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Weighted Residual Approach for Bending Analysis of Nanobeam Using by Modified Couple Stress Theory

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Abstract

With the development of nanotechnology, interest in nanomaterials has increased significantly in recent years. This study examines the bending analysis of a nanobeam with modified couple stress theory and weighted residual methods. The formulas derived for calculating bending analysis results in the article has been found by using Weighted Residual Method. The results have compared to show effects on nanobeam and the calculated values are shown in the graphs and tables. The results which is obtained in the article are compared with the results already found in the literature and it was observed that they are consistent.

Keywords: Nanotechnology, Bending, Nanobeam, Modified Couple Stress Theory, Weighted Residual Methods.

1. Introduction

Recently, the use of nanomaterials has been very wide with a high interest. For instance; nanorods are used in electronics, material and manufacturing industry, medical and healthcare, aviation and space research, environment and energy, defense sector, biomedical industries and various engineering structures. Nanotechnology is an evolving technology, and we can see more applications in nanotechnology. Nowadays, it is clear that it will spread much wider in future. Nanotechnology can be said to be a golden key for technology, as technology developments continue and the for human needs is now being ensured by smaller materials and tools.

It is anticipated that nanotechnology will benefit humanity in all areas as a result of its correct and logical use, and will allow a different size of technology to be considered. Knowing the factors that influence nanomaterials will make using nanotechnology easier to implement, this is the main focus of this study.

Several studies have been conducted in the literature about vibration and bending of nanorods and the factors which are affecting on nanorods. Ecsedi and Baksa [1] published studies about axial vibration of nanorods based on non-local elasticity. Şimşek [2] examined axial vibration of nanorod using non-local strain gradient theory. Yayli [3] investigated axial vibration analysis of a Rayleigh nanorod with deformable boundaries. Aydoğdu [4] published study about axial vibration analysis of carbon nanotubes embedded in an elastic medium. Axial vibration analysis of a tapered nanorod with a differential quadrature method was examined by Danesh et al. [5]. Guo and Yang [6] conducted axial vibration analysis of nanocones based on non-local elasticity theory. Murmu and Pradhan [7] published studies about vibration analysis of nanoplates under uniaxial pre stressed conditions via non-local elasticity. Comparison of various refined non-local beam theories for bending, vibration and buckling analysis of beams investigated by Berrabah et al. [8]. Fakher et al [9] published study about bending and free vibration analysis of nanobeams by differential and integral forms of non-local strain gradient with Rayleigh-Ritz method. Akgöz and Civalek [10] indicated buckling analysis of functionally graded microbeams based on the strain gradient theory. Attia and Mahmoud [11] published studies modeling and analysis of nanobeams based on non-local couple-stress elasticity and surface energy theories. Static analysis of nanobeams including surface effects by non-local finite element published by Mahmoud et al. [12]. Norouzzadeh et al. [13] indicated non-linear bending analysis of nanobeams based on non-local strain gradient model using an isogeometric finite element approach. Alshorbagy et al. [14] published study about static analysis of nanobeams using non-local FEM. Coupling effects of non-local and surface energy on vibration analysis of nanobeams investigated by Eltaher et al. [15]. Studies so far show how diverse the vibration and bending analysis is, and then there are many different theories to be revealed. Arefi et al. [16] examined the non-local bending analysis of curved nanobeams reinforced by grapheme nanoplatelets. Lu et al. [17] indicated size-dependent vibration analysis of nanobeams based on the non-local strain gradient theory. A unified non-local formulation for bending, buckling and free vibration analysis of nanobeams published by Nikam and Sayyad [18]. Determination of the appropriate gradient elasticity theory for bending analysis of nanobeams by considering boundary conditions effect investigated by Shokrieh and Zibaei [19]. Beni [20] published study about size-dependent electromechanical bending, buckling and free vibration analysis of functionally graded piezoelectric nanobeams. While this is similar in terms of the subject, it shows us how much factors are effecting on nanomaterials. Ghadiri et al. [21] indicated non-linear forced vibration analysis of nanobeams subjected to moving concentrated load resting on a viscoelastic foundation considering thermal and surface effects. Behera and Chakravery [22] published study about application of differential quadrature method in free vibration analysis of nanobeams based on various non-local theories. In this type of works we frequently use non-local theory and that can be used in many aspects in terms of usage width. Nejad and Hadi [23] investigated non-local analysis of free vibration of bidirectional functionally graded Euler-Bernoulli nanobeams. Ebrahimi and Salari [24] indicated thermal buckling and free vibration analysis of size dependent Timoshenko FG nanobeams in thermal enviroments. In studies like this, main purpose is observing the outside effects on nanomaterials. Lin et al. [25] investigated assessment of first and third order shear deformation beam theories for the buckling and vibration analysis of nanobeams incorporating surface stress effects. Vibration analysis of smart piezoelectrically actuated nanobeams subjected to magneto-electrical field in thermal environment studied by Ebrahimi and Barati [26]. Ansari et al. [27] conducted non-linear vibration analysis of Timoshenko nanobeams based on surface stress elasticity theory. Free vibration analysis of Euler-Bernoulli nanobeam using differential transform method published by Jena and Chakraverty [28]. Forced vibration analysis of functionally graded nanobeams studied by Akbas [29]. Shafiei et al. [30] investigated nonlinear vibration of axially functionally graded non-uniform nanobeams. One reason for the diversification of literature studies is that multiple factors can be addressed in nanorods. Complex modal analysis of transverse free vibrations for axially moving nanobeams based on the non-local strain gradient theory published by Wang et al. [31]. Vibration analysis of Euler-Bernoulli nanobeams by using finite element method examined by Eltaher et al. [32]. One other analysis done by Oskouie and Ansari [33] about linear and nonlinear vibrations of fractional viscoelastic Timoshenko nanobeams considering surface energy effects. Zhang et al. [34] investigated analyses of transverse vibrations of axially pre tensioned viscoelastic nanobeams with small size and surface effects. Yang et al. [35] studied nonlinear bending, buckling and vibration of bidirectional functionally graded nanobeams. Another study published by Lv and Liu [36] about nonlinear bending response of functionally graded nanobeams with material uncertainties. As we can see, many researchers have often included factors that affect nanomaterials in their studies. Exact solutions of bending deflections for nanobeams and nanoplates based on non-local elasticity theory published by Yan et al. [37]. Sciarra and Baretta [38] indicated a new non-local bending model for Euler-Bernoulli nanobeams. Finite Element analysis of plates and shells investigated by Civalek [39]. One of the studies about Timoshenko beams was done by Civalek and Kiracioglu in their work called free vibration analysis of Timoshenko beams by DSC method [40]. Mercan et al. examined vibration analysis of FG cylindrical shells with power-law index using discrete singular convolution technique [41].

In this study, it has been examined how nanostructures influence the bending analysis of their changing characteristics with the change in their properties of the particular type. Because boundary conditions can be deformed, it is possible to have a closed solution of this differential equation separately, but it is intended to create general self-value problem using by Modified Couple Stress Theory, and bending analysis of a free-end nanobeam can be found, giving very small values to these elastic boundary conditions. At the same time, the bending analysis of nanobeam and which key parameters effects on nanobeam's bending has been studied in this article. The parameters that have an effect on the bending of the nanobeams has been compared with the calculated values and evaluations has been made on these results.

2. Bending Analysis of A Nanobeam

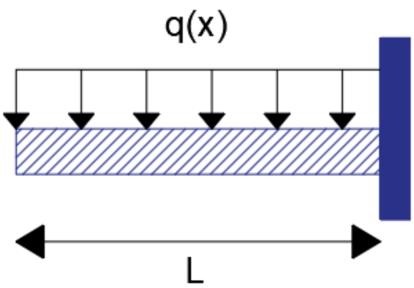

Fig. 1. A cantilever nanobeam which is uniformly loaded

In this section, bending analysis of the uniformly loaded cantilever nanobeam shown in Fig 1 has been made. While performing these analyzes, weighted residual function methods are used. We can define the components for the displacement of Euler-Bernoulli beam as follows:

$$u = -z\psi(x),\ v = 0,\ w = w(x) \tag{1}$$

u, v, w are u components of the displacement vectors of x, y, z respectively and ψ is the angle of rotation of the center axis.

$$\psi \approx \frac{dw(x)}{dx} \tag{2}$$

Using Eqs. (1,2):

$$\varepsilon_{xx} = \frac{du}{dx} = -z\frac{d^2w(x)}{dx^2},\ \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0 \tag{3}$$

equaliton is obtained [47]. From the Eqs. (1,2) given above:

$$\theta_y = \frac{dw(x)}{dx},\ \theta_x = \theta_z = 0 \tag{4}$$

calculation is acquired [47]. From the Eq. (4) given above, the following relation is received:

$$\chi_{xy} = -\frac{1}{2}\frac{d^2w(x)}{dx^2},\ \chi_{xx} = \chi_{yy} = \chi_{zz} = \chi_{yz} = \chi_{zx} = 0 \tag{5}$$

Using the equations obtained above:

$$\sigma_{xx} = \frac{E(1-\upsilon)}{(1+\upsilon)(1-2\upsilon)}\left(-z\frac{d^2w(x)}{dx^2}\right), \sigma_{yy} = \sigma_{zz} = \frac{E\upsilon}{(1+\upsilon)(1-2\upsilon)}\left(-z\frac{d^2w(x)}{dx^2}\right)$$

$$\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \tag{6}$$

this equation is acquired.

In the equation obtained above, E represents Young's modulus and υ represents the Poisson ratio. E and υ are related to the Lame constants λ and μ.

$$\lambda = \frac{E\upsilon}{(1+\upsilon)(1-2\upsilon)} \tag{7}$$

$$\mu = \frac{E}{2(1+\upsilon)} \tag{8}$$

As defined in Eq. (8), μ is known as the material constant. This expression is also known as shear modulus (G). For a beam with a high aspect ratio, the Poisson effect can be neglected to more easily express the simple beam theory formulation. And the equation created in this way is as follows:

$$\sigma_{xx} = -Ez\frac{d^2w(x)}{dx^2},\ \sigma_{ij} = 0 \tag{9}$$

If the equation is replaced in the relevant relations:

$$m_{xy} = \mu l_2^2\frac{d^2w(x)}{dx^2}, m_{xx} = m_{yy} = m_{zz} = m_{yz} = m_{zx} = 0 \tag{10}$$

When this calculation is acquired [47]. Using Eqs. (3, 6, 9, 10) the following equation is obtained.

$$U = -\frac{1}{2}\int_{x=0}^{L} M_x \frac{d^2w(x)}{dx^2}dx - \frac{1}{2}\int_{x=0}^{L} Y_{xy}\frac{d^2w(x)}{dx^2}dx \tag{11}$$

The closed equation created after the calculations is as follows [47]:

$$w(x) = \frac{1}{(EI+\mu A l_2^2)}\left[-\frac{qx^4}{72} + c_1x + c_2\right] \tag{12}$$

3. Weighted Residual Methods

3.1. Sub-domain method

The application of this method is achieved by taking the residual function of the integral over the selected areas as zero. The residual function in modified couple stress theory is given below:

$$R = (EI + \mu * A * l_2^2) * \frac{\partial^2 W}{\partial x^2} + \frac{q*x^2}{2} = 0 \tag{13}$$

Following approximation function is used in this study [14]:

$$W = a_1 * (x-L)^2 + a_2 * (x-L)^3 + a_3 * (x-L)^4 \tag{14}$$

$$\mathrm{R} = (\mathrm{EI} + \mu * \mathrm{A} * l_2^2) * (2 * \mathrm{a}_1 + 6 * \mathrm{a}_2 * (\mathrm{x}-\mathrm{L}) + 12 * \mathrm{a}_3 * (\mathrm{x}-\mathrm{L})^2) + \frac{\mathrm{q}*\mathrm{x}^2}{2} \tag{15}$$

$$\int_0^{\frac{L}{3}}((EI + \mu * A * l_2^2) * (2 * a_1 + 6 * a_2 * (x-L) + 12 * a_3 * (x-L)^2) + \frac{q*x^2}{2})\,dx = 0 \tag{16}$$

$$\int_{\frac{L}{3}}^{\frac{2L}{3}}((EI + \mu * A * l_2^2) * (2 * a_1 + 6 * a_2 * (x-L) + 12 * a_3 * (x-L)^2) + \frac{q*x^2}{2})\,dx = 0 \tag{17}$$

$$\int_{\frac{2L}{3}}^{L}((EI + \mu * A * l_2^2) * (2 * a_1 + 6 * a_2 * (x-L) + 12 * a_3 * (x-L)^2) + \frac{q*x^2}{2})\,dx = 0 \tag{18}$$

After the integrals are taken, following equations are obtained:

$$\frac{1}{162}L(L^2q + 6(\mathrm{EI} + \mu * A * I_2^2)(18a_1 + L(-45a_2 + 76\mathrm{L}a_3))) = 0 \tag{19}$$

$$\frac{1}{162}L\left(7L^2q + 6(EI + \mu * A * l_2^2)\big(18a_1 + L(-27a_2 + 28La_3)\big)\right) = 0 \tag{20}$$

$$\frac{1}{162}L\left(19L^2q + 6(EI + \mu * A * l_2^2)\big(18a_1 + L(-9a_2 + 4La_3)\big)\right) = 0 \tag{21}$$

Once this equation system is solved, the unknown coefficients are found:

$$a_1 = -\frac{L^2q}{4(EI+\mu*A*l_2^2)} \tag{22}$$

$$a_2 = -\frac{Lq}{6(EI+\mu*A*l_2^2)} \tag{23}$$

$$a_3 = -\frac{q}{24(EI+\mu*A*l_2^2)} \tag{24}$$

These coefficients are written in the calculation and the W function becomes as in Eq. (25):

$$W = -\frac{L^2q}{4(EI+\mu*A*l_2^2)}*(x-L)^2 - \frac{Lq}{6(EI+\mu*A*l_2^2)}*(x-L)^3 - \frac{q}{24(EI+\mu*A*l_2^2)}*(x-L)^4 \tag{25}$$

3.2. Galerkin method

When solving with the Galerkin method, three boundary condition relations are chosen for three unknown coefficients:

$$\int_0^L R*W_i\,dx = 0 i = 0,1,2....N \tag{26}$$

$$W_1 = (x-L) \tag{27}$$

$$W_2 = (x-L)^2 \tag{28}$$

$$W_3 = (x-L)^3 \tag{29}$$

Using the approximation functions given above, the following equations are obtained when substituted sequentially in Eq. (26):

$$\int_0^L \left((EI+\mu*A*l_2^2)*(2*a_1+6*a_2*(x-L)+12*a_3*(x-L)^2)+\frac{q*x^2}{2}\right)*(x-L)^2\,dx = 0 \tag{30}$$

$$\int_0^L ((EI+\mu*A*l_2^2)*(2*a_1+6*a_2*(x-L)+12*a_3*(x-L)^2)+\frac{q*x^2}{2})*(x-L)^2\,dx = 0 \tag{31}$$

$$\int_0^L ((\mathrm{EI}+\mu*\mathrm{A}*l_2^2)*(2*a_1+6*a_2*(x-L)+12*a_3*(x-L)^2)+\frac{q*x^2}{2})*(x-L)^3\,dx = 0 \tag{32}$$

The integrals given above are calculated, the following system of calculations is formed:

$$-\frac{L^4q}{24} - L^2(EI+\mu*A*l_2^2)(a_1+L(-2a_2+3La_3)) = 0 \tag{33}$$

$$\frac{1}{60}L\left(L^4q+6L^2(EI+\mu*A*l_2^2)(20a_1+9L(-5a_2+8La_3))\right) = 0 \tag{34}$$

$$\frac{1}{120}L^4\left(-L^2q-36(EI+\mu*A*l_2^2)\left(5a_1+4L(-3a_2+5La_3)\right)\right)=0 \quad (35)$$

The equation system is solved, the unknowns a_1, a_2, a_3 are found as follows:

$$a_1=-\frac{L^2q}{4(EI+\mu*A*l_2^2)} \quad (36)$$

$$a_2=-\frac{Lq}{6(EI+\mu*A*l_2^2)} \quad (37)$$

$$a_3=-\frac{q}{24(EI+\mu*A*l_2^2)} \quad (38)$$

When the found unknown coefficients are replaced in the W function, the equation to be formed is as:

$$W=-\frac{L^2q}{4(EI+\mu*A*l_2^2)}*(x-L)^2-\frac{Lq}{6(EI+\mu*A*l_2^2)}*(x-L)^3-\frac{q}{24(EI+\mu*A*l_2^2)}*(x-L)^4 \quad (39)$$

2. Analysis and Numerical Results

The values obtained as a result of the three-parameter solution method using the weighted residual methods are given in Table 1. The graphic in Fig.2 was created using Table 1. The values acquired by solving the two-parameter solution method by using the weighted residual method are given in Table 2. And by using Table 2, the results received by using the two-parameter solution method and the weighted residual methods are compared in Fig.3. While drawing the graphs in Figures 2 and 3, the numerical values in Tables 1 and 2 were used.

Table 1. Comparison of the calculated values with the weighted residual methods of the three-parameter solution that we used in the study

	x=0	x=2	x=4	x=6	x=8	x=10
$W_{Subdomain}$	-1246,26	-914,59	-592,223	-303,091	-87,0721	0
$W_{Galerkin}$	-1246,26	-914,59	-592,223	-303,091	-87,0721	0
$W_{General}$ [47]	-1246,26	-914,59	-592,223	-303,091	-87,0721	0

The displacement values of an axially loaded nanobeams are given by using different weighted residual methods given in the table. It has been shown that the values calculated by applying different solution methods and also the calculated implicit solution value are the same.

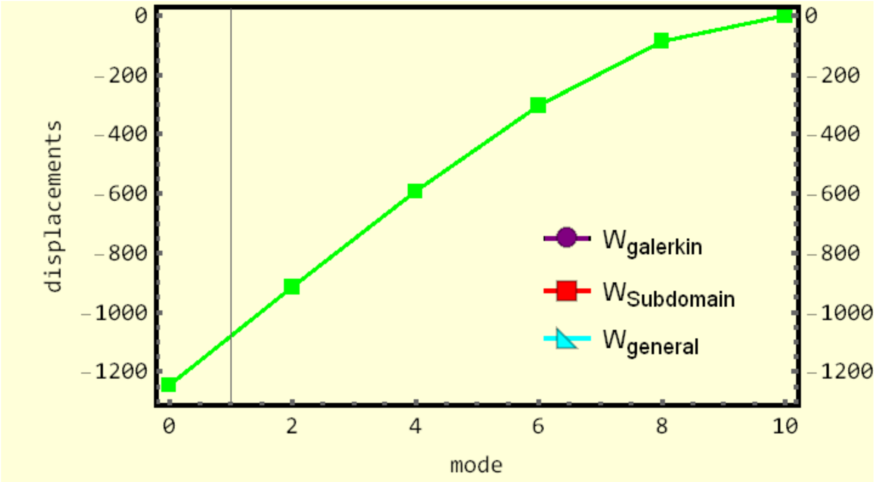

Fig. 2. Comparison of the three-parameter solution with weighted residual methods

Table 2. Comparison of the calculated values of the two-parameter solution with the weighted residual methods

	x=0	x=2	x=4	x=6	x=8	x=10
$W_{Subdomain}$	-1246,26	-914,59	-592,223	-303,091	-87,0721	0
$W_{Galerkin}$	-1080,09	-776,33	-484,546	-236,623	-64,473	0
$W_{General}$ [47]	-1246,26	-914,59	-592,223	-303,091	-87,0721	0

The displacement values of an axially loaded nanobeams are given by using different weighted residual methods given in Table 2. In the previous study, it has been observed that all values are the same in the three-parameter solution calculation, but not all values are the same in the two-parameter solution.

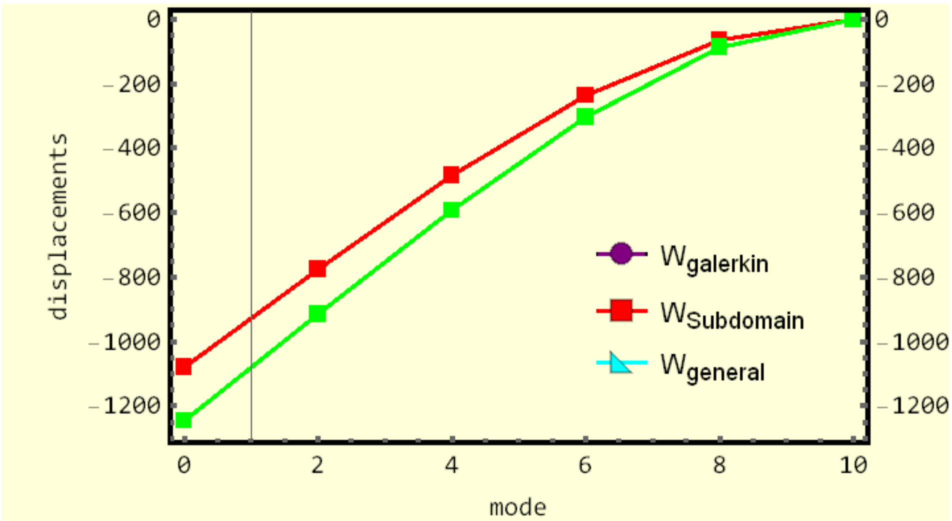

Fig. 3. Comparison of two parameter solution with weighted residual function solution methods

3. Conclusion

Within the scope of this study, the bending analysis of nanobeams has been analyzed using different weighted residual methods. The results of the three-parameter solution method gave the same result as the implicit equation to all weighted residual methods applied. In the two-parameter method, the closed equation gave the same result with all methods except for the Galerkin method. In this direction, the values of the two-parameter solution method and the three-parameter solution method are presented in the article with tables and graphs after the solutions of the bending analysis are completed.

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Estimation of Vessel Passage through Bosphorus

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Abstract

The Bosphorus is a heritage from the past to the present with its natural, historical and cultural features. Vessels passing through the Bosphorus, which is a narrow waterway, create a risk in Istanbul Strait. Thousands of vessels carrying dangerous cargoes pass through the Bosphorus every year. In this study, the changes in the cargos carried by these vessels and number of vessels over years are examined. It is seen that approximately 44000 vessels per year have passed through considering the last decade. The data of Ministry of Transport and Infrastructure, General Directorate of Marine between the years 1995 and 2020 has been used to estimate the change of the number of the vessels for five years period. Although the estimation show that the vessels numbers will decrease with an estimation of ~32,250 vessels for the year 2025, the total gross tonnage will increase in the future years. The results show that, while the number of bulk carrier vessels, livestock carriers, passenger ships and refrigerated cargo carriers will increase, the number of general cargo vessels will decrease in the future years.

Keywords: Bosphorus, Istanbul Strait, Estimation, Vessels Number, Maritime Traffic.

1. Introduction

Istanbul, where Asia and Europe continents meet, has an intense maritime traffic in the Bosphorus, which is a narrow waterway between the Black Sea and the Marmara open through Mediterranean and international waters. Approximately 48,000 vessels/year passed through the Bosphorus according to the data of Ministry of Transport and Infrastructure, General Directorate of Marine between the years 1995 and 2020 as shown in Table 1 [1, 2], besides the local maritime traffic of the region. There are around 2000 scheduled ship traffic daily only in the southern part of the Istanbul Strait [3]. As can be seen from numbers, Bosphorus has a busy maritime traffic. Especially in the last decade many researchers study maritime safety and risk assessment for the region. The main subject is accident statistics and risk assessment over all the studies [4-8]. There are passage schedule and navigation management researches for reducing the maritime risk of this busy region [9-11]. Uğurlu et al. [12] studied economic loss of marine accidents of the region. Therefore, in order to reduce the maritime accidents and enhance the navigation safety, General Directorate of Coastal Safety and related institutions do research and give couple of services in the region.

Table 1. Number of vessels passing through Bosphorus

Years	Number of Vessels	Years	Number of Vessels	Total Gross Tonnage	Years	Number of Vessels	Total Gross Tonnage
1995	46,954	2004	54,564	-	2013	46,532	551,771,780
1996	49,952	2005	54,794	-	2014	45,529	582,468,334
1997	50,942	2006	54,880	475,796,880	2015	43,544	565,216,784
1998	49,304	2007	56,606	484,867,696	2016	42,553	565,282,287
1999	47,906	2008	54,396	513,639,614	2017	42,978	599,324,748
2000	48,079	2009	51,422	514,656,446	2018	41,103	613,088,166
2001	42,637	2010	50,871	505,615,881	2019	41,112	638,892,062
2002	47,283	2011	49,798	523,543,509	2020	38,404	619,758,776
2003	46,939	2012	48,329	550,526,579			

Vessels can request pilot (through Bosphorus) to increase the safety of passage. Average ~53.5% of the passed vessels have requested pilot between the years 2006 and 2020 according to the data of Ministry of Transport and Infrastructure, General Directorate of Marine. Number of passed vessels have been archived on 6 categories in the data. The categories are according to vessel lengths, as shorter than 100 m, between 100-150 m, between 150-200 m, between 200-250 m, between 250-300 m, longer than 300 m as shown in Table 2 [2].

Table 2. Number of vessels passing through Bosphorus according to their length

Years	**Number of Vessels**								
	Longer than 300 m	Btw. 250-300 m	Btw. 200-250 m	Btw. 150-200 m	Btw. 100-150 m	Shorter than 100 m	**Total**	**Total With Pilot**	Percentage of Vessels with Pilot
2006	0	957	2,696	7,216	22,427	21,584	54,880	26,589	48.45%
2007	25	1,114	2,514	6,840	23,889	22,224	56,606	26,685	47.14%
2008	25	1,256	2,630	7,931	22,050	20,504	54,396	27,001	49.64%
2009	8	1,051	2,812	8,256	20,683	18,612	51,422	24,977	48.57%
2010	6	1,216	2,401	7,881	20,990	18,377	50,871	26,035	51.18%
2011	6	1,283	2,511	8,419	20,176	17,403	49,798	26,011	52.23%
2012	14	1,285	2,567	9,278	18,976	16,209	48,329	24,812	51.34%
2013	14	1,268	2,519	9,307	18,341	15,083	46,532	24,023	51.63%
2014	2	1,364	2,929	10,154	16,734	14,346	45,529	24,508	53.83%
2015	0	1,283	2,647	10,235	16,178	13,201	43,544	23,349	53.62%
2016	0	1,143	2,730	10,363	16,077	12,240	42,553	22,356	52.54%
2017	5	1,318	2,682	10,965	16,101	11,907	42,978	24,059	55.98%
2018	3	1,377	2,726	11,640	14,466	10,891	41,103	23,565	57.33%
2019	0	1,324	3,076	11,873	15,497	9,342	41,112	26,632	64.78%
2020	2	1,299	3,651	10,592	14,441	8,419	38,404	24,754	64.46%

Almost all the passed vessels longer than 150 m length have requested pilot after the year 2019. However, before the year 2018 almost all the passed vessels longer than 200 m length have requested pilot. Average ~85.38% of passed vessels between 150-200 m length, average ~44.91% of passed vessels between 100-150 m length and average ~30.43% of passed vessels shorter than 100 m length have requested pilot between the years 2006 and 2020. Where almost half of the passed vessels between 100-150 m length request pilot, only one third of the passed vessels shorter than 100 m requested pilot for that period (Fig. 1, Table 3).

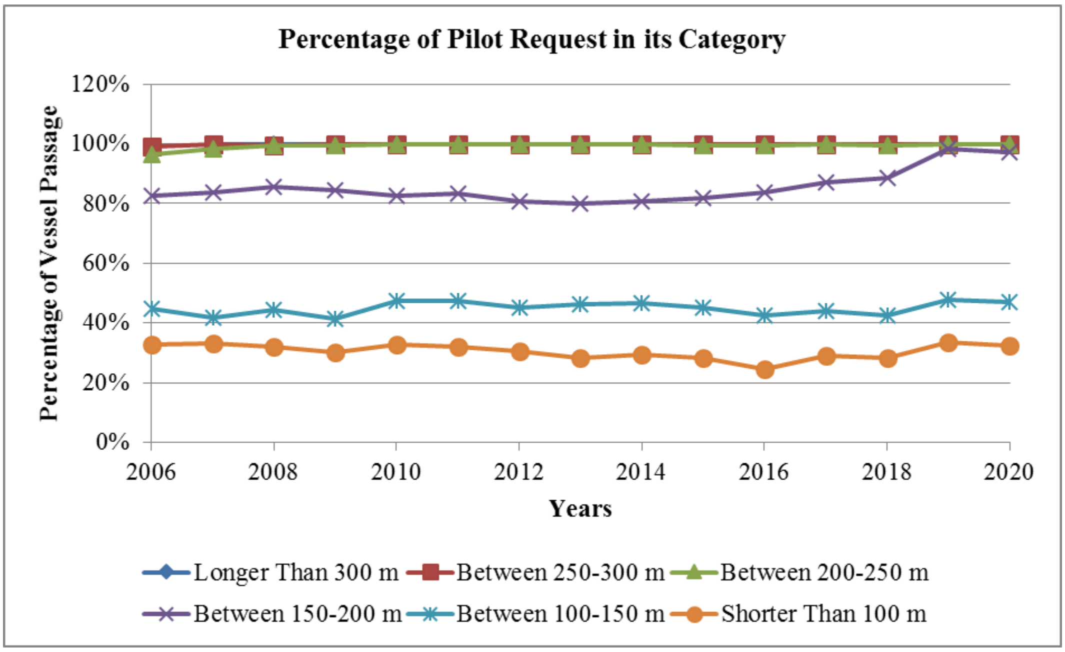

Fig. 1. Percentage of vessels passing through Bosphorus according to vessels length

Table 3. Percentage of vessels passing through Bosphorus with Pilot on vessels category

Years	Percentage of Vessels with Pilot					
	Longer than 300 m	Btw. 250-300 m	Btw. 200-250 m	Btw. 150-200 m	Btw. 100-150 m	Shorter than 100 m
2006		99.06%	96.48%	82.44%	44.70%	32.74%
2007	100.00%	99.64%	98.49%	83.76%	41.84%	33.07%
2008	100.00%	99.60%	99.28%	85.74%	44.39%	31.82%
2009	100.00%	99.90%	99.40%	84.50%	41.45%	29.95%
2010	100.00%	100.00%	100.00%	82.60%	47.17%	32.65%
2011	100.00%	100.00%	100.00%	83.45%	47.51%	32.18%
2012	100.00%	100.00%	99.65%	80.63%	45.03%	30.42%
2013	100.00%	100.00%	99.84%	80.05%	46.38%	28.31%
2014	100.00%	100.00%	99.86%	80.83%	46.49%	29.49%
2015		100.00%	99.32%	81.92%	45.16%	28.38%
2016		100.00%	99.45%	83.56%	42.47%	24.60%
2017	100.00%	99.92%	99.96%	87.23%	43.80%	28.87%
2018	100.00%	99.85%	99.34%	88.43%	42.36%	28.08%
2019		100.00%	99.97%	98.25%	47.92%	33.63%
2020	100.00%	100.00%	100.00%	97.26%	46.95%	32.31%

When the percentage of passed vessel have been calculated according to the length categories for the year 2006; 39.33% of passed vessels are in the shorter than 100 m length category, 40.87 % of passed vessels are between 100-150 m, 13.15% of passed vessels are between 150-200 m, 4.91% of passed vessels are between 200-250 m, 1.74% of passed vessels are between 250-300 m. However, for the year 2020, 21.92% of passed vessels are in the category of shorter than 100 m length, 37.60 % of passed vessels are between 100-150 m, 27.58% of passed vessels are between 150-200 m, 9.51% of passed vessels are between 200-250 m, 3.38% of passed vessels are between 250-300 m, 0.01% of passed vessels are longer than 300 m (Fig. 2). It can be seen from the graph comparing the years 2006 and 2020; that there is no considerable change for longer than 300m length vessels and a slight decrease for the 100-150 m length vessels. However, the percentage of passed vessel doubled for 150-200 m length and 250-300 m length vessels. The percentage of passed vessel also doubled for the 200-250 m length vessels in the last few years comparing year 2006. Furthermore, the percentage of the passed vessels shorter than 100 m dropped to almost half. If the dominant percentage of the passed vessels generally evaluated, the vessels between 150-200 m length doubled, however the vessels shorter than 100 m dropped to almost half.

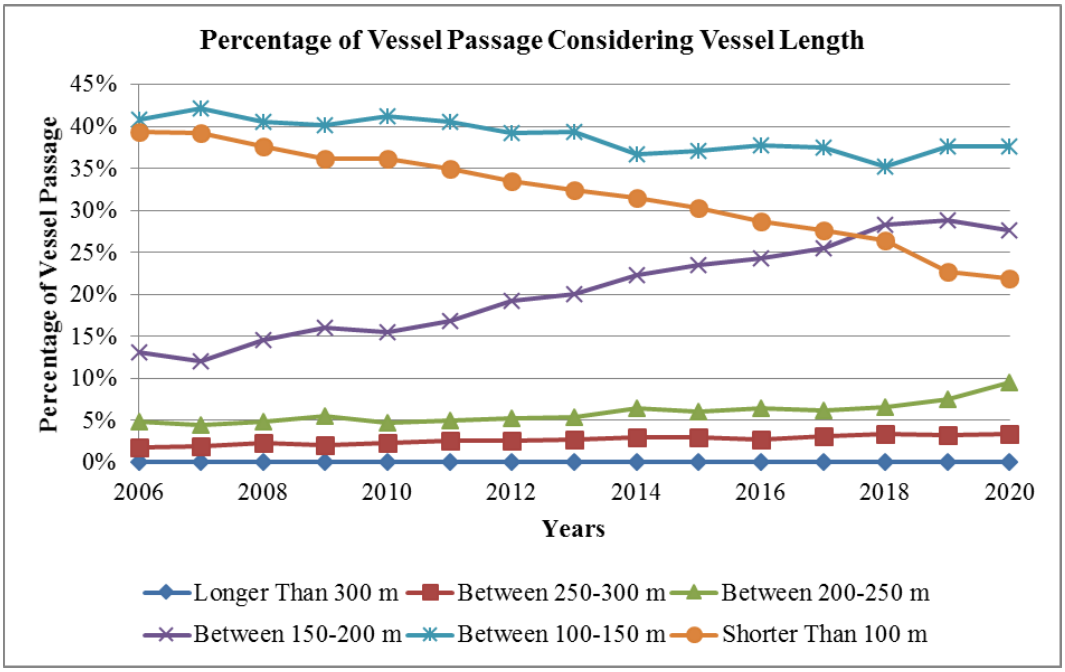

Fig. 2. Percentage of vessels passing through Bosphorus according to vessels length

Passed vessels through Bosphorus have been also archived between the years 2006 and 2020 according to their types. The data consists of 17 types of vessels: Barge / Barge Carrier; Bulk Carrier; Cement Carrier; Container Ship; Ferry; General Cargo Ship; Livestock Carrier; Naval; Passenger Ship; Refrigerated Cargo Carrier; Roll on Roll of Vessel; Other Tanker, TTA; Chemical Tanker, TCH; Liquefied Petroleum Gas/Natural Gas Tanker, LPG/LNG; Tug; Vehicle Carrier and Other (Table 4).

Table 4. Number of vessels passing through Bosphorus according to vessels types

	2006	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**
Barge / Barge Carrier	63	47	52	53	28	17	2	19	12	17	6	18	3	9	15
Bulk Carrier	5,419	5,145	5,978	6,635	5,863	6,341	7,163	6,898	7,263	7,485	7,664	8,206	8,501	8,811	8,592
Cement Carrier	13	6			3	4	2	1	4	8	4	6	12	9	18
Container Ship	2,401	2,727	2,773	2,014	2,292	2,718	2,707	2,868	3,073	2,664	2,734	2,659	2,561	2,642	2,633
Ferry	4	1	1		1	3	1	1	4	2	1	1	1	2	1
General Cargo Ship	33,082	34,822	32,735	30,840	30,876	29,288	27,126	25,521	24,107	22,412	21,344	21,163	19,269	18,637	16,864
Livestock Carrier	141	136	70	147	243	238	390	432	391	434	585	544	508	530	555
Naval	168	166	200	180	114	94	129	196	237	318	342	237	176	178	205
Passenger Ship	1,658	1,702	1,147	786	631	481	583	474	649	444	291	336	367	250	74
Refrigerated Cargo Carrier	908	819	805	623	602	441	248	204	65	24	40	46	34	59	52
Roll on Roll of Vessel	436	441	713	350	457	599	492	406	431	377	352	396	245	266	222
Other Tanker, TTA	7,659	7,204	6,564	6,557	6,464	6,216	5,912	5,685	5,587	5,825	6,033	6,212	6,014	5,934	5,252
Chemical Tanker, TCH	1,680	2,050	1,975	1,876	1,711	1,660	1,779	1,561	1,618	1,576	1,681	1,878	1,950	2,462	2,653
Liquefied Petroleum Gas/Natural Gas Tanker, LPG/LNG	814	800	764	866	1,099	1,227	1,336	1,760	1,540	1,232	989	742	623	561	530
Tug	294	253	313	304	293	245	274	241	231	282	237	262	384	270	175
Vehicle Carrier	14	92	189	78	42	47	37	47	93	17	16	45	88	113	87
Other	126	195	117	113	152	179	148	218	224	427	234	227	367	379	476
Total	54,880	56,606	54,396	51,422	50,871	49,798	48,329	46,532	45,529	43,544	42,553	42,978	41,103	41,112	38,404

2. Methodology

Data can be examined via statistical methods. Regression analysis can be used to see the data change [13]. Generally, linear regression analysis is used to see the trend of the data. It is also possible to use other regression analysis to obtain more suitable regression lines. However, to see the trend and estimate the vessel passage roughly, linear regression analysis has been preferred in the study. Although polynomial regression analysis has been performed in the study, the results are not given for all trials.

2.1. Linear regression analysis

Linear regression is the simplest regression analysis can be applied to the data (x_i, y_i). Mathematical expression of the regression line is shown in Eq. (1).

$$y = a_0 + a_1 x \tag{1}$$

To calculate slope (a_1) Eq. (2) and to calculate constant (a_0) Eq. (3) can be used, where n is the number of the data, $\overline{y}$ mean value of the data y, $\overline{x}$ mean value of the data x.

$$a_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \tag{2}$$

$$a_0 = \overline{y} - a_1 \overline{x} \tag{3}$$

Coefficient of determination (R^2) in Eq. (4) is used to quantify the error of the regression line where S_t is the total sum of the squares around the mean as shown in Eq. (5) and S_r is the total sum of the squares of the residuals as shown in Eq. (6).

$$R^2 = \frac{S_t - S_r}{S_t} \tag{4}$$

$$S_t = \sum_{i=1}^{n} (y_i - \overline{y})^2 \tag{5}$$

$$S_r = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2 \tag{6}$$

Coefficient of determination of the model is the explanation percentage of the original uncertainty.

2.2. Polynomial regression analysis

A second order polynomial regression analysis can be also applied to the data (x_i, y_i). Mathematical expression of the regression line is shown in Eq. (7).

$$y = a_0 + a_1 x + a_2 x^2 \tag{7}$$

The terms (a_0, a_1, a_2) as shown in Eq. (8) can be solved with any method. To determine coefficient of determination (R^2) Eq. (4) and Eq. (5) can be used. However S_r the total sum of the squares of the residuals should be calculated using Eq. (9).

$$\begin{aligned}(n)a_0 + (\textstyle\sum_{i=1}^{n} x_i)a_1 + (\sum_{i=1}^{n} x_i^2)a_2 &= \textstyle\sum_{i=1}^{n} y_i \\ (\textstyle\sum_{i=1}^{n} x_i)a_0 + (\sum_{i=1}^{n} x_i^2)a_1 + (\sum_{i=1}^{n} x_i^3)a_2 &= \textstyle\sum_{i=1}^{n} x_i y_i \\ (\textstyle\sum_{i=1}^{n} x_i^2)a_0 + (\sum_{i=1}^{n} x_i^3)a_1 + (\sum_{i=1}^{n} x_i^4)a_2 &= \textstyle\sum_{i=1}^{n} x_i^2 y_i\end{aligned} \quad (8)$$

$$S_r = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2 \quad (9)$$

3. Application and Results

The number of vessels passed through Bosphorus data (Table 1) have been analyzed by linear regression between the years 1995 and 2020 (Fig. 3). However the coefficient of determination (R^2) the explanation of data is calculated an inadequate value as 24.85%.

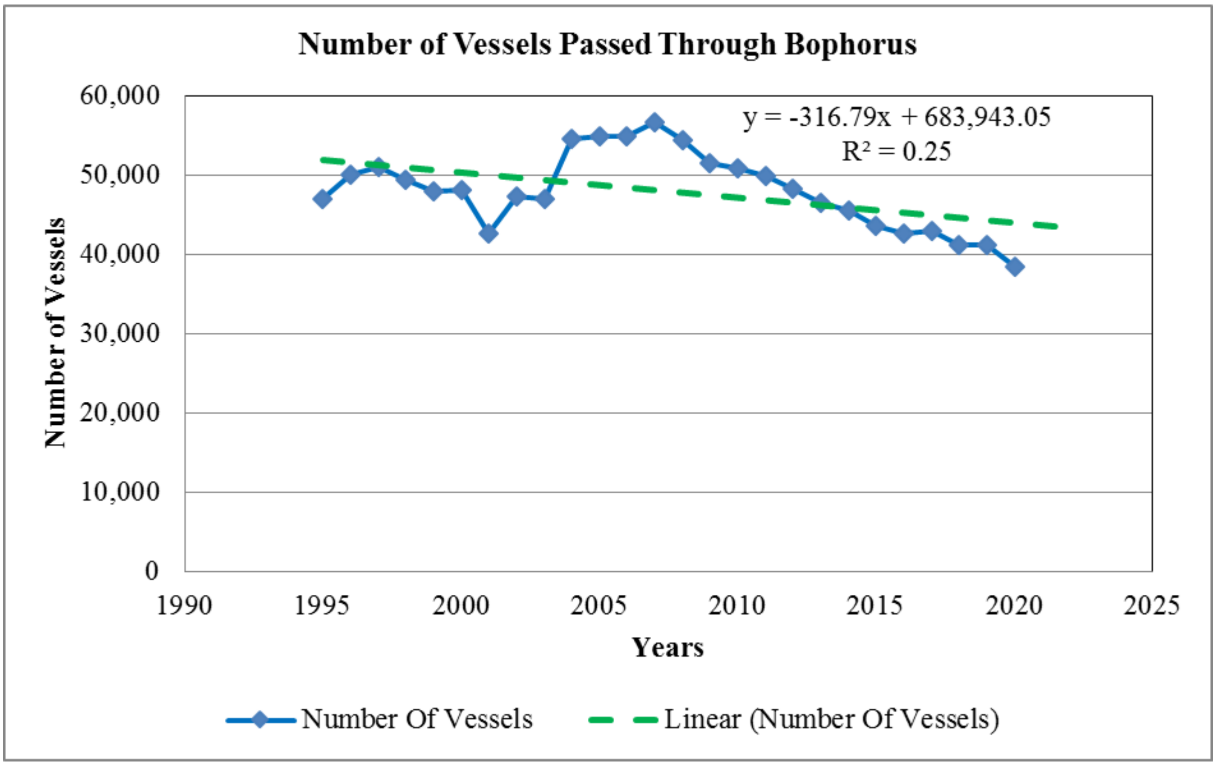

Fig. 3. Number of vessels passing through Bosphorus

Therefore, the data of number of vessels passed through Bosphorus have been reanalyzed by linear regression between the years 2006 and 2020 (Fig. 4). The coefficient of determination value is calculated as 97.38%. Thus the linear regression line can be used to estimate forward 5 years period. The projected estimation gives 32,255 vessels will pass through Bosphorus in 2025 (Table 5).

Table 5. Estimation of forward 5 years period of number of vessels passing through Bosphorus

Years	Number of Vessels
2021	37,238
2022	35,992
2023	34,747
2024	33,501
2025	32,255

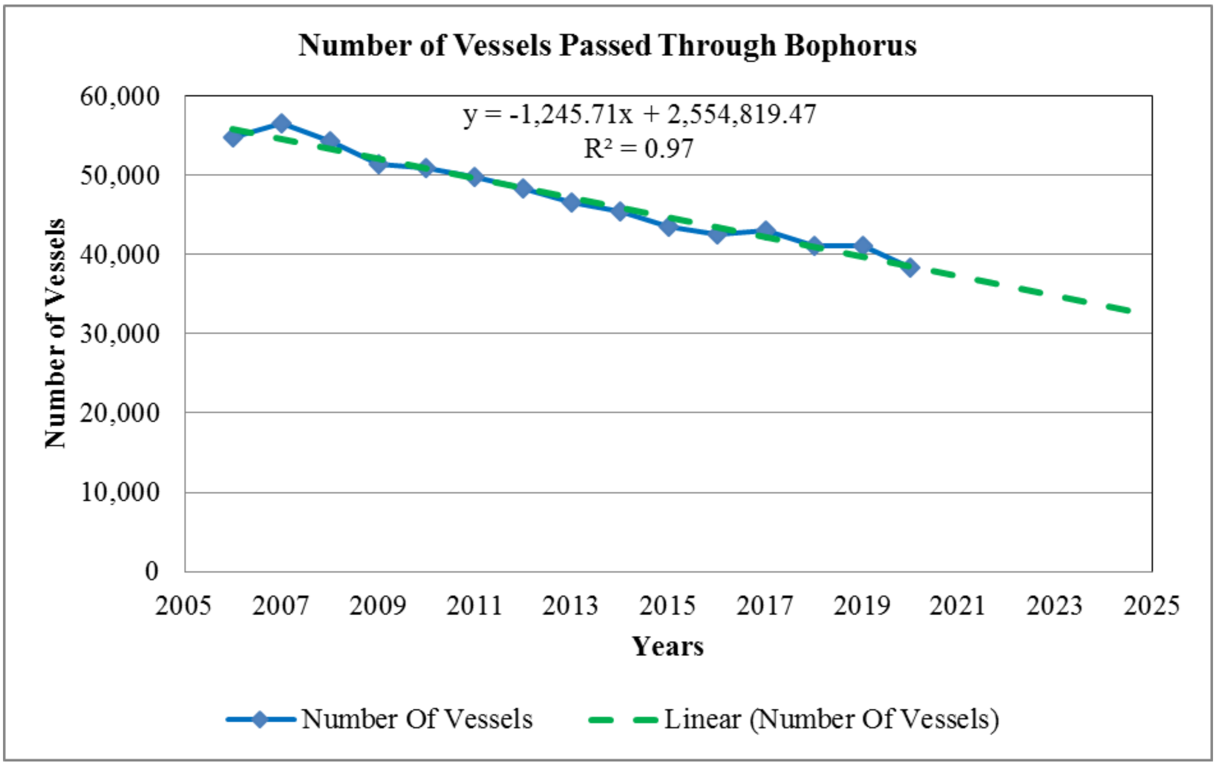

Fig. 4. Number of vessels passing through Bosphorus

The total gross tonnage (GT) data of vessels passed through Bosphorus have been analyzed by linear regression between years 2006 and 2020 (Fig. 5). The coefficient of determination value is calculated as 94.34%. Thus the linear regression line can be used to estimate forward 5 years period. The estimation shows that 684,862,096 total gross tonnage will pass through Bosphorus in year 2025 (Table 6).

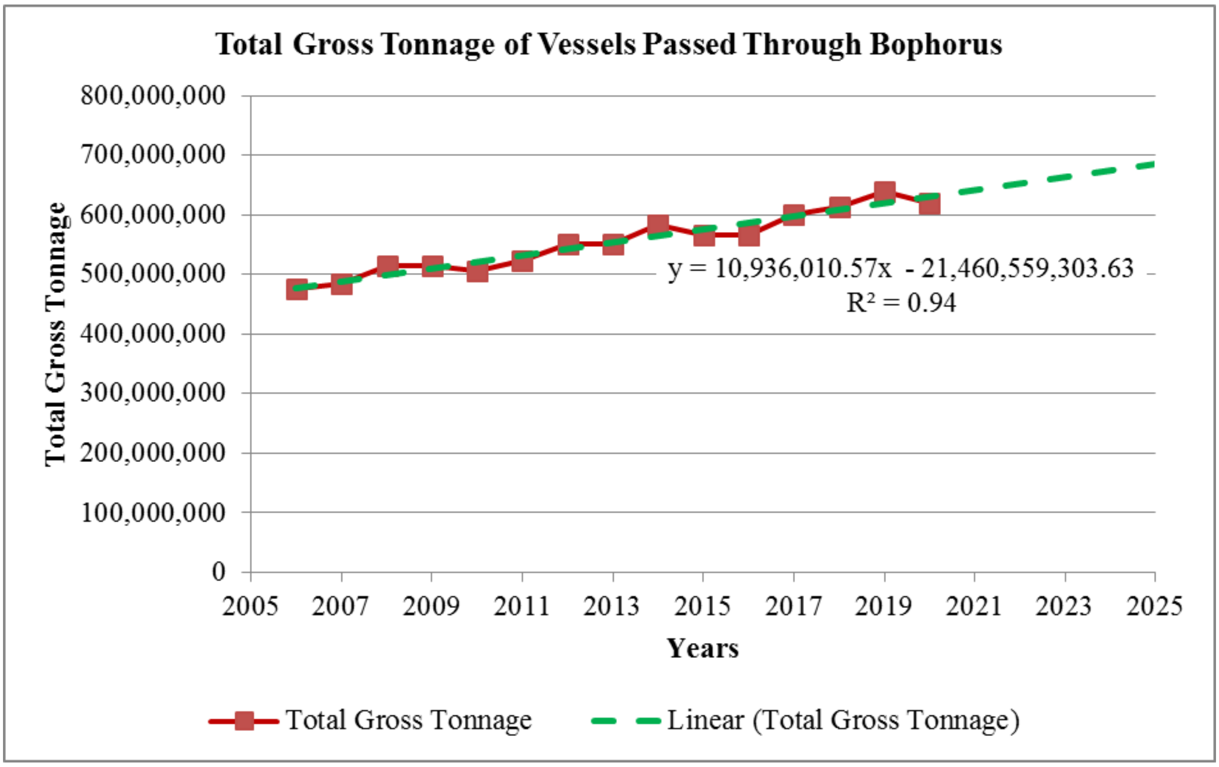

Fig. 5. Total gross tonnage of vessels passing through Bosphorus

Table 6. Estimation of forward 5 years period of total gross tonnage of vessels passing through Bosphorus

Years	Total Gross Tonnage of Vessels
2021	641,118,054
2022	652,054,065
2023	662,990,075
2024	673,926,086
2025	684,862,096

The gross tonnage per vessel has been calculated by dividing total gross tonnage to number of vessels (Fig. 6) and the estimation show that 18,582 gross tonnages per vessel will pass through Bosphorus in year 2025 (Table 7).

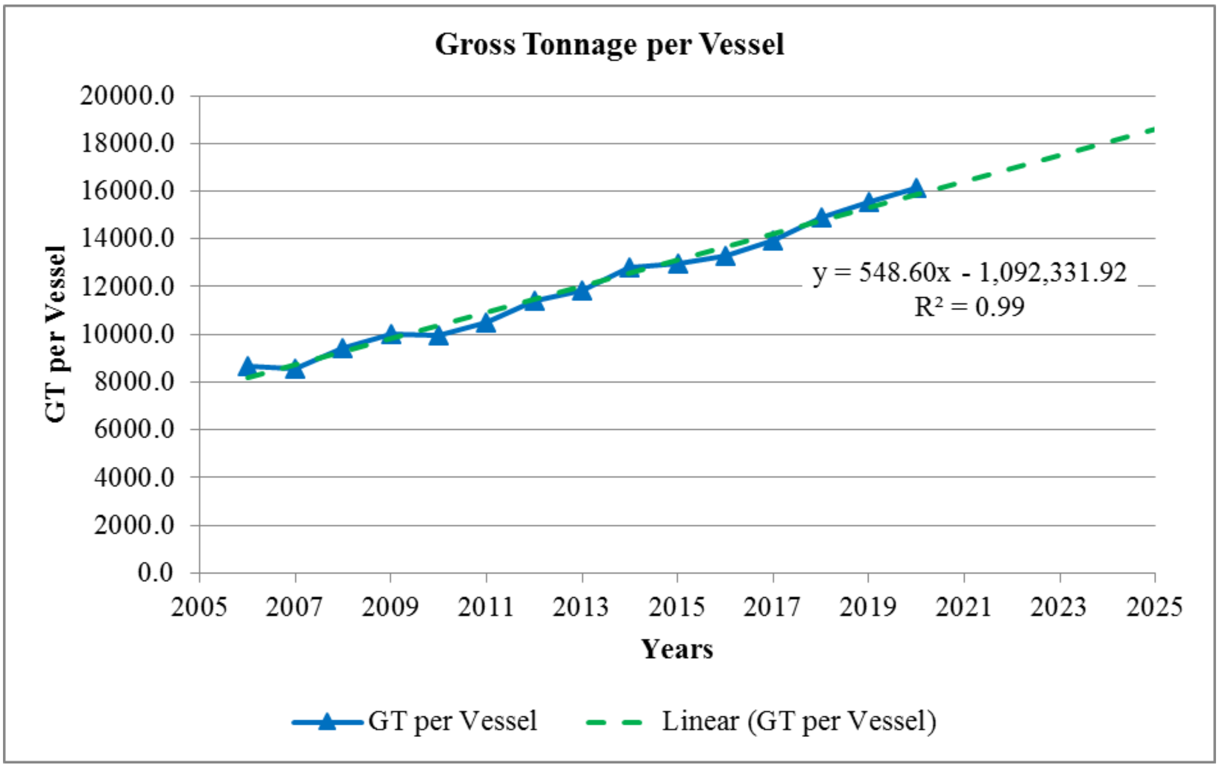

Fig. 6. Total gross tonnage per vessel passing through Bosphorus

Table 7. Estimation of forward 5 years period of total gross tonnage per vessel passing through Bosphorus

Years	Total Gross Tonnage per Vessels
2021	16,388
2022	16,936
2023	17,485
2024	18,034
2025	18,582

The data of number of vessels according to vessel type (Table 4) has been analyzed and selected results of regression analysis considering the coefficient of determination, have been given below. Vessel type of bulk carrier have been analyzed by linear regression between years 2006 and 2020 (Fig. 7). The coefficient of determination value is 94.05%. Thus the linear regression line can be used to project for 5 years. The estimation show that 10,103 bulk carrier type vessels will pass the Istanbul Strait in year 2025 (Table 8).

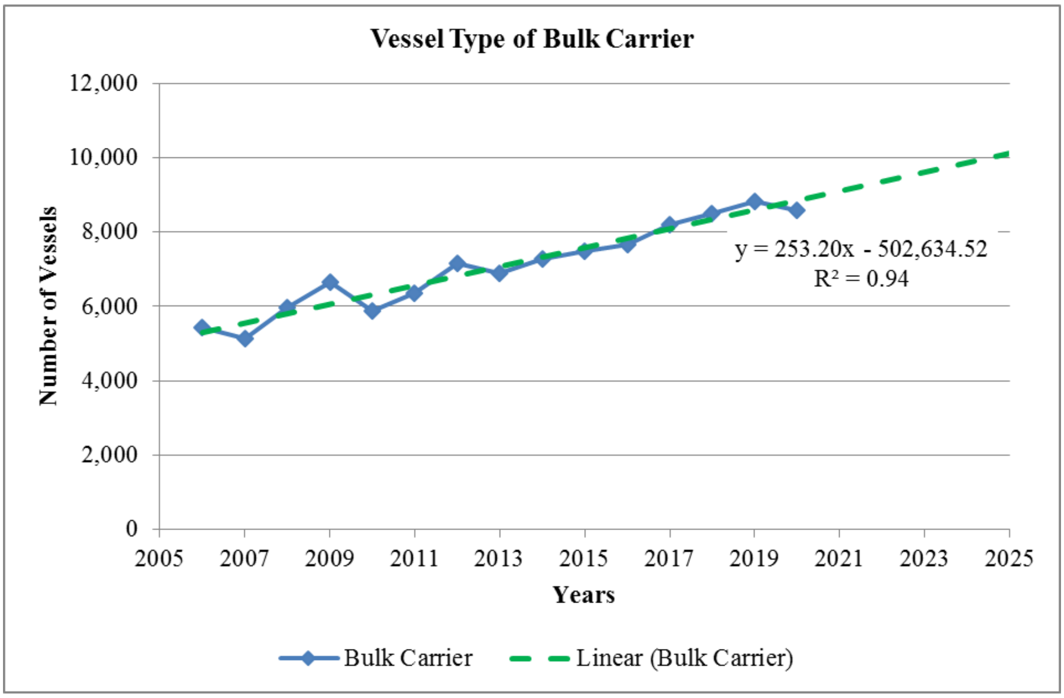

Fig. 7. Number of bulk carrier vessel type passing through Bosphorus

Table 8. Estimation of forward 5 years period of number of bulk carrier vessel type

Years	Bulk Carrier Vessel Type
2021	9,090
2022	9,343
2023	9,596
2024	9,850
2025	10,103

Vessel type of general cargo have been analyzed by linear regression between the years 2006 and 2020 (Fig. 8). The coefficient of determination value is calculated as 98.11%. Thus the linear regression line can be used to estimate forward 5 years period. The estimation show that 10,356 general cargo type vessels will pass through Bosphorus in year 2025 (Table 9).

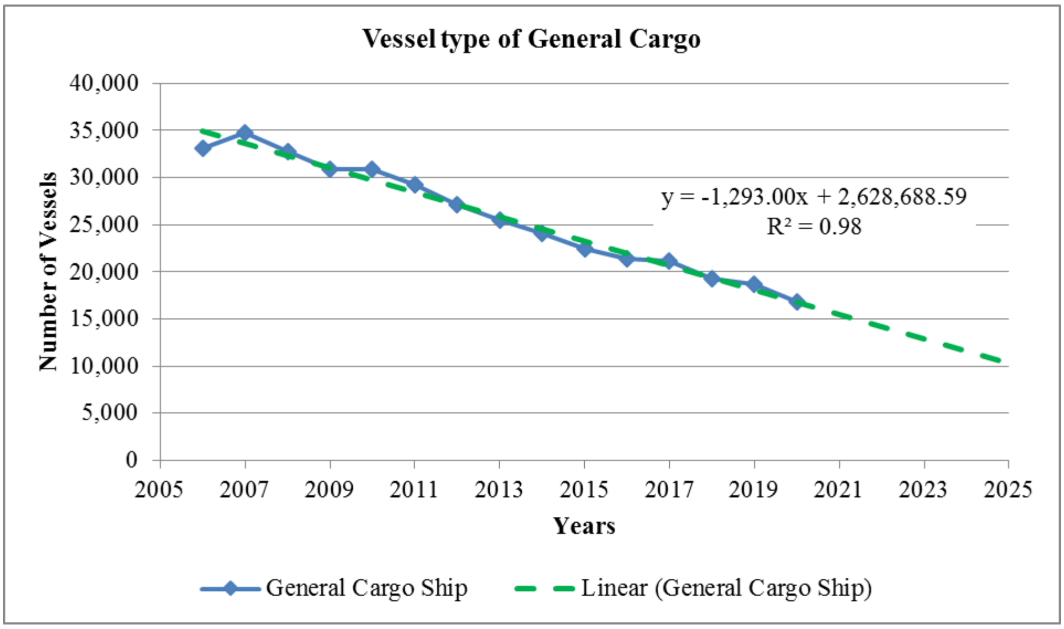

Fig. 8. Number of general cargo vessel type passing through Bosphorus

Table 9. Estimation of forward 5 years period of number of general cargo vessel type

Years	General Cargo Vessel Type
2021	15,528
2022	14,235
2023	12,942
2024	11,649
2025	10,356

Number of vessel type of livestock carrier passed through Bosphorus have been analyzed by linear regression between the years 2006 and 2020 (Fig. 9).

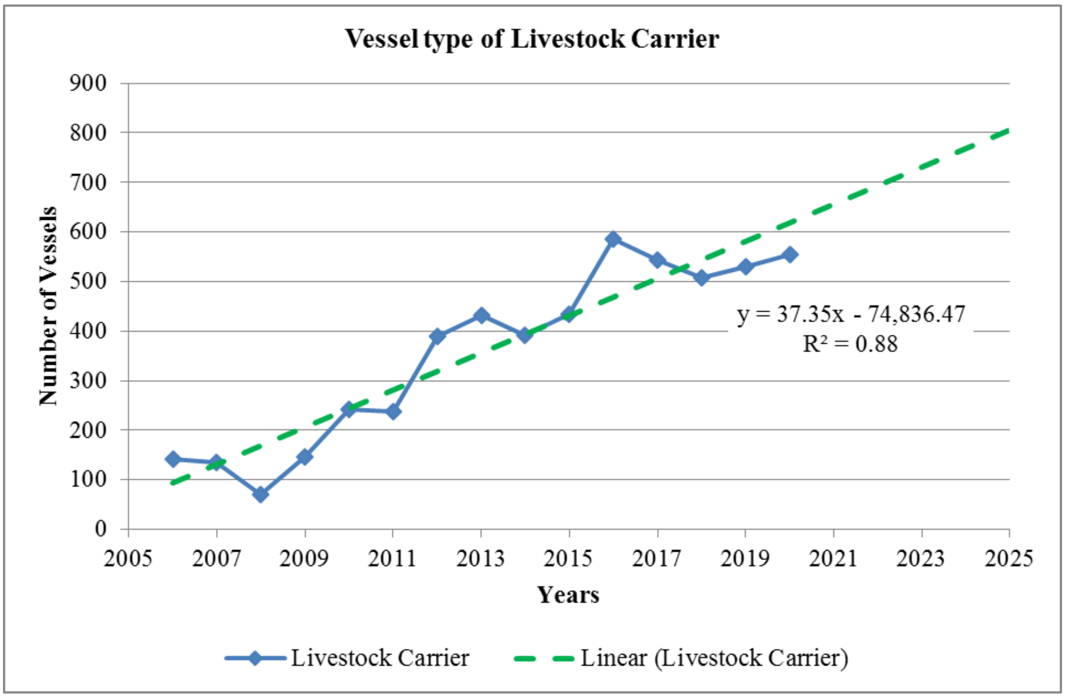

Fig. 9. Number of livestock carrier vessel type passing through Bosphorus

The coefficient of determination value is 88.43%. Thus the regression line can be used to estimate forward 5 years period. The estimation show that 805 livestock carrier ship type vessels will pass in the year 2025 (Table 10).

Table 10. Estimation of forward 5 years period of number of livestock carrier vessel type

Years	Livestock Carrier Vessel Type
2021	655
2022	692
2023	730
2024	767
2025	805

Number of passenger ship vessels have been analyzed by second order polynomial regression between the years 2006 and 2020 (Fig. 10). The coefficient of determination value is calculated as 88.02%. Thus the polynomial regression line can be used to estimate forward 5 years period. The estimation show that 733 passenger ship type vessels will pass through Bosphorus in 2025 (Table 11).

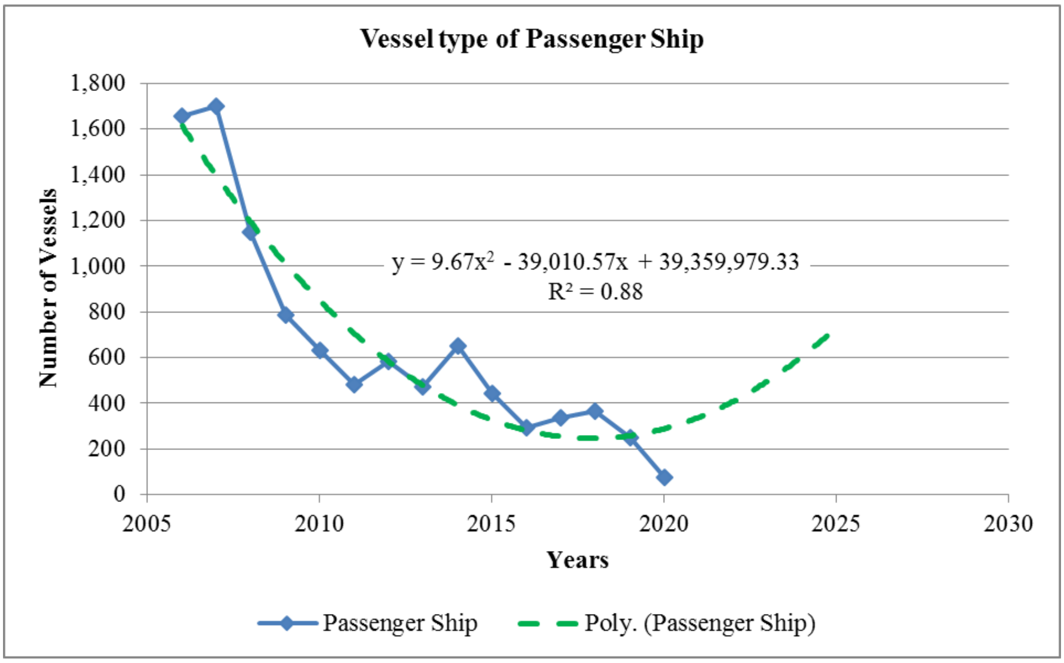

Fig. 10. Number of passenger ship vessel type passing through Bosphorus

Table 11. Estimation of forward 5 years period of number of passenger ship vessel type

Years	Passenger Ship Vessel Type
2021	338
2022	408
2023	497
2024	605
2025	733

Number of refrigerated cargo carrier vessel type have been analyzed by second order polynomial regression between the years 2006 and 2020 (Fig. 11). The coefficient of determination value is calculated as 96.40%. Thus the polynomial regression line can be used to estimate forward 5 years period. The estimation show that 274 refrigerated cargo carrier vessels will pass through Bosphorus for the year 2025 (Table 12).

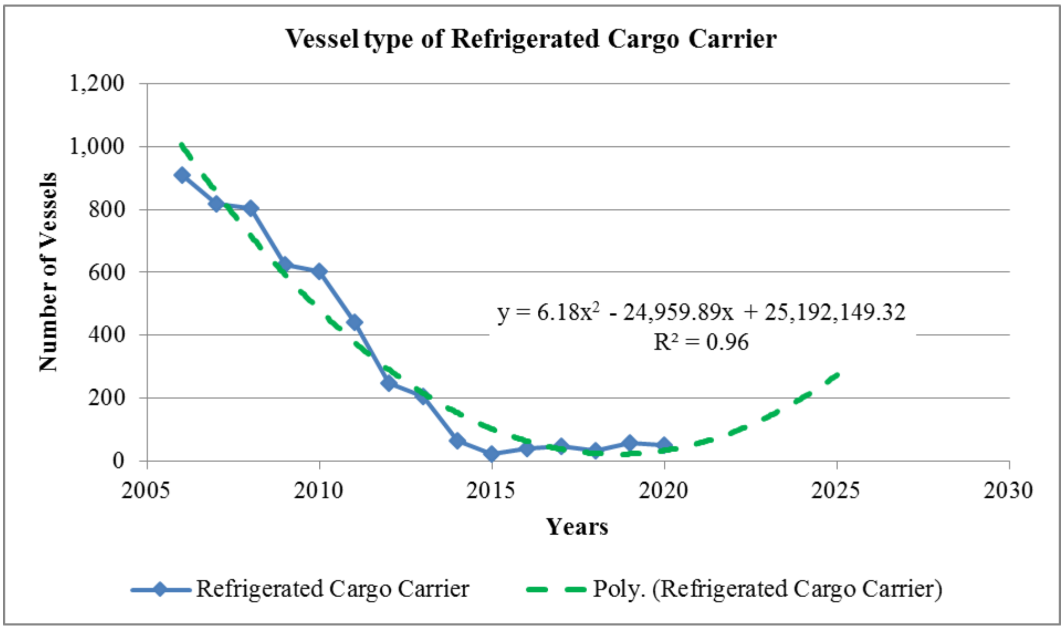

Fig. 11. Number of refrigerated cargo carrier vessel type passing through Bosphorus

Table 12. Estimation of forward 5 years period of number of refrigerated cargo carrier vessel type

Years	Refrigerated Cargo Carrier Vessel Type
2021	57
2022	93
2023	141
2024	201
2025	274

4. Conclusions

The data of Ministry of Transport and Infrastructure, General Directorate of Marine between the years 1995 and 2020 has been used to estimate the change of the number of vessels in the future years. Although the estimation show that the number of vessels passed through Bosphorus will decrease in the future years (Fig. 4, Table 5), the total gross tonnage will increase (Fig. 5, Table 6). Furthermore, the gross tonnage per vessel will also increase in the future (Fig. 6, Table 7).

The data of number of vessels according to vessel type (Table 4) has been analyzed and selected projectable results of regression analysis have been given in this study. The estimation results show that the number of bulk carrier vessel type (Fig. 7, Table 8), the number of livestock carrier vessel type (Fig. 9, Table 10), the number of passenger ship vessel type (Fig. 10, Table 11) and the number of refrigerated cargo carrier vessel type (Fig. 11, Table 12) will increase in the future. However, the number of general cargo vessel type (Fig. 8, Table 9) will decrease in the future years.

It is expected that these estimations will be apply with a high degree of accuracy if present conditions in the region will continue. These conditions may include international trade, economy, maritime transport, local regulations...etc. In order to improve the estimations all influencing variables should be study. More precise estimations can be made with statistical models.

Notations

m	Meter
Btw.	Between
y	Dependent variable of the data
x	Independent variable of the data
a_1	Slope
a_0	Constant
$\overline{y}$	Mean value of the data y
$\overline{x}$	Mean value of the data x
R^2	Coefficient of determination
GT	Gross Tonnage

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The Modified Palomba Economic Model by Difference Equations and its Stability Analysis

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Abstract

In this study, Palomba economic model was analyzed through difference equations. For these products given as capital and consumer goods in the model, their non-marketable loss rates were also taken into account. Tirivial and non-tirivial equilibrium points were found and local asymptotic stability (LAS) conditions of these equilibrium points were investigated. Although the non-trivial equilibrium point is always unstable, the stability conditions of the trivial equilibrium point were found. In this way, in addition to the thought put forward by Palomba, the absence of periodicity was expressed too. The findings were supported by numerical studies.

Keywords: Palomba Economic Model, Difference Equation System, Stability Analysis.

1. Introduction

Both engineers, naturalists and researchers and practitioners working in the economic and social sciences use mathematical models of the systems, in which they study. These models, which are expected to be solved in any way, or which are predicted to be solved under certain conditions, and which are expressed through mathematical equations, give a simplified description of real-life problems [1]. It is seen that this modeling process is mostly done by using differential equations and difference equations in the literature. The application areas of these equations are widely used in the mathematical modeling studies in fields such as finance, accounting and economics in social sciences.
The difference equation theory has been greatly developed over the last three hundred years. Generally, the difference equations describe the change in a variable between two periods. By using these equations, the relevant factors that cause changes in the values of functions in different time periods can be examined [2].
Mathematical models are widely used in economics to represent relationships between various quantities such as price, level of production, demand, employment and investment [3]. Palomba (1939) considered an economy in which only two types of goods (products) existed as capital and consumer goods. Palomba's proposed model has the following assumptions [4]:

- Two types of goods as a and b exist. Type-a goods are goods that enter directly and these are ready for immediate consumption. Type-b goods are capital goods in which the products of other capital goods enter directly and only the products of final goods enter indirectly.
- The economy is in a dynamic state, tending to increase capital equipment. A part of type-a commodities is diverted from their normal destination and allocated to type -b.
- Let ε_1 and ε_2 denote positive constants. ε_1 shows the increase coefficient of type-a goods and $-\varepsilon_2$ shows the increase coefficient of type-b goods.
- The increase coefficient of type-a goods through the type-b goods is $-\gamma_1$ and the increase coefficient of type-b goods through type-a goods is γ_2. In here, γ_1 and γ_2 are positive constants.

Let consider C_1 and C_2 to show the volumes of type-a and type-b goods at time t $(t \geq 0)$. The model consisting of two nonlinear autonomous differential equation proposed by Palomba is

$$\begin{aligned} \frac{dC_1}{dt} &= C_1\,(\varepsilon_1 - \gamma_1 C_2) = \varepsilon_1 C_1 - \gamma_1 C_1 C_2 \\ \frac{dC_2}{dt} &= -C_2(\varepsilon_2 - \gamma_2 C_1) = -\varepsilon_2 C_2 + \gamma_2 C_1 C_2. \end{aligned} \tag{1}$$

Palomba created his model using ordinary differential equations system and showed that a cyclical situation occurred in his analysis.

Daşbaşı and Boztosun [5] analyzed Palomba's model using incommensurate fractional-order differential equations system and explained the stability of equilibrium points according to different states of derivative orders.
According to the results of their model, there are followings:

- Trivial equilibrium point is always unstable.
- The positive equilibrium point is stable, when the sum of the derivative orders is between 0 and 2.
- If the sum of the derivative orders is 2 or more, then the positive equilibrium point is unstable.
- They showed graphically the existence of cyclical state in case of instability.

In here, the Palomba model was analyzed using the difference equation system. Also, The non-marketable loss rate of the first and second type goods were taken into account as parameters μ_1 and μ_2, respectively. Therefore, we have proposed the model,

$$\begin{aligned} x_{t+1} &= f(x_t, y_t) = x_t + \delta\big(x_t\,(\varepsilon_1 - \gamma_1 y_t - \mu_1)\big) \\ y_{t+1} &= g(x_t, y_t) = y_t - \delta\big(y_t(\varepsilon_2 - \gamma_2 x_t + \mu_2)\big) \end{aligned} \tag{2}$$

for $t \geq 0$ by means of the difference equations. In here, it is

$$\varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2, \delta, \mu_1, \mu_2 > 0 \tag{3}$$

2. Existence and Stability of Fixed Point

In this section, firstly, some definitions and theorems about the stability of equilibrium points of difference equation systems are given, then the fixed points (equilibrium points) of the system (2) were investigated and the stability of these points was analyzed.

Definition 2.1 That the equilibrium point of the first-order difference equation system given as

$$X_{t+1} = F(X_t) \tag{4}$$

is the point $\bar{X}$ that satisfies the equations $\bar{X} = F(\bar{X})$. Also, let us consider $J(\bar{X})$ to be the Jacobian matrix calculated at this equilibrium point. If the eigenvalues obtained from the equation $det(J(\bar{X}) - \lambda I) = 0$ satisfy the conditions $\lambda_i \neq 1$ for $i = 1,2,\dots,n$, then this point is called hyperbolic equilibrium, otherwise it is called non-hyperbolic equilibrium [6].

Theorem 2.1 (Jury Conditions, Schur-Cohn Criterion) Considering the n –th degree characteristic equation of the system in (4) given as

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n, \tag{5}$$

it is presumed λ_i for $i = 1,2,\dots,n$ as the eigenvalues obtained from the equation $P(\lambda) = 0$. In this case, the local asymptotically stability (LAS) conditions of the equilibrium point are

$$|\lambda_i| < 1. \tag{6}$$

In addition, if the inequalities in Eq. (6) are satisfied, then we have followings:

- $P(1) = 1 + a_1 + a_2 + \cdots + a_n > 0$,
- $(-1)^n P(-1) = 1 - a_1 + a_2 - \cdots + (-1)^n a_n > 0$,
- $|a_n| < 1$ [7].

Definition 2.2 Let's consider the difference equation system

$$\begin{aligned} x_{t+1} &= f(x_t, y_t) \\ y_{t+1} &= g(x_t, y_t) \end{aligned} \tag{7}$$

For this system, the point $(\bar{x}, \bar{y})$ that satisfies the condition

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{y}) \\ \bar{y} &= g(\bar{x}, \bar{y}), \end{aligned} \tag{8}$$

is called as the equilibrium point [7].

Definition 2.3 Let us assumed that the functions $f(x, y)$ and $g(x, y)$ given in the system (7) have continuous partial derivatives for x and y on an open set of $\mathbb{R}_+^2$. Also, let $(\bar{x}, \bar{y})$ be the equilibrium point of the system and $\bar{J} = J(\bar{x}, \bar{y}) = \begin{pmatrix} f_x(\bar{x}, \bar{y}) & f_y(\bar{x}, \bar{y}) \\ g_x(\bar{x}, \bar{y}) & g_y(\bar{x}, \bar{y}) \end{pmatrix}$ be the Jacobian matrix at this point. The characteristic equation of the eigenvalues λ_i for $i = 1,2$ obtained from the equation $det(J(\bar{X}) - \lambda I_{2x2}) = 0$ is [8]

$$\lambda^2 - Tr(\bar{J})\lambda + Det(\bar{J}) = 0 \tag{9}$$

Corollary 2.1 If the characteristic equation in Eq. (5) is taken into account for $n = 2$, then

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 \tag{10}$$

is found. In here, it is $a_1 = -Tr(J)$ and $a_2 = Det(J)$. According to Theorem 2.1., if

$$|Tr(J)| < 1 + Det(J) < 2 \tag{11}$$

is satisfied, then the point $(\bar{x}, \bar{y})$ is LAS [9].

Proposition 2.1 The system (2) have

- the trivial equilibrium point $E_0(0,0)$, and
- the positive equilibrium point $E_1\left(\frac{(\varepsilon_2+\mu_2)}{\gamma_2}, \frac{(\varepsilon_1-\mu_1)}{\gamma_1}\right)$, when $\left(\varepsilon_1 - \mu_1\right) > 0$.

Proposition 2.2 The system (2) has two equilibrium points with the following dynamical properties:

- the trivial equilibrium point $E_0(0,0)$ is LAS, when $\left(\varepsilon_1 - \mu_1\right) < 0$.
- let $\left(\varepsilon_1 - \mu_1\right) > 0$. the nontrivial equilibrium point $E_1\left(\frac{(\varepsilon_2+\mu_2)}{\gamma_2}, \frac{(\varepsilon_1-\mu_1)}{\gamma_1}\right)$ is always unstable point.

Proof The Jacobian matrix of (2) is

$$J(x,y) = \begin{pmatrix} 1 + \delta\left((\varepsilon_1 - \mu_1) - \gamma_1 y\right) & -\delta\gamma_1 x \\ \delta\gamma_2 y & 1 - \delta\left((\varepsilon_2 + \mu_2) - \gamma_2 x\right) \end{pmatrix} \tag{12}$$

through of partial derivatives.

- The Jacobian matrix in Eq. (12) calculated at the equilibrium point $E_0(0,0)$ is $J\left(E_0(0,0)\right) = \begin{pmatrix} \left(1 + \delta\left(\varepsilon_1 - \mu_1\right)\right) & 0 \\ 0 & \left(1 - \delta(\varepsilon_2 + \mu_2)\right) \end{pmatrix}$. In addition, the characteristic equation for this point is

$$\lambda^2 - \left(\left(1 + \delta\left(\varepsilon_1 - \mu_1\right)\right) + \left(1 - \delta(\varepsilon_2 + \mu_2)\right)\right)\lambda + \left(1 + \delta\left(\varepsilon_1 - \mu_1\right)\right)\left(1 - \delta(\varepsilon_2 + \mu_2)\right) = 0 \tag{13}$$

and the eigenvalues are found as $\lambda_1 = \left(1 + \delta\left(\varepsilon_1 - \mu_1\right)\right)$ and $\lambda_2 = \left(1 - \delta(\varepsilon_2 + \mu_2)\right)$. It is obvious that the eigenvalues are positive real numbers due to Eq. (3). According to the stability conditions of the equilibrium point in Eq. (6), if

$$-\frac{2}{\delta} < \left(\varepsilon_1 - \mu_1\right) < 0 \tag{14a}$$

$$(\varepsilon_2 + \mu_2) < \frac{2}{\delta} \tag{14b}$$

then it is $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Consequently, E_0 is LAS. When the inequalities in (14a) and (14b) are evaluated together, the stability condition of this point can be considered as

$$0 < \{(\mu_1 - \varepsilon_1) \text{ and } (\mu_2 + \varepsilon_2)\} < \frac{2}{\delta}. \tag{15}$$

- Let $(\varepsilon_1 - \mu_1) > 0$. In this case, E_1 exists. The Jacobian matrix evaluated at $E_1\left(\frac{(\varepsilon_2+\mu_2)}{\gamma_2}, \frac{(\varepsilon_1-\mu_1)}{\gamma_1}\right)$ becomes $J(E_1) = \begin{pmatrix} 1 & -\delta\gamma_1 \frac{(\varepsilon_2+\mu_2)}{\gamma_2} \\ \delta\gamma_2 \frac{(\varepsilon_1-\mu_1)}{\gamma_1} & 1 \end{pmatrix}$ and thus the characteristic equation is

$$\lambda^2 - 2\lambda + \left(1 + \delta^2(\varepsilon_1 - \mu_1)(\varepsilon_2 + \mu_2)\right) = 0. \tag{16}$$

According to Corollary 2.1, the stability conditions are not met. Therefore, E_1 is unstable point. Proof is completed. As a result of the stability analysis, the following Table can be reached.

Table 1. Existence and LAS conditions of equilibrium points of system (2)

Equilibrium point	Existence condition	LAS condition
$E_0(0,0)$	Always exists	If $0 < \{(\mu_1 - \varepsilon_1) \text{ and } (\mu_2 + \varepsilon_2)\} < \frac{2}{\delta}$
$E_1\left(\frac{(\varepsilon_2 + \mu_2)}{\gamma_2}, \frac{(\varepsilon_1 - \mu_1)}{\gamma_1}\right)$	$(\varepsilon_1 - \mu_1) > 0$	Unstable

Corollary 2.2 Let $\varepsilon_1 = \mu_1$. By Eq. (16), characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0. \tag{17}$$

Therefore, the eigenvalues are $\lambda_1 = \lambda_2 = 1$, and so, $E_1\left(\frac{(\varepsilon_2+\mu_2)}{\gamma_2}, 0\right)$ hyperbolic equilibrium point.

Corollary 2.3 E_0 is unstable point, when E_1 exists. Therefore, these equilibrium points can not be stable at together. This situation is also seen in the Table 1.

3. Numerical Studies

In this section, the system (2) is graphically shown by giving the values to the parameters used in the system to support the qualitative analysis results shown in Table 1. These values are shown in Table 2.

Table 2. The considered values of the parameters.

Parameters	Values For Fig. 1	Values For Fig. 2
δ	.4	.4
ε_1	.9	1.1
γ_1	.01	.01
μ_1	1	1
ε_2	.1	.1
γ_2	.1	.1
μ_2	.5	.5
Initial conditions	$x(1) = 3$ and $y(1) = 2$	

From first column values in Table 2, E_1 is not exists due to $(\varepsilon_1 - \mu_1) = .9 - 1 = -.1 < 0$. It is clear that $0 < \left\{\overbrace{(\mu_1 - \varepsilon_1)}^{.1} \text{ and } \overbrace{(\mu_2 + \varepsilon_2)}^{.6}\right\} < \overbrace{\frac{2}{\delta}}^{5}$ in Eq. (15) is satisfied. Therefore, the trivial equilibrium point $E_0(0,0)$ is LAS. Fig. 1 shows this situation.

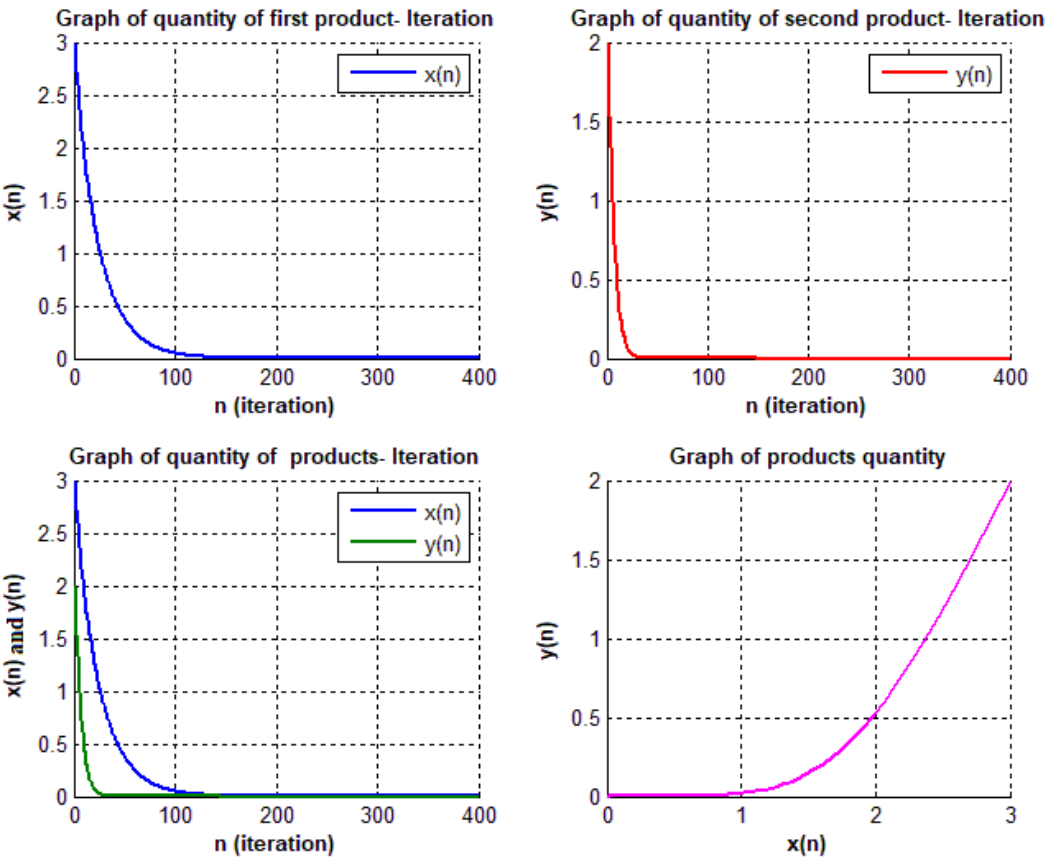

Fig. 1. For first column values in Table 2., time-dependent changes in quantities of the first and second products

From second column values in Table 2, the nontrivial equilibrium point is found as $E_1(6,10)$. Inequalities in Eq. (15) is not satisfied. Therefore, E_0 is unstable point. The existence of periodic orbits can be seen in Fig. 2.

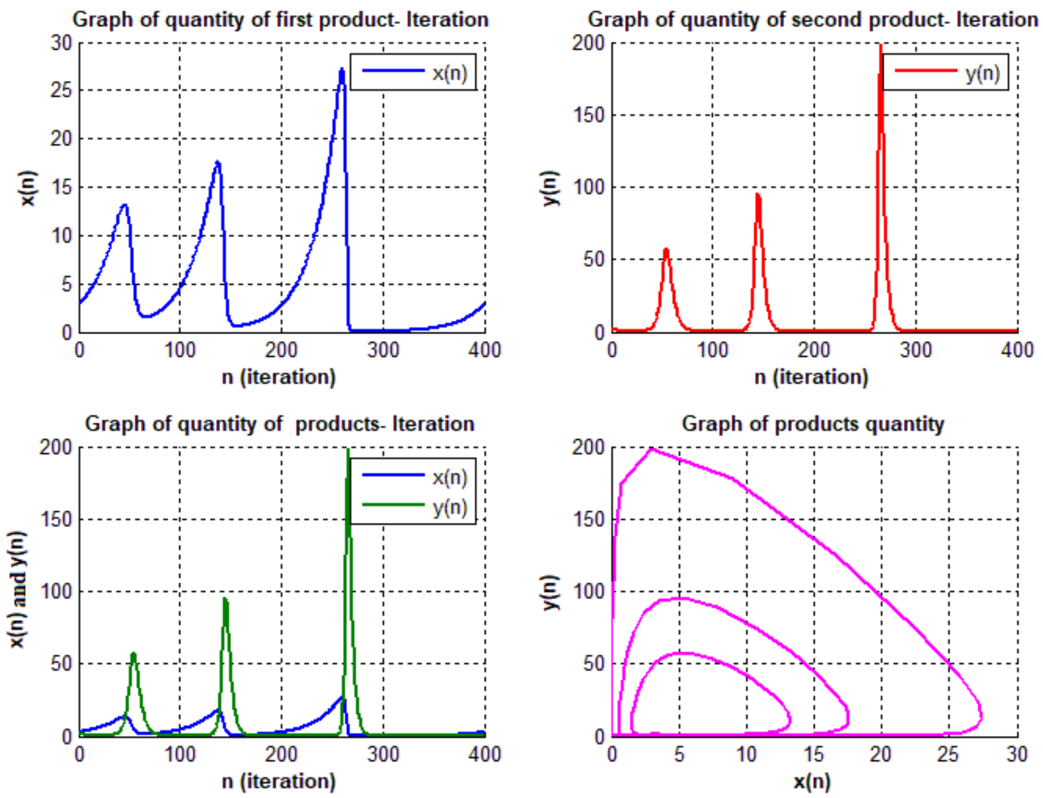

Fig.2. For second column values in Table 2, time-dependent changes in quantities of the first and second products

4. Result and Discussion

Palomba was the first to use the Lotka-Volterra equations. According to the results of the analysis of the model where he proposed through the ordinary differential equations, he explained that the volumes of goods will be in a cyclical state [4].

Daşbaşı and Boztosun [5] studied the Palomba model using incommensurate fractional-order differential equation system in Caputo meaning. They showed the order of derivatives in the equations expressing the first and second type goods with α_1 and α_2, respectively. Also, they expressed the existence and stability of the fixed points of the system. In the case of $0 < \alpha_1, \alpha_2 < 1$, the volumes of these products approach positive values depending on time. Contrary to the cyclical situation, stability analysis of the positive equilibrium point under certain conditions was made and shown with graphs.

In here, the Palomba economic model was modeled by difference equation systems and then analyzed. In addition, the followings were taken into account in the model: the non-marketable loss rate of the first and second type goods are parameters μ_1 and μ_2, respectively. Trivial and positive (Non-trivial) equilibrium points were found in the analysis. The stability of the trivial equilibrium point under certain conditions has been expressed. Although the positive equilibrium point exists under certain conditions, it has been shown to be an unstable point. Therefore, it can be said that there is a cyclical situation in harmony with Palomba.

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