0 + \overline{\mathcal{K}}(\boldsymbol{H}_0)\,\overline{\zeta}_0,$$

it comes

$$\begin{aligned} d\boldsymbol{H}_0 \wedge \rho_0 \,&=\, 0 - \mathcal{L}_1(\boldsymbol{H}_0)\,\rho_0 \wedge \kappa_0 - \mathcal{K}(\boldsymbol{H}_0)\,\rho_0 \wedge \zeta_0 - \overline{\mathcal{L}}_1(\boldsymbol{H}_0)\,\rho_0 \wedge \overline{\kappa}_0 - \overline{\mathcal{K}}(\boldsymbol{H}_0)\,\rho_0 \wedge \overline{\zeta}_0 \\ &=\, -\mathcal{L}_1(\boldsymbol{H}_0)\,\rho_0 \wedge \Big(\kappa_0' - \frac{i}{3}\,\boldsymbol{B}_0\,\rho_0\Big) - \mathcal{K}(\boldsymbol{H}_0)\,\rho_0 \wedge \frac{\zeta_0'}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \overline{\mathcal{L}}_1(\boldsymbol{H}_0)\,\rho_0 \wedge \Big(\overline{\kappa}_0' + \frac{i}{3}\,\overline{\boldsymbol{B}}_0\,\rho_0\Big) \\ &\quad -\, \overline{\mathcal{K}}(\boldsymbol{H}_0)\,\rho_0 \wedge \frac{\overline{\zeta}_0'}{\mathcal{L}_1(\overline{\boldsymbol{k}})} \\ &=\, -\mathcal{L}_1(\boldsymbol{H}_0)\,\rho_0 \wedge \kappa_0' - \frac{\mathcal{K}(\boldsymbol{H}_0)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\rho_0 \wedge \zeta_0' - \overline{\mathcal{L}}_1(\boldsymbol{H}_0)\,\rho_0 \wedge \overline{\kappa}_0' - \frac{\overline{\mathcal{K}}(\boldsymbol{H}_0)}{\mathcal{L}_1(\overline{\boldsymbol{k}})}\,\rho_0 \wedge \overline{\zeta}_0', \end{aligned}$$

hence multiplying by i, we get the expression of the second term

$$i\,d\boldsymbol{H}_0 \wedge \rho_0 \,=\, -\,i\,\mathcal{L}_1(\boldsymbol{H}_0)\,\rho_0 \wedge \kappa_0' - i\,\frac{\mathcal{K}(\boldsymbol{H}_0)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\rho_0 \wedge \zeta_0'' - i\,\overline{\mathcal{L}}_1(\boldsymbol{H}_0)\,\rho_0 \wedge \overline{\kappa}_0' - i\,\frac{\overline{\mathcal{K}}(\boldsymbol{H}_0)}{\mathcal{L}_1(\overline{\boldsymbol{k}})}\,\rho_0 \wedge \overline{\zeta}_0''.$$

Summing and collecting the three computed terms yields

$$
\begin{aligned}
d\zeta_0'' &= \rho_0\wedge\kappa_0'\Big[\underbrace{i\,\boldsymbol{Z}_0^{5\prime}H_0 - i\,\mathcal{L}_1(H_0) + i\,H_0\,\boldsymbol{R}_0^{1\prime}}_{=:\,Z_0^{1\prime\prime}}\Big] + \rho_0\wedge\zeta_0''\Big[\underbrace{\boldsymbol{Z}_0^{2\prime} - i\,\boldsymbol{Z}_0^{9\prime}\overline{H}_0 - i\,\frac{\mathcal{K}(H_0)}{\overline{\mathcal{L}}_1(k)} + i\,H_0\,\boldsymbol{R}_0^{2\prime}}_{=:\,Z_0^{2\prime\prime}}\Big] \\
&\quad + \rho_0\wedge\overline{\kappa}_0'\Big[\underbrace{-i\,\boldsymbol{Z}_0^{8\prime}H_0 - i\,\overline{\mathcal{L}}_1(H_0) + i\,H_0\,\overline{\boldsymbol{R}}_0^{1\prime}}_{=:\,Z_0^{3\prime\prime}}\Big] + \rho_0\wedge\overline{\zeta}_0''\Big[\underline{-i\,\boldsymbol{Z}_0^{9\prime}H_0 - i\,\frac{\overline{\mathcal{K}}(H_0)}{\mathcal{L}_1(\overline{k})} + i\,H_0\,\overline{\boldsymbol{R}}_0^{2\prime}}\Big] \\
&\quad + \boldsymbol{Z}_0^{5\prime}\,\kappa_0'\wedge\zeta_0'' + \kappa_0'\wedge\overline{\kappa}_0'\Big[\underbrace{-H_0}_{=:\,Z_0^{6\prime\prime}}\Big] + \boldsymbol{Z}_0^{8\prime}\,\zeta_0''\wedge\overline{\kappa}_0' + \boldsymbol{Z}_0^{9\prime}\,\zeta_0''\wedge\overline{\zeta}_0''.
\end{aligned}
$$

Lemma 9.1. *One has the identical vanishing of the coefficient of* $\rho_0\wedge\overline{\zeta}_0''$ *in* $d\zeta_0''$

$$
\begin{aligned}
\boldsymbol{Z}_0^{4\prime\prime} &:= -\,i\,\boldsymbol{Z}_0^{9\prime}H_0 - i\,\frac{\overline{\mathcal{K}}(H_0)}{\mathcal{L}_1(\overline{k})} + i\,H_0\,\overline{\boldsymbol{R}}_0^{2\prime} \\
&\equiv 0.
\end{aligned}
$$

Proof. This is equivalent to

$$
\overline{\mathcal{K}}(H_0) \overset{?}{\equiv} \mathcal{L}_1(\overline{k})\,H_0\Big(-\boldsymbol{Z}_0^{9\prime} + \overline{\boldsymbol{R}}_0^{2\prime}\Big)
$$

and after a replacement using Proposition 7.2, to

$$
\overline{\mathcal{K}}(H_0) \overset{?}{\equiv} \mathcal{L}_1(\overline{k})\,H_0\left(-\frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})} - \frac{\overline{\mathcal{L}}_1(k)}{\mathcal{L}_1(\overline{k})}\right),
$$

an identity which was already seen by Lemma 8.8. □

In summary

$$
\begin{aligned}
d\rho_0 &= \boldsymbol{R}_0^{1\prime}\,\rho_0\wedge\kappa_0' + \boldsymbol{R}_0^{2\prime}\,\rho_0\wedge\zeta_0'' + \overline{\boldsymbol{R}}_0^{1\prime}\,\rho_0\wedge\overline{\kappa}_0' + \overline{\boldsymbol{R}}_0^{2\prime}\,\rho_0\wedge\overline{\zeta}_0'' + i\,\kappa_0'\wedge\overline{\kappa}_0', \\
d\kappa_0' &= \boldsymbol{K}_0^{1\prime\prime}\,\rho_0\wedge\kappa_0' + \boldsymbol{K}_0^{2\prime}\,\rho_0\wedge\zeta_0'' + \boldsymbol{K}_0^{3\prime\prime}\,\rho_0\wedge\overline{\kappa}_0' \\
&\quad + \boldsymbol{K}_0^{5\prime}\,\kappa_0'\wedge\zeta_0'' + \boldsymbol{K}_0^{6\prime}\,\kappa_0'\wedge\overline{\kappa}_0' + \zeta_0''\wedge\overline{\kappa}_0', \\
d\zeta_0'' &= \boldsymbol{Z}_0^{1\prime\prime}\,\rho_0\wedge\kappa_0' + \boldsymbol{Z}_0^{2\prime\prime}\,\rho_0\wedge\zeta_0'' + \boldsymbol{Z}_0^{3\prime\prime}\,\rho_0\wedge\overline{\kappa}_0' \\
&\quad + \boldsymbol{Z}_0^{5\prime}\,\kappa_0'\wedge\zeta_0'' + \boldsymbol{Z}_0^{6\prime\prime}\,\kappa_0'\wedge\overline{\kappa}_0' + \boldsymbol{Z}_0^{8\prime}\,\zeta_0''\wedge\overline{\kappa}_0' + \boldsymbol{Z}_0^{9\prime}\,\zeta_0''\wedge\overline{\zeta}_0''.
\end{aligned}
$$

Notice that new coefficients $\boldsymbol{Z}_0^{2\prime\prime}, \boldsymbol{Z}_0^{3\prime\prime}, \boldsymbol{Z}_0^{4\prime\prime}$ appear in $d\zeta_0''$, which were absent in $d\zeta_0'$, as they are coming from the second term $i\,dH_0\wedge\rho_0$.

10. Absorption and apparition of two 1-forms π^1, π^2

With the 4-dimensional group parametrized by $(\mathsf{c},\overline{\mathsf{c}},\mathsf{e},\overline{\mathsf{e}})$, the lifted coframe writes:

$$
\begin{pmatrix}\rho\\ \kappa\\ \zeta\end{pmatrix} := \begin{pmatrix}\mathsf{c}\overline{\mathsf{c}} & 0 & 0\\ -i\,\overline{\mathsf{c}}\mathsf{e} & \mathsf{c} & 0\\ -\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}}\end{pmatrix}\begin{pmatrix}\rho_0\\ \kappa_0'\\ \zeta_0''\end{pmatrix} \qquad\Longleftrightarrow\qquad \begin{cases}\rho := \mathsf{c}\overline{\mathsf{c}}\,\rho_0,\\ \kappa := -\,i\,\overline{\mathsf{c}}\mathsf{e}\,\rho_0 + \mathsf{c}\,\kappa_0',\\ \zeta := -\,\dfrac{i}{2}\,\dfrac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}}\,\rho_0 + \mathsf{e}\,\kappa_0' + \dfrac{\mathsf{c}}{\overline{\mathsf{c}}}\,\zeta_0'',\end{cases}
$$

with inverse formulas

$$\begin{aligned}
\rho_0 &= \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,\rho, \\
\kappa_0' &= i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\rho + \frac{1}{\mathsf{c}}\,\kappa, \qquad (10.1) \\
\zeta_0'' &= -\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}\mathsf{c}}\,\rho - \frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\kappa + \frac{\overline{\mathsf{c}}}{\mathsf{c}}\,\zeta.
\end{aligned}$$

The Maurer-Cartan matrix becomes

$$\begin{aligned}
dg\cdot g^{-1} &= \begin{pmatrix} \overline{\mathsf{c}}\,d\mathsf{c} + \mathsf{c}d\overline{\mathsf{c}} & 0 & 0 \\ -i\,\mathsf{e}d\overline{\mathsf{c}} - i\,\overline{\mathsf{c}}d\mathsf{e} & d\mathsf{c} & 0 \\ -\frac{i}{2}\frac{\mathsf{e}\mathsf{e}\,d\overline{\mathsf{c}}}{\mathsf{c}} - i\,\frac{\overline{\mathsf{c}}\mathsf{e}\,d\mathsf{e}}{\mathsf{c}} + \frac{i}{2}\frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}\,d\mathsf{c}}{\mathsf{c}\mathsf{c}} & d\mathsf{e} & \frac{d\mathsf{c}}{\overline{\mathsf{c}}} - \frac{\mathsf{c}\,d\overline{\mathsf{c}}}{\overline{\mathsf{c}}\overline{\mathsf{c}}} \end{pmatrix} \begin{pmatrix} \frac{1}{\mathsf{c}\overline{\mathsf{c}}} & 0 & 0 \\ i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}} & \frac{1}{\mathsf{c}} & 0 \\ -\frac{i}{2}\frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}\mathsf{c}} & -\frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}\mathsf{c}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}} \end{pmatrix} \\
&=: \begin{pmatrix} \alpha+\overline{\alpha} & 0 & 0 \\ \beta & \alpha & 0 \\ 0 & i\,\beta & \alpha-\overline{\alpha} \end{pmatrix},
\end{aligned}$$

in terms of the group-invariant 1-forms

$$\begin{aligned}
\alpha &:= \frac{d\mathsf{c}}{\mathsf{c}}, \\
\beta &:= i\,\frac{\mathsf{e}\,d\mathsf{c}}{\mathsf{c}\mathsf{c}} - i\,\frac{\mathsf{e}\,d\overline{\mathsf{c}}}{\mathsf{c}\overline{\mathsf{c}}} - i\,\frac{d\mathsf{e}}{\mathsf{c}}.
\end{aligned}$$

Now, if we exterior-differentiate the lifted coframe on the product manifold equipped with coordinates

$$\big(z_1, z_2, \overline{z}_1, \overline{z}_2, v\big) \times \big(\mathsf{c}, \overline{\mathsf{c}}, \mathsf{e}, \overline{\mathsf{e}}\big) \;\in\; M^5 \times G^4,$$

after hard computations, we may come to structure equations of the abstract shape

$$\begin{aligned}
d\rho &= \big(\alpha+\overline{\alpha}\big)\wedge\rho + R^1\,\rho\wedge\kappa + R^2\,\rho\wedge\zeta + \overline{R}^1\,\rho\wedge\overline{\kappa} + \overline{R}^2\,\rho\wedge\overline{\zeta} + i\,\kappa\wedge\overline{\kappa}, \\
d\kappa &= \beta\wedge\rho + \alpha\wedge\kappa + K^1\,\rho\wedge\kappa + K^2\,\rho\wedge\zeta + K^3\,\rho\wedge\overline{\kappa} + K^4\,\rho\wedge\overline{\zeta} \\
&\quad + K^5\,\kappa\wedge\zeta + K^6\,\kappa\wedge\overline{\kappa} + \zeta\wedge\overline{\kappa}, \\
d\zeta &= \gamma\wedge\rho + i\,\beta\wedge\kappa + \big(\alpha-\overline{\alpha}\big)\wedge\zeta \\
&\quad + Z^1\,\rho\wedge\kappa + Z^2\,\rho\wedge\zeta + Z^3\,\rho\wedge\overline{\kappa} + Z^4\,\rho\wedge\overline{\zeta} \\
&\quad + Z^5\,\kappa\wedge\zeta + Z^6\,\kappa\wedge\overline{\kappa} + Z^7\,\kappa\wedge\overline{\zeta} + Z^8\,\zeta\wedge\overline{\kappa} + Z^9\,\zeta\wedge\overline{\zeta}.
\end{aligned}$$

A moment of reflection convinces of the truth of

Assertion 10.2. *The relations coming from the normalizations of the group parameters* f, b, c *are preserved*

$$\begin{aligned}
1 &= \big[\zeta\wedge\overline{\kappa}\big]\big\{d\kappa\big\}, \\
0 &= \overline{R}^1 - 2\,K^6 + Z^8, \\
0 &= i\,K^3 - Z^6,
\end{aligned}$$

as well as the auxiliary relations

$$\begin{aligned}
K^5 &= R^2, \\
Z^7 &= i\,K^4, \\
Z^9 &= -\overline{R}^2.
\end{aligned}$$

□

Now, we want to *absorb* as many as possible of these torsion coefficients. So we introduce *modified Maurer-Cartan forms* — mind notations

$$\begin{aligned}\pi^1 &:= \alpha - a_1\,\rho - a_2\,\kappa - a_3\,\zeta - a_4\,\overline{\kappa} - a_5\,\overline{\zeta},\\ \pi^2 &:= \beta - b_1\,\rho - b_2\,\kappa - b_3\,\zeta - b_4\,\overline{\kappa} - b_5\,\overline{\zeta}\end{aligned}$$

and we try to determine (fix) the unknown coefficients a_i, b_i. By replacement, setting $c_i := 0$ in the formula seen above for $d\zeta$, we obtain without pain

$$\begin{aligned}d\rho &= (\pi^1+\overline{\pi}^1) + \rho\wedge\kappa\Big(R^1 - a_2 - \overline{a}_4\Big) + \rho\wedge\zeta\Big(R^2 - a_3 - \overline{a}_5\Big)\\ &\quad + \rho\wedge\overline{\kappa}\Big(\overline{R}^1 - a_4 - \overline{a}_2\Big) + \rho\wedge\overline{\zeta}\Big(\overline{R}^2 - a_5 - \overline{a}_3\Big) + i\,\kappa\wedge\overline{\kappa},\\ d\kappa &= \pi^2\wedge\rho + \pi^1\wedge\kappa + \rho\wedge\kappa\Big(K^1 + a_1 - b_2\Big) + \rho\wedge\zeta\Big(K^2 - b_3\Big)\\ &\quad + \rho\wedge\overline{\kappa}\Big(K^3 - b_4\Big) + \rho\wedge\overline{\zeta}\Big(K^4 - b_5\Big) + \kappa\wedge\zeta\Big(K^5 - a_3\Big)\\ &\quad + \kappa\wedge\overline{\kappa}\Big(K^6 - a_4\Big) + \kappa\wedge\overline{\zeta}\,(-a_5) + \zeta\wedge\overline{\kappa},\\ d\zeta &= i\,\pi^2\wedge\kappa + (\pi^1 - \overline{\pi}^1)\wedge\zeta + \rho\wedge\kappa\Big(Z^1 + i\,b_1\Big) + \rho\wedge\zeta\Big(Z^2 + a_1 - \overline{a}_1\Big)\\ &\quad + \rho\wedge\overline{\kappa}\Big(Z^3\Big) + \rho\wedge\overline{\zeta}\Big(Z^4\Big) + \kappa\wedge\zeta\Big(Z^5 - i\,b_3 + a_2 - \overline{a}_4\Big) + \kappa\wedge\overline{\kappa}\Big(Z^6 - i\,b_4\Big)\\ &\quad + \kappa\wedge\overline{\zeta}\Big(Z^7 - i\,b_5\Big) + \zeta\wedge\overline{\kappa}\Big(Z^8 - a_4 + \overline{a}_2\Big) + \zeta\wedge\overline{\zeta}\Big(Z^9 - a_5 + \overline{a}_3\Big).\end{aligned}$$

Now, replacing from Assertion 10.2

$$Z^8 := -\overline{R}^1 + 2\,K^6, \qquad Z^6 := i\,K^3, \qquad K^5 := R^2, \qquad Z^7 := i\,K^4, \qquad Z^9 := -\overline{R}^2,$$

the absorption equations write out as

$$\begin{aligned} a_2 + \overline{a}_4 &= R^1,\\ \underline{a_3 + \overline{a}_5} &= R^2, \end{aligned} \qquad \begin{aligned} -a_1 + b_2 &= K^1,\\ b_3 &= K^2,\\ b_4 &= K^3,\\ b_5 &= K^4,\\ \underline{a_3} &= R^2,\\ a_4 &= K^6,\\ \underline{-a_5} &= 0, \end{aligned} \qquad \begin{aligned} i\,b_1 &= -Z^1,\\ -a_1 + \overline{a}_1 &= Z^2,\\ 0 &= \boxed{Z^3},\\ 0 &= \boxed{Z^4},\\ -a_2 + \overline{a}_4 + i\,b_3 &= Z^5,\\ i\,b_4 &= i\,K^3,\\ i\,b_5 &= i\,K^4,\\ -\overline{a}_2 + a_4 &= -\overline{R}^1 + 2\,K^6,\\ \underline{-\overline{a}_3 + a_5} &= -\overline{R}^2. \end{aligned}$$

The boxed Z^3 and Z^4 are clearly essential torsions, since they cannot be annihilated by any choice of a_i, b_i. We will compute them explicitly a bit later.

At the end of the second colon, $a_5 = 0$, whence at the ends of the other two colons, we get $a_3 := R^2$, hence all the 4 underlined equations drop. Also, unique assignments exist for

$$\begin{aligned} b_3 &:= K^2, & b_1 &:= i\,Z^1, \\ b_4 &:= K^3, & b_4 &:= K^3, \\ b_5 &:= K^4, & b_5 &:= K^4 \\ a_4 &:= K^6, \end{aligned}$$

and it remains to solve

$$a_2 + \overline{K}^6 \overset{*}{=} R^1, \quad -a_1 + b_2 = K^1, \qquad \begin{aligned} -a_1 + \overline{a}_1 &= Z^2, \\ -a_2 + \overline{K}^6 + i\,K^2 &\overset{?}{=} Z^5, \\ -\overline{a}_2 + K^6 &\overset{*}{=} -\overline{R}^1 + 2\,K^6. \end{aligned}$$

Certainly

$$b_2 := K^1 + a_1$$

and the two equations $\overset{*}{=}$ for a_2 are equivalent — this comes from the normalization relation $0 = \overline{R}^1 - 2\,K^6 + Z^8$ already taken account of —, yielding

$$a_2 := R^1 - \overline{K}^6.$$

However, the equation $\overset{?}{=}$ cannot be satisfied automatically, and this provides an essential torsion combination

$$-R^1 + \overline{K}^6 + \overline{K}^6 + i\,K^2 = Z^5 \qquad \Longleftrightarrow \qquad -i\,K^2 + Z^5 - \overline{Z}^8 = 0,$$

which was already seen in Lemma 8.6. The last remaining equation

$$-a_1 + \overline{a}_1 = Z^2$$

shows that one can annihilate $\mathsf{Im}\,Z^2$ by choosing

$$\mathsf{Im}\,a_1 := -\frac{1}{2}\,\mathsf{Im}\,Z^2$$

and it still remains precisely *one* real degree of freedom, a free variable that we will re-denote

$$\mathsf{t} := \mathsf{Re}\,a_1.$$

In summary, we have established a fundamental

Proposition 10.3. *With* $\mathsf{t} \in \mathbb{R}$ *being a free variable, by defining the precise modified Maurer-Cartan forms*

$$\begin{aligned} \pi^1 &:= \alpha - \Big(\mathsf{t} - \frac{i}{2}\,\mathsf{Im}\,Z^2\Big)\,\rho - \Big(R^1 - \overline{K}^6\Big)\,\kappa - R^2\,\zeta - K^6\,\overline{\kappa} - 0, \\ \pi^2 &:= \beta - i\,Z^1\,\rho - \Big(\mathsf{t} - \frac{i}{2}\,\mathsf{Im}\,Z^2 + K^1\Big)\,\kappa - K^2\,\zeta - K^3\,\overline{\kappa} - K^4\,\overline{\zeta}, \end{aligned}$$

it holds

$$\begin{aligned} d\rho &= \big(\pi^1 + \overline{\pi}^1\big) \wedge \rho + i\,\kappa \wedge \overline{\kappa}, \\ d\kappa &= \pi^2 \wedge \rho + \pi^1 \wedge \kappa + \zeta \wedge \overline{\kappa}, \\ d\zeta &= \big(\pi^1 - \overline{\pi}^1\big) \wedge \zeta + i\,\pi^2 \wedge \kappa \\ &\quad + \big(\mathsf{Re}\,Z^2\big)\,\rho \wedge \zeta + Z^3\,\rho \wedge \overline{\kappa} + Z^4\,\rho \wedge \overline{\zeta} + \Big(Z^5 + R^1 - 2\,\overline{K}^6 - i\,K^2\Big)\,\kappa \wedge \zeta. \end{aligned}$$

□

We yet have to compute the remaining 4 essential torsion coefficients

$$\mathsf{Re}\, Z^2, \qquad Z^3, \qquad Z^4, \qquad Z^5 + R^1 - 2\,\overline{K}^6 - i\,K^2.$$

Fortunately, by anticipation, we have already explored and finalized

$$\begin{aligned} Z^5 + R^1 - 2\,\overline{K}^6 - i\,K^2 &= -\,i\,K^2 + Z^5 - \overline{Z}^8 \\ &= \frac{1}{\mathsf{c}}\,\boldsymbol{W}_0. \end{aligned}$$

Assertion 10.4. *One torsion coefficient vanishes identically*

$$0 \equiv Z^4.$$

Proof. Recall

$$\begin{aligned} Z^4 &= \big[\rho\wedge\overline{\zeta}\big]\big\{d\zeta\big\} \\ &= \big[\rho\wedge\overline{\zeta}\big]\Big\{-\,\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}}\,d\rho_0 + \mathsf{e}\,d\kappa_0' + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,d\zeta_0''\Big\}. \end{aligned}$$

Compute separately

$$\begin{aligned} -\,\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}}\,\big[\rho\wedge\overline{\zeta}\big]\big\{d\rho_0\big\} &= -\,\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}}\,\overline{\boldsymbol{R}}_0^{2\prime}\Big(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Big)\Big(\frac{\mathsf{c}}{\overline{\mathsf{c}}}\Big) = -\,\frac{i}{2}\,\frac{\mathsf{ee}}{\mathsf{c}\overline{\mathsf{c}}}\,\overline{\boldsymbol{R}}_0^{2\prime}, \\ \mathsf{e}\,\big[\rho\wedge\overline{\zeta}\big]\big\{d\kappa_0'\big\} &= 0, \\ \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\big[\rho\wedge\overline{\zeta}\big]\big\{d\zeta_0''\big\} &= \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\underline{\boldsymbol{Z}_0^{4\prime\prime}}_{\circ}\Big(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Big)\Big(\frac{\mathsf{c}}{\overline{\mathsf{c}}}\Big) + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\boldsymbol{Z}_0^{9\prime}\Big(-\,\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}\mathsf{c}\mathsf{c}}\Big)\Big(\frac{\mathsf{c}}{\overline{\mathsf{c}}}\Big) \\ &= 0 - \frac{i}{2}\,\frac{\mathsf{ee}}{\mathsf{c}\overline{\mathsf{c}}}\,\boldsymbol{Z}_0^{9\prime} \end{aligned}$$

and since we have already seen in Lemma 9.1 that $\boldsymbol{Z}_0^{4\prime\prime} \equiv 0$, in the proof of which we have used $\overline{\boldsymbol{R}}_0^{2\prime} + \boldsymbol{Z}_0^{9\prime} \equiv 0$, the sum of these 3 terms is indeed zero, and we done. □

It remains to analyze Z^3 and $\mathsf{Re}\, Z^2$, a substantial task to which the two next sections are devoted. At least, we know that

$$\begin{aligned} d\zeta &= \big(\pi^1 - \overline{\pi}^1\big)\wedge\rho + i\,\kappa\wedge\overline{\kappa} \\ &\quad + \big(\mathsf{Re}\, Z^2\big)\,\rho\wedge\zeta + Z^3\,\rho\wedge\overline{\kappa} + \frac{1}{\mathsf{c}}\,\boldsymbol{W}_0\,\kappa\wedge\zeta. \end{aligned}$$

11. Computation of Pocchiola's invariant $\boldsymbol{J}_0$

We now determine

$$\begin{aligned} Z^3 &= \big[\rho\wedge\overline{\kappa}\big]\big\{d\zeta\big\} \\ &= -\,\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}}\,\big[\rho\wedge\overline{\kappa}\big]\big\{d\rho_0\big\} + \mathsf{e}\,\big[\rho\wedge\overline{\kappa}\big]\big\{d\kappa_0'\big\} + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\big[\rho\wedge\overline{\kappa}\big]\big\{d\zeta_0''\big\} \\ &= -\,\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}}\Big[\overline{\boldsymbol{R}}_0^{1\prime}\Big(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Big)\Big(\frac{1}{\overline{\mathsf{c}}}\Big) + \overline{\boldsymbol{R}}_0^{2\prime}\Big(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Big)\Big(-\,\frac{\mathsf{c}\overline{\mathsf{e}}}{\overline{\mathsf{c}}\overline{\mathsf{c}}}\Big) + \underline{i\Big(i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\Big)\Big(\frac{1}{\overline{\mathsf{c}}}\Big)}_{\circ}\Big] \\ &\quad + \mathsf{e}\Big[\boldsymbol{K}_0^{3\prime\prime}\Big(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Big)\Big(\frac{1}{\overline{\mathsf{c}}}\Big) + \boldsymbol{K}_0^{6\prime}\Big(i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\Big)\Big(\frac{1}{\overline{\mathsf{c}}}\Big) + \underline{\Big(-\,\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}\mathsf{c}\mathsf{c}}\Big)\Big(\frac{1}{\overline{\mathsf{c}}}\Big)}_{\circ}\Big] \\ &\quad + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\Big[\boldsymbol{Z}_0^{3\prime\prime}\Big(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Big)\Big(\frac{1}{\overline{\mathsf{c}}}\Big) + \boldsymbol{Z}_0^{6\prime\prime}\Big(i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\Big)\Big(\frac{1}{\overline{\mathsf{c}}}\Big) + \boldsymbol{Z}_0^{8\prime}\Big(-\,\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}\mathsf{c}\mathsf{c}}\Big)\Big(\frac{1}{\overline{\mathsf{c}}}\Big) + \boldsymbol{Z}_0^{9\prime}\Big(-\,\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{ee}}{\mathsf{c}\mathsf{c}\mathsf{c}}\Big)\Big(\frac{\mathsf{c}\overline{\mathsf{e}}}{\overline{\mathsf{c}}\overline{\mathsf{c}}}\Big)\Big], \end{aligned}$$

hence after collecting

$$\begin{aligned} Z^3 &= \frac{\mathsf{ee}}{\mathsf{cc\overline{c}}} \bigg[-\frac{i}{2}\,\overline{\boldsymbol{R}}_0^{1\prime} + i\,\boldsymbol{K}_0^{6\prime} - \frac{i}{2}\,\boldsymbol{Z}_0^{8\prime} \bigg] + \frac{\mathsf{ee\overline{e}}}{\mathsf{cc\overline{c}}} \bigg[\underline{\frac{i}{2}\,\overline{\boldsymbol{R}}_0^{2\prime} + \frac{i}{2}\,\boldsymbol{Z}_0^{9\prime}}_{\circ} \bigg] \\ &+ \frac{\mathsf{e}}{\mathsf{c\overline{c}\overline{c}}} \Big[\boldsymbol{K}_0^{3\prime\prime} + i\,\boldsymbol{Z}_0^{6\prime\prime} \Big] + \frac{1}{\mathsf{c\overline{c}\overline{c}}}\,\boldsymbol{Z}_0^{3\prime\prime}. \end{aligned}$$

As we already know, the second term vanishes, the third one as well

$$\boldsymbol{K}_0^{3\prime\prime} + i\,\boldsymbol{Z}_0^{6\prime\prime} = 2i\,\boldsymbol{H}_0 - i\,\boldsymbol{H}_0 - i\,\boldsymbol{H}_0,$$

and also the first one

$$\begin{aligned} -\frac{i}{2}\,\overline{\boldsymbol{R}}_0^{1\prime} + i\,\boldsymbol{K}_0^{6\prime} - \frac{i}{2}\,\boldsymbol{Z}_0^{8\prime} &= -\frac{i}{2} \left(\frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} + \frac{2}{3}\,\overline{\boldsymbol{P}} \right) + i \left(-\frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} + \frac{1}{3}\,\overline{\boldsymbol{P}} \right) \\ &- \frac{i}{2} \left(-\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} \right). \end{aligned}$$

It remains only one term

$$\begin{aligned} Z^3 &= \frac{1}{\mathsf{c\overline{c}\overline{c}}}\,\boldsymbol{Z}_0^{3\prime\prime} \\ &= \frac{1}{\mathsf{c\overline{c}\overline{c}}} \Big(-i\,\boldsymbol{Z}_0^{8\prime}\,\boldsymbol{H}_0 - i\,\overline{\mathcal{L}}_1\big(\boldsymbol{H}_0\big) + i\,\boldsymbol{H}_0\,\overline{\boldsymbol{R}}_0^{1\prime} \Big) \\ &= \frac{i}{\mathsf{c\overline{c}\overline{c}}} \left(\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\boldsymbol{H}_0 - \overline{\mathcal{L}}_1\big(\boldsymbol{H}_0\big) + \frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\boldsymbol{H}_0 + \frac{2}{3}\,\boldsymbol{H}_0\,\overline{\boldsymbol{P}} \right) \\ &= \frac{i}{\mathsf{c\overline{c}\overline{c}}} \Bigg(\underbrace{\frac{4}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\boldsymbol{H}_0 + \frac{2}{3}\,\overline{\boldsymbol{P}}_0\,\boldsymbol{H}_0 - \overline{\mathcal{L}}_1\big(\boldsymbol{H}_0\big)}_{=:\,\overline{J}_0} \Bigg). \end{aligned}$$

Then a direct expansion of the derivative $\overline{\mathcal{L}}_1\big(\boldsymbol{H}_0\big)$ which uses neither Lemma 2.5, nor Lemma 2.9, provides (exercise) exactly the same expression as the one of Pocchiola

$$\begin{aligned} \overline{\boldsymbol{J}}_0 &= \frac{1}{6}\,\frac{\overline{\mathcal{L}}_1\Big(\overline{\mathcal{L}}_1\Big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\Big)\Big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \frac{5}{6}\,\frac{\overline{\mathcal{L}}_1\Big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\Big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})^2} - \frac{1}{6}\,\frac{\overline{\mathcal{L}}_1\Big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\Big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\overline{\boldsymbol{P}} \\ &+ \frac{20}{27}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)^3}{\overline{\mathcal{L}}_1(\boldsymbol{k})^3} + \frac{5}{18}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)^2}{\overline{\mathcal{L}}_1(\boldsymbol{k})^2}\,\overline{\boldsymbol{P}} + \frac{1}{6}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)\,\overline{\mathcal{L}}_1\big(\overline{\boldsymbol{P}}\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} - \frac{1}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\overline{\boldsymbol{P}}\,\overline{\boldsymbol{P}} \\ &- \frac{1}{6}\,\overline{\mathcal{L}}_1\Big(\overline{\mathcal{L}}_1\big(\overline{\boldsymbol{P}}\big)\Big) + \frac{1}{3}\,\overline{\mathcal{L}}_1\big(\overline{\boldsymbol{P}}\big)\,\overline{\boldsymbol{P}} - \frac{2}{27}\,\overline{\boldsymbol{P}}\,\overline{\boldsymbol{P}}\,\overline{\boldsymbol{P}}. \end{aligned}$$

12. Computation of the derived invariant $R := \mathsf{Re}\, Z^2$

Next, we determine

$$
\begin{aligned}
Z^2 &= \big[\rho\wedge\zeta\big]\big\{d\zeta\big\}\\
&= -\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}}\,\big[\rho\wedge\zeta\big]\big\{d\rho_0\big\}+\mathsf{e}\,\big[\rho\wedge\zeta\big]\big\{d\kappa_0'\big\}+\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\big[\rho\wedge\zeta\big]\big\{d\zeta_0''\big\}\\
&= -\frac{i}{2}\,\frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}}\,\bigg[\boldsymbol{R}_0^{2\prime}\Big(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Big)\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big)\bigg]\\
&\quad+\mathsf{e}\bigg[\boldsymbol{K}_0^{2\prime}\Big(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Big)\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big)+\boldsymbol{K}_0^{5\prime}\Big(i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\Big)\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big)-\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big)\Big(-i\,\frac{\overline{\mathsf{e}}}{\overline{\mathsf{c}\mathsf{c}}}\Big)\bigg]\\
&\quad+\frac{\mathsf{c}}{\overline{\mathsf{c}}}\bigg[\boldsymbol{Z}_0^{2\prime\prime}\Big(\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Big)\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big)+\boldsymbol{Z}_0^{5\prime}\Big(i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\Big)\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big)-\boldsymbol{Z}_0^{8\prime}\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big)\Big(-i\,\frac{\overline{\mathsf{e}}}{\overline{\mathsf{c}\mathsf{c}}}\Big)-\boldsymbol{Z}_0^{9\prime}\Big(\frac{\overline{\mathsf{c}}}{\mathsf{c}}\Big)\Big(\frac{i}{2}\,\frac{\mathsf{c}\overline{\mathsf{e}\mathsf{e}}}{\overline{\mathsf{c}\mathsf{c}}\mathsf{c}}\Big)\bigg]
\end{aligned}
$$

hence after collecting

$$
\begin{aligned}
Z^2 &= i\,\frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}+\frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}\mathsf{c}}\Big(-\frac{i}{2}\,\boldsymbol{R}_0^{2\prime}+i\,\boldsymbol{K}_0^{5\prime}\Big)+\frac{\mathsf{c}\overline{\mathsf{e}\mathsf{e}}}{\overline{\mathsf{c}\mathsf{c}\mathsf{c}}}\Big(-\frac{i}{2}\,\boldsymbol{Z}_0^{9\prime}\Big)\\
&\quad+\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\Big(\boldsymbol{K}_0^{2\prime}+i\,\boldsymbol{Z}_0^{5\prime}\Big)+\frac{\overline{\mathsf{e}}}{\overline{\mathsf{c}\mathsf{c}}}\Big(i\,\boldsymbol{Z}_0^{8\prime}\Big)+\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,\boldsymbol{Z}_0^{2\prime\prime},
\end{aligned}
$$

that is to say

$$
\begin{aligned}
Z^2 &= i\,\frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}+\frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}\mathsf{c}}\bigg(\frac{i}{2}\,\frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)}-i\,\frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)}\bigg)+\frac{\mathsf{c}\overline{\mathsf{e}\mathsf{e}}}{\overline{\mathsf{c}\mathsf{c}\mathsf{c}}}\bigg(-\frac{i}{2}\,\frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})}\bigg)\\
&\quad+\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\bigg(-\frac{i}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)^2}+\frac{i}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)^3}-\frac{i}{3}\,\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{k})\big)}{\mathcal{L}_1(\overline{k})}\\
&\quad-\frac{i}{3}\,\frac{\overline{\mathcal{L}}_1\big(\mathcal{L}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}-\frac{2}{3}\,\frac{\mathcal{T}(k)}{\overline{\mathcal{L}}_1(k)}+i\,\frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\bigg)\\
&\quad+\frac{\overline{\mathsf{e}}}{\overline{\mathsf{c}\mathsf{c}}}\bigg(-i\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\bigg)+\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\bigg(\underbrace{-\,i\,\boldsymbol{Z}_0^{9\prime}\,\overline{\boldsymbol{H}}_0+i\,\boldsymbol{H}_0\,\boldsymbol{R}_0^{2\prime}}_{\text{on hold}}+\boldsymbol{Z}_0^{2\prime}-i\,\frac{\mathcal{K}(\boldsymbol{H}_0)}{\overline{\mathcal{L}}_1(k)}\bigg).
\end{aligned}
$$

Now, observe firstly that when we consider

$$
2\,\mathsf{Re}\,Z^2 \,=\, Z^2+\overline{Z}^2,
$$

the real part of the sum of the first three terms of Z^2

$$
i\,\frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}+\frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}\mathsf{c}\mathsf{c}}\bigg(-\frac{i}{2}\,\frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)}\bigg)+\frac{\mathsf{c}\overline{\mathsf{e}\mathsf{e}}}{\overline{\mathsf{c}\mathsf{c}\mathsf{c}}}\bigg(-\frac{i}{2}\,\frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})}\bigg)
$$

vanishes, visibly. Secondly, in the sum $Z^2+\overline{Z}^2$, if the terms multiples of $\frac{\mathsf{e}}{\mathsf{cc}}$ are grouped together, we realize that we recover W_0 exactly

$$
\begin{aligned}
& i\,\frac{\mathsf{e}}{\mathsf{cc}}\Bigg(-\frac{1}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)^2}+\frac{1}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)^3}-\frac{1}{3}\,\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{k})\big)}{\mathcal{L}_1(\overline{k})} \\
&\qquad -\frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\mathcal{L}_1(k)\big)}{\mathcal{L}_1(k)}+\frac{2i}{3}\,\frac{\mathcal{T}(k)}{\overline{\mathcal{L}}_1(k)}+\frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}+\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{k})\big)}{\mathcal{L}_1(\overline{k})}\Bigg) \\
&\quad = i\,\frac{\mathsf{e}}{\mathsf{cc}}\Bigg(-\frac{1}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)^2}+\frac{1}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)^3}+\frac{2}{3}\,\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{k})\big)}{\mathcal{L}_1(\overline{k})} \\
&\qquad +\frac{2}{3}\,\frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}+\frac{i}{3}\,\frac{\mathcal{T}(k)}{\overline{\mathcal{L}}_1(k)}\Bigg) \\
&\quad = i\,\frac{\mathsf{e}}{\mathsf{cc}}\,W_0,
\end{aligned}
$$

as we remember its explicit expression from Section 8.

In addition thirdly, using the explicit expressions from Proposition 7.2

$$
R_0^{2\prime} = -\frac{\mathcal{L}_1(k)}{\overline{\mathcal{L}}_1(k)} \qquad\qquad \text{and} \qquad\qquad Z_0^{9\prime} = \frac{\overline{\mathcal{L}}_1(\overline{k})}{\mathcal{L}_1(\overline{k})},
$$

and the explicit expression of

$$
H_0 = -\frac{1}{6}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)}+\frac{2}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^2}{\overline{\mathcal{L}}_1(k)^2}+\frac{1}{18}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\,\overline{P}+\frac{1}{6}\,\overline{\mathcal{L}}_1\big(\overline{P}\big)-\frac{1}{9}\,\overline{P}\,\overline{P},
$$

we verify by a direct computation the identical vanishing

$$
0 \equiv -i\,Z_0^{9\prime}\,\overline{H}_0+i\,H_0\,R_0^{2\prime}+\overline{-i\,Z_0^{9\prime}\,\overline{H}_0+i\,H_0\,R_0^{2\prime}},
$$

which means that the term '*on hold*' underbraced above disappears when taking $2\,\mathsf{Re}\,Z^2$, and we receive

$$
2\,\mathsf{Re}\,Z^2 = i\,\frac{\mathsf{e}}{\mathsf{cc}}\,W_0-i\,\frac{\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}}\,\overline{W}_0+\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Bigg(Z_0^{2\prime}-i\,\frac{\mathcal{K}(H_0)}{\overline{\mathcal{L}}_1(k)}+\overline{Z}_0^{2\prime}+i\,\frac{\overline{\mathcal{K}}(\overline{H}_0)}{\mathcal{L}_1(\overline{k})}\Bigg).
$$

Fourthly and lastly, by replacing

$$
H_0 = -\frac{i}{2}\,K_0^{3\prime},
$$

we get

$$
\tag{12.1} 2\,\mathsf{Re}\,Z^2 = 2\,\mathsf{Re}\Bigg(i\,\frac{\mathsf{e}}{\mathsf{cc}}\,W_0+\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Bigg(Z_0^{2\prime}\underbrace{-\frac{1}{2}\,\frac{\mathcal{K}\big(K_0^{3\prime}\big)}{\overline{\mathcal{L}}_1(k)}}_{\text{on hold}}\Bigg)\Bigg).
$$

A miraculous re-expression of $2\,\mathsf{Re}\,Z^2$ was discovered by Pocchiola on his computer, and was shown in [25, 18], but without any details of proof.

Lemma 12.2. *One has in fact*

$$
2\,\mathsf{Re}\,Z^2 = 2\,\mathsf{Re}\left[i\,\frac{\mathsf{e}}{\mathsf{cc}}\,W_0+\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Bigg(-\frac{i}{2}\,\overline{\mathcal{L}}_1\big(W_0\big)+\frac{i}{2}\,\Bigg(-\frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}+\frac{1}{3}\,\overline{P}\Bigg)\,W_0\Bigg)\right].
$$

This expression shows that $\mathsf{Re}(Z^2)$ depends on the first jet of W_0, that it vanishes when $W_0 = 0$, and therefore, $\mathsf{Re}\, Z^2$ is *not* a primary invariant. We provide details of proof, with no computer help.

Proof. To transform the term '*on hold*' above, we need a technical lemma, whose proof, to be done afterwards, uses mainly the Poincaré relation $d \circ d = 0$ applied to the structure equations (7.1).

Lemma 12.3. *The following two identities hold identically*

$$\frac{\mathcal{K}(K_0^{3\prime})}{\overline{\mathcal{L}}_1(k)} = \overline{\mathcal{L}}_1(K_0^{2\prime}) - K_0^{2\prime} K_0^{6\prime} - K_0^{1\prime} + \overline{K}_0^{1\prime} + Z_0^{2\prime}, \tag{12.4}$$

$$\overline{\mathcal{L}}_1(Z_0^{5\prime}) + \mathcal{L}_1(Z_0^{8\prime}) = Z_0^{5\prime} K_0^{6\prime} + Z_0^{8\prime} \overline{K}_0^{6\prime} + i\, Z_0^{2\prime}. \tag{12.5}$$

Admitting these identities temporarily, let us prove the proposition. In order to replace the term '*on hold*' in (12.1) above, let us multiply by $-\frac{1}{2}$ the first identity (12.4), and take $2\,\mathsf{Re}(\bullet)$

$$2\,\mathsf{Re}\bigg(-\frac{1}{2}\,\frac{\mathcal{K}(K_0^{3\prime})}{\overline{\mathcal{L}}_1(k)}\bigg) = 2\,\mathsf{Re}\bigg(-\frac{1}{2}\,\boxed{\overline{\mathcal{L}}_1(K_0^{2\prime})} + \frac{1}{2}\,K_0^{2\prime} K_0^{6\prime} + 0 - \frac{1}{2}\,Z_0^{2\prime}\bigg).$$

We yet have to transform the boxed term. To this aim, we conjugate the second identity (12.5)

$$\mathcal{L}_1(\overline{Z}_0^{5\prime}) + \overline{\mathcal{L}}_1(\overline{Z}_0^{8\prime}) = \overline{Z}_0^{5\prime} \overline{K}_0^{6\prime} + \overline{Z}_0^{8\prime} K_0^{6\prime} - i\, \overline{Z}_0^{2\prime},$$

and to this identity multiplied by i, we subtract (12.4) also multiplied by i, to get

$$-i\,\overline{\mathcal{L}}_1\Big(Z_0^{5\prime} - \overline{Z}_0^{8\prime}\Big) + i\,\mathcal{L}_1\Big(\overline{Z}_0^{5\prime} - Z_0^{8\prime}\Big) = -i\,K_0^{6\prime}\,\Big(Z_0^{5\prime} - \overline{Z}_0^{8\prime}\Big) + i\,\overline{K}_0^{6\prime}\,\Big(\overline{Z}_0^{5\prime} - Z_0^{8\prime}\Big) + Z_0^{2\prime} + \overline{Z}_0^{2\prime}.$$

But here, remembering that, by definition of W_0

$$Z_0^{5\prime} - \overline{Z}_0^{8\prime} = W_0 + i\,K_0^{2\prime},$$

we can replace to get

$$-i\,\overline{\mathcal{L}}_1(W_0) + \overline{\mathcal{L}}_1(K_0^{2\prime}) + i\,\mathcal{L}_1(\overline{W}_0) + \mathcal{L}_1(\overline{K}_0^{2\prime}) = -i\,K_0^{6\prime}\,W_0 + K_0^{6\prime} K_0^{2\prime} + i\,\overline{K}_0^{6\prime}\,\overline{W}_0 + \overline{K}_0^{6\prime}\,\overline{K}_0^{2\prime} + Z_0^{2\prime} + \overline{Z}_0^{2\prime},$$

that is to say for the mentioned boxed term

$$2\,\mathsf{Re}\Big(\overline{\mathcal{L}}_1(K_0^{2\prime})\Big) = 2\,\mathsf{Re}\Big(i\,\overline{\mathcal{L}}_1(W_0) - i\,K_0^{6\prime}\,W_0 + K_0^{2\prime} K_0^{6\prime} + Z_0^{2\prime}\Big).$$

Multiplying this result by $-\frac{1}{2}$, and replacing above yields

$$2\,\mathsf{Re}\bigg(-\frac{1}{2}\,\frac{\mathcal{K}(K_0^{3\prime})}{\overline{\mathcal{L}}_1(k)}\bigg) = 2\,\mathsf{Re}\bigg(-\frac{i}{2}\,\overline{\mathcal{L}}_1(W_0) + \frac{i}{2}\,K_0^{6\prime}\,W_0 - \underline{\frac{1}{2}\,K_0^{2\prime} K_0^{6\prime}}_{\circ} - \frac{1}{2}\,Z_0^{2\prime} + \underline{\frac{1}{2}\,K_0^{2\prime} K_0^{6\prime}}_{\circ} - \frac{1}{2}\,Z_0^{2\prime}\bigg)$$

and a final replacement in (12.1) concludes, if one remembers that

$$K_0^{6\prime} = -\frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\mathcal{L}_1(k)\big)}{\overline{\mathcal{L}}_1(k)} + \frac{1}{3}\,\overline{P}. \qquad \square$$

Proof of Lemma 12.3. To treat the first identity (12.4), apply the exterior differentiation operator d to the structure equation for $d\kappa_0'$ from (7.1)

$$\begin{aligned}
0 &= d^2\kappa_0' \\
&= d\boldsymbol{K}_0^{1\prime} \wedge \rho_0 \wedge \kappa_0' + \boldsymbol{K}_0^{1\prime}\, d\rho_0 \wedge \kappa_0' - \boldsymbol{K}_0^{1\prime}\, \rho_0 \wedge d\kappa_0' \\
&\quad + d\boldsymbol{K}_0^{2\prime} \wedge \rho_0 \wedge \zeta_0' + \boldsymbol{K}_0^{2\prime}\, d\rho_0 \wedge \zeta_0' - \boldsymbol{K}_0^{2\prime}\, \rho_0 \wedge d\zeta_0' \\
&\quad + \underbrace{d\boldsymbol{K}_0^{3\prime} \wedge \rho_0 \wedge \overline{\kappa}_0'}_{\text{needed}} + \boldsymbol{K}_0^{3\prime}\, d\rho_0 \wedge \overline{\kappa}_0' - \boldsymbol{K}_0^{3\prime}\, \rho_0 \wedge d\overline{\kappa}_0' \\
&\quad + d\boldsymbol{K}_0^{5\prime} \wedge \kappa_0' \wedge \zeta_0' + \boldsymbol{K}_0^{5\prime}\, d\kappa_0' \wedge \zeta_0' - \boldsymbol{K}_0^{5\prime}\, \kappa_0' \wedge d\zeta_0' \\
&\quad + d\boldsymbol{K}_0^{6\prime} \wedge \kappa_0' \wedge \overline{\kappa}_0' + \boldsymbol{K}_0^{6\prime}\, d\kappa_0' \wedge \overline{\kappa}_0' - \boldsymbol{K}_0^{6\prime}\, \kappa_0' \wedge d\overline{\kappa}_0' \\
&\qquad + d\zeta_0' \wedge \overline{\kappa}_0' - \zeta_0' \wedge d\overline{\kappa}_0'.
\end{aligned}$$

Because we are dealing with $\mathcal{K}\big(\boldsymbol{K}_0^{3\prime}\big)$, we can wedge throughout with $\kappa_0' \wedge \overline{\zeta}_0'$ to obtain $\mathcal{K}\big(\boldsymbol{K}_0^{3\prime}\big) / \overline{\mathcal{L}}_1(\boldsymbol{k})$ from the term marked '*needed*', and we get

$$\begin{aligned}
0 = 0 &\qquad + 0 &\qquad - \boldsymbol{K}_0^{1\prime}\, \rho_0 \wedge d\kappa_0' \wedge \kappa_0' \wedge \overline{\zeta}_0' \\
+ d\boldsymbol{K}_0^{2\prime} \wedge \rho_0 \wedge \zeta_0' \wedge \kappa_0' \wedge \overline{\zeta}_0' &\qquad + \boldsymbol{K}_0^{2\prime}\, d\rho_0 \wedge \zeta_0' \wedge \kappa_0' \wedge \overline{\zeta}_0' &\qquad - \boldsymbol{K}_0^{2\prime}\, \rho_0 \wedge d\zeta_0' \wedge \kappa_0' \wedge \overline{\zeta}_0' \\
+ d\boldsymbol{K}_0^{3\prime} \wedge \rho_0 \wedge \overline{\kappa}_0' \wedge \kappa_0' \wedge \overline{\zeta}_0' &\qquad + \boldsymbol{K}_0^{3\prime}\, d\rho_0 \wedge \overline{\kappa}_0' \wedge \kappa_0' \wedge \overline{\zeta}_0' &\qquad - \boldsymbol{K}_0^{3\prime}\, \rho_0 \wedge d\overline{\kappa}_0' \wedge \kappa_0' \wedge \overline{\zeta}_0' \\
+ 0 &\qquad + \boldsymbol{K}_0^{5\prime}\, d\kappa_0' \wedge \zeta_0' \wedge \kappa_0' \wedge \overline{\zeta}_0' &\qquad - 0 \\
+ 0 &\qquad + \boldsymbol{K}_0^{6\prime}\, d\kappa_0' \wedge \overline{\kappa}_0' \wedge \kappa_0' \wedge \overline{\zeta}_0' &\qquad - 0 \\
+ 0 &\qquad + d\zeta_0' \wedge \overline{\kappa}_0' \wedge \kappa_0' \wedge \overline{\zeta}_0' &\qquad - \zeta_0' \wedge d\overline{\kappa}_0' \wedge \kappa_0' \wedge \overline{\zeta}_0'.
\end{aligned}$$

In the left column, observe that two exterior differentials appear, $d\boldsymbol{K}_0^{2\prime}$, $d\boldsymbol{K}_0^{3\prime}$. Already in Section 9, we have implicitly used the following companion of Lemma 2.6.

Lemma 12.6. *The exterior differential of any function $\boldsymbol{G} = \boldsymbol{G}\big(z_1, z_2, \overline{z}_1, \overline{z}_2, v\big)$ on M expresses as*

$$d\boldsymbol{G} = \bigg(\mathcal{T}(\boldsymbol{G}) - \frac{i}{3}\,\boldsymbol{B}_0 + \frac{i}{3}\,\overline{\boldsymbol{B}}_0\bigg)\,\rho_0 + \mathcal{L}_1(\boldsymbol{G})\,\kappa_0' + \frac{\mathcal{K}(\boldsymbol{G})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\zeta_0' + \overline{\mathcal{L}}_1(\boldsymbol{G})\,\overline{\kappa}_0' + \frac{\overline{\mathcal{K}}(\boldsymbol{G})}{\mathcal{L}_1(\overline{\boldsymbol{k}})}\,\overline{\zeta}_0'.$$

Proof. Replacing κ_0 by $\kappa_0' - \frac{i}{3}\,\boldsymbol{B}_0\,\rho_0$ from (6.8), and ζ_0 by $\frac{\zeta_0'}{\overline{\mathcal{L}}_1(\boldsymbol{k})}$ from (5.4), we indeed obtain

$$\begin{aligned}
d\boldsymbol{G} &= \mathcal{T}(\boldsymbol{G})\,\rho_0 + \mathcal{L}_1(\boldsymbol{G})\,\kappa_0 + \mathcal{K}(\boldsymbol{G})\,\zeta_0 + \overline{\mathcal{L}}_1(\boldsymbol{G})\,\overline{\kappa}_0 + \overline{\mathcal{K}}(\boldsymbol{G})\,\overline{\zeta}_0 \\
&= \mathcal{T}(\boldsymbol{G})\,\rho_0 + \mathcal{L}_1(\boldsymbol{G})\,\Big(\kappa_0' - \frac{i}{3}\,\boldsymbol{B}_0\,\rho_0\Big) + \mathcal{K}(\boldsymbol{G})\,\frac{\zeta_0'}{\overline{\mathcal{L}}_1(\boldsymbol{k})} \\
&\quad + \overline{\mathcal{L}}_1(\boldsymbol{G})\,\Big(\overline{\kappa}_0' + \frac{i}{3}\,\overline{\boldsymbol{B}}_0\,\rho_0\Big) + \overline{\mathcal{K}}(\boldsymbol{G})\,\frac{\overline{\zeta}_0'}{\mathcal{L}_1(\overline{\boldsymbol{k}})}.
\end{aligned}$$

□

Using this lemma for $dK_0^{2\prime}$, $dK_0^{3\prime}$, and replacing also $d\rho_0$, $d\kappa_0'$, $d\zeta_0'$, $d\overline{\kappa}_0'$, $d\overline{\zeta}_0'$ by means of (7.1), we have

$$\begin{aligned}
0 \,=\, 0 \qquad\qquad & +0 \qquad\qquad -\boldsymbol{K}_0^{1\prime}\,\rho_0\wedge\zeta_0'\wedge\overline{\kappa}_0'\wedge\kappa_0'\wedge\overline{\zeta}_0' \\
+\overline{\mathcal{L}}_1\big(\boldsymbol{K}_0^{2\prime}\big)\,\overline{\kappa}_0'\wedge\rho_0\wedge\zeta_0'\wedge\kappa_0'\wedge\overline{\zeta}_0' & +\boldsymbol{K}_0^{2\prime}\,\overline{\boldsymbol{R}}_0^{1\prime}\,\rho_0\wedge\overline{\kappa}_0'\wedge\zeta_0'\wedge\kappa_0'\wedge\overline{\zeta}_0' - \boldsymbol{K}_0^{2\prime}\,\rho_0\wedge\boldsymbol{Z}_0^{8\prime}\,\zeta_0'\wedge\overline{\kappa}_0'\wedge\overline{\zeta}_0' \\
+\frac{\mathcal{K}(\boldsymbol{K}_0^{3\prime})}{\overline{\mathcal{L}}_1(\boldsymbol{k})}\,\zeta_0'\wedge\rho_0\wedge\overline{\kappa}_0'\wedge\kappa_0'\wedge\overline{\zeta}_0' & +\boldsymbol{K}_0^{3\prime}\,\boldsymbol{R}_0^{2\prime}\,\rho_0\wedge\zeta_0'\wedge\overline{\kappa}_0'\wedge\kappa_0'\wedge\overline{\zeta}_0' - 0 \\
+0 \qquad\qquad & +\boldsymbol{K}_0^{5\prime}\,\boldsymbol{K}_0^{3\prime}\,\rho_0\wedge\overline{\kappa}_0'\wedge\zeta_0'\wedge\kappa_0'\wedge\overline{\zeta}_0' - 0 \\
+0 \qquad\qquad & +\boldsymbol{K}_0^{6\prime}\,\boldsymbol{K}_0^{2\prime}\,\rho_0\wedge\zeta_0'\wedge\overline{\kappa}_0'\wedge\kappa_0'\wedge\overline{\zeta}_0' - 0 \\
+0 \qquad\qquad & +\boldsymbol{Z}_0^{2\prime}\,\rho_0\wedge\zeta_0'\wedge\overline{\kappa}_0'\wedge\kappa_0'\wedge\overline{\zeta}_0' \qquad -\zeta_0'\wedge\overline{\boldsymbol{K}}_0^{1\prime}\wedge\rho_0\wedge\overline{\kappa}_0'\wedge\kappa_0'\wedge\overline{\zeta}_0',
\end{aligned}$$

hence caring about signs when factoring by the naturally appearing 5-form

$$\begin{aligned}
0 \,=\, \rho_0\wedge\kappa_0'\wedge\zeta_0'\wedge\overline{\kappa}_0'\wedge\overline{\zeta}_0'\,\Big(0 \qquad & +0 \qquad -\boldsymbol{K}_0^{1\prime} \\
& \overline{\mathcal{L}}_1\big(\boldsymbol{K}_0^{2\prime}\big) - \boldsymbol{K}_0^{2\prime}\,\overline{\boldsymbol{R}}_0^{1\prime} - \boldsymbol{K}_0^{2\prime}\,\boldsymbol{Z}_0^{8\prime} \\
& -\frac{\mathcal{K}(\boldsymbol{K}_0^{3\prime})}{\overline{\mathcal{L}}_1(\boldsymbol{k})} + \boldsymbol{K}_0^{3\prime}\,\boldsymbol{R}_0^{2\prime} - 0 \\
& +0 \qquad -\boldsymbol{K}_0^{5\prime}\,\boldsymbol{K}_0^{3\prime} - 0 \\
& +0 \qquad +\boldsymbol{K}_0^{6\prime}\,\boldsymbol{K}_0^{2\prime} - 0 \\
& +0 \qquad +\boldsymbol{Z}_0^{2\prime} \qquad +\overline{\boldsymbol{K}}_0^{1\prime}\ \Big),
\end{aligned}$$

whence we arrive at the announced first identity (12.4) by remembering some useful relations

$$\frac{\mathcal{K}(\boldsymbol{K}_0^{3\prime})}{\overline{\mathcal{L}}_1(\boldsymbol{k})} = \overline{\mathcal{L}}_1\big(\boldsymbol{K}_0^{2\prime}\big) + \boldsymbol{K}_0^{2\prime}\,\boldsymbol{K}_0^{6\prime} - \boldsymbol{K}_0^{2\prime}\,\Big(\underbrace{\overline{\boldsymbol{R}}_0^{1\prime} + \boldsymbol{Z}_0^{8\prime}}_{=\,2\,\boldsymbol{K}_0^{6\prime}}\Big) + \boldsymbol{K}_0^{3\prime}\,\Big(\underbrace{\boldsymbol{R}_0^{2\prime} - \boldsymbol{K}_0^{5\prime}}_{=\,0!}\Big) - \boldsymbol{K}_0^{1\prime} + \overline{\boldsymbol{K}}_0^{1\prime} + \boldsymbol{Z}_0^{2\prime}.$$

For the second identity (12.5), we proceed similarly, applying the exterior differentiation operator d to the structure equation for $d\zeta_0'$ from (7.1)

$$\begin{aligned}
0 &= d^2\zeta_0' \\
&= \underbrace{d(\boldsymbol{Z}_0^{2\prime})\wedge\rho_0\wedge\zeta_0'}_{\text{don't want}} + \boldsymbol{Z}_0^{2\prime}\,d\rho_0\wedge\zeta_0' - \boldsymbol{Z}_0^{2\prime}\,\rho_0\wedge d\zeta_0' \\
&\quad + \underbrace{d(\boldsymbol{Z}_0^{5\prime})\wedge\kappa_0'\wedge\zeta_0'}_{\text{want}} + \boldsymbol{Z}_0^{5\prime}\,d\kappa_0'\wedge\zeta_0' - \boldsymbol{Z}_0^{5\prime}\,\kappa_0'\wedge d\zeta_0' \\
&\quad + \underbrace{d(\boldsymbol{Z}_0^{8\prime})\wedge\zeta_0'\wedge\overline{\kappa}_0'}_{\text{want}} + \boldsymbol{Z}_0^{8\prime}\,d\zeta_0'\wedge\overline{\kappa}_0' - \boldsymbol{Z}_0^{8\prime}\,\zeta_0'\wedge d\overline{\kappa}_0' \\
&\quad + \underbrace{d(\boldsymbol{Z}_0^{9\prime})\wedge\zeta_0'\wedge\overline{\zeta}_0'}_{\text{don't want}} + \boldsymbol{Z}_0^{9\prime}\,d\zeta_0'\wedge\overline{\zeta}_0' - \boldsymbol{Z}_0^{9\prime}\,\zeta_0'\wedge d\overline{\zeta}_0'.
\end{aligned}$$

Observe that the desired identity involves the derivatives of $\boldsymbol{Z}_0^{5\prime}$ and $\boldsymbol{Z}_0^{8\prime}$. Hence we may conserve those terms marked '*want*' by wedging with the appropriate 2-form $\rho_0 \wedge \overline{\zeta}_0'$

$$
\begin{aligned}
0 = 0 \qquad & + \boldsymbol{Z}_0^{2\prime}\, d\rho_0 \wedge \zeta_0' \wedge \rho_0 \wedge \overline{\zeta}_0' + 0 \\
+ d\boldsymbol{Z}_0^{5\prime} \wedge \kappa_0' \wedge \zeta_0' \wedge \rho_0 \wedge \overline{\zeta}_0' & + \boldsymbol{Z}_0^{5\prime}\, d\kappa_0' \wedge \zeta_0' \wedge \rho_0 \wedge \overline{\zeta}_0' - \boldsymbol{Z}_0^{5\prime}\, \kappa_0' \wedge d\zeta_0' \wedge \rho_0 \wedge \overline{\zeta}_0' \\
+ d\boldsymbol{Z}_0^{8\prime} \wedge \zeta_0' \wedge \overline{\kappa}_0' \wedge \rho_0 \wedge \overline{\zeta}_0' & + \boldsymbol{Z}_0^{8\prime}\, d\zeta_0' \wedge \overline{\kappa}_0' \wedge \rho_0 \wedge \overline{\zeta}_0' - \boldsymbol{Z}_0^{8\prime}\, \zeta_0' \wedge d\overline{\kappa}_0' \wedge \rho_0 \wedge \overline{\zeta}_0' \\
+ 0 \qquad & + 0 \qquad - \boldsymbol{Z}_0^{9\prime}\, \zeta_0' \wedge d\overline{\zeta}_0' \wedge \rho_0 \wedge \overline{\zeta}_0'.
\end{aligned}
$$

Using Lemma 12.6 for $d\boldsymbol{Z}_0^{5\prime}$, $d\boldsymbol{Z}_0^{8\prime}$, and replacing also $d\rho_0$, $d\kappa_0'$, $d\zeta_0'$, $d\overline{\kappa}_0'$, $d\overline{\zeta}_0'$ by means of (7.1), we have

$$
\begin{aligned}
0 \;=\; 0 \qquad & + \boldsymbol{Z}_0^{2\prime}\, i\, \kappa_0' \wedge \overline{\kappa}_0' \wedge \zeta_0' \wedge \rho_0 \wedge \overline{\zeta}_0' \quad + 0 \\
+ \overline{\mathcal{L}}_1\big(\boldsymbol{Z}_0^{5\prime}\big)\, \overline{\kappa}_0' \wedge \kappa_0' \wedge \zeta_0' \wedge \rho_0 \wedge \overline{\zeta}_0' & + \boldsymbol{Z}_0^{5\prime}\, \boldsymbol{K}_0^{6\prime}\, \kappa_0' \wedge \overline{\kappa}_0' \wedge \zeta_0' \wedge \rho_0 \wedge \overline{\zeta}_0' + \boldsymbol{Z}_0^{5\prime}\, \kappa_0' \wedge \boldsymbol{Z}_0^{8\prime}\, \zeta_0' \wedge \overline{\kappa}_0' \wedge \rho_0 \wedge \overline{\zeta}_0' \\
+ \mathcal{L}_1\big(\boldsymbol{Z}_0^{8\prime}\big)\, \kappa_0' \wedge \zeta_0' \wedge \overline{\kappa}_0' \wedge \rho_0 \wedge \overline{\zeta}_0' & + \boldsymbol{Z}_0^{8\prime}\, \boldsymbol{Z}_0^{5\prime}\, \kappa_0' \wedge \zeta_0' \wedge \overline{\kappa}_0' \wedge \rho_0 \wedge \overline{\zeta}_0' + \boldsymbol{Z}_0^{8\prime}\, \zeta_0' \wedge \overline{\boldsymbol{K}}_0^{6\prime} \wedge \overline{\kappa}_0' \wedge \kappa_0' \wedge \rho_0 \wedge \overline{\zeta}_0' \\
+ 0 \qquad & + 0 \qquad - 0,
\end{aligned}
$$

hence caring about signs when factoring by the naturally appearing 5-form, we arrive at the announced second identity (12.5)

$$
\begin{aligned}
0 \;=\; \rho_0 \wedge \kappa_0' \wedge \zeta_0' \wedge \overline{\kappa}_0' \wedge \overline{\zeta}_0' \Big(0 \qquad & + i\, \boldsymbol{Z}_0^{2\prime} \quad + 0 \\
- \overline{\mathcal{L}}_1\big(\boldsymbol{Z}_0^{5\prime}\big) & + \boldsymbol{Z}_0^{5\prime}\, \boldsymbol{K}_0^{6\prime} + \underline{\boldsymbol{Z}_0^{5\prime}\, \boldsymbol{Z}_0^{8\prime}}_{\circ} \\
- \mathcal{L}_1\big(\boldsymbol{Z}_0^{8\prime}\big) & - \underline{\boldsymbol{Z}_0^{8\prime}\, \boldsymbol{Z}_0^{5\prime}}_{\circ} + \boldsymbol{Z}_0^{8\prime}\, \overline{\boldsymbol{K}}_0^{6\prime} \\
+ 0 \qquad & + 0 \qquad - 0 \qquad \Big).
\end{aligned}
$$

□

13. Summarized structure equations

All this work conducted us to finalize the statement of Proposition 10.3, but before, let us make an ample summary.

After normalizations of the group parameters $\mathsf{f}, \mathsf{b}, \mathsf{d}$, the equivalence problem for 2-nondegenerate (constant) Levi rank 1 $\mathcal{C}^\omega$ or $\mathcal{C}^\infty$ real hypersurfaces $M^5 \subset \mathbb{C}^3$ conducts to a 4-dimensional G-structure

$$
\begin{pmatrix}
\mathsf{c}\overline{\mathsf{c}} & 0 & 0 \\
-i\, \overline{\mathsf{c}}\mathsf{e} & \mathsf{c} & 0 \\
-\frac{i}{2}\, \frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{c}} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}}
\end{pmatrix},
$$

where $\mathsf{c} \in \mathbb{C}^*$ and $\mathsf{e} \in \mathbb{C}$, with Maurer-Cartan forms (conjutates are not written)

$$
\begin{aligned}
\alpha &:= \frac{d\mathsf{c}}{\mathsf{c}}, \\
\beta &:= i\, \frac{\mathsf{e}\, d\mathsf{c}}{\mathsf{c}\mathsf{c}} - i\, \frac{\mathsf{e}\, d\overline{\mathsf{c}}}{\mathsf{c}\overline{\mathsf{c}}} - i\, \frac{d\mathsf{e}}{\mathsf{c}}.
\end{aligned}
$$

Furthermore, 2 fundamental primary differential invariants occur

$$
J \;=\; \frac{i}{\overline{\mathsf{c}}\mathsf{c}\mathsf{c}}\, \overline{\boldsymbol{J}}_0 \qquad \text{and} \qquad W \;=\; \frac{1}{\mathsf{c}}\, \boldsymbol{W}_0,
$$

where $\boldsymbol{J}_0$ and $\boldsymbol{W}_0$ are explicit functions on M, together with 1 secondary invariant

$$
\begin{aligned}
R &:= \mathsf{Re}\, Z^2 \\
&= \mathsf{Re}\left[i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\boldsymbol{W}_0 + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\bigg(-\frac{i}{2}\,\overline{\mathcal{L}}_1(\boldsymbol{W}_0) + \frac{i}{2}\bigg(-\frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} + \frac{1}{3}\,\overline{\boldsymbol{P}}\bigg)\,\boldsymbol{W}_0\bigg)\right].
\end{aligned}
$$

On the 10-dimensional manifold $M^5 \times G^4 \times \mathbb{R}$ equipped with coordinates

$$
\big(z_1, z_2, \overline{z}_1, \overline{z}_2, v\big) \times \big(\mathsf{c}, \overline{\mathsf{c}}, \mathsf{e}, \overline{\mathsf{e}}\big) \times (\mathsf{t}),
$$

there are two modified-prolonged Maurer-Cartan forms

$$
\begin{aligned}
\pi^1 &:= \alpha - \Big(\mathsf{t} - \frac{i}{2}\,\mathsf{Im}\, Z^2\Big)\,\rho - \Big(R^1 - \overline{K}^6\Big)\,\kappa - R^2\,\zeta - K^6\,\overline{\kappa} - 0, \\
\pi^2 &:= \beta - i\,Z^1\,\rho - \Big(\mathsf{t} - \frac{i}{2}\,\mathsf{Im}\, Z^2 + K^1\Big)\,\kappa - K^2\,\zeta - K^3\,\overline{\kappa} - K^4\,\overline{\zeta},
\end{aligned}
$$

where R^i, K^i, Z^i are explicit functions on $M^5 \times G^4$.

Theorem 13.1. *After finalization of absorption, the structure equations read*

$$
\begin{aligned}
d\rho &= \big(\pi^1 + \overline{\pi}^1\big) \wedge \rho + i\,\kappa \wedge \overline{\kappa}, \\
d\kappa &= \pi^2 \wedge \rho + \pi^1 \wedge \kappa + \zeta \wedge \overline{\kappa}, \\
d\zeta &= \big(\pi^1 - \overline{\pi}^1\big) \wedge \zeta + i\,\pi^2 \wedge \kappa \\
&\quad + R\,\rho \wedge \zeta + J\,\rho \wedge \overline{\kappa} + W\,\kappa \wedge \zeta.
\end{aligned}
$$

□

14. The final $\{e\}$-structure

Let Ω_1 and Ω_2 be the two 2-forms defined by:

$$
\begin{aligned}
\Omega_1 &:= d\pi^1 - i\,\kappa \wedge \overline{\pi}^2 - \zeta \wedge \overline{\zeta}, \\
\Omega_2 &:= d\pi^2 - \pi^2 \wedge \overline{\pi}^1 - \zeta \wedge \overline{\pi}^2.
\end{aligned}
$$

When the two fundamental invariants $\boldsymbol{J}_0 \equiv 0 \equiv \boldsymbol{W}_0$ vanish identically, since we know that

$$
\begin{aligned}
R &= \mathsf{Re}\left[i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\boldsymbol{W}_0 + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\bigg(-\frac{i}{2}\,\overline{\mathcal{L}}_1(\boldsymbol{W}_0) + \frac{i}{2}\bigg(-\frac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(\boldsymbol{k})\big)}{\overline{\mathcal{L}}_1(\boldsymbol{k})} + \frac{1}{3}\,\overline{\boldsymbol{P}}\bigg)\,\boldsymbol{W}_0\bigg)\right], \\
J &= \frac{i}{\overline{\mathsf{c}\mathsf{c}\mathsf{c}}}\,\overline{\boldsymbol{J}}_0, \\
W &= \frac{1}{\mathsf{c}}\,\boldsymbol{W}_0,
\end{aligned}
$$

it comes

$$
0 \equiv R \equiv J \equiv W.
$$

Independently, the addendum to [18] shows that in the case where all invariants vanish, these auxiliary 2-forms Ω_1 and Ω_2 satisfy

$$
\begin{aligned}
\big(\Omega_1 + \overline{\Omega}_1\big) \wedge \rho &= 0, \\
\Omega_2 \wedge \rho + \Omega_1 \wedge \kappa &= 0, \\
\big(\Omega_1 - \overline{\Omega}_1\big) \wedge \zeta + i\,\Omega_2 \wedge \kappa &= 0.
\end{aligned}
$$

In general, the right-hand sides of these structure equations are not necessarily zero, and they depend on the invariants R, J, W.

Proposition 14.1. *The two 2-forms Ω_1 and Ω_2 satisfy*

$$
\begin{aligned}
&(14.2) && \big(\Omega_1 + \overline{\Omega}_1\big) \wedge \rho = 0,\\
&(14.3) && \Omega_2 \wedge \rho + \Omega_1 \wedge \kappa = -R\,\rho\wedge\zeta\wedge\overline{\kappa} - W\,\kappa\wedge\zeta\wedge\overline{\kappa},\\
&(14.4) && i\,\Omega_2\wedge\kappa + \big(\Omega_1 - \overline{\Omega}_1\big)\wedge\zeta = -dR\wedge\rho\wedge\zeta - R\,\big(\pi^1+\overline{\pi}^1\big)\wedge\rho\wedge\zeta - i\,R\,\pi^2\wedge\rho\wedge\kappa\\
&&& \quad + i\,R\,\kappa\wedge\zeta\wedge\overline{\zeta} - dJ\wedge\rho\wedge\overline{\kappa} - 3\,J\,\overline{\pi}^1\wedge\rho\wedge\overline{\kappa}\\
&&& \quad - J\,\rho\wedge\kappa\wedge\overline{\zeta} - dW\wedge\kappa\wedge\zeta - W\,\pi^2\wedge\rho\wedge\zeta\\
&&& \quad - W\,\pi^1\wedge\kappa\wedge\zeta - W J\,\rho\wedge\kappa\wedge\overline{\kappa}.
\end{aligned}
$$

Proof. These relations come from Poincaré's identities

$$0 \equiv d\circ d\rho \equiv d\circ d\kappa \equiv d\circ d\zeta,$$

applied to the finalized structure equations of Theorem 13.1, in which $d\rho$, $d\kappa$, $d\zeta$ should be replaced again using Theorem 13.1, followed by a reorganization of the obtained 3-forms.

For the first line (14.2)

$$
\begin{aligned}
0 &= d\circ d\rho\\
&= \big(d\pi^1 + d\overline{\pi}^1\big)\wedge\rho - \big(\pi^1+\overline{\pi}^1\big)\wedge d\rho + i\,d\kappa\wedge\overline{\kappa} - i\,\kappa\wedge d\overline{\kappa}\\
&= \big(d\pi^1 + d\overline{\pi}^1\big)\wedge\rho - \big(\pi^1+\overline{\pi}^1\big)\wedge\Big(\underline{\big(\pi^1+\overline{\pi}^1}_{\circ}\big)\wedge\rho + i\,\kappa\wedge\overline{\kappa}\Big)\\
&\quad + i\Big(\pi^2\wedge\rho + \pi^1\wedge\rho + \underline{\zeta\wedge\overline{\kappa}}_{\circ}\Big)\wedge\overline{\kappa} - i\,\kappa\wedge\Big(\overline{\pi}^2\wedge\rho + \overline{\pi}^1\wedge\overline{\kappa} + \underline{\overline{\zeta}\wedge\kappa}_{\circ}\Big).
\end{aligned}
$$

Afer simplification, this becomes

$$0 = \Big(d\pi^1 - i\,\kappa\wedge\overline{\pi}^2\Big)\wedge\rho + \Big(d\overline{\pi}^1 + i\,\overline{\kappa}\wedge\pi^2\Big)\wedge\rho,$$

and after insertion of twice $-\zeta\wedge\overline{\zeta}$ which is purely imaginary — hence disappears —, we obtain (14.2)

$$
\begin{aligned}
0 &= \Big(d\pi^1 - i\,\kappa\wedge\overline{\pi}^2 - \zeta\wedge\overline{\zeta}\Big)\wedge\rho + \Big(d\overline{\pi}^1 + i\,\overline{\kappa}\wedge\pi^2 - \overline{\zeta}\wedge\zeta\Big)\wedge\rho\\
&= \Omega_1\wedge\rho + \overline{\Omega}_1\wedge\rho.
\end{aligned}
$$

For (14.3), we proceed analogously, starting from the second structure equation of Theorem 13.1

$$
\begin{aligned}
0 &= d\circ d\kappa\\
&= d\pi^2\wedge\rho - \pi^2\wedge d\rho + d\pi^1\wedge\kappa - \pi^1\wedge d\kappa + d\zeta\wedge\overline{\kappa} - \zeta\wedge d\overline{\kappa}\\
&= d\pi^2\wedge\rho - \pi^2\wedge\Big(\big(\pi^1+\overline{\pi}^1\big)\wedge\rho + i\,\kappa\wedge\overline{\kappa}\Big) + d\pi^1\wedge\kappa - \pi^1\wedge\Big(\pi^2\wedge\rho + \zeta\wedge\overline{\kappa}\Big)\\
&\quad + \Big(\big(\pi^1-\overline{\pi}^1\big)\wedge\zeta + i\,\pi^2\wedge\kappa + R\,\rho\wedge\zeta + W\,\kappa\wedge\zeta\Big)\wedge\overline{\kappa} - \zeta\wedge\Big(\overline{\pi}^2\wedge\rho + \overline{\pi}^1\wedge\overline{\kappa} + \overline{\zeta}\wedge\kappa\Big).
\end{aligned}
$$

After four annihilations by pairs and a reorganization, this becomes

$$
\begin{aligned}
0 &= d\pi^2\wedge\rho - \underline{\pi^2\wedge\pi^1\wedge\rho}_1 - \pi^2\wedge\overline{\pi}^1\wedge\rho - \underline{i\,\pi^2\wedge\kappa\wedge\overline{\kappa}}_2 + d\pi^1\wedge\kappa - \underline{\pi^1\wedge\pi^2\wedge\rho}_1 - \underline{\pi^1\wedge\zeta\wedge\overline{\kappa}}_3\\
&\quad + \underline{\pi^1\wedge\zeta\wedge\overline{\kappa}}_3 - \underline{\overline{\pi}^1\wedge\zeta\wedge\overline{\kappa}}_4 + \underline{i\,\pi^2\wedge\kappa\wedge\overline{\kappa}}_2 + R\,\rho\wedge\zeta\wedge\overline{\kappa} + W\,\kappa\wedge\zeta\wedge\overline{\kappa} - \zeta\wedge\overline{\pi}^2\wedge\rho\\
&\quad - \underline{\zeta\wedge\overline{\pi}^1\wedge\overline{\kappa}}_4 - \zeta\wedge\overline{\zeta}\wedge\kappa\\
&= \Big(d\pi^2 - \pi^2\wedge\overline{\pi}^1 - \zeta\wedge\overline{\pi}^2\Big)\wedge\rho + \big(d\pi^1 - \zeta\wedge\overline{\zeta}\big)\wedge\kappa\\
&\quad + R\,\rho\wedge\zeta\wedge\overline{\kappa} + W\,\kappa\wedge\zeta\wedge\overline{\kappa},
\end{aligned}
$$

which is (14.3), since we can insert $\big(-\,i\,\kappa\wedge\overline{\pi}^2\big)\wedge\kappa = 0$. Lastly

$$
\begin{aligned}
0 \,&=\, d\circ d\zeta \\
&=\, i\,d\pi^2\wedge\kappa - i\,\pi^2\wedge d\kappa + d\pi^1\wedge\zeta - \pi^1\wedge d\zeta - d\overline{\pi}^1\wedge\zeta + \overline{\pi}^1\wedge d\zeta \\
&\quad + dR\wedge\rho\wedge\zeta + R\,d\rho\wedge\zeta - R\,\rho\wedge d\zeta \\
&\quad + dJ\wedge\rho\wedge\overline{\kappa} + J\,d\rho\wedge\overline{\kappa} - J\,\rho\wedge d\overline{\kappa} \\
&\quad + dW\wedge\kappa\wedge\zeta + W\,d\kappa\wedge\zeta - W\,\kappa\wedge d\zeta,
\end{aligned}
$$

whence by replacements

$$
\begin{aligned}
0 \,=\,& i\,d\pi^2\wedge\kappa - i\,\pi^2\wedge\big(\pi^1\wedge\kappa + \zeta\wedge\overline{\kappa}\big) + d\pi^1\wedge\zeta - \pi^1\wedge\Big(i\,\pi^2\wedge\kappa - \overline{\pi}^1\wedge\zeta + R\,\rho\wedge\zeta \\
& + J\,\rho\wedge\overline{\kappa} + W\,\kappa\wedge\zeta\Big) - d\overline{\pi}^1\wedge\zeta + \overline{\pi}^1\wedge\Big(i\,\pi^2\wedge\kappa + \pi^1\wedge\zeta + R\,\rho\wedge\zeta + J\,\rho\wedge\overline{\kappa} + W\,\kappa\wedge\zeta\Big) \\
& + dR\wedge\rho\wedge\zeta + R\Big(\big(\pi^1+\overline{\pi}^1\big)\wedge\rho + i\,\kappa\wedge\overline{\kappa}\Big)\wedge\zeta - R\,\rho\wedge\Big(i\,\pi^2\wedge\kappa + \big(\pi^1-\overline{\pi}^1\big)\wedge\zeta + W\,\kappa\wedge\zeta\Big) \\
& + dJ\wedge\rho\wedge\overline{\kappa} + J\,\big(\pi^1+\overline{\pi}^1\big)\wedge\rho\wedge\overline{\kappa} - J\,\rho\wedge\big(\overline{\pi}^1\wedge\overline{\kappa} + \overline{\zeta}\wedge\kappa\big) \\
& + dW\wedge\kappa\wedge\zeta + W\,\big(\pi^2\wedge\rho + \pi^1\wedge\kappa\big)\wedge\zeta - W\,\kappa\wedge\Big(\big(\pi^1-\overline{\pi}^1\big)\wedge\zeta + R\,\rho\wedge\zeta + J\,\rho\wedge\overline{\kappa}\Big).
\end{aligned}
$$

Let us expand this and underline the eight annihilating pairs

$$
\begin{aligned}
0 \,=\,& i\,d\pi^2\wedge\kappa - \underline{i\,\pi^2\wedge\pi^1\wedge\kappa}_1 - i\,\pi^2\wedge\zeta\wedge\overline{\kappa} + d\pi^1\wedge\zeta - \underline{i\,\pi^1\wedge\pi^2\wedge\kappa}_1 + \underline{\pi^1\wedge\overline{\pi}^1\wedge\zeta}_2 \\
& - \underline{R\,\pi^1\wedge\rho\wedge\zeta}_3 - \underline{J\,\pi^1\wedge\rho\wedge\overline{\kappa}}_6 - \underline{W\,\pi^1\wedge\kappa\wedge\zeta}_7 - d\overline{\pi}^1\wedge\zeta + i\,\overline{\pi}^1\wedge\pi^2\wedge\kappa + \underline{\overline{\pi}^1\wedge\pi^1\wedge\zeta}_2 \\
& + \underline{R\,\overline{\pi}^1\wedge\rho\wedge\zeta}_4 + J\,\overline{\pi}^1\wedge\rho\wedge\overline{\kappa} + \underline{W\,\overline{\pi}^1\wedge\kappa\wedge\zeta}_8 + dR\wedge\rho\wedge\zeta + \underline{R\,\pi^1\wedge\rho\wedge\zeta}_3 + R\,\overline{\pi}^1\wedge\rho\wedge\zeta \\
& + i\,R\,\kappa\wedge\overline{\kappa}\wedge\zeta - i\,R\,\rho\wedge\pi^2\wedge\kappa - R\,\rho\wedge\pi^1\wedge\zeta + \underline{R\,\rho\wedge\overline{\pi}^1\wedge\zeta}_4 - \underline{RW\,\rho\wedge\kappa\wedge\zeta}_5 \\
& + dJ\wedge\rho\wedge\overline{\kappa} + \underline{J\,\pi^1\wedge\rho\wedge\overline{\kappa}}_6 + J\,\overline{\pi}^1\wedge\rho\wedge\overline{\kappa} - J\,\rho\wedge\overline{\pi}^1\wedge\overline{\kappa} - J\,\rho\wedge\overline{\zeta}\wedge\kappa + dW\wedge\kappa\wedge\zeta \\
& + W\,\pi^2\wedge\rho\wedge\zeta + \underline{W\,\pi^1\wedge\kappa\wedge\zeta}_7 - W\,\kappa\wedge\pi^1\wedge\zeta + \underline{W\,\kappa\wedge\overline{\pi}^1\wedge\zeta}_8 - \underline{W R\,\kappa\wedge\rho\wedge\zeta}_5 \\
& - W J\,\kappa\wedge\rho\wedge\overline{\kappa}.
\end{aligned}
$$

After simplification and reorganization

$$
\begin{aligned}
0 \,=\,& i\Big(d\pi^2 - \pi^2\wedge\overline{\pi}^1\Big)\wedge\kappa + \Big(d\pi^1 - d\overline{\pi}^1 - i\,\overline{\kappa}\wedge\pi^2\Big)\wedge\zeta \\
& + dR\wedge\rho\wedge\zeta + R\,\overline{\pi}^1\wedge\rho\wedge\zeta - i\,R\,\kappa\wedge\zeta\wedge\overline{\kappa} + i\,R\,\pi^2\wedge\rho\wedge\kappa + R\,\pi^1\wedge\rho\wedge\zeta \\
& + dJ\wedge\rho\wedge\overline{\kappa} + 3\,J\,\overline{\pi}^1\wedge\rho\wedge\overline{\kappa} + J\,\rho\wedge\kappa\wedge\overline{\zeta} \\
& + dW\wedge\kappa\wedge\zeta + W\,\pi^2\wedge\rho\wedge\zeta + W\,\pi^1\wedge\kappa\wedge\zeta + W J\,\rho\wedge\kappa\wedge\overline{\kappa}.
\end{aligned}
$$

To reach (14.4) completely, only the first line must yet be transformed, and it suffices to insert into it two terms which cancel together

$$
i\Big(d\pi^2 - \pi^2\wedge\overline{\pi}^1 - \underline{\zeta\wedge\overline{\pi}^2}_{\circ}\Big)\wedge\kappa + \Big(d\pi^1 - \underline{i\,\kappa\wedge\overline{\pi}^2}_{\circ} - d\overline{\pi}^1 - i\,\overline{\kappa}\wedge\pi^2\Big)\wedge\zeta. \qquad \square
$$

Remind that all present considerations hold on the 9-dimensional manifold $M^5\times G^4$ equipped with the coordinates

$$
\big(z_1, z_2, \overline{z}_1, \overline{z}_2, v\big) \times \big(\mathsf{c}, \mathsf{e}, \overline{\mathsf{c}}, \overline{\mathsf{e}}\big),
$$

the supplementary real variable $\mathsf{t}\in\mathbb{R}$ being considered as a parameter until it becomes a variable at the very end of the process for an $\{e\}$-structure on the 10-dimensional manifold

$M^5 \times G^4 \times \mathbb{R}$. In order to build up such an $\{e\}$-structure, the goal now is to fully determine the two 2-forms Ω_1, Ω_2, and precisely, to determine how they express in terms of the coframe

$$\big\{\pi^1, \pi^2, \overline{\pi}^1, \overline{\pi}^2, \rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta}\big\}.$$

To begin with, suppose that there are two ways of solving for $\{\Omega_1, \Omega_2\}$ the structure equations of Proposition 14.1, leading to another set of solutions $\{\Omega_1', \Omega_2'\}$. Then their differences $\Gamma_1 := \Omega_1' - \Omega_1$ and $\Gamma_2 := \Omega_2' - \Omega_2$ must necessarily satisfy the homogeneous equations

$$\begin{aligned}
\big(\Gamma_1 + \overline{\Gamma}_1\big) \wedge \rho &= 0, \\
\Gamma_2 \wedge \rho + \Gamma_1 \wedge \kappa &= 0, \\
i\,\Gamma_2 \wedge \kappa + \big(\Gamma_1 - \overline{\Gamma}_1\big) \wedge \zeta &= 0.
\end{aligned}$$

The addendum to the article [18] provides a detailed proof of the elementary

Proposition 14.5. *The general solution $\{\Gamma_1, \Gamma_2\}$ to these homogeneous equations is given by*

$$\Gamma_1 := \Lambda \wedge \rho, \qquad\qquad \Gamma_2 := \Lambda \wedge \kappa + h\,\rho \wedge \kappa,$$

where Λ is a real 1-form and h is purely imaginary function. □

This means that the two sets of solutions are related to each other by

$$\Omega_1' = \Omega_1 + \Lambda \wedge \rho, \qquad\qquad \Omega_2' = \Omega_2 + \Lambda \wedge \kappa + h\,\rho \wedge \kappa.$$

Due to this flexibility represented by Λ, h, it will be necessary to prolong the structure equations by adding this real 1-form:

$$\Lambda = d\mathrm{t} + \cdots,$$

the remainder terms being very complicated, while the function h *could* be some new invariant. However, it will be later shown that h expresses in terms of the 3rd-order jets of W and J, thus eliminating the possibility of appearance of new primary CR invariants. On the other hand, the existence of Λ can be explained by an application (not detailed here) of Cartan's test, due to the fact that there is one degree of real-valued indeterminancy during the fourth absorption.

It therefore suffices to find a particular set of solution Ω_1 and Ω_2, and then to parametrize the solution space by means of Λ, h. We will adopt the following strategy. First, we will find the simplest forms for Ω_1 and Ω_2 restrained by the first two equations (14.2), (14.3) of the starting Proposition 14.1. Then we will simplify these 2-forms by means of Cartan's lemma to eliminate as many unknown variables as possible using the third, more subtle, equation (14.4). At the end of the elimination, those remaining unknowns which cannot be computed due to the lack of information turn out to behave like Λ and h, and hence we will terminate the process of solving for solutions.

In $M^5 \times G^4$, it will be useful to adopt the following notations for the covariant derivatives

$$\begin{aligned}
dR &= R_{\pi^1}\,\pi^1 + R_{\pi^2}\,\pi^2 + R_{\overline{\pi}^1}\,\overline{\pi}^1 + R_{\overline{\pi}^2}\,\overline{\pi}^2 + R_\rho\,\rho + R_\kappa\,\kappa + R_\zeta\,\zeta + R_{\overline{\kappa}}\,\overline{\kappa} + R_{\overline{\zeta}}\,\overline{\zeta}, \\
dJ &= J_{\pi^1}\,\pi^1 + J_{\pi^2}\,\pi^2 + J_{\overline{\pi}^1}\,\overline{\pi}^1 + J_{\overline{\pi}^2}\,\overline{\pi}^2 + J_\rho\,\rho + J_\kappa\,\kappa + J_\zeta\,\zeta + J_{\overline{\kappa}}\,\overline{\kappa} + J_{\overline{\zeta}}\,\overline{\zeta}, \\
dW &= W_{\pi^1}\,\pi^1 + W_{\pi^2}\,\pi^2 + W_{\overline{\pi}^1}\,\overline{\pi}^1 + W_{\overline{\pi}^2}\,\overline{\pi}^2 + W_\rho\,\rho + W_\kappa\,\kappa + W_\zeta\,\zeta + W_{\overline{\kappa}}\,\overline{\kappa} + W_{\overline{\zeta}}\,\overline{\zeta}.
\end{aligned} \tag{14.6}$$

Some of these coefficients will be revealed during the course of solving the structure equations. We first turn ourselves to finding the simplest form of Ω_1, Ω_2 satisfying *only* the first two equations (14.2), (14.3).

Proposition 14.7. *There exists a real-valued function p and two differential 1-forms Π, Ψ such that*

$$\begin{aligned}
\Omega_1 &= \Pi \wedge \rho + p\,\kappa \wedge \overline{\kappa} - \overline{W}\,\kappa \wedge \overline{\zeta} - W\,\zeta \wedge \overline{\kappa}, \\
\Omega_2 &= \Psi \wedge \rho + \Pi \wedge \kappa - R\,\zeta \wedge \overline{\kappa}.
\end{aligned}$$

Proof. We can rearrange the terms in (14.3)

$$0 = \big(\Omega_1 + W\,\zeta\wedge\overline{\kappa}\big)\wedge\kappa + \big(\Omega_2 + R\,\zeta\wedge\overline{\kappa}\big)\wedge\rho, \tag{14.8}$$

in order that an application of the Cartan Lemma yield functions Δ, Θ, Π'', Ψ so that

$$\begin{aligned}\Omega_1 + W\,\zeta\wedge\overline{\kappa} &= \Delta\wedge\kappa + \Theta\wedge\rho,\\ \Omega_2 + R\,\zeta\wedge\overline{\kappa} &= \Pi''\wedge\kappa + \Psi\wedge\rho,\end{aligned}$$

with a double prime on Π'' meaning that we will soon modify it two times.

In fact, substituting these representations back into (14.8), we see that there are constraints on Θ and Π''

$$\begin{aligned}0 &= \big(\underline{\Delta\wedge\kappa}_{\circ} + \Theta\wedge\rho\big)\wedge\kappa + \big(\Pi''\wedge\kappa + \underline{\Psi\wedge\rho}_{\circ}\big)\wedge\rho\\ &= \big(\Theta - \Pi''\big)\wedge\rho\wedge\kappa.\end{aligned}$$

By the Cartan Lemma again, this implies the existence of two functions a, b so that Θ and Π'' are related to each other by

$$\Theta = \Pi'' + a\,\rho + b\,\kappa.$$

Next, putting this into the expression of Ω_1, while letting $\Pi' := \Pi'' + b\,\kappa$, it follows that

$$\begin{aligned}\Omega_1 &= \Delta\wedge\kappa + \Theta\wedge\rho - W\,\zeta\wedge\overline{\kappa}\\ &= \Delta\wedge\kappa + \big(\Pi'' + \underline{a\,\rho}_{\circ} + b\,\kappa\big)\wedge\rho - W\,\zeta\wedge\overline{\kappa}\\ &= \Delta\wedge\kappa + \Pi'\wedge\rho - W\,\zeta\wedge\overline{\kappa},\end{aligned}$$

while Ω_2 becomes

$$\begin{aligned}\Omega_2 &= \Pi''\wedge\kappa + \Psi\wedge\rho - R\,\zeta\wedge\overline{\kappa}\\ &= \big(\Pi'' + b\,\kappa\big)\wedge\kappa + \Psi\wedge\rho - R\,\zeta\wedge\overline{\kappa}\\ &= \Pi'\wedge\kappa + \Psi\wedge\rho - R\,\zeta\wedge\overline{\kappa}.\end{aligned}$$

The next observation is that Δ can be further simplified. Indeed, let us replace Ω_1 in (14.2)

$$\begin{aligned}0 &= \big(\Omega_1 + \overline{\Omega}_1\big)\wedge\rho\\ &= \Delta\wedge\kappa\wedge\rho - W\,\zeta\wedge\overline{\kappa}\wedge\rho + \overline{\Delta}\wedge\overline{\kappa}\wedge\rho - \overline{W}\,\overline{\zeta}\wedge\kappa\wedge\rho.\end{aligned}$$

Then decomposing Δ as a linear combination along the coframe

$$\Delta = d_1\,\pi^1 + d_2\,\pi^2 + d_3\,\overline{\pi}^1 + d_4\,\overline{\pi}^2 + d_5\,\rho + d_6\,\kappa + d_7\,\zeta + d_8\,\overline{\kappa} + d_9\,\overline{\zeta},$$

we obtain the following values for these coefficients

$$d_1 = d_2 = d_3 = d_4 = 0, \qquad d_8 = \overline{d}_8, \qquad d_9 = \overline{W},$$

except for d_5 and d_6 which on which no constraint is deduced so, and hence

$$\Delta = d_5\,\rho + d_6\,\kappa + d_8\,\overline{\kappa} + \overline{W}\,\overline{\zeta}.$$

Finally, if we write $p := -\,d_8$ and if we set $\Pi := \Pi' - d_5\,\kappa$, we obtain by reorganization

$$\begin{aligned}\Omega_1 &= \Delta\wedge\kappa + \Pi'\wedge\rho - W\,\zeta\wedge\overline{\kappa}\\ &= \Big(d_5\,\rho + \underline{d_6\,\kappa}_{\circ} + d_8\,\overline{\kappa} + \overline{W}\,\overline{\zeta}\Big)\wedge\kappa + \Pi'\wedge\rho - W\,\zeta\wedge\overline{\kappa}\\ &= -\,d_8\,\kappa\wedge\overline{\kappa} + \big(\Pi' - d_5\,\kappa\big)\wedge\rho - \overline{W}\,\kappa\wedge\overline{\zeta} - W\,\zeta\wedge\overline{\kappa}\\ &= p\,\kappa\wedge\overline{\kappa} + \Pi\wedge\rho - \overline{W}\,\kappa\wedge\overline{\zeta} - W\,\zeta\wedge\overline{\kappa},\end{aligned}$$

and moreover

$$\begin{aligned}\Omega_2 &= \Psi \wedge \rho + \Pi' \wedge \kappa - R\,\zeta \wedge \overline{\kappa}\\ &= \Psi \wedge \rho + \big(\Pi' - d_5\,\kappa\big) \wedge \kappa - R\,\zeta \wedge \overline{\kappa}\\ &= \Psi \wedge \rho + \Pi \wedge \kappa - R\,\zeta \wedge \overline{\kappa}.\end{aligned}$$ □

Now, using the representations of Ω_1 and of Ω_2 offered by this Proposition 14.7, we can therefore rewrite the third (still not taken account of) equation (14.4) as

$$\begin{aligned}&i\,\Psi \wedge \rho \wedge \kappa - i\,R\,\zeta \wedge \overline{\kappa} \wedge \kappa + \big(\Pi - \overline{\Pi}\big) \wedge \rho \wedge \zeta + 2\,p\,\kappa \wedge \overline{\kappa} \wedge \zeta - 2\,\overline{W}\,\kappa \wedge \overline{\zeta} \wedge \zeta\\ &\qquad = -\,dR \wedge \rho \wedge \zeta - R\,\big(\pi^1 + \overline{\pi}^1\big) \wedge \rho \wedge \zeta - i\,R\,\pi^2 \wedge \rho \wedge \kappa + i\,R\,\kappa \wedge \zeta \wedge \overline{\kappa}\\ &\qquad\quad -\,dJ \wedge \rho \wedge \overline{\kappa} - 3J\,\overline{\pi}^1 \wedge \rho \wedge \overline{\kappa} - J\,\rho \wedge \kappa \wedge \overline{\zeta}\\ &\qquad\quad -\,dW \wedge \kappa \wedge \zeta - W\,\pi^2 \wedge \rho \wedge \zeta - W\,\pi^1 \wedge \kappa \wedge \zeta - W\,J\,\rho \wedge \kappa \wedge \overline{\kappa}.\end{aligned} \tag{14.9}$$

But before we commence with analyzing this equation (a long task), we make a side remark. As we can rewrite

$$\begin{aligned}\Omega_1 &= \tfrac{1}{2}\,\big(\Pi + \overline{\Pi}\big) \wedge \rho + \tfrac{1}{2}\,\big(\Pi - \overline{\Pi}\big) \wedge \rho + p\,\kappa \wedge \overline{\kappa} - \overline{W}\,\kappa \wedge \zeta - W\,\zeta \wedge \overline{\kappa},\\ \Omega_2 &= \Psi \wedge \rho + \tfrac{1}{2}\,\big(\Pi + \overline{\Pi}\big) \wedge \kappa + \tfrac{1}{2}\,\big(\Pi - \overline{\Pi}\big) \wedge \kappa - R\,\zeta \wedge \overline{\kappa},\end{aligned}$$

we remark that Proposition 14.5 already tells us that the real part $\frac{1}{2}\,\big(\Pi + \overline{\Pi}\big)$ of Π is *a priori* not fully determined, as can be formulated by an

Observation 14.10. *For an arbitrary real* 1*-form* Λ*, the* 2*-forms*

$$\Omega_1' := \Omega_1 + \Lambda \wedge \rho \qquad \text{and} \qquad \Omega_2' := \Omega_2 + \Lambda \wedge \kappa$$

still satisfy the structure equations of Proposition 14.1.

Proof. For the sake of completeness, let us detail the arguments. The first equation (14.2) is clear

$$\big(\Omega_1' + \overline{\Omega}_1'\big) \wedge \rho = \Big(\Omega_1 + \underline{\Lambda \wedge \rho}_{\circ} + \overline{\Omega}_1 + \underline{\Lambda \wedge \rho}_{\circ}\Big) \wedge \rho = \big(\Omega_1 + \overline{\Omega}_1\big) \wedge \rho.$$

The second equation (14.3) also

$$\begin{aligned}\Omega_2' \wedge \rho + \Omega_1' \wedge \kappa &= \big(\Omega_2 + \Lambda \wedge \kappa\big) \wedge \rho + \big(\Omega_1 + \Lambda \wedge \rho\big) \wedge \kappa\\ &= \Omega_2 \wedge \rho + \underline{\Lambda \wedge \kappa \wedge \rho}_{\circ} + \Omega_1 \wedge \kappa + \underline{\Lambda \wedge \rho \wedge \kappa}_{\circ}\\ &= \Omega_2 \wedge \rho + \Omega_1 \wedge \kappa,\end{aligned}$$

and the third one as well

$$\begin{aligned}i\,\Omega_2' \wedge \kappa + \big(\Omega_1' - \overline{\Omega}_1'\big) \wedge \zeta &= i\,\big(\Omega_2 + \underline{\Lambda \wedge \kappa}_{\circ}\big) \wedge \kappa + \Big(\Omega_1 + \underline{\Lambda \wedge \rho}_{\circ} - \overline{\Omega}_1 - \underline{\Lambda \wedge \rho}_{\circ}\Big) \wedge \zeta\\ &= i\,\Omega_2 \wedge \kappa + \big(\Omega_1 - \overline{\Omega}_1\big) \wedge \zeta.\end{aligned}$$ □

Now, coming back to (14.9), we remember that we should insert the covariant derivatives dR, dJ, dW from (14.6), and we will do this *in a progressive way*, not in one stroke.
Indeed, by wedging $(\bullet) \wedge \rho$ both sides of (14.9), we get rid of dJ, dR and it remains only

$$\begin{aligned}&-\,i\,R\,\zeta \wedge \overline{\kappa} \wedge \kappa \wedge \rho + 2p\,\kappa \wedge \overline{\kappa} \wedge \zeta \wedge \rho - 2\overline{W}\,\kappa \wedge \overline{\zeta} \wedge \zeta \wedge \rho\\ &\quad = i\,R\,\kappa \wedge \zeta \wedge \overline{\kappa} \wedge \rho - dW \wedge \kappa \wedge \zeta \wedge \rho - W\,\pi^1 \wedge \kappa \wedge \zeta \wedge \rho,\end{aligned}$$

that is to say after putting everything to the right

$$0 = -\,dW \wedge \rho \wedge \kappa \wedge \zeta - \big(2\,p + 2\,i\,R\big)\,\rho \wedge \kappa \wedge \zeta \wedge \overline{\kappa} + 2\overline{W}\,p \wedge \kappa \wedge \zeta \wedge \overline{\zeta} - W\,\pi^1 \wedge \rho \wedge \kappa \wedge \zeta.$$

Thus, inserting the expansion of dW from (14.6)

$$\begin{aligned} -\,dW\wedge\rho\wedge\kappa\wedge\zeta \;&=\; -\,W_{\pi^1}\,\pi^1\wedge\rho\wedge\kappa\wedge\zeta - W_{\pi^2}\,\pi^2\wedge\rho\wedge\kappa\wedge\zeta - W_{\overline{\pi}^1}\,\overline{\pi}^1\wedge\rho\wedge\kappa\wedge\zeta \\ &\quad - W_{\overline{\pi}^2}\,\overline{\pi}^2\wedge\rho\wedge\kappa\wedge\zeta - W_{\overline{\kappa}}\,\overline{\kappa}\wedge\rho\wedge\kappa\wedge\zeta - W_{\overline{\zeta}}\,\overline{\zeta}\wedge\rho\wedge\kappa\wedge\zeta, \end{aligned}$$

we get

$$\begin{aligned} 0 \;=\; &-\big(W_{\pi^1}+W\big)\,\pi^1\wedge\rho\wedge\kappa\wedge\zeta - W_{\pi^2}\,\pi^2\wedge\rho\wedge\kappa\wedge\zeta - W_{\overline{\pi}^1}\,\overline{\pi}^1\wedge\rho\wedge\kappa\wedge\zeta - W_{\overline{\pi}^2}\,\overline{\pi}^2\wedge\rho\wedge\kappa\wedge\zeta \\ &-\big(2\,p+2\,i\,R-W_{\overline{\kappa}}\big)\,\rho\wedge\kappa\wedge\zeta\wedge\overline{\kappa} - \big(2\,\overline{W}+W_{\overline{\zeta}}\big)\,\rho\wedge\kappa\wedge\zeta\wedge\overline{\zeta}, \end{aligned}$$

whence by identification of coefficients of these independent 4-forms

$$\begin{aligned} &W_{\pi^1} \;=\; -\,W, \qquad W_{\pi^2} \;=\; 0, \qquad W_{\overline{\pi}^1} \;=\; 0, \qquad W_{\overline{\pi}^2} \;=\; 0, \\ &W_{\overline{\kappa}} \;=\; 2\,p+2\,i\,R, \qquad W_{\overline{\zeta}} \;=\; -\,2\,\overline{W}, \end{aligned}$$

while no condition is imposed so on W_ρ, W_κ, W_ζ, and thus

$$dW \;=\; -\,W\,\pi^1 + W_\rho\,\rho + W_\kappa\,\kappa + W_\zeta\,\zeta + \big(2\,p+2\,i\,R\big)\,\overline{\kappa} - 2\,\overline{W}\,\overline{\zeta}.$$

Next, putting this expression of dW back into (14.9) allows us to eliminate p so that we can focus only on $\Pi-\overline{\Pi}$ and Ψ, which we place on the left

$$\begin{aligned} i\,\Psi\wedge\rho\wedge\kappa + \big(\Pi-\overline{\Pi}\big)\wedge\rho\wedge\zeta \;=\; &\underline{i\,R\,\zeta\wedge\overline{\kappa}\wedge\kappa}_{1} - \underline{2\,p\,\kappa\wedge\overline{\kappa}\wedge\zeta}_{2} - \underline{2\,\overline{W}\,\kappa\wedge\overline{\zeta}\wedge\zeta}_{3} - dR\wedge\rho\wedge\zeta \\ &- R\,\big(\pi^1+\overline{\pi}^1\big)\wedge\rho\wedge\zeta - i\,R\,\pi^2\wedge\rho\wedge\kappa + \underline{i\,R\,\kappa\wedge\zeta\wedge\overline{\kappa}}_{1} \\ &- dJ\wedge\rho\wedge\overline{\kappa} - 3J\,\overline{\pi}^1\wedge\rho\wedge\overline{\kappa} - J\,\rho\wedge\kappa\wedge\overline{\zeta} \\ &+ \underline{W\,\pi^1\wedge\kappa\wedge\zeta}_{4} - W_\rho\,\rho\wedge\kappa\wedge\zeta - \big(\underline{2\,p}_{2} + \underline{2\,i\,R}_{1}\big)\,\overline{\kappa}\wedge\kappa\wedge\zeta \\ &+ \underline{2\,\overline{W}\,\overline{\zeta}\wedge\kappa\wedge\zeta}_{3} - W\,\pi^2\wedge\rho\wedge\zeta - \underline{W\,\pi^1\wedge\kappa\wedge\zeta}_{4} - W\,J\,\rho\wedge\kappa\wedge\overline{\kappa}. \end{aligned}$$

Here, four simplifications by pairs are underlined, in which we observe that p eliminates itself, and if we collect at first the terms divisible by $\rho\wedge\kappa$, we get

$$\begin{aligned} i\,\Psi\wedge\rho\wedge\kappa + \big(\Pi-\overline{\Pi}\big)\wedge\rho\wedge\zeta \;=\; &\Big(-\,i\,R\,\pi^2 - J\,\overline{\zeta} - W_\rho\,\zeta - W\,J\,\overline{\kappa}\Big)\wedge\rho\wedge\kappa \\ &- dR\wedge\rho\wedge\zeta - R\,\big(\pi^1+\overline{\pi}^1\big)\wedge\rho\wedge\zeta \\ &- dJ\wedge\rho\wedge\overline{\kappa} - 3J\,\overline{\pi}^1\wedge\rho\wedge\overline{\kappa} - W\,\pi^2\wedge\rho\wedge\zeta. \end{aligned}$$

By introducing the modified 1-form

$$\Psi' \;:=\; \Psi - i\,\Big(i\,R\,\pi^2 + J\,\overline{\zeta} + W_\rho\,\zeta + W\,J\,\overline{\kappa}\Big),$$

the equation becomes

$$\begin{aligned} i\,\Psi'\wedge\rho\wedge\kappa + \big(\Pi-\overline{\Pi}\big)\wedge\rho\wedge\zeta \;=\; &-\,dR\wedge\rho\wedge\zeta - R\,\big(\pi^1+\overline{\pi}^1\big)\wedge\rho\wedge\zeta \\ &- dJ\wedge\rho\wedge\overline{\kappa} - 3J\,\overline{\pi}^1\wedge\rho\wedge\overline{\kappa} - W\,\pi^2\wedge\rho\wedge\zeta. \end{aligned} \tag{14.11}$$

Now, let us wedge $(\bullet)\wedge\kappa\wedge\zeta$ all this to make Ψ and $\Pi-\overline{\Pi}$ disappear, replacing simultaneously

$$dJ \;=\; J_{\pi^1}\,\pi^1 + J_{\pi^2}\,\pi^2 + J_{\overline{\pi}^1}\,\overline{\pi}^1 + J_{\overline{\pi}^2}\,\overline{\pi}^2 + J_\rho\,\rho + J_\kappa\,\kappa + J_\zeta\,\zeta + J_{\overline{\kappa}}\,\overline{\kappa} + J_{\overline{\zeta}}\,\overline{\zeta},$$

to obtain

$$\begin{aligned} 0 \;&=\; -\,J_{\pi^1}\,\pi^1\wedge\rho\wedge\overline{\kappa}\wedge\kappa\wedge\zeta - J_{\pi^2}\,\pi^2\wedge\rho\wedge\overline{\kappa}\wedge\kappa\wedge\zeta - J_{\overline{\pi}^1}\,\overline{\pi}^1\wedge\rho\wedge\overline{\kappa}\wedge\kappa\wedge\zeta - J_{\overline{\pi}^2}\,\overline{\pi}^2\wedge\rho\wedge\overline{\kappa}\wedge\kappa\wedge\zeta \\ &\qquad - J_{\overline{\zeta}}\,\overline{\zeta}\wedge\rho\wedge\overline{\kappa}\wedge\kappa\wedge\zeta - 3J\,\overline{\pi}^1\wedge\rho\wedge\overline{\kappa}\wedge\kappa\wedge\zeta \\ &=\; -\,J_{\pi^1}\,\pi^1\wedge\rho\wedge\kappa\wedge\zeta\wedge\overline{\kappa} - J_{\pi^2}\,\pi^2\wedge\rho\wedge\kappa\wedge\zeta\wedge\overline{\kappa} - \big(J_{\overline{\pi}^1}+3\,J\big)\,\overline{\pi}^1\wedge\rho\wedge\kappa\wedge\zeta\wedge\overline{\kappa} - J_{\overline{\pi}^2}\,\overline{\pi}^2\wedge\rho\wedge\kappa\wedge\zeta\wedge\overline{\kappa} \\ &\qquad - J_{\overline{\zeta}}\,\rho\wedge\kappa\wedge\zeta\wedge\overline{\kappa}\wedge\overline{\zeta}, \end{aligned}$$

and since these 5-forms are linearly independent, we get by identification

$$J_{\pi^1} = 0, \qquad J_{\pi^2} = 0, \qquad J_{\overline{\pi}^1} = -3J, \qquad J_{\overline{\pi}^2} = 0, \qquad J_{\overline{\zeta}} = 0,$$

while no condition is imposed in this way on J_ρ, J_κ, J_ζ, $J_{\overline{\kappa}}$. Consequently, the 1-form dJ contracts as

$$dJ = -3J\,\overline{\pi}^1 + J_\rho\,\rho + J_\kappa\,\kappa + J_\zeta\,\zeta + J_{\overline{\kappa}}\,\overline{\kappa},$$

hence putting this expression back into (14.11), we obtain

$$\begin{aligned} i\,\Psi' \wedge \rho \wedge \kappa + \big(\Pi - \overline{\Pi}\big) \wedge \rho \wedge \zeta = & -dR \wedge \rho \wedge \zeta - R\,\big(\pi^1 + \overline{\pi}^1\big) \wedge \rho \wedge \zeta \\ & + \underline{3J\,\overline{\pi}^1 \wedge \rho \wedge \overline{\kappa}}_{\circ} - J_\kappa\,\kappa \wedge \rho \wedge \overline{\kappa} - J_\zeta\,\zeta \wedge \rho \wedge \overline{\kappa} - \underline{3J\,\overline{\pi}^1 \wedge \rho \wedge \overline{\kappa}}_{\circ} - W\,\pi^2 \wedge \rho \wedge \zeta. \end{aligned}$$

We can yet absorb in Ψ' one term from the right-hand side by introducing

$$\Psi'' := \Psi' + i\,J_\kappa\,\overline{\kappa},$$

so that our equation becomes

$$\begin{aligned} i\,\Psi'' \wedge \rho \wedge \kappa + \big(\Pi - \overline{\Pi}\big) \wedge \rho \wedge \zeta = & -dR \wedge \rho \wedge \zeta - R\,\big(\pi^1 + \overline{\pi}^1\big) \wedge \rho \wedge \zeta \\ & + J_\zeta\,\rho \wedge \zeta \wedge \overline{\kappa} - W\,\pi^2 \wedge \rho \wedge \zeta. \end{aligned}$$

Now, observe that all terms except the first one $i\,\Psi'' \wedge \rho \wedge \kappa$ are multiple of $\rho \wedge \zeta$. Consequently, wedging on both sides by $(\bullet) \wedge \zeta$, we annihilate everything except

$$i\,\Psi'' \wedge \rho \wedge \kappa \wedge \zeta = 0.$$

Thanks to the Cartan Lemma, there exist function e, f, g so that

$$\Psi'' = e\,\rho + f\,\kappa + g\,\zeta.$$

For later use, we also observe in passing that

$$\begin{aligned} \Psi & = \Psi' + i\,W_\rho\,\zeta + i\,W J\,\overline{\kappa} - R\,\pi^2 + i\,J\,\overline{\zeta} \\ & = \Psi'' - i\,J_\kappa\,\overline{\kappa} + i\,W_\rho\,\zeta + i\,W J\,\overline{\kappa} - R\,\pi^2 + i\,J\,\overline{\zeta} \\ & = -R\,\pi^2 + e\,\rho + f\,\kappa + \big(i\,W_\rho + g\big)\,\zeta + i\,\big(WJ - J_\kappa\big)\,\overline{\kappa} + i\,J\,\overline{\zeta}. \end{aligned} \tag{14.12}$$

Inserting this just above conducts to an identity

$$\begin{aligned} i\,g\,\rho \wedge \kappa \wedge \zeta + \big(\Pi - \overline{\Pi}\big) \wedge \rho \wedge \zeta = & -dR \wedge \rho \wedge \zeta - R\,\big(\pi^1 + \overline{\pi}^1\big) \wedge \rho \wedge \zeta \\ & + J_\zeta\,\rho \wedge \zeta \wedge \overline{\kappa} - W\,\pi^2 \wedge \rho \wedge \zeta, \end{aligned}$$

in which *all* terms are now multiples of $\rho \wedge \zeta$. Consequently, the Cartan Lemma implies the existence of functions r and s such that

$$\Pi - \overline{\Pi} = i\,g\,\kappa - dR - R\,\pi^1 - R\,\overline{\pi}^1 + J_\zeta\,\overline{\kappa} - W\,\pi^2 + r\,\rho + s\,\zeta.$$

But here, we can take advantage of the fact that $\Pi - \overline{\Pi}$ is purely imaginary to obtain some information about g, r, s. Indeed, conjugating

$$\overline{\Pi} - \Pi = -i\,\overline{g}\,\overline{\kappa} - dR - R\,\overline{\pi}^1 - R\,\pi^1 - \overline{J_\zeta}\,\kappa - \overline{W}\,\overline{\pi}^2 + \overline{r}\,\rho + \overline{s}\,\overline{\zeta},$$

and summing, we eliminate $\Pi - \overline{\Pi}$, hence we are left after reorganization with

$$\begin{aligned} 0 = & -2\,dR - 2R\,\pi^1 - W\,\pi^2 - 2R\,\overline{\pi}^1 - \overline{W}\,\overline{\pi}^2 \\ & + (r + \overline{r})\,\rho + \big(i\,g + \overline{J_\zeta}\big)\,\kappa + s\,\zeta + \big(-i\,\overline{g} + J_\zeta\big)\,\overline{\kappa} + \overline{s}\,\overline{\zeta}. \end{aligned}$$

Naturally, one has to use the expansion of dR from (14.6) to continue the computation

$$\begin{aligned} 0 = & -\big(2\,R_{\pi^1} + 2\,R\big)\,\pi^1 - \big(2\,R_{\pi^2} + W\big)\,\pi^2 - \big(2\,R_{\overline{\pi}^1} + 2\,R\big)\,\overline{\pi}^1 - \big(2\,R_{\overline{\pi}^2} + \overline{W}\big)\,\overline{\pi}^2 \\ & - \big(2\,R_\rho - r - \overline{r}\big)\,\rho - \big(2\,R_\kappa - i\,g - \overline{J_\zeta}\big)\,\kappa - \big(2\,R_\zeta - s\big)\,\zeta - \big(2\,R_{\overline{\kappa}} + i\,\overline{g} - J_\zeta\big)\,\overline{\kappa} - \big(2\,R_{\overline{\zeta}} - \overline{s}\big)\,\overline{\zeta}. \end{aligned}$$

An identification to zero of all the nine coefficients of π^1, π^2, $\overline{\pi}^1$, $\overline{\pi}^2$, ρ, κ, ζ, $\overline{\kappa}$, $\overline{\zeta}$ gives

$$R_{\pi^1} = -R, \qquad R_{\pi^2} = -\tfrac{1}{2}W, \qquad R_{\overline{\pi}^1} = -R, \qquad R_{\overline{\pi}^2} = -\tfrac{1}{2}\overline{W},$$
$$R_\rho = \tfrac{1}{2}\big(r+\overline{r}\big), \qquad R_\kappa = \tfrac{1}{2}\big(i\,g+\overline{J_\zeta}\big), \qquad R_\zeta = \tfrac{1}{2}\,s, \qquad R_{\overline{\kappa}} = \tfrac{1}{2}\big(-i\,\overline{g}+J_\zeta\big), \qquad R_{\overline{\zeta}} = \tfrac{1}{2}\,\overline{s},$$

and so:

$$dR = -R\,\pi^1 - \tfrac{1}{2}\,W\,\pi^2 - R\,\overline{\pi}^1 - \tfrac{1}{2}\,\overline{W}\,\overline{\pi}^2 + R_\rho\,\rho + R_\kappa\,\kappa + R_\zeta\,\zeta + R_{\overline{\kappa}}\,\overline{\kappa} + R_{\overline{\zeta}}\,\overline{\zeta}.$$

Inserting this back into what precedes, we can therefore obtain both

$$\Pi - \overline{\Pi} = -\tfrac{1}{2}W\pi^2 + \tfrac{1}{2}\overline{W}\overline{\pi}^2 + R_\zeta\zeta - R_{\overline{\zeta}}\overline{\zeta} + (R_\kappa - \overline{J_\zeta})\kappa - (R_{\overline{\kappa}} - J_\zeta)\overline{\kappa} + \tfrac{1}{2}(g_\rho - \overline{g}_\rho)\rho,$$

and replacing $g = -2i\,R_\kappa + i\,\overline{J_\zeta}$ in (14.12)

$$\Psi = -R\pi^2 + e\rho + f\kappa + i(W_\rho - 2R_\kappa + \overline{J_\zeta})\zeta + i(WJ - J_\kappa)\overline{\kappa} + iJ\overline{\zeta}.$$

Thus

$$\begin{aligned}\Omega_1 &= p\kappa\wedge\overline{\kappa} + \Pi\wedge\rho + \overline{W\zeta}\wedge\kappa - W\zeta\wedge\overline{\kappa}\\
&= p\kappa\wedge\overline{\kappa} + \tfrac{1}{2}(\Pi - \overline{\Pi})\wedge\rho + \overline{W\zeta}\wedge\kappa - W\zeta\wedge\overline{\kappa} + \tfrac{1}{2}(\Pi + \overline{\Pi})\wedge\rho\\
&= -\tfrac{1}{4}W\pi^2\wedge\rho + \tfrac{1}{4}\overline{W}\overline{\pi}^2\wedge\rho - \tfrac{1}{2}(R_\kappa - \overline{J_\zeta})\rho\wedge\kappa - \tfrac{1}{2}R_\zeta\rho\wedge\zeta + \tfrac{1}{2}(R_{\overline{\kappa}} - J_\zeta)\rho\wedge\overline{\kappa}\\
&\quad + \tfrac{1}{2}R_{\overline{\zeta}}\rho\wedge\overline{\zeta} + (\tfrac{1}{2}W_{\overline{\kappa}} - iR)\kappa\wedge\overline{\kappa} - \overline{W}\kappa\wedge\overline{\zeta} - W\zeta\wedge\overline{\kappa} + \tfrac{1}{2}(\Pi + \overline{\Pi})\wedge\rho,\end{aligned}$$

and

$$\begin{aligned}\Omega_2 &= -R\pi^2\wedge\rho - \tfrac{1}{4}W\pi^2\wedge\kappa + \tfrac{1}{4}\overline{W}\overline{\pi}^2\wedge\kappa - i(W_\rho - 2R_\kappa + \overline{J_\zeta})\rho\wedge\zeta\\
&\quad - i(WJ - J_\kappa)\rho\wedge\overline{\kappa} - iJ\rho\wedge\overline{\zeta} - \tfrac{1}{2}R_\zeta\kappa\wedge\zeta + \tfrac{1}{2}(R_{\overline{\kappa}} - J_\zeta)\kappa\wedge\overline{\kappa} + \tfrac{1}{2}R_{\overline{\zeta}}\kappa\wedge\overline{\zeta}\\
&\quad - R\zeta\wedge\overline{\kappa} + \tfrac{1}{2}(\Pi + \overline{\Pi})\wedge\kappa + (\tfrac{1}{2}(r - \overline{r}) - f)\rho\wedge\kappa.\end{aligned}$$

If we define

$$\Lambda := \tfrac{1}{2}(\Pi + \overline{\Pi}) + \text{real part of}\ \big(\tfrac{1}{2}(g_\rho - \overline{g_\rho}) - d_\kappa\big)\rho$$

and

$$h := \text{imaginary part of}\ \big(\tfrac{1}{2}(g_\rho - \overline{g_\rho}) - d_\kappa\big),$$

we conclude that

$$\begin{aligned}\Omega_1 &= -\tfrac{1}{4}W\pi^2\wedge\rho + \tfrac{1}{4}\overline{W}\overline{\pi}^2\wedge\rho - \tfrac{1}{2}(R_\kappa - \overline{J_\zeta})\rho\wedge\kappa - \tfrac{1}{2}R_\zeta\rho\wedge\zeta + \tfrac{1}{2}(R_{\overline{\kappa}} - J_\zeta)\rho\wedge\overline{\kappa}\\
&\quad + \tfrac{1}{2}R_{\overline{\zeta}}\rho\wedge\overline{\zeta} + \big(\tfrac{1}{2}W_{\overline{\kappa}} - iR\big)\kappa\wedge\overline{\kappa} - \overline{W}\kappa\wedge\overline{\zeta} - W\zeta\wedge\overline{\kappa} + \Lambda\wedge\rho,\\
\Omega_2 &= -R\pi^2\wedge\rho - \tfrac{1}{4}W\pi^2\wedge\kappa + \tfrac{1}{4}\overline{W}\overline{\pi}^2\wedge\kappa - i(W_\rho - 2R_\kappa + \overline{J_\zeta})\rho\wedge\zeta\\
&\quad - i(WJ - J_\kappa)\rho\wedge\overline{\kappa} - iJ\rho\wedge\overline{\zeta} - \tfrac{1}{2}R_\zeta\kappa\wedge\zeta + \tfrac{1}{2}(R_{\overline{\kappa}} - J_\zeta)\kappa\wedge\overline{\kappa} + \tfrac{1}{2}R_{\overline{\zeta}}\kappa\wedge\overline{\zeta}\\
&\quad - R\zeta\wedge\overline{\kappa} + \Lambda\wedge\kappa + h\rho\wedge\kappa.\end{aligned}$$

Notice that all coefficients of 2-forms — except only h — depend on R, J, W and their coframe derivatives.

We are now close to the termination towards an $\{e\}$-structure. In summary, we have obtained the following structure equations

$$\begin{aligned}
d\rho &= \pi^1 \wedge \rho + \overline{\pi}^1 \wedge \rho + i\kappa \wedge \overline{\kappa}, \\
d\kappa &= \pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa}, \\
d\zeta &= i\pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi}^1 \wedge \zeta + W\kappa \wedge \zeta + R\rho \wedge \zeta + J\rho \wedge \overline{\kappa}, \\
d\pi^1 &= \Lambda \wedge \rho - \tfrac{1}{4}W\pi^2 \wedge \rho + \tfrac{1}{4}\overline{W}\overline{\pi}^2 \wedge \rho - i\overline{\pi}^2 \wedge \kappa \\
&\quad - \tfrac{1}{2}(R_\kappa - \overline{J_\zeta})\rho \wedge \kappa - \tfrac{1}{2}R_\zeta \rho \wedge \zeta + \tfrac{1}{2}(R_{\overline{\kappa}} - J_\zeta)\rho \wedge \overline{\kappa} + \tfrac{1}{2}R_{\overline{\zeta}}\rho \wedge \overline{\zeta} \\
&\quad + \big(\tfrac{1}{2}W_{\overline{\kappa}} - iR\big)\kappa \wedge \overline{\kappa} - \overline{W}\kappa \wedge \overline{\zeta} - W\zeta \wedge \overline{\kappa} + \zeta \wedge \overline{\zeta}, \\
d\pi^2 &= \Lambda \wedge \kappa + \pi^2 \wedge \overline{\pi}^1 - \overline{\pi}^2 \wedge \zeta - R\pi^2 \wedge \rho - \tfrac{1}{4}W\pi^2 \wedge \kappa + \tfrac{1}{4}\overline{W}\overline{\pi}^2 \wedge \kappa \\
&\quad + h\rho \wedge \kappa - i(W_\rho - 2R_\kappa + \overline{J_\zeta})\rho \wedge \zeta - i(WJ - J_\kappa)\rho \wedge \overline{\kappa} - iJ\rho \wedge \overline{\zeta} \\
&\quad - \tfrac{1}{2}R_\zeta \kappa \wedge \zeta + \tfrac{1}{2}(R_{\overline{\kappa}} - J_\zeta)\kappa \wedge \overline{\kappa} + \tfrac{1}{2}R_{\overline{\zeta}}\kappa \wedge \overline{\zeta} - R\zeta \wedge \overline{\kappa}.
\end{aligned}$$

But at this stage, we cannot directly deduce from these equations an appropriate expression for h. For example, any attempt to isolate h by wedging the equation $d\pi^2 = \cdots$ with any appropriate differential form will include a component of Maurer-Cartan type. This is to be expected, because h will soon be shown below to depend on higher order jets of R, J, W, while the torsions above only depend up to the 2^{nd}-order jets of these invariants. Therefore, an application of the exterior differentiation on both sides of the equation $d\pi^2 = \cdots$ appears necessary to reach an expression for h from the Poincaré relation $d \circ d = 0$.

To facilitate the discussion, we set

$$\begin{aligned}
\widehat{\Omega}_1 &= -\tfrac{1}{4}W\pi^2 \wedge \rho + \tfrac{1}{4}\overline{W}\overline{\pi}^2 \wedge \rho - \tfrac{1}{2}(R_\kappa - \overline{J_\zeta})\rho \wedge \kappa - \tfrac{1}{2}R_\zeta \rho \wedge \zeta \\
&\quad + \tfrac{1}{2}(R_{\overline{\kappa}} - J_\zeta)\rho \wedge \overline{\kappa} + \tfrac{1}{2}R_{\overline{\zeta}}\rho \wedge \overline{\zeta} + \left(\tfrac{1}{2}W_{\overline{\kappa}} - iR\right)\kappa \wedge \overline{\kappa} - \overline{W}\kappa \wedge \overline{\zeta} - W\zeta \wedge \overline{\kappa}, \\
\widehat{\Omega}_2 &= -R\pi^2 \wedge \rho - \tfrac{1}{4}W\pi^2 \wedge \kappa + \tfrac{1}{4}\overline{W}\overline{\pi}^2 \wedge \kappa - i(W_\rho - 2R_\kappa + \overline{J_\zeta})\rho \wedge \zeta \\
&\quad - i(WJ - J_\kappa)\rho \wedge \overline{\kappa} - iJ\rho \wedge \overline{\zeta} - \tfrac{1}{2}R_\zeta \kappa \wedge \zeta + \tfrac{1}{2}(R_{\overline{\kappa}} - J_\zeta)\kappa \wedge \overline{\kappa} + \tfrac{1}{2}R_{\overline{\zeta}}\kappa \wedge \overline{\zeta} \\
&\quad - R\zeta \wedge \overline{\kappa},
\end{aligned}$$

so that

$$\begin{aligned}
d\pi^1 &= \Lambda \wedge \rho - i\overline{\pi}^2 \wedge \kappa + \zeta \wedge \overline{\zeta} + \widehat{\Omega}_1, \\
d\pi^2 &= \Lambda \wedge \kappa + \pi^2 \wedge \overline{\pi}^1 - \overline{\pi}^2 \wedge \zeta + \widehat{\Omega}_2 + h\rho \wedge \kappa.
\end{aligned}$$

Proposition 14.13. *The function h is a function of the 3^{rd}-order jets of W and J.*

Proof. By applying exterior differentiation d to the equation of $d\pi^2$, while wedging on both sides with $\kappa \wedge \pi^1 \wedge \overline{\pi}^1 \wedge \pi^2 \wedge \overline{\pi}^2$, we obtain

$$\begin{aligned}
2h\, \rho \wedge \kappa \wedge \overline{\kappa} \wedge \zeta \wedge \pi^1 \wedge \overline{\pi}^1 \wedge \pi^2 \wedge \overline{\pi}^2 &= -\overline{\widehat{\Omega}}_2 \wedge \kappa \wedge \zeta \wedge \pi^1 \wedge \overline{\pi}^1 \wedge \pi^2 \wedge \overline{\pi}^2 \\
&\quad - d\widehat{\Omega}_2 \wedge \kappa \wedge \pi^1 \wedge \overline{\pi}^1 \wedge \pi^2 \wedge \overline{\pi}^2. \quad \square
\end{aligned}$$

At this point, let Φ be the auxiliary real 2-form

$$\Phi := d\Lambda - \Lambda \wedge \pi^1 - \Lambda \wedge \overline{\pi}^1 - i\pi^2 \wedge \overline{\pi}^2.$$

Again this comes from the consideration of the model case. The structure equations therefore become

$$\begin{aligned} d\rho &= \pi^1 \wedge \rho + \overline{\pi}^1 \wedge \rho + i\kappa \wedge \overline{\kappa}, \\ d\kappa &= \pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa}, \\ d\zeta &= i\pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi}^1 \wedge \zeta + W\kappa \wedge \zeta + R\rho \wedge \zeta + J\rho \wedge \overline{\kappa}, \\ d\pi^1 &= \Lambda \wedge \rho - i\overline{\pi}^2 \wedge \kappa + \zeta \wedge \overline{\zeta} + \widehat{\Omega}_1, \\ d\pi^2 &= \Lambda \wedge \kappa + \pi^2 \wedge \overline{\pi}^1 - \overline{\pi}^2 \wedge \zeta + \widehat{\Omega}_2 + h\rho \wedge \kappa, \\ d\Lambda &= \Lambda \wedge \pi^1 + \Lambda \wedge \overline{\pi}^1 + i\pi^2 \wedge \overline{\pi}^2 + \Phi. \end{aligned}$$

Proposition 14.14. *The real 2-form Φ is a function of the 4^{th}-order jets of W and J.*

Proof. By taking exterior derivative of $d\pi^1$ and $d\pi^2$ again, this time using the expression of $d\Lambda$, we have

$$\begin{aligned} \Phi \wedge \rho &= i\overline{\widehat{\Omega}}_2 \wedge \kappa + ih\rho \wedge \kappa \wedge \overline{\kappa} - W\kappa \wedge \zeta \wedge \overline{\zeta} + \overline{W}\zeta \wedge \overline{\kappa} \wedge \overline{\zeta} - 2R\rho \wedge \zeta \wedge \overline{\zeta} \\ &\quad - J\rho \wedge \overline{\kappa} \wedge \overline{\zeta} + \overline{J}\rho \wedge \kappa \wedge \zeta - d\widehat{\Omega}_1, \\ \Phi \wedge \kappa &= -\widehat{\Omega}_2 \wedge \overline{\pi}^1 - h\rho \wedge \kappa \wedge \overline{\pi}^1 + \pi^2 \wedge \overline{\widehat{\Omega}}_1 + \overline{\widehat{\Omega}}_2 \wedge \zeta - h\rho \wedge \overline{\kappa} \wedge \zeta - W\overline{\pi}^2 \wedge \kappa \wedge \zeta \\ &\quad - R\overline{\pi}^2 \wedge \rho \wedge \zeta - J\overline{\pi}^2 \wedge \rho \wedge \overline{\kappa} - d\widehat{\Omega}_2 - d(h\rho \wedge \kappa). \end{aligned}$$

Writing Φ as

$$\Phi = \widehat{\Omega}_3 + u\rho \wedge \kappa,$$

where $\widehat{\Omega}_3$ is the 2-form not containing $\rho \wedge \kappa$, then each of the coefficients in $\widehat{\Omega}_3$ is a function of the 4^{th}-order jet of W and J. Since Φ is real, taking conjugate on both sides, we must have

$$\widehat{\Omega}_3 + u\rho \wedge \kappa = \overline{\widehat{\Omega}}_3 + \overline{u}\rho \wedge \overline{\kappa}.$$

Therefore by inspection, $\overline{u}$ is also a function of the 4^{th}-order jets of W and J, and therefore so is u. This finishes the proof. □

With this proposition, we have therefore fully constructed an $\{e\}$-structure.

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