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WEAKLY LOCALLY ARTINIAN SUPPLEMENTED MODULES

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Abstract. In this study, by using the concept of locally artinian supplemented modules, we have obtained weakly locally artinian supplemented modules as a proper generalization of these modules in module theory. Our results generalize and extend various comparable results in the existing literature. We have proved that the notion of weakly locally artinian supplemented modules inherited by factor modules, finite sums and small covers. We have obtained that weakly locally artinian supplemented modules with small radical coincide with weakly (radical) supplemented modules which have locally artinian radical. Also, we have shown that if N and $\frac{M}{N}$ are weakly locally artinian supplemented for some submodule $N \subseteq M$ which has a weak locally artinian supplement in M then M is weakly locally artinian supplemented.

1. Introduction

Throughout this paper, the ring R will denote an associative ring with identity element and modules will be left unital. We will use the notation $U \ll M$ to stress that U is a *small* submodule of M. A submodule $N \subseteq M$ is said to be *essential* in M, denoted as $N \trianglelefteq M$, if $N \cap K \neq 0$ for every non-zero submodule $K \subseteq M$. By $Rad(M)$ we denote the sum of all small submodules of M or, equivalently the intersection of all maximal submodules of M. $Soc(M)$ will indicate socle of M which is sum of all semisimple submodules of M. A non-zero module M is called *hollow* if every proper submodule of M is small in M, and M is called *local* if the sum of all proper submodules of M is also a proper submodule of M. A module M is called *semilocal* if $\frac{M}{Rad(M)}$ is semisimple. A ring R is said to be *semilocal* if $\frac{R}{Rad(R)}$ is semisimple. By [5, Proposition 20.2], a commutative ring R is semilocal if and only if R has only finitely many maximal ideals. M is called *locally artinian* if every finitely generated submodule of M is artinian [10, 31]. A submodule V of M is called *a supplement* of U in M if $M = U + V$ and $U \cap V \ll V$. A submodule V of M is called *a weak supplement* of U in M if $M = U + V$ and $U \cap V \ll M$. The module M is called *(weakly) supplemented* if every submodule of M has a (weak) supplement in M. In [1], it is proved that the class of weakly supplemented modules need not be

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closed under extensions, that is if U and M/U are weakly supplemented for some submodule U of M then M need not be weakly supplemented. A submodule U of M has ample supplements in M if every submodule V of M such that $M = U + V$ contains a supplement $V^{'}$ of U in M. The module M is called *amply supplemented* if every submodule of M has ample supplements in M [10].

Let R be a a principal ideal domain (PID) with exactly one non-zero maximal ideal, then R is said to be *a Discrete valuation ring (DVR)*. By [13, Lemma 2.1] that every module with small radical over a Discrete Valuation Ring, is the direct sum of a finitely generated free module and a bounded module. In [12], he generalized the concept of modules with small radical to radical supplemented modules. M is called *radical supplemented* if $Rad(M)$ has a supplement in M. These modules are also a proper generalization of supplemented modules. Then, in [2], it is defined as a module M *strongly radical supplemented* (or briefly srs) if every submodule N of M with $Rad(M) \subseteq N$ has a supplement in M. Then it is introduced that modules whose every submodule containing the radical has a weak supplement (in particular, over dedekind domains the radical has a weak supplement) in the module as *weakly radical supplemented module (wrs)* which is a generalization of strongly radical supplemented modules [7].

In [11], a generalization of concept of socle as a $Soc_s(M) = \sum\{ U \ll M \,|\, U$ is simple$\}$. Here $Soc_s(M) \subseteq Rad(M)$ and $Soc_s(M) \subseteq Soc(M)$. In [3], a module M is called *strongly local* if it is local and $Rad(M)$ is semisimple. A submodule U of M is called an *ss-supplement* of U in M if $M = U+V$ and $U \cap V \subseteq Soc_s(V)$. The module M is called *ss-supplemented* if every submodule of M has an ss-supplement in M. A submodule U of M has ample ss-supplements in M if every submodule V of M such that $M = U + V$ contains an ss-supplement $V^{'}$ of U in M. The module M is called *amply ss-supplemented* if every submodule of M has ample ss-supplements in M. In [8], strongly local and (amply) ss-supplemented modules are generalized as RLA-local and (amply) locally artinian supplemented modules, respectively. A local module M is called *RLA-local* if $Rad(M)$ is a locally artinian submodule of M. A module M is called *locally artinian supplemented* if every submodule U of M has a locally artinian supplement in M, that is, V is a supplement of U in M such that $U \cap V$ is locally artinian. M is called *amply locally artinian supplemented* if every submodule U of M has ample locally artinian supplements in M. Here a submodule U of M has ample locally artinian supplements in M if every submodule V of M such that $M = U + V$ contains a locally artinian supplement $V^{'}$ of U in M.

In Section 2, it is proved that a module with a small radical is weakly locally artinian supplemented if and only if M is weakly supplemented and $Rad(M)$ is locally artinian. It is also proved that finite sum of weakly locally artinian supplemented modules is weakly locally artinian supplemented and every factor module of a weakly locally artinian supplemented module is weakly locally artinian supplemented. It is shown that a notion of weakly locally artinian supplemented modules inherited by small cover. It is also shown that if N and $\frac{M}{N}$ are weakly locally artinian supplemented for some submodule $N \subseteq M$ which has a weak locally artinian supplement in M, then M is weakly locally artinian supplemented.

2. Weakly Locally Artinian Supplemented Modules

Definition 1. Let M be a module. Then M is called *weakly locally artinian supplemented* if every submodule N of M has a weak supplement K in M with $N \cap K$ is locally artinian, i.e. N has a weak locally artinian supplement K in M.

By the definition, it is clear that every weakly locally artinian supplemented module is weakly supplemented. The following example shows that the converse is not always true.

Example 1. Consider the $\mathbb{Z}$-module $\mathbb{Q}$. By [1, Lemma 2.8], $M =_{\mathbb{Z}} \mathbb{Q}$ is weakly supplemented. So $\frac{\mathbb{Q}}{\mathbb{Z}}$ is weakly supplemented because of $\mathbb{Q}$ is weakly supplemented. But $\frac{\mathbb{Q}}{\mathbb{Z}}$ is not locally artinian by [9, Theorem 3]. Since $Rad(\frac{\mathbb{Q}}{\mathbb{Z}}) = \frac{\mathbb{Q}}{\mathbb{Z}}$, $\frac{\mathbb{Q}}{\mathbb{Z}}$ is not weakly locally artinian supplemented.

Lemma 1. *Let M be a weakly supplemented module and $Rad(M)$ be a locally artinian submodule of M. Then M is weakly locally artinian supplemented.*

Proof. Let $N \subseteq M$. By the hypothesis, there exists a submodule K of M such that $M = N + K$, $N \cap K \ll M$. So $N \cap K \subseteq Rad(M)$. Since $Rad(M)$ is a locally artinian submodule of M, $N \cap K$ is a locally artinian submodule of M by [10, 31.2 (ii)]. Thus M is weakly locally artinian supplemented. □

Theorem 1. *Let M be a module with small radical. Then the following statements are equivalent.*

1. *M is weakly locally artinian supplemented;*
2. *M is weakly supplemented and $Rad(M)$ has a weak locally artinian supplement in M;*
3. *M is weakly supplemented and $Rad(M)$ is locally artinian.*

Proof. (1) ⇒ (2) Since M is weakly locally artinian supplemented, M is weakly supplemented and $Rad(M)$ has a weak locally artinian supplement in M.

(2) ⇒ (3) Since $Rad(M) \ll M$, M is a weak locally artinian supplement of $Rad(M)$ in M. Thus we have $M = Rad(M) + M$, $Rad(M) = Rad(M) \cap M \ll M$ and $Rad(M)$ is locally artinian.

(3) ⇒ (1) Clear by Lemma 1. □

Since every finitely generated module has a small radical, we have:

Corollary 1. Let M be a finitely generated module. Then M is weakly locally artinian supplemented if and only if M is weakly supplemented with locally artinian radical.

Proposition 1. Let M be a weakly locally artinian supplemented module and $N, K \subseteq M$ be submodules with $M = N + K$. Then K contains a weak locally artinian supplement $K^{'}$ of N in M.

Proof. Let $N \cap K = U$. Since M is weakly locally artinian supplemented, there exists a submodule V of M such that $M = U + V$, $U \cap V \ll M$ and $U \cap V$ is locally artinian. Then $K = K \cap (U + V) = U + (K \cap V)$ and $M = N + K = N + [U + (K \cap V)] = N + (K \cap V)$. It follows that $N \cap (K \cap V) = U \cap V \ll M$ and $U \cap V$ is locally artinian, say $K^{'} = K \cap V$. Then we obtain that $K^{'}$ is a weak locally artinian supplement of N in M, as required. □

Proposition 2. Let M be a weakly locally artinian supplemented module and N be a small submodule of M. Then N is locally artinian.

Proof. By the hypothesis, there exists a submodule K of M such that $M = N + K$, $N \cap K \ll M$ and $N \cap K$ is locally artinian. Since $N \ll M$, $M = K$. So $N \cap K = N \cap M = N$ is locally artinian. □

Corollary 2. Let M be a weakly locally artinian supplemented module and $Rad(M) \ll M$. Then $Rad(M)$ is locally artinian.

Proof. Clear by Proposition 2. □

With the help of the next theorem, we verify that under special conditions, notions of weakly locally artinian supplemented modules and weakly radical supplemented modules are the same.

Theorem 2. *Let M be a module with $Rad(M) \ll M$. Then the following statements are equivalent.*

(1) *M is weakly locally artinian supplemented;*
(2) *M is weakly supplemented and $Rad(M)$ has a weak locally artinian supplement in M;*
(3) *M is weakly supplemented and $Rad(M)$ is locally artinian;*
(4) *M is weakly radical supplemented and $Rad(M)$ is locally artinian.*

Proof. (1) ⇔ (2) ⇔ (3) Clear by Theorem 1.

(3) ⇒ (4) Obvious.

(4) ⇒ (1) Let $N \subseteq M$. By the hypothesis, $N + Rad(M)$ has a weak supplement K in M. Then we have $M = (N+Rad(M))+K$. Since $Rad(M) \ll M$, $M = N+K$, $N \cap K \subseteq (N+Rad(M)) \cap K \ll M$. So $N \cap K \ll M$. Thus $N \cap K \subseteq Rad(M)$. Since $Rad(M)$ is locally artinian, $N \cap K$ is locally artinian by [10, 31.2 (ii)]. Therefore K is a weak locally artinian supplement of N in M, as desired. □

We will show that in the factor modules, the property is preserved in weakly locally artinian supplemented modules just as it is in weakly supplemented modules.

Proposition 3. If M is a weakly locally artinian supplemented module, then every factor module of M is weakly locally artinian supplemented.

Proof. Let M be a weakly locally artinian supplemented module and $\frac{M}{N}$ be a factor module of M. By the assumption, for any submodule $N \subseteq U \subseteq M$, there exists a submodule V of M such that $M = U+V$, $U \cap V \ll M$ and $U \cap V$ is locally artinian. Let $\pi : M \longrightarrow \frac{M}{N}$ be the canonical projection. Then we have $\frac{M}{N} = \frac{U}{N} + \frac{V+N}{N}$, $\frac{U}{N} \cap \frac{V+N}{N} = \pi(U \cap V) \ll \frac{M}{N}$ and $\frac{U}{N} \cap \frac{V+N}{N} = \pi(U \cap V)$ is locally artinian by [10, 31.2 (i)], as required. □

The following lemma plays a key role in showing that the notion of weakly locally artinian supplemented modules is inherited by finite sum.

Lemma 2. *Let M be a module, $M_1 \subseteq M$, $N \subseteq M$ and M_1 be weakly locally artinian supplemented. If $M_1 + N$ has a weak locally artinian supplement in M, then N has a weak locally artinian supplement in M.*

Proof. Let K be a weak locally artinian supplement of $M_1 + N$ in M. Then we can write $M = (M_1+N)+K$, $(M_1+N) \cap K \ll M$ and $(M_1+N) \cap K$ is locally artinian.

Since M_1 is weakly locally artinian supplemented, $(N+K)\cap M_1$ has a weak locally artinian supplement L in M_1, i.e. $M_1 = (N+K)\cap M_1 + L$, $(N+K)\cap L \ll M_1$ and $(N+K)\cap L$ is locally artinian. Then $M = M_1 + (N+K) = [(N+K)\cap M_1 + L] + (N+K) = N + (K+L)$ and $N\cap(K+L) \subseteq K\cap(N+L) + L\cap(N+K) \subseteq K\cap(N+M_1) + L\cap(N+K) \ll M$. By [10, 31 (2) (i), (ii)], $N\cap(K+L)$ is locally artinian and so $K+L$ is a weak locally artinian supplement of N in M. □

Corollary 3. Let M be an R-module, $N \subseteq M$ and $M_i \subseteq M$ for $i = 1,2,\dots,n$. If $N + M_1 + \cdots + M_n$ has a weak locally artinian supplement in M and M_i is a weakly locally artinian supplemented module for every $i = 1,2,\dots,n$, then N has a weak locally artinian supplement in M.

Corollary 4. Let $M = M_1 + M_2$. If M_1 and M_2 are weakly locally artinian supplemented modules, then M is a weakly locally artinian supplemented module.

The following corollary is obtained from the previous result by applying induction.

Corollary 5. A finite sum of weakly locally artinian supplemented modules is weakly locally artinian supplemented.

Recall from [6] that N is a *small cover* of a module M if there exists an epimorphism $f: N \longrightarrow M$ such that $Ker(f) \ll M$.

Lemma 3. *Let M be a weakly locally artinian supplemented module. Then every small cover of M is weakly locally artinian supplemented.*

Proof. Let N be a small cover of M. Then there exists an epimorphism $f: N \longrightarrow M$ such that $Ker(f) \ll N$. Note that $f^{-1}(K) \ll N$ for every $K \ll M$ holds since $Ker(f) \ll N$. Let $L \subset N$. Then $f(L)$ has a weak locally artinian supplement of X in M. Note that $M = X + f(L)$, $X \cap f(L) \ll M$ and $X\cap f(L)$ is locally artinian. Again it is easy to check that $f^{-1}(X)$ is a weak locally artinian supplement of L in N. □

Proposition 4. Let M be a weakly locally artinian supplemented module. Then every locally artinian supplement in M is weakly locally artinian supplemented.

Proof. Let K be a locally artinian supplement of N in M. Then we have $M = N + K$, $N\cap K \ll K$ and $N \cap K$ is locally artinian. $\frac{M}{N} \cong \frac{K}{(N\cap K)}$ is weakly locally artinian supplemented by Proposition 3. By Lemma 3, K is weakly locally artinian supplemented. □

Corollary 6. Let M be a weakly locally artinian supplemented module. Then every locally artinian direct summand in M is weakly locally artinian supplemented.

Proof. Since every locally artinian direct summand is locally artinian supplement, the proof follows from Proposition 4. □

Recall that a submodule $N \subseteq M$ is called *closed* in M if $N \trianglelefteq K$ for some $K \subseteq M$ implies $K = N$. A submodule $N \subseteq M$ is called *coclosed* in M if $\frac{N}{K} \ll \frac{M}{K}$ for some $K \subseteq M$ implies $K = N$.

Theorem 3. *Let $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence. If K and N are weakly locally artinian supplemented and K has a weak locally artinian supplement in M then M is weakly locally artinian supplemented. If K*

is coclosed locally artinian submodule in M then the converse holds, that is if M is weakly locally artinian supplemented then K and N are weakly locally artinian supplemented.

Proof. Without restriction of generality we will assume that $K \subseteq M$. Let T be a weak locally artinian supplement of K in M i.e. $M = K+T$, $K \cap T \ll M$ and $K \cap T$ is locally artinian. Then we have, $\frac{M}{(K\cap T)} = \frac{K}{(K\cap T)} \oplus \frac{T}{(K\cap T)}$. Since $\frac{K}{(K\cap T)}$ is a factor module of K, $\frac{K}{(K\cap T)}$ is weakly locally artinian supplemented by Proposition 3. On the other hand, $\frac{T}{(K\cap T)} \cong \frac{M}{K} \cong N$ is weakly locally artinian supplemented by the hypothesis. Then, by Corollary 5, $\frac{M}{(K\cap T)}$ is weakly locally artinian supplemented as a finite sum of weakly locally artinian supplemented modules. It follows from Lemma 3 that M is weakly locally artinian supplemented.

Suppose that M is weakly locally-artinian supplemented and K is a coclosed locally-artinian submodule in M. Then $K \cap T \ll K$ by [4, Lemma 1.1] and $K \cap T$ is locally artinian by [10, 31.2 (ii)] i.e. K is a locally artinian supplement of T in M. Therefore K is weakly locally artinian supplemented by Proposition 4. □

Recall from [6, Theorem 3.5] that a ring R is semilocal if and only if every R-module with small radical is weakly supplemented. By using Theorem 1, we have the following Proposition.

Proposition 5. Let R be a semilocal ring and M be an R-module. Suppose $N \subseteq M$ such that $\frac{M}{N}$ is finitely generated and $Rad(\frac{M}{N})$ is locally artinian. If N is weakly locally artinian supplemented then M is weakly locally artinian supplemented.

Proof. Suppose $\frac{M}{N}$ is generated by $m_1+N, m_2+N, \ldots, m_n+N$. For the submodule $K = Rm_1 + Rm_2 + \cdots + Rm_n$, we have $M = N + K$. Then M is weakly locally artinian supplemented by Corollary 4. □

3. Conclusion

The aim of this paper is to reveal the existence of the concept of weakly locally artinian supplemented modules. Our results improve and generalize some known results on locally artinian supplemented modules.

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RULED SURFACE WITH CONSTANT SLOPE ACCORDING TO OSCULATING PLANE OF BASE CURVE IN GALILEAN 3−SPACE

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Abstract. The object of this paper is to investigate the properties of the ruled surface which direction vector has a constant slope with osculating plane of the base curve in Galiean 3−space. We obtain some properties of this kind of ruled surface by calculating the geometric invariants. Also, we give an application on the example and their graphs are visualized by using the Mathematica program.

1. Introduction

Inertial reference frame is defined as a coordinate system moving at a constant velocity. In 1632, Galileo first described the principle "the laws of motion are the same in all inertial frames" using the example of a ship travelling at constant velocity. According to this principle, any observer below the deck would not be able to tell whether the ship was moving or stationary. The Galilean transformation between two inertial frames (x, y, z) and (x', y', z') is defined as

$$\begin{aligned} x' &= a + x, \\ y' &= b + cx + (\cos\varphi)\, y + (\sin\varphi)\, z, \\ z' &= d + ex - (\sin\varphi)\, y + (\cos\varphi)\, z, \end{aligned}$$

where a, b, c, d, e, and φ are some constants. In Galilean space, since two inertial frames are related by a Galilean transformation, all physical laws are the same in all inertial reference frames.

In differential geometry, various surfaces have been extensively studied by the authors in the special spaces: extrinsically and intrinsically [3, 5, 6, 7, 8, 11, 12]. Ruled surface is one of these surfaces and is defined as a surface formed by moving the generating vector along a base curve [14]. Many authors studied on the characaterizations of the ruled surfaces [1, 4, 9, 10, 13].

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In this paper, we investigate the ruled surface whose generator vector has a constant slope according to osculating plane of the base curve in Galilean 3−space and we obtained some important results of this ruled surface. Also, we give some properties of this kind of ruled surface using its invariant curvatures. Finally, we present an example of such a ruled surface in Galilean 3−space.

2. Preliminaries

The standard metric of Galilean 3−space $\mathbb{G}_3$ is defined as

$$\langle x,y\rangle=\begin{cases} x_1y_1, & x_1\neq 0 \text{ or } y_1\neq 0\\ x_2y_2+x_3y_3, & x_1=0=y_1,\end{cases} \tag{2.1}$$

where x_i and y_j $(i,j=1,2,3)$ are shown the coefficients of the vectors x and y, respectively. The cross product in Galilean 3−space is defined by

$$x\times y=\begin{cases} \begin{vmatrix} 0 & e_2 & e_3\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix}, & x_1\neq 0 \text{ or } y_1\neq 0,\\ \\ \begin{vmatrix} e_1 & e_2 & e_3\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix}, & x_1=0=y_1,\end{cases} \tag{2.2}$$

[12].

Let $\alpha: I\subset\mathbb{R}\to\mathbb{G}_3$ be a unit speed curve with the parametrization $\alpha(s)=(s,y(s),z(s))$. The Frenet frame is defined $\{T(s)=\alpha'(s),N(s),B(s)\}$ for the curve $\alpha(s)$ in Galilean 3−space. The Frenet equations are given by

$$\begin{aligned} T'(s) &= \kappa(s)N(s),\\ N'(s) &= \tau(s)B(s),\\ B'(s) &= -\tau(s)N(s),\end{aligned} \tag{2.3}$$

with the curvature $\kappa(s)=\|\alpha''(s)\|$ and the torsion $\tau(s)=\frac{1}{\kappa^2(s)}\det(\alpha',\alpha'',\alpha''')$ [2].

Let $X(u,v)=(x(u,v),y(u,v),z(u,v))$ be a parametric surface in Galilean 3−space. The interior geometry of the parametric surface $X(u,v)$ at the point $X(u_0,v_0)$ is obtained by the first fundamental form. The first fundamental form of the surface is

$$I=(g_1du+g_2dv)^2+\varepsilon(h_{uu}du^2+2h_{uv}dudv+h_{vv}dv^2) \tag{2.4}$$

where $g_1:=x_u=\frac{\partial x}{\partial u}$, $g_2:=x_v$, $h_{uv}:=y_uy_v+z_uz_v$, $h_{uu}:=y_u^2+z_u^2$, $h_{vv}:=y_v^2+z_v^2$, and

$$\varepsilon=\begin{cases} 0, & \text{if the direction } du:dv \text{ is non-isotropic},\\ 1, & \text{if the direction } du:dv \text{ is isotropic}.\end{cases}$$

The Gauss map of the surface $X(u,v)$ is defined as

$$U=\frac{1}{W}(0,-x_uz_v+x_vz_u,x_uy_v-x_vy_u) \tag{2.5}$$

where $W = \sqrt{(x_u z_v - x_v z_u)^2 + (x_v y_u - x_u y_v)^2}$. The second fundamental form is given by

$$II = L_{11}(du)^2 + 2L_{12}dudv + L_{22}(dv)^2$$

where

$$L_{ij} = \frac{1}{g_1} \langle g_1 (0, y_{,ij}, z_{,ij}) + g_{i,j} (0, y_u, z_u), U \rangle \text{ for } g_1 \neq 0 \tag{2.6}$$

or

$$L_{ij} = \tfrac{1}{g_2} \langle g_2 (0, y_{,ij}, z_{,ij}) + g_{i,j} (0, y_v, z_v), U \rangle \text{ for } g_2 \neq 0$$

where $y_{,ij} = \frac{\partial y}{\partial u_i u_j}$, $j = 1, 2$ and $u_1 := u$, $u_2 := v$. The invariant curvatures K and H of the surface are calculated as:

$$K := \frac{L_{11}L_{22} - L_{12}^2}{W^2} \text{ and } H := \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{2W^2} \tag{2.7}$$

where K, H are called as Gaussian curvature and mean curvature of the surface, respectively. A surface in Galilean $3-$space is called as flat (resp. minimal) surface if its Gaussian (resp. mean) curvature is zero [3, 11]. The principal curvatures k_1 and k_2 of the surface X are given as

$$k_1 = 2H \text{ and } k_2 = \frac{L_{11}L_{22} - L_{12}^2}{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}. \tag{2.8}$$

3. Constant Slope Ruled Surface in Galilean 3−space

In this section, we will analyze the properties of the ruled surfaces whose director vector make a constant slope with the osculating plane of the base curve α. Then, we will obtain some properties of this kind of surfaces.

In Galilean $3-$space, we construct the ruled surface with constant slope according to the osculating plane of the base curve as follows:

$$X(s, \lambda) = \alpha(s) + \lambda D(s) \tag{3.1}$$

where $\alpha(s) = (s, y(s), z(s))$ is the director curve and $D(s) = \cos(\theta(s)) T(s) + \sin(\theta(s)) N(s) + \omega B(s)$ is the generator vector of the ruled surface $X(s, \lambda)$. The coefficients of the principal fundamental form are given by

$$\begin{gathered} g_1 = 1 - \lambda\theta'(s)\sin(\theta(s)),\ g_2 = \cos(\theta(s)), \\ g_{1,1} = \tfrac{d}{ds}(1 - \lambda\theta'(s)\sin(\theta(s))), g_{1,2} = -\theta'(s)\sin(\theta(s)),\ \text{and } g_{2,2} = 0. \end{gathered} \tag{3.2}$$

The normal vector U of the surface $X(s, \lambda)$ is calculated as

$$\begin{aligned} U = &\frac{1}{\kappa W}\{0, -A(z''\sin\theta + \omega y'') + \lambda\cos\theta\,(Bz'' + \tau\sin\theta y''), \\ &A(y''\sin\theta - \omega z'') - \lambda\cos\theta\,(By'' - \tau\sin\theta z'')\} \end{aligned} \tag{3.3}$$

where $W = \left(A^2(\sin^2\theta + \omega^2) + \lambda^2\cos^2\theta(B^2 + \tau^2\sin^2\theta)\right)^{1/2}$, $A = 1 - \lambda\theta'\sin\theta$, and $B = \kappa\cos\theta + \theta'\cos\theta - \omega\tau$. The coefficients of the second fundamental form are calculated as follows:

$$L_{11}=\frac{1}{\kappa W}\left\{\begin{array}{c}\left[-A(z''\sin\theta+\omega y'')+\lambda\cos\theta\left(Bz''+\tau\sin\theta y''\right)\right]\\ \left[2A_sy'+Ay''+\left(\frac{\lambda A_s}{\kappa A}-\frac{\lambda\kappa'}{\kappa^2}\right)\left(By''-\tau\sin\theta z''\right)+\frac{\lambda}{\kappa}\left(By''-\tau\sin\theta z''\right)_s\right]\\ +\left[A(y''\sin\theta-\omega z'')-\lambda\cos\theta\left(By''-\tau\sin\theta z''\right)\right]\\ \left[2A_sz'+Az''+\left(\frac{\lambda A_s}{\kappa A}-\frac{\lambda\kappa'}{\kappa^2}\right)\left(Bz''+\tau\sin\theta y''\right)+\frac{\lambda}{\kappa}\left(Bz''+\tau\sin\theta y''\right)_s\right]\end{array}\right\},$$

$$L_{12}=\frac{1}{\kappa WA}\left\{\begin{array}{c}\left[-A(z''\sin\theta+\omega y'')+\lambda\cos\theta\left(Bz''+\tau\sin\theta y''\right)\right]\\ \left[-2Ay'\theta'\sin\theta+\frac{1}{\kappa}(A-\lambda\theta'\sin\theta)\left(By''-\tau\sin\theta z''\right)\right]\\ +\left[A(y''\sin\theta-\omega z'')-\lambda\cos\theta\left(By''-\tau\sin\theta z''\right)\right]\\ \left[-2Az'\theta'\sin\theta+\frac{1}{\kappa}(A-\lambda\theta'\sin\theta)\left(Bz''+\tau\sin\theta y''\right)\right],\end{array}\right\},$$

and $L_{22}=0$, where the derivatives are taken according to the parameter s.

Now, we will examine some properties of the ruled surfaces with constant slope relative to special curves in Galilean space.

Case 1. If the base curve $\alpha(s)$ of the ruled surface is a **plane** curve with the parametric equation $\alpha(s)=(s,y(s),0)$, then the ruled surface with constant slope is given by

$$X(s,\lambda)=\alpha(s)+\lambda D(s)$$

where $D(s)=\cos\theta(s)T(s)+\sin\theta(s)N(s)+\omega\overrightarrow{e}_3$ is the generator vector, ω is an arbitrary constant, and $\overrightarrow{e}_3=(0,0,1)$. The Gauss map of the ruled surface is calculated as follows:

$$U(s,\lambda)=\frac{1}{W}(0,-\omega(1-\lambda\theta'\sin\theta),(1-\lambda\theta'\sin\theta)\sin\theta-\lambda\cos^2\theta(\theta'+\kappa))$$

where $W=\left(\omega^2(1-\lambda\theta'\sin\theta)^2+(\sin\theta-\lambda\theta'-\lambda\kappa\cos^2\theta)^2\right)^{1/2}$.

Theorem 3.1. *The ruled surface $X(s,\lambda)$ with the base curve $\alpha(s)=(s,y(s),0)$ is developable if and only if the following equation is satisfied*

$$(1-\lambda\theta'\sin\theta)(\cos\theta(\theta'+\kappa)-2\theta'\sin\theta y')-\lambda\theta'\sin\theta\cos\theta(\theta'+\kappa)=0.$$

Proof. From Eq. (2.6), the coefficients of the first and second fundamental forms of $X(s,\lambda)$ are calculated as

$$g_1=1-\lambda\theta'\sin\theta \qquad \text{and} \qquad g_2=\cos\theta,$$

$$(3.4)\quad L_{11}=-\frac{\omega}{W}(1-\lambda\theta'\sin\theta)\left(\begin{array}{c}(1-\lambda\theta'\sin\theta)\kappa+2y'\frac{d(1-\lambda\theta'\sin\theta)}{ds}\\ -\lambda\theta'\sin\theta(\theta'+\kappa)+\lambda\cos\theta(\theta''+\kappa')\\ +\frac{\lambda\cos\theta(\theta'+\kappa)\frac{d(1-\lambda\theta'\sin\theta)}{ds}}{1-\lambda\theta'\sin\theta}\end{array}\right),$$

$$L_{12}=-\frac{\omega}{W}(1-\lambda\theta'\sin\theta)\left(\begin{array}{c}-2\theta'\sin\theta y'+\cos\theta(\theta'+\kappa)\\ -\frac{\lambda\theta'\cos\theta\sin\theta(\theta'+\kappa)}{1-\lambda\theta'\sin\theta}\end{array}\right),$$

$$L_{22}=0.$$

The Gauss curvature of the surface $X(s,\lambda)$ is

$$K=-\frac{\omega^2}{W^4}(1-\lambda\theta'\sin\theta)^2\left(\begin{array}{c}-2\theta'\sin\theta y'+\cos\theta(\theta'+\kappa)\\ -\frac{\lambda\theta'\cos\theta\sin\theta(\theta'+\kappa)}{1-\lambda\theta'\sin\theta}\end{array}\right)^2.$$

The ruled surface $X(s,\lambda)$ with the base curve $\alpha(s)=(s,y(s),0)$ is to be developable, its Gauss curvature must be zero. So, we obtain the desired result. □

Theorem 3.2. *The ruled surface $X(s,\lambda)$ with the base curve $\alpha(s)=(s,y(s),0)$ is minimal if and only if*

$$\cos\theta\begin{bmatrix} -(1-\lambda\theta'\sin\theta)\kappa \\ +2y'\frac{d(1-\lambda\theta'\sin\theta)}{ds} \\ +\lambda\theta'\sin\theta(\theta'+\kappa) \\ +\lambda\cos\theta(\theta''+\kappa') \\ +\frac{\lambda\cos\theta(\theta'+\kappa)\frac{d(1-\lambda\theta'\sin\theta)}{ds}}{1-\lambda\theta'\sin\theta}\end{bmatrix}-2\theta'(1-\lambda\theta'\sin\theta)\left(-2\sin\theta y'+\cos\theta\right)=0.$$

Proof. If we substitute the components of Eq. (3.4) into the Eq. (2.7), we obtain the mean curvature of the ruled surface $X(s,\lambda)$. For the surface to be minimal, its mean curvature is equal to zero. So, we get the desired differential equation for $1-\lambda\theta'\sin\theta\neq 0$. □

Corollary 3.3. *If the function θ is a constant, then the ruled surface $X(s,\lambda)$ has the Gauss curvature and the mean curvature as follows:*

$$K=-\frac{\omega^2}{W^4}(\kappa^2(s)\cos^2\theta)\qquad \text{and}\qquad H=-\frac{\omega\cos^2\theta}{2W^3}(-\kappa(s)+\lambda\kappa'(s)\cos\theta).$$

Corollary 3.4. *If the function θ is a constant and the surface $X(s,\lambda)$ is developable, then the base curve is the straight line in Galilean space.*

Corollary 3.5. *If the function θ is a constant and the surface $X(s,\lambda)$ is minimal, then the base curve α has the curvature $\kappa(s)=e^{s/\lambda\cos\theta}+c$.*

There exists a common perpendicular to two constructive rulings in the ruled surface, then the foot of the common perpendicular on the main rulings is called a central point. The locus of the central point is called the striction curve.

Theorem 3.6. *The following conditions are satified for the striction curve of the ruled surface $X(s,\lambda)$:*
(i) If the function θ is an arbitrary constant, then the striction curve is
$\beta(s)=\left(s-\frac{1}{\kappa(s)},y(s)-\frac{1}{\kappa(s)\cos\theta}(\cos\theta y'(s)+\sin\theta),-\frac{\omega}{\kappa(s)\cos\theta}\right)$.
(ii) If the function θ is not a constant, then the striction curve is
$\beta(s)=\left(s-\frac{1}{(\theta'(s))^2}\cot^2\theta(s),y(s)-\frac{\cos\theta(s)}{(\theta'(s))^2\sin^2\theta(s)}(\cos\theta(s)y'(s)+\sin\theta(s)),-\frac{\omega\cos\theta}{(\theta'(s))^2\sin^2\theta(s)}\right)$.

Proof. The striction curve of the ruled surface is two types depending on whether D' is isotropic, or non-isotropic vector in Galilean space. The striction curve of the ruled surface is calculated the following formula

$$\beta(s)=\alpha(s)-\frac{\langle T(s),D(s)\rangle}{\langle D'(s),D'(s)\rangle}D(s).$$

If the derivative of the generator vector with respect to s is an isotropic vector, then the striction curve is calculated in (i) and if $D'(s)$ is a non isotropic vector, then the striction curve is calculated as in (ii). □

Case 2. If the base curve $\alpha(s)$ of the ruled surface is a **plane** curve with the parametric expression $\alpha(s)=(s,0,z(s))$, then the ruled surface with constant slope is given by

$$X(s,\lambda)=\alpha(s)+\lambda D(s)$$

where $D(s) = \cos\phi(s)T(s) + \sin\phi(s)N(s) - \sigma\overrightarrow{e}_2$ is the generator vector, σ is an arbitrary constant, and $\overrightarrow{e}_2 = (0,1,0)$. The Gauss map of the ruled surface is calculated as follows:

$$U(s,\lambda) = \frac{1}{W}(0, -(1-\lambda\phi'\sin\phi)\sin\phi - \lambda\cos^2\phi(\phi'+\kappa), -\sigma(1-\lambda\phi'\sin\phi))$$

where $W = \left(\sigma^2(1-\lambda\phi'\sin\phi)^2 + ((1-\lambda\phi'\sin\phi)\sin\phi + \lambda\cos^2\phi(\phi'+\kappa))^2\right)^{1/2}$.

Theorem 3.7. *The ruled surface $X(s,\lambda)$ with the base curve $\alpha(s) = (s,0,z(s))$ is developable if and only if the differential equation is satisfied*

$$(1-\lambda\phi'\sin\phi)(\cos\phi(\phi'+\kappa) - 2\phi'(\sin\phi)z') - \lambda\phi'\sin\phi\cos\phi(\phi'+\kappa) = 0.$$

Proof. From Eq.(2.6), the coefficients of the first and second fundamental forms of $X(s,\lambda)$ are calculated as

$$g_1 = 1-\lambda\phi'\sin\phi \qquad \textit{and} \qquad g_2 = \cos\phi,$$

$$\begin{aligned}
(3.5)\; L_{11} &= -\frac{\sigma}{W}(1-\lambda\phi'\sin\phi)\begin{pmatrix} (1-\lambda\phi'\sin\phi)\kappa + 2z'\frac{d(1-\lambda\phi'\sin\phi)}{ds} \\ -\lambda\phi'\sin\phi(\phi'+\kappa) + \lambda\cos\phi(\phi''+\kappa') \\ +\frac{\lambda\cos\phi(\phi'+\kappa)\frac{d(1-\lambda\phi'\sin\phi)}{ds}}{1-\lambda\phi'\sin\phi} \end{pmatrix}, \\
L_{12} &= -\frac{\sigma}{W}(1-\lambda\phi'\sin\phi)\begin{pmatrix} -2\phi'\sin\phi z' + \cos\phi(\phi'+\kappa) \\ -\frac{\lambda\phi'\cos\phi\sin\phi(\phi'+\kappa)}{1-\lambda\phi'\sin\phi} \end{pmatrix}, \\
L_{22} &= 0.
\end{aligned}$$

The Gauss curvature of the surface $X(s,\lambda)$ is

$$K = -\frac{\sigma^2}{W^4}(1-\lambda\phi'\sin\phi)^2\begin{pmatrix} -2\phi'\sin\phi z' + \cos\phi(\phi'+\kappa) \\ -\frac{\lambda\phi'\cos\phi\sin\phi(\phi'+\kappa)}{1-\lambda\phi'\sin\phi} \end{pmatrix}^2.$$

The ruled surface $X(s,\lambda)$ with the base curve $\alpha(s) = (s,0,z(s))$ is to be developable, its Gauss curvature must be zero. So, we obtain the desired result. □

Theorem 3.8. *The ruled surface $X(s,\lambda)$ with the base curve $\alpha(s) = (s,0,z(s))$ is minimal if and only if the following differential eqaution is fulfilled*

$$\cos\phi\begin{bmatrix} -(1-\lambda\phi'\sin\phi)\kappa \\ +2z'\frac{d(1-\lambda\phi'\sin\phi)}{ds} \\ +\lambda\phi'\sin\phi(\phi'+\kappa) \\ +\lambda\cos\phi(\phi''+\kappa') \\ +\frac{\lambda\cos\phi(\phi'+\kappa)\frac{d(1-\lambda\phi'\sin\phi)}{ds}}{1-\lambda\phi'\sin\phi} \end{bmatrix} - 2\phi'(1-\lambda\phi'\sin\phi)(-2\sin\phi z' + \cos\phi) = 0.$$

Proof. If we substitute the components of Eq. (3.5) into the Eq. (2.7), we obtain the mean curvature of the ruled surface $X(s,\lambda)$. For the surface to be minimal, its mean curvature is zero. From here, we get the desired differential equation for $1-\lambda\phi'\sin\phi \neq 0$. □

Corollary 3.9. *If the function ϕ is a constant, then the ruled surface $X(s,\lambda)$ has the Gauss curvature and the mean curvature as follows:*

$$K = -\frac{\sigma^2}{W^4}(\kappa^2(s)\cos^2\phi) \qquad \textit{and} \qquad H = -\frac{\sigma\cos^2\phi}{2W^3}(-\kappa(s) + \lambda\kappa'(s)\cos\phi).$$

Corollary 3.10. *If the function ϕ is a constant and the surface $X(s,\lambda)$ is developable, then the base curve is the straight line in Galilean space.*

Corollary 3.11. *If the function ϕ is a constant and the surface $X(s,\lambda)$ is minimal, then the base curve α has the curvature $\kappa(s) = e^{s/\lambda\cos\phi} + c_1$.*

Theorem 3.12. *The following conditions are satified for the striction curve of the ruled surface $X(s,\lambda)$:*

(i) If the function ϕ is an arbitrary constant, then the striction curve is $\beta(s) = \left(s - \frac{1}{\kappa(s)}, \frac{\sigma}{\kappa(s)\cos\phi}, z(s) - \frac{1}{\kappa(s)\cos\phi}(\cos\phi z'(s) + \sin\phi)\right)$.

(ii) If the function ϕ is not a constant, then the striction curve is $\beta(s) = \left(s - \frac{1}{(\phi'(s))^2}\cot^2\phi(s), \frac{\sigma\cos\phi}{(\phi'(s))^2\sin^2\phi(s)}, z(s) + \frac{\cos\phi(s)}{(\phi'(s))^2\sin^2\phi(s)}(\cos\phi(s)z'(s) + \sin\phi(s))\right)$.

Proof. The striction curve of ruled surface is two types depending on whether the vector D' is isotropic or non-isotropic in Galilean space. The striction curve of the ruled surface is calculated from the formula

$$\beta(s) = \alpha(s) - \frac{\langle T(s), D(s)\rangle}{\langle D'(s), D'(s)\rangle} D(s).$$

If the derivative of the generator vector with respect to s is isotropic vector, then the striction curve is calculated in (i) and if $D'(s)$ is non isotropic vector, then the striction curve is calculated as in (ii). □

Example

In this section, we give the ruled surface whose base curve is $\alpha(s)$ and generator vector $D(s)$. Let $\alpha = \alpha(s)$ be an admissible unit speed curve with the parametrization

$$\alpha(s) = \left(s, \frac{1}{4}\left[(3-4s)\cos(2\sqrt{s}) + 6\sqrt{s}\sin(2\sqrt{s})\right], \frac{1}{4}\left[(3-4s)\sin(2\sqrt{s}) - 6\sqrt{s}\cos(2\sqrt{s})\right]\right)$$

with the Frenet frame apparatus

$$\begin{aligned} T(s) &= \left(1, \frac{1}{2}\cos(2\sqrt{s}) + \sqrt{s}\sin(2\sqrt{s}), \frac{1}{2}\sin(2\sqrt{s}) - \sqrt{s}\cos(2\sqrt{s})\right), \\ N(s) &= \left(0, \cos(2\sqrt{s}), \sin(2\sqrt{s})\right), \\ B(s) &= \left(0, -\sin(2\sqrt{s}), \cos(2\sqrt{s})\right), \end{aligned}$$

$\kappa(s) = 1$ and $\tau(s) = 1/\sqrt{s}$. The ruled surface $X(s,\lambda)$ with constant slope according to osculating plane of the curve $\alpha(s)$ is given as follows

$$X(s,\lambda) = \alpha(s) + \lambda D(s),$$

where $D(s) = \cos s^3 T(s) + \sin s^3 N(s) + \omega B(s)$ and ω is an arbitrary constant in Figure 1.

4. Conclusion

This study is important in terms of finding invariants of the ruled surface with constant slope in Galilean 3−space. The striction curve of this surface is calculated. Also, the conditions for the surface to be minimal and developable are obtained. It is also examined the special cases and the results are obtained.

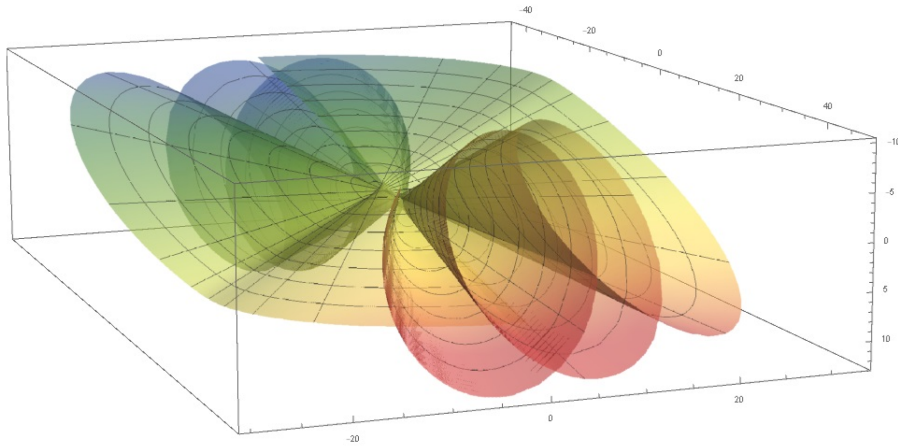

FIGURE 1. The ruled surface with constant slope $\omega = 3$.

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BLOW UP AT INFINITY OF WEAK SOLUTIONS FOR A HIGHER-ORDER PARABOLIC EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. The main goal of this work is to study the inital boundary value problem for a higher-order parabolic equation with logarithmic source term

$$u_t + (-\Delta)^m u = u \ln |u| .$$

We obtain blow-up at $+\infty$ of weak solutions, by employing potential well technique. This improves and extends some previous studies.

1. Introduction

In this paper, we cosider the following higher-order parabolic problem with logarithmic nonlinearity

$$\begin{cases} u_t + Au = u \ln |u|, & x \in \Omega, \quad t > 0, \\ D^\gamma u(x,t) = 0, \quad |\gamma| \le m-1, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $A = (-\Delta)^m$, $m \ge 1$ a positive integer, Ω is a bound domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$ is multi-index, γ_i $(i = 1, 2, ..., n)$ are non-negative integers, $|\gamma| = \gamma_1 + \gamma_2 + ... + \gamma_n$, $D^\gamma = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \partial x_2^{\gamma_2} ... \partial x_n^{\gamma_n}}$ are multi-index derivative operator, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator.

When $m = 1$, the equation (1.1) becomes a heat equation as follows

$$u_t - \Delta u = u \ln |u| . \tag{1.2}$$

In the equation (1.2), Chen et al. [2] obtained under some suitable conditions for the global existence, decay estimate and blow-up at $+\infty$ of weak solutions, via the logarithmic Sobolev inequality and potential well technique. Also, Han [5] obtained the blow-up at infinity of solutions, via the logarithmic Sobolev inequality. Additionally, Chen and Tian [3] obtained the global existence of solution, blow-up at $+\infty$ of solution, by adding strong damping term to the equation (1.2).

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Peng and Zhou [10] studied the following semilinear heat equation with logarithmic nonlinearity

$$u_t - \Delta u = u^{p-2} u \ln |u|\,,$$

where $2 < p$. They studied the existence of the unique global weak solutions and blow-up in the finite time of weak solutions, via potential well technique and energy technique.

Li and Liu [8] established a class of fourth-order parabolic equation with logarithmic source term as follows

$$u_t + \Delta^2 u = u^{p-2} u \ln |u|\,,$$

where $2 < p$. They studied the existence of global solutions, by using potential well technique. In addition, they also studied result of decay and finite time blow-up for weak solutions.

Nhan and Truong [9] studied the following nonlinear pseudo-parabolic equation

$$u_t - \Delta u_t - div\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p-2} u \log |u|\,.$$

They obtained results as regard the existence or non-existence of global solutions. Also, He et al. [6] proved the decay and the finite time blow-up for weak solutions of the equation.

Resently many other authors investigated higher-order hyperbolic and parabolic type equation [4, 7, 11, 12, 13, 14, 15]. Ishige et al. [7] studied the Cauchy problem for nonlinear higher-order heat equation as follows

$$u_t + (-\Delta)^m u = |u|^p\,.$$

They obtained existence of solutions of the Cauchy problem by introducing a new majorizing kernel. In addition, they studied the local existence of solutions under the different conditions.

Xiao and Li [13] considered initial boundary value problem for nonlinear higher-order heat equations of

$$u_t + (-\Delta)^m u_t + (-\Delta)^m u = f(u).$$

They established the existence of weak solution to the static problem, by using the potential well technique.

The remainder of our work is organized as follows. In Section2, some important Lemmas are given. In Section 3, the main result is proved.

2. Preliminaries

Let $\|u\|_{H^m(\Omega)} = \left(\sum\limits_{|\gamma|\leq m} \|D^\gamma u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$ denote $H^m(\Omega)$ norm, let $H_0^m(\Omega)$ denote the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. Let $\|.\|_r$ and $\|.\|$ denote the usual $L^r(\Omega)$ norm and $L^2(\Omega)$ norm.

For $u \in H_0^m(\Omega)\backslash\{0\}$, we define the energy functional

$$J(u) = \frac{1}{2}\left\|A^{\frac{1}{2}}u\right\|^2 - \frac{1}{2}\int_\Omega |u|^2 \ln |u|\,dx + \frac{1}{4}\|u\|^2\,, \tag{2.1}$$

and Nehari functional

$$I(u) = \left\|A^{\frac{1}{2}}u\right\|^2 - \int_\Omega |u|^2 \ln |u|\,dx. \tag{2.2}$$

By (2.1) and (2.2), we obtain

$$J(u)=\frac{1}{2}I(u)+\frac{1}{4}\|u\|^2. \tag{2.3}$$

Further, let

$$d=\inf_{u\in\mathcal{N}}J(u), \tag{2.4}$$

denote the potential depth, where $\mathcal{N}$ is the Nehari manifold, which is defined by

$$\mathcal{N}=\{u\in H_0^m(\Omega)\backslash\{0\}:I(u)=0\}.$$

Lemma 2.1. [1]. *Let k be a number with $2\leq k<+\infty$, $n\leq 2m$ and $2\leq k\leq\frac{2n}{n-2m}$, $n>2m$. Then there is a constant C depending*

$$\|u\|_k\leq C\left\|A^{\frac{1}{2}}u\right\|,\quad \forall u\in H_0^m(\Omega).$$

Lemma 2.2. *$J(t)$ is a nonincreasing function for $t\geq 0$ and*

$$J'(u)=-\int\limits_\Omega u_t^2dx\leq 0. \tag{2.5}$$

Proof. Multiplying the equation (1.1) by u_t and integrating on Ω, we get

$$\int\limits_\Omega u_t^2dx+\int\limits_\Omega Auu_tdx=\int\limits_\Omega uu_t\ln|u|\,dx.$$

By straightforward calculation, we obtain

$$\int\limits_\Omega u_t^2dx+\frac{1}{2}\frac{d}{dt}\left\|A^{\frac{1}{2}}u\right\|^2=\frac{1}{2}\frac{d}{dt}\int_\Omega|u|^2\ln|u|\,dx-\frac{1}{4}\frac{d}{dt}\|u\|^2,$$

which yields that

$$\frac{1}{2}\frac{d}{dt}\left\|A^{\frac{1}{2}}u\right\|^2-\frac{1}{2}\frac{d}{dt}\int_\Omega|u|^2\ln|u|\,dx+\frac{1}{4}\frac{d}{dt}\|u\|_2^2=-\int\limits_\Omega u_t^2dx.$$

Thus, we get

$$\frac{d}{dt}\left(\frac{1}{2}\left\|A^{\frac{1}{2}}u\right\|^2-\frac{1}{2}\int_\Omega|u|^2\ln|u|\,dx+\frac{1}{4}\|u\|_2^2\right)=-\int\limits_\Omega u_t^2dx. \tag{2.6}$$

By 2.1 and 2.6, we obtain

$$\frac{d}{dt}J(u)=-\int\limits_\Omega u_t^2dx. \tag{2.7}$$

Moreover, Integrating (2.7) with respect to t on $[0,t]$, we arrive at

$$\int_0^t\|u_s(s)\|^2ds+J(u(t))=J(u_0). \tag{2.8}$$

□

Lemma 2.3. *Let $u\in H_0^m(\Omega)\backslash\{0\}$ and $j(\lambda)=J(\lambda u)$. Then we get*

(i) $\lim_{\lambda\to 0^+}j(\lambda)=0$ *and* $\lim_{\lambda\to+\infty}j(\lambda)=-\infty$,

(ii) *there is a unique $\lambda^*>0$ such that $j'(\lambda^*)=0$,*

(iii) $j(\lambda)$ *is increasing on* $(0,\lambda^*)$, *decreasing on* $(\lambda^*,+\infty)$ *and taking the maximum at* λ^*,

(iv) $I(\lambda u)>0$ *for* $\lambda\in(0,\lambda^*)$, $I(\lambda u)<0$ *for* $\lambda\in(\lambda^*,+\infty)$ *and* $I(\lambda^* u)=0$.

Proof. By the definition of j, for $u\in H_0^1(\Omega)\backslash\{0\}$, we get

$$j(\lambda)=\frac{\lambda^2}{2}\left\|A^{\frac{1}{2}}u\right\|^2-\frac{\lambda^2}{2}\int_\Omega |u|^2\ln|u|\,dx-\frac{\lambda^2}{2}\ln\lambda\,\|u\|_2^2+\frac{\lambda^2}{4}\|u\|^2. \tag{2.9}$$

By (2.9), we have

$$\begin{aligned}\frac{d}{d\lambda}j(\lambda) &= \lambda\left\|A^{\frac{1}{2}}u\right\|^2-\lambda\int_\Omega |u|^2\ln|u|\,dx\\ &\quad -\lambda\ln\lambda\,\|u\|^2-\frac{\lambda}{2}\|u\|^2+\frac{\lambda}{2}\|u\|^2\\ &= \lambda\left(\left\|A^{\frac{1}{2}}u\right\|^2-\int_\Omega |u|^2\ln|u|\,dx-\ln\lambda\,\|u\|^2\right).\end{aligned}$$

Moreover, by taking

$$\lambda^*=\lambda^*(u)=\exp\left(\frac{\left\|A^{\frac{1}{2}}u\right\|^2-\int_\Omega |u|^2\ln|u|\,dx}{\|u\|^2}\right)$$

By (2.2), we get

$$\begin{aligned}I(\lambda u) &= \left\|A^{\frac{1}{2}}(\lambda u)\right\|^2-\int_\Omega |\lambda u|^2\ln|\lambda u|\,dx\\ &= \lambda^2\left\|A^{\frac{1}{2}}u\right\|^2-\lambda^2\int_\Omega |u|^2\ln|u|\,dx-\lambda^2\ln\lambda\,\|u\|^2\\ &= \lambda j'(\lambda).\end{aligned}$$

So, $I(\lambda u)>0$ for $\lambda\in(0,\lambda^*)$, $I(\lambda u)<0$ for $\lambda\in(\lambda^*,+\infty)$ and $I(\lambda^* u)=0$. Therefore, the proof is completed. □

Lemma 2.4. *d defined by (2.4) is positive and there exists a positive function $u\in\mathcal{N}$ such that $J(u)=d$.*

Proof. Let $\{u_r\}_r^\infty\subset\mathcal{N}$ be a minimizing sequence for J, which means that

$$\lim_{r\to\infty}J(u_r)=d. \tag{2.10}$$

We can easy show that $\{|u_r|\}_r\subset\mathcal{N}$ is also a minimizing sequence for J due to $|u_r|\in\mathcal{N}$ and $J(|u_r|)=J(u_r)$. Therefore, we can suppose that $u_r>0$ a.e. Ω for all $r\in N$.

Moreover, we have already observed that J is coercive on $\mathcal{N}$ which satisfies that $\{u_r\}_r^\infty$ is bounded in $H_0^m(\Omega)$. Let $\mu>0$ be small enough such that $2+\mu<\frac{2n}{n-2}$. Since $H_0^m(\Omega)\hookrightarrow L^{2+\mu}(\Omega)$ is compact, so there exists a function u and a subsequence of $\{u_r\}_r^\infty$, still denote by $\{u_r\}_r^\infty$, such that

$$u_r\to u \ \text{ weakly in } \ H_0^m(\Omega),$$

$$u_r\to u \ \text{ strongly in } \ L^{2+\mu}(\Omega),$$

$$u_r(x)\to u(x) \ \text{ a.e. in } \ \Omega.$$

Also, $u \geq 0$ a.e. in Ω. First, we prove $u \neq 0$. From the dominated convergence theorem, we have

$$\int_\Omega |u|^2 \ln |u| \, dx = \lim_{r\to\infty} \int_\Omega |u_r|^2 \ln |u_r| \, dx, \tag{2.11}$$

and

$$\int_\Omega |u|^2 \, dx = \lim_{r\to\infty} \int_\Omega |u_r|^2 \, dx. \tag{2.12}$$

From the weak lower semicontinuity of $H_0^m(\Omega)$, we get

$$\left\| A^{\frac{1}{2}} u \right\|^2 \leq \liminf_{r\to\infty} \left\| A^{\frac{1}{2}} u_r \right\|^2. \tag{2.13}$$

Then it follows from (2.1), (2.10), (2.11), (2.12) and (2.13) that

$$\begin{aligned} J(u) &= \frac{1}{2} \left\| A^{\frac{1}{2}} u \right\|^2 - \frac{1}{2} \int_\Omega |u|^2 \ln |u| \, dx + \frac{1}{4} \|u\|^2 \\ &\leq \liminf_{r\to\infty} \frac{1}{2} \left\| A^{\frac{1}{2}} u_r \right\|^2 - \lim_{r\to\infty} \frac{1}{2} \int_\Omega |u_r|^2 \ln |u_r| \, dx + \lim_{r\to\infty} \frac{1}{4} \|u_r\|^2 \\ &= \liminf_{r\to\infty} \left(\frac{1}{2} \left\| A^{\frac{1}{2}} u_r \right\|^2 - \frac{1}{2} \int_\Omega |u_r|^2 \ln |u_r| \, dx + \frac{1}{4} \|u_r\|^2 \right) \\ &= \liminf_{r\to\infty} J(u_r) = d. \end{aligned} \tag{2.14}$$

Using (2.2), (2.11) and (2.13), we have

$$\begin{aligned} I(u) &= \left\| A^{\frac{1}{2}} u \right\|^2 - \int_\Omega |u|^2 \ln |u| \, dx \\ &\leq \liminf_{r\to\infty} \left\| A^{\frac{1}{2}} u_r \right\|^2 - \lim_{r\to\infty} \int_\Omega |u_r|^2 \ln |u_r| \, dx \\ &= \liminf_{r\to\infty} \left(\left\| A^{\frac{1}{2}} u_r \right\|^2 - \int_\Omega |u_r|^2 \ln |u_r| \, dx \right) \\ &= \liminf_{r\to\infty} I(u_r) = 0. \end{aligned} \tag{2.15}$$

Since $u_r \in \mathcal{N}$, we have $I(u_r) = 0$. So, by Lemma 1 and the fact $x^{-\mu} \ln x \leq (e\mu)^{-1}$ for $x \geq 1$, we get

$$\begin{aligned} \left\| A^{\frac{1}{2}} u_r \right\|^2 &= \int_\Omega |u_r|^2 \ln |u_r| \, dx \\ &\leq (e\mu)^{-1} \int_\Omega |u_r|^{2+\mu} \, dx \\ &= (e\mu)^{-1} \|u_r\|_{2+\mu}^{2+\mu} \\ &\leq C \left\| A^{\frac{1}{2}} u_r \right\|_2^{2+\mu}, \end{aligned}$$

where C is Sobolev emdedding constant. This satisfies that

$$\int_\Omega |u_r|^2 \ln |u_r| \, dx = \left\| A^{\frac{1}{2}} u_r \right\|^2 \geq C. \tag{2.16}$$

By (2.11) and (2.16), we have

$$\int_\Omega |u|^2 \ln |u| \, dx \geq C.$$

Thus, we have $u \in H_0^m(\Omega)\backslash\{0\}$.

If $I(u_r) < 0$, from Lemma 3, there exists a λ^* such that $I(\lambda^* u) = 0$ and $0 < \lambda^* < 1$. Thus, $\lambda^* u \in \mathcal{N}$. It follows from (2.3), (2.4), (2.12) and (2.13) that

$$\begin{aligned} d &\leq J(\lambda^* u) \\ &= \frac{1}{2} I(\lambda^* u) + \frac{1}{4} \|\lambda^* u\|^2 \\ &= \frac{(\lambda^*)^2}{4} \|u\|^2 \\ &\leq (\lambda^*)^2 \liminf_{r\to\infty} \frac{1}{4} \|u_r\|^2 \\ &= (\lambda^*)^2 \liminf_{r\to\infty} J(u_r) \\ &= (\lambda^*)^2 d, \end{aligned}$$

which indicates $\lambda^* \geq 1$ by $d > 0$. It contradicts $0 < \lambda^* < 1$. By (2.15), we have $I(u) = 0$. For this reason, $u \in \mathcal{N}$. From (2.10), we have $J(u) \geq d$. From (2.14), we have $J(u) \leq d$. So, $J(u) = d$. □

3. Main results

Definition 3.1. (Maximal Existence Time). Assume that $u(t)$ be weak solutions of problem (1.1). We define the maximal existence time $T_{\max}$ as follows

(i) If $u(t)$ exists for all $0 \leq t < \infty$, then $T_{\max} = +\infty$;
(ii) If there exists a $t_0 \in (0, \infty)$ such that $u(t)$ exists for $0 \leq t < t_0$, but doesn't exists at $t = t_0$, then $T_{\max} = t_0$.

Definition 3.2. (Blow-up at $+\infty$). Let $u(t)$ be a weak solution of (1.1). We call $u(t)$ blow-up at $+\infty$ if the maximal existence time $T_{\max} = +\infty$ and

$$\lim_{t\to+\infty} \|u(t)\|^2 = +\infty.$$

Theorem 3.3. *Assume that $u_0 \in H_0^m(\Omega)\backslash\{0\}$, $J(u_0) < d$ and $I(u_0) < 0$. Let $u(t)$ be a weak solution to the problem (1.1). Then $u(t)$ blows up at $+\infty$ such that*

$$\lim_{t\to+\infty} \|u(t)\|^2 = \infty.$$

Proof. Let $u(t)$ be weak solution of (1.1) with $J(u_0) < d$ and $I(u_0) < 0$. Let $F : [0, \infty) \to \mathbb{R}^+$, and

$$F(t) = \int_0^t \|u(s)\|^2 \, ds. \tag{3.1}$$

Then, a direct calculation gives

$$F'(t) = \|u(t)\|^2 . \tag{3.2}$$

From (2.2) and (3.2), we get

$$\begin{aligned} F''(t) &= 2\int_\Omega u u_t dx \\ &= 2\int_\Omega u^2 \ln|u| \, dx - 2\int_\Omega A u^2 dx \\ &= -2I(u). \end{aligned} \tag{3.3}$$

By (3.2) and (3.3), we get

$$\begin{aligned} F'(t)\ln F'(t) - F''(t) &= \|u(t)\|^2 \ln \|u(t)\|^2 + 2I(u) \\ &= 2\|u(t)\|^2 \ln \|u(t)\| + 2\left\|A^{\frac{1}{2}}u\right\|^2 - 2\int_\Omega |u|^2 \ln |u|\, dx \\ &\geq 0, \end{aligned}$$

which, in turn, yields that

$$(\ln F'(t))' \leq \ln F'(t).$$

This means

$$\ln F'(t) \leq e^t \ln F'(0) = e^t \ln \|u_0\|^2 .$$

Then

$$\|u(t)\|^2 \leq \|u_0\|^{e^t}, \quad \forall t \geq 0,$$

which yields that $u(t)$ does not blow up in finite time.

On the other hand, using the Hölder inequality and combining (3.3), we have

$$\begin{aligned} \frac{1}{4}(F'(t))^2 &= \frac{1}{4}\left(\int_0^t F''(s)ds\right)^2 \\ &= \left(\int_0^t \int_\Omega u u_s dx ds\right)^2 \\ &\leq \int_0^t \|u(s)\|^2 ds \int_0^t \|u_s\|^2 ds. \end{aligned} \tag{3.4}$$

From (2.3) and (3.3), it follows

$$\begin{aligned} F''(t) &= -2I(u) \\ &= -4J(u) + \|u\|^2 \\ &\geq -4J(u_0) + 4\int_0^t \|u_s(s)\|^2 ds + \|u\|^2 . \end{aligned} \tag{3.5}$$

By Lemma 3, there exists a $\lambda^* \in (0,1)$ such that $I(\lambda^* u(t)) = 0$. Thus, by the definition of d, it follows that

$$\begin{aligned} d &= \inf_{u\in\mathcal{N}} J(u) \leq J(\lambda^* u(t)) \\ &= \frac{1}{2}I(\lambda^* u(t)) + \frac{1}{4}\|\lambda^* u(t)\|^2 \\ &= \frac{1}{2}I(\lambda^* u(t)) + \frac{(\lambda^*)^2}{4}\|u(t)\|^2 \\ &\leq \frac{1}{4}\|u(t)\|^2 . \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\begin{aligned} F''(t) &\geq -4J(u_0) + 4\int_0^t \|u_s(s)\|^2 ds + \|u\|^2 \\ &\geq 4(d - J(u_0)) + 4\int_0^t \|u_s(s)\|^2 ds. \end{aligned} \tag{3.7}$$

Using (3.1), (3.4) and (3.7), we get

$$\begin{aligned} F(t)F''(t) &\geq 4\left(d - J(u_0)\right)F(t) + 4\int_0^t \|u(s)\|^2 \|u_s(s)\|^2 \, ds \\ &\geq 4\left(d - J(u_0)\right)F(t) + \left(F'(t)\right)^2. \end{aligned} \tag{3.8}$$

Then, we see that

$$F(t)F''(t) - \left(F'(t)\right)^2 \geq 4\left(d - J(u_0)\right)F(t).$$

By $J(u_0) < d$ and $I(u) < 0$, then we know

$$F(t)F''(t) - \left(F'(t)\right)^2 > 0.$$

On the other hand, by straightforward calculation, it is clear that

$$(\ln F(t))' = \frac{F'(t)}{F(t)}, \tag{3.9}$$

and

$$(\ln F(t))'' = \frac{F(t)F''(t) - \left(F'(t)\right)^2}{\left(F(t)\right)^2} > 0. \tag{3.10}$$

From (3.10), we know that $(\ln F(t))'$ is increasing with respect to t, using this fact, integrating (3.9) from t_0 to t, we get

$$(\ln F(t))' = (\ln F(t_0))' + \int_{t_0}^t \frac{F(s)F''(s) - \left(F'(s)\right)^2}{\left(F(s)\right)^2} ds,$$

and

$$\begin{aligned} \ln F(t) - \ln F(t_0) &= \int_{t_0}^t (\ln F(s))' \, ds \\ &= \int_{t_0}^t \frac{F'(s)}{F(s)} ds \\ &\geq \frac{F'(t_0)}{F(t_0)}\left(t - t_0\right), \end{aligned}$$

where $0 \leq t_0 \leq t$. Then

$$F(t) \geq F(t_0) \exp\left(\frac{F'(t_0)}{F(t_0)}\left(t - t_0\right)\right).$$

Since $F(0) = 0$ and $F'(0) > 0$, we can take t_0 small enough such that $F'(t_0) > 0$ and $F(t_0) > 0$. Then for sufficiently large t,

$$\begin{aligned} \|u(t)\|^2 &= F'(t) \\ &\geq \frac{F'(t_0)}{F(t_0)} F(t) \\ &\geq F'(t_0) \exp\left(\frac{F'(t_0)}{F(t_0)}\left(t - t_0\right)\right) \\ &= \|u(t_0)\|^2 \exp\left(\frac{F'(t_0)}{F(t_0)}\left(t - t_0\right)\right) \\ &\geq \|u_0\|^2 \exp\left(\frac{F'(t_0)}{F(t_0)}\left(t - t_0\right)\right), \quad t \geq t_0, \end{aligned}$$

i.e.,

$$\lim_{t\to+\infty} \|u(t)\|^2 = +\infty.$$

This shows that weak solution $u(t)$ of the problem (1.1) blows up at $+\infty$. □

4. Conclusion

In this paper, we examined the initial boundary value problem for a higher-order parabolic equation with logarithmic nonlinearity. We obtained blow-up at infinity of weak solution, by using the potential well method and logarithmic convexity method.

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BLOW-UP RESULTS FOR A VISCOELASTIC PLATE EQUATION WITH DISTRIBUTED DELAY

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Abstract. In this paper, we consider a nonlinear viscoelastic plate equation with distributed delay. Under suitable conditions, we obtain the blow-up of solutions with distributed delay and source terms. Time delays often appear in many practical problems such as thermal, economic phenomena, biological, chemical, physical, electrical engineering systems, mechanical applications and medicine.

1. Introduction

In this paper, we consider the following viscoelastic plate equation with distributed delay and source terms

$$\begin{cases} u_{tt}+\Delta^2 u-\int_0^t g(t-s)\,\Delta^2 u(s)\,ds\ +\mu_1 u_t & \\ +\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,u_t(x,t-q)\,dq & \\ =b\,|u|^{p-2}\,u, & (x,t)\in\Omega\times(0,\infty), \\ u(x,t)=\frac{\partial u(x,t)}{\partial \upsilon}=0, & x\in\partial\Omega, \\ u_t(x,-t)=f_0(x,t), & (x,t)\in\Omega\times(0,\tau_2), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where b, $\mu_1>0$, $p>2$ and τ_1, τ_2 are the time delay with $0\leq\tau_1<\tau_2$, μ_2 is an L^∞ function, and g is a differentiable function under the assumptions (A1), (A2), and (A3). υ is the unit outward normal vector.

Problems about the mathematical behavior of solutions for PDEs with time delay effects have become interesting for many authors mainly because time delays often appear in many practical problems such as thermal, economic phenomena, biological, chemical, physical, electrical engineering systems, mechanical applications and medicine. Moreover, it is well known that delay effects may destroy the stabilizing properties of a well-behaved system. In the literature, there are several examples that illustrate how time delays destabilize some internal or boundary control system [6, 7]. Viscous materials are the opposite of elastic materials that posses the ability to dissipate and store the mechanical energy. The mechanical

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properties of these viscous substances are of great importance when they seem in many applications of natural sciences [3].

In 1986, Datko et al. [5] indicated that delay is a source of instability. In [12], Nicaise and Pignotti considered the following wave equation with a linear damping and delay term

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0. \tag{1.2}$$

They obtained some stability results in the case $0 < \mu_2 < \mu_1$. In the absence of delay, Zuazua [26] looked into exponentially stability for the equation (1.2).

Cavalcanti et al. [2], studied the model as follows:

$$u_{tt} + \gamma \Delta u_{tt} + \Delta^2 u - \int_0^t g(t-s)\,\Delta^2 u(s)\,ds + a(t)\,u_t = 0, \tag{1.3}$$

in $\Omega \times (0,\infty)$, where $a(t)$ is a nonlocal nonlinearity type function. They established the exponential decay result when $\gamma = 0$, of the energy in general domains of (1.3). Rivera et al. [25], coupled (1.3) with a dynamic boundary condition and indicated that the sum of the first and second energies decay polynomially and exponentially, according as the relaxation function g decays polynomially or exponentially. Also, for more results on (1.3), see also Lagnese [8].

Mukiawa [9], considered the viscoelastic plate equation as follows

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\,\Delta^2 u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0, \tag{1.4}$$

with a constant time delay and partially hinged boundary condition. The author proved a general decay result of the equation (1.4).

In [10], Mustafa and Kafini studied the infinite memory-type plate equation in the presence of constant time delay as follows

$$u_{tt} + \Delta^2 u - \int_0^\infty g(s)\,\Delta^2 u(t-s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = u\,|u|^{\gamma}. \tag{1.5}$$

The authors proved an explicit and general decay result for the energy, under the condition that $|\mu_2| \le \mu_1$, without restrictive assumptions on the behavior of the relaxation function g at infinity of the equation (1.5).

In [3], Choucha et al. considered the following equation

$$\begin{aligned} & u_{tt} - \Delta u - \omega \Delta u_t + \textstyle\int_0^t g(t-s)\,\Delta u(s)\,ds \\ & + \mu_1 u_t + \textstyle\int_{\tau_1}^{\tau_2} |\mu_2(\rho)|\, u_t(x,t-\rho)\,d\rho = b\,|u|^{p-2}\,u. \end{aligned} \tag{1.6}$$

The authors obtained the blow-up of solutions under appropriate conditions of the equation (1.6). In [4], the authors showed the exponential growth of solution for the equation (1.6).

The authors obtained the blow-up of solutions under appropriate conditions of the equation (1.6). In [4], the authors showed the exponential growth of solution for the equation (1.6). In recent years, some other authors investigate hyperbolic type equations (see [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]).

In this paper, we consider the nonlinear viscoelastic plate equation (1.1) with distributed delay ($\int_{\tau_1}^{\tau_2} |\mu_2(q)|\, u_t(x,t-q)\,dq$) and source ($b\,|u|^{p-2}\,u$) terms. Our aim is to get the blow-up results under appropriate conditions for the problem (1.1).

The paper is organized as follows: In section 2, we give some materials that will be used later. In section 3, we state and prove our main result.

2. Preliminaries

In this part, we prepare some materials for the proof of our result. As usual, the notation $\|.\|_p$ denotes L^p norm, and $(.,.)$ is the L^2 inner product. In particular, we write $\|.\|$ instead of $\|.\|_2$.

Now, we give some assumptions used later:

(A1) $g: R_+ \to R_+$ is a decreasing and differentiable function satisfies

$$g(t) \geq 0,\ 1 - \int_0^\infty g(s)\,ds = l > 0. \tag{2.1}$$

(A2) There exists a constant $\xi > 0$ such that

$$g'(t) \leq -\xi g(t),\ t \geq 0. \tag{2.2}$$

(A3) $\mu_2 : [\tau_1, \tau_2] \to R$ is an L^∞ function such that

$$\left(\frac{2\delta - 1}{2}\right) \int_{\tau_1}^{\tau_2} |\mu_2(q)|\,dq \leq \mu_1,\ \delta > \frac{1}{2}. \tag{2.3}$$

Let $B_p > 0$ be the constant satisfying [1]

$$\|\nabla v\|_p \leq B_p \|\Delta v\|_p,\ \text{for } v \in H_0^2(\Omega). \tag{2.4}$$

It holds

$$\begin{aligned} &\int_0^t g(t-s)(\Delta u(s), \Delta u_t(t))\,ds \\ &= -\tfrac{1}{2} g(t) \|\Delta u(t)\|^2 + \tfrac{1}{2}(g' o \Delta u)(t) \\ &-\tfrac{1}{2}\tfrac{d}{dt}\left[(g o \Delta u)(t) - \left(\int_0^t g(s)\,ds\right)\|\Delta u(t)\|^2\right], \end{aligned} \tag{2.5}$$

where

$$(g o \Delta u)(t) = \int_\Omega \int_0^t g(t-s)\,|\Delta u(t) - \Delta u(s)|^2\,ds. \tag{2.6}$$

Firstly, similar to [11], we introduce the new variable

$$y(x, \rho, q, t) = u_t(x, t - q\rho),$$

thus, we get

$$\begin{cases} q y_t(x, \rho, q, t) + y_\rho(x, \rho, q, t) = 0, \\ y(x, 0, q, t) = u_t(x, t). \end{cases} \tag{2.7}$$

Hence, problem (1.1) is equivalent to:

$$\begin{cases} u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\,ds \\ +\mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(q)|\,|y(x,1,q,t)|\,dq \qquad x \in \Omega,\ t > 0, \\ = b\,|u|^{p-2} u, \\ q y_t(x, \rho, q, t) + y_\rho(x, \rho, q, t) = 0, \end{cases} \tag{2.8}$$

with initial and boundary conditions

$$\begin{cases} u(x,t) = \frac{\partial u(x,t)}{\partial \upsilon} = 0, \qquad x \in \partial\Omega, \\ y(x, \rho, q, 0) = f_0(x, q\rho), \\ u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x), \end{cases} \tag{2.9}$$

where

$$(x, \rho, q, t) \in \Omega \times (0,1) \times (\tau_1, \tau_2) \times (0, \infty).$$

Theorem 2.1. *Suppose that (2.1), (2.2) and (2.3) hold. Let*

$$\left\{ \begin{array}{c} p \geq 2,\ n = 1,2,3,4, \\ 2 < p < \frac{2(n-2)}{n-4},\ n \geq 5. \end{array} \right. \tag{2.10}$$

Thus, for any initial data

$$(u_0, u_1, f_0) \in H_0^2(\Omega) \times H_0^2(\Omega) \times L^2(\ \Omega \times (0,1) \times (\tau_1, \tau_2)),$$

the problem (2.8)-(2.9) has a unique solution

$$u \in C\left([0,T]; H_0^2(\Omega)\right),$$

for some $T > 0$.

Now, we define the energy functional as follows:

Lemma 2.2. *Assume that (2.1), (2.2), (2.3) and (2.10) hold. Let u be a solution of (2.8)-(2.9). Then, $E(t)$ is nonincreasing, such that*

$$\begin{array}{c} E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\Delta u\|^2 + \frac{1}{2}(go\Delta u)(t) \\ +\frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q\,|\mu_2(q)|\left|y^2(x,\rho,q,t)\right| dq d\rho dx - \frac{b}{p}\|u\|_p^p, \end{array} \tag{2.11}$$

which satisfies

$$E'(t) \leq -c_1\left(\|u_t\|^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)|\left|y^2(x,1,q,t)\right| dq dx\right). \tag{2.12}$$

Proof. By multiplying the first equation of (2.8) by u_t and integrating over Ω, we obtain

$$\begin{array}{l} \frac{d}{dt}\left\{ \begin{array}{c} \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\Delta u\|^2 \\ +\frac{1}{2}(go\Delta u)(t) - \frac{b}{p}\|u\|_p^p \end{array} \right\} \\ \quad = -\mu_1\|u_t\|^2 - \int_\Omega u_t \int_{\tau_1}^{\tau_2} |\mu_2(q)|\,|y(x,1,q,t)|\,dq dx \\ \quad +\frac{1}{2}(g' o\Delta u)(t) - \frac{1}{2}g(t)\|\Delta u\|^2, \end{array} \tag{2.13}$$

and

$$\begin{array}{l} \frac{d}{dt}\frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q\,|\mu_2(q)|\left|y^2(x,\rho,q,t)\right| dq d\rho dx \\ \quad = -\frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} 2\,|\mu_2(q)|\,y y_\rho dq d\rho dx \\ \quad = \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)|\left|y^2(x,0,q,t)\right| dq dx \\ \quad -\frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)|\left|y^2(x,1,q,t)\right| dq dx \\ \quad = \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(q)|\,dq\right)\|u_t\|^2 \\ \quad -\frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)|\left|y^2(x,1,q,t)\right| dq dx. \end{array} \tag{2.14}$$

Thus,

$$\begin{aligned}\frac{d}{dt}E(t) &= -\mu_1\|u_t\|^2-\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,|u_t y(x,1,q,t)|\,dqdx \\ &\quad -\frac{1}{2}g(t)\|\Delta u\|^2+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,dq\right)\|u_t\|^2 \\ &\quad +\frac{1}{2}(g'o\Delta u)(t)-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|\left|y^2(x,1,q,t)\right|dqdx. \qquad (2.15)\end{aligned}$$

By using (2.13) and (2.14), we obtain (2.11). Utilizing Young's inequality, (2.1), (2.2), (2.3) and (2.15), we get (2.12). Consequently, the proof is completed. □

Lemma 2.3. [3] *There exists $c>0$, depending on Ω only, such that*

$$\left(\int_\Omega|u|^p\,dx\right)^{s/p}\le c\left[\|\nabla u\|^2+\|u\|_p^p\right], \qquad (2.16)$$

for all $u\in L^{p+1}(\Omega)$ and $2\le s\le p$.

From above lemma and by using Sobolev Embedding theorem, we have the following corollary:

Corollary 2.3.1. There exists $c>0$, depending on Ω only, such that

$$\left(\int_\Omega|u|^p\,dx\right)^{s/p}\le c\left[\|\Delta u\|^2+\|u\|_p^p\right], \qquad (2.17)$$

for all $u\in L^{p+1}(\Omega)$ and $2\le s\le p$.

Using the fact that $\|u\|_2^2\le c\|u\|_p^2\le c\left(\|u\|_p^p\right)^{2/p}$, we have the following corollary:

Corollary 2.3.2. There exists $C>0$, depending on Ω only, such that

$$\|u\|_2^2\le c\left[\|\Delta u\|_2^{4/p}+\left(\|u\|_p^p\right)^{2/p}\right]. \qquad (2.18)$$

Lemma 2.4. [3] *There exists $C>0$, depending on Ω only, such that*

$$\|u\|_p^s\le C\left[\|\nabla u\|^2+\|u\|_p^p\right], \qquad (2.19)$$

for all $u\in L^{p+1}(\Omega)$ and $2\le s\le p$.

From above lemma and by using Sobolev Embedding theorem, we have the following corollary:

Corollary 2.4.1. There exists $C>0$, depending on Ω only, such that

$$\|u\|_p^s\le c\left[\|\Delta u\|^2+\|u\|_p^p\right], \qquad (2.20)$$

for all $u\in L^{p+1}(\Omega)$ and $2\le s\le p$.

Now, we define the functional as follows:

$$\begin{aligned}H(t) &= -E(t) \\ &= \frac{b}{p}\|u\|_p^p-\frac{1}{2}\|u_t\|^2-\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\Delta u\|^2 \\ &\quad -\frac{1}{2}(go\Delta u)(t)-\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}q\,|\mu_2(q)|\left|y^2(x,\rho,q,t)\right|dqd\rho dx. \qquad (2.21)\end{aligned}$$

3. Blow-up results

In this part, we establish the blow-up of solutions for the problem (2.8)-(2.9).

Theorem 3.1. *Suppose that (2.1)-(2.3) and (2.10) hold. Suppose further that* $E(0) < 0$ *holds. Then, the solution of the problem (2.8)-(2.9) blows up in finite time.*

Proof. By (2.11), we have

$$E(t) \leq E(0) \leq 0. \tag{3.1}$$

Hence,

$$\begin{aligned} H'(t) &= -E'(t) \geq c_1 \left(\|u_t\|^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| \left| y^2(x,1,q,t) \right| dq dx \right) \\ &\geq c_1 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| \left| y^2(x,1,q,t) \right| dq dx \geq 0, \end{aligned} \tag{3.2}$$

and

$$0 \leq H(0) \leq H(t) \leq \frac{b}{p} \|u\|_p^p. \tag{3.3}$$

Set

$$\mathcal{K}(t) = H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t dx + \frac{\varepsilon \mu_1}{2} \int_\Omega u^2 dx, \tag{3.4}$$

here $\varepsilon > 0$ to be specified later and

$$\frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{2p} < 1. \tag{3.5}$$

We multiply the first equation of (2.8) by u and with a derivative of (3.4), we obtain

$$\begin{aligned} \mathcal{K}'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) \\ &\quad + \varepsilon \|u_t\|^2 + \varepsilon \int_\Omega \Delta u \int_0^t g(t-s) \Delta u(s) ds dx \\ &\quad - \varepsilon \|\Delta u\|^2 + \varepsilon b \int_\Omega |u|^p dx \\ &\quad - \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| |u y(x,1,q,t)| dq dx. \end{aligned} \tag{3.6}$$

By using

$$\begin{aligned} &\varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| |u y(x,1,q,t)| dq dx \\ \leq\ & \varepsilon \left\{ \delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(q)| dq \right) \|u\|^2 \right. \\ &\left. + \frac{1}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| \left| y^2(x,1,q,t) \right| dq dx \right\}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
& \varepsilon \int_0^t g(t-s)\,ds \int_\Omega \Delta u \Delta u(s)\,dxds \\
= & \varepsilon \int_0^t g(t-s)\,ds \int_\Omega \Delta u\left(\Delta u(s) - \Delta u(t)\right)dxds \\
& + \varepsilon \int_0^t g(s)\,ds \left\|\Delta u\right\|^2 \\
\geq & \frac{\varepsilon}{2} \int_0^t g(s)\,ds \left\|\Delta u\right\|^2 - \frac{\varepsilon}{2}\left(go\Delta u\right)(t), \qquad (3.8)
\end{aligned}$$

combining with (3.6), we get

$$\begin{aligned}
\mathcal{K}'(t) \geq & (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \left\|u_t\right\|^2 \\
& - \varepsilon \left(1 - \frac{1}{2}\int_0^t g(s)\,ds\right) \left\|\Delta u\right\|^2 \\
& + \varepsilon b \left\|u\right\|_p^p - \varepsilon \delta_1 \left(\int_{\tau_1}^{\tau_2} \left|\mu_2(q)\right| dq\right) \left\|u\right\|^2 \\
& - \frac{\varepsilon}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} \left|\mu_2(q)\right| \left|y^2(x,1,q,t)\right| dqdx \\
& + \frac{\varepsilon}{2}\left(go\Delta u\right)(t). \qquad (3.9)
\end{aligned}$$

By using (3.2) and setting δ_1 such that, $\frac{1}{4\delta_1 c_1} = \kappa H^{-\alpha}(t)$, we obtain

$$\begin{aligned}
\mathcal{K}'(t) \geq & \left[(1-\alpha) - \varepsilon\kappa\right] H^{-\alpha}(t) H'(t) + \varepsilon \left\|u_t\right\|^2 \\
& - \varepsilon \left[\left(1 - \frac{1}{2}\int_0^t g(s)\,ds\right)\right] \left\|\Delta u\right\|^2 + \varepsilon b \left\|u\right\|_p^p \\
& - \varepsilon \frac{H^\alpha(t)}{4c_1\kappa} \left(\int_{\tau_1}^{\tau_2} \left|\mu_2(q)\right| dq\right) \left\|u\right\|^2 + \frac{\varepsilon}{2}\left(go\Delta u\right)(t). \qquad (3.10)
\end{aligned}$$

From (2.21), for $0 < a < 1$

$$\begin{aligned}
\varepsilon b \left\|u\right\|_p^p = & \varepsilon p(1-a) H(t) + \frac{\varepsilon p(1-a)}{2} \left\|u_t\right\|^2 + \varepsilon b a \left\|u\right\|_p^p \\
& + \frac{\varepsilon p(1-a)}{2} \left(1 - \int_0^t g(s)\,ds\right) \left\|\Delta u\right\|^2 \\
& + \frac{\varepsilon}{2} p(1-a) \left(go\Delta u\right)(t) \\
& + \frac{\varepsilon p(1-a)}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q \left|\mu_2(q)\right| \left|y^2(x,\rho,q,t)\right| dqd\rho dx, \qquad (3.11)
\end{aligned}$$

with (3.10) implies

$$\begin{aligned}\mathcal{K}'(t) \geq{}& [(1-\alpha)-\varepsilon\kappa] H^{-\alpha}(t) H'(t) + \varepsilon\left[\frac{p(1-a)}{2}+1\right]\|u_t\|^2 \\ &+\varepsilon\left[\left(\frac{p(1-a)}{2}\right)\left(1-\int_0^t g(s)\,ds\right)-\left(1-\frac{1}{2}\int_0^t g(s)\,ds\right)\right]\|\Delta u\|^2 \\ &-\varepsilon\frac{H^\alpha(t)}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,dq\right)\|u\|^2+\varepsilon p(1-a)H(t)+\varepsilon ba\|u\|_p^p \\ &+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} q|\mu_2(q)|\left|y^2(x,\rho,q,t)\right|dqd\rho dx \\ &+\frac{\varepsilon}{2}(p(1-a)+1)(go\Delta u)(t). \end{aligned} \tag{3.12}$$

By using (2.18), (3.3) and Young's inequality, we obtain

$$\begin{aligned} H^\alpha(t)\|u\|_2^2 &\leq \left(b\int_\Omega |u|^p\,dx\right)^\alpha \|u\|_2^2 \\ &\leq c\left\{\left(\int_\Omega |u|^p\,dx\right)^{\alpha+2/p}+\left(\int_\Omega |u|^p\,dx\right)^\alpha \|\Delta u\|_2^{4/p}\right\} \\ &\leq c\left\{\left(\int_\Omega |u|^p\,dx\right)^{(p\alpha+2)/p}+\|\Delta u\|_2^2+\left(\int_\Omega |u|^p\,dx\right)^{p\alpha/(p-2)}\right\}. \end{aligned} \tag{3.13}$$

By exploiting (3.5), we obtain

$$2<\alpha p+2\leq p \text{ and } 2<\frac{\alpha p^2}{p-2}\leq p.$$

Consequently, by Lemma 2.2

$$H^\alpha(t)\|u\|_2^2 \leq c\left(\|u\|_p^p+\|\Delta u\|_2^2\right). \tag{3.14}$$

By combining (3.12) and (3.14), we have

$$\begin{aligned}\mathcal{K}'(t) \geq{}& [(1-\alpha)-\varepsilon\kappa] H^{-\alpha}(t) H'(t) \\ &+\varepsilon\left[\frac{p(1-a)}{2}+1\right]\|u_t\|^2+\frac{\varepsilon}{2}(p(1-a)+1)(go\Delta u)(t) \\ &+\varepsilon\left\{\left(\frac{p(1-a)}{2}-1\right)-\int_0^t g(s)\,ds\left(\frac{p(1-a)-1}{2}\right)\right. \\ &\left.-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,dq\right)\right\}\|\Delta u\|^2 \\ &+\varepsilon\left[ab-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,dq\right)\right]\|u\|_p^p+\varepsilon p(1-a)H(t) \\ &+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} q|\mu_2(q)|\left|y^2(x,\rho,q,t)\right|dqd\rho dx. \end{aligned} \tag{3.15}$$

Taking $a>0$ small enough such that

$$\alpha_1=\frac{p(1-a)}{2}-1>0$$

and suppose

$$\int_0^\infty g(s)\,ds < \frac{\frac{p(1-a)}{2}-1}{\left(\frac{p(1-a)}{2}-\frac{1}{2}\right)} = \frac{2\alpha_1}{2\alpha_1+1}. \tag{3.16}$$

Choosing κ such that,

$$\begin{aligned}\alpha_2 &= \left(\frac{p(1-a)}{2}-1\right) - \int_0^t g(s)\,ds\left(\frac{p(1-a)-1}{2}\right)\\ &\quad -\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,dq\right)\\ &> 0\end{aligned}$$

and

$$\alpha_3 = ab - \frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,dq\right) > 0.$$

Fixing κ and a, we have ε small enough

$$\alpha_4 = (1-\alpha) - \varepsilon\kappa > 0.$$

Hence, for some $\beta > 0$, (3.15) becomes

$$\begin{aligned}\mathcal{K}'(t) &\geq \beta\Big\{H(t) + \|u_t\|^2 + \|\Delta u\|^2 + (go\Delta u)(t) + \|u\|_p^p\\ &\quad + \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} q\,|\mu_2(q)|\left|y^2(x,\rho,q,t)\right| dq\,d\rho\,dx\Big\}.\end{aligned} \tag{3.17}$$

Thus, it follows that

$$\mathcal{K}(t) \geq \mathcal{K}(0) > 0,\ t > 0. \tag{3.18}$$

Now, utilizing Holder's and Young's inequalities, we obtain

$$\begin{aligned}\|u\|_2 &= \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}}\\ &\leq \left[\left(\int_\Omega \left(|u|^2\right)^{p/2} dx\right)^{\frac{2}{p}}\left(\int_\Omega 1dx\right)^{1-\frac{2}{p}}\right]^{\frac{1}{2}}\\ &\leq C\,\|u\|_p\end{aligned} \tag{3.19}$$

and

$$\left|\int_\Omega uu_t dx\right| \leq \|u_t\|_2\,\|u\|_2 \leq c\,\|u_t\|_2\,\|u\|_p\,.$$

Therefore,

$$\begin{aligned}\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} &\leq c\,\|u_t\|_2^{\frac{1}{1-\alpha}}\,\|u\|_p^{\frac{1}{1-\alpha}}\\ &\leq c\left[\|u_t\|_2^{\frac{\theta}{1-\alpha}} + \|u\|_p^{\frac{\mu}{1-\alpha}}\right],\end{aligned} \tag{3.20}$$

here $\frac{1}{\mu}+\frac{1}{\theta}=1.$ Taking $\theta = 2(1-\alpha)$, we obtain

$$\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq p.$$

For $s = \frac{2}{(1-2\alpha)}$, we get

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} \leq c\left(\|u_t\|_2^2 + \|u\|_p^s\right).$$

Hence, Corollary 2.3.1 gives that

$$\begin{aligned} \left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} &\leq c\left[\|u_t\|_2^2 + \|u\|_p^p + \|\Delta u\|_2^2\right] \\ &\leq c\left[\|u_t\|_2^2 + \|u\|_p^p + \|\Delta u\|_2^2 + (go\Delta u)(t)\right]. \end{aligned} \tag{3.21}$$

Therefore,

$$\begin{aligned} \mathcal{K}^{\frac{1}{1-\alpha}}(t) &= \left(H(t)^{1-\alpha} + \varepsilon\int_\Omega uu_t dx + \frac{\varepsilon\mu_1}{2}\int_\Omega u^2 dx\right)^{\frac{1}{1-\alpha}} \\ &\leq c\left[H(t) + \left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} + \|u\|_2^{\frac{2}{1-\alpha}} + \|\Delta u\|_2^{\frac{2}{1-\alpha}}\right] \\ &\leq c\left[H(t) + \|u_t\|^2 + \|u\|_p^p + \|\Delta u\|^2 + (go\Delta u)(t)\right]. \end{aligned} \tag{3.22}$$

By (3.17) and (3.22), we get

$$\mathcal{K}'(t) \geq \lambda\mathcal{K}^{\frac{1}{1-\alpha}}(t), \tag{3.23}$$

here $\lambda > 0$, which depends on β and c. An integration of (3.23), we have

$$\mathcal{K}^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{\mathcal{K}^{\frac{-\alpha}{1-\alpha}}(0) - \lambda\frac{\alpha}{(1-\alpha)}t}.$$

Thus, $\mathcal{K}(t)$ blows up in a finite time

$$T \leq T^* = \frac{1-\alpha}{\lambda\alpha\mathcal{K}^{\alpha/(1-\alpha)}(0)}.$$

As a result, we complete the proof. □

4. Conclusion

In recent years, there has been published much work concerning the wave equations (Kirchhoff, Petrovsky, Bessel,... etc.) with different state of delay time (constant delay, time-varying delay,... etc.). However, to the best of our knowledge, there were no blow-up results for the nonlinear viscoelastic plate equation with distributed delay. We have been obtained the blow-up of solutions with distributed delay and source terms under suitable conditions.

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SMARANDACHE CURVES ACCORDING TO ALTERNATIVE FRAME IN $\mathbb{E}^3$

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Abstract. In this study, we focus on Smarandache curves which are a special class of curves. These curves have previously been studied by many authors in different spaces. We will re-characterize these curves with the help of an alternative frame different from Frenet frame. Also, we will obtain frame vectors curvature and torsion of these curves.

1. Introduction

Curves, which have an important position in differential geometry, have enabled many studies. Many theories have been developed by establishing relations between Frenet frame. One of the special curves studied in differential geometry is Smarandache curve. Smarandache curve is defined as the regular curve drawn by these vectors, when the Frenet vectors of the unit speed regular curve are taken as position vectors [2]. A.T. Ali introduce special Smarandache curves in the Euclidean space. Some special Smarandache curves are expressed in 3-dimensional Euclidean space and introduced the Serret-Frenet elements of a special case [3]. NC-Smarandache curve with Frenet vectors {T,N,B} and unit Darboux vector C of the curve α is defined in the study titled "An application of Smarandache curves" [4]. In [5], authors obtain results about the characterization of Smarandache curves according to the Sabban frame formed on the S^2 unit sphere. In [7], authors classify general results of Smarandache curves with respect to the causal character of the curve. In her master's thesis named "Smarandache Curves of Bertrand Curve Pair According to Frenet Frame", she define Smarandache curves according to the Frenet vectors of the Bertrand partner curve and found some characterizations belonging to these curves [8]. In the study titled "Smarandache Curves According to Bishop Frame in Euclidean 3-Space", Smarandache curves belonging to Bishop frame are examined and they give some characterizations of these curves [6].
In this present paper, we introduce Smarandache curves according to the alternate frame defined by Uzunoglu et al. of a unit speed curve in Euclidean 3-Space.

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Firstly, we give Frenet frame, alternative frame and its properties. After that we mentione the relationship with alternative frame and Frenet frame. Then we define the special Smarandache curves according to alternative frame and we calculate the curvature, torsion, Frenet frame elements and alternative frame elements of this curves.

2. Preliminaries

In this section, basic definitions and theories about the Frenet frame and the Serret-Frenet formulas and the alternative frame will be given.

Definition 2.1. Let $\alpha : I \subset \mathbb{R} \to E^3$ be a unit speed curve. The vectors {T,N,B} Frenet frame along the α can be defined as follows

$$T(s) = \alpha^{'}(s), \quad N(s) = \frac{T^{'}(s)}{||T^{'}(s)||}, \quad B(s) = T(s) \times N(s) \tag{2.1}$$

where T is the unit tangent vector field, N is the principal normal vector field, B is the binormal vector field. Frenet derivative formulas can be given as follows

$$\begin{bmatrix} T^{'}(s) \\ N^{'}(s) \\ B^{'}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} \tag{2.2}$$

where κ is the curvature and τ is the torsion of the curve α [1]. The curvature and the torsion of the curve α are calculated as follows

$$\begin{cases} \kappa(s) = ||\alpha''(s)|| \\ \tau(s) = \frac{<\alpha^{'} \wedge \alpha^{''}, \alpha^{'''}>}{\|\alpha^{'} \wedge \alpha^{''}\|^2} \end{cases} . \tag{2.3}$$

Definition 2.2. Let $\alpha : I \subset \mathbb{R} \to E^3$ be a unit speed curve. Each unit speed curve has at least four continuous derivatives one can associate three orthogonal unit vector field. T, N and B are tangent, the principal normal and the binormal vector fields, respectively. Uzunoğlu et al. [9] defined the alternative moving frame denote by $\{N, C, W\}$ along the curve α in Euclidean 3-space as

$$N(s) = N(s), \quad C(s) = \frac{N^{'}(s)}{\|N^{'}(s)\|}, \quad W(s) = N(s) \times C(s). \tag{2.4}$$

For the derivatives of the alternative moving frame, we have

$$\begin{bmatrix} N^{'}(s) \\ C^{'}(s) \\ W^{'}(s) \end{bmatrix} = \begin{bmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ C(s) \\ W(s) \end{bmatrix} \tag{2.5}$$

where f and g are curvatures of the curve α as

$$\begin{cases} f = \sqrt{\kappa^2 + \tau^2} \\ g = \frac{(\tau/\kappa)^{'}}{1+\tau^2/\kappa^2} \end{cases} . \tag{2.6}$$

Definition 2.3. Let $\alpha : I \to E^3$ be a unit speed curve denote by {T,N,B} the moving Frenet frame. Smarandache curve is called the regular curve drawn by the vector whose position vector is

$$\beta(s) = \frac{a(s)T(s) + b(s)N(s) + c(s)B(s)}{\sqrt{a^2(s) + b^2(s) + c^2(s)}}$$

where a,b,c are real functions [4].

3. Smarandache Curves In Euclidean 3-Space

In this section, TN, TB, NB and TNB-Smarandache curves will be introduced and their curvature and torsion will be expressed in Euclidean 3-space.

Definition 3.1. [3] Let $\alpha(s)$ be a unit speed regular curve in E^3 and {T,N,B} be its moving Frenet-Serret frame. TN-Smarandache curve is defined by

$$\beta_{TN}(s) = \frac{1}{\sqrt{2}}(T + N). \tag{3.1}$$

Theorem 3.2. [3] *Let $\alpha(s)$ be a unit speed regular curve in E^3. The curvature and torsion of the TN-Smarandache curve are as follows, respectively.*

$$\begin{cases} \kappa_{\beta_{TN}} = \frac{\sqrt{2}}{(2\kappa^2+\tau^2)^2}\sqrt{\delta_1^2 + \mu_1^2 + \eta_1^2} \\ \\ \tau_{\beta_{TN}} = \frac{\sqrt{2}[(\tau^3+2\kappa^2\tau-\tau\kappa'+\kappa\tau')\bar{\delta}_1+(\kappa\tau'-\kappa'\tau)\bar{\mu}_1+(2\kappa^3+\kappa\tau^2)\bar{\eta}_1]}{(\tau^3+2\kappa^2\tau-\tau\kappa'+\kappa\tau')^2+(\kappa\tau'-\kappa'\tau)^2+(2\kappa^3+\kappa\tau^2)^2} \end{cases} \tag{3.2}$$

where

$$\begin{cases} \delta_1 = -[\kappa^2(2\kappa^2 + \tau^2) + \tau(\tau\kappa' - \kappa\tau')] \\ \mu_1 = -[\kappa^2(2\kappa^2 + 3\tau^2) - \tau(\tau^3 + \kappa\tau' - \tau\kappa')] \\ \eta_1 = \kappa[\tau(2\kappa^2 + \tau^2) - 2\tau\kappa' - \kappa\tau'] \end{cases} \tag{3.3}$$

$$\begin{cases} \bar{\delta}_1 = \kappa^3 + \kappa(\tau^2 - 3\kappa') - \kappa'' \\ \bar{\mu}_1 = -\kappa^3 - \kappa(\tau^2 + 3\kappa') - 3\tau\tau' + \kappa'' \\ \bar{\eta}_1 = -\kappa^2\tau - \tau^3 + 2\tau\kappa' + \kappa\tau' + \tau'' \end{cases} \tag{3.4}$$

Definition 3.3. [3] Let $\alpha(s)$ be a unit speed regular curve in E^3 and {T,N,B} be its moving Frenet-Serret frame. TB-Smarandache curve is defined by

$$\beta_{TB}(s) = \frac{1}{\sqrt{2}}(T + B). \tag{3.5}$$

Theorem 3.4. [3] *Let $\alpha(s)$ be a unit speed regular curve in E^3. The curvature and torsion of the TB-Smarandache curve are as follows, respectively.*

$$\begin{cases} \kappa_{\beta_{TB}} = \frac{\sqrt{2(\delta_2^2+\mu_2^2)}}{(\kappa-\tau)^4} \\ \\ \tau_{\beta_{TB}} = \frac{\sqrt{2}[\kappa^2\tau\bar{\delta}_2-2\kappa\tau^2\bar{\delta}_2+\tau^3\bar{\delta}_2+\kappa^3\bar{\eta}_2-2\kappa^2\tau\bar{\eta}_2+\kappa\tau^2\bar{\eta}_2]}{(\tau(\kappa-\tau)^2)^2+(\kappa(\kappa-\tau)^2)^2} \end{cases} \tag{3.6}$$

where

$$\begin{cases} \delta_2 = -\kappa^4 + 3\kappa^3\tau - 3\kappa^2\tau^2 + \kappa\tau^3 \\ \mu_2 = 0 \\ \eta_2 = \kappa^3\tau - 3\kappa^2\tau^2 + 3\kappa\tau^3 - \tau^4 \end{cases} \tag{3.7}$$

$$\left\{\begin{array}{l} \bar{\delta}_2 = -3\kappa\kappa' + 2\kappa\tau' + \kappa^{'}\tau \\ \bar{\mu}_2 = (\tau - \kappa)(\tau^2 + \kappa^2) + \kappa^{''} - \tau'' \\ \bar{\eta}_2 = -3\tau\tau' + 2\tau\kappa' + \kappa\tau' \end{array}\right. \tag{3.8}$$

Definition 3.5. [3] Let $\alpha(s)$ be a unit speed regular curve in E^3 and {T,N,B} be its moving Frenet-Serret frame. NB-Smarandache curve is defined by

$$\beta_{NB}(s) = \frac{1}{\sqrt{2}}(N + B). \tag{3.9}$$

Theorem 3.6. [3] *Let $\alpha(s)$ be a unit speed regular curve in E^3. The curvature and torsion of the NB-Smarandache curve are as follows, respectively.*

$$\left\{\begin{array}{l} \kappa_{\beta_{NB}} = \frac{\sqrt{2}}{(\kappa^2+2\tau^2)^2}\sqrt{\delta_3^2 + \mu_3^2 + \eta_3^2} \\ \\ \tau_{\beta_{NB}} = \frac{\sqrt{2}[(2\tau^3+\tau\kappa^2)\bar{\delta}_3+(\tau^{'}\kappa-\tau\kappa')\bar{\mu}_3+(\kappa^3+2\kappa\tau^2+\kappa\tau'-\tau\kappa')\bar{\eta}_3]}{(2\tau^3+\tau\kappa^2)^2+(\tau'\kappa-\tau\kappa')^2+(\kappa^3+2\kappa\tau^2+\kappa\tau'-\tau\kappa')^2} \end{array}\right. \tag{3.10}$$

where

$$\left\{\begin{array}{l} \delta_3 = (\kappa^2 + 2\tau^2)\kappa\tau + 2\tau(\kappa\tau^{'} - \tau\kappa^{'}) \\ \mu_3 = -(\kappa^2 + 2\tau^2)(\kappa^2 + \tau^2) + \kappa(\kappa'\tau - \tau^{'}\kappa) \\ \eta_3 = (\kappa^2 + 2\tau^2)(-\tau^2) + \kappa(\kappa\tau' - \kappa'\tau) \end{array}\right. \tag{3.11}$$

$$\left\{\begin{array}{l} \bar{\delta}_3 = \kappa^3 + \kappa(\tau^2 - 3\kappa^{'}) - \kappa^{''} \\ \bar{\mu}_3 = -\kappa^3 - \kappa(\tau^2 + 3\kappa^{'}) - 4\tau\tau' + \kappa^{''} \\ \bar{\eta}_3 = -\kappa^2\tau - \tau^3 + 2\tau\kappa^{'} + \kappa\tau^{'} + \tau'' \end{array}\right. \tag{3.12}$$

Definition 3.7. [3] Let $\alpha(s)$ be a unit speed regular curve in E^3 and {T,N,B} be its moving Frenet-Serret frame. TNB-Smarandache curve is defined by

$$\beta_{TNB}(s) = \frac{1}{\sqrt{3}}(T + N + B). \tag{3.13}$$

Theorem 3.8. [3] *Let $\alpha(s)$ be a unit speed regular curve in E^3. The curvature and torsion of the TNB-Smarandache curve are as follows, respectively.*

$$\left\{\begin{array}{l} \kappa_{\beta_{TNB}} = \frac{\sqrt{3}}{(2\kappa^2+2\tau^2-2\kappa\tau)^2}\sqrt{\delta_4^2 + \mu_4^2 + \eta_4^2} \\ \\ \tau_{\beta_{TNB}} = \frac{\sqrt{3}[(\kappa^2\tau+\kappa\tau'-2\kappa\tau^2-\tau\tau^{'}+2\tau^3-\tau\kappa^{'}+\tau\kappa^2)\bar{\delta}_4+(\kappa\tau^{'}-\tau\kappa')\bar{\mu}_4+(2\kappa^3-\tau\kappa^{'})\bar{\eta}_4]}{(\kappa^2\tau+\kappa\tau'-2\kappa\tau^2-\tau\tau^{'}+2\tau^3-\tau\kappa^{'}+\tau\kappa^2)^2+(\kappa\tau^{'}-\tau\kappa')^2+(2\kappa^3-\tau\kappa^{'})^2} \end{array}\right. \tag{3.14}$$

where

$$\left\{\begin{array}{l} \delta_4 = \kappa\tau[4\kappa(\kappa - \tau) + 2(\tau' + \tau^2) + \kappa^{'}] - \kappa^2(2\kappa^2 + \tau^{'}) - 2\kappa'\tau^2 \\ \delta_4 = 2\kappa\tau[(\kappa - \tau)^2 + 2\tau - 2\tau'] - 2(\kappa^4 + \tau^4) + \kappa^{'}\tau^2 \;\; - \kappa^2\tau' \\ \delta_4 = \tau[2\kappa(\kappa^2 + 4\tau^2 - \kappa' - 2\kappa\tau) + (\tau\kappa' + \tau' - 2\tau^3)] \end{array}\right. \tag{3.15}$$

$$\left\{\begin{array}{l} \bar{\delta}_4 = \kappa^3 + \kappa(\tau^2 - 3\kappa^{'}) - \kappa^{''} \\ \bar{\mu}_4 = -\kappa^3 - \kappa(\tau^2 + 3\kappa^{'}) - 3\tau\tau' + \kappa^{''} \\ \bar{\eta}_4 = -\kappa^2\tau - \tau^3 + 2\tau\kappa^{'} + \kappa\tau^{'} + \tau'' \end{array}\right. \tag{3.16}$$

4. Smarandache Curves According to Alternative Frame In E^3

In this section, these special curves will be re-examined on an alternative frame inspired by Smarandache curves defined according to the Frenet frame in Euclidean 3-space.

Definition 4.1. Let $\beta(s)$ be a unit speed regular curve in E^3 and {N,C,W} be its moving alternative frame. NC-Smarandache curve is defined by

$$\beta_{NC}(s) = \frac{1}{\sqrt{2}}(N + C). \tag{4.1}$$

Theorem 4.2. *Let $\beta(s)$ be a unit speed regular curve in E^3. The curvature and torsion of NC-Smarandache curve are as follows, respectively.*

$$\begin{cases} f = \sqrt{[\frac{\sqrt{2}\cdot\sqrt{\delta_5^2+\mu_5^2+\eta_5^2}}{(2f^2+g^2)^2}]^2 + [\frac{\sqrt{2}\cdot(\tilde{\delta_5}\widehat{\delta_5}+\tilde{\mu_5}\widehat{\mu_5}+\tilde{\eta_5}\widehat{\eta_5})}{\tilde{\delta}_5^2+\tilde{\mu_5}^2+\tilde{\eta_5}^2}]^2} \\ \\ g = \dfrac{[\frac{\frac{\sqrt{2}\cdot(\tilde{\delta_5}\widehat{\delta_5}+\tilde{\mu_5}\widehat{\mu_5}+\tilde{\eta_5}\widehat{\eta_5})}{\tilde{\delta}_5^2+\tilde{\mu}_5^2+\tilde{\eta}_5^2}}{\frac{\sqrt{2}\cdot\sqrt{\delta_5^2+\mu_5^2+\eta_5^2}}{(2f^2+g^2)^2}}]'}{1+[\frac{\frac{\sqrt{2}\cdot(\tilde{\delta_5}\widehat{\delta_5}+\tilde{\mu_5}\widehat{\mu_5}+\tilde{\eta_5}\widehat{\eta_5})}{\tilde{\delta}_5^2+\tilde{\mu}_5^2+\tilde{\eta}_5^2}}{\frac{\sqrt{2}\cdot\sqrt{\delta_5^2+\mu_5^2+\eta_5^2}}{(2f^2+g^2)^2}}]^2} \end{cases} \tag{4.2}$$

where

$$\begin{cases} \delta_5 = -[f^2(2f^2+g^2) + g(gf^{'} - fg^{'})] \\ \mu_5 = -[f^2(2f^2+3g^2) - g(g^3 + fg' - gf')] \\ \eta_5 = f[g(2f^2+g^2) - 2(gf^{'} - fg^{'})] \end{cases} \tag{4.3}$$

$$\begin{cases} \bar{\delta_5} = [(\delta_5^{'} - f\mu_5)(\delta_5^2+\mu_5^2+\eta_5^2) - \delta_5(\delta_5\delta_5^{'} + \mu_5\mu_5^{'} + \eta_5\eta_5^{'})] \\ \bar{\mu_5} = [(f\delta_5 + \mu_5^{'} - g\eta_5)(\delta_5^2+\mu_5^2+\eta_5^2) - \mu_5(\delta_5\delta_5^{'} + \mu_5\mu_5^{'} + \eta_5\eta_5^{'})] \\ \bar{\eta_5} = [(g\mu_5 + \eta_5^{'})(\delta_5^2+\mu_5^2+\eta_5^2) - \eta_5(\delta_5\delta_5^{'} + \mu_5\mu_5^{'} + \eta_5\eta_5^{'})] \end{cases} \tag{4.4}$$

$$\begin{cases} \widehat{\delta_5} = (-2ff^{'} - f^{''} + f^3 - ff^{'} + fg^2) \\ \widehat{\mu_5} = (-f^3 - ff^{'} - 2ff^{'} + f^{''} - 2gg^{'} - fg^2 - gg^{'}) \\ \widehat{\eta_5} = (-f^2g - g^3 + 2gf^{'} + fg^{'} + g^{''}) \end{cases} \tag{4.5}$$

$$\begin{cases} \tilde{\delta_5} = (g^3 + 2f^2g - gf^{'} + fg^{'}), \\ \tilde{\mu_5} = (fg^{'} - f^{'}g), \\ \tilde{\eta_5} = (-f^2g - g^3 + 2gf^{'} + fg^{'} + g^{''}) \end{cases} \tag{4.6}$$

Proof. Let $\beta(s)$ be a unit speed regular NC-Smarandache curve as in (4.1). If we take the derivative of the Smarandache curve according to arclenght parameter, we have

$$\frac{d\beta_{NC}}{ds_\beta}\frac{ds_\beta}{ds} = \frac{1}{\sqrt{2}}(-fN + fC + gW), \tag{4.7}$$

and since

$$\left\| \frac{d\beta_{NC}}{ds_\beta} \right\| = 1,$$

we can see

$$\frac{ds_\beta}{ds} = \sqrt{\frac{1}{2}(f^2 + f^2 + g^2)} = \sqrt{\frac{2f^2 + g^2}{2}}. \tag{4.8}$$

From the equations (4.7) and (4.8), the tangent vector of β_{NC} is

$$T_{\beta_{NC}} = \frac{-fN + fC + gW}{\sqrt{2f^2 + g^2}}. \tag{4.9}$$

If we take derivate this expression is again, we can see that

$$T'_{\beta_{NC}} = \frac{\sqrt{2}}{(2f^2 + g^2)^2}(\delta_5 N + \mu_5 C + \eta_5 W) \tag{4.10}$$

where

$$\begin{cases} \delta_5 = -[f^2(2f^2 + g^2) + g(gf' - fg')], \\ \mu_5 = -[f^2(2f^2 + 3g^2) - g(g^3 + fg' - gf')], \\ \eta_5 = f[g(2f^2 + g^2) - 2(gf' - fg')]. \end{cases}$$

The curvature of the β_{NC} is indicated by the $\kappa_{\beta_{NC}}$ taking the norm of equation (4.10).

$$\kappa_{\beta_{NC}} = \frac{\sqrt{2}}{(2f^2 + g^2)^2}\sqrt{\delta_5^2 + \mu_5^2 + \eta_5^2} \tag{4.11}$$

If the principal normal of β_{NC} is indicated by $N_{\beta_{NC}}$, it is found in the form of

$$N_{\beta_{NC}} = \frac{\delta_5 N + \mu_5 C + \eta_5 W}{\sqrt{\delta_5^2 + \mu_5^2 + \eta_5^2}}. \tag{4.12}$$

If we take the derivative of the equation (4.12), we obtain

$$N' = \frac{\sqrt{2}}{\sqrt{2f^2 + g^2}} \frac{\bar{\delta}_5 N + \bar{\mu}_5 C + \bar{\eta}_5 W}{(2f^2 + g^2)^{\frac{3}{2}}}. \tag{4.13}$$

where

$$\begin{cases} \bar{\delta_5} = [(\delta_5' - f\mu_5)(\delta_5^2 + \mu_5^2 + \eta_5^2) - \delta_5(\delta_5\delta_5' + \mu_5\mu_5' + \eta_5\eta_5')], \\ \bar{\mu_5} = [(f\delta_5 + \mu_5' - g\eta_5)(\delta_5^2 + \mu_5^2 + \eta_5^2) - \mu_5(\delta_5\delta_5' + \mu_5\mu_5' + \eta_5\eta_5')], \\ \bar{\eta_5} = [(g\mu_5 + \eta_5')(\delta_5^2 + \mu_5^2 + \eta_5^2) - \eta_5(\delta_5\delta_5' + \mu_5\mu_5' + \eta_5\eta_5')]. \end{cases}$$

If we take the norm of the equation (4.13), we get

$$\|N'_{\beta_{NC}}\| = \frac{\sqrt{2}}{\sqrt{2f^2 + g^2}} \frac{\sqrt{\bar{\delta}_5^2 + \bar{\mu}_5^2 + \bar{\eta}_5^2}}{(\bar{\delta}_5^2 + \bar{\mu}_5^2 + \bar{\eta}_5^2)^{\frac{3}{2}}}. \tag{4.14}$$

Since $C_{\beta_{NC}} = \frac{N'_{\beta_{NC}}}{\|N'_{\beta_{NC}}\|}$, if necessary calculations are made from the equations (4.13) and (4.14)

$$C_{\beta_{NC}} = \frac{\bar{\delta}_5 N + \bar{\mu}_5 C + \bar{\eta}_5 W}{\sqrt{\bar{\delta}_5^2 + \bar{\mu}_5^2 + \bar{\eta}_5^2}}.$$

From the definition of Darboux vector, we know $W_{\beta_{NC}} = N_{\beta_{NC}} \times C_{\beta_{NC}}$. So we have

$$W_{\beta_{NC}} = \frac{1}{\sqrt{\delta_5^2 + \mu_5^2 + \eta_5^2} \cdot \sqrt{\bar{\delta_5}^2 + \bar{\mu_5}^2 + \bar{\eta_5}^2}} \begin{vmatrix} N & C & W \\ \delta_5 & \mu_5 & \eta_5 \\ \bar{\delta_5} & \bar{\mu_5} & \bar{\eta_5} \end{vmatrix}$$

and so on

$$W_{\beta_{NC}} = \frac{(\mu_5\bar{\eta_5} - \eta_5\bar{\mu_5})N - (\delta_5\bar{\eta_5} - \eta_5\bar{\delta_5})C + (\delta_5\bar{\mu_5} - \mu_5\delta_5)W}{\sqrt{\delta_5^2 + \mu_5^2 + \eta_5^2} \cdot \sqrt{\bar{\delta_5}^2 + \bar{\mu_5}^2 + \bar{\eta_5}^2}}. \tag{4.15}$$

To find the torsion, we need to find the second and third derivates of the β_{NC} curve. These derivates are available below.

$$\beta_{NC}(s) = \frac{1}{\sqrt{2}}(N + C), \tag{4.16}$$

$$\beta'_{NC} = \frac{1}{\sqrt{2}}(fC - fN + gW), \tag{4.17}$$

$$\beta''_{NC} = \frac{1}{\sqrt{2}}(-(f^2 + f')N + (-f^2 + f' - g^2)C + (fg + g')W), \tag{4.18}$$

$$\beta'''_{NC} = \frac{1}{\sqrt{2}}(\widehat{\delta_5}N + \widehat{\mu_5}C + \widehat{\eta_5}W) \tag{4.19}$$

where

$$\begin{cases} \widehat{\delta_5} = (-2ff' - f'' + f^3 - ff' + fg^2), \\ \widehat{\mu_5} = (-f^3 - ff' - 2ff' + f'' - 2gg' - fg^2 - gg'), \\ \widehat{\eta_5} = (-f^2g - g^3 + 2gf' + fg' + g''). \end{cases}$$

In equation (2.3), if the expressions (4.17), (4.18) and (4.19) are written in their places and the necessary calculations are made, torsion is found as

(4.20)

$$\tau_{\beta_{NC}} = \frac{\sqrt{2} \cdot \left[\ (g^3 + 2f^2g - gf' + fg')\widehat{\delta_5} + (fg' - f'g)\widehat{\mu_5} + (2f^3 + fg^2)\widehat{\eta_5}\ \right]}{(g^3 + 2f^2g - gf' + fg')^2 + (fg' - f'g)^2 + (2f^3 + fg^2)^2}$$

In equation (2.6), if the expressions (4.11) and (4.20) are written in their places and the necessary calculations are made, curvature and torsion according to alternative frame are obtained as

$$f = \sqrt{[\frac{\sqrt{2} \cdot \sqrt{\delta_5^2 + \mu_5^2 + \eta_5^2}}{(2f^2 + g^2)^2}]^2 + [\frac{\sqrt{2} \cdot (\tilde{\delta_5}\widehat{\delta_5} + \tilde{\mu_5}\widehat{\mu_5} + \tilde{\eta_5}\widehat{\eta_5})}{\tilde{\delta_5^2} + \tilde{\mu_5}^2 + \tilde{\eta_5}^2}]^2} \tag{4.21}$$

and

$$g = \frac{[\frac{\frac{\sqrt{2} \cdot (\tilde{\delta_5}\widehat{\delta_5} + \tilde{\mu_5}\widehat{\mu_5} + \tilde{\eta_5}\widehat{\eta_5})}{\tilde{\delta_5^2} + \tilde{\mu_5^2} + \tilde{\eta_5^2}}}{\frac{\sqrt{2} \cdot \sqrt{\delta_5^2 + \mu_5^2 + \eta_5^2}}{(2f^2 + g^2)^2}}]'}{1 + [\frac{\frac{\sqrt{2} \cdot (\tilde{\delta_5}\widehat{\delta_5} + \tilde{\mu_5}\widehat{\mu_5} + \tilde{\eta_5}\widehat{\eta_5})}{\tilde{\delta_5^2} + \tilde{\mu_5^2} + \tilde{\eta_5^2}}}{\frac{\sqrt{2} \cdot \sqrt{\delta_5^2 + \mu_5^2 + \eta_5^2}}{(2f^2 + g^2)^2}}]^2} \tag{4.22}$$

where

$$\begin{cases} \tilde{\delta_5} = (g^3 + 2f^2g - gf' + fg'), \\ \tilde{\mu_5} = (fg' - f'g), \\ \tilde{\eta_5} = (-f^2g - g^3 + 2gf' + fg' + g''). \end{cases}$$

□

Definition 4.3. Let $\beta(s)$ be a unit speed regular curve in E^3 and {N,C,W} be its moving alternative frame. NW-Smarandache curve is defined by

$$\beta_{NW}(s) = \frac{1}{\sqrt{2}}(N + W). \tag{4.23}$$

Theorem 4.4. *Let $\beta(s)$ be a unit speed regular curve in E^3. The curvature and torsion of NW-Smarandache curve are as follows, respectively.*

$$\begin{cases} f = \sqrt{[\frac{\sqrt{2}\cdot\sqrt{f^2+g^2}}{(f-g)}]^2 + [\frac{\sqrt{2}\cdot(\tilde{\delta_6}\widehat{\delta_6}+\tilde{\eta_6}\widehat{\eta_6})}{\tilde{\delta_6^2}+\tilde{\eta_6}^2}]^2} \\ g = \dfrac{[\frac{\frac{\sqrt{2}\cdot(\tilde{\delta_6}\widehat{\delta_6}+\tilde{\eta_6}\widehat{\eta_6})}{\tilde{\delta_6^2}+\tilde{\eta_6^2}}}{\frac{\sqrt{2}\cdot\sqrt{f^2+g^2}}{(f-g)}}]'}{1+[\frac{\frac{\sqrt{2}\cdot(\tilde{\delta_6}\widehat{\delta_6}+\tilde{\eta_6}\widehat{\eta_6})}{\tilde{\delta_6^2}+\tilde{\eta_6^2}}}{\frac{\sqrt{2}\cdot\sqrt{f^2+g^2}}{(f-g)}}]^2} \end{cases} \tag{4.24}$$

where

$$\begin{cases} \bar{\delta_6} = (-f^{'}f^2 - f^{'}g^2 + f^2f^{'} + fgg^{'}) \\ \bar{\mu_6} = (-f^4 - f^2g^2 - g^2f^2 - g^4) \\ \bar{\eta_6} = (g^{'}f^2 + g^{'}g^2 - gff^{'} - g^2g^{'}) \end{cases} \tag{4.25}$$

$$\begin{cases} \widehat{\delta_6} = (-3ff^{'} + 2fg^{'} + gf^{'}) \\ \widehat{\mu_6} = (f^2+g^2)(-f+g) + f^{''} - g^{''} \\ \widehat{\eta_6} = (-3gg^{'} + 2gf^{'} + fg^{'}) \end{cases} \tag{4.26}$$

$$\begin{cases} \tilde{\delta_6} = (f^2g - 2fg^2 + g^3), \\ \tilde{\mu_6} = 0, \\ \tilde{\eta_6} = (f^3 - 2f^2g + fg^2). \end{cases} \tag{4.27}$$

Proof. Let $\beta(s)$ be a unit speed regular NW-Smarandache curve as in (4.23). If we take the derivative of Smarandache curve according to arclenght parameter, we have

$$\frac{d\beta_{NW}}{ds_\beta}\frac{ds_\beta}{ds} = \frac{(f-g)C}{\sqrt{2}}, \tag{4.28}$$

and since

$$\left\|\frac{d\beta_{NW}}{ds_\beta}\right\| = 1,$$

we can see

$$\frac{ds_\beta}{ds} = \sqrt{\frac{(f-g)^2}{2}} = \frac{|f-g|}{\sqrt{2}}. \tag{4.29}$$

From the equations (4.28) and (4.29), tangent vector of β_{NW} is

$$T_{\beta_{NW}} = \begin{cases} C & f > g \\ -C & f < g \end{cases}. \tag{4.30}$$

If we take derivate this expression is again, we can see that

$$T^{'}_{\beta_{NW}} = \frac{\sqrt{2}(-fN + gW)}{|f-g|} \tag{4.31}$$

The curvature of the β_{NW} is indicated by the $\kappa_{\beta_{NW}}$ taking the norm of equation (4.31).

$$\kappa_{\beta_{NW}} = \frac{\sqrt{2}}{(f-g)}\sqrt{f^2+g^2} \tag{4.32}$$

If the β_{NW} is indicated by principal normal $N_{\beta_{NW}}$, it is found in the form of

$$N_{\beta_{NW}} = \frac{1}{\sqrt{f^2+g^2}}(-fN+gW) \tag{4.33}$$

If we take the derivative of the equation (4.33), we obtain that

$$N^{'} = \frac{\sqrt{2}}{|f-g|}\cdot\frac{\bar{\delta}_6 N+\bar{\mu}_6 C+\bar{\eta}_6 W}{(f^2+g^2)^{\frac{3}{2}}}. \tag{4.34}$$

where

$$\begin{cases} \bar{\delta}_6 = (-f^{'}f^2 - f^{'}g^2 + f^2 f^{'} + fgg^{'}) \\ \bar{\mu}_6 = (-f^4 - f^2g^2 - g^2f^2 - g^4) \\ \bar{\eta}_6 = (g^{'}f^2 + g^{'}g^2 - gff^{'} - g^2g^{'}) \end{cases}$$

If we take the norm of the equation (4.34), we get

$$\|N^{'}_{\beta_{NW}}\| = \frac{\sqrt{2}}{(f^2+g^2)^{\frac{3}{2}}|f-g|}\cdot\sqrt{\bar{\delta}_6^2+\bar{\mu}_6^2+\bar{\eta}_6^2}. \tag{4.35}$$

Since $C_{\beta_{NW}} = \frac{N^{'}_{\beta_{NW}}}{\|N^{'}_{\beta_{NW}}\|}$, if necessary calculations are made from the equations (4.34) and (4.35),

$$C_{\beta_{NW}} = \frac{N^{'}_{\beta_{NW}}}{\|N^{'}_{\beta_{NW}}\|} = \frac{\bar{\delta}_6 N+\bar{\mu}_6 C+\bar{\eta}_6 W}{\sqrt{\bar{\delta}_6^2+\bar{\mu}_6^2+\bar{\eta}_6^2}}.$$

From the definition of Darboux vector, we know $W_{\beta_{NW}} = N_{\beta_{NW}}\times C_{\beta_{NW}}$,

$$W_{\beta_{NW}} = \frac{1}{\sqrt{\bar{\delta}_6^2+\bar{\mu}_6^2+\bar{\eta}_6^2}\cdot\sqrt{f^2+g^2}}\begin{vmatrix} N & C & W \\ -f & 0 & g \\ \bar{\delta}_6 & \bar{\mu}_6 & \bar{\eta}_6 \end{vmatrix}$$

and so on

$$W_{\beta_{NW}} = \frac{-g\bar{\mu}_6 N+(f\bar{\eta}_6+g\bar{\delta}_6)C-f\bar{\mu}_6 W}{\sqrt{\bar{\delta}_6^2+\bar{\mu}_6^2+\bar{\eta}_6^2}\cdot\sqrt{f^2+g^2}}.$$

To find the torsion, we need to find the second and third derivatives of the β_{NW} curve. The derivates are available below.

$$\beta_{NW}(s) = \frac{1}{\sqrt{2}}(N+W), \tag{4.36}$$

$$\beta^{'}_{NW} = \frac{1}{\sqrt{2}}(fC-gC), \tag{4.37}$$

$$\beta^{''}_{NW} = \frac{1}{\sqrt{2}}(-f^2+gf)N+(f^{'}-g^{'})C+(fg-g^2)W), \tag{4.38}$$

$$\beta^{'''}_{NW} = \frac{1}{\sqrt{2}}(\widehat{\delta_6}N+\widehat{\mu_6}C+\widehat{\eta_6}W) \tag{4.39}$$

where

$$\begin{cases} \widehat{\delta_6} = (-3ff^{'} + 2fg^{'} + gf^{'}) \\ \widehat{\mu_6} = (f^2 + g^2)(-f + g) + f^{''} - g^{''} \\ \widehat{\eta_6} = (-3gg^{'} + 2gf^{'} + fg^{'}) \end{cases}$$

In equation (2.3), if the expressions (4.37), (4.38) and (4.39) are written in their places and the necessary calculations are made, torsion of β_{NW} is found as

$$\tau_{\beta_{NW}} = \frac{\sqrt{2} \cdot [\ (f^2g - 2fg^2 + g^3)\bar{\delta_5} + 0 + (f^3 - 2f^2g + fg^2)\bar{\eta_5},\ ]}{(f^2g - 2fg^2 + g^3)^2 + (f^3 - 2f^2g + fg^2)^2} \tag{4.40}$$

In equation (2.6), if the expressions (4.32) and (4.40) are written in their places and the necessary calculations are made, curvature and torsion according to alternative frame are obtained as

$$f = \sqrt{[\frac{\sqrt{2} \cdot \sqrt{f^2 + g^2}}{(f - g)}]^2 + [\frac{\sqrt{2} \cdot (\tilde{\delta_6}\widehat{\delta_6} + \tilde{\eta_6}\widehat{\eta_6})}{\tilde{\delta_6^2} + \tilde{\eta_6}^2}]^2} \tag{4.41}$$

and

$$g = \frac{[\frac{\frac{\sqrt{2} \cdot (\tilde{\delta_6}\widehat{\delta_6} + \tilde{\eta_6}\widehat{\eta_6})}{\tilde{\delta_6^2} + \tilde{\eta_6^2}}}{\frac{\sqrt{2} \cdot \sqrt{f^2 + g^2}}{(f - g)}}]'}{1 + [\frac{\frac{\sqrt{2} \cdot (\tilde{\delta_6}\widehat{\delta_6} + \tilde{\eta_6}\widehat{\eta_6})}{\tilde{\delta_6^2} + \tilde{\eta_6^2}}}{\frac{\sqrt{2} \cdot \sqrt{f^2 + g^2}}{(f - g)}}]^2} \tag{4.42}$$

where

$$\begin{cases} \tilde{\delta_6} = (f^2g - 2fg^2 + g^3), \\ \tilde{\mu_6} = 0, \\ \tilde{\eta_6} = (f^3 - 2f^2g + fg^2). \end{cases}$$

□

Definition 4.5. Let $\beta(s)$ be a unit speed regular curve in E^3 and {N,C,W} be its moving alternative frame. CW-Smarandache curve is defined by

$$\beta_{CW}(s) = \frac{1}{\sqrt{2}}(C + W). \tag{4.43}$$

Theorem 4.6. *Let $\beta(s)$ be a unit speed regular curve in E^3. The curvature and torsion of CW-Smarandache curve are as follows, respectively.*

$$\begin{cases} f = \sqrt{[\frac{\sqrt{2} \cdot \sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2}}{(f^2 + 2g^2)^2}]^2 + [\frac{\sqrt{2} \cdot (\tilde{\delta_7}\widehat{\delta_7} + \tilde{\mu_7}\widehat{\mu_7} + \tilde{\eta_7}\widehat{\eta_7})}{\tilde{\delta_7^2} + \tilde{\mu_7}^2 + \tilde{\eta_7}^2}]^2} \\ \\ g = \frac{[\frac{\frac{\sqrt{2} \cdot (\tilde{\delta_7}\widehat{\delta_7} + \tilde{\mu_7}\widehat{\mu_7} + \tilde{\eta_7}\widehat{\eta_7})}{\tilde{\delta_7^2} + \tilde{\mu_7^2} + \tilde{\eta_7^2}}}{\frac{\sqrt{2} \cdot \sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2}}{(f^2 + 2g^2)^2}}]'}{1 + [\frac{\frac{\sqrt{2} \cdot (\tilde{\delta_7}\widehat{\delta_7} + \tilde{\mu_7}\widehat{\mu_7} + \tilde{\eta_7}\widehat{\eta_7})}{\tilde{\delta_7^2} + \tilde{\mu_7^2} + \tilde{\eta_7^2}}}{\frac{\sqrt{2} \cdot \sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2}}{(f^2 + 2g^2)^2}}]^2} \end{cases} \tag{4.44}$$

where

$$\begin{cases} \delta_7 = (fg(f^2+2g^2)) + 2g(fg' - gf') \\ \mu_7 = -(f^2+2g^2)(f^2+g^2) + f(f'g - g'f) \\ \eta_7 = -g^2(f^2+2g^2) + f(fg' - gf') \end{cases} \tag{4.45}$$

$$\begin{cases} \bar{\delta_7} = [(\delta_7' - f\mu_7)(\delta_7^2+\mu_7^2+\eta_7^2) - \delta_7(\delta_7\delta_7' + \mu_7\mu_7' + \eta_7\eta_7')] \\ \bar{\mu_7} = [(f\delta_7 + \mu_7' - g\eta_7)(\delta_7^2+\mu_7^2+\eta_7^2) - \mu_7(\delta_7\delta_7' + \mu_7\mu_7' + \eta_7\eta_7')] \\ \bar{\eta_7} = [(g\mu_7 + \eta_7')(\delta_7^2+\mu_7^2+\eta_7^2) - \eta_7(\delta_7\delta_7' + \mu_7\mu_7' + \eta_7\eta_7')] \end{cases} \tag{4.46}$$

$$\begin{cases} \widehat{\delta_7} = (-f'' + f(2g' + f^2) + g(f' + gf)) \\ \widehat{\mu_7} = (f(-3f' + gf) + g(-3g' + g^2) - g'') \\ \widehat{\eta_7} = -g(f^2+g^2+3g') + g'' \end{cases} \tag{4.47}$$

$$\begin{cases} \tilde{\delta_7} = (2g^3 + gf^2) \\ \tilde{\mu_7} = (g'f - gf') \\ \tilde{\eta_7} = (f^3 + 2fg^2 + fg' - gf') \end{cases} \tag{4.48}$$

Proof. Let $\beta(s)$ be a unit speed regular CW-Smarandache curve as in (4.43). If we take the derivative of Smarandache curve according to arclenght parameter, we have

$$\frac{d\beta_{CW}}{ds_\beta}\frac{ds_\beta}{ds} = \frac{1}{\sqrt{2}}(-fN + gW - gC), \tag{4.49}$$

and since

$$\left\| \frac{d\beta_{CW}}{ds_\beta} \right\| = 1,$$

we can see

$$\frac{ds_\beta}{ds} = \sqrt{\frac{1}{2}(f^2+g^2+g^2)} = \sqrt{\frac{f^2+2g^2}{2}} \tag{4.50}$$

From the equations (4.49) and (4.50), tangent vector of β_{CW} is

$$T_{\beta_{CW}} = \frac{-fN + gW - gC}{\sqrt{f^2+2g^2}}. \tag{4.51}$$

If we take derivate this expression is again, we can see that

$$T'_{\beta_{CW}} = \frac{\delta_7 N + \mu_7 C + \eta_7 W}{(f^2+2g^2)^{\frac{3}{2}}} \cdot \frac{\sqrt{2}}{\sqrt{(f^2+2g^2)}} \tag{4.52}$$

where

$$\begin{cases} \delta_7 = (fg(f^2+2g^2)) + 2g(fg' - gf') \\ \mu_7 = -(f^2+2g^2)(f^2+g^2) + f(f'g - g'f) \\ \eta_7 = -g^2(f^2+2g^2) + f(fg' - gf') \end{cases}$$

The curvature of the β_{CW} is indicated by the $\kappa_{\beta_{CW}}$ taking the norm of equation (4.52).

$$\kappa_{\beta_{CW}} = \frac{\sqrt{2}}{(f^2+2g^2)^2}\sqrt{\delta_7^2+\mu_7^2+\eta_7^2} \tag{4.53}$$

If the principal normal of β_{CW} is indicated by $N_{\beta_{CW}}$, it is found in the form of

$$N_{\beta_{CW}} = \frac{\delta_7 N + \mu_7 C + \eta_7 W}{\sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2}} \tag{4.54}$$

If we take the derivative of the equation (4.54), we obtain

$$N^{'} = \frac{\sqrt{2}}{\sqrt{f^2 + 2g^2}} \frac{\bar{\delta_7} N + \bar{\mu_7} C + \bar{\eta_7} W}{(2f^2 + g^2)^{\frac{3}{2}}} \tag{4.55}$$

where

$$\begin{cases} \bar{\delta_7} = [(\delta_7^{'} - f\mu_7)(\delta_7^2 + \mu_7^2 + \eta_7^2) - \delta_7(\delta_7\delta_7^{'} + \mu_7\mu_7^{'} + \eta_7\eta_7^{'})] \\ \bar{\mu_7} = [(f\delta_7 + \mu_7^{'} - g\eta_7)(\delta_7^2 + \mu_7^2 + \eta_7^2) - \mu_7(\delta_7\delta_7^{'} + \mu_7\mu_7^{'} + \eta_7\eta_7^{'})] \\ \bar{\eta_7} = [(g\mu_7 + \eta_7^{'})(\delta_7^2 + \mu_7^2 + \eta_7^2) - \eta_7(\delta_7\delta_7^{'} + \mu_7\mu_7^{'} + \eta_7\eta_7^{'})] \end{cases}$$

If we take the norm of the equation (4.55), we get

$$\|N_{\beta_{CW}}^{'}\| = \frac{\sqrt{2}}{\sqrt{f^2 + 2g^2}} \frac{\sqrt{\bar{\delta}_7^2 + \bar{\mu}_7^2 + \bar{\eta}_7^2}}{(\bar{\delta}_7^2 + \bar{\mu}_7^2 + \bar{\eta}_7^2)^{\frac{3}{2}}} \tag{4.56}$$

Since $C_{\beta_{CW}} = \frac{N_{\beta_{CW}}^{'}}{\|N_{\beta_{CW}}^{'}\|}$, if necessary calculations are made from the equations (4.55) and (4.56)

$$C_{\beta_{CW}} = \frac{\bar{\delta_7} N + \bar{\mu_7} C + \bar{\eta_7} W}{\sqrt{\bar{\delta}_7^2 + \bar{\mu}_7^2 + \bar{\eta}_7^2}}.$$

From the definition of Darboux vector, we know $W_{\beta_{CW}} = N_{\beta_{CW}} \times C_{\beta_{CW}}$. So we have

$$W_{\beta_{CW}} = \frac{1}{\sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2} \cdot \sqrt{\bar{\delta_7}^2 + \bar{\mu_7}^2 + \bar{\eta_7}^2}} \begin{vmatrix} N & C & W \\ \delta_7 & \mu_5 & \eta_7 \\ \bar{\delta_7} & \bar{\mu_7} & \bar{\eta_7} \end{vmatrix}$$

and so on

$$W_{\beta_{CW}} = \frac{(\mu_7\bar{\eta_7} - \eta_5\bar{\mu_7})N - (\delta_7\bar{\eta_7} - \eta_7\bar{\delta_7})C + (\delta_7\bar{\mu_7} - \mu_7\delta_7)W}{\sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2} \cdot \sqrt{\bar{\delta_7}^2 + \bar{\mu_7}^2 + \bar{\eta_7}^2}}. \tag{4.57}$$

To find the torsion, we need to find the second and third derivates of the β_{CW} curve. These derivates are available below.

$$\beta_{CW}(s) = \frac{1}{\sqrt{2}}(C + W), \tag{4.58}$$

$$\beta_{CW}^{'} = \frac{1}{\sqrt{2}}(-fN + gW - gC), \tag{4.59}$$

$$\beta_{CW}^{''} = \frac{1}{\sqrt{2}}(-f^{'} + gf)N + (-f^2 - g^2 - g^{'})C + (g^{'} - g^2)W), \tag{4.60}$$

$$\beta_{CW}^{'''} = \frac{1}{\sqrt{2}}(\widehat{\delta_7} N + \widehat{\mu_7} C + \widehat{\eta_7} W) \tag{4.61}$$

where

$$\begin{cases} \widehat{\delta_7} = (-f^{''} + f(2g^{'} + f^2) + g(f^{'} + gf)) \\ \widehat{\mu_7} = (f(-3f^{'} + gf) + g(-3g^{'} + g^2) - g^{''}) \\ \widehat{\eta_7} = -g(f^2 + g^2 + 3g^{'}) + g^{''} \end{cases}$$

In equation (2.3), if the expressions (4.59), (4.60) and (4.61) are written in their places and the necessary calculations are made, torsion is found as

(4.62)

$$\tau_{\beta_{CW}} = \frac{\sqrt{2} \cdot \left[\ (2g^3 + gf^2)\widehat{\delta_7} + (g^{'}f - gf^{'})\widehat{\mu_7} + (f^3 + 2fg^2 + fg^{'} - gf^{'})\widehat{\eta_7} \ \right]}{(2g^3 + gf^2)^2 + (g^{'}f - gf^{'})^2 + (f^3 + 2fg^2 + fg^{'} - gf^{'})^2}$$

In equation (2.6), if the expressions (4.53) and (4.62) are written in their places and the necessary calculations are made, curvature and torsion according to alternative frame are obtained as

$$f = \sqrt{[\frac{\sqrt{2} \cdot \sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2}}{(f^2 + 2g^2)^2}]^2 + [\frac{\sqrt{2} \cdot (\tilde{\delta_7}\widehat{\delta_7} + \tilde{\mu_7}\widehat{\mu_7} + \tilde{\eta_7}\widehat{\eta_7})}{\tilde{\delta}_7^2 + \tilde{\mu_7}^2 + \tilde{\eta_7}^2}]^2} \tag{4.63}$$

and

$$g = \frac{[\frac{\frac{\sqrt{2} \cdot (\tilde{\delta_7}\widehat{\delta_7} + \tilde{\mu_7}\widehat{\mu_7} + \tilde{\eta_7}\widehat{\eta_7})}{\tilde{\delta}_7^2 + \tilde{\mu}_7^2 + \tilde{\eta}_7^2}}{\frac{\sqrt{2} \cdot \sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2}}{f^2 + 2g^2}}]'}{1 + [\frac{\frac{\sqrt{2} \cdot (\tilde{\delta_7}\widehat{\delta_7} + \tilde{\mu_7}\widehat{\mu_7} + \tilde{\eta_7}\widehat{\eta_7})}{\tilde{\delta}_7^2 + \tilde{\mu}_7^2 + \tilde{\eta}_7^2}}{\frac{\sqrt{2} \cdot \sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2}}{f^2 + 2g^2}}]^2} \tag{4.64}$$

where

$$\begin{cases} \tilde{\delta_7} = (2g^3 + gf^2), \\ \tilde{\mu_7} = (g^{'}f - gf^{'}), \\ \tilde{\eta_7} = (f^3 + 2fg^2 + fg^{'} - gf^{'}). \end{cases}$$

□

Definition 4.7. Let $\beta(s)$ be a unit speed regular curve in E^3 and {N,C,W} be its moving alternative frame. NCW-Smarandache curve is defined by

$$\beta_{NCW}(s) = \frac{1}{\sqrt{3}}(N + C + W). \tag{4.65}$$

Theorem 4.8. *Let $\beta(s)$ be a unit speed regular curve in E^3. The curvature and torsion of NCW-Smarandache curve are as follows, respectively.*

$$\begin{cases} f = \sqrt{[\frac{\sqrt{3} \cdot \sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}}{(2f^2 + 2g^2 - 2gf)^2}]^2 + [\frac{\sqrt{3} \cdot (\tilde{\delta_8}\widehat{\delta_8} + \tilde{\mu_8}\widehat{\mu_8} + \tilde{\eta_8}\widehat{\eta_8})}{\tilde{\delta}_8^2 + \tilde{\mu_8}^2 + \tilde{\eta_8}^2}]^2} \\ \\ g = \frac{[\frac{\frac{\sqrt{3} \cdot (\tilde{\delta_8}\widehat{\delta_8} + \tilde{\mu_8}\widehat{\mu_8} + \tilde{\eta_8}\widehat{\eta_8})}{\tilde{\delta}_8^2 + \tilde{\mu}_8^2 + \tilde{\eta}_8^2}}{\frac{\sqrt{3} \cdot \sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}}{(2f^2 + 2g^2 - 2gf)^2}}]'}{1 + [\frac{\frac{\sqrt{3} \cdot (\tilde{\delta_8}\widehat{\delta_8} + \tilde{\mu_8}\widehat{\mu_8} + \tilde{\eta_8}\widehat{\eta_8})}{\tilde{\delta}_8^2 + \tilde{\mu}_8^2 + \tilde{\eta}_5^2}}{\frac{\sqrt{3} \cdot \sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}}{(2f^2 + 2g^2 - 2gf)^2}}]^2} \end{cases} \tag{4.66}$$

where
(4.67)

$$\begin{cases} \delta_8 = gff' - 2f'g^2 - 2f^4 - 4f^2g^2 + 4f^3g + 2g^3f + 2fgg' - f^2g' \\ \mu_8 = f^2(-2f^2 - 4g^2 - 2fg - g') + g^2(-2g^4 + 2fg - g' + fg(f' - g')) \\ \eta_8 = 2f^2(fg - 2g^2 + g') + g^2(4fg - 2g^2 + f') - fg(g' + 2f') \end{cases}$$

$$(4.68) \qquad \begin{cases} \bar{\delta_8} = [(\delta_8' - f\mu_8)(\delta_8^2 + \mu_8^2 + \eta_8^2) - \delta_8(\delta_8\delta_8' + \mu_8\mu_8' + \eta_8\eta_8')] \\ \bar{\mu_8} = [(f\delta_8 + \mu_8' - g\eta_8)(\delta_8^2 + \mu_8^2 + \eta_8^2) - \mu_8(\delta_8\delta_8' + \mu_8\mu_8' + \eta_8\eta_8')] \\ \bar{\eta_8} = [(g\mu_8 + \eta_8')(\delta_8^2 + \mu_8^2 + \eta_8^2) - \eta_8(\delta_8\delta_8' + \mu_8\mu_8' + \eta_8\eta_8')] \end{cases}$$

$$(4.69) \qquad \begin{cases} \widehat{\delta_8} = (f^3 + fg^2 - 3ff' - f'' + 2g'f + gf') \\ \widehat{\mu_8} = (g^3 - f^3 - 3(ff' + gg') - (-f'' + g'') + fg(f - g)) \\ \widehat{\eta_8} = (g'' - f^2g - 3gg' - g^3 + 2gf' + fg') \end{cases}$$

$$(4.70) \qquad \begin{cases} \tilde{\delta_8} = (2f^2g - 2fg^2 + fg' - gf') \\ \tilde{\mu_8} = (fg' - f'g) \\ \tilde{\eta_8} = (2f^3 + 2fg^2 - 2gf^2 - gf' + fg') \end{cases}$$

Proof. Let $\beta(s)$ be a unit speed reguler NCW-Smarandache curve as in (4.65). If we take the derivative of the Smarandache curve according to arclenght parameter, we have

$$(4.71) \qquad \frac{d\beta_{NCW}}{ds_\beta}\frac{ds_\beta}{ds} = \frac{1}{\sqrt{3}}(fC - fN + gW - gC),$$

and since

$$\left\| \frac{d\beta_{NCW}}{ds_\beta} \right\| = 1,$$

we can see

$$(4.72) \qquad \frac{ds_\beta}{ds} = \sqrt{\frac{2}{3}(f^2 + g^2 - gf)}.$$

From the equations (4.71) and (4.72) tangent vector of β_{NCW} is

$$(4.73) \qquad T_{\beta_{NCW}} = \frac{fC - fN + gW - gC}{\sqrt{2(f^2 + g^2 - gf)}}$$

If we take derivate this expression is again, we can see that

$$(4.74) \qquad T'_{\beta_{NCW}} = \frac{\delta_8 N + \mu_8 C + \eta_8 W}{(2f^2 + 2g^2 - 2gf)^{\frac{3}{2}}} \frac{\sqrt{3}}{\sqrt{(2f^2 + 2g^2 - 2gf)}}$$

where

$$\begin{cases} \delta_8 = gff' - 2f'g^2 - 2f^4 - 4f^2g^2 + 4f^3g + 2g^3f + 2fgg' - f^2g' \\ \mu_8 = f^2(-2f^2 - 4g^2 - 2fg - g') + g^2(-2g^4 + 2fg - g' + fg(f' - g')) \\ \eta_8 = 2f^2(fg - 2g^2 + g') + g^2(4fg - 2g^2 + f') - fg(g' + 2f') \end{cases}$$

The curvature of the β_{NCW} is indicated by the $\kappa_{\beta_{NCW}}$ taking the norm of equation (4.74)

$$(4.75) \qquad \kappa_{\beta_{NCW}} = \frac{\sqrt{3}}{(2f^2 + 2g^2 - 2gf)^2}\sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}.$$

If the principal normal of β_{NCW} is indicated by $N_{\beta_{NCW}}$, it is found in the form of

$$N_{\beta_{NCW}} = \frac{\delta_8 N + \mu_8 C + \eta_8 W}{\sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}}. \tag{4.76}$$

If we take the derivative of the equation (4.76), we obtain

$$N^{'} = \frac{\sqrt{3}}{\sqrt{2f^2 + 2g^2 - 2gf}} \frac{\bar{\delta}_8 N + \bar{\mu}_8 C + \bar{\eta}_8 W}{(\bar{\delta}_8^2 + \bar{\mu}_8^2 + \bar{\eta}_8^2)^{\frac{3}{2}}} \tag{4.77}$$

where

$$\begin{cases} \bar{\delta_8} = [(\delta_8^{'} - f\mu_8)(\delta_8^2 + \mu_8^2 + \eta_8^2) - \delta_8(\delta_8\delta_8^{'} + \mu_8\mu_8^{'} + \eta_8\eta_8^{'})] \\ \bar{\mu_8} = [(f\delta_8 + \mu_8^{'} - g\eta_8)(\delta_8^2 + \mu_8^2 + \eta_8^2) - \mu_8(\delta_8\delta_8^{'} + \mu_8\mu_8^{'} + \eta_8\eta_8^{'})] \\ \bar{\eta_8} = [(g\mu_8 + \eta_8^{'})(\delta_8^2 + \mu_8^2 + \eta_8^2) - \eta_8(\delta_8\delta_8^{'} + \mu_8\mu_8^{'} + \eta_8\eta_8^{'})] \end{cases}$$

If we take the norm of the equation (4.77), we get

$$\|N_{\beta_{NCW}}^{'}\| = \frac{\sqrt{3}}{\sqrt{2f^2 + 2g^2 - 2gf}} \frac{\sqrt{\bar{\delta}_8^2 + \bar{\mu}_8^2 + \bar{\eta}_8^2}}{(\bar{\delta}_8^2 + \bar{\mu}_8^2 + \bar{\eta}_8^2)^{\frac{3}{2}}} \tag{4.78}$$

Since $C_{\beta_{NCW}} = \frac{N_{\beta_{NCW}}^{'}}{\|N_{\beta_{NCW}}^{'}\|}$, if necessary calculations are made from the equations (4.77) and (4.78)

$$C_{\beta_{NCW}} = \frac{\bar{\delta}_8 N + \bar{\mu}_8 C + \bar{\eta}_8 W}{\sqrt{\bar{\delta}_8^2 + \bar{\mu}_8^2 + \bar{\eta}_8^2}}.$$

From the definition of Darboux vector, we know $W_{\beta_{NCW}} = N_{\beta_{NCW}} \times C_{\beta_{NCW}}$. So we have

$$W_{\beta_{NCW}} = \frac{1}{\sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}\sqrt{\bar{\delta_8}^2 + \bar{\mu_8}^2 + \bar{\eta_8}^2}} \begin{vmatrix} N & C & W \\ \delta_8 & \mu_8 & \eta_8 \\ \bar{\delta_8} & \bar{\mu_8} & \bar{\eta_8} \end{vmatrix}$$

and so on

$$W_{\beta_{NCW}} = \frac{(\mu_8\bar{\eta_8} - \eta_8\bar{\mu_8})N - (\delta_8\bar{\eta_8} - \eta_8\bar{\delta_8})C + (\delta_8\bar{\mu_8} - \mu_8\delta_8)W}{\sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2} \cdot \sqrt{\bar{\delta_8}^2 + \bar{\mu_8}^2 + \bar{\eta_8}^2}}$$

To find the torsion, we need to find the second and third derivates of the β_{NCW} curve. These derivates are available below.

$$\beta_{NCW}(s) = \frac{1}{\sqrt{3}}(N + C + W), \tag{4.79}$$

$$\beta_{NCW}^{'} = \frac{1}{\sqrt{3}}(fC - fN + gW - gC), \tag{4.80}$$

$$\beta_{NCW}^{''} = \frac{1}{\sqrt{3}}((-f^{'} - f^2 + gf)N + (-f^2 + f^{'} - g^{'} - g^2)C + (fg - g^2 + g^{'})W), \tag{4.81}$$

$$\beta_{NCW}^{'''} = \frac{1}{\sqrt{3}}(\widehat{\delta_8}N + \widehat{\mu_8}C + \widehat{\eta_8}W) \tag{4.82}$$

where

$$\begin{cases} \widehat{\delta_8} = (f^3 + fg^2 - 3ff^{'} - f^{''} + 2g^{'}f + gf^{'}) \\ \widehat{\mu_8} = (g^3 - f^3 - 3(ff^{'} + gg^{'}) - (-f^{''} + g^{''}) + fg(f-g)) \\ \widehat{\eta_8} = (g^{''} - f^2 g - 3gg^{'} - g^3 + 2gf^{'} + fg^{'}) \end{cases}$$

In equation (2.3), if the expressions (4.80), (4.81) and (4.82) are written in their places and the necessary calculations are made, torsion is found as

(4.83)

$$\tau_{\beta_{NCW}} = \frac{\sqrt{3}\cdot\left[\ (2f^2g - 2fg^2 + fg^{'} - gf^{'})\widehat{\delta_8} + (fg^{'} - f^{'}g)\widehat{\mu_8} + (2f^3 + 2fg^2 - 2gf^2 - gf^{'} + fg^{'})\widehat{\eta_8}\ \right]}{(2f^2g - 2fg^2 + fg^{'} - gf^{'})^2 + (fg^{'} - f^{'}g)^2 + (2f^3 + 2fg^2 - 2gf^2 - gf^{'} + fg^{'})^2}$$

In equation (2.6), if the expressions (4.75) and (4.83) are written in their places and the necessary calculations are made, curvature and torsion according to alternative frame are obtained as

$$f = \sqrt{[\frac{\sqrt{3}\cdot\sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}}{(2f^2 + 2g^2 - 2gf)^2}]^2 + [\frac{\sqrt{3}\cdot(\tilde{\delta_8}\widehat{\delta_8} + \tilde{\mu_8}\widehat{\mu_8} + \tilde{\eta_8}\widehat{\eta_8})}{\tilde{\delta_8}^2 + \tilde{\mu_8}^2 + \tilde{\eta_8}^2}]^2} \tag{4.84}$$

and

$$g = \frac{[\frac{\frac{\sqrt{3}\cdot(\tilde{\delta_8}\widehat{\delta_8} + \tilde{\mu_8}\widehat{\mu_8} + \tilde{\eta_8}\widehat{\eta_8})}{\tilde{\delta_8}^2 + \tilde{\mu_8}^2 + \tilde{\eta_8}^2}}{\frac{\sqrt{3}\cdot\sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}}{2f^2 + 2g^2 - 2gf}}]'}{1 + [\frac{\frac{\sqrt{3}\cdot(\tilde{\delta_8}\widehat{\delta_8} + \tilde{\mu_8}\widehat{\mu_8} + \tilde{\eta_8}\widehat{\eta_8})}{\tilde{\delta_8}^2 + \tilde{\mu_8}^2 + \tilde{\eta_8}^2}}{\frac{\sqrt{3}\cdot\sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}}{2f^2 + 2g^2 - 2gf}}]^2} \tag{4.85}$$

where

$$\begin{cases} \tilde{\delta_8} = (2f^2g - 2fg^2 + fg^{'} - gf^{'}), \\ \tilde{\mu_8} = (fg^{'} - f^{'}g), \\ \tilde{\eta_8} = (2f^3 + 2fg^2 - 2gf^2 - gf^{'} + fg^{'}). \end{cases}$$

□

5. Conclusion

Smarandache curves have been studied many times since they were defined. The importance of this study is that, unlike the studies in the literature, these curves are re-characterized with the help of an alternative frame different from Frenet frame.

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CONNECTEDNESS IN TEMPORAL INTUITIONISTIC FUZZY TOPOLOGY IN CHANG'S SENSE

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Abstract. In this paper, connectedness in temporal intuitionistic fuzzy topology in Chang's sense is introduced and investigated. In the content of the paper, basic definitions, theorems and propositions about connectedness in temporal intuitionistic fuzzy topology in Chang's sense are given.

1. Introduction

Fuzzy logic was firstly defined by Zadeh in 1965 [21]. Then,Intuitionistic fuzzy set (shortly IFS) was defined by K.Atanassov [1, 2]. Intuitionistic fuzzy logic comes into play in situations where fuzzy logic cannot respond or is insufficient. The intuitionistic fuzzy set theory is useful in various application areas such as; medicine, medical diagnosis, medical application, career determination, real life situations, education, decision making, multi criteria decision making, artificial intellligence, networking, computer,smart systems, economy and various fields. The concept of fuzzy topology was defined by Chang in 1968 [4]. Çoker generalized the concept of fuzzy topology in the sense of intuitionistic fuzzy set theory in 1997. The fuzzifying of the concept of topology was made by Šostak in 1985 [19]. Coker and Demirci [6] defined the concept of intuitionistic fuzzy set in Šostak's sense in 1996. Temporal intuitionistic fuzzy set, another approach in which temporal variables also participated in calculating the membership and non-membership degrees, was defined by Atanassov in 1991 [3]. This is one of the most important extensions of IFS. In recent years, Šostak's mean temporal intuitionistic fuzzy topology was defined by Kutlu and Bilgin [9]. Also, the other fundamental concepts of Šostak's mean temporal intuitionistic fuzzy topology defined by the author in [10, 9, 11]. The concepts of temporal and overall intuitionistic fuzzy topology in Chang's sense firstly defined by Kutlu in 2019 [13]. In this study, Kutlu gave basic definitions and theorems and explained them in detail. The concept of temporal intuitionistic fuzzy has recently started to attract the attention of researchers [13, 12, 14]. The concept of connectedness in intuitionistic fuzzy topological spaces in Šostak's sense is investigated by

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El-Latif and Khalaf [7]. Connectedness in intuitionistic fuzzy special topological spaces is researched by Özçağ and Çoker [17]. Connectedness in intuitionistic fuzzy topological spaces is investigated by Kim and Abbas [8]. In this paper, connectedness in temporal intuitionistic fuzzy topology in Chang's sense is introduced and investigated. Basic definition, theorem and propositions about connectedness in temporal intuitionistic fuzzy topology in Chang's sense are given.

2. Preliminaries

Definition 2.1. [1] An intuitionistic fuzzy set in a non-empty set X given by a set of ordered triples $A=\{(x,\mu_A(x),\eta_A(x)):\ x\in X\}$ where $\mu_A(x):\ X\to I$,$\eta_A(x):$ $X\to I$ and $I=[0,1]$, are functions such that $0\le\mu(x)+\eta(x)\le 1$for all $x\in X$. For $x\in X$, $\mu_A(x)$ and $\eta_A(x)$ represent the degree of membership and degree of non-membership of x to A respectively. For each $x\in X$; intuitionistic fuzzy index of x in A can be defined as follows $\pi_A(x)=1-\mu_A(x)-\eta_A(x)$. π_A is the called degree of hesitation or indeterminacy.

By $IFS(X)$, we denote to the set of all intuitionistic fuzzy sets.

Definition 2.2. [1] Let$A,B\in IFS(X)$. Then,

(i) $A\subseteq B\Leftrightarrow\mu_A(x)\le\mu_B(x)$ and $\eta_A(x)\ge\eta_B(x)$for$\forall x\in X$,

(ii) $A=B\Leftrightarrow A\subseteq B$ and $B\subseteq A$,

(iii) $A^c=\{(x,\eta_A(x),\mu_A(x)):\ x\in X\}$,

(iv) $\bigcap A_i=\{(x,\wedge\mu_{A_i}(x),\vee\eta_{A_i}(x)):\ x\in X\}$,

(v) $\bigcup A_i=\{(x,\vee\mu_{A_i}(x),\wedge\eta_{A_i}(x)):x\in X\}$,

(vi) $\tilde{0}=\{(x,0,1):\ x\in X\}$ and $\tilde{1}=\{(x,1,0):\ x\in X\}$.

Definition 2.3. [5, 1] Let a and b be two real numbers in $[0,1]$ satisfying the inequality$a+b\le 1$. Then, the pair $\langle a,b\rangle$ is called an intuitionistic fuzzy pair. Let $\langle a_1,b_1\rangle$ and $\langle a_2,b_2\rangle$be two intuitionistic fuzzy pair (briefly IF-pair). Then define

(i) $\langle a_1,b_1\rangle\le\langle a_2,b_2\rangle\Leftrightarrow a_1\le a_2$ and $b_1\ge b_2$,

(ii) $\langle a_1,b_1\rangle=\langle a_2,b_2\rangle\Leftrightarrow a_1=a_2$ and $b_1=b_2$,

(iii) If $\{\langle a_i,b_i\rangle\,;i\in J\}$ is a family of intuitionistic fuzzy pairs, then $\vee\langle a_i,b_i\rangle=\langle\vee a_i,\wedge b_i\rangle$ and $\wedge\langle a_i,b_i\rangle=\langle\wedge a_i,\vee b_i\rangle$,

(iv) The complement of $\langle a,b\rangle$is defined by $\overline{\langle a,b\rangle}=\langle b,a\rangle$,

(v) $1^{\sim}=\langle 1,0\rangle$ and $0^{\sim}=\langle 0,1\rangle$. .

Definition 2.4. [6] An intuitionistic fuzzy topology in Chang's sense (briefly, CT-IFS) on a non-empty set X is a family τ_t of TIFSs satisfying the following axioms:

I. $\tilde{0}\in\tau$ and $\tilde{1}\in\tau$,

II. $A_1\cap A_2\in\tau$ for each $A_1,\ A_2\in\tau$,

III. $\bigcup\limits_{i\in I}A_i\in\tau$ For any arbitrary family $\{A_i;\ i\in I\}\in\tau$,

Definition 2.5. [3]. Let E be an universe and T be a non-empty time-moment set. We call the elements of T "time moments". Based on the definition of IFS, a temporal intuitionistic fuzzy set (breifly TIFS) A is defined as the following: $A(T)=\{(x,\mu_A(x,t)\,,\,\eta_A(x,t)\,):\ (x,t)\in E\times T\}$ where:

(a) $A\subseteq E$ is a fixed set

(b) $\mu_A(x,t)+\ \eta_A(x,t)\le 1$ for every $(x,t)\in E\times T$

(c) $\mu_A(x,t)$ and $\eta_A(x,t)$ are the degrees of membership and non-membership, respectively, of the element $x\in E$ at the time moment $t\in T$

By $TIFS^{(X,T)}$, we denote to the set of all TIFSs over nonempty set X and time-moment set T. For brevity, we write A instead of$A(T)$. The hesitation degree of a TIFS is defined as$\pi_A(x,t) = 1 - \mu_A(x,t) - \eta_A(x,t)$. Obviously, every ordinary IFS can be regarded as TIFS for which T is a singleton set. All operations and operators on IFS can be defined for TIFSs.

Definition 2.6. [13] Let $A(T') = \{(x, \mu_A(x,t), \eta_A(x,t)) : (x,t) \in X \times T'\}$ and $B(T'') = \{(x, \mu_B(x,t), \eta_B(x,t)) : (x,t) \in X \times T''\}$ where$T'$and T''have finite number of distinct time-elements or they are time intervals. Then,

$$A(T') \cap B(T'') = \{(x,\ \min(\bar{\mu}_A(x,t), \bar{\mu}_B(x,t)), \max(\bar{\eta}_A(x,t), \bar{\eta}_B(x,t)) : (x,t) \in X \times (T' \cup T''))$$

$$A(T') \cup B(T'') = \{(x,\ \max(\bar{\mu}_A(x,t), \bar{\mu}_B(x,t)), \min(\bar{\eta}_A(x,t), \bar{\eta}_B(x,t)) : (x,t) \in X \times (T' \cup T''))$$

Also from definition of subset in IFS theory, Subsets of TIFS can be defined as the follow: $A(T') \subseteq B(T'') \Leftrightarrow \bar{\mu}_A(x,t) \leq \bar{\mu}_B(x,t)$ and $\bar{\eta}_A(x,t) \geq \bar{\eta}_B(x,t)$ for every $(x,t) \in X \times (T' \cup T'')$ where

$$\bar{\mu}_A(x,t) = \begin{cases} \mu_A(x,t), & if\ t \in T' \\ 0, & if\, t \in T'' - T' \end{cases}$$

$$\bar{\mu}_B(x,t) = \begin{cases} \mu_B(x,t) & if\, t \in T'' \\ 0, & if\, t \in T' - T'' \end{cases}$$

$$\bar{\eta}_A(x,t) = \begin{cases} \eta_A(x,t), & if\, t \in T' \\ 1, & if\, t \in T'' - T' \end{cases}$$

$$\bar{\eta}_B(x,t) = \begin{cases} \eta_B(x,t), & if\, t \in T'' \\ 1, & if\, t \in T'' - T' \end{cases}$$

It is obviously seen that $\bar{\mu}_A(x,t) = \mu_A(x,t)$, $\bar{\mu}_B(x,t) = \mu_B(x,t)$, $\bar{\eta}_A(x,t) = \eta_A(x,t)$, $\bar{\eta}_B(x,t) = \eta_B(x,t)$ when $T' = T''$.

Let Jbe an arbitrary index set. Then we define that $T = \bigcup_{i \in J} T_i$where T_i is a time set for each $i \in J$. Thus, we can extend the definition of union and intersection of TIFSs family $F = \{A_i(T_i) = (x, \mu_{A_i}(x,t), \eta_{A_i}(x,t)) : x \in X \times T_i,\ i \in J\}$ as follows:

$$\bigcup_{i \in J} A(T_i) = \left\{ \left(x,\ \max_{i \in J} (\bar{\mu}_{A_i}(x,t)), \min_{i \in J} (\bar{\eta}_{A_i}(x,t)) :\ (x,t) \in X \times T \right) \right.,$$

$$\bigcap_{i \in J} A(T_i) = \left\{ \left(x,\ \min_{i \in J} (\bar{\mu}_{A_i}(x,t)), \max_{i \in J} (\bar{\eta}_{A_i}(x,t)) :\ (x,t) \in X \times T \right) \right.,$$

where

$$\bar{\mu}_{A_j}(x,t) = \begin{cases} \mu_{A_j}(x,t), & if\, t \in T_j \\ 0, & if\, t \in T - T_j \end{cases}$$

and

$$\bar{\eta}_{A_j}(x,t) = \begin{cases} \eta_{A_j}(x,t), & if\, t \in T_j \\ 1, & if\, t \in T - T_j \end{cases} .$$

The operations defined above are defined over all of the time moments. In the following definition, these operations will be defined for an individual time moment.

Definition 2.7. [13] Let

$$A(T') = \{(x, \mu_A(x,t), \eta_A(x,t)) : (x,t) \in X \times T'\}$$
$$B(T'') = \{(x, \mu_B(x,t), \eta_B(x,t)) : (x,t) \in X \times T''\}$$

where T' and T''have finite number of distinct time-elements or they are time intervals. Then, the definitions of instant intersection and instant union of TIFSs are defined as follows:

$$\begin{aligned} A(T') \cap_{t_0} B(T'') &= \{(x, \min(\bar{\mu}_A(x,t_0), \bar{\mu}_B(x,t_0)), \max(\bar{\eta}_A(x,t_0), \bar{\eta}_B(x,t_0)) : \\ &\quad (x,t_0) \in X \times (T' \cup T'')\}, \end{aligned}$$

$$\begin{aligned} A(T') \cup_{t_0} B(T'') &= \{(x, \max(\bar{\mu}_A(x,t_0), \bar{\mu}_B(x,t_0)), \min(\bar{\eta}_A(x,t_0), \bar{\eta}_B(x,t_0)) : \\ &\quad (x,t_0) \in X \times (T' \cup T'')\}. \end{aligned}$$

Also from definition of subset in IFS theory, instant subsets of TIFS can be defined as the following: $A(T') \subseteq_{t_0} B(T'') \Leftrightarrow \bar{\mu}_A(x,t_0) \leq \bar{\mu}_B(x,t_0)$ and $\bar{\eta}_A(x,t) \geq \bar{\eta}_B(x,t)$ for every $(x,t_0) \in X \times (T' \cup T'')$ where

$$\bar{\mu}_A(x,t_0) = \begin{cases} \mu_A(x,t_0), & if\, t_0 \in T' \\ 0, & if\, t_0 \in T'' - T' \end{cases}$$

$$\bar{\mu}_B(x,t_0) = \begin{cases} \mu_B(x,t_0), & if\, t_0 \in T'' \\ 0, & if\, t_0 \in T' - T'' \end{cases}$$

$$\bar{\eta}_A(x,t_0) = \begin{cases} \eta_A(x,t_0), & if\, t_0 \in T' \\ 1, & if\, t_0 \in T'' - T' \end{cases}$$

$$\bar{\eta}_B(x,t_0) = \begin{cases} \eta_B(x,t_0), & if\, t_0 \in T'' \\ 1, & if\, t_0 \in T'' - T' \end{cases}$$

[13] Let J be an arbitrary index set. Then we define that $T = \bigcup_{i \in J} T_i$where T_i is a time set for each $i \in J$. Thus, we can extend the definition of union and intersection of TIFSs family $F_{t_0} = \{A_i(T_i) = (x, \mu_{A_i}(x,t_0), \eta_{A_i}(x,t_0)) : (x,t_0) \in X \times T_i, i \in J\}$ as follows:

$$\bigcup_{i \in J}{}^{t_0} A(T_i) = \left\{\left(x, \max_{i \in J}(\bar{\mu}_{A_i}(x,t_0)), \min_{i \in J}(\bar{\eta}_{A_i}(x,t_0)) : (x,t_0) \in X \times T\right)\right.$$

$$\bigcap_{i \in J}{}^{t_0} A(T_i) = \left\{\left(x, \min_{i \in J}(\bar{\mu}_{A_i}(x,t_0)), \max_{i \in J}(\bar{\eta}_{A_i}(x,t_0)) : (x,t_0) \in X \times T\right)\right.$$

where

$$\bar{\mu}_{A_j}(x,t_0) = \begin{cases} \mu_{A_j}(x,t_0), & if\, t_0 \in T_j \\ 0, & if\, t_0 \in T - T_j \end{cases}$$

$$\bar{\eta}_{A_j}(x,t_0) = \begin{cases} \eta_{A_j}(x,t_0), & if\, t_0 \in T_j \\ 1, & if\, t_0 \in T - T_j \end{cases}.$$

In fact, these TIFS operators can be seen as IFS operators over TIFSs, since they are defined for a single time moment.

Definition 2.8. [13] $\underset{\sim}{0}^t$ and $\underset{\sim}{1}^t \in TIFS^{(X,T)}$ are defined as:

$$\underset{\sim}{0}^t = \{(x, 0, 1) : (x,t) \in X \times T\}$$

and

$$\underset{\sim}{1}^t = \{(x, 1, 0) : (x,t) \in X \times T\}$$

for each time moment t, i.e. $\mu_{\underset{\sim}{0}^t}(x,t) = 0$, $\eta_{\underset{\sim}{0}^t}(x,t) = 1$ and $\mu_{\underset{\sim}{1}^t}(x,t) = 1$, $\eta_{\underset{\sim}{1}^t}(x,t) = 0$ for each $(x,t) \in X \times T$.

Definition 2.9. $\underset{\sim}{0}^{t_0}$ and $\underset{\sim}{1}^{t_0} \in TIFS^{(X,T)}$ are defined as:

$$\underset{\sim}{0}^{t_0} = \left\{ \left(x, \mu_{\underset{\sim}{0}^{t_0}}(x,t), \eta_{\underset{\sim}{0}^{t_0}}(x,t) \right) : \ (x,t) \in X \times T \right\}$$

and

$$\underset{\sim}{1}^{t_0} = \left\{ \left(x, \mu_{\underset{\sim}{1}^{t_0}}(x,t), \eta_{\underset{\sim}{1}^{t_0}}(x,t) \right) : \ (x,t) \in X \times T \right\}$$

for individual time moment $t_0 \in T$, i.e. $\mu_{\underset{\sim}{0}^t}(x,t_0) = 0$, $\eta_{\underset{\sim}{0}^t}(x,t_0) = 1$ and $\mu_{\underset{\sim}{1}^{t_0}}(x,t_0) = 1$, $\eta_{\underset{\sim}{1}^{t_0}}(x,t_0) = 0$ for each $(x,t_0) \in X \times \{t_0\}$.

3. MAIN RESULTS

Definition 3.1. [13] An temporal intuitionistic fuzzy topology in Chang's sense (briefly, CT-TIFS) on a non-empty set X is a family τ_{t_0} of TIFSs satisfying the following axioms for fixed time moment t_0

I. $\underset{\sim}{0}^{t_0} \in \tau_{t_0}$ and $\underset{\sim}{1}^{t_0} \in \tau_{t_0}$,

II. For each A_1, $A_2 \in \tau_{t_0}$, there exist a $F \in \tau_{t_0}$ such that $\mu_F(x,t_0) = \mu_{A_1 \cap_{t_0} A_2}(x,t_0)$, $\eta_F(x,t_0) = \eta_{A_1 \cap_{t_0} A_2}(x,t_0)$ for each $(x,t_0) \in X \times \{t_0\}$.

III. For any arbitrary family $\{A_i;\, i \in I\} \in \tau_{t_0}$, there exist a $D \in \tau_{t_0}$ such that $\mu_D(x,t_0) = \mu_{\bigcup\limits_{i \in I}{}^{t_0} A_i}(x,t_0)$ and $\eta_D(x,t_0) = \eta_{\bigcup\limits_{i \in I}{}^{t_0} A_i}(x,t_0)$ for each $(x,t_0) \in X \times \{t_0\}$.

The pair $((X,T), \tau_{t_0})$ is called temporal intuitionistic fuzzy topological space in Chang's sense. Any member of τ_{t_0} is called temporal intuitionistic fuzzy open set (TIFOS). On the other hand, the complement of any member of τ_{t_0} is called intuitionistic fuzzy closed set (TIFCS). It is obtained intuitionistic fuzzy topological space in Chang's sense from every temporal intuitionistic fuzzy topological space in Chang's sense by the following method.

Proposition 1. [13] Let τ_{t_0} is an temporal intuitionistic fuzzy topological space in Chang's sense on non-empty set X and time moment set T, Then we define IFS's from every $A \in \tau_{t_0}$TIFSs by following way: $\mu_{\hat{A}}(x) = \mu_{A_i}(x,t_0)$ and $\eta_{\hat{A}}(x) = \eta_A(x,t_0)$. So that the new family $\tau^{t_0} = \left\{ \hat{A} : A \in \tau_{t_0} \right\}$ obtained from τ_{t_0} is a intuitionistic fuzzy topology in Chang's sense.

Definition 3.2. [13] Let τ_{t_0} is an temporal intuitionistic fuzzy topological space in Chang's sense on non-empty set X and time moment set $T$and $A \in \tau_{t_0}$. Then temporal intuitionistic fuzzy interior and temporal intuitionistic fuzzy closure of A defined as follows: $\text{int}_{t_0}(A) = \cup \{G\,;\, G \in \tau_{t_0},\ G \subseteq A\}$, $cl_{t_0}(A) = \cap \left\{C\,;\, \bar{C} \in \tau_{t_0},\ A \subseteq C\right\}$. Following propositions are valid for both of fuzzy and intuitionistic fuzzy case [19, 4, 5, 6, 18, 16, 15], it can be proved as in the above-mentioned articles.

Definition 3.3. Let τ_{t_0} be a temporal intuitionistic fuzzy topological space in Chang's sense on non-empty set X and time moment set. Then,

i. A is a TIFCS in $\tau_{t_0} \Leftrightarrow cl_{t_0}(A) = A$,

ii. A is a TIFOS in $\tau_{t_0} \Leftrightarrow \text{int}_{t_0}(A) = A$,

iii. $cl_{t_0}(A) = \overline{\text{int}_{t_0}(A)}$ for any $A \in TIFS^{(X,T)}$,

iv. $\text{int}_{t_0}(A) = \overline{cl_{t_0}(A)}$ for any $A \in TIFS^{(X,T)}$,

v. $\operatorname{int}_{t_0}(A) \subseteq A$ for any $A \in TIFS^{(X,T)}$,
vi. $A \subseteq cl_{t_0}(A)$ for any $A \in TIFS^{(X,T)}$,
vii. $A \subseteq B \Rightarrow \operatorname{int}_{t_0}(A) \subseteq \operatorname{int}_{t_0}(B)$ for any $A, B \in TIFS^{(X,T)}$,
viii. $A \subseteq B \Rightarrow cl_{t_0}(A) \subseteq cl_{t_0}(B)$ for any $A, B \in TIFS^{(X,T)}$,
ix. $cl_{t_0}(cl_{t_0}(A)) = cl_{t_0}(A)$ for any $A \in TIFS^{(X,T)}$,
x. $\operatorname{int}_{t_0}(\operatorname{int}_{t_0}(A)) = \operatorname{int}_{t_0}(A)$ for any $A \in TIFS^{(X,T)}$,
xi. $\operatorname{int}_{t_0}(A \cap B) = \operatorname{int}_{t_0}(A) \cap \operatorname{int}_{t_0}(B)$ for any $A, B \in TIFS^{(X,T)}$,
xii. $cl_{t_0}(A \cup B) = cl_{t_0}(A) \cup cl_{t_0}(B)$ for any $A, B \in TIFS^{(X,T)}$,
xiii. $\operatorname{int}_{t_0}\left(\underset{\sim}{1}^{t_0}\right) = \underset{\sim}{1}^{t_0}$,
xiv. $cl_{t_0}\left(\underset{\sim}{0}^{t_0}\right) = \underset{\sim}{0}^{t_0}$. We will give definitions of temporal intuitionistic fuzzy continuous functions and open function definitions, which are defined for fuzzy and intuitionistic fuzzy sets in [19, 4, 5, 6, 18, 16, 15], respectively.

Theorem 3.4. $((X,T), \tau_{t_0})$ *is an temporal intuitionistic fuzzy topology in Chang's sense on nonempty set* x *an time moment* T *and* $X_1, X_2 \neq \emptyset$, $X = X_1 \cup_{t_0} X_2$ *and* $X_1 \cap_{t_0} X_2 = \emptyset_{t_0}$ *are subsets; the following expressions are equivalent:*
(i) $((X,T), \tau_{t_0})$ *temporal intuitionistic fuzzy topology in Chang's sense is the topological sum of* X_1 *and* X_2 *spaces.*
(ii) X_1 *and* X_2 *sets are both temporal intuitionistic fuzzy open set (TIFOS) and temporal intuitionistic fuzzy closed set (TIFCS) in* X*.*
(iii) X_1 *(or* X_2*) is both TIFOS and TIFCS.*
(iv) $cl_{t_0}(X_1) \cap_{t_0} X_2 = \emptyset_{t_0}$ *and* $X_1 \cap_{t_0} cl_{t_0}(X_2) = \emptyset_{t_0}$

Definition 3.5. Let τ_{t_0} is a temporal intuitionistic fuzzy topology in Chang's sense on nonempty set X and time moment set $t_0 \in T$ and $A, B \in \tau_{t_0}$. $((X,T), \tau_{t_0})$ is called to be temporal disconnected at time moment t_0 if there are sets of A and B nonempty set with;

$$\begin{aligned} A \cup_{t_0} B &= X \\ A \cap_{t_0} cl_{t_0}(B) &= \emptyset_{t_0} \\ cl_{t_0}(A) \cap_{t_0} B &= \emptyset_{t_0} \end{aligned}$$

otherwise $((X,T), \tau_{t_0})$ is called to be temporal connected.

Proposition 2. Let $((X,T), \tau_{t_0})$ is a temporal intuitionistic fuzzy topology in Chang's sense, the following expressions are equivalent:
(i) $((X,T), \tau_{t_0})$ temporal intuitionistic fuzzy topological space is temporal disconnected.
(ii) $((X,T), \tau_{t_0})$ has at least both open and closed subset that is nonempty and distinct from itself.

Proposition 3. A temporal intuitionistic fuzzy topological space $((X,T), \tau_{t_0})$ is temporal connected at time moment t_0 if and only if it has no subset, both open and closed, other than empty and itself.

Proof. (i)$\Rightarrow$ (ii) If the temporal intuitionistic fuzzy topological space $((X,T), \tau_{t_0})$ is temporal disconnected, there are sets A and B different from the empty set as $A \cup_{t_0} B = X$, $A \cap_{t_0} cl_{t_0}(B) = \emptyset_{t_0}$ and $cl_{t_0}(A) \cap_{t_0} B = \emptyset_{t_0}$. Since $B = (cl_{t_0}(A))^c$ and $A = (cl_{t_0}(B))^c$; A and B sets are open. But since $A = B^c$; A is both open and closed and $A \neq \emptyset_{t_0}, A \neq X$
(ii)$\Rightarrow$ (i) If $A \in X$ is subset of both open and closed, which is different from the

empty and itself, $A \cup_{t_0} B = X$ and $A \cap_{t_0} B = \emptyset_{t_0}$ as $B = A^c$. Since $A \neq X$; $B \neq \emptyset_{t_0}$. Since A is closed; $cl_{t_0}(A) = A$ and $cl_{t_0}(A) \cap_{t_0} B = \emptyset_{t_0}$. Since A is open, B is closed and $A \cap_{t_0} cl_{t_0}(B) = \emptyset_{t_0}$. Then $((X,T), \tau_{t_0})$ temporal intuitionistic fuzzy topological space is temporal disconnected. □

Proposition 4. $((X,T), \tau_{t_0})$ temporal intuitionistic fuzzy topological space in Chang's sense is temporal connected at time moment t_0, if and only if it has no subset, both open and closed, other than empty and itself.

Theorem 3.6. $((X,T), \tau_{t_0})$ *temporal intuitionistic fuzzy topological space in Chang's sense, the following expressions are equivalent:*
(i) $((X,T), \tau_{t_0})$ *temporal intuitionistic fuzzy topological space in Chang's sense is temporal disconnected at time moment* t_0*.*
(ii) $((X,T), \tau_{t_0})$ *temporal intuitionistic fuzzy topological space in Chang's sense is the topological sum.*
(iii) X *is the union of two distinct open sets other than empty.*
(iv) X *is the union of two distinct closed sets other than empty.*
(v) X *has at least one subset, both open and closed that is distinct from empty and itself.*

Theorem 3.7. $((X,T), \tau_{t_0})$ *temporal intuitionistic fuzzy topological space in Chang's sense, the following expressions are equivalent:*
(i) $((X,T), \tau_{t_0})$ *temporal intuitionistic fuzzy topological space in Chang's sense is temporal connected at time moment* t_0*.*
(ii) $((X,T), \tau_{t_0})$ *temporal intuitionistic fuzzy topological space in Chang's sense cannot be any topological sum.*
(iii) X *cannot be written as the union of two distinct open sets other than empty.*
(iv) X *cannot be written as the union of two distinct closed sets other than empty.*
(v) Both open and closed subsets of X *are only* X *and the empty set.*

4. Conclusion

In this paper, connectedness in temporal intuitionistic fuzzy topology in Chang's sense is introduced and investigated. Basic definition, theorem and propositions about connectedness in temporal intuitionistic fuzzy topology in Chang's sense are given.

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NEW TYPES OF SETS IN ČECH CLOSURE SPACES

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Abstract. In this paper, we analysis and introduce the concepts of regular closed (open) sets and regular generalized closed (open) sets in Čech closure spaces. Also, we investigate the properties such as intersection, union, subspaces of regular generalized closed (open) sets of a Čech closure spaces. Moreover, by giving counter examples of one-sided theorems, it has been shown that the converse situation is not realized.

1. INTRODUCTION

In a topological space, many types of sets such as open set, closed set, generalized closed set, generalized open set, regular generalized closed set, regular open, regular closed are defined. Firstly, studies in this area started with the generalized closed set model that Levine [8] put forward by generalizing closed sets of any topological space. For example, it was shown that completeness, normality, compactness in a uniform space are inherited by generalized closed subsets. Balachandran et al. [9] introduced the concept of generalized continuous maps by using generalized closed sets. In the following years, Palaniappan and Chandrasekhara Rao [10] introduced regular generalized closed sets in topological spaces.

There are many methods researchers can use to define a topology. Čech closure space, one of these methods, are a set of axioms used to define a topology on a set other than any null set by E. Čech in [3]. After defining Čech closure spaces, it has managed to attract the attention of many researchers and then studied on these spaces, see e.g. [4, 5, 6, 7].

Thanks to this paper, regular generalized closed (open) sets and regular closed (open) sets, which are two new concepts for Čech closure spaces, are brought into literature. Moreover, the given sets were analyzed in detail and their properties such as subspaces, intersection, union were examined. In addition, the types of sets such as closed sets, generalized closed sets given previously for Čech closure spaces

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and the new set types given in this paper were compared and the relationships between each other were studied.

2. PRELIMINARIES

In this section, we recall some basic notions in Čech closure spaces.

Throughout this paper, let $\mathcal{U} \neq \emptyset$ be a set, $2^{\mathcal{U}}$ denotes the power set of $\mathcal{U}$ and X, Y be non-empty subsets of U.

Definition 2.1. [1] An operator $c : 2^{\mathcal{U}} \to 2^{\mathcal{U}}$ defined satisfying the axioms:
[$c1$] $c(\emptyset) = \emptyset$,
[$c2$] $X \subseteq c(X)$ for all $X \subseteq \mathcal{U}$,
[$c3$] $c(X \cup Y) = c(X) \cup c(Y)$ for all $X, Y \subseteq \mathcal{U}$
is called a Čech closure operator (briefly closure operator) and the pair $(\mathcal{U}, c)$ is called a Čech closure space (briefly closure space). Here, for $X \subset \mathcal{U}$, we call $c(X)$ the closure of X.

Definition 2.2. [1] Let c be a closure operator and $(\mathcal{U}, c)$ be a closure space. Then, for $\emptyset \neq X \subseteq \mathcal{U}$,
(i) A c on $\mathcal{U}$ is called idempotent if $c(X) = c(c(X))$.
(ii) X is closed set (briefly c-set) in $(\mathcal{U}, c)$ if $X = c(X)$.
(iii) X is open set (briefly o-set) in $(\mathcal{U}, c)$ if its complement is c-set.
(iv) The $\emptyset$ and $\mathcal{U}$ are both o-set and c-set.

Definition 2.3. [1] Let $(\mathcal{U}, c)$ be a closure space. A closure space $(\mathcal{V}, c_{\mathcal{V}})$ is called a subspace of $(\mathcal{U}, c)$ if $\mathcal{V} \subseteq \mathcal{U}$ and $c_{\mathcal{V}}(X) = c(X) \cap \mathcal{V}$, for all $X \subseteq \mathcal{V}$.

Definition 2.4. [1] Let $(\mathcal{V}, c_{\mathcal{V}})$ be a Čech closure subspace of $(\mathcal{U}, c)$. If K is a closed subset of $(\mathcal{V}, c_{\mathcal{V}})$, then K is a closed subset of $(\mathcal{U}, c)$.

Definition 2.5. [2] Let $(\mathcal{U}, c)$ be a closure space. Then,
(i) A $X \subseteq \mathcal{U}$ is called a generalized closed set (briefly gc-set), if $c(X) \subseteq K$ whenever K is an open subset of $(\mathcal{U}, c)$ with $X \subseteq Y$.
(ii) A $X \subseteq \mathcal{U}$ is called a generalized open set (briefly go-set), if its complement is gc-set.
(iii) If X and Y are generalized closed subsets of $(\mathcal{U}, c)$, then $X \cup Y$ is gc-set. Moreover, the $X \cap Y$ need not be a gc-set.

Remark 2.6. [2] Every c-set is gc-set. The converse need not be a c-set.

Definition 2.7. [1] An interior operator on $\mathcal{U}$ is a map $int : 2^{\mathcal{U}} \to 2^{\mathcal{U}}$ which satisfies
(i) $int(\mathcal{U}) = \mathcal{U}$,
(ii) $int(X) \subseteq X$ for all $X \subseteq \mathcal{U}$,
(iii) $int(X \cap Y) = int(X) \cap int(Y)$ for all $X, Y \subseteq \mathcal{U}$.

In other words, the set $int(X)$ with respect to the closure operator c is defined as $int(X) = \mathcal{U} – c(\mathcal{U} - X)$.

3. REGULER GENERALIZED CLOSED SETS

Definition 3.1. Let $(\mathcal{U}, c)$ be a closure space. A $X \subseteq \mathcal{U}$ is called a regular closed set (briefly rc-set) [regular open set (briefly ro-set)], if $X = c(int(X))$ [$X = int(c(X))$].

Remark 3.2. Every ro-set is o-set. The converse need not be a ro-set as can be seen from the following example.

Example 3.3. Let $\mathcal{U} = \{m, n\}$ and define a closure operator c on $\mathcal{U}$ by $c(\emptyset) = \emptyset$ and $c(\{m\}) = c(\{n\}) = c(\mathcal{U}) = \mathcal{U}$. Then $\{m\}$ and $\{n\}$ are o-sets but they are not ro-sets.

Definition 3.4. Let $(\mathcal{U}, c)$ be a closure space. A $X \subseteq \mathcal{U}$ is called a regular generalized closed set (briefly rgc-set) iff $c(X) \subseteq K$ whenever $X \subseteq K$, where K is ro-set.

Proposition 1. Let $(\mathcal{U}, c)$ be a closure space. If X and Y are regular generalized closed subsets of $(\mathcal{U}, c)$, then $X \cup Y$ is rgc-set.

Proof. Let K be a regular open subset of $(\mathcal{U}, c)$ such that $X \cup Y \subseteq K$. Then $X \subseteq K$ and $Y \subseteq K$. Since X and Y are rgc-set, $c(X) \subseteq K$ and $c(Y) \subseteq K$. Therefore $c(X) \cup c(Y) \subseteq K$ and hence $c(X \cup Y) \subseteq K$. Consequently $X \cup Y$ is rgc-set. □

Remark 3.5. The intersection of two rgc-sets is generally not a rgc set.

Example 3.6. Let $\mathcal{U} = \{m, n, r\}$ and define a closure operator c on $\mathcal{U}$ by $c(\emptyset) = \emptyset$, $c(\{m\}) = \{m, n\}$, $c(\{n\}) = c(\{r\}) = c(\{n, r\}) = \{n, r\}$, $c(\{m, n\}) = c(\{m, r\}) = c(\mathcal{U}) = \mathcal{U}$. Then $\{m, n\}$ and $\{m, r\}$ are rgc-set but $\{m, n\} \cap \{m, r\} = \{m\}$ is not rgc-set.

Proposition 2. Let $(\mathcal{U}, c)$ be a closure space. If X is a rgc-set and K is a c-set in $(\mathcal{U}, c)$, then $X \cap K$ is rgc-set.

Proof. Let G be an regular open subset of $(\mathcal{U}, c)$ such that $X \cap K \subseteq G$. Then $X \subseteq G \cup (\mathcal{U} - K)$ and so $c(X) \subseteq G \cup (\mathcal{U} - K)$. Therefore $c(X) \cap K \subseteq G$. Since K is a c-set, $c(X \cap K) \subseteq G$. Hence, $X \cap K$ is a rgc-set. □

Proposition 3. Let $(\mathcal{V}, c_{\mathcal{V}})$ be a closed subspace of $(\mathcal{U}, c)$. If K is a regular generalized closed subset of $(\mathcal{V}, c_{\mathcal{V}})$, then K is a regular generalized closed subset of $(\mathcal{U}, c)$.

Proof. Let G be an regular open subset of $(\mathcal{U}, c)$ such that $K \subseteq G$. Then $K \subseteq G \cap \mathcal{V}$. Since K is a rgc-set and $G \cap \mathcal{V}$ is a ro-set in $(\mathcal{V}, c_{\mathcal{V}})$, $c(K) \cap \mathcal{V} = c_{\mathcal{V}}(K) \subseteq G$. But $\mathcal{V}$ is a closed subset of $(\mathcal{U}, c)$ and $c(K) \subseteq G$. Hence, K is a regular generalized closed subset of $(\mathcal{U}, c)$. □

Theorem 3.7. *Let $(\mathcal{U}, c)$ be a closure space and c be idempotent. If X is a regular generalized closed subset of $(\mathcal{U}, c)$ and $X \subseteq Y \subseteq c(X)$, then $c(Y) - Y$ contains no nonempty rc-set.*

Proof. Since $Y \subseteq c(X)$ and c is idempotent, then $c(Y) \subseteq c(c(X)) = c(X)$. That is $c(Y) \subseteq c(X)$. Since $X \subseteq Y$, we obtain $\mathcal{U} - Y \subseteq \mathcal{U} - X$. Form $c(Y) \subseteq c(X)$ and $\mathcal{U} - Y \subseteq \mathcal{U} - X$, $(u(Y) \cap (\mathcal{U} - Y)) \subseteq (c(X) \cap (\mathcal{U} - X))$ which implies $(c(Y) - Y) \subseteq (c(X) - X)$. Now X is a rgc-set. Hence $c(X) - X$ has no nonempty regular generalized closed subset, neither does $c(Y) - Y$. □

Theorem 3.8. *Let $(\mathcal{U}, c)$ be a closure space and $X \subseteq \mathcal{U}$. If X is a rgc-set, then $c(X) - X$ contains no nonempty rc-set.*

Proof. Suppose that X is a rgc-set. Let Y be a regular generalized closed subset of $c(X) - X$. Then $Y \subseteq c(X) \cap (\mathcal{U} - X)$ and so $X \subseteq \mathcal{U} - Y$. But X is a rgc-set. Therefore $c(X) \subseteq \mathcal{U} - Y$. Consequently $Y \subseteq \mathcal{U} - c(X)$. Since $Y \subseteq c(X)$, $Y \subseteq c(X) \cap (\mathcal{U} - c(X)) = \emptyset$. Thus $B = \emptyset$. Therefore $c(X) - X$ contains no nonempty rc-set. □

The converse of this theorem is not true in general as can be seen from the following example.

Example 3.9. Let $\mathcal{U} = \{x, y, z\}$ and define a closure operator c on $\mathcal{U}$ by $c(\emptyset) = \emptyset$, $c(\{x\}) = \{x, y\}$, $c(\{y\}) = c(\{z\}) = c(\{y, z\}) = \{y, z\}$, $c(\{x, y\}) = c(\{x, z\}) = c(\mathcal{U}) = \mathcal{U}$. Then $c(\{x\}) - \{x\} = \{y\}$ does not contain nonempty rc-set. But $\{x\}$ is not rgc-set.

Corollary 1. Let $(\mathcal{U}, c)$ be a closure space and X be a rgc-subset of $(\mathcal{U}, c)$. Then X is a rc-set if and only if $c(int(X)) - X$ is a rc-set.

Proof. Let X be regular generalized closed subset of $(\mathcal{U}, c)$. If X is a rc-set, then $c(int(X)) - X = \emptyset$. But $\emptyset$ is always a rc-set. Therefore $c(int(X)) - X$ is a rc-set.

Conversely, suppose that $c(int(X)) - X$ is a rc-set. But X is a rgc-set. Also $c(X) - X$ contains the rc-set $c(int(X)) - X$. By Theorem 3.8, we have $c(int(X)) - X = \emptyset$. Hence $c(int(X)) = X$. Therefore X is a rc-set. □

Theorem 3.10. *Let $(\mathcal{U}, c)$ be a closure space and $X \subseteq \mathcal{U}$. If X is a gc-set, then X is a rgc-set.*

Proof. Suppose that $X \subseteq K$, where K is a ro-set. Now K is a ro-set, implies that K is a open. Since X is a gc-set, then $c(X) \subseteq K$. Therefore X is a rgc-set. □

The converse of this theorem is not true in general as can be seen from the following example.

Example 3.11. Let $\mathcal{U} = \{1, 2, 3, 4\}$ and define a closure operator c on $\mathcal{U}$ by

$$c(\emptyset) = \emptyset, \quad c(\{1\}) = c(\{1, 2\}) = \{1, 2\}, \quad c(\{2\}) = \{2\},$$
$$c(\{3\}) = c(\{2, 3\}) = \{2, 3\}, \quad c(\{4\}) = \{4\}, \quad c(\{2, 4\}) = \{2, 4\},$$
$$c(\{1, 3\}) = c(\{1, 2, 3\}) = \{1, 2, 3\}, \quad c(\{1, 4\}) = c(\{1, 2, 4\}) = \{1, 2, 4\},$$
$$c(\{3, 4\}) = c(\{2, 3, 4\}) = \{2, 3, 4\}, \quad c(\{1, 3, 4\}) = c(\mathcal{U}) = \mathcal{U}.$$

Then $\{1, 3\}$ is a rgc-set but it is not gc-set.

Definition 3.12. Let $(\mathcal{U}, c)$ be a closure space. A $X \subseteq \mathcal{U}$ is called a regular generalized open set (briefly a rgo-set) if and only if its complement is a rgc-set.

Theorem 3.13. *Let $(\mathcal{U}, c)$ be a closure space. A set $X \subseteq \mathcal{U}$ is a rgo-set if and only if $H \subseteq int(X)$ whenever H is a rc-set and $H \subseteq X$.*

Proof. Let $H \subseteq int(X)$ whenever H is a rc-set, $H \subseteq X$ and $K = \mathcal{U} - X$. Suppose that $K \subseteq G$ where G is a ro-set.

Now $T \subseteq G$ implies $H = \mathcal{U} - G \subseteq X$ and H is a rc-set which implies $H \subseteq int(X)$. Also $H \subseteq int(X)$ implies $\mathcal{U} - int(X) \subseteq \mathcal{U} - H = G$. This inturn implies $\mathcal{U} - int(\mathcal{U} - K) \subseteq G$. Or equivalently $c(K) \subseteq G$. Thus K is a rgc-set. Hence we obtain X is a a rgo-set.

Conversely, suppose that X is a rgo-set, $H \subseteq X$ and H is a rc-set. Moreover $\mathcal{U}-H$ is a ro-set. Then $\mathcal{U}-X \subseteq \mathcal{U}-H$. Since $\mathcal{U}-X$ is a rgc-set, $\mathcal{U}-c(X) \subseteq \mathcal{U}-H$. Therefore $H \subseteq \mathcal{U}-(\mathcal{U}-c(X)) = int(X)$. □

Theorem 3.14. *Let $(\mathcal{U}, c)$ be a closure space. If X is a rgo-subset of $(\mathcal{U}, c)$, then $G = \mathcal{U}$ whenever G is a ro-set and $int(X) \cup (\mathcal{U} - X) \subseteq G$.*

Proof. Suppose that X is a rgo-set in $(\mathcal{U}, c)$. Let G be a ro-set and $int(X) \cup (\mathcal{U} - X) \subseteq G$. This implies $\mathcal{U} - G \subseteq (\mathcal{U} - int(X)) \cap (\mathcal{U} - (\mathcal{U} - X))$. That is $\mathcal{U} - G \subseteq (\mathcal{U} - int(X)) \cap X$ or equivalently $\mathcal{U} - G \subseteq (\mathcal{U} - int(X)) - (\mathcal{U} - X) = (\mathcal{U} - (\mathcal{U} - c(\mathcal{U} - X))) - (\mathcal{U} - X) = c(\mathcal{U} - X) - (\mathcal{U} - X) = c(\mathcal{U} - X) - (\mathcal{U} - X)$. Now $\mathcal{U} - G$ is a rc-set and $\mathcal{U} - X$ is rgc-set. By Theorem 3.8, it follows taht $\mathcal{U} - G = \emptyset$. Hence we obtain $G = \mathcal{U}$. □

Theorem 3.15. *Let $(\mathcal{U}, c)$ be a closure space. If X is a rgc-subset of $(\mathcal{U}, c)$, then $c(X) - X$ is a rgo-set.*

Proof. Suppose that X is a rgc-set and $H \subseteq c(X) - X$, where H is a rc-set. By Theorem 3.8, $H = \emptyset$ and so $H \subseteq int(c(X) - X)$. By Theorem 3.13, $c(X) - X$ is a rgo-set. □

The converse of this theorem is not true in general as can be seen from the following example.

Example 3.16. Let $\mathcal{U} = \{m, n, r, s\}$ and define a closure operator c on $\mathcal{U}$ by

$$c(\emptyset) = \emptyset, \quad c(\{m\}) = c(\{m, n\}) = \{m, n\}, \quad c(\{n\}) = \{n\},$$
$$c(\{r\}) = c(\{n, r\}) = \{n, r\}, \quad c(\{s\}) = \{s\}, \quad c(\{n, s\}) = \{n, s\},$$
$$c(\{m, r\}) = c(\{m, n, r\}) = \{m, n, r\}, \quad c(\{m, s\}) = c(\{m, n, s\}) = \{m, n, s\},$$
$$c(\{r, s\}) = c(\{n, r, s\}) = \{n, r, s\}, \quad c(\{m, r, s\}) = c(\mathcal{U}) = \mathcal{U}.$$

Then $c(\{r\}) - \{r\} = \{n\}$ is a rgo-set. But $\{r\}$ is not a rgc-set in $(\mathcal{U}, c)$.

4. Conclusion

In the present paper, we have introduced regular closed (open) sets and regular generalized closed (open) sets in Čech closure spaces. In addition, some basic properties of new concepts for Čech closure spaces were examined. We have investigated behavior relative to union, intersection, subspaces of regular closed (open) sets and regular generalized closed (open) sets. We hope that the findings in this paper will help researcher enhance and promote the further study on Čech closure spaces to carry out a general framework.

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GLOBAL NONEXISTENCE OF SOLUTIONS FOR THE HIGHER ORDER KIRCHHOFF TYPE SYSTEM WITH LOGARITHMIC NONLINEARITIES

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Abstract. This paper deals with the system of a class of nonlinear higher-order Kirchhoff-type equations with logarithmic nonlinearities. Under the appropriate assumptions, the theorem of global nonexistence is established at positive initial energy levels.

1. Introduction

In this paper, we study the following initial-boundary value problem

(1.1)
$$\begin{cases} u_{tt}+M\left(\left\|D^m u\right\|^2+\left\|D^m v\right\|^2\right)(-\triangle)^m u+(-\triangle)^m u_t=|u|^{r-2}u\ln|u|, & x\in\Omega,\ t>0,\\ v_{tt}+M\left(\left\|D^m u\right\|^2+\left\|D^m v\right\|^2\right)(-\triangle)^m v+(-\triangle)^m v_t=|v|^{r-2}v\ln|v|, & x\in\Omega,\ t>0,\\ u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x),\ \ v(x,0)=v_0(x),\ \ v_t(x,0)=v_1(x), & x\in\Omega,\\ \frac{\partial^i}{\partial\nu^i}u(x,t)=0,\ \ \frac{\partial^i}{\partial\nu^i}v(x,t)=0,\ i=0,1,2,...m-1, & x\in\partial\Omega,\ t\geq 0,\end{cases}$$

where $Du=\nabla u=\left(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},...\frac{\partial u}{\partial x_n}\right)$ and $r\geq 2\gamma+2$ are real numbers and $m\geq 1$ are positive integers. The Kirchhoff term $M(s)=\beta_1+\beta_2 s^\gamma$, $\gamma>0$, $\beta_1\geq 1$, $\beta_2\geq 0$. We will take $\beta_1=\beta_2=1$ for simplify. $\Omega\subset R^n$ is a regular and bounded domain with smooth boundary $\partial\Omega$. And v denotes the outer normal.

Problem (1.1) is a generalization of a model considered by Kirchhoff [9]. Kirchhoff type equation has in the mathematical description of small amplitude vibrations of an elastic string. In the case $M(s)=1$, $m=1$ and $p\geq 2$, a problem of the single wave equation of the (1.1) form becomes

$$u_{tt}-\triangle u+f(u_t)=|u|^{p-2}u\ln|u|. \tag{1.2}$$

Several results of the problem (1.2) concerning local or global existence and qualitative theory have been studied by many mathematicians(see [1, 2, 4, 5, 6, 7, 10, 13, 19]). In the case $M(s)\neq 1$, $m=1$ and $p\geq 2$, a problem of the single wave

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equation of (1.1) becomes the Kirchhoff-type equation which has been investigated by many authors [3, 14, 18].

In the case $M(s) \neq 1$, $m > 1$ the single form of the problem (1.1) without logarithmic source terms have been discussed by many authors (see [12, 16, 15, 11]).

Let us finally mention that wave equation system with logarithmic nonlinearies was studied by Wang et al [17].They proved global existence and finite time blow up under the different conditions by employing the potential well method and concavity method. In [8], the authors studied (1.1) problem with nonlinear damping terms. They established global existence and decay estimates.

The rest of this work is organized as follows. In Section 3, our aim is to prove the blow up of solution for $E(0) > 0$. In section 2, we give some lemmas which will be useful.

2. Preliminaries

Now we define the potential energy functional of problem (1.1)

$$\begin{aligned} J(u,v) &= \frac{1}{2}\left(\|D^m u\|^2 + \|D^m v\|^2\right) + \frac{1}{2\gamma+2}\left(\|D^m u\|^2 + \|D^m v\|^2\right)^{\gamma+1} \\ &\quad -\frac{1}{r}\left(\int_\Omega |u|^r \ln|u|\,dx + \int_\Omega |v|^r \ln|v|\,dx\right) + \frac{1}{r^2}\left(\|u\|_r^r + \|v\|_r^r\right) \end{aligned} \tag{2.1}$$

and the Nehari functional

$$\begin{aligned} I(u,v) &= \left(\|D^m u\|^2 + \|D^m v\|^2\right) + \left(\|D^m u\|^2 + \|D^m v\|^2\right)^{\gamma+1} \\ &\quad -\left(\int_\Omega |u|^r \ln|u|\,dx + \int_\Omega |v|^r \ln|v|\,dx\right). \end{aligned} \tag{2.2}$$

By (2.1) and (2.2) we obtain

$$\begin{aligned} J(u,v) &= \frac{I(u,v)}{r} + \frac{(r-2)}{2r}\left(\|D^m u\|^2 + \|D^m v\|^2\right) \\ &\quad + \frac{(r-2\gamma-2)}{2\gamma+2}\left(\|D^m u\|^2 + \|D^m v\|^2\right)^{\gamma+1} \\ &\quad + \frac{1}{r^2}\left(\|u\|_r^r + \|v\|_r^r\right). \end{aligned} \tag{2.3}$$

Then we can introduce the stable set

$$W = \left\{(u,v) \in H_0^m(\Omega) \times H_0^m(\Omega) : I(u,v) > 0\right\} \cup \{0\},$$

the outer space of the potential well

$$V = \left\{(u,v) \in H_0^m(\Omega) \times H_0^m(\Omega) : I(u,v) < 0\right\}.$$

We introduce the total energy

$$E(u,v) = \frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{1}{2}\left(\|D^m u\|^2+\|D^m v\|^2\right) + \frac{1}{2\gamma+2}\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma+1} - \frac{1}{r}\left(\int_\Omega |u|^r \ln|u|\,dx + \int_\Omega |v|^r \ln|v|\,dx\right) + \frac{1}{r^2}\left(\|u\|_r^r+\|v\|_r^r\right). \tag{2.4}$$

For $(u,v)\in H_0^m(\Omega)\times H_0^m(\Omega)$, $t\geq 0$

$$E(0) = \frac{1}{2}\left(\|u_1\|^2+\|v_1\|^2\right)+\frac{1}{2}\left(\|D^m u_0\|^2+\|D^m v_0\|^2\right) + \frac{1}{2\gamma+2}\left(\|D^m u_0\|^2+\|D^m v_0\|^2\right)^{\gamma+1} - \frac{1}{r}\left(\int_\Omega |u_0|^r \ln|u_0|\,dx + \int_\Omega |v_0|^r \ln|v_0|\,dx\right) + \frac{1}{r^2}\left(\|u_0\|_r^r+\|v_0\|_r^r\right). \tag{2.5}$$

is the initial total energy. We introduce by (2.4) and (2.3)

$$E(u,v) = \frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+J(u,v), \tag{2.6}$$

Lemma 2.1. *Let k be a number with $2\leq k<\infty$ if $n\leq 2s$ and $2\leq k\leq \frac{2n}{n-2k}$ if $n>2s$. Then there is a constant such that*

$$\|u\|_k \leq C\|D^m u\|,\ \forall (u,v)\in H_0^m(\Omega)\times H_0^m(\Omega).$$

Lemma 2.2. *$E(t)$ is a nonincreasing function for $t\geq 0$ and*

$$E'(t) = -\left(\|D^m u_t\|^2+\|D^m v_t\|^2\right)\leq 0. \tag{2.7}$$

Proof. Multiplying the first equation of (1.1) by u_t and the second equation of (1.1) by v_t, and integrating on Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2+\frac{d}{dt}\left(\frac{1}{r^2}\|u\|_r^r-\frac{1}{r}\int_\Omega |u|^r\ln|u|\,dx\right) \frac{1}{2}\left(1+\left(\|D^m u\|^2+\|D^m v\|^2\right)^\gamma\right)\frac{d}{dt}\|D^m u\|^2 = -\int_\Omega |D^m u_t|^2\,dx, \tag{2.8}$$

and

$$\frac{1}{2}\frac{d}{dt}\|v_t\|^2+\frac{d}{dt}\left(\frac{1}{r^2}\|v\|_r^r-\frac{1}{r}\int_\Omega |v|^r\ln|v|\,dx\right) \frac{1}{2}\left(1+\left(\|D^m u\|^2+\|D^m v\|^2\right)^\gamma\right)\frac{d}{dt}\|D^m v\|^2 = -\int_\Omega |D^m v_t|^2\,dx. \tag{2.9}$$

A summarization of (2.8) and (2.9) hence gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u_t\|^2+\|v_t\|^2\right)\\
&+\frac{1}{2}\left(1+\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma}\right)\frac{d}{dt}\left(\|D^m u\|^2+\|D^m v\|^2\right)\\
&\frac{d}{dt}\left(-\frac{1}{r}\left(\int_\Omega |u|^r\ln|u|\,dx+\int_\Omega |v|^r\ln|v|\,dx\right)+\frac{1}{r^2}\left(\|u\|_r^r+\|v\|_r^r\right)\right)\\
(2.10)\quad =\;& -\left(\int_\Omega |D^m u_t|^2\,dx+\int_\Omega |D^m v_t|^2\,dx\right).
\end{aligned}
$$

Integrating (2.10) with respect to t on $[0,t]$, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{1}{2}\left(\|D^m u\|^2+\|D^m v\|^2\right)\\
&+\frac{1}{2\gamma+2}\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma+1}+\frac{1}{r^2}\left(\|u\|_r^r+\|v\|_r^r\right)\\
&-\frac{1}{r}\left(\int_\Omega |u|^r\ln|u|\,dx+\int_\Omega |v|^r\ln|v|\,dx\right)\\
&+\left(\int_0^t \|D^m u_\tau\|^2\,d\tau+\int_\Omega \|D^m v_\tau\|^2\,d\tau\right)\\
=\;&\frac{1}{2}\left(\|u_1\|^2+\|v_1\|^2\right)+\frac{1}{2}\left(\|D^m u_0\|^2+\|D^m v_0\|^2\right)\\
&+\frac{1}{2\gamma+2}\left(\|D^m u_0\|^2+\|D^m v_0\|^2\right)^{\gamma+1}+\frac{1}{r^2}\left(\|u_0\|_r^r+\|v_0\|_r^r\right)\\
(2.11)\quad &-\frac{1}{r}\left(\int_\Omega |u_0|^r\ln|u_0|\,dx+\int_\Omega |v_0|^r\ln|v_0|\,dx\right).
\end{aligned}
$$

By using the definition of total energy and initial total energy, we restate (2.11) as

$$
(2.12)\qquad E(t)+\left(\int_0^t \|D^m u_\tau\|^2\,d\tau+\int_\Omega \|D^m v_\tau\|^2\,d\tau\right)=E(0).
$$

□

Now, we give some properties related with $J(u,v)$ and $I(u,v)$, respectively.

Lemma 2.3. *For any* $(u,v)\in H_0^m(\Omega)\times H_0^m(\Omega)$, $\|D^m u\|\neq 0$ *and* $\|D^m u\|\neq 0$,*let* $g(\lambda)=J(\lambda u,\lambda v)$. *Then we have*

i) $\lim\limits_{\lambda\to 0} g(\lambda)=0$, $\lim\limits_{\lambda\to\infty} g(\lambda)=-\infty$,
ii) There is a unique λ^* such that $g'(\lambda)=0$,

iii) Then we have

$$I(\lambda u, \lambda v) = \lambda g'(\lambda) \begin{cases} > 0, & 0 \leq \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda. \end{cases}$$

Proof. By the definition of $J(u, v)$, we obtain

$$\begin{aligned} g(\lambda) &= J(\lambda u, \lambda v) \\ &= \frac{1}{2}\lambda^2 \left(\|D^m u\|^2 + \|D^m v\|^2 \right) + \frac{1}{r^2}\lambda^r \left(\|u\|_r^r + \|v\|_r^r \right) \\ &\quad - \frac{1}{r}\ln|\lambda| \lambda^r \left(\|u\|_r^r + \|v\|_r^r \right) - \frac{1}{r}\lambda^r \left(\int_\Omega u^r \ln|u|\, dx + \int_\Omega v^r \ln|v|\, dx \right) \\ &\quad + \frac{1}{2\gamma+2}\lambda^{2\gamma+2} \left(\|D^m u\|^2 + \|D^m v\|^2 \right)^{2\gamma+2}. \end{aligned} \tag{2.13}$$

Since $\|D^m u\| \neq 0$, and $\|D^m v\| \neq 0$, $\lim\limits_{\lambda \to 0} g(\lambda) = 0$, $\lim\limits_{\lambda \to \infty} g(\lambda) = -\infty$. Now, differentiating $g(\lambda)$ with respect to λ, we have

$$\begin{aligned} g'(\lambda) &= \lambda \left(\|D^m u\|^2 + \|D^m v\|^2 \right) + \lambda^{2\gamma+1} \left(\|D^m u\|^2 + \|D^m v\|^2 \right)^{2\gamma+2} \\ &\quad - \lambda^{r-1} \left(\int_\Omega u^r \ln|u|\, dx + \int_\Omega v^r \ln|v|\, dx \right) - \lambda^{r-1} \ln|\lambda| \left(\|u\|_r^r + \|v\|_r^r \right) \\ &= \lambda \Bigg(\left(\|D^m u\|^2 + \|D^m v\|^2 \right) + \beta_2 \lambda^{2\gamma} \left(\|D^m u\|^2 + \|D^m v\|^2 \right)^{2\gamma+2} \\ &\quad - \lambda^{r-2} \left(\int_\Omega u^r \ln|u|\, dx + \int_\Omega v^r \ln|v|\, dx \right) - \lambda^{r-2} \ln|\lambda| \left(\|u\|_r^r + \|v\|_r^r \right) \Bigg). \end{aligned} \tag{2.14}$$

Let

$$\begin{aligned} \psi(\lambda) &= \lambda^{2\gamma} \left(\|D^m u\|^2 + \|D^m v\|^2 \right)^{2\gamma+2} \\ &\quad - \lambda^{r-2} \left(\int_\Omega u^r \ln|u|\, dx + \int_\Omega v^r \ln|v|\, dx \right) \\ &\quad - \lambda^{r-2} \ln|\lambda| \left(\|u\|_r^r + \|v\|_r^r \right). \end{aligned}$$

Then from $2\gamma \leq r-2$ we can deduce that $\lim\limits_{\lambda \to \infty} \psi(\lambda) = -\infty$, $\psi(\lambda)$ is monotone decreasing when $\lambda > \lambda^1$ and there exists a unique λ^1 such that $\psi(\lambda^1) = 0$. Then we obtain there is a $\lambda^* > \lambda^1$ such that $\lambda \left[\left(\|D^m u\|^2 + \|D^m v\|^2 \right) + \psi(\lambda) \right] = 0$, which means $g'(\lambda) = 0$.

The last property (iii), is only a simple corollary of the fact that

$$\lambda \frac{dJ(\lambda u, \lambda v)}{d\lambda} = \lambda g'(\lambda) = I(\lambda u, \lambda v). \tag{2.15}$$

□

Lemma 2.4. *i) The definition of the potential well depth*

$$d = \inf_{u\in N} J(u,v), \tag{2.16}$$

where

$$N = \{(u,v) : (u,v) \in H_0^m(\Omega) \times H_0^m(\Omega) \setminus \{0\} : I(u,v) = 0\},$$

is equivalent to

$$d = \inf \left\{ \sup_{\lambda\geq 0} J(\lambda u, \lambda v) \mid (u,v) \in H_0^m(\Omega) \times H_0^m(\Omega), \|D^m u\|^2 \neq 0, \|D^m v\|^2 \neq 0 \right\}. \tag{2.17}$$

ii) The constant d in (2.16) satisfies

$$d = \frac{(r-2)}{2r} \left(\frac{1}{C_1^{r+1}} \right)^{\frac{2}{r-1}},$$

where C_1 is the optimal constant of Lemma 2.1 ($H_0^m(\Omega) \hookrightarrow L^{r+1}$) and

$$\begin{cases} 2\gamma + 2 \leq r \leq \frac{n+2m}{n-2m}, & n > 2m, \\ 2\gamma + 2 \leq r \leq \infty, & n \leq 2m. \end{cases} \tag{2.18}$$

Proof. i) The definition of d from (iii) of Lemma 2.3 it implies that for any $(u,v) \in H_0^m(\Omega) \times H_0^m(\Omega)$, there exist a λ^* such that $I(\lambda^* u, \lambda^* v) = 0$, that is $(\lambda^* u, \lambda^* v) \in N$. By the definition of d we obtain

$$J(\lambda^* u, \lambda^* v) \geq d \text{ for any } (u,v) \in H_0^m(\Omega) \times H_0^m(\Omega) / \{0\}. \tag{2.19}$$

And because of Lemma 2.3

$$\sup_{\lambda\geq 0} J(\lambda u, \lambda v) = J(\lambda^* u, \lambda^* v),$$

which by virtue of (2.19) means

$$\inf_{(u,v)\in H_0^m(\Omega)\times H_0^m(\Omega)} \sup_{\lambda\geq 0} J(\lambda u, \lambda v) = \inf_{(u,v)\in H_0^m(\Omega)\times H_0^m(\Omega)} J(\lambda^* u, \lambda^* v) \geq d, \tag{2.20}$$

As $(u,v) \in H_0^m(\Omega) \times H_0^m(\Omega) / \{0\}$, we obtain d is not equivalent to 0, which gives (2.17). On the other hand, from the definition of d given by (2.17) it implies that there exists λ^1 such that

$$\sup_{\lambda\geq 0} J(\lambda u, \lambda v) = \sup J\left(\lambda^1 u, \lambda^{1*} v\right).$$

Then from Lemma 2.3 we can deduce $\lambda^* = \lambda^1$. And it shows that

$$I\left(\lambda^1 u, \lambda^1 v\right) = I(\lambda^* u, \lambda^* v) = 0,$$

which means $\left(\lambda^1 u, \lambda^1 v\right) \in N$. By the definition of d, we get,

$$d = \inf_{(\lambda^* u, \lambda^* v)\in N} J\left(\lambda^1 u, \lambda^1 v\right),$$

that is

$$d = \inf_{(u,v)\in N} J(u,v). \tag{2.21}$$

This complete our proof for (i).

ii) By virtue of $I(u,v)=0$ and definition of $I(u,v)$ and the embedding theorems we obtain

$$\left(\|D^m u\|^2+\|D^m v\|^2\right)+\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma+1}=\int_\Omega |u|^r\ln|u|\,dx+\int_\Omega |v|^r\ln|v|\,dx,$$

$$\begin{aligned}\left(\|D^m u\|^2+\|D^m v\|^2\right) &\leq \int_\Omega |u|^r\ln|u|\,dx+\int_\Omega |v|^r\ln|v|\,dx\\ &\leq \|u\|_{r+1}^{r+1}+\|v\|_{r+1}^{r+1}\\ &\leq C_1^{r+1}\left(\|D^m u\|^{r+1}+\|D^m v\|^{r+1}\right)\\ &\leq C_1^{r+1}\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\frac{r-1}{2}}\left(\|D^m u\|^2+\|D^m v\|^2\right), \qquad (2.22)\end{aligned}$$

which means

$$\|D^m u\|^2+\|D^m v\|^2\geq\left(\frac{1}{C_1^{r+1}}\right)^{\frac{2}{r-1}}. \qquad (2.23)$$

From the definition of d, we have $(u,v)\in N$. By the definition of $J(u,v)$, (2.22), (2.3) and $I(u,v)=0$, we get

$$\begin{aligned}J(u,v) &= \frac{I(u,v)}{r}+\frac{(r-2)}{2r}\left(\|D^m u\|^2+\|D^m v\|^2\right)\\ &\quad+\frac{(r-2\gamma-2)}{2\gamma+2}\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma+1}+\frac{1}{r^2}\left(\|u\|_r^r+\|v\|_r^r\right)\\ &\geq \frac{(r-2)}{2r}\left(\|D^m u\|^2+\|D^m v\|^2\right)\\ &\geq \frac{(r-2)}{2r}\left(\frac{1}{C_1^{r+1}}\right)^{\frac{2}{r-1}},\end{aligned}$$

where $2\gamma\leq r-2$. Combining of (2.21) and (2.23), we can see clearly that

$$d=\frac{(r-2)}{2r}\left(\frac{1}{C_1^{r+1}}\right)^{\frac{2}{r-1}}.$$

□

Lemma 2.5. *Let (u,v) be a weak solution problem of (1.1) and $(u_0,v_0)\in H_0^{r_1}(\Omega)\times H_0^{r_2}(\Omega),(u_1,v_1)\in L^2(\Omega)\times L^2(\Omega)$. Suppose that $E(0)<d$*
i) if $(u_0,v_0)\in W$, then $(u,v)\in W$ for $0\leq t\leq T$;
ii) if $(u_0,v_0)\in V$, then $(u,v)\in V$ for $0\leq t\leq T$,
where T is the maximum existence time of $(u(t),v(t))$.

Proof. We only prove case (i), case (ii) is similar. Let $(u(t),v(t))$ be a weak solution problem of (1.1) under the conditions and $(u_0,v_0)\in W$ and T can define of the maximum existence time of $(u(x,t),v(x,t))$. Then by (2.7) the energy functional is nonincreasing about t. So that, we have $E((u(t),v(t)))<E(0)<d$ which means $I((u(t),v(t)))>0$ for $0<t<T$. We will use contradiction and we suppose that; there is a $t_1\in(0,T)$ such that $I(u(t_1),v(t_1))<0$. In this way there is a $t^* \in (0,T)$ to make $I(u(t^*), v(t^*)) = 0$ because of continuity of $I(u(t), v(t))$ about time. Then by (2.16), we get

$$d > E(0) \geq E(u(t^*), v(t^*)) \geq J(u(t^*), v(t^*)) \geq d,$$

which is a contradiction. □

Lemma 2.6. *Under the condition of Lemma 2.5 (ii), we get*

$$d < \frac{(r-2)}{2r}\left(\|D^m u\|^2 + \|D^m v\|^2\right).$$

Proof. By using definition of the d, we get

$$d = \frac{(r-2)}{2r}\left(\frac{1}{C_1^{r+1}}\right)^{\frac{2}{r-1}},$$

which together $I(u,v) < 0$. Then similar calculations at (2.22), we get

$$\|D^m u\|^2 + \|D^m v\|^2 \geq \left(\frac{1}{C_1^{r+1}}\right)^{\frac{2}{r-1}}, \tag{2.24}$$

which means

$$d < \frac{(r-2)}{2r}\left(\frac{1}{C_1^{r+1}}\right)^{\frac{2}{r-1}}.$$

□

3. Finite time blow up of solutions for positive initial energy

In tis part we introduce the finite time blow up solution to problem (1.1) with $E(0) > 0$. Now we give some lemmas which will be used the proof of the Theorem 3.3.

Lemma 3.1. *Let (u,v) be a weak solution problem of (1.1) and $(u_0, v_0) \in H_0^m(\Omega) \times H_0^m(\Omega)$, $(u_1, v_1) \in H_0^m(\Omega) \times H_0^m(\Omega)$. Suppose that $E(0) > 0$ and initial data supplies*

$$\|D^m u_0\|^2 + \|D^m v_0\|^2 + 2(u_0, u_1) + 2(v_0, v_1) > \frac{2r(C+2)}{(r-2)C} E(0) > 0, \tag{3.1}$$

where C is the best constant of Lemma 2.1.

By $(u,v) \in V$, the map

$$\left\{t \mapsto \|D^m u\|^2 + \|D^m v\|^2 + 2(u, u_t) + 2(v, v_t)\right\}$$

is strictly increasing.

Proof. Defining the following auxiliary function

$$G(t) = \|D^m u\|^2 + \|D^m v\|^2 + 2(u, u_t) + 2(v, v_t), \tag{3.2}$$

where

$$G(0) = \|D^m u_0\|^2 + \|D^m v_0\|^2 + 2(u_0, u_1) + 2(v_0, v_1). \tag{3.3}$$

By taking derivative of above function, we get

$$\begin{aligned} G'(t) &= 2\left(D^m u, D^m u_t\right) + 2\left(D^m v, D^m v_t\right) \\ &\quad +2\left(\|u_t\|^2 + \|v_t\|^2\right) + 2\left[(u, u_{tt}) + (v, v_{tt})\right] \\ &= 2\left(\|u_t\|^2 + \|v_t\|^2\right) - 2I(u,v). \end{aligned}$$

By $I(u,v) < 0$, for all $t \in [0, \infty)$ it gives that

$$G'(t) > 0. \tag{3.4}$$

From (3.1), (3.3) and (3.4) we obtain

$$G(t) > G(0) > 0,$$

which gives that the map

$$\left\{t \mapsto \|D^m u\|^2 + \|D^m v\|^2 + 2(u, u_t) + 2(v, v_t)\right\}$$

is strictly increasing. □

Lemma 3.2. *Under the conditions of Lemma 3.1 (u,v) is the solution of problem (1.1) with the maximum existence time interval $[0,T)$ and $T \le \infty$. If $(u_0, v_0) \in V$, then the all solutions (u,v) belong to V.*

Proof. Our purpose is to show that $(u,v) \in V$. Arguing by contradiction, we consider that $t^* \in (0,T)$ is the first time which satisfies

$$I(u(t^*), v(t^*)) = 0,$$

and

$$I(u(t), v(t)) < 0 \text{ for } t \in [0, t^*).$$

Then from Lemma 3.1 and the continuity of (u,v) and (u_t, v_t) in t, for $t \in (0, t^*)$ we get

$$\begin{aligned} & \|D^m u\|^2 + \|D^m v\|^2 + 2(u, u_t) + 2(v, v_t) \\ > & \|D^m u_0\|^2 + \|D^m v_0\|^2 + 2(u_0, u_1) + 2(v_0, v_1) \\ > & \frac{2r(C+2)}{(r-2)C} E(0). \end{aligned} \tag{3.5}$$

By (2.3), (2.6) and (2.12) we arrive at

$$\begin{aligned} E(0) &= E(t) + \left(\int_0^t \|D^m u_\tau\|^2 d\tau + \int_\Omega \|D^m v_\tau\|^2 d\tau\right) \\ &= \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{I(t)}{r} + \frac{(r-2)}{2r}\left(\|D^m u\|^2 + \|D^m v\|^2\right) \\ &\quad + \frac{(r-2\gamma-2)}{2\gamma+2}\left(\|D^m u\|^2 + \|D^m v\|^2\right)^{\gamma+1} + \frac{1}{r^2}\left(\|u\|_r^r + \|v\|_r^r\right) \\ &\quad + \left(\int_0^t \|D^m u_\tau\|^2 d\tau + \int_\Omega \|D^m v_\tau\|^2 d\tau\right) \\ &\ge \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{I(t)}{r} + \frac{(r-2)}{2r}\left(\|D^m u\|^2 + \|D^m v\|^2\right). \end{aligned} \tag{3.6}$$

By using $r \geq 2\gamma + 2$, $I(u(t^*), v(t^*)) = 0$, Young's inequality and Lemma 211, we conclude that

$$\begin{aligned}
E(0) &\geq E(t^*) \\
&\geq \frac{1}{2}\left(\|u_t(t^*)\|^2 + \|v_t(t^*)\|^2\right) + \frac{I(t^*)}{r} + \frac{(r-2)}{2r}\left(\|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2\right) \\
&\geq \left(\frac{1}{2} - \frac{1}{r}\right)\left(\|u_t(t^*)\|^2 + \|v_t(t^*)\|^2\right) + \frac{(r-2)}{2r}\left(\|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2\right) \\
&\geq \frac{(r-2)\,C}{2r\,(C+2)}\left(\|u_t(t^*)\|^2 + \|v_t(t^*)\|^2\right) + \frac{(r-2)}{2r}\left(\|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2\right) \\
&= \frac{(r-2)\,C}{2r\,(C+2)}\left(\|u_t(t^*)\|^2 + \|v_t(t^*)\|^2 + \|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2\right) \\
&\quad + \frac{(r-2)}{r\,(C+2)}\left(\|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2\right) \\
&\geq \frac{(r-2)\,C}{2r\,(C+2)}\left(\|u_t(t^*)\|^2 + \|v_t(t^*)\|^2 + \|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2\right) \\
&\quad + \frac{(r-2)\,C}{r\,(C+2)}\left(\|u(t^*)\|^2 + \|v(t^*)\|^2\right) \\
&\geq \frac{(r-2)\,C}{2r\,(C+2)}\Big[\|u_t(t^*)\|^2 + \|v_t(t^*)\|^2 \\
&\quad + \|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2 + \left(\|u(t^*)\|^2 + \|v(t^*)\|^2\right)\Big] \\
&\geq \frac{(r-2)\,C}{2r\,(C+2)}\Big\{\left[2\,(u_t(t^*), u(t^*)) + 2\,(v_t(t^*), v(t^*))\right] \\
&\quad + \|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2\Big\}. \qquad (3.7)
\end{aligned}$$

Clearly, we show that (3.7) contradicts (3.5). This completes the proof of lemma. □

Theorem 3.3. *Let (u, v) be a weak solution of problem of (1.1) and $(u_0, v_0) \in H_0^m(\Omega) \times H_0^m(\Omega), (u_1, v_1) \in H_0^m(\Omega) \times H_0^m(\Omega)$. Suppose that (3.1) holds. Therefore the solution of problem (1.1) blows up in finite time as long as $E(0) > 0$ and $(u_0, v_0) \in V$.*

Proof. We prove the finite time blow up of solution to (1.1). If it is not this case , we suppose existence time $T = \infty$. For any $T_0 > 0$, we define the auxiliary function

$$\begin{aligned}
\Phi(t) &= \|u\|^2 + \|v\|^2 + \int_0^t \left(\|D^m u\|^2 + \|D^m v\|^2\right) d\tau \\
&\quad (T_0 - t)\left(\|D^m u\|^2 + \|D^m v\|^2\right). \qquad (3.8)
\end{aligned}$$

It is clear that $\Phi(t) > 0$ for all $t \in [0, T_0]$. In view of continuity of $\Phi(t)$ in t, we obtain that there is a $\xi > 0$ which is independent on T_0 such that

$$\Phi(t) > \xi. \qquad (3.9)$$

Then by $t \in [0, T_0]$,we derive

$$\begin{aligned}\Phi'(t) &= 2\left(\int_\Omega uu_t dx + \int_\Omega vv_t dx\right) \\ &+ \left(\|D^m u\|^2 + \|D^m v\|^2\right) - \left(\|D^m u_0\|^2 + \|D^m v_0\|^2\right) \\ &= 2\left(\int_\Omega uu_t dx + \int_\Omega vv_t dx\right) \\ &+2\left(\int_0^t \left(D^m u(\tau), D^m u_\tau(\tau)\right) + \left(D^m v(\tau), D^m v_\tau(\tau)\right)\right), \end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\Phi''(t) &= 2\left(\|u_t\|^2 + \|v_t\|^2\right) + 2(u, u_{tt}) + 2(v, v_{tt}) \\ & \quad 2(D^m u, D^m u_t) + 2(D^m v, D^m v_t) \\ &= 2\left(\|u_t\|^2 + \|v_t\|^2\right) - 2I(u, v). \end{aligned} \tag{3.11}$$

From (3.10) it implies

$$\begin{aligned}(B'(t))^2 &= 4\left((u, u_t)^2 + (v, v_t)^2\right) \\ &+4\left(\int_0^t \left(D^m u(\tau), D^m u_\tau(\tau)\right) + \left(D^m v(\tau), D^m v_\tau(\tau)\right)\right)^2 \\ &+8\left(\begin{array}{c}\left(\int_0^t \left(D^m u(\tau), D^m u_\tau(\tau)\right) + \left(D^m v(\tau), D^m v_\tau(\tau)\right) d\tau\right) \\ ((u, u_t) + (v, v_t))\end{array}\right). \end{aligned} \tag{3.12}$$

Our aim is to estimate each terms in (3.12) by Cauchy-Schwarz and Young's inequalities. We obtain the first and second terms as follow

$$\begin{aligned}(u, u_t)^2 + (v, v_t)^2 &\leq \left(\|u\| \|u_t\| + \|v\| \|v_t\|\right)^2 \\ &\leq \left(\|u\|^2 + \|v\|^2\right)\left(\|u_t\|^2 + \|v_t\|^2\right), \end{aligned} \tag{3.13}$$

and

$$
\begin{aligned}
&\left(\int_0^t \left(D^m u(\tau), D^m u_\tau(\tau)\right) + \left(D^m v(\tau), D^m v_\tau(\tau)\right) d\tau\right)^2 \\
\leq\ &\left(\int_0^t \|D^m u(\tau)\| \, \|D^m u_\tau(\tau)\| + \|D^m v(\tau)\| \, \|D^m v_\tau(\tau)\| \, d\tau\right)^2 \\
\leq\ &\left(\int_0^t \left(\|D^m u(\tau)\|^2 + \|D^m v(\tau)\|^2\right)^{\frac{1}{2}} + \left(\|D^m u_\tau(\tau)\|^2 \, \|D^m v_\tau(\tau)\|^2\right)^{\frac{1}{2}} d\tau\right)^2 \\
\leq\ &\int_0^t \left(\|D^m u(\tau)\|^2 + \|D^m v(\tau)\|^2\right) d\tau \\
&\int_0^t \left(\|D^m u_\tau(\tau)\|^2 + \|D^m v_\tau(\tau)\|^2\right) d\tau. \qquad (3.14)
\end{aligned}
$$

For the last term by using again Cauchy-Schwarz and Young's inequalities we obtain

$$
\begin{aligned}
&2\left(\left((u, u_t) + (v, v_t)\right) \int_0^t \left(D^m u(\tau), D^m u_\tau(\tau)\right) + \left(D^m v(\tau), D^m v_\tau(\tau)\right) d\tau\right) \\
\leq\ &2\left(\left(\|u\|^2 + \|v\|^2\right)^{\frac{1}{2}} \left(\|u_t\|^2 + \|v_t\|^2\right)^{\frac{1}{2}}\right) \\
&\left(\int_0^t \left(\|D^m u(\tau)\|^2 + \|D^m v(\tau)\|^2\right) d\tau \int_0^t \left(\|D^m u_\tau(\tau)\|^2 + \|D^m v_\tau(\tau)\|^2\right) d\tau\right)^{\frac{1}{2}} \\
\leq\ &\left(\|u_t\|^2 + \|v_t\|^2\right) \int_0^t \left(\|D^m u(\tau)\|^2 + \|D^m v(\tau)\|^2\right) d\tau \\
&+ \left(\|u\|^2 + \|v\|^2\right) \int_0^t \left(\|D^m u_\tau(\tau)\|^2 + \|D^m v_\tau(\tau)\|^2\right) d\tau. \qquad (3.15)
\end{aligned}
$$

Substituting (3.13)-(3.15) into (3.12) becomes

(3.16)

$$
\left(\Phi'(t)\right)^2 \leq 4\Phi(t)\left(\left(\|u_t\|^2 + \|v_t\|^2\right) + \int_0^t \left(\|D^m u_\tau(\tau)\|^2 + \|D^m v_\tau(\tau)\|^2\right) d\tau\right).
$$

Combining (3.11) and (3.16) we obtain

$$\begin{aligned}
&\Phi''(t)\Phi(t)-\frac{\zeta}{4}\left(\Phi'(t)\right)^2\\
\geq\ &\Phi(t)\left(\Phi''(t)-\zeta\left(\begin{array}{c}\left(\|u_t\|^2+\|v_t\|^2\right)\\ +\int_0^t\left(\|D^m u_\tau(\tau)\|^2+\|D^m v_\tau(\tau)\|^2\right)d\tau\end{array}\right)\right)\\
\geq\ &\Phi(t)\left(2\left(\|u_t\|^2+\|v_t\|^2\right)-2I(u,v)\right.\\
&\left.-\zeta\left(\left(\|u_t\|^2+\|v_t\|^2\right)+\int_0^t\left(\|D^m u_\tau(\tau)\|^2+\|D^m v_\tau(\tau)\|^2\right)d\tau\right)\right).
\end{aligned}\tag{3.17}$$

Let

$$\begin{aligned}
\eta(t) &= (2-\zeta)\left(\|u_t\|^2+\|v_t\|^2\right)-2I(u,v)\\
&\quad-\zeta\left(\int_0^t\left(\|D^m u_\tau(\tau)\|^2+\|D^m v_\tau(\tau)\|^2\right)d\tau\right).
\end{aligned}\tag{3.18}$$

By Lemma 2.2 we get

$$\begin{aligned}
E(0) &= E(t)+\left(\int_0^t\|D^m u_\tau\|^2 d\tau+\int_\Omega\|D^m v_\tau\|^2 d\tau\right)\\
&= \frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{r-2}{2r}\left(\|D^m u\|^2+\|D^m v\|^2\right)\\
&\quad+\frac{r-2\gamma-2}{2\gamma+2}\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma+1}\\
&\quad\frac{I(u,v)}{r}+\frac{1}{r^2}\left(\|u\|_r^r+\|v\|_r^r\right)\\
&\quad+\left(\int_0^t\|D^m u_\tau\|^2 d\tau+\int_\Omega\|D^m v_\tau\|^2 d\tau\right).
\end{aligned}\tag{3.19}$$

Then by combining (3.18) and (3.19), noting $\zeta = \frac{4C+2r+4}{C+2}$, which guarantees $2 < \zeta < r+2$, and using Lemma 2.1 again, it gives that

$$\begin{aligned}
\zeta(t) &= (r+2-\zeta)\left(\|u_t\|^2+\|v_t\|^2\right)-2rE(0) \\
&\quad +(2r-\zeta)\int_0^t\left(\|D^m u_\tau(\tau)\|^2+\|D^m v_\tau(\tau)\|^2\right)d\tau \\
&\quad +(r-2)\left(\|D^m u\|^2+\|D^m v\|^2\right) \\
&\quad +\frac{r-2\gamma-2}{\gamma+1}\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma+1} \\
&\quad +\frac{2}{r}\left(\|u\|_r^r+\|v\|_r^r\right) \\
&\geq (r+2-\zeta)\left(\|u_t\|^2+\|v_t\|^2\right)-2rE(0) \\
&\quad +(r-2)\left(\|D^m u\|^2+\|D^m v\|^2\right) \\
&\geq (r+2-\zeta)\left(\|u_t\|^2+\|v_t\|^2\right)-2rE(0) \\
&\quad +\frac{2(r+2-\zeta)}{C}\left(\|D^m u\|^2+\|D^m v\|^2\right) \\
&\quad +\left((r-2)-\frac{2(r+2-\zeta)}{C}\right)\left(\|D^m u\|^2+\|D^m v\|^2\right) \\
&\geq (r+2-\zeta)\left(\|u_t\|^2+\|v_t\|^2+2\left(\|u\|^2+\|v\|^2\right)\right)-2rE(0) \\
&\quad +\left((r-2)-\frac{2(r+2-\zeta)}{C}\right)\left(\|D^m u\|^2+\|D^m v\|^2\right) \\
&\geq \frac{C(r-2)}{C+2}\Big[\|u_t\|^2+\|v_t\|^2+2\left(\|u\|^2+\|v\|^2\right) \\
&\quad +\|D^m u\|^2+\|D^m v\|^2\Big]-2rE(0) \\
&\geq \frac{C(r-2)}{C+2}\left[2(u,u_t)+2(v,v_t)+\|D^m u\|^2+\|D^m v\|^2\right] \\
&\quad -2rE(0). \qquad (3.20)
\end{aligned}$$

Therefore by Lemma 3.1 and Lemma 3.2, we conclude that

$$\begin{aligned}
\zeta(t) &\geq \frac{C(r-2)}{C+2}\left[2(u,u_t)+2(v,v_t)+\|D^m u\|^2+\|D^m v\|^2\right]-2rE(0) \\
&= \frac{C(r-2)}{C+2}\left[2(u,u_t)+2(v,v_t)+\|D^m u\|^2+\|D^m v\|^2-\frac{2r(C+2)}{C(r-2)}\right] \\
&\geq \frac{C(r-2)}{C+2}\left[2(u_0,u_1)+2(v_0,v_1)+\|D^m u_0\|^2+\|D^m v_0\|^2-\frac{2r(C+2)}{C(r-2)}\right] \\
&> \sigma_2 > 0,
\end{aligned}$$

which shows that

$$\Phi''(t)\Phi(t)-\frac{\zeta}{4}\left(\Phi'(t)\right)^2 > \Phi(t)\sigma_2 > 0.$$

Let $y(t) = \Phi(t)^{-\frac{\zeta-4}{4}}$, then we obtain

$$y''(t) \leq -\frac{\zeta-4}{4}\sigma_2 y(t)^{\frac{\zeta}{\zeta-4}}, \ t \in [0, T_0],$$

where $\zeta = \frac{4C+2r+4}{C+2} \geq 4$.

That is

$$\lim_{t \to T^*} y(t) = 0,$$

where T^* is independent of initial choice of T_0 and $T^* < T_0$. Therefore, we can conclude that

$$\lim_{t \to T^*} \Phi(t) = \infty.$$

□

4. Conclusion

This paper has been able to prove the blow up result for a higher order Kirchhoff type system with logarithmic nonlinearities. This result is new for these types of systems, and it generalises many related problems in the literature.

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ON FUNCTION SPACES CHARACTERIZED BY THE WIGNER TRANSFORM

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Abstract. Let ω_i be weight functions on $\mathbb{R}$, (i=1,2,3,4). In this work, we define $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ to be vector space of $(f,g) \in \left(L^p_{\omega_1} \times L^q_{\omega_2}\right)(\mathbb{R})$ such that the $\tau-$Wigner transforms $W_\tau(f,.)$ and $W_\tau(.,g)$ belong to $L^r_{\omega_3}\left(\mathbb{R}^2\right)$ and $L^s_{\omega_4}\left(\mathbb{R}^2\right)$ respectively for $1 \leq p,q,r,s < \infty$, $\tau \in (0,1)$. We endow this space with a sum norm and prove that $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ is a Banach space. We also show that $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ becomes an essential Banach module over $\left(L^1_{\omega_1} \times L^1_{\omega_2}\right)(\mathbb{R})$. We then consider approximate identities.

1. Introduction

In this paper $S(\mathbb{R})$ denotes the space of complex-valued continuous functions on $\mathbb{R}$ rapidly decreasing at infinity, respectively. The space $L^p(\mathbb{R}), (1 \leq p < \infty)$ denotes the usual Lebesgue space. Let ω be weight function on $\mathbb{R}$, i.e., positive real valued, measurable and locally bounded function which satisfy $\omega(x) \geq 1$, $\omega(x+y) \leq \omega(x)\omega(y)$ for all $x, y \in \mathbb{R}$. For $a \geq 0$, a weight $\omega(x,t) = (1+|x|+|t|)^a$ which is defined on $\mathbb{R}^2$ is called weight of polynomial type. The weighted Lebesgue space is defined by $L^p_\omega(\mathbb{R}) = \{f : \ f\omega \in L^p(\mathbb{R})\}$ for $1 \leq p < \infty$. It is known that $L^p_\omega(\mathbb{R})$ is a Banach space under the norm $\|f\|_{p,\omega} = \|f\omega\|_p$, [9]. For any function $f : \mathbb{R} \longrightarrow \mathbb{C}$, the translation, modulation and dilation operators T_x, M_ω and D_s are given by $T_x f(t) = f(t-x)$, $M_\omega f(t) = e^{2\pi i\omega t} f(t)$ and $D_s f(t) = |s|^{-\frac{1}{2}} f\left(\frac{t}{s}\right)$ for all $x, \omega \in \mathbb{R}$, $0 \neq s \in \mathbb{R}$, respectively. The parameters in wavelet theory are "time" x and "scale" s. Dilation operator D_s preserves the shape of f, but it changes the scale, [7].

Given any fixed $0 \neq g \in L^2(\mathbb{R})$(called the window function), the short-time Fourier transform (STFT) of a function $f \in L^2(\mathbb{R})$ with respect to g is defined by

$$V_g f(x,\omega) = \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi i t\omega}dt,$$

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for $x,\,\omega\in\mathbb{R}$. The short-time Fourier transform is written as convolution $V_gf\,(x,\omega)=e^{-2\pi ix\omega}\,(f*M_\omega g^*)\,(x)$, where $g^*\,(t)=\overline{g\,(-t)}$. It is easy to see that $V_gf\,(x,\omega)=e^{-2\pi ix\omega}V_fg\,(-x,-\omega)$. If g is a compact supported function with its support centered at the origin, then the short-time Fourier transform $V_gf\,(x,.)$ is the Fourier Transform of a segment of f centered in a neighborhood of x, [7]. Let $\tau\in(0,1)$ and let $0\neq g\in L^2\,(\mathbb{R})$ be any fixed window function. The $\tau-$short-time Fourier transform of a function $f\in L^2\,(\mathbb{R})$ with respect to g is given by $V_g^\tau f\,(x,\omega)=V_gf\left(\frac{x}{1-\tau},\frac{\omega}{\tau}\right)$ for $x,\,\omega\in\mathbb{R}$, [1,2,10]

The cross-Wigner distribution of $f,g\in L^2\,(\mathbb{R})$ is defined to be

$$W\,(f,g)\,(x,\omega)=\int\limits_{\mathbb{R}}f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}e^{-2\pi it\omega}dt$$

for $x,\,\omega\in\mathbb{R}$. If $f=g$, then $W\,(f,f)=Wf$ is said the Wigner distribution of $f\in L^2\,(\mathbb{R})$. The Wigner distribution is a quadratic time-frequency representation and it measures how much of the signal energy during the any time period which is concentrated in a frequency band. In this way, information about the energy density in the time-frequency plane is taken. It also gives the joint probability density function of the position and momentum variables, [7]. Let $\tau\in[0,1]$ and let f,g be in $L^2\,(\mathbb{R})$, the $\tau-$Wigner transform is given by

$$W_\tau\,(f,g)\,(x,\omega)=\int\limits_{\mathbb{R}}f\,(x+\tau t)\,\overline{g\,(x-(1-\tau)\,t)}e^{-2\pi it\omega}dt,\;x,\omega\in\mathbb{R}$$

[1,2,10]. Let $(X,\|.\|_X)$ be a Banach space and let $(Y,\|.\|_Y)$ be a Banach algebra. If X is an algebric $Y-$module, and $\|yx\|_X\leq\|y\|_Y\,\|x\|_X$ for all $y\in Y$, $x\in X$, then X is called a Banach Y-module, [12]. If a net $(e_\alpha)_{\alpha\in I}$ in a Banach algebra $(E,\|.\|_E)$ satisfies $\lim\limits_{\alpha\in I}e_\alpha x=x$ for all $x\in E$, then $(e_\alpha)_{\alpha\in I}$ is called a left approximate identity. Also if a net $(e_\alpha)_{\alpha\in I}$ in a Banach algebra $(E,\|.\|_E)$ satisfies $\lim\limits_{\alpha\in I}xe_\alpha=x$ for all $x\in E$, then $(e_\alpha)_{\alpha\in I}$ is called a right approximate identity. If a net $(e_\alpha)_{\alpha\in I}$ is a left approximate identity and right approximate identity, then $(e_\alpha)_{\alpha\in I}$ is called an approximate identity. Moreover if there exists $C>0$ such that $\|e_\alpha\|_E\leq C$ for all $\alpha\in I$, then $(e_\alpha)_{\alpha\in I}$ is said a bounded approximate identity, [3].

2. Main Results

Definition 2.1. Let ω_i $(i=1,2,3,4)$ be weight functions on $\mathbb{R}$ and let $1\leq p,q,r,s<\infty$, $\tau\in(0,1)$. The space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}\,(\mathbb{R})$ consists of all $(f,g)\in\left(L^p_{\omega_1}\times L^q_{\omega_2}\right)(\mathbb{R})$ such that their binary $\tau-$Wigner transforms $(W_\tau\,(f,.)\,,W_\tau\,(.,g))$ are in $\left(L^r_{\omega_3}\times L^s_{\omega_4}\right)\left(\mathbb{R}^2\right)$. It is easy to see that

$$\|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}=\|(f,g)\|_{L^p_{\omega_1}\times L^q_{\omega_2}}+\|(W_\tau\,(f,.)\,,W_\tau\,(.,g))\|_{L^r_{\omega_3}\times L^s_{\omega_4}}$$

is a norm on the vector space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}\,(\mathbb{R})$. Also, there exist sum and maximum norms on the spaces $\left(L^p_{\omega_1}\times L^q_{\omega_2}\right)(\mathbb{R})$ and $\left(L^r_{\omega_3}\times L^s_{\omega_4}\right)\left(\mathbb{R}^2\right)$.

Theorem 2.2. $\left(CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}\,(\mathbb{R})\,,\|.\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}\right)$ *is a Banach space.*

Proof. Assume that $((f_n,g_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}\,(\mathbb{R})$. Clearly $((f_n,g_n))_{n\in\mathbb{N}}$ and $((W_\tau\,(f_n,.)\,,W_\tau\,(.,g_n)))_{n\in\mathbb{N}}$ are Cauchy sequences in $\left(L^p_{\omega_1}\times L^q_{\omega_2}\right)$

$(\mathbb{R})$ and $\left(L^r_{\omega_3} \times L^s_{\omega_4}\right)(\mathbb{R}^2)$, respectively. Since $\left(L^p_{\omega_1} \times L^q_{\omega_2}\right)(\mathbb{R})$ and $\left(L^r_{\omega_3} \times L^s_{\omega_4}\right)(\mathbb{R}^2)$ are Banach spaces, there exist $(f,g) \in \left(L^p_{\omega_1} \times L^q_{\omega_2}\right)(\mathbb{R})$ and $(h,k) \in \left(L^r_{\omega_3} \times L^s_{\omega_4}\right)(\mathbb{R}^2)$ such that $\|f_n - f\|_{p,\omega_1} \longrightarrow 0$, $\|g_n - g\|_{q,\omega_2} \longrightarrow 0$, $\|W_\tau(f_n,.) - h\|_{r,\omega_3} \longrightarrow 0$ and $\|W_\tau(.,g_n) - k\|_{s,\omega_4} \longrightarrow 0$. This implies $\|W_\tau(f_n,.) - h\|_r \longrightarrow 0$ and $\|W_\tau(.,g_n) - k\|_s \longrightarrow 0$. Then $\left((W_\tau(f_n,.), W_\tau(.,g_n))\right)_{n\in\mathbb{N}}$ has a subsequence $\left((W_\tau(f_{n_k},.), W_\tau(.,g_{n_k}))\right)_{n_k\in\mathbb{N}}$ which converges pointwise to (h,k) almost everywhere. Also it is easy to show that $\|f_{n_k} - f\|_p \longrightarrow 0$ and $\|g_{n_k} - g\|_q \longrightarrow 0$. On the other hand, if we use the Hölder inequality, then for any $u \in S(\mathbb{R})$ we find

$$\begin{aligned}
&|W_\tau(f_{n_k},u)(x,\omega) - W_\tau(f,u)(x,\omega)| = \\
&= \left| \int_{\mathbb{R}} f_{nk}(x+\tau t)\overline{u(x-(1-\tau)t)}e^{-2\pi i\omega t}dt - \int_{\mathbb{R}} f(x+\tau t)\overline{u(x-(1-\tau)t)}e^{-2\pi it\omega}dt \right| \\
&\le \int_{\mathbb{R}} \left|(f_{nk} - f)(x+\tau t)\overline{u(x-(1-\tau)t)}e^{-2\pi it\omega}\right| dt
\end{aligned}$$

$$\le \left(\frac{1}{\tau}\right)^{\frac{1}{p}} \left(\frac{1}{1-\tau}\right)^{\frac{1}{p'}} \|f_{nk} - f\|_p \|u\|_{p'}, \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Then by (2.1), we obtain

$$\begin{aligned}
&|W_\tau(f,u)(x,\omega) - h(x,\omega)| \le |W_\tau(f_{n_k},u)(x,\omega) - W_\tau(f,u)(x,\omega)| + \\
&+ |W_\tau(f_{n_k},u)(x,\omega) - h(x,\omega)| \le
\end{aligned}$$

$$\le \left(\frac{1}{\tau}\right)^{\frac{1}{p}} \left(\frac{1}{1-\tau}\right)^{\frac{1}{p'}} \|f_{nk} - f\|_p \|u\|_{p'} + |W_\tau(f_{n_k},.)(x,\omega) - h(x,\omega)| \tag{2.2}$$

for any $u \in S(\mathbb{R})$. By using the inequality (2.2), it is easily seen that $W_\tau(f,.) = h$ almost everywhere. So the equivalence classes of $W_\tau(f,.)$ and h are equal. Using a similar method, we find that $W_\tau(.,g) = k$ almost everywhere. Then the equivalence classes of $W_\tau(.,g)$ and k are equal. Hence

$$\begin{aligned}
&\|(f_n,g_n) - (f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} = \|(f_n - f, g_n - g)\|_{L^p_{\omega_1}\times L^q_{\omega_2}} + \\
&+ \|(W_\tau(f_n - f,.), W_\tau(.,g_n - g))\|_{L^r_{\omega_3}\times L^s_{\omega_4}} \longrightarrow 0
\end{aligned}$$

and $(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. That means $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ is a Banach space. □

Theorem 2.3. *Let ω_i $(i = 1,2,3,4)$ be weight functions of polynomial type. Then $(S \times S)(\mathbb{R})$ is dense in $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$.*

Proof. Take any $(f,g) \in (S \times S)(\mathbb{R})$. Then $(W_\tau(f,.), W_\tau(.,g)) \in (S \times S)(\mathbb{R}^2)$. Since ω_i $(i = 1,2,3,4)$ are weight functions of polynomial type, we have $(f,g) \in \left(L^p_{\omega_1} \times L^q_{\omega_2}\right)(\mathbb{R})$ and $(W_\tau(f,.), W_\tau(.,g)) \in \left(L^r_{\omega_3} \times L^s_{\omega_4}\right)(\mathbb{R}^2)$. That means $(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. Hence we have $(S \times S)(\mathbb{R}) \subset CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$.

Now take any $(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. Then we have $(f,g) \in \left(L^p_{\omega_1} \times L^q_{\omega_2}\right)(\mathbb{R})$ and $(W_\tau(f,.), W_\tau(.,g)) \in \left(L^r_{\omega_3} \times L^s_{\omega_4}\right)(\mathbb{R}^2)$. Since $\overline{(S \times S)(\mathbb{R})} = \left(L^p_{\omega_1} \times L^q_{\omega_2}\right)(\mathbb{R})$ and $\overline{(S \times S)(\mathbb{R}^2)} = \left(L^r_{\omega_3} \times L^s_{\omega_4}\right)(\mathbb{R}^2)$, there exist $((h_n,k_n))_{n\in\mathbb{N}} \subset (S \times S)(\mathbb{R})$ and

$((H_n, K_n))_{n\in\mathbb{N}} \subset (S \times S)\left(\mathbb{R}^2\right)$ such that

$$\|(f,g) - (h_n,k_n)\|_{L^p_{\omega_1}\times L^q_{\omega_2}} \to 0 \tag{2.3}$$

and

$$\|(W_\tau(f,.), W_\tau(.,g)) - (H_n,K_n)\|_{L^r_{\omega_3}\times L^s_{\omega_4}} \to 0. \tag{2.4}$$

Then by (2.4), we have $\|W_\tau(f,.) - H_n\|_r \to 0$ and $\|W_\tau(.,g) - K_n\|_s \to 0$. So $(H_n)_{n\in\mathbb{N}}$ and $(K_n)_{n\in\mathbb{N}}$ have subsequences $(H_{n_k})_{n_k\in\mathbb{N}}$ and $(K_{n_k})_{n_k\in\mathbb{N}}$which converge pointwise to $W_\tau(f,.)$ and $W_\tau(.,g)$ almost everywhere, respectively. Then, we easily show that

$$\|W_\tau(f,.) - H_{n_k}\|_{r,\omega_3} \to 0,\ \|W_\tau(.,g) - K_{n_k}\|_{s,\omega_4} \to 0. \tag{2.5}$$

Using Hölder inequality, we have for any $u \in S(\mathbb{R})$

$$\begin{aligned}
&|W_\tau(h_n,u)(x,\omega) - H_{n_k}(x,\omega)| \le |W_\tau(f,u)(x,\omega) - H_{n_k}(x,\omega)| + \\
&+ |W_\tau(f,u)(x,\omega) - W_\tau(h_n,u)(x,\omega)| \\
&\le |W_\tau(f,u)(x,\omega) - H_{n_k}(x,\omega)| + \int_{\mathbb{R}} |(f-h_n)(x+\tau t)|\,|u(x-(1-\tau)t)|\,dt
\end{aligned}$$

$$\le |W_\tau(f,u)(x,\omega) - H_{n_k}(x,\omega)| + \left(\frac{1}{\tau}\right)^{\frac{1}{p}}\left(\frac{1}{1-\tau}\right)^{\frac{1}{p'}} \|f-h_n\|_p \|u\|_{p'}, \tag{2.6}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. By (2.3) and (2.6), we achieve $W_\tau(h_n,.) = H_{n_k}$. Similarly, we can write $W_\tau(.,k_n) = K_{n_k}$. Then by (2.5), we find

$$\|W_\tau(f,.) - W_\tau(h_n,.)\|_{r,\omega_3} \to 0,\ \|W_\tau(.,g) - W_\tau(.,k_n)\|_{s,\omega_4} \to 0.$$

This implies

$$\|(W_\tau(f,.), W_\tau(.,g)) - (W_\tau(h_n,.), W_\tau(.,k_n))\|_{L^r_{\omega_3}\times L^s_{\omega_4}} \to 0. \tag{2.7}$$

Finally combining (2.3) and (2.7), we get

$$\begin{aligned}
\|(f,g) - (h_n,k_n)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} &= \|(f,g) - (h_n,k_n)\|_{L^p_{\omega_1}\times L^q_{\omega_2}} + \\
+ \|(W_\tau(f,.), W_\tau(.,g)) &- (W_\tau(h_n,.), W_\tau(.,k_n))\|_{L^r_{\omega_3}\times L^s_{\omega_4}} \longrightarrow 0.
\end{aligned}$$

Therefore the proof is completed. □

Definition 2.4. Let ω_1 and ω_3 be weight functions on $\mathbb{R}$ and let be $1 \le p, r < \infty$, $\tau \in (0,1)$. The space $CW^{p,r,\tau}_{\omega_1,\omega_3}(\mathbb{R})$ consists of all $f \in L^p_{\omega_1}(\mathbb{R})$ such that their $\tau-$Wigner transforms $W_\tau(f,.)$ are in $L^r_{\omega_3}\left(\mathbb{R}^2\right)$. We endow this space with the sum norm

$$\|f\|_{CW^{p,r,\tau}_{\omega_1,\omega_3}} = \|f\|_{p,\omega_1} + \|W_\tau(f,.)\|_{r,\omega_3}.$$

Let ω_2 and ω_4 be weight functions on $\mathbb{R}$ and let be $1 \le q, s < \infty$, $\tau \in (0,1)$. The space $CW^{q,s,\tau}_{\omega_2,\omega_4}(\mathbb{R})$ consists of all $g \in L^q_{\omega_2}(\mathbb{R})$ such that their $\tau-$Wigner transforms $W_\tau(.,g)$ are in $L^s_{\omega_4}\left(\mathbb{R}^2\right)$. This space is equipped with the sum norm

$$\|g\|_{CW^{q,s,\tau}_{\omega_2,\omega_4}} = \|g\|_{q,\omega_2} + \|W_\tau(.,g)\|_{s,\omega_4}.$$

By using the method in Theorem 1, it is easy to see that this spaces $CW^{p,r,\tau}_{\omega_1,\omega_3}(\mathbb{R})$ and $CW^{q,s,\tau}_{\omega_2,\omega_4}(\mathbb{R})$ are Banach space with these sum norm.

Lemma 2.5. *The space* $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ *is isomorphic to* $\left(CW^{p,r,\tau}_{\omega_1,\omega_3} \times CW^{q,s,\tau}_{\omega_2,\omega_4}\right)(\mathbb{R})$.

Proof. Take the mapping $I : CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \to \left(CW^{p,r,\tau}_{\omega_1,\omega_3} \times CW^{q,s,\tau}_{\omega_2,\omega_4}\right)(\mathbb{R})$, $I((f,g)) = (f,g)$. It is clear that this mapping is linear and bijective. Also, since

$$\begin{aligned}
&\|H((f,g))\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} = \|(f,g)\|_{L^p_{\omega_1}\times L^q_{\omega_2}} + \|(W_\tau(f,.),W_\tau(.,g))\|_{L^r_{\omega_3}\times L^s_{\omega_4}}\\
&= \|f\|_{p,\omega_1} + \|g\|_{q,\omega_2} + \|W_\tau(f,.)\|_{r,\omega_3} + \|W_\tau(.,g)\|_{s,\omega_4}\\
&= \|f\|_{p,\omega_1} + \|W_\tau(f,.)\|_{r,\omega_3} + \|g\|_{q,\omega_2} + \|W_\tau(.,g)\|_{s,\omega_4}\\
&= \|f\|_{CW^{p,r,\tau}_{\omega_1,\omega_3}} + \|g\|_{CW^{q,s,\tau}_{\omega_2,\omega_4}} = \|(f,g)\|_{CW^{p,r,\tau}_{\omega_1,\omega_3}\times CW^{q,s,\tau}_{\omega_2,\omega_4}},
\end{aligned}$$

The mapping I is isometry of $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ into $\left(CW^{p,r,\tau}_{\omega_1,\omega_3} \times CW^{q,s,\tau}_{\omega_2,\omega_4}\right)(\mathbb{R})$. Therefore, we obtain that $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \cong \left(CW^{p,r,\tau}_{\omega_1,\omega_3} \times CW^{q,s,\tau}_{\omega_2,\omega_4}\right)(\mathbb{R})$. □

Definition 2.6. Let f and g be any functions on $\mathbb{R}$. The binary translation mapping is defined by

$$T_x(f,g)(t) = (T_x f(t), T_x g(t)) = (f(t-x), g(t-x)),\ x,t \in \mathbb{R}.$$

The following lemma is written easily from Proposition4 in [11]

Lemma 2.7. *For $\tau \in (0,1)$ and $z \in \mathbb{R}$, we have*

$$W_\tau(T_z f, h)(x,\omega) = e^{-2\pi i\omega z} T_{(z(1-\tau),0)} W_\tau(f,h)(x,\omega)$$

and

$$W_\tau(k, T_z g)(x,\omega) = e^{2\pi i\omega z} T_{(z\tau,0)} W_\tau(k,g).$$

Theorem 2.8. *Assume that ω_3 is symmetric weight function. The space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ is invariant under binary translations. Moreover,*

$$\|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} \le (u(z) + v((z,0))\,v((z\tau,0)))\,\|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}},$$

where $u = \max\{\omega_1, \omega_2\}$ and $v = \max\{\omega_3, \omega_4\}$.

Proof. Let $(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. Then, we write $(f,g) \in \left(L^p_{\omega_1} \times L^q_{\omega_2}\right)(\mathbb{R})$ and $(W_\tau(f,.), W_\tau(.,g)) \in \left(L^r_{\omega_3} \times L^s_{\omega_4}\right)(\mathbb{R}^2)$. Also, since $\|T_z f\|_{p,\omega_1} \le \omega_1(z)\|f\|_{p,\omega_1}$ and $\|T_z g\|_{q,\omega_2} \le \omega_2(z)\|f\|_{q,\omega_2}$ [5], we have

$$\begin{aligned}
\|T_z(f,g)\|_{L^p_{\omega_1}\times L^q_{\omega_2}} &= \|T_z f\|_{p,\omega_1} + \|T_z g\|_{q,\omega_2}\\
&\le \omega_1(z)\|f\|_{p,\omega_1} + \omega_2(z)\|f\|_{q,\omega_2}\\
&\le u(z)\|(f,g)\|_{L^p_{\omega_1}\times L^q_{\omega_2}}, \qquad (2.8)
\end{aligned}$$

where $u = \max\{\omega_1, \omega_2\}$. Then, we write $T_z(f,g) \in \left(L^p_{\omega_1} \times L^q_{\omega_2}\right)(\mathbb{R})$ for all $z \in \mathbb{R}$. By Lemma 2.7, we have

$$\begin{aligned}
&\|(W_\tau(T_z f,.), W_\tau(.,T_z g))\|_{L^r_{\omega_3}\times L^s_{\omega_4}} = \|W_\tau(T_z f,.)\|_{r,\omega_3} + \|W_\tau(.,T_z g)\|_{s,\omega_4}\\
&= \left\|e^{-2\pi i\omega z} T_{(z(1-\tau),0)} W_\tau(f,.)\right\|_{r,\omega_3} + \left\|e^{2\pi i\omega z} T_{(z\tau,0)} W_\tau(.,g)\right\|_{s,\omega_4}\\
&\le \omega_3((z(1-\tau),0))\|W_\tau(f,.)\|_{r,\omega_3} + \omega_4((z\tau,0))\|W_\tau(.,g)\|_{s,\omega_4}\\
&\le \omega_3((z,0))\,\omega_3((z\tau,0))\|W_\tau(f,.)\|_{r,\omega_3} + \omega_4((z,0))\,\omega_4((z\tau,0))\|W_\tau(.,g)\|_{s,\omega_4}\\
&\le v((z,0))\,v((z\tau,0))\,\|(W_\tau(f,.), W_\tau(.,g))\|_{L^r_{\omega_3}\times L^s_{\omega_4}}, \qquad (2.9)
\end{aligned}$$

where $v = \max\{\omega_3, \omega_4\}$. Combining (2.8) and (2.9),

$$\begin{aligned}
&\|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} = \|(T_z f, T_z g)\|_{L^p_{\omega_1}\times L^q_{\omega_2}} + \|(W_\tau(T_z f,.), W_\tau(., T_z g))\|_{L^r_{\omega_3}\times L^s_{\omega_4}} \\
&\le u(z)\|(f,g)\|_{L^p_{\omega_1}\times L^q_{\omega_2}} + v((z,0))\, v((z\tau,0))\, \|(W_\tau(f,.), W_\tau(.,g))\|_{L^r_{\omega_3}\times L^s_{\omega_4}} \\
&\le u(z)\|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} + v((z,0))\, v((z\tau,0))\, \|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} \\
&= (u(z) + v((z,0))\, v((z\tau,0)))\, \|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}.
\end{aligned}$$

Finally, we say $T_z(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. □

Definition 2.9. Let f, g, h, k be Borel measurable functions on $\mathbb{R}$. The binary convolution is defined by $(f,g)*(h,k) = (f*h, g*k)$, where "$*$" denotes usual convolution. The following conditions must be required for the binary convolution to be defined;

$$\int_{\mathbb{R}} |f(y)\, h(x-y)|\, dy < \infty$$

and

$$\int_{\mathbb{R}.} |g(y)\, k(x-y)|\, dy < \infty$$

Theorem 2.10. ***a)*** *Assume that ω_3 is symmetric weight function. The binary translation mapping $(f,g) \longrightarrow T_z(f,g)$ is continuous from $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ into $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ for every fixed $z \in \mathbb{R}$.*

b) *The binary translation mapping $z \to T_z(f,g)$ is continuous from $\mathbb{R}$ into $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$.*

Proof. **a)** Let $(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ be given. It is enough to prove the theorem for $(f,g) = (0,0)$. Let $\varepsilon > 0$ be given. Choose an $\delta > 0$ such that $\delta = \frac{\varepsilon}{u(z)+v((z,0))v((z\tau,0))}$. Thus, if $\|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} < \delta$, then by (2.9)

$$\begin{aligned}
\|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} &\le (u(z) + v((z,0))\, v((z\tau,0)))\, \|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} \\
&< \delta\,(u(z) + v((z,0))\, v((z\tau,0))) = \varepsilon.
\end{aligned}$$

b) Take any $(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. It is known that the translation mapping is continuous from $\mathbb{R}$ into $L^p_{\omega_1}(\mathbb{R})$ and $L^q_{\omega_2}(\mathbb{R})$, [5]. So for any given $\varepsilon > 0$, there exists $\delta_1(\varepsilon) > 0$ such that if $|z-u| < \delta_1$ for z, $u \in \mathbb{R}$, then

$$\begin{aligned}
\|T_z(f,g) - T_u(f,g)\|_{L^p_{\omega_1}\times L^q_{\omega_2}} &= \|(T_z f - T_u f -, T_z g - T_u g)\|_{L^p_{\omega_1}\times L^q_{\omega_2}} \\
&= \max\left\{\|T_z f - T_u f\|_{p,\omega_1}, \|T_z g - T_u g\|_{q,\omega_2}\right\} < \frac{\varepsilon}{2}
\end{aligned} \tag{2.10}$$

Also since the modulation mapping is continuous from $\mathbb{R}$ into $L^r_{\omega_3}(\mathbb{R}^2)$ and $L^s_{\omega_4}(\mathbb{R}^2)$ [5], for the same $\varepsilon > 0$, there exists $\delta_2(\varepsilon) > 0$ such that if $|z-u| < \delta_2$ for z, $u \in \mathbb{R}$,

then

$$\begin{aligned}
&\left\|\left(W_\tau\left(T_z f-T_u f, .\right), W_\tau\left(., T_z g-T_u g\right)\right)\right\|_{L^r_{\omega_3}\times L^s_{\omega_4}}= \\
&=\left\|W_\tau\left(T_z f-T_u f, .\right)\right\|_{r,\omega_3}+\left\|W_\tau\left(., T_z g-T_u g\right)\right\|_{s,\omega_4}= \\
&=\max\{\left\|e^{-2\pi i\omega z}T_{(z(1-\tau),0)}W_\tau(f,.)-e^{-2\pi i\omega u}T_{(u(1-\tau),0)}W_\tau(f,.)\right\|_{r,\omega_3}, \\
&\left\|e^{2\pi i\omega z}T_{(z\tau,0)}W_\tau(.,g)-e^{2\pi i\omega u}T_{(u\tau,0)}W_\tau(.,g)\right\|_{s,\omega_4}\}= \\
&=\max\{\left\|M_{(0,-z)}T_{(z(1-\tau),0)}W_\tau(f,.)-M_{(0,-u)}T_{(u(1-\tau),0)}W_\tau(f,.)\right\|_{r,\omega_3}, \\
&\left\|M_{(0,z)}T_{(z\tau,0)}W_\tau(.,g)-M_{(0,u)}T_{(u\tau,0)}W_\tau(.,g)\right\|_{s,\omega_4}\}<\frac{\varepsilon}{2} \qquad (2.11)
\end{aligned}$$

Set $\delta=\min\{\delta_1,\delta_2\}$. From (2.10) and (2.11), if $|z-u|<\delta$ for $z,\ u\in\mathbb{R}$, then

$$\begin{aligned}
&\left\|T_z(f,g)-T_u(f,g)\right\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}=\left\|\left(T_z f-T_u f-, T_z g-T_u g\right)\right\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} \\
&=\left\|\left(T_z f-T_u f-, T_z g-T_u g\right)\right\|_{L^p_{\omega_1}\times L^q_{\omega_2}}+ \\
&+\left\|\left(W_\tau\left(T_z f-T_u f, .\right), W_\tau\left(., T_z g-T_u g\right)\right)\right\|_{L^r_{\omega_3}\times L^s_{\omega_4}}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

□

Corollary 1. a) *The binary translation mapping* $z\to T_z(f,g)$ *is continuous from* $\mathbb{R}$ into $CW^{p,r,\tau}_{\omega_1,\omega_3}(\mathbb{R})$.

b) *The binary translation mapping* $z\to T_z(f,g)$ *is continuous from* $\mathbb{R}$ into $CW^{q,s,\tau}_{\omega_2,\omega_4}(\mathbb{R})$.

Lemma 2.11. *Let* $f,\ g\in S(\mathbb{R})$. *If* $\tau\in(0,1)$, *then*

$$W_\tau(f,g)(x,\omega)=e^{\frac{2\pi i x\omega}{\tau}}\frac{1}{\sqrt{\tau(1-\tau)}}V^{\tau}_{D_{\frac{\tau}{\tau-1}}g}f(x,\omega)$$

holds for all $x,\ \omega\in\mathbb{R}$.

Proof. Assume that $f,\ g\in S(\mathbb{R})$. If we make the substitution $u=x+\tau t$, then we have

$$\begin{aligned}
W_\tau(f,g)(x,\omega)&=\int_{\mathbb{R}} f(x+\tau t)\overline{g(x-(1-\tau)t)}e^{-2\pi i\omega t}dt \\
&=\int_{\mathbb{R}} f(u)\overline{g\left(u-\frac{u-x}{\tau}\right)}e^{-2\pi i\omega\left(\frac{u-x}{\tau}\right)}\frac{du}{\tau} \\
&=\int_{\mathbb{R}} f(u)\overline{g\left(u\left(\frac{\tau-1}{\tau}\right)+\frac{x}{\tau}\right)}e^{-2\pi i\omega\left(\frac{u-x}{\tau}\right)}\frac{du}{\tau} \\
&=\frac{1}{\tau}e^{2\pi i\frac{x\omega}{\tau}}\int_{\mathbb{R}} f(u)\overline{g\left(\left(\frac{\tau-1}{\tau}\right)\left(u-\frac{x}{1-\tau}\right)\right)}e^{-2\pi i\frac{\omega u}{\tau}}du \\
&=\frac{1}{\tau}e^{2\pi i\frac{x\omega}{\tau}}\left|\frac{\tau}{\tau-1}\right|^{\frac{1}{2}}\int_{\mathbb{R}} f(u)\overline{D_{\frac{\tau}{\tau-1}}g\left(u-\frac{x}{1-\tau}\right)}e^{-2\pi i\frac{\omega u}{\tau}}du \\
&=e^{\frac{2\pi i x\omega}{\tau}}\frac{1}{\sqrt{\tau(1-\tau)}}V^{\tau}_{D_{\frac{\tau}{\tau-1}}g}f(x,\omega)
\end{aligned}$$

for all $x,\ \omega\in\mathbb{R}$. □

Theorem 2.12. *Let $f,\ h,\ g,\ k,\ f_1,\ f_2 \in S(\mathbb{R})$. If $\tau \in (0,1)$, then*

$$W_\tau(f*h,f_1)(x,\omega) = \frac{1}{\sqrt{\tau(1-\tau)}} e^{\frac{-2\pi i x\omega}{1-\tau}} \left(h*\left(f*M_{\frac{\omega}{\tau}}\left(D_{\frac{\tau}{\tau-1}}f_1\right)^*\right)\right)\left(\frac{x}{1-\tau}\right)$$

and

$$W_\tau(f_2,g*k)(x,\omega) = \frac{1}{\tau} e^{\frac{2\pi i x\omega(1+\tau)}{\tau(1-\tau)}} \overline{\left(D_{\frac{\tau}{\tau-1}}k*\left(D_{\frac{\tau}{\tau-1}}g*M_{\frac{-\omega}{\tau}}f_2^*\right)\right)\left(\frac{-x}{1-\tau}\right)}$$

holds for all $x,\ \omega \in \mathbb{R}$.

Proof. Take any $f,\ h,\ f_1 \in S(\mathbb{R})$. Then by Lemma 2.11, we have

$$\begin{aligned} W_\tau(f*h,f_1)(x,\omega) &= e^{\frac{2\pi i x\omega}{\tau}} \frac{1}{\sqrt{\tau(1-\tau)}} V^\tau_{D_{\frac{\tau}{\tau-1}}f_1}(f*h)(x,\omega) \\ &= e^{\frac{2\pi i x\omega}{\tau}} \frac{1}{\sqrt{\tau(1-\tau)}} V_{D_{\frac{\tau}{\tau-1}}f_1}(f*h)\left(\frac{x}{1-\tau},\frac{\omega}{\tau}\right) \\ &= e^{\frac{2\pi i x\omega}{\tau}} \frac{1}{\sqrt{\tau(1-\tau)}} e^{\frac{-2\pi i x\omega}{\tau(1-\tau)}} \left((f*h)*M_{\frac{\omega}{\tau}}\left(D_{\frac{\tau}{\tau-1}}f_1\right)^*\right)\left(\frac{x}{1-\tau}\right) \\ &= \frac{1}{\sqrt{\tau(1-\tau)}} e^{\frac{-2\pi i x\omega}{1-\tau}} \left(h*\left(f*M_{\frac{\omega}{\tau}}\left(D_{\frac{\tau}{\tau-1}}f_1\right)^*\right)\right)\left(\frac{x}{1-\tau}\right). \end{aligned}$$

Now take any $g,\ k,\ f_2 \in S(\mathbb{R})$. Again by Lemma 2.11, we get

$$\begin{aligned} W_\tau(f_2,g*k)(x,\omega) &= e^{\frac{2\pi i x\omega}{\tau}} \frac{1}{\sqrt{\tau(1-\tau)}} V^\tau_{D_{\frac{\tau}{\tau-1}}(g*k)}f_2(x,\omega) \\ &= e^{\frac{2\pi i x\omega}{\tau}} \frac{1}{\sqrt{\tau(1-\tau)}} V_{D_{\frac{\tau}{\tau-1}}(g*k)}f_2\left(\frac{x}{1-\tau},\frac{\omega}{\tau}\right) \\ &= e^{\frac{2\pi i x\omega}{\tau}} \frac{1}{\sqrt{\tau(1-\tau)}} e^{\frac{-2\pi i x\omega}{\tau(1-\tau)}} \overline{V_{f_2}D_{\frac{\tau}{\tau-1}}(g*k)\left(\frac{-x}{1-\tau},\frac{-\omega}{\tau}\right)} \\ &= e^{\frac{2\pi i x\omega}{\tau}} \frac{1}{\sqrt{\tau(1-\tau)}} e^{\frac{-2\pi i x\omega}{\tau(1-\tau)}} \overline{e^{\frac{-2\pi i(-x)\omega}{\tau(1-\tau)}}\left(D_{\frac{\tau}{\tau-1}}(g*k)*M_{\frac{-\omega}{\tau}}f_2^*\right)\left(\frac{-x}{1-\tau}\right)} \\ &= \frac{1}{\sqrt{\tau(1-\tau)}} e^{\frac{2\pi i x\omega(1+\tau)}{\tau(1-\tau)}} \overline{\left|\frac{\tau}{\tau-1}\right|^{\frac{-1}{2}}\left(\left(D_{\frac{\tau}{\tau-1}}g*D_{\frac{\tau}{\tau-1}}k\right)*M_{\frac{-\omega}{\tau}}f_2^*\right)\left(\frac{-x}{1-\tau}\right)} \\ &= \frac{1}{\tau} e^{\frac{2\pi i x\omega(1+\tau)}{\tau(1-\tau)}} \overline{\left(D_{\frac{\tau}{\tau-1}}k*\left(D_{\frac{\tau}{\tau-1}}g*M_{\frac{-\omega}{\tau}}f_2^*\right)\right)\left(\frac{-x}{1-\tau}\right)}. \end{aligned}$$

□

Theorem 2.13. *Suppose that $\omega_3 = k_1$ and $\omega_4 = k_2$ such that k_1 and k_2 are constant numbers. Then $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ is an essential Banach module over $\left(L^1_{\omega_1}\times L^1_{\omega_2}\right)(\mathbb{R})$.*

Proof. Let ω_3 and ω_4 be constant weight functions. It is known that $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ is a Banach space by Theorem 2.2. Now we take any $(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ and $(h,k) \in \left(L^1_{\omega_1}\times L^1_{\omega_2}\right)(\mathbb{R})$. Since $L^p_{\omega_1}(\mathbb{R})$ and $L^q_{\omega_2}(\mathbb{R})$ are Banach convolution module over $L^1_{\omega_1}(\mathbb{R})$ and $L^1_{\omega_2}(\mathbb{R})$ respectively, we have

$$\|f*h\|_{p,\omega_1} \le \|f\|_{p,\omega_1}\|h\|_{1,\omega_1} \tag{2.12}$$

and

$$\|g*k\|_{q,\omega_2} \leq \|g\|_{q,\omega_2}\,\|k\|_{1,\omega_2}\,. \tag{2.13}$$

Take any $f_1 \in S(\mathbb{R})$. By Theorem 2.12, we get

$$\begin{aligned}
\|W_\tau(f*h,f_1)\|_{r,\omega_3} &= \left\|\frac{1}{\sqrt{\tau(1-\tau)}}e^{\frac{-2\pi i x\omega}{1-\tau}}\left(h*\left(f*M_{\frac{\omega}{\tau}}\left(D_{\frac{\tau}{\tau-1}}f_1\right)^*\right)\right)\left(\frac{x}{1-\tau}\right)\right\|_{r,\omega_3}\\
&= \frac{1}{\sqrt{\tau(1-\tau)}}\left\|\int_{\mathbb{R}} h(u)\,T_u\left(f*M_{\frac{\omega}{\tau}}\left(D_{\frac{\tau}{\tau-1}}f_1\right)^*\right)\left(\frac{x}{1-\tau}\right)du\right\|_{r,\omega_3}\\
&= \frac{1}{\sqrt{\tau(1-\tau)}}\int_{\mathbb{R}} |h(u)|\left\|T_u\left(f*M_{\frac{\omega}{\tau}}\left(D_{\frac{\tau}{\tau-1}}f_1\right)^*\right)\left(\frac{x}{1-\tau}\right)\right\|_{r,\omega_3}du\\
&= \frac{1}{\sqrt{\tau(1-\tau)}}\int_{\mathbb{R}} |h(u)|\left\|\left(f*M_{\frac{\omega}{\tau}}\left(D_{\frac{\tau}{\tau-1}}f_1\right)^*\right)\left(\frac{x}{1-\tau}\right)\right\|_{r,\omega_3}du\\
&= \frac{1}{\sqrt{\tau(1-\tau)}}\int_{\mathbb{R}} |h(u)|\left\|e^{\frac{2\pi i x\omega}{\tau(1-\tau)}}e^{\frac{-2\pi i x\omega}{\tau(1-\tau)}}\left(f*M_{\frac{\omega}{\tau}}\left(D_{\frac{\tau}{\tau-1}}f_1\right)^*\right)\left(\frac{x}{1-\tau}\right)\right\|_{r,\omega_3}du\\
&= \frac{1}{\sqrt{\tau(1-\tau)}}\int_{\mathbb{R}} |h(u)|\left\|e^{\frac{2\pi i x\omega}{\tau(1-\tau)}}V_{D_{\frac{\tau}{\tau-1}}f_1}f\left(\frac{x}{1-\tau},\frac{\omega}{\tau}\right)\right\|_{r,\omega_3}du\\
&= \frac{1}{\sqrt{\tau(1-\tau)}}\int_{\mathbb{R}} |h(u)|\left\|V^\tau_{D_{\frac{\tau}{\tau-1}}f_1}f(x,\omega)\right\|_{r,\omega_3}du\\
&= \int_{\mathbb{R}} |h(u)|\left\|e^{\frac{-2\pi i x\omega}{\tau}}\frac{1}{\sqrt{\tau(1-\tau)}}e^{\frac{2\pi i x\omega}{\tau}}V^\tau_{D_{\frac{\tau}{\tau-1}}f_1}f(x,\omega)\right\|_{r,\omega_3}du\\
&= \int_{\mathbb{R}} |h(u)|\left\|e^{\frac{-2\pi i x\omega}{\tau}}W_\tau(f,f_1)\right\|_{r,\omega_3}du\\
&= \|W_\tau(f,f_1)\|_{r,\omega_3}\,\|h\|_1\\
&\leq \|W_\tau(f,f_1)\|_{r,\omega_3}\,\|h\|_{1,\omega_3} < \infty. \qquad (2.14)
\end{aligned}$$

Thus $W_\tau(f*h,.) \in L^r_{\omega_3}(\mathbb{R}^2)$. Now take $f_2 \in S(\mathbb{R})$. Again by Theorem 2.12, we have

$$\begin{aligned}
\|W_\tau(f_2,g*k)\|_{s,\omega_4} &= \left\|\frac{1}{\tau}e^{\frac{2\pi i x\omega(1+\tau)}{\tau(1-\tau)}}\overline{\left(D_{\frac{\tau}{\tau-1}}k*\left(D_{\frac{\tau}{\tau-1}}g*M_{\frac{-\omega}{\tau}}f_2^*\right)\right)\left(\frac{-x}{1-\tau}\right)}\right\|_{s,\omega_4}\\
&= \frac{1}{\tau}\left\|\left(D_{\frac{\tau}{\tau-1}}k*\left(D_{\frac{\tau}{\tau-1}}g*M_{\frac{-\omega}{\tau}}f_2^*\right)\right)\left(\frac{-x}{1-\tau}\right)\right\|_{s,\omega_4}\\
&= \frac{1}{\tau}\left\|\int_{\mathbb{R}} D_{\frac{\tau}{\tau-1}}k(u)\,T_u\left(D_{\frac{\tau}{\tau-1}}g*M_{\frac{-\omega}{\tau}}f_2^*\right)\left(\frac{-x}{1-\tau}\right)du\right\|_{s,\omega_4}\\
&= \frac{1}{\tau}\int_{\mathbb{R}}\left\|D_{\frac{\tau}{\tau-1}}k(u)\,T_u\left(D_{\frac{\tau}{\tau-1}}g*M_{\frac{-\omega}{\tau}}f_2^*\right)\left(\frac{-x}{1-\tau}\right)\right\|_{s,\omega_4}du
\end{aligned}$$

$$= \frac{1}{\tau}\int_{\mathbb{R}} \left|D_{\frac{\tau}{\tau-1}}k\left(u\right)\right| \left\| e^{\frac{2\pi i x\omega}{\tau(1-\tau)}} e^{\frac{-2\pi i x\omega}{\tau(1-\tau)}} \left(D_{\frac{\tau}{\tau-1}}g * M_{\frac{-\omega}{\tau}}f_2^*\right)\left(\frac{-x}{1-\tau}\right)\right\|_{s,\omega_4} du$$

$$= \frac{1}{\tau}\int_{\mathbb{R}} \left|D_{\frac{\tau}{\tau-1}}k\left(u\right)\right| \left\| e^{\frac{2\pi i x\omega}{\tau(1-\tau)}} V_{f_2}D_{\frac{\tau}{\tau-1}}g\left(\frac{-x}{1-\tau},\frac{-\omega}{\tau}\right)\right\|_{s,\omega_4} du$$

$$= \frac{1}{\tau}\int_{\mathbb{R}} \left|D_{\frac{\tau}{\tau-1}}k\left(u\right)\right| \left\| e^{\frac{4\pi i x\omega}{\tau(1-\tau)}} e^{\frac{-2\pi i x\omega}{\tau(1-\tau)}} V_{f_2}D_{\frac{\tau}{\tau-1}}g\left(\frac{-x}{1-\tau},\frac{-\omega}{\tau}\right)\right\|_{s,\omega_4} du$$

$$= \frac{1}{\tau}\int_{\mathbb{R}} \left|D_{\frac{\tau}{\tau-1}}k\left(u\right)\right| \left\| e^{\frac{4\pi i x\omega}{\tau(1-\tau)}} V_{D_{\frac{\tau}{\tau-1}}g}f_2\left(\frac{x}{1-\tau},\frac{\omega}{\tau}\right)\right\|_{s,\omega_4} du$$

$$= \frac{1}{\tau}\int_{\mathbb{R}} \left|D_{\frac{\tau}{\tau-1}}k\left(u\right)\right| \left\| \sqrt{\tau\left(1-\tau\right)}e^{\frac{-2\pi i x\omega}{\tau}} \frac{1}{\sqrt{\tau\left(1-\tau\right)}} e^{\frac{2\pi i x\omega}{\tau}} V^{\tau}_{D_{\frac{\tau}{\tau-1}}g}f_2\left(x,\omega\right)\right\|_{s,\omega_4} du$$

$$= \frac{1}{\tau}\sqrt{\tau\left(1-\tau\right)}\int_{\mathbb{R}} \left|D_{\frac{\tau}{\tau-1}}k\left(u\right)\right| \left\|W_\tau\left(f_2,g\right)\right\|_{s,\omega_4} du$$

$$= \sqrt{\frac{1-\tau}{\tau}}\left\|W_\tau\left(f_2,g\right)\right\|_{s,\omega_4}\left\|D_{\frac{\tau}{\tau-1}}k\right\|_1$$

$$= \sqrt{\frac{1-\tau}{\tau}}\left\|W_\tau\left(f_2,g\right)\right\|_{s,\omega_4}\sqrt{\frac{\tau}{1-\tau}}\left\|k\right\|_1$$

(2.15)

$$\leq \left\|W_\tau\left(f_2,g\right)\right\|_{s,\omega_4}\left\|k\right\|_{1,\omega_4} < \infty.$$

So $W_\tau\left(.,g*k\right) \in L^s_{\omega_4}\left(\mathbb{R}^2\right)$. Combining (2.12), (2.13), (2.14) and (2.15), we achieve

$$\left\|(f,g)*(h,k)\right\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} = \left\|(f*h,g*k)\right\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} =$$

$$= \left\|(f*h,g*k)\right\|_{L^p_{\omega_1}\times L^q_{\omega_2}} + \left\|\left(W_\tau\left(f*h,.\right),W_\tau\left(.,g*k\right)\right)\right\|_{L^r_{\omega_3}\times L^s_{\omega_4}}$$

$$= \max\left\{\left\|f*h\right\|_{p,\omega_1},\left\|g*k\right\|_{q,\omega_2}\right\} + \max\left\{\left\|W_\tau\left(f*h,.\right)\right\|_{r,\omega_3},\left\|W_\tau\left(.,g*k\right)\right\|_{s,\omega_4}\right\}$$

$$= \max\left\{\left\|f\right\|_{p,\omega_1}\left\|h\right\|_{1,\omega_1},\left\|g\right\|_{q,\omega_2}\left\|k\right\|_{1,\omega_2}\right\} + \max\left\{\left\|W_\tau\left(f,.\right)\right\|_{r,\omega_3}\left\|h\right\|_{1,\omega_3},\left\|W_\tau\left(.,g\right)\right\|_{s,\omega_4}\left\|k\right\|_{1,\omega_4}\right\}$$

$$= \max\left\{\left\|f\right\|_{p,\omega_1},\left\|g\right\|_{q,\omega_2}\right\}\max\left\{\left\|h\right\|_{1,\omega_1},\left\|k\right\|_{1,\omega_2}\right\} +$$

$$+ \max\left\{\left\|W_\tau\left(f,.\right)\right\|_{r,\omega_3},\left\|W_\tau\left(.,g\right)\right\|_{s,\omega_4}\right\}\max\left\{\left\|h\right\|_{1,\omega_3},\left\|k\right\|_{1,\omega_4}\right\}$$

$$= \max\left\{\left\|f\right\|_{p,\omega_1},\left\|g\right\|_{q,\omega_2}\right\}\max\left\{\left\|h\right\|_{1,\omega_1},\left\|k\right\|_{1,\omega_2}\right\} +$$

$$+ \max\left\{\left\|W_\tau\left(f,.\right)\right\|_{r,\omega_3},\left\|W_\tau\left(.,g\right)\right\|_{s,\omega_4}\right\}\max\left\{\left\|h\right\|_{1,\omega_1},\left\|k\right\|_{1,\omega_2}\right\}$$

$$= \left\{\max\left\{\left\|f\right\|_{p,\omega_1},\left\|g\right\|_{q,\omega_2}\right\} + \max\left\{\left\|W_\tau\left(f,.\right)\right\|_{r,\omega_3},\left\|W_\tau\left(.,g\right)\right\|_{s,\omega_4}\right\}\right\}\max\left\{\left\|h\right\|_{1,\omega_1},\left\|k\right\|_{1,\omega_2}\right\}$$

$$= \left\{\left\|(f,g)\right\|_{L^p_{\omega_1}\times L^q_{\omega_2}} + \left\|\left(W_\tau\left(f,.\right),W_\tau\left(.,g\right)\right)\right\|_{L^r_{\omega_3}\times L^s_{\omega_4}}\right\}\left\|(h,k)\right\|_{L^1_{\omega_1}\times L^1_{\omega_2}}$$

(2.16)

$$= \left\|(f,g)\right\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}\left\|(h,k)\right\|_{L^1_{\omega_1}\times L^1_{\omega_2}}.$$

Therefore we obtain that $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}\left(\mathbb{R}\right)$ is a Banach module over $\left(L^1_{\omega_1}\times L^1_{\omega_2}\right)\left(\mathbb{R}\right)$.

Now we will show that $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ is an essential Banach module over $\left(L^1_{\omega_1} \times L^1_{\omega_2}\right)(\mathbb{R})$ and use Module Factorization Theorem [12]. For this, it suffices to prove that $\left(L^1_{\omega_1} \times L^1_{\omega_2}\right)(\mathbb{R}) * CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ is dense in $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. By (2.16), we can write

$$\left(L^1_{\omega_1} \times L^1_{\omega_2}\right)(\mathbb{R}) * CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \subset CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}).$$

Also it is known that $L^1_{\omega_1}(\mathbb{R})$ and $L^1_{\omega_2}(\mathbb{R})$ have bounded approximate identity, [6]. Let U and V be compact neighbourhoods of the unit element of $\mathbb{R}$. We can choose approximate identities $(e_\alpha)_{\alpha\in I}$ and $(e_\beta)_{\beta\in I}$ which are positive bounded and supp$e_\alpha \subset U$, supp$e_\beta \subset V$, $\|e_\alpha\|_1 = 1$ and $\|e_\beta\|_1 = 1$ for all α, $\beta \in I$. Let $(h,k) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. For fixed α_0, $\beta_0 \in I$, we get

$$\begin{aligned}
\|(e_{\alpha_0}, e_{\beta_0}) * (h,k) - (h,k)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} &= \|(e_{\alpha_0} * h - h, e_{\beta_0} * k - k)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} \\
&\approx \|(e_{\alpha_0} * h - h, e_{\beta_0} * k - k)\|_{CW^{p,r,\tau}_{\omega_1,\omega_3} \times CW^{q,s,\tau}_{\omega_2,\omega_4}} \\
&= \|e_{\alpha_0} * h - h\|_{CW^{p,r,\tau}_{\omega_1,\omega_3}} + \|e_{\beta_0} * k - k\|_{CW^{q,s,\tau}_{\omega_2,\omega_4}}. \qquad (2.17)
\end{aligned}$$

On the other hand, since the translation mapping is continuous by Corollary 1, we have $\|T_z h - h\|_{CW^{p,r,\tau}_{\omega_1,\omega_3}} < \frac{\varepsilon}{2}$ for given any $\varepsilon > 0$. Hence

$$\begin{aligned}
\|e_{\alpha_0} * h - h\|_{CW^{p,r,\tau}_{\omega_1,\omega_3}} &= \left\| \int_{\mathbb{R}} e_{\alpha_0}(z)\left(T_z h(y) - h(y)\right) dz \right\|_{CW^{p,r,\tau}_{\omega_1,\omega_3}} \\
&\leq \int_{\mathbb{R}^d} e_{\alpha_0}(z) \|T_z h - h\|_{CW^{p,r,\tau}_{\omega_1,\omega_3}} dz < \frac{\varepsilon}{2}. \qquad (2.18)
\end{aligned}$$

Similarly we write for the same $\varepsilon > 0$,we can make

$$\|e_{\beta_0} * k - k\|_{CW^{q,s,\tau}_{\omega_2,\omega_4}} < \frac{\varepsilon}{2}. \qquad (2.19)$$

Then, by (2.17), (2.18) and (2.19), we obtain

$$\|(e_{\alpha_0}, e_{\beta_0}) * (h,k) - (h,k)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

That means $\left(L^1_{\omega_1} \times L^1_{\omega_2}\right)(\mathbb{R}) * CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ is dense in $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. Therefore from Module Factorization Theorem, the proof is completed. □

By using Theorem 6, it easy to prove following Corollary

Corollary 2. Assume that $\omega_3 = k_1$ and $\omega_4 = k_2$ such that k_1 and k_2 are constant numbers. Then,

a) $CW^{p,r,\tau}_{\omega_1,\omega_3}(\mathbb{R})$ is an essential Banach module over $L^1_{\omega_1}(\mathbb{R})$.
b) $CW^{q,s,\tau}_{\omega_2,\omega_4}(\mathbb{R})$ is an essential Banach module over $L^1_{\omega_2}(\mathbb{R})$.

Theorem 2.14. *Let $\omega_3 = k_1$ and $\omega_4 = k_2$ be constant weight functions. Then there exists $\left((e_\alpha, e_\beta)\right)_{\alpha,\beta\in I}$ is an approximate identity of the space $\left(L^1_{\omega_1} \times L^1_{\omega_2}\right)(\mathbb{R})$ such that*

$$\lim_{\alpha,\beta\in I} (e_\alpha, e_\beta) * (f,g) = (f,g)$$

for all $(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$.

Proof. Using the fact that $L^1_{\omega_1}(\mathbb{R})$ and $L^1_{\omega_2}(\mathbb{R})$ have bounded approximate identities $(e_\alpha)_{\alpha\in I}$ and $(e_\beta)_{\beta\in I}$, respectively, we easily obtain that $((e_\alpha, e_\beta))_{\alpha,\beta\in I}$ is an approximate identity in $\left(L^1_{\omega_1}\times L^1_{\omega_2}\right)(\mathbb{R})$. On the other hand, from Corollary 2 and by [3], we get

$$\lim_{\alpha\in I} e_\alpha * f = f,\ \lim_{\beta\in I} e_\beta * g = g \tag{2.20}$$

for all $f\in CW^{p,r,\tau}_{\omega_1,\omega_3}(\mathbb{R})$ and $g\in CW^{q,s,\tau}_{\omega_2,\omega_4}(\mathbb{R})$. Therefore by (2.20), we obtain

$$\lim_{\alpha,\beta\in I}(e_\alpha, e_\beta) * (f,g) = \lim_{\alpha,\beta\in I}(e_\alpha * f, e_\beta * g) = (f,g)$$

$(f,g)\in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. □

3. Conclusion

Wigner transform, which is a quadratic time-frequency representation; it is very ideal in the mathematical description of the time-frequency information of the signals. The reasons for its preference can be summarized as follows: The Wigner transform measures the energy of a given signal in any frequency band and in any time period. Thus, in signal analysis, the information of the energy density in the time-frequency plane is obtained. It also gives the joint probability distribution for position and momentum variables in physics, [7]. In our previous papers, we have characterized function spaces using the wavelet transform and fractional wavelet transform, [4,8]. In this study, we defined a new function space using the $\tau-$Wigner transform, which is a quadric time-frequency transform. We then have studied the Banach module structure of this space, the continuity of the translation mapping and its approximate units. In this way, a new function space with rich features was characterized thanks to the $\tau-$Wigner transform, which is frequently used in harmonic analysis, signal analysis and operator theory.

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RICCI SOLITONS AND GRADIENT RICCI SOLITONS ON NEARLY COSYMPLECTIC MANIFOLDS

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Abstract. In this article, a number of properties have been obtained by examining Ricci solitons and gradient Ricci solitons on nearly cosymplectic manifolds.

1. Introduction

In 1959, Liberman [22] and in 1967 Blair [8] have described as odd-dimensional cosymplectic manifolds similar to Kähler manifolds. Later in 1970, nearly Kähler manifolds with the structure (M, J, g) have been introduced as almost Hermitian manifolds by Gray. Based on this study, almost complex structure's covariant derivative is skew symmetric operator according to the Levi-Civita connection. Also the covariant derivative operator satisfies

$$(\nabla_X J) X = 0,$$

for every vector field X on M [18]. Following year, Blair has defined an almost contact manifold with Killing structure tensors which is a nearly cosymplectic manifold [7]. Nearly cosymplectic manifolds have defined by the same viewpoint as cosymplectic (and also called coKähler) manifolds. Almost contact metric structure (φ, ξ, η, g) that provides the condition

$$(\nabla_X \varphi) X = 0, \tag{1.1}$$

is called a nearly cosymplectic structure. Also a smooth manifold M with nearly cosymplectic structure which endowed with almost contact metric structure (φ, ξ, η, g) is said to be nearly cosymplectic manifold. Recently, nearly cosymplectic manifolds have been studied by many researchers (e.g. [1], [15], [26], [27]). Ricci solitons have recently become an important research topic due to the Ricci flow on many manifolds. Firstly, the definition of Ricci soliton, Ricci solitons have been introduced by Hamilton [19] and can be obtained as a generalization of Einstein metrics. By the way on a manifold M, a Ricci flow is defined as

$$\partial g/\partial t = -2Ric(g)$$

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(we will use S instead of Ric) in the space of metrics.We can express that Ricci solitons move under the Ricci flow easily with difeomorphisms in the first metric, which are the static points of the Ricci flow. In a Riemannian manifold (M, g) admits a smooth vector field V, a Ricci soliton provides the following condition: [20]

$$(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0, \tag{1.2}$$

where S is the Ricci tensor, $\pounds$ is the Lie derivative and λ is a constant. Depends on the announcements of λ, that is $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, the Ricci soliton is said to be shrinking, steady or expanding. Ricci solitons have an important place not only in mathematics but also in physics. Theoretical physicists have studied the equations of Ricci soliton in relation to string theory. At the same time, in physics, metrics which satisfy some special conditions (1.2) are mainly practical and generally use as quasi-Einstein metrics (e.g. [17], [12]). The first contribution to these studies have come from Friedan, who has conducted research on some aspects of Ricci solitons [17]. According to this study, a Ricci soliton is said to be a gradient Ricci soliton (called the potential function), if the vector field Y can be replaced by the gradient of some smooth function f on M. Thus, the concept of gradient Ricci soliton emerged and equation (1.2) consider as the form

$$\nabla\nabla f = S + \lambda g. \tag{1.3}$$

After the study of Ricci solitons and gradient Ricci solitons on contact metric manifolds [23], Ricci solitons and gradient Ricci solitons have been studied on several manifolds. Some of those are: in 2008, Sharma [25] studied Ricci solitons in $K-$contact manifolds with the structure field ξ is killing. In [14],Ricci solitons in $P-$Sasakian manifolds have been studied by De and recently in [5], Barua and De have studied Ricci solitons in Riemannian manifolds. Subsequent studies have been on nearly contact manifolds as nearly Sasakian and nearly Kenmotsu manifolds. In 2019, Ayar and Yıldırım have studied Ricci solitons and gradient Ricci solitons on nearly Kenmotsu manifolds and they have reached important results ([3], [4]). In the light of these studies, by taking these works into consideration, we study Ricci solitons and gradient Ricci solitons on nearly cosymplectic manifolds.In this study, after the intoduction section, the definition and basic curvature properties of the nearly cosymplectic manifolds are given. In the next section, according to the conditions provided by Ricci soliton and Gradient Ricci soliton, manifolds where a nearly cosymplectic manifold is locally isomorphic are processed and we have shown that if a metric of a nearly cosymplectic manifold is a Ricci soliton, then either it is an Einstein or a $\eta-$Einstein manifold. Finaly we give some important results and theorems related to this topic.

2. Nearly Cosymplectic Manifolds

In this section, first of all, let us give some information about the nearly cosymplectic manifolds that we examined on Ricci solitons and gradient Ricci solitons. Let $(M, \varphi, \xi, \eta, g)$ be an $(n = 2m + 1)-$ dimensional almost contact Riemannian manifold, an endomorphism φ of tangent bundle of $\Gamma(M)$, a vector field ξ, called structure vector field, η dual form of ξ and g is the Riemannian metric. Under the above condition almost contact structure (φ, ξ, η, g) satisfies following: [6],

$$\varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta(X) = g(X,\xi), \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.3}$$

for any X, Y tangent on M. Almost contact manifold is called nearly cosymplectic manifold if the equality

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 0, \tag{2.4}$$

$X, Y \in \Gamma(M)$.It is known that in a nearly cosymplectic manifold with the Reeb vector field ξ is Killing and satisfies the $\nabla_\xi \xi = 0$ and $\nabla_\xi \eta = 0$ conditions. Also the tensor field h of type $(1,1)$ defined by

$$\nabla_X \xi = hX, \tag{2.5}$$

is skew symmetric and anti-commutative with φ. Also h providing $h\xi = 0$, $\eta \circ h = 0$ features and

$$(\nabla_\xi \varphi)X = \varphi h X = \frac{1}{3}(\nabla_\xi \varphi)X.$$

In a nearly cosymplectic manifolds some formulas given by ([15], [16]):

$$g((\nabla_X \varphi)Y, hZ) = \eta(Y)g(h^2X, \varphi Z) - \eta(X)g(h^2Y, \varphi Z), \tag{2.6}$$

$$(\nabla_X h)Y = g(h^2X, Y)\xi - \eta(Y)h^2X, \tag{2.7}$$

$$tr(h^2) = constant, \tag{2.8}$$

$$R(Y,Z)\xi = \eta(Y)h^2Z - \eta(Z)h^2Y, \tag{2.9}$$

$$S(\xi, Z) = -\eta(Z)tr(h^2), \tag{2.10}$$

$$S(\varphi Y, Z) = S(Y, \varphi Z), \quad \varphi Q = Q\varphi, \tag{2.11}$$

$$S(\varphi Y, \varphi Z) = S(Y,Z) + \eta(Y)\eta(Z)tr(h^2). \tag{2.12}$$

where R is Riemann curvature tensor and S is Ricci tensor.

Definition 2.1. An $n-$dimensional Riemann manifold (M, g) is said to be Einstein Manifold if the Ricci Tensor satisfies;

$$S(X,Y) = \rho g(X,Y), \tag{2.13}$$

for every $X, Y \in \chi(M)$, where $\rho : M \to R$ is a function [11].

Definition 2.2. Let M be a nearly cosymplectic manifold, for every $X, Y \in \chi(M)$, if M satisfies the condition [11].

$$S(X,Y) = \alpha g(X,Y) + \beta \eta(X)\eta(Y), \tag{2.14}$$

then M is an $\eta-$Einstein manifold where $\alpha, \beta : M \longrightarrow R$ is a function.

Lemma 2.3. *If M a $n-$dimensional $\eta-$Einstein nearly cosymplectic manifold, the $\eta-$Einstein condition for nearly cosymplectic manifolds is characterized by the following equality* [2]*,*

$$S(X,Y) = \left\{\frac{r + tr(h^2)}{n-1}\right\} g(X,Y) + \left\{\frac{-n\ tr(h^2) - r}{n-1}\right\} \eta(X)\eta(Y). \tag{2.15}$$

Proposition 1. ([15])Let $(M, \varphi, \xi, \eta, g)$ be a nearly cosymplectic manifold. Then $h = 0$ if and only if M is locally isometric to the Riemannian product $\mathbb{R} \times N$, where N is a nearly Kähler manifold.

3. Ricci Solitons and Gradient Ricci Solitons on Nearly Cosymplectic Manifolds

Theorem 3.1. *In a nearly cosymplectic manifold if the metric g is a Ricci soliton and Y is point-wise collinear with ξ, then Y is a constant multiple of ξ provided $\lambda = tr(h^2)$.*

Proof. In particular, let Y be point-wise collinear with ξ i.e. $V = f\xi$ in where f is a function on the nearly cosymplectic manifold Then

$$(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0 \tag{3.1}$$

which states that

$$g(\nabla_X f\xi, Y) + g(\nabla_Y f\xi, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{3.2}$$

Using (2.5) and (2.2)

$$X(f)\eta(Y) + Y(f)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{3.3}$$

Putting ξ in the equation instead of Y in (3.3) we get

$$X(f) + \xi(f)\eta(X) + 2S(X, Y) + 2\lambda\eta(X) = 0,$$

$$X(f) + \xi(f)\eta(X) - 2tr(h^2) + 2\lambda\eta(X) = 0. \tag{3.4}$$

Again putting $X = \xi$ in (3.4)

$$2\xi(f) - 2tr(h^2) + 2\lambda = 0,$$

$$\xi(f) = tr(h^2) - \lambda, \tag{3.5}$$

from (3.4) and (3.5)

$$X(f) = (tr(h^2) - \lambda)\eta(X).$$

Consequently we get $\lambda = tr(h^2)$ if and only if $X(f) = 0$,under the condition $X \notin \ker(\eta)$. □

Theorem 3.2. *In a nearly cosymplectic manifold with the Ricci soliton metric g and vector field Y which is point-wise collinear with ξ, then the manifold is an Einstein and Ricci soliton depends on $tr(h^2)$ which is*

i) if $tr(h^2) < 0$ then Ricci soliton is shrinking,
ii) if $tr(h^2) = 0$ then Ricci soliton is steady,
iii) if $tr(h^2) > 0$ then Ricci soliton is expanding.

Proof. In particular, let $Y = \xi$ in (3.1), then from (3.3)

$$S(X, Y) = -\lambda g(X, Y). \tag{3.6}$$

Putting $Y = \xi$ in (3.6) ,

$$S(X, \xi) = -\lambda\eta(X), \tag{3.7}$$

from (2.10)

$$-trh^2\eta(X) = -\lambda\eta(X),$$

then we get

$$\lambda = tr(h^2).$$

Hence we get desired results. Also, using (3.6) equation takes the form

$$S(X,Y) = tr(h^2)g(X,Y),$$

that is, an Einstein manifold. □

Corollary 1. Let a nearly cosymplectic manifold with the Ricci soliton metric g and vector field Y which is point-wise collinear with ξ. If the Ricci soliton is steadily then the manifold is locally isometric to the Riemannian product $R \times N$, where N is a nearly Kähler manifold.

Proof. From Theorem 3.2 and Theorem 1 the proof is clear. □

Theorem 3.3. *An $\eta-$Einstein nearly cosymplectic manifold admits a Ricci soliton $(g, \xi, tr(h^2))$ and Ricci soliton is shrinking.*

Proof. Let M be an $\eta-$Einstein nearly cosymplectic manifold then,

$$S(X,Y) = \left\{\frac{r + tr(h^2)}{n-1}\right\} g(X,Y) + \left\{\frac{-n\ tr(h^2) - r}{n-1}\right\} \eta(X)\eta(Y). \tag{3.8}$$

Taking $Y = \xi$ in (3.1) and from (3.3) we get

$$2S(X,Y) + 2\lambda g(X,Y) = 0,$$

and

$$\left\{\frac{r + tr(h^2)}{n-1}\right\} g(X,Y) + \left\{\lambda + \frac{-n\ tr(h^2) - r}{n-1}\right\} \eta(X)\eta(Y) = 0. \tag{3.9}$$

Putting $Y = \xi$ in (3.9)

$$\frac{r + tr(h^2)}{n-1} + \lambda + \frac{-n\ tr(h^2) - r}{n-1} = 0,$$

so we have $\lambda = tr(h^2)$. Since $tr(h^2) < 0$, $\lambda < 0$ so the Ricci soliton is shrinking. □

Theorem 3.4. *If an $\eta-$Einstein nearly cosymplectic manifold admits a gradient Ricci soliton then the manifold is locally isometric to the Riemann product $\mathbb{R} \times N$,where N is a nearly Kähler manifold.*

Proof. When the vector field Y is the gradient of a potential function $-f$, then we can call g as a gradient Ricci soliton. (1.2) And here we can give the following equation,

$$\nabla\nabla f = S + \lambda g. \tag{3.10}$$

From (3.10)

$$\nabla_Y Df = QY + \lambda Y, \tag{3.11}$$

in where D symbolize the gradient operator of g. From (3.11) it is obviously that

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X. \tag{3.12}$$

Putting $X = \xi$ with g this implies

$$g(R(\xi,Y)Df, \xi) = g((\nabla_\xi Q)Y, \xi) - g((\nabla_Y Q)\xi, \xi). \tag{3.13}$$

Then

(3.14)
$$g(QX,Y)=S(X,Y)=\left\{\frac{r+tr(h^2)}{n-1}\right\}g(X,Y)+\left\{\frac{-n\ tr(h^2)-r}{n-1}\right\}\eta(X)\eta(Y),$$

from above equation

$$QX=\left\{\frac{r+tr(h^2)}{n-1}\right\}X+\left\{\frac{-n\ tr(h^2)-r}{n-1}\right\}\eta(X)\xi, \tag{3.15}$$

taking $\frac{r+tr(h^2)}{n-1}=A$, $\frac{-n\ tr(h^2)-r}{n-1}=f$ and also using (2.1),(2.2) and (2.5),

$$(\nabla_Y Q)X=fg(X,\nabla_Y\xi)\xi+f\eta(X)\nabla_Y\xi, \tag{3.16}$$

$$(\nabla_Y Q)X=fg(X,hY)\xi+f\eta(X)hY. \tag{3.17}$$

Putting $Y=\xi$ in (3.17) we have

$$(\nabla_\xi Q)(X)=0, \tag{3.18}$$

and putting $X=\xi$ in (3.17) we have

$$(\nabla_Y Q)\xi=fhY. \tag{3.19}$$

Furthermore from (3.13) we get

$$g(R(\xi,Y)\nabla f,\xi)=0.$$

Since $g(R(\xi,Y)\nabla f,\xi)=g(R(Df,\xi)\xi,Y)$ one can easily obtain that

$$g(R(Df,\xi)\xi,Y)=g(\eta(Df)h^2\xi-\eta(\xi)h^2Df,Y)=0,$$

so, we have

$$g(h^2Df,Y)=0.$$

Consequently we get $tr(h^2)=0$ and and it is clear from here $h=0$. So the manifold is locally isometric to the Riemann product $\mathbb{R}\times N$, where N is a nearly Kähler manifold. The proof is complete. □

4. Conclusion

Ricci solitons have an important application for many sciences such as physics and mathematical physics. Researchers have increased studies on this field from different areas in recent years. In this paper, the idea of examining Ricci solitons and gradient Ricci solitons on nearly cosymplectic manifolds is emphasized. The works on this subject will be useful tools for the applications of Ricci Solitons on different manifolds.

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ON SPHERICAL INDICATRICES AND THEIR SPHERICAL IMAGE OF NULL CURVES IN MINKOWSKI 3-SPACE

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Abstract. In this paper, we investigate the spherical images of null curves and null helixes in Minkowski 3-space. We provide the spherical indicatrices of null curves in Minkowski 3-space with their causal characteristics. We also show the conditions of spherical indicatrices of null curves to be a curve lying on pseudo-sphere in Minkowski 3-space. In addition, we give the properties of spherical indicatrices of null curves satisfying generalized helices and lying on pseudo-sphere in Minkowski 3-space.

1. Introduction

Since the second mid of 20th-century mathematicians and physicist have actively studied about differential geometry of Riemannian manifold and its applications. It is because theories in differential geometry connect mathematics with real problems, especially physics. Many topics in classical differential geometry of Riemannian space are then extended into those of Lorentz-Minkowski space since its important use in physics especially related to general relativity theory. Some literatures providing an explanation about differential geometry in Riemannian space can be seen in [2, 12, 13] while the theory of differential geometry in the semi-Riemannian manifold can be seen in [5].

One theory of differential geometry in Riemannian space that can be extended to Lorentz-Minkowski space is the spherical indicatrix of curves. The idea has been existed for a long time ago to the tie of Gauss. The idea is essentially simple. if there is some group of the set of lines in space in some organized relationship with another, one might construct and examine the relevant spherical indicatrix [14]. The theory of spherical indicatrix of curves in Riemannian space can be found in [1, 15, 17]while in the case of Lorentz-Minkowski space can be seen in [16].

In Lorentz-Minkowski space, a curve can locally be timelike, spacelike or null depending on the casual character of the tangent vector along the curves. Some studies about the theory of curves in Minkowski space and its applications have been studied by [3, 4, 6]. In Lorentzian geometry, the properties of spacelike curves and

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timelike curves can be studied by approaches similar to those in Riemann geometry. However, it does not work for null curves or it can be said that the theory of null curves has many results which have no Riemannian analogues. It is because, in the case of the null curves, the arc length vanishes so that it is impossible to normalize the tangent vector in the usual way as in spacelike and timelike curve cases.

In the mathematical study of relativity theory, it is known that a lightlike particle is a future-pointing null geodesic in spacetime which is a connected and time-oriented 3-dimensional Lorentzian manifold [5]. The study of null curves also plays an important role in the physical theories that the classical relativistic string is a surface or world-sheet in Minkowski space which satisfies the Lorentzian analogue of the minimal surface equations [10]. In another finding, Nersian and Ramos [11] show that there exists a geometrical particle model based entirely on the geometry of null curves in Minkowski 4-dimensional spacetime.

Since its important roles both in mathematics and physics, many mathematicians and physicist are interested in studying the theory of null curves. For instance, Duggal and Jin [8] write a comprehensive book related to the theory of null from its introduction, properties until its applications. Inoguchi and Lee also explained the theory of null curves comprehensively in another article [3].

In this paper, we study the spherical indicatrices of null curves parametrized by distinguished parameter in Minkowski 3-space. In this work, we assume that the null curve is a space curve such that its curvature and torsion are not vanish. After the preliminary section, we give the Frenet frames of the spherical indicatrices of a null curve in term of the Frenet frame of the null curve. We also provide the curvatures and torsions of the spherical indicatrices. We also then show the conditions of spherical indicatrices of null curves to be a curve lying on pseudo-sphere in Minkowski 3-space. In addition, we give the properties of spherical indicatrices of null curves satisfying generalized helices and lying on pseudo-sphere in Minkowski 3-space.

2. Preliminaries

Minkowski space $\mathbf{E}_1^3$ is the real vector space $\mathbf{R}^3$ equipped with the standard indefinite Lorentzian metric g defined by

$$g(x,y) = -x_1y_1 + x_2y_2 + x_3y_3 \tag{2.1}$$

for any vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbf{E}_1^3$. The cross product in Minkowski 3-space is defined by

$$x \times y = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_3). \tag{2.2}$$

In Minkowski 3-space, v is timelike if $g(v,v) < 0$, spacelike if $g(v,v) > 0$ or $v = 0$, or null (lightlike) if $g(v,v) = 0$ and $v \neq 0$. The norm of a vector in $\mathbf{E}_1^3$ is defined by $||v|| = \sqrt{|g(v,v)|}$.

Let $\alpha : I \to \mathbf{E}_1^3$ be a curve in Minkowski 3-space. Locally, α can be timelike, spacelike or null if its tangent vectors along the curve are timelike, spacelike or null, respectively. For non null curves, the arc length s is defined by $s = \int_0^t \sqrt{|g(\alpha', \alpha')|}dt$. If $g(\alpha', \alpha) = \pm 1$, the non null curves are called curves parametrized by arc length. For a null curve, since $g(\alpha', \alpha') = 0$ then the pseudo-arc length s is defined by $s = \int_0^t g(\alpha'', \alpha'')^{\frac{1}{4}} dt$ and if $g(\alpha'', \alpha'') = 1$ the the null curve is parametrized by pseudo-arc length.

Let $\{T, N, B\}$ is the Frenet frame of α in $\mathbf{E}_1^3$. T, N and B are called tangent vector, principal normal vector and binormal vector of α, respectively.

If α is a non null curve with non null normals parametrized by arch length, then the Frenet equations of α are given by [21]

$$T' = \kappa N, \quad N' = -\varepsilon_0\varepsilon_1\kappa T + \tau B, \quad B' = -\varepsilon_1\varepsilon_2\tau N \tag{2.3}$$

where

$$g(T,T) = \varepsilon_0 = \pm 1, \quad g(N,N) = \varepsilon_1 = \pm 1, \quad g(B,B) = \varepsilon_2 = \pm 1,$$
$$g(T,N) = g(T,B) = g(N,B) = 0.$$

The vector products of Frenet vectors of α in Minkowski 3-space are given by

$$T \times N = B, \quad N \times B = -\epsilon_1 T, \quad B \times T = -\varepsilon_0 N. \tag{2.4}$$

If α is a pseudo null curve, that is α is a spacelike curve with a null principal normal N, then the Frenet equations of α are given by [20]

$$T' = \kappa N, \quad N' = \tau N, \quad B' = -\kappa T - \tau B \tag{2.5}$$

where

$$g(T,T) = g(N,B) = 1, \quad g(N,N) = g(B,B) = g(T,N) = g(T,B) = 0$$

and

$$T \times N = N, \quad N \times B = T, \quad T \times B = -B. \tag{2.6}$$

Here, κ can only two values:, $\kappa = 0$ if α is a straight line and $\kappa = 0$, otherwise.

If α is a null curve parametrized by distingushed parameter, then the Frenet equations of α are given by [18]

$$T' = \kappa N, \quad N' = \tau T - \kappa B, \quad B' = -\tau N \tag{2.7}$$

where

$$g(T,T) = g(B,B) = 0, \quad g(T,B) = g(N,B) = 0, \quad g(N,N) = g(T,B) = 1$$

and

$$T \times B = N, \quad T \times N = -T, \quad N \times B = -B. \tag{2.8}$$

Here, κ and τ are called the curvature and the torsion if α is a timelike curve or a spacelike curve with non null Frener frame. In case α is a pseudo null curve or a null curve then τ is called pseudo torsion.

Let $C : I \to \mathbf{E}_1^3$ be a null curve paremetrized by pseudo arc length s. A curve $\alpha : I \to \mathbf{E}_1^3$ generated by the unit tangent vector along a curve $C(s)$, i.e., $\alpha(s) = T(s)$ on the sphere of radius 1 about the origin is called tangent indicatrix of $C(s)$. Similarly, $\alpha(s) = N(s)$ and $\alpha(s) = B(s)$ are called the principal indicatrix and binormal indicatrix of $C(s)$

3. Spherical Indicatrices of Null Curves

In this section, we provide the causal characteristics of spherical indicatrices of null curves in Minkowski 3-space. In this section we assume that the null curve is not a straight line so that the null curve has non null curvature anywhere.

3.1. **Tangent indicatrix.**

Theorem 3.1. *Let $\alpha(s) = T(s)$ be a tangent indicatrix of a null curve parametrized by distinguished parameter s. Then α is a spacelike curve.*

Proof. From equation (2.7), we have $\alpha'(s) = \kappa(s)N(s)$. Therefore, $g(\alpha', \alpha') = \kappa^2(s) > 0$. It implies that α is a spacelike curve. □

Theorem 3.2. *Let $\alpha(s) = T(s)$ be a tangent indicatrix of null curves parametrized by distinguished parameter s. If the null curve is not a plane curve, then $\alpha(s)$ is a spacelike curve with non null Frenet frame satisfying*

$$\begin{bmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\tau}{\sqrt{|2\kappa\tau|}} & 0 & \frac{-\kappa}{\sqrt{|2\kappa\tau|}} \\ \frac{\tau}{\sqrt{|2\kappa\tau|}} & 0 & \frac{\kappa}{\sqrt{|2\kappa\tau|}} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{3.1}$$

Proof. Let $\bar{s}$ be arc length of the $\alpha(s)$. Then, since α is spacelike curve with non null Frenet frame, then by taking derivative of α with respect to the pseudo arc length s using equations (2.5) and (2.7), we have

$$\frac{d\alpha}{d\bar{s}}\frac{d\bar{s}}{ds} = \kappa N \Rightarrow \bar{T} \cdot \frac{d\bar{s}}{ds} = \kappa N. \tag{3.2}$$

Taking the norm of equation (3.2), we have $\frac{d\bar{s}}{ds} = \pm\kappa$. Take $\frac{d\bar{s}}{ds} = \kappa$ so that

$$\bar{T} = N. \tag{3.3}$$

Differentiating equation (3.3), yields

$$\frac{d\bar{T}}{d\bar{s}}\frac{d\bar{s}}{ds} = \tau T - \kappa B \Rightarrow \bar{\kappa}\bar{N}\kappa = \tau T - \kappa B. \tag{3.4}$$

Since the null curve is not a straight line and not a plane curve then $\kappa \neq 0$ and $\tau \neq 0$, by taking the norm of equation (3.4), we have

$$\bar{\kappa}\kappa = \sqrt{|-2\kappa\tau|} = \sqrt{|2\kappa\tau|}. \tag{3.5}$$

Therefore, from equation (3.4), we find

$$\bar{N} = \frac{\tau T - \kappa B}{\sqrt{|2\kappa\tau|}}. \tag{3.6}$$

Consequently, $\bar{N}$ is timelike or spacelike if $\kappa\tau > 0$ or $\kappa\tau < 0$, respectively. Therefore, from equations (2.6) and (2.8), we have

$$\begin{aligned} \bar{B} &= \cdot\, \bar{T} \times \bar{N} \\ &= \cdot\, N \times \left(\frac{\tau T - \kappa B}{\sqrt{|2\kappa\tau|}} \right) \\ &= \frac{\tau T + \kappa B}{\sqrt{|2\kappa\tau|}}. \end{aligned}$$

Hence, the proof is completed. □

Theorem 3.3. *Let $\alpha(s) = T(s)$ be a tangent indicatrix of a null curve parametrized by distinguished parameter s. If $\alpha(s) = T(s)$ has non null Frenet frame then the curvature and torsion of $\alpha(s)$ are respectively given by*

$$\bar{\kappa} = \frac{\sqrt{|2\kappa\tau|}}{\kappa} \quad \textit{and} \quad \bar{\tau} = -\frac{\kappa'\tau - \kappa\tau'}{2\kappa^2\tau}. \tag{3.7}$$

Proof. It is clear from equation (3.5) that

$$\bar{\kappa} = \frac{\sqrt{|2\kappa\tau|}}{\kappa}.$$

Taking derivative of $\bar{B}$ in equation (3.1) with respect to the pseudo arc length s yields

$$\begin{aligned}
\frac{d\bar{B}}{d\bar{s}}\frac{d\bar{s}}{ds} =& \frac{\tau' T + \kappa\tau N + \kappa' B - \kappa\tau N}{|-2\kappa\tau|^{\frac{1}{2}}} - \frac{(-2\kappa'\tau - 2\kappa\tau')(\tau T + \kappa B)}{2|-2\kappa\tau|^{\frac{3}{2}}} \\
\frac{d\bar{B}}{d\bar{s}}\kappa =& \frac{(-2\kappa\tau)(\tau' T + \kappa' B) + (\kappa'\tau + \kappa\tau')(\tau T + \kappa B)}{|-2\kappa\tau|^{\frac{3}{2}}} \\
\frac{d\bar{B}}{d\bar{s}} =& \frac{\tau(\kappa'\tau - \kappa\tau')T - \kappa(\kappa'\tau - \kappa\tau')B}{\kappa|-2\kappa\tau|^{\frac{3}{2}}} \\
=& \frac{(\kappa'\tau - \kappa\tau')(\tau T - \kappa B)}{\kappa|-2\kappa\tau|^{\frac{3}{2}}}.
\end{aligned}$$

By applying equations (2.5) and (2.7), we get

$$\begin{aligned}
\bar{\tau} =& \cdot g\left(\frac{d\bar{B}}{d\bar{s}}, \bar{N}\right) \\
=& \cdot g\left(\frac{(\kappa'\tau - \kappa\tau')(\tau T - \kappa B)}{\kappa|-2\kappa\tau|^{\frac{3}{2}}}, \frac{\tau T - \kappa B}{|-2\kappa\tau|^{\frac{1}{2}}}\right) \\
=& \frac{(\kappa'\tau - \kappa\tau')(-2\kappa\tau)}{\kappa|-2\kappa\tau|^{2}} \\
=& -\frac{\kappa'\tau - \kappa\tau'}{2\kappa^2\tau}.
\end{aligned}$$

□

3.2. **Principal Normal Indicatrix.**

Theorem 3.4. *Let $\alpha(s) = N(s)$ be a principal normal indicatrix of a null curve parametrized by pseudo arc length s. Then if $\alpha(s)$ is not a plane curve then it is a spacelike or a timelike curve and if $\alpha(s)$ is a plane curve then it is a null curve.*

Proof. From equation (2.7), we have $\alpha'(s) = -\tau(s)T(s) + \kappa(s)B(s)$. Therefore, $g(\alpha'(s), \alpha'(s)) = 2\kappa(s)\tau(s)$. As a consequence, if $\alpha(s)$ is not a plane curve, then it is a spacelike or a timelike curve if κ and τ have different sign or same sign, respectively. If $\alpha(s)$ is a plane curve then $\tau(s) = 0$ which implies $\alpha(s)$ is a null curve. □

Theorem 3.5. *Let $\alpha(s) = N(s)$ be a principal normal indicatrix of a non plane null curve parametrized by pseudo arc length s. Then the Frenet frame of $\alpha(s)$ is given by*

$$\begin{bmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{bmatrix} = \begin{bmatrix} \frac{\tau}{\sqrt{|\lambda|}} & 0 & \frac{-\kappa}{\sqrt{|\lambda|}} \\ \frac{\tau\mu}{\sqrt{\mu^2\lambda+\lambda^4}} & \frac{-\lambda^2}{\sqrt{\mu^2\lambda+\lambda^4}} & \frac{\kappa\mu}{\sqrt{\mu^2\lambda+\lambda^4}} \\ \frac{\tau\lambda}{\sqrt{\mu^2+\lambda^2}} & \frac{\mu}{\sqrt{\mu^2+\lambda^2}} & \frac{\kappa\lambda}{\sqrt{\mu^2+\lambda^2}} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{3.8}$$

where $\lambda = 2\kappa\tau$ and $\mu = \kappa'\tau - \kappa\tau'$.

Proof. Let $\bar{s}$ be the arc length of the curve $\alpha(s)$. Taking derivative of α with respect to the pseudo arc length s, we have

$$\frac{d\alpha}{d\bar{s}}\frac{d\bar{s}}{ds} = \tau T - \kappa B \Rightarrow \bar{T}\frac{d\bar{s}}{ds} = \tau T - \kappa B. \tag{3.9}$$

Taking the inner product of equation (3.9), we get

$$\frac{d\bar{s}}{ds} = \sqrt{|-2\kappa\tau|}. \tag{3.10}$$

Therefore,

$$\bar{T} = \frac{\tau T - \kappa B}{\sqrt{|-2\kappa\tau|}}. \tag{3.11}$$

Differentiating equation (3.11) and using equation (3.10), we have

$$\begin{aligned}
\frac{d\bar{T}}{d\bar{s}}\frac{d\bar{s}}{ds} =& \frac{\tau' T + \kappa\tau N - \kappa' B + \kappa\tau N}{|-2\kappa\tau|^{\frac{1}{2}}} - \frac{(-2\kappa'\tau - 2\kappa\tau')(\tau T - \kappa B)}{2|-2\kappa\tau|^{\frac{3}{2}}} \\
\bar{\kappa}\bar{N} =& \frac{(-2\kappa\tau)(\tau' T + 2\kappa\tau N - \kappa' B) + (\kappa'\tau + \kappa\tau')(\tau T - \kappa B)}{|-2\kappa\tau|^2} \\
=& \frac{(-2\kappa\tau\tau' + \kappa'\tau^2 + \kappa\tau\tau')T - 4\kappa^2\tau^2 N + (2\kappa\kappa'\tau - \kappa'\kappa\tau - \kappa^2\tau')B}{|-2\kappa\tau|^2} \\
=& \frac{\tau(\kappa'\tau - \kappa\tau')T - 4\kappa^2\tau^2 N + \kappa(\kappa'\tau - \kappa\tau')B}{|-2\kappa\tau|^2} \\
=& \frac{(\kappa'\tau - \kappa\tau')(\tau T + \kappa B) - 4\kappa^2\tau^2 N}{4\kappa^2\tau^2}.
\end{aligned}$$

Taking the norm of the equation above yields

$$\bar{\kappa} = \frac{|2\kappa\tau(\kappa'\tau - \kappa\tau')^2 + 16\kappa^4\tau^4|^{\frac{1}{2}}}{4\kappa^2\tau^2}. \tag{3.12}$$

Therefore,

$$\bar{N} = \frac{(\kappa'\tau - \kappa\tau')(\tau T + \kappa B) - 4\kappa^2\tau^2 N}{|2\kappa\tau(\kappa'\tau - \kappa\tau')^2 + 16\kappa^4\tau^4|^{\frac{1}{2}}}. \tag{3.13}$$

As a consequence,

$$\begin{aligned}
\bar{B} =& \bar{T} \times \bar{N} \\
=& \left(\frac{\tau T - \kappa B}{\sqrt{|-2\kappa\tau|}}\right) \times \left(\frac{(\kappa'\tau - \kappa\tau')(\tau T + \kappa B) - 4\kappa^2\tau^2 N}{4\kappa^2\tau^2}\right) \\
=& \frac{\kappa\tau(\kappa'\tau - \kappa\tau')(T \times B) - 4\kappa^2\tau^3(T \times N) - \kappa\tau(\kappa'\tau - \kappa\tau')(B \times T) + 4\kappa^3\tau^2(B \times N)}{2\kappa\tau|(\kappa'\tau - \kappa\tau')^2 + 8\kappa^3\tau^3)|^{\frac{1}{2}}} \\
=& \frac{2\kappa\tau(\kappa'\tau - \kappa\tau')N + 4\kappa^2\tau^2(\tau T + \kappa B)}{2\kappa\tau|(\kappa'\tau - \kappa\tau')^2 + 8\kappa^3\tau^3)|^{\frac{1}{2}}} \\
=& \frac{(\kappa'\tau - \kappa\tau')N + 2\kappa\tau(\tau T + \kappa B)}{|(\kappa'\tau - \kappa\tau')^2 + 8\kappa^3\tau^3)|^{\frac{1}{2}}}.
\end{aligned}$$

Setting $\lambda = 2\kappa\tau$ and $\mu = \kappa'\tau - \kappa\tau'$ completes the proof. □

Theorem 3.6. *Let $\alpha(s) = N(s)$ be a principal normal indicatrix of a null curve parametrized by distinguished parameter s. If the null curve is not a plane curve, then the curvature and torsion of α are given by*

$$\bar{\kappa} = \frac{|\lambda\mu^2 + \lambda^4|^{\frac{1}{2}}}{\lambda^2} \tag{3.14}$$

and

$$\begin{aligned}\bar{\tau} =& \frac{-(\mu^2+\lambda^3)(\lambda^2\kappa'\tau + \lambda^2\kappa\tau' + 2\kappa\tau\lambda\lambda') + (2\mu\mu' + 3\lambda^2\lambda')\kappa\tau\lambda^2}{(\lambda\mu^2+\lambda^4)^{\frac{3}{2}}(\mu+\lambda)^{\frac{1}{2}}} \\ &+ \frac{(\mu^2\lambda^3)\mu^2\mu' - \mu^3(2\mu\mu' + 3\lambda^2\lambda')}{2(\lambda\mu^2+\lambda^4)^{\frac{3}{2}}(\mu+\lambda)^{\frac{1}{2}}}.\end{aligned} \tag{3.15}$$

Proof. From equation (3.11), we have

$$\bar{\kappa} = \frac{|2\kappa\tau(\kappa\tau' - \kappa'\tau)^2 + 16\kappa^4\tau^4|^{\frac{1}{2}}}{4\kappa^2\tau^2} = \frac{|\lambda\mu^2 + \lambda^4|^{\frac{1}{2}}}{\lambda^2}.$$

Taking derivative of $\bar{B}$ in equation (3.8) with respect to the pseudo arc length s, we have

$$\begin{aligned}\frac{d\bar{B}}{d\bar{d}}\frac{d\bar{s}}{ds} =& \frac{2(\mu^2+\lambda^3)(\tau'\lambda + \tau\lambda' + \mu\tau) - (2\mu\mu' + 3\lambda^2\lambda')\tau\lambda}{2(\mu^2+\lambda^3)^{\frac{3}{2}}}T \\ &+ \frac{2(\mu^2+\lambda^3)\mu' - (2\mu\mu' + 3\lambda\lambda')\mu}{2(\mu^2+\lambda^3)^{\frac{3}{2}}}N \\ &\frac{2(\mu^2+\lambda^3)(\kappa'\lambda + \kappa\lambda' - \kappa\mu) - (2\mu\mu' + 3\lambda^2\lambda')\kappa\lambda}{2(\mu^2+\lambda^3)^{\frac{3}{2}}}B \\ \frac{d\bar{B}}{ds} =& \frac{2(\mu^2+\lambda^3)(\tau'\lambda + \tau\lambda' + \mu\tau) - (2\mu\mu' + 3\lambda^2\lambda')\tau\lambda}{2\lambda^{\frac{1}{2}}(\mu^2+\lambda^3)^{\frac{3}{2}}}T \\ &+ \frac{2(\mu^2+\lambda^3)\mu' - (2\mu\mu' + 3\lambda\lambda')\mu}{2\lambda^{\frac{1}{2}}(\mu^2+\lambda^3)^{\frac{3}{2}}}N \\ &\frac{2(\mu^2+\lambda^3)(\kappa'\lambda + \kappa\lambda' - \kappa\mu) - (2\mu\mu' + 3\lambda^2\lambda')\kappa\lambda}{2\lambda^{\frac{1}{2}}(\mu^2+\lambda^3)^{\frac{3}{2}}}B.\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{\tau} =& -g\left(\frac{d\bar{B}}{d\bar{s}}, \bar{N}\right) \\ =& \frac{-(\mu^2+\lambda^3)(\lambda^2\kappa'\tau + \lambda^2\kappa\tau' + 2\kappa\tau\lambda\lambda') + (2\mu\mu' + 3\lambda^2\lambda')\kappa\tau\lambda^2}{(\lambda\mu^2+\lambda^4)^{\frac{3}{2}}(\mu+\lambda)^{\frac{1}{2}}} \\ &+ \frac{(\mu^2\lambda^3)\mu^2\mu' - \mu^3(2\mu\mu' + 3\lambda^2\lambda')}{2(\lambda\mu^2+\lambda^4)^{\frac{3}{2}}(\mu+\lambda)^{\frac{1}{2}}}.\end{aligned}$$

□

3.3. **Binormal Indicatrix.**

Theorem 3.7. *Let $\alpha(s) = B(s)$ be a binormal indicatrix of a null curve parametrized by distinguished parameter s. Then α is a spacelike or a null curve.*

Proof. From equation (2.7), we have $\alpha'(s) = -\tau(s)N(s)$. Therefore, $g(\alpha', \alpha') = \tau^2(s) > 0$. It implies that α is a spacelike curve if the null curve is not a plane curve and α is a null curve is the null curve is a plane curve. □

Theorem 3.8. *Let $\alpha(s) = B(s)$ be a binormal indicatrix of a non plane null curves parametrized by distinguished parameter s. If α is a spacelike curve with non null Frenet frames, then the Frenet frame of $\alpha(s)$ is given by*

$$\begin{bmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ \frac{-\tau}{\sqrt{|2\kappa\tau|}} & 0 & \frac{\kappa}{\sqrt{|2\kappa\tau|}} \\ \frac{-\tau}{\sqrt{|2\kappa\tau|}} & 0 & \frac{-\kappa}{\sqrt{|2\kappa\tau|}} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{3.16}$$

where $\varepsilon = 1$ or $\varepsilon = -1$ when α is a spacelike curve or timelike principal normal, respectively.

Proof. Let $\bar{s}$ be arc length of the $\alpha(s)$. Then, since α is a spacelike curve with non null Frenet frame, then by taking derivative of α with respect to the pseudo arc length s using equations (2.5) and (2.7), we have

$$\frac{d\alpha}{d\bar{s}}\frac{d\bar{s}}{ds} = -\tau N \Rightarrow \bar{T} \cdot \frac{d\bar{s}}{ds} = -\tau N. \tag{3.17}$$

Taking the norm of equation (3.17), we have $\frac{d\bar{s}}{ds} = \pm\tau$. Take $\frac{d\bar{s}}{ds} = \tau$ so that

$$\bar{T} = -N. \tag{3.18}$$

Differentiating equation (3.18), yields

$$\frac{d\bar{T}}{d\bar{s}}\frac{d\bar{s}}{ds} = -\tau T + \kappa B \Rightarrow \bar{\kappa}\bar{N}\tau = -\tau T + \kappa B. \tag{3.19}$$

Since α is not a straight line and not a plane curve then $\kappa \neq 0$ and $\tau \neq 0$, by taking the norm of equation (3.19), we have

$$\bar{\kappa}\tau = \sqrt{|-2\kappa\tau|} = \sqrt{|2\kappa\tau|}. \tag{3.20}$$

Therefore, from equation (3.19), we find

$$\bar{N} = \frac{-\tau T + \kappa B}{\sqrt{|2\kappa\tau|}}. \tag{3.21}$$

Consequently, $\bar{N}$ is timelike or spacelike if $\kappa\tau > 0$ or $\kappa\tau < 0$, respectively. Therefore, from equations (2.6) and (2.8), we have

$$\begin{aligned} \bar{B} &= \cdot\, \bar{T} \times \bar{N} \\ &= \cdot\, N \times \left(\frac{-\tau T + \kappa B}{\sqrt{|2\kappa\tau|}} \right) \\ &= \frac{-\tau T - \kappa B}{\sqrt{|2\kappa\tau|}}. \end{aligned}$$

Hence, the proof is completed. □

Theorem 3.9. *Let $\alpha(s) = B(s)$ be a tangent indicatrix of a null curve parametrized by distinguished parameter s. If $\alpha(s) = B(s)$ is not a plane curve then the curvature and torsion of $\alpha(s)$ are respectively given by*

$$\bar{\kappa} = \frac{\sqrt{|2\kappa\tau|}}{\tau} \quad \text{and} \quad \bar{\tau} = \frac{\kappa'\tau - \kappa\tau'}{2\kappa^2\tau}. \tag{3.22}$$

Proof. It is clear from equation (3.20) that

$$\bar{\kappa} = \frac{\sqrt{|2\kappa\tau|}}{\tau}.$$

Taking derivative of $\bar{B}$ in equation (3.1) with respect to the pseudo arc length s yields

$$\begin{aligned}
\frac{d\bar{B}}{d\bar{s}}\frac{d\bar{s}}{ds} &= \frac{-\tau' T - \kappa\tau N - \kappa' B + \kappa\tau N}{|-2\kappa\tau|^{\frac{1}{2}}} - \frac{(-2\kappa'\tau - 2\kappa\tau')(-\tau T - \kappa B)}{2|-2\kappa\tau|^{\frac{3}{2}}}\\
\frac{d\bar{B}}{d\bar{s}}\kappa &= \frac{(-2\kappa\tau)(-\tau' T - \kappa' B) + (\kappa'\tau + \kappa\tau')(-\tau T - \kappa B)}{|-2\kappa\tau|^{\frac{3}{2}}}\\
\frac{d\bar{B}}{d\bar{s}} &= \frac{\tau(\kappa\tau' - \kappa'\tau)T - \kappa(\kappa\tau' - \kappa'\tau)B}{\kappa|-2\kappa\tau|^{\frac{3}{2}}}\\
&= \frac{(\kappa\tau' - \kappa'\tau)(\tau T - \kappa B)}{\kappa|-2\kappa\tau|^{\frac{3}{2}}}.
\end{aligned}$$

By applying equations (2.5) and (2.7), we get

$$\begin{aligned}
\bar{\tau} &= \cdot\, g\left(\frac{d\bar{B}}{d\bar{s}}, \bar{N}\right)\\
&= \cdot\, g\left(\frac{(\kappa\tau' - \kappa'\tau)(\tau T - \kappa B)}{\kappa|-2\kappa\tau|^{\frac{3}{2}}}, \frac{\tau T - \kappa B}{|-2\kappa\tau|^{\frac{1}{2}}}\right)\\
&= \frac{(\kappa\tau' - \kappa'\tau)(-2\kappa\tau)}{\kappa|-2\kappa\tau|^{2}}\\
&= \frac{\kappa'\tau - \kappa\tau'}{2\kappa^2\tau}.
\end{aligned}$$

□

4. Spherical Image of Spherical Indicatrices

In this section, we provide the properties of spherical indicatrices of null curves on pseudo sphere in Minkowski 3-space. In this section we assume the null curve is neither a plane curve nor a straight line.

Definition 4.1. [5] Pseudo sphere in semi-Riemannian space of center c and radius r is defined by

$$\mathbf{S}_1^2 = \{\alpha \in \mathbf{E}_1^3 : g(\alpha - c, \alpha - c) = r^2\}. \tag{4.1}$$

Theorem 4.2. *Let $\alpha(s) = T(s)$ be a unit speed tangent indicatrix of a null curve. If α lies on the pseudo sphere of center c and radius r, then*

$$\alpha - c = \rho\bar{N} + \sigma\bar{B}, \tag{4.2}$$

where $\rho = -\frac{1}{\bar{\kappa}}$ and $\sigma = \frac{\bar{\kappa}'\bar{\tau}}{\bar{\kappa}^2}$.

Proof. Let $\alpha(s) = T(s)$ is a unit speed curve lying on a pseudo sphere $\mathbf{S}_1^2$ of center c and radius r. Therefore, it satisfies $g(\alpha - c, \alpha - c) = r^2$. Differentiating this equation yields

$$g(\bar{T}, \alpha - c) = 0. \tag{4.3}$$

Taking the derivative of equation (4.3)

$$g(\bar{T},\bar{T})+\bar{\kappa}g(\alpha-c,\bar{N})=0\Rightarrow g(\alpha-c,\bar{N})=-\frac{1}{\bar{\kappa}}. \tag{4.4}$$

Differentiating equation (4.4) and using the fact that $\alpha - c$ is perpendicular to $\bar{T}$, we have

$$\begin{aligned} g(\bar{T},\bar{N})+g(\alpha-c,-\varepsilon_0\varepsilon_1\bar{\kappa}\bar{T}+\bar{\tau}\bar{B}) &= \frac{\bar{\kappa}'}{\bar{\kappa}^2} \\ \bar{\tau}g(\alpha-c,\bar{B}) &= \frac{\bar{\kappa}'}{\bar{\kappa}^2} \\ g(\alpha-c,\bar{B}) &= \frac{\bar{\kappa}'\bar{\tau}}{\bar{\kappa}^2}. \end{aligned} \tag{4.5}$$

On the other hand, since $\alpha - c$ is perpendicular to $\bar{T}$, then we can express

$$\alpha - c = \rho\bar{N} + \sigma\bar{B}$$

where $\rho = g(\alpha - c, \bar{N})$ and $\sigma = g(\alpha - c, B)$. Consequently, by equation (4.4) and (4.5), we find equation (4.2). Hence the proof is completed □

Corollary 4.1. Let $\alpha(s) = T(s)$ be unit tangent indicatrix of a space null curve. If α lies on the pseudo sphere, then the center c and the radius r f the curve α are respectively given by

$$c=\alpha+\frac{1}{\bar{\kappa}}\bar{N}-\frac{\bar{\kappa}'\bar{\tau}}{\bar{\kappa}^2}\bar{B} \quad \textit{and} \quad r=\frac{1}{\bar{\kappa}^2}\sqrt{|\bar{\kappa}^2+\epsilon_0(\bar{\kappa}'\bar{\tau})^2|}. \tag{4.6}$$

Theorem 4.3. *Let $\alpha(s) = T(s)$ be unit speed tangent indicatrix of a space null curve. If α lies on pseudo sphere of center c, radius r and positive curvature, then α has curvature $\bar{\kappa} \geq \frac{1}{r}$.*

Proof. Let α lies on the pseudo sphere of center c and radius r. Then, we have $g(\alpha - c, \alpha - c) = r^2$. From equation (4.4) we have

$$\bar{\kappa} = \frac{1}{g(\alpha - c, \bar{N}}.$$

By Schwarz inequality, $||g(\alpha - c, \bar{N}|| \leq ||\alpha - c||||\bar{N}|| = a$, we have $\bar{\kappa} \geq \frac{1}{r}$ and the proof is completed. □

Remark 4.4. The theorem 4.2 and 4.3 is similar in case α is the principal normal indicatrix or binormal indicatrix of a space null curve.

Definition 4.5. [7] A null curve $\alpha : I \to \mathbf{E}_1^3$ is called generalized helix is there exist a non-zero vector v in $\mathbf{E}_1^3$ such that $g(\alpha', v) = constant$.

Theorem 4.6. [7] *A non-geodesic null Frenet curve is a null generalized helix if and only if its slope $\frac{\tau}{\kappa}$ is constant.*

Theorem 4.7. *Let $C : I \to \mathbf{E}_1^3$ be a null generalized helix in $\mathbf{E}_1^3$. Then the tangent indicatrix $\alpha(s) = T(s)$ of the generalized null helix C lies on the osculating plane of radius $r = \frac{1}{2}\sqrt{\frac{2\kappa}{\tau}}$ and center $c = \bar{T} + r\bar{N}$.*

Proof. Let $C : I \to \mathbf{E}_1^3$ be a null generalized helix, then $\frac{\tau}{\kappa} = constant$. From equation (3.7), we have

$$\bar{\kappa} = \frac{\sqrt{|2\kappa\tau|}}{\kappa} = constant$$

and

$$\bar{\tau} = -\frac{\kappa'\tau - \kappa\tau'}{2\kappa^2\tau} = -\left[\frac{d}{ds}\left(\frac{\tau}{\kappa}\right)'\right]\frac{1}{2\tau} = 0.$$

Therefore, $\alpha(s) = T(s)$ is a circle in $\mathbf{E}_1^3$ which lies on the plane spanned by $\{\bar{T}, \bar{N}\}$ or osculating plane. From equation (4.6), the radius and the center of α are given by

$$r = \frac{1}{\bar{\kappa}} = \frac{1}{2}\sqrt{\frac{2\kappa}{\tau}}$$

and

$$c = \alpha + \frac{1}{\bar{\kappa}}\bar{N} = \alpha + r\bar{N}.$$

□

Remark 4.8. The theorem 4.7 is similar in case α is the principal normal indicatrix or binormal of indicatrix lying on pseudo sphere.

Example 4.9. Define a null curve $C : I \to \mathbf{E}_1^3$ parametrized by distinguished parameter s defined by

$$\alpha(s) = (s, \cos s, \sin s).$$

With simple calculation, we have

$$T = (1, -\sin s, \cos s), \quad N = (0, -\cos s, -\sin s), \quad B = \left(-\frac{1}{2}, -\frac{\sin s}{2}, \frac{\cos s}{2}\right)$$

and

$$\kappa = 1 \quad and \quad \tau = -\frac{1}{2}.$$

Since $\frac{\tau}{\kappa} = -\frac{1}{2} = constant$, the curve C is a helix in $\mathbf{E}_1^3$.
1. Tangent indicatrix
The curve $\alpha(s) = T(s) = (1, -\sin s, \cos s)$ is the tangent indicatrix of C. By using equations (3.1) and (3.7) we get

$$\bar{T} = (0, -\cos s, -\sin s), \quad \bar{N} = (0, \sin s, -\cos s), \quad \bar{B} = (-1, 0, 0)$$

and

$$\bar{\kappa} = 1, \quad \bar{\tau} = 0.$$

It can be seen that α is a spacelike helix with spacelike principal normal vector. Furthermore, by theorem 4.7, α lies on $\mathbf{S}_1^2$ with radius 1 and centered in (1, 0, 0).
2. Principal normal indicatrix
The curve $\alpha(s) = N(s) = (0, -\cos s, -\sin s)$ is the principal normal of C. By using equations (3.8), (3.14) and (3.15) we get

$$\bar{T} = (0, \sin s, -\cos s), \quad \bar{N} = (0, \cos s, -\sin s), \quad \bar{B} = (-1, 0, 0)$$

and

$$\bar{\kappa} = 1, \quad \bar{\tau} = 0.$$

It can be seen that α is a spacelike helix with spacelike principal normal vector. Furthermore, by theorem 4.7, α lies on $\mathbf{S}_1^2$ of radius 1 and centered in $(1, 0, 0)$.
3. Binormal indicatrix
The curve $\alpha(s) = B(s) = \left(-\frac{1}{2}, -\frac{\sin s}{2}, \frac{\cos s}{2}\right)$ is the binormal indicatrix of C. By using equations (3.16) and (3.22) we get

$$\bar{T} = (0, \cos s, \sin s), \quad \bar{N} = (0, -\sin s, \cos s), \quad \bar{B} = (1, 0, 0)$$

and

$$\bar{\kappa} = 1, \quad \bar{\tau} = 0.$$

It can be seen that α is a spacelike helix with spacelike principal normal vector. Furthermore, by theorem 4.7, α lies on $\mathbf{S}_1^2$ with radius 1 and centered in (1, 0, 0).

5. Conclusion

Spherical indicatrices of a space null curve in Minkowski 3-space are spacelike curves with non null Frenet frame. Sphaerical indicatrices lying on the pseudo sphere of a space null curve with positive curvature has a curvature $\bar{\kappa} \geq \frac{1}{r}$ where r is the radius of the pseudo sphere.

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C-PURE SUBMODULES AND C-FLAT MODULES

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Abstract. Let R be a ring. A right R-module A is said to be C-flat if the kernel of any epimorphism $B \to A$ is C-pure in B, i.e. the induced map $\mathrm{Hom}(C,B) \to \mathrm{Hom}(C,A)$ is surjective for any cyclic right R-module C. Projective modules are C-flat and C-flat modules are weakly-flat and neat-flat. In this article, it is discussed the connections between C-flat, weakly-flat, neat-flat and singly flat modules. It is shown that C-flat modules coincide with singly-projective modules over arbitrary rings. Next, several characterizations of certain classes of rings and modules via C-purity are considered. We prove that every C-flat module is injective if and only if R is a QF ring. Moreover, we show that R is a CF ring if and only if every FP-injective right R-module is C-flat.

1. Introduction

Throughout, R will denote an associative ring with identity, and modules will be unital R-modules unless otherwise stated.

There are many submodule structures, but the most commonly studied structures are closed submodule and pure submodule, due to their important role played in Module and Ring Theory. A submodule B of a right R-module A is called closed (in A) provided B has no proper essential extension in A. Let $\varepsilon : 0 \to B \to A \to C \to 0$ be an exact sequence of right R-modules. ε is called (Cohn) pure exact if, every finitely presented right R-module F is projective with respect to ε (see [22]). The sequence ε is called C-pure (resp. neat) if every cyclic (resp. simple) right R-module is projective with respect to ε (see [19, 16], respectively). C-pure (resp. neat) and pure are in general inequivalent, none implies the other. In general, C-pure submodules are closed and closed submodules are neat, and the converses are true if R is a right CPS ring, i.e. every cyclic right R-module is a direct sum of a projective module and a semisimple module (see [9]).

Recently, there is a significant interest in some classes of modules that are defined via closed submodules, neat submodules and C-pure submodules (see [1, 4, 6, 23]). A right R-module A is called weakly-flat [23](resp. neat-flat [4]) if the kernel of any epimorphism $B \to A$ is closed (resp. neat) in B.

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In this article, motivated by the weakly-flat and neat-flat modules, we continue the study and investigation of modules A, for which any short exact sequence ending with A is C-pure. Namely, a right R-module A is said to be C-flat if the kernel of any epimorphism $B \to A$ is C-pure in B, i.e. the induced map $\mathrm{Hom}(C, B) \to \mathrm{Hom}(C, A)$ is surjective for any cyclic right R-module C ([11]). Projective modules are C-flat and C-flat modules are weakly-flat and neat-flat. It is discussed the connections between C-flat, weakly-flat and neat-flat modules. In [3], a right R-module A is called singly-projective if for any cyclic right R-module C, every homomorphism $f : C \to A$ factors through a finitely generated free right R-module F. It is shown that C-flat modules coincide with singly-projective modules over arbitrary rings. Next, several characterizations of certain classes of rings and modules via C-purity are considered. We prove that every C-flat module is injective if and only if R is a QF ring. Moreover, we show that R is a CF ring if and only if every FP-injective right R-module is C-flat.

2. C-flat modules

Let $\varepsilon : 0 \to B \xrightarrow{f} A \xrightarrow{g} C \to 0$ be an exact sequence of right R-modules. ε is called *C-pure* exact if, $f(B)$ is a C-pure submodule of A (see [19]). In this case, f and g are called C-pure monomorphism and C-pure epimorphism, respectively. By definition, the class of C-pure exact sequences is projectively generated by the class of cyclic right R-modules. Hence C-pure exact sequences form a proper class in the sense of Bushbaum, (see [11, Proposition 1.7]).

Proposition 1. The following are equivalent for a right R-module A.

(1) A is C-flat.
(2) Every exact sequence $0 \to C \to B \to A \to 0$ is C-pure.
(3) There exists a C-pure exact sequence $0 \to C \to P \to A \to 0$ with P projective.
(4) There exists a C-pure exact sequence $0 \to C \to F \to A \to 0$ with F C-flat.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1) Let $0 \to C \to B \xrightarrow{g} A \to 0$ be any short exact sequence. We claim that g is a C-pure epimorphism, i.e., $Ker(g)$ is a C-pure submodule of B. By (4), there exists a C-pure exact sequence $0 \to C \to F \xrightarrow{h} A \to 0$ with F C-flat. Considering the pullback of g and h, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & D & \xrightarrow{\beta} & F & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & C & \longrightarrow & B & \xrightarrow{g} & A & \longrightarrow & 0 \end{array}$$

Since F is C-flat, β is a C-pure epimorphism. Also, since h is C-pure epimorphism, $g\alpha = h\beta$ is again C-pure epimorphism. This means that g is a C-pure epimorphism by [11, Proposition 1.7], and this completes the proof.

□

Remark 2.1. (1) If an R-module A is C-flat then A is torsion-free by [11, Proposition 4.3] but not conversely. If R is a commutative integral domain, then every torsion-free module is C-flat by [11, Proposition 4.4].

(2) Obviously, projective modules are C-flat. But the converse is not true in general. Let $R = k[X,Y]$ be a polynomial ring in two variables X, Y over a field k. Here the ideal (X,Y) of R is torsion-free, and so is C-flat by (1) since R is a commutative integral domain. But it is not flat as R-module by [2, Chapter I, Exercise 2.3], and so is not projective.

(3) C-flat and flat are in general inequivalent and none implies the other by [11, Remarks 4.6 and 4.7].

(4) Note that a cyclic right R-module is C-flat if and only if it is projective. Thus R is a semisimple Artinian ring if and only if every right R-module is C-flat.

The following observation is easy to show by Proposition 1 and useful for the further characterization of C-flat modules.

Lemma 2.2. *Let R be a ring. An R-module A is singly projective if and only if A is C-flat.*

Let A be a finitely presented right R-module. Then there exists a short exact sequence $0 \to G \to F \to A \to 0$ with F and G finitely generated free. If we apply the functor $\mathrm{Hom}_R(-, R)$ to this exact sequence, we obtain the sequence $0 \to A^* \to F^* \to G^* \to Tr(A) \to 0$ where $Tr(A)$ is the cokernel of the dual map $F^* \to G^*$. Note that $Tr(A)$ is a finitely presented left R-module. The left R-module $Tr(A)$ is called an Auslander-Bridger Transpose of the right R-module A (see [20, §5]). Over a right Noetherian ring, every cyclic right R-module C and its transpose $Tr(C)$ are finitely presented.

Proposition 2. Let R be a right Noetherian ring and A a right R-module. Then A is C-flat if and only if $\mathrm{Tor}_1(A, Tr(C)) = 0$ for each cyclic right R-module C.

Proof. Let $0 \to G \to F \to A \to 0$ be an exact sequence with F projective. If we assume that A is C-flat right R-module, then $0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(C, F) \to \mathrm{Hom}(C, A) \to 0$ is exact for any cyclic right R-module C. Since R is right Noetherian, C is finitely presented, and so $0 \to G \otimes Tr(C) \to F \otimes Tr(C)$ is left exact by [20, Theorem 8.3]. Hence $\mathrm{Tor}_1^R(A, Tr(C)) = 0$. Conversely, suppose $\mathrm{Tor}_1^R(A, Tr(C)) = 0$ for each cyclic right R-module C. Thus $0 \to G \otimes Tr(C) \to F \otimes Tr(C)$ is left exact, and so $0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(C, F) \to \mathrm{Hom}(C, A) \to 0$ is exact again by [20, Theorem 8.3]. This means that A is C-flat by Proposition 1. □

Recall that a right R-module A is called singly injective if $\mathrm{Ext}_R^1(F/K, A) = 0$ for any cyclic submodule K of any finitely generated free right R-module F. A right R-module A is called singly flat if $\mathrm{Tor}_R^1(A, F/K) = 0$ for any cyclic submodule K of any finitely generated free right R-module F (see [14]).

Proposition 3. Let R be a right Noetherian ring and A a left R-module. A is singly injective if and only if $\mathrm{Ext}_R^1(Tr(C), A) = 0$ for any cyclic right R-module C.

Proof. Let $0 \to A \to E(A) \to E(A)/A \to 0$ be an exact sequence. If we assume that A is singly injective left R-module, then $0 \to C \otimes A \to C \otimes E(A) \to C \otimes E(A)/A \to 0$ is exact by [7, Lemma 2.1 and Proposition 2.2]. So $\mathrm{Hom}(Tr(C), E(A)) \to \mathrm{Hom}(Tr(C), E(A)/A)$ is epic by [20, Theorem 8.3], whence $\mathrm{Ext}_R^1(Tr(C), A) = 0$. Conversely, if we assume that $\mathrm{Ext}_R^1(Tr(C), A) = 0$ for each cyclic right R-module C, then $\mathrm{Hom}(Tr(C), E(A)) \to \mathrm{Hom}(Tr(C), E(A)/A)$ is epic. So $0 \to C \otimes A \to C \otimes E(A)$ is left exact by [20, Theorem 8.3], whence A is singly injective by [7, Lemma 2.1 and Corollary 2.6]. □

Corollary 1. Let R be a right Noetherian ring. Then the following are true:

(1) A is C-flat right R-module if and only if A^+ is singly injective.
(2) A is singly-injective left R-module if and only if A^+ is C-flat.
(3) A is C-flat right R-module if and only if A is singly flat.
(4) A is C-flat right R-module if and only if A^{++} is C-flat.

Proof. (1) For any cyclic right R-module C, $Tr(C)$ is finitely presented. Thus the result follows by the standard isomorphism $\mathrm{Ext}^1_R(Tr(C), A^+) \cong \mathrm{Tor}_1(A, Tr(C))^+$ and Propositions 2 and 3.

(2) Let A be a left R-module and C a cyclic right R-module. Then we have $\mathrm{Tor}_1(A^+, Tr(C)) \cong \mathrm{Ext}^1_R(Tr(C), A)^+$ by [18, Theorem 9.51]. Hence the result follows also by Propositions 2 and 3.

(3) follows by (1) and [14, Lemma 2.4].

(4) follows by (1) and (2). □

Recall that a ring R is said to be left hereditary (respectively, left semihereditary, left PP) if every left ideal (respectively, finitely generated left ideal, principal left ideal) of R is projective. A right R-module A is called FP-injective if $\mathrm{Ext}^1_R(F/K, A) = 0$ for any finitely generated submodule K of any finitely generated free right R-module F (see [13]).

Corollary 2. The following are equivalent for a right Noetherian ring R.

(1) Every singly injective left R-module is FP-injective.
(2) Every C-flat right R-module is flat.
Moreover, if R is a commutative PP ring, then the above conditions are equivalent to:
(3) R is hereditary.

Proof. (1) ⇔ (2) follows by Corollary 1 and [14, Corollary 2.11].

(1) ⇔ (3) follows by [7, Theorem 3.9] and by the fact that Noetherian semihereditary rings are hereditary. □

In [12], a ring R is called right CPS if every cyclic right R-module is a direct sum of a projective module and a semisimple module.

Remark 2.3. (1) Following [9], C-pure submodules are closed, but not conversely.

(2) Since closed submodules are neat by [21], C-pure submodules are neat.

(3) A ring R is right CPS ring if and only if neat submodules are C-pure. In particular, if R is a right CPS ring, then closed submodules are also C-pure (see [9]).

Since C-pure submodules are closed and closed submodules are neat, we have the following implications in our concepts:

C-flat ⇒ weakly flat ⇒ neat-flat.

Recall that a ring R is said to be a right C-ring if $\mathrm{Soc}(R/I) \neq 0$ for every essential right ideal I of R. Right CPS rings, left perfect rings and right semiartinian rings are well known examples of right C-rings ([5, 10.10]). Together with Remark 2.3(3) and [4, Proposition 2.9], we obtain the following.

Corollary 3. Let R be a right CPS ring and A be a right R-module. The following statements are equivalent:

(1) A is C-flat.

(2) A is weakly-flat.
(3) A is neat-flat.
(4) $Soc(A) = A.Soc(R_R)$.

A right R-module M is called CS if every closed submodule of M is a direct summand of M and a ring R called *right* CS if R_R is CS. A ring R is called *right* Σ-CS (respectively, *right finitely* Σ-CS) if every (respectively, finite) direct sum of copies of R_R is CS (the reader might consult [8]).

Proposition 4. If every neat-flat right R-modules is C-flat, then R is a right CS and right C-ring.

Proof. The hypothesis implies that every neat-flat right R-module weakly-flat, and so R is a C-ring by [4, Proposition 2.7]. Let I be a closed right ideal of R. Then I is neat in R, and so R/I is neat-flat by [4, Lemma 2.1]. Thus by the hypothesis, R/I is C-flat and also $0 \to I \to R \to R/I \to 0$ is C-pure. Since R/I is projective with respect to C-pure exact sequences, the exact sequence $0 \to I \to R \to R/I \to 0$ splits. Thus I is a direct summand of R, that is R is a right CS ring. □

Recall by [17] that a ring R is a right SC-ring if every cyclic singular right R-module is semisimple. By Remark 2.3(3) and [4, Proposition 2.9], we obtain the following.

Corollary 4. Let R be a right SC-ring. The following are equivalent:

(1) R is right CPS.
(2) Every neat-flat right R-module is C-flat.
(3) Every weakly-flat right R-module is C-flat.

Proof. (1) $\Rightarrow$ (2) This follows directly by Corollary 3. (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) Let I be a closed right ideal of R. Then R/I is weakly-flat. Thus by the hypothesis, R/I is C-flat and also $0 \to I \to R \to R/I \to 0$ is C-pure. Similar to the proof of Proposition 4, R is a right CS ring. Thus R is a right CPS ring by the fact that R is a right SC-ring (see [12, Corollary 4.4]). □

Remark 2.4. If R is a right Σ-CS ring, then every C-flat right R-module is projective by [4, Theorem 2.10.]. The converse is also true if R is a right CPS-ring by Corollary 3 and [4, Theorem 2.10].

Proposition 5. If R is right finitely Σ-CS, then every finitely generated C-flat right R-module is projective.

Proof. Let A be a finitely generated C-flat right R-module and consider the short exact sequence $\varepsilon : 0 \to K \to F \to A \to 0$ with F finitely generated free. By Proposition 1, ε is C-pure, and so is closed. This means that ε splits by the hypothesis. Thus A is projective. □

A ring R is called right CF if every cyclic right R-module embeds in a free module. R is said to be a left AFG ring in case the left annihilator of every nonempty subset of R is a finitely generated left ideal, equivalently every right R-module has a singly projective preenvelope (see [15]).

Proposition 6. Let R be a ring. The following are equivalent:

(1) R is right CF.
(2) Every FP-injective right R-module is C-flat.

(3) Every injective right R-module is C-flat.
Moreover, if R is left AFG, then the above conditions are equivalent to:
(4) Every right R-module has a monic C-flat preenvelope.

Proof. (1) ⇒ (2) Let E be an FP-injective right R-module and C a cyclic right R-module. Since R is a CF ring, C can be embedded in a finitely generated free right R-module F. Consider the inclusion map $i : C \hookrightarrow F$ and a homomorphism $f : C \to E$. As E is FP-injective, there exists a homomorphism $g : F \to E$ such that $gi = f$. Thus E is C-flat by Lemma 2.2 and [15, Lemma 2.1].

(2) ⇒ (3) Since injective modules are FP-injective, it is clear.

(3) ⇔ (4) ⇒ (1) Follows by Lemma 2.2 and [15, Lemma 3.6]. □

Recall that R is said to be a *QF-ring* if R is left Noetherian and left self-injective, or equivalently every injective (resp. projective) right R-module is projective (resp. injective) (see [10]). In the following result, we give a new characterization of a QF ring.

Proposition 7. Let R be a ring. The following are equivalent:

(1) R is a QF ring.
(2) R is a right CF ring and every C-flat right R-module is projective.
(3) Every C-flat right R-module is injective.

Proof. (1) ⇒ (2) It is clear that R is a right CF and right Σ-CS ring. Thus (2) follows by Remark 2.4.

(2) ⇒ (1) is clear by Proposition 6.

(1) ⇒ (3) Let A be a C-flat right R-module. Since R is QF, R is right Σ-CS, and so A is projective by Remark 2.4. Being R is QF implies that A is injective.

(3) ⇒ (1) Clear since projective right modules are C-flat. □

3. Conclusion

In this paper, we continue the study and investigation of C-flat modules and we discuss the connections between C-flat, weakly-flat, neat-flat and singly flat modules. Then we investigate basic properties of the C-flat modules and some characterizations of CF and QF rings. We show that C-flat modules coincide with singly-projective modules over arbitrary rings. This work provides a new approach to singly projective modules in terms of C-pure submodules.

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ON SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE nTH DERIVATIVES ARE (η_1, η_2)-STRONGLY CONVEX

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Abstract. The aim of this paper we establish some new inequalities of Hermite-Hadamard type by using (η_1, η_2) −strongly convex function whose nth derivatives in absolute value at certain powers. Moreover, we also consider their relevances for other related known results.

1. Introduction

In the following integral inequalities which are well known in the literature as the Hermite-Hadamard inequality.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

Many authors have studied and generalized the Hermite-Hadamard inequality in several ways via different classes functions. For some recent result related to the Hermite-Hadamard inequality, we refer the interested reader to the papers. $[4-15]$. Convex functions have played an important role in the development of various fields in pure and applied sciences. A significant class of convex functions is strongly convex functions. The strongly convex functions also play an important role in optimization theory and mathematical economics.

Now let's state the definitions necessary for our work.

Definition 1.1. [11]A set $I \subseteq \mathbb{R}$ is invex with respect to a real bifunction $\eta : I \times I \to \mathbb{R}$, if

$$x, y \in I,\ \lambda \in [0,1] \Longrightarrow y + \lambda\eta(x,y) \in I. \tag{1.2}$$

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If I is an invex set with respect to η, then a function $f : I \to \mathbb{R}$ is called preinvex , if $x, y \in I$ and $\lambda \in [0,1]$.

$$f(y + \lambda\eta(x,y)) \leq \lambda f(x) + (1-\lambda) f(y). \tag{1.3}$$

In 2016,Gordji et al. [11] introduced the concept $\eta-$convexity as follows:

Definition 1.2. A function $f : I \to \mathbb{R}$ is called convex with respect to $\eta-$convex, if

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)) \tag{1.4}$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 1.3. [24] Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \to \mathbb{R}$. Consider $f : I \to \mathbb{R}$ and $\eta_2 : f(I) \times f(I) \to \mathbb{R}$. The function f is said to be $(\eta_1, \eta_2)-$ convex, if

$$f(x + \lambda\eta_1(y,x)) \leq f(x) + \lambda\eta_2(f(y), f(x)) \tag{1.5}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.4. Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \to \mathbb{R}$. Consider $f : I \to \mathbb{R}$ and $\eta_2 : f(I) \times f(I) \to \mathbb{R}$. The function f is said to be $(\eta_1, \eta_2)-$ strongly convex, if $c \geq 0$,

$$\begin{aligned} &f(x + \lambda\eta_1(y,x)) \\ &\leq f(x) + \lambda\eta_2(f(y), f(x)) - c\lambda(1-\lambda)\eta_1(y,x)\eta_2(y,x) \end{aligned} \tag{1.6}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.5. An $(\eta_1, \eta_2)-$ strongly convex function reduces to

Remark 1.6. **(i)** If we choose $c = 0$ in definition 1.4 we obtain $(\eta_1, \eta_2)-$ convex function.

(ii) If we choose $c = 0$ and $\eta_1(x,y) = x - y$ for all $x, y \in I$ in definition 1.4 we obtain $\eta-$ convex function.

(iii) If we choose $c = 0$ and $\eta_2(x,y) = x - y$ for all $x, y \in f(I)$ in definition 1.4 we obtain preinvex function.

(iv) If we choose $c = 0$ and $\eta_1(x,y) = \eta_2(x,y) = x - y$ in definition 1.4 we obtain classical convex function.

(v) If we choose $\eta_1(x,y) = \eta_2(x,y) = x - y$ in definition 1.4 we obtain strongly convex function.

(vi) If we choose $\eta_1(x,y) = x - y$ for all $x, y \in I$ in definition 1.4 we obtain $\eta -$ *strongly* convex function.

2. Main Results

In this section, we establish some new inequalities of Hermite-Hadamard type by using $(\eta_1, \eta_2)-$strongly convex function whose n th derivatives in absolute value at certain powers. Moreover, we also consider their relevances for other related known results.

Lemma 2.1. *Let $I \subseteq \mathbb{R}$ be an invex set with respect to η_1 such that for all $x \in I$ and $t \in [0,1]$. Also let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-times differentiable functions on*

I° with $a<b$, and $n\in N^{+}$. For any $a,b\in I^{\circ}$ with $\eta_1(b,a)>0$, suppose that $f^n\in L_1[a,a+\eta_1(b,a)]$. Then for $\alpha>0$, the following equality holds;

$$\begin{aligned}
&\tfrac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)\,dx\\
&-\sum_{k=1}^{n}\tfrac{\eta_1(b,a)^k\left[f^{(k-1)}(a+\eta_1(b,a))+(-1)^k f^{(k-1)}(a)\right]}{2(k!)}\\
&=\tfrac{\eta_1(b,a)^{nthis}}{2(n!)}\int_0^1 t^n f^{(n)}(a+t\eta_1(b,a))\,dt
\end{aligned}\tag{2.1}$$

Proof. By integration by parts, it follows that

(2.2)

$$\begin{aligned}
&\tfrac{\eta_1(b,a)^{n+1}}{2(n!)}\int_0^1 t^n f^{(n)}(a+t\eta_1(b,a))\,dt\\
&=-\tfrac{\eta_1(b,a)^n}{2(n!)}f^{(n-1)}(a+\eta_1(b,a))+\tfrac{\eta_1(b,a)^n}{2[(n-1)!]}\int_0^1 t^{n-1}f^{(n-1)}(a+t\eta_1(b,a))\,dt\\
&=-\tfrac{\eta_1(b,a)^n}{2(n!)}f^{(n-1)}(a+\eta_1(b,a))-\tfrac{\eta_1(b,a)^{n-1}}{2[(n-1)!]}f^{(n-2)}(a+\eta_1(b,a))\\
&+\tfrac{\eta_1(b,a)^{n-1}}{2[(n-2)!]}\int_0^1 t^{n-2}f^{(n-2)}(a+t\eta_1(b,a))\,dt\\
&=-\sum_{k=1}^{n-1}\tfrac{\eta_1(b,a)^{k+1}f^{(k)}(a+\eta_1(b,a))}{2(k!)}+\tfrac{\eta_1(b,a)^2}{2}\int_0^1 tf'(a+t\eta_1(b,a))\,dt\\
&=-\sum_{k=1}^{n}\tfrac{\eta_1(b,a)^k f^{(k-1)}(a+\eta_1(b,a))}{2(k!)}+\tfrac{1}{2}\int_a^{a+\eta_1(b,a)} f(x)\,dx.
\end{aligned}$$

with the same argument as the above we have

(2.3)

$$\begin{aligned}
&\tfrac{\eta_1(b,a)^{n+1}}{2(n!)}\int_0^1 (t-1)^n f^{(n)}(a+t\eta_1(b,a))\,dt\\
&=-\tfrac{\eta_1(b,a)^n}{2(n!)}(-1)^n f^{(n-1)}(a)+\tfrac{\eta_1(b,a)^n}{2[(n-1)!]}\int_0^1 (t-1)^{n-1}f^{(n-1)}(a+t\eta_1(b,a))\,dt\\
&=-\tfrac{\eta_1(b,a)^n}{2(n!)}(-1)^n f^{(n-1)}(a)-\tfrac{\eta_1(b,a)^{n-1}}{2[(n-1)!]}(-1)^n f^{(n-2)}(a)\\
&+\tfrac{\eta_1(b,a)^{n-1}}{2[(n-2)!]}\int_0^1 (t-1)^{n-2}f^{(n-2)}(a+t\eta_1(b,a))\,dt\\
&=-\sum_{k=1}^{n}\tfrac{\eta_1(b,a)^k(-1)^k f^{(k-1)}(a)}{2(k!)}+\tfrac{1}{2}\int_a^{a+\eta_1(b,a)} f(x)\,dx.
\end{aligned}$$

Adding these two equations leads to Lemma 2.1. □

Lemma 2.2. *Let $I\subseteq\mathbb{R}$ be an invex set with respect to η_1 such that for all $x\in I$ and $t\in[0,1]$. Also let $f:I\subset\mathbb{R}\to\mathbb{R}$ be n-times differentiable functions on I° with $a<b$, and $n\in N^{+}$. For any $a,b\in I^{\circ}$ with $\eta_1(b,a)>0$, suppose that $f^n\in L_1[a,a+\eta_1(b,a)]$ .Then for $\alpha>0$, the following equality holds;*

$$\begin{aligned}
&\tfrac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)\,dx+\tfrac{1}{\eta_1(b,a)}\int_{b+\frac{1}{2}\eta_1(a,b)}^{b+\eta_1(a,b)} f(x)\,dx\\
&-\sum_{k=1}^{n}\tfrac{\eta_1(b,a)^k\left[1+(-1)^k\right]}{2^k(k!)}\\
&\times\left[f^{(k-1)}\left(a+\tfrac{1}{2}\eta_1(b,a)\right)+f^{(k-1)}\left(b+\tfrac{1}{2}\eta_1(a,b)\right)\right]\\
&=\tfrac{\eta_1(b,a)^n}{2(n!)}\left[\int_0^{\frac{1}{2}}(-t)^n f^{(n)}(a+t\eta_1(b,a))\,dt\right.\\
&\left.+\int_{\frac{1}{2}}^1(1-t)^n f^{(n)}(b+t\eta_1(a,b))\,dt\right].
\end{aligned}\tag{2.4}$$

Proof. This follows from integration by parts immediately. □

Theorem 2.3. *Let $I\subseteq\mathbb{R}$ be an invex set with respect to η_1 such that for all $x\in I$ and $t\in[0,1]$. Also let $f:I\subset\mathbb{R}\to\mathbb{R}$ be n-times differentiable functions on I° with $a<b$, and $n\in N^{+}$ $(\eta_1,\eta_2)-$strongly convex function where η_2 is an integrable bi functionon $f(I)\times f(I)$ with modulus $c\geq 0$. For any $a,b\in I^{\circ}$ with $\eta_1(b,a)>0$, suppose that $f^n\in L_1[a,a+\eta_1(b,a)]$ and $|f^n|^q$ for $q\geq 1$.Then for*

$\alpha > 0$, *the following inequality holds;*

$$\begin{aligned}
&\left|\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)\,dx - \sum_{k=1}^{n}\frac{\eta_1(b,a)^k\left[f^{(k-1)}(a+\eta_1(b,a))+(-1)^k f^{(k-1)}(a)\right]}{2(k!)}\right| \\
&\le \frac{\eta_1(b,a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{2}{n+1}\left(|f^n(a)|^q\right)+\frac{1}{n+1}\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)-\frac{2c\eta_1(b,a)\eta_2(b,a)}{(n+2)(n+3)}\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.5}$$

Proof. By using Lemma 1, the power mean inequality and the (η_1, η_2) −strongly convex function of $|f^n|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)\,dx - \sum_{k=1}^{n}\frac{\eta_1(b,a)^k\left[f^{(k-1)}(a+\eta_1(b,a))+(-1)^k f^{(k-1)}(a)\right]}{2(k!)}\right| \\
&\le \frac{\eta_1(b,a)^n}{2(n!)}\int_0^1\left[t^n+(1-t)^n\right]\left|f^{(n)}(a+t\eta_1(b,a))\right|dt \\
&\le \frac{\eta_1(b,a)^n}{2(n!)}\left(\int_0^1\left[t^n+(1-t)^n\right]dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left[t^n+(1-t)^n\right]\left|f^{(n)}(a+t\eta_1(b,a))\right|dt\right)^{\frac{1}{q}} \\
&\le \frac{\eta_1(b,a)^n}{2(n!)}\left(\int_0^1\left[t^n+(1-t)^n\right]\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\left[t^n+(1-t)^n\right]\left[|f^n(a)|^q+t\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)-ct(1-t)\eta_1(b,a)\eta_2(b,a)\right]dt\right)^{\frac{1}{q}} \\
&= \frac{\eta_1(b,a)^n}{2(n!)}\left(\int_0^1\left[t^n+(1-t)^n\right]\right)^{1-\frac{1}{q}} \\
&\times\left(\left(|f^n(a)|^q\right)\int_0^1\left[t^n+(1-t)^n\right]dt+\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)\left(\int_0^1 t\left[t^n+(1-t)^n\right]dt\right)\right. \\
&\left.-c\eta_1(b,a)\eta_2(b,a)\int_0^1 t(1-t)\left[t^n+(1-t)^n\right]dt\right)^{\frac{1}{q}} \\
&= \frac{\eta_1(b,a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{1-\frac{1}{q}}\left(\frac{2}{n+1}\left(|f^n(a)|^q\right)+\frac{1}{n+1}\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)-\frac{2c\eta_1(b,a)\eta_2(b,a)}{(n+2)(n+3)}\right)^{\frac{1}{q}}
\end{aligned} \tag{2.6}$$

where

$$\int_0^1\left[t^n+(1-t)^n\right]dt=\frac{2}{n+1} \tag{2.7}$$

$$\int_0^1 t\left[t^n+(1-t)^n\right]dt=\frac{1}{n+1} \tag{2.8}$$

and

$$\int_0^1 t(1-t)\left[t^n+(1-t)^n\right]dt=\frac{2}{(n+2)(n+3)} \tag{2.9}$$

This completes the proof of the theorem. □

We will give some special cases of Theorem 2.3 which show that our result generalize several results obtained previous works.

Remark 2.4. As can be seen from the special elections below, our results are more general.

(i) If we choose $c = 0$ in *Theorem* 2.3 the results are we obtain also provided for (η_1, η_2) −convex functions, is proved by S. Kermausuor et. al. [25].

(ii) If we choose $c = 0$ and $\eta_1(x, y) = x - y$ for all $x, y \in I$ in *Theorem* 2.3 the results are we obtain also provided for $\eta-$ convex function.

(iii) If we choose $c = 0$ and $\eta_2(x, y) = x - y$ for all $x, y \in f(I)$ in *Theorem* 2.3 the results are we obtain also provided for preinvex function.

(iv) If we choose $c = 0$ and $\eta_1(x, y) = \eta_2(x, y) = x - y$ in *Theorem* 2.3 the results are we obtain also provided for classical convex function.

(v) If we choose $\eta_1(x,y)=\eta_2(x,y)=x-y$ in *Theorem* 2.3 the results are we obtain also provided for strongly convex function.

(vi) If we choose $\eta_1(x,y)=x-y$ for all $x,y\in I$ in *Theorem* 2.3 we obtain $\eta-strongly$ convex function.

Theorem 2.5. *Let $I\subset\mathbb{R}$ be an invex set with respect to η_1 such that for all $x\in I$ and $t\in[0,1]$. Also let $f:I\subset\mathbb{R}\to\mathbb{R}$ be n-times differantiable functions on I° with $a<b$, and $n\in N^+$ $(\eta_1,\eta_2)-$strongly convex function where η_2 is an integrable bi functionon $f(I)\times f(I)$ with modulus $c\geq 0$. For any $a,b\in I^\circ$ with $\eta_1(b,a)>0$, suppose that $f^n\in L_1[a,a+\eta_1(b,a)]$ and $|f^n|^q$ for $q\geq 1$. Then for $\alpha>0$, the following inequality holds;*

(2.10)
$$\begin{aligned}&\left|\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(x)\,dx-\sum_{k=1}^{n}\frac{\eta_1(b,a)^k\left[f^{(k-1)}(a+\eta_1(b,a))+(-1)^k f^{(k-1)}(a)\right]}{2(k!)}\right|\\&\leq\frac{\eta_1(b,a)^n}{2(n!)}\left(\int_0^1[t^n+(1-t)^n]^p\,dt\right)^{\frac{1}{p}}\\&\times\left(|f^n(a)|^q+\frac{1}{2}\eta_2(|f^n(b)|^q,|f^n(a)|^q)-\frac{c\eta_1(b,a)\eta_2(b,a)}{6}\right)^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 1, the Hölder's inequality and the $(\eta_1,\eta_2)-$strongly convexity of $|f^n|^q$, we have

(2.11)
$$\begin{aligned}&\left|\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(x)\,dx-\sum_{k=1}^{n}\frac{\eta_1(b,a)^k\left[f^{(k-1)}(a+\eta_1(b,a))+(-1)^k f^{(k-1)}(a)\right]}{2(k!)}\right|\\&\leq\frac{\eta_1(b,a)^n}{2(n!)}\int_0^1[t^n+(1-t)^n]\left|f^{(n)}(a+t\eta_1(b,a))\right|dt\\&\leq\frac{\eta_1(b,a)^n}{2(n!)}\left(\int_0^1[t^n+(1-t)^n]^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}(a+t\eta_1(b,a))\right|^q\right)^{\frac{1}{q}}\\&\leq\frac{\eta_1(b,a)^n}{2(n!)}\left(\int_0^1[t^n+(1-t)^n]^p\,dt\right)^{\frac{1}{p}}\\&\times\left(\int_0^1[|f^n(a)|^q+t\eta_2(|f^n(b)|^q,|f^n(a)|^q)-ct(1-t)\eta_1(b,a)\eta_2(b,a)]\,dt\right)^{\frac{1}{q}}\\&=\frac{\eta_1(b,a)^n}{2(n!)}\left(\int_0^1[t^n+(1-t)^n]^p\,dt\right)^{\frac{1}{p}}\\&\times\left(|f^n(a)|^q\int_0^1 1dt+\eta_2(|f^n(b)|^q,|f^n(a)|^q)\int_0^1 tdt-c\eta_1(b,a)\eta_2(b,a)\int_0^1 t(1-t)\,dt\right)^{\frac{1}{q}}\\&=\frac{\eta_1(b,a)^n}{2(n!)}\left(\int_0^1[t^n+(1-t)^n]^p\,dt\right)^{\frac{1}{p}}\left(|f^n(a)|^q+\frac{1}{2}\eta_2(|f^n(b)|^q,|f^n(a)|^q)-\frac{c\eta_1(b,a)\eta_2(b,a)}{6}\right)^{\frac{1}{q}}.\end{aligned}$$

It can easily be verified that $t^n+(1-t)^n\leq 1$ for $t\in[0,1]$. So, it follows that

$$\int_0^1[t^n+(1-t)^n]^p\,dt\leq\int_0^1[t^n+(1-t)^n]\,dt=\frac{2}{n+1}\tag{2.12}$$

Hence, the desired inequality follows from 2.11 and 2.12. This completes the proof of the theorem. □

We will give some special cases of Theorem 2.5 which show that our result generalize several results obtained previous works.

Remark 2.6. As can be seen from the special elections below, our results are more general.

(i) If we choose $c=0$ in *Theorem* 2.5 the results are we obtain also provided for $(\eta_1,\eta_2)-$convex functions, is proved by S. Kermausuor et. al. [25].

(ii) If we choose $c=0$ and $\eta_1(x,y)=x-y$ for all $x,y\in I$ in *Theorem* 2.5 the results are we obtain also provided for $\eta-$ convex function.

(iii) If we choose $c=0$ and $\eta_2(x,y)=x-y$ for all $x,y\in f(I)$ in *Theorem* 2.5 the results are we obtain also provided for preinvex function.

(iv) If we choose $c=0$ and $\eta_1(x,y)=\eta_2(x,y)=x-y$ in *Theorem* 2.5 the results are we obtain also provided for classical convex function.

(v) If we choose $\eta_1(x,y)=\eta_2(x,y)=x-y$ in *Theorem* 2.5 the results are we obtain also provided for strongly convex function.

(vi) If we choose $\eta_1(x,y)=x-y$ for all $x,y\in I$ in *Theorem* 2.5 we obtain $\eta-$ *strongly* convex function.

Theorem 2.7. *Let $I\subset\mathbb{R}$ be an invex set with respect to $\eta_1$such that for all $x\in I$ and $t\in[0,1]$. Also let $f:I\subset\mathbb{R}\to\mathbb{R}$ be n-times differantiable functions on I° with $a<b$, and $n\in N^+$ $(\eta_1,\eta_2)-$strongly convex function where η_2 is an integrable bi functionon $f(I)\times f(I)$ with modulus $c\geq 0$. For any $a,b\in I^\circ$ with $\eta_1(b,a)>0$, suppose that $f^n\in L_1[a,a+\eta_1(b,a)]$ and $|f^n|^q$ for $q\geq 1$.Then for $\alpha>0$, the following inequality holds;*

$$\Big|\tfrac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(x)\,dx+\tfrac{1}{\eta_1(b,a)}\int_{b+\frac{1}{2}\eta_1(a,b)}^{b+\eta_1(a,b)}f(x)\,dx$$
$$-\sum_{k=1}^n\tfrac{\eta_1(b,a)^k\left[1+(-1)^k\right]}{2^k(k!)}$$
$$\Big|\times\left[f^{(k-1)}\left(a+\tfrac{1}{2}\eta_1(b,a)\right)+f^{(k-1)}\left(b+\tfrac{1}{2}\eta_1(a,b)\right)\right]\Big|$$

$$\begin{aligned}(2.13)\qquad &\leq\tfrac{\eta_1(b,a)^n}{(n!)}\left(\tfrac{1}{2^{n+1}(n+1)}\right)^{1-\frac{1}{q}}\\&\times\Big[\Big(\tfrac{1}{2^{n+1}(n+1)}|f^n(a)|^q+\tfrac{1}{2^{n+2}(n+2)}\eta_2(|f^n(b)|^q,|f^n(a)|^q)\\&-\tfrac{c\eta_1(b,a)\eta_2(b,a)(n+4)}{2^{n+3}(n+2)(n+3)}\Big)^{\frac{1}{q}}\Big]\\&+\tfrac{\eta_1(b,a)^n}{(n!)}\left(\tfrac{1}{2^{n+1}(n+1)}\right)^{1-\frac{1}{q}}\\&\Big[\Big(\tfrac{1}{2^{n+1}(n+1)}|f^n(b)|^q+\tfrac{n+3}{2^{n+2}(n+2)(n+1)}\eta_2(|f^n(a)|^q,|f^n(b)|^q)\\&-\tfrac{c\eta_1(b,a)\eta_2(b,a)(n+4)}{2^{n+3}(n+2)(n+3)}\Big)^{\frac{1}{q}}\Big].\end{aligned}$$

Proof. By using Lemma 2, the Power mean inequality and the $(\eta_1,\eta_2)-$strongly convexity of $|f^n|^q$, we have

$$\begin{aligned}&\Big|\tfrac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(x)\,dx+\tfrac{1}{\eta_1(b,a)}\int_{b+\frac{1}{2}\eta_1(a,b)}^{b+\eta_1(a,b)}f(x)\,dx\\&-\sum_{k=1}^n\tfrac{\eta_1(b,a)^k\left[1+(-1)^k\right]}{2^k(k!)}\left[f^{(k-1)}\left(a+\tfrac{1}{2}\eta_1(b,a)\right)+f^{(k-1)}\left(b+\tfrac{1}{2}\eta_1(a,b)\right)\right]\Big|\\&\leq\tfrac{\eta_1(b,a)^n}{(n!)}\left[\int_0^{\frac{1}{2}}(t)^n\left|f^{(n)}(a+t\eta_1(b,a))\right|dt+\int_{\frac{1}{2}}^1(1-t)^n\left|f^{(n)}(b+t\eta_1(a,b))\right|dt\right]\\&\leq\tfrac{\eta_1(b,a)^n}{(n!)}\left[\left(\int_0^{\frac{1}{2}}(t)^n\,dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}(t)^n\left|f^{(n)}(a+t\eta_1(b,a))\right|^q\right)^{\frac{1}{q}}dt\right]\\&+\tfrac{\eta_1(b,a)^n}{(n!)}\left[\left(\int_{\frac{1}{2}}^1(1-t)^n\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1(1-t)^n\left|f^{(n)}(b+t\eta_1(a,b))\right|^q dt\right)^{\frac{1}{q}}\right]\end{aligned}$$

$$\leq \frac{\eta_1(b,a)^n}{(n!)}\left[\left(\int_0^{\frac{1}{2}}(t)^n dt\right)^{1-\frac{1}{q}}\right.$$
$$\left.\times\left(\int_0^{\frac{1}{2}}(t)^n\left[|f^n(a)|^q+t\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)-ct(1-t)\eta_1(b,a)\eta_2(b,a)\right]dt\right)^{\frac{1}{q}}\right]$$
$$+\frac{\eta_1(b,a)^n}{(n!)}\left[\left(\int_{\frac{1}{2}}^1(1-t)^n dt\right)^{1-\frac{1}{q}}\right.$$
$$\left.\times\left(\int_{\frac{1}{2}}^1(1-t)^n\left[|f^n(b)|^q+t\eta_2\left(|f^n(a)|^q,|f^n(b)|^q\right)-ct(1-t)\eta_1(b,a)\eta_2(b,a)\right]dt\right)^{\frac{1}{q}}\right]$$

(2.14)

$$\leq \frac{\eta_1(b,a)^n}{(n!)}\left[\left(\int_0^{\frac{1}{2}}(t)^n dt\right)^{1-\frac{1}{q}}\left(\left(|f^n(a)|^q\int_0^{\frac{1}{2}}(t)^n dt\right)+\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)\int_0^{\frac{1}{2}}t^{n+1}dt\right.\right.$$
$$\left.\left.-c\eta_1(b,a)\eta_2(b,a)\int_0^{\frac{1}{2}}t^{n+1}(1-t)\,dt\right)^{\frac{1}{q}}\right]$$
$$+\frac{\eta_1(b,a)^n}{(n!)}\left[\left(\int_{\frac{1}{2}}^1(1-t)^n dt\right)^{1-\frac{1}{q}}\left(\left(|f^n(b)|^q\int_{\frac{1}{2}}^1(1-t)^n\right)+\eta_2\left(|f^n(a)|^q,|f^n(b)|^q\right)\int_{\frac{1}{2}}^1 t(1-t)^n\right.\right.$$
$$\left.\left.-c\eta_1(b,a)\eta_2(b,a)\int_{\frac{1}{2}}^1 t(1-t)^{n+1}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{\eta_1(b,a)^n}{(n!)}\left(\frac{1}{2^{n+1}(n+1)}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2^{n+1}(n+1)}|f^n(a)|^q+\frac{1}{2^{n+2}(n+2)}\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)\right.\right.$$
$$\left.\left.-\frac{c\eta_1(b,a)\eta_2(b,a)(n+4)}{2^{n+3}(n+2)(n+3)}\right)^{\frac{1}{q}}\right]$$
$$+\frac{\eta_1(b,a)^n}{(n!)}\left(\frac{1}{2^{n+1}(n+1)}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2^{n+1}(n+1)}|f^n(b)|^q+\frac{n+3}{2^{n+2}(n+2)(n+1)}\eta_2\left(|f^n(a)|^q,|f^n(b)|^q\right)\right.\right.$$
$$\left.\left.-\frac{c\eta_1(b,a)\eta_2(b,a)(n+4)}{2^{n+3}(n+2)(n+3)}\right)^{\frac{1}{q}}\right].$$

This completes the proof of the theorem. □

We will give some special cases of Theorem 2.7 which show that our result generalize several results obtained previous works.

Remark 2.8. As can be seen from the special elections below, our results are more general.

(i) If we choose $c=0$ in $Theorem$ 2.7 the results are we obtain also provided for $(\eta_1,\eta_2)-$convex functions, is proved by S. Kermausuor et. al. [25].

(ii) If we choose $c=0$ and $\eta_1(x,y)=x-y$ for all $x,y\in I$ in $Theorem$ 2.7 the results are we obtain also provided for $\eta-$ convex function.

(iii) If we choose $c=0$ and $\eta_2(x,y)=x-y$ for all $x,y\in f(I\ )$ in $Theorem$ 2.7 the results are we obtain also provided for preinvex function.

(iv) If we choose $c=0$ and $\eta_1(x,y)=\eta_2(x,y)=x-y$ in $Theorem$ 2.7 the results are we obtain also provided for classical convex function.

(v) If we choose $\eta_1(x,y)=\eta_2(x,y)=x-y$ in $Theorem$ 2.7 the results are we obtain also provided for strongly convex function.

(vi) If we choose $\eta_1(x,y)=x-y$ for all $x,y\in I$ in $Theorem$ 2.7we obtain $\eta-strongly$ convex function.

Theorem 2.9. *Let $I\subset\mathbb{R}$ be an invex set with respect to $\eta_1$such that for all $x\in I$ and $t\in[0,1]$. Also let $f:I\subset\mathbb{R}\to\mathbb{R}$ be n-times differantiable functions on I° with $a<b$, and $n\in N^{+}$ $(\eta_1,\eta_2)-$strongly convex function where η_2 is an integrable bi functionon $f(I)\times f(I)$ with modulus $c\geq 0$. For any $a,b\in I^{\circ}$ with*

$\eta_1(b,a) > 0$, *suppose that* $f^n \in L_1[a, a+\eta_1(b,a)]$ *and* $|f^n|^q$ *for* $q \geq 1$.*Then for* $\alpha > 0$, *the following inequality holds;*

$$\left| \frac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x)\,dx + \frac{1}{\eta_1(b,a)} \int_{b+\frac{1}{2}\eta_1(a,b)}^{b+\eta_1(a,b)} f(x)\,dx \right.$$
$$-\sum_{k=1}^{n} \frac{\eta_1(b,a)^k \left[1+(-1)^k\right]}{2^k (k!)}$$
$$\left. \times \left[f^{(k-1)}\left(a+\frac{1}{2}\eta_1(b,a)\right) + f^{(k-1)}\left(b+\frac{1}{2}\eta_1(a,b)\right)\right] \right|$$

$$\begin{aligned} (2.15) \quad \leq\ & \frac{\eta_1(b,a)^n}{2(n!)} \left(\frac{1}{2^{np}(np+1)}\right)^{\frac{1}{p}} \\ & \times \left[\left(|f^n(a)|^q + \frac{1}{4}\eta_2(|f^n(b)|^q, |f^n(a)|^q) - \frac{c\eta_1(b,a)\eta_2(b,a)}{6} \right)^{\frac{1}{q}} \right. \\ & \left. + \left(|f^n(b)|^q + \frac{3}{4}\eta_2(|f^n(a)|^q, |f^n(b)|^q) - \frac{c\eta_1(b,a)\eta_2(b,a)}{6} \right)^{\frac{1}{q}} \right], \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Again, using Lemma 2, the Hölder's inequality and the $(\eta_1, \eta_2)-$strongly convexity of $|f^n|^q$, we have

$$\begin{aligned} & \left| \tfrac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x)\,dx + \tfrac{1}{\eta_1(b,a)} \int_{b+\frac{1}{2}\eta_1(a,b)}^{b+\eta_1(a,b)} f(x)\,dx \right. \\ & \left. - \textstyle\sum_{k=1}^{n} \tfrac{\eta_1(b,a)^k\left[1+(-1)^k\right]}{2^k(k!)} \left[f^{(k-1)}\left(a+\tfrac{1}{2}\eta_1(b,a)\right) + f^{(k-1)}\left(b+\tfrac{1}{2}\eta_1(a,b)\right)\right] \right| \\ & \leq \tfrac{\eta_1(b,a)^n}{(n!)} \left[\int_0^{\frac{1}{2}} (t)^n \left|f^{(n)}(a+t\eta_1(b,a))\right| dt + \int_{\frac{1}{2}}^1 (1-t)^n \left|f^{(n)}(b+t\eta_1(a,b))\right| dt \right] \\ & \leq \tfrac{\eta_1(b,a)^n}{(n!)} \left[\left(\int_0^{\frac{1}{2}} (t)^{np}\,dt\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left|f^{(n)}(a+t\eta_1(b,a))\right|^q dt\right)^{\frac{1}{q}} \right] \\ & + \tfrac{\eta_1(b,a)^n}{(n!)} \left[\left(\int_{\frac{1}{2}}^1 (1-t)^{np}\,dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left|f^{(n)}(b+t\eta_1(a,b))\right| dt\right)^{\frac{1}{q}} \right] \end{aligned}$$

$$\begin{aligned} & \leq \tfrac{\eta_1(b,a)^n}{(n!)} \left[\left(\int_0^{\frac{1}{2}} (t)^{np}\,dt\right)^{\frac{1}{p}} \right. \\ & \left. \times \left(\int_0^{\frac{1}{2}} \left[|f^n(a)|^q + t\eta_2(|f^n(b)|^q, |f^n(a)|^q) - ct(1-t)\eta_1(b,a)\eta_2(b,a)\right] dt\right)^{\frac{1}{q}} \right] \\ & + \tfrac{\eta_1(b,a)^n}{(n!)} \left[\left(\int_{\frac{1}{2}}^1 (1-t)^{np}\,dt\right)^{\frac{1}{p}} \right. \\ & \left. \times \left(\int_{\frac{1}{2}}^1 \left[|f^n(b)|^q + t\eta_2(|f^n(a)|^q, |f^n(b)|^q) - ct(1-t)\eta_1(b,a)\eta_2(b,a)\right] dt\right)^{\frac{1}{q}} \right] \end{aligned}$$

(2.16)

$$
\begin{aligned}
&\leq \tfrac{\eta_1(b,a)^n}{(n!)}\left[\left(\int_0^{\frac{1}{2}}(t)^{np}\,dt\right)^{\frac{1}{p}}\right.\\
&\times\left.\left(|f^n(a)|^q\int_0^{\frac{1}{2}}1dt+\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)\int_0^{\frac{1}{2}}tdt-c\eta_1(b,a)\,\eta_2(b,a)\int_0^{\frac{1}{2}}t(1-t)\,dt\right)^{\frac{1}{q}}\right]\\
&+\tfrac{\eta_1(b,a)^n}{(n!)}\left[\left(\int_{\frac{1}{2}}^{1}(1-t)^{np}\,dt\right)^{\frac{1}{p}}\right.\\
&\times\left.\left(|f^n(b)|^q\int_{\frac{1}{2}}^{1}1dt+\eta_2\left(|f^n(a)|^q,|f^n(b)|^q\right)\int_{\frac{1}{2}}^{1}tdt-c\eta_1(b,a)\,\eta_2(b,a)\int_{\frac{1}{2}}^{1}t(1-t)\,dt\right)^{\frac{1}{q}}\right]\\
&\leq \tfrac{\eta_1(b,a)^n}{(n!)}\left(\tfrac{1}{2^{np}(np+1)}\right)^{\frac{1}{p}}\left[\left(\tfrac{1}{2}|f^n(a)|^q+\tfrac{1}{8}\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)-\tfrac{c\eta_1(b,a)\eta_2(b,a)}{12}\right)^{\frac{1}{q}}\right]\\
&+\tfrac{\eta_1(b,a)^n}{(n!)}\left(\tfrac{1}{2^{np}(np+1)}\right)^{\frac{1}{p}}\left[\left(\tfrac{1}{2}|f^n(b)|^q+\tfrac{3}{8}\eta_2\left(|f^n(a)|^q,|f^n(b)|^q\right)-\tfrac{c\eta_1(b,a)\eta_2(b,a)}{12}\right)^{\frac{1}{q}}\right]\\
&=\tfrac{\eta_1(b,a)^n}{2(n!)}\left(\tfrac{1}{2^{np}(np+1)}\right)^{\frac{1}{p}}\left[\left(|f^n(a)|^q+\tfrac{1}{4}\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)-\tfrac{c\eta_1(b,a)\eta_2(b,a)}{6}\right)^{\frac{1}{q}}\right.\\
&+\left.\left(|f^n(b)|^q+\tfrac{3}{4}\eta_2\left(|f^n(a)|^q,|f^n(b)|^q\right)-\tfrac{c\eta_1(b,a)\eta_2(b,a)}{6}\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

This completes the proof of the theorem. □

We will give some special cases of Theorem 2.9 which show that our result generalize several results obtained previous works.

Remark 2.10. As can be seen from the special elections below, our results are more general.

(i) If we choose $c = 0$ in *Theorem* 2.9 the results are we obtain also provided for (η_1, η_2) −convex functions, is proved by S. Kermausuor et. al. [25].

(ii) If we choose $c = 0$ and $\eta_1(x, y) = x - y$ for all $x, y \in I$ in *Theorem* 2.9 the results are we obtain also provided for $\eta-$ convex function.

(iii) If we choose $c = 0$ and $\eta_2(x, y) = x - y$ for all $x, y \in f(I)$ in *Theorem* 2.9 the results are we obtain also provided for preinvex function.

(iv) If we choose $c = 0$ and $\eta_1(x, y) = \eta_2(x, y) = x - y$ in *Theorem* 2.9 the results are we obtain also provided for classical convex function.

(v) If we choose $\eta_1(x, y) = \eta_2(x, y) = x - y$ in *Theorem* 2.9 the results are we obtain also provided for strongly convex function.

(vi) If we choose $\eta_1(x, y) = x - y$ for all $x, y \in I$ in *Theorem* 2.9 we obtain $\eta - strongly$ convex function.

3. Conclusion

In this study, we present some inequalities for (η_1, η_2) −strongly convex functions involving whose nth derivatives in absolute value at certain powers. It is also shown that the results proved here are the strong generalization of some already published ones. It is an interesting and new problem that the forthcoming researchers can use the techniques of this study and obtain similar inequalities for different kinds of strongly convexity in their future work.

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ON *-BOUNDEDNESS AND *-LOCAL BOUNDEDNESS OF NON-NEWTONIAN SUPERPOSITION OPERATORS IN $c_{0,\alpha}$ AND c_α TO $\ell_{1,\beta}$

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Abstract. Many investigations have been made about of non-Newtonian calculus and superposition operators until today. Non-Newtonian superposition operator was defined by Sağır and Erdoğan in [9]. In this study, we have defined *- boundedness and *-locally boundedness of operator. We have proved that the non-Newtonian superposition operator ${}_NP_f : c_{0,\alpha} \to \ell_{1,\beta}$ is *-locally bounded if and only if f satisfies the condition (NA_2'). Then we have shown that the necessary and sufficient conditions for the *-boundedness of ${}_NP_f : c_{0,\alpha} \to \ell_{1,\beta}$. Finally, the similar results have been also obtained for ${}_NP_f : c_\alpha \to \ell_{1,\beta}$.

1. Introduction and Preliminaries

Non-Newtonian calculus was firstly introduced and worked by Michael Grossman and Robert Katz between years 1967 and 1970. They published the book about fundamentals of non-Newtonian calculus and which includes some special calculus such as geometric, harmonic, quadratic. Çakmak and Başar [5] obtained some results on sequence spaces with respect to non-Newtonian calculus. Duyar and Erdogan [7] worked on non-Newtonian real number series. Also, Güngör [11] studied on some geometric properties of $\ell_p(N)$.

Many studies are done until today on superposition operator which is one of the non-linear operators. Dedagich and Zabreiko [2] studied on the superposition operators in the space ℓ_p. After, some properties of superposition operator, such as boundedness, continuity, were studied by Tainchai [3], Sama-ae [4], Sağır and Güngör [6] and many others. Non-Newtonian superposition operator was defined and characterized in some non-Newtonian sequence spaces by Sağır and Erdoğan in [9]. In this article, we define *- boundedness and *-locally boundedness of operator. We prove that the non-Newtonian superposition operator ${}_NP_f : c_{0,\alpha} \to \ell_{1,\beta}$ is *-locally bounded if and only if f satisfies the condition (NA_2'). Then we show that

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the necessary and sufficient conditions for the *-boundedness of ${}_{N}P_f : c_{0,\alpha} \to \ell_{1,\beta}$. Also the similar results are obtained for ${}_{N}P_f : c_\alpha \to \ell_{1,\beta}$.

A *generator* is defined as an injective function with domain $\mathbb{R}$ and the range of generator is a subset of $\mathbb{R}$. Let take any α generator with range $A = \mathbb{R}_\alpha$. Let define $\alpha-$addition, $\alpha-$subtraction, $\alpha-$multiplication, $\alpha-$division and $\alpha-$order as follows;

$$\begin{array}{ll}
\alpha-\text{addition} & x\dot{+}y = \alpha\left(\alpha^{-1}(x) + \alpha^{-1}(y)\right) \\
\alpha-\text{subtraction} & x\dot{-}y = \alpha\left(\alpha^{-1}(x) - \alpha^{-1}(y)\right) \\
\alpha-\text{multiplication} & x\dot{\times}y = \alpha\left(\alpha^{-1}(x) \times \alpha^{-1}(y)\right) \\
\alpha-\text{division} & x\dot{/}y = \alpha\left(\alpha^{-1}(x) / \alpha^{-1}(y)\right) \ \ (y \neq \dot{0}) \\
\alpha-\text{order} & x\dot{<}y \ \ \left(x\dot{\leq}y\right) \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y) \ \ \left(\alpha^{-1}(x) \leq \alpha^{-1}(y)\right)
\end{array}$$

for $x, y \in \mathbb{R}_\alpha$ [1].

$(\mathbb{R}_\alpha, \dot{+}, \dot{\times}, \dot{\leq})$ is totally ordered field [5].

The numbers $x\dot{>}\dot{0}$ are $\alpha-$positive numbers and the numbers $x\dot{<}\dot{0}$ are $\alpha-$negative numbers in $\mathbb{R}_\alpha$. $\alpha-$integers are obtained by successive $\alpha-$addition of $\dot{1}$ to $\dot{0}$ and successive $\alpha-$subtraction of $\dot{1}$ from $\dot{0}$. For each integer n, we set $\dot{n} = \alpha(n)$.

$\alpha-$absolute value of a number $x \in \mathbb{R}_\alpha$ is defined by

$$|x|_\alpha = \alpha\left(\left|\alpha^{-1}(x)\right|\right) = \begin{cases} x & if \quad x\dot{>}\dot{0} \\ \dot{0} & if \quad x = \dot{0} \\ \dot{0}\dot{-}x & if \quad x\dot{<}\dot{0} \end{cases}.$$

For $x \in \mathbb{R}_\alpha$, $\sqrt[p]{x}^{\alpha} = \alpha\left(\sqrt[p]{\alpha^{-1}(x)}\right)$ and $x^{p_\alpha} = \alpha\left\{\left[\alpha^{-1}(x)\right]^p\right\}$.

Grossman and Katz described the *-calculus with the help of two arbitrary selected generators. In this paper, we study according to *-calculus. Let take any generators α and β and let * ("star") is shown the ordered pair of arithmetics ($\alpha-$arithmetic, $\beta-$arithmetic). The following notations will be used.

	$\alpha-$arithmetic	$\beta-$ arithmetic
Realm	$A\,(=\mathbb{R}_\alpha)$	$B\,(=\mathbb{R}_\beta)$
Summation	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}$	$\ddot{/}$
Ordering	$\dot{<}$	$\ddot{<}$

In the $*-$calculus, $\alpha-$arithmetic is used on arguments and $\beta-$arithmetic is used on values.

The isomorphism from $\alpha-$arithmetic to $\beta-$arithmetic is the unique function $\imath$(iota) that possesses the following three properties.

1. $\imath$ is one-to-one.
2. $\imath$ is on A and onto B.
3. For any numbers u and v in A,

$$\begin{aligned}
\iota\left(u\dot{+}v\right) &= \iota(u)\ddot{+}\iota(v), \ \iota\left(u\dot{-}v\right) = \iota(u)\ddot{-}\iota(v), \\
\iota\left(u\dot{\times}v\right) &= \iota(u)\ddot{\times}\iota(v), \ \iota\left(u\dot{/}v\right) = \iota(u)\ddot{/}\iota(v), v \neq \dot{0} \\
u \ &\dot{<} \ v \Longleftrightarrow \iota(u)\ddot{<}\iota(v).
\end{aligned}$$

It turns out that $\iota(x) = \beta\left\{\alpha^{-1}(x)\right\}$ for every number x in A and that $\iota(\dot{n}) = \ddot{n}$ for every integer n [1].

In non-Newtonian metric space, the definitions of $\alpha-$accumulation point of a set, $\alpha-$convergence of a sequence and $\alpha-$bounded sequence have been given in the studies which are numbered[5, 10]. The definitions of *-limit and *-continuity of the function $f : X \subset \mathbb{R}_\alpha \to \mathbb{R}_\beta$ have been introduced by Sağır and Erdogan[10]. Duyar and Erdogan introduced $\alpha-$series and its $\alpha-$convergence[7].

Let X be a vector space over the field $\mathbb{R}_\alpha$ and $\|.\|_{X,\alpha}$ be a function from X to $\mathbb{R}_\alpha^+ \cup \{\dot{0}\}$ satisfying the following non-Newtonian norm axioms. For $x, y \in X$ and $\lambda \in \mathbb{R}_\alpha$,

(NN1) $\|x\|_{X,\alpha} = \dot{0} \Leftrightarrow x = \dot{0}$,

(NN2) $\left\|\lambda \dot{\times} x\right\|_{X,\alpha} = |\lambda|_\alpha \dot{\times} \|x\|_{X,\alpha}$,

(NN3) $\left\|x \dot{+} y\right\|_{X,\alpha} \dot{\leq} \|x\|_{X,\alpha} \dot{+} \|y\|_{X,\alpha}$.

Then $\left(X, \|.\|_{X,\alpha}\right)$ is said to be a non-Newtonian normed space.

The non-Newtonian sequence spaces S_α, $\ell_{\infty,\alpha}$, c_α, $c_{0,\alpha}$ and $\ell_{p,\alpha}$ over the non-Newtonian real field $\mathbb{R}_\alpha$ are defined as following:

$$S_\alpha = \{x = (x_k) : \forall k \in \mathbb{N}, x_k \in \mathbb{R}_\alpha\}$$

$$\ell_{\infty,\alpha} = \left\{x = (x_k) \in S_\alpha : \ {}_\alpha\sup_{k\in\mathbb{N}} |x_k|_\alpha \dot{<} \dot{+}\infty\right\},$$

$$c_\alpha = \left\{x = (x_k) \in S_\alpha : \exists l \in \mathbb{R}_\alpha \ni \ {}^\alpha\lim_{k\to\infty} \left|x_k \dot{-} l\right|_\alpha = \dot{0}\right\},$$

$$c_{0,\alpha} = \left\{x = (x_k) \in S_\alpha : \ {}^\alpha\lim_{k\to\infty} |x_k|_\alpha = \dot{0}\right\},$$

$$\ell_{p,\alpha} = \left\{x = (x_k) \in S_\alpha : \ {}_\alpha\sum_{k=1}^{\infty} |x_k|_\alpha^{p_\alpha} \dot{<} \dot{+}\infty\right\} \ (1 \leq p < \infty).$$

The sequence spaces $\ell_{\infty,\alpha}$, c_α, $c_{0,\alpha}$ are non-Newtonian normed spaces with the non-Newtonian norm $\|.\|_{\ell_{\infty,\alpha}}$ which is defined as $\|x\|_{\ell_{\infty,\alpha}} = {}^\alpha\sup_{k\in\mathbb{N}} |x_k|_\alpha$ and the sequence space $\ell_{p,\alpha}$ is a non-Newtonian normed space with the non-Newtonian norm $\|.\|_{\ell_{p,\alpha}}$ which is defined as $\|x\|_{\ell_{p,\alpha}} = \left({}_\alpha\sum_{k=1}^{\infty} |x_k|_\alpha^{p_\alpha}\right)^{\left(\frac{1}{p}\right)_\alpha}$ [5]. The $\alpha-$sequence $e_n^{(k)}$ is defined as $e_n^{(k)} = \begin{cases} \dot{1}, & k = n \\ \dot{0}, & k \neq n \end{cases}$.

Let S_N be space of non-Newtonian real number sequences, X_α be a sequence space on $\mathbb{R}_\alpha$ and Y_β be a sequence space on $\mathbb{R}_\beta$. A non-Newtonian superposition operator ${}_N P_f$ on X_α is a mapping from X_α into S_N defined by ${}_N P_f(x) = (f(k, x_k))_{k=1}^{\infty}$ where $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ satisfies condition (NA_1) as follows;

(NA_1) $f(k, \dot{0}) = \ddot{0}$ for all $k \in \mathbb{N}$.

If ${}_N P_f(x) \in Y_\beta$ for all $x = (x_k) \in X_\alpha$, we say that ${}_N P_f$ acts from X_α into Y_β and write ${}_N P_f : X_\alpha \to Y_\beta$ [9].

Also, we shall assume the following conditions:

(NA_2) $f(k, .)$ is *-continuous for all $k \in \mathbb{N}$.

(NA_2') $f(k, .)$ is $\beta-$bounded on every $\alpha-$bounded subset of $\mathbb{R}_\alpha$ for all $k \in \mathbb{N}$.

Sağır and Erdoğan [9] have characterized the non-Newtonian superposition operators ${}_N P_f$ on $c_{0,\alpha}$ and c_α as the following.

Theorem 1.1. *Let $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ satisfies the condition (NA_2'). Then ${}_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$ if and only if there exist a $\alpha-$number $\mu \dot{>} \dot{0}$ and a $\beta-$sequence $(c_k) \in \ell_{1,\beta}$ such that $|f(k,t)|_\beta \ddot{\leq} c_k$ when $|t|_\alpha \dot{\leq} \mu$ for all $k \in \mathbb{N}$.*

Theorem 1.2. *Let $f:\mathbb{N}\times\mathbb{R}_\alpha\to\mathbb{R}_\beta$ satisfies the condition (NA_2'). Then ${}_N P_f : c_\alpha \to \ell_{1,\beta}$ if and only if there exist a $\alpha-$number $\mu\dot{>}\dot{0}$ and a $\beta-$sequence $(c_k)\in\ell_{1,\beta}$ such that $|f(k,t)|_\beta \ddot{\leq} c_k$ when $\left|t\dot{-}z\right|_\alpha \dot{\leq}\mu$ for all $z\in\mathbb{R}_\alpha$ and for all $k\in\mathbb{N}$.*

2. Main Results

Definition 2.1. Let (X_α, d_α) and $\left(Y_\beta, d_\beta'\right)$ be non-Newtonian sequence spaces. An operator $F:X_\alpha\to Y_\beta$ is *-bounded if $F(A)$ is $\beta-$bounded for every $\alpha-$bounded subset A of X_α .

Definition 2.2. Let (X_α, d_α) and $\left(Y_\beta, d_\beta'\right)$ be non-Newtonian sequence spaces. An operator $F:X_\alpha\to Y_\beta$ is *-locally bounded at $x_0\in X_\alpha$ if there exist $\alpha-$number $\mu\dot{>}\dot{0}$ and $\beta-$number $\eta\ddot{>}\ddot{0}$ such that $F(x)\in B_{d_\beta'}\left[F(x_0),\eta\right]$ for $x\in B_{d_\alpha}[x_0,\mu]$. F is *-locally bounded if it is *-locally bounded for every $x\in X_\alpha$.

Theorem 2.3. *Let (X_α, d_α) and $\left(Y_\beta, d_\beta'\right)$ be non-Newtonian metric sequence spaces. An operator $F:X_\alpha\to Y_\beta$ is *-locally bounded if F is *-bounded.*

Proof. Let $x\in X_\alpha$ with $x\in B_{d_\alpha}[x_0,\mu]$ for $x_0\in X_\alpha$ and $\mu\dot{>}\dot{0}$. Since F is *-bounded, $F\left(B_{d_\alpha}[x_0,\mu]\right)$ is $\beta-$bounded set. Then there exists a $\beta-$number $\eta\ddot{>}\ddot{0}$ such that $d_\beta'\left(F(x),F(x_0)\right)\ddot{\leq}\eta$. So we obtain that $F(x)\in B_{d_\beta'}\left[F(x_0),\eta\right]$. Thus F is *-locally bounded at $x_0\in X_\alpha$. □

Corollary 2.4. Let X_α be an $\alpha-$sequence space. $F:X_\alpha\to\ell_{1,\beta}$ is *-locally bounded if F is *-bounded.

Theorem 2.5. *If the function $f:\mathbb{N}\times\mathbb{R}_\alpha\to\mathbb{R}_\beta$ is *-locally bounded, it is satisfies the condition (NA_2').*

Proof. Let A be an $\alpha-$bounded subset of $\mathbb{R}_\alpha$. Then there exists $\dot{[}a,b\dot{]}\subset\mathbb{R}_\alpha$ such that $A\subset\dot{[}a,b\dot{]}$. Let $c\in\dot{[}a,b\dot{]}$. Since f is *-locally bounded, there exists $\delta_c\dot{>}\dot{0}$ and $\gamma_c\ddot{>}\ddot{0}$ such that

$$\left|f(x)\ddot{-}f(c)\right|_\beta\ddot{\leq}\gamma_c \text{ with } \left|x\dot{-}c\right|_\alpha\dot{\leq}\delta_c\ .$$

Then it is written that $f(x)\in B_\beta\left[f(c),\gamma_c\right]$ for $x\in B_\alpha[c,\delta_c]$. Since

$$\left||f(x)|_\beta\ddot{-}|f(c)|_\beta\right|_\beta\ddot{\leq}\left|f(x)\ddot{-}f(c)\right|_\beta\ddot{\leq}\gamma_c\ ,$$

we get

$$|f(x)|_\beta\ddot{\leq}\gamma_c\ddot{+}|f(c)|_\beta$$

when $x\in B_\alpha[c,\delta_c]$. Every $\alpha-$closed interval $\dot{[}a,b\dot{]}$ on $\mathbb{R}_\alpha$ is $\alpha-$compact by *-Heine Borel Theorem in [9]. Then there exist $c_1,c_2,...,c_n\in\dot{[}a,b\dot{]}$ such that $\dot{[}a,b\dot{]}\subset\bigcup_{k=1}^{n}B_\alpha[c_k,\delta_{c_k}]$, since $\dot{[}a,b\dot{]}\subset\bigcup_{c\in\dot{[}a,b\dot{]}}B_\alpha[c,\delta_c]$. So we have $|f(x)|_\beta\ddot{\leq}\iota(c_k)\ddot{+}|f(c_k)|_\beta$ for each $x\in B_\alpha[c_k,\delta_{c_k}]$ where $1\leq k\leq n$. If $M={}^{\beta}\max\left\{\iota(c_k)\ddot{+}|f(c_k)|_\beta : 1\leq k\leq n\right\}$, then $|f(x)|_\beta\ddot{\leq}M$ for $x\in\bigcup_{k=1}^{n}B_\alpha[c_k,\delta_{c_k}]$. Since $A\subset\dot{[}a,b\dot{]}\subset\bigcup_{k=1}^{n}B_\alpha[c_k,\delta_{c_k}]$, we get $|f(x)|_\beta\ddot{\leq}M$ for $x\in A$. □

Theorem 2.6. *Let $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$. Then the non-Newtonian superposition operator ${}_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$ is *-locally bounded if and only if f satisfies the condition (NA'_2).*

Proof. Assume that f satisfies the condition (NA'_2). Let $z = (z_k) \in c_{0,\alpha}$. Since ${}_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$ and f satisfies (NA'_2), by Theorem 1.1, there exist $\mu \dot{>} \dot{0}$ and $(c_k) \in \ell_{1,\beta}$ such that

$$|f(k,t)|_\beta \ddot{\leq} c_k \text{ whenever } |t|_\alpha \dot{\leq} \mu \tag{2.1}$$

for all $k \in \mathbb{N}$. Let $\varphi = \frac{\mu}{\dot{2}}\alpha$ and $x \in c_{0,\alpha}$ such that $\left\| x \dot{-} z \right\|_{c_{0,\alpha}} \dot{\leq} \varphi$. Since ${}^\alpha \lim_{k\to\infty} |z_k|_\alpha = \dot{0}$, there exists a positive integer r such that $|z_k|_\alpha \dot{\leq} \varphi$ for all $k \geq r$. Then

$$\|z_\lambda\|_{c_{0,\alpha}} = {}^\alpha \sup_{k\geq r} |z_k|_\alpha \dot{\leq} \varphi \tag{2.2}$$

for $\lambda \in \{r, r+1, ...\}$. Since $\left\| x \dot{-} z \right\|_{c_{0,\alpha}} \dot{\leq} \varphi$, we get that

$$ {}^\alpha \sup_k \left| x_k \dot{-} z_k \right|_\alpha \dot{\leq} \varphi \tag{2.3}$$

By (2.2) and (2.3), it is written that

$$\begin{aligned}
|x_k|_\alpha &\dot{\leq} {}^\alpha \sup_{n\geq r} |x_n|_\alpha \\
&= {}^\alpha \sup_{n\geq r} \left| x_n \dot{-} z_n \dot{+} z_n \right|_\alpha \\
&\dot{\leq} {}^\alpha \sup_{n\geq r} \left| x_n \dot{-} z_n \right|_\alpha \dot{+} {}^\alpha \sup_{n\geq r} |z_n|_\alpha \\
&\dot{\leq} \varphi \dot{+} \varphi \\
&= \mu
\end{aligned}$$

for all $k \geq r$. From (2.1), we have $|f(k,x_k)|_\beta \ddot{\leq} c_k$ for all $k \geq r$. Then

$$ {}_\beta \sum_{k=r}^{\infty} |f(k,x_k)|_\beta \ddot{\leq} {}_\beta \sum_{k=r}^{\infty} c_k = {}_\beta \sum_{k=r}^{\infty} |c_k|_\beta \ddot{\leq} {}_\beta \sum_{k=1}^{\infty} |c_k|_\beta = \|(c_k)\|_{\ell_{1,\beta}} . \tag{2.4}$$

Let $m_k = {}^\beta \sup_{\left|t \dot{-} z_k\right|_\alpha \dot{\leq} \varphi} |f(k,t)|_\beta$ for all $k \in \mathbb{N}$. Since f satisfies the condition (NA'_2), it is seen that $m_k \ddot{<} \ddot{+}\infty$ for all $k \in \mathbb{N}$. So we get $\left| x_k \dot{-} z_k \right|_\alpha \dot{\leq} \varphi$ for all $k \in \mathbb{N}$ by (2.3). Then we have

$$|f(k,x_k)|_\beta \ddot{\leq} m_k \tag{2.5}$$

for all $k \in \mathbb{N}$. Using the relations (2.4) and (2.5), it is obtained that

$$\begin{aligned}
\|{}_N P_f(x)\|_{\ell_{1,\beta}} &= {}_\beta \sum_{k=1}^{\infty} |f(k,x_k)|_\beta \\
&= {}_\beta \sum_{k=1}^{r-1} |f(k,x_k)|_\beta \ddot{+} {}_\beta \sum_{k=r}^{\infty} |f(k,x_k)|_\beta \\
&\ddot{\leq} {}_\beta \sum_{k=1}^{r-1} m_k \ddot{+} \|(c_k)\|_{\ell_{1,\beta}} .
\end{aligned}$$

Then we have

$$\begin{aligned}\left\|{}_N P_f(x) \ddot{-} {}_N P_f(z)\right\|_{\ell_{1,\beta}} &\ddot{\leq} \left\|{}_N P_f(x)\right\|_{\ell_{1,\beta}} \ddot{+} \left\|{}_N P_f(z)\right\|_{\ell_{1,\beta}} \\ &\ddot{\leq} {}_\beta\sum_{k=1}^{r-1} m_k \ddot{+} \left\|(c_k)\right\|_{\ell_{1,\beta}} \ddot{+} \left\|{}_N P_f(z)\right\|_{\ell_{1,\beta}}.\end{aligned}$$

Therefore we get that

$$\left\|{}_N P_f(x) \ddot{-} {}_N P_f(z)\right\|_{\ell_{1,\beta}} \ddot{\leq} \gamma \text{ when } \gamma = \left\|{}_N P_f(z)\right\|_{\ell_{1,\beta}} \ddot{+} {}_\beta\sum_{k=1}^{r-1} m_k \ddot{+} \left\|(c_k)\right\|_{\ell_{1,\beta}}.$$

Hence, the non-Newtonian operator ${}_N P_f$ *-locally bounded at z.

Conversely assume that ${}_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$ is *-locally bounded. Let $k \in \mathbb{N}$ and $b \in \mathbb{R}_\alpha$. Let $y = (y_n)$ be defined as $y_n = \begin{cases} b\ , & n = k \\ \dot{0}\ , & n \neq k \end{cases}$. Then $(y_n) \in c_{0,\alpha}$. By assumption, there exist $\mu \dot{>} \dot{0}$ and $\eta \ddot{>} \ddot{0}$ such that

$$\left\|{}_N P_f(x) \ddot{-} {}_N P_f(y)\right\|_{\ell_{1,\beta}} \ddot{\leq} \eta \text{ whenever } \left\|x \dot{-} y\right\|_{c_{0,\alpha}} \dot{\leq} \mu. \tag{2.6}$$

Let $a \in \mathbb{R}_\alpha$ with $\left|a \dot{-} b\right|_\alpha \dot{\leq} \mu$ and let $x = (x_n)$ with $x_n = \begin{cases} a\ , & n = k \\ \dot{0}\ , & n \neq k \end{cases}$. Then $x \in c_{0,\alpha}$. Since

$$\left\|x \dot{-} y\right\|_{c_{0,\alpha}} = {}^\alpha\sup_n \left|x_n \dot{-} y_n\right|_\alpha = \left|a \dot{-} b\right|_\alpha \dot{\leq} \mu,$$

we get $\left\|{}_N P_f(x) \ddot{-} {}_N P_f(y)\right\|_{\ell_{1,\beta}} \ddot{\leq} \eta$ by (2.6). Then we have

$$\begin{aligned}\left|f(k,a) \ddot{-} f(k,b)\right|_\beta &\ddot{\leq} {}_\beta\sum_{n=1}^{\infty} \left|f(n,x_n) \ddot{-} f(n,y_n)\right|_\beta \\ &= \left\|{}_N P_f(x) \ddot{-} {}_N P_f(y)\right\|_{\ell_{1,\beta}} \\ &\ddot{\leq} \eta\end{aligned}$$

Hence $f(k,.)$ is *-locally bounded at b. Since $b \in \mathbb{R}_\alpha$ is arbitrary, $f(k,.)$ is *-locally bounded. Thus $f(k,.)$ satisfies the condition (NA_2'). □

Corollary 2.7. Let $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ satisfies the condition (NA_2). The non-Newtonian superposition operator ${}_N P_f$ is *-locally bounded if ${}_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$.

Corollary 2.8. Let $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$. If ${}_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$ is *-bounded, f satisfies the condition (NA_2').

Proposition 2.9. Assume that $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ satisfies the condition (NA_2'). If for each $\mu \dot{>} \dot{0}$ there exists a $\beta-$number $\eta(\mu) \ddot{>} \ddot{0}$ such that

$${}_\beta\sum_{k=1}^{\infty} \left|f(k,x_k)\right|_\beta \ddot{\leq} \eta(\mu) \text{ whenever } \left|x_k\right|_\alpha \dot{\leq} \mu$$

for all $k \in \mathbb{N}$, then there exists a $c(\mu) = (c_k(\mu)) \in \ell_{1,\beta}$ with $c_k(\mu) \ddot{\geq} \ddot{0}$ and $\left\|c(\mu)\right\|_{\ell_{1,\beta}} \ddot{\leq} \eta(\mu)$ for all $k \in \mathbb{N}$ such that

$$\left|f(k,t)\right|_\beta \ddot{\leq} c_k(\mu) \text{ whenever } |t|_\alpha \dot{\leq} \mu\ .$$

Proof. Let $\mu \dot{>} \dot{0}$. We define

$$A(\mu) = \left\{ t \in \mathbb{R}_\alpha : |t|_\alpha \dot{\leq} \mu \right\} \text{ and } c_k(\mu) = {}^{\beta}\sup \left\{ |f(k,t)|_\beta : t \in A(\mu) \right\}$$

for all $k \in \mathbb{N}$. Then $|f(k,t)|_\beta \ddot{\leq} c_k(\mu)$ where $|t|_\alpha \dot{\leq} \mu$. Since f satisfies the condition (NA_2'), it is obtained that $\ddot{0} \ddot{\leq} c_k(\mu) \ddot{<} \ddot{+}\infty$ for all $k \in \mathbb{N}$. For each $\varepsilon \ddot{>} \ddot{0}$. there exists an α-sequence $x = (x_k)$ when $|x_k|_\alpha \dot{\leq} \mu$ such that

$$c_k(\mu) \ddot{\leq} |f(k, x_k)|_\beta \ddot{+} \frac{\varepsilon}{\ddot{2}^{k_\beta}} \beta \tag{2.7}$$

for all $k \in \mathbb{N}$. By (2.7), we have

$$_\beta\sum_{k=1}^{\infty} c_k(\mu) = {}_\beta\sum_{k=1}^{\infty} |c_k(\mu)|_\beta \ddot{\leq} {}_\beta\sum_{k=1}^{\infty} |f(k, x_k)|_\beta \ddot{+} {}_\beta\sum_{k=1}^{\infty} \frac{\varepsilon}{\ddot{2}^{k_\beta}} \beta \ddot{\leq} \eta(\mu) \ddot{+} \varepsilon \ .$$

Thus, $\|c_k(\mu)\|_{\ell_{1,\beta}} \ddot{\leq} \eta(\mu) \ddot{+} \varepsilon$. Since ε is arbitrary, it is written that $\|c(\mu)\|_{\ell_{1,\beta}} \ddot{\leq} \eta(\mu)$ with $c(\mu) = (c_k(\mu))$. So there exists a β-sequence $c(\mu) = (c_k(\mu)) \in \ell_{1,\beta}$ with $c_k(\mu) \ddot{\geq} \ddot{0}$ and $\|c(\mu)\|_{\ell_{1,\beta}} \ddot{\leq} \eta(\mu)$ such that $|f(k,t)|_\beta \ddot{\leq} c_k(\mu)$ whenever $|t|_\alpha \dot{\leq} \mu$ for each $k \in \mathbb{N}$. □

Theorem 2.10. *Let $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$. The non-Newtonian superposition operator $_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$ is *-bounded if and only if for all $\mu \dot{>} \dot{0}$ there exists a β-sequence $c(\mu) = (c_k(\mu)) \in \ell_{1,\beta}$ such that*

$$|f(k,t)|_\beta \ddot{\leq} c_k(\mu) \ \textit{whenever} \ |t|_\alpha \dot{\leq} \mu$$

for each $k \in \mathbb{N}$.

Proof. Let $x \in c_{0,\alpha}$ and $\mu \dot{>} \dot{0}$ with $\|x\|_{c_{0,\alpha}} \ddot{\leq} \mu$. Then $|x_k|_\alpha \dot{\leq} \mu$ for all $k \in \mathbb{N}$. By the hypothesis, there exists a β-sequence $c(\mu) = (c_k(\mu)) \in \ell_{1,\beta}$ such that $|f(k,t)|_\beta \ddot{\leq} c_k(\mu)$ for each $k \in \mathbb{N}$. Then

$$\|_N P_f(x)\|_{\ell_{1,\beta}} = {}_\beta\sum_{k=1}^{\infty} |f(k, x_k)|_\beta \ddot{\leq} {}_\beta\sum_{k=1}^{\infty} c_k(\mu) = {}_\beta\sum_{k=1}^{\infty} |c_k(\mu)|_\beta = \|c(\mu)\|_{\ell_{1,\beta}} \ .$$

Thus, $_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$ is *-bounded.

Conversely, assume that $_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$ is *-bounded. Let $\mu \dot{>} \dot{0}$. Then for each $x \in c_{0,\alpha}$ with $\|x\|_{c_{0,\alpha}} \ddot{\leq} \mu$, it is obtained that

$$\|_N P_f(x)\|_{\ell_{1,\beta}} = {}_\beta\sum_{k=1}^{\infty} |f(k, x_k)|_\beta \ddot{\leq} \eta(\mu) \ddot{<} \ddot{+}\infty$$

for a β-positive integer $\eta(\mu)$. By Corollary 2.8, f satisfies the condition (NA_2'). In view of Proposition 2.9, there exists a β-sequence $c(\mu) = (c_k(\mu)) \in \ell_{1,\beta}$ with $\|c(\mu)\|_{\ell_{1,\beta}} \ddot{\leq} \eta(\mu)$ such that $|f(k,t)|_\beta \ddot{\leq} c_k(\mu)$ whenever $|t|_\alpha \dot{\leq} \mu$ for each $k \in \mathbb{N}$. □

Example 2.11. Let function $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ be defined by $f(k,t) = \dfrac{|\iota(t)|_\beta}{\ddot{5}^{k_\beta}} \beta$ for all $k \in \mathbb{N}$ and $t \in \mathbb{R}_\alpha$. Since there exist $\gamma = \dot{1}$ and $(c_k) = \left(\dfrac{\ddot{1}}{\ddot{5}^{k_\beta}} \beta \right) \in \ell_{1,\beta}$ such that $|f(k,t)|_\beta \ddot{\leq} c_k$ whenever $|t|_\alpha \dot{\leq} \dot{1}$ for each $k \in \mathbb{N}$, the non-Newtonian superposition operator ${}_{N}P_f$ acts from $c_{0,\alpha}$ to $\ell_{1,\beta}$. Let $\mu\dot{>}\dot{0}$ and $t\in\mathbb{R}_\alpha$ with $|t|_\alpha\dot{\leq}\mu$. Then, for all $k\in\mathbb{N}$

$$|f(k,t)|_\beta=\frac{|\iota(t)|_\beta}{\ddot{5}^{k_\beta}}\beta\ddot{\leq}\frac{\iota(\mu)}{\ddot{5}^{k_\beta}}\beta \text{ and } {}_{\beta}\sum_{k=1}^{\infty}\frac{\iota(\mu)}{\ddot{5}^{k_\beta}}\beta=\left(\frac{\iota(\mu)}{\ddot{4}^{k_\beta}}\beta\right).$$

Hence we obtaine that $|f(k,t)|_\beta\ddot{\leq}c_k(\mu)$ whenever $(c_k(\mu))=\left(\frac{\iota(\mu)}{\ddot{5}^{k_\beta}}\beta\right)\in\ell_{1,\beta}$ for all $k\in\mathbb{N}$. Then, ${}_{N}P_f:c_{0,\alpha}\to\ell_{1,\beta}$ is *-bounded by Theorem 2.10.

Theorem 2.12. *Let $f:\mathbb{N}\times\mathbb{R}_\alpha\to\mathbb{R}_\beta$. The non-Newtonian superposition operator ${}_{N}P_f:c_\alpha\to\ell_{1,\beta}$ is *-locally bounded if and only if f satisfies the condition (NA_2').*

Proof. Assume that f satisfies the condition (NA_2'). Let $z=(z_k)\in c_\alpha$. By Theorem 1.2 there exist $\mu\dot{>}\dot{0}$ and $(c_k)\in\ell_{1,\beta}$ such that

$$|f(k,t)|_\beta\ddot{\leq}c_k \text{ whenever } \left|t\dot{-}a\right|_\alpha\dot{\leq}\mu \tag{2.8}$$

for each $a\in\mathbb{R}_\alpha$ and for all $k\in\mathbb{N}$. Let $\eta\dot{>}\dot{0}$ and $x\in c_\alpha$ with $\left\|x\dot{-}z\right\|_{c,\alpha}\dot{\leq}\eta$. Since $x\in c_\alpha$, there exists $a\in\mathbb{R}_\alpha$ such that

$${}^{\alpha}\lim_{k\to\infty}\left|x_k\dot{-}a\right|_\alpha=\dot{0}\ . \tag{2.9}$$

From (2.8), there exist a $\rho\dot{>}\dot{0}$ and a $(c_k)\in\ell_{1,\beta}$ such that

$$|f(k,t)|_\beta\ddot{\leq}c_k \text{ whenever } \left|t\dot{-}a\right|_\alpha\dot{\leq}\rho \tag{2.10}$$

for all $k\in\mathbb{N}$. By (2.9), there exists $i\in\mathbb{N}$

$$\left|x_k\dot{-}a\right|_\alpha\dot{\leq}\rho \tag{2.11}$$

for all $k\geq i$. By (2.10) and (2.11), we obtaine that $|f(k,x_k)|_\beta\ddot{\leq}c_k$ for all $k\geq i$. Then

$${}_{\beta}\sum_{k=i}^{\infty}|f(k,x_k)|_\beta\ddot{\leq}{}_{\beta}\sum_{k=i}^{\infty}c_k={}_{\beta}\sum_{k=i}^{\infty}|c_k|_\beta\ddot{\leq}{}_{\beta}\sum_{k=1}^{\infty}|c_k|_\beta=\|c_k\|_{\ell_{1,\beta}} \tag{2.12}$$

Let $m_k={}^{\beta}\sup_{\left|t\dot{-}z_k\right|_\alpha\dot{\leq}\eta}|f(k,t)|_\beta$ for each $k\in\mathbb{N}$. Since f satisfies the condition (NA_2'), $m_k\ddot{<}\ddot{+}\infty$ for all $k\in\mathbb{N}$. Since $\left\|x\dot{-}z\right\|_{c,\alpha}\dot{\leq}\eta$, we have that $\left|x_k\dot{-}z_k\right|_\alpha\dot{\leq}\eta$ for all $k\in\mathbb{N}$. Then, for all $k\in\mathbb{N}$

$$|f(k,x_k)|_\beta\ddot{\leq}m_k \tag{2.13}$$

By (2.12) and (2.13),

$$\begin{aligned}\|{}_{N}P_f(x)\|_{\ell_{1,\beta}} &= {}_{\beta}\sum_{k=1}^{\infty}|f(k,x_k)|_\beta\\ &= {}_{\beta}\sum_{k=1}^{i-1}|f(k,x_k)|_\beta\ddot{+}{}_{\beta}\sum_{k=i}^{\infty}|f(k,x_k)|_\beta\\ &\ddot{\leq} {}_{\beta}\sum_{k=1}^{i-1}m_k\ddot{+}\|(c_k)\|_{\ell_{1,\beta}}\ .\end{aligned}$$

Then

$$\left\|{}_{N}P_{f}\left(x\right)\ddot{-}{}_{N}P_{f}\left(z\right)\right\|_{\ell_{1,\beta}} \ddot{\leq} \left\|{}_{N}P_{f}\left(x\right)\right\|_{\ell_{1,\beta}}\ddot{+}\left\|{}_{N}P_{f}\left(z\right)\right\|_{\ell_{1,\beta}}$$
$$\ddot{\leq} {}_{\beta}\sum_{k=1}^{i-1} m_{k}\ddot{+}\left\|\left(c_{k}\right)\right\|_{\ell_{1,\beta}}\ddot{+}\left\|{}_{N}P_{f}\left(z\right)\right\|_{\ell_{1,\beta}} .$$

Therefore we have that

$$\left\|{}_{N}P_{f}\left(x\right)\ddot{-}{}_{N}P_{f}\left(z\right)\right\|_{\ell_{1,\beta}}\ddot{\leq}\gamma \text{ when } \gamma=\left\|{}_{N}P_{f}\left(z\right)\right\|_{\ell_{1,\beta}}\ddot{+}{}_{\beta}\sum_{k=1}^{i-1}m_{k}\ddot{+}\left\|\left(c_{k}\right)\right\|_{\ell_{1,\beta}}.$$

Hence ${}_{N}P_{f}$ *-locally bounded at z.

Conversely, assume that ${}_{N}P_{f}:c_{\alpha}\rightarrow\ell_{1,\beta}$ is *-locally bounded. Let $k\in\mathbb{N}$ and $b\in\mathbb{R}_{\alpha}$. Let $y=(y_{n})$ be as follows

$$y_{n}=\begin{cases} b\ , & n=k \\ \dot{0}\ , & n\neq k\end{cases}$$

for all $k\in\mathbb{N}$ and $b\in\mathbb{R}_{\alpha}$. Then $y\in c_{\alpha}$. By the hypothesis, there exist $\mu\dot{>}\dot{0}$ and $\varphi\ddot{>}\ddot{0}$ such that

$$\left\|{}_{N}P_{f}\left(x\right)\ddot{-}{}_{N}P_{f}\left(y\right)\right\|_{\ell_{1,\beta}}\ddot{\leq}\varphi \text{ whenever } \left\|x\dot{-}y\right\|_{c,\alpha}\dot{\leq}\mu . \tag{2.14}$$

Let $a\in\mathbb{R}_{\alpha}$ with $\left|a\dot{-}b\right|_{\alpha}\dot{\leq}\mu$ and $x=(x_{n})$ with $x_{n}=\begin{cases} a\ , & n=k \\ \dot{0}\ , & n\neq k\end{cases}$. Then $x\in c_{\alpha}$. Since

$$\left\|x\dot{-}y\right\|_{c,\alpha}={}^{\alpha}\sup_{n}\left|x_{n}\dot{-}y_{n}\right|_{\alpha}=\left|a\dot{-}b\right|_{\alpha}\dot{\leq}\mu,$$

by virtue of (2.14), it is written that $\left\|{}_{N}P_{f}\left(x\right)\ddot{-}{}_{N}P_{f}\left(y\right)\right\|_{\ell_{1,\beta}}\ddot{\leq}\varphi$. Then we have

$$\left|f\left(k,a\right)\ddot{-}f\left(k,b\right)\right|_{\beta} \ddot{\leq} {}_{\beta}\sum_{n=1}^{\infty}\left|f\left(n,x_{n}\right)\ddot{-}f\left(n,y_{n}\right)\right|_{\beta}$$
$$= \left\|{}_{N}P_{f}\left(x\right)\ddot{-}{}_{N}P_{f}\left(y\right)\right\|_{\ell_{1,\beta}}$$
$$\ddot{\leq} \varphi$$

Therefore $f\left(k,.\right)$ is *-locally bounded at b. Since $b\in\mathbb{R}_{\alpha}$ is arbitrary, $f\left(k,.\right)$ is *-locally bounded. Hence $f\left(k,.\right)$ satisfies the condition (NA_{2}'). □

Corollary 2.13. Let the function $f:\mathbb{N}\times\mathbb{R}_{\alpha}\rightarrow\mathbb{R}_{\beta}$ satisfy the condition (NA_{2}). Then ${}_{N}P_{f}:c_{\alpha}\rightarrow\ell_{1,\beta}$ is *-locally bounded.

Corollary 2.14. Let $f:\mathbb{N}\times\mathbb{R}_{\alpha}\rightarrow\mathbb{R}_{\beta}$. If ${}_{N}P_{f}:c_{\alpha}\rightarrow\ell_{1,\beta}$ is *-bounded, f satisfies the condition (NA_{2}').

Theorem 2.15. *Let $f:\mathbb{N}\times\mathbb{R}_{\alpha}\rightarrow\mathbb{R}_{\beta}$. ${}_{N}P_{f}:c_{\alpha}\rightarrow\ell_{1,\beta}$ is *-bounded if and only if for every $\mu\dot{>}\dot{0}$ there exists a sequence $c\left(\mu\right)=\left(c_{k}\left(\mu\right)\right)\in\ell_{1,\beta}$ such that*

$$\left|f\left(k,t\right)\right|_{\beta}\ddot{\leq}c_{k}\left(\mu\right) \text{ whenever } \left|t\right|_{\alpha}\dot{\leq}\mu$$

for all $k\in\mathbb{N}$.

Proof. Let $\mu \dot{>} \dot{0}$ and $x \in c_\alpha$ with $\|x\|_{c,\alpha} \dot{\leq} \mu$. Then $|x_k|_\alpha \dot{\leq} \mu$ for all $k \in \mathbb{N}$. By the hypothesis, for each $k \in \mathbb{N}$ there exists a sequence $c(\mu) = (c_k(\mu)) \in \ell_{1,\beta}$ such that $|f(k, x_k)|_\beta \ddot{\leq} c_k(\mu)$. Then it is written that

$$\|{}_N P_f(x)\|_{\ell_{1,\beta}} = {}_\beta\sum_{k=1}^{\infty} |f(k, x_k)|_\beta \ddot{\leq} {}_\beta\sum_{k=1}^{\infty} c_k(\mu) = {}_\beta\sum_{k=1}^{\infty} |c_k(\mu)|_\beta = \|c(\mu)\|_{\ell_{1,\beta}} .$$

Thus ${}_N P_f : c_\alpha \to \ell_{1,\beta}$ is *-bounded.

Conversely, assume that ${}_N P_f : c_\alpha \to \ell_{1,\beta}$ is *-bounded. Let $\mu \dot{>} \dot{0}$. There exists a positive $\beta-$number $\eta(\mu)$ such that

$$\|{}_N P_f(x)\|_{\ell_{1,\beta}} = {}_\beta\sum_{k=1}^{\infty} |f(k, x_k)|_\beta \ddot{\leq} \eta(\mu)$$

for each $x \in c_\alpha$ with $\|x\|_{c,\alpha} \dot{\leq} \mu$. From Corollary 2.13, f satisfies the condition (NA_2'). By Proposition 2.9, there exists a β-sequence $c(\mu) = (c_k(\mu)) \in \ell_{1,\beta}$ with $\|c(\mu)\|_{\ell_{1,\beta}} \ddot{\leq} \eta(\mu)$ such that $|f(k,t)|_\beta \ddot{\leq} c_k(\mu)$ whenever $|t|_\alpha \dot{\leq} \mu$ for all $k \in \mathbb{N}$. □

Example 2.16. Let $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ be as follows

$$f(k,t) = \frac{(\iota(t))^{2_\beta}}{\ddot{5}^{k_\beta}} \beta$$

for all $k \in \mathbb{N}$. Let $\mu \dot{>} \dot{0}$ and $t \in \mathbb{R}_\alpha$ with $|t|_\alpha \dot{\leq} \mu$. Then

$$|f(k,t)|_\beta = \frac{(\iota(t))^{2_\beta}}{\ddot{5}^{k_\beta}} \beta \ddot{\leq} \frac{(\iota(\mu))^{2_\beta}}{\ddot{5}^{k_\beta}} \beta$$

for each $k \in \mathbb{N}$. Since

$${}_\beta\sum_{k=1}^{\infty} \frac{(\iota(\mu))^{2_\beta}}{\ddot{5}^{k_\beta}} \beta = (\iota(\mu))^{2_\beta} \ddot{\times} {}_\beta\sum_{k=1}^{\infty} \frac{\ddot{1}}{\ddot{5}^{k_\beta}} \beta = (\iota(\mu))^{2_\beta} \ddot{\times} \frac{\ddot{1}}{\ddot{5}} \beta \ddot{\times} \frac{\ddot{1}}{\ddot{1} \ddot{-} \frac{\ddot{1}}{\ddot{5}} \beta} \beta = \frac{(\iota(\mu))^{2_\beta}}{\ddot{4}} \beta \ddot{<} \ddot{+}\infty,$$

we have $|f(k,t)|_\beta \ddot{\leq} c_k$ when $c_k = \dfrac{(\iota(\mu))^{2_\beta}}{\ddot{5}^{k_\beta}} \beta$ for each $k \in \mathbb{N}$. Hence ${}_N P_f : c_\alpha \to \ell_{1,\beta}$ is *-bounded by Theorem 2.15.

3. Conclusion

In this paper, the well-known boundedness and locally boundedness in classical calculus were extended to non-Newtonian calculus. Also their properties on some non-Newtonian sequence spaces were investigated.

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ON GENERALIZATIONS OF LOCALLY ARTINIAN SUPPLEMENTED MODULES

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Abstract. The aim of this paper is to investigate generalizations of locally artinian supplemented modules in module theory, namely locally artinian radical supplemented modules and strongly locally artinian radical supplemented modules. We have obtained elementary features of them. Also, we have characterized strongly locally artinian radical supplemented modules by left perfect rings. Finally, we have proved that the reduced part of a strongly locally artinian radical supplemented R-module has the same property over a Dedekind domain R.

1. Introduction

Throughout this paper, the ring R will denote an associative ring with identity element and modules will be left unital. We will use the notation $U \ll M$ to stress that U is a *small* submodule of M. $Rad(M)$ will indicate radical of M which is sum of all small submodules of M, and $Soc(M)$ will indicate socle of M which is sum of all semisimple submodules of M. A non-zero module M is called *hollow* if every proper submodule of M is small in M, and M is called *local* if the sum of all proper submodules of M is also a proper submodule of M. A module M is called *semilocal* if $\frac{M}{Rad(M)}$ is semisimple. M is called *locally artinian* if every finitely generated submodule of M is artinian [8, 31]. A submodule V of M is called *a supplement* of U in M if $M = U + V$ and $U \cap V \ll V$. The module M is called *supplemented* if every submodule of M has a supplement in M. A submodule U of M has ample supplements in M if every submodule V of M with $M = U + V$ contains a supplement $V^{'}$ of U in M. The module M is called *amply supplemented* if every submodule of M has ample supplements in M [8]. Moreover, it is called $\oplus$-supplemented if every submodule of M has a supplement in the form of a direct summand of M. Clearly, the $\oplus$-supplemented modules are supplemented.

In [10], Zöschinger introduced a notion of modules with radical which has supplements and called them *radical supplemented*. In the same paper and in [12],

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the structure of radical supplemented modules is determined. Motivated by this, Büyükaşık and Türkmen call a module M *strongly radical supplemented* (or, briefly, a srs-module) if every submodule containing the radical has a supplement [2]. In [4], it is introduced another notion of $\oplus$-radical supplemented modules. A module M is called $\oplus$-radical supplemented if $Rad(M)$ has a supplement which is a direct summand of M. In this paper, a module M is called *strongly $\oplus$-radical supplemented* provided that every submodule containing the radical has a supplement which is a direct summand of M.

In [9], a generalization of concept of socle as a $Soc_s(M) = \sum\{U \ll M \,|\, U \text{ is simple}\}$ is defined. Here $Soc_s(M) \subseteq Rad(M)$ and $Soc_s(M) \subseteq Soc(M)$. In [3], a module M is called *strongly local* if it is local and $Rad(M)$ is semisimple. A submodule U of M is called an *ss-supplement* of U in M if $M = U + V$ and $U \cap V \subseteq Soc_s(V)$. The module M is called *ss-supplemented* if every submodule of M has an ss-supplement in M. A submodule U of M has ample ss-supplements in M if every submodule V of M such that $M = U + V$ contains an ss-supplement V' of U in M. The module M is called *amply ss-supplemented* if every submodule of M has ample ss-supplements in M. In [6], strongly local and (amply) ss-supplemented modules are generalized as RLA-local and (amply) locally artinian supplemented modules, respectively. A local module M is called *RLA-local* if $Rad(M)$ is a locally artinian submodule of M. A module M is called *locally artinian supplemented* if every submodule U of M has a locally artinian supplement in M, that is, V is a supplement of U in M such that $U \cap V$ is locally artinian. M is called *amply locally artinian supplemented* if every submodule U of M has ample locally artinian supplements in M. Here a submodule U of M has ample locally artinian supplements in M if every submodule V of M such that $M = U + V$ contains a locally artinian supplement V' of U in M.

Motivated by this, we define locally artinian radical supplemented modules as a generalization of locally artinian supplemented modules and also define the concept of strongly locally artinian radical supplemented modules which is contained in the concept of locally artinian radical supplemented modules. In Section 2, it is shown that a module M with small radical is strongly locally artinian radical supplemented if and only if M is strongly radical supplemented and $Rad(M)$ is locally artinian if and only if M is locally artinian supplemented. It is also shown that every factor module of a strongly locally artinian radical supplemented module is strongly locally artinian radical supplemented. It is proved that any finite sum of strongly locally artinian radical supplemented module is strongly locally artinian radical supplemented. It is also proved that R is a left perfect ring and $Rad(M)$ is locally artinian if and only if every R-module is a strongly locally artinian radical supplemented module. Finally, it is obtained that over a Dedekind domain R, an R-module M is strongly locally artinian radical supplemented if and only if the reduced part N of M is strongly locally artinian radical supplemented.

2. Strongly Locally Artinian Radical Supplemented Modules

Definition 1. Let M be a module. Then M is called a *locally artinian radical supplemented module* if $Rad(M)$ has a locally artinian supplement in M. A module M is called *strongly locally artinian radical supplemented* if every submodule which contains $Rad(M)$ in M has a locally artinian supplement in M.

Proposition 1. Let M be a module with $Rad(M) = 0$. Then M is a locally artinian radical supplemented module.

Proof. Since M is a locally artinian supplement of $Rad(M)$ in M, the proof is clear. □

Recall that a module M is called *radical* if $Rad(M) = M$.

Proposition 2. Let M be a radical module. Then M is strongly locally artinian radical supplemented.

Proof. Let U be a submodule with $Rad(M) \subseteq U$. Since $Rad(M) = M$, $U = M$. So 0 is a locally artinian supplement of U in M. Therefore M is strongly locally artinian radical supplemented. □

Recall that $P(M)$ is the sum of all radical submodule of a module M and $P(M)$ is a largest radical submodule of M. So, note that $Rad(P(M)) = P(M)$.

Proposition 3. $P(M)$ is a strongly locally artinian radical supplemented module for every module M.

Proof. Since $Rad(P(M)) = P(M)$, the proof follows from Proposition 2. □

It is clear that every locally artinian supplemented modules are locally artinian radical supplemented. Definition 1, notice that every strongly locally artinian supplemented module is locally artinian radical supplemented. The following example shows that the converse of these situations are not always true.

Recall that an integral domain R is a *Dedekind domain* if every non-zero ideal of R is invertible.

Example 1. (i) Let $M =_{\mathbb{Z}} \mathbb{Z}$. Since $Rad(\mathbb{Z}) = 0$, M is locally artinian radical supplemented by Proposition 1. But M is not a locally artinian supplemented module.

(ii) Let R be a local Dedekind domain and K be a quotient field of R. Since $Rad(K) = K$, K is strongly locally artinian radical supplemented by Proposition 2. It follows from [6, Example 2.7] that K is not locally artinian supplemented.

Proposition 4. Let M be a module with small radical. Then M is locally artinian radical supplemented if and only if $Rad(M)$ is a locally artinian submodule of M.

Proof. ($\Rightarrow$) Since M is locally artinian radical supplemented, there exists a submodule N of M such that $M = Rad(M) + N$, $Rad(M) \cap N \ll N$ and $Rad(M) \cap N$ is locally artinian. Since $Rad(M) \ll M$, then $N = M$. So $Rad(M) \cap M = Rad(M)$ is locally artinian.

($\Leftarrow$) By the hypothesis, M is a locally artinian supplement of $Rad(M)$ in M, as desired. □

Corollary 1. Let M be a finitely generated module. Then M is locally artinian radical supplemented if and only if $Rad(M)$ is a locally artinian submodule of M.

Proof. Since M is finitely generated, M has a small radical. So the proof follows from Proposition 4. □

Example 2. (see [6, Example 2.2]) Consider $\mathbb{Z}$-module $M = \mathbb{Z}_8$. Since $Rad(M) = \langle \overline{2} \rangle \ll M$ and M is locally artinian, M is an RLA-local module. It follows from [6, Theorem 2.11] that M is locally artinian supplemented. Then $Rad(M)$ is locally artinian by [8, 31.2.(1)(i)]. So M is locally artinian radical supplemented by Proposition 4. In addition, as M is locally artinian supplemented, M is strongly locally artinian radical supplemented. But $Rad(M)$ has not an ss-supplement in M.

Recall that a ring R is called *a left max ring* if every non-zero R-module has a maximal submodule.

Corollary 2. Let R be a left max ring and M be an R-module. Then M is locally artinian radical supplemented if and only if $Rad(M)$ is a locally artinian submodule of M.

Proof. By the hypothesis, there exists a submodule N of M such that $M = Rad(M) + N$. It follows that $Rad(\frac{M}{N}) = \frac{M}{N}$. Since R is a left max ring, $\frac{M}{N} = 0$. So $M = N$. Thus $Rad(M) \ll M$. The proof follows from Proposition 4. □

Proposition 5. Every factor module of a strongly locally artinian radical supplemented module is strongly locally artinian radical supplemented.

Proof. Let M be a strongly locally artinian radical supplemented module with $N \subseteq K \subseteq M$ and $Rad(\frac{M}{N}) \subseteq \frac{K}{N}$. Let $\pi : M \longrightarrow \frac{M}{N}$ be a canonical projection. Then $\pi(Rad(M)) = \frac{Rad(M)+N}{N} \subseteq Rad(\frac{M}{N}) \subseteq \frac{K}{N}$. So $Rad(M) \subseteq K$. By the hypothesis, there exists a submodule T of M such that $M = K + T$, $K \cap T \ll T$ and $K \cap T$ is locally artinian. Then $\frac{M}{N} = \frac{K}{N} + \frac{(T+N)}{N}$, $\frac{K}{N} \cap \frac{(T+N)}{N} \ll \frac{(T+N)}{N}$. By [8, 31.2 (1)(i)], $\frac{K}{N} \cap \frac{(T+N)}{N}$ is locally artinian. Therefore $\frac{M}{N}$ is strongly locally artinian radical supplemented. □

Corollary 3. Every homomorphic image of a strongly locally artinian radical supplemented module is strongly locally artinian radical supplemented.

Proposition 6. Let M be a module and $N \subseteq M$. If N is a strongly locally artinian radical supplemented module and $Rad(\frac{M}{N}) = \frac{M}{N}$, then M is a strongly locally artinian radical supplemented module.

Proof. Let U be a submodule of M with $Rad(M) \subseteq U$. Since $Rad(\frac{M}{N}) = \frac{M}{N}$, $M = Rad(M) + N$. So $M = U + N$. Then $Rad(N) \subseteq Rad(M) \subseteq U$ and $Rad(N) \subseteq N$. Note that $Rad(N) \subseteq U \cap N$. Since N is strongly locally artinian radical supplemented, $N = (U \cap N) + K$, $(U \cap N) \cap K = U \cap K \ll K$ and $U \cap K$ is locally artinian for some submodule K of M. Then we have $M = Rad(M) + (U \cap N) + K = U + (U \cap N) + K = U + K$. Thus M is strongly locally artinian radical supplemented. □

Lemma 1. *Let M be a module, M_1 and K be submodules of M and $Rad(M) \subseteq K$. If M_1 is a strongly locally artinian radical supplemented and $M_1 + K$ has a locally artinian radical supplement in M, then K has a locally artinian supplement in M.*

Proof. Let N be a locally artinian supplement of $M_1 + K$ in M and T be a locally artinian supplement of $(N + K) \cap M_1$ in M_1. Then we have $M = N + K + T$, $(M_1 + K) \cap N$ is locally artinian. Also we have $(N + K) \cap T \ll T$ and $(N + K) \cap T$ is locally artinian. Since $(M_1 + K) \cap N$ is locally artinian, $N \cap (K + T)$ is locally artinian by [8, 31.2(1)(i)]. It follows from $(M_1 + K) \cap N \ll N$ and $(N + K) \cap T \ll T$ that $K \cap (N+T) \subseteq N \cap (K+T) + T \cap (K+N) \subseteq N \cap (K+M_1) + T \cap (K+N) \ll N+T$. So $T \cap (K + N)$ is locally artinian by [8, 31.2(2)], as required. □

Proposition 7. Let $M = M_1 + M_2$ be a module with submodules $M_1, M_2 \subseteq M$. If M_1 and M_2 are strongly locally artinian radical supplemented, then M is strongly locally artinian radical supplemented.

Proof. Let K be a module with $Rad(M) \subseteq K$. Since $M_1 + M_2 + K$ has a locally artinian radical supplement 0 in M, $M_1 + K$ has a locally artinian supplement in M by Lemma 1. Applying again Lemma 1, we obtain that M is strongly locally artinian supplemented. □

Corollary 4. Every finite sum of strongly locally artinian radical supplemented modules is a strongly locally artinian radical supplemented module.

Proposition 8. Let M be a module with $Rad(M) \ll M$. Then M is strongly locally artinian radical supplemented if and only if M is locally artinian supplemented.

Proof. ($\Rightarrow$) Let N be a submodule of M. Then $Rad(M) \subseteq Rad(M) + N$. By the hypothesis, $Rad(M) + N$ has a locally artinian supplement K in M. So, $M = Rad(M) + N + K$, $(Rad(M) + N) \cap K \ll K$ and $(Rad(M) + N) \cap K$ is locally artinian. Since $Rad(M) \ll M$, then $M = N + K$. It is clear that $N \cap K \ll K$. By [8, 31.1(i)] $N \cap K$ is locally artinian. Thus M is locally artinian supplemented.

($\Leftarrow$) It is clear. □

Recall from a module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M, equivalently, for a submodule N of M, whenever $Rad(\frac{M}{N}) = \frac{M}{N}$, then $M = N$. Since every coatomic module has small radical, the following corollary is obtained clearly.

Corollary 5. Let M be a coatomic module. Then M is locally artinian supplemented if and only if M is strongly locally artinian radical supplemented.

Corollary 6. Let M be a module with $Rad(M) \ll M$. Then the following statements are equivalent.

(1) M is locally artinian supplemented;
(2) M is supplemented and M is locally artinian radical supplemented;
(3) M is strongly radical supplemented and $Rad(M)$ is locally artinian;
(4) M is strongly locally artinian radical supplemented.

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Clear by Proposition 4.

(3) $\Rightarrow$ (4) Let K be a module with $Rad(M) \subseteq K$. Since M is strongly radical supplemented, there exists a submodule L of M such that $M = K + L$, $K \cap L \ll L$. Then $K \cap L \subseteq Rad(L) \subseteq Rad(M)$. It follows from [8, 31.2(1)(i)] that $K \cap L$ is locally artinian, as desired.

(4) $\Rightarrow$ (1) Since M is strongly locally artinian radical supplemented, M is locally artinian radical supplemented. The proof follows from Proposition 8. □

It follows from [8, 43.9] that a ring R is *left perfect* if and only if R is semilocal and $Rad(R)$ is right T-nilpotent if and only if every R-module has a projective cover, that is, for any R-module M, there exists a projective module P and an epimorphism $f : P \longrightarrow M$ with small kernel.

Theorem 1. *Let R be a ring. Then the following statements are equivalent.*

(1) *R is a left perfect ring and $Rad(R)$ is locally artinian;*
(2) *every free R-module is strongly locally artinian radical supplemented;*
(3) *every R-module is strongly locally artinian radical supplemented.*

Proof. (1) $\Rightarrow$ (2) Let F be free R-module $R^{(I)}$ for some index set I. It follows from [8, 31.2(2)and 43.9] that $Rad(F) = Rad(R)^{(I)}$ is locally artinian and F is supplemented. Since $Rad(F) \ll F$, F is locally artinian supplemented by [6, Theorem 2.9]. We obtain that F is strongly locally artinian radical supplemented by Proposition 8.

(2) $\Rightarrow$ (3) Since every R-module is a homomorphic image of a free R-module, the proof is obvious by Proposition 5.

(3) $\Rightarrow$ (1) Clear by Proposition 8 and [8, 43.9]. □

Recall that $P(M)$ is the divisible part of M for an R-module M over a Dedekind domain R. According to [1, Lemma 4.4], $P(M)$ is (divisible) injective, and so there exists a submodule N of M such that $M = P(M) \oplus N$. Here, N is called *the reduced part* of M. Note that $P(M) \subseteq Rad(M)$. By Proposition 3, $P(M)$ is strongly locally artinian radical supplemented. Using these facts, we obtain the following result.

Proposition 9. Let R be a Dedekind domain and M be an R-module. Then M is strongly locally artinian radical supplemented if and only if the reduced part N of M is strongly locally artinian radical supplemented.

Proof. ($\Rightarrow$) Since N is a homomorphic image of M, N is strongly locally artinian radical supplemented by Proposition 5.

($\Leftarrow$) Clear by Proposition 7. □

3. Conclusion

In this paper, we obtain new classes of modules from locally artinian supplemented modules. To obtain these class of modules, we have associated with radical of the module and every submodule that contains radical of the module. Also, we study on the algebraic structure of these modules. We characterize strongly locally artinian radical supplemented modules over a left perfect ring.

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BIPOLAR SOFT ORDERED TOPOLOGY AND A NEW DEFINITION FOR BIPOLAR SOFT TOPOLOGY

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Abstract. In our study, we gave a new definition for bipolar soft topology and we were able to examine the concept of bipolar soft ordered topology using the base concept we defined on this new bipolar soft topology. We also define the concept of bipolar soft set relation by defining an R relation on a bipolar soft set. Thus, we have defined the concept of bipolar soft interval and presented the bipolar soft ordered topology structure using these intervals in our study. Then, we expressed some applications of bipolar soft order topology.

1. Introduction

Traditional methods fail to solve many complex problems especially in decision making due to uncertainty problems encountered in fields such as economy, engineering, environment. One of the theories put forward to eliminate uncertainty is the soft (briefly s-)set theory introduced by Molodtsov [2]. Then, Maji et al. [1] introduced some new concepts such as subset and complement to s-set theory. The studies on this set theory are increasing and there are many applications especially on s-sets in recent years [8, 11, 12, 13].

In 2010, Babitha and Sunil [14] defined the relation and the ordering in s-sets. Moreover, Park et. al [15] studied the equivalence relations, partitions and functions. In the following years, the notions of symmetric kernel, anti-reflexive kernel, symmetric clousure and reflexive clousure of a s-set relationship was given by Yang and Guo [16] and they proposed s-set relation mappings and inverse s-set relation mappings. The definition of supremum and infimum of the s-set, directed complete s-set were given by Tanay and Yaylalı [17].

The topology structure of s-set has been studied by many researchers and different definitions have been made: Shabir and Naz [18] introduced the soft topological spaces. They studied many concepts such as s-open set, s-neighbourhood of a point in s-topological spaces. As a different approach to s-topology; Cağman et al. [19] defined the concepts of s-closure, s-Hausdorff space, s-limit point, s-interior, s-open

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set and the structure of s-topology was improved by Roy and Samanta [8]. In addition, many researchers such as Aygünoğlu and Aygün [20], Min [24], Zorlutuna et al.[21], Hussain and Ahmad [22], Varol and Aygün [23] studied s-topological spaces.

In last years, the concept of bipolar soft (briefly bs-)set defined by Shabir and Naz [25] and Karaaslan and Karatas[26]. After this set theory has been proposed, the structure of bs-topology has been defined by many researchers: Shabir M. and Bakhtawar A.[27] introduced the bs-topological spaces. Then, Öztürk Y.T.[28] introduced some properties of the bs-topological space.

In our study, we first defined a new bs-topological structure. Using this definition, we gave the concepts of bs-base and bs-intervals on this structure. Then, the bs-order topology is introduced and some applications of bs-order topology are expressed. At the same time, examples are added for easy understanding of the concepts and structures given in our study.

2. Preliminaries

In this section, we recall some basic notions in s-sets and bs-sets. Let U be an initial universe, $E = \{e_1, e_2, ..., e_n\}$ be a set of parameters, $\emptyset \neq V, Y, Z \subseteq E$ and $P(U)$ denotes the power set of U.

Definition 2.1. [2] A pair (Γ, V) is called a s-set over U, where Γ is a mapping given by $\Gamma : V \to P(U)$.

From now on, $S(U)$ denotes the family of all s-sets over U.

Definition 2.2. [1] Let $(\Gamma, V), (\Lambda, Y) \in S(U)$. Then, (Γ, V) is a s-subset of (Λ, Y) if $V \subseteq Y$ and $\Gamma(e) \subseteq \Lambda(e)$; $\forall e \in V$.
We write $(\Gamma, V)\tilde{\subseteq}(\Lambda, Y)$.

Definition 2.3. [1] Let $(\Gamma, V), (\Lambda, Y) \in S(U)$. Then,
(i) the union of (Γ, V) and (Λ, Y) over U is the s-set (Ω, Z), where $Z = V \cup Y$ and

$$\Omega(e) = \begin{cases} \Gamma(e) & \text{if } e \in V - Y \\ \Lambda(e) & \text{if } e \in Y - V \\ \Gamma(e) \cup \Lambda(e) & \text{if } e \in V \cap Y \end{cases}$$

for each $e \in Z$. We write $(\Gamma, V)\tilde{\cup}(\Lambda, Y) = (\Omega, Z)$.

(ii) the intersection of (Γ, V) and (Λ, Y) over U is the s-set (Ω, Z), where $Z = V \cap Y$ and $\Omega(e) = \Gamma(e) \cap \Lambda(e)$; $\forall e \in Z$. We write $(\Gamma, V)\tilde{\cap}(\Lambda, Y) = (\Omega, Z)$.

Definition 2.4. [5] Let $(\Gamma, V) \in S(U)$. Then, the complement of (Γ, V) is a s-set $(\Gamma, V)^c = (\Lambda, V)$ where $\Lambda(e) = U - \Gamma(e)$; $\forall e \in V$.

Definition 2.5. [1] Let $(\Gamma, E) \in S(U)$. Then, (Γ, E) is called:
(i) A null s-set, denoted by $\tilde{\emptyset}$, if $\Gamma(e) = \emptyset$; $\forall e \in E$.
(ii) An absolute s-set, denoted by $\tilde{U}$, if $\Gamma(e) = U$; $\forall e \in E$.

Definition 2.6. [1] The NOT set of E denoted by $\neg E$ is defined by $\neg E = \{\neg e_1, \neg e_2, ..., \neg e_n\}$ where $\neg e_i = \textit{not } e_i$; $\forall i$.

Definition 2.7. [3] A (Γ, Λ, V) is called a bs-set over U where Γ, Λ are mappings given by $\Gamma : V \to P(U)$, $\Lambda : \neg V \to P(U)$ such that $\Gamma(e) \cap \Lambda(\neg e) = \emptyset$; $\forall e \in V$.

From now on, $BS(U)$ denotes the family of all bs-sets over U.

Definition 2.8. [3] Let $(\Gamma,\Lambda,V),(\Gamma_1,\Lambda_1,Y)\in BS(U)$. Then; (Γ,Λ,V) is a bs-subset of (Γ_1,Λ_1,Y), if
(i) $V\subseteq Y$,
(ii) $\Gamma(e)\subseteq\Gamma_1(e)$, $\Lambda_1(\neg e)\subseteq\Lambda(\neg e)$; $\forall e\in V$.
We write $(\Gamma,\Lambda,V)\tilde{\subseteq}(\Gamma_1,\Lambda_1,Y)$.

Definition 2.9. [3] Let $(\Gamma,\Lambda,V)\in BS(U)$. Then, the complement of a bs-set (Γ,Λ,V) is denoted by $(\Gamma,\Lambda,V)^c$ and is defined by $(\Gamma,\Lambda,V)^c=(\Gamma^c,\Lambda^c,V)$ where Γ^c, Λ^c are mappings given by $\Gamma^c(e)=\Lambda(\neg e)$, $\Lambda^c(\neg e)=\Gamma(e)$; $\forall e\in V$.

Definition 2.10. [9] Let $(\Gamma,\Lambda,V)\in BS(U)$. Then,
(i) (Γ,Λ,V) is said to be relative null bs-set, denoted by $(\Phi,\tilde{U},V)$, if $\Gamma(e)=\emptyset$; $\forall e\in V$ and $\Lambda(\neg e)=U$; $\forall\neg e\in\neg V$.

(ii) the relative null bs-set with respect to U of E is called a NULL bs-set over U and is denoted by $(\Phi,\tilde{U},E)$.

(iii) (Γ,Λ,V) over U is said to be relative absolute bs-set, denoted by $(\tilde{U},\Phi,V)$, if $\Gamma(e)=U$; $\forall e\in V$ and $\Lambda(\neg e)=\emptyset$; $\forall\neg e\in\neg V$.

(iv) the relative absolute bs-set with respect to U of E is called a ABSOLUTE bs-set over U and is denoted by $(\tilde{U},\Phi,E)$.

Definition 2.11. [3] Let $(\Gamma,\Lambda,V),(\Gamma_1,\Lambda_1,Y)\in BS(U)$. Then;
(i) extended union of (Γ,Λ,V) and (Γ_1,Λ_1,Y) over U is the bs-set $(\Omega,\mho,Z)$ over U, where $Z=V\cup Y$ and

$$\Omega(e)=\begin{cases}\Gamma(e) & \text{if } e\in V-Y\\ \Gamma_1(e) & \text{if } e\in Y-V\\ \Gamma(e)\cup\Gamma_1(e) & \text{if } e\in V\cap Y\end{cases}$$

and

$$\mho(\neg e)=\begin{cases}\Lambda(\neg e) & \text{if } \neg e\in(\neg V)-(\neg Y)\\ \Lambda_1(\neg e) & \text{if } \neg e\in(\neg Y)-(\neg V)\\ \Lambda(\neg e)\cap\Lambda_1(\neg e) & \text{if } \neg e\in(\neg V)\cap(\neg Y)\end{cases}$$

for all $e\in Z$. We denote it by $(\Gamma,\Lambda,V)\tilde{\cup}(\Gamma_1,\Lambda_1,Y)=(\Omega,\mho,Z)$.

(ii) extended intersection of (Γ,Λ,V) and (Γ_1,Λ_1,Y) over U is the bs-set $(\Omega,\mho,Z)$ over U, where $Z=V\cup Y$ and

$$\Omega(e)=\begin{cases}\Gamma(e) & \text{if } e\in V-Y\\ \Gamma_1(e) & \text{if } e\in Y-V\\ \Gamma(e)\cap\Gamma_1(e) & \text{if } e\in V\cap Y\end{cases}$$

and

$$\mho(\neg e)=\begin{cases}\Lambda(e) & \text{if } e\in(\neg V)-(\neg Y)\\ \Lambda_1(e) & \text{if } e\in(\neg Y)-(\neg V)\\ \Lambda(e)\cup\Lambda_1(e) & \text{if } e\in(\neg V)\cap(\neg Z)\end{cases}$$

for all $e\in Z$. We denote it by $(\Gamma,\Lambda,V)\tilde{\cap}(\Gamma_1,\Lambda_1,Y)=(\Omega,\mho,Z)$.

(iii) restricted union of (Γ,Λ,V), (Γ_1,Λ_1,Y) over U is the bs-set $(\Omega,\mho,Z)$, where $\emptyset\neq Z=V\cap Y$ and

$$\Omega(e)=\Gamma(e)\cup\Lambda(e) \quad \textit{and} \quad \mho(\neg e)=\Gamma_1(\neg e)\cap\Lambda_1(\neg e)$$

for all $e \in Z$. We denote it by $(\Gamma, \Lambda, V) \cup_{\mathfrak{R}} (\Gamma_1, \Lambda_1, Y) = (\Omega, \mho, Z)$.

(iv) restricted intersection of (Γ, Λ, V) and (Γ_1, Ω_1, Y) over U is the bs-set $(\Omega, \mho, Z)$, where $\emptyset \neq Z = V \cap Y$ and

$$\Omega(e) = \Gamma(e) \cap \Lambda(e) \quad and \quad \mho(\neg e) = \Gamma_1(\neg e) \cup \Lambda_1(\neg e)$$

for all $e \in Z$. We denote it by $(\Gamma, \Lambda, V) \cap_{\mathfrak{R}} (\Gamma_1, \Lambda_1, Y) = (\Omega, \mho, Z)$.

Definition 2.12. [8] Let $(\Gamma, V) \in S(U)$. Then; a s-topology $\tilde{\tau}$ on (Γ, V) is a family of s-subsets of (Γ, V) if
(i) $(\Gamma, V), \Phi \in \tilde{\tau}$,
(ii) if $(\Omega, Z), (\Lambda, Y) \in \tilde{\tau}$, then $(\Omega, Z) \tilde{\cap} (\Lambda, Y) \in \tilde{\tau}$,
(iii) if $(\Gamma_\alpha, V_\alpha) \in \tilde{\tau}$, $\forall \alpha \in \wedge$; then $(\Gamma_\alpha, V_\alpha)_{\alpha \in \wedge} \in \tilde{\tau}$.

If $\tilde{\tau}$ is a s-topology on (Γ, V), then $(\Gamma, V, \tilde{\tau})$ is called the soft topological space. Moreover; the member of $\tilde{\tau}$ is called an open s-set in $(\Gamma, V, \tilde{\tau})$. Then, (Γ, V) is said to be closed s-set if the complement of (Γ, V) is open s-set.

Definition 2.13. [10] Let $\tilde{\tilde{\tau}}$ be the collection of bs-sets over U with E. If
(i) $(\Phi, \tilde{U}, E), (\tilde{U}, \Phi, E) \in \tilde{\tilde{\tau}}$,
(ii) the union of any number of bs-sets in $\tilde{\tilde{\tau}}$ belong to $\tilde{\tilde{\tau}}$,
(iii) the intersection of finite number of bs-ets in $\tilde{\tilde{\tau}}$ belong to $\tilde{\tilde{\tau}}$;
then $\tilde{\tilde{\tau}}$ is said to be a bs-topology over U and $(U, \tilde{\tilde{\tau}}, E, \neg E)$ is called a bs-topological space over U.

Moreover; the members of $\tilde{\tilde{\tau}}$ are said to be bs-open sets in U. A bs-set (Γ, Λ, E) over X is said to be a bs-closed set in X, if its bs-complemet $(\Gamma, \Lambda, E)^c$ belongs to $\tilde{\tilde{\tau}}$.

Definition 2.14. [10] Let $(\Gamma, \Lambda, E) \in BS(U)$ and $(X, \tilde{\tilde{\tau}}, E, \neg E)$ be a bs-topological space over $X \subseteq U$. (Γ, Λ, E) over X is said to be a bs-clopen set in X, if it is both a bs-open set and a bs-closed set over X.

3. A New Definition for Bipolar Soft Topology

Definition 3.1. Let $(\Gamma, \Lambda, E) \in BS(U)$. Then, a bs-topology $\tilde{\tilde{\tau}}$ on (Γ, Λ, V) is a family of bs-subsets of (Γ, Λ, V) if it satisfies the following properties
$i)$ $(\Phi, \tilde{U}, V)$, $(\Gamma, \Lambda, V) \in \tilde{\tilde{\tau}}$,
$ii)$ If $(\Gamma_1, \Lambda_1, Y), (\Gamma_2, \Lambda_2, Z) \in \tilde{\tilde{\tau}}$, then $(\Gamma_1, \Lambda_1, Y) \tilde{\cap} (\Gamma_2, \Lambda_2, Z) \in \tilde{\tilde{\tau}}$,
$iii)$ If $(\Gamma_\alpha, \Lambda_\alpha, V_\alpha) \in \tilde{\tilde{\tau}}$, $\forall \alpha \in \wedge$; then $(\Gamma_\alpha, \Lambda_\alpha, V_\alpha)_{\alpha \in \wedge} \in \tilde{\tilde{\tau}}$.

If $\tilde{\tilde{\tau}}$ is a bs-topology on (Γ, Λ, V), then $(\Gamma, \Lambda, V, \tilde{\tilde{\tau}})$ is called the bs-topological space.

Definition 3.2. Let $(\Gamma, \Lambda, E) \in BS(U)$. Then, if $\tilde{\tilde{\tau}}$ is a bs-topology on (Γ, Λ, V), then the member of $\tilde{\tilde{\tau}}$ is called an open bs-set in $(\Gamma, \Lambda, V, \tilde{\tilde{\tau}})$.

Definition 3.3. Let $(\Gamma, \Lambda, V, \tilde{\tilde{\tau}})$ be a bs-topological space and $(\Gamma_1, \Lambda_1, Y) \subseteq (\Gamma, \Lambda, V)$. Then, (Γ_1, Λ_1, Y) is said to be closed bs-set if the complement of (Γ_1, Λ_1, Y) is open bs-set.

Definition 3.4. A collection $\tilde{\tilde{\beta}}$ of some bs-subsets of (Γ, Λ, V) is called a bs-base for some bs-topology on (Γ, Λ, V) if
(i) $(\Phi, \tilde{U}, V) \in \tilde{\tilde{\beta}}$,

(ii) $\tilde{\tilde{\beta}} = (\Gamma, \Lambda, V)$ i.e. "$x \in \Gamma(e)$ and $y \in \Lambda(\neg e)$", $\forall e \in V$; there exits $(K, L, Y) \in \tilde{\tilde{\beta}}$ such that $x \in K(e)$ and $y \in L(\neg e)$, where $Y \subseteq V$,
(iii) If (Γ_1, Λ_1, Y), $(\Gamma_2, \Lambda_2, Z) \in \tilde{\tilde{\beta}}$ then $x \in \Gamma_1(e) \cap \Gamma_2(e)$ and $y \in \Lambda_1(\neg e) \cup \Lambda_2(\neg e)$, $\forall e \in Y \cap Z$; there exists $(\Gamma_3, \Lambda_3, D) \subseteq \tilde{\tilde{\beta}}$ such that

$$(\Gamma_3, \Lambda_3, D) \tilde{\subseteq} (\Gamma_1, \Lambda_1, Y) \tilde{\cap} (\Gamma_2, \Lambda_2, Z)$$

and "$x \in \Gamma_3(e)$ and $y \in \Lambda_3(\neg e)$", where $D \subseteq Y \cap Z$.

Theorem 3.5. *Let $(\Gamma, \Lambda, V, \tilde{\tilde{\tau}})$ be a bs-topological space. $\tilde{\tilde{\beta}}$ is a bs-base if and only if*
(i) $\tilde{\tilde{\beta}} \tilde{\subseteq} \tilde{\tilde{\tau}}$,
(ii) $(M, N, Y) = \tilde{\bigcup}_{for\ some (K,L,Z) \in \tilde{\tilde{\beta}}} (K, L, Z)$; $\forall (M, N, Y) \in \tilde{\tilde{\tau}}$.

Proof. $(\Rightarrow)$ (i) By Definition 3.4.
(ii) Let $(M, N, Y) \in \tilde{\tilde{\tau}}$. If $(M, N, Y) = (\Phi, \tilde{U}, B)$ then $(M, N, Y) = \tilde{\bigcup}_{i \in \emptyset} (K_i, L_i, Z_i)$. If $(M, N, Y) \neq (\Phi, \tilde{U}, Y)$ then $y \in N(\neg e)$, $\forall x \in M(e)$; there exists a bs-set $(K, L, Z) \in \tilde{\tilde{\beta}}$ such that $(K, L, Z) \tilde{\subseteq} (M, N, Y)$ and "$x \in K(e)$ and $y \in L(\neg e)$" where $Z \subset Y$ then $(M, N, Y) = \tilde{\bigcup}_{for\ some (K,L,Z) \in \tilde{\tilde{\beta}}} (K, L, Z)$.
$(\Leftarrow)$ (i) $(\Phi, U, Y) = \tilde{\bigcup}_{i \in \emptyset} (K_i, L_i, Z_i)$,
(ii) Since $\tilde{\tilde{\tau}}$ is a bs-topology then $(\Gamma, \Lambda, V) \in \tilde{\tilde{\tau}}$ and by (2) $(\Gamma, \Lambda, V) = \tilde{\bigcup} \tilde{\tilde{\beta}}$,
(iii) Let $(M_1, G_1, Y_1), (M_2, G_2, Y_2) \in \tilde{\tilde{\beta}}$ then $(M_1, G_1, Y_1), (M_2, G_2, Y_2) \in \tilde{\tilde{\tau}}$ since $(M_1, G_1, Y_1) \tilde{\cap} (M_2, G_2, Y_2) \in \tilde{\tilde{\tau}}$ then by (ii)

$$(M_1, G_1, Y_1) \tilde{\cap} (M_2, G_2, Y_2) = \tilde{\bigcup}_{for\ some (K,L,Z) \in \tilde{\tilde{\beta}}} (K, L, Z).$$

Then for $e \in Y_1 \cap Y_2$ that $(K, L, Z) \tilde{\subseteq} (M_1, G_1, Y_1) \tilde{\cap} (M_2, G_2, Y_2)$ and "$x \in K(e)$ and $y \in L(\neg e)$" where $Z \subset Y_1 \cap Y_2$. □

4. Main Results

Definition 4.1. Let $(\Gamma_1, \Lambda_1, V), (\Gamma_2, \Lambda_2, Y) \in BS(U)$. Then, $(\Gamma_1, \Lambda_1, V) \times (\Gamma_2, \Lambda_2, Y) = (\Omega, \mho, V \times Y)$ is the cartesian product of (Γ_1, Λ_1, V) and (Γ_2, Λ_2, Y), such that $(v, y) \in V \times Y$, $\Omega : V \times Y \to P(U \times U)$ and $(\neg v, \neg y) \in \neg V \times \neg Y$, $\mho : \neg V \times \neg Y \to P(U \times U)$ where $\Omega(v, y) = \Gamma_1(v) \times \Gamma_2(y) = \{(h_i, h_j) : h_i \in \Gamma_1(v), h_j \in \Gamma_2(y)\}$ and $\mho(\neg v, \neg y) = \Lambda_1(\neg v) \times \Lambda_2(\neg y) = \{(t_i, t_j) : t_i \in \Lambda_1(\neg v), t_j \in \Lambda_2(\neg y)\}$.

Definition 4.2. Let $(\Gamma_1, \Lambda_1, V), (\Gamma_2, \Lambda_2, Y) \in BS(U)$. Then, a bs-set relation $\tilde{R}$ from (Γ_1, Λ_1, V) to (Γ_2, Λ_2, Y) is a bs-subset of $(\Gamma_1, \Lambda_1, V) \times (\Gamma_2, \Lambda_2, Y)$. In other words, a bs-set relation $\tilde{R}$ from (Γ_1, Λ_1, V) to (Γ_2, Λ_2, Y) is of the form $\tilde{R} = (\Omega_1, \mho_1, S)$ where $S \subset V \times Y$ and $\Omega_1(v, y) = \Omega(v, y)$, $\mho_1(\neg v, \neg y) = \mho(\neg v, \neg y)$, $\forall (v, y) \in S$ where $(\Omega, \mho, V \times Y) = (\Gamma_1, \Lambda_1, V) \times (\Gamma_2, \Lambda_2, Y)$.

Definition 4.3. Let $(\Gamma, \Lambda, V) \in BS(U)$ and $\tilde{R}$ be a bs-set relation on (Γ, Λ, V), then
(1) $\tilde{R}$ is called reflexive if $\Omega_1(v, v) \in \tilde{R}$, $\mho_1(\neg v, \neg v) \in \tilde{R}$; $\forall v \in V$.
(2) $\tilde{R}$ is called symmetric if $\Omega_1(v, y) \in \tilde{R} \Rightarrow \Omega_1(y, v) \in \tilde{R}$, $\mho_1(\neg v, \neg y) \in \tilde{R} \Rightarrow \mho_1(\neg y, \neg v) \in \tilde{R}$; $\forall v, y \in V$.
(3) $\tilde{R}$ is called transitive if $\Omega_1(v, y) \in \tilde{R}$, $\Omega_1(y, z) \in \tilde{R} \Rightarrow \Omega_1(v, z) \in \tilde{R}$, $\mho_1(\neg v, \neg z) \in \tilde{R}$, $\mho_1(\neg y, \neg z) \in \tilde{R} \Rightarrow \mho_1(\neg v, \neg z) \in \tilde{R}$; $\forall v, y, z \in V$.

Definition 4.4. Let $(\Gamma, \Lambda, V) \in BS(U)$ and the a binary bs-set relation $\tilde{R}$ on (Γ, Λ, V) is called an antisymmetric relation if $\Gamma(v) \times \Gamma(y) \in \tilde{R}$, $\Gamma(y) \times \Gamma(v) \in \tilde{R}$, $\Lambda(\neg v) \times \Lambda(\neg y) \in \tilde{R}$ and $\Lambda(\neg y) \times \Lambda(\neg v) \in \tilde{R}$, $\forall \Gamma(v), \Gamma(y) \in (\Gamma, A)$ and $\Lambda(\neg v), \Lambda(\neg y) \in (\Lambda, \neg A)$ imply $\Gamma(y) = \Gamma(v)$ and $\Lambda(\neg v) = \Lambda(\neg y)$.

Definition 4.5. Let $(\Gamma, \Lambda, V) \in BS(U)$ and the a binary bs-set relation $\leq$ on (Γ, Λ, V) which is reflexive, transitive and antisymmetric is called a partial ordering on a bs-set (Γ, Λ, V). The quadruple $(\Gamma, \Lambda, V, \leq)$ is called a partially ordered bs-set.

Definition 4.6. Let $(\Gamma, \Lambda, V) \in BS(U)$ and $\leq$ be an ordering of (Γ, Λ, V) and $(\Gamma(v), \Lambda(\neg v))$, $(\Gamma(y), \Lambda(\neg y))$ be any two elements in (Γ, Λ, V). If "$\Gamma(v) \leq \Gamma(y)$ and $\Lambda(\neg y) \leq \Lambda(\neg v)$" or "$\Gamma(y) \leq \Gamma(v)$ and $\Lambda(\neg v) \leq \Lambda(\neg y)$" then $(\Gamma(v), \Lambda(\neg v))$ and $(\Gamma(y), \Lambda(\neg y))$ are comparable in the ordering. If they are not comparable, then $(\Gamma(v), \Lambda(\neg v))$ and $(\Gamma(y), \Lambda(\neg y))$ are incomparable.

Definition 4.7. Let $(\Gamma, \Lambda, Y) \in BS(U)$. Then, if $(\Gamma, \Lambda, Y, \leq)$ is a partially ordered bs-set then,

a) For $y \in Y$; if $\Gamma(y) \leq \Gamma(\beta)$, $\Lambda(\neg\beta) \leq \Lambda(\neg y)$, $\forall \beta \in Y$; then $(\Gamma(y), \Lambda(\neg y))$ is the least element of (Γ, Λ, Y) in the ordering "$\leq$".

b) For $y \in Y$; if there exists no $\beta \in Y$ such that "$\Gamma(\beta) \leq \Gamma(y)$ and $\Lambda(\neg y) \leq \Lambda(\neg\beta)$" and "$\Gamma(\beta) \neq \Gamma(y)$ and $\Lambda(\neg y) \neq \Lambda(\neg\beta)$", then $(\Gamma(y), \Lambda(\neg y))$ is a minimal element of (Γ, Λ, Y) in the ordering "$\leq$".

a′) For $y \in Y$; if $\Lambda(\neg y) \leq \Lambda(\neg\beta)$, $\forall \beta \in Y$ $\Gamma(\beta) \leq \Gamma(y)$; then $(\Gamma(y), \Lambda(\neg y))$ is the greatest element of (Γ, Λ, Y) in the ordering "$\leq$".

b′) For $y \in Y$; if there exists no $\beta \in Y$ such that "$\Gamma(y) \leq \Gamma(\beta)$ and $\Lambda(\neg\beta) \leq \Lambda(\neg y)$" and "$\Gamma(\beta) \neq \Gamma(y)$ and $\Lambda(\neg\beta) \neq \Lambda(\neg y)$", then $(\Gamma(y), \Lambda(\neg y))$ is a maximal element of (Γ, Λ, Y) in the ordering "$\leq$".

Definition 4.8. Let $(\Gamma_1, \Lambda_1, V), (\Gamma_2, \Lambda_2, Y) \in BS(U)$, $\leq$ be an ordering of (Γ_1, Λ_1, V) and $(\Gamma_2, \Lambda_2, Y) \subseteq (\Gamma_1, \Lambda_1, V)$.

a) For $v \in V$, $(\Gamma_1(v), \Lambda_1(\neg v))$ is a lower bound of (Γ_2, Λ_2, Y) in the partially ordered bs-set $(\Gamma_1, \Lambda_1, V, \leq)$ if $\Gamma_1(v) \leq \Gamma_2(\beta)$ and $\Lambda_2(\neg\beta) \leq \Lambda_1(\neg v)$; $\forall \beta \in Y$.

b) For $v \in V$, $(\Gamma_1(v), \Lambda_1(\neg v))$ is called infimum of (Γ_2, Λ_2, Y) in $(\Gamma_1, \Lambda_1, V, \leq)$ if it is the greatest element of the set of all lower bounds of the bs-subset (Γ_2, Λ_2, Y) in $(\Gamma_1, \Lambda_1, V, \leq)$.

Similarly,

a′) For $v \in V$, $(\Gamma_1(v), \Lambda_1(\neg v))$ is an upper bound of (Γ_2, Λ_2, Y) in the partially ordered bs-set $(\Gamma_1, \Lambda_1, V, \leq)$ if $\Gamma_2(\beta) \leq \Gamma_1(v)$, $\Lambda_1(\neg v) \leq \Lambda_2(\neg\beta)$; $\forall \beta \in Y$.

b′) For $v \in V$, $(\Gamma_1(v), \Lambda_1(\neg v))$ is called supremum of (Γ_2, Λ_2, Y) in $(\Gamma_1, \Lambda_1, V, \leq)$ if it is the least element of the set of all upper bounds of the s-subset (Γ_2, Λ_2, Y) in $(\Gamma_1, \Lambda_1, V, \leq)$.

4.1. **Bipolar Soft Intervals.**

Definition 4.9. Let $(\Gamma, \Lambda, V) \in BS(U)$ and $\tilde{R}$ be a bs-set relation on (Γ, Λ, V). If for no $v \in V$, $\Gamma(v)$ and $\Lambda(\neg v)$ the s-set relation $\Gamma(v)\tilde{R}\Gamma(v)$ and $\Lambda(\neg v)\tilde{R}\Lambda(\neg v)$ hold, the bs-set relation $\tilde{R}$ is called nonreflexive.

Definition 4.10. Let $(\Gamma, \Lambda, V) \in BS(U)$ and a bs-set relation $\tilde{R}$ on (Γ, Λ, V) is called simple order bs-set relation if it is comparable, nonreflexive and transitive. (Γ, Λ, V) is called a simple ordered bs-set a the simple order bs-set relation $\tilde{R}$.

Definition 4.11. Let $\leq$ be a bs-set relation on (Γ, Λ, V), then restriction of a bs-set relation $\leq$ to a bs-subset (Γ_1, Λ_1, Y) is defined as follows:
We denote $\Gamma_1(v) \leq_{(\Gamma_1,\Lambda_1,Y)} \Gamma_1(y)$ and $\Lambda_1(\neg y) \leq_{(\Gamma_1,\Lambda_1,Y)} \Lambda_1(\neg v)$: if and only if $\Gamma(v) \leq \Gamma(y)$ and $\Lambda(\neg y) \leq \Lambda(\neg v)$; $\forall v, y \in B$.

Example 4.12. Let $V = \{m_1, m_2, m_3\}$ be a parameter set and $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be a universe set. $\Gamma(m_1) = \{u_1\}$, $\Gamma(m_2) = \{u_2, u_4, u_5\}$, $\Gamma(m_3) = \{u_3\}$ $\Lambda(\neg m_1) = \{u_3, u_4\}$, $\Lambda(\neg m_2) = \{u_1, u_5\}$, $\Lambda(\neg m_3) = \{u_2\}$; $Y = \{m_1, m_2\}$, $\Gamma_1(m_1) = \{u_1\}$, $\Gamma_1(m_2) = \{u_2, u_4\}$, $\Lambda_1(\neg m_1) = \{u_4\}$, $\Lambda_1(\neg m_2) = \{u_1\}$. Then $(\Gamma_1, \Lambda_1, Y) \subseteq (\Gamma, \Lambda, V)$.

$$\begin{aligned} \leq_{(\Gamma,\Lambda,V)} &= \left\{ \begin{array}{c} \Gamma(m_1) \times \Gamma(m_2), \Gamma(m_2) \times \Gamma(m_3), \\ \Lambda(\neg m_1) \times \Lambda(\neg m_2), \Lambda(\neg m_2) \times \Lambda(\neg m_3) \end{array} \right\} \\ &= \left\{ \begin{array}{c} (u_1, u_2), (u_1, u_4), (u_1, u_5), (u_2, u_3), (u_4, u_3), (u_5, u_3), \\ (u_3, u_1), (u_3, u_5), (u_4, u_1), (u_4, u_5), (u_1, u_2), (u_5, u_2) \end{array} \right\} \end{aligned}$$

$$\text{Then } \leq_{(\Gamma_1,\Lambda_1,Y)} = \left\{ \begin{array}{c} \Gamma_1(m_1) \times \Gamma_1(m_2), \\ \Lambda_1(\neg m_1) \times \Lambda_1(\neg m_2) \end{array} \right\} = \{(u_1, u_2), (u_1, u_4), (u_4, u_1)\}.$$

Definition 4.13. Assume that (Γ, Λ, V) is a bs-set having a simple order bs-set relation $<$ and $(\Gamma(v), \Lambda(\neg v))$ and $(\Gamma(y), \Lambda(\neg y))$ be elements of (Γ, Λ, V) such that $\Gamma(v) < \Gamma(y)$ and $\Lambda(\neg y) < \Lambda(\neg v)$. Then we can define following four bs-subsets of (Γ, Λ, V) which are called bs-intervals (respectively; bs-closed interval, bs-half open intervals, bs-open interval) determined by $(\Gamma(v), \Lambda(\neg v))$ and $(\Gamma(y), \Lambda(\neg y))$:

a) **bs-Open Interval:** The bs-open interval is a bs-subset (Γ_1, Λ_1, Y) of (Γ, Λ, V) where "$Y_1 = \{x : \Gamma(v) < \Gamma(x) < \Gamma(y)\}$ and $Y_2 = \{\neg x : \Lambda(\neg y) < \Lambda(\neg x) < \Lambda(\neg v)\}$", "$\Gamma_1 = \Gamma|_Y$ and $\Lambda_1 = \Lambda|_Y$" and denoted by $(\Gamma(v), \Gamma(y)) = \{\Gamma(x) : \Gamma(v) < \Gamma(x) < \Gamma(y)\}$ and $(\Lambda(\neg y), \Lambda(\neg v)) = \{\Lambda(\neg x) : \Lambda(\neg y) < \Lambda(\neg x) < \Lambda(\neg v)\}$.

b) **bs-Half Open Interval:**

(*i*) The bs-open interval is a bs-subset (Γ_1, Λ_1, Y) of (Γ, Λ, V) where $B_1 = \{x : \Gamma(v) < \Gamma(x) < \Gamma(y) \text{ or } \Gamma(x) = \Gamma(y)\}$ and $Y_2 = \{\neg x : \Lambda(\neg y) < \Lambda(\neg x) < \Lambda(\neg a) \text{ or } \Lambda(\neg x) = \Lambda(\neg y)\}$, "$\Gamma_1 = \Gamma|_Y$ and $\Lambda_1 = \Lambda|_Y$" denoted by $(\Gamma(v), \Gamma(y)] = \{\Gamma(x) : \Gamma(v) < \Gamma(x) < \Gamma(y) \text{ or } \Gamma(x) = \Gamma(y)\}$ and $(\Lambda(\neg y), \Lambda(\neg v)] = \{\Lambda(\neg x) : \Lambda(\neg y) < \Lambda(\neg x) < \Lambda(\neg v) \text{ or } \Lambda(\neg x) = \Lambda(\neg y)\}$.

(*ii*) The bs-open interval is a bs-subset (Γ_1, Λ_1, Y) of (Γ, Λ, V) where $Y_1 = \{x : \Gamma(v) < \Gamma(x) < \Gamma(y) \text{ or } \Gamma(x) = \Gamma(v)\}$ and $Y_2 = \{\neg x : \Lambda(\neg y) < \Lambda(\neg x) < \Lambda(\neg v) \text{ or } \Lambda(\neg x) = \Lambda(\neg v)\}$, "$\Gamma_1 = \Gamma|_Y$ and $\Lambda_1 = \Lambda|_Y$" and denoted by $[\Gamma(v), \Gamma(y)) =$

$\{\Gamma(x) : \Gamma(v) < \Gamma(x) < \Gamma(y) \ or \ \Gamma(x) = \Gamma(v)\}$ and $[\Lambda(\neg y), \Lambda(\neg v)) = \{\Lambda(\neg x) : \Lambda(\neg y) < \Lambda(\neg x) < \Lambda(\neg v) \ or \ \Lambda(\neg x) = \Lambda(\neg v)\}$.

d) **bs-Closed Interval:** The bs-open interval is a bs-subset (Γ_1, Λ_1, Y) of (Γ, Λ, V) where $Y_1 = \{x : \Gamma(v) < \Gamma(x) < \Gamma(y) \ or \ \Gamma(x) = \Gamma(v) \ or \ \Gamma(x) = \Gamma(y)\}$ and $Y_2 = \{\neg x : \Lambda(\neg y) < \Lambda(\neg x) < \Lambda(\neg v) \ or \ \Lambda(\neg x) = \Lambda(\neg y) \ or \ \Lambda(\neg x) = \Lambda(\neg v)\}$, "$\Gamma_1 = \Gamma|_Y$ and $\Lambda_1 = \Lambda|_Y$" and denoted by $[\Gamma(v), \Gamma(y)] = \{\Gamma(x) : \Gamma(v) < \Gamma(x) < \Gamma(y) \ or \ \Gamma(x) = \Gamma(v) \ or \ \Gamma(x) = \Gamma(y)\}$ and $[\Lambda(\neg y), \Lambda(\neg v)] = \{\Lambda(\neg x) : \Lambda(\neg y) < \Lambda(\neg x) < \Lambda(\neg v) \ or \ \Lambda(\neg x) = \Lambda(\neg v) \ or \ \Lambda(\neg x) = \Lambda(\neg y)\}$.

These are the bs-intervals on an arbitrary simple ordered bs-set.

Example 4.14. Let $V = \{v_1, v_2, v_3, v_4, v_5\}$ be the parameter set and $U = \{m_1, m_2, m_3, m_4, m_5, m_6\}$ be the universe set. Lets define a bs-set (Γ, Λ, V) such that $\Gamma(v_1) = \{m_1, m_4\}$, $\Gamma(v_2) = \{m_1, m_2, m_6\}$, $\Gamma(v_3) = \{m_2, m_3, m_5\}$, $\Gamma(v_4) = \{m_2, m_3\}$, $\Gamma(v_5) = \{m_1\}$, $\Lambda(\neg v_1) = \{m_2, m_3\}$, $\Lambda(\neg v_2) = \{m_3, m_4, m_5\}$, $\Lambda(\neg v_3) = \{m_1, m_4, m_6\}$, $\Lambda(\neg v_4) = \{m_1, m_4\}$, $\Lambda(\neg v_5) = \{m_2\}$. Consider a bs-set relation on (Γ, Λ, V) defined by

$$<_{(\Gamma,\Lambda,V)}= \left\{ \begin{array}{c} \Gamma(v_2) \times \Gamma(v_3), \Gamma(v_1) \times \Gamma(v_2), \Gamma(v_2) \times \Gamma(v_4), \\ \Gamma(v_2) \times \Gamma(v_5), \Gamma(v_1) \times \Gamma(v_3), \Gamma(v_3) \times \Gamma(v_4), \\ \Gamma(v_3) \times \Gamma(v_5), \Gamma(v_1) \times \Gamma(a_4), \Gamma(v_1) \times \Gamma(v_5), \\ \Gamma(v_4) \times \Gamma(v_5), \Lambda(\neg v_3) \times \Lambda(\neg v_2), \Lambda(\neg v_2) \times \Lambda(\neg v_1), \\ \Lambda(\neg v_4) \times \Lambda(\neg v_2), \Lambda(\neg v_5) \times \Lambda(\neg v_2), \Lambda(\neg v_3) \times \Lambda(\neg v_1), \\ \Lambda(\neg v_4) \times \Lambda(\neg v_3), \Lambda(\neg v_5) \times \Lambda(\neg v_3), \Lambda(\neg v_4) \times \Lambda(\neg v_1), \\ \Lambda(\neg v_5) \times \Lambda(\neg v_1), \Lambda(\neg v_5) \times \Lambda(\neg v_4) \end{array} \right\}.$$

The bs-set relation "$<$" is comparable, nonreflexive, transitive so it is simple ordered bs-set relation.

Definition 4.15. Let (Γ, Λ, V) be an simple ordered bs-set with a bs-set relation $<$ and $(\Gamma(v), \Lambda(\neg v))$ be in (Γ, Λ, V). Then there are four soft subsets of (Γ, Λ, V) which are called bs-rays determined $(\Gamma(a), \Lambda(\neg v))$. They are following:

i) $((\Gamma(v), *) \cup (*, \Lambda(\neg v)) = \{(\Gamma(x), \Lambda(\neg x)) : \Gamma(v) < \Gamma(x) \ and \ \Lambda(\neg x) < \Lambda(\neg v)\}$ is called bs-open ray,

ii) $(*, \Gamma(v)) \cup (\Lambda(\neg v), *) = \{(\Gamma(x), \Lambda(\neg x)) : \Gamma(x) < \Gamma(v) \ and \ \Lambda(\neg v) < \Lambda(\neg x)\}$ is called bs-open ray,

iii) $[\Gamma(v), *) \cup (\Lambda(\neg v), *] = \left\{ (\Gamma(x), \Lambda(\neg x)) : \begin{array}{c} \left[\begin{array}{c} \Gamma(v) < \Gamma(x) \\ or \\ \Gamma(v) = \Gamma(x) \end{array} \right] \\ and \\ \left[\begin{array}{c} \Lambda(\neg x) < \Lambda(\neg v) \\ or \\ \Lambda(\neg x) = \Lambda(\neg v) \end{array} \right] \end{array} \right\}$ is called bs-closed ray,

$$iv)\ (\Gamma(v),*]\cup[\Lambda(\neg v),*)=\left\{(\Gamma(x),\Lambda(\neg x)):\begin{array}{c}\left[\begin{array}{c}\Gamma(x)<\Gamma(v)\\ or\\ \Gamma(x)=\Gamma(v)\end{array}\right]\\ and\\ \left[\begin{array}{c}\Lambda(\neg v)<\Lambda(\neg x)\\ or\\ \Lambda(\neg v)=\Lambda(\neg x)\end{array}\right]\end{array}\right\}$$ is called bs-closed ray.

4.2. **Bipolar Soft Ordered Topology.**

Theorem 4.16. *Let (Γ,Λ,V) be a bs-set with a simple ordered bs-set relation; assume that (Γ,Λ,V) has more then one element. Let $\tilde{\tilde{\beta}}$ be a collection of all bs-subsets of (Γ,Λ,V) of the following types:*
(1) $(\Phi,\tilde{U},V)$,
(2) *All bs-open intervals $(\Gamma(v),\Gamma(y))$ and $(\Lambda(\neg y),\Lambda(\neg v))$ in (Γ,Λ,V),*
(3) *All bs-intervals of from $[\Gamma(v_0),\Gamma(y))$ and $(\Lambda(\neg y),\Lambda(\neg v_0)]$, where $\Gamma(v_0)$ is the least element of (Γ,V) and $\Lambda(\neg v_0)$ is the greatest element of $(\Lambda,\neg V)$,*
(4) *All bs-intervals of the form $(\Gamma(v),\Gamma(y_0)]$ and $[\Lambda(\neg y_0),\Lambda(\neg v))$, where $\Gamma(y_0)$ is the greatest element of (Γ,V) and $\Lambda(y_0)$ is the least element of $(\Lambda,\neg V)$.*

Then the collection $\tilde{\tilde{\beta}}$ is a bs-base for a bs-topology on (Γ,Λ,V).

If (Γ,V) has no least element or $(\Lambda,\neg V)$ has no greatest element, there is no s-sets in type (3). Moreover; if (Γ,V) has no greatest element or $(\Lambda,\neg V)$ has no least element, there is no s-sets in type (4).

Proof. Lets check $\tilde{\tilde{\beta}}$ satisfies the requirements for being a bs-base given in the Definition 3.4.
(1) $(\Phi,\tilde{U},V)\in\tilde{\tilde{\beta}}$,
(2) Take $e\in V$, $x\in\Gamma(e)$ and $y\in\Lambda(\neg e)$. By comparability there exists a bs-interval (Γ_1,Λ_1,B), where $x\in Y\subset V$ and $x\in\Gamma_1(e)$ and $y\in\Lambda_1(\neg e)$.
(3) Let $(\Gamma_1,G_1,Y),(\Gamma_2,\Lambda_2,Z)\in\tilde{\tilde{\beta}}$, where $(\Gamma_1,\Lambda_1,Y)=\begin{pmatrix}(\Gamma(v),\Gamma(y)),\\ (\Lambda(\neg y),\Lambda(\neg v))\end{pmatrix}$, $(\Gamma_2,\Lambda_2,Z)=\begin{pmatrix}(\Gamma(z),\Gamma(d)),\\ (\Lambda(\neg d),\Lambda(\neg z))\end{pmatrix}$. Then

$$(\Gamma_1,G_1,Y)\tilde{\cap}(\Gamma_2,\Lambda_2,Z)=\begin{pmatrix}(\Gamma(v),\Gamma(y)),\\ (\Lambda(\neg y),\Lambda(\neg v))\end{pmatrix}\tilde{\cap}\begin{pmatrix}(\Gamma(z),\Gamma(d)),\\ (\Lambda(\neg d),\Lambda(\neg z))\end{pmatrix}=$$

$$\begin{cases}(\Phi,\tilde{U},V) & (\Gamma(y)<\Gamma(z))\vee(\Lambda(\neg z)<\Lambda(\neg y))\\ (\Phi,\tilde{U},V) & (\Gamma(d)<\Gamma(v))\vee(\Lambda(\neg v)<\Lambda(\neg d))\\ (\Gamma(z),\Gamma(y))\wedge(\Lambda(\neg y),\Lambda(\neg z)) & (\Gamma(z)<\Gamma(y))\vee(\Lambda(\neg y)<\Lambda(\neg z))\\ (\Gamma(v),\Gamma(d))\wedge(\Lambda(\neg d),\Lambda(\neg v)) & (\Gamma(v)<\Gamma(d))\vee(\Lambda(\neg d)<\Lambda(\neg v))\\ (\Gamma(z),\Gamma(d))\wedge(\Lambda(\neg d),\Lambda(\neg z)) & ((\Gamma(v)<\Gamma(z))\wedge(\Gamma(y)<\Gamma(d)))\\ (\Gamma(z),\Gamma(d))\wedge(\Lambda(\neg d),\Lambda(\neg z)) & ((\Lambda(\neg z)<\Lambda(\neg v))\wedge(\Lambda(\neg d)<\Lambda(\neg y)))\\ (\Gamma(v),\Gamma(d))\wedge(\Lambda(\neg d),\Lambda(\neg v)) & ((\Gamma(z)<\Gamma(v))\wedge(\Gamma(d)<\Gamma(y)))\\ (\Gamma(v),\Gamma(d))\wedge(\Lambda(\neg d),\Lambda(\neg v)) & ((\Lambda(\neg v)<\Lambda(\neg z))\wedge(\Lambda(\neg y)<\Lambda(\neg d)))\\ (\Gamma(v'),\Gamma(d'))\wedge(\Lambda(\neg d'),\Lambda(\neg v')) & \end{cases}$$ □

Example 4.17. Let $V = \{v_1, v_2, v_3\}$ be a parameter set and $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a universe set. $K(v_1) = \{u_1, u_2\}$, $K(v_2) = \{u_2\}$, $K(v_3) = \{u_3, u_4, u_5\}$;

$$<=_{(K,L,M)}= \left\{ \begin{array}{c} K(v_1) \times K(v_2), K(v_2) \times K(v_3), K(v_1) \times K(v_3), \\ L(\neg v_2) \times L(v_1), L(\neg v_3) \times L(\neg v_2), L(\neg v_3) \times L(\neg v_1) \end{array} \right\}.$$

Then the bs-ordered topology is;

$$\tilde{\tilde{\tau}} = \left\{ \begin{array}{c} (\Phi, \tilde{U}, A), [\Gamma(v_1), \Gamma(v_2)), (\Gamma(v_1), \Gamma(v_2)), [\Gamma(v_1), \Gamma(v_3)), \\ (\Gamma(v_1), \Gamma(v_3)), [\Gamma(v_1), \Gamma(v_3)], (\Gamma(v_2), \Gamma(v_3)), (\Gamma(v_2), \Gamma(v_3)], \\ (\Lambda(\neg v_2), \Lambda(\neg v_1)], (\Lambda(\neg v_2), \Lambda(\neg v_1)), (\Lambda(\neg v_3), \Lambda(\neg v_1)], \\ (\Lambda(\neg v_3), \Lambda(\neg v_1)), [\Lambda(\neg v_3), \Lambda(\neg v_1)], \\ (\Lambda(\neg v_3), \Lambda(\neg v_2)), [\Lambda(\neg v_3), \Lambda(\neg v_2)) \end{array} \right\}.$$

Example 4.18. Let $U = \mathbb{R}$, $V = [\frac{3}{2}, \infty)$, $\neg V = (\neg\infty, \neg\frac{3}{2}]$ and (Γ, Λ, V) be a bs-set where $\Gamma(v) = (\frac{3}{2}, v]$ and $\Lambda(\neg v) = (\neg v, \neg - \frac{3}{2}]$; $\forall v \in V$. Lets define simple ordered on (Γ, Λ, V) as follows: $\Gamma(v) < \Gamma(y) :\Leftrightarrow v < y :\Leftrightarrow (\frac{3}{2}, v] \subseteq (\frac{3}{2}, y]$ and $\Lambda(\neg y) < \Lambda(\neg v) :\Leftrightarrow \neg y < \neg v :\Leftrightarrow (\neg v, \neg - \frac{3}{2}] \subseteq (\neg y, \neg - \frac{3}{2}]$. $\Gamma(\frac{3}{2})$ is the smallest element and ,$\Lambda(\neg - \frac{3}{2})$ is the biggest element, so

$$\tilde{\tilde{\beta}} = \left\{ \begin{array}{c} [\Gamma(\frac{3}{2}), \Gamma(v)), [\Lambda(\neg v), \Lambda(\neg - \frac{3}{2})), \\ (\Gamma(v), \Gamma(y)), (\Lambda(\neg v), \Lambda(\neg y)) \end{array} : \begin{array}{c} \Gamma(v), \Gamma(y) \in (\Gamma, V) \\ and \\ \Lambda(\neg v), \Lambda(\neg y) \in (\Lambda, \neg V) \end{array} \right\}$$

is a bs-base for the bs-ordered topology on (Γ, Λ, V).

Example 4.19. Let $U = (-\infty, -1] \cup [1, \infty)$ be the initial universe and $V = Z^-$ be the parameter set and $\neg V = Z^+$, let (Γ, Λ, V) be a bs-set, defined by $(\Gamma, V) = \{\Gamma(v) = (v, -1] : v \in V\}$ and $(\Lambda, V) = \{\Lambda(\neg v) = (\neg v, \neg 1] : \neg v \in \neg V\}$. Consider the bs-set relation $<$ on (Γ, Λ, V), which is defined by $\Gamma(v) < \Gamma(y) :\Leftrightarrow v < y$ and $\Lambda(\neg y) < \Lambda(\neg v) :\Leftrightarrow y < v$. (Γ, Λ, V) is a simple ordered bs-set with the relation $<$. By examining the bs-subsets of (Γ, Λ, V), $F(-1)$ is the biggest element and $\Lambda(\neg 1)$ is the smallest element . $\tilde{\tilde{\beta}} = \{(\Gamma(v), \Gamma(-1)], (\Gamma(v), \Gamma(y)), [\Lambda(\neg 1), \Lambda(\neg v)), (\Lambda(\neg y), \Lambda(\neg v)) : \Gamma(v), \Gamma(y) \in (\Gamma, V)\ and\ \Lambda(\neg v), \Lambda(\neg y) \in (\Lambda, \neg V)\}$ is a base for the bs-ordered topology.

Definition 4.20. Let $(\Gamma, \Lambda, V, \tilde{\tilde{\tau}})$ be a bs-topological space and $\tilde{\mathcal{BS}}$ be a collection of nonnull bs-open subsets of (Γ, Λ, V). If finite intersection of the elements of $\tilde{\mathcal{BS}}$ is a base for $\tilde{\tilde{\tau}}$ then $\tilde{\tilde{\mathcal{BS}}}$ is called bs-subbase, ie.;

$$\tilde{\mathcal{Y}}_{\tilde{\mathcal{BS}}} = \{\tilde{\cap}_{j \in J}(Y_j, S_j, V_j) : J\ is\ a\ finite\ and\ for\ all\ j \in J, (Y_j, S_j, V_j) \in \tilde{\mathcal{BS}}\}$$

Theorem 4.21. *Let (Γ, Λ, V) be a nonnull bs-set and $\tilde{\mathcal{BS}}$ be a collection of bs-subsets of (Γ, Λ, V). Then there exists a bs-topology on (Γ, Λ, V) which has as a subbase $\tilde{\mathcal{BS}}$.*

Proof. Lets show that $\tilde{\mathcal{Y}}_{\tilde{\mathcal{BS}}} = \{\tilde{\cap}_{j \in J}(Y_j, S_j, V_j) : J\ is\ a\ finite\ and\ for\ all\ j \in J, (Y_j, S_j V_j) \in \tilde{\mathcal{BS}}\}$ satisfies the conditions of being a bs-base
(1) $(\Gamma, \Lambda, V) = \tilde{\cap}_{j \in J}(Y_j, S_j, V_j)$ then $(\Gamma, \Lambda, V) \in \tilde{\mathcal{Y}}_{\tilde{\mathcal{BS}}}$.
(2) Let $(\Gamma_1, \Lambda_1, Y), (\Gamma_2, \Lambda_2, B) \in \tilde{\mathcal{Y}}_{\tilde{\mathcal{BS}}} \Rightarrow$ If $(\Gamma_1, \Lambda_1, Y)\tilde{\cap}(\Gamma_2, \Lambda_2, Y) = (\Phi, \tilde{U}, Y) \Rightarrow (\Gamma_1, v_1, Y)\tilde{\cap}(\Gamma_2, \Lambda_2, Y) = \tilde{\cup}_{j \in \emptyset}(\Gamma_j, \Lambda_j, Y_j)$.

If $(\Gamma_1,\Lambda_1,Y)\tilde{\cap}(\Gamma_2,\Lambda_2,Y) \neq (\Phi,U,Y) \Rightarrow$ since $(\Gamma_1,\Lambda_1,Y) = \tilde{\cup}_{i=1}^{n}(Y_i,S_i,V_i)$, $(\Gamma_2,\Lambda_2,Y) = \tilde{\cup}_{j=1}^{m}(Y_j,S_j,V_j)$,

$$(\Gamma_1,\Lambda_1,Y)\tilde{\cap}(\Gamma_2,\Lambda_2,Y) = \tilde{\cup}_{i=1}^{n}(Y_i,S_i,V_i) \;\tilde{\cap}\; \tilde{\cup}_{j=1}^{m}(Y_j,S_j,V_j) = \tilde{\cup}_{j=1}^{m}(Y_j,S_j,V_j).$$

This is finite intersection of elements of $\tilde{\mathcal{BS}}$ so in $\tilde{\mathcal{Y}}_{\tilde{\mathcal{BS}}}$. Therefore $\tilde{\mathcal{Y}}_{\tilde{\mathcal{BS}}}$ is a bs-base. □

5. Conclusion

The aim of this study is to give some applications by defining the concept of bipolar soft ordered topology and to lead the studies that can be done on this bipolar soft ordered topological structure. For this, we first gave a new concept for the bipolar soft topology. We also established a relationship $\tilde{R}$ on a bipolar soft set by completing the concept of bipolar soft interval. Finally, thanks to $\tilde{R}$, the concept of bipolar soft ordered topology and some examples on this bipolar soft ordered topology are given.

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NONEXISTENCE OF GLOBAL SOLUTIONS FOR A KIRCHHOFF-TYPE VISCOELASTIC EQUATION WITH DISTRIBUTED DELAY

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Abstract. In this paper, we consider a Kirchhoff-type viscoelastic equation with distributed delay and source terms. We obtain the nonexistence of global solutions under suitable conditions.

1. Introduction

In this paper, we consider the following Kirchhoff-type viscoelastic equation with distributed delay and source terms

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds & \\ +\mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(q)|\, u_t(x,t-q)\,dq & \\ = b|u|^{p-2}u, & (x,t)\in\Omega\times(0,\infty), \\ u(x,t)=0, & x\in\partial\Omega, \\ u_t(x,-t) = f_0(x,t), & (x,t)\in\Omega\times(0,\tau_2), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where b, $\mu_1 > 0$, $p > 2$ and τ_1, τ_2 are the time delay with $0 \le \tau_1 < \tau_2$, μ_2 is an L^∞ function, and g is a differentiable function under the assumptions (A1), (A2), and (A3). $M(s)$ is a nonnegative function of C^1 for $s \ge 0$ satisfy, $M(s) = m_0 + \alpha s^\gamma$, $m_0 > 0$, $\alpha \ge 0$ and $\gamma \ge 0$, specially we take $M(s) = 1 + s^\gamma$ where $m_0 = 1$, $\alpha = 1$.

Problems about the mathematical behavior of solutions for PDEs with time delay effects have become interesting for many authors mainly because time delays often appear in many practical problems such as thermal, economic phenomena, biological, chemical, physical, electrical engineering systems, mechanical applications and medicine. Moreover, it is well known that delay effects may destroy the stabilizing properties of a well-behaved system. In the literature, there are several examples that illustrate how time delays destabilize some internal or boundary control system [5, 6]. Viscous materials are the opposite of elastic materials that posses the ability to dissipate and store the mechanical energy. The mechanical

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properties of these viscous substances are of great importance when they seem in many natural sciences applications [2]. The problem (1.1) is a general form of a model introduced by Kirchhoff [7]. To be more precise, Kirchhoff recommended a model denoted by the equation for $f = g = 0$,

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} + g\left(\frac{\partial u}{\partial t}\right) = \left\{\rho_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right\}\frac{\partial^2 u}{\partial x^2} + f(u), \tag{1.2}$$

for $0 < x < L$, $t \geq 0$, where $u(x,t)$ is the lateral displacement, E is the Young modulus, ρ is the mass density, h is the cross-section area, L is the lenght, ρ_0 is the initial axial tension, δ is the resistance modulus, and f and g are the external forces. Furthermore, (1.2) is called a degenerate equation when $\rho_0 = 0$ and nondegenerate one when $\rho_0 > 0$.

In 1986, Datko et al. [4] indicated that delay is a source of instability. In [9], Nicaise and Pignotti considered the following wave equation with a linear damping and delay term

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = 0. \tag{1.3}$$

They obtained some stability results in the case $0 < \mu_2 < \mu_1$. In the absence of delay, Zuazua [23] looked into exponentially stability for the equation (1.3).

Wu and Tsai [24], considered the following Kirchhoff-type equation

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + |u_t|^{r-2} u_t = |u|^{p-2} u, \tag{1.4}$$

with the positive upper bounded initial energy and they obtained the blow-up of solutions for the equation (1.4). In 2013, Ye [22], considered the global existence results by constructing a stable set in $H_0^1(\Omega)$ and showed the decay by using a lemma of Komornik for the nonlinear Kirchhoff-type equation (1.4) with dissipative term.

When $M(s) = 1$, the equation (1.1) becomes the following form

$$\begin{aligned} & u_{tt} - \Delta u - \omega \Delta u_t + \textstyle\int_0^t g(t-s)\Delta u(s)\, ds \\ & + \mu_1 u_t + \textstyle\int_{\tau_1}^{\tau_2} |\mu_2(\rho)|\, u_t(x, t-\rho)\, d\rho \\ & = b|u|^{p-2} u. \end{aligned} \tag{1.5}$$

In [2], Choucha et al. obtained the blow-up of solutions under appropriate conditions of the equation (1.5). In [3], the authors showed the exponential growth of solution for the equation (1.5). In recent years, some other authors investigate hyperbolic type equations (see [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]).

In this paper, we consider the Kirchhoff-type ($M\left(\|\nabla u\|^2\right)$) viscoelastic equation (1.1) with distributed delay ($\int_{\tau_1}^{\tau_2} |\mu_2(q)|\, u_t(x, t-q)\, dq$) and source ($b|u|^{p-2} u$) terms. Our aim is to obtain the nonexistence of global solutions for the equation (1.1).

The paper is organized as follows: In section 2, we give some materials that will be used later. In section 3, we state and prove our main result.

2. Preliminaries

In this part, we give materials for the proof of our result. As usual, the notation $\|.\|_p$ denotes L^p norm, and $(.,.)$ is the L^2 inner product. In particular, we write $\|.\|$ instead of $\|.\|_2$.

Now, we denote some assumptions used in this paper:

(A1) $g : R_+ \to R_+$ is a decreasing and differentiable function, that

$$g(t) \geq 0,\ 1 - \int_0^\infty g(s)\, ds = l > 0. \tag{2.1}$$

(A2) There exists a constant $\xi > 0$, that

$$g'(t) \leq -\xi g(t),\ t \geq 0. \tag{2.2}$$

(A3) $\mu_2 : [\tau_1, \tau_2] \to R$ is an L^∞ function, that

$$\left(\frac{2\delta - 1}{2}\right) \int_{\tau_1}^{\tau_2} |\mu_2(\rho)|\, d\rho \leq \mu_1,\ \delta > \frac{1}{2}. \tag{2.3}$$

Let $B_p > 0$ be the constant satisfies [1]

$$\|v\|_p \leq B_p \|\nabla v\|_p,\ \text{ for } v \in H_0^1(\Omega). \tag{2.4}$$

It holds

(2.5)

$$\int_0^t g(t-s)(\nabla u(s), \nabla u_t(t))\, ds = -\tfrac{1}{2} g(t) \|\nabla u(t)\|^2 + \tfrac{1}{2}(g' o \nabla u)(t)$$
$$-\tfrac{1}{2}\tfrac{d}{dt}\left[(g o \nabla u)(t) - \left(\int_0^t g(s)\, ds\right)\|\nabla u(t)\|^2\right],$$

where

$$(g o \nabla u)(t) = \int_\Omega \int_0^t g(t-s)\, |\nabla u(t) - \nabla u(s)|^2\, ds. \tag{2.6}$$

Firstly, as in [8], we introduce the new variable

$$y(x, \rho, q, t) = u_t(x, t - q\rho),$$

thus, we get

$$\begin{cases} q y_t(x, \rho, q, t) + y_\rho(x, \rho, q, t) = 0, \\ y(x, 0, q, t) = u_t(x, t). \end{cases} \tag{2.7}$$

Hence, problem (1.1) is equivalent to:

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)\, ds \\ +\mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(q)|\, |y(x,1,q,t)|\, dq \\ = b|u|^{p-2}u, \qquad x \in \Omega,\ t > 0, \\ q y_t(x, \rho, q, t) + y_\rho(x, \rho, q, t) = 0, \end{cases} \tag{2.8}$$

with initial and boundary conditions

$$\begin{cases} u(x,t) = 0, \qquad x \in \partial\Omega, \\ y(x, \rho, q, 0) = f_0(x, q\rho), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \end{cases} \tag{2.9}$$

where

$$(x, \rho, q, t) \in \Omega \times (0,1) \times (\tau_1, \tau_2) \times (0, \infty).$$

Theorem 2.1. *Suppose that (2.1), (2.2) and (2.3) hold. Let*

$$\begin{cases} p \geq 2, & n = 1, 2, \\ 2 < p < \frac{2n-2}{n-2}, & n \geq 3. \end{cases} \tag{2.10}$$

Thus, for any initial data

$$(u_0, u_1, f_0) \in H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega \times (0,1) \times (\tau_1, \tau_2)),$$

the problem (2.8)-(2.9) has a unique solution

$$u \in C\left([0,T]; H_0^1(\Omega)\right),$$

for some $T > 0$.

Now, we define the energy functional as follows:

Lemma 2.2. *Suppose that (2.1), (2.2), (2.3) and (2.10) hold. Let u be a solution of (2.8). Then, $E(t)$ is nonincreasing, such that*

$$\begin{aligned} E(t) = {} & \tfrac{1}{2}\|u_t\|^2 + \tfrac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|^2 \\ & + \tfrac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \tfrac{1}{2}(go\nabla u)(t) \\ & + \tfrac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q\,|\mu_2(q)|\left|y^2(x,\rho,q,t)\right| dq\,d\rho\,dx - \tfrac{b}{p}\|u\|_p^p, \end{aligned} \tag{2.11}$$

which satisfies

$$E'(t) \leq -c_1\left(\|u_t\|^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)|\left|y^2(x,1,q,t)\right| dq\,dx\right). \tag{2.12}$$

Proof. By multiplying the first equation of (2.8) by u_t and integrating over Ω, we obtain

$$\begin{aligned} & \tfrac{d}{dt}\Big\{\tfrac{1}{2}\|u_t\|^2 + \tfrac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|^2 \\ & + \tfrac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \tfrac{1}{2}(go\nabla u)(t) - \tfrac{b}{p}\|u\|_p^p\Big\} \\ & = -\mu_1\|u_t\|^2 - \int_\Omega u_t \int_{\tau_1}^{\tau_2} |\mu_2(q)|\,|y(x,1,q,t)|\,dq\,dx \\ & + \tfrac{1}{2}(g'o\nabla u)(t) - \tfrac{1}{2}g(t)\|\nabla u\|^2, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} & \tfrac{d}{dt}\tfrac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q\,|\mu_2(q)|\left|y^2(x,\rho,q,t)\right| dq\,d\rho\,dx \\ & = -\tfrac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} 2\,|\mu_2(q)|\,y y_\rho\,dq\,d\rho\,dx \\ & = \tfrac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)|\left|y^2(x,0,q,t)\right| dq\,dx \\ & - \tfrac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)|\left|y^2(x,1,q,t)\right| dq\,dx \\ & = \tfrac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(q)|\,dq\right)\|u_t\|^2 \\ & - \tfrac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)|\left|y^2(x,1,q,t)\right| dq\,dx. \end{aligned} \tag{2.14}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}E(t) &= -\mu_1\|u_t\|^2-\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,|u_t y(x,1,q,t)|\,dqdx+\frac{1}{2}(g'o\nabla u)(t)\\ &\quad -\frac{1}{2}g(t)\|\nabla u\|^2+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,dq\right)\|u_t\|^2\\ &\quad -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|\left|y^2(x,1,q,t)\right|dqdx. \qquad (2.15)\end{aligned}$$

By using (2.13) and (2.14), we obtain (2.11). Utilizing Young's inequality, (2.1), (2.2), (2.3) and (2.15), we get (2.12). Hence, we complete the proof. □

Lemma 2.3. [2] *There exists $c>0$, depending on Ω only, such that*

$$\left(\int_\Omega |u|^p\,dx\right)^{s/p}\leq c\left[\|\nabla u\|^2+\|u\|_p^p\right], \qquad (2.16)$$

for all $u\in L^{p+1}(\Omega)$ and $2\leq s\leq p$.

Using the fact that $\|u\|_2^2\leq c\|u\|_p^2\leq c\left(\|u\|_p^p\right)^{2/p}$, we have the corollary as follows:

Corollary 2.3.1. There exists $C>0$, depending on Ω only, that

$$\|u\|_2^2\leq c\left[\|\nabla u\|_2^{4/p}+\left(\|u\|_p^p\right)^{2/p}\right]. \qquad (2.17)$$

Lemma 2.4. [2] *There exists $C>0$, depending on Ω only, such that*

$$\|u\|_p^s\leq C\left[\|\nabla u\|^2+\|u\|_p^p\right], \qquad (2.18)$$

for all $u\in L^{p+1}(\Omega)$ and $2\leq s\leq p$.

Now, we define the functional as follows:

$$\begin{aligned}H(t) &= -E(t)\\ &= \frac{b}{p}\|u\|_p^p-\frac{1}{2}\|u_t\|^2\\ &\quad -\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|^2\\ &\quad -\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}-\frac{1}{2}(go\nabla u)(t)\\ &\quad -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}q|\mu_2(q)|\left|y^2(x,\rho,q,t)\right|dqd\rho dx. \qquad (2.19)\end{aligned}$$

3. Nonexistence of solutions

In this part, we obtain the nonexistence of global solutions for the problem (2.8)-(2.9).

Theorem 3.1. *Suppose that (2.1)-(2.3) and (2.10) hold. Suppose further that $E(0)<0$ holds. Then, the solution of the problem (2.8)-(2.9) blows up in finite time.*

Proof. By (2.11), we get

$$E(t) \leq E(0) \leq 0. \tag{3.1}$$

Hence

$$\begin{aligned} H'(t) &= -E'(t) \\ &\geq c_1 \left(\|u_t\|^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| \left| y^2(x,1,q,t) \right| dqdx \right) \\ &\geq c_1 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| \left| y^2(x,1,q,t) \right| dqdx \geq 0, \end{aligned} \tag{3.2}$$

and

$$0 \leq H(0) \leq H(t) \leq \frac{b}{p} \|u\|_p^p. \tag{3.3}$$

Set

$$\mathcal{K}(t) = H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t dx + \frac{\varepsilon \mu_1}{2} \int_\Omega u^2 dx, \tag{3.4}$$

here $\varepsilon > 0$ to be specified later and

$$\frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{2p} < 1. \tag{3.5}$$

We multiply the first equation of (2.8) by u and with a derivative of (3.4), to obtain

$$\begin{aligned} \mathcal{K}'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) \\ &\quad + \varepsilon \|u_t\|^2 + \varepsilon \int_\Omega \nabla u \int_0^t g(t-s) \nabla u(s) dsdx \\ &\quad - \varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|^{2(\gamma+1)} + \varepsilon b \int_\Omega |u|^p dx \\ &\quad - \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| |uy(x,1,q,t)| dqdx. \end{aligned} \tag{3.6}$$

By using

$$\begin{aligned} &\varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| |uy(x,1,q,t)| dqdx \\ &\leq \varepsilon \left\{ \delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(q)| dq \right) \|u\|^2 \right. \\ &\quad \left. + \frac{1}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| \left| y^2(x,1,q,t) \right| dqdx \right\}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} & \varepsilon \int_0^t g(t-s)\, ds \int_\Omega \nabla u \nabla u(s)\, dx ds \\ = & \varepsilon \int_0^t g(t-s)\, ds \int_\Omega \nabla u (\nabla u(s) - \nabla u(t))\, dx ds \\ & + \varepsilon \int_0^t g(s)\, ds \|\nabla u\|^2 \\ \geq & \frac{\varepsilon}{2} \int_0^t g(s)\, ds \|\nabla u\|^2 - \frac{\varepsilon}{2} (go\nabla u)(t) . \end{aligned} \tag{3.8}$$

By (3.6), we get

$$\begin{aligned} \mathcal{K}'(t) \geq & (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \left(1 - \frac{1}{2}\int_0^t g(s)\, ds\right) \|\nabla u\|^2 \\ & - \varepsilon \|\nabla u\|^{2(\gamma+1)} + \varepsilon b \|u\|_p^p - \varepsilon \delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(q)|\, dq\right) \|u\|^2 \\ & - \frac{\varepsilon}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| \left|y^2(x,1,q,t)\right| dq dx + \frac{\varepsilon}{2} (go\nabla u)(t) . \end{aligned} \tag{3.9}$$

By using (3.2) and setting δ_1 such that, $\frac{1}{4\delta_1 c_1} = \kappa H^{-\alpha}(t)$, we obtain

$$\begin{aligned} \mathcal{K}'(t) \geq & [(1-\alpha) - \varepsilon\kappa] H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 \\ & - \varepsilon \left[\left(1 - \frac{1}{2}\int_0^t g(s)\, ds\right)\right] \|\nabla u\|^2 - \varepsilon \|\nabla u\|^{2(\gamma+1)} \\ & + \varepsilon b \|u\|_p^p - \varepsilon \frac{H^\alpha(t)}{4c_1\kappa} \left(\int_{\tau_1}^{\tau_2} |\mu_2(q)|\, dq\right) \|u\|^2 + \frac{\varepsilon}{2} (go\nabla u)(t) . \end{aligned} \tag{3.10}$$

For $0 < a < 1$, by (2.19)

$$\begin{aligned} \varepsilon b \|u\|_p^p = & \varepsilon p (1-a) H(t) + \frac{\varepsilon p (1-a)}{2} \|u_t\|^2 + \varepsilon b a \|u\|_p^p \\ & + \frac{\varepsilon p (1-a)}{2} \left(1 - \int_0^t g(s)\, ds\right) \|\nabla u\|^2 \\ & + \frac{\varepsilon p (1-a)}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} + \frac{\varepsilon}{2} p (1-a) (go\nabla u)(t) \\ & + \frac{\varepsilon p (1-a)}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q |\mu_2(q)| \left|y^2(x,\rho,q,t)\right| dq d\rho dx, \end{aligned} \tag{3.11}$$

with (3.10), it gives

$$\begin{aligned}
\mathcal{K}'(t) \geq\ & [(1-\alpha)-\varepsilon\kappa] H^{-\alpha}(t) H'(t) + \varepsilon\left[\frac{p(1-a)}{2}+1\right]\|u_t\|^2 \\
&+\varepsilon\left[\left(\frac{p(1-a)}{2}\right)\left(1-\int_0^t g(s)\,ds\right)-\left(1-\frac{1}{2}\int_0^t g(s)\,ds\right)\right]\|\nabla u\|^2 \\
&+\varepsilon\left(\frac{p(1-a)}{2(\gamma+1)}-1\right)\|\nabla u\|^{2(\gamma+1)} - \varepsilon\frac{H^\alpha(t)}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,dq\right)\|u\|^2 \\
&+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} q|\mu_2(q)|\left|y^2(x,\rho,q,t)\right| dq\,d\rho\,dx \\
&+\varepsilon p(1-a) H(t) + \varepsilon ba\|u\|_p^p + \frac{\varepsilon}{2}(p(1-a)+1)(go\nabla u)(t). \qquad (3.12)
\end{aligned}$$

By using (2.17), (3.3) and Young's inequality, we obtain

$$\begin{aligned}
H^\alpha(t)\|u\|_2^2 &\leq \left(b\int_\Omega |u|^p\,dx\right)^\alpha \|u\|_2^2 \\
&\leq c\left\{\left(\int_\Omega |u|^p\,dx\right)^{\alpha+2/p} + \left(\int_\Omega |u|^p\,dx\right)^\alpha \|\nabla u\|_2^{4/p}\right\} \\
&\leq c\left\{\left(\int_\Omega |u|^p\,dx\right)^{(p\alpha+2)/p} + \|\nabla u\|_2^2 + \left(\int_\Omega |u|^p\,dx\right)^{p\alpha/(p-2)}\right\}. \qquad (3.13)
\end{aligned}$$

By exploiting (3.5), we obtain

$$2<\alpha p+2\leq p \text{ and } 2<\frac{\alpha p^2}{p-2}\leq p.$$

As a result, by Lemma 2.2, such that

$$H^\alpha(t)\|u\|_2^2 \leq c\left(\|u\|_p^p + \|\nabla u\|_2^2\right). \qquad (3.14)$$

By combining (3.12) and (3.14), we have

$$\begin{aligned}
\mathcal{K}'(t) \geq\ & [(1-\alpha)-\varepsilon\kappa] H^{-\alpha}(t) H'(t) \\
&+\varepsilon\left[\frac{p(1-a)}{2}+1\right]\|u_t\|^2 + \frac{\varepsilon}{2}(p(1-a)+1)(go\nabla u)(t) \\
&+\varepsilon\left\{\left(\frac{p(1-a)}{2}-1\right) - \int_0^t g(s)\,ds\left(\frac{p(1-a)-1}{2}\right)\right. \\
&\left.-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,dq\right)\right\}\|\nabla u\|^2 \\
&+\varepsilon\left[ab-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,dq\right)\right]\|u\|_p^p \\
&+\varepsilon\left(\frac{p(1-a)}{2(\gamma+1)}-1\right)\|\nabla u\|^{2(\gamma+1)} + \varepsilon p(1-a)H(t) \\
&+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} q|\mu_2(q)|\left|y^2(x,\rho,q,t)\right| dq\,d\rho\,dx. \qquad (3.15)
\end{aligned}$$

Taking $a>0$ small enough, that

$$\alpha_1=\frac{p(1-a)}{2}-1>0$$

and suppose

$$\int_0^\infty g(s)\,ds<\frac{\frac{p(1-a)}{2}-1}{\left(\frac{p(1-a)}{2}-\frac{1}{2}\right)}=\frac{2\alpha_1}{2\alpha_1+1}. \tag{3.16}$$

Choosing κ such that,

$$\begin{aligned}\alpha_2 &= \left(\frac{p(1-a)}{2}-1\right)-\int_0^t g(s)\,ds\left(\frac{p(1-a)-1}{2}\right)\\ &\quad -\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,dq\right)\\ &> 0\end{aligned}$$

and

$$\alpha_3=ab-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|\,dq\right)>0 \text{ and } \frac{p(1-a)}{2(\gamma+1)}-1>0.$$

Fixing κ and a, we have ε small enough,

$$\alpha_4=(1-\alpha)-\varepsilon\kappa>0.$$

Hence, for some $\beta>0$, (3.15) becomes

$$\begin{aligned}\mathcal{K}'(t) &\geq \beta\Big\{H(t)+\|u_t\|^2+\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+(go\nabla u)(t)\\ &\quad +\|u\|_p^p+\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} q\,|\mu_2(q)|\left|y^2(x,\rho,q,t)\right|dq\,d\rho\,dx\Big\}.\end{aligned} \tag{3.17}$$

Therefore,

$$\mathcal{K}(t)\geq\mathcal{K}(0)>0,\ t>0. \tag{3.18}$$

Now, utilizing Holder's and Young's inequalities, we obtain

$$\begin{aligned}\|u\|_2 &= \left(\int_\Omega u^2dx\right)^{\frac{1}{2}}\\ &\leq \left[\left(\int_\Omega\left(|u|^2\right)^{p/2}dx\right)^{\frac{2}{p}}\left(\int_\Omega 1dx\right)^{1-\frac{2}{p}}\right]^{\frac{1}{2}}\\ &\leq C\|u\|_p\end{aligned} \tag{3.19}$$

and

$$\left|\int_\Omega uu_tdx\right|\leq\|u_t\|_2\|u\|_2\leq c\|u_t\|_2\|u\|_p.$$

Hence,

$$\begin{aligned}\left|\int_\Omega uu_tdx\right|^{\frac{1}{1-\alpha}} &\leq c\|u_t\|_2^{\frac{1}{1-\alpha}}\|u\|_p^{\frac{1}{1-\alpha}}\\ &\leq c\left[\|u_t\|_2^{\frac{\theta}{1-\alpha}}+\|u\|_p^{\frac{\mu}{1-\alpha}}\right],\end{aligned} \tag{3.20}$$

here $\frac{1}{\mu}+\frac{1}{\theta}=1$. Taking $\theta=2(1-\alpha)$, we have

$$\frac{\mu}{1-\alpha}=\frac{2}{1-2\alpha}\leq p.$$

For $s=\frac{2}{(1-2\alpha)}$, we get

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}}\leq c\left(\|u_t\|_2^2+\|u\|_p^s\right).$$

Hence, Lemma 2.3 gives

$$\begin{aligned}\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} &\leq c\left[\|u_t\|_2^2+\|u\|_p^p+\|\nabla u\|_2^2\right]\\ &\leq c\left[\|u_t\|_2^2+\|u\|_p^p+\|\nabla u\|_2^2+\|\nabla u\|^{2(\gamma+1)}+(go\nabla u)(t)\right]. \qquad (3.21)\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{K}^{\frac{1}{1-\alpha}}(t) &= \left(H(t)^{1-\alpha}+\varepsilon\int_\Omega uu_t dx+\frac{\varepsilon\mu_1}{2}\int_\Omega u^2dx\right)^{\frac{1}{1-\alpha}}\\ &\leq c\left[H(t)+\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}}+\|u\|_2^{\frac{2}{1-\alpha}}+\|\nabla u\|_2^{\frac{2}{1-\alpha}}\right]\\ &\leq c\left[H(t)+\|u_t\|^2+\|u\|_p^p+\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+(go\nabla u)(t)\right]. \qquad (3.22)\end{aligned}$$

By (3.17) and (3.22), we obtain

$$\mathcal{K}'(t)\geq\lambda\mathcal{K}^{\frac{1}{1-\alpha}}(t), \qquad (3.23)$$

here $\lambda>0$, which depends on β and c. An integration of (3.23), we get

$$\mathcal{K}^{\frac{\alpha}{1-\alpha}}(t)\geq\frac{1}{\mathcal{K}^{\frac{-\alpha}{1-\alpha}}(0)-\lambda\frac{\alpha}{(1-\alpha)}t}.$$

Therefore, $\mathcal{K}(t)$ blows up in time

$$T\leq T^*=\frac{1-\alpha}{\lambda\alpha\mathcal{K}^{\alpha/(1-\alpha)}(0)}.$$

Then, the proof is completed. □

4. Conclusion

In recent years, there has been published much work concerning the wave equations (Kirchhoff, Petrovsky, Bessel,... etc.) with different state of delay time (constant delay, time-varying delay,... etc.). However, to the best of our knowledge, there were no nonexistence of global results for the Kirchhoff-type viscoleastic equation with distributed delay and source terms. We have been obtained the nonexistence of global solutions under suitable conditions.

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STATISTICAL RELATIONS MEASURES

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ABSTRACT. In this study, starting with the researches method and studies, which scale used and in accordance with this scale, which data statistical relationship measures to be applicated were given. In accordance with this aim, 18 tests were taken into consideration which consisted of statistical relationship measures tests. Also, data sets which were taken from2148 students from three different high schools in Kahramanmaraş were examined practically on statistical relationship measures. With the aim of this, knowledge that were taken from the students from Science High School and Vocational High School were used. In the study, the effects of the factors like gender, education status of the father, private room status, number of siblings, computer status, living place,taking private lesson status, age of the student, body weight of the student and the income of the family on the effect of high school type were evaluated with the statistical relationship measures. Parametric test were applied to the data sets which are intermittent scale or proportional scale. And non parametric testa were applied to the data sets which are classifier scale or sequential scale. As a result, in the thesis study, the effect of factors that were thought to effect the student's high school type, effect degree and direction were tried to be determined with the relationship measures test.

1. INTRODUCTION

The word statistics comes from the root of the Latin word "status". Statistics; collecting data, summarizing and presenting in the form of figures and graphics, tables, texts is the science that creates methods and theories about data analysis, evaluation, interpretation, decision making. Also, to observe, count and measure a large number of units to investigate collective (collective) events; It is a method of analyzing the results in order to group and interpret them (Alpar, 1995). As a result of field studies, tests are needed to interpret scientific and objective results. No matter how good the theoretical part of a study is, if the statistical tests used for the application of the study are not scientifically appropriate, the expected results from the studies are not obtained. In this case, when deciding on the statistical tests of a study, it should be well determined which statistical techniques can be analyzed for the data obtained. The degree of relationship between the data is shown by the correlation coefficients. Parametric tests are used if the received data is intermittent or proportional, non-parametric tests are used if it is classifier or sequencer.

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Parametric tests are inflexible statistical methods applied according to the relevant parameter, an appropriate distribution and variance. Non-parametric tests, on the other hand, are statistical methods that do not depend on the relevant parameter, an appropriate distribution and variance, and are performed by taking their ranking scores instead of the data. In order to apply parametric tests, the data should be normally distributed and the variances should be homogeneous. On the other hand, non-parametric tests do not need these requirements and do not make assumptions about sample distribution [14].
In the study, information is given about which scale the data will be handled and which statistical method will be applied to the data in line with this scale when starting the researchers' methods and studies. For this purpose, a total of 18 tests consisting of statistical measures tests were examined. In addition, using the data obtained from 2148 students in three different types of high schools in Kahramanmaras, the correlation coefficients that can determine the relationship between the variables are emphasized. As an application, the information obtained from the students studying at Science High School, Anatolian High School and Vocational High School in Kahramanmaras was used. In the study, correlation coefficients were applied to determine whether factors such as gender, father's education level, private room status, number of siblings, computer status, place of residence, taking private tutoring, student's age, student's body weight and family income are related to high school type. Different test statistics should be applied according to the data obtained from students studying in different high school types. Parametric tests are used if the received data is intermittent or proportional, and non-parametric tests are used if it is classifier or sequencer [7,12].

2. MATERIAL

In the study, preliminary information was given about the relationship measures tests, and then a questionnaire about these tests was applied. From the results of the survey, it is shown whether there is a relationship in the correlation coefficients. In addition, hand-solved examples of relationship measure tests are included in the appendix.

In the study, it was determined whether factors such as gender, father's education level, private room status, number of siblings, computer status, place of residence, taking private lessons, student's age, student's body weight and family income are related to high school type, and if there is a relationship, the direction of this relationship. and its degree are shown with the correlation coefficients.

The survey study consists of students who have been educated in three different types of high schools, residing in the province, district and surrounding villages of Kahramanmaras. Students who did not want to participate in the application were excluded from the scope of the research and the questionnaire was applied to 2148 students in total.

Frequency and percentage distributions for the demographic data of the participants are shown as follows. Percent Residence Frequency Percent

Table 1. Distributions by Demographic Characteristics of Participants

Gender	Frequency	Percent	Residence	Frequency	Percent
Female	1198	55,3	Provincial	1304	60,7
Male	960	44,7	Town	308	14,3
Father Educational Status			Village	536	25,0
Primary school	764	35,6	Private Lesson Status		
Middle School	616	28,7	Yes	160	7,4

High school	492	22,9	No	1988	92,6
University	276	12,8	High School Type		
Private Room Status			Vocational high School	556	25,9
Yes	1340	62,4	Anatolian High School	1068	49,7
No	800	37,6	Science High School	524	24,4
Number of siblings			Family Income Status		
0	36	1,7	2000 and below	1032	48,0
1	136	6,3	2000-3000	672	31,3
2	432	20,1	3000-4000	276	12,8
3 and more	1544	71,9	4000 and over	168	7,8
Computer Status					
Yes	1184	55,1			
No	964	44,9			

3.METHOD

In this study, statistical terms and descriptive statistics are defined and then given in detail on statistical measure tests. The solution of a numerical example for each test is also shown.

In general, a statistical test generalized to the determination of a hypothesis (H_0),

$$\text{Test statistic} = \frac{\text{statistics} - \text{parameter}}{\sqrt{\text{var(statistics)}}} \qquad (1)$$

is in the form. As shown in the equation, it is created for the distribution of the statistics and is determined by comparing it with the critical values showing the rejection regions of H_0 with a certain probability (type error) [4,8].

Relationship Measures Tests

It is desired to know how the relationship between two different data sets is. It is necessary to create a separate test statistic according to the data taken from the data set [10,13]. The data are analyzed under the headings of classifier, ordinal, interval, interval or proportional scale, and classifier or ordinal scale variables.

Classifier Scale Variables

Goodman and Kruskal Gamma Statistics

Goodman and Kruskal Gamma statistics, known as the gamma test, creates a symmetric measure of the measurement link in two ordinal variables. Goodmann and Kruskal Gamma statistics indicate a difference between the (P) congruent and (Q) discordant pair. When the value of the Goodman and Kruskal Gamma statistic is 1 or close to 1, the connection level increases, and when it is close to 0, the connection level decreases. The same method is used to calculate the Somers D statistics [1,3].

Cramer V Statistics

Although the probability coefficient is widely used in social sciences and sciences, it also has a disadvantage. In order to calculate the Cramer V statistic, nxn of the tables, that is, rows and columns, must be equal. For example, it is applied to tables created as 3x3, 4x4, 5x5 … instead of 2x3, 3x4, 5x4 tables. [2,9].

Phi Coefficient

Phi coefficient is explained as a non-parametric test applied to find the correlation coefficient of data in 2x2 size tables created with nominal scales. It is also known as a correlation coefficient that calculates the size of the link between two variables. Phi coefficient is also known as Kendall's correlation coefficient. It explains the relationship dimension of the variables that are qualitatively dichotomous (hardworking-lazy, bad-good, thin-fat) between two variables. Phi coefficient is determined by the symbol .

Lambda λ Statistic

Lambda λ statistics, also called Gutman's estimation coefficient, in which dependent and independent variables affect each other, are applied in error rates. Lambda λ statistics, which is a classifying scale, is applied as a correlation statistic that compares more than one group or category, as in Cramer V, Phi and Probability coefficient. The feature that distinguishes the lambda λ statistic from other statistics is that it has an asymmetric structure. Lambda λ statistic is also applied symmetrically. In the Lambda λ statistic, it takes values between [5].

Probability Coefficient C Statistic

Probability coefficient C statistic, created with a classifier scale, is defined as a non-parametric statistics created to indicate the correlation coefficient of groups or figures rather than 2x2 tables. The probability coefficient C statistic is determined as a symmetrical structure [3,4].

Relative Risk

It shows the measure of the connection between the occurrence or absence of a situation or an event. Relative Risk also creates risk estimates for the future.

Odds Ratio

The 'odds ratio', also known as the relative odds ratio or estimated relative risk, is defined as a measure of effect size. Odds ratio is applied to determine the risk of the population as a result of retrospective studies of the variables [2,6].

Sorter Scale Variables

Somers D Statistics

Somers D statistic is preferred for asymmetric measurements calculated to show the compatibility or connection between two ordinal variables such as x and y. Somers D Coefficient is expressed between (-1,1) values. That is, it takes a value between $-1 < S_d < +1$. Somers D Statistics model is created as at least 2x2.

Kendall Tau b Statistics

It is among the non-parametric tests. Bi-order variables are also statistical measures that indicate the strength and wool of the connection. Kendall Tau b statistic takes a value between $-1 \leq \tau \leq +1$. In Kendall Tau b statistics, when the number of samples is more than 10, it approaches the normal distribution [9,10].

Kendall Tau c Statistics

The Kendall Tau c statistic is calculated in cases that are rectangular or square. In cases where the Kendall Tau b statistic is not calculated, it is the statistical test that can be calculated. The Kendall tau c statistic is also called Stuart's Tauc or Kendall-Stuart Tau c. In addition, in this test, at least one of the data must be ordinal [9].

Kendall Goodness of Fit Coefficient W Statistic

Spearman, Kendall Tau b and Kendall Tau c statistics are applied to calculate the correlation between two ordinal variables. However, when the ordinal variable value is greater than 3, Kendall W statistics is applied. The Kendall W statistic can be calculated until the ordinal variable value, that is, is $n \leq 7$. In case the ordinal variable value is greater than 7, the distribution of the sample constitutes the Chi-square distribution. Therefore, it is calculated with the chi-square ruler. Kendall W statistics are formed between $0 \leq W \leq 1$ values [3,12].

Spearman Rank Correlation Coefficient

Spearman Rho is applied when there is a linear relationship between the two ordinal variables or between the variables for which the connection is investigated, when one of the variables moves away from the normal distribution. That is, it is used to investigate the connection between two ordinal variables. Spearman Rho is explained as the non-parametric Pearson Correlation coefficient. The most important difference between the two tests is that Spearman deals with Rho's ordinal numbers and Pearson's raw values.

Linear by Linear Relationship Statistics

Variables must be obtained in ordinal scale and created in double-order rxc size. Chi-square test statistics are applied until the sample number of linear statistics is 0. Linear by Linear relationship statistics are also defined as Mantel-Haenszel test [9].

Cohen Kappa Statistics

Evaluation at different locations or by a different observer also examines the similarities between the observer or two different places. Cohen Kappa coefficient takes values between -1 and +1. When the value of Cohen Kappa coefficient approaches 1, it explains a complete fit, a value close to 0 explains inconsistency, and a value close to -1 explains the reverse fit [13,15].

Interval Scale Variables

Pearson Correlation Coefficient

It is used to give information about the strength and direction of the linear relationship between two variables indicated by measurement. When both variables are normally distributed, the graph is continuously variable. To properly use the Pearson correlation coefficient, the data between variables should be applied with an interval scale, both variables should be normally distributed, and both variables should be randomly selected from the population. In other words, it is applied when the relationship between two different x and y variables is linear [4].

Intermittent and Proportional Scale Variables

Eta Coefficient

The eta coefficient is a nonlinear correlation coefficient. It is used for two continuous variables that do not have a linear relationship between them. For this reason, it is also called the relationship ratio. It is also applied for data created with interval and proportional scale. Eta coefficient takes a value between 0 and 1. When the value of Eta is close to 1, the relationship level increases, and when it takes a value close to 0, the relationship level decreases. The eta coefficient is a special case of the Pearson coefficient.

Classifier or Orderer Scale Variables

Yule Q Statistics

The Yule Q statistic is explained as a symmetric measure based on the difference in congruent and incompatible pairs. Yule Q statistic is calculated in 2x2 size tables. Unlike the Phi coefficient, the data can be calculated on a scaled or ordinal scale. Yule takes a value in the range of $-1 \leq \gamma \leq +1$ [11].

Yule Y Statistics

The Yule Y statistic is shown as the rank coefficient. In this test, it is obtained by taking the geometric mean of Yule. In Yule Y statistics, the difference between the marginal distribution (which can be variable in unit sense) between two variables is weaker than Yule Q statistics [3,14].

4. FINDINGS AND DISCUSSION

In the study, correlation coefficients were applied to determine whether factors such as gender, father's education level, private room status, number of siblings, computer status, place of residence, taking private tutoring, student's age, student's body weight and family income are related to high school type. Analyzes were created according to the 5% significance level. Therefore, the results are stated as 'relation' if the sign value is less than 0.05, and 'no relationship' if the sign value is greater than 0.05.

Table 2. Analysis of High School Type by Gender

Relationship Measures	Coefficients	P
Pearson Correlation Coefficient	899,215	0,000
Linear by Linear Relation	759,545	0,000
Lambda Statistics	0,343	0,000
Goodman and Kruskal Statistics	0,301	0,000
Uncertainty Coefficient	0,559	0,000
Somers D Statistics	0,633	0,000
Eta Coefficient	0,595	0,000
Phi Coefficient	0,647	0,000
Cramer V Statistics	0,647	0,000
Kendal Tau b Statistics	0,562	0,000
Kendal Tau c Statistics	0,626	0,000
Spearman Rank Correlation Coefficient	0,593	0,000
Cohen Koppa Statistics	0,023	0,000

As a result of the analysis of the relationship measures tests according to Table 2, it was decided that the type of high school was dependent on gender ($p < 0{,}01$).

Table 3. Analysis of High School Type by Father's Educational Status

Relationship Measures	Coefficients	P
Pearson Correlation Coefficient	416,557	0,000
Linear by Linear Relation	280,227	0,000
Lambda Statistics	0,089	0,000
Goodman and Kruskal Statistics	0,76	0,000
Uncertainty Coefficient	0,312	0,000
Somers D Statistics	0,292	0,000
Eta Coefficient	0,365	0,000
Phi Coefficient	0,440	0,000
Cramer V Statistics	0,311	0,000
Kendal Tau b Statistics	0,313	0,000
Kendal Tau c Statistics	0,316	0,000
Spearman Rank Correlation Coefficient	0,351	0,000
Cohen Koppa Statistics	0,131	0,106

According to Table 3, it is seen that there is an independent relationship between father's education level and high school type in Cohen Koppa statistical analysis ($p > 0{,}05$). However, in the analysis of other relationship measures, it is seen that there is a positive significant relationship between father's education status and high school type($p < 0{,}01$).

Table 4. Analysis of High School Type by Private Room Status

Relationship Measures	Coefficients	P
Pearson Correlation Coefficient	17,328	0,008
Linear by Linear Relation	0,980	0,322
Lambda Statistics	0,002	0,045
Goodman and Kruskal Statistics	0,005	0,007
Uncertainty Coefficient	0,002	0,915
Somers D Statistics	0,002	0,915

Eta Coefficient	0,036	0,000
Phi Coefficient	0,90	0,008
Cramer V Statistics	0,64	0,008
Kendal Tau b Statistics	0,002	0,915
Kendal Tau c Statistics	0,002	0,915
Spearman Rank Correlation Coefficient	0,21	0,916
Cohen Koppa Statistics	0,11	0,322

According to Table 4, in the statistical analysis of Eta Coefficient, it is seen that there is a positive significant relationship between private room status and high school type ($p < 0{,}01$). However, in the analysis of other relationship measure tests, it is seen that there is no relationship between private room status and high school type ($p > 0{,}05$).

Table 5. Analysis of High School Type by Number of Siblings

Relationship Measures	Coefficients	P
Pearson Correlation Coefficient	31,211	0,000
Linear by Linear Relation	14,164	0,000
Lambda Statistics	0,002	0,689
Goodman and Kruskal Statistics	0,008	0,000
Uncertainty Coefficient	-0,069	0,001
Somers D Statistics	-0,058	0,001
Eta Coefficient	0,105	0,000
Phi Coefficient	0,121	0,000
Cramer V Statistics	0,085	0,000
Kendal Tau b Statistics	-0,070	0,001
Kendal Tau c Statistics	-0,055	0,001
Spearman Rank Correlation Coefficient	-0,130	0,001
Cohen Koppa Statistics	-0,076	0,000

According to Table 5, it is seen that there is no relationship between the number of siblings and the type of high school in the Lambda statistical analysis ($p > 0{,}05$). However, in the analysis of other relationship measure tests, it is seen that there is a positive significant relationship between the number of siblings and high school type ($p < 0{,}01$).

Table 6. Analysis of High School Type by Computer Status

Relationship Measures	Coefficients	P
Pearson Correlation Coefficient	75,232	0,000
Linear by Linear Relation	69,742	0,000
Lambda Statistics	0,041	0,000
Goodman and Kruskal Statistics	0,021	0,000
Uncertainty Coefficient	-0,170	0,000
Somers D Statistics	-0,153	0,000
Eta Coefficient	0,180	0,000
Phi Coefficient	0,187	0,000
Cramer V Statistics	0,132	0,000
Kendal Tau b Statistics	-0,172	0,000
Kendal Tau c Statistics	-0,144	0,000
Spearman Rank Correlation Coefficient	-0,304	0,000
Cohen Koppa Statistics	-0,181	0,000

According to Table 6, it is seen that there is a positive significant relationship between the number of siblings and the type of high school in the analysis of all relationship measure tests ($p < 0{,}01$).

Table 7. Analysis of High School Type by Place of Residence

Relationship Measures	Coefficients	P
Pearson Correlation Coefficient	27,926	0,000
Linear by Linear Relation	0,957	0,328
Lambda Statistics	0,000	0,000
Goodman and Kruskal Statistics	0,007	0,000
Uncertainty Coefficient	-0,011	0,576
Somers D Statistics	-0,010	0,576
Eta Coefficient	0,063	0,000

Phi Coefficient	0,114	0,000
Cramer V Statistics	0,081	0,000
Kendal Tau b Statistics	-0,011	0,576
Kendal Tau c Statistics	-0,010	0,576
Spearman Rank Correlation Coefficient	-0,019	0,576
Cohen Koppa Statistics	-0,011	0,595

According to Table 7, in the statistical analysis of Pearson Correlation Coefficient, Lambda statistic, Goodman and Kruskal statistic, Eta Coefficient and Phi Coefficient, it is seen that there is a positive significant relationship between the place of residence and the type of high school ($p < 0{,}01$). However, in the analysis of other relationship measures, it is seen that there is no relationship between the place of residence and the type of high school ($p > 0{,}05$)

Table 8. Analysis of High School Type by Taking Private Lessons

Relationship Measures	Coefficients	P
Pearson Correlation Coefficient	5,525	0,063
Linear by Linear Relation	1,242	0,265
Lambda Statistics	0,000	0,000[b]
Goodman and Kruskal Statistics	0,002	0,056
Uncertainty Coefficient	0,017	0,234
Somers D Statistics	0,011	0,234
Eta Coefficient	0,051	0,000
Phi Coefficient	0,051	0,063
Cramer V Statistics	0,051	0,063
Kendal Tau b Statistics	0,023	0,234
Kendal Tau c Statistics	0,013	0,234
Spearman Rank Correlation Coefficient	0,079	0,234
Cohen Koppa Statistics	0,024	0,271

According to Table 8, in the statistical analysis of Lambda Statistics and Eta Coefficient, it is seen that there is a positive significant relationship between taking private lessons and high school type($p < 0{,}01$). However, in the analysis of other relationship measure tests, it is seen that there is no relationship between taking private lessons and high school type($p > 0{,}05$).

Table 9. Analysis of High School Type by Family Income Status

Relationship Measures	Coefficients	P
Pearson Correlation Coefficient	244,789	0,000
Linear by Linear Relation	182,873	0,000
Lambda Statistics	0,027	0,002
Goodman and Kruskal Statistics	0,055	0,000
Uncertainty Coefficient	0,230	0,000
Somers D Statistics	0,234	0,000
Eta Coefficient	0,316	0,000
Phi Coefficient	0,338	0,000
Cramer V Statistics	0,239	0,000
Kendal Tau b Statistics	0,230	0,000
Kendal Tau c Statistics	0,220	0,000
Spearman Rank Correlation Coefficient	0,352	0,000
Cohen Koppa Statistics	0,081	0,000

According to Table 9, it is seen that there is a positive significant relationship between family income status and high school type in the analysis of all relationship measure tests ($p < 0{,}01$).

5. CONCLUSIONS

In this study, correlation measures tests were examined under the headings of classifier scaled variables, ordinal scaled variables, intermittent scaled variables, interval scaled-ratio scaled variables, and classifier-ordered scaled variables. For this purpose, a total of 18 tests consisting of statistical measures tests were examined. In the study, the correlation coefficients that can determine the relationship between the variables were emphasized by using the data obtained from 2148 students in three different types of high schools in Kahramanmaras. As an application, the information obtained from the students studying at Science High School, Anatolian High School and Vocational High School in Kahramanmaras was used.

In line with this study, factors such as gender, father's education level, private room status, number of siblings, computer status, place of residence, taking private lessons, student's age, student's body weight, and family income were determined correlation coefficients were applied to determine whether it is related to the type of high school. Different test statistics were applied according to the data obtained from students studying in different types of high schools. Parametric tests were used if the data were intermittent or proportional, and non-parametric tests were used if they were classifiers or sequencers.

As a result, in the analysis of all statistical relationship measures between the type of high school and gender, computer status and income status of the family, it was seen that there was a positive significant relationship ($p < 0{,}05$). In the Cohen Koppa statistic analysis made between high school type and father's education level, it was found that there was no relationship. However, in the analysis of other statistical measures tests, it was observed that there was a positive significant relationship ($p < 0{,}05$). In the statistical analysis of the Eta coefficient between the high school type and the private room status of the student, it was seen that there was a positive significant relationship ($p < 0{,}05$). No relationship was found in the analysis of other statistical measures tests ($p > 0{,}05$). It was revealed that there was no relationship in the Lambda statistic analysis made between the type of high school and the number of siblings, but there was a positive significant relationship in the analysis of other statistical measures tests. In the statistical analysis of Pearson Correlation Coefficient, Lambda statistic, Goodman and Kruskal statistic, Eta Coefficient and Phi Coefficient between high school type and place of residence, there was a positive significant relationship ($p < 0{,}05$). However, in the analysis of other relationship measure tests, no relationship was found between the place of residence and the type of high school ($p > 0{,}05$). In the statistical analysis of Lambda statistics and Eta coefficient between the high school type and the student's taking private lessons, it was seen that there was a positive significant relationship ($p < 0{,}05$). In the analysis of other relationship measure tests, it was seen that there was no relationship ($p > 0{,}05$).

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DECISION MAKING PROCESS FOR SERVING RESTAURANTS USING INTUITIONISTIC FUZZY SET THEORY VIA CONTROLLED SETS

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ABSTRACT. The Intuitionistic fuzzy set theory gives quite successful results in decision-making processes when compared to other set theories. For this reason, it finds application areas in many areas of daily life such as political science, robotic systems, economic research, medical studies. The success of the IFS concept in decision-making processes in these areas has also been proven. In this study, it is aimed to create an IFS model that can make suggestions to support the end user in the process of choosing a product. Hamming measure will be used to achieve this goal. As per the definition of this measure, the degree of non-membership of the data is as important as the degree of membership. These values also determine the degree of intuition. However, the easier it is to estimate the yield value of a property of an object, the more difficult it is to estimate the non-fulfillment value. For this reason, in this study, the data will be intuitiveized by the controlled set theory and the relationships with the results will be determined. In all these processes The Microsoft SQL Server data structure for coding was used and algorithms were created according to this coding. A healthy evaluation of the data is as important as the value of the data's feature of not having that feature. This situation is the most important factor determining the value of intuition.

1. INTRODUCTION

The concept of fuzzy cluster, membership degrees and the degree of non-membership given by the fuzzy logic rules are easily obtained. Thus, the classification of an object is easily made over a chain, specifically unit range. However, the heuristic fuzzy set theory revealed that this situation would not be so clear and the degree of intuition was also important. The biggest problem in this theory is determining the degree of non-membership rather than determining the degree to which an object has a feature. Because the degree of intuition also appears depending on these two values. In fact, in heuristic fuzzy set theory, the fact that an object has a property is explained by two independent variables and the third dependent variable connected to them. The biggest challenge faced by many scientists working in this field is choosing two independent variables separately. Because this choice determines the degree of intuition. For example, in determining the height degree of a person whose height is 180 cm as 0.825 and the degree of not being as 0.175, what criterion has been set for the value 0.175? The intuition level for this person is 0.1.

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Is this intuition a human intuition? It is clear that the answer is no. Because, some different methods can be followed in determining the degree of membership, but the degree of non-membership should be linked to our intuition. It is our experience that strengthens our intuition. Then the degree of non-membership of a person whose height is 180 cm should vary according to the group to which he belongs. For example, a value of 0.175 may be acceptable according to people in a city, but in the group with basketball athletes this value is 0 or the membership level should be re-examined.

To solve the above-mentioned situation, controlled sets were defined by Çuvalcıoğlu [3] in 2014, and the fact that an object has a property is expressed by one independent two dependent variables.

Due to the conflicts in the results obtained in the decision-making processes made so far, some improvements have been made or given as they are. The reason for this is that the degree of non-membership and the degree of intuition cannot be determined exactly. However, distance measures use all three values. In that case, the relationship results to be obtained from the distance measures by not being a member closest to the line and using the degree of intuition will be successful. The resulting models will have a more similar structure to human intuition. For distance measurements, see [4,5] respectively.

In this study, the purpose of using this method is to successfully complete the relation of hundreds of features of thousands of products with hundreds of products in a short time and successfully.
The article is organized as follows. In Chapter 2, some basic concepts are given, and in Chapter 3, the controlled set theory to be used in heurization is mentioned. In addition, information was given about the dimensions and properties used in the study. In the section, coding, obtaining the results and evaluating the results were made.

2. PRELIMINARY PREPARATIONS

We recall some basic concepts of IFS, [1, 2].

Definition 2.1. A fuzzy set A in a nonempty set X is an object having the form $A = \{(x, \mu_A(x), 1 - \mu_A(x))xX\}$
where the function $\mu_A : X \to [0,1]$ denoted the degree of membership and 1-μ_A denoted degree of non membership degree of x.

Definition 2.2. An Intuitionistic fuzzy set (briefly I F S see[1]) A in a nonempty set X is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x))xX\}$ where the functions $\mu_A, \gamma_A : X \to [0,1]$ denote the degree of membership and degree of non membership, respectively and $0 \leq \mu_A(x), \gamma_A(x) \leq 1$ for all x∈X. The value $\pi_A(x) = 1 - \mu_A(x) - \gamma_A(x)$ is called hesitation degree of x.

Example 2.3. Let X = {a,b,c,d} an universal and

$$A = \{(a, 0.875, 0.125), (b, 0.54, 0.16), (c, 0.25, 0.35), (d, 0.95, 0.0)\}$$

Intuitionistic fuzzy set on X.The hesitation degrees of a,b,c,d∈X are as follow, respectively

$$\pi_A(a) = 0, \pi_A(b) = 0.3, \pi_A(c) = 0.4, \qquad \pi_A(d) = 0.05$$

Definition 2.4. Let A, B ∈ IFS (X).
i. $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ ve $\gamma_A(x)\ \gamma_B(x)$ $\forall x \in X$.
ii. $A = B \Leftrightarrow A \subseteq B$ ve $B \subseteq A$
iii. $A^c = \{(x, \gamma_A(x), \mu_A(x))xX\}$
iv. $AB = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\gamma_A(x), \gamma_B(x)\}) \mid x \in X\}$

v. $AB = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\gamma_A(x), \gamma_B(x)\}) \mid x \in X\}$

Örnek 2.5. $X = \{a, b, c\}$ be an universal.
$A = \{(a, 0.45, 0.25), (b, 0.75, 0.25), (c, 0.50, 0.15)\}$,
$B = \{(a, 0.35, 0.15), (b, 0.90, 0.05), (c, 0.65, 0.45)\}$
Intuitionistic fuzzy sets on X. Then,
$A^c = \{(a, 0.25, 0.45), (b, 0.25, 0.75), (c, 0.15, 0.50)\}$
$AB = \{(a, 0.45, 0.15), (b, 0.90, 0.05), (c, 0.65, 0.15)\}$
$AB = \{(a, 0.35, 0.25), (b, 0.75, 0.25), (c, 0.50, 0.45)\}$

3. CONTROLLED SET THEORY

Hesitation value is of great importance in applications made using more data than theoretical studies on intuitionistic fuzzy sets, in the general name of decision-making processes. As can be seen from its definition, the calculation of the hesitation value is directly related to the membership degree and non-membership degree. In conventional methods, whether the degree of hesitation is effective on the degree of membership or non-membership is not taken into account. In many studies, the degree of non-membership is tried to be given without any justification. That is, the more concrete the determination of the degree of membership, the more abstract the degree of non-membership or, provided that the condition is met, it is random. This situation negatively affects the role of the hesitation value of the data in the decision-making process.

The Hamming measure to be used in this study works on all these values. Therefore, not only membership degree but also non-membership and hesitation value will be decisive for interpreting the decision-making process in a way that is close to correct.
Due to the situations discussed above, the method of forming controlled sets will be used to consistently determine the non-membership and hesitation value using the membership degrees of the obtained data. Controlled sets were first described by Çuvalcıoğlu [3] in 2014.
Zadeh's example about long peoples, the membership degree of one whose lenght 170 cm is almost 1, say 0.8. But, if we choose the universal as the people whose lenght is longer than 170 cm then the membership degree of the person 171 cm tall is subject of discussion. Because, in this universal, while the membership degree of the person 190 cm tall is almost 1, how can we say taht the membership degree of the person 171 cm tall is 0.8? In Zadeh's example $\mu(171)=0.8$, $1-\mu(171)=0.2$. But, if we choose the universal as above, then we are in the expectations that the universal have an element which membership degree is 0.2. Hence, the membership degree of the persons which 171cm is 0.8. This is a contradiction. In this statement, the best solution can be the membership degree of the persons which 171cm is 0.5. Also in this case, the membership degree of the persons which 180cm is 0.9 thus the non-membership degree of the persons which 180cm is 0.1. This is a contradiction, too.

This problem can be solve by defining a bijective and order preserving function between the image of fuzzy set on Zadeh's universal as above and the image of fuzzy set on subset of Zadeh's same universal. However, in this case, problems associated with the concept appears to be taller. So, in order to find any element in universal with its membership and non-membersip degree, there must also have an element such that it control the other. With this idea, if we use the Zadeh's universal for the long people then the membership degree of the persons which 171cm is 0.8 and thus the non-membership degree of the persons which 171cm is 0.2. But, If we use the subuniversal of the Zadeh's universal then the membership degree of the persons which 171cm is 0.8 and thus the non-membership degree of the persons which 171cm is 0.0. Because, there is not an element which control the non-membership degree of the persons which 171cm.

An extension of the fuzzy theory is the intuitionistic fuzzy theory which have the hessitation degree that is not belong the fuzzy theory. But, there are same problems in Intuitionistic fuzzy theory like Fuzzy theory's. Because, there is not any criterion for non-membership degree of an element. For example, the set A={(x, 0.8, 0.2), (y, 0.4, 0.3)} is an intuitionistic fuzzy set on U={x,y}. But, there is not any controller element for x, like y.

As a result of these discussions, controlled sets were defined by Çuvalcıoğlu [3] as follows.

Definition 4.1: Let E be an universal, μ is a fuzzy set on E. The set E is called μ-controlled set if $\forall x\in E, \exists\, y\in E \ni 1-\mu(x) = \mu(y)$.
If E is μ-controlled set then we write $E\in CS(\mu)$.

Example4.2: Let $X = \{a, b, c, d, e, f\}$ be an universal.
$A = \{(a, 0.45), (b, 0.05), (c, 0.50), (d, 0.35), (e, 0.90), (f, 0.65)\}$ is not a controlled set. But,
$B = \{(a, 0.45), (b, 0.10), (c, 0.50), (d, 0.45), (e, 0.90), (f, 0.55)\}$ is a controlled set. In B, a is controled by f, c is controlled by itself, f controlled by a and d, etc.

4.1 Controlled sets on classical set theory

In this study, it has been shown that the $\mathcal{C} =\{X\subset E: X \in CS(\mu)\}$ family has the maximal element property. We define the set $\tilde{a} = \{b \in E : 1- \mu(a) = \mu(b)\}$.

Proposition4.1.1: Let E be an universal, μ is fuzzy set, $\mathcal{C} =\{X\subset E: X \in CS(\mu)\}$ and $A\in \mathcal{C}$. For $a\in A$, we define $\mathcal{C}' = \{Y_a : a\in A\} \subset \mathcal{C}$ where $\breve{a} = \{c\tilde{a} : cA\}$, $Y_a=\{a\}\cup\breve{a}$ then, we get $A=\cup\, \mathcal{C}'$

With the above proposition, it can be easly seen the family $\mathcal{C}$ has a base as following,

$$\underline{A} = A \cup \widetilde{A}$$

where $\widetilde{A} = \bigcup_{a\in A}\tilde{a}$.

Theorem 4.1.2: Let E be an universal, μ is fuzzy set, $A\in\mathcal{P}(E)$. The mapping $J: P(E) \longrightarrow P(E)$ defined by $J(X) = \underline{X}$ is a closure operator.

From this theorem, it is concluded that the $\mathcal{C}$ family is a closed system, $(\mathcal{C},\subseteq)$ is complete lattice, and every closed subsystems C' of $\mathcal{C}$ is complete lattice. As a result of these, it was obtained that $\mathcal{C}$ is a Moore family. It has also been shown that the $\mathcal{C}$ closed system is algebraic. According to Schimid's theorem $\mathcal{C}$ closed system is inductive, as a result every chain in $\mathcal{C}$ have a supremum in $\mathcal{C}$.
As a result of these properties, it can be easily seen that cluster theoretic properties are workable on controlled sets.

4.2. (α, α^*)-Controlled sets

Another problem is whether a fuzzy set can be created as a controlled set. Considering the studies on intuitionistic fuzzy sets, the main reason for this problem can be explained as follows, The membership degree is very important for an element in any set. But the non-membership degree is very important, too. We can not claim that all sets are controlled set. However, it is possible to construct a set in such a way that it can have a controlled set property. We can introduce controlled set using the membership degrees of elements. The study on the solution of this problem is given by Çuvalcıoğlu [3] as follows.

Definition 4.2.1: Let E be an α-set. We define the following mapping on E as following,

$$\alpha^*(x) = \{\sup_{y\in E}\alpha(y), \alpha(x) \leq 1-\alpha(y) \ \ 0 \qquad \text{, otherwise}$$

It is clear that α^* is a mapping from E to I. In addition, it can be easily seen that the sum of α and α^* is less or equal than 1. From this properties, we can give the following definition

Definition 4.2.2: Let E be α-set. Then the set $A = \{< x, \alpha(x), \alpha^*(x) x \in E\}$ is called (α,α^*)-controlled set.

Example 4.2.3: If we use the set A used in above example, A is not a controlled set. But if we use the definition5. We get a new set A^* of which element's membership degrees have the same membership with the same elements of A

$A^* = \{(a, 0.45, 0.50), (b, 0.05, 0.90), (c, 0.50, 0.50), (d, 0.35, 0.65), (e, 0.90, 0.05), (f, 0.65, 0.35)\}$

In this set, a and c are controlled by c, e is controlled by b, etc.

If we examine the controlled set B using Definition 5., then we get, $B^* = \{(a, 0.45, 0.55), (b, 0.10, 0.9), (c, 0.50, 0.50), (d, 0.45, 0.55), (e, 0.90, 0.10), (f, 0.55, 0.45)\}$ is a controlled set. In B^*, like as B, a is controled by f, c is controlled by itself, f controlled by a and d, etc.

From the definition, it can be easily seen that every (α,α^*)-controlled set is an intuitionistic fuzzy set. But the converse of this is not true generaly.

5. HAMMING DISTANCE BETWEEN IFSS

In this section, we will provide information about the Hamming measure between IFSs, which we will use in the process of deciding the relationship between our data. The hamming measure, like other measures, can be associated with a similarity measure between IFSs..

Tanım 5.1: Let X be a nonempty set and A, B, C IFS (X). The distance measure between A and B is a function d: IFS × IFS → [0, 1].

i. $0 \ d(A, B) \ 1$

ii. $d(A, B) = 0 \ A = B$

iii. $d(A, B) = d(B, A)$

iv. $d(A, C) + d(B, C) \geq d(A, B)$

v. If $A \subseteq B \subseteq C$ then $d(A, C) \geq d(A, B)$ and $d(A, C) \geq d(B, C)$.

Hamming measure is defined as follow;

$$d(A, B) = \sum_{i=1}^{n} \left(|\mu_A(x) - \mu_B(x)| + |\gamma_A(x) - \gamma_B(x)| + |\pi_A(x) - \pi_B(x)| \right)$$

6. MAIN RESULTS

In this study, the Hamming measure will be run on controlled sets. Therefore, in this section, first of all, controlled sets will be created with the data obtained by experts in the field. For this purpose, blurring will be done with SQL (Structured Query Language) using Microsoft SQL Server program.

6.1. Case study.

Let A = {GR1, GR2, GR3, GR4, GR5} grape varieties,
Let C = {x │x property of beverage product} be properties of product obtained from grapes.
Let S = {x│x types of food} be the set of foods to be consumed according to product characteristics.
The membership degrees of the relations of these data with each other were determined by experts as follows.

	WHITE	RED	D. ACID	O. ACID	Y. ACID	D. ALCO	O. ALCO	Y. ALCO	D. SUGA	O. SUGA	Y. SUGA	NARENCİYE	D. HARD	O. HARD	Y. HARD	GRAMINEO	SPICE	BİBER
DRIN	1	0	0,001	0,255	0,985	0,755	0,625	0,245	0,255	0,355	0,785	0,025	0,155	0,955	0,125	0,925	0,785	0,91
DRIN	1	0	0,865	0,625	0,165	0,125	0,945	0,345	0,125	0,255	0,852	0,125	0,755	0,455	0,352	0,652	0,698	0,569
DRIN	0	1	0,125	0,355	0,854	0,055	0,255	0,945	0,855	0,455	0,145	0,256	0,325	0,955	0,255	0,254	0,985	0,254
DRIN	0	1	0,125	0,855	0,245	0,155	0,455	0,845	0,125	0,255	0,865	0,452	0,855	0,455	0,015	0,845	0,254	0,125
DRIN	0	1	0,015	0,985	0,255	0,255	0,455	0,652	0,008	0,125	0,912	0,024	0,125	0,125	0,867	0,256	0,456	0,854
DRIN	0	1	0,005	0,955	0,455	0,355	0,5	0,736	0,125	0,255	0,721	0,362	0,855	0,455	0,236	0,352	0,845	0,785

Tablo 1

	RED MEAT	OILY RED MEAT	VENISON	CHICKEN	TURKEY	FISH	CHEESE	FRUIT	VEGETABLES	PORK	SHRIMP	SPICE
WHITE	0,245	0,015	0,15	0,853	0,952	0,895	0,85	0,785	0,825	0,865	0,845	0,845
RED	0,855	0,855	0,925	0,555	0,245	0,855	0,955	0,655	0,15	0,525	0,425	0,852
D. ACID	0,015	0,015	0,015	0,125	0,015	0,325	0,225	0,245	0,455	0,225	0,225	0,225
O. ACID	0,225	0,225	0,225	0,425	0,225	0,555	0,425	0,585	0,325	0,425	0,425	0,425
Y. ACID	0,952	0,952	0,952	0,853	0,952	0,75	0,85	0,785	0,015	0,865	0,845	0,845
D. ALCOHOL	0,125	0,015	0,015	0,625	0,625	0,625	0,545	0,625	0,525	0,015	0,625	0,125
O. ALCOHOL	0,425	0,225	0,225	0,425	0,425	0,425	0,45	0,425	0,125	0,225	0,425	0,425

Y. ALCOHOL	0,853	0,952	0,952	0,125	0,125	0,125	0,785	0,125	0,15	0,952	0,125	0,853
D. SUGARY	0,745	0,525	0,525	0,325	0,325	0,325	0,825	0,845	0,425	0,425	0,325	0,825
O. SUGARY	0,525	0,325	0,325	0,825	0,825	0,825	0,455	0,545	0,125	0,15	0,845	0,425
Y. SUGARY	0,125	0,125	0,125	0,455	0,495	0,855	0,35	0,45	0,15	0,015	0,255	0,355
NARENCİYE	0,35	0,55	0,55	0,853	0,952	0,895	0,85	0,785	0,425	0,865	0,845	0,845
D. HARD	0,125	0,125	0,125	0,125	0,125	0,125	0,225	0,125	0,425	0,125	0,125	0,545
O. HARD	0,255	0,425	0,425	0,255	0,255	0,745	0,455	0,345	0,125	0,255	0,255	0,345
Y. HARD	0,855	0,753	0,753	0,952	0,955	0,125	0,75	0,85	0,015	0,855	0,85	0,245
GRAMINEOUS	0,425	0,75	0,75	0,853	0,952	0,895	0,85	0,785	0,925	0,865	0,845	0,845
SPICE	0,755	0,75	0,89	0,745	0,745	0,85	0,985	0,125	0,855	0,95	0,725	0,56
RED FRUIT	0,75	0,55	0,455	0,855	0,745	0,854	0,855	0,125	0,255	0,125	0,125	0,325
PIPER	0,855	0,865	0,985	0,845	0,845	0,752	0,985	0,125	0,755	0,95	0,825	0,125
FRUIT FLAVORS	0,546	0,325	0,256	0,325	0,325	0,125	0,725	0,125	0,125	0,526	0	0,125

Table2

6.2 Data Structure

The Microsoft SQL Server data structure is created as follows.
Beverage Products Chart
Food Types Chart
Table of features of the product

```
CREATE TABLE [dbo].[TBL_PROPERTY](
	[ID] [bigint] IDENTITY(1,1) NOT NULL,
	[DESCRIPTION] [nvarchar](50) NULL,
 CONSTRAINT [PK_TBL_PROPERTY] PRIMARY KEY CLUSTERED
(
	[ID] ASC
)WITH (PAD_INDEX = OFF, STATISTICS_NORECOMPUTE = OFF, IGNORE_DUP_KEY = OFF,
ALLOW_ROW_LOCKS = ON, ALLOW_PAGE_LOCKS = ON) ON [PRIMARY]
) ON [PRIMARY]
```

```
CREATE TABLE [dbo].[TBL_DATA](
            [ID] [bigint] IDENTITY(1,1) NOT NULL,
            [TYPE] [int] NULL,
            [TYPE_ID] [bigint] NULL,
            [PROPERTY_ID] [bigint] NULL,
            [MU] [float] NULL,
            [NU] [float] NULL,
            [PI] [float] NULL,
 CONSTRAINT [PK_TBL_DATA] PRIMARY KEY CLUSTERED
(
            [ID] ASC
)WITH (PAD_INDEX = OFF, STATISTICS_NORECOMPUTE = OFF, IGNORE_DUP_KEY = OFF,
ALLOW_ROW_LOCKS = ON, ALLOW_PAGE_LOCKS = ON) ON [PRIMARY]
) ON [PRIMARY]
```

Veri Tablosu

6.3 Data Arrangement

Data entry on Microsoft SQL Server is done as followsTBL_DRINK

ID	DESCRIPTION
1	DRINK NO.1
2	DRINK NO.2
3	DRINK NO.3
4	DRINK NO.4
5	DRINK NO.5
6	DRINK NO.6

Table-3

TBL_FOOD

ID	DESCRIPTION
1	RED MEAT
2	OILY RED MEAT
3	VENISON
4	CHICKEN
5	TURKEY
6	FISH
7	CHEESE
8	FRUIT
9	VEGETABLES
10	PORK
11	SHRIMP
12	SPICE

Table-4

TBL_PROPERTY

ID	DESCRIPTION
1	RED
2	WHITE
3	ACIDITYL
4	ACIDITYM
5	ACIDITYH
6	ALCOHOLL
7	ALCOHOLM
8	ALCOHOLH
9	SWEETL
10	SWEETM
11	SWEETH
12	CITRUS

Table-5

Each value that gives the relationship between the elements of the A-C and C-S sets is determined by the degree of membership, degree of non-membership, and degree of intuition.

The C property of each element of A is entered one by one.

S relation of each element of C is entered one by one.

TBL_DATA

ID	TYPE	TYPE_ID	PROPERTY_ID	MU
2	1	1	1	0
3	1	1	2	1
4	1	1	3	0,001
5	1	1	4	0,255
6	1	1	5	0,985
7	1	1	6	0,755
8	1	1	7	0,625
9	1	1	8	0,245
10	1	1	9	0,255
11	1	1	10	0,355
12	1	1	11	0,785
13	1	1	12	0,025
14	1	1	13	0,155
15	1	1	14	0,955
16	1	1	15	0,125
17	1	1	16	0,925
18	1	1	17	0,785
19	1	1	18	0,001
20	1	1	19	0,91
21	1	1	20	0,895
22	1	2	1	0
23	1	2	2	1
24	1	2	3	0,865
25	1	2	4	0,625
26	1	2	5	0,165

27	1	2	6	0,125
28	1	2	7	0,945
29	1	2	8	0,345
30	1	2	9	0,125
31	1	2	10	0,255
32	1	2	11	0,852
33	1	2	12	0,125
34	1	2	13	0,755
35	1	2	14	0,455
36	1	2	15	0,352
37	1	2	16	0,652
38	1	2	17	0,698
39	1	2	18	0,021
40	1	2	19	0,569
41	1	2	20	0,825
42	1	3	1	1
43	1	3	2	0
44	1	3	3	0,125
45	1	3	4	0,355
46	1	3	5	0,854
47	1	3	6	0,055
48	1	3	7	0,255
49	1	3	8	0,945
50	1	3	9	0,855
51	1	3	10	0,455
52	1	3	11	0,145
53	1	3	12	0,256
54	1	3	13	0,325
55	1	3	14	0,955
56	1	3	15	0,255
57	1	3	16	0,254
58	1	3	17	0,985
59	1	3	18	0,985
60	1	3	19	0,254
61	1	3	20	0,785
62	1	4	1	1
63	1	4	2	0
64	1	4	3	0,125
65	1	4	4	0,855
66	1	4	5	0,245
67	1	4	6	0,155
68	1	4	7	0,455
69	1	4	8	0,845
70	1	4	9	0,125
71	1	4	10	0,255
72	1	4	11	0,865
73	1	4	12	0,452
74	1	4	13	0,855
75	1	4	14	0,455

76	1	4	15	0,015
77	1	4	16	0,845
78	1	4	17	0,254
79	1	4	18	0,654
80	1	4	19	0,125
81	1	4	20	0,645
82	1	5	1	1
83	1	5	2	0
84	1	5	3	0,015
85	1	5	4	0,985
86	1	5	5	0,255
87	1	5	6	0,255
88	1	5	7	0,455
89	1	5	8	0,652
90	1	5	9	0,008
91	1	5	10	0,125
92	1	5	11	0,912
93	1	5	12	0,024
94	1	5	13	0,125
95	1	5	14	0,125
96	1	5	15	0,867
97	1	5	16	0,256
98	1	5	17	0,456
99	1	5	18	0,985
100	1	5	19	0,854
101	1	5	20	0,856
102	1	6	1	1
103	1	6	2	0
104	1	6	3	0,005
105	1	6	4	0,955
106	1	6	5	0,455
107	1	6	6	0,355
108	1	6	7	0,5
109	1	6	8	0,736
110	1	6	9	0,125
111	1	6	10	0,255
112	1	6	11	0,721
113	1	6	12	0,362
114	1	6	13	0,855
115	1	6	14	0,455
116	1	6	15	0,236
117	1	6	16	0,352
118	1	6	17	0,845
119	1	6	18	0,995
120	1	6	19	0,785
121	1	6	20	0,886
122	2	1	1	0,855
123	2	1	2	0,245
124	2	1	3	0,015

125	2	1	4	0,225
126	2	1	5	0,952
127	2	1	6	0,125
128	2	1	7	0,425
129	2	1	8	0,853
130	2	1	9	0,745
131	2	1	10	0,525
132	2	1	11	0,125
133	2	1	12	0,35
134	2	1	13	0,125
135	2	1	14	0,255
136	2	1	15	0,855
137	2	1	16	0,425
138	2	1	17	0,755
139	2	1	18	0,75
140	2	1	19	0,855
141	2	1	20	0,546
142	2	2	1	0,855
143	2	2	2	0,015
144	2	2	3	0,015
145	2	2	4	0,225
146	2	2	5	0,952
147	2	2	6	0,015
148	2	2	7	0,225
149	2	2	8	0,952
150	2	2	9	0,525
151	2	2	10	0,325
152	2	2	11	0,125
153	2	2	12	0,55
154	2	2	13	0,125
155	2	2	14	0,425
156	2	2	15	0,753
157	2	2	16	0,75
158	2	2	17	0,75
159	2	2	18	0,55
160	2	2	19	0,865
161	2	2	20	0,325
162	2	3	1	0,925
163	2	3	2	0,15
164	2	3	3	0,015
165	2	3	4	0,225
166	2	3	5	0,952
167	2	3	6	0,015
168	2	3	7	0,225
169	2	3	8	0,952
170	2	3	9	0,525
171	2	3	10	0,325
172	2	3	11	0,125
173	2	3	12	0,55

174	2	3	13	0,125
175	2	3	14	0,425
176	2	3	15	0,753
177	2	3	16	0,75
178	2	3	17	0,89
179	2	3	18	0,455
180	2	3	19	0,985
181	2	3	20	0,256
182	2	4	1	0,555
183	2	4	2	0,853
184	2	4	3	0,125
185	2	4	4	0,425
186	2	4	5	0,853
187	2	4	6	0,625
188	2	4	7	0,425
189	2	4	8	0,125
190	2	4	9	0,325
191	2	4	10	0,825
192	2	4	11	0,455
193	2	4	12	0,853
194	2	4	13	0,125
195	2	4	14	0,255
196	2	4	15	0,952
197	2	4	16	0,853
198	2	4	17	0,745
199	2	4	18	0,855
200	2	4	19	0,845
201	2	4	20	0,325
202	2	5	1	0,245
203	2	5	2	0,952
204	2	5	3	0,015
205	2	5	4	0,225
206	2	5	5	0,952
207	2	5	6	0,625
208	2	5	7	0,425
209	2	5	8	0,125
210	2	5	9	0,325
211	2	5	10	0,825
212	2	5	11	0,495
213	2	5	12	0,952
214	2	5	13	0,125
215	2	5	14	0,255
216	2	5	15	0,955
217	2	5	16	0,952
218	2	5	17	0,745
219	2	5	18	0,745
220	2	5	19	0,845
221	2	5	20	0,325
222	2	6	1	0,855

223	2	6	2	0,895
224	2	6	3	0,325
225	2	6	4	0,555
226	2	6	5	0,75
227	2	6	6	0,625
228	2	6	7	0,425
229	2	6	8	0,125
230	2	6	9	0,325
231	2	6	10	0,825
232	2	6	11	0,855
233	2	6	12	0,895
234	2	6	13	0,125
235	2	6	14	0,745
236	2	6	15	0,125
237	2	6	16	0,895
238	2	6	17	0,85
239	2	6	18	0,854
240	2	6	19	0,752
241	2	6	20	0,125
242	2	7	1	0,955
243	2	7	2	0,85
244	2	7	3	0,225
245	2	7	4	0,425
246	2	7	5	0,85
247	2	7	6	0,545
248	2	7	7	0,45
249	2	7	8	0,785
250	2	7	9	0,825
251	2	7	10	0,455
252	2	7	11	0,35
253	2	7	12	0,85
254	2	7	13	0,225
255	2	7	14	0,455
256	2	7	15	0,75
257	2	7	16	0,85
258	2	7	17	0,985
259	2	7	18	0,855
260	2	7	19	0,985
261	2	7	20	0,725
262	2	8	1	0,655
263	2	8	2	0,785
264	2	8	3	0,245
265	2	8	4	0,585
266	2	8	5	0,785
267	2	8	6	0,625
268	2	8	7	0,425
269	2	8	8	0,125
270	2	8	9	0,845
271	2	8	10	0,545

272	2	8	11	0,45
273	2	8	12	0,785
274	2	8	13	0,125
275	2	8	14	0,345
276	2	8	15	0,85
277	2	8	16	0,785
278	2	8	17	0,125
279	2	8	18	0,125
280	2	8	19	0,125
281	2	8	20	0,125
282	2	9	1	0,15
283	2	9	2	0,825
284	2	9	3	0,455
285	2	9	4	0,325
286	2	9	5	0,015
287	2	9	6	0,525
288	2	9	7	0,125
289	2	9	8	0,15
290	2	9	9	0,425
291	2	9	10	0,125
292	2	9	11	0,15
293	2	9	12	0,425
294	2	9	13	0,425
295	2	9	14	0,125
296	2	9	15	0,015
297	2	9	16	0,925
298	2	9	17	0,855
299	2	9	18	0,255
300	2	9	19	0,755
301	2	9	20	0,125
302	2	10	1	0,525
303	2	10	2	0,865
304	2	10	3	0,225
305	2	10	4	0,425
306	2	10	5	0,865
307	2	10	6	0,015
308	2	10	7	0,225
309	2	10	8	0,952
310	2	10	9	0,425
311	2	10	10	0,15
312	2	10	11	0,015
313	2	10	12	0,865
314	2	10	13	0,125
315	2	10	14	0,255
316	2	10	15	0,855
317	2	10	16	0,865
318	2	10	17	0,95
319	2	10	18	0,125
320	2	10	19	0,95

321	2	10	20	0,526
322	2	11	1	0,425
323	2	11	2	0,845
324	2	11	3	0,225
325	2	11	4	0,425
326	2	11	5	0,845
327	2	11	6	0,625
328	2	11	7	0,425
329	2	11	8	0,125
330	2	11	9	0,325
331	2	11	10	0,845
332	2	11	11	0,255
333	2	11	12	0,845
334	2	11	13	0,125
335	2	11	14	0,255
336	2	11	15	0,85
337	2	11	16	0,845
338	2	11	17	0,725
339	2	11	18	0,125
340	2	11	19	0,825
341	2	11	20	0
342	2	12	1	0,852
343	2	12	2	0,845
344	2	12	3	0,225
345	2	12	4	0,425
346	2	12	5	0,845
347	2	12	6	0,125
348	2	12	7	0,425
349	2	12	8	0,853
350	2	12	9	0,825
351	2	12	10	0,425
352	2	12	11	0,355
353	2	12	12	0,845
354	2	12	13	0,545
355	2	12	14	0,345
356	2	12	15	0,245
357	2	12	16	0,845
358	2	12	17	0,56
359	2	12	18	0,325
360	2	12	19	0,125
361	2	12	20	0,125

Table-6

These values are as in Table-1 and Table-2 and as in Table-6, “TYPE=1” Beverage Products, “TYPE=2” Food Types are arranged with TYPE_ID connections on Microsoft SQL Server.

By using the controlled set feature, these values are controlled through Description-1 and Description-2.

TYPE	TYPE_ID	PROPERTY_ID	MU	NU	PI
1	2	15	0,352	0,352	0,296
1	5	15	0,867	0,125	0,008
1	1	17	0,785	0	0,215
1	2	10	0,255	0,455	0,29
1	6	15	0,236	0,352	0,412
2	10	19	0,95	0	0,05
1	3	18	0,985	0,001	0,014
1	2	5	0,165	0,455	0,38
2	3	17	0,89	0	0,11
2	12	15	0,245	0,753	0,002
2	8	4	0,585	0,325	0,09
2	6	20	0,125	0,725	0,15
2	9	4	0,325	0,585	0,09
2	4	20	0,325	0,546	0,129
1	4	9	0,125	0,855	0,02
2	9	2	0,825	0,15	0,025
2	11	5	0,845	0,015	0,14
1	3	7	0,255	0,625	0,12
2	5	16	0,952	0	0,048
2	12	2	0,845	0,15	0,005
2	10	20	0,526	0,325	0,149
2	11	2	0,845	0,15	0,005
1	6	4	0,955	0	0,045
2	9	8	0,15	0,785	0,065
1	1	1	0	1	0
1	2	8	0,345	0,652	0,003

```
WITH M_SUBCALCULATE AS
(SELECT TBL_DATA.ID, TBL_DATA.TYPE, TBL_DATA.PROPERTY_ID, TBL_DATA.TYPE_ID, TBL_DATA.MU,
TBL_DATA_1.MU AS MU_S, TBL_DATA_1.ID AS ID_S, TBL_DATA_1.TYPE_ID AS TYPE_ID_S,
TBL_DATA_1.PROPERTY_ID AS PROPERTY_ID_S, CASE WHEN (TBL_DATA.MU + TBL_DATA_1.MU) > 1 THEN 0 ELSE
TBL_DATA_1.MU END AS CONTROL FROM TBL_DATA INNER JOIN TBL_DATA AS TBL_DATA_1 ON TBL_DATA.TYPE =
TBL_DATA_1.TYPE AND TBL_DATA.PROPERTY_ID = TBL_DATA_1.PROPERTY_ID)
,

M_CALCULATE AS(SELECT TOP (100) PERCENT TYPE, TYPE_ID, PROPERTY_ID, MU, MAX(CONTROL) AS NU, 1 -
MU - MAX(CONTROL) AS PI FROM M_SUBCALCULATE GROUP BY TYPE, TYPE_ID, MU, PROPERTY_ID)

SELECT * FROM M_CALCULATE
```

2	4	11	0,455	0,495	0,05
1	2	4	0,625	0,355	0,02
2	12	16	0,845	0	0,155
1	3	3	0,125	0,865	0,01
1	5	12	0,024	0,452	0,524
2	10	8	0,952	0	0,048
2	1	5	0,952	0,015	0,033
2	1	13	0,125	0,545	0,33
2	5	2	0,952	0,015	0,033
2	9	19	0,755	0,125	0,12
2	6	5	0,75	0,015	0,235
2	7	16	0,85	0	0,15
2	2	11	0,125	0,855	0,02
2	3	12	0,55	0,425	0,025
2	8	6	0,625	0,125	0,25
1	1	10	0,355	0,455	0,19
2	2	9	0,525	0,425	0,05
2	11	16	0,845	0	0,155
2	1	17	0,755	0,125	0,12
2	7	7	0,45	0,45	0,1
2	8	1	0,655	0,245	0,1
2	7	10	0,455	0,545	-1,11E-16
1	5	20	0,856	0	0,144
2	4	4	0,425	0,555	0,02
2	5	15	0,955	0,015	0,03
1	5	8	0,652	0,345	0,003
2	5	13	0,125	0,545	0,33
1	2	2	1	0	0
1	3	10	0,455	0,455	0,09
1	4	6	0,155	0,755	0,09
2	6	4	0,555	0,425	0,02
1	1	13	0,155	0,755	0,09
2	10	12	0,865	0	0,135
1	2	20	0,825	0	0,175
1	3	6	0,055	0,755	0,19
1	5	10	0,125	0,455	0,42
2	2	1	0,855	0	0,145
2	6	18	0,854	0,125	0,021
2	8	18	0,125	0,855	0,02
1	5	13	0,125	0,855	0,02
1	6	9	0,125	0,855	0,02
2	3	14	0,425	0,455	0,12
2	9	15	0,015	0,955	0,03

1	6	14	0,455	0,455	0,09
1	1	4	0,255	0,625	0,12
1	3	14	0,955	0	0,045
2	5	9	0,325	0,525	0,15
2	3	2	0,15	0,85	0
2	4	5	0,853	0,015	0,132
2	5	3	0,015	0,455	0,53
2	5	5	0,952	0,015	0,033
2	7	11	0,35	0,495	0,155
1	6	8	0,736	0,245	0,019
2	11	3	0,225	0,455	0,32
1	1	16	0,925	0	0,075
1	6	1	1	0	0
1	6	11	0,721	0,145	0,134
2	10	14	0,255	0,745	0
1	3	9	0,855	0,125	0,02
2	7	18	0,855	0,125	0,02
1	3	16	0,254	0,652	0,094
1	1	20	0,895	0	0,105
1	6	19	0,785	0,125	0,09
2	8	7	0,425	0,45	0,125
2	10	17	0,95	0	0,05
2	12	17	0,56	0,125	0,315
1	4	20	0,645	0	0,355
2	6	8	0,125	0,853	0,022
2	9	13	0,425	0,545	0,03
2	1	3	0,015	0,455	0,53
2	8	13	0,125	0,545	0,33
2	1	1	0,855	0	0,145
2	4	14	0,255	0,745	0
2	7	15	0,75	0,245	0,005
2	10	15	0,855	0,125	0,02
2	9	17	0,855	0,125	0,02
2	1	7	0,425	0,45	0,125
1	5	14	0,125	0,455	0,42
1	6	6	0,355	0,355	0,29
2	3	4	0,225	0,585	0,19
2	5	6	0,625	0,125	0,25
2	5	7	0,425	0,45	0,125
2	12	13	0,545	0,425	0,03
1	4	15	0,015	0,867	0,118
1	6	10	0,255	0,455	0,29
2	4	6	0,625	0,125	0,25
2	5	1	0,245	0,655	0,1

2	3	20	0,256	0,725	0,019
2	4	13	0,125	0,545	0,33
2	8	19	0,125	0,865	0,01
2	2	12	0,55	0,425	0,025
2	8	12	0,785	0	0,215
2	7	2	0,85	0,15	2,78E-17
2	12	18	0,325	0,55	0,125
1	6	13	0,855	0,125	0,02
2	1	11	0,125	0,855	0,02
2	4	12	0,853	0	0,147
2	8	14	0,345	0,455	0,2
2	7	19	0,985	0	0,015
1	1	19	0,91	0	0,09
1	3	19	0,254	0,569	0,177
2	2	6	0,015	0,625	0,36
2	2	18	0,55	0,325	0,125
2	12	9	0,825	0	0,175
1	1	14	0,955	0	0,045
2	1	15	0,855	0,125	0,02
1	4	17	0,254	0,698	0,048
2	7	8	0,785	0,15	0,065
2	5	18	0,745	0,255	0
2	9	3	0,455	0,455	0,09
2	12	19	0,125	0,865	0,01
1	4	4	0,855	0	0,145
1	5	2	0	1	0
1	1	6	0,755	0,155	0,09
1	3	4	0,355	0,625	0,02
2	2	19	0,865	0,125	0,01
2	5	11	0,495	0,495	0,01
2	6	2	0,895	0,015	0,09
2	1	16	0,425	0,425	0,15
2	4	10	0,825	0,15	0,025
1	4	14	0,455	0,455	0,09
1	6	2	0	1	0
1	2	18	0,021	0,654	0,325
1	5	6	0,255	0,355	0,39
1	6	18	0,995	0,001	0,004
2	3	19	0,985	0	0,015
2	12	8	0,853	0,125	0,022
2	11	6	0,625	0,125	0,25
2	7	3	0,225	0,455	0,32
1	1	18	0,001	0,995	0,004

2	12	5	0,845	0,015	0,14
1	5	16	0,256	0,652	0,092
1	1	7	0,625	0,255	0,12
2	10	3	0,225	0,455	0,32
2	12	1	0,852	0	0,148
2	8	9	0,845	0	0,155
1	5	1	1	0	0
1	5	9	0,008	0,855	0,137
2	2	14	0,425	0,455	0,12
2	3	7	0,225	0,45	0,325
2	5	8	0,125	0,853	0,022
2	9	14	0,125	0,745	0,13
2	8	10	0,545	0,455	-5,55E-17
2	11	9	0,325	0,525	0,15
2	7	5	0,85	0,015	0,135
1	2	6	0,125	0,755	0,12
2	2	20	0,325	0,546	0,129
2	4	9	0,325	0,525	0,15
2	9	20	0,125	0,725	0,15
1	4	11	0,865	0	0,135
2	6	1	0,855	0	0,145
2	2	17	0,75	0,125	0,125
2	9	11	0,15	0,495	0,355
2	11	18	0,125	0,855	0,02
2	9	6	0,525	0,125	0,35
2	5	14	0,255	0,745	0
2	9	5	0,015	0,952	0,033
1	6	3	0,005	0,865	0,13
2	1	8	0,853	0,125	0,022
2	4	1	0,555	0,425	0,02
1	1	5	0,985	0	0,015
2	8	3	0,245	0,455	0,3
2	11	1	0,425	0,555	0,02
1	2	9	0,125	0,855	0,02
2	12	12	0,845	0	0,155
2	12	20	0,125	0,725	0,15
1	2	11	0,852	0,145	0,003
2	10	9	0,425	0,525	0,05
1	5	19	0,854	0,125	0,021
1	3	1	1	0	0
1	5	5	0,255	0,455	0,29
2	2	15	0,753	0,245	0,002
2	3	6	0,015	0,625	0,36

2	5	12	0,952	0	0,048
2	2	3	0,015	0,455	0,53
2	6	10	0,825	0,15	0,025
2	11	7	0,425	0,45	0,125
2	3	13	0,125	0,545	0,33
1	4	16	0,845	0	0,155
2	10	5	0,865	0,015	0,12
2	12	10	0,425	0,545	0,03
2	8	17	0,125	0,855	0,02
2	11	19	0,825	0,125	0,05
2	11	13	0,125	0,545	0,33
1	3	5	0,854	0	0,146
2	3	16	0,75	0	0,25
2	5	19	0,845	0,125	0,03
2	11	11	0,255	0,495	0,25
2	12	11	0,355	0,495	0,15
2	2	4	0,225	0,585	0,19
2	7	4	0,425	0,555	0,02
2	6	6	0,625	0,125	0,25
2	7	17	0,985	0	0,015
1	4	2	0	1	0
2	3	5	0,952	0,015	0,033
2	7	14	0,455	0,455	0,09
1	1	2	1	0	0
1	3	12	0,256	0,452	0,292
2	3	3	0,015	0,455	0,53
2	11	14	0,255	0,745	0
2	1	19	0,855	0,125	0,02
1	3	13	0,325	0,325	0,35
2	1	6	0,125	0,625	0,25
2	1	20	0,546	0,325	0,129
1	2	12	0,125	0,452	0,423
2	6	14	0,745	0,255	0
1	1	12	0,025	0,452	0,523
2	11	4	0,425	0,555	0,02
2	10	6	0,015	0,625	0,36
1	4	10	0,255	0,455	0,29
2	12	14	0,345	0,455	0,2
1	1	8	0,245	0,736	0,019
2	11	12	0,845	0	0,155
2	6	13	0,125	0,545	0,33
2	8	8	0,125	0,853	0,022
2	10	16	0,865	0	0,135
2	3	1	0,925	0	0,075

2	3	9	0,525	0,425	0,05
1	2	16	0,652	0,256	0,092
2	2	7	0,225	0,45	0,325
1	3	20	0,785	0	0,215
2	4	18	0,855	0,125	0,02
1	4	19	0,125	0,854	0,021
2	4	8	0,125	0,853	0,022
2	12	7	0,425	0,45	0,125
2	1	14	0,255	0,745	0
1	4	1	1	0	0
1	4	5	0,245	0,455	0,3
2	7	6	0,545	0,125	0,33
2	8	20	0,125	0,725	0,15
1	1	3	0,001	0,865	0,134
2	10	2	0,865	0,015	0,12
2	6	3	0,325	0,455	0,22
2	6	16	0,895	0	0,105
2	10	7	0,225	0,45	0,325
2	10	18	0,125	0,855	0,02
1	4	13	0,855	0,125	0,02
1	5	18	0,985	0,001	0,014
2	3	8	0,952	0	0,048
2	10	1	0,525	0,425	0,05
2	9	18	0,255	0,745	0
2	3	10	0,325	0,545	0,13
2	4	19	0,845	0,125	0,03
2	8	11	0,45	0,495	0,055
1	3	8	0,945	0	0,055
1	6	17	0,845	0	0,155
2	5	10	0,825	0,15	0,025
2	8	5	0,785	0,015	0,2
1	6	7	0,5	0,5	0
2	1	9	0,745	0	0,255
1	4	3	0,125	0,865	0,01
2	9	9	0,425	0,525	0,05
1	2	3	0,865	0,125	0,01
2	1	10	0,525	0,455	0,02
1	1	15	0,125	0,867	0,008
2	5	17	0,745	0,125	0,13
2	7	1	0,955	0	0,045
1	1	11	0,785	0,145	0,07
1	5	3	0,015	0,865	0,12
2	5	4	0,225	0,585	0,19
2	9	7	0,125	0,45	0,425

1	4	7	0,455	0,5	0,045
2	1	2	0,245	0,245	0,51
1	4	12	0,452	0,452	0,096
1	5	11	0,912	0	0,088
1	6	20	0,886	0	0,114
2	10	11	0,015	0,855	0,13
2	12	4	0,425	0,555	0,02
2	5	20	0,325	0,546	0,129
1	2	13	0,755	0,155	0,09
2	11	15	0,85	0,125	0,025
2	7	13	0,225	0,545	0,23
2	4	15	0,952	0,015	0,033
2	10	13	0,125	0,545	0,33
1	3	15	0,255	0,352	0,393
2	3	11	0,125	0,855	0,02
2	8	16	0,785	0	0,215
1	6	12	0,362	0,452	0,186
2	11	17	0,725	0,125	0,15
1	3	17	0,985	0	0,015
2	4	3	0,125	0,455	0,42
2	4	7	0,425	0,45	0,125
2	7	20	0,725	0,256	0,019
2	10	10	0,15	0,845	0,005
1	6	16	0,352	0,352	0,296
2	8	2	0,785	0,15	0,065
1	2	17	0,698	0,254	0,048
1	2	19	0,569	0,254	0,177
2	2	16	0,75	0	0,25
2	2	5	0,952	0,015	0,033
2	2	13	0,125	0,545	0,33
2	7	12	0,85	0	0,15
2	1	18	0,75	0,125	0,125
1	2	1	0	1	0
1	4	8	0,845	0	0,155
2	1	4	0,225	0,585	0,19
2	4	16	0,853	0	0,147
1	5	7	0,455	0,5	0,045
2	6	19	0,752	0,125	0,123
2	10	4	0,425	0,555	0,02
1	6	5	0,455	0,455	0,09
1	1	9	0,255	0,255	0,49
2	4	2	0,853	0,015	0,132
2	9	12	0,425	0,55	0,025
2	9	10	0,125	0,845	0,03

2	9	16	0,925	0	0,075
1	2	7	0,945	0	0,055
2	7	9	0,825	0	0,175
1	2	14	0,455	0,455	0,09
2	4	17	0,745	0,125	0,13
2	2	2	0,015	0,952	0,033
2	11	10	0,845	0,15	0,005
2	6	15	0,125	0,855	0,02
2	6	9	0,325	0,525	0,15
2	11	8	0,125	0,853	0,022
2	3	18	0,455	0,455	0,09
2	6	11	0,855	0,125	0,02
2	12	6	0,125	0,625	0,25
2	3	15	0,753	0,245	0,002
2	8	15	0,85	0,125	0,025
2	1	12	0,35	0,55	0,1
1	5	17	0,456	0,456	0,088
1	3	11	0,145	0,852	0,003
1	5	4	0,985	0	0,015
2	9	1	0,15	0,655	0,195
1	4	18	0,654	0,021	0,325
1	3	2	0	1	0
2	6	12	0,895	0	0,105
2	2	10	0,325	0,545	0,13
2	6	17	0,85	0,125	0,025
2	11	20	0	0,725	0,275
2	12	3	0,225	0,455	0,32
2	2	8	0,952	0	0,048
2	6	7	0,425	0,45	0,125

Table-7

The clusters obtained as in Table-7 are intuitive fuzzy sets.
Description-3
When Hamming measure is applied to these clusters using definition-3, the relationship between A and S can be seen as in Table-8.

DRINK_ID	DRINK	FOOD_ID	FOOD	DISTANCE
4	DRINK NO.4	4	CHICKEN	21,062
2	DRINK NO.2	4	CHICKEN	18,612
3	DRINK NO.3	4	CHICKEN	18,516
5	DRINK NO.5	4	CHICKEN	17,674
6	DRINK	4	CHICKEN	17,41

	NO.6			
1	DRINK NO.1	4	CHICKEN	15,002
2	DRINK NO.2	7	CHEESE	19,302
4	DRINK NO.4	7	CHEESE	18,028
5	DRINK NO.5	7	CHEESE	16,242
1	DRINK NO.1	7	CHEESE	15,434
6	DRINK NO.6	7	CHEESE	14,214
3	DRINK NO.3	7	CHEESE	13,11
5	DRINK NO.5	6	FISH	18,3
4	DRINK NO.4	6	FISH	18,088
2	DRINK NO.2	6	FISH	17,864
3	DRINK NO.3	6	FISH	17,774
6	DRINK NO.6	6	FISH	16,218
1	DRINK NO.1	6	FISH	13,61
6	DRINK NO.6	8	FRUIT	20,962
5	DRINK NO.5	8	FRUIT	19,944
2	DRINK NO.2	8	FRUIT	19,186
4	DRINK NO.4	8	FRUIT	19,174
3	DRINK NO.3	8	FRUIT	19,122
1	DRINK NO.1	8	FRUIT	17,284
2	DRINK NO.2	3	VENISON	20,774
1	DRINK NO.1	3	VENISON	18,044
4	DRINK NO.4	3	VENISON	17,184
5	DRINK NO.5	3	VENISON	16,798
6	DRINK NO.6	3	VENISON	14,89

3	DRINK NO.3	3	VENISON	12,33
2	DRINK NO.2	2	OILY RED MEAT	20,386
1	DRINK NO.1	2	OILY RED MEAT	18,404
4	DRINK NO.4	2	OILY RED MEAT	16,012
5	DRINK NO.5	2	OILY RED MEAT	15,696
6	DRINK NO.6	2	OILY RED MEAT	14,228
3	DRINK NO.3	2	OILY RED MEAT	11,758
4	DRINK NO.4	10	PORK	20,64
5	DRINK NO.5	10	PORK	19,416
6	DRINK NO.6	10	PORK	18,72
2	DRINK NO.2	10	PORK	18,206
1	DRINK NO.1	10	PORK	16,454
3	DRINK NO.3	10	PORK	16,08
2	DRINK NO.2	1	RED MEAT	20,526
4	DRINK NO.4	1	RED MEAT	18,65
1	DRINK NO.1	1	RED MEAT	18,396
5	DRINK NO.5	1	RED MEAT	15,464
6	DRINK NO.6	1	RED MEAT	15,1
3	DRINK NO.3	1	RED MEAT	11,666
4	DRINK NO.4	11	SHRIMP	22,502
3	DRINK NO.3	11	SHRIMP	20,264
5	DRINK NO.5	11	SHRIMP	19,93
6	DRINK NO.6	11	SHRIMP	19,506
2	DRINK NO.2	11	SHRIMP	17,832
1	DRINK	11	SHRIMP	14,144

	NO.1			
5	DRINK NO.5	12	SPICE	20,286
2	DRINK NO.2	12	SPICE	18,256
1	DRINK NO.1	12	SPICE	17,602
6	DRINK NO.6	12	SPICE	17,322
4	DRINK NO.4	12	SPICE	15,002
3	DRINK NO.3	12	SPICE	14,406
4	DRINK NO.4	5	TURKEY	22,59
3	DRINK NO.3	5	TURKEY	20,296
5	DRINK NO.5	5	TURKEY	19,426
6	DRINK NO.6	5	TURKEY	19,124
2	DRINK NO.2	5	TURKEY	18,764
1	DRINK NO.1	5	TURKEY	13,802
3	DRINK NO.3	9	VEGETABLES	21,66
5	DRINK NO.5	9	VEGETABLES	21,222
4	DRINK NO.4	9	VEGETABLES	20,454
6	DRINK NO.6	9	VEGETABLES	19,152
2	DRINK NO.2	9	VEGETABLES	16,362
1	DRINK NO.1	9	VEGETABLES	14,832

Table-8

When the results in Table-8 are evaluated, the beverage consumed with "Chicken" food should be Drink no.4. It is clear that Drink no.1 and Drink no.2 drinks can be found in a service with "oily red meat" and "Venison", considering that there will be no fish and red meat in the same service, but fish and vegetable dishes.

As a different evaluation of the results, considering the Table-8, it can be considered that the Drink no.1 drink and the Drink no.2 drink have similar properties.

7. CONCLUSION

It is considered as an important problem that data with different characteristics reach the same results in studies conducted through the Hamming measure. In this study, it will be more difficult to associate data with different characteristics with the same data, since the data that does not provide a characteristic by means of controlled sets is expressed with data within its own universal region. At the very least, it is not possible for elements that control each other to match the same data. The method followed in this study can be tested by using it in old applications. In addition, these criteria can be taken into account so that they can be easily applied to multi-criteria decision making problems in future studies.

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The Declaration of Ethics Committee Approval

This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics

The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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