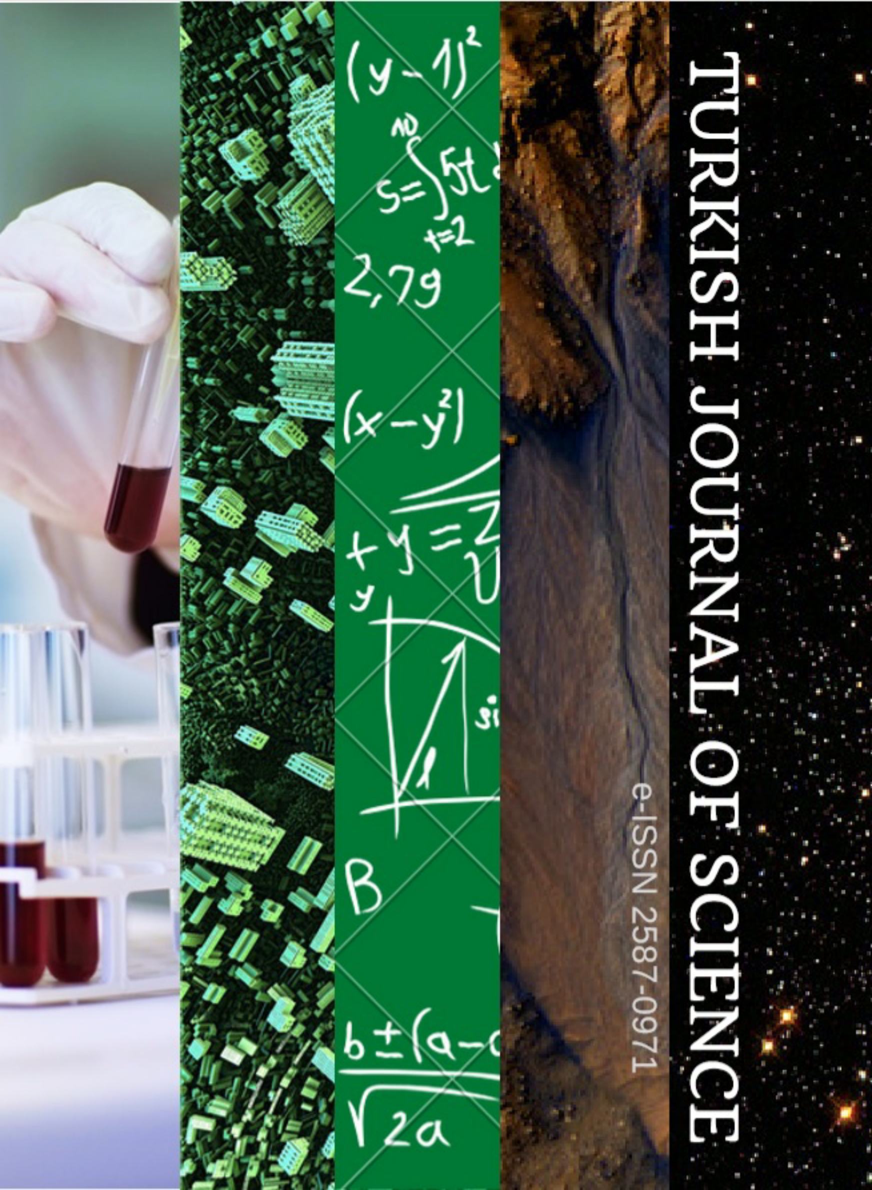
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Fourth Derivative Block Method for Solving Two-point Singular Boundary Value Problems and Related Stiff Problems

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Abstract. This paper contains the formulation of an algorithm for solving two-point singular nonlinear boundary value problems of ordinary differential equations. This method is basically a fourth derivative block method obtained from the collocation and interpolation of an assumed derivatives and functional of a basis function. Its implementation was on the evaluation of derivatives of the given smooth first derivative function $u'(t)$ up to the fourth derivative, at some points t. It is proved that the algorithm is consistent, zero-stable and convergent. Errors for uniform step lengths are also investigated and presented. Numerical examples are provided to show the efficiency of the algorithm.

1. Introduction

Considering the following singular non-linear two-point boundary value problem

$$a(t)u''(t) + b(t)u'(t) = f(t, u, u'), \quad t \in [0, 1], \; u'(0) = 0, \; u(1) = u_b \tag{1}$$

with assumption that

$$a(0) = 0, \; a(t) > 0, \; t \in (0, 1), b(0) \neq 0, f(0, u(0), u'(0)) = 0 \tag{2}$$

with coefficients $a(t)$ and $b(t)$ are differentiable functions on $[0, 1]$ and $f(t, u, u')$ is assumed continuous on $\omega := [0, 1] \times \Re^2$. It could be observed that the problem is singular at the initial point $t = 0$. If $a(t)$ and $b(t)$ satisfy

$$a(1) = 0, b(1) \neq 0 \text{and} f(1, u(1), u'(1)) = 0, \tag{3}$$

it is also singular at point $t = 1$.

Problems of the form (1) satisfying conditions (2) to (3) posses property which make the solutions difficult to obtain or the numerical solutions are poor, and as such special techniques are required for their effective solution. Problems (1) with condition (2) are one-point singular in nature while (1) with conditions (2) and (3) are two-point singular problems. These singular two-point problems happen much of the time in

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numerous models, for example, electro-hydrodynamics and some warm blasts, and in the recent times, have been researched by utilizing some numerical techniques by Baxley ([3] and [4]), Qu and Agarwal [5], Chawla and Subramanian ([6] and Chawla, Subramanian and Sathi [7]). The numerical approach explored in these literature include cubic and quintic spline methods, and the collocation methods. An even-order two-point boundary value problem solutions were obtained by Liu[8]. A continuous generic algorithm was used by Arqub, Abo-Hammour, Momani and Shawegfeh [9] for solving the form of problem in (1). A subdivision collocation method for solving two point boundary value problems of order three was considered by Mustafa and Ejaz [10]. Parametric difference method was used by Pandey [11] in the solution of two-point boundary value problem. An alternative approach was considered by Ghomanjani and Shateyi [12] using the Bezoer curve method with an orthogonal based Bernstein polynomials constructed by the Gram-Schmidt technique. Solutions of one-point singular Lane-endem equations and related stiff problems were effectively solved using some new numerical techniques by Ogunniran, Haruna & Adeniyi [13] and also Ogunniran [14] obtained a class of multi-derivative method for solution of some singular Advection equations of partial differential equations. An extensive linear analysis was carried out on some Runge-kutta methods on their possibilities in the solution of one point singular Lane-Endem equations by Ogunniran, Tayo, Haruna and Adebisi [15]. Extensive analysis for the possibility of existence and uniqueness of solution for a two-point boundary value problems for ordinary differential equations was carried out by Eloe and Henderson [16] in their paper titled; two-point boundary value problems for ordinary differential equations, uniqueness implies existence. However, two-point boundary value problems may exist in problems of order greater than two as found in Agarwal and Kelevedjiev [17]. This paper presents a unique approach on the solution of fourth-order two-point boundary value problems.

2. Method

Recently, lots of attention has been on obtaining more effective and proficient methods for solving stiff problems and subsequently a wide class of methods have been proposed. A possibly decent numerical method for solving stiff systems of ordinary differential equations need to have good accuracy and some reasonably wide region of absolute stability (Dahlquist, 1963). According to Hairer and Wanner (1996), the search for high order A-stable multi-step methods is carried out in two main ways: using high derivatives of solutions and including some additional stages, such as off-step points or super-future points. And this transforms into the many field of general multi-step methods.
Throughout the formulation of this method, except where stated otherwise, the transformation

$$T = \frac{2(t - t_n)}{kh} - 1, \tag{4}$$

where $k = 3$ is the step number and h is the step length, a small distance taken that does not entirely leave the interval.
For purpose of obtaining an approximation for (1), we assume a continuous approximation for $u_n(t)$ of a three step fourth derivative method of the form:

$$u(x) \approx \alpha(t)u_n + \sum_{i=1}^{4} h^i \beta_i(t) f_n^{(i-1)} + \sum_{i=1}^{4}\sum_{j=1}^{3} h^i \gamma_{ij}(t) f_{(n+j)}^{(i-1)} \tag{5}$$

for $u' = f(t, u)$ where $f(t, u)$ is continuous and differentiable, u_n is an approximation to $u(t_n)$, $t_n = nh; h > 0$ and $f_m^{(j)} = f^{(j)}(t_m, u_m)$ such that:

$$f^{(0)}(t_m, u_m) \tag{6}$$

$$f^{(j)}(t_m, u_m) = \frac{\partial f^{(j-1)}(t, u)}{\partial t} + f(t, u)\frac{\partial f^{(j-1)}(t, u)}{\partial u} \tag{7}$$

To this end, approximation of the exact solution $u(t)$ was sought by evaluating the function:

$$u(t) = \sum_{j=0}^{16} a_j t^j \tag{8}$$

where $a_j,\ j = 0(1)16$ are coefficients determined, t^j are the basis functions of degree 16.
While ensuring that (5) corresponds with the analytical solution at the end point t_n, the following conditions were imposed on $u(x)$ and its derivatives; $u^{(k)}(t),\ \ k = 1(1)4$

$$\begin{gathered} u(t_{n+j}) = u_{n+j},\ j = 0 \\ u'(t_{n+j}) = f_{n+j},\ j = 0,1,2,3. \\ u''(t_{n+j}) = g_{n+j},\ j = 0,1,2,3. \\ u'''(t_{n+j}) = h_{n+j},\ j = 0,1,2,3. \\ u^{(iv)}(t_{n+j}) = i_{n+j},\ j = 0,1,2,3. \end{gathered} \tag{9}$$

while the conditions of (9) are imposed on (8), the following system equations were obtained;

$$\begin{gathered}
a_0 = y_n \\
a_1 = f_n \\
a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + 8a_8 \\
+9a_9 + 10a_{10} + 11a_{11} + 12a_{12} + 13a_{13} + 14a_{14} + 15a_{15} + 16a_{16} = f_{n+1} \\
a_1 + 4a_2 + 12a_3 + 32a_4 + 80a_5 + 192a_6 + 448a_7 + 1024a_8 \\
+2304a_9 + 5120a_{10} + 11264a_{11} + 24576a_{12} + 53248a_{13} \\
+114688a_{14} + 245760a_{15} + 524288a_{16} = f_{n+2} \\
a_1 + 6a_2 + 27a_3 + 108a_4 + 405a_5 + 1458a_6 + 5103a_7 + 17496a_8 \\
+59049a_9 + 196830a_{10} + 649539a_{11} + 2125764a_{12} + 6908733a_{13} \\
+22320522a_{14} + 71744535a_{15} + 229582512a_{16} = f_{n+3} \\
2a_2 = g_n \\
2a_2 + 6a_3 + 12a_4 + 20a_5 + 30a_6 + 42a_7 + 56a_8 + 72a_9 \\
+90a_{10} + 110a_{11} + 132a_{12} + 156a_{13} + 182a_{14} + 210a_{15} + 240a_{16} = g_{n+1} \\
2a_2 + 12a_3 + 48a_4 + 160a_5 + 480a_6 + 1344a_7 + 3584a_8 \\
+9216a_9 + 23040a_{10} + 56320a_{11} + 135168a_{12} + 319488a_{13} + 745472a_{14} \\
+1720320a_{15} + 3932160a_{16} = g_{n+2} \\
2a_2 + 18a_3 + 108a_4 + 540a_5 + 2430a_6 + 10206a_7 + 40824a_8 \\
+157464a_9 + 590490a_{10} + 2165130a_{11} + 7794468a_{12} + 27634932a_{13} \\
+96722262a_{14} + 334807830a_{15} + 1147912560a_{16} = g_{n+3} \\
6a_3 = h_n \\
6a_3 + 24a_4 + 60a_5 + 120a_6 + 210a_7 + 336a_8 + 504a_9 \\
+720a_{10} + 990a_{11} + 1320a_{12} + 1716a_{13} + 2184a_{14} \\
+2730a_{15} + 3360a_{16} = h_{n+1} \\
6a_3 + 48a_4 + 240a_5 + 960a_6 + 3360a_7 + 10752a_8 + 32256a_9 \\
+92160a_{10} + 253440a_{11} + 675840a_{12} + 1757184a_{13} + 4472832a_{14} \\
+11182080a_{15} + 27525120a_{16} = h_{n+2} \\
6a_3 + 72a_4 + 540a_5 + 3240a_6 + 17010a_7 + 81648a_8 + 367416a_9 \\
+1574640a_{10} + 6495390a_{11} + 25981560a_{12} + 101328084a_{13} \\
+386889048a_{14} + 1450833930a_{15} + 5356925280a_{16} = h_{n+3} \\
24a_4 = i_n \\
24a_4 + 120a_5 + 360a_6 + 840a_7 + 1680a_8 + 3024a_9 \\
+5040a_{10} + 7920a_{11} + 11880a_{12} + 17160a_{13} + 24024a_{14} \\
+32760a_{15} + 43680a_{16} = i_{n+1} \\
24a_4 + 240a_5 + 1440a_6 + 6720a_7 + 26880a_8 + 96768a_9 \\
+322560a_{10} + 1013760a_{11} + 3041280a_{12} + 8785920a_{13} + 24600576a_{14} \\
+67092480a_{15} + 178913280a_{16} = i_{n+2}
\end{gathered}$$

$$
\begin{gathered}
24\,a_4 + 360\,a_5 + 3240\,a_6 + 22680\,a_7 + 136080\,a_8 + 734832\,a_9 \\
+3674160\,a_{10} + 17321040\,a_{11} + 77944680\,a_{12} + 337760280\,a_{13} + 1418593176\,a_{14} \\
+5803335720\,a_{15} + 23213342880\,a_{16} = i_{n+3}
\end{gathered} \tag{10}
$$

Solving (10), $a_j,\ \ j = 0(1)16$ were obtained, the values were substituted in (8) and related term were collected in $u_n, f_n, f_{n+1}, f_{n+2}, f_{n+3}, g_n, g_{n+1}, g_{n+2}, g_{n+3}, h_n, h_{n+1}, h_{n+2}, h_{n+3}, i_n, i_{n+1}, i_{n+2}, i_{n+3}$ to obtain:

$$
\begin{gathered}
u(t) = \alpha_n u_n + h\beta_1(t) f_n + h^2\beta_2(t) g_n + h^3\beta_3(t) h_n + h^4\beta_4(t) i_n + h\left[\gamma_{11}(t) f_{n+1} + \gamma_{12}(t) f_{n+2} + \gamma_{13}(t) f_{n+3}\right] \\
+h^2\left[\gamma_{21}(t) g_{n+1} + \gamma_{22}(t) g_{n+2} + \gamma_{23}(t) g_{n+3}\right] + h^3\left[\gamma_{31}(t) h_{n+1} + \gamma_{32}(t) h_{n+2} + \gamma_{33}(t) h_{n+3}\right] \\
+h^4\left[\gamma_{41}(t) i_{n+1} + \gamma_{42}(t) i_{n+2} + \gamma_{43}(t) i_{n+3}\right]
\end{gathered} \tag{11}
$$

where

$$
\begin{gathered}
\alpha_n(t) = 1 \\
\beta_1(t) = t - \frac{54391\,t^5}{1296} + \frac{713717\,t^6}{3888} - \frac{62588555\,t^7}{163296} + \frac{276696055\,t^8}{559872} - \frac{1259675\,t^9}{2916} + \frac{9332263\,t^{10}}{34992} - \frac{5049247\,t^{11}}{42768} \\
+\frac{5227135\,t^{12}}{139968} - \frac{1251365\,t^{13}}{151632} + \frac{18545\,t^{14}}{15309} - \frac{2477\,t^{15}}{23328} + \frac{4711\,t^{16}}{1119744} \\
\beta_2(t) = -81\,t^5 + \frac{2007\,t^6}{4} - \frac{19035\,t^7}{14} + \frac{68955\,t^8}{32} - \frac{320945\,t^9}{144} + \frac{101285\,t^{10}}{64} \\
-\frac{138705\,t^{11}}{176} + \frac{35265\,t^{12}}{128} - \frac{13795\,t^{13}}{208} + \frac{4695\,t^{14}}{448} - \frac{47\,t^{15}}{48} + \frac{21\,t^{16}}{512} \\
\beta_3(t) = \frac{1863\,t^5}{16} - \frac{10359\,t^6}{16} + \frac{367875\,t^7}{224} - \frac{637125\,t^8}{256} + \frac{89695\,t^9}{36} - \frac{27605\,t^{10}}{16} + \frac{148197\,t^{11}}{176} \\
-\frac{18525\,t^{12}}{64} + \frac{14285\,t^{13}}{208} - \frac{75\,t^{14}}{7} + \frac{95\,t^{15}}{96} - \frac{21\,t^{16}}{512} \\
\beta_4(t) = \frac{529\,t^5}{81} - \frac{36821\,t^6}{972} + \frac{1026995\,t^7}{10206} - \frac{11217545\,t^8}{69984} + \frac{1974065\,t^9}{11664} - \frac{17350807\,t^{10}}{139968} \\
+\frac{2742691\,t^{11}}{42768} - \frac{6550475\,t^{12}}{279936} + \frac{894155\,t^{13}}{151632} - \frac{957245\,t^{14}}{979776} + \frac{1117\,t^{15}}{11664} - \frac{4711\,t^{16}}{1119744} \\
\gamma_{11}(t) = 1/2\,t^2 - \frac{4711\,t^5}{270} + \frac{61909\,t^6}{864} - \frac{72497\,t^7}{504} + \frac{33704285\,t^8}{186624} - \frac{338717\,t^9}{2187} + \frac{549763\,t^{10}}{5832} - \\
\frac{147109\,t^{11}}{3564} + \frac{603953\,t^{12}}{46656} - \frac{5983\,t^{13}}{2106} + \frac{67715\,t^{14}}{163296} - \frac{211\,t^{15}}{5832} + \frac{533\,t^{16}}{373248} \\
\gamma_{12}(t) = -\frac{891\,t^5}{10} + \frac{1881\,t^6}{4} - \frac{32103\,t^7}{28} + \frac{13485\,t^8}{8} - \frac{474731\,t^9}{288} + \frac{71729\,t^{10}}{64} - \frac{189809\,t^{11}}{352} \\
+\frac{23465\,t^{12}}{128} - \frac{17941\,t^{13}}{416} + \frac{2995\,t^{14}}{448} - \frac{59\,t^{15}}{96} + \frac{13\,t^{16}}{512} \\
\gamma_{13}(t) = -\frac{2511\,t^5}{40} + \frac{11277\,t^6}{32} - \frac{101169\,t^7}{112} + \frac{354405\,t^8}{256} - \frac{25249\,t^9}{18} + \frac{3935\,t^{10}}{4} - \frac{42811\,t^{11}}{88} \\
+\frac{10849\,t^{12}}{64} - \frac{4241\,t^{13}}{104} + \frac{1445\,t^{14}}{224} - \frac{29\,t^{15}}{48} + \frac{13\,t^{16}}{512} \\
\gamma_{21}(t) = -\frac{193\,x^5}{90} + \frac{4037\,x^6}{324} - \frac{32233\,x^7}{972} + \frac{617465\,x^8}{11664} - \frac{3921371\,x^9}{69984} + \frac{1920017\,x^{10}}{46656} \\
-\frac{202945\,x^{11}}{9504} + \frac{729457\,x^{12}}{93312} - \frac{199861\,x^{13}}{101088} + \frac{107395\,x^{14}}{326592} - \frac{755\,x^{15}}{23328} + \frac{533\,x^{16}}{373248} \\
\gamma_{22}(t) = 1/6\,t^3 - \frac{533\,t^5}{180} + \frac{7015\,t^6}{648} - \frac{123413\,t^7}{6048} + \frac{766211\,t^8}{31104} - \frac{119969\,t^9}{5832} + \frac{119401\,t^{10}}{9720} - \\
\frac{25199\,t^{11}}{4752} + \frac{12791\,t^{12}}{7776} - \frac{335\,t^{13}}{936} + \frac{353\,t^{14}}{6804} - \frac{35\,t^{15}}{7776} + \frac{11\,t^{16}}{62208} \\
\gamma_{23}(t) = -\frac{81\,t^5}{10} + \frac{99\,t^6}{2} - \frac{3753\,t^7}{28} + \frac{6801\,t^8}{32} - \frac{63241\,t^9}{288} + \frac{6219\,t^{10}}{40} - \frac{27139\,t^{11}}{352} \\
+\frac{5149\,t^{12}}{192} - \frac{2671\,t^{13}}{416} + \frac{113\,t^{14}}{112} - \frac{3\,t^{15}}{32} + \frac{t^{16}}{256} \\
\gamma_{31}(t) = \frac{81\,t^5}{8} - \frac{909\,t^6}{16} + \frac{32589\,t^7}{224} - \frac{28497\,t^8}{128} + \frac{16205\,t^9}{72} - \frac{3147\,t^{10}}{20} + \frac{13639\,t^{11}}{176} \\
-\frac{2579\,t^{12}}{96} + \frac{167\,t^{13}}{26} - \frac{113\,t^{14}}{112} + \frac{3\,t^{15}}{32} - \frac{t^{16}}{256} \\
\gamma_{32}(t) = \frac{23\,t^5}{90} - \frac{241\,t^6}{162} + \frac{8999\,t^7}{2268} - \frac{49373\,t^8}{7776} + \frac{157207\,t^9}{23328} - \frac{6032\,t^{10}}{1215} + \frac{8189\,t^{11}}{3168} \\
-\frac{14773\,t^{12}}{15552} + \frac{8129\,t^{13}}{33696} - \frac{1097\,t^{14}}{27216} + \frac{31\,t^{15}}{7776} - \frac{11\,t^{16}}{62208} \\
\gamma_{33}(t) = 1/24\,t^4 - \frac{11\,t^5}{45} + \frac{145\,t^6}{216} - \frac{365\,t^7}{324} + \frac{79441\,t^8}{62208} - \frac{1495\,t^9}{1458} + \frac{11591\,t^{10}}{19440} - \frac{25\,t^{11}}{99} \\
+\frac{1201\,t^{12}}{15552} - \frac{35\,t^{13}}{2106} + \frac{65\,t^{14}}{27216} - \frac{t^{15}}{4860} + \frac{t^{16}}{124416} \\
\gamma_{41}(t) = -\frac{27\,t^5}{10} + \frac{57\,t^6}{4} - \frac{963\,t^7}{28} + \frac{199\,t^8}{4} - \frac{41233\,t^9}{864} + \frac{30527\,t^{10}}{960} - \frac{5279\,t^{11}}{352} \\
+\frac{1921\,t^{12}}{384} - \frac{37\,t^{13}}{32} + \frac{79\,t^{14}}{448} - \frac{23\,t^{15}}{1440} + \frac{t^{16}}{1536} \\
\gamma_{42}(t) = -\frac{27\,t^5}{20} + \frac{123\,t^6}{16} - \frac{2241\,t^7}{112} + \frac{7979\,t^8}{256} - \frac{3469\,t^9}{108} + \frac{5501\,t^{10}}{240} - \frac{1015\,t^{11}}{88} \\
+\frac{785\,t^{12}}{192} - t^{13} + \frac{9\,t^{14}}{56} - \frac{11\,t^{15}}{720} + \frac{t^{16}}{1536} \\
\gamma_{43}(t) = -\frac{t^5}{90} + \frac{7\,t^6}{108} - \frac{131\,t^7}{756} + \frac{1081\,t^8}{3888} - \frac{6905\,t^9}{23328} + \frac{3403\,t^{10}}{15552} - \frac{1087\,t^{11}}{9504} \\
+\frac{1313\,t^{12}}{31104} - \frac{121\,t^{13}}{11232} + \frac{197\,t^{14}}{108864} - \frac{7\,t^{15}}{38880} + \frac{t^{16}}{124416}
\end{gathered} \tag{12}
$$

Evaluating (12) at t_{n+1}, t_{n+2} and t_{n+3} yield the desired discrete block method below:

$$
\begin{array}{c}
u_{n+1} = -\frac{34637\, i_{n+3}}{1868106240} h^4 - \frac{2617\, i_{n+2}}{1064448} h^4 - \frac{37603\, i_{n+1}}{5322240} h^4 + \frac{50857\, i_n}{373621248} h^4 \\
+ \frac{401183\, h_{n+3}}{934053120} h^3 + \frac{224473\, h_{n+2}}{11531520} h^3 + \frac{965\, h_{n+1}}{2306304} h^3 + \frac{4135199\, h_n}{934053120} h^3 \\
- \frac{20331329\, g_{n+3}}{5604318720} h^2 - \frac{398083\, g_{n+2}}{3294720} h^2 - \frac{7097579\, g_{n+1}}{23063040} h^2 + \frac{331133249\, g_n}{5604318720} h^2 \\
+ h \frac{12457877\, f_{n+3}}{1120863744} + h \frac{1071529\, f_{n+2}}{4612608} + h \frac{1730473\, f_{n+1}}{4612608} + h \frac{427519381\, f_n}{1120863744} + u_n \\
u_{n+2} = -h^4 \frac{16\, i_{n+3}}{729729} - h^4 \frac{i_{n+2}}{297} - h^4 \frac{64\, i_{n+1}}{10395} + h^4 \frac{509\, i_n}{3648645} \\
+ h^3 \frac{1864\, h_{n+3}}{3648645} + h^3 \frac{206\, h_{n+2}}{6435} + h^3 \frac{584\, h_{n+1}}{45045} + h^3 \frac{470\, h_n}{104247} \\
- h^2 \frac{47504\, g_{n+3}}{10945935} - h^2 \frac{10441\, g_{n+2}}{45045} - h^2 \frac{8864\, g_{n+1}}{45045} + h^2 \frac{654539\, g_n}{10945935} \\
+ h \frac{29272\, f_{n+3}}{2189187} + h \frac{6577\, f_{n+2}}{9009} + h \frac{7864\, f_{n+1}}{9009} + h \frac{839939\, f_n}{2189187} + u_n \\
u_{n+3} = -h^4 \frac{81\, i_{n+3}}{512512} + h^4 \frac{729\, i_{n+2}}{197120} - h^4 \frac{729\, i_{n+1}}{197120} + h^4 \frac{81\, i_n}{512512} \\
+ h^3 \frac{6327\, h_{n+3}}{1281280} + h^3 \frac{41553\, h_{n+2}}{1281280} + h^3 \frac{41553\, h_{n+1}}{1281280} + h^3 \frac{6327\, h_n}{1281280} \\
- h^2 \frac{162531\, g_{n+3}}{2562560} + h^2 \frac{194643\, g_{n+2}}{2562560} - h^2 \frac{194643\, g_{n+1}}{2562560} + h^2 \frac{162531\, g_n}{2562560} \\
+ h \frac{28905\, f_{n+3}}{73216} + h \frac{80919\, f_{n+2}}{73216} + h \frac{80919\, f_{n+1}}{73216} + h \frac{28905\, f_n}{73216} + u_n
\end{array}
\tag{13}
$$

2.1. *Order and Error Constant*

Applying Taylor's series expansion on (5) and collecting like terms, we have the difference equation

$$
l[u(t); h] = c_0 u(t) + c_1 h y^{(1)}(t) + c_2 h y^{(2)}(t) + \cdots + c_q h y^{(q)}(t) + \cdots \tag{14}
$$

where

$$
\left.
\begin{array}{c}
c_0 = 1 - \alpha_n \\
c_1 = 3 - \beta_i - \sum_{j=1}^{3} \gamma_{j3} \\
\vdots \\
c_q = \frac{3^q}{q!} - \frac{3^{q-1}}{(q-1)!} \sum_{j=1}^{3} \gamma_{j3}
\end{array}
\right\} \tag{15}
$$

According to Henrici (1962), a method has order p if

$$
l[u(t); h] = o(h^{p+1}) \tag{16}
$$

where

$$
c_0 = c_1 = \cdots = c_p = 0 \text{ but } c_{p+1} \neq 0. \tag{17}
$$

Using this principle, the order and error constant of (13) are shown below

Evaluating Point	Order	Error Constant
t_{n+1}	16	$\frac{183}{4883993354240000}$
t_{n+2}	16	$\frac{1231}{62585703900720000}$
t_{n+3}	16	$\frac{570391}{32043880397168640000}$

The method (13) is consistent since order of the method is 16 which is greater than 1.

2.2. *Zero-stability*

This relates to a phenomenon where the step size $h \to 0$. Taking limit of (13) as $h \to 0$, we have:

$$
u_{n+1} = u_{n+2} = u_{n+3} = u_n \tag{18}
$$

which can be written in matrix form as

$$
IU_i - B_0 U_{i-1} = 0 \tag{19}
$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_i = \begin{pmatrix} u_{n+1} \\ u_{n+2} \\ u_{n+3} \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{i-1} = \begin{pmatrix} u_i \\ u_i \\ u_i \end{pmatrix}$$

According to Lambert (1975), a block method is zero-stable if the roots r_k of the first characteristic polynomial $\xi(r) = det|Ir - B_0|$ does not exceed 1 i.e. $|r_k| \leq 1$. The first characteristic polynomial of method (13) is given by

$$r^2(r-1) = 0 \tag{20}$$

The roots of (20) are $r = 0, 0, 1$ in which all is < 1, thus Method (13) is zero-stable.

2.3. *Convergence*

According to Henrici (1962), we can establish the convergence of the block method since consistency and zero-stability are necessary and sufficient reasons for convergence.

2.4. *Linear Stability*

Practically, the robustness of a method is reliably found with $h > 0$, this implies that the convergence of a method is a necessary but not a sufficient condition for a method to be useful. Linear stability is a conceptional behaviour of numerical methods concerned with the behaviour of the method when $h > 0$ and its region of absolute stability. This is a concept different from zero-stability. The linear stability properties of the derived method is determined by expressing it in a form applicable to the test problem:

$$u' = \lambda u, \ \text{ for which } \ u_n' = \lambda u_n, \ u_n'' = \lambda^2 u_n, \cdots, u_n^{(n)} = \lambda^n u_n., \ \ \lambda < 0 \tag{21}$$

to yield:

$$U_{\mu+1} = M(z) U_\mu, \ \ z = h\lambda \tag{22}$$

where the amplification matrix $M(z)$ is given by:

$$M(z) = (A^{(0)} - zB^{(0)} - z^2C^{(0)} - z^3D^{(0)} - z^4E^{(0)})^{-1}(A^{(1)} + zB^{(1)} + z^2C^{(1)} + z^3D^{(1)} + z^4E^{(1)}). \tag{23}$$

where $z = h\lambda$.

$$A^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B^{(0)} = \begin{pmatrix} \frac{1730473}{4612608} & \frac{1071529}{4612608} & \frac{12457877}{1120863744} \\ \frac{7864}{9009} & \frac{6577}{9009} & \frac{29272}{2189187} \\ \frac{80919}{73216} & \frac{80919}{73216} & \frac{28905}{73216} \end{pmatrix} \quad B^{(1)} = \begin{pmatrix} 0 & 0 & \frac{427519381}{1120863744} \\ 0 & 0 & \frac{839939}{2189187} \\ 0 & 0 & \frac{28905}{73216} \end{pmatrix}$$

$$C^{(0)} = \begin{pmatrix} -\frac{7097579}{23063040} & -\frac{398083}{3294720} & -\frac{20331329}{5604318720} \\ -\frac{8864}{45045} & -\frac{10441}{45045} & -\frac{47504}{10945935} \\ -\frac{194643}{2562560} & \frac{194643}{2562560} & -\frac{162531}{2562560} \end{pmatrix} \quad C^{(1)} = \begin{pmatrix} 0 & 0 & \frac{331133249}{5604318720} \\ 0 & 0 & \frac{654539}{10945935} \\ 0 & 0 & \frac{162531}{2562560} \end{pmatrix}$$

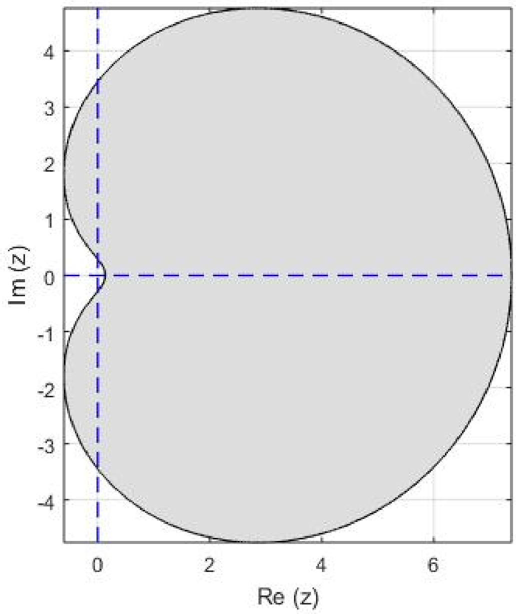

Figure 1: Region of Absolute Stability of Method (13)

$$D^{(0)} = \begin{pmatrix} \frac{965}{2306304} & \frac{224473}{11531520} & \frac{401183}{934053120} \\ \frac{584}{45045} & \frac{206}{6435} & \frac{1864}{3648645} \\ \frac{41553}{1281280} & \frac{41553}{1281280} & \frac{6327}{1281280} \end{pmatrix} \quad D^{(1)} = \begin{pmatrix} 0 & 0 & \frac{4135199}{934053120} \\ 0 & 0 & \frac{470}{104247} \\ 0 & 0 & \frac{6327}{1281280} \end{pmatrix}$$

$$E^{(0)} = \begin{pmatrix} -\frac{37603}{5322240} & -\frac{2617}{1064448} & -\frac{34637}{1868106240} \\ -\frac{64}{10395} & -\frac{1}{297} & -\frac{16}{729729} \\ -\frac{729}{197120} & \frac{729}{197120} & -\frac{81}{512512} \end{pmatrix} \quad E^{(1)} = \begin{pmatrix} 0 & 0 & \frac{50857}{373621248} \\ 0 & 0 & \frac{509}{3648645} \\ 0 & 0 & \frac{81}{512512} \end{pmatrix}$$

The matrix $M(z)$ has eigenvalues $\xi_1, \xi_2, \cdots, \xi_m = 0, 0, \cdots, \xi_m$, where the dominant eigenvalue ξ_m is the stability function $R(z)$ which is a rational function with real coefficients, m is the order of $R(z)$,

$$R(z) = \frac{\begin{array}{c} 28561\,z^{12} + 1164410\,z^{11} + 25844325\,z^{10} + 401535225\,z^{9} + 4765597305\,z^{8} + 44819838000\,z^{7} \\ +338397658200\,z^{6} + 2046767184000\,z^{5} + 9765253436400\,z^{4} + 35606883312000\,z^{3} \\ +93666717144000\,z^{2} + 158855192496000\,z + 130821923232000 \end{array}}{\begin{array}{c} z^{12} - 170\,z^{11} + 14325\,z^{10} - 754425\,z^{9} + 26611305\,z^{8} - 648043200\,z^{7} \\ +11234575800\,z^{6} - 141313788000\,z^{5} + 1292613260400\,z^{4} - 8445396960000\,z^{3} \\ +37600178616000\,z^{2} - 102788653968000\,z + 130821923232000 \end{array}} \tag{24}$$

The stability function and plot for the method is as given below:

3. Numerical Examples

This section contains some two-point singular boundary value problems, their conditions and true solutions as found in literature.

Test Problem 1 [5]

$$\left.\begin{array}{c} u''(t) + \frac{2}{t}u'(t) = \beta t^{\beta-2}(1 + \beta + \beta t^{\beta}) \\ t \in (0, 1), u'(0) = 0, u(1) = e \\ u(t) = e^{t^{\beta}} \end{array}\right\} \tag{25}$$

Test Problem 2 [5]

$$\left.\begin{array}{c} u''(t) + \frac{2}{t}u'(t) = 3cos(t) - tsin(t) \\ t \in (0, 1), u'(0) = 0, u(1) = cos1 + sin1 \\ u(t) = cost + tsint. \end{array}\right\} \tag{26}$$

Test Problem 3 [5]

$$\left.\begin{array}{c} u''(t) + \frac{2}{t}u'(t) = -2(e^{u} + e^{\frac{u}{2}}) \\ t \in (0, 1), u'(0) = 0, u(1) = 0 \\ u(t) = 2log\dfrac{2}{1 + t^2} \end{array}\right\} \tag{27}$$

Test Problem 4 [11]

The boundary value problem below arose from the analysis of the confinement of a plasma column by radiation pressure with different boundary conditions,

$$\left.\begin{array}{c} u''(t) = \lambda sinh(\lambda u(t)), \;\; 0 < t < 1 \\ \text{subject to boundary conditions} \\ u'(0) = 1, \text{ and } u(1) = 0. \\ u(t) = sinh(t) \end{array}\right\} \tag{28}$$

4. Discussion of Results and Conclusion

The following formula

$$\lim_{maxt} |u(t) - u_i(t)| \tag{29}$$

where $u(t)$ is the exact solution and $u_i(t)$ is the numerical solution evaluated at some $t \in [0, 1]$, was used in the computation of maximum errors. Numerical methods were programmed on Windows 10 operating system in MATLAB 9.2 environment on 8.00GB RAM HP Pavilion x360 Convertible, 64-bits Operating System, x64-based processor Intel(R) Core(TM) i3-7100U CPU @ 2.40GHz.

The following table display the comparison of performances for new method against existing methods with Computational time.

Table 1: Table of Comparison of Maximum Errors with Existing Methods Using Different h, $\beta = 1$ for Test Problem 1

Test Problem	Method	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
1	Qu & Agarwal (1997)	1.720000 E(-2)	2.800000 E(-3)	2.660000 E(-4)
	Method (13)	5.714330 E(-5)	5.714330 E(-5)	5.716299 E(-5)
2	Qu & Agarwal (1997)	1.090000 E(-5)	1.080000 E(-6)	7.890000 E(-8)
	Method (13)	5.403023 E(-9)	5.403023 E(-9)	5.461926 E(-9)
3	Qu & Agarwal (1997)	1.200000 E(-3)	1.070000 E(-4)	8.030000 E(-6)
	Method (13)	2.021798 E(-8)	2.021798 E(-8)	1.200000 E(-8)

Table 2: Table of Comparison of Maximum Errors with Existing Method Using Different h and λ, for Test Problem 4

λ	Method	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$
0.1	Pandey (2018)	1.1055857 E(-1)	1.1045857 E(-1)	1.1032658 E(-1)
	Method (13)	1.387182 E(-4)	1.387225 E(-5)	1.387229 E(-8)
0.15	Pandey (2018)	1.7244926 E(-1)	1.7190818 E(-1)	1.7155264 E(-1)
	Method (13)	1.559298 E(-4)	1.559312 E(-5)	1.559399 E(-8)
0.2	Pandey (2018)	2.374965 E(-1)	2.358833 E(-1)	0.0000 E(0)
	Method (13)	0.000000 E(0)	0.000000 E(0)	0.000000 E(0)

[1)]

Table 3: Table of Computational Time of Method (13) Measured in *seconds*

Test Problem	h	Computation Time
1	$\frac{1}{8}$	0.3438
	$\frac{1}{16}$	0.4063
	$\frac{1}{32}$	0.4844
2	$\frac{1}{8}$	0.2500
	$\frac{1}{16}$	0.3250
	$\frac{1}{32}$	0.3350
3	$\frac{1}{8}$	0.2110
	$\frac{1}{16}$	0.2520
	$\frac{1}{32}$	0.3200
4	$\frac{1}{8}$	0.3250
	$\frac{1}{16}$	0.3255
	$\frac{1}{32}$	0.3525

[1)] Tables 1 and 2 show the numerical computational results on test problems considered. Extensive comparison was done with existing methods in literature. The method was demonstrated on some Examples and the superiority of the derived method over existing methods was established. It is worthy to say that the derived methods exhibit stronger computational strength executed under very low computational times as shown in Table 3. Figures 2-9 show the graphical comparison for solutions of derived method with exact and error distribution curve across the selected interval.

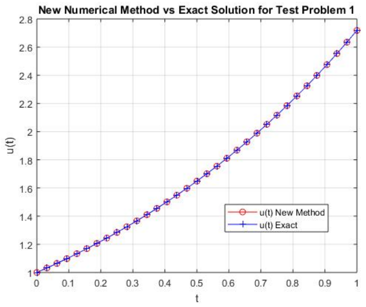

Figure 2: Method (13) vs Exact for Test Problem 1

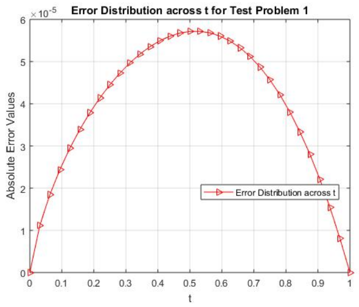

Figure 3: Error Distribution along t of Method (13) for Test Problem 1

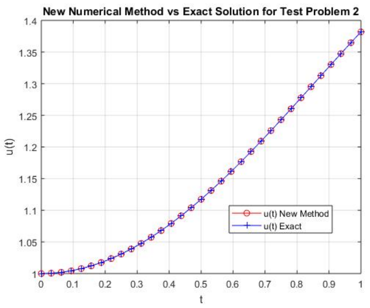

Figure 4: Method (13) vs Exact for Test Problem 2

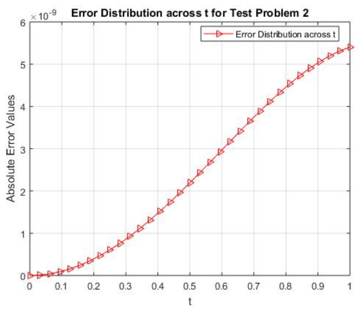

Figure 5: Error Distribution along t of Method (13) for Test Problem 2

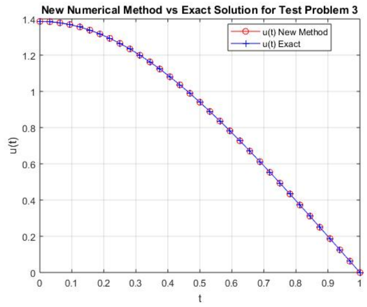

Figure 6: Method (13) vs Exact for Test Problem 3

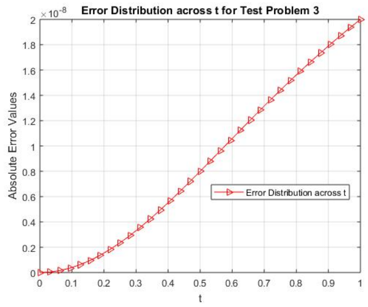

Figure 7: Error Distribution along t of Method (13) for Test Problem 3

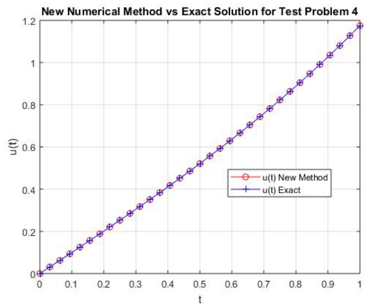

Figure 8: Method (13) vs Exact for Test Problem 4

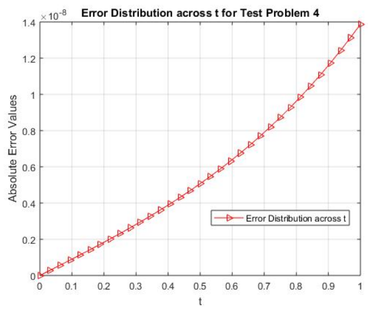

Figure 9: Error Distribution along t of Method (13) for Test Problem 4

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Classification of Ruled Surfaces Family with Common Characteristic Curve in Euclidean 3-space

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Abstract. Classification of ruled surfaces that satisfying certain geometric conditions has been studied by many researchers in the past years. The purpose of this paper is to study and classify the family of ruled surfaces whose common directrix satisfies the requirements of characteristic curves in three-dimensional Euclidean space. The family of ruled surfaces is parameterized by its directrix curve and director vector that is expressed by a linear combination of Frenet frame with angular functions as coefficients. According to the type of characteristic directrix curve, the family of ruled surfaces is classified into three types, and one type when the family is developable.

1. Introduction

A ruled surface is constructed by continuous motion of a straight line called the ruling (or generator) through a given curve which is called the base (or directrix) curve. The ruled surfaces are classical subject in differential geometry and they have many recent applications in different areas of sciences including computer-aided geometric design (CAGD), computer graphic, architectural designing, mechanics, robotics, product design and manufacturing [4, 10].

Developable surfaces are a special class of ruled surfaces that have vanishing Gaussian curvature. The developable surface is locally isometric to a plane, this means that the developable surface can be developed (flattened) onto a plane without stretching and tearing. In manufacturing, a developable surface can be produced from paper or sheet metal with no distortion. Therefore, the developable surfaces are commonly used in industrial design and modeling [3, 12].

Geodesic, asymptotic, and line of curvature are characteristic curves that lie on the surface and have been used in surface analysis. The geodesic curve gives the shortest path between two given points on curved spaces. All straight lines in the plane are geodesics, as are the rulings of any ruled surface [9]. A curve is an asymptotic if its normal curvature is equal to zero. All straight lines on a surface are asymptotic lines. Finally, a line of curvature is a curve whose tangent always points along with a principal directions i.e. a direction in which the normal curvature is maximum or minimum. In shape analysis, the line of curvature is one of the most important characteristic curves indicates the directions in which a shape bends extremely.

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There are several articles for designing the surfaces family that possess the given curve as a characteristic curve. Wang et al. [13] and Li et al. [6] studied the parametric representation of a surface pencil with a common spatial geodesic and line of curvature respectively. Bayram et.al. [1] studied surface pencil with a common asymptotic curve.

In this article, a family of ruled surfaces with common characteristic curve is studied and classified in Euclidean 3- space. A ruled surfaces family is constructed by its directrix curve and director vector that is expressed by a linear combination of Frenet frame with angular functions as coefficients. According to the characteristic directrix curve, three types of ruled surfaces family are classified. The main theorem shows that the common directrix curve is a geodesic or an asymptotic if and only if the ruled surfaces family is a rectifying or an osculating. And the family of generalized ruled surfaces with a special condition has a directrix curve as a line of curvature. Finally, the developability condition of such families is studied.

The paper is organized as follows. In section 2, some basic concepts about space curves and ruled surfaces are given. The main results are studied in section 3, where the family of ruled surfaces based on its common directrix curve are classified into three categories (rectifying, osculating, generalized) whose common directrix curve is (geodesic, asymptotic, line of curvature) respectively. The classification classes are investigated under developability condition in section 4. Finally, the conclusion and future works are given in section 5.

2. Preliminaries

This section introduces some basic facts about the differential geometry of space curves and ruled surfaces in three-dimensional Euclidean space, as well as some basic definitions and notions that are required subsequently. More details can be found in such standard references as [2, 9].

2.1. Curves in Euclidean 3-space

A smooth space curve in 3-dimensional Euclidean space E^3 is parameterized by a map $\gamma : I \subseteq \mathbb{R} \to E^3$, γ is called a regular if $\gamma' \neq 0$ for every point of an interval $I \subseteq \mathbb{R}$, and if $|\gamma'(s)| = 1$ where $|\gamma'(s)| = \sqrt{\langle \gamma'(s), \gamma'(s) \rangle}$, then γ is said to be of unit speed (or parameterized by arc-length s). For a unit speed regular curve $\gamma(s)$ in E^3, the unit tangent vector $t(s)$ of γ at $\gamma(s)$ is given by $t(s) = \gamma'(s)$. If $\gamma''(s) \neq 0$, the unit principal normal vector n(s) of the curve at $\gamma(s)$ is given by $n(s) = \frac{\gamma''(s)}{\|\gamma''\|}$. The unit vector $b(s) = t(s) \times n(s)$ is called the unit binormal vector of γ at $\gamma(s)$. For each point of $\gamma(s)$ where $\gamma''(s) \neq 0$, we associate the Serret-Frenet frame $\{t, n, b\}$ along the curve γ. As the parameter s traces out the curve, the Serret-Frenet frame moves along γ and satisfies the following Frenet-Serret formula.

$$\begin{cases} t'(s) &= \kappa(s)n(s), \\ n'(s) &= -\kappa(s)t(s) + \tau b(s), \\ b'(s) &= -\tau(s)n(s). \end{cases} \tag{1}$$

where $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and torsion functions.The planes spanned by $\{t(s), n(s)\}$, $\{t(s), b(s)\}$ and $\{n(s), b(s)\}$ are respectively called the osculating plane, the rectifying plane and the normal plane. When the point moves along the unit speed curve with non-vanishing curvature and torsion, the Serret-Frenet frame $\{t, n, b\}$ is drawn to the curve at each position of the moving point, this motion consists of translation with rotation and described by the following Darboux vector

$$\omega = \tau t + \kappa b. \tag{2}$$

The direction of Darboux vector is the direction of rotational axis and its magnitude gives the angular velocity of rotation. The unit Darboux vector field is defined by

$$\hat{\omega} = \frac{\tau}{\sqrt{\tau^2 + \kappa^2}} t + \frac{\kappa}{\sqrt{\tau^2 + \kappa^2}} b. \tag{3}$$

For a regular curve on a surface, there exists another frame which is called Darboux frame and denoted by $\{t(s), g(s), N(s)\}$. In this frame, $t(s)$ is the unit tangent of the curve, $N(s)$ is the unit normal of the surface and g is a unit vector given by $g = N \times t$. Derivative of the Darboux frame according to arc-length parameter is governed by the following relations

$$\begin{cases} t'(s) & = \kappa_g g(s) + \kappa_n N(s), \\ g'(s) & = -\kappa_g t(s) + \tau_g N(s), \\ N'(s) & = -\kappa_n t(s) - \tau_g g(s). \end{cases} \tag{4}$$

where κ_g is the geodesic curvature, κ_n is the normal curvature and τ_g is the geodesic torsion at each point of the curve $\gamma(s)$ which are given by

$$\kappa_g =< \gamma''(s), g >, \qquad \kappa_n =< \gamma''(s), N > \text{ and } \qquad \tau_g =< N', g > . \tag{5}$$

The relation between Darboux frame and Serret-Frenet frame can be given by the following matrix representation

$$\begin{pmatrix} t \\ g \\ N \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}. \tag{6}$$

Where

$$\begin{cases} g(s) & = \cos\phi(s) n(s) + \sin\phi(s) b(s), \\ N(s) & = -\sin\phi(s) n(s) + \cos\phi(s) b(s). \end{cases} \tag{7}$$

Differentiating (7), using (4) and (1), we get the relation between geodesic curvature, normal curvature, and geodesic torsion with curvature and torsion as follows

$$\kappa_g = \kappa\cos\phi, \qquad \kappa_n = \kappa\sin\phi, \qquad \text{and } \tau_g = \tau + \frac{d\phi}{ds}. \tag{8}$$

Definition 2.1. *A curve lying on a surface is*

1. *a geodesic if and only if its geodesic curvature vanishes ($\kappa_g = 0$). By (8) and (7), it is equivalent to*

$$N = \pm n. \tag{9}$$

2. *an asymptotic if and only if its normal curvature vanishes ($\kappa_n = 0$). By (8) and (7), it is equivalent to*

$$N = \pm b. \tag{10}$$

3. *a line of curvature if and only if its geodesic torsion vanishes ($\tau_g = 0$). By (8), it is equivalent to*

$$\tau + \frac{d\phi}{ds} = 0. \tag{11}$$

2.2. *Ruled surfaces*

A ruled surface is generated by the motion of a straight line on a given curve and parameterized by

$$X(s, v) = \gamma(s) + vD(s), 0 \le s \le \ell,\ v \in \mathbb{R}. \tag{12}$$

A unit regular curve $\gamma(s)$ is called a base curve (or directrix), and the line passing through $\gamma(s)$ that is parallel to $D(s)$ is called the ruling (or generator) of the ruled surface at $\gamma(s)$. $D(s)$ is a unit director vector field that gives the direction of the ruling. Different ruled surfaces are constructed based on different $\gamma(s)$ and $D(s)$. The unit normal vector field to the ruled surface is defined by

$$N(s, v) = \frac{X_s \times X_v}{|X_s \times X_v|} = \frac{(\gamma' \times D) + v(D' \times D)}{|(\gamma' \times D) + v(D' \times D)|}. \tag{13}$$

A point on a ruled surface that satisfies $X_s \times X_v = 0$ is called a singular point, where the surface normal is not defined, a point that is not singular is called a regular. In general, a ruled surface may have singular points, they are located (if exist) on the striction curve which parameterized by [2]

$$C(s) = \gamma(s) - \frac{\langle \gamma'(s), D'(s) \rangle}{\langle D'(s), D'(s) \rangle} D(s), \qquad D'(s) \neq 0. \tag{14}$$

A ruled surface where all rulings tangent the directrix i.e. $\gamma'(s) = D(s)$, has singularities (edge of regression) along the directrix curve. The Gaussian curvature is non-positive for a ruled surface, it vanishes identically for special classes called the developable surfaces. Equivalently, a ruled surface (12) is developable if and only if [2]

$$det\langle \gamma'(s), D(s), D'(s) \rangle = 0. \tag{15}$$

The vector field $D(s)$ lies in the space formed by moving frame $\{t, n, b\}$ of $\gamma(s)$ and using (6) is defined by

$$D(s) = \cos\theta(s)t(s) + \sin\theta(s)g(s), \ \text{ where } \ g(s) = \cos\phi(s)n(s) + \sin\phi(s)b(s). \tag{16}$$

Therefore $D(s)$ can be given by[11]

$$D(s) = \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s)). \tag{17}$$

The functions $\phi(s)$ and $\theta(s)$ are two scalar functions that are called the first and second angular functions [5]. When $\phi(s)$ and $\theta(s)$ take special choices, the director vector $D(s)$ lies in a rectifying, an osculating, or a normal plane of the directrix curve.

Definition 2.2. *The surface defined by*

$$\begin{cases} X(s,v) & = \gamma(s) + vD(s),\ 0 \leq s \leq \ell,\ v \in \mathbb{R}, \quad \textit{where,} \\ D(s) & = \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s)). \end{cases} \tag{18}$$

is called the family of ruled surfaces with a common directrix curve.

Through this paper, the singular points on the constructed ruled surface are avoided, therefore the vectors $\gamma'(s)$ and $D(s)$ are not collinear, it requires that $\sin\theta(s) \neq 0$ which can be used as a regularity condition, i.e. $0 < \theta(s) < \pi$. The surface normal along the directrix using (6) is given by

$$N(s, 0) = -\sin\phi(s)n(s) + \cos\phi(s)b(s). \tag{19}$$

By choosing different values of $\phi(s)$ and $\theta(s)$ we obtain not only the members of ruled surfaces family with common directrix curve but also yields different families. For example, when $\cos\phi(s) = 0$, $\sin\phi(s) = 0$, or $\cos\theta(s) = 0$, the corresponding families of ruled surfaces are parameterized respectively by:

$$X_{rec}(s,v) = \gamma(s) + v(\cos\theta(s)t(s) + \sin\theta(s)b(s)), \tag{20}$$

$$X_{osc}(s,v) = \gamma(s) + v(\cos\theta(s)t(s) + \sin\theta(s)n(s)), \tag{21}$$

$$X_{nor}(s,v) = \gamma(s) + v(\cos\phi(s)n(s) + \sin\phi(s)b(s)). \tag{22}$$

which are called the rectifying, osculating and normal ruled surfaces family respectively, and:

$$D_{rec}(s) = \cos\theta(s)t(s) + \sin\theta(s)b(s), \tag{23}$$

$$D_{osc}(s) = \cos\theta(s)t(s) + \sin\theta(s)n(s), \tag{24}$$

$$D_{nor}(s) = \cos\phi(s)n(s) + \sin\phi(s)b(s). \tag{25}$$

are called the rectifying, the osculating and the normal director vector respectively. This means that, according to type of director vector field, three different families of ruled surfaces are constructed. Especially, when the director vector field $D(s)$ has the same direction of the tangent vector field $t(s)$, the principal normal vector field $n(s)$, the binormal vector field $b(s)$, and the unit Darboux vector field $\omega(s)$, then the corresponding ruled surfaces are parameterized respectively by

$$\begin{aligned} X_t(s,v) &= \gamma(s) + vt(s), && (26)\\ X_n(s,v) &= \gamma(s) + vn(s), && (27)\\ X_b(s,v) &= \gamma(s) + vb(s), && (28)\\ X_{Dar}(s,v) &= \gamma(s) + v\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}t(s) + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}b(s)\Big). && (29) \end{aligned}$$

From the above discussion, we can easily obtain the following lemma which constructs the ruled surfaces families by choosing different values of angular functions $\phi(s)$ and $\theta(s)$.

Lemma 2.3. *The ruled surfaces family (18) with common directrix curve is*

1. *A generalized ruled surfaces family $X_{gen}(s,v)$ (18), if and only if* $\cos\theta(s) \neq 0$, $\cos\phi(s) \neq 0$ *and* $\sin\phi(s) \neq 0$.
2. *A rectifying ruled surfaces family $X_{rec}(s,v)$ (20), if and only if* $\cos\phi(s) = 0$.
3. *An osculating ruled surfaces family $X_{osc}(s,v)$ (21), if and only if* $\sin\phi(s) = 0$.
4. *A normal ruled surfaces family $X_{nor}(s,v)$ (22), if and only if* $\cos\theta(s) = 0$.

Similarly, the following members are constructed by choosing different values for angular functions.

Lemma 2.4. *The ruled surfaces family (18) have the following members :*

1. *Principal normal ruled surface $X_{pn}(s,v)$ (27), if and only if* $\cos\theta(s) = 0$ *and* $\sin\phi(s) = 0$.
2. *Binormal ruled surface $X_b(s,v)$ (28), if and only if* $\cos\theta(s) = 0$ *and* $\cos\phi(s) = 0$.
3. *Darboux ruled surface $X_{Dar}(s,v)$ (29), if and only if* $\cos\theta = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}$ *and* $\sin\theta(s) = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}$.

Excluding the tangent ruled surface (26) via the regularity condition ($\sin\theta(s) \neq 0$), the other ruled surfaces families $X_{gen}(s,v)$, $X_{rec}(s,v)$, $X_{osc}(s,v)$, and $X_{nor}(s,v)$, also the members $X_n(s,v)$, $X_b(s,v)$, and $X_{Dar}(s,v)$ will be studied in the next section and classified depending on the type of characteristic curve. It is worth noting that during writing this paper, the geometry of $X_{rec}(s,v)$ (20) is studied in [8].

3. Classification of ruled surfaces family with common characteristic curve

In this section, we classify the ruled surfaces family whose common directrix satisfies the requirements of characteristic curve in three-dimensional Euclidean space. We show that the ruled surfaces family parameterized by (18) whose directrix curve is characteristic can be classified into three types: (rectifying, osculating, or generalized) ruled surfaces family. This is the main result of this article and will be given in the following main theorem.

Theorem 3.1 (Main Theorem). *Let $X(s,v)$ be a ruled surfaces family parameterized by (18), and let $\gamma(s)$ be its common characteristic directrix curve with non-vanishing curvature and torsion, then $\gamma(s)$ is :*

1. *Geodesic directrix curve, if and only if $X(s,v)$ is a rectifying ruled surfaces family $X_{osc}(s,v)$.*
2. *Asymptotic directrix curve, if and only if $X(s,v)$ is an osculating ruled surfaces family $X_{rec}(s,v)$.*
3. *Line of curvature directrix curve, if $X(s,v)$ is a generalized ruled surfaces family $X_{gen}(s,v)$ satisfying $\tau + \frac{d\phi}{ds} = 0$.*

Proof.
1. From (19), $N = \pm n$, if and only if $\cos\phi(s) = 0$, by lemma(2.4) and (9) this is equivalent to $\gamma(s)$ is a geodesic, if and only if $X(s,v)$ is a rectifying ruled surfaces family.
2. From (19), it is clear that $N = \pm b$, if and only if $\sin\phi(s) = 0$, by lemma(2.4) and (10) this equivalent to $\gamma(s)$ is an asymptotic, if and only if $X(s,v)$ is an osculating ruled surfaces family.

3. Let $X(s,v)$ be the ruled surfaces family (18) where the common directrix curve $\gamma(s)$ is a line of curvature, hence $\gamma(s)$ satisfies the constraint $\tau + \frac{d\phi}{ds} = 0$ (11), it follows that

$$\phi(s) = -\int_0^s \tau ds. \tag{30}$$

By using (30) in (18), we obtain

$$X(s,v) = \gamma(s) + v[\cos\theta(s)t(s) + \sin\theta(s)(\cos\int_0^s \tau ds\, n(s) - \sin\int_0^s \tau ds.\, b(s))]. \tag{31}$$

Therefore, the family (31) is generalized ruled surfaces family $X_{gen}(s,v)$.

□

Remark 3.2. *The proof of the main theorem showed how to construct the family of ruled surfaces whose directrix curve is a line of curvature,* $\cos\theta(s)$ *is the family parameter when varies the family members are constructed. The normal ruled surfaces family $X_{nor}(s,v)$ (22) degenerates to a member of this family via the line of curvature condition (30) and can be produced by substitution* $\cos\theta(s) = 0$ *as the following*

$$X_{nor}(s,v) = \gamma(s) + v[\cos\int_0^s \tau ds\, n(s) - \sin\int_0^s \tau ds\, b(s)]. \tag{32}$$

The main theorem can be decomposed into the following three theorems which study the constructed families separately and can be used as tools together with the main theorem to build the classification theorem.

Theorem 3.3. *Let $X(s,v)$ be a ruled surfaces family given by (18), where the common directrix curve $\gamma(s)$ is a line of curvature. Then, the following statements hold:*

1. *$X(s,v)$ is neither rectifying ruled surfaces family $X_{rec}(s,v)$ nor osculating ruled surfaces family $X_{osc}(s,v)$.*
2. *$X(s,v)$ is a generalized ruled surfaces family $X_{gen}(s,v)$.*
3. *$X_{nor}(s,v)$ (13) is a member of this family.*

Theorem 3.4. *Let $X(s,v)$ be a ruled surfaces family given by (18), where the common directrix curve $\gamma(s)$ is a geodesic. Then, the following statements hold:*

1. *$X(s,v)$ is neither generalized ruled surfaces family $X_{gen}(s,v)$ nor osculating ruled surfaces family $X_{osc}(s,v)$.*
2. *$X(s,v)$ is a rectifying ruled surfaces family $X_{rec}(s,v)$.*
3. *$X_b(s,v)$ (28) is a member of this family.*

Theorem 3.5. *Let $X(s,v)$ be a ruled surfaces family given by (18), where the common directrix curve $\gamma(s)$ is asymptotic. Then, the following statements hold:*

1. *$X(s,v)$ is neither generalized ruled surfaces family $X_{gen}(s,v)$ nor rectifying ruled surfaces family $X_{rec}(s,v)$.*
2. *$X(s,v)$ is an osculating ruled surfaces family $X_{osc}(s,v)$.*
3. *$X_n(s,v)$ (27) is a member of this family.*

Based on the main Theorem and Theorems (3.3), (3.4) and (3.5), we can get the following theorem of classification.

Theorem 3.6. *(Classification of ruled surfaces family) Let $X(s,v)$ be a ruled surfaces family parametrized by (18), where the common directrix is characteristic curve. Then $X(s,v)$ is either a rectifying ruled surfaces family $X_{rec}(s,v)$, an osculating ruled surfaces family $X_{osc}(s,v)$ or a generalized ruled surfaces family $X_{gen}(s,v)$ satisfying (30).*

The explicit classification can be obtained in following equivalent theorem.

Theorem 3.7. *Let $X(s, v)$ be a ruled surfaces family whose common directrix is a geodesic, an asymptotic or a line of curvature. Then $X(s, v)$ is a rectifying ruled surfaces family $X_{rec}(s, v)$, an osculating ruled surfaces family $X_{osc}(s, v)$ or a generalized ruled surfaces family $X_{gen}(s, v)$ satisfying (30) respectively.*

According to the above classification theorems, the ruled surfaces family whose common directrix is a characteristic curve can be classified into three different type $X_{rec}(s, v)$, $X_{osc}(s, v)$ or $X_{gen}(s, v)$ based on the type of the common directrix curve. Also, without any conditions, the families $X_{rec}(s, v)$ and $X_{osc}(s, v)$ have common geodesic and asymptotic directrix curve respectively, whereas the family $X_{gen}(s, v)$ has a common line of curvature when satisfying (30).

Corollary 3.8. *Let $X(s, v)$ be a ruled surfaces family of type $X_{rec}(s, v)$, $X_{osc}(s, v)$ or $X_{gen}(s, v)$ satisfying (30). Then, the common directrix curve is geodesic, asymptotic or line of curvature .*

The existence of such families $X_{rec}(s, v)$, $X_{osc}(s, v)$, or $X_{gen}(s, v)$ can be ensured via the proof of main Theorem, so we obtain the following corollary.

Corollary 3.9. *Given a unit speed regular curve $\gamma(s)$ with non vanishing curvature and torsion. Then, there exists a rectifying, an osculating or a generalized ruled surfaces family in which $\gamma(s)$ its a common geodesic, asymptotic or line of curvature directrix curve respectively.*

Remark 3.10. *According to the previous discussion, the ruled surfaces family parameterized by (18) whose common directrix is a characteristic curve is one of the following:*

1. *A rectifying ruled surfaces family $X_{rec}(s, v)$ (20) where the common directrix is a geodesic.*
2. *An osculating ruled surfaces family $X_{osc}(s, v)$ (21) where the common directrix is an asymptotic.*
3. *A generalized ruled surfaces family $X_{gen}(s, v)$ (18) where the common directrix is a line of curvature.*

Determining the type of a unit director vector field $D(s)$ is enough to design ruled surfaces family with a common geodesic or asymptotic directrix curve, this is the subject of the following proposition which its proof is straightforward based on the main Theorem (3.1).

Proposition 3.11. *Let $X(s, v)$ be a ruled surfaces family (18), and let $\gamma(s)$ be its common directrix curve and $D(s)$ is the unit director vector, then $\gamma(s)$ is a common*

1. *geodesic directrix curve on $X(s, v)$, if and only if $D(s)$ is a rectifying director vector.*
2. *asymptotic directrix curve on $X(s, v)$, if and only if $D(s)$ is an osculating director vector.*

Finally, this section ended with the following three theorems which clarify that there is an equivalent between the type of ruled surfaces family, directrix curve, angular functions, and the unit director vector field $D(s)$.

Theorem 3.12. *Let $X(s, v)$ be a ruled surfaces family (18), and let $\gamma(s)$ be its common directrix curve with non-vanishing curvature and torsion. Then the followings are equivalent:*

1. *$X(s, v)$ is a rectifying ruled surface family $X_{rec}(s, v)$,*
2. *$\gamma(s)$ is a common geodesic directrix curve,*
3. $\cos\phi(s) = 0$,
4. *$D(s)$ is a rectifying director vector.*

Theorem 3.13. *Let $X(s, v)$ be a ruled surfaces family (18), and let $\gamma(s)$ be its common directrix curve with non-vanishing curvature and torsion. Then the followings are equivalent:*

1. *$X(s, v)$ is an osculating ruled surface family $X_{osc}(s, v)$,*
2. *$\gamma(s)$ is a common asymptotic directrix curve,*
3. $\sin\phi(s) = 0$,
4. *$D(s)$ is an osculating director vector.*

Theorem 3.14. *Let $X(s, v)$ be a ruled surfaces family (18), and let $\gamma(s)$ be its common directrix curve with non-vanishing curvature and torsion. Then the followings are equivalent:*

1. *$\gamma(s)$ is a common line of curvature directrix curve,*
2. $\phi(s) = -\int_0^s \tau ds$,
3. *$X(s, v)$ is a generalized ruled surface family $X_{gen}(s, v)$.*

4. Classification of developable ruled surfaces family with common characteristic curve

In this section, we give a classification of the developable ruled surfaces families whose directrix is characteristic curve. In particular, we study under what conditions the three different families $X_{rec}(s,v)$, $X_{osc}(s,v)$ and $X_{gen}(s,v)$ satisfying (30) are to be developable. Firstly, the developability condition (15) can be explicitly written for a ruled surfaces family (18) as the following

Lemma 4.1. *A ruled surfaces family (18) is a developable, if and only if the following condition is satisfied,*

$$\sin\theta(s)(\frac{d\phi}{ds}+\tau(s))-\kappa(s)\sin\phi(s)\cos\theta(s)=0. \tag{33}$$

Proof. Using the developability condition (15), a ruled surfaces family (18) is developable, if and only if $det\langle\gamma', D, D'\rangle = 0$, since $\gamma'(s) = t$ and $D(s) = \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s))$, by taking the derivative of $D(s)$ and using the Frenet-Serret formula of $\gamma(s)$, we get $D'(s) = -\sin\theta(s)[\kappa(s)\cos\phi + \frac{d\theta}{ds}]t(s) + [\cos\theta(s)(\kappa(s) + \cos\phi\frac{d\theta}{ds}) - \sin\theta(s)\sin\phi(s)(\frac{d\phi}{ds} + \tau)]n + [\sin\phi(s)\cos\theta(s)\frac{d\theta}{ds} + \sin\theta(s)\cos\phi(\frac{d\phi}{ds} + \tau)]b$. Then we obtain $det\langle\gamma', D, D'\rangle = \sin\theta(s)(\frac{d\phi}{ds} + \tau(s)) - \kappa(s)\sin\phi(s)\cos\theta(s)$, this completes the proof of the lemma. □

In [11], the author gave the above condition (33) by using a different technique. The following definition and lemma are needed to study the developability of rectifying ruled surfaces family $X_{rec}(s,v)$ (20).

Definition 4.2. *For a rectifying vector $D(s) = \cos\theta(s)t(s) + \sin\theta(s)b(s)$ defined along a unit speed regular curve $\gamma(s)$, we define a scalar function $H(s) = \kappa(s)\cos\theta(s) - \tau(s)\sin\theta(s)$ and we call it Darboux function of $D(s)$ along $\gamma(s)$.*

Lemma 4.3. *Suppose that $D(s) = \cos\theta(s)t(s) + \sin\theta(s)b(s)$ is a rectifying vector field defined along a unit speed regular curve $\gamma(s)$, then $D(s)$ is a unit Darboux vector, if and only if $H(s)$ vanishes.*

Proof. Let $D(s) = \cos\theta(s)t(s) + \sin\theta(s)b(s)$ be a unit Darboux vector. From (3), $\cos\theta = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}$, $\sin\theta(s) = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}$. This implies that $H(s) = \kappa\cos\theta - \tau\sin\theta = 0$, and vice versa. □

The following theorem shows the developability condition for the $X_{rec}(s,v)$, $X_{osc}(s,v)$ and $X_{gen}(s,v)$.

Theorem 4.4. *Let $X(s,v)$ be the ruled surfaces family of type $X_{rec}(s,v)$, $X_{osc}(s,v)$ or $X_{gen}(s,v)$ satisfying (30). Then,*

1. *A rectifying ruled surfaces family $X_{rec}(s,v)$ is developable, if and only if the director vector is a unit Darboux vector.*
2. *An osculating ruled surfaces family $X_{osc}(s,v)$ is developable, if and only if the directrix curve is a plane curve .*
3. *A generalized ruled surfaces family $X_{gen}(s,v)$ satisfying (30) is developable, if and only if $\cos\theta(s) = 0$.*

Proof. Using Lemma (2.4),

1. A ruled surfaces family (18) with a common directrix curve is of type $X_{rec}(s,v)$ (20), if and only if $\cos\phi(s) = 0$. Then the developability condition (33) turns into

$$\tau(s)\sin\theta(s)-\kappa(s)\cos\theta(s)=0. \tag{34}$$

   Using lemma (4.3), $X_{rec}(s,v)$ is developable, if and only if the director vector is a unit Darboux vector.

2. A ruled surfaces family (18) with a common directrix is of type $X_{osc}(s,v)$ (21), if and only if $\sin\phi(s)=0$, this implies that the developability condition (33) becomes
$$\sin\theta(s)\tau(s)=0. \tag{35}$$
Since $\sin\theta(s)\neq 0$ (regularity condition), then $X_{osc}(s,v)$ is developable if and only if $\tau=0$, that is the directrix curve is a plane curve.
3. Let $X(s,v)$ be a ruled surfaces family of type $X_{gen}(s,v)$ that satisfying (30). After substitution, the developabity condition (33) can be expressed as
$$\kappa(s)\sin\phi(s)\cos\theta(s)=0. \tag{36}$$
Since $\sin\phi(s)\neq 0$, and $\kappa(s)\neq 0$ (regular curve), then $X_{gen}(s,v)$ (18) is developable, if and only if $\cos\theta(s)=0$.

□

Based on the above theorem and its proof, we conclude the following equivalent theorem.

Theorem 4.5. *Let $X(s,v)$ be a ruled surfaces family of type $X_{rec}(s,v)$, $X_{osc}(s,v)$, or $X_{gen}(s,v)$ satisfying (30) . Then,*

1. *Darboux ruled surface $X_{Dar}(s,v)$ (29) is the only member of the family of rectifying ruled surfaces $X_{rec}(s,v)$ (20), that is developable.*
2. *The family of osculating ruled surfaces $X_{osc}(s,v)$ (21) that has plane directrix curve is developable. .*
3. *Normal ruled surface $X_{nor}(s,v)$ (22) is the only member of the family of generalized ruled surfaces $X_{gen}(s,v)$ satisfying (30), that is developable. .*

Corollary 4.6. *There is no family of developable ruled surfaces having parametrization (18) whose common directrix curve is geodesic or line of curvature.*

Remark 4.7. *It is important to note that this result is restricted to parameterization (18) that uses the angular functions as coefficients. When other different parameterizations are used, the family of developable ruled surfaces whose common directrix curve is geodesic [14] or line of curvature [7] can be constructed by using marching-scale functions instead of angular functions in (18).*

Theorem 4.8. *A family of developable surfaces whose common directrix is characteristic curve is an osculating ruled surfaces family $X_{osc}(s,v)$ (21) whose common directrix is asymptotic plane curve.*

Remark 4.9. *When the common asymptotic of the osculating ruled surfaces family $X_{osc}(s,v)$ (21) is a plane curve, then this family is the osculating planes of asymptotic plane curve.*

Theorem 4.10. *Let $X(s,v)$ be a developable ruled surfaces family whose common directrix $\gamma(s)$ is a characteristic curve. Then, the following statements hold:*

1. *$X(s,v)$ is neither generalized ruled surfaces family $X_{gen}(s,v)$ nor a rectifying ruled surfaces family $X_{rec}(s,v)$.*
2. *$X(s,v)$ is an osculating ruled surfaces family $X_{osc}(s,v)$.*
3. *The directrix curve $\gamma(s)$ is a plane curve .*
4. *The common directrix curve $\gamma(s)$ is asymptotic.*
5. *$X_{osc}(s,v)$ is the family of osculating planes of $\gamma(s)$.*

Finally, we review a known result which characterizes $X_{pn}(s,v)$ and $X_b(s,v)$ with developability property.

Corollary 4.11. *Let X be a ruled surface member of type $X_{pn}(s,v)$ (27) or $X_b(s,v)$ (28) then $X_{pn}(s,v)$ and $X_b(s,v)$ are developable if and only if the directrix curve is a plane curve.*

5. Conclusion and future works

In this study, using parameterization (18), the family of ruled surfaces has been classified based on its common characteristic directrix curve into three families: Rectifying ruled surfaces whose common directrix is a geodesic, osculating ruled surfaces whose common directrix is an asymptotic, or generalized ruled surfaces whose common directrix is a line of curvature. The three families converted into one family under the developability condition. For future works, we will investigate how to extend these results to other ambient spaces with different dimensions and using other frames.

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An Application of Least Squares Method in Nonlinear Models-Solid Waste Sample

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Abstract. Nowadays, the usage of nonlinear regression models is common. However, non-linear models are more difficult to teach than linear models. The ordinary least squares method is effective in teaching nonlinear regression models. This study aims to teach the subjects of non-linear regression in statistics for students. The transformed form of non-linear regression models are used for this. Therefore the ordinary least squares estimators of regression models are obtained and the comparison of these is made. Besides, an explanatory application is made on this subject.

1. Introduction

Regression analysis technique is one of the statistical analysis used to determine the relationship between variables that have a cause-effect relationship and to make predictions about this relationship [5]. In this technique, a mathematical model is first established to find the relationship between the variables, and then the validity of the established model is examined [3]. If the established regression model doesn't fit to the data, misleading results will occur in the future [7]. The classical linear regression is used to modelling the relationship between two variables such as income-education level, height-weight of people, dose-effect of a drug, or height-boiling point of water [8]. Sometimes the dependent variable can be affected by more than one independent variable. In this case, the multiple linear regression is used to modelling the relationship between the variables [8].

Nowadays, the linear models are preferred so that calculations such as hypothesis testing and parameter estimation can be made more easily. The linear model works well in many cases, but in some cases this isn't possible. It cannot be said that the relationship can best be expressed with a linear model. Therefore, it can also be decided more accurately by trying non-linear models as well as linear models.

Various estimation methods have been developed in linear and non-linear regression models. The ordinary least squares (OLS) method is one of the most frequently used methods in regression analysis [6]. Therefore, the OLS method plays an important role in teaching regression analysis. In addition, OLS method is an optimal method according to the Gauss-Markov Theorem, since it aims to minimize the sum of squares of error.

It is possible for the stochastic term to be additive or unpredictable depending on the dependent variable. Depending on the variability of the stochastic term, nonlinear models can be linear regression models with

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appropriate transformation in parameters (log) and nonlinear regression models. Similarly, OLS method can be used for parameter estimation in transformed models. If the OLS criterion for linear and non-linear models in the variables was applied to the initial variables, the OLS criterion for non-linear models in the parameters should also be applied to the transformed variables, for example $\ln \nu$. However, the OLS parameter estimation for the transformed models is biased. Although non-linear models can be converted to linear regression models and estimated with OLS, it is useful to be careful about the characteristics of the stochastic error term that enters these models. Otherwise, the application of the OLS for the transformed model will not produce a model with correct statistical properties.

The teaching of nonlinear models is quite complex compared to linear models. Students often have problems interpreting the transformed regression model in regression analysis [1]. The OLS method plays an important role in teaching regression analysis to students. In this study, the visualized approach to teaching nonlinear regression subjects for graduate students is presented, based on the population-based solid waste amount experiment in Ordu Metropolitan Municipality. Thus, the OLS method is applied in non-linear models. In addition, the estimation parameters of the models are also compared.

2. Main Results

Consider the regression model (1)

$$\omega_i = \theta_1 \nu_i^{\theta_2} \varepsilon_i, (i = 1, ..., n) \tag{1}$$

where ω_i is a dependent variable, ν_i is an independent variable, θ_1 and θ_2 are unknown parameters, and ε_i is an error variable.

The regression model (1) is inherently linear regression model since it can be made linear according to the parameters θ_1 and θ_2 by suitable transformations.

Then it is assumed that the error isn't additive. In other words, the error variability isn't constant for all ν_i. There is likely to be unpredictable and random fluctuation at the different levels of ν_i [2]. In this case, in [2], it is assumed that

$$\prod_{i=1}^{n} \varepsilon_i = 1 \ \ or \ \ \prod_{i=1}^{n} \ln \varepsilon_i = 0. \tag{2}$$

By taking the logarithm of both sides of the model (1), the equation (3) is obtained:

$$\ln \omega_i = \ln \theta_1 + \theta_2 \ln \nu_i + \ln \varepsilon_i. \tag{3}$$

Under the minimum criterion (4),

$$\phi_1 = \Sigma_{i=1}^{n} (\ln \omega_i - \ln \theta_1 - \theta_2 \ln \nu_i)^2 \tag{4}$$

using OLS method, differentiation with respect to θ_1 and θ_2 yields the normal equations (5):

$$\Sigma_{i=1}^{n} \ln \omega_i = n \ln \theta_1 + \theta_2, \ \ and \ \ \Sigma_{i=1}^{n} \ln \nu_i \ln \omega_i = \ln \theta_1 \Sigma_{i=1}^{n} \ln \nu_i + \theta_2 \Sigma_{i=1}^{n} (\ln \nu_i)^2. \tag{5}$$

The parameter estimates θ_1 and θ_2 are obtained when the normal equations are solved simultaneously. To use the ordinary linear regression model, $\ln \varepsilon_i \sim N(0, \sigma^2)$ is supposed. When the regression model (3) run, normality tests are applied to the error estimates obtained from this regression [4].

Now, let consider the regression model (6). This model is actually non-linear with respect to the parameters θ_1 and θ_2. The error variability is assumed to be independent of ν_i, that is, the regression model (6) is in the form:

$$\omega_i = \theta_1 \nu_i^{\theta_2} + \varepsilon_i. \tag{6}$$

Table 1: Population of districts and total average solid waste amount.

Districts	Population	Total average solid waste amount (kg)
Altınordu	202310	160000
Ünye	120720	80000
Fatsa	111072	73000
Perşembe	31094	15000
Kumru	31064	21000
Korgan	29349	9000
Gölköy	28952	18000
Aybastı	25900	17000
Akkuş	23064	13000
Ulubey	18239	6000
Mesudiye	15759	6000
İkizce	14969	3000
Gürgentepe	13821	7000
Çatalpınar	13786	7000
Çaybaşı	13127	4000
Kabataş	10604	11000
Çamaş	8594	6000
Kabadüz	8531	6000
Gülyalı	7994	4000

Under the minimum criterion (7),

$$\phi_2 = \Sigma_{i=1}^{n}(\omega_i - \theta_1 {\nu_i}^{\theta_2})^2 \tag{7}$$

using OLS method, differentiation with respect to θ_1 and θ_2 yields the normal equations (8):

$$2\Sigma_{i=1}^{n}(\omega_i - \theta_1 {\nu_i}^{\theta_2}){\nu_i}^{\theta_2} = 0, \ \ and \ \ 2\Sigma_{i=1}^{n}(\omega_i - \theta_1 {\nu_i}^{\theta_2})\theta_1 {\nu_i}^{\theta_2} \ln \nu_i = 0. \tag{8}$$

Since it isn't possible to solve these normal equations analytically, the OLS estimation can be made iteratively by using a linearization of the model with respect to θ_1 and θ_2. The estimation of the parameters θ_1 and θ_2 that obtained from the criterion (7) is biased from the estimation of the parameters θ_1 and θ_2 that obtained from the criterion (4).

It is also necessary to be careful about the properties of the stochastic error term. For hypothesis testing, it is supposed that the stochastic error term ε_i of the regression model (6) fits the normal distribution, but the stochastic error term ε_i of the regression model (1) and its statistical counterpart (3) fits the log-normal distribution with mean $\exp(\sigma^2/2)$ and variance $\exp(\sigma^2)(\exp(\sigma^2) - 1)$ [2].

3. An Explanatory Application

In this section, the impact of the population of Ordu Metropolitan Municipality on the amount of solid waste is examined. The analysis is based on the amount of solid waste collected by the district municipalities for the winter period of 2015 in 19 different districts of Ordu province and the population in the districts. The data are given in (**??**).

The following hypothesis is chosen for the regression analysis:

"The amount of solid waste depends on the population."

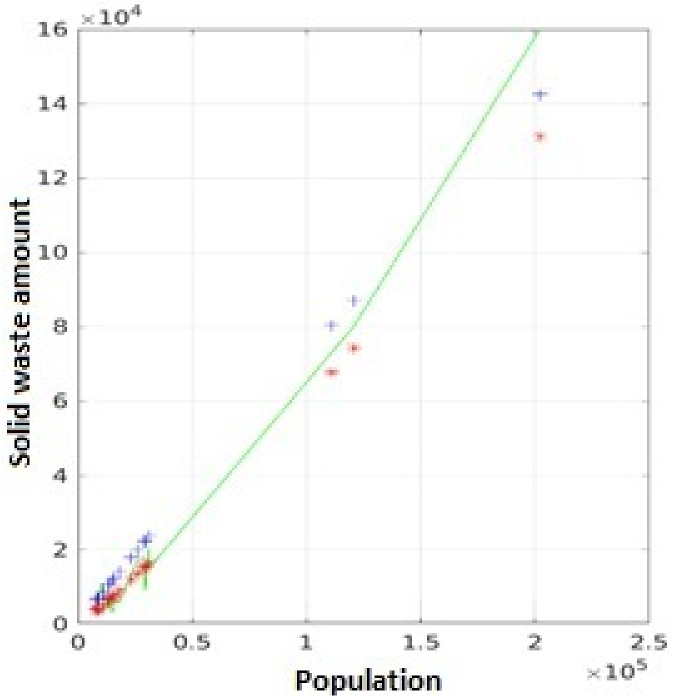


Figure 1: Relationship between population and solid waste by using models with additive and unpredictable error terms.

Let the regression model (1) be chosen for population-based solid waste modeling, where ω_i is the total average amount of solid waste in the winter period in 2015 and ν_i is the population in these municipalities. Under the minimum criterion (4), estimates of θ_1 and θ_2 are obtained by using OLS method:

$$\hat{\theta}_1\text{=0,185142} \quad \text{and} \quad \hat{\theta}_2\text{=1,102617}.$$

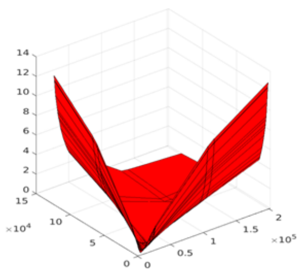
The regression results show that the $\ln\theta_1$ and θ_2 parameters are both significant (p-value< 0.05) and the population is dependent on waste. θ_2 is the slope parameter and it measures the percentage change in ω for a given percentage change in ν (1). Now, let the model (6) be chosen for population-based solid waste modeling. Under the minimum criterion (7), estimates of θ_1 and θ_2 are obtained by iteration (with the generalized Newton-Raphson iteration):

$$\hat{\theta}_1\text{=1,212444} \quad \text{and} \quad \hat{\theta}_2\text{=0,955536}.$$

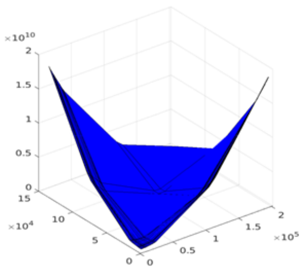
The green line indicates the amount of solid waste corresponding to the population. And the red and blue points indicate the values corresponding to the regression models (1) and (6), respectively (1).

Both $\phi_1(0,185142;1,102617)$ and $\phi_2(1,212444;0,955536)$ minimum criterions have biased minimum solutions (2).

As a result, when the (1) regression is run, the OLS method is applied for the linear regression model according to the parameters. When the regression (6) is run, the OLS method is applied for the non-linear model according to the parameters.



(a) 2a



(b) 2b

Figure 2: Comparison of $\phi_1(0,185142;1,102617)$ and $\phi_2(1,212444;0,955536)$ minimum criterions solutions.

4. Conclusion

The application of the OLS helps students to better understand the interpretation of parameters in the transformed models. It also allows them to obtain clear interpretation of the statistical models. They get explanation about the statistical properties of transformed models and the using of the OLS in the nonlinear regression. The above analysis explains the properties of the stochastic error term to students with application.

Acknowledgments

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Conformal Generic Riemannian Maps from Almost Hermitian Manifolds

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Abstract. In the present paper, the notion of conformal generic Riemannian maps from almost Hermitian manifolds onto Riemannian manifolds is defined. Examples for this type conformal maps are given. The concept of pluriharmonic map is used to get conditions defining totally geodesic foliations for certain distributions and being horizontally homothetic map on the base manifold.

1. Introduction

The notion of submersion was introduced by O'Neill [10] and Gray [6]. Then, this notion was widely studied [4] and new kind of Riemannian submersions like invariant submersion, anti-invariant submersion, slant submersion, generic submersion were introduced [1, 2, 11–13]. Riemannian maps between Riemannian manifolds are generalization of isometric immersions and Riemannian submersions [4–6, 10]. Let $F : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < rankF < min\{\dim M_1, \dim M_2\}$. Then the tangent bundle TM_1 of M_1 has the following decomposition:

$$TM_1 = kerF_* \oplus (kerF_*)^{\perp}.$$

We always have $(rangeF_*)^{\perp}$ because of $rankF < min\{\dim M_1, \dim M_2\}$. Therefore tangent bundle TM_2 of M_2 has the following decomposition:

$$TM_2 = (rangeF_*) \oplus (rangeF_*)^{\perp}.$$

A smooth map $F : (M_1^m, g_1) \longrightarrow (M_2^m, g_2)$ is called Riemannian map at $p_1 \in M_1$ if the horizontal restriction $F^h_{*p_1} : (kerF_{*p_1})^{\perp} \longrightarrow (rangeF_*)$ is a linear isometry. Hence a Riemannian map satisfies the equation

$$g_1(X, Y) = g_2(F_*(X), F_*(Y)) \tag{1}$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$. So that isometric immersions and Riemannian submersions are particular Riemannian maps, respectively, with $kerF_* = \{0\}$ and $(rangeF_*)^{\perp} = \{0\}$ [5].

We say that $F : (M^m, g_M) \longrightarrow (N^n, g_N)$ is a conformal Riemannian map at $p \in M$ if $0 < rankF_{*p} \leq min\{m, n\}$ and F_{*p} maps the horizontal space $(ker(F_{*p})^{\perp})$ conformally onto $range(F_{*p})$, i.e., there exist a number $\lambda^2(p) \neq 0$ such that

$$g_N(F_{*p}(X), F_{*p}(Y)) = \lambda^2(p) g_M(X, Y) \tag{2}$$

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for $X, Y \in \Gamma((ker(F_{*p})^{\perp})$. Also F is called conformal Riemannian if F is conformal Riemannian at each $p \in M$ [14, 15]. Here, λ is the dilation of F at a point $p \in M$ and it is a continuous function as $\lambda : M \to [0, \infty)$.

An even-dimensional Riemannian manifold (M, g_M, J) is called an almost Hermitian manifold if there exists a tensor field J of type $(1, 1)$ on M such that $J^2 = -I$ where I denotes the identity transformation of TM and

$$g_M(X, Y) = g_M(JX, JY), \forall X, Y \in \Gamma(TM). \tag{3}$$

Let (M, g_M, J) be an almost Hermitian manifold and its Levi-Civita connection is ∇ with respect to g_M. If J is parallel with respect to ∇, i.e.

$$(\nabla_X J)Y = 0, \tag{4}$$

we say M is a Kaehlerian manifold [3, 21].

Riemannian maps would provide relationship between Riemannian maps, harmonic maps and Lagrangian field theory on the mathematical side and Maxwell's equation, Schrodinger's equation on the physical side [5]. Some application areas of conformal Riemannian maps are computer vision [7], geometric modelling [18] and medical imaging [19].

In this paper, conformal generic Riemannian maps from almost Hermitian manifolds to Riemannian manifolds were introduced, geometric properties of the base manifold and the total manifold by the existence of such maps were investigated and examples were given. Also, certain geodesicity conditions for conformal generic Riemannian maps were obtained. Moreover, several conditions for conformal generic Riemannian maps to be horizontally homothetic maps by using the adapted version of the notion of pluriharmonic maps were obtained.

2. Preliminaries

In this section, some definitions and useful results for conformal generic Riemannian maps are given. Let $(M, g_{_M})$ and $(N, g_{_N})$ be Riemannian manifolds and $F : M \longrightarrow N$ is a smooth map between them. The second fundamental form of F is given by

$$(\nabla F_*)(X, Y) = \overset{N}{\nabla}{}^F_X F_*(Y) - F_*(\overset{M}{\nabla}_X Y) \tag{5}$$

for $X, Y \in \Gamma(TM)$. The second fundamental form ∇F_* is symmetric [8].

Let F be a Riemannian map from a Riemannian manifold (M^m, g_M) to a Riemannian manifold (N^n, g_N). Then we define O'Neill's tensor fields $\mathcal{T}$ and $\mathcal{A}$ for Riemannian submersions as

$$\mathcal{A}_X Y = h\overset{M}{\nabla}_{hX} vY + v\overset{M}{\nabla}_{hX} hY, \tag{6}$$

$$\mathcal{T}_X Y = h\overset{M}{\nabla}_{vX} vY + v\overset{M}{\nabla}_{vX} hY \tag{7}$$

for vector fields $X, Y \in \Gamma(TM)$, where $\overset{M}{\nabla}$ is the Levi-Civita connection of g_M [10]. For any $X \in \Gamma(TM)$, $\mathcal{T}_X$ and $\mathcal{A}_X$ are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. It is also easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_X = \mathcal{T}_{vX}$, and $\mathcal{A}$ is horizontal, $\mathcal{A}_X = \mathcal{A}_{hX}$. The tensor field $\mathcal{T}$ is symmetric on the vertical distribution [10, 20]. On the other hand, from (6) and (7) we have

$$\overset{M}{\nabla}_U V = \mathcal{T}_U V + \hat{\nabla}_U V, \tag{8}$$

$$\overset{M}{\nabla}_U X = h\overset{M}{\nabla}_U X + \mathcal{T}_U X, \tag{9}$$

$$\overset{M}{\nabla}_X V = \mathcal{A}_X V + v\overset{M}{\nabla}_X V, \tag{10}$$

$$\overset{M}{\nabla}_X Y = h\overset{M}{\nabla}_X Y + \mathcal{A}_X Y \tag{11}$$

for $X, Y \in \Gamma((ker\, F_*)^{\perp})$ and $U, V \in \Gamma(kerF_*)$, where $\hat{\nabla}_U V = v\overset{M}{\nabla}_U V$ [11, 12].

A vector field on M is called a projectable vector field if it is related to a vector field on N. Thus, we say a vector field is basic on M if it is both a horizontal and a projectable vector field. Hereafter, when we mention a horizontal vector field, we always consider a basic vector field [3].

On the other hand, let F be a conformal Riemannian map between Riemannian manifolds $(M^m, g_{_M})$ and $(N^n, g_{_N})$. Then, we have

$$\begin{aligned} (\nabla F_*)(X,Y)\mid_{rangeF_*} &= X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X) \\ &- g_M(X,Y)F_*(grad(\ln\lambda)) \end{aligned} \tag{12}$$

where $X, Y \in \Gamma((kerF_*)^\perp)$. Hence from (12), we obtain $\overset{N}{\nabla}{}^F_X F_*(Y)$ as

$$\begin{aligned} \overset{N}{\nabla}{}^F_X F_*(Y) &= F_*(h\overset{M}{\nabla}_X Y) + X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X) \\ &- g_M(X,Y)F_*(grad(\ln\lambda)) + (\nabla F_*)^\perp(X,Y) \end{aligned} \tag{13}$$

where $(\nabla F_*)^\perp(X,Y)$ is the component of $(\nabla F_*)(X,Y)$ on $(rangeF_*)^\perp$ for $X, Y \in \Gamma((kerF_*)^\perp)$ [16, 17].

Now, a map F from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is a pluriharmonic map if F satisfies the following equation

$$(\nabla F_*)(X,Y) + (\nabla F_*)(JX,JY) = 0 \tag{14}$$

for $X, Y \in \Gamma(TM)$ [9].

3. Conformal Generic Riemannian Maps

Now, we define the notion of conformal generic Riemannian map and give its tangent space's decomposition.

Let F be a conformal Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, the complex subspace of the vertical subspace $\mathcal{V}_p$ at $p \in M$ is

$$\mathcal{D}_p = (kerF_{*p} \cap J(kerF_{*p})).$$

Definition 3.1. *Let F be a conformal Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). If the dimension of $\mathcal{D}_p$ is constant along M and it defines a differentiable distribution on M then we say that F is a conformal generic Riemannian map.*

Let F be a conformal generic Riemannian map. Then, we say F is purely real (respectively, complex) if $\mathcal{D}_p = \{0\}$ (respectively, $\mathcal{D}_p = kerF_{*p}$). Orthogonal complementary distribution $\mathcal{D}^\perp$ of a conformal generic Riemannian map F is called purely real distribution and it satisfies

$$kerF_* = \mathcal{D} \oplus \mathcal{D}^\perp \tag{15}$$

and

$$\mathcal{D} \cap \mathcal{D}^\perp = \{0\}. \tag{16}$$

Let F be a conformal Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). For $U \in \Gamma(kerF_*)$, we write

$$JU = \phi U + \omega U \tag{17}$$

where $\phi U \in \Gamma(kerF_*)$ and $\omega U \in \Gamma((kerF_*)^\perp)$. We contemplate the complementary orthogonal distribution μ to $\omega\mathcal{D}^\perp$ in $(kerF_*)^\perp$. Therefore we have

$$\phi\mathcal{D}^\perp \subseteq \mathcal{D}^\perp, (kerF_*)^\perp = \omega\mathcal{D}^\perp \oplus \mu. \tag{18}$$

In addition, for $X \in \Gamma((kerF_*)^{\perp})$, we write

$$JX = BX + CX \tag{19}$$

where $BX \in \Gamma(\mathcal{D}^{\perp})$ and $CX \in \Gamma(\mu)$. Clearly, we get

$$B((kerF_*)^{\perp}) = \mathcal{D}^{\perp}. \tag{20}$$

From (15) for $U \in \Gamma(kerF_*)$, we can write

$$JU = \Phi_1 U + \Phi_2 U + \omega U \tag{21}$$

where Φ_1 and Φ_2 are the projections from $kerF_*$ to $\mathcal{D}$ and $\mathcal{D}^{\perp}$, respectively.

We say that a conformal generic Riemannian map is proper if $\mathcal{D}^{\perp}$ is neither complex nor purely real. Now, we give examples to conformal generic Riemannian maps.

Example 3.2. *Every conformal semi-invariant Riemannian map [17] F from an almost Hermitian manifold to a Riemannian manifold is a conformal generic Riemannian map with $\mathcal{D}^{\perp}$ is a totally real distribution.*

Example 3.3. *Let $F : (\mathbb{R}^8, g_{\mathbb{R}^8}, J) \longrightarrow (\mathbb{R}^5, g_{\mathbb{R}^8})$ be a map defined by*

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \longrightarrow (\frac{x_1 - x_2 + x_6}{\sqrt{3}}, \frac{x_1 + x_2}{\sqrt{2}}, 0, x_4, x_3)$$

for any point $x \in \mathbb{R}^8$. We obtain the horizontal distribution and the vertical distributions

$$\mathcal{H} = (kerF_*)^{\perp} = \{H_1 = \frac{1}{\sqrt{3}}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}), H_2 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), H_3 = \frac{\partial}{\partial x_4}, H_4 = \frac{\partial}{\partial x_3}\}$$

and

$$\mathcal{V} = (kerF_*) = \{V_1 = \frac{\partial}{\partial x_5}, V_2 = \frac{\partial}{\partial x_7}, V_3 = \frac{\partial}{\partial x_8}, V_4 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \frac{2}{\sqrt{3}}\frac{\partial}{\partial x_6}\},$$

respectively. Thus, using (2) we have

$$g_{\mathbb{R}^5}(F_*(H_i), F_*(H_i)) = \lambda^2 g_{\mathbb{R}^8}(H_i, H_i), i = 1, 2, 3, 4$$

and

$$g_{\mathbb{R}^5}(F_*(H_i), F_*(H_j)) = \lambda^2 g_{\mathbb{R}^8}(H_i, H_j) = 0, i \neq j.$$

It follows that F is a conformal Riemannian map at any point $x \in \mathbb{R}^8$ with $0 < rankF_* = 4 \leq min\{\dim(\mathbb{R}^8), \dim(\mathbb{R}^5)\}$ and $\lambda = 1$. On the other hand, by using the standard complex structure $J = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7)$ on $\mathbb{R}^8$, one can see that

$$\begin{aligned} JV_1 &= \frac{3}{2+\sqrt{3}}H_1 - \frac{3}{3+2\sqrt{3}}V_4, \\ JV_4 &= aH_1 + \sqrt{2}H_2 + \frac{2}{\sqrt{3}}V_1 - \frac{a}{\sqrt{3}}V_4, a \in \mathbb{R}, \\ JV_2 &= V_3, \quad JH_3 = -H_4. \end{aligned}$$

Hence, F is a conformal generic Riemannian map with $\mathcal{D} = span\{V_2, V_3\}$, $\mathcal{D}^{\perp} = span\{V_1, V_4\}$ and $\mu = span\{H_3, H_4\}$.

Now, we examine some geometric properties on the total manifold and the base manifold of a proper conformal generic Riemannian map.

Lemma 3.4. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the distribution $\mathcal{D}$ is integrable if and only if the following condition is satisfied*

$$(\nabla F_*)(U, JV) = (\nabla F_*)(JU, V) \tag{22}$$

for $U, V \in \Gamma(\mathcal{D})$.

Proof. Since M is a Kaehlerian manifold, from (4), (8), (19) and (21) we have

$$\mathcal{T}_U JV + v\overset{M}{\nabla}_U JV = B\mathcal{T}_U V + C\mathcal{T}_U V + \Phi_1 v\overset{M}{\nabla}_U V + \Phi_2 v\overset{M}{\nabla}_U V + \omega v\overset{M}{\nabla}_U V \tag{23}$$

and changing the role of U and V in (23) we have

$$\mathcal{T}_V JU + v\overset{M}{\nabla}_V JU = B\mathcal{T}_V U + C\mathcal{T}_V U + \Phi_1 v\overset{M}{\nabla}_V U + \Phi_2 v\overset{M}{\nabla}_V U + \omega v\overset{M}{\nabla}_V U. \tag{24}$$

Since $\mathcal{T}$ is symmetric on $kerF_*$, taking horizontal parts of (23) and (24) we get

$$\mathcal{T}_U JV - \mathcal{T}_V JU = \omega\{v\overset{M}{\nabla}_U V - v\overset{M}{\nabla}_V U\}. \tag{25}$$

From equation (5) we obtain

$$-(\nabla F_*)(U, JV) + (\nabla F_*)(JU, V) = F_*(\omega v[U, V]). \tag{26}$$

The proof is clear from (26). □

Lemma 3.5. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the distribution $\mathcal{D}^\perp$ is integrable if and only if the following condition is satisfied*

$$v\overset{M}{\nabla}_{V_1}\Phi_2 V_2 - v\overset{M}{\nabla}_{V_2}\Phi_2 V_1 + \mathcal{T}_{V_2}\omega V_1 - \mathcal{T}_{V_1}\omega V_2 \in \Gamma(\mathcal{D}^\perp) \tag{27}$$

for $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$.

Proof. The real distribution $\mathcal{D}^\perp$ is integrable if and only if $g_M([V_1, V_2], U) = 0$ and $g_M([V_1, V_2], X) = 0$ for $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$, $U \in \Gamma(\mathcal{D})$ and $X \in \Gamma(kerF_*)^\perp$.Since $kerF_*$ is always integrable we have $g_M([V_1, V_2], X) = 0$. Hence, we only examine $g_M([V_1, V_2], U) = 0$. For $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$ we have

$$\begin{aligned}\overset{M}{\nabla}_{V_1} V_2 &= -B\mathcal{T}_{V_1}\Phi_2 V_2 - C\mathcal{T}_{V_1}\Phi_2 V_2 + \Phi_1 v\overset{M}{\nabla}_{V_1}\Phi_2 V_2 + \Phi_2 v\overset{M}{\nabla}_{V_1}\Phi_2 V_2 \\ &+ \omega v\overset{M}{\nabla}_{V_1}\Phi_2 V_2 - \Phi_1\mathcal{T}_{V_1}\omega V_2 - \Phi_2\mathcal{T}_{V_1}\omega V_2 - \omega\mathcal{T}_{V_1}\omega V_2 \\ &- Bh\overset{M}{\nabla}_{V_1}\omega V_2 - Ch\overset{M}{\nabla}_{V_1}\omega V_2. \end{aligned} \tag{28}$$

Interchanging the role of V_1 and V_2 in (28) we have

$$\begin{aligned}\overset{M}{\nabla}_{V_2} V_1 &= -B\mathcal{T}_{V_2}\Phi_2 V_1 - C\mathcal{T}_{V_2}\Phi_2 V_1 + \Phi_1 v\overset{M}{\nabla}_{V_2}\Phi_2 V_1 + \Phi_2 v\overset{M}{\nabla}_{V_2}\Phi_2 V_1 \\ &+ \omega v\overset{M}{\nabla}_{V_2}\Phi_2 V_1 - \Phi_1\mathcal{T}_{V_2}\omega V_1 - \Phi_2\mathcal{T}_{V_2}\omega V_1 - \omega\mathcal{T}_{V_2}\omega V_1 \\ &- Bh\overset{M}{\nabla}_{V_2}\omega V_1 - Ch\overset{M}{\nabla}_{V_2}\omega V_1. \end{aligned} \tag{29}$$

Now, using (28) and (29) we get

$$g_M([V_1, V_2], U) = g_M(\Phi_1\{v\overset{M}{\nabla}_{V_1}\Phi_2 V_2 - v\overset{M}{\nabla}_{V_2}\Phi_2 V_1 + \mathcal{T}_{V_2}\omega V_1 - \mathcal{T}_{V_1}\omega V_2\}, U). \tag{30}$$

The proof is complete from (30). □

Lemma 3.6. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the horizontal distribution $(kerF_*)^\perp$ is integrable if and only if the following condition is satisfied*

$$\begin{aligned}&\frac{1}{\lambda^2} g_N((\nabla F_*)(Y, BX) - (\nabla F_*)(X, BY) + F_*(h\overset{M}{\nabla}_X CY - h\overset{M}{\nabla}_Y CX), F_*(\omega U)) \\ &= g_M(v\overset{M}{\nabla}_Y BX - v\overset{M}{\nabla}_X BY + \mathcal{A}_Y CX - \mathcal{A}_X CY, \phi U)\end{aligned} \tag{31}$$

for $X, Y \in \Gamma((kerF_*)^\perp)$.

Proof. The horizontal distribution $(kerF_*)^\perp$ is integrable if and only if $g_M([X,Y],U) = 0$ for $X, Y \in \Gamma((kerF_*)^\perp)$ and $U \in \Gamma(kerF_*)$. From (4) we have

$$J\nabla_X Y = \mathcal{A}_X BY + v\overset{M}{\nabla}_X BY + \mathcal{A}_X CY + h\overset{M}{\nabla}_X CY. \tag{32}$$

After changing the roles of X and Y, we get

$$\begin{aligned} J[X,Y] &= \mathcal{A}_X BY - \mathcal{A}_Y BX + v\overset{M}{\nabla}_X BY - v\overset{M}{\nabla}_Y BX \\ &+ \mathcal{A}_X CY - \mathcal{A}_Y CX + h\overset{M}{\nabla}_X CY - h\overset{M}{\nabla}_Y CX. \end{aligned} \tag{33}$$

Now, from (17) we get for $U \in \Gamma(kerF_*)$

$$\begin{aligned} 0 = -g_M([X,Y],U) &= -g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX + h\overset{M}{\nabla}_X CY - h\overset{M}{\nabla}_Y CX, \omega U) \\ &- g_M(v\overset{M}{\nabla}_X BY - v\overset{M}{\nabla}_Y BX + \mathcal{A}_X CY - \mathcal{A}_Y CX, \phi U). \end{aligned} \tag{34}$$

Hence, from (2) and (5) we obtain

$$\begin{aligned} &\frac{1}{\lambda^2} g_N((\nabla F_*)(Y,BX) - (\nabla F_*)(X,BY) + F_*(h\overset{M}{\nabla}_X CY - h\overset{M}{\nabla}_Y CX), F_*(\omega U)) \\ &= g_M(v\overset{M}{\nabla}_Y BX - v\overset{M}{\nabla}_X BY + \mathcal{A}_Y CX - \mathcal{A}_X CY, \phi U). \end{aligned} \tag{35}$$

The proof is complete from (35). □

Now, we remark some useful notions.

Definition 3.7. *Let $F : M \longrightarrow N$ be a conformal Riemannian map. Then, if*

$$\mathcal{H}(grad(\ln\lambda)) = 0, \tag{36}$$

we say F is a horizontally homothetic map [3].

Definition 3.8. *Let F be a map from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then F is called a $kerF_*$-pluriharmonic map if F satisfies the following equation*

$$(\nabla F_*)(U_1, U_2) + (\nabla F_*)(JU_1, JU_2) = 0 \tag{37}$$

for $U_1, U_2 \in \Gamma(kerF_*)$ [16, 17].

Theorem 3.9. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any two conditions below imply the third condition:*

i- $C\{\mathcal{T}_{U_1}\phi_{U_2} + h\overset{M}{\nabla}_{U_1}\omega U_2\} = \mathcal{T}_{\phi U_1}\phi U_2 + \mathcal{A}_{\omega U_1}\phi U_2 + \mathcal{A}_{\omega U_2}\phi U_1$,

ii- F is a $kerF_*$-pluriharmonic map,

iii- F is a horizontally homothetic map and $(\nabla F_*)^\perp(\omega U_1, \omega U_2) = 0$

for any $U_1, U_2 \in \Gamma(kerF_*)$.

Proof. We only show the proof of (iii). The proof of (i) and (ii) are clear. From (5), (13), (14) and (37), we get

$$\begin{aligned} 0 &= F_*(\mathcal{T}_{\phi U_1}\phi U_2 + \mathcal{A}_{\omega U_1}\phi U_2 + \mathcal{A}_{\omega U_2}\phi U_1) + F_*(C\mathcal{T}_{U_1}\phi_{U_2} + Ch\overset{M}{\nabla}_{U_1}\omega U_2) \\ &+ (\nabla F_*)^{\perp}(\omega U_1, \omega U_2) + \omega U_1(\ln\lambda)F_*(\omega U_2) \\ &+ \omega U_2(\ln\lambda)F_*(\omega U_1) - g_M(\omega U_1, \omega U_2)F_*(grad(\ln\lambda)) \end{aligned} \tag{38}$$

for any $U_1, U_2 \in \Gamma(kerF_*)$. Suppose that (i) and (ii) are satisfied in (38). Then, we have $C\{\mathcal{T}_{U_1}\phi_{U_2} + h\overset{M}{\nabla}_{U_1}\omega U_2\} = \mathcal{T}_{\phi U_1}\phi U_2 + \mathcal{A}_{\omega U_1}\phi U_2 + \mathcal{A}_{\omega U_2}\phi U_1$ and F is a $kerF_*$-pluriharmonic map for any $U_1, U_2 \in \Gamma(kerF_*)$, respectively. Thus, we have

$$\begin{aligned} 0 &= (\nabla F_*)^{\perp}(\omega U_1, \omega U_2) + \omega U_1(\ln\lambda)F_*(\omega U_2) \\ &+ \omega U_2(\ln\lambda)F_*(\omega U_1) - g_M(\omega U_1, \omega U_2)F_*(grad(\ln\lambda)). \end{aligned} \tag{39}$$

It is clear from (39) that $(\nabla F_*)^{\perp}(\omega U_1, \omega U_2) = 0$. Now, we obtain from (2), (18) and (39)

$$0 = \lambda^2\omega U_2(\ln\lambda)g_M(\omega U_1, \omega U_1) \tag{40}$$

for $\omega U_1 \in \Gamma(\omega(\mathcal{D}^{\perp}))$. So, we get $\omega U_2(\ln\lambda) = 0$. It means λ is a constant on $\omega(\mathcal{D}^{\perp})$. Similarly, we obtain from (39)

$$0 = -\lambda^2 CX(\ln\lambda)g_M(\omega U_1, \omega U_2) \tag{41}$$

with $\omega U_1 = \omega U_2$ for $CX \in \Gamma(\mu)$. So, we get $CX(\ln\lambda) = 0$. It means λ is a constant on μ. Thus, F is a horizontally homothetic map from (40) and (41). The proof is complete. □

Definition 3.10. *Let F be a map from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then F is called a $(kerF_*)^{\perp}$-pluriharmonic map if F satisfies the following equation*

$$(\nabla F_*)(Z_1, Z_2) + (\nabla F_*)(JZ_1, JZ_2) = 0 \tag{42}$$

for $Z_1, Z_2 \in \Gamma((kerF_*)^{\perp})$ [16, 17].

Theorem 3.11. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any three conditions below imply the fourth condition:*

i- $\overset{N}{\nabla}{}^F_{Z_1}F_*(Z_2) = F_*(\mathcal{T}_{BZ_1}BZ_2 + \mathcal{A}_{CZ_2}BZ_1 + \mathcal{A}_{CZ_1}BZ_2)$,

ii- F is a $(kerF_*)^{\perp}$-pluriharmonic map,

iii- F is a horizontally homothetic map and $(\nabla F_*)^{\perp}(CZ_1, CZ_2) = 0$,

iv- The distribution $(kerF_*)^{\perp}$ defines a totally geodesic foliation in M

for any $Z_1, Z_2 \in \Gamma((kerF_*)^{\perp})$.

Proof. We only show the proof of (iii) and (iv). The proof of (i) and (ii) are clear. From (5), (13), (14) and (42), we get

$$\begin{aligned} F_*(\overset{M}{\nabla}_{Z_1}Z_2) &= \overset{N}{\nabla}{}^F_{Z_1}F_*(Z_2) + (\nabla F_*)^{\perp}(CZ_1, CZ_2) \\ &- F_*(\mathcal{T}_{BZ_1}BZ_2 + \mathcal{A}_{CZ_2}BZ_1 + \mathcal{A}_{CZ_1}BZ_2) \\ &+ CZ_1(\ln\lambda)F_*(CZ_2) + CZ_2(\ln\lambda)F_*(CZ_1) \\ &- g_M(CZ_1, CZ_2)F_*(grad(\ln\lambda)) \end{aligned} \tag{43}$$

for any $Z_1, Z_2 \in \Gamma((kerF_*)^\perp)$. Suppose that (i), (ii) and (iii) are satisfied in (43). Then, we have

$$\begin{aligned}
&\overset{N}{\nabla}{}^F_{Z_1} F_*(Z_2) = F_*(\mathcal{T}_{BZ_1} BZ_2 + \mathcal{A}_{CZ_2} BZ_1 + \mathcal{A}_{CZ_1} BZ_2),\\
&(\nabla F_*)(Z_1, Z_2) + (\nabla F_*)(JZ_1, JZ_2) = 0,\\
&CZ_1(\ln\lambda)F_*(CZ_2) + CZ_2(\ln\lambda)F_*(CZ_1) - g_M(CZ_1, CZ_2)F_*(grad(\ln\lambda)) = 0,\\
&(\nabla F_*)^\perp(CZ_1, CZ_2) = 0,
\end{aligned}$$

respectively. Thus, we have $F_*(\overset{M}{\nabla}_{Z_1} Z_2) = 0$ for $Z_1, Z_2 \in \Gamma((kerF_*)^\perp)$. Therefore, the distribution $(kerF_*)^\perp$ defines a totally geodesic foliation in M. Suppose that (i), (ii) and (iv) are satisfied in (43). Then, it is clear from (43) that $(\nabla F_*)^\perp(CZ_1, CZ_2) = 0$ and we obtain

$$0 = CZ_1(\ln\lambda)F_*(CZ_2) + CZ_2(\ln\lambda)F_*(CZ_1) - g_M(CZ_1, CZ_2)F_*(grad(\ln\lambda)) \tag{44}$$

for any $Z_1, Z_2 \in \Gamma((kerF_*)^\perp)$. From (2) and (44), we get

$$0 = \lambda^2 CZ_2(\ln\lambda)g_M(CZ_1, CZ_1) \tag{45}$$

for $CZ_1 \in \Gamma(\mu)$. So, we get $CZ_2(\ln\lambda) = 0$. It means λ is a constant on μ. Similarly, we obtain from (18) and (44)

$$0 = -\lambda^2 \omega U_1(\ln\lambda)g_M(CZ_1, CZ_2) \tag{46}$$

with $CZ_1 = CZ_2$ for $\omega U_1 \in \Gamma(\omega(\mathcal{D}^\perp))$. So, we get $\omega U_1(\ln\lambda) = 0$. It means λ is a constant on $\omega(\mathcal{D}^\perp)$. Thus, F is a horizontally homothetic map from (45) and (46). The proof is complete. □

Definition 3.12. *Let F be a map from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then F is called a $\mathcal{D}^\perp$-pluriharmonic map if F satisfies the following equation*

$$(\nabla F_*)(V_1, V_2) + (\nabla F_*)(JV_1, JV_2) = 0 \tag{47}$$

for $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$ [16, 17].

Theorem 3.13. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any three conditions below imply the fourth condition:*

- *i-* $\mathcal{T}_{\phi V_1}\phi V_2 + \mathcal{A}_{\omega V_2}\phi V_1 + \mathcal{A}_{\omega V_1}\phi V_2 = 0,$
- *ii- F is a $\mathcal{D}^\perp$-pluriharmonic map,*
- *iii- F is a horizontally homothetic map and $(\nabla F_*)^\perp(\omega V_1, \omega V_2) = 0$,*
- *iv- The distribution $\mathcal{D}^\perp$ defines a totally geodesic foliation in M*

for any $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$.

Proof. We only show the proof of (iii) and (iv). The proof of (i) and (ii) are clear. From (5), (13), (14) and (47), we get

$$\begin{aligned}
F_*(\overset{M}{\nabla}_{V_1} V_2) &= -F_*(\mathcal{T}_{\phi V_1}\phi V_2 + \mathcal{A}_{\omega V_2}\phi V_1 + \mathcal{A}_{\omega V_1}\phi V_2)\\
&+ \omega V_1(\ln\lambda)F_*(\omega V_2) + \omega V_2(\ln\lambda)F_*(\omega V_1)\\
&- g_M(\omega V_1, \omega V_2)F_*(grad(\ln\lambda)) + (\nabla F_*)^\perp(\omega V_1, \omega V_2)
\end{aligned} \tag{48}$$

for any $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$. Suppose that (i), (ii) and (iii) are satisfied in (48). Then, we have

$$\begin{aligned}
&\mathcal{T}_{\phi V_1}\phi V_2 + \mathcal{A}_{\omega V_2}\phi V_1 + \mathcal{A}_{\omega V_1}\phi V_2 = 0,\\
&(\nabla F_*)(V_1, V_2) + (\nabla F_*)(JV_1, JV_2) = 0,\\
&\omega V_1(\ln\lambda)F_*(\omega V_2) + \omega V_2(\ln\lambda)F_*(\omega V_1) - g_M(\omega V_1, \omega V_2)F_*(grad(\ln\lambda)) = 0,\\
&(\nabla F_*)^\perp(\omega V_1, \omega V_2) = 0,
\end{aligned}$$

respectively. Thus, we have $F_*(\overset{M}{\nabla}_{V_1}V_2) = 0$ for $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$. Therefore, the distribution $\mathcal{D}^\perp$ defines a totally geodesic foliation in M. Suppose that (i), (ii) and (iv) are satisfied in (48). Then, it is clear from (48) that $(\nabla F_*)^\perp(\omega V_1, \omega V_2) = 0$ and we obtain

$$0 = \omega V_1(\ln\lambda)F_*(\omega V_2) + \omega V_2(\ln\lambda)F_*(\omega V_1) - g_M(\omega V_1, \omega V_2)F_*(grad(\ln\lambda)) \tag{49}$$

for any $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$. From (2) and (49), we get

$$0 = \lambda^2\omega V_2(\ln\lambda)g_M(\omega V_1, \omega V_1) \tag{50}$$

for $\omega V_1 \in \Gamma(\omega(\mathcal{D}^\perp))$. So, we get $\omega V_2(\ln\lambda) = 0$. It means λ is a constant on $\omega(\mathcal{D}^\perp)$. Similarly, we obtain from (18) and (49)

$$0 = -\lambda^2 CX(\ln\lambda)g_M(\omega V_1, \omega V_2) \tag{51}$$

with $\omega V_1 = \omega V_2$ for $CX \in \Gamma(\mu)$. So, we get $CX(\ln\lambda) = 0$. It means λ is a constant on μ. Thus, F is a horizontally homothetic map from (50) and (51). The proof is complete. □

Definition 3.14. *Let F be a map from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then F is called a $\mathcal{D}$-pluriharmonic map if F satisfies the following equation*

$$(\nabla F_*)(V_1, V_2) + (\nabla F_*)(JV_1, JV_2) = 0 \tag{52}$$

for $V_1, V_2 \in \Gamma(\mathcal{D})$ [16, 17].

Theorem 3.15. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any two conditions below imply the third condition:*

i- $C\mathcal{T}_{\phi V_1}\phi^2 V_2 + \omega v\overset{M}{\nabla}_{\phi V_1}\phi^2 V_2 = 0,$

ii- F is a $\mathcal{D}$-pluriharmonic map,

iii- The distribution $\mathcal{D}$ defines a totally geodesic foliation in M

for any $V_1, V_2 \in \Gamma(\mathcal{D})$.

Proof. We only show the proof of (iii). The proof of (i) and (ii) are clear. From (5), (14), (17), (18), and (52), we get

$$F_*(\overset{M}{\nabla}_{V_1}V_2) = F_*(C\mathcal{T}_{\phi V_1}\phi^2 V_2 + \omega v\overset{M}{\nabla}_{\phi V_1}\phi^2 V_2) \tag{53}$$

for any $V_1, V_2 \in \Gamma(\mathcal{D})$. Suppose that (i) and (ii) are satisfied in (53). Then, we have

$$\begin{aligned} &C\mathcal{T}_{\phi V_1}\phi^2 V_2 + \omega v\overset{M}{\nabla}_{\phi V_1}\phi^2 V_2 = 0, \\ &(\nabla F_*)(V_1, V_2) + (\nabla F_*)(JV_1, JV_2) = 0, \end{aligned}$$

respectively. Thus, we have $F_*(\overset{M}{\nabla}_{V_1}V_2) = 0$ for $V_1, V_2 \in \Gamma(\mathcal{D})$. Therefore, the distribution $\mathcal{D}$ defines a totally geodesic foliation in M. □

Definition 3.16. *Let F be a map from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then F is called a $\{(kerF_*)^\perp - kerF_*\}$-pluriharmonic map if F satisfies the following equation*

$$(\nabla F_*)(X, V) + (\nabla F_*)(JX, JV) = 0 \tag{54}$$

for $X \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(kerF_*)$ [17].

Theorem 3.17. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N). *Then any two conditions below imply the third condition:*

i- $C\{\mathcal{A}_X\phi V + h\overset{M}{\nabla}_X\omega V\} + \omega\{\mathcal{A}_X\omega V + v\overset{M}{\nabla}_X\phi V\} = -\{\mathcal{T}_{BX}\phi V + \mathcal{A}_{\omega V}BX + \mathcal{A}_{CX}\phi V\}$,

ii- F is a $\{(kerF_*)^\perp - kerF_*\}$*-pluriharmonic map,*

iii- F is a horizontally homothetic map and $(\nabla F_*)^\perp(CX, \omega V) = 0$

for any $X \in \Gamma((kerF_*)^\perp)$ *and* $V \in \Gamma(kerF_*)$.

Proof. We only show the proof of (iii). The proof of (i) and (ii) are clear. Since second fundamental form of a map (∇F_*) is symmetric from (5), (12), (13), (14), (18) and (54), we get

$$\begin{aligned} 0 &= F_*(C\mathcal{A}_X\phi V + \omega v\overset{M}{\nabla}_X\phi V + \omega\mathcal{A}_X\omega V + Ch\overset{M}{\nabla}_X\omega V) \\ &- F_*(\mathcal{T}_{BX}\phi V + \mathcal{A}_{\omega V}BX + \mathcal{A}_{CX}\phi V) + (\nabla F_*)^\perp(CX, \omega V) \\ &+ CX(\ln\lambda)F_*(\omega V) + \omega V(\ln\lambda)F_*(CX) \end{aligned} \tag{55}$$

for any $X \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(kerF_*)$. Suppose that (i) and (ii) are satisfied in (55). Then, we have

$$\begin{aligned} &C\{\mathcal{A}_X\phi V + h\overset{M}{\nabla}_X\omega V\} + \omega\{\mathcal{A}_X\omega V + v\overset{M}{\nabla}_X\phi V\} = -\{\mathcal{T}_{BX}\phi V + \mathcal{A}_{\omega V}BX + \mathcal{A}_{CX}\phi V\}, \\ &(\nabla F_*)(X, V) + (\nabla F_*)(JX, JV) = 0, \end{aligned}$$

respectively. Then, it is clear from (55) that $(\nabla F_*)^\perp(CX, \omega V) = 0$. Thus, we have

$$0 = CX(\ln\lambda)F_*(\omega V) + \omega V(\ln\lambda)F_*(CX) \tag{56}$$

for any $X \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(kerF_*)$. From (2) and (56), we get

$$0 = \lambda^2\omega V(\ln\lambda)g_M(CX, CX) \tag{57}$$

for $CX \in \Gamma(\mu)$. So, we get $\omega V(\ln\lambda) = 0$. It means λ is a constant on $\omega(\mathcal{D}^\perp)$. Similarly, we obtain from (18) and (56)

$$0 = \lambda^2 CX(\ln\lambda)g_M(\omega V, \omega V) \tag{58}$$

for $\omega V \in \Gamma(\omega(\mathcal{D}^\perp))$. It means λ is a constant on μ. Thus, F is a horizontally homothetic map from (57) and (58). The proof is complete. □

Now, we investigate totally geodesicness of distributions in M.

Theorem 3.18. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N). *Then,* $kerF_*$ *defines a totally geodesic foliation in M if and only if*

i- $g_N((\nabla F_*)(U, V), F_*(\omega\phi Z)) - g_N((\nabla F_*)(U, \phi V), F_*(\omega Z))$
$= \lambda^2\{g_M(\hat{\nabla}_U V, \phi^2 Z) - g_M(h\overset{M}{\nabla}_U\omega V, \omega Z)\}$,

ii- $g_N((\nabla F_*)(U, V), F_*(\omega BX)) + g_N((\nabla F_*)(U, \phi V), F_*(CX))$
$= \lambda^2\{g_M(\hat{\nabla}_U V, \phi BX) + g_M(h\overset{M}{\nabla}_U\omega V, CX)\}$

are satisfied for any $U, V \in \Gamma(kerF_*)$, $X \in \Gamma(\mu)$ *and* $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. Firstly, we show (i). Since M is a Kaehlerian manifold from (17), we have

$$g_M(\overset{M}{\nabla}_U V, Z) = g_M(\overset{M}{\nabla}_U \phi V + \omega V, \phi Z + \omega Z)$$

for any $U, V \in \Gamma(kerF_*)$ and $Z \in \Gamma(\mathcal{D}^\perp)$. Then, from (2), (8) and (9) we have

$$= g_M(\overset{M}{\nabla}_U JV, \phi Z) + g_M(\mathcal{T}_U \phi V, \omega Z) + g_M(h\overset{M}{\nabla}_U \omega Z, \omega Z).$$

Since $(\nabla F_*)(U, \phi V) = -F_*(\mathcal{T}_U \phi V)$, we obtain

$$= g_M(\overset{M}{\nabla}_U JV, \phi Z) + g_M(h\overset{M}{\nabla}_U \omega V, \omega Z) - \frac{1}{\lambda^2} g_N((\nabla F_*)(U, \phi V), F_*(\omega Z)) \tag{59}$$

for any $U, V \in \Gamma(kerF_*)$. On the other hand, we have from (8)

$$\begin{aligned} g_M(\overset{M}{\nabla}_U JV, \phi Z) &= -g_M(\overset{M}{\nabla}_U V, J\phi Z) \\ &= -g_M(\mathcal{T}_U V, \omega\phi Z) - g_M(\hat{\nabla}_U V, \phi^2 Z) \\ &= \frac{1}{\lambda^2} g_N((\nabla F_*)(U, V), F_*(\omega\phi Z)) - g_M(\hat{\nabla}_U V, \phi^2 Z). \end{aligned} \tag{60}$$

Now, using (60) in (59) we get

$$\begin{aligned} 0 &= \frac{1}{\lambda^2}\{g_N((\nabla F_*)(U, V), F_*(\omega\phi Z)) - g_N((\nabla F_*)(U, \phi V), F_*(\omega Z))\} \\ &+ g_M(h\overset{M}{\nabla}_U \omega V, \omega Z) - g_M(\hat{\nabla}_U V, \phi^2 Z). \end{aligned} \tag{61}$$

Therefore, we obtain (i). Now, we show (ii). Thus, from (8), (9), (17) and (19) we get

$$\begin{aligned} g_M(\overset{M}{\nabla}_U V, X) &= g_M(\overset{M}{\nabla}_U V, JBX) + g_M(\overset{M}{\nabla}_U \phi V + \omega V, CX) \\ &= g_M(\mathcal{T}_U V, \omega BX) + g_M(\hat{\nabla}_U V, \phi BX) \\ &+ g_M(\mathcal{T}_U \phi V, CX) + g_M(h\overset{M}{\nabla}_U \omega V, CX) \\ &= -\frac{1}{\lambda^2} g_N((\nabla F_*)(U, V), F_*(\omega BX)) + g_M(\hat{\nabla}_U V, \phi BX) \\ &- \frac{1}{\lambda^2} g_N((\nabla F_*)(U, \phi V), F_*(CX)) + g_M(h\overset{M}{\nabla}_U \omega V, CX) \end{aligned} \tag{62}$$

for any $U, V \in \Gamma(kerF_*)$ and $X \in \Gamma(\mu)$. Hence, we obtain (ii) from (62). The proof is complete. □

Theorem 3.19. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, $(kerF_*)^\perp$ defines a totally geodesic foliation in M if and only if*

$$g_N((\nabla F_*)(X, BY), F_*(\omega U)) = \lambda^2\{g_M(h\overset{M}{\nabla}_X CY, \omega U) + g_M(v\overset{M}{\nabla}_X BY + \mathcal{A}_X CY, \phi U)\}$$

is satisfied for any $X, Y \in \Gamma((kerF_*)^\perp)$ and $U \in \Gamma(kerF_*)$.

Proof. From (17) and (19), we have

$$g_M(\overset{M}{\nabla}_X Y, U) = g_M(\overset{M}{\nabla}_X BY + CY, \phi U + \omega U)$$

for any $X, Y \in \Gamma((kerF_*)^\perp)$ and $U \in \Gamma(kerF_*)$. Since $(\nabla F_*)(X, BY) = -F_*(\mathcal{A}_X BY)$ we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_X Y, U) &= -\frac{1}{\lambda^2} g_N((\nabla F_*)(X, BY), F_*(\omega U)) + g_M(h\overset{M}{\nabla}_X CY, \omega U) \\ &+ g_M(v\overset{M}{\nabla}_X BY + \mathcal{A}_X CY, \phi U). \end{aligned} \tag{63}$$

We obtain the proof from (63). □

Theorem 3.20. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. Then, the distribution* $\mathcal{D}$ *defines a totally geodesic foliation in M if and only if*

i- $g_N((\nabla F_*)(U_1, \phi U_2), F_*(\omega V)) = \lambda^2 g_M(v\overset{M}{\nabla}_{U_1}\phi U_2, \phi V)$,

ii- $g_N((\nabla F_*)(U_1, \phi U_2), F_*(CX)) = \lambda^2 g_M(v\overset{M}{\nabla}_{U_1}\phi U_2, BX)$

are satisfied for any $U_1, U_2 \in \Gamma(\mathcal{D})$, $X \in \Gamma((kerF_*)^\perp)$ *and* $V \in \Gamma(\mathcal{D}^\perp)$.

Proof. From (16) and (17) we know $\omega U_2 = 0$. Then, we get

$$\begin{aligned} g_M(\overset{M}{\nabla}_{U_1}U_2, V) &= g_M(\overset{M}{\nabla}_{U_1}\phi U_2, \phi V + \omega V) \\ &= g_M(\mathcal{T}_{U_1}\phi U_2, \omega V) + g_M(v\overset{M}{\nabla}_{U_1}\phi U_2, \phi V) \end{aligned}$$

for any $U_1, U_2 \in \Gamma(\mathcal{D})$ and $V \in \Gamma(\mathcal{D}^\perp)$. Since $(\nabla F_*)(U_1, \phi U_2) = -F_*(\mathcal{T}_{U_1}\phi U_2)$, we have

$$g_M(\overset{M}{\nabla}_{U_1}U_2, V) = -\frac{1}{\lambda^2} g_N((\nabla F_*)(U_1, \phi U_2), F_*(\omega V)) + g_M(v\overset{M}{\nabla}_{U_1}\phi U_2, \phi V). \tag{64}$$

From (64) we have (i). Similarly, we get

$$\begin{aligned} g_M(\overset{M}{\nabla}_{U_1}U_2, X) &= g_M(\overset{M}{\nabla}_{U_1}\phi U_2, BX + CX) \\ &= g_M(\mathcal{T}_{U_1}\phi U_2, CX) + g_M(v\overset{M}{\nabla}_{U_1}\phi U_2, BX) \\ &= -\frac{1}{\lambda^2} g_N((\nabla F_*)(U_1, \phi U_2), F_*(CX)) + g_M(v\overset{M}{\nabla}_{U_1}\phi U_2, BX) \end{aligned} \tag{65}$$

for any $U_1, U_2 \in \Gamma(\mathcal{D})$ and $X \in \Gamma((kerF_*)^\perp)$. From (65) we have (ii). The proof is complete. □

In a similar way, we get the following theorem.

Theorem 3.21. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. Then, the distribution* $\mathcal{D}^\perp$ *defines a totally geodesic foliation in M if and only if*

i- $g_N((\nabla F_*)(V_1, \phi U), F_*(\omega V_2)) = \lambda^2 g_M(v\overset{M}{\nabla}_{V_1}\phi U, \phi V_2)$,

ii- $g_N((\nabla F_*)(V_1, BX), F_*(\omega V_2)) = \lambda^2 \{g_M(h\overset{M}{\nabla}_{V_1}CX, \omega V_2) + g_M(v\overset{M}{\nabla}_{V_1}BX + \mathcal{T}_{V_1}CX, \phi V_2)\}$

are satisfied for any $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$, $X \in \Gamma((kerF_*)^\perp)$ *and* $U \in \Gamma(\mathcal{D})$.

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An examination on $N-D^{*}$ partner curves with common principal normal and Darboux vector in $\mathbf{E}^3$.

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Abstract. In this paper, we define and examine curves with common principal normal and Darboux vectors such that the principal normal vector of first curve and the Darboux vector of second curve are linearly dependent. We name the first curve as $N-D^{*}$ curve, and the second curve as the $N-D^{*}$ partner curve. These new curves are called $N-D^{*}$ pair curves. Also we give the Frenet-Serret apparatus of the second curve based on the Frenet-Serret apparatus of first curve.

1. Introduction and Preliminaries

The evolute and involute curve are the curves whose tangent lines intersect orthogonally, hence the principal normal vector of the first curve and tangent vector of second curve are linearly dependent. So if that is the case, then first curve is named to be evolute, and the second curve is called as involute curve. For more detail see in [2], [3].

Mannheim curve examined first by A. Mannheim in 1878 is a curve if and only if $\kappa/\left(\kappa^2+\tau^2\right)$ is a nonzero constant, where κ is the curvature and τ is the torsion. Also, a new definition of these associated curves was given by Liu and Wang (2008); if the principal normal vector of the first curve and binormal vector of the second curve are linearly dependent, then the first curve is called Mannheim curve, and the second curve is called Mannheim partner curve. As a result they called these new curves as Mannheim pair. For more detail see in [4]. Bertrand pair curves are another special curves with common principal normal lines. A curve is Bertrand curve, if and only if there exist nonzero real constant numbers λ and β such that $\lambda\kappa+\beta\tau=1$. For more detail see in [5]. Before in [6, 7], we produced some other new partner curves by using similar way.

By this study, it is of interest to us to define a new curve pair such that there exist a linear dependence between the principal normal and the Darboux vectors. By doing so, we introduce a new concept such that $N-D^{*}$ partner curves and examine some of their invariants.

2. $N-D^{*}$ pair curves

Let α and α^{*} be the curves with Frenet-Serret apparatus $\{T,N,B,D,\kappa,\tau\}$ and $\{T^{*},N^{*},B^{*},D^{*},\kappa^{*},\tau^{*}\}$, where $\kappa,\kappa^{*}$and τ,τ^{*} are the curvature functions of the first and the second curve, respectively, and $D=\dfrac{\tau T+\kappa B}{\sqrt{\kappa^2+\tau^2}}$ is

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unit Darbux vector field discovered by Gaston Darboux as the areal velocity vector of the Frenet frame of a space curve. The Darboux vector field of any arclengthed curve α has symmetrical properties [1]: $D\times T = T'$; $D \times N = N'$; $D \times B = B'$. Similarly, $D^* = \dfrac{\tau^* T^* + \kappa^* B^*}{\sqrt{\kappa^{*2} + \tau^{*2}}}$ as the unit Darboux vector field of second curve α^*, in E^3.

Definition 2.1. *If the principal normal vector of first curve and unit Darboux vector D^* of second curve are linearly dependent, then first curve is called $N - D^*$ curve, and the second curve is called $N - D^*$partner curve. As a result we called these new curves as $N - D^*$ pair curves with the following equations:*

$$\alpha^*(s) = \alpha(s) + \lambda(s)D^*(s),$$

and under the condition $N = D^*$

$$\alpha^*(s) = \alpha(s) + \lambda(s)N(s).$$

Corollary 2.2. *The distance between $N - D^*$ pair curves is $|\lambda|$.*

2.1. Tangent vector field of $N - D^*$ partner curve

Theorem 2.3. *If the tangent vector field of $N - D^*$ partner curve is T^*, then it can be given based on the Frenet apparatus of the first curve as*

$$T^* = \left(\frac{1-\lambda\kappa}{\lambda'}T + N + \frac{\lambda\tau}{\lambda'}B\right)\cos\theta,$$

where $\dfrac{ds}{ds^*} = \dfrac{1}{\sqrt{\delta}}$, and $\delta = (1-\kappa\lambda)^2 + \lambda'^2 + \tau^2\lambda^2$. Also $\dfrac{\lambda'}{\sqrt{\delta}} = \cos\theta$, $\theta = \sphericalangle(T^*, D^*)$, $0 < \theta < \pi/2$.

Proof. Since $\alpha^* = \alpha + \lambda N$, and taking its derivative with respect to it's arclength parameter s^* we have

$$\begin{aligned}\frac{d\alpha^*}{ds^*} &= \frac{d(\alpha + \lambda N)}{ds}\frac{ds}{ds^*},\\ &= \left((1-\kappa\lambda)T + \lambda' N + \lambda\tau B\right)\frac{ds}{ds^*}.\end{aligned}$$

and $\left\|\dfrac{d\alpha^*}{ds}\right\| = \sqrt{(1-\lambda\kappa)^2 + \lambda'^2 + \lambda^2\tau^2} = \sqrt{\delta}$. Also α^* is an arc-lengthed curve with the s^*; $\left\langle \dfrac{d\alpha^*}{ds^*}, \dfrac{d\alpha^*}{ds^*}\right\rangle = 1$, hence

$$\frac{ds}{ds^*} = \frac{1}{\sqrt{(1-\kappa\lambda)^2 + \lambda'^2 + (\lambda\tau)^2}} = \frac{1}{\sqrt{\delta}}.$$

Now, we can write the tangent vector field as

$$T^* = \frac{(1-\kappa\lambda)T + \lambda' N + \tau\lambda B}{\sqrt{\delta}}.$$

Let $\sphericalangle(T^*, D^*) = \theta$, $0 < \theta < \pi/2$, so

$$\langle T^*, N\rangle = \langle D^*, T^*\rangle = \|T^*\|\,\|D^*\| = \cos\theta.$$

Since

$$\langle T^*, N\rangle = \left\langle \frac{(1-\kappa\lambda)T + \lambda' N + \tau\lambda B}{\sqrt{(1-\kappa\lambda)^2 + \lambda'^2 + \tau^2\lambda^2}}, N\right\rangle = \frac{\lambda'}{\sqrt{\delta}},$$

we have

$$\left\langle T^*, \tilde{D}^*\right\rangle = \|T^*\|\,\|D^*\|\cos\theta = \cos\theta.$$

So there is the relationship among the curvatures and θ as in the following way

$$\frac{\lambda'}{\cos\theta} = \sqrt{\delta}.$$

By utilizing this latter relation, we have the proof as in the following

$$T^* = \left(\frac{1-\lambda\kappa}{\lambda'}T + N + \frac{\lambda\tau}{\lambda'}B\right)\cos\theta.$$

□

Theorem 2.4. *There is the relationship among the λ, curvatures of $N - D^*$ curve and θ , based on the Frenet-Serret apparatus as in the following way*

$$\tan\theta = \frac{\sqrt{(1-\kappa\lambda)^2 + \tau^2\lambda^2}}{\lambda'}.$$

Proof. Since $\delta = \dfrac{\lambda'^2}{\cos^2\theta}$ and $\delta = (1-\kappa\lambda)^2 + \lambda'^2 + \tau^2\lambda^2$, it is trivial

$$(1-\kappa\lambda)^2 + \lambda'^2 + \tau^2\lambda^2 = \frac{\lambda'^2}{\cos^2\theta},$$

$$\lambda'^2\left(1-\sec^2\theta\right) + (1-\kappa\lambda)^2 + \tau^2\lambda^2 = 0,$$

$$\tan^2\theta = \frac{(1-\kappa\lambda)^2 + \tau^2\lambda^2}{\lambda'^2}.$$

□

Theorem 2.5. *There is the relationship among the curvatures of $N - D^*$ curve λ, and angle θ based on the Frenet-Serret apparatus as in the following way*

$$\delta' = \frac{2\left(\lambda'' - \left(\kappa^2+\tau^2\right)\lambda + \kappa\right)}{\cos\theta}.$$

Proof. Since $\langle N^*, T^*\rangle = 0$, $\langle N^*, B^*\rangle = 0$, the principal normal vector N^* of the second $N - D^*$ partner curve is perpendicular to its Darboux vector; $\langle N^*, D^*\rangle = 0$, Note that we also have $N = D^*$. Hence for the principal normal vector fields N we have that $\langle N, N^*\rangle = 0$. Since $\dfrac{dT^*}{ds^*} = \kappa^* N^*$ we have

$$\frac{dT^*}{ds} = \frac{1}{\delta}\left([(1-\kappa\lambda)T + \lambda' N + \tau\lambda B]'\ \sqrt{\delta} - [(1-\kappa\lambda)T + \lambda' N + \tau\lambda B]\ \sqrt{\delta}'\right).$$

Since $\dfrac{1}{\kappa^*} \neq 0$, $\dfrac{ds}{ds^*} \neq 0$, $\dfrac{1}{\delta} \neq 0$, then $\left\langle N, \dfrac{dT^*}{ds}\right\rangle = 0$, hence

$$\left\langle N, \frac{((1-\kappa\lambda)T + \lambda' N + \tau\lambda B')'\ \sqrt{\delta}}{\delta}\right\rangle - \left\langle N, \frac{((1-\kappa\lambda)T + \lambda' N + \tau\lambda B')}{\delta}\ \sqrt{\delta}'\right\rangle = 0,$$

$$\left\langle N, (1-\kappa\lambda)' T + (1-\kappa\lambda) T' + \lambda'' N + \lambda' N' + (\tau\lambda)' B + \tau\lambda B'\right\rangle - \lambda'\ \sqrt{\delta}' = 0.$$

$$\left\langle N, (1-\kappa\lambda)\,\kappa N + \lambda'' N - \lambda\tau^2 N\right\rangle - \lambda' \sqrt{\delta}' = 0,$$

$$\lambda'' - \left(\kappa^2 + \tau^2\right)\lambda + \kappa = \frac{\cos\theta}{2}\delta',$$

$$\frac{\lambda'}{\sqrt{\delta}} = \cos\theta,$$

$$\frac{2\left(\lambda'' - \left(\kappa^2 + \tau^2\right)\lambda + \kappa\right)}{\lambda'} = \frac{\delta'}{\sqrt{\delta}}.$$

□

2.2. *First curvature of $N - D^*$ partner curve*

Theorem 2.6. *If the first curvature of $N - D^*$ partner curve is κ^*, then it can be given based on the Frenet apparatus of the first curve as in the following way*

$$\kappa^* = \frac{\cos\theta}{2\lambda'}\left(\begin{array}{c} \left((\kappa'\lambda + 2\lambda'\kappa) + (1-\kappa\lambda)\left(\frac{\lambda'}{\cos\theta}\right)'\right)^2 + \left(\left(\kappa - \lambda\left(\kappa^2+\tau^2\right) + \lambda''\right) - \lambda'\left(\frac{\lambda'}{\cos\theta}\right)'\right)^2 \\ + \left((2\tau'\lambda + \tau\lambda') - \lambda\tau\left(\frac{\lambda'}{\cos\theta}\right)'\right)^2 \end{array}\right)^{\frac{1}{2}}$$

Proof. Since $\kappa^* N^* = \dfrac{dT^*}{ds}\dfrac{ds}{ds^*}$ and $\dfrac{ds}{ds^*} = \dfrac{1}{\sqrt{\delta}}$ it can be calculated as

$$\kappa^* N^* = \frac{1}{\sqrt{\delta^3}}\left[\begin{array}{c} \left[((1-\kappa\lambda)' - \lambda'\kappa)\ \sqrt{\delta} - (1-\kappa\lambda)\ \sqrt{\delta}'\right]T \\ + \left[\left((1-\kappa\lambda)\,\kappa + \lambda'' - \lambda\tau^2\right)\sqrt{\delta} - \lambda'\sqrt{\delta}'\right]N \\ + \left[(\lambda'\tau + (\tau\lambda')')\ \sqrt{\delta} - \tau\lambda\sqrt{\delta}'\right]B \end{array}\right].$$

Also $\kappa^{*2} = \langle \kappa^* N^*, \kappa^* N^* \rangle$, so we have

$$\kappa^* = \frac{1}{\sqrt{\delta^3}}\left(\begin{array}{c} \left((\kappa'\lambda + 2\lambda'\kappa)\ \sqrt{\delta} + (1-\kappa\lambda)\ \sqrt{\delta}'\right)^2 + \left(\left(\kappa - \lambda\left(\kappa^2+\tau^2\right) + \lambda''\right)\sqrt{\delta} - \lambda'\sqrt{\delta}'\right)^2 \\ + \left((2\tau'\lambda + \tau\lambda')\ \sqrt{\delta} - \lambda\tau\sqrt{\delta}'\right)^2 \end{array}\right)^{\frac{1}{2}}.$$

For $N - D^*$ *partner* curve, under the condition $\dfrac{\lambda'}{\cos\theta} = \sqrt{\delta}$ and $2\left(\dfrac{\lambda'}{\cos\theta}\right)'\dfrac{\lambda'}{\cos\theta} = \delta'$ we have the proof. □

2.3. *Normal vector field of $N - D^*$ partner curve*

Theorem 2.7. *If the normal vector field of $N - D^*$ partner curve is N^*, then it can be given based on the Frenet apparatus of the first curve as*

$$N^* = \frac{1}{\nabla}\left(\begin{array}{c} \left((\kappa'\lambda + 2\lambda'\kappa)\ \sqrt{\delta} + (1-\kappa\lambda)\ \sqrt{\delta}'\right)T + \left(\left(\kappa - \lambda\left(\kappa^2+\tau^2\right) + \lambda''\right)\sqrt{\delta} - \lambda'\sqrt{\delta}'\right)N \\ + \left((2\tau'\lambda + \tau\lambda')\ \sqrt{\delta} - \lambda\tau\sqrt{\delta}'\right)B \end{array}\right),$$

where

$$\nabla=\left(\begin{array}{c}\left((1-\kappa\lambda)'\sqrt{\delta}-\lambda'\kappa-(1-\kappa\lambda)\sqrt{\delta'}\right)^2+\left(\left(\kappa-\lambda\left(\kappa^2+\tau^2\right)+\lambda'\right)\sqrt{\delta}-\lambda'\sqrt{\delta'}\right)^2\\+\left((\lambda'\tau+(\tau\lambda)')\sqrt{\delta}-\tau\lambda\sqrt{\delta'}\right)^2\end{array}\right)^{\frac{1}{2}}.$$

Proof. Since $\kappa^*N^*=\frac{dT^*}{ds}\frac{ds}{ds^*}$, we have the general form as following:

$$\kappa^*N^*=\frac{1}{\sqrt{\delta^3}}\left(\begin{array}{c}\left((\kappa'\lambda+2\lambda'\kappa)\sqrt{\delta}+(1-\kappa\lambda)\sqrt{\delta'}\right)T+\left(\left(\kappa-\lambda\left(\kappa^2+\tau^2\right)+\lambda''\right)\sqrt{\delta}-\lambda'\sqrt{\delta'}\right)N\\+\left((2\tau'\lambda+\tau\lambda')\sqrt{\delta}-\lambda\tau\sqrt{\delta'}\right)B\end{array}\right). \tag{1}$$

Under the condition that $\frac{1}{\sqrt{\delta}}=\frac{\cos\theta}{\lambda'}$, we have

$$N^*=\frac{1}{2\nabla\sqrt{\delta}}\left[\begin{array}{c}((\kappa'\lambda+2\lambda'\kappa)2\delta+(1-\kappa\lambda)\delta')T+\left(\left(\kappa-\lambda\left(\kappa^2+\tau^2\right)+\lambda''\right)2\delta-\lambda'\delta'\right)N\\+((2\tau'\lambda+\tau\lambda')2\delta-\lambda\tau\delta')B\end{array}\right],$$

where

$$\nabla=\left(\begin{array}{c}\left((\kappa'\lambda+2\lambda'\kappa)\sqrt{\delta}+(1-\kappa\lambda)\sqrt{\delta'}\right)^2+\left(\left(\kappa-\lambda\left(\kappa^2+\tau^2\right)+\lambda''\right)\sqrt{\delta}-\lambda'\sqrt{\delta'}\right)^2\\+\left((2\tau'\lambda+\tau\lambda')\sqrt{\delta}-\lambda\tau\sqrt{\delta'}\right)^2\end{array}\right)^{\frac{1}{2}},$$

which completes the proof. □

2.4. Binormal vector field of $N-D^*$ partner curve

Theorem 2.8. *If the binormal vector field of $N-D^*$ partner curve is B^*, then it can be given based on the Frenet apparatus of the first curve as*

$$B^*=\frac{1}{\nabla}\left(\begin{array}{c}\left(\lambda^2\tau^3+\lambda'(\lambda\tau)'+\tau(\lambda')^2+\kappa^2\lambda^2\tau-\kappa\lambda\tau-\lambda\tau\lambda''\right)T\\+\left(-\left((\lambda\tau)'+\tau\lambda'-\lambda\tau(1-\kappa\lambda)'-\kappa\lambda(\lambda\tau)'\right)\right)N\\+\left(\kappa+\lambda''-\lambda'(1-\kappa\lambda)'+\kappa^3\lambda^2-2\kappa^2\lambda-\lambda\tau^2+\kappa(\lambda')^2+\kappa\lambda^2\tau^2-\kappa\lambda\lambda''\right)B\end{array}\right).$$

Proof. It is clear that $B^*=T^*\Lambda N^*$, hence

$$B^*=\frac{1}{\nabla\sqrt{\delta}}\left(\begin{array}{c}\left(\sqrt{\delta}\left(\lambda^2\tau^3+\lambda'(\lambda\tau)'+\tau(\lambda')^2+\kappa^2\lambda^2\tau-\kappa\lambda\tau-\lambda\tau\lambda''\right)\right)T\\+\left(-\sqrt{\delta}\left((\lambda\tau)'+\tau\lambda'-\lambda\tau(1-\kappa\lambda)'-\kappa\lambda(\lambda\tau)'\right)\right)N\\+\left(\sqrt{\delta}\left(\kappa+\lambda''-\lambda'(1-\kappa\lambda)'+\kappa^3\lambda^2-2\kappa^2\lambda-\lambda\tau^2+\kappa(\lambda')^2+\kappa\lambda^2\tau^2-\kappa\lambda\lambda''\right)\right)B\end{array}\right)$$

$$=\frac{1}{\nabla\sqrt{\delta}}\left(\begin{array}{c}\left(\sqrt{\delta}\left(\lambda^2\tau^3+\lambda'(\lambda\tau)'+\tau(\lambda')^2+\kappa^2\lambda^2\tau-\kappa\lambda\tau-\lambda\tau\lambda''\right)\right)T\\+\left(-\sqrt{\delta}\left((\lambda\tau)'+\tau\lambda'-\lambda\tau(1-\kappa\lambda)'-\kappa\lambda(\lambda\tau)'\right)\right)N\\+\left(\sqrt{\delta}\left(\kappa+\lambda''-\lambda'(1-\kappa\lambda)'+\kappa^3\lambda^2-2\kappa^2\lambda-\lambda\tau^2+\kappa(\lambda')^2+\kappa\lambda^2\tau^2-\kappa\lambda\lambda''\right)\right)B\end{array}\right).$$

□

Corollary 2.9. *There is the relationship among the curvatures of* $N-D^*$ *curve* λ*,and angle* θ *based on the Frenet-Serret apparatus as in the following way*

$$\nabla\theta'\cos\theta = \kappa\tau + \tau\lambda'' - \lambda\tau^3 - \kappa^2\lambda\tau - \kappa\lambda\lambda'\tau' + \lambda\tau\kappa'\lambda'.$$

Proof. We know that $\langle N, B^*\rangle = \langle D^*, B^*\rangle = \|B^*\|\,\|D^*\|\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$, and derivation of both sides give us

$$\begin{aligned}
\langle N', B^*\rangle + \left\langle N, B^{*'}\right\rangle &= \theta'\cos\theta,\\
\langle -\kappa T + \tau B, B^*\rangle + \left\langle N, -\tau^* N^* \frac{ds^*}{ds}\right\rangle &= \theta'\cos\theta,\\
-\kappa\langle T, B^*\rangle + \tau\langle B, B^*\rangle &= \theta'\cos\theta.
\end{aligned}$$

As a result we have

$$\begin{aligned}
\nabla\sqrt{\delta}\theta'\cos\theta = &-\kappa\left(\sqrt{\delta}\left(\lambda^2\tau^3 + \lambda'(\lambda\tau)' + \tau(\lambda')^2 + \kappa^2\lambda^2\tau - \kappa\lambda\tau - \lambda\tau\lambda''\right)\right)\\
&+ \tau\left(\sqrt{\delta}\left(\kappa + \lambda'' - \lambda'(1-\kappa\lambda)' + \kappa^3\lambda^2 - 2\kappa^2\lambda - \lambda\tau^2 + \kappa(\lambda')^2 + \kappa\lambda^2\tau^2 - \kappa\lambda\lambda''\right)\right).
\end{aligned}$$

Hence the result of these products completes the proof with the equality

$$-\sqrt{\delta}\left(\lambda\tau^3 - \tau\lambda'' - \kappa\tau + \kappa^2\lambda\tau + \tau\lambda'(1-\kappa\lambda)' + \kappa\lambda'(\lambda\tau)'\right) = \nabla\sqrt{\delta}\theta'\cos\theta,$$

$$\nabla\theta'\cos\theta = -\left(\lambda\tau^3 - \tau\lambda'' - \kappa\tau + \kappa^2\lambda\tau + \tau\lambda'(1-\kappa\lambda)' + \kappa\lambda'(\lambda\tau)'\right).$$

□

2.5. Second curvature of $N - D^*$ *partner curve*

Theorem 2.10. *If the second curvature of* $N - D^*$ *partner curve is* τ^**, then it can be given based on the Frenet apparatus of the first curve as*

$$\tau^* = \frac{\nabla^2\lambda'\cos\theta + \left(2\kappa^2\lambda' - \tau^2\lambda' + \kappa\lambda\kappa' - 2\lambda\tau\tau'\right)\lambda'\sqrt{\delta^3} + \left(\kappa - \kappa^2\lambda + \lambda\tau^2\right)\cos\theta\sqrt{\delta^3}\sqrt{\delta}'}{\sqrt{\delta^3}\left(2\lambda'\tau + \lambda\tau' - \lambda^2\kappa\tau'\right)\lambda'}.$$

Proof. Since the definition of $N - D^*$ partner curve we know that $\langle N^*, N\rangle = 0$. So derivation of both sides give us

$$\left\langle \frac{dN^*}{ds}, N\right\rangle + \langle N^*, N'\rangle = 0,$$

$$\left\langle \left(-\kappa^* T^* + \tau^* B^*\right)\frac{ds^*}{ds}, N\right\rangle + \langle N^*, N'\rangle = 0.$$

As a result we have

$$\tau^*\langle B^*, N\rangle = \kappa^*\langle T^*, N\rangle - \frac{1}{\sqrt{\delta}}\langle N^*, -\kappa T + \tau B\rangle.$$

Hence the result of these products completes the proof with the equality

$$\tau^* = \frac{\kappa^*\nabla\lambda'\cos\theta + \left(2\kappa^2\lambda' - \tau^2\lambda' + \kappa\lambda\kappa' - 2\lambda\tau\tau'\right)\lambda' + \left(\kappa - \kappa^2\lambda + \lambda\tau^2\right)\cos\theta\sqrt{\delta}'}{\left(2\lambda'\tau + \lambda\tau' - \lambda^2\kappa\tau'\right)\lambda'}.$$

□

3. Conclusion

In this study the principal normal vector of the curve α and unit Darboux vector of second curve β have been taken linearly dependent, then we get a new partner curve wich has been called $N - D^*$ curve as a way of generate the new curves. Also Frenet-Serret apparatus of $N - D^*$ curve have been given based on the Frenet-Serret apparatus of first curve α. In a similar way, using alternative frame vectors new associated curves can be defined. Further, Frenet-Serret apparatus of these curves can be given based on the Frenet-Serret apparatus of first curve α.

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Inequalities for strongly convex functions via Atangana-Baleanu Integral Operators

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Abstract. In this study, new results are generated for strongly convex functions with the help of Atangana-Baleanu integral operators.

1. Introduction

We will start by remembering the definitions of convex and strongly convex functions respectively.
Let I be an interval in $\mathbb{R}$. Then $\rho : I \to \mathbb{R}$ is said to be convex if for all $n_1, n_2 \in I$ and all $t \in [0,1]$,

$$\rho(tn_1 + (1-t)n_2) \leq t\rho(n_1) + (1-t)\rho(n_2) \tag{1}$$

holds. If the inequality in (1) is reversed, then ρ is said to be concave (See [36]).
Recall that a function $\rho : I \subset \mathbb{R} \to \mathbb{R}$ is called strongly convex with modulus $c > 0$ if

$$\rho(tn_1 + (1-t)n_2) \leq t\rho(n_1) + (1-t)\rho(n_2) - ct(1-t)(n_1 - n_2)^2, \tag{2}$$

for all $n_1, n_2 \in I$ and $t \in [0,1]$ (See [37]).

The following inequality is the Hermite-Hadamard inequality that has an important place for convex functions.

Theorem 1.1. *Suppose that $\rho : I \subset \mathbb{R} \to \mathbb{R}$ is convex function on $I \subseteq \mathbb{R}$ where $n_1, n_2 \in I$, with $n_1 < n_2$. The following double inequality is called Hermite-Hadamard's inequality for convex functions (See [36]):*

$$\rho\left(\frac{n_1+n_2}{2}\right) \leq \frac{1}{n_2-n_1}\int_{n_1}^{n_2} \rho(n)\,dn \leq \frac{\rho(n_1)+\rho(n_2)}{2}. \tag{3}$$

In [27], Merentes and Nikodem obtained the following inequality. This inequality is important as being a counterpart of the Hermite-Hadamard inequality for strongly convex functions.

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Theorem 1.2. *If a function $\rho : I \to \mathbb{R}$ is strongly convex with modulus c then*

$$\rho\left(\frac{n_1+n_2}{2}\right)+\frac{c}{12}(n_1-n_2)^2 \le \frac{1}{n_2-n_1}\int_{n_1}^{n_2}\rho(n)\,dn \le \frac{\rho(n_1)+\rho(n_2)}{2}-\frac{c}{6}(n_1-n_2)^2, \tag{4}$$

for all $n_1, n_2 \in I$, $n_1 < n_2$.

In [30], Ostrowski proved Ostrowski's inequality which is another important inequality in the theory of inequalities as the following:

Theorem 1.3. *Let ρ be a differentiable function on (n_1, n_2) and let, on (n_1, n_2), $\left|\rho'(n)\right| \le K$. Then, for every $n \in (n_1, n_2)$*

$$\left|\rho(n)-\frac{1}{n_2-n_1}\int_{n_1}^{n_2}\rho(t)dt\right| \le \left[\frac{1}{4}+\frac{\left(n-\frac{n_1+n_2}{2}\right)^2}{(n_2-n_1)^2}\right](n_2-n_1)\,K. \tag{5}$$

We recommend to see the studies [11], [12], [14], [16], [17], [21]-[29], [32]-[36] and [41]-[43] for results that include convex functions, strongly convex functions, Hermite-Hadamard and Ostrowski inequalities.

Now, we will give some of the derivative and integral operators.

Definition 1.4. *(See [15]) Let $\rho \in H^1(0, n_2)$, $n_2 > n_1$, $\xi \in [0,1]$ then, the definition of the new Caputo fractional derivative is:*

$${}^{CF}D^{\xi}\rho(t)=\frac{M(\xi)}{1-\xi}\int_{n_1}^{t}\rho'(s)exp\left[-\frac{\xi}{(1-\xi)}(t-s)\right]ds \tag{6}$$

where $M(\xi)$ is normalization function.

The integral operator associated to this fractional derivative has been given with a non-singular kernel structure as follows.

Definition 1.5. *(See [2]) Let $\rho \in H^1(0, n_2)$, $n_2 > n_1$, $\xi \in [0,1]$ then, the definition of the left and right side of Caputo-Fabrizio fractional integral is:*

$$\left({}^{CF}_{n_1}I^{\xi}\rho\right)(t)=\frac{1-\xi}{B(\xi)}\rho(t)+\frac{\xi}{B(\xi)}\int_{n_1}^{t}\rho(y)dy,$$

and

$$\left({}^{CF}I^{\xi}_{n_2}\rho\right)(t)=\frac{1-\xi}{B(\xi)}\rho(t)+\frac{\xi}{B(\xi)}\int_{t}^{n_2}\rho(y)dy$$

where $B(\xi)$ is normalization function.

Atangana and Baleanu have defined the following fractional derivative and integral operators.

Definition 1.6. *(See [7]) Let $\rho \in H^1(n_1, n_2)$, $n_2 > n_1$, $\xi \in [0,1]$ then, the definition of the new fractional derivative is given:*

$${}^{ABC}_{n_1}D^{\xi}_{t}\left[\rho(t)\right]=\frac{B(\xi)}{1-\xi}\int_{n_1}^{t}\rho'(x)E_{\xi}\left[-\xi\frac{(t-x)^{\xi}}{(1-\xi)}\right]dx. \tag{7}$$

Definition 1.7. *(See [7]) Let $\rho \in H^1(n_1, n_2)$, $n_2 > n_1$, $\xi \in [0,1]$ then, the definition of the new fractional derivative is given:*

$${}^{ABR}_{n_1}D^{\xi}_{t}\left[\rho(t)\right]=\frac{B(\xi)}{1-\xi}\frac{d}{dt}\int_{n_1}^{t}\rho(x)E_{\xi}\left[-\xi\frac{(t-x)^{\xi}}{(1-\xi)}\right]dx. \tag{8}$$

The associated integral operator is presented as follows.

Definition 1.8. *(See [7]) The fractional integral associate to the new fractional derivative with non-local kernel of a function $\rho \in H^1(n_1, n_2)$ as defined:*

$$ {}^{AB}_{n_1}I^{\xi}_{t}\left\{\rho(t)\right\} = \frac{1-\xi}{B(\xi)}\rho(t) + \frac{\xi}{B(\xi)\Gamma(\xi)}\int_{n_1}^{t}\rho(u)(t-u)^{\xi-1}du $$

where $n_2 > n_1, \xi \in [0,1]$.

In [1], the authors have given the right hand side of integral operator as following;

$$ \left({}^{AB}I^{\xi}_{n_2}\right)\left\{\rho(t)\right\} = \frac{1-\xi}{B(\xi)}\rho(t) + \frac{\xi}{B(\xi)\Gamma(\xi)}\int_{t}^{n_2}\rho(u)(u-t)^{\xi-1}du. $$

Here, $\Gamma(\xi)$ is the Gamma function. Since the normalization function $B(\xi) > 0$ is positive, it immediately follows that the fractional Atangana-Baleanu integral of a positive function is positive. It should be noted that, when the order $\xi \longrightarrow 1$, we recover the classical integral. Also, the initial function is recovered whenever the fractional order $\xi \longrightarrow 0$.

We recommend to see the studies [2]-[6], [8], [9], [13], [18]-[20], [31] and [38]-[40] for results that include fractional operators.

In this study, we obtained Hermite-Hadamard and Ostrowski type inequalities for strongly convex functions with the help of Atangana-Baleanu integral operators. In addition, we obtained inequalities on the product of convex and strongly convex functions and the product of two strongly convex functions via Atangana-Baleanu integral operators.

2. New results for strongly convex functions

Theorem 2.1. *Let $\rho : I \to \mathbb{R}$ be a strongly convex function with modulus c $(c > 0)$. If $\rho \in L[n_1, n_2]$, for all $n_1, n_2 \in I$, $n_1 < n_2$ following inequality which involves Atangana-Baleanu integral operators holds*

$$ \begin{aligned} & \frac{1}{(n_2-n_1)^{\xi}}\left({}^{AB}_{n_1}I^{\xi}_{n_2}\left\{\rho(n_2)\right\}\right) \\ \leq\ & \frac{\xi}{B(\xi)\Gamma(\xi)}\left[\frac{\rho(n_1)}{\xi+1} + \frac{\rho(n_2)}{\xi(\xi+1)} - \frac{c(n_2-n_1)^2}{(\xi+1)(\xi+2)}\right] + \frac{1-\xi}{(n_2-n_1)^{\xi}B(\xi)}\rho(n_2) \end{aligned} \tag{9} $$

where $\xi \in (0,1]$, $B(\xi)$ and $\Gamma(\xi)$ are normalization function and Euler gamma function respectively.

Proof. Since ρ is strongly convex function, we can write

$$ \rho(tn_1 + (1-t)n_2) \leq t\rho(n_1) + (1-t)\rho(n_2) - ct(1-t)(n_2-n_1)^2 \tag{10} $$

for all $n_1, n_2 \in I$, $n_1 < n_2$ and $t \in [0,1]$. If we multiply the both sides of (10) with $t^{\xi-1}$, and after that if we integrate the resulting inequality on $[0,1]$ over t, we obtain

$$ \begin{aligned} & \int_0^1 t^{\xi-1}\rho(tn_1 + (1-t)n_2)dt \\ \leq\ & \int_0^1 t^{\xi-1}\left[t\rho(n_1) + (1-t)\rho(n_2) - ct(1-t)(n_2-n_1)^2\right]dt \\ =\ & \frac{\rho(n_1)}{\xi+1} + \frac{\rho(n_2)}{\xi(\xi+1)} - \frac{c(n_2-n_1)^2}{(\xi+1)(\xi+2)}. \end{aligned} \tag{11} $$

By changing the variable $tn_1 + (1-t)n_2 = u$, we can write the inequality in (11) as

$$\begin{aligned} & \frac{1}{(n_2-n_1)^{\xi}} \int_{n_1}^{n_2} (n_2-u)^{\xi-1} \rho(u)du \\ \leq\ & \frac{\rho(n_1)}{\xi+1} + \frac{\rho(n_2)}{\xi(\xi+1)} - \frac{c(n_2-n_1)^2}{(\xi+1)(\xi+2)}. \end{aligned} \tag{12}$$

If we multiply the both sides of (12) by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and if we add the term $\frac{1-\xi}{(n_2-n_1)^{\xi}B(\xi)}\rho(n_2)$ to the both sides of (12), we get the inequality in (9). □

Remark 2.2. *If we choose $\xi = 1$ in Theorem 2.1, we obtain the right hand side of (4).*

Theorem 2.3. *Suppose that $\rho, \sigma : I \subset \mathbb{R} \to [0,\infty)$ are convex and strongly convex (with modulus c, $c > 0$) functions on I respectively where $n_1, n_2 \in I$, $n_1 < n_2$. If $\rho\sigma \in L[n_1, n_2]$, we have the following inequality*

$$\begin{aligned} & \frac{1}{(n_2-n_1)^{\xi}} \left({}^{AB}_{n_1}I^{\xi}_{n_2} \{\rho\sigma(n_2)\} \right) \\ \leq\ & \frac{\xi}{B(\xi)\Gamma(\xi)} \left\{ \frac{1}{\xi+2} \left[\rho(n_1)\sigma(n_1) + \frac{2}{\xi(\xi+1)} \rho(n_2)\sigma(n_2) \right] \right. \\ & + \frac{1}{(\xi+1)(\xi+2)} \left[\rho(n_1)\sigma(n_2) + \rho(n_2)\sigma(n_1) \right] \\ & \left. -c(n_2-n_1)^2 \frac{1}{(\xi+2)(\xi+3)} \left[\rho(n_1) + \frac{2\rho(n_2)}{\xi+1} \right] \right\} + \frac{1-\xi}{(n_2-n_1)^{\xi}B(\xi)} \rho\sigma(n_2) \end{aligned} \tag{13}$$

where $\xi \in (0,1]$, $B(\xi)$ and $\Gamma(\xi)$ are normalization function and Euler gamma function respectively.

Proof. If we consider the definitions of convex function and strongly convex function we can write

$$\rho(tn_1 + (1-t)n_2) \leq t\rho(n_1) + (1-t)\rho(n_2) \tag{14}$$

and

$$\sigma(tn_1 + (1-t)n_2) \leq t\sigma(n_1) + (1-t)\sigma(n_2) - ct(1-t)(n_2-n_1)^2 \tag{15}$$

for all $n_1, n_2 \in I$ and $t \in [0,1]$. If we multiply the inequalities in (14) and (15) side by side, we obtain

$$\begin{aligned} & \rho(tn_1 + (1-t)n_2)\sigma(tn_1 + (1-t)n_2) \\ \leq\ & t^2\rho(n_1)\sigma(n_1) + t(1-t)\rho(n_1)\sigma(n_2) - ct^2(1-t)(n_2-n_1)^2\rho(n_1) \\ & +t(1-t)\sigma(n_1)\rho(n_2) + (1-t)^2\rho(n_2)\sigma(n_2) - ct(1-t)^2(n_2-n_1)^2\rho(n_2). \end{aligned} \tag{16}$$

Similar to the steps in the proof of the Theorem 2.1, if we multiply both sides of (16) by $t^{\xi-1}$, and after that if we integrate the resulting inequality on $[0,1]$ over t, we obtain

$$\begin{aligned} & \int_0^1 t^{\xi-1} \rho(tn_1 + (1-t)n_2)\sigma(tn_1 + (1-t)n_2)dt \\ \leq\ & \frac{1}{\xi+2} \left[\rho(n_1)\sigma(n_1) + \frac{2}{\xi(\xi+1)} \rho(n_2)\sigma(n_2) \right] \\ & + \frac{1}{(\xi+1)(\xi+2)} \left[\rho(n_1)\sigma(n_2) + \rho(n_2)\sigma(n_1) \right] \\ & -c(n_2-n_1)^2 \frac{1}{(\xi+2)(\xi+3)} \left[\rho(n_1) + \frac{2\rho(n_2)}{\xi+1} \right]. \end{aligned} \tag{17}$$

If we change the variable for the left hand side of inequality in (17), and after this operation if we multiply the both sides of resulting inequality with $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and if we add the term $\frac{1-\xi}{(n_2-n_1)^{\xi}B(\xi)}\rho\sigma(n_2)$, we get the inequality in (13). □

Remark 2.4. *If we choose $\xi = 1$ in Theorem 2.3, we obtain the inequality in Theorem 2.14 in [41].*

Theorem 2.5. *Suppose that $\rho, \sigma : I \subset \mathbb{R} \to [0, \infty)$ are strongly convex functions with modulus c ($c > 0$) on I respectively where $n_1, n_2 \in I$, $n_1 < n_2$. If $\rho\sigma \in L[n_1, n_2]$, we have the following inequality*

$$
\begin{aligned}
&\frac{1}{(n_2-n_1)^{\xi}}\left({}^{AB}_{n_1}I^{\xi}_{n_2}\{\rho\sigma(n_2)\}\right) \\
\leq\ & \frac{\xi}{B(\xi)\Gamma(\xi)}\left\{\frac{1}{\xi+2}\left[\rho(n_1)\sigma(n_1)+\frac{2}{\xi(\xi+1)}\rho(n_2)\sigma(n_2)\right]\right. \\
&+\frac{1}{(\xi+1)(\xi+2)}\left[\rho(n_1)\sigma(n_2)+\rho(n_2)\sigma(n_1)\right] \\
&\left.-\frac{c(n_2-n_1)^2}{(\xi+2)(\xi+3)}\left[\rho(n_1)+\frac{2\rho(n_2)}{\xi+1}+\sigma(n_1)+\frac{2\sigma(n_2)}{\xi+1}\right]+c^2(n_2-n_1)^4\frac{2}{(\xi+2)(\xi+3)(\xi+4)}\right\} \\
&+\frac{1-\xi}{(n_2-n_1)^{\xi}B(\xi)}\rho\sigma(n_2)
\end{aligned}
\tag{18}
$$

where $\xi \in (0, 1]$, $B(\xi)$ and $\Gamma(\xi)$ are normalization function and Euler gamma function respectively.

Proof. Theorem 2.5 can be proved similar to the proof of Theorem 2.3. It is left to the interested reader. □

Remark 2.6. *If we choose $\xi = 1$ in Theorem 2.5, we obtain the inequality in Theorem 2.11 in [41].*

Now, we will give some results by using the following lemma which is the first lemma of Ostrowski type that includes Atangana-Baleanu operator.

Lemma 2.7. *(See [10]) Let $n_1 < n_2$, $n_1, n_2 \in I^{\circ}$ and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I°. If $\rho' \in L[n_1, n_2]$, the following identity for Atangana-Baleanu integral operators is valid for all $n \in [n_1, n_2]$, $\xi \in (0, 1]$ and $t \in [0, 1]$:*

$$
\begin{aligned}
&\frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right] \\
&-\frac{1}{(n_2-n_1)}\left[{}^{AB}I^{\xi}_{n}\{\rho(n_1)\}+{}^{AB}_{n}I^{\xi}_{n_2}\{\rho(n_2)\}\right] \\
&+\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right] \\
=\ & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\int_0^1 t^{\xi}\rho'(tn+(1-t)n_1)dt \\
&-\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\int_0^1 t^{\xi}\rho'(tn+(1-t)n_2)dt.
\end{aligned}
\tag{19}
$$

Here $B(\xi) > 0$ and $\Gamma(\xi)$ are normalization function and Euler gamma function respectively.

By using this lemma, let us arrange results for first order differentiable strongly convex functions.

Theorem 2.8. *Let $n_1 < n_2$, $n_1, n_2 \in I^{\circ}$ and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\rho' \in L[n_1, n_2]$. If $|\rho'|$ is strongly convex function with modulus $c > 0$ on $[n_1, n_2]$, $|\rho'| \leq M$ and $\frac{M}{\xi+1} \geq \max\left\{\frac{c(n-n_1)^2}{(\xi+2)(\xi+3)}, \frac{c(n_2-n)^2}{(\xi+2)(\xi+3)}\right\}$, for* *all $n \in [n_1, n_2]$, $\xi \in (0,1]$ we obtain the inequality below:*

$$\begin{aligned}
&\left|\frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right]\right. \\
&-\frac{1}{(n_2-n_1)}\left[{}^{AB}I_n^{\xi}\{\rho(n_1)\}+{}^{AB}_{n}I_{n_2}^{\xi}\{\rho(n_2)\}\right] \\
&\left.+\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right]\right| \\
\leq\; & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{M}{\xi+1}-\frac{c(n-n_1)^2}{(\xi+2)(\xi+3)}\right) \\
&+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{M}{\xi+1}-\frac{c(n_2-n)^2}{(\xi+2)(\xi+3)}\right).
\end{aligned} \tag{20}$$

Here $B(\xi) > 0$.

Proof. By using the equality in (19), we have

$$\begin{aligned}
&\left|\frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right]\right. \\
&-\frac{1}{(n_2-n_1)}\left[{}^{AB}I_n^{\xi}\{\rho(n_1)\}+{}^{AB}_{n}I_{n_2}^{\xi}\{\rho(n_2)\}\right] \\
&\left.+\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right]\right| \\
\leq\; & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\int_0^1 t^{\xi}\left|\rho'(tn+(1-t)n_1)\right|dt \\
&+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\int_0^1 t^{\xi}\left|\rho'(tn+(1-t)n_2)\right|dt.
\end{aligned} \tag{21}$$

If we use the strongly convexity of $\left|\rho'\right|$ and the fact that $\left|\rho'\right| \leq M$ in (21), we can deduce

$$\begin{aligned}
&\left|\frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right]\right. \\
&-\frac{1}{(n_2-n_1)}\left[{}^{AB}I_n^{\xi}\{\rho(n_1)\}+{}^{AB}_{n}I_{n_2}^{\xi}\{\rho(n_2)\}\right] \\
&\left.+\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right]\right| \\
&\leq \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\int_0^1 t^{\xi}\left[t\left|\rho'(n)\right|+(1-t)\left|\rho'(n_1)\right|-ct(1-t)(n-n_1)^2\right]dt \\
&+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\int_0^1 t^{\xi}\left[t\left|\rho'(n)\right|+(1-t)\left|\rho'(n_2)\right|-ct(1-t)(n_2-n)^2\right]dt \\
\leq\; & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{M}{\xi+1}-\frac{c(n-n_1)^2}{(\xi+2)(\xi+3)}\right) \\
&+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{M}{\xi+1}-\frac{c(n_2-n)^2}{(\xi+2)(\xi+3)}\right).
\end{aligned}$$

The proof is done. □

Corollary 2.9. *In Theorem 2.8, if we choose $n = \frac{n_1+n_2}{2}$, we have the following inequality:*

$$\begin{aligned}
&\left| \frac{(n_2-n_1)^{\xi-1}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\rho\left(\frac{n_1+n_2}{2}\right) \right. \\
&-\frac{1}{n_2-n_1}\left[{}^{AB}I^{\xi}_{\frac{n_1+n_2}{2}}\{\rho(n_1)\} + {}^{AB}_{\frac{n_1+n_2}{2}}I^{\xi}_{n_2}\{\rho(n_2)\}\right] \\
&\left. +\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right] \right| \\
\leq\ & \frac{(n_2-n_1)^{\xi}}{2^{\xi}B(\xi)\Gamma(\xi)}\left(\frac{M}{\xi+1} - \frac{c\,(n_2-n_1)^2}{4\,(\xi+2)\,(\xi+3)}\right).
\end{aligned}$$

In the rest of the this section, for the simplicity we will use the following notations:

$$\begin{aligned}
N_1 \quad = \quad & \frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right] \\
& -\frac{1}{(n_2-n_1)}\left[{}^{AB}I^{\xi}_{n}\{\rho(n_1)\} + {}^{AB}_{n}I^{\xi}_{n_2}\{\rho(n_2)\}\right] \\
& +\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right],
\end{aligned}$$

$$\begin{aligned}
N_2 = & \frac{(n_2-n_1)^{\xi-1}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\rho\left(\frac{n_1+n_2}{2}\right) \\
&-\frac{1}{n_2-n_1}\left[{}^{AB}I^{\xi}_{\frac{n_1+n_2}{2}}\{\rho(n_1)\} + {}^{AB}_{\frac{n_1+n_2}{2}}I^{\xi}_{n_2}\{\rho(n_2)\}\right] \\
&+\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right].
\end{aligned}$$

It will also not be repeated in the rest of the study that $B > 0$ is the normalization function and Γ is the gamma function.

Theorem 2.10. *Let $n_1 < n_2$, $n_1, n_2 \in I^{\circ}$and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\rho' \in L[n_1, n_2]$. If $\left|\rho'\right|^q$ is strongly convex function with modulus $c > 0$ on $[n_1, n_2]$ and $\left|\rho'\right| \leq M$, $M^q \geq \max\left\{\frac{c(n-n_1)^2}{6}, \frac{c(n_2-n)^2}{6}\right\}$, for all $n \in [n_1, n_2]$, $\xi \in (0, 1]$ we obtain the inequality below:*

$$\begin{aligned}
|N_1| \quad \leq \quad & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi p+1}\right)^{\frac{1}{p}}\left(M^q - \frac{c(n-n_1)^2}{6}\right)^{\frac{1}{q}} \\
& +\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi p+1}\right)^{\frac{1}{p}}\left(M^q - \frac{c(n_2-n)^2}{6}\right)^{\frac{1}{q}}
\end{aligned} \tag{22}$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. To prove Theorem 2.10; we will use Lemma 2.7, property of modulus, Hölder inequality, strongly convexity of $\left|\rho'\right|^q$ and the fact that $\left|\rho'\right| \leq M$. So, we can write

$$\begin{aligned}
|N_1| \leq\ & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\rho'(tn+(1-t)n_1)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\rho'(tn+(1-t)n_2)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi p+1}\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1 \left[t\left|\rho'(n)\right|^q+(1-t)\left|\rho'(n_1)\right|^q-ct(1-t)(n-n_1)^2\right]dt\right)^{\frac{1}{q}} \\
&+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi p+1}\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1 \left[t\left|\rho'(n)\right|^q+(1-t)\left|\rho'(n_2)\right|^q-ct(1-t)(n_2-n)^2\right]dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi p+1}\right)^{\frac{1}{p}}\left(M^q-\frac{c(n-n_1)^2}{6}\right)^{\frac{1}{q}} \\
&+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi p+1}\right)^{\frac{1}{p}}\left(M^q-\frac{c(n_2-n)^2}{6}\right)^{\frac{1}{q}}
\end{aligned}$$

which is the inequality in (22). □

Corollary 2.11. *In Theorem 2.10, if we choose $n = \frac{n_1+n_2}{2}$, we have the following inequality:*

$$|N_2| \leq \frac{(n_2-n_1)^{\xi}}{2^{\xi}B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi p+1}\right)^{\frac{1}{p}}\left(M^q-\frac{c(n_2-n_1)^2}{24}\right)^{\frac{1}{q}}.$$

Theorem 2.12. *Let $n_1 < n_2$, $n_1, n_2 \in I^\circ$and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\rho' \in L[n_1, n_2]$. If $\left|\rho'\right|^q$ is strongly convex function with modulus $c > 0$ on $[n_1, n_2]$ and $\left|\rho'\right| \leq M$, $\frac{M^q}{\xi+1} \geq \max\left\{\frac{c(n-n_1)^2}{(\xi+2)(\xi+3)}, \frac{c(n_2-n)^2}{(\xi+2)(\xi+3)}\right\}$, for all $n \in [n_1, n_2]$, $\xi \in (0, 1]$ we obtain the inequality below:*

$$\begin{aligned}
|N_1| \leq\ & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi+1}\right)^{\frac{1}{p}}\left(\frac{M^q}{\xi+1}-\frac{c(n-n_1)^2}{(\xi+2)(\xi+3)}\right)^{\frac{1}{q}} \\
&+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi+1}\right)^{\frac{1}{p}}\left(\frac{M^q}{\xi+1}-\frac{c(n_2-n)^2}{(\xi+2)(\xi+3)}\right)^{\frac{1}{q}}
\end{aligned} \tag{23}$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. In the proof of Theorem 2.12, we will use the Hölder's inequality in a different way as following:

$$\begin{aligned}
|N_1| \leq\ & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi}dt\right)^{\frac{1}{p}}\left(\int_0^1 t^{\xi}\left|\rho'(tn+(1-t)n_1)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi}dt\right)^{\frac{1}{p}}\left(\int_0^1 t^{\xi}\left|\rho'(tn+(1-t)n_2)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

If we use the strongly convexity of $\left|\rho'\right|^q$ and the fact that $\left|\rho'\right| \leq M$ in above and if we make the necessary calculations in obtained new inequality we will reach the inequality in (23). □

Corollary 2.13. *In Theorem 2.12, if we choose $n = \frac{n_1+n_2}{2}$, we have the following inequality:*

$$|N_2| \leq \frac{(n_2-n_1)^{\xi}}{2^{\xi}B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi+1}\right)^{\frac{1}{p}}\left(\frac{M^q}{\xi+1}-\frac{c(n_2-n_1)^2}{4(\xi+2)(\xi+3)}\right)^{\frac{1}{q}}.$$

Remark 2.14. *Theorems 2.8-2.12 and Corollaries 2.9-2.13 are generalizations of Theorem 2.2, Theorem 2.5, Theorem 2.8, Corollary 2.4, Corollary 2.7 and Corollary 2.10 respectively which are obtained by Set et al. in [41].*

Theorem 2.15. *Let $n_1 < n_2$, $n_1, n_2 \in I^\circ$ and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\rho' \in L[n_1, n_2]$. If $\left|\rho'\right|^q$ is strongly convex function with modulus $c > 0$ on $[n_1, n_2]$ and $\left|\rho'\right| \leq M$, $\frac{M^q}{\xi p+1} \geq \max\left\{\frac{c(n-n_1)^2}{(\xi p+2)(\xi p+3)}, \frac{c(n_2-n)^2}{(\xi p+2)(\xi p+3)}\right\}$, for all $n \in [n_1, n_2]$, $\xi \in (0, 1]$ we obtain the inequality below:*

$$\begin{aligned} &|N_1| \\ \leq\ & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{q-1}{\xi(q-p)+q-1}\right)^{1-\frac{1}{q}}\left(\frac{M^q}{\xi p+1}-\frac{c(n-n_1)^2}{(\xi p+2)(\xi p+3)}\right)^{\frac{1}{q}} \\ &+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{q-1}{\xi(q-p)+q-1}\right)^{1-\frac{1}{q}}\left(\frac{M^q}{\xi p+1}-\frac{c(n_2-n)^2}{(\xi p+2)(\xi p+3)}\right)^{\frac{1}{q}} \end{aligned} \tag{24}$$

where $q \geq p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. Applying Hölder's inequality in a different way, we have

$$\begin{aligned} &|N_1| \\ \leq\ & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi\left(\frac{q-p}{q-1}\right)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\xi p}\left|\rho'(tn+(1-t)n_1)\right|^q dt\right)^{\frac{1}{q}} \\ &+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi\left(\frac{q-p}{q-1}\right)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\xi p}\left|\rho'(tn+(1-t)n_2)\right|^q dt\right)^{\frac{1}{q}} \\ \leq\ & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi\left(\frac{q-p}{q-1}\right)}dt\right)^{1-\frac{1}{q}} \\ &\times\left(\int_0^1 t^{\xi p}\left[t\left|\rho'(n)\right|^q+(1-t)\left|\rho'(n_1)\right|^q-ct(1-t)(n-n_1)^2\right]dt\right)^{\frac{1}{q}} \\ &+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi\left(\frac{q-p}{q-1}\right)}dt\right)^{1-\frac{1}{q}} \\ &\times\left(\int_0^1 t^{\xi p}\left[t\left|\rho'(n)\right|^q+(1-t)\left|\rho'(n_2)\right|^q-ct(1-t)(n_2-n)^2\right]dt\right)^{\frac{1}{q}}. \end{aligned}$$

If we use the fact that $\left|\rho'\right| \leq M$ and if we calculate the integrals above, we complete the proof. □

Corollary 2.16. *In Theorem 2.15, if we choose $n = \frac{n_1+n_2}{2}$, we have the following inequality:*

$$|N_2| \leq \frac{(n_2-n_1)^{\xi}}{2^{\xi}B(\xi)\Gamma(\xi)}\left(\frac{q-1}{\xi(q-p)+q-1}\right)^{1-\frac{1}{q}}\left(\frac{M^q}{\xi p+1}-\frac{c(n_2-n_1)^2}{4(\xi p+2)(\xi p+3)}\right)^{\frac{1}{q}}.$$

To obtain new results for second order differentiable strongly convex functions we will use the following Ostrowski-like lemma.

Lemma 2.17. *(See [10]) Let $n_1 < n_2$, $n_1, n_2 \in I^\circ$and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I°. If $\rho'' \in L[n_1, n_2]$, identity for Atangana-Baleanu integral operators in equation (25) is valid for all $n \in [n_1, n_2]$, $t, \xi \in [0, 1]$:*

$$\begin{aligned} &\frac{1}{(n_2 - n_1)}\left[{}^{AB}I_n^{\xi}\{\rho(n_1)\} +_n^{AB} I_{n_2}^{\xi}\{\rho(n_2)\}\right] - \frac{1-\xi}{(n_2 - n_1)B(\xi)}\left[\rho(n_1) + \rho(n_2)\right] \\ &-\frac{\rho(n)}{(n_2 - n_1)B(\xi)\Gamma(\xi)}\left[(n_2 - n)^{\xi} + (n - n_1)^{\xi}\right] + \frac{(n - n_1)^{\xi+1} - (n_2 - n)^{\xi+1}}{(n_2 - n_1)B(\xi)\Gamma(\xi)(\xi+1)}\rho'(n) \\ = \; &\frac{(n - n_1)^{\xi+2}}{(n_2 - n_1)B(\xi)\Gamma(\xi)(\xi+1)}\int_0^1 t^{\xi+1}\rho''(tn + (1-t)n_1)dt \\ &+\frac{(n_2 - n)^{\xi+2}}{(n_2 - n_1)B(\xi)\Gamma(\xi)(\xi+1)}\int_0^1 t^{\xi+1}\rho''(tn + (1-t)n_2)dt. \end{aligned} \tag{25}$$

Theorem 2.18. *Let $n_1 < n_2$, $n_1, n_2 \in I^\circ$and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\rho'' \in L[n_1, n_2]$. If $\left|\rho''\right|$ is strongly convex function with modulus $c > 0$ on $[n_1, n_2]$, $\left|\rho''\right| \leq M_1$ and $\frac{M_1}{\xi+2} \geq \max\left\{\frac{c(n-n_1)^2}{(\xi+3)(\xi+4)}, \frac{c(n_2-n)^2}{(\xi+3)(\xi+4)}\right\}$, for all $n \in [n_1, n_2]$, $\xi \in [0, 1]$ we obtain the inequality below:*

$$\begin{aligned} &\left|\frac{1}{(n_2 - n_1)}\left[{}^{AB}I_n^{\xi}\{\rho(n_1)\} +_n^{AB} I_{n_2}^{\xi}\{\rho(n_2)\}\right] - \frac{1-\xi}{(n_2 - n_1)B(\xi)}\left[\rho(n_1) + \rho(n_2)\right]\right. \\ &\left.-\frac{\rho(n)}{(n_2 - n_1)B(\xi)\Gamma(\xi)}\left[(n_2 - n)^{\xi} + (n - n_1)^{\xi}\right] + \frac{(n - n_1)^{\xi+1} - (n_2 - n)^{\xi+1}}{(n_2 - n_1)B(\xi)\Gamma(\xi)(\xi+1)}\rho'(n)\right| \\ \leq \; &\frac{(n - n_1)^{\xi+2}}{(n_2 - n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{M_1}{\xi+2} - \frac{c(n-n_1)^2}{(\xi+3)(\xi+4)}\right) \\ &+\frac{(n_2 - n)^{\xi+2}}{(n_2 - n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{M_1}{\xi+2} - \frac{c(n_2-n)^2}{(\xi+3)(\xi+4)}\right). \end{aligned} \tag{26}$$

Proof. By using the equality in (25), property of modulus and strongly convexity of $\left|\rho''\right|$ we have

$$\begin{aligned} &\left|\frac{1}{(n_2 - n_1)}\left[{}^{AB}I_n^{\xi}\{\rho(n_1)\} +_n^{AB} I_{n_2}^{\xi}\{\rho(n_2)\}\right] - \frac{1-\xi}{(n_2 - n_1)B(\xi)}\left[\rho(n_1) + \rho(n_2)\right]\right. \\ &\left.-\frac{\rho(n)}{(n_2 - n_1)B(\xi)\Gamma(\xi)}\left[(n_2 - n)^{\xi} + (n - n_1)^{\xi}\right] + \frac{(n - n_1)^{\xi+1} - (n_2 - n)^{\xi+1}}{(n_2 - n_1)B(\xi)\Gamma(\xi)(\xi+1)}\rho'(n)\right| \\ \leq \; &\frac{(n - n_1)^{\xi+2}}{(n_2 - n_1)B(\xi)\Gamma(\xi)(\xi+1)}\int_0^1 t^{\xi+1}\left[t\left|\rho''(n)\right| + (1-t)\left|\rho''(n_1)\right| - ct(1-t)(n-n_1)^2\right]dt \\ &+\frac{(n_2 - n)^{\xi+2}}{(n_2 - n_1)B(\xi)\Gamma(\xi)(\xi+1)}\int_0^1 t^{\xi+1}\left[t\left|\rho''(n)\right| + (1-t)\left|\rho''(n_2)\right| - ct(1-t)(n_2-n)^2\right]dt. \end{aligned}$$

We complete the proof by making the necessary calculations in above and by taking into consideration that $\left|\rho''\right| \leq M_1$. □

Corollary 2.19. *In Theorem 2.18, if we choose $n = \frac{n_1+n_2}{2}$, we have the following inequality:*

$$|N_2| \leq \frac{(n_2 - n_1)^{\xi+1}}{2^{\xi+1}B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{M_1}{\xi+2} - \frac{c(n_2-n_1)^2}{4(\xi+3)(\xi+4)}\right).$$

In the rest of this section, for simplicity we will use

$$\begin{aligned} N_3 &= \frac{1}{(n_2-n_1)}\left[{}^{AB}I_n^{\xi}\{\rho(n_1)\}+{}_n^{AB}I_{n_2}^{\xi}\{\rho(n_2)\}\right]-\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right] \\ &-\frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right]+\frac{(n-n_1)^{\xi+1}-(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\rho'(n). \end{aligned}$$

Theorem 2.20. *Let $n_1 < n_2$, $n_1, n_2 \in I^{\circ}$ and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\rho'' \in L[n_1, n_2]$. If $\left|\rho''\right|^q$ is strongly convex function with modulus $c > 0$ on $[n_1, n_2]$ and $\left|\rho''\right| \le M_1$, $M_1^q \ge \max\left\{\frac{c(n-n_1)^2}{6}, \frac{c(n_2-n)^2}{6}\right\}$, for all $n \in [n_1, n_2]$, $\xi \in [0, 1]$ we obtain the inequality below:*

$$\begin{aligned} &|N_3| \qquad (27) \\ \le\ & \frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{1}{(\xi+1)p+1}\right)^{\frac{1}{p}}\left(M_1^q-\frac{c(n-n_1)^2}{6}\right)^{\frac{1}{q}} \\ &+\frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{1}{(\xi+1)p+1}\right)^{\frac{1}{p}}\left(M_1^q-\frac{c(n_2-n)^2}{6}\right)^{\frac{1}{q}} \end{aligned}$$

where $q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. To prove this theorem, we will use similar operations that we used when proving Theorem 2.10. So, we have

$$\begin{aligned} &|N_3| \\ \le\ & \frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\int_0^1 t^{(\xi+1)p}dt\right)^{\frac{1}{p}} \\ &\times\left(\int_0^1\left[t\left|\rho''(n)\right|^q+(1-t)\left|\rho''(n_1)\right|^q-ct(1-t)(n-n_1)^2\right]dt\right)^{\frac{1}{q}} \\ &+\frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\int_0^1 t^{(\xi+1)p}dt\right)^{\frac{1}{p}} \\ &\times\left(\int_0^1\left[t\left|\rho''(n)\right|^q+(1-t)\left|\rho''(n_2)\right|^q-ct(1-t)(n_2-n)^2\right]dt\right)^{\frac{1}{q}}. \end{aligned}$$

If we calculate the integrals above and if we consider the fact that $\left|\rho''\right| \le M_1$, we get the inequality in (27). □

Corollary 2.21. *In Teorem 2.20, if we choose $n = \frac{n_1+n_2}{2}$, we have the following inequality:*

$$|N_2| \le \frac{(n_2-n_1)^{\xi+1}}{2^{\xi+1}B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{1}{(\xi+1)p+1}\right)^{\frac{1}{p}}\left(M_1^q-\frac{c(n_2-n_1)^2}{24}\right)^{\frac{1}{q}}.$$

Theorem 2.22. *Let $n_1 < n_2$, $n_1, n_2 \in I^{\circ}$ and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\rho'' \in L[n_1, n_2]$. If $\left|\rho''\right|^q$ is strongly convex function with modulus $c > 0$ on $[n_1, n_2]$ and $\left|\rho''\right| \le M_1$, $\frac{M_1^q}{\xi+2} \ge \max\left\{\frac{c(n-n_1)^2}{(\xi+3)(\xi+4)}, \frac{c(n_2-n)^2}{(\xi+3)(\xi+4)}\right\}$, for all $n \in [n_1, n_2]$, $\xi \in [0, 1]$ we obtain the inequality below:*

$$\begin{aligned} &|N_3| \qquad (28) \\ \le\ & \frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{1}{\xi+2}\right)^{\frac{1}{p}}\left(\frac{M_1^q}{\xi+2}-\frac{c(n-n_1)^2}{(\xi+3)(\xi+4)}\right)^{\frac{1}{q}} \\ &+\frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{1}{\xi+2}\right)^{\frac{1}{p}}\left(\frac{M_1^q}{\xi+2}-\frac{c(n_2-n)^2}{(\xi+3)(\xi+4)}\right)^{\frac{1}{q}} \end{aligned}$$

where $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Via Hölder's inequality and strongly convexity of $\left|\rho''\right|^q$ we can write

$$\begin{aligned} |N_3| \leq & \frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\int_0^1 t^{\xi+1}dt\right)^{\frac{1}{p}} \\ & \times\left(\int_0^1 t^{\xi+1}\left[t\left|\rho''(n)\right|^q + (1-t)\left|\rho''(n_1)\right|^q - ct(1-t)(n-n_1)^2\right]dt\right)^{\frac{1}{q}} \\ & + \frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\int_0^1 t^{\xi+1}dt\right)^{\frac{1}{p}} \\ & \times\left(\int_0^1 t^{\xi+1}\left[t\left|\rho''(n)\right|^q + (1-t)\left|\rho''(n_2)\right|^q - ct(1-t)(n_2-n)^2\right]dt\right)^{\frac{1}{q}}. \end{aligned}$$

If we consider the fact that $\left|\rho''\right| \leq M_1$ and calculate the integrals, we get the inequality in (28). □

Corollary 2.23. *In Theorem 2.22, if we choose* $n = \frac{n_1+n_2}{2}$, *we have the following inequality:*

$$|N_2| \leq \frac{(n_2-n_1)^{\xi+1}}{2^{\xi+1}B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{1}{\xi+2}\right)^{\frac{1}{p}}\left(\frac{M_1^q}{\xi+2} - \frac{c(n_2-n_1)^2}{4(\xi+3)(\xi+4)}\right)^{\frac{1}{q}}.$$

Theorem 2.24. *Let* $n_1 < n_2$, $n_1, n_2 \in I^\circ$ *and* $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ *be a differentiable function on* I° *and* $\rho'' \in L[n_1, n_2]$. *If* $\left|\rho''\right|^q$ *is strongly convex function with modulus* $c > 0$ *on* $[n_1, n_2]$ *and* $\left|\rho''\right| \leq M_1$, $\frac{M_1^q}{\xi p+p+1} \geq \max\left\{\frac{c(n-n_1)^2}{(\xi p+p+2)(\xi p+p+3)}, \frac{c(n_2-n)^2}{(\xi p+p+2)(\xi p+p+3)}\right\}$, *for all* $n \in [n_1, n_2]$, $\xi \in [0,1]$ *we obtain the inequality below:*

$$\begin{aligned} & |N_3| \\ \leq & \frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{q-1}{(\xi+1)(q-p)+q-1}\right)^{1-\frac{1}{q}}\left(\frac{M_1^q}{\xi p+p+1} - \frac{c(n-n_1)^2}{(\xi p+p+2)(\xi p+p+3)}\right)^{\frac{1}{q}} \\ & + \frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{q-1}{(\xi+1)(q-p)+q-1}\right)^{1-\frac{1}{q}}\left(\frac{M_1^q}{\xi p+p+1} - \frac{c(n_2-n)^2}{(\xi p+p+2)(\xi p+p+3)}\right)^{\frac{1}{q}} \end{aligned} \tag{29}$$

where $q \geq p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Via a version of the Hölder inequality that we have used in the proof of Theorem 2.15, we can write

$$\begin{aligned} |N_3| \leq & \frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\int_0^1 t^{(\xi+1)\left(\frac{q-p}{q-1}\right)}dt\right)^{1-\frac{1}{q}} \\ & \times\left(\int_0^1 t^{(\xi+1)p}\left|\rho''(tn+(1-t)n_1)\right|^q dt\right)^{\frac{1}{q}} \\ & + \frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\int_0^1 t^{(\xi+1)\left(\frac{q-p}{q-1}\right)}dt\right)^{1-\frac{1}{q}} \\ & \times\left(\int_0^1 t^{(\xi+1)p}\left|\rho''(tn+(1-t)n_2)\right|^q dt\right)^{\frac{1}{q}}. \end{aligned}$$

If we use strongly convexity of $\left|\rho''\right|^q$ with $\left|\rho''\right| \leq M_1$, and if we calculate the necessary integrals, we obtain the inequality in (29). □

Corollary 2.25. *In Theorem 2.24, if we choose $n = \frac{n_1+n_2}{2}$, we have the following inequality:*

$$|N_2| \leq \frac{(n_2-n_1)^{\xi+1}}{2^{\xi+1}B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{q-1}{(\xi+1)(q-p)+q-1}\right)^{1-\frac{1}{q}} \times\left(\frac{M_1^q}{\xi p+p+1}-\frac{c(n_2-n_1)^2}{4(\xi p+p+2)(\xi p+p+3)}\right)^{\frac{1}{q}}.$$

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