UJMA

Universal Journal of Mathematics and Applications

VOLUME IV
ISSUE IV

ISSN 2619-9653

http://dergipark.gov.tr/ujma

VOLUME IV ISSUE IV
ISSN 2619-9653

December 2021
http://dergipark.gov.tr/ujma

UNIVERSAL JOURNAL OF MATHEMATICS AND APPLICATIONS

Contents

Universal Journal of Mathematics and Applications, 4 (4) (2021) 125-131
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1008148

Characterizations of Framed Curves in Four-Dimensional Euclidean Space

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Article Info

Keywords: *Bishop-type framed curves, Euler angles, Framed curves in* $\mathbb{R}^4$
2010 AMS: *53A04, 58K05*
Received: *11 October 2021*
Accepted: *29 December 2021*
Available online: *30 December 2021*

Abstract

Framed curves in Euclidean space are used to investigate singular curves and are important for singularity theory. In this study, we investigate framed curves in four-dimensional Euclidean space. We introduce Bishop-type frame of framed curves is with the help of Euler angles. Also, we give framed rectifying curves, framed osculating curves and framed normal curves with the help of Bishop-type framed curves in four-dimensional Euclidean space. Also, we obtain some characterizations depending on framed curvatures.

1. Introduction

Framed curves in n-dimensional Euclidean space were first introduced by Honda and Takahashi [1]. Framed curves in Euclidean space are used to investigate singular curves and are important for singularity theory. A framed curve in the Euclidean space is a curve with a moving frame. It is a generalization not only of regular curves with linear independent condition but also of Legendre curves in the unit tangent bundle. There are many studies in the literature for framed curves in three-dimensional Euclidean space. In three-dimensional Euclidean space, there are studies such as existence and uniqueness conditions of framed curves [2], evolutes of framed immersions [3], framed rectifying curves [4, 5], framed normal curves [6], Bertrand and Mannheim curves of framed curves [7].

Frenet and Bishop frames are important in classical differential geometry. Frenet frames cannot be built as the curvatures vanishes at some points and the Bishop frame is used [8]. In [9], this situation is extended to the four-dimensional Euclidean space and a parallel frame is formed. In four-dimensional Euclidean space, this frame is called the parallel transport frame and it can be [10]- [13] etc. studies are available.

In this paper, we introduce framed curves in four-dimensional Euclidean space. Also, we give some new results for the relation of framed curves with Frenet curves in four-dimensional Euclidean space. Moreover, we introduce Bishop-type frame of framed curves with the help of Euler angles. Also, by using Bishop-type framed curves in four-dimensional Euclidean space, we introduce framed rectifying curves, framed osculating curves and framed normal curves.

2. Framed curves in n-Euclidean space

A framed curve in the n-dimensional Euclidean space is a space curve with a moving frame which may have singular points, in detail see [1]. Let $\gamma : I \to \mathbb{R}^n$ be a curve with singular points. The set

$$\Delta_{n-1} = \{\mu = (\mu_1, \mu_2, ..., \mu_{n-1}) \in \mathbb{R}^n \times ... \times \mathbb{R}^n \mid \langle \mu_i, \mu_j \rangle = \delta_{ij}, \quad i, j = 1, 2..., n-1\}$$

is an $(n, n-1)-$type Stiefel manifold. Let $\mu = (\mu_1, \mu_2, ..., \mu_{n-1}) \in \Delta_{n-1}$. Take the unit vector defined by $\nu = \mu_1 \wedge ... \wedge \mu_{n-1}$.

Definition 2.1 (Framed curve). $(\gamma, \mu) : I \to \mathbb{R}^n \times \Delta_{n-1}$ *is called a framed curve if* $\langle \gamma'(s), \mu_i(s) \rangle = 0$ *for all* $s \in I$ *and* $i = 1, 2, ..., n-1$ *[1].*

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Let $(\gamma,\mu): I \to \mathbb{R}^n \times \Delta_{n-1}$ be a framed curve and $\nu(s) = \mu_1(s) \wedge \ldots \wedge \mu_{n-1}$. By definition $(\mu(s), \nu(s)) \in SO(n)$ for each $s \in I$ and $\{\mu(s), \nu(s)\}$ is called a moving frame along the framed base curve $\gamma(s)$. Then, Frenet-Serret type formula is given by

$$\begin{pmatrix} \mu'(s) \\ \nu'(s) \end{pmatrix} = A(s) \begin{pmatrix} \mu(s) \\ \nu(s) \end{pmatrix}$$

where $A(s) = (a_{i,j}) \in o(n)$ for $i,j = 1,2,\ldots,n$ and $o(n)$ is the set of all skew-symmetric matrices. Moreover, there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that:

$$\gamma'(s) = \alpha(s)\nu(s)$$

In addition, s_0 is a singular point of the framed curve γ if and only if $\alpha(s_0) = 0$. $(A,\alpha) : I \to o(n) \times \mathbb{R}$ is called the curvature of the framed curve.

3. Framed curves in $\mathbb{R}^4 \times \Delta_3$

Let us take the vectors $x = (x_1,x_2,x_3,x_4), y = (y_1,y_2,y_3,y_4)$ and $z = (z_1,z_2,z_3,z_4)$ in four-dimensional Euclidean space $\mathbb{R}^4$ with Euclidean inner product. The ternary product (or vector product) is defined

$$x \wedge y \wedge z = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

where $\{e_1,e_2,e_3,e_4\}$ is the standard basis of $\mathbb{R}^4$ [14].
The set

$$\Delta_3 = \{\mu = (\mu_1,\mu_2,\mu_3) \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 | \quad \langle \mu_i, \mu_j \rangle = \delta_{ij}, i,j = 1,2,3\}$$

is a six-dimensional smooth manifold. We define a unit vector $\nu : \mu_1 \wedge \mu_2 \wedge \mu_3$ of $\mathbb{R}^4$.

Definition 3.1. $(\gamma,\mu_1,\mu_2,\mu_3) : I \to \mathbb{R}^4 \times \Delta_3$ *is called a framed curve if* $\langle \gamma'(s), \mu_i(s) \rangle = 0$ *for all* $s \in I$ *and* $i = 1,2,3$.

The Frenet-Serret type formula is given by

$$\begin{pmatrix} \nu'(s) \\ \mu_1'(s) \\ \mu_2'(s) \\ \mu_3'(s) \end{pmatrix} = \begin{pmatrix} 0 & -l(s) & -m(s) & -n(s) \\ l(s) & 0 & p(s) & q(s) \\ m(s) & -p(s) & 0 & r(s) \\ n(s) & -q(s) & -r(s) & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \mu_1(s) \\ \mu_2(s) \\ \mu_3(s) \end{pmatrix} \tag{3.1}$$

where

$$\begin{array}{ll} l(s) = \langle \mu_1'(s), \nu(s) \rangle & m(s) = \langle \mu_2'(s), \nu(s) \rangle \\ n(s) = \langle \mu_3'(s), \nu(s) \rangle & p(s) = \langle \mu_1'(s), \mu_2(s) \rangle \\ q(s) = \langle \mu_1'(s), \mu_3(s) \rangle & r(s) = \langle \mu_2'(s), \mu_3(s) \rangle \end{array}$$

In addition, there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that $\gamma'(s) = \alpha(s)\nu(s)$. If s_0 is a singular point of γ, $\alpha(s_0) = 0$. If s_0 is a singular point of (γ,μ), $(l,m,n,p,q,r,\alpha)(s_0) = 0$. Note that $(\gamma,\mu_1,\mu_2,\mu_3)$ is a framed immersion if and only if $(l(s),m(s),n(s),p(s),q(s),r(s),\alpha(s)) \neq (0,0,0,0)$ for all $s \in I$.

Example 3.2. *Regular curves at* $\mathbb{R}^4$ *with linear independent conditions are a natural example of framed curves (i.e.* $\gamma'(s), \gamma''(s), \gamma'''(s)$ *are linear independent for all* $s \in I$*).* $\gamma : I \to \mathbb{R}^4$ *regular curve with linear independent conditions. If we take* $\mu_1(s) = N_1(s)$, $\mu_2(s) = N_2(s)$ *and* $\mu_3(s) = N_3(s)$*, then* $(\gamma,N_1,N_2,N_3) : I \to \mathbb{R}^4 \times \Delta_3$ *is a framed curve. Also,* $\nu(s) = \mu_1 \wedge \mu_2 \wedge \mu_3 = T(s)$*. Therefore,*

$$\begin{array}{ll} T(s) = \dfrac{\gamma'(s)}{\|\gamma'(s)\|} & N_1(s) = \dfrac{\|\gamma'(s)\|^2\gamma''(s) - \langle \gamma'(s),\gamma''(s)\rangle\gamma'(s)}{\|\|\gamma'(s)\|^2\gamma''(s) - \langle \gamma'(s),\gamma''(s)\rangle\gamma'(s)\|} \\ N_2(s) = N_3(s) \times T(s) \times N_1(s) & N_3(s) = \dfrac{T(s) \times N_1(s) \times \gamma'''(s)}{\|T(s) \times N_1(s) \times \gamma'''(s)\|} \end{array}$$

3.1. Framed curves $\mathbb{R}^4 \times \Delta_3$ with Bishop frame

In this section, adapted frame for framed curves are obtained by using Euler angles and these frame is called Bishop-type frame of framed curves. By using Euler angles an arbitrary rotation matrix is given by

$$\begin{pmatrix} \cos\theta\cos\psi & -\cos\varphi\sin\psi + \sin\varphi\cos\psi\sin\theta & \sin\varphi\sin\psi + \cos\varphi\cos\psi\sin\theta \\ \cos\theta\sin\psi & \cos\varphi\cos\psi + \sin\varphi\sin\psi\sin\theta & -\sin\varphi\cos\psi + \cos\varphi\sin\psi\sin\theta \\ -\sin\theta & \sin\varphi\cos\theta & \cos\varphi\cos\theta \end{pmatrix}$$

where θ, ψ, φ are Euler angles [9]. We define $(\overline{\mu}_1(s), \overline{\mu}_2(s), \overline{\mu}_3(s)) \in \Delta_3$ by

$$\begin{aligned}
\overline{\mu}_1(s) &= (\cos\theta(s)\cos\psi(s))\mu_1(s) + (-\cos\varphi(s)\sin\psi(s) + \sin\varphi(s)\cos\psi(s)\sin\theta(s))\mu_2(s) \\
&+ (\sin\varphi(s)\sin\psi(s) + \cos\varphi(s)\cos\psi(s)\sin\theta(s))\mu_3(s) \\
\overline{\mu}_2(s) &= (\cos\theta(s)\sin\psi(s))\mu_1(s) + (\cos\varphi(s)\cos\psi(s) + \sin\varphi(s)\sin\psi(s)\sin\theta(s))\mu_2(s) \\
&+ (-\sin\varphi(s)\cos\psi(s) + \cos\varphi(s)\sin\psi(s)\sin\theta(s))\mu_3(s) \\
\overline{\mu}_3(s) &= -\sin\theta(s)\mu_1(s) + \sin\varphi(s)\cos\theta(s)\mu_2(s) + \cos\varphi(s)\cos\theta(s)\mu_3(s)
\end{aligned} \tag{3.2}$$

Therefore, $(\gamma, \overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3) : I \to \mathbb{R}^4 \times \Delta_3$ is a framed curve and

$$\overline{\nu}(s) = \overline{\mu}_1(s) \wedge \overline{\mu}_2(s) \wedge \overline{\mu}_3(s) = \nu(s)$$

By differentiating equations 3.1 and 3.2, we get

$$\begin{aligned}
\overline{\mu}_1^{'} &= (l\cos\theta\cos\psi - m\cos\varphi\sin\psi + m\sin\varphi\cos\psi\sin\theta + n\sin\varphi\sin\psi + n\cos\varphi\cos\psi\sin\theta)\nu \\
&+(-\theta^{'}\sin\theta\cos\psi - \psi^{'}\cos\theta\sin\psi + p\cos\varphi\sin\psi - p\sin\varphi\cos\psi\sin\theta \\
&-q\sin\varphi\sin\psi - q\cos\varphi\sin\theta\cos\psi)\mu_1 + (p\cos\theta\cos\psi + \varphi^{'}\sin\varphi\sin\psi - \psi^{'}\cos\varphi\cos\psi \\
&+\varphi^{'}\sin\theta\cos\varphi\cos\psi + \theta^{'}\cos\theta\sin\varphi\cos\psi - \psi^{'}\sin\theta\sin\varphi\sin\psi - r\sin\varphi\sin\psi \\
&-r\cos\varphi\cos\psi\sin\theta)\mu_2 + (q\cos\theta\cos\psi - r\cos\varphi\sin\psi + r\sin\varphi\cos\psi\sin\theta \\
&+\varphi^{'}\cos\varphi\sin\psi + \psi^{'}\sin\varphi\cos\psi - \varphi^{'}\sin\theta\sin\varphi\cos\psi + \theta^{'}\cos\theta\cos\varphi\cos\psi \\
&-\psi^{'}\sin\theta\cos\varphi\sin\psi)\mu_3,
\end{aligned}$$

$$\begin{aligned}
\overline{\mu}_2^{'} &= (l\cos\theta\sin\psi + m\cos\varphi\cos\psi + m\sin\varphi\sin\psi\sin\theta - n\sin\varphi\cos\psi + n\cos\varphi\sin\psi\sin\theta)\nu \\
&+(-\theta^{'}\sin\theta\sin\psi + \psi^{'}\cos\theta\cos\psi - p\cos\varphi\cos\psi - p\sin\varphi\sin\psi\sin\theta \\
&+q\sin\varphi\cos\psi - q\cos\varphi\sin\theta\sin\psi)\mu_1 + (p\cos\theta\sin\psi - \psi^{'}p\cos\varphi\sin\psi + \psi^{'}\sin\varphi\cos\psi\sin\theta \\
&-\varphi^{'}\cos\psi\sin\varphi + \varphi^{'}\sin\theta\cos\varphi\sin\psi + \theta^{'}\cos\theta\sin\varphi\sin\psi + r\sin\varphi\cos\psi \\
&-r\cos\varphi\sin\psi\sin\theta)\mu_2 + (q\cos\theta\sin\psi + r\cos\varphi\cos\psi + r\sin\varphi\sin\psi\sin\theta \\
&-\varphi^{'}\cos\varphi\cos\psi + \psi^{'}\sin\varphi\sin\psi - \varphi^{'}\sin\theta\sin\varphi\sin\psi + \theta^{'}\cos\theta\cos\varphi\sin\psi \\
&+\psi^{'}\sin\theta\cos\varphi\cos\psi)\mu_3,
\end{aligned}$$

$$\begin{aligned}
\overline{\mu}_3^{'} &= (-l\sin\theta + m\sin\varphi\cos\theta + n\cos\varphi\cos\theta)\nu + (-\theta^{'}\cos\theta - p\sin\varphi\cos\theta - q\cos\varphi\cos\theta)\mu_1 \\
&+(-p\sin\theta + \varphi^{'}\cos\theta\cos\varphi - \theta^{'}\sin\theta\sin\varphi - r\cos\varphi\cos\theta)\mu_2 \\
&+(-q\sin\theta + r\sin\varphi\cos\theta - \varphi^{'}\sin\varphi\cos\theta + \theta^{'}\sin\theta\cos\varphi)\mu_3, \\
\overline{\nu}^{'} &= \nu^{'} = -l\mu_1 - m\mu_2 - n\mu_3.
\end{aligned}$$

Corollary 3.3. *If we take Euler angles $\theta, \psi, \varphi : I \to \mathbb{R}$ which satisfies*

$$\begin{aligned}
\theta^{'}(s) &= -p(s)\sin\varphi(s) - q(s)\cos\varphi(s) \\
\varphi^{'}(s) &= \tan\theta(s)(p(s)\cos\varphi(s) - q(s)\sin\varphi(s)) + r \\
\psi^{'}(s) &= \sec\theta(s)(p(s)\cos\varphi(s) - q(s)\sin\varphi(s))
\end{aligned} \tag{3.3}$$

we call the frame $\{\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3, \nu\}$ a Bishop-type frame along framed base curve $\mathbb{R}^4$. Also, the formula is given by

$$\begin{pmatrix} \nu^{'}(s) \\ \overline{\mu}_1^{'}(s) \\ \overline{\mu}_2^{'}(s) \\ \overline{\mu}_3^{'}(s) \end{pmatrix} = \begin{pmatrix} 0 & -L(s) & -M(s) & -N(s) \\ L(s) & 0 & 0 & 0 \\ M(s) & 0 & 0 & 0 \\ N(s) & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \overline{\mu}_1(s) \\ \overline{\mu}_2(s) \\ \overline{\mu}_3(s) \end{pmatrix} \tag{3.4}$$

where

$$\begin{aligned}
L(s) &= l(s)\cos\theta(s)\cos\psi(s) - m(s)\cos\varphi(s)\sin\psi(s) + m(s)\sin\varphi(s)\cos\psi(s)\sin\theta(s) \\
&+n(s)\sin\varphi(s)\sin\psi(s) + n(s)\cos\varphi(s)\cos\psi(s)\sin\theta(s) \\
M(s) &= l(s)\cos\theta(s)\sin\psi(s) + m(s)\cos\varphi(s)\cos\psi(s) + m(s)\sin\varphi(s)\sin\psi(s)\sin\theta(s) \\
&-n(s)\sin\varphi(s)\cos\psi(s) + n(s)\cos\varphi(s)\sin\psi(s)\sin\theta(s) \\
N(s) &= -l(s)\sin\theta(s) + m(s)\sin\varphi(s)\cos\theta(s) + n(s)\cos\varphi(s)\cos\theta(s)
\end{aligned}$$

and with equation 3.3.

Corollary 3.4. *According to the equation 3.3, there is*

$$\varphi^{'}(s) = \psi^{'}(s)\sin\theta(s) + r(s)$$

relation between the Euler angles.

4. Special framed curves in $\mathbb{R}^4$ with Bishop frame

4.1. Framed rectifying curves

Definition 4.1. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3): I\to\mathbb{R}^4\times\Delta_3$ be a framed curve. γ is called framed rectifying curve in $\mathbb{R}^4$ if its position vector γ satisfies:*

$$\gamma(s)=\lambda_1(s)\nu(s)+\lambda_2(s)\overline{\mu}_2(s)+\lambda_3(s)\overline{\mu}_3(s) \tag{4.1}$$

where $\lambda_1(s),\lambda_2(s)$ and $\lambda_3(s)$ are differentiable functions.

Theorem 4.2. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3): I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with non-zero framed curvatures. Then, γ is a framed rectifying curve if and only if $\lambda_2 M(s)+\lambda_3 N(s)-\alpha(s)=0$, where λ_2 and λ_3 are real constants.*

Proof. Assume γ is a framed rectifying curve. By differentiating equation 4.1, we get

$$\begin{aligned}\alpha(s)\nu(s) \quad = \quad & (\lambda_1'(s)+\lambda_2(s)M(s)+\lambda_3(s)N(s))\nu(s)+(-L(s)\lambda_1(s))\overline{\mu}_1(s)\\ & +(-M(s)\lambda_1(s)+\lambda_2'(s))\overline{\mu}_2(s)+(-N(s)\lambda_1(s)+\lambda_3'(s))\overline{\mu}_3(s)\end{aligned} \tag{4.2}$$

Then according to equation 4.2, we have

$$\begin{aligned} L(s)\lambda_1(s) &= 0\\ M(s)\lambda_1(s)-\lambda_2'(s) &= 0\\ N(s)\lambda_1(s)-\lambda_3'(s) &= 0\\ \lambda_1'(s)+\lambda_2(s)M(s)+\lambda_3(s)N(s) &= \alpha(s)\end{aligned} \tag{4.3}$$

Since framed curvatures are non-zero, we get

$$\lambda_1(s)=0,\quad \lambda_2(s)=\lambda_2(const.),\lambda_3(s)=\lambda_3(const.) \tag{4.4}$$

Therefore, by using last equation of 4.3 and equation 4.4, we have

$$\lambda_2 M(s)+\lambda_3 N(s)-\alpha(s)=0 \tag{4.5}$$

where λ_2 and λ_3 are real constants. Conversely, assume that the curvatures (L,M,N,α) satisfies the equation 4.5. By considering, the vector $X\in\mathbb{R}^4$ given by

$$X(s)=\gamma(s)-\lambda_2\overline{\mu}_2(s)-\lambda_3\overline{\mu}_3(s) \tag{4.6}$$

By differentiating equation 4.6, we have

$$X'(s)=\alpha(s)\nu(s)-(\lambda_2 M(s)+\lambda_3 N(s))\nu(s)$$

By using equation 4.5, we get

$$X'(s)=0 \tag{4.7}$$

Based on the equation 4.7, we conclude that γ is congruent to a framed rectifying curve in $\mathbb{R}^4$. $\square$

Corollary 4.3. *If γ is a framed rectifying curve with non-zero curvatures in $\mathbb{R}^4$, then the curvatures of γ have the following relationships*

$$\frac{\left(\frac{\alpha}{N}\right)'}{\left(\frac{M}{N}\right)'}=constant \quad or \quad \frac{\left(\frac{\alpha}{M}\right)'}{\left(\frac{N}{M}\right)'}=constant \tag{4.8}$$

Proof. By using equation 4.5 and the derivative of equation 4.5, we get 4.8. $\square$

4.2. Framed first osculating curves

Definition 4.4. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3): I\to\mathbb{R}^4\times\Delta_3$ be a framed curve. γ is called framed first osculating curve in $\mathbb{R}^4$ if its position vector γ satisfies:*

$$\gamma(s)=\varepsilon_1(s)\nu(s)+\varepsilon_2(s)\overline{\mu}_1(s)+\varepsilon_3(s)\overline{\mu}_3(s) \tag{4.9}$$

for some functions $\varepsilon_1(s),\varepsilon_2(s),\varepsilon_3(s)$.

Theorem 4.5. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3): I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with non-zero framed curvatures. Then, γ is a framed first osculating curve if and only if $\varepsilon_2 L(s)+\varepsilon_3 N(s)-\alpha(s)=0$ where ε_2 and ε_3 are real constants.*

Proof. Suppose that γ is a framed first osculating curve. By differentiating equation 4.9, we get

$$\begin{aligned}\alpha(s)\nu(s) &= (\varepsilon_1^{'}(s)+\varepsilon_2(s)L(s)+\varepsilon_3(s)N(s))\nu(s)+(-L(s)\varepsilon_1(s)+\varepsilon_2^{'}(s))\overline{\mu}_1(s)\\ &\quad -\varepsilon_1(s)M(s)\overline{\mu}_2(s)+(-N(s)\varepsilon_1(s)+\varepsilon_3^{'}(s))\overline{\mu}_3(s)\end{aligned}$$

Therefore, we have

$$\begin{aligned} L(s)\varepsilon_1(s)-\varepsilon_2^{'}(s) &= 0\\ M(s)\varepsilon_1(s) &= 0\\ N(s)\varepsilon_1(s)+\varepsilon_3^{'}(s) &= 0\\ \varepsilon_1^{'}(s)+\varepsilon_2(s)L(s)+\varepsilon_3(s)N(s) &= \alpha(s)\end{aligned} \tag{4.10}$$

Since framed curvatures are non-zero, we get

$$\varepsilon_1(s)=0,\quad \varepsilon_2=\varepsilon_2(const.),\varepsilon_3=\varepsilon_3(const.) \tag{4.11}$$

Consequently, by using last equation of 4.10 and equation 4.11, we have

$$\varepsilon_2L(s)+\varepsilon_3N(s)-\alpha(s)=0 \tag{4.12}$$

where ε_2 and ε_3 are real constants. Conversely, assume that the curvatures (L,M,N,α) satisfies the equation 4.12. By considering, the vector $X\in\mathbb{R}^4$ given by

$$X(s)=\gamma(s)-\varepsilon_2\overline{\mu}_1(s)-\varepsilon_3\overline{\mu}_3(s) \tag{4.13}$$

By differentiating equation 4.13, we have

$$X^{'}(s)=\alpha(s)\nu(s)-(\varepsilon_2L(s)+\varepsilon_3N(s))\nu(s) \tag{4.14}$$

By using equation 4.14, we get

$$X^{'}(s)=0 \tag{4.15}$$

By according to equation 4.15, we conclude that γ is congruent to a framed first osculating curve in $\mathbb{R}^4$. □

Corollary 4.6. *If γ is a framed first osculating curve with non-zero curvatures in $\mathbb{R}^4$, then the curvatures of γ have the following relationships*

$$\frac{\left(\frac{\alpha}{N}\right)^{'}}{\left(\frac{L}{N}\right)^{'}}=constant\quad or\quad \frac{\left(\frac{\alpha}{L}\right)^{'}}{\left(\frac{N}{L}\right)^{'}}=constant \tag{4.16}$$

Proof. By using equation 4.12 and the derivative of equation 4.12, we get 4.16. □

4.3. Framed second osculating curves

Definition 4.7. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3):I\to\mathbb{R}^4\times\Delta_3$ be a framed curve. γ is called framed second osculating curve in $\mathbb{R}^4$ if its position vector γ satisfies:*

$$\gamma(s)=\eta_1(s)\nu(s)+\eta_2(s)\overline{\mu}_1(s)+\eta_3(s)\overline{\mu}_2(s)$$

for some functions $\eta_1(s),\eta_2(s),\eta_3(s)$.

Theorem 4.8. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3):I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with non-zero framed curvatures. Then, γ is a framed second osculating curve if and only if $\eta_2L(s)+\eta_3M(s)-\alpha(s)=0$ where η_2 and η_3 are real constants.*

Proof. Its proof is done in a similar way to Theorem 4.5. □

Corollary 4.9. *If γ is a framed second osculating curve with non-zero curvatures in $\mathbb{R}^4$, then the curvatures of γ have the following relationships*

$$\frac{\left(\frac{\alpha}{M}\right)^{'}}{\left(\frac{L}{M}\right)^{'}}=constant\quad or\quad \frac{\left(\frac{\alpha}{L}\right)^{'}}{\left(\frac{M}{L}\right)^{'}}=constant$$

4.4. Framed normal curves

Definition 4.10. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3): I\to\mathbb{R}^4\times\Delta_3$ be a framed curve. γ is called framed normal curve in $\mathbb{R}^4$ if its position vector γ satisfies:*

$$\gamma(s)=\delta_1(s)\overline{\mu}_1(s)+\delta_2(s)\overline{\mu}_2(s)+\delta_3(s)\overline{\mu}_3(s) \tag{4.17}$$

for some functions $\delta_1(s),\delta_2(s),\delta_3(s)$.

Theorem 4.11. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3): I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with non-zero framed curvatures. Then, γ is a framed normal curve if and only if $\delta_1 L(s))+\delta_2 M(s)+\delta_3 N(s)-\alpha(s)=0$ where δ_1, δ_2, δ_3 are real constants.*

Proof. Suppose that γ is a framed normal curve. By differentiating equation 4.17, we get

$$\begin{aligned}\alpha(s)\nu(s) \quad = \quad & (\delta_1(s)L(s)+\delta_2(s)M(s)+\delta_3(s)N(s))\nu(s)+\delta_1^{'}(s)\overline{\mu}_1(s)\\ & +\delta_2^{'}\overline{\mu}_2(s)+\delta_3^{'}(s)\overline{\mu}_3(s)\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\delta_1^{'}(s)=0\\&\delta_2^{'}(s)=0\\&\delta_3^{'}(s)=0\\&\delta_1(s)L(s)+\delta_2(s)M(s)+\delta_3(s)N(s)=\alpha(s)\end{aligned} \tag{4.18}$$

Then, we have

$$\delta_1(s)=\delta_1(const.),\delta_2=\delta_2(const.),\delta_3=\delta_3(const.) \tag{4.19}$$

Consequently, by using last equation of 4.18 and equation 4.19, we have

$$\delta_1 L(s)+\delta_2 M(s)+\delta_3 N(s)-\alpha(s)=0 \tag{4.20}$$

where δ_1,δ_2 and δ_3 are real constants. Conversely, assume that the curvatures (L,M,N,α) satisfies the equation 4.20. By considering, the vector $X\in\mathbb{R}^4$ given by

$$X(s)=\gamma(s)-\delta_1\overline{\mu}_1(s)-\delta_2\overline{\mu}_2(s)-\delta_3\overline{\mu}_3(s) \tag{4.21}$$

By differentiating equation 4.21, we have

$$X^{'}(s)=\alpha(s)\nu(s)-(\delta_1 L(s)+\delta_2 M(s)+\delta_3 N(s))\nu(s)$$

Then, we get

$$X^{'}(s)=0 \tag{4.22}$$

By according to equation 4.22, we conclude that γ is congruent to a framed normal curve in $\mathbb{R}^4$. □

5. Conclusion

In this study, we defined framed curves in four-dimensional Euclidean space. In addition, we gave Bishop-type frame of framed curves in four-dimensional Euclidean space. Actually, the Frenet-type frame of framed curves can give in four-dimensional Euclidean space. In addition, we investigate framed rectifying, normal and osculating curves. Thus, since the four-dimensional frame of framed curve, all framed curve studies can be extended to this space.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Universal Journal of Mathematics and Applications, 4 (4) (2021) 132-135
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.973671

The Bounds for the First General Zagreb Index of a Graph

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Article Info

Keywords: *The first general Zagreb index, the chromatic number, the clique number, the independent number.*
2010 AMS: *05C09*
Received: *21 July 2021*
Accepted: *22 November 2021*
Available online: *30 December 2021*

Abstract

The first general Zagreb index of a graph G is defined as the sum of the αth powers of the vertex degrees of G, where α is a real number such that $\alpha \neq 0$ and $\alpha \neq 1$. In this note, for $\alpha > 1$, we present upper bounds involving chromatic and clique numbers for the first general Zagreb index of a graph; for an integer $\alpha \geq 2$, we present a lower bound involving the independence number for the first general Zagreb index of a graph.

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. Let $G = (V(G), E(G))$ be a graph with n vertices and e edges, where $V = \{v_1, v_2, ..., v_n\}$. We assume that the vertices in G are arranged such that $\Delta(G) = d_G(v_1) \geq d_G(v_2) \geq \cdots \geq d_G(v_n) = \delta(G)$, where $d_G(v_i)$, for each i with $1 \leq i \leq n$, is the degree of vertex v_i in G. The chromatic number, denoted $\chi(G)$, of a graph G is the smallest number of colors which can be assigned to $V(G)$ so that the adjacent vertices in G are colored differently. A clique of a graph G is a complete subgraph of G. A clique of largest possible size is called a maximum clique. The clique number, denoted $\omega(G)$, of a graph G is the number of vertices in a maximum clique of G. A set of vertices in a graph G is independent if the vertices in the set are pairwise nonadjacent. A maximum independent set in a graph G is an independent set of largest possible size. The independence number, denoted $\beta(G)$, of a graph G is the cardinality of a maximum independent set in G. If H is any graph of order n with degree sequence $d_H(v_1) \geq d_H(v_2) \geq \cdots \geq d_H(v_n)$, and if H^* is any graph of order n with degree sequence $d^*_H(v_1) \geq d^*_H(v_2) \geq \cdots \geq d^*_H(v_n)$, such that $d_H(v_i) \leq d^*_H(v_i)$ (for each i with $1 \leq i \leq n$), then H^* is said to dominate H. We use $C(n,r)$ to denote the number of r-element subsets of a set of size n, where n and r are nonnegative integers such that $0 \leq r \leq n$.

The first Zagreb index was introduced by Gutman and Trinajstić in [8]. For a graph G, its first Zagreb index is defined as $\sum_{i=1}^{n} d_G^2(v_i)$. Li and Zheng in [9] further extended the first Zagreb index of a graph and introduced the concept of the first general Zagreb index of a graph. The first general Zagreb index, denoted $M_\alpha(G)$, of a graph G is defined as $\sum_{i=1}^{n} d_G^\alpha(v_i)$, where α is a real number such that $\alpha \neq 0$ and $\alpha \neq 1$.

In this note, we will present upper bounds involving chromatic and cliques numbers for the first general Zagreb index of a graph when $\alpha > 1$ and a lower bound involving the independent number for the first general Zagreb index of a graph when α is an integer at least 2. The main results of this note are as follows.

Theorem 1.1. *Let G be a graph of order n. Assume α is a real number such that $\alpha > 1$. Then*

$$(1)\ M_\alpha \leq n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\chi}\right).$$

Equality holds if and only if G is K_n.

$$(2)\ M_\alpha \leq n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\omega}\right).$$

Equality holds if and only if G is K_n.

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Theorem 1.2. *Let G be a graph of order n. Assume α is an integer which is at least 2. Then*

$$M_\alpha \geq \frac{n^{\alpha+1}}{\beta^\alpha} + n(\Delta^\alpha - (1+\Delta)^\alpha).$$

Equality holds if and only if G is a disjoint union of β complete graphs of order $\Delta+1$.

2. Lemmas

In order to prove Theorem 1 and Theorem 2, we need the following results as our lemmas. The first one is a theorem proved by Erdős in [6]. Its proofs in English can be found in [1].

Lemma 2.1. *If H is any graph of order n, then there exists a graph H^* of order n, where $\chi(H^*) \leq \omega(H)$, such that H^* dominates H.*

The second one can be found in [4] and [10].

Lemma 2.2. *If G is a graph, then*

$$\beta \geq \sum_{v \in V} \frac{1}{d(v)+1}.$$

Equality holds if and only if each component of G is complete.

3. Proofs

Next, we will prove Theorem 1.1. The ideas from the proofs of Theorem 3 on Page 53 in [5] will be used in the proofs of Theorem 1.1 below.

Proof of (1) in Theorem 1.1 Let us partition the vertex set V of G into the pairwise disjoint nonempty subsets of V_1, V_2, ..., V_χ such that V_i is independent for each i with $1 \leq i \leq \chi$. Set $|V_i| := n_i$ for each i with $1 \leq i \leq \chi$. Then we have that $n = \sum_{i=1}^{\chi} n_i$ and $d(x) \leq n - n_i$ for each vertex x in V_i and each i with $1 \leq i \leq \chi$. Without loss of generality, we assume that $n_1 \leq n_2 \leq \cdots \leq n_\chi$. From Cauchy-Schwarz inequality, we have that

$$\sum_{i=1}^{\chi} n_i^2 \geq \frac{\left(\sum_{i=1}^{\chi} n_i\right)^2}{\chi} = \frac{n^2}{\chi}.$$

Now

$$\begin{aligned}
M_\alpha &= \sum_{v \in V} d^\alpha(v), \\
&= \sum_{i=1}^{\chi} \sum_{v \in V_i} d^\alpha(v), \\
&\leq \sum_{i=1}^{\chi} n_i (n-n_i)^\alpha, \\
&= \sum_{i=1}^{\chi} n_i (n-n_i)(n-n_i)^{\alpha-1} \leq \sum_{i=1}^{\chi} n_i (n-n_i)(n-n_1)^{\alpha-1}, \\
&= (n-n_1)^{\alpha-1} \sum_{i=1}^{\chi} n_i(n-n_i) \leq (n-1)^{\alpha-1}(n^2 - \sum_{i=1}^{\chi} n_i^2) \leq (n-1)^{\alpha-1}\left(n^2 - \frac{n^2}{\chi}\right) = n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\chi}\right).
\end{aligned}$$

If

$$M_\alpha = n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\chi}\right),$$

we, from the above proofs, have that $n_1 = n_2 = \cdots = n_\chi = 1$ and $d(v) = n-1$ for each vertex v in V. Thus G is K_n. If G is K_n, it is easy to verify that

$$M_\alpha = n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\chi}\right).$$

This completes the proof of (1) in Theorem 1.1.

Proof of (2) in Theorem 1.1 From Lemma 2.1, we can find a graph G^* dominating G and $\chi(G^*) \leq \omega(G)$. From (1) of this theorem, we have that

$$M_\alpha(G) \leq M_\alpha(G^*) \leq n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\chi(G^*)}\right) \leq n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\omega(G)}\right).$$

If

$$M_\alpha(G) = n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\omega}\right),$$

then

$$M_\alpha(G^*) = n^2(n-1)^{\alpha-1}\left(1-\frac{1}{\chi(G^*)}\right).$$

From (1) of this theorem, we have that G^* is K_n and $\chi(G^*) = n$. Thus $\omega(G) \geq \chi(G^*) = n$. Hence G is K_n. If G is K_n, it is again easy to verify that

$$M_\alpha(G) = n^2(n-1)^{\alpha-1}\left(1-\frac{1}{\omega}\right).$$

This completes the proof of (2) in Theorem 1.1.

Next, we will prove Theorem 1.2 which is motivated by Theorem 3.1 on Page 309 in [7].

Proof of Theorem 1.2 From Lemma 2.2 and the inequalities on the power means, arithmetic means, and harmonic means of n positive integers, we have that

$$\left(\frac{(1+d_1)^\alpha+(1+d_2)^\alpha+\cdots+(1+d_n)^\alpha}{n}\right)^{\frac{1}{\alpha}} \geq \frac{(1+d_1)+(1+d_2)+\cdots+(1+d_n)}{n} \geq \frac{n}{\frac{1}{1+d_1}+\frac{1}{1+d_2}+\cdots+\frac{1}{1+d_n}} \geq \frac{n}{\beta}.$$

Then

$$(1+d_1)^\alpha+(1+d_2)^\alpha+\cdots+(1+d_n)^\alpha \geq \frac{n^{\alpha+1}}{\beta^\alpha}.$$

It is easy to check that for each i with $1 \leq i \leq n$ we have

$$(1+d_i)^\alpha = \sum_{k=0}^{\alpha} C(\alpha,k)d_i^k \leq \sum_{k=0}^{\alpha} C(\alpha,k)\Delta^k - \Delta^\alpha + d_i^\alpha = (1+\Delta)^\alpha - \Delta^\alpha + d_i^\alpha.$$

Equality holds if and only if $d_i = \Delta$. Thus

$$(1+\Delta)^\alpha-\Delta^\alpha+d_1^\alpha+(1+\Delta)^\alpha-\Delta^\alpha+d_2^\alpha+\cdots+(1+\Delta)^\alpha-\Delta^\alpha+d_n^\alpha \geq (1+d_1)^\alpha+(1+d_2)^\alpha+\cdots+(1+d_n)^\alpha \geq \frac{n^{\alpha+1}}{\beta^\alpha}.$$

Therefore

$$M_\alpha \geq \frac{n^{\alpha+1}}{\beta^\alpha} + n(\Delta^\alpha - (1+\Delta)^\alpha).$$

If

$$M_\alpha = \frac{n^{\alpha+1}}{\beta^\alpha} + n(\Delta^\alpha - (1+\Delta)^\alpha),$$

then $d_1 = d_2 = \cdots = d_n = \Delta$. From Lemma 2, we have that G is a union of β complete graphs of order $\Delta+1$. If G is a union of β complete graphs of order $\Delta+1$, then $(\Delta+1)\beta = n$. It is easy to verify that

$$M_\alpha = \frac{n^{\alpha+1}}{\beta^\alpha} + n(\Delta^\alpha - (1+\Delta)^\alpha).$$

This completes the proof of Theorem 1.2.

Remark 3.1. *Let G be a graph with n vertices and e edges. Notice that*

$$n+4e+M_2 = \sum_{i=1}^{n}(1+d_i)^2 \geq \frac{n^3}{\beta^2}.$$

We have that

$$M_2 \geq \frac{n^3}{\beta^2} - n - 4e.$$

It can be verified that $M_2 = \frac{n^3}{\beta^2} - n - 4e$ if and only if G is a disjoint union of β complete graphs of order $\Delta+1$.

Remark 3.2. *Let G be a graph with n vertices and e edges. Notice that*

$$n+6e+3M_2+M_3 = \sum_{i=1}^{n}(1+d_i)^3 \geq \frac{n^4}{\beta^3}.$$

We have that

$$M_3 \geq \frac{n^4}{\beta^3} - n - 6e - 3U,$$

where U is an upper bound for M_2. A variety of concrete expressions for U can be found in [3].

Acknowledgements

The author would like to thank the referees for their suggestions and comments which improve the original version of the paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Universal Journal of Mathematics and Applications, 4 (4) (2021) 136-146
Research paper

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.984462

Singular Minimal Surfaces which are Minimal

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Article Info

Keywords: *Minimal surface, Semi-symmetric connection, Singular minimal surface, Translation surface*
2010 AMS: *53A10, 53C42, 53C05*
Received: *18 August 2021*
Accepted: *23 December 2021*
Available online: *30 December 2021*

Abstract

In the present paper, we discuss the singular minimal surfaces in Euclidean $3-$space $\mathbb{R}^3$ which are minimal. Such a surface is nothing but a plane, a trivial outcome. However, a non-trivial outcome is obtained when we modify the usual condition of singular minimality by using a special semi-symmetric metric connection instead of the Levi-Civita connection on $\mathbb{R}^3$. With this new connection, we prove that, besides planes, the singular minimal surfaces which are minimal are the generalized cylinders, providing their explicit equations. A trivial outcome is observed when we use a special semi-symmetric non-metric connection. Furthermore, our discussion is adapted to the Lorentz-Minkowski 3-space.

1. Introduction

Let $\left(\mathbb{R}^3, \langle\cdot,\cdot\rangle\right)$ be a Euclidean $3-$space and $\mathbf{v}$ a fixed unit vector in $\mathbb{R}^3$. Let $\mathbf{r}: M^2 \to \mathbb{R}^3_+(\mathbf{v})$ be a smooth immersion of an oriented compact surface M^2 into the halfspace

$$\mathbb{R}^3_+(\mathbf{v}) : \left\{ p \in \mathbb{R}^3 : \langle p, \mathbf{v}\rangle > 0 \right\}.$$

Denote H and $\mathbf{n}$ the mean curvature and unit normal vector field on M^2. Let $\alpha \in \mathbb{R}$. The potential $\alpha-$energy of $\mathbf{r}$ in the direction of $\mathbf{v}$ is defined by [32]

$$E(\mathbf{r}) = \int\limits_{M^2} \langle p, \mathbf{v}\rangle^{\alpha} dM^2,$$

where dM^2 is the measure on M^2 with respect to the induced metric tensor from the Euclidean metric $\langle\cdot,\cdot\rangle$ and $p = \mathbf{r}(\tilde{p})$, $\tilde{p} \in M^2$.
Let $\Sigma : M^2 \times (-\theta,\theta) \to \mathbb{R}^3_+(\mathbf{v})$ be a compactly supported variation of $\mathbf{r}$ with variaton vector field ξ. The first variation of E becomes

$$E'(0) = -\int\limits_{M^2} \left(2H\langle \mathbf{r},\mathbf{v}\rangle - \alpha\langle \mathbf{n},\mathbf{v}\rangle\right)\langle \xi,\mathbf{n}\rangle^{\alpha-1} dM^2.$$

For all compactly supported variations, the immersion $\mathbf{r}$ is a critical point of E if and only if

$$2H(\tilde{p}) = \alpha \frac{\langle \mathbf{n}(\tilde{p}), \mathbf{v}\rangle}{\langle \mathbf{r}(\tilde{p}), \mathbf{v}\rangle}, \tag{1.1}$$

for some point $\tilde{p} \in M^2$.
A surface M^2 is referred to as a *singular minimal surface* or $\alpha-$*minimal surface* with respect to the vector $\mathbf{v}$, if holds Eq. (1.1) (see [11, 12]). In the particular case $\alpha = 1$ and $\mathbf{v} = (0,0,1)$, the surface M^2 represents *two-dimensional analogue of the catenary* which is known as a model for the surfaces with the lowest gravity center, in other words, one has minimal potential energy under gravitational forces [6, 13, 18].

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A *translation surface* M^2 in $\mathbb{R}^3$ is a surface that can be written as the sum of two so-called *translating curves* [9]. When the translating curves lie in orthogonal planes, up to a change of coordinates, the surface M^2 can be locally given in the explicit form $z = p(x) + q(y)$, where (x, y, z) is the rectangular coordinates and p, q smooth functions. In such case, if M^2 is minimal (H vanishes identically [27, p. 17]), it describes a plane or the *Scherk surface* [43]

$$z(x,y) = \frac{1}{\lambda}\log\left|\frac{\cos\lambda x}{\cos\lambda y}\right|,\ \lambda\in\mathbb{R},\ \lambda\neq 0.$$

If the translating curves lie in non-orthogonal planes, the translation surface M^2 is locally given by $z = p(x) + q(y + \mu x)$, $\mu \in \mathbb{R}$, $\mu \neq 0$, and so-called an *affine translation surface* or a *translation graph* [26, 45]. A minimal affine translation surface is so-called *affine Scherk surface* and is given in the explicit form

$$z(x,y) = \frac{1}{\lambda}\log\left|\frac{\cos\lambda\sqrt{1+\mu^2}x}{\cos\lambda(y+\mu x)}\right|.$$

López [32] obtained the singular minimal translation surfaces in $\mathbb{R}^3$ of type $z = p(x) + q(y)$ with respect to horizontal and vertical directions. This result was generalized to higher dimensions in [5]. For further study of singular minimal surfaces, we refer to the López's series of interesting papers on the solutions of the Dirichlet problem for the $\alpha-$singular minimal surface equation [33], the Lorentz-Minkowski counterpart of the condition of singular minimality [34], the compact singular minimal surfaces [35] and the singular minimal surfaces with density [36].

In this paper, we approach a singular minimal surface M^2 in $\mathbb{R}^3$ which is minimal. We hereinafter assume that $\alpha \neq 0$ in Eq. (1.1), otherwise any minimal surface obeys our approach, which is trivial. Under this circumstance, Eq. (1.1) gives $\langle \mathbf{n}(\tilde{p}), \mathbf{v}\rangle = 0$, that is, the tangent plane of M^2 at any point $\tilde{p}$ is parallel to $\mathbf{v}$. In such case, the surface M^2 belongs to the class of so-called *constant angle surfaces* and has to be a plane parallel to $\mathbf{v}$ (see [37, Proposition 9]), yielding the following outcome.

Proposition 1.1. *Let M^2 be a singular minimal surface in $\mathbb{R}^3$ with respect to an arbitrary vector $\mathbf{v}$. If M^2 is minimal, then it is a plane parallel to $\mathbf{v}$.*

This result is changed when we modify Eq. (1.1) by using a special semi-symmetric metric connection ∇ (see Eq. (3.1)) on $\mathbb{R}^3$. In Section 3, we prove that, besides planes, the singular minimal surfaces which are minimal with respect to ∇ are the generalized cylinders, providing their explicit equations. It is also observed, in Section 3, that this approach produces only trivial example when a special semi-symmetric non-metric connection D (see Eq. (3.19)) is used.

We find the motivation in Wang's paper [44] whose minimal translation surfaces were obtained with respect to the connections ∇ and D. The notion of a semi-symmetric metric (resp. non-metric) connection on a Riemannian manifold were defined by Hayden [22] (resp. Agashe [1]) and since then has been studied by many authors. Without giving a complete list, we may refer to [2–4, 7, 10, 14, 15, 19, 25, 38–42, 47–50]. The present authors also obtained singular minimal translation surfaces in $\mathbb{R}^3$ with respect to the connections ∇ and D [16].

Let $\mathbb{R}^3_1$ be a Lorentz-Minkowski 3−space endowed with the canonical Lorentzian metric $\langle\cdot,\cdot\rangle_L = dx^2 + dy^2 - dz^2$. Then we have [34, Definition 1.1]

Definition 1.1. *Let $\mathbf{r}$ be a smooth immersion of a spacelike surface M^2 in the halfspace $z > 0$ of $\mathbb{R}^3_1$ and $\mathbf{n}$ unit normal vector field on M^2 and H the mean curvature. M^2 is called $\alpha-$singular maximal surface if satisfies*

$$H = -\alpha\frac{\langle\mathbf{n},(0,0,1)\rangle_L}{z},\ \alpha\neq 0. \tag{1.2}$$

Due to the fact that the $z-$coordinate represents the time coordinate, the concept of gravity has no meaning. Therefore, unlike the Riemannian case, Eq. (1.2) describes only spacelike surfaces with prescribed angle between $\mathbf{n}$ and the $z-$axis. Point out that H is non-vanishing in Eq. (1.2) if $\alpha \neq 0$ because $\langle\mathbf{n},(0,0,1)\rangle_L \neq 0$ for timelike vectors $\mathbf{n}$ and $(0,0,1)$ and so we can not adapt Eq. (1.2) to our study as is. For this reason, we modify the concept as follows:

Definition 1.2. *Let $\mathbf{r}$ be a smooth immersion of an oriented timelike surface M^2 in $\mathbb{R}^3_1$ and $\mathbf{n}$ unit normal vector field on M^2 and H the mean curvature. Let $\mathbf{v} \in \mathbb{R}^3_1$, $\mathbf{v} \neq \mathbf{0}$, a spacelike vector non-parallel to $\mathbf{n}$ such that $\mathbf{n}$ and $\mathbf{v}$ span a spacelike 2-space. Then M^2 is called singular minimal surface with respect to $\mathbf{v}$ if satisfies*

$$2H = \alpha\frac{\langle\mathbf{n},\mathbf{v}\rangle_L}{\langle\mathbf{r},\mathbf{v}\rangle_L},\ \alpha\in\mathbb{R},\ \alpha\neq 0. \tag{1.3}$$

With Definition 1.2, we may view the singular minimal surface M^2 as a timelike surface in $\mathbb{R}^3_1$ with prescribed Lorentz spacelike angle between $\mathbf{n}$ and $\mathbf{v}$. If M^2 is minimal, it follows from Eq. (1.3) that $\langle\mathbf{n},\mathbf{v}\rangle_L = 0$, namely the angle is $\frac{\pi}{2}$, and, as in Riemannian case, M^2 becomes a timelike constant angle surface which has to be a plane (see [21, Theorem 3.1]), yielding the following trivial outcome.

Proposition 1.2. *Let M^2 be a singular minimal surface in $\mathbb{R}^3_1$ with respect to a spacelike vector $\mathbf{v}$. If M^2 is minimal, then it is a plane parallel to $\mathbf{v}$.*

In Section 4, we also state non-trivial results in $\mathbb{R}^3_1$ for singular minimal surfaces which are minimal with respect to the connections ∇ and D given by Eqs. (4.1) and (4.19), respectively.

2. Preliminaries

Most of following notions can be found [8,40,46].
Let $(\bar{M},\bar{g})$ be a semi-Riemannian manifold and $\bar{\nabla}$ a linear connection on $\bar{M}$. The *torsion tensor field* T of $\bar{\nabla}$ is defined by

$$T(\bar{\mathbf{x}},\bar{\mathbf{y}})=\bar{\nabla}_{\bar{\mathbf{x}}}\bar{\mathbf{y}}-\bar{\nabla}_{\bar{\mathbf{x}}}\bar{\mathbf{y}}-[\bar{\mathbf{x}},\bar{\mathbf{y}}],$$

where $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are vector fields on $\bar{M}$. A linear connection is called a *semi-symmetric* (resp. *non-*) *metric connection* if there exist a $1-$form π such that

$$T(\bar{\mathbf{x}},\bar{\mathbf{y}})=\pi(\bar{\mathbf{y}})\bar{\mathbf{x}}-\pi(\bar{\mathbf{x}})\bar{\mathbf{y}},\ \bar{\nabla}\bar{g}=0\ (\text{resp. } \bar{\nabla}\bar{g}\neq 0).$$

The linear connection $\bar{\nabla}$ is called *Levi-Civita connection* if $T=0$ and $\bar{\nabla}\bar{g}=0$. We denote the Levi-Civita connection by $\bar{\nabla}^L$.
Let M be a semi-Riemannian submanifold of $\bar{M}$ and ∇^L and g the induced Levi-Civita connection and metric tensor, respectively. Then the *Gauss formula* follows

$$\bar{\nabla}^L_{\mathbf{x}}\mathbf{y}=\nabla^L_{\mathbf{x}}\mathbf{y}+h(\mathbf{x},\mathbf{y}),$$

where h is so-called *second fundamental form* of M and $\mathbf{x}$ and $\mathbf{y}$ tangent vector fields to M. Let $\{\mathbf{f}_1,...,\mathbf{f}_n\}$ be an orthonormal frame on M at any point $p\in M$. Then the *mean curvature vector* $\mathbf{H}(p)$ at p is defined by

$$\mathbf{H}(p)=\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i h(\mathbf{f}_i,\mathbf{f}_i),$$

where $\varepsilon_i=g(\mathbf{f}_i,\mathbf{f}_i)$ and $n=\dim M$. The length of mean curvature vector is called *mean curvature*. A semi-Riemannian submanifold is called *minimal* if its mean curvature vanishes identically.
Let $\bar{M}=\mathbb{R}^3_1$ be the Lorentz-Minkowski $3-$space and $\bar{g}=\langle\cdot,\cdot\rangle_L=dx^2+dy^2-dz^2$. A vector field $\mathbf{x}$ on $\mathbb{R}^3_1$ is said to be *spacelike* (resp. *timelike*) if $\mathbf{x}=0$ or $\langle\mathbf{x},\mathbf{x}\rangle_L>0$ (resp. $\langle\mathbf{x},\mathbf{x}\rangle_L<0$). A vector field $\mathbf{x}$ is said to be *null* if $\langle\mathbf{x},\mathbf{x}\rangle_L=0$ and $\mathbf{x}\neq 0$. A timelike vector $\mathbf{x}=(a,b,c)$ is said to be *future pointing* (resp. *past pointing*) if $c>0$ (resp. $c<0$). A *Lorentz timelike angle* θ between two future (past) pointing timelike vectors $\mathbf{x}$ and $\mathbf{y}$ is associated with [17]

$$|\langle\mathbf{x},\mathbf{y}\rangle_L|=\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle_L|}\sqrt{|\langle\mathbf{y},\mathbf{y}\rangle_L|}\cosh\theta.$$

A *Lorentz spacelike angle* θ between two spacelike vectors $\mathbf{x}$ and $\mathbf{y}$ spanning a spacelike vector subspace ($\mathbb{R}^3_1$ induces a Riemannian metric on it) is associated with [17]

$$|\langle\mathbf{x},\mathbf{y}\rangle_L|=\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle_L|}\sqrt{|\langle\mathbf{y},\mathbf{y}\rangle_L|}\cos\theta.$$

Let M^2 be an immersed surface into $\mathbb{R}^3_1$. The surface M^2 is said to be *spacelike* (resp. *timelike*) if all tangent planes of M^2 are spacelike (resp. timelike). For such a spacelike (resp. timelike) surface, we have the decomposition $\mathbb{R}^3_1=T_pM^2\oplus\left(T_pM^2\right)^{\perp}$, where T_pM^2 is the tangent plane of M^2 at the point p. Notice that $\left(T_pM^2\right)^{\perp}$ is a timelike (resp. spacelike) $1-$space of $\mathbb{R}^3_1$. A *Gauss map* $\mathbf{n}$ of M^2 is a smooth map $\mathbf{n}:M^2\to\mathbb{R}^3_1$, $|\langle\mathbf{n},\mathbf{n}\rangle_L|=1$.
We finish this section remarking that a spacelike (resp. timelike) surface in $\mathbb{R}^3_1$ is locally a graph of a smooth function $u(x,y)$ (resp. $u(x,z)$ or $u(y,z)$) [28, Proposition 3.3].

3. Singular minimal surfaces in $\mathbb{R}^3$

3.1. $\nabla-$Singular minimal surfaces

Let ∇^L be the Levi-Civita connection on $\mathbb{R}^3$ and $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ the standard basis on $\mathbb{R}^3$ and $\mathbf{x},\mathbf{y}$ tangent vector fields to $\mathbb{R}^3$. Consider the following semi-symmetric metric connection on $\mathbb{R}^3$ [44]

$$\nabla_{\mathbf{x}}\mathbf{y}=\nabla^L_{\mathbf{x}}\mathbf{y}+\langle\mathbf{y},\mathbf{e}_3\rangle\mathbf{x}-\langle\mathbf{x},\mathbf{y}\rangle\mathbf{e}_3. \tag{3.1}$$

The nonzero derivatives are

$$\nabla_{\mathbf{e}_1}\mathbf{e}_1=-\mathbf{e}_3,\ \nabla_{\mathbf{e}_1}\mathbf{e}_3=\mathbf{e}_1,\ \nabla_{\mathbf{e}_2}\mathbf{e}_2=-\mathbf{e}_3,\ \nabla_{\mathbf{e}_2}\mathbf{e}_3=\mathbf{e}_2.$$

Definition 3.1. *Let $\mathbf{r}$ be a smooth immersion of an oriented surface M^2 into $\mathbb{R}^3$ and $\mathbf{n}$ unit normal vector field on M^2 and H^{∇} the mean curvature with respect to ∇. Let $\mathbf{v}\in\mathbb{R}^3$, $\mathbf{v}\neq\mathbf{0}$, a unit fixed vector non-parallel to $\mathbf{n}$. The surface M^2 is called $\nabla-$singular minimal surface with respect to $\mathbf{v}$ if holds*

$$2H^{\nabla}=\alpha\frac{\langle\mathbf{n},\mathbf{v}\rangle}{\langle\mathbf{r},\mathbf{v}\rangle},\ \alpha\in\mathbb{R},\ \alpha\neq 0. \tag{3.2}$$

In particular, the surface M^2 is said to be $\nabla-$*minimal* if $H^{\nabla}=0$. With Definition 3.1, we first observe the $\nabla-$singular minimal surfaces of type $z=u(x,y)$ which are $\nabla-$minimal.

Theorem 3.1. *Let M^2 be a $\nabla-$singular minimal surface in $\mathbb{R}^3$ of type $z=u(x,y)$ with respect to a unit vector $\mathbf{v}=(a,b,c)$, $a^2+b^2\neq 0$. If M^2 is $\nabla-$minimal, then one of the following happens*

1. $\mathbf{v}=(0,b\neq 0,c)$ *and*

$$u(x,y)=\frac{c}{b}y+\frac{1}{2b^2}\ln\left[\cos\left(2bx+\lambda_1\right)\right]+\lambda_2;$$

2. $\mathbf{v}=(a\neq 0,0,c)$ *and*

$$u(x,y)=\frac{c}{a}x+\frac{1}{2a^2}\ln\left[\cos\left(2ay+\lambda_3\right)\right]+\lambda_4;$$

3. $\mathbf{v}=(a,b,c)$, $ab\neq 0$, *and*

$$u(x,y)=\frac{c}{a}x-\frac{1}{2\left(a^2+b^2\right)}\ln\left[\cos\left(-2\left|a\right|\left(y-\frac{b}{a}x\right)+\lambda_5\right)\right]+\frac{bc}{a^2+b^2}\left(y-\frac{b}{a}x\right)+\lambda_6,$$

where $\lambda_1,...,\lambda_6\in\mathbb{R}$.

Proof. The unit normal vector field on M^2 is

$$\mathbf{n}=\frac{-u_x\mathbf{e}_1-u_y\mathbf{e}_2+\mathbf{e}_3}{\sqrt{1+(u_x)^2+\left(u_y\right)^2}},$$

where $u_x=\frac{\partial u}{\partial x}$ and so. Suppose that M^2 is $\nabla-$minimal. Due to $\alpha\neq 0$, Eq. (3.2) gives $\langle\mathbf{n},\mathbf{v}\rangle=0$ and

$$au_x+bu_y=c. \tag{3.3}$$

The condition of $\nabla-$minimality yields

$$\left[1+\left(u_y\right)^2\right]u_{xx}-2u_xu_yu_{xy}+\left[1+(u_x)^2\right]u_{yy}-2\left[1+(u_x)^2+\left(u_y\right)^2\right]=0. \tag{3.4}$$

We distinguish several cases: the first case is that $a=0$. Then Eq. (3.3) follows $u(x,y)=f(x)+\frac{c}{b}y$, for an arbitrary smooth function f. Considering this into Eq. (3.4) leads to

$$\frac{bf''}{1+(bf')^2}=2b, \tag{3.5}$$

where $f'=\frac{df}{dx}$ and so. The first statement of the theorem is obtained by integrating Eq. (3.5). The roles of x and y in Eq. (3.4) are symmetric and hence we may conclude the second statement of the theorem by similar steps when $a\neq 0$ and $b=0$. The last case is that $ab\neq 0$. Then the solution to Eq. (3.3) is given by

$$u(x,y)=\frac{c}{a}x+g\left(y-\frac{b}{a}x\right), \tag{3.6}$$

for a smooth function g. Substituting Eq. (3.6) into Eq. (3.4) follows

$$g''-2\left[a^2+\left(c-bg'\right)^2+\left(ag'\right)^2\right]=0, \tag{3.7}$$

for $g'=\frac{dg}{d\tilde{y}}$, $g''=\frac{d^2g}{d\tilde{y}^2}$, $\tilde{y}=y-\frac{b}{a}x$. Eq. (3.7) can be rewritten as

$$\frac{\left(a^2+b^2\right)g''}{a^2+\left(bc-\left(a^2+b^2\right)g'\right)^2}=2. \tag{3.8}$$

The proof is completed by integrating Eq. (3.8). □

Remark 3.1. *The surface given in the first statement of Theorem 3.1 is a generalized cylinder (see [20, p. 439]) and may be written parametrically*

$$\mathbf{r}(x,y)=\left(x,0,\frac{1}{2b^2}\ln\left[\cos\left(2bx+\lambda_1\right)\right]+\lambda_2\right)+y\left(0,1,\frac{c}{b}\right).$$

This is a $\nabla-$*minimal translation surface of type* $z=p(x)+q(y)$ *which was already found by Wang [44]. The same may be concluded for the above second statement. However, the surface described in the last statement of Theorem 3.1 is the generalized cylinder parametrically written by*

$$\mathbf{r}(x,\tilde{y})=x\left(1,\frac{b}{a},\frac{c}{a}\right)+(0,\tilde{y},g(\tilde{y})),$$

where $\tilde{y}=y-\frac{b}{a}x$. *Due to* $b\neq 0$, *it belongs to the class of affine translation surfaces and a new example of* $\nabla-$*minimal surfaces.*

In the following we classify $\nabla-$singular minimal surfaces in $\mathbb{R}^3$ of type $y=u(x,z)$ which are $\nabla-$minimal.

Theorem 3.2. *Let* M^2 *be a* $\nabla-$*singular minimal surface in* $\mathbb{R}^3$ *of type* $y=u(x,z)$ *with respect to a unit vector* $\mathbf{v}=(a,b,c)$, $a^2+c^2\neq 0$. *If* M^2 *is* $\nabla-$*minimal, then one of the following happens*

1. *M^2 is a plane parallel to the vector $(0,0,1)$;*
2. $\mathbf{v}=(0,b,c)$, $bc\neq 0$, *and*
$$u(x,z)=\frac{b}{c}z+\frac{1}{2bc}\ln\left[\cos(2bx+\lambda_1)\right]+\lambda_2;$$
3. $\mathbf{v}=(a,b,0)$, $a\neq 0$, *and*
$$u(x,z)=\frac{b}{a}x\pm\frac{1}{2|a|}\arctan\left(\frac{1}{|a\lambda_2|}\sqrt{e^{4z}-a^2}\right)+\lambda_3;$$
4. $\mathbf{v}=(a,0,c)$, $ac\neq 0$, *and*
$$u(x,z)=\pm\frac{1}{2|a|}\arctan\left(\frac{1}{|\lambda_4|}\sqrt{e^{4a^2\left(z-\frac{c}{a}x\right)}-\lambda_4^2}\right)+\lambda_5,\ \lambda_4\neq 0;$$
5. $\mathbf{v}=(a,b,c)$, $ac\neq 0$, *and*
$$u(x,z)=\frac{b}{a}x+h\left(z-\frac{c}{a}x\right),$$
where h is a smooth function satisfying
$$z-\tfrac{c}{a}x=\tfrac{1}{2|a|(a^2+c^2)(a^2+b^2c^2)}\{bc(2|a|h+\lambda_6)-$$
$$-|a|\ln\left[bc\cos(2|a|h+\lambda_6)-|a|\sin(2|a|h+\lambda_6)\right]\}+\lambda_7,$$
for $\lambda_1,\dots,\lambda_7\in\mathbb{R}$.

Proof. Let M^2 be locally given by
$$(x,z)\longmapsto \mathbf{r}(x,z)=(x,u(x,z),z),$$
for a smooth function $u=u(x,z)$. The normal vector field on M^2 is
$$\mathbf{n}=\frac{u_x\mathbf{e}_1-\mathbf{e}_2+u_z\mathbf{e}_3}{\sqrt{1+(u_x)^2+(u_z)^2}}. \tag{3.9}$$
Because M^2 is $\nabla-$singular minimal, we get Eq. (3.2). Assume that M^2 is $\nabla-$minimal. Due to $\alpha\neq 0$, Eqs. (3.2) and (3.9) follow $\langle\mathbf{n},\mathbf{v}\rangle=0$ and
$$au_x+cu_z=b. \tag{3.10}$$
Remark also that we may write $\mathbf{v}=a\mathbf{r}_x+c\mathbf{r}_z$, which means that the tangent plane of M at any point is parallel to $\mathbf{v}$. The condition of $\nabla-$minimality leads to
$$\left[1+(u_z)^2\right]u_{xx}-2u_xu_zu_{xz}+\left[1+(u_x)^2\right]u_{zz}+2\left[1+(u_x)^2+(u_z)^2\right]u_z=0. \tag{3.11}$$
We distinguish several cases:

1. $a=0$, $c\neq 0$. Then Eq. (3.10) gives $u_z=\frac{b}{c}$ and so Eq. (3.11) turns M^2 to a plane parallel to $\mathbf{v}$ if $b=0$. Otherwise, $b\neq 0$, the solution to Eq. (3.10) is given by $u(x,z)=\frac{b}{c}z+f(x)$, for an arbitrary smooth function f. Hence Eq. (3.11) reduces to
$$\frac{cf''}{1+(cf')^2}=-2b, \tag{3.12}$$
where $f'=\frac{df}{dx}$, etc. The second statement of the theorem is obtained by integrating Eq. (3.12).
2. $a\neq 0$, $c=0$. Then Eq. (3.10) gives $u(x,z)=\frac{b}{a}x+g(z)$ for an arbitrary smooth function g and so Eq. (3.11) may be written as
$$\frac{g''}{g'}-\frac{a^2g'g''}{1+(ag')^2}=-2, \tag{3.13}$$
where $g'=\frac{dg}{dz}$, etc. Integrating Eq. (3.13), we obtain the third statement of the theorem.
3. $ac\neq 0$. The solution to Eq. (3.10) is
$$u(x,z)=\frac{b}{a}x+h\left(z-\frac{c}{a}x\right), \tag{3.14}$$
for an arbitrary smooth function h. By plugging the partial derivatives of Eq. (3.14) into Eq. (3.11), we write
$$h''+2\left[a^2+(b-ch')^2+(ah')^2\right]h'=0, \tag{3.15}$$
where $h'=\frac{dh}{d\tilde{z}}$, $h''=\frac{d^2h}{d\tilde{z}^2}$, $\tilde{z}=z-\frac{c}{a}x$. We have two subcases: the first subcase is that $b=0$. Then Eq. (3.15) may be rewritten as
$$\frac{h''}{h'}-\frac{h'h''}{a^2+(h')^2}=-2a^2. \tag{3.16}$$

The fourth statement of the theorem is proved by integrating Eq. (3.16). The second subcase is $b \neq 0$. Hence, we may write Eq. (3.15) as

$$\frac{-\left(a^2+c^2\right)h''}{a^2+\left(bc-\left(a^2+c^2\right)h'\right)^2}=2h'. \tag{3.17}$$

A first integration of Eq. (3.17) yields

$$\frac{\left(a^2+c^2\right)dh}{-|a|\tan\left(2|a|h+\lambda\right)+bc}=d\tilde{z}, \tag{3.18}$$

for $\lambda \in \mathbb{R}$. By a first integration of Eq. (3.18), we finish the proof.

$\square$

Remark 3.2. *The surfaces given in the second and third statements of Theorem 3.2 are $\nabla-$minimal generalized cylinders and are examples of $\nabla-$minimal translation surfaces of type $y=p(x)+q(z)$, which was found by Wang [44]. However, the surfaces given in the last two statements of Theorem 3.2 are a $\nabla-$minimal affine translation surface.*

Lastly, we deal with a surface M^2 of type $x=u(y,z)$. The unit normal vector field on M^2 is

$$\mathbf{n}=\frac{\mathbf{e}_1-u_y\mathbf{e}_2-u_z\mathbf{e}_3}{\sqrt{1+\left(u_y\right)^2+\left(u_z\right)^2}}.$$

Suppose that M^2 is $\nabla-$singular minimal with respect to the vector $\mathbf{v}=(a,b,c)$. The mean curvature is same as that of the surface of type $y=u(x,z)$. If M^2 is also $\nabla-$minimal, then Eq. (1.3) gives

$$bu_y+cu_z=a,$$

where $b^2+c^2\neq 0$. Therefore, without giving a proof, we may state a similar result for those surfaces of type $x=u(y,z)$ to Theorem 3.2 by replacing x with y and a with b.

3.2. $D-$Singular minimal surfaces

Let D be the semi-symmetric non-metric connection on $\mathbb{R}^3$ given by [44]

$$D_{\mathbf{x}}\mathbf{y}=\nabla^L_{\mathbf{x}}\mathbf{y}+\langle\mathbf{y},\mathbf{e}_3\rangle\mathbf{x}, \tag{3.19}$$

where $\mathbf{x},\mathbf{y}$ are tangent vector fields to $\mathbb{R}^3$. The nonzero derivatives are

$$D_{\mathbf{e}_1}\mathbf{e}_3=\mathbf{e}_1,\ D_{\mathbf{e}_2}\mathbf{e}_3=\mathbf{e}_2,\ D_{\mathbf{e}_3}\mathbf{e}_3=\mathbf{e}_3.$$

Definition 3.2. *Let $\mathbf{r}$ be a smooth immersion of an oriented surface M^2 into $\mathbb{R}^3$ and $\mathbf{n}$ unit normal vector field on M^2 and H^D denote the mean curvature with respect to D. Let $\mathbf{v}\in\mathbb{R}^3$, $\mathbf{v}\neq\mathbf{0}$, a unit fixed vector non-parallel to $\mathbf{n}$. The surface M^2 is called $D-$singular minimal surface with respect to $\mathbf{v}$ if holds*

$$2H^D=\alpha\frac{\langle\mathbf{n},\mathbf{v}\rangle}{\langle\mathbf{r},\mathbf{v}\rangle},\ \alpha\in\mathbb{R},\ \alpha\neq 0. \tag{3.20}$$

In particular, the surface M^2 is said to be *$D-$minimal* if $H^D=0$. We first consider the $D-$singular minimal surfaces of type $z=u(x,y)$ which are $D-$minimal. Hence Eq. (3.20) gives $\langle\mathbf{n},\mathbf{v}\rangle=0$ and

$$au_x+bu_y=c, \tag{3.21}$$

where $\mathbf{v}=(a,b,c)$ and $a^2+b^2\neq 0$. Morever the condition of $D-$minimality yields

$$\left[1+\left(u_y\right)^2\right]u_{xx}-2u_xu_yu_{xy}+\left[1+\left(u_x\right)^2\right]u_{yy}=0, \tag{3.22}$$

where the roles of x and y are symmetric. If $a=0$, then Eq. (3.21) follows $u(x,y)=f(x)+\frac{c}{b}y$, for an arbitrary smooth function f. Considering this into Eq. (3.22) yields $\frac{1}{b^2}\frac{d^2f}{dx^2}=0$, which leads M^2 to be a plane parallel to $\mathbf{v}$. By symmetry, we may obtain same obtain when $a\neq 0$ and $b=0$. Let $ab\neq 0$. Then the solution to Eq. (3.21) is $u(x,y)=\frac{c}{a}x+g\left(y-\frac{b}{a}x\right)$, for an arbitrary smooth function f. After substituting its partial derivatives into Eq. (3.22), we conclude $\frac{1}{a^2}\frac{d^2g}{d\tilde{y}^2}=0$, $\tilde{y}=y-\frac{b}{a}x$, yielding that M is a plane parallel to $\mathbf{v}$.
Therefore we state the following

Theorem 3.3. *Let M^2 be a $D-$singular minimal surface in $\mathbb{R}^3$ of type $z=u(x,y)$ with respect to a unit vector $\mathbf{v}=(a,b,c)$, $a^2+b^2\neq 0$. If M^2 is $D-$minimal, then it is a plane parallel to $\mathbf{v}$.*

When we take surfaces of type $y=u(x,z)$ and $x=u(y,z)$, we get similar equations to Eqs. (3.21) and (3.22) and thus the above result remains true for those surfaces as well.

4. Singular minimal surfaces in $\mathbb{R}_1^3$

4.1. $\nabla-$Singular minimal surfaces

Let ∇^L be the Levi-Civita connection $\mathbb{R}_1^3$ and $\mathbf{x},\mathbf{y}$ tangent vector fields to $\mathbb{R}_1^3$. Consider the following semi-symmetric metric connection on $\mathbb{R}_1^3$ [44]

$$\nabla_{\mathbf{x}}\mathbf{y} = \nabla^L_{\mathbf{x}}\mathbf{y} + \langle \mathbf{y},\mathbf{e}_3\rangle_L \mathbf{x} - \langle \mathbf{x},\mathbf{y}\rangle_L \mathbf{e}_3. \tag{4.1}$$

The nonzero derivatives are

$$\nabla_{\mathbf{e}_1}\mathbf{e}_1 = -\mathbf{e}_3,\ \nabla_{\mathbf{e}_1}\mathbf{e}_3 = -\mathbf{e}_1,\ \nabla_{\mathbf{e}_2}\mathbf{e}_2 = -\mathbf{e}_3,\ \nabla_{\mathbf{e}_2}\mathbf{e}_3 = -\mathbf{e}_2.$$

Definition 4.1. *Let $\mathbf{r}$ be a smooth immersion of an oriented timelike surface M^2 in $\mathbb{R}_1^3$ and $\mathbf{n}$ unit normal vector field on M^2 and H^∇ the mean curvature of M^2 with respect to ∇. Let $\mathbf{v}\neq\mathbf{0}\in\mathbb{R}_1^3$ a unit fixed spacelike vector non-parallel to $\mathbf{n}$ such that $\mathbf{n}$ and $\mathbf{v}$ span a spacelike 2-space. M^2 is called $\nabla-$singular minimal surface with respect to $\mathbf{v}$ if satisfies*

$$2H^\nabla = \alpha\frac{\langle \mathbf{n},\mathbf{v}\rangle_L}{\langle \mathbf{r},\mathbf{v}\rangle_L},\ \alpha\in\mathbb{R},\ \alpha\neq 0. \tag{4.2}$$

The surface M^2 is called $\nabla-$*minimal* if $H^\nabla=0$. With Definition 4.1, we classify the $\nabla-$singular minimal surfaces of type $z=u(x,y)$, which are $\nabla-$minimal.

Theorem 4.1. *Let M^2 be a $\nabla-$singular minimal surface in $\mathbb{R}_1^3$ of type $z=u(x,y)$ with respect to a unit spacelike vector $\mathbf{v}=(a,b,c)$. If M^2 is $\nabla-$minimal, then one of the following happens*

1. $\mathbf{v}=(0,b\neq 0,c)$ *and*
$$u(x,y)=\frac{c}{b}y+\frac{1}{2b^2}\ln\left[\cosh(2bx+\lambda_1)\right]+\lambda_2;$$
2. $\mathbf{v}=(a\neq 0,0,c)$ *and*
$$u(x,y)=\frac{c}{a}x+\frac{1}{2a^2}\ln\left[\cosh(2ay+\lambda_3)\right]+\lambda_4;$$
3. $\mathbf{v}=(a,b,c)$, $ab\neq 0$, *and*
$$u(x,y)=\frac{c}{a}x+\frac{bc}{a^2+b^2}\left(y-\frac{b}{a}x\right)+\frac{1}{2(a^2+b^2)}\ln\left[\cosh\left(-2|a|\left\{y-\frac{b}{a}x\right\}+\lambda_5\right)\right]+\lambda_6,$$

   where $\lambda_1,\dots,\lambda_6\in\mathbb{R}$, $\lambda_5\neq 0$.

Proof. The unit spacelike normal vector field on M^2 is

$$\mathbf{n}=\frac{-u_x\mathbf{e}_1-u_y\mathbf{e}_2-\mathbf{e}_3}{\sqrt{-1+(u_x)^2+(u_y)^2}},$$

Suppose that M^2 is $\nabla-$minimal. Due to $\alpha\neq 0$, Eq. (4.2) gives $\langle\mathbf{n},\mathbf{v}\rangle=0$ and

$$au_x+bu_y=c. \tag{4.3}$$

The condition of $\nabla-$minimality yields

$$\left[1-(u_y)^2\right]u_{xx}+2u_xu_yu_{xy}+\left[1-(u_x)^2\right]u_{yy}+2\left[-1+(u_x)^2+(u_y)^2\right]=0. \tag{4.4}$$

We distinguish several cases: the first case is that $a=0$. Then Eq. (4.3) follows $u(x,y)=f(x)+\frac{c}{b}y$, for an arbitrary smooth function f. Considering this into Eq. (4.4) yields

$$\frac{bf''}{1-(bf')^2}=2b, \tag{4.5}$$

where $f'=\frac{df}{dx}$ and so. The first statement of the theorem is derived by integrating Eq. (4.5). The roles of x and y in Eq. (4.4) are symmetric and hence we may conclude the second statement of the theorem by similar steps when $a\neq 0$ and $b=0$. The last case is that $ab\neq 0$. Then the solution to Eq. (4.3) is given by

$$u(x,y)=\frac{c}{a}x+g\left(y-\frac{b}{a}x\right), \tag{4.6}$$

for a smooth function g. Substituting Eq. (4.6) into Eq. (4.4) follows

$$g''+2\left[-a^2+(c-bg')^2+(ag')^2\right]=0, \tag{4.7}$$

where $g'=\frac{dg}{d\tilde{y}}$, $g''=\frac{d^2g}{d\tilde{y}^2}$, $\tilde{y}=y-\frac{b}{a}x$. Eq. (4.7) may be rewritten as

$$\frac{-(a^2+b^2)g''}{a^2-\left[bc-(a^2+b^2)g'\right]^2}=-2. \tag{4.8}$$

The proof is completed by integrating Eq. (4.8). □

Remark 4.1. *The last statement of Theorem 4.1 is a new example in $\mathbb{R}_1^3$ of $\nabla-$minimal surfaces while the first two statements are $\nabla-$minimal translation surfaces, introduced by Wang [44].*

In the following we classify $\nabla-$singular minimal surfaces in $\mathbb{R}_1^3$ of type $y=u(x,z)$ which are $\nabla-$minimal.

Theorem 4.2. *Let M^2 be a $\nabla-$singular minimal surface in $\mathbb{R}_1^3$ of type $y=u(x,z)$ with respect to a unit spacelike vector $\mathbf{v}=(a,b,c)$, $a^2+c^2\neq 0$. If M^2 is $\nabla-$minimal, then one of the following happens*

1. *M^2 is a plane parallel to $\mathbf{v}=(a,b,0)$, $a\neq 0$;*
2. *$\mathbf{v}=(0,b,c)$, $bc\neq 0$ and*
$$u(x,y)=\frac{b}{c}z+\frac{1}{2bc}\ln\left[\cosh\left(2bx+\lambda_1\right)\right]+\lambda_2;$$
3. *$\mathbf{v}=(a,b,0)$, $a\neq 0$, and*
$$u(x,z)=\frac{b}{a}x\pm\frac{1}{2|a|}\sinh^{-1}\left(\lambda_3 e^{2z}\right)+\lambda_4,\ \lambda_3\neq 0;$$
4. *$\mathbf{v}=(a,0,c)$, $ac\neq 0$, and*
$$u(x,z)=\pm\frac{1}{2|a|}\sinh^{-1}\left[|\lambda_5|\, e^{2a^2\left(z-\frac{c}{a}x\right)}\right]+\lambda_6,\ \lambda_5\neq 0;$$
5. *$\mathbf{v}=(a,\pm 1,c)$, $a=\pm c$, $c\neq 0$, and*
$$u(x,z)=\frac{\pm 1}{c}x\pm\frac{1}{4c}\ln\left[1\pm 2\lambda_7 e^{2\left(1+c^2\right)(z\pm x)}\right]+\lambda_8,\ \lambda_7\neq 0;$$
6. *$\mathbf{v}=(a,b,c)$, $abc\neq 0$, and*
$$u(x,z)=\frac{b}{a}x+h\left(z-\frac{c}{a}x\right),$$
where h is a smooth function satisfying
$$z-\frac{c}{a}x=\frac{-bc\left(c^2-a^2\right)}{2|a|\left(a^2-b^2c^2\right)}\left(2|a|h+\lambda_9\right)-$$
$$-\frac{c^2-a^2}{2\left(a^2-b^2c^2\right)}\ln\left[bc\cosh\left(2|a|h+\lambda_9\right)-|a|\sinh\left(2|a|h+\lambda_9\right)\right]+\lambda_{10},$$
for $\lambda_1,\ldots,\lambda_{10}\in\mathbb{R}$.

Proof. Let M^2 be locally given by
$$(x,z)\longmapsto \mathbf{r}(x,z)=(x,u(x,z),z),$$
for a smooth function $u=u(x,z)$. The normal vector field on M^2 is
$$\mathbf{n}=\frac{u_x\mathbf{e}_1-\mathbf{e}_2-u_z\mathbf{e}_3}{\sqrt{1+(u_x)^2-(u_z)^2}}.$$

Because M^2 is $\nabla-$singular minimal, we get Eq. (4.1). Assume that M^2 is $\nabla-$minimal. Due to $\alpha\neq 0$, Eq. (4.1) gives $\langle\mathbf{n},\mathbf{v}\rangle=0$ and
$$au_x+cu_z=b. \tag{4.9}$$

Remark also that we may write $\mathbf{v}=a\mathbf{r}_x+c\mathbf{r}_z$, implying the tangent plane of M at any point is parallel to $\mathbf{v}$. The condition of $\nabla-$minimality yields
$$\left[(u_z)^2-1\right]u_{xx}-2u_xu_zu_{xz}+\left[1+(u_x)^2\right]u_{zz}-2\left[1+(u_x)^2-(u_z)^2\right]u_z=0. \tag{4.10}$$

We distinguish several cases:

1. $a=0$, $c\neq 0$. Then $b\neq 0$ because $\mathbf{v}$ is spacelike. The solution to Eq. (4.9) is given by $u(x,z)=\frac{b}{c}z+f(x)$, for an arbitrary smooth function f. Hence Eq. (4.10) turns to
$$\frac{cf''}{1-(cf')^2}=-2b, \tag{4.11}$$
where $f'=\frac{df}{dx}$, etc. Because M^2 is non-degenerate, $1-(cf')^2\neq 0$. Therefore the second statement of the theorem is proved by integrating Eq. (4.11).

2. $a \neq 0$, $c = 0$. Then Eq. (4.9) gives $u(x,z) = \frac{b}{a}x + g(z)$ for an arbitrary smooth function g and so Eq. (4.10) leads to

$$g'' - 2\left[1 - \left(ag'\right)^2\right] g' = 0, \tag{4.12}$$

where $g' = \frac{dg}{dz}$, etc. That $g' = 0$ is a trivial solution to Eq. (4.12), implying the first statement of the theorem. Otherwise, $g' \neq 0$, Eq. (4.12) may be rewritten as

$$\frac{g''}{g'} + \frac{a}{2}\left(\frac{g''}{1-ag'} - \frac{g''}{1+ag'}\right) = 2. \tag{4.13}$$

The third statement of the theorem is obtained by integrating Eq. (4.13).

3. $ac \neq 0$. The solution to Eq. (4.9) is

$$u(x,z) = \frac{b}{a}x + h\left(z - \frac{c}{a}x\right),$$

for an arbitrary smooth function h. Therefore Eq. (4.10) reduces to

$$h'' - 2\left[a^2 + \left(b - ch'\right)^2 - \left(ah'\right)^2\right] h' = 0, \tag{4.14}$$

where $h' = \frac{dh}{d\tilde{z}}$, $h' = \frac{d^2h}{d\tilde{z}^2}$, $\tilde{z} = z - \frac{c}{a}x$. We have three subcases: the first one is that $b = 0$. Then Eq. (4.14) may be rewritten as

$$\frac{h''}{h'} + \frac{h''}{2(a-h')} - \frac{h''}{2(a+h')} = 2a^2. \tag{4.15}$$

Integrating Eq. (4.15) gives the fourth statement of the theorem. The second subcase is that $a^2 = c^2$ and $b = \pm 1$. Then Eq. (4.14) may be rewritten as

$$\frac{\pm 2ch''}{1+c^2 \mp 2ch'} + \frac{h''}{h'} = 2\left(1+c^2\right). \tag{4.16}$$

After integrating Eq. (4.16), we obtain the fifth statement of the theorem. The third subcase is that $a^2 \neq c^2$. Then Eq. (4.14) may be rewritten as

$$\frac{-\left(c^2-a^2\right)h''}{a^2 - \left[bc - \left(c^2-a^2\right)h'\right]^2} = 2h'. \tag{4.17}$$

A first integration of Eq. (4.17) yields

$$\frac{\left(c^2-a^2\right)dh}{-|a|\tanh\left(2|a|h+\lambda\right)+bc} = d\tilde{z}, \tag{4.18}$$

for $\lambda \in \mathbb{R}$. The proof is completed by a first integration of Eq. (4.18).

□

Remark 4.2. *The last three statements of Theorem 4.1 are new examples in $\mathbb{R}^3_1$ of $\nabla-$minimal surfaces while the second and third statements are $\nabla-$minimal translation surfaces, introduced by Wang [44].*

Let M^2 be a timelike surface $\mathbb{R}^3_1$ of type $x = u(y,z)$. The spacelike unit normal vector field on M^2 is

$$\mathbf{n} = \frac{\mathbf{e}_1 - u_y\mathbf{e}_2 + u_z\mathbf{e}_3}{\sqrt{1+\left(u_y\right)^2 - \left(u_z\right)^2}}.$$

Suppose that M^2 is $\nabla-$singular minimal with respect to the vector $\mathbf{v} = (a,b,c)$. If M^2 is also $\nabla-$minimal, then Eq. (4.2) gives

$$bu_y + cu_z = a,$$

where $b^2 + c^2 \neq 0$. Notice that the mean curvature is same as that of the surface of type $y = u(x,z)$. Therefore, without giving a proof, we may state a similar result for those surfaces of type $x = u(y,z)$ to Theorem 4.1 by replacing x with y and a with b.

4.2. $D-$Singular minimal surfaces

Consider the following semi-symmetric non-metric connection on $\mathbb{R}^3_1$ [44]

$$D_{\mathbf{x}}\mathbf{y} = \nabla^L_{\mathbf{x}}\mathbf{y} + \langle \mathbf{y}, \mathbf{e}_3 \rangle \mathbf{x}, \tag{4.19}$$

where $\mathbf{x}, \mathbf{y}$ are tangent vector fields to $\mathbb{R}^3$. The nonzero derivatives are

$$D_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1,\ D_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2,\ D_{\mathbf{e}_3}\mathbf{e}_3 = \mathbf{e}_3.$$

Definition 4.2. *Let $\mathbf{r}$ be a smooth immersion of an oriented timelike surface M^2 in $\mathbb{R}^3_1$ and $\mathbf{n}$ unit normal vector field on M^2 and H^D the mean curvature of M^2 with respect to D. Let $\mathbf{v} \neq \mathbf{0} \in \mathbb{R}^3_1$ a unit fixed spacelike vector non-parallel to $\mathbf{n}$ such that $\mathbf{n}$ and $\mathbf{v}$ span a spacelike 2-space. M^2 is called $D-$singular minimal surface with respect to $\mathbf{v}$ if satisfies*

$$2H^D = \alpha \frac{\langle \mathbf{n}, \mathbf{v} \rangle_L}{\langle \mathbf{r}, \mathbf{v} \rangle_L},\ \alpha \in \mathbb{R},\ \alpha \neq 0. \tag{4.20}$$

The surface M^2 is called *$D-$minimal* if $H^D = 0$. With Definition 4.2, we first observe the $D-$singular minimal surfaces of type $z = u(x,y)$ which are $D-$minimal. Hence Eq. (4.20) gives $\langle \mathbf{n}, \mathbf{v} \rangle = 0$ and

$$au_x + bu_y = c, \tag{4.21}$$

where $\mathbf{v} = (a,b,c)$. The condition of $D-$minimality yields

$$\left[1 - (u_y)^2\right] u_{xx} + 2u_x u_y u_{xy} + \left[1 - (u_x)^2\right] u_{yy} = 0, \tag{4.22}$$

where the roles of x and y are symmetric. If $a = 0$, then Eq. (4.21) follows $u(x,y) = f(x) + \frac{c}{b}y$, for an arbitrary smooth function f. Considering this into Eq. (4.22) yields $\frac{1}{b^2}\frac{d^2 f}{dx^2} = 0$, which leads M^2 to be a plane parallel to $\mathbf{v}$. By symmetry, we can obtain same result when $a \neq 0$ and $b = 0$. Let $ab \neq 0$. Then the solution to Eq. (4.21) is $u(x,y) = \frac{c}{a}x + g\left(y - \frac{b}{a}x\right)$, for a smooth function g. After substituting its partial derivatives into Eq. (4.22), we conclude $\frac{1}{a^2}\frac{d^2 g}{d\tilde{y}^2} = 0$, $\tilde{y} = y - \frac{c}{a}x$, yielding that M is a plane parallel to $\mathbf{v}$.
Therefore, we state the following

Theorem 4.3. *Let M^2 be a $D-$singular minimal surface in $\mathbb{R}^3_1$ of type $z = u(x,y)$ with respect to a unit spacelike vector $\mathbf{v}$. If M^2 is $D-$minimal, then it is a plane parallel to $\mathbf{v}$.*

When we take the surfaces of type $y = u(x,z)$ or $x = u(y,z)$, we may state a similar result to Theorem 4.3.

5. Conclusions and further remarks

In this study, we discussed the singular minimal surfaces in $\mathbb{R}^3$ (resp. $\mathbb{R}^3_1$) which are minimal and expressed a trivial outcome, Proposition 1.1 (resp. Proposition 1.2). Nevertheless, the non-trivial outcomes, Theorems 3.1 and 3.2 (resp. Theorems 4.1 and 4.2), were obtained by using the modified version, Definition 3.1 (resp. Definition 4.1), of singular minimality. With this definition, we observed that the singular minimal surfaces which are minimal are a generalized cylinder. Since the generalized cylinders belong to a subcase of translation surfaces, the $\nabla-$minimal translation surfaces introduced by Wang [44] were presented by some of our results. Still, we also exhibited new examples of $\nabla-$minimal surfaces, as explained in Remarks 3.1 and 3.2 (resp. Remarks 4.1 and 4.2). Morever, a trivial outcome, Theorem 3.3 (resp. Theorem 4.3), was found by using the semi-symmetric non-metric connection D given by Eq. (3.19) (resp. Eq. (4.19)).
On the other hand, let M^2 be locally a graph surface in $\mathbb{R}^3$ of a smooth function $u(x,y)$ and H and H^∇ denote the mean curvatures with respect to the Levi-Civita connection and the semi-symmetric metric connection ∇ given by Eq. (3.1), respectively. Then, the following relation occurs

$$H^\nabla = H - \langle \mathbf{n}, (0,0,1) \rangle, \tag{5.1}$$

where $\mathbf{n}$ is the unit normal vector field on M^2. Notice also that Eq. (5.1) remains true for a graph of the forms $u(x,z)$ or $u(y,z)$ up to a sign. Therefore, $\nabla-$minimal graph surfaces turn to the *translating solitons* whose the mean curvature satisfies

$$H = \langle \mathbf{n}, (0,0,1) \rangle \tag{5.2}$$

Eq. (5.2) appears in the theories of mean curvature flow and manifolds with density, for details see ([23, 24, 29–31]). Eventually, the above discussion imply that $\nabla-$singular minimal surfaces which are $\nabla-$minimal are a cylindrical translating soliton. Such surfaces were considered in [23, 31]. Nevertheless, this paper provides a novel approach.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Universal Journal of Mathematics and Applications, 4 (4) (2021) 147-153
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.983509

Geometric Structure of the Set of Pairs of Matrices under Simultaneous Similarity

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Article Info

Keywords: *Bundles, simultaneous similarity, connection, curvature*
2010 AMS: *15A18, 57S25, 15A22*
Received: *16 August 2021*
Accepted: *19 November 2021*
Available online: *30 December 2021*

Abstract

In this paper pairs of matrices under similarity are considered because of their scientific applications, especially pairs of matrices being simultaneously diagonalizable. For example, a problem in quantum mechanics is the position and momentum operators, because they do not have a shared base representing the system's states. They do not commute, and that is why switching operators form a crucial element in quantum physics.
A study of the set of linear operators consisting of pairs of simultaneously diagonalizable matrices is done using geometric constructions such as the principal bundles. The main goal of this work is to construct connections that allow us to establish a relationship between the local geometry around a point with the local geometry around another point. The connections give us a way to help distinguish bundle sections along tangent vectors.

1. Introduction

Let $\mathfrak{M}$ be the manifold of pairs of n-order real matrices $T=(X_1,X_2)$. A frequent question, in both mathematics and physics, is to ask if it is possible to find a base in the space $\mathbb{R}^n$ in which both matrices diagonalize, that is, to ask if they diagonalize simultaneously. Concretely, the simultaneous diagonalization of pairs of symmetric matrices has a particular interest, (see [5], [7] and [8], for example), due to its applications in sciences. For example, they appear when we must give the "instanton"solution of Yang-Mills field presented in an octonion form, and it can be represented by triples of traceless matrices [1], [6], [13]. Another application of simultaneous diagonalization is found when studying, for example, thermal transmissivity, whose study is different depending on whether the interaction matrices diagonalize simultaneously [12].
In order to formalize the simultaneous diagonalization problem, it is necessary to start by defining an equivalence relation called similarity, which allows establishing criteria for simultaneous diagonalization.
It is well known that, in the space of n-square real matrices, the subset of diagonalizable matrices is generic. Then, any non-diagonalizable matrix is arbitrarily close to a diagonalizable matrix and reduced to a diagonal form by a small perturbation of its entries. This property cannot be generalized to the case of simultaneous diagonalization of a pair (or m-tuple) of n-order real square matrices. Necessary or sufficient conditions for simultaneous diagonalization have been studied. These studies have been realized under different points of view, for example, analysing the spectra of families of pairs of matrices [8] computing versal deformations [2].
A good tool for distinguishing one subset from another within a differentiable variety could be by trying to identify it from the zeros of bundle sections built on the variety, then, the characteristic classes allow to identify its obstructions. In this particular setup, the interest is about the set of the m-tuples of simultaneously diagonalizable real matrices. Some results about families of pairs of matrices that are simultaneously diagonalizable can be found in [7], [8].
Principal bundles [10] have significant applications in different mathematical areas as topology and differential geometry, in special bundles given by a Lie group action. The first attempts to apply the theory of fiber bundles in the field of physics were made by E. Lubkin [11], who pointed out that the caliber fields had a fiber bundle structure. Further, they form part of the basic framework of gauge theories describing the interaction of forces by differentiating connections [14], and quantum theory [3].
An important object in principal bundles theory is that of connection. Visually, a connection gives us a way to move through the fibers of a principal bundle through isomorphisms between them, which leads us to curvature invariants. In this article, a connection on a specific main bundle is defined as well as the curvature derived from the connection.

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2. Preliminaries

2.1. Simultaneous equivalence of pairs of matrices

The purpose of this section is to give necessary and sufficient conditions that for two pairs of matrices,$T=(X_1,X_2),T'=(Y_1,Y_2)$ are simultaneously diagonalizable. First of all, we define the simultaneous similarity equivalence relating the elements of $\mathfrak{M}$.

Definition 2.1. *Let $T=(X_1,X_2),T'=(Y_1,Y_2)$ be two pairs of matrices in $\mathfrak{M}$. Then, T is simultaneously similar to T' if and only if there exists $P\in\mathfrak{G}=Gl(n;\mathbb{R})$ such that*

$$T'=(Y_1,Y_2)=(PX_1P^{-1},PX_2P^{-1})=PTP^{-1}. \tag{2.1}$$

A particular case of pairs of matrices is that those that are similar to a pair of matrices which are both diagonal, that is, they diagonalize simultaneously.

Definition 2.2. *The pair of matrices $T=(X_1,X_2)$ is simultaneously diagonalizable if and only if there exists an equivalent pair formed by diagonal matrices.*

Necessary conditions for simultaneous diagonalizable pairs can be found in the following propositions(see [7], [9]):

Proposition 2.3. *Let $T=(X_1,X_2)$ be a simultaneously diagonalizable pair. Then both matrices X_i must be diagonalizable. (The converse is false).*

Proposition 2.4. *Let $T=(X_1,X_2)$ be a simultaneously diagonalizable pair. Then, the Lie bracket $[X_1,X_2]=0$.*

Regarding sufficient conditions, we have the following results.

Theorem 2.5. *Let $T=(X_1,X_2)$ be a pair of commuting n-order square matrices and suppose that the matrix X_i, for some $i=1,2$, is diagonalizable with simple eigenvalues ($\lambda_j\neq\lambda_k$ for all $j\neq k$; $k,j=1,\dots n$). Then T is a pair of simultaneously diagonalizable matrices.*

Proof. Without loss of generality, we can assume that X_1 is diagonalizable.
Let $v_1,\dots,v_n$ be a basis such that $X_1(v_i)=\lambda_i v_i$ for $i=1,\dots,n$.
Then, $X_1(X_2v_i)=X_2(X_1v_i)=\lambda_iX_2v_i$.
So, if $X_2v_i\neq 0$, it is an eigenvector of X_1 of eigenvalue λ_i, but condition $\lambda_k\neq\lambda_\ell$ implies that $\dim\mathrm{Ker}(X_1-\lambda_jI)=1$, then, $X_2v_i=\mu_2v_i$, that is to say v_i is an eigenvector for X_2 of eigenvalue μ_2. If $X_2v_i=0$ the vector v_i is an eigenvector of X_2 of eigenvalue zero. That is to say, $\{v_1,\dots,v_n\}$ is a basis of eigenvectors for X_2, and T is a pair of simultaneous diagonalizable matrices with $P=\begin{pmatrix}v_1^t & \dots & v_n^t\end{pmatrix}^{-1}$. □

Remark 2.6. *The matrix X_2 does not necessarily have simple eigenvalues.*

Theorem 2.7. *Let X_1,X_2 be two n-order square matrices. If X_1 and X_2 are commuting and diagonalizable matrices, then, $T=(X_1,X_2)$ is simultaneously diagonalizable.*

Proof. Let P_1 be an invertible matrix such that $D_1=P_1X_1P_1^{-1}=\begin{pmatrix}D_1^1 & & \\ & \ddots & \\ & & D_r^1\end{pmatrix}$ with $D_i^1=\lambda_i^1I\in M_{n_i}(\mathbb{C})$, $1\leq i\leq r$ and $n_1+\ldots+n_r=n$.
Let us consider $v_{1_1},\dots,v_{n_1},\dots,v_{1_r},\dots,v_{n_r}$ the vector columns of P_1^{-1}, then

$$\begin{aligned}X_2X_1v_{i_\ell}&=X_2\lambda_\ell v_{i_\ell}=\lambda_\ell X_2v_{i_\ell}\\ X_2X_1v_{i_\ell}&=X_1X_2v_{i_\ell}\end{aligned}$$

Consequently $X_2v_{i_\ell}$ is an eigenvector of X_1 of eigenvalue λ_ℓ or $X_2v_{i_\ell}=0$, in any case we have that $X_2v_{i_\ell}\in[v_{1_\ell},\dots,v_{n_\ell}]=F_\ell$, consequently, $X_2F_\ell\subset F_\ell$. That is, the subspace F_ℓ is X_2 invariant for all $1\leq\ell\leq r$, and $P_1X_2P_1^{-1}$ is block-diagonal matrix

$$P_1X_2P_1^{-1}=\begin{pmatrix}Y_1^2 & & \\ & \ddots & \\ & & Y_r^2\end{pmatrix}$$

where the size of each block Y_j^2 is the same of the corresponding block D_j^1 in the matrix $P_1X_aP_1^{-1}$.
If all matrices Y_k^2 are diagonal the proof is concluded. If any submatrix Y_k^2 is not diagonal, then taking into account that the matrix X_2 diagonalizes, all submatrices Y_k^2 diagonalize.

Consider $P_2=\begin{pmatrix}P_2^1 & & \\ & \ddots & \\ & & P_2^r\end{pmatrix}$ where P_2^j diagonalizes Y_2^2 for $1\leq j\leq r$.
Clearly P_2 diagonalizes D_1:

$$\begin{pmatrix}P_2^1 & & \\ & \ddots & \\ & & P_2^r\end{pmatrix}\begin{pmatrix}D_1^1 & & \\ & \ddots & \\ & & D_r^1\end{pmatrix}\begin{pmatrix}P_2^1 & & \\ & \ddots & \\ & & P_2^r\end{pmatrix}^{-1}=\begin{pmatrix}P_2^1D_1^1(P_2^1)^{-1} & & \\ & \ddots & \\ & & P_2^rD_r^1(P_2^r)^{-1}\end{pmatrix}=\begin{pmatrix}P_2^1\lambda_1^1I(P_2^1)^{-1} & & \\ & \ddots & \\ & & P_2^r\lambda_r^1I(P_2^r)^{-1}\end{pmatrix}=\begin{pmatrix}D_1^1 & & \\ & \ddots & \\ & & D_r^1\end{pmatrix}$$

Then P_2P_1 diagonalizes X_1 and X_2.

□

2.1.1. Invariant polynomials associated to a pair of matrices under similarity

We are going to construct polynomials $\mathscr{P}(T)$ with $2n^2$ variables x^1_{ij}, x^2_{ij} $1 \leq i, j \leq n$, corresponding to the elements of the pair of matrices $T = (X_1, X_2) = ((x^1_{ij}), (x^2_{ij}))$.

Example 2.8. *Let $T = ((x^1_{ij}), (x^2_{ij}))$ be a pair of matrices. We can define the polynomial*

$$\mathscr{P}(T) = trace\, X_1 + trace\, X_2 = x^1_{11} + \ldots + x^1_{nn} + x^2_{11} + \ldots + x^2_{nn}.$$

We are interested in those which will be invariant under simultaneous similarity in the following sense.

Definition 2.9. *Let $T \in \mathfrak{M}$. A polynomial $\mathscr{P}(T)$ is called invariant under similarity, if $\mathscr{P}(T) = \mathscr{P}(PTP^{-1})$ for all $P \in GL(n;\mathbb{R})$.*

For this study, we will use the characteristic polynomials associated with each matrix of the pair.
Given the pair of matrices $T = (X_1, X_2)$, we can associate it with the following polynomial:

$$\sigma_T(t) = \det(tI - X_1) \cdot \det(tI - X_2) = \prod_{j=1}^{2} \det(tI - X_j) \tag{2.2}$$

Proposition 2.10. *The polynomial (2.2) is invariant under simultaneous similarity.*

Proof.

$$\sigma_{PTP^{-1}}(t) = \prod_{j=1}^{2} \det(tI - PX_jP^{-1}) = (\det P)^2 (\det P^{-1})^2 \prod_{j=1}^{2} \det(tI - X_j) = \sigma_T(t)$$

□

The polynomial $\sigma_T(t)$ can be written in the following manner

$$\sigma_T(t) = \prod_{j=1}^{2} \left(\Sigma_{i=0}^{n} \sigma_i^j(X_j) t^i \right) = \sum_{i=0}^{2n} \sigma_i(T) t^i$$

where, clearly, $\sigma_0(T) = \prod_{j=1}^{2} \det X_j$ and $\sigma_{2n}(T) = 1$.
For the set of variables x^1_{ij}, x^2_{ij} $1 \leq i, j \leq n$, we consider the corresponding pair $T = ((x^1_{ij}), (x^2_{ij}))$. Then, the polynomials $\sigma_i(T)$ are homogeneous polynomials in the given variables.

Proposition 2.11. *Each polynomial $\sigma_i(T)$ is an invariant polynomial.*

Proof. It suffices to note that $\sigma_i^j(X_j)$ is invariant. □

Let be now $T = (X_1, X_2)$, a simultaneously diagonalizable pair, then and taking into account the invariance of the characteristic polynomial we have that $\sigma_i(T)$ are expressed in terms of the eigenvalues of X_i, $i = 1, 2$:

$$\prod_{j=1}^{2} \det(tI - X_j) = \prod_{j=1}^{2} \prod_{k=1}^{n} (t - \lambda_k^j) = \Sigma_{i=0}^{2n} \sigma_i(\lambda_1^1, \ldots, \lambda_n^1, \lambda_1^2, \ldots, \lambda_n^2) t^i$$

Then, in this class of pairs of matrices, these polynomials can be expressed with $2n$ variables instead of the $2n^2$ variables intervening in the general case.

2.2. Fiber Bundles

Following Husmoller [10], a fiber bundle is a structure (E, B, π, F), where E, B, and F are topological spaces called the total space, base space of the bundle, and the fiber, respectively, and $\pi : E \to B$ is a continuous surjection called the bundle projection, satisfying the following local triviality condition: for every $x \in E$, there is an open neighborhood $U \subset B$ of $\pi(x)$ (called a trivializing neighborhood) such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ in such a way that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\pi_1"] \\
U &
\end{tikzcd}$$

where $\pi_1 : U \times F \to U$ is the natural projection. The set of all $\{(U_i, \varphi_i)\}$ is called a local trivialization of the bundle.
Thus for any $p \in B$, $\pi^{-1}(\{p\})$ is homeomorphic to F and is called the fiber over p.
A trivial example of a bundle is the one given by

$$(B \times F, \pi, B, F)$$

where $\pi : B \times F \longrightarrow B$ is the projection on the first factor, in this case, the fibers are $\{p\} \times F$ for all $p \in B$.
A fiber bundle (E', B', π', F') is a subbundle of (E, B, π, F) provided E' is a subspace of E, B' is a subspace of B, and π' is the restriction of π to E', $\pi' = \pi_{E'} : E' \longrightarrow B'$,

In the special case where the fiber is a group G, the fiber bundle is called the principal bundle. In this case, any fiber $\pi^{-1}(b)$ is a space isomorphic to G. More specifically, G acts freely without fixed points on the fibers.
In the case where E, B and F are smooth manifolds and all the functions above are smooth maps, the fiber bundle is called a smooth fiber bundle.
It is possible to induce bundles in the following manner.
Let $\pi : E \longrightarrow B$ be a fiber bundle with fiber F and let $f : B' \longrightarrow B$ be a continuous map. Then, a fiber bundle over B' can be deduced as follows:

$$f^*E = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\} \subseteq B' \times E$$

and equip it with the subspace topology and the projection map $\pi' : f^*E \longrightarrow B'$ defined as the projection onto the first factor:

$$\pi'(b', e) = b'$$

Defining f' so that the following diagram is commutative

$$\begin{tikzcd} f^*E \arrow[r, "f'"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

we have that (f^*E, B', π') is a fiber bundle so that the fibers on $b \in B$ correspond to the fibers on $f^{-1}(b)$.
An important concept on fiber bundles is the cross-section notion.

Definition 2.12. *A cross section of a bundle (E, B, π, F) is a map $s : B \longrightarrow E$ such that $\pi s = I_B$. In other words, a cross section is a map $s : B \longrightarrow E$ such that $s(b) \in \pi^{-1}(b)$, the fibre over b, for each $b \in B$.*

Let (E', B, π', F') be a subbundle of (E, B, π, F), and let s be a cross section of (E, B, π, F). Then s is a cross section of (E', B, π', F') if and only if $s(b) \in E'$ for each $b \in B$.
One of the main goals of studying cross sections is to account for the existence or non-existence of global sections. When there are some problems with constructing a global section, one says that there are some obstructions.

3. Bundle of pairs of matrices

3.1. Lie group actions

The simultaneous equivalence relation defined in (2.1), can be seen as the action of a Lie group $\mathfrak{G}$ over $\mathfrak{M}$ in the following manner:
Let us consider the following map:

$$\begin{aligned} \alpha : \mathfrak{G} \times \mathfrak{M} &\longrightarrow \mathfrak{M} \\ (P, T) &\longrightarrow PTP^{-1} = (PX_1P^{-1}, PX_2P^{-1}) \end{aligned}$$

that verifies

i) If $I \in \mathfrak{G}$ is the identity element, then $\alpha(I, T) = T$ for all $T \in \mathfrak{M}$.
ii) If P_1 and P_2 are in $\mathfrak{G}$, then $\alpha(P_1, \alpha(P_2, T)) = \alpha(P_1P_2, T)$ for all $T \in \mathfrak{M}$.
Indeed: $\alpha(P_1, \alpha(P_2, T)) = \alpha(P_1, P_2TP_2^{-1}) = P_1P_2TP_2^{-1}P_1^{-1} = (P_1P_2)T(P_1P_2)^{-1} = \alpha(P_1P_2, T)$

So, the map α defines an action of $\mathfrak{G}$ over the differentiable manifold $\mathfrak{M}$ that allows seeing the equivalent classes as differentiable manifolds providing a hard link between geometry and algebra.
Analogously we can define an action of $\mathfrak{G}$ over $\mathfrak{G} \times \mathfrak{M}$ in the following manner:

$$\begin{aligned} \beta : \mathfrak{G} \times (\mathfrak{G} \times \mathfrak{M}) &\longrightarrow \mathfrak{G} \times \mathfrak{M} \\ (Q, (P, T)) &\longrightarrow (PQ^{-1}, \alpha(Q^{-1}, T)). \end{aligned}$$

It will be denoted by β_Q the restriction of β to the set $\{Q\} \times (\mathfrak{G} \times \mathfrak{M})$ and by $\beta_{(P,T)}$ the restriction of β to the set $\mathfrak{G} \times \{(P, T)\}$.

Proposition 3.1. *The $\mathfrak{G}$-action β is free, transitive and its orbits are diffeomorphic to $\mathfrak{G}$*

Proof. Suppose that $\beta(Q, (P, T)) = (P, T)$, so

$$\begin{aligned} \beta(Q, (P, T)) &= (PQ^{-1}, \alpha(Q^{-1}, T)) \\ &= (PQ^{-1}, Q^{-1}TQ) \\ &= (P, T) \end{aligned}$$

then, $PQ^{-1} = P$ and $Q^{-1}TQ = T$ and $Q = I$.

$$\begin{aligned}
\beta(R,\beta(Q,(P,T))) &= \beta(R,(PQ^{-1},\alpha(Q,T)))\\
&= \beta(R,(PQ^{-1},QTQ^{-1}))\\
&= (PQ^{-1}R^{-1},\alpha(R,QTQ^{-1}))\\
&= (PQ^{-1}R^{-1},RQTQ^{-1}R^{-1})\\
&= (P(RQ)^{-1},\alpha(RQ,T))\\
&= \beta(RQ,(P,T)),
\end{aligned}$$

Let us denote by $\mathscr{O}(P,T)$ the orbit of T under $\mathfrak{G}$-action $\mathscr{O}(P,T)=\{(\overline{P},\overline{T})=\beta(Q,(P,T)),\forall Q\in\mathfrak{G}\}$

$$\begin{aligned}
\varphi:\mathfrak{G} &\longrightarrow \mathscr{O}(P,T)\\
Q &\longrightarrow (\overline{P},\overline{T})=\beta(Q,(P,T))
\end{aligned}$$

φ is a diffeomorphism:
If $\varphi(Q)=\varphi(\overline{Q})$, then $PQ=P\overline{Q}$ consequently $Q=\overline{Q}$
And, for $(\overline{P},\overline{T})\in\mathscr{O}(P,T)$, there exists $Q\in\mathfrak{G}$ with $(\overline{P},\overline{T})=(PQ^{-1},QTQ^{-1})$, so $\varphi(Q)=(\overline{P},\overline{T})$. $\square$

The set $\mathfrak{M}$ is identified as the set of orbits class $\mathfrak{G}\times\mathfrak{M}/\beta$.

Proposition 3.2. *There exists a bijection between $\mathfrak{M}$ and $\mathfrak{G}\times\mathfrak{M}/\beta$.*

Proof. We define f as

$$\begin{aligned}
\mathfrak{G}\times\mathfrak{M}/\beta &\longrightarrow \mathfrak{M}\\
(P,T)\circ\mathfrak{G} &\longrightarrow T'
\end{aligned}$$

where T' is in such a way that there exists $Q\in\mathfrak{G}$ such that $\beta(Q,(P,T))=(I,T')$

1) It suffices to take $Q=P$ to obtain $T'=P^{-1}TP$.
2) f is well-defined because of unicity of T':
   Let $(I,T')\sim(I,T'')$, then, there exists Q such that

   $$\beta(Q,(I,T'))=(IQ^{-1},\alpha(Q^{-1},T'))=(I,T'').$$

   So, $IQ^{-1}=I$ and $Q^{-1}=I=Q$ and $IQ^{-1}\alpha(Q^{-1},T')=\alpha(I,T')=T'$.
3) f is bijective:
   If $f(I,T')\circ\mathfrak{G}=f(I,T'')\circ\mathfrak{G}$, then $T'=T''$ and $f(I,T')\circ\mathfrak{G}=f(I,T'')\circ\mathfrak{G}$, so f is injective.
   And, clearly, for all $T\in\mathfrak{M}$, $f(I,T)\circ\mathfrak{G}=T$ and f is surjective.

$\square$

Proposition 3.3. *The $\mathfrak{G}$-action preserves the fibers $F_T=\alpha^{-1}(T)$ of $\alpha:\mathfrak{G}\times\mathfrak{M}\longrightarrow\mathfrak{M}$.*

Proof. Let $(P,\overline{T})\in\alpha^{-1}(T)$, then

$$\alpha(Q,(P,\overline{T}))=(PQ^{-1},QTQ^{-1})=PQ^{-1}Q\overline{T}Q^{-1}QP^{-1}=P\overline{T}P^{-1}=T.$$

So, $(PQ^{-1},QTQ^{-1})\in\alpha^{-1}(T)$. $\square$

From propositions 3.1 and 3.3 we can deduce the following result.

Proposition 3.4. $(\mathfrak{G}\times\mathfrak{M},\mathfrak{M},\alpha,\mathfrak{G})$ *is a principal fiber bundle.*

Clearly, we observe that F_T is diffeomorphic to $\mathfrak{G}$:

$$\begin{aligned}
\psi:F_T &\longrightarrow \mathfrak{G}\\
(Q,\overline{T}) &\longrightarrow Q
\end{aligned}$$

If $\psi(Q,\overline{T})=(\overline{Q},\overline{\overline{Q}})$, then $Q=\overline{Q}$ and $Q\overline{T}Q^{-1}=\overline{Q}\overline{\overline{T}}\overline{Q}^{-1}=Q\overline{\overline{T}}Q^{-1}$, so $\overline{T}=\overline{\overline{T}}$ and the map ψ is injective.
On the other hand, for all $Q\in\mathfrak{G}$, there exists $(Q,Q^{-1}TQ)\in F_T$ such that $\psi(Q,Q^{-1}TQ)=T$, so, the map ψ is surjective.

4. Connections and curvature

A connection is a mathematical object defined over a differentiable manifold that allows the local geometry around a point to be related to local geometry around another point in the manifold. The connection is an object that shows us how to derive local sections and thus compare the fibers on different points of the base space [4].
Curvature is useful to obtain characteristic classes that are global invariants that measure the deviation of the local product structure from a global product structure. The theory of characteristic classes generalizes the idea of obstructions to construct cross-sections of fiber bundles.
Let us use the notation $T_I\mathfrak{G}$ for the tangent space to the manifold $\mathfrak{G}$ at the unit element I. Since $\mathfrak{G}$ is an open subset of $M_n(\mathbb{R})$, we have that $T_I(\mathfrak{G}) = M_n(\mathbb{R})$, and, since $\mathfrak{M}$ is a linear space, $T_T(\mathfrak{M}) = \mathfrak{M}$, then $T_{(I,T)}(\mathfrak{G}\times\mathfrak{M}) = M_n \times \mathfrak{M}$.
The action β of $\mathfrak{G}$ over $\mathfrak{G}\times\mathfrak{M}$ permits us to construct a vertical subspace $T_{(P,T)}\mathscr{O}(P,T) \subset T(\mathfrak{G}\times\mathfrak{M})$.

$$T_{(P,T)}\mathscr{O}(P,T) = \mathscr{I}mg\, d\beta_{(P,T)} = \{(-PQ,[T,Q]) \mid \forall Q \in T\mathfrak{G}\}$$

where $[T,Q] = ([X_1,Q],[X_2,Q])$.
(To describe $\mathscr{I}mg\, d\beta_{(P,T)}$ it suffices to compute the linear approximation of $\beta_{(P,T)}(I+\varepsilon Q)^{-1} \sim \beta_{(P,T)}(I-\varepsilon Q)$).
The subspace $T_{(P,T)}\mathscr{O}(P,T)$ is generated by $d\beta(A_{ij})$ with $\{A_{ij}\}$ a basis for $T\mathfrak{G} = M_n(\mathbb{R})$.
Consider the Euclidean scalar product in the space $T_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ defined as:

$$\langle (P,T_1),(Q,T_2\rangle = \langle (P,(X_1,X_2)),(Q,(Y_1,Y_2))\rangle = \operatorname{tr} P\overline{Q}^t + \operatorname{tr} X_1\overline{Y}_1^t + \operatorname{tr} X_2\overline{Y}_2^t = \operatorname{tr} P\overline{Q}^t + \operatorname{tr} T_1\overline{T}_2^t.$$

An orthogonal element $(X,Y) \in T_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ to $T_{(P,T)}\mathscr{O}(P,T)$ is a solution of the equation:

$$\langle (-PQ,[T,Q]),(X,Y)\rangle = \operatorname{tr}(-PQ\overline{X}^t) + \operatorname{tr}([T,Q]\overline{Y}^t) = 0.$$

It is possible to construct a horizontal subspace

$$\begin{aligned} T_{(P,T)}\mathscr{O}(P,T)^{\perp} &= \{(A,B) \in T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \mid -\overline{A}^t P + \overline{B}^t T - T\overline{B}^t = 0\} \\ &= \{(A,B) \in T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \mid -\overline{A}^t P + [\overline{B}^t, T] = 0\}, \end{aligned}$$

where $[\overline{B}^t,T]$ denotes $[\overline{B}_1^t,X_1] + [\overline{B}_2^t,X_2]$.

Definition 4.1. *Given a principal bundle $(\mathfrak{G}\times\mathfrak{M},\mathfrak{M},\alpha,\mathfrak{G})$, a differentiable distribution $\mathscr{H}$ of fields over $\mathfrak{G}\times\mathfrak{M}$ such that for each point $(P,T) \in \mathfrak{G}\times\mathfrak{M}$, the subspace $H_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \subset T_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ is called connection if it verifies:*

a) $T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) = V_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \oplus H_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$
b) For each $(P,T) \in \mathfrak{G}\times\mathfrak{M}$ and for each $Q \in \mathfrak{G}$, for the translation $\beta_Q(P,T) = (PQ^{-1},Q^{-1}TQ)$, the space $H_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ fulfills $H_{\beta_Q(P,T)}(\mathfrak{G}\times\mathfrak{M}) = H_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$

So, the connections allow us to decompose the vectors into a vertical part $V_u(E)$ and a horizontal part $H_u(E)$, which we will call the vertical and horizontal subspace, respectively, of $T_u(E)$ concerning the connection $\mathscr{H}$.

Proposition 4.2. *The subgroup $T_{(P,(X_1,X_2))}\mathscr{O}(P,(X_1,X_2))^{\perp}$ verifies the conditions of definition 4.1.*

Proof. Let $(C,D) \in T_{\beta_Q(P,T)}\mathscr{O}(\beta_Q(P,T))^{\perp}$, then,

$$-\overline{C}^t PQ + \overline{D}_1^t Q^{-1}X_1 Q - Q^{-1}X_1 Q\overline{D}_1^t + \overline{D}_2^t Q^{-1}X_2 Q - Q^{-1}X_2 Q\overline{D}_2^t = 0$$

or, equivalently:

$$-Q\overline{C}^t P + Q\overline{D}_1^t Q^{-1}X_1 - X_1 Q\overline{D}_1^t Q^{-1} + Q\overline{D}_2^t Q^{-1}X_2 - X_2 Q\overline{D}_2^t Q^{-1} = 0$$

Then, setting $\overline{E}^t = Q\overline{C}^t$, $\overline{F}_1^t = Q\overline{D}_1^t Q^{-1}$ and $\overline{F}_2^t = Q\overline{D}_2^t Q^{-1}$, we have that equivalently $(E,F) \in T_{(P,T)}\mathscr{O}(P,T)^{\perp}$. □

The bijectivity of the map

$$\begin{aligned} d\beta_{(P,T)}(\mathfrak{G}) &\longrightarrow T_{(P,T)}\mathscr{O}(P,T) \\ Q &\longrightarrow (-PQ, X_1Q - QX_1, X_2Q - QX_2) \end{aligned}$$

permit us to define a 1-form ω over $\mathfrak{G}\times\mathfrak{M}$ with values in the Lie algebra $\mathfrak{G}$ as follows: Given $(A,B) \in T_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ there exists a unique element $Q \in T\mathfrak{G}$ such that the vertical component of (A,B) is $A_v = d\beta_{(P,T)}Q$, thus we define $\omega(A,B) = Q$. It is clear that $\omega(A,B) = 0$ if and only if (A,B) is horizontal.
From this 1-form, it is possible to build a 2-form Ω in the following maner:

$$\begin{aligned} \Omega : T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \times T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) &\longrightarrow T_I\mathfrak{G} \\ (X,Y) = ((P_1,T_1),(P_2,T_2)) &\longrightarrow \Omega((P_1,T_1),(P_2,T_2)) = \Omega(X,Y) \end{aligned}$$

verifying:

$$\Omega(X,Y) = d\omega(X,Y) + \frac{1}{2}[\omega(X),\omega(Y)] \tag{4.1}$$

and (4.1) is called curvature of the connection.

5. Conclusion

In this work, we build bundles to obtain algebraic objects providing more geometric information about the space of pairs of matrices. With them, we consider an operator called connection, and we define the curvature associated with it. These are ingredients to obtain invariants that measure in a certain way how the local product structure of the bundle separates from a global product structure.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Universal Journal of Mathematics and Applications, 4 (4) (2021) 154-163
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.978875

Multi-Parametric Families of Real and Non Singular Solutions of the Kadomtsev-Petviasvili I Equation

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Article Info

Keywords: *Kadomtsev-Petviashvili equation, rational solutions*
2010 AMS: *33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd*
Received: *4 August 2021*
Accepted: *16 December 2021*
Available online: *30 December 2021*

Abstract

Multi-parametric solutions to the Kadomtsev-Petviashvili equation (KPI) in terms of Fredholm determinants are constructed in function of exponentials. A representation of these solutions as a quotient of wronskians of order $2N$ in terms of trigonometric functions is deduced. All these solutions depend on $2N-1$ real parameters.
A third representation in terms of a quotient of two real polynomials depending on $2N-2$ real parameters is given; the numerator is a polynomial of degree $2N(N+1)-2$ in x, y and t and the denominator is a polynomial of degree $2N(N+1)$ in x, y and t. The maximum absolute value is equal to $2(2N+1)^2-2$.
We explicitly construct the expressions for the first third orders and we study the patterns of their absolute value in the plane (x,y) and their evolution according to time and parameters. It is relevant to emphasize that all these families of solutions are real and non singular.

1. Introduction

We consider the Kadomtsev-Petviashvili I equation (KPI)

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \tag{1.1}$$

where subscripts x, y and t denote partial derivatives.
This equation was introduced by Kadomtsev and Petviashvili [1] in 1970. It is considered as a model in hydrodynamic for surface and internal water waves [2] or in nonlinear optics [3]. Dryuma showed in 1974 how the KP equation could be written in Lax form [4].
Manakov, Zakharov, Bordag and Matveev first constructed rational solutions in 1977 [5] and two month later Krichever published other solutions [6].
In the frame of algebraic geometry, Krichever constructed for the first time in 1976 [7] the solutions to KPI equation in terms of Riemann theta functions and a little later, it was done by Dubrovin [8].
Others rational solutions of the KPI equation were obtained. For example, one can quote of the studies of Krichever in 1978 [9], Satsuma and Ablowitz in 1979 [10], Matveev in 1979 [11], Freeman and Nimmo in 1983 [12, 13], Matveev in 1987 [14], Pelinovsky and Stepanyants in 1993 [15], Pelinovsky in 1994 [16], Ablowitz, Villarroel, Chakravarty, Trubatch [17–19] in 1997-2000, Biondini and Kodama [20–22] in 2003-2007.

We give in the following three types of representations of the solutions to the KPI equation : first, in terms of Fredholm determinants of order $2N$ depending on $2N-1$ real parameters in function of exponentials, then in terms of wronskians of order $2N$ with $2N-1$ real parameters in function of some trigonometric functions.
In a third representation, real rational solutions of order N depending on $2N-2$ real parameters are constructed and they can be written as a ratio of two polynomials; the numerator is a polynomial in x, y and t of degree $2N(N+1)-2$ and the denominator a polynomial in x, y and t of degree $2N(N+1)$.
So we get rational real and non singular solutions to the KPI equation at each order N depending on $2N-2$ real parameters. We present explicit rational solutions and the representations of their absolute value in the plane of the coordinates (x,y) according to the $2N-2$ real parameters a_i and b_i $(1 \leq i \leq N-1)$ and time t for the first three orders.

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2. Families of solutions of order N depending on $2N-1$ real parameters in terms of Fredholm determinants to the KPI equation

We define the numbers λ_ν, κ_ν, δ_ν, γ_ν, $x_{r,\nu}$, e_ν depending on a real number ε by

$$\begin{array}{l}
\lambda_j = 1-2\varepsilon^2 j^2, \lambda_{N+j} = -\lambda_j, \qquad \kappa_j = 2\sqrt{1-\lambda_j^2}, \qquad \delta_j = \kappa_j \lambda_j, \qquad \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}, \\
x_{r,j} = (r-1)\ln\frac{\gamma_j - i}{\gamma_j + i}, r = 1,3, \qquad \tau_j = -12i\lambda_j^2\sqrt{1-\lambda_j^2} - 4i(1-\lambda_j^2)\sqrt{1-\lambda_j^2}, \\
\kappa_{N+j} = \kappa_j, \qquad \delta_{N+j} = -\delta_j, \qquad \gamma_{N+j} = \gamma_j^{-1}, \qquad x_{r,N+j} = -x_{r,j},, \qquad \tau_{N+j} = \tau_j \\
e_j = 2i\left(\sum_{k=1}^{1/2M-1} a_k (je)^{2k+1} - i\sum_{k=1}^{1/2M-1} b_K (je)^{2k+1}\right), \\
e_{N+j} = 2i\left(\sum_{k=1}^{1/2M-1} a_k (je)^{2k+1} + i\sum_{k=1}^{1/2M-1} b_k (je)^{2k+1}\right), \\
a_j, b_j \text{ real numbers }, \qquad \varepsilon_j = 1, \qquad \varepsilon_{N+j} = 0, \quad \varphi \text{ a real number} \\
1 \le j \le N.
\end{array} \tag{2.1}$$

Then we have the following statement:

Theorem 2.1. *Let $v(x,y,t)$ be the expression defined by*

$$v(x,y,t) = \frac{\det(I + D_3(x,y,t))}{\det(I + D_1(x,y,t))}, \tag{2.2}$$

with I the unit matrix and $D_r = (d_{jk}^{(r)})_{1\le j,k\le 2N}$ the matrix

$$d_{\nu\mu}^{(r)} = (-1)^{\varepsilon_\nu} \prod_{\eta\neq\mu}\left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu}\right)\exp(i\kappa_\nu x - 2\delta_\nu y + \tau_\nu t + x_{r,\nu} + e_\nu). \tag{2.3}$$

Then the function defined by

$$u(x,y,t) = -2(|v(x+3t,y,t)|^2 - 1) \tag{2.4}$$

is a solution to the KPI equation (1.1) depending on $2N-1$ real parameters a_k, b_k, $1\le k\le N-1$ and ε.

Proof. We have proven in [23] that the function w defined by (2.5)

$$w(x,y) = \frac{\det(I + D_3(x,y,0))}{\det(I + D_1(x,y,0))}\exp(2iy - i\varphi) \tag{2.5}$$

is a solution to the nonlinear Schrödinger equation (2.6)

$$iw_y + w_{xx} + 2|w|^2 w = 0. \tag{2.6}$$

It can then be similarly proven that the function v defined by

$$\tilde{v}(x,y,t) = v(x,y,t)\times\exp(2iy - i\varphi) = \frac{\det(I + D_3(x,y,t))}{\det(I + D_1(x,y,t))}\times\exp(2iy - i\varphi)$$

is a solution to the NLS equation (2.6) by considering t as a parameter. We can then deduce that the function u defined by (3)

$$u(x,t) = -2(|v(x+3t,y,t)|^2 - 1) = -2(|\tilde{v}(x+3t,y,t)|^2 - 1)$$

is a solution to the KPI equation, which proves the result. □

3. Families of solutions of order N depending on $2N-1$ real parameters in terms of wronskians to the KPI equation

We denote $W_r(w)$ the wronskian of the functions $\phi_{r,1},\dots,\phi_{r,2N}$ defined by

$$W_r(w) = \det[(\partial_w^{\mu-1}\phi_{r,\nu})_{\nu,\mu\in[1,\dots,2N]}]. \tag{3.1}$$

We consider the matrix $D_r = (d_{\nu\mu}^{(r)})_{\nu,\mu\in[1,\dots,2N]}$ defined in (2.3).
We consider the real parameters a_k, b_k $1\le k\le N-1$ and ε, and κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν defined in the previous section.
Then we have the following statement

Theorem 3.1. *Let $\Phi_{r,\nu}$ be the functions defined by*

$$\phi_{r,\nu}(x,y,t,w) = \sin\left(\frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{r,\nu}}{2} - i\frac{\tau_\nu}{2}t + \gamma_\nu w - i\frac{e_\nu}{2}\right), \quad 1 \le \nu \le N,$$
$$\phi_{r,\nu}(x,y,t,w) = \cos\left(\frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{r,\nu}}{2} - i\frac{\tau_\nu}{2}t + \gamma_\nu w - i\frac{e_\nu}{2}\right), \quad N+1 \le \nu \le 2N, \quad r = 1,3,$$

Let v be the expression defined by

$$v(x,y,t) = \frac{W_3(\phi_{3,1},\ldots,\phi_{3,2N})(x,y,t,0)}{W_1(\phi_{1,1},\ldots,\phi_{1,2N})(x,y,t,0)} \tag{3.2}$$

Then the function defined by

$$u(x,y,t) = -2(|v(x+3t,y,t)|^2 - 1) \tag{3.3}$$

is a solution to the KPI equation (1.1) depending on $2N-1$ real parameters a_k, b_k, $1 \le k \le N-1$ and ε.

Proof. We have proven in [24] that the function v defined by (3.4)

$$w(x,y) = \frac{W_3(\phi_{3,1},\ldots,\phi_{3,2N})(x,y,0,0)}{W_1(\phi_{1,1},\ldots,\phi_{1,2N})(x,y,0,0)}\exp(2iy - i\varphi) \tag{3.4}$$

is a solution to the nonlinear Schrödinger equation (3.5)

$$iw_y + w_{xx} + 2|w|^2 w = 0. \tag{3.5}$$

We can similarly prove that the function v defined by

$$\tilde{v}(x,y,t) = v(x,y,t) \times \exp(2iy - i\varphi)$$

is a solution of the NLS equation by considering t as a parameter. We can then deduce that the function u defined by

$$u(x,t) = -2(|v(x+3t,y,t)|^2 - 1) = -2(|\tilde{v}(x+3t,y,t)|^2 - 1)$$

is a solution to the KPI equation which proves the result. □

4. Real and non singular rational solutions to the KPI equation of order N depending on $2N-2$ real parameters

We construct in this section rational solutions to the KPI equation as a quotient of two determinants.
We define functions of the following arguments:

$$X_\nu = \frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{3,\nu}}{2} - i\frac{\tau_\nu}{2}t - i\frac{e_\nu}{2}, \tag{4.1}$$
$$Y_\nu = \frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{1,\nu}}{2} - i\frac{\tau_\nu}{2}t - i\frac{e_\nu}{2}, \tag{4.2}$$

for $1 \le \nu \le 2N$, with κ_ν, δ_ν, $x_{r,\nu}$ defined in the first section.
We consider the following functions:

$$\begin{aligned}
&\varphi_{4j+1,k} = \gamma_k^{4j-1}\sin X_k, \quad \varphi_{4j+2,k} = \gamma_k^{4j}\cos X_k,\\
&\varphi_{4j+3,k} = -\gamma_k^{4j+1}\sin X_k, \quad \varphi_{4j+4,k} = -\gamma_k^{4j+2}\cos X_k,\\
&\varphi_{4j+1,N+k} = \gamma_k^{2N-4j-2}\cos X_{N+k}, \quad \varphi_{4j+2,N+k} = -\gamma_k^{2N-4j-3}\sin X_{N+k},\\
&\varphi_{4j+3,N+k} = -\gamma_k^{2N-4j-4}\cos X_{N+k}, \quad \varphi_{4j+4,N+k} = \gamma_k^{2N-4j-5}\sin X_{N+k},\\
&\psi_{4j+1,k} = \gamma_k^{4j-1}\sin Y_k, \quad \psi_{4j+2,k} = \gamma_k^{4j}\cos Y_k,\\
&\psi_{4j+3,k} = -\gamma_k^{4j+1}\sin Y_k, \quad \psi_{4j+4,k} = -\gamma_k^{4j+2}\cos Y_k,\\
&\psi_{4j+1,N+k} = \gamma_k^{2N-4j-2}\cos Y_{N+k}, \quad \psi_{4j+2,N+k} = -\gamma_k^{2N-4j-3}\sin Y_{N+k},\\
&\psi_{4j+3,N+k} = -\gamma_k^{2N-4j-4}\cos Y_{N+k}, \quad \psi_{4j+4,N+k} = \gamma_k^{2N-4j-5}\sin Y_{N+k},\\
&1 \le k \le N
\end{aligned} \tag{4.3}$$

Then we get the following result

Theorem 4.1. *Let v be the expression defined by*

$$v(x,y,t) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} \tag{4.4}$$

where

$$
\begin{aligned}
&n_{j1} = \varphi_{j,1}(x,y,t,0), \quad n_{jk} = \frac{\partial^{2k-2}\varphi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&n_{jN+1} = \varphi_{j,N+1}(x,y,t,0), \quad n_{jN+k} = \frac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&d_{j1} = \psi_{j,1}(x,y,t,0), \quad d_{jk} = \frac{\partial^{2k-2}\psi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&d_{jN+1} = \psi_{j,N+1}(x,y,t,0), \quad d_{jN+k} = \frac{\partial^{2k-2}\psi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&2 \leq k \leq N, \quad 1 \leq j \leq 2N,
\end{aligned}
\tag{4.5}
$$

the functions φ and ψ being defined in (4.3).
Then the function defined by

$$
u(x,t) = -2(|v(x+3t,y,t)|^2 - 1) \tag{4.6}
$$

is a solution to the KPI equation (1.1) depending on $2N-2$ parameters a_k, b_k, $1 \leq k \leq N-1$.

Proof. It is still a consequence of our previous works. Precisely, we have proven in [25] that the function w defined by

$$
w(x,y) = \frac{\det(n_{jk})_{j,k\in[1,2N]_{t=0}}}{\det(d_{jk})_{j,k\in[1,2N]_{t=0}}} \times \exp(2iy - i\varphi) \tag{4.7}
$$

is a solution to the nonlinear Schrödinger equation (4.8)

$$
iw_y + w_{xx} + 2|w|^2 w = 0. \tag{4.8}
$$

We can prove in the same way that the function v defined by

$$
\tilde{v}(x,y,t) = v(x,y,t) \times \exp(2iy - i\varphi)
$$

is a solution of the NLS equation by considering t as a parameter. Then, we can deduce that the function u defined by

$$
u(x,t) = -2(|v(x+3t,y,t)|^2 - 1) = -2(|\tilde{v}(x+3t,y,t)|^2 - 1)
$$

is a solution to the KPI equation which proves the result. □

5. The structure of the solutions to the KPI equation

The structure of the rational solutions to the KPI equation is given by the following result

Theorem 5.1. *Let u the function defined by*

$$
u(x,t) = -2(|v(x+3t,y,t)|^2 - 1) = \frac{n(x,y,t)}{d(x,y,t)},
$$

with

$$
v(x,y,t) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})}
$$

$$
\begin{aligned}
&n_{j1} = \varphi_{j,1}(x,y,t,0), \quad n_{jk} = \frac{\partial^{2k-2}\varphi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&n_{jN+1} = \varphi_{j,N+1}(x,y,t,0), \quad n_{jN+k} = \frac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&d_{j1} = \psi_{j,1}(x,y,t,0), \quad d_{jk} = \frac{\partial^{2k-2}\psi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&d_{jN+1} = \psi_{j,N+1}(x,y,t,0), \quad d_{jN+k} = \frac{\partial^{2k-2}\psi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&2 \leq k \leq N, \quad 1 \leq j \leq 2N,
\end{aligned}
$$

Then the function v is a rational solution to the KPI equation (1.1) quotient of two polynomials $n(x,y,t)$ and $d(x,y,t)$ depending on $2N-2$ real parameters a_j and b_j, $1 \leq j \leq N-1$.
n is a polynomial of degree $2N(N+1)-2$ in x, y and t.
d is a polynomial of degree $2N(N+1)$ in x, y and t.

Proof. It is already proven in the previous section that this function is a solution to the KPI equation.
The proof of the structure of the solution is similar to this given in [26]. The difference in this present case is due to the reduction of the fraction which cancel the terms in $x^{2N(N+1)}$, $y^{2N(N+1)}$, $t^{2N(N+1)}$ in the numerator, these terms having the same maximal power in numerator and denominator, and the fact that the elevation by the power 2 makes that the succeeding terms in x, y and t are to the power $2N(N+1)-2$. □

Theorem 5.2. *Let u the function defined by*

$$u(x,t) = -2(|v(x+3t,y,t)|^2 - 1) = \frac{n(x,y,t)}{d(x,y,t)},$$

with

$$v(x,y,t) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})}$$

$$\begin{array}{l}
n_{j1} = \varphi_{j,1}(x,y,t,0), \quad n_{jk} = \dfrac{\partial^{2k-2}\varphi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
n_{jN+1} = \varphi_{j,N+1}(x,y,t,0), \quad n_{jN+k} = \dfrac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
d_{j1} = \psi_{j,1}(x,y,t,0), \quad d_{jk} = \dfrac{\partial^{2k-2}\psi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
d_{jN+1} = \psi_{j,N+1}(x,y,t,0), \quad d_{jN+k} = \dfrac{\partial^{2k-2}\psi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
2 \le k \le N, \quad 1 \le j \le 2N,
\end{array}$$

Then the function v_0 defined by

$$v_0(x,y,t) = v(x,y,t)_{(a_j=b_j=0,\, 1\le j\le N-1)} \tag{5.1}$$

is the solution of order N solution to the KPI equation (1.1) whose highest amplitude in modulus is equal to $2(2N+1)^2-2$.

Proof. The proof of this result is similar to this given in [26]. We do not give more details. The reader can do by himself the rewriting of this proof. □

6. Explicit expressions and patterns of the rational solutions to the KPI equation in function of parameters and time

We have explicitly constructed rational solutions to the KPI equation of order N depending on $2N-2$ parameters for $1\le N\le 3$.
In the following, we only give patterns of the modulus of the solutions in the plane (x,y) of coordinates in function of the parameters a_i, and b_i, for $1\le i\le N-1$ for $2\le N\le 3$, and time t.
We present the solutions using the following notations $X=2x$, $Y=4y$, $T=2t$,

$$u_N(X,Y,T) = 1 - \frac{G_N(X,Y,T)}{Q_N(X,Y,T)}$$

with

$$\begin{array}{l}
G_N(X,Y,T) = \sum_{k=0}^{2N(N+1)} g_k(Y,T)X^k,\\
Q_N(X,Y,T) = \sum_{k=0}^{2N(N+1)} q_k(Y,T)X^k.
\end{array}$$

By construction, all these solutions constructed in this study are real. Moreover, we know from the study of the NLS equation that the solutions constructed by ourself were non singular. From the construction, the denominators of the solutions to the KPI equation being the square of those of the solutions of the NLS equation, we get the non singularity of all these families of solutions to the KPI equation.

6.1. Case $N=1$

The polynomials Q_1 and G_1 are given by
$\mathbf{q}_4 = 1, \quad \mathbf{q}_3 = -12T, \quad \mathbf{q}_2 = 54T^2+2Y^2+2, \quad \mathbf{q}_1 = -108T^3+(-12Y^2-12)T, \quad \mathbf{q}_0 = 81T^4+Y^4+\left(18Y^2+18\right)T^2+2Y^2+1$
$\mathbf{g}_4 = 1, \quad \mathbf{g}_3 = -12T, \quad \mathbf{g}_2 = 54T^2+2Y^2-14, \quad \mathbf{g}_1 = -108T^3+(-12Y^2+84)T, \quad \mathbf{g}_0 = 81T^4+Y^4+\left(18Y^2-126\right)T^2+18Y^2+17$
This type of solution to the KPI equation is different from our previous works.
In our previous works [26–31], we constructed solution of order 1 to KPI equation and got
$\tilde{v}_1(X,Y,T)$
$= -2\,\frac{9-6X^2+72XT+X^4+1296T^4+216X^2T^2-216T^2+10Y^2+Y^4-24XTY^2-24X^3T+2X^2Y^2-864XT^3+72T^2Y^2}{(X^2-12XT+36T^2+Y^2+1)^2}.$
The solution of order 1 obtained in this paper can be rewritten as

$v_1(X,Y,T) = 16\,\frac{-1+X^2-6XT+9T^2-Y^2}{X^2-6XT+9T^2+Y^2+1)^2} = 16\,\frac{-1+(X-3T)^2-Y^2}{(1+(X-3T)^2+Y^2)^2}.$

It can be easily seen in this example that these two solutions are different and non singular. Moreover, we can verify that the maximum of the absolute value of v_1 is equal to $2(2N+1)^2-2=16$ obtained when $X=Y=T=0$.

In the case $N=1$, one obtains a peak which the height decreases very quickly as t increases.

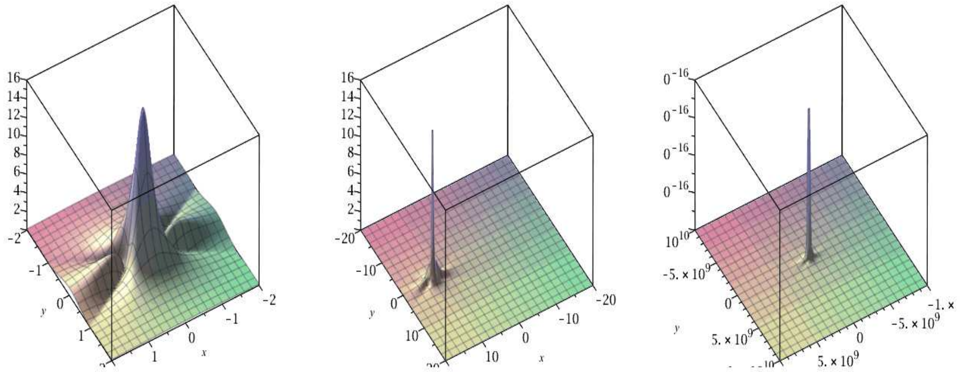

Figure 1. Solution of order 1 to the KPI equation, on the left for $t = 0$; in the center for $t = 4$; on the right for $t = 10^8$.

6.2. Case $N = 2$

In the case $N = 2$, the polynomials G_2 and G_2 more complex are given by

$\mathbf{q}_{12} =, 1 \quad \mathbf{f}_{11} = -36T, \quad \mathbf{q}_{10} = 594T^2+6Y^2+6, \quad \mathbf{q}_9 = -5940T^3+(-180Y^2+12)T-12a_1, \quad \mathbf{q}_8 = 40095T^4+6Y^4+(2430Y^2-$
$2754)T^2+324Ta_1-36Y^2+36Yb_1+(3Y^2+3)^2+54, \quad \mathbf{q}_7 = -192456T^5+(-19440Y^2+42768)T^3-3888T^2a_1+36Y^2a_1+(-144Y^4+$
$288Y^2-864Yb_1-1872+2(-36Y^2+60)(3Y^2+3))T+36a_1-12(3Y^2+3)a_1, \quad \mathbf{q}_6 = 673596T^6+2Y^6+(102060Y^2-333396)T^4+$
$27216T^3a_1+54Y^4-12Y^3b_1+(1512Y^4+3024Y^2+9072Yb_1+30312+2(162Y^2-702)(3Y^2+3)+(-36Y^2+60)^2)T^2+198Y^2-$
$108Yb_1+54{a_1}^2+18{b_1}^2+(-756Y^2a_1-1332a_1-12(-36Y^2+60)a_1+108(3Y^2+3)a_1)T+2(3Y^4-18Y^2+18Yb_1+27)(3Y^2+$
$3)+18, \quad \mathbf{q}_5 = -1732104T^7+(-367416Y^2+1592136)T^5-122472T^4a_1+(-9072Y^4-54432Y^2-54432Yb_1-273456+2(162Y^2-$
$702)(-36Y^2+60)+2(-324Y^2+2268)(3Y^2+3))T^3+(6804Y^2a_1+17172a_1-324(3Y^2+3)a_1-12(162Y^2-702)a_1+108(-36Y^2+$
$60)a_1)T^2+(-36Y^6-972Y^4+216Y^3b_1-3564Y^2+1944Yb_1-972{a_1}^2-324{b_1}^2+2(-18Y^4-180Y^2-108Yb_1-450)(3Y^2+3)-$
$324+2(3Y^4-18Y^2+18Yb_1+27)(-36Y^2+60))T-12(3Y^4-18Y^2+18Yb_1+27)a_1+2(18Y^2a_1+18a_1)(3Y^2+3), \quad \mathbf{q}_4 = 3247695T^8+$
$(918540Y^2-4960116)T^6+367416T^5a_1+(34020Y^4+340200Y^2+204120Yb_1+1472580+(162Y^2-702)^2+2(243Y^2-2349)(3Y^2+$
$3)+2(-324Y^2+2268)(-36Y^2+60))T^4+(-34020Y^2a_1-12(-324Y^2+2268)a_1-111780a_1-324(-36Y^2+60)a_1+108(162Y^2-$
$702)a_1+324(3Y^2+3)a_1)T^3+(270Y^6+7290Y^4-1620Y^3b_1+26730Y^2-14580Yb_1+7290{a_1}^2+2430{b_1}^2+2430+2(27Y^4+702Y^2+$
$162Yb_1+3411)(3Y^2+3)+2(-18Y^4-180Y^2-108Yb_1-450)(-36Y^2+60)+2(3Y^4-18Y^2+18Yb_1+27)(162Y^2-702))T^2+$
$(-12(-18Y^4-180Y^2-108Yb_1-450)a_1+108(3Y^4-18Y^2+18Yb_1+27)a_1+2(-54Y^2a_1-342a_1)(3Y^2+3)+2(18Y^2a_1+18a_1)(-36Y^2+$
$60))T-12(18Y^2a_1+18a_1)a_1+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9{a_1}^2+9{b_1}^2+9)(3Y^2+3)+(3Y^4-18Y^2+18Yb_1+27)^2, \quad \mathbf{q}_3 =$
$-4330260T^9+(-1574640Y^2+10182672)T^7-734832T^6a_1+(-81648Y^4-1143072Y^2-489888Yb_1-4856112+2(243Y^2-2349)(-36Y^2+$
$60)+2(-324Y^2+2268)(162Y^2-702))T^5+(102060Y^2a_1+413100a_1-324(162Y^2-702)a_1+108(-324Y^2+2268)a_1-12(243Y^2-$
$2349)a_1+324(-36Y^2+60)a_1)T^4+(-1080Y^6-29160Y^4+6480Y^3b_1-106920Y^2+58320Yb_1-29160{a_1}^2-9720{b_1}^2-9720+2(27Y^4+$
$702Y^2+162Yb_1+3411)(-36Y^2+60)+2(-18Y^4-180Y^2-108Yb_1-450)(162Y^2-702)+2(-324Y^2+2268)(3Y^4-18Y^2+18Yb_1+$
$27))T^3+(-12(27Y^4+702Y^2+162Yb_1+3411)a_1+108(-18Y^4-180Y^2-108Yb_1-450)a_1-324(3Y^4-18Y^2+18Yb_1+27)a_1+$
$2(-54Y^2a_1-342a_1)(-36Y^2+60)+2(18Y^2a_1+18a_1)(162Y^2-702))T^2+(-12(-54Y^2a_1-342a_1)a_1+108(18Y^2a_1+18a_1)a_1+$
$2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9{a_1}^2+9{b_1}^2+9)(-36Y^2+60)+2(-18Y^4-180Y^2-108Yb_1-450)(3Y^4-18Y^2+18Yb_1+$
$27))T-12(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9{a_1}^2+9{b_1}^2+9)a_1+2(18Y^2a_1+18a_1)(3Y^4-18Y^2+18Yb_1+27), \quad \mathbf{q}_2 = 3897234T^{10}+$
$(1771470Y^2-13345074)T^8+944784T^7a_1+(122472Y^4+2204496Y^2+734832Yb_1+(-324Y^2+2268)^2+9640296+2(243Y^2-2349)(162Y^2-$
$702))T^6+(-183708Y^2a_1-883548a_1+324(162Y^2-702)a_1+108(243Y^2-2349)a_1-324(-324Y^2+2268)a_1)T^5+(2430Y^6+65610Y^4-$
$14580Y^3b_1+240570Y^2-131220Yb_1+65610{a_1}^2+21870{b_1}^2+21870+2(27Y^4+702Y^2+162Yb_1+3411)(162Y^2-702)+2(243Y^2-$
$2349)(3Y^4-18Y^2+18Yb_1+27)+2(-18Y^4-180Y^2-108Yb_1-450)(-324Y^2+2268))T^4+(324(3Y^4-18Y^2+18Yb_1+27)a_1-$
$324(-18Y^4-180Y^2-108Yb_1-450)a_1+108(27Y^4+702Y^2+162Yb_1+3411)a_1+2(-54Y^2a_1-342a_1)(162Y^2-702)+2(18Y^2a_1+$
$18a_1)(-324Y^2+2268))T^3+(-324(18Y^2a_1+18a_1)a_1+108(-54Y^2a_1-342a_1)a_1+(-18Y^4-180Y^2-108Yb_1-450)^2+2(Y^6+$
$27Y^4-6Y^3b_1+99Y^2-54Yb_1+9{a_1}^2+9{b_1}^2+9)(162Y^2-702)+2(27Y^4+702Y^2+162Yb_1+3411)(3Y^4-18Y^2+18Yb_1+27))T^2+$
$(108(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9{a_1}^2+9{b_1}^2+9)a_1+2(-54Y^2a_1-342a_1)(3Y^4-18Y^2+18Yb_1+27)+2(18Y^2a_1+$
$18a_1)(-18Y^4-180Y^2-108Yb_1-450))T+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9{a_1}^2+9{b_1}^2+9)(3Y^4-18Y^2+18Yb_1+27)+$
$(18Y^2a_1+18a_1)^2, \quad \mathbf{q}_1 = -2125764T^{11}+(-1180980Y^2+10156428)T^9-708588T^8a_1+(-104976Y^4-2309472Y^2-629856Yb_1+$
$2(243Y^2-2349)(-324Y^2+2268)-10602576)T^7+(183708Y^2a_1+324(-324Y^2+2268)a_1-324(243Y^2-2349)a_1+1023516a_1)T^6+$
$(-2916Y^6-78732Y^4+17496Y^3b_1-288684Y^2+157464Yb_1-78732{a_1}^2-26244{b_1}^2+2(243Y^2-2349)(-18Y^4-180Y^2-108Yb_1-$
$450)+2(27Y^4+702Y^2+162Yb_1+3411)(-324Y^2+2268)-26244)T^5+(-324(27Y^4+702Y^2+162Yb_1+3411)a_1+324(-18Y^4-$
$180Y^2-108Yb_1-450)a_1+2(-54Y^2a_1-342a_1)(-324Y^2+2268)+2(243Y^2-2349)(18Y^2a_1+18a_1))T^4+(-324(-54Y^2a_1-342a_1)a_1+$
$324(18Y^2a_1+18a_1)a_1+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9{a_1}^2+9{b_1}^2+9)(-324Y^2+2268)+2(27Y^4+702Y^2+162Yb_1+$
$3411)(-18Y^4-180Y^2-108Yb_1-450))T^3+(-324(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9{a_1}^2+9{b_1}^2+9)a_1+2(-54Y^2a_1-342a_1)(-18Y^4-$
$180Y^2-108Yb_1-450)+2(27Y^4+702Y^2+162Yb_1+3411)(18Y^2a_1+18a_1))T^2+(2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9{a_1}^2+$
$9{b_1}^2+9)(-18Y^4-180Y^2-108Yb_1-450)+2(-54Y^2a_1-342a_1)(18Y^2a_1+18a_1))T+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+$
$9{a_1}^2+9{b_1}^2+9)(18Y^2a_1+18a_1), \quad \mathbf{q}_0 = 531441T^{12}+Y^{12}+(354294Y^2-3424842)T^{10}+236196T^9a_1+54Y^{10}-12Y^9b_1+(98415Y^4-$
$118098Y^2+236196Yb_1+10491039)T^8-1259712T^7a_1+927Y^8-432Y^7b_1+18Y^6{a_1}^2+54Y^6{b_1}^2+(14580Y^6+253692Y^4+69984Y^3b_1-$
$1495908Y^2-839808Yb_1+39366{a_1}^2+13122{b_1}^2-16011756)T^6+5364Y^6-4104Y^5b_1+486Y^4{a_1}^2+1134Y^4{b_1}^2-108Y^3{a_1}^2b_1-$
$108Y^3{b_1}^3+(-17496Y^4a_1+314928Y^2a_1+52488Ya_1b_1+2711880a_1)T^5+(1215Y^8+46332Y^6+5832Y^5b_1+598266Y^4+229392Y^3b_1-$
$13122Y^2{a_1}^2+30618Y^2{b_1}^2+4328316Y^2+1358856Yb_1-153090{a_1}^2-42282{b_1}^2+11592639)T^4+10287Y^4-10800Y^3b_1+1782Y^2{a_1}^2+$
$4698Y^2{b_1}^2-972Y{a_1}^2b_1-972Y{b_1}^3+81{a_1}^4+162{a_1}^2{b_1}^2+81{b_1}^4+(-2592Y^6a_1-85536Y^4a_1-19440Y^3a_1b_1-816480Y^2a_1-128304Ya_1b_1+$

$2916a_1{}^3+2916a_1b_1{}^2-2330208a_1)T^3+(54Y^{10}+2862Y^8+50076Y^6-2592Y^5b_1+3402Y^4a_1{}^2-1458Y^4b_1{}^2+323676Y^4-84672Y^3b_1+49572Y^2a_1{}^2-4860Y^2b_1{}^2+2916Ya_1{}^2b_1+2916Yb_1{}^3+688014Y^2-365472Yb_1+178362a_1{}^2+61398b_1{}^2+61398)T^2+1782Y^2-972Yb_1+162a_1{}^2+162b_1{}^2+(-108Y^8a_1-3600Y^6a_1+648Y^5a_1b_1-29160Y^4a_1+9936Y^3a_1b_1-972Y^2a_1{}^3-972Y^2a_1b_1{}^2-68688Y^2a_1+36936Ya_1b_1-6156a_1{}^3-6156a_1b_1{}^2-6156a_1)T+81$

$\mathbf{g}_{12}=1,\quad \mathbf{g}_{12}=1,\quad \mathbf{g}_{11}=-36T,\quad \mathbf{g}_{10}=594T^2+6Y^2-42,\quad \mathbf{g}_9=-5940T^3+(-180Y^2+1452)T-12a_1,\quad \mathbf{g}_8=40095T^4+15Y^4+(2430Y^2-22194)T^2+324Ta_1-162Y^2+36Yb_1-81,\quad \mathbf{g}_7=-192456T^5+(-19440Y^2+198288)T^3-3888T^2a_1+(-360Y^4+3888Y^2-864Yb_1+1944)T,\quad \mathbf{g}_6=673596T^6+20Y^6+(102060Y^2-1149876)T^4+27216T^3a_1-132Y^4+96Y^3b_1+(3780Y^4-40824Y^2+9072Yb_1-6588)T^2-1728Ta_1-1476Y^2+1152Yb_1+54a_1{}^2+18b_1{}^2+1620,\quad \mathbf{g}_5=-1732104T^7+(-367416Y^2+4531464)T^5-122472T^4a_1+72Y^4a_1+(-22680Y^4+244944Y^2-54432Yb_1-126360)T^3+31104T^2a_1+3888Y^2a_1-216Ya_1b_1+(-360Y^6+1224Y^4-1728Y^3b_1-35640Y^2-17280Yb_1-972a_1{}^2-324b_1{}^2-81000)T+3240a_1,\ quad\mathbf{g}_4=3247695T^8+15Y^8+(918540Y^2-12308436)T^6+367416T^5a_1+156Y^6+72Y^5b_1+(85050Y^4-918540Y^2+204120Yb_1+1406970)T^4-233280T^3a_1+8442Y^4-3312Y^3b_1-162Y^2a_1{}^2+378Y^2b_1{}^2+(2700Y^6-540Y^4+12960Y^3b_1+692388Y^2+103680Yb_1+7290a_1{}^2+2430b_1{}^2+1286604)T^2+23004Y^2-13176Yb_1+1134a_1{}^2+2214b_1{}^2+(-1080Y^4a_1-53136Y^2a_1+3240Ya_1b_1-84888a_1)T+1647,\quad \mathbf{g}_3=-4330260T^9+(-1574640Y^2+22779792)T^7-734832T^6a_1+96Y^6a_1+(-204120Y^4+2204496Y^2-489888Yb_1-6362712)T^5+933120T^4a_1+1440Y^4a_1+720Y^3a_1b_1+(-10800Y^6-32400Y^4-51840Y^3b_1-4304016Y^2-311040Yb_1-29160a_1{}^2-9720b_1{}^2-8581680)T^3-864Y^2a_1+4752Ya_1b_1-108a_1{}^3-108a_1b_1{}^2+(6480Y^4a_1+287712Y^2a_1-19440Ya_1b_1+644112a_1)T^2+(-180Y^8-3408Y^6-864Y^5b_1-124344Y^4+28224Y^3b_1+1944Y^2a_1{}^2-4536Y^2b_1{}^2-262224Y^2+82080Yb_1-8424a_1{}^2-24840b_1{}^2-144180)T+7776a_1,\quad \mathbf{g}_2=3897234T^{10}+6Y^{10}+(1771470Y^2-27516834)T^8+944784T^7a_1+270Y^8+(306180Y^4-3306744Y^2+734832Yb_1+15142788)T^6-2099520T^5a_1+9468Y^6-2592Y^5b_1+378Y^4a_1{}^2-162Y^4b_1{}^2+(24300Y^6+150660Y^4+116640Y^3b_1+12763332Y^2+466560Yb_1+65610a_1{}^2+21870b_1{}^2+27660204)T^4+26460Y^4-1728Y^3b_1+2916Y^2a_1{}^2-4860Y^2b_1{}^2+324Ya_1{}^2b_1+324Yb_1{}^3+(-19440Y^4a_1-769824Y^2a_1+58320Ya_1b_1-2087856a_1)T^3+(810Y^8+22248Y^6+3888Y^5b_1+759996Y^4-75168Y^3b_1-8748Y^2a_1{}^2+20412Y^2b_1{}^2+1864296Y^2+55728Yb_1+14580a_1{}^2+104004b_1{}^2+2079594)T^2+51678Y^2-33696Yb_1+3402a_1{}^2+3078b_1{}^2+(-864Y^6a_1-25056Y^4a_1-6480Y^3a_1b_1-85536Y^2a_1-53136Ya_1b_1+972a_1{}^3+972a_1b_1{}^2-178848a_1)T-8586,\quad \mathbf{g}_1=-2125764T^{11}+(-1180980Y^2+19604268)T^9-708588T^8a_1+36Y^8a_1+(-262440Y^4+2834352Y^2-629856Yb_1-18738216)T^7+2519424T^6a_1-144Y^6a_1-216Y^5a_1b_1+(-29160Y^6-274104Y^4-139968Y^3b_1-18563256Y^2-279936Yb_1-78732a_1{}^2-26244b_1{}^2-42766056)T^5+1080Y^4a_1-5616Y^3a_1b_1+324Y^2a_1{}^3+324Y^2a_1b_1{}^2+(29160Y^4a_1+1014768Y^2a_1-87480Ya_1b_1+2991816a_1)T^4+(-1620Y^8-58320Y^6-7776Y^5b_1-2114424Y^4+46656Y^3b_1+17496Y^2a_1{}^2-40824Y^2b_1{}^2-7917264Y^2-1127520Yb_1+17496a_1{}^2-192456b_1{}^2+527148)T^3-27216Y^2a_1+8424Ya_1b_1-2268a_1{}^3-2268a_1b_1{}^2+(2592Y^6a_1+111456Y^4a_1+19440Y^3a_1b_1+785376Y^2a_1+190512Ya_1b_1-2916a_1{}^3-2916a_1b_1{}^2-878688a_1)T^2+(-36Y^{10}-2196Y^8-54504Y^6+19008Y^5b_1-2268Y^4a_1{}^2+972Y^4b_1{}^2-176040Y^4+100224Y^3b_1-33048Y^2a_1{}^2+23976Y^2b_1{}^2-1944Ya_1{}^2b_1-1944Yb_1{}^3+125388Y^2+67392Yb_1+88452a_1{}^2+17820b_1{}^2+212220)T-10044a_1,\quad \mathbf{g}_0=,531441T^{12}+Y^{12}+(354294Y^2-6259194)T^{10}+236196T^9a_1+102Y^{10}-12Y^9b_1+(98415Y^4-1062882Y^2+236196Yb_1+9546255)T^8-1259712T^7a_1+1935Y^8-432Y^7b_1+18Y^6a_1{}^2+54Y^6b_1{}^2+(14580Y^6+183708Y^4+69984Y^3b_1+10681308Y^2+39366a_1{}^2+13122b_1{}^2+25348788)T^6+2772Y^6+2808Y^5b_1-1674Y^4a_1{}^2-162Y^4b_1{}^2-108Y^3a_1{}^2b_1-108Y^3b_1{}^3+(-17496Y^4a_1-524880Y^2a_1+52488Ya_1b_1-1487160a_1)T^5+(1215Y^8+54108Y^6+5832Y^5b_1+2176794Y^4+42768Y^3b_1-13122Y^2a_1{}^2+30618Y^2b_1{}^2+12189852Y^2+1732104Yb_1-48114a_1{}^2+132678b_1{}^2-11229921)T^4-6129Y^4+30672Y^3b_1-5994Y^2a_1{}^2-13446Y^2b_1{}^2+1620Ya_1{}^2b_1+1620Yb_1{}^3+81a_1{}^4+162a_1{}^2b_1{}^2+81b_1{}^4+(-2592Y^6a_1-147744Y^4a_1-19440Y^3a_1b_1-1562976Y^2a_1-221616Ya_1b_1+2916a_1{}^3+2916a_1b_1{}^2+2708640a_1)T^3+(54Y^{10}+4158Y^8+82908Y^6-33696Y^5b_1+3402Y^4a_1{}^2-1458Y^4b_1{}^2-138564Y^4-312768Y^3b_1+72900Y^2a_1{}^2-28188Y^2b_1{}^2+2916Ya_1{}^2b_1+2916Yb_1{}^3-2375730Y^2+515808Yb_1-171558a_1{}^2-39690b_1{}^2-186138)T^2+41958Y^2-21708Yb_1+1458a_1{}^2+4050b_1{}^2+(-108Y^8a_1-144Y^6a_1+648Y^5a_1b_1+50328Y^4a_1+20304Y^3a_1b_1-972Y^2a_1{}^3-972Y^2a_1b_1{}^2+273456Y^2a_1-77112Ya_1b_1+1620a_1{}^3+1620a_1b_1{}^2-16524a_1)T+3969$

For $N=2$, the formation of three peaks is obtained when the parameters a_1 or b_1 are not equal to 0.

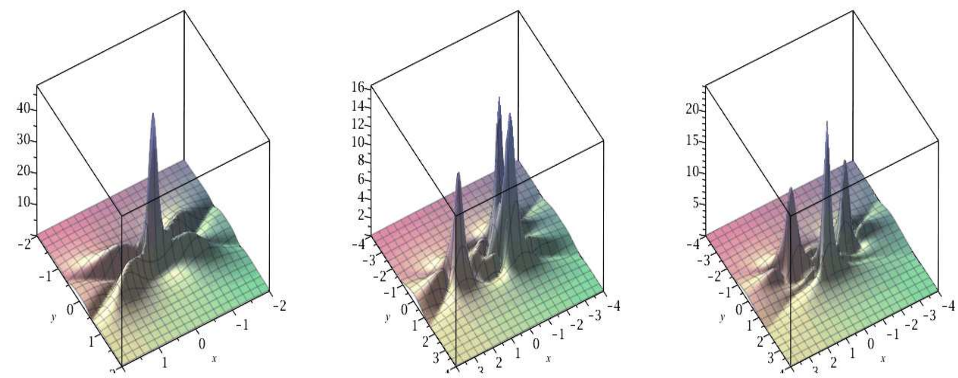

Figure 2. Solution of order 2 to the KPI equation for $t=0$, on the left $a_1=0$, $b_1=0,$; in the center $a_1=10$, $b_1=0,$; on the right $a_1=10$, $b_1=10$.

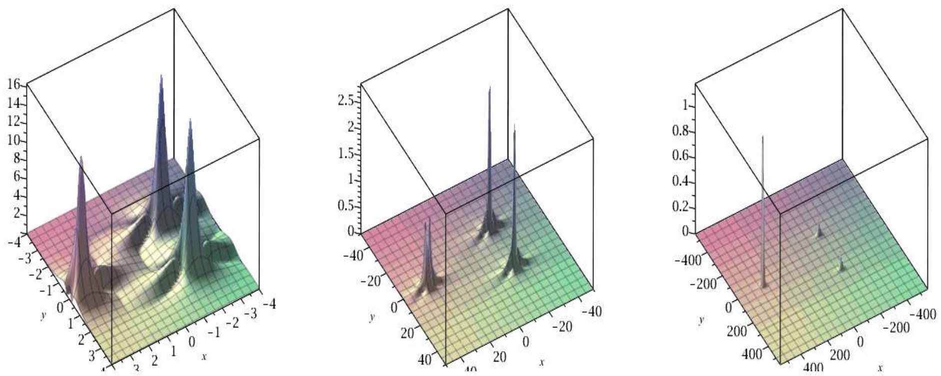

Figure 3. Solution of order 2 to the KPI equation for $t = 0$, on the left $a_1 = 10^2$, $b_1 = 0$; in the center $a_1 = 10^4$, $b_1 = 0$; on the right for $t = 10$, $a_1 = 10^8$, $b_1 = 0$.

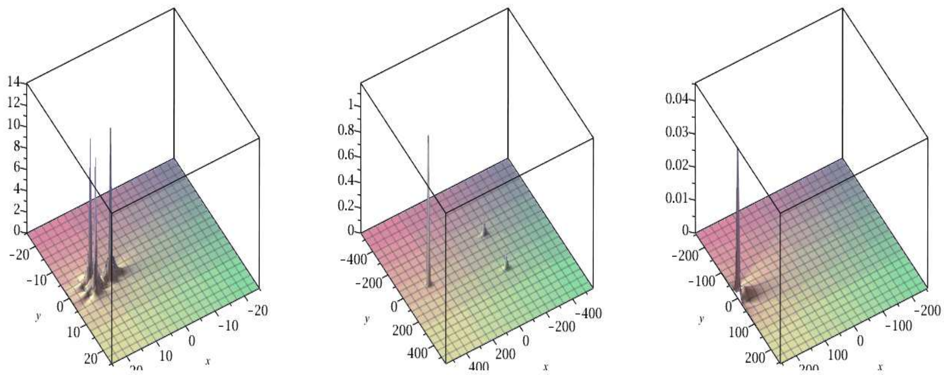

Figure 4. Solution of order 2 to the KPI equation, on the left for $t = 5$, $a_1 = 0$, $b_1 = 0$; in the center for $t = 10$, $a_1 = 0$, $b_1 = 0$; on the right for $t = 100$, $a_1 = 0$, $b_1 = 0$.

6.3. Case $N = 3$

In this case, polynomials G_3 and Q_3 depending on 4 parameters being too complex, we cannot give their explicit expressions. Even without parameters, due to the length of the solution, the explicit expression cannot be given here.

In the case $N = 3$, for $a_1 \neq 0$ or $b_1 \neq 0$ and the other parameters equal to zero, we obtain a triangle with 6 peaks; for $a_2 \neq 0$ or $b_2 \neq 0$, and other parameters equal to zero, we obtain a concentric rings of 5 peaks with a peak in the center.

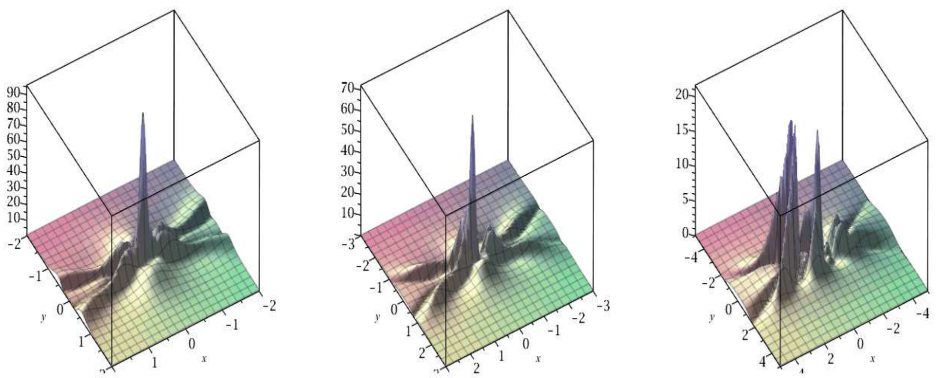

Figure 5. Solution of order 3 to the KPI equation, on the left for $t = 0$; in the center for $t = 0,01$; on the right for $t = 0,1$; all the parameters are equal to 0.

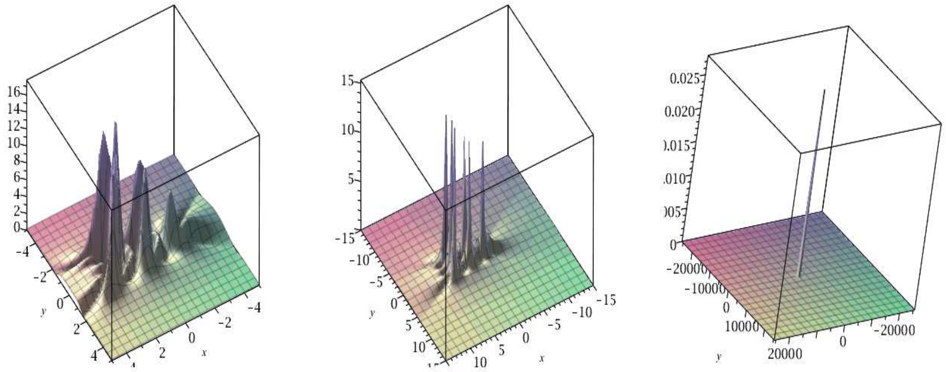

Figure 6. Solution of order 3 to the KPI equation, on the left for $t = 0,2$; in the center for $t = 10^0$; on the right for $t = 10^1$; all the parameters are equal to 0.

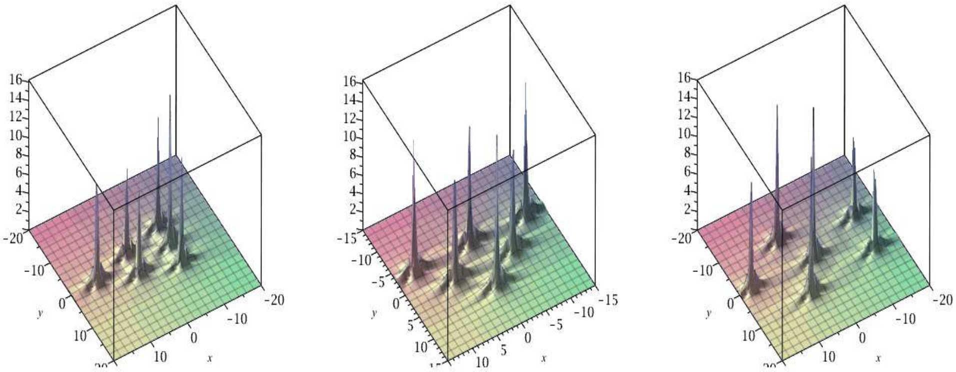

Figure 7. Solution of order 3 to the KPI equation, on the left for $a_1 = 10^3$; in the center for $b_1 = 10^3$; on the right for $a_2 = 10^6$; here $t = 0$.

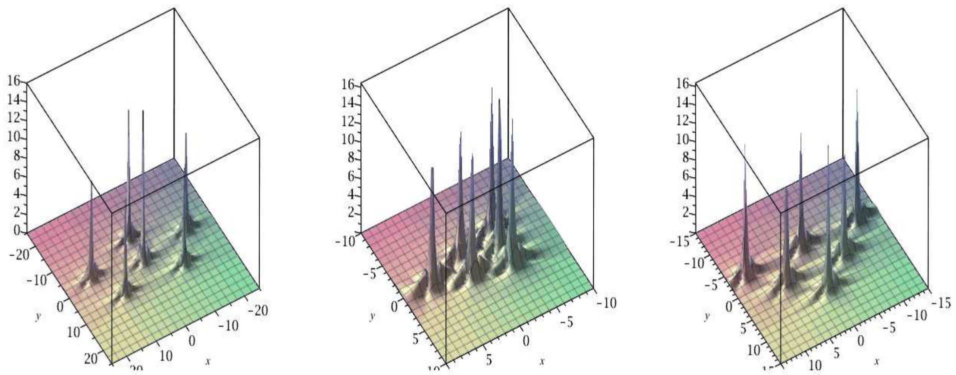

Figure 8. Solution of order 3 to the KPI equation, on the left for $t = 0$, $b_2 = 10^6$; in the center for $t = 0,01$, $a_1 = 10^3$ all the other parameters are equal to 0; on the right for $t = 0,1$, $b_1 = 10^3$ all the parameters are equal to 0.

7. Conclusion

We have given three representations of the solutions to the KPI equation: in terms of Fredholm determinants of order $2N$ depending on $2N-1$ real parameters in function of exponentials; in terms of wronskians of order $2N$ depending on $2N-1$ real parameters by means of trigonometric functions; in terms of real rational solutions as a quotient of two polynomials $n(x,y,t)$ and $d(x,y,t)$ of degrees $2N(N+1)-2$ in x, y, t and $2N(N+1)$ in x, y, t respectively and depending on $2N-2$ real parameters a_j and b_j, $1 \leq j \leq N-1$.
The maximum of the modulus of those solutions is equal to $2(2N+1)^2-2$. That gives a new approach to find explicit solutions for higher orders and try to describe the structure of those rational solutions.
In the (x,y) plane of coordinates, different structures appear.
All the solutions described in this study are different from those constructed in previous works [26–31] .
It will be relevant to go on this study for higher orders to try to understand the structure of those rational solutions.

Acknowledgements

The author would like to thank the referees for their suggestions and comments which improve the original version of the paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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VOLUME IV ISSUE IV
ISSN 2619-9653

December 2021
http://dergipark.gov.tr/ujma

UNIVERSAL JOURNAL OF MATHEMATICS AND APPLICATIONS

Contents

Universal Journal of Mathematics and Applications, 4 (4) (2021) 125-131
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1008148

Characterizations of Framed Curves in Four-Dimensional Euclidean Space

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Article Info

Keywords: *Bishop-type framed curves, Euler angles, Framed curves in* $\mathbb{R}^4$
2010 AMS: *53A04, 58K05*
Received: *11 October 2021*
Accepted: *29 December 2021*
Available online: *30 December 2021*

Abstract

Framed curves in Euclidean space are used to investigate singular curves and are important for singularity theory. In this study, we investigate framed curves in four-dimensional Euclidean space. We introduce Bishop-type frame of framed curves is with the help of Euler angles. Also, we give framed rectifying curves, framed osculating curves and framed normal curves with the help of Bishop-type framed curves in four-dimensional Euclidean space. Also, we obtain some characterizations depending on framed curvatures.

1. Introduction

Framed curves in n-dimensional Euclidean space were first introduced by Honda and Takahashi [1]. Framed curves in Euclidean space are used to investigate singular curves and are important for singularity theory. A framed curve in the Euclidean space is a curve with a moving frame. It is a generalization not only of regular curves with linear independent condition but also of Legendre curves in the unit tangent bundle. There are many studies in the literature for framed curves in three-dimensional Euclidean space. In three-dimensional Euclidean space, there are studies such as existence and uniqueness conditions of framed curves [2], evolutes of framed immersions [3], framed rectifying curves [4, 5], framed normal curves [6], Bertrand and Mannheim curves of framed curves [7].
Frenet and Bishop frames are important in classical differential geometry. Frenet frames cannot be built as the curvatures vanishes at some points and the Bishop frame is used [8]. In [9], this situation is extended to the four-dimensional Euclidean space and a parallel frame is formed. In four-dimensional Euclidean space, this frame is called the parallel transport frame and it can be [10]- [13] etc. studies are available.
In this paper, we introduce framed curves in four-dimensional Euclidean space. Also, we give some new results for the relation of framed curves with Frenet curves in four-dimensional Euclidean space. Moreover, we introduce Bishop-type frame of framed curves with the help of Euler angles. Also, by using Bishop-type framed curves in four-dimensional Euclidean space, we introduce framed rectifying curves, framed osculating curves and framed normal curves.

2. Framed curves in n-Euclidean space

A framed curve in the n-dimensional Euclidean space is a space curve with a moving frame which may have singular points, in detail see [1].
Let $\gamma : I \to \mathbb{R}^n$ be a curve with singular points. The set

$$\Delta_{n-1} = \{\mu = (\mu_1, \mu_2, ..., \mu_{n-1}) \in \mathbb{R}^n \times ... \times \mathbb{R}^n \mid \langle \mu_i, \mu_j \rangle = \delta_{ij}, \quad i, j = 1, 2..., n-1\}$$

is an $(n, n-1)-$type Stiefel manifold. Let $\mu = (\mu_1, \mu_2, ..., \mu_{n-1}) \in \Delta_{n-1}$. Take the unit vector defined by $\nu = \mu_1 \wedge ... \wedge \mu_{n-1}$.

Definition 2.1 (Framed curve). $(\gamma, \mu) : I \to \mathbb{R}^n \times \Delta_{n-1}$ *is called a framed curve if* $\langle \gamma'(s), \mu_i(s) \rangle = 0$ *for all* $s \in I$ *and* $i = 1, 2, ..., n-1$ *[1].*

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Let $(\gamma,\mu): I \to \mathbb{R}^n \times \Delta_{n-1}$ be a framed curve and $\nu(s) = \mu_1(s) \wedge ... \wedge \mu_{n-1}$. By definition $(\mu(s), \nu(s)) \in SO(n)$ for each $s \in I$ and $\{\mu(s), \nu(s)\}$ is called a moving frame along the framed base curve $\gamma(s)$. Then, Frenet-Serret type formula is given by

$$\begin{pmatrix} \mu'(s) \\ \nu'(s) \end{pmatrix} = A(s) \begin{pmatrix} \mu(s) \\ \nu(s) \end{pmatrix}$$

where $A(s) = (a_{i,j}) \in o(n)$ for $i, j = 1, 2, ..., n$ and $o(n)$ is the set of all skew-symmetric matrices. Moreover, there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that:

$$\gamma'(s) = \alpha(s)\nu(s)$$

In addition, s_0 is a singular point of the framed curve γ if and only if $\alpha(s_0) = 0$. $(A, \alpha) : I \to o(n) \times \mathbb{R}$ is called the curvature of the framed curve.

3. Framed curves in $\mathbb{R}^4 \times \Delta_3$

Let us take the vectors $x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4)$ and $z = (z_1, z_2, z_3, z_4)$ in four-dimensional Euclidean space $\mathbb{R}^4$ with Euclidean inner product. The ternary product (or vector product) is defined

$$x \wedge y \wedge z = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

where $\{e_1, e_2, e_3, e_4\}$ is the standard basis of $\mathbb{R}^4$ [14].
The set

$$\Delta_3 = \{\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 | \quad \langle \mu_i, \mu_j \rangle = \delta_{ij}, i, j = 1, 2, 3\}$$

is a six-dimensional smooth manifold. We define a unit vector $\nu : \mu_1 \wedge \mu_2 \wedge \mu_3$ of $\mathbb{R}^4$.

Definition 3.1. *$(\gamma, \mu_1, \mu_2, \mu_3) : I \to \mathbb{R}^4 \times \Delta_3$ is called a framed curve if $\langle \gamma'(s), \mu_i(s) \rangle = 0$ for all $s \in I$ and $i = 1, 2, 3$.*

The Frenet-Serret type formula is given by

$$\begin{pmatrix} \nu'(s) \\ \mu_1'(s) \\ \mu_2'(s) \\ \mu_3'(s) \end{pmatrix} = \begin{pmatrix} 0 & -l(s) & -m(s) & -n(s) \\ l(s) & 0 & p(s) & q(s) \\ m(s) & -p(s) & 0 & r(s) \\ n(s) & -q(s) & -r(s) & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \mu_1(s) \\ \mu_2(s) \\ \mu_3(s) \end{pmatrix} \tag{3.1}$$

where

$$\begin{array}{ll} l(s) = \langle \mu_1'(s), \nu(s) \rangle & m(s) = \langle \mu_2'(s), \nu(s) \rangle \\ n(s) = \langle \mu_3'(s), \nu(s) \rangle & p(s) = \langle \mu_1'(s), \mu_2(s) \rangle \\ q(s) = \langle \mu_1'(s), \mu_3(s) \rangle & r(s) = \langle \mu_2'(s), \mu_3(s) \rangle \end{array}$$

In addition, there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that $\gamma'(s) = \alpha(s)\nu(s)$. If s_0 is a singular point of γ, $\alpha(s_0) = 0$. If s_0 is a singular point of (γ, μ), $(l, m, n, p, q, r, \alpha)(s_0) = 0$. Note that $(\gamma, \mu_1, \mu_2, \mu_3)$ is a framed immersion if and only if $(l(s), m(s), n(s), p(s), q(s), r(s), \alpha(s)) \neq (0, 0, 0, 0)$ for all $s \in I$.

Example 3.2. *Regular curves at $\mathbb{R}^4$ with linear independent conditions are a natural example of framed curves (i.e. $\gamma'(s), \gamma''(s), \gamma'''(s)$ are linear independent for all $s \in I$). $\gamma : I \to \mathbb{R}^4$ regular curve with linear independent conditions. If we take $\mu_1(s) = N_1(s)$, $\mu_2(s) = N_2(s)$ and $\mu_3(s) = N_3(s)$, then $(\gamma, N_1, N_2, N_3) : I \to \mathbb{R}^4 \times \Delta_3$ is a framed curve. Also, $\nu(s) = \mu_1 \wedge \mu_2 \wedge \mu_3 = T(s)$. Therefore,*

$$\begin{array}{ll} T(s) = \dfrac{\gamma'(s)}{\|\gamma'(s)\|} & N_1(s) = \dfrac{\|\gamma'(s)\|^2\gamma''(s) - \langle \gamma'(s), \gamma''(s) \rangle \gamma'(s)}{\|\|\gamma'(s)\|^2\gamma''(s) - \langle \gamma'(s), \gamma''(s) \rangle \gamma'(s)\|} \\ N_2(s) = N_3(s) \times T(s) \times N_1(s) & N_3(s) = \dfrac{T(s) \times N_1(s) \times \gamma'''(s)}{\|T(s) \times N_1(s) \times \gamma'''(s)\|} \end{array}$$

3.1. Framed curves $\mathbb{R}^4 \times \Delta_3$ with Bishop frame

In this section, adapted frame for framed curves are obtained by using Euler angles and these frame is called Bishop-type frame of framed curves. By using Euler angles an arbitrary rotation matrix is given by

$$\begin{pmatrix} \cos\theta\cos\psi & -\cos\varphi\sin\psi + \sin\varphi\cos\psi\sin\theta & \sin\varphi\sin\psi + \cos\varphi\cos\psi\sin\theta \\ \cos\theta\sin\psi & \cos\varphi\cos\psi + \sin\varphi\sin\psi\sin\theta & -\sin\varphi\cos\psi + \cos\varphi\sin\psi\sin\theta \\ -\sin\theta & \sin\varphi\cos\theta & \cos\varphi\cos\theta \end{pmatrix}$$

where θ, ψ, φ are Euler angles [9]. We define $(\overline{\mu}_1(s), \overline{\mu}_2(s), \overline{\mu}_3(s)) \in \Delta_3$ by

$$\begin{aligned} \overline{\mu}_1(s) &= (\cos\theta(s)\cos\psi(s))\mu_1(s) + (-\cos\varphi(s)\sin\psi(s) + \sin\varphi(s)\cos\psi(s)\sin\theta(s))\mu_2(s) \\ &+ (\sin\varphi(s)\sin\psi(s) + \cos\varphi(s)\cos\psi(s)\sin\theta(s))\mu_3(s) \\ \overline{\mu}_2(s) &= (\cos\theta(s)\sin\psi(s))\mu_1(s) + (\cos\varphi(s)\cos\psi(s) + \sin\varphi(s)\sin\psi(s)\sin\theta(s))\mu_2(s) \\ &+ (-\sin\varphi(s)\cos\psi(s) + \cos\varphi(s)\sin\psi(s)\sin\theta(s))\mu_3(s) \\ \overline{\mu}_3(s) &= -\sin\theta(s)\mu_1(s) + \sin\varphi(s)\cos\theta(s)\mu_2(s) + \cos\varphi(s)\cos\theta(s)\mu_3(s) \end{aligned} \tag{3.2}$$

Therefore, $(\gamma, \overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3) : I \to \mathbb{R}^4 \times \Delta_3$ is a framed curve and

$$\overline{\nu}(s) = \overline{\mu}_1(s) \wedge \overline{\mu}_2(s) \wedge \overline{\mu}_3(s) = \nu(s)$$

By differentiating equations 3.1 and 3.2, we get

$$\begin{aligned} \overline{\mu}_1' &= (l\cos\theta\cos\psi - m\cos\varphi\sin\psi + m\sin\varphi\cos\psi\sin\theta + n\sin\varphi\sin\psi + n\cos\varphi\cos\psi\sin\theta)\nu \\ &+(-\theta'\sin\theta\cos\psi - \psi'\cos\theta\sin\psi + p\cos\varphi\sin\psi - p\sin\varphi\cos\psi\sin\theta \\ &-q\sin\varphi\sin\psi - q\cos\varphi\sin\theta\cos\psi)\mu_1 + (p\cos\theta\cos\psi + \varphi'\sin\varphi\sin\psi - \psi'\cos\varphi\cos\psi \\ &+\varphi'\sin\theta\cos\varphi\cos\psi + \theta'\cos\theta\sin\varphi\cos\psi - \psi'\sin\theta\sin\varphi\sin\psi - r\sin\varphi\sin\psi \\ &-r\cos\varphi\cos\psi\sin\theta)\mu_2 + (q\cos\theta\cos\psi - r\cos\varphi\sin\psi + r\sin\varphi\cos\psi\sin\theta \\ &+\varphi'\cos\varphi\sin\psi + \psi'\sin\varphi\cos\psi - \varphi'\sin\theta\sin\varphi\cos\psi + \theta'\cos\theta\cos\varphi\cos\psi \\ &-\psi'\sin\theta\cos\varphi\sin\psi)\mu_3, \end{aligned}$$

$$\begin{aligned} \overline{\mu}_2' &= (l\cos\theta\sin\psi + m\cos\varphi\cos\psi + m\sin\varphi\sin\psi\sin\theta - n\sin\varphi\cos\psi + n\cos\varphi\sin\psi\sin\theta)\nu \\ &+(-\theta'\sin\theta\sin\psi + \psi'\cos\theta\cos\psi - p\cos\varphi\cos\psi - p\sin\varphi\sin\psi\sin\theta \\ &+q\sin\varphi\cos\psi - q\cos\varphi\sin\theta\sin\psi)\mu_1 + (p\cos\theta\sin\psi - \psi' p\cos\varphi\sin\psi + \psi'\sin\varphi\cos\psi\sin\theta \\ &-\varphi'\cos\psi\sin\varphi + \varphi'\sin\theta\cos\varphi\sin\psi + \theta'\cos\theta\sin\varphi\sin\psi + r\sin\varphi\cos\psi \\ &-r\cos\varphi\sin\psi\sin\theta)\mu_2 + (q\cos\theta\sin\psi + r\cos\varphi\cos\psi + r\sin\varphi\sin\psi\sin\theta \\ &-\varphi'\cos\varphi\cos\psi + \psi'\sin\varphi\sin\psi - \varphi'\sin\theta\sin\varphi\sin\psi + \theta'\cos\theta\cos\varphi\sin\psi \\ &+\psi'\sin\theta\cos\varphi\cos\psi)\mu_3, \end{aligned}$$

$$\begin{aligned} \overline{\mu}_3' &= (-l\sin\theta + m\sin\varphi\cos\theta + n\cos\varphi\cos\theta)\nu + (-\theta'\cos\theta - p\sin\varphi\cos\theta - q\cos\varphi\cos\theta)\mu_1 \\ &+(-p\sin\theta + \varphi'\cos\theta\cos\varphi - \theta'\sin\theta\sin\varphi - r\cos\varphi\cos\theta)\mu_2 \\ &+(-q\sin\theta + r\sin\varphi\cos\theta - \varphi'\sin\varphi\cos\theta + \theta'\sin\theta\cos\varphi)\mu_3, \\ \overline{\nu}' &= \nu' = -l\mu_1 - m\mu_2 - n\mu_3. \end{aligned}$$

Corollary 3.3. *If we take Euler angles $\theta, \psi, \varphi : I \to \mathbb{R}$ which satisfies*

$$\begin{aligned} \theta'(s) &= -p(s)\sin\varphi(s) - q(s)\cos\varphi(s) \\ \varphi'(s) &= \tan\theta(s)(p(s)\cos\varphi(s) - q(s)\sin\varphi(s)) + r \\ \psi'(s) &= \sec\theta(s)(p(s)\cos\varphi(s) - q(s)\sin\varphi(s)) \end{aligned} \tag{3.3}$$

we call the frame $\{\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3, \nu\}$ a Bishop-type frame along framed base curve $\mathbb{R}^4$. Also, the formula is given by

$$\begin{pmatrix} \nu'(s) \\ \overline{\mu}_1'(s) \\ \overline{\mu}_2'(s) \\ \overline{\mu}_3'(s) \end{pmatrix} = \begin{pmatrix} 0 & -L(s) & -M(s) & -N(s) \\ L(s) & 0 & 0 & 0 \\ M(s) & 0 & 0 & 0 \\ N(s) & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \overline{\mu}_1(s) \\ \overline{\mu}_2(s) \\ \overline{\mu}_3(s) \end{pmatrix} \tag{3.4}$$

where

$$\begin{aligned} L(s) &= l(s)\cos\theta(s)\cos\psi(s) - m(s)\cos\varphi(s)\sin\psi(s) + m(s)\sin\varphi(s)\cos\psi(s)\sin\theta(s) \\ &+n(s)\sin\varphi(s)\sin\psi(s) + n(s)\cos\varphi(s)\cos\psi(s)\sin\theta(s) \\ M(s) &= l(s)\cos\theta(s)\sin\psi(s) + m(s)\cos\varphi(s)\cos\psi(s) + m(s)\sin\varphi(s)\sin\psi(s)\sin\theta(s) \\ &-n(s)\sin\varphi(s)\cos\psi(s) + n(s)\cos\varphi(s)\sin\psi(s)\sin\theta(s) \\ N(s) &= -l(s)\sin\theta(s) + m(s)\sin\varphi(s)\cos\theta(s) + n(s)\cos\varphi(s)\cos\theta(s) \end{aligned}$$

and with equation 3.3.

Corollary 3.4. *According to the equation 3.3, there is*

$$\varphi'(s) = \psi'(s)\sin\theta(s) + r(s)$$

relation between the Euler angles.

4. Special framed curves in $\mathbb{R}^4$ with Bishop frame

4.1. Framed rectifying curves

Definition 4.1. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3):I\to\mathbb{R}^4\times\Delta_3$ be a framed curve. γ is called framed rectifying curve in $\mathbb{R}^4$ if its position vector γ satisfies:*

$$\gamma(s)=\lambda_1(s)\nu(s)+\lambda_2(s)\overline{\mu}_2(s)+\lambda_3(s)\overline{\mu}_3(s) \tag{4.1}$$

where $\lambda_1(s),\lambda_2(s)$ and $\lambda_3(s)$ are differentiable functions.

Theorem 4.2. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3):I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with non-zero framed curvatures. Then, γ is a framed rectifying curve if and only if $\lambda_2M(s)+\lambda_3N(s)-\alpha(s)=0$, where λ_2 and λ_3 are real constants.*

Proof. Assume γ is a framed rectifying curve. By differentiating equation 4.1, we get

$$\begin{aligned}\alpha(s)\nu(s) \quad = \quad & (\lambda_1'(s)+\lambda_2(s)M(s)+\lambda_3(s)N(s))\nu(s)+(-L(s)\lambda_1(s))\overline{\mu}_1(s)\\ & +(-M(s)\lambda_1(s)+\lambda_2'(s))\overline{\mu}_2(s)+(-N(s)\lambda_1(s)+\lambda_3'(s))\overline{\mu}_3(s)\end{aligned} \tag{4.2}$$

Then according to equation 4.2, we have

$$\begin{aligned} L(s)\lambda_1(s) &= 0\\ M(s)\lambda_1(s)-\lambda_2'(s) &= 0\\ N(s)\lambda_1(s)-\lambda_3'(s) &= 0\\ \lambda_1'(s)+\lambda_2(s)M(s)+\lambda_3(s)N(s) &= \alpha(s)\end{aligned} \tag{4.3}$$

Since framed curvatures are non-zero, we get

$$\lambda_1(s)=0,\quad \lambda_2(s)=\lambda_2(const.),\lambda_3(s)=\lambda_3(const.) \tag{4.4}$$

Therefore, by using last equation of 4.3 and equation 4.4, we have

$$\lambda_2M(s)+\lambda_3N(s)-\alpha(s)=0 \tag{4.5}$$

where λ_2 and λ_3 are real constants. Conversely, assume that the curvatures (L,M,N,α) satisfies the equation 4.5. By considering, the vector $X\in\mathbb{R}^4$ given by

$$X(s)=\gamma(s)-\lambda_2\overline{\mu}_2(s)-\lambda_3\overline{\mu}_3(s) \tag{4.6}$$

By differentiating equation 4.6, we have

$$X'(s)=\alpha(s)\nu(s)-(\lambda_2M(s)+\lambda_3N(s))\nu(s)$$

By using equation 4.5, we get

$$X'(s)=0 \tag{4.7}$$

Based on the equation 4.7, we conclude that γ is congruent to a framed rectifying curve in $\mathbb{R}^4$. □

Corollary 4.3. *If γ is a framed rectifying curve with non-zero curvatures in $\mathbb{R}^4$, then the curvatures of γ have the following relationships*

$$\frac{\left(\frac{\alpha}{N}\right)'}{\left(\frac{M}{N}\right)'}=constant \quad or \quad \frac{\left(\frac{\alpha}{M}\right)'}{\left(\frac{N}{M}\right)'}=constant \tag{4.8}$$

Proof. By using equation 4.5 and the derivative of equation 4.5, we get 4.8. □

4.2. Framed first osculating curves

Definition 4.4. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3):I\to\mathbb{R}^4\times\Delta_3$ be a framed curve. γ is called framed first osculating curve in $\mathbb{R}^4$ if its position vector γ satisfies:*

$$\gamma(s)=\varepsilon_1(s)\nu(s)+\varepsilon_2(s)\overline{\mu}_1(s)+\varepsilon_3(s)\overline{\mu}_3(s) \tag{4.9}$$

for some functions $\varepsilon_1(s),\varepsilon_2(s),\varepsilon_3(s)$.

Theorem 4.5. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3):I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with non-zero framed curvatures. Then, γ is a framed first osculating curve if and only if $\varepsilon_2L(s)+\varepsilon_3N(s)-\alpha(s)=0$ where ε_2 and ε_3 are real constants.*

Proof. Suppose that γ is a framed first osculating curve. By differentiating equation 4.9, we get

$$\begin{aligned}\alpha(s)\nu(s) &= (\varepsilon_1'(s)+\varepsilon_2(s)L(s)+\varepsilon_3(s)N(s))\nu(s)+(-L(s)\varepsilon_1(s)+\varepsilon_2'(s))\overline{\mu}_1(s)\\ &\quad -\varepsilon_1(s)M(s)\overline{\mu}_2(s)+(-N(s)\varepsilon_1(s)+\varepsilon_3'(s))\overline{\mu}_3(s)\end{aligned}$$

Therefore, we have

$$\begin{aligned} L(s)\varepsilon_1(s)-\varepsilon_2'(s) &= 0\\ M(s)\varepsilon_1(s) &= 0\\ N(s)\varepsilon_1(s)+\varepsilon_3'(s) &= 0\\ \varepsilon_1'(s)+\varepsilon_2(s)L(s)+\varepsilon_3(s)N(s) &= \alpha(s)\end{aligned} \tag{4.10}$$

Since framed curvatures are non-zero, we get

$$\varepsilon_1(s)=0, \quad \varepsilon_2=\varepsilon_2(const.), \varepsilon_3=\varepsilon_3(const.) \tag{4.11}$$

Consequently, by using last equation of 4.10 and equation 4.11, we have

$$\varepsilon_2 L(s)+\varepsilon_3 N(s)-\alpha(s)=0 \tag{4.12}$$

where ε_2 and ε_3 are real constants. Conversely, assume that the curvatures (L,M,N,α) satisfies the equation 4.12. By considering, the vector $X\in\mathbb{R}^4$ given by

$$X(s)=\gamma(s)-\varepsilon_2\overline{\mu}_1(s)-\varepsilon_3\overline{\mu}_3(s) \tag{4.13}$$

By differentiating equation 4.13, we have

$$X'(s)=\alpha(s)\nu(s)-(\varepsilon_2 L(s)+\varepsilon_3 N(s))\nu(s) \tag{4.14}$$

By using equation 4.14, we get

$$X'(s)=0 \tag{4.15}$$

By according to equation 4.15, we conclude that γ is congruent to a framed first osculating curve in $\mathbb{R}^4$. □

Corollary 4.6. *If γ is a framed first osculating curve with non-zero curvatures in $\mathbb{R}^4$, then the curvatures of γ have the following relationships*

$$\frac{\left(\frac{\alpha}{N}\right)'}{\left(\frac{L}{N}\right)'}=constant \quad or \quad \frac{\left(\frac{\alpha}{L}\right)'}{\left(\frac{N}{L}\right)'}=constant \tag{4.16}$$

Proof. By using equation 4.12 and the derivative of equation 4.12, we get 4.16. □

4.3. Framed second osculating curves

Definition 4.7. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3):I\to\mathbb{R}^4\times\Delta_3$ be a framed curve. γ is called framed second osculating curve in $\mathbb{R}^4$ if its position vector γ satisfies:*

$$\gamma(s)=\eta_1(s)\nu(s)+\eta_2(s)\overline{\mu}_1(s)+\eta_3(s)\overline{\mu}_2(s)$$

for some functions $\eta_1(s),\eta_2(s),\eta_3(s)$.

Theorem 4.8. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3):I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with non-zero framed curvatures. Then, γ is a framed second osculating curve if and only if $\eta_2 L(s)+\eta_3 M(s)-\alpha(s)=0$ where η_2 and η_3 are real constants.*

Proof. Its proof is done in a similar way to Theorem 4.5. □

Corollary 4.9. *If γ is a framed second osculating curve with non-zero curvatures in $\mathbb{R}^4$, then the curvatures of γ have the following relationships*

$$\frac{\left(\frac{\alpha}{M}\right)'}{\left(\frac{L}{M}\right)'}=constant \quad or \quad \frac{\left(\frac{\alpha}{L}\right)'}{\left(\frac{M}{L}\right)'}=constant$$

4.4. Framed normal curves

Definition 4.10. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3): I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve. γ is called framed normal curve in $\mathbb{R}^4$ if its position vector γ satisfies:*

$$\gamma(s) = \delta_1(s)\overline{\mu}_1(s) + \delta_2(s)\overline{\mu}_2(s) + \delta_3(s)\overline{\mu}_3(s) \tag{4.17}$$

for some functions $\delta_1(s), \delta_2(s), \delta_3(s)$.

Theorem 4.11. *Let $(\gamma,\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3): I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve with non-zero framed curvatures. Then, γ is a framed normal curve if and only if $\delta_1 L(s)) + \delta_2 M(s) + \delta_3 N(s) - \alpha(s) = 0$ where δ_1, δ_2, δ_3 are real constants.*

Proof. Suppose that γ is a framed normal curve. By differentiating equation 4.17, we get

$$\begin{aligned} \alpha(s)\nu(s) \quad = \quad & (\delta_1(s)L(s) + \delta_2(s)M(s) + \delta_3(s)N(s))\nu(s) + \delta_1^{'}(s)\overline{\mu}_1(s) \\ & +\delta_2^{'}\overline{\mu}_2(s) + \delta_3^{'}(s)\overline{\mu}_3(s) \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \delta_1^{'}(s) = 0 \\ & \delta_2^{'}(s) = 0 \\ & \delta_3^{'}(s) = 0 \\ & \delta_1(s)L(s) + \delta_2(s)M(s) + \delta_3(s)N(s) = \alpha(s) \end{aligned} \tag{4.18}$$

Then, we have

$$\delta_1(s) = \delta_1(const.), \delta_2 = \delta_2(const.), \delta_3 = \delta_3(const.) \tag{4.19}$$

Consequently, by using last equation of 4.18 and equation 4.19, we have

$$\delta_1 L(s) + \delta_2 M(s) + \delta_3 N(s) - \alpha(s) = 0 \tag{4.20}$$

where δ_1, δ_2 and δ_3 are real constants. Conversely, assume that the curvatures (L,M,N,α) satisfies the equation 4.20. By considering, the vector $X \in \mathbb{R}^4$ given by

$$X(s) = \gamma(s) - \delta_1\overline{\mu}_1(s) - \delta_2\overline{\mu}_2(s) - \delta_3\overline{\mu}_3(s) \tag{4.21}$$

By differentiating equation 4.21, we have

$$X^{'}(s) = \alpha(s)\nu(s) - (\delta_1 L(s) + \delta_2 M(s) + \delta_3 N(s))\nu(s)$$

Then, we get

$$X^{'}(s) = 0 \tag{4.22}$$

By according to equation 4.22, we conclude that γ is congruent to a framed normal curve in $\mathbb{R}^4$. □

5. Conclusion

In this study, we defined framed curves in four-dimensional Euclidean space. In addition, we gave Bishop-type frame of framed curves in four-dimensional Euclidean space. Actually, the Frenet-type frame of framed curves can give in four-dimensional Euclidean space. In addition, we investigate framed rectifying, normal and osculating curves. Thus, since the four-dimensional frame of framed curve, all framed curve studies can be extended to this space.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Universal Journal of Mathematics and Applications, 4 (4) (2021) 132-135
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.973671

The Bounds for the First General Zagreb Index of a Graph

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Article Info

Keywords: *The first general Zagreb index, the chromatic number, the clique number, the independent number.*
2010 AMS: *05C09*
Received: *21 July 2021*
Accepted: *22 November 2021*
Available online: *30 December 2021*

Abstract

The first general Zagreb index of a graph G is defined as the sum of the αth powers of the vertex degrees of G, where α is a real number such that $\alpha \neq 0$ and $\alpha \neq 1$. In this note, for $\alpha > 1$, we present upper bounds involving chromatic and clique numbers for the first general Zagreb index of a graph; for an integer $\alpha \geq 2$, we present a lower bound involving the independence number for the first general Zagreb index of a graph.

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. Let $G = (V(G), E(G))$ be a graph with n vertices and e edges, where $V = \{v_1, v_2, ..., v_n\}$. We assume that the vertices in G are arranged such that $\Delta(G) = d_G(v_1) \geq d_G(v_2) \geq \cdots \geq d_G(v_n) = \delta(G)$, where $d_G(v_i)$, for each i with $1 \leq i \leq n$, is the degree of vertex v_i in G. The chromatic number, denoted $\chi(G)$, of a graph G is the smallest number of colors which can be assigned to $V(G)$ so that the adjacent vertices in G are colored differently. A clique of a graph G is a complete subgraph of G. A clique of largest possible size is called a maximum clique. The clique number, denoted $\omega(G)$, of a graph G is the number of vertices in a maximum clique of G. A set of vertices in a graph G is independent if the vertices in the set are pairwise nonadjacent. A maximum independent set in a graph G is an independent set of largest possible size. The independence number, denoted $\beta(G)$, of a graph G is the cardinality of a maximum independent set in G. If H is any graph of order n with degree sequence $d_H(v_1) \geq d_H(v_2) \geq \cdots \geq d_H(v_n)$, and if H^* is any graph of order n with degree sequence $d^*_H(v_1) \geq d^*_H(v_2) \geq \cdots \geq d^*_H(v_n)$, such that $d_H(v_i) \leq d^*_H(v_i)$ (for each i with $1 \leq i \leq n$), then H^* is said to dominate H. We use $C(n,r)$ to denote the number of r-element subsets of a set of size n, where n and r are nonnegative integers such that $0 \leq r \leq n$.

The first Zagreb index was introduced by Gutman and Trinajstić in [8]. For a graph G, its first Zagreb index is defined as $\sum_{i=1}^{n} d_G^2(v_i)$. Li and Zheng in [9] further extended the first Zagreb index of a graph and introduced the concept of the first general Zagreb index of a graph. The first general Zagreb index, denoted $M_\alpha(G)$, of a graph G is defined as $\sum_{i=1}^{n} d_G^{\alpha}(v_i)$, where α is a real number such that $\alpha \neq 0$ and $\alpha \neq 1$.

In this note, we will present upper bounds involving chromatic and cliques numbers for the first general Zagreb index of a graph when $\alpha > 1$ and a lower bound involving the independent number for the first general Zagreb index of a graph when α is an integer at least 2. The main results of this note are as follows.

Theorem 1.1. *Let G be a graph of order n. Assume α is a real number such that $\alpha > 1$. Then*

$$(1)\ M_\alpha \leq n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\chi}\right).$$

Equality holds if and only if G is K_n.

$$(2)\ M_\alpha \leq n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\omega}\right).$$

Equality holds if and only if G is K_n.

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Theorem 1.2. *Let G be a graph of order n. Assume α is an integer which is at least 2. Then*

$$M_\alpha \geq \frac{n^{\alpha+1}}{\beta^\alpha} + n(\Delta^\alpha - (1+\Delta)^\alpha).$$

Equality holds if and only if G is a disjoint union of β complete graphs of order $\Delta+1$.

2. Lemmas

In order to prove Theorem 1 and Theorem 2, we need the following results as our lemmas. The first one is a theorem proved by Erdős in [6]. Its proofs in English can be found in [1].

Lemma 2.1. *If H is any graph of order n, then there exists a graph H^* of order n, where $\chi(H^*) \leq \omega(H)$, such that H^* dominates H.*

The second one can be found in [4] and [10].

Lemma 2.2. *If G is a graph, then*

$$\beta \geq \sum_{v \in V} \frac{1}{d(v)+1}.$$

Equality holds if and only if each component of G is complete.

3. Proofs

Next, we will prove Theorem 1.1. The ideas from the proofs of Theorem 3 on Page 53 in [5] will be used in the proofs of Theorem 1.1 below.

Proof of (1) in Theorem 1.1 Let us partition the vertex set V of G into the pairwise disjoint nonempty subsets of V_1, V_2, ..., V_χ such that V_i is independent for each i with $1 \leq i \leq \chi$. Set $|V_i| := n_i$ for each i with $1 \leq i \leq \chi$. Then we have that $n = \sum_{i=1}^{\chi} n_i$ and $d(x) \leq n - n_i$ for each vertex x in V_i and each i with $1 \leq i \leq \chi$. Without loss of generality, we assume that $n_1 \leq n_2 \leq \cdots \leq n_\chi$. From Cauchy-Schwarz inequality, we have that

$$\sum_{i=1}^{\chi} n_i^2 \geq \frac{\left(\sum_{i=1}^{\chi} n_i\right)^2}{\chi} = \frac{n^2}{\chi}.$$

Now

$$\begin{aligned}
M_\alpha &= \sum_{v \in V} d^\alpha(v), \\
&= \sum_{i=1}^{\chi} \sum_{v \in V_i} d^\alpha(v), \\
&\leq \sum_{i=1}^{\chi} n_i (n-n_i)^\alpha, \\
&= \sum_{i=1}^{\chi} n_i (n-n_i)(n-n_i)^{\alpha-1} \leq \sum_{i=1}^{\chi} n_i (n-n_i)(n-n_1)^{\alpha-1}, \\
&= (n-n_1)^{\alpha-1} \sum_{i=1}^{\chi} n_i(n-n_i) \leq (n-1)^{\alpha-1}(n^2 - \sum_{i=1}^{\chi} n_i^2) \leq (n-1)^{\alpha-1}\left(n^2 - \frac{n^2}{\chi}\right) = n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\chi}\right).
\end{aligned}$$

If

$$M_\alpha = n^2(n-1)^{\alpha-1}\left(1-\frac{1}{\chi}\right),$$

we, from the above proofs, have that $n_1 = n_2 = \cdots = n_\chi = 1$ and $d(v) = n-1$ for each vertex v in V. Thus G is K_n. If G is K_n, it is easy to verify that

$$M_\alpha = n^2(n-1)^{\alpha-1}\left(1-\frac{1}{\chi}\right).$$

This completes the proof of (1) in Theorem 1.1.

Proof of (2) in Theorem 1.1 From Lemma 2.1, we can find a graph G^* dominating G and $\chi(G^*) \leq \omega(G)$. From (1) of this theorem, we have that

$$M_\alpha(G) \leq M_\alpha(G^*) \leq n^2(n-1)^{\alpha-1}\left(1-\frac{1}{\chi(G^*)}\right) \leq n^2(n-1)^{\alpha-1}\left(1-\frac{1}{\omega(G)}\right).$$

If

$$M_\alpha(G) = n^2(n-1)^{\alpha-1}\left(1-\frac{1}{\omega}\right),$$

then

$$M_\alpha(G^*) = n^2(n-1)^{\alpha-1}\left(1-\frac{1}{\chi(G^*)}\right).$$

From (1) of this theorem, we have that G^* is K_n and $\chi(G^*) = n$. Thus $\omega(G) \geq \chi(G^*) = n$. Hence G is K_n. If G is K_n, it is again easy to verify that

$$M_\alpha(G) = n^2(n-1)^{\alpha-1}\left(1-\frac{1}{\omega}\right).$$

This completes the proof of (2) in Theorem 1.1.

Next, we will prove Theorem 1.2 which is motivated by Theorem 3.1 on Page 309 in [7].

Proof of Theorem 1.2 From Lemma 2.2 and the inequalities on the power means, arithmetic means, and harmonic means of n positive integers, we have that

$$\left(\frac{(1+d_1)^\alpha+(1+d_2)^\alpha+\cdots+(1+d_n)^\alpha}{n}\right)^{\frac{1}{\alpha}} \geq \frac{(1+d_1)+(1+d_2)+\cdots+(1+d_n)}{n} \geq \frac{n}{\frac{1}{1+d_1}+\frac{1}{1+d_2}+\cdots+\frac{1}{1+d_n}} \geq \frac{n}{\beta}.$$

Then

$$(1+d_1)^\alpha+(1+d_2)^\alpha+\cdots+(1+d_n)^\alpha \geq \frac{n^{\alpha+1}}{\beta^\alpha}.$$

It is easy to check that for each i with $1 \leq i \leq n$ we have

$$(1+d_i)^\alpha = \sum_{k=0}^{\alpha} C(\alpha,k)d_i^k \leq \sum_{k=0}^{\alpha} C(\alpha,k)\Delta^k - \Delta^\alpha + d_i^\alpha = (1+\Delta)^\alpha - \Delta^\alpha + d_i^\alpha.$$

Equality holds if and only if $d_i = \Delta$. Thus

$$(1+\Delta)^\alpha - \Delta^\alpha + d_1^\alpha + (1+\Delta)^\alpha - \Delta^\alpha + d_2^\alpha + \cdots + (1+\Delta)^\alpha - \Delta^\alpha + d_n^\alpha \geq (1+d_1)^\alpha + (1+d_2)^\alpha + \cdots + (1+d_n)^\alpha \geq \frac{n^{\alpha+1}}{\beta^\alpha}.$$

Therefore

$$M_\alpha \geq \frac{n^{\alpha+1}}{\beta^\alpha} + n(\Delta^\alpha - (1+\Delta)^\alpha).$$

If

$$M_\alpha = \frac{n^{\alpha+1}}{\beta^\alpha} + n(\Delta^\alpha - (1+\Delta)^\alpha),$$

then $d_1 = d_2 = \cdots = d_n = \Delta$. From Lemma 2, we have that G is a union of β complete graphs of order $\Delta+1$. If G is a union of β complete graphs of order $\Delta+1$, then $(\Delta+1)\beta = n$. It is easy to verify that

$$M_\alpha = \frac{n^{\alpha+1}}{\beta^\alpha} + n(\Delta^\alpha - (1+\Delta)^\alpha).$$

This completes the proof of Theorem 1.2.

Remark 3.1. *Let G be a graph with n vertices and e edges. Notice that*

$$n + 4e + M_2 = \sum_{i=1}^{n}(1+d_i)^2 \geq \frac{n^3}{\beta^2}.$$

We have that

$$M_2 \geq \frac{n^3}{\beta^2} - n - 4e.$$

It can be verified that $M_2 = \frac{n^3}{\beta^2} - n - 4e$ if and only if G is a disjoint union of β complete graphs of order $\Delta+1$.

Remark 3.2. *Let G be a graph with n vertices and e edges. Notice that*

$$n + 6e + 3M_2 + M_3 = \sum_{i=1}^{n}(1+d_i)^3 \geq \frac{n^4}{\beta^3}.$$

We have that

$$M_3 \geq \frac{n^4}{\beta^3} - n - 6e - 3U,$$

where U is an upper bound for M_2. A variety of concrete expressions for U can be found in [3].

Acknowledgements

The author would like to thank the referees for their suggestions and comments which improve the original version of the paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Universal Journal of Mathematics and Applications, 4 (4) (2021) 136-146
Research paper

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.984462

Singular Minimal Surfaces which are Minimal

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Article Info

Keywords: *Minimal surface, Semi-symmetric connection, Singular minimal surface, Translation surface*
2010 AMS: *53A10, 53C42, 53C05*
Received: *18 August 2021*
Accepted: *23 December 2021*
Available online: *30 December 2021*

Abstract

In the present paper, we discuss the singular minimal surfaces in Euclidean $3-$space $\mathbb{R}^3$ which are minimal. Such a surface is nothing but a plane, a trivial outcome. However, a non-trivial outcome is obtained when we modify the usual condition of singular minimality by using a special semi-symmetric metric connection instead of the Levi-Civita connection on $\mathbb{R}^3$. With this new connection, we prove that, besides planes, the singular minimal surfaces which are minimal are the generalized cylinders, providing their explicit equations. A trivial outcome is observed when we use a special semi-symmetric non-metric connection. Furthermore, our discussion is adapted to the Lorentz-Minkowski 3-space.

1. Introduction

Let $\left(\mathbb{R}^3, \langle\cdot,\cdot\rangle\right)$ be a Euclidean $3-$space and $\mathbf{v}$ a fixed unit vector in $\mathbb{R}^3$. Let $\mathbf{r}: M^2 \to \mathbb{R}^3_+(\mathbf{v})$ be a smooth immersion of an oriented compact surface M^2 into the halfspace

$$\mathbb{R}^3_+(\mathbf{v}) : \left\{ p \in \mathbb{R}^3 : \langle p, \mathbf{v}\rangle > 0 \right\}.$$

Denote H and $\mathbf{n}$ the mean curvature and unit normal vector field on M^2. Let $\alpha \in \mathbb{R}$. The potential $\alpha-$energy of $\mathbf{r}$ in the direction of $\mathbf{v}$ is defined by [32]

$$E(\mathbf{r}) = \int_{M^2} \langle p, \mathbf{v}\rangle^{\alpha} dM^2,$$

where dM^2 is the measure on M^2 with respect to the induced metric tensor from the Euclidean metric $\langle\cdot,\cdot\rangle$ and $p = \mathbf{r}(\tilde{p}), \tilde{p} \in M^2$.
Let $\Sigma : M^2 \times (-\theta, \theta) \to \mathbb{R}^3_+(\mathbf{v})$ be a compactly supported variation of $\mathbf{r}$ with variaton vector field ξ. The first variation of E becomes

$$E'(0) = -\int_{M^2} \left(2H\langle \mathbf{r}, \mathbf{v}\rangle - \alpha\langle \mathbf{n}, \mathbf{v}\rangle\right)\langle \xi, \mathbf{n}\rangle^{\alpha-1} dM^2.$$

For all compactly supported variations, the immersion $\mathbf{r}$ is a critical point of E if and only if

$$2H(\tilde{p}) = \alpha \frac{\langle \mathbf{n}(\tilde{p}), \mathbf{v}\rangle}{\langle \mathbf{r}(\tilde{p}), \mathbf{v}\rangle}, \tag{1.1}$$

for some point $\tilde{p} \in M^2$.
A surface M^2 is referred to as a *singular minimal surface* or *$\alpha-$minimal surface* with respect to the vector $\mathbf{v}$, if holds Eq. (1.1) (see [11, 12]). In the particular case $\alpha = 1$ and $\mathbf{v} = (0,0,1)$, the surface M^2 represents *two-dimensional analogue of the catenary* which is known as a model for the surfaces with the lowest gravity center, in other words, one has minimal potential energy under gravitational forces [6, 13, 18].

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A *translation surface* M^2 in $\mathbb{R}^3$ is a surface that can be written as the sum of two so-called *translating curves* [9]. When the translating curves lie in orthogonal planes, up to a change of coordinates, the surface M^2 can be locally given in the explicit form $z = p(x) + q(y)$, where (x, y, z) is the rectangular coordinates and p, q smooth functions. In such case, if M^2 is minimal (H vanishes identically [27, p. 17]), it describes a plane or the *Scherk surface* [43]

$$z(x,y) = \frac{1}{\lambda}\log\left|\frac{\cos\lambda x}{\cos\lambda y}\right|, \ \lambda \in \mathbb{R}, \ \lambda \neq 0.$$

If the translating curves lie in non-orthogonal planes, the translation surface M^2 is locally given by $z = p(x) + q(y + \mu x)$, $\mu \in \mathbb{R}$, $\mu \neq 0$, and so-called an *affine translation surface* or a *translation graph* [26, 45]. A minimal affine translation surface is so-called *affine Scherk surface* and is given in the explicit form

$$z(x,y) = \frac{1}{\lambda}\log\left|\frac{\cos\lambda\sqrt{1+\mu^2}x}{\cos\lambda(y+\mu x)}\right|.$$

López [32] obtained the singular minimal translation surfaces in $\mathbb{R}^3$ of type $z = p(x) + q(y)$ with respect to horizontal and vertical directions. This result was generalized to higher dimensions in [5]. For further study of singular minimal surfaces, we refer to the López's series of interesting papers on the solutions of the Dirichlet problem for the $\alpha-$singular minimal surface equation [33], the Lorentz-Minkowski counterpart of the condition of singular minimality [34], the compact singular minimal surfaces [35] and the singular minimal surfaces with density [36].

In this paper, we approach a singular minimal surface M^2 in $\mathbb{R}^3$ which is minimal. We hereinafter assume that $\alpha \neq 0$ in Eq. (1.1), otherwise any minimal surface obeys our approach, which is trivial. Under this circumstance, Eq. (1.1) gives $\langle \mathbf{n}(\tilde{p}), \mathbf{v}\rangle = 0$, that is, the tangent plane of M^2 at any point $\tilde{p}$ is parallel to $\mathbf{v}$. In such case, the surface M^2 belongs to the class of so-called *constant angle surfaces* and has to be a plane parallel to $\mathbf{v}$ (see [37, Proposition 9]), yielding the following outcome.

Proposition 1.1. *Let M^2 be a singular minimal surface in $\mathbb{R}^3$ with respect to an arbitrary vector $\mathbf{v}$. If M^2 is minimal, then it is a plane parallel to $\mathbf{v}$.*

This result is changed when we modify Eq. (1.1) by using a special semi-symmetric metric connection ∇ (see Eq. (3.1)) on $\mathbb{R}^3$. In Section 3, we prove that, besides planes, the singular minimal surfaces which are minimal with respect to ∇ are the generalized cylinders, providing their explicit equations. It is also observed, in Section 3, that this approach produces only trivial example when a special semi-symmetric non-metric connection D (see Eq. (3.19)) is used.

We find the motivation in Wang's paper [44] whose minimal translation surfaces were obtained with respect to the connections ∇ and D. The notion of a semi-symmetric metric (resp. non-metric) connection on a Riemannian manifold were defined by Hayden [22] (resp. Agashe [1]) and since then has been studied by many authors. Without giving a complete list, we may refer to [2–4, 7, 10, 14, 15, 19, 25, 38–42, 47–50]. The present authors also obtained singular minimal translation surfaces in $\mathbb{R}^3$ with respect to the connections ∇ and D [16].

Let $\mathbb{R}^3_1$ be a Lorentz-Minkowski 3$-$space endowed with the canonical Lorentzian metric $\langle\cdot,\cdot\rangle_L = dx^2 + dy^2 - dz^2$. Then we have [34, Definition 1.1]

Definition 1.1. *Let $\mathbf{r}$ be a smooth immersion of a spacelike surface M^2 in the halfspace $z > 0$ of $\mathbb{R}^3_1$ and $\mathbf{n}$ unit normal vector field on M^2 and H the mean curvature. M^2 is called $\alpha-$singular maximal surface if satisfies*

$$H = -\alpha\frac{\langle \mathbf{n}, (0,0,1)\rangle_L}{z}, \ \alpha \neq 0. \tag{1.2}$$

Due to the fact that the $z-$coordinate represents the time coordinate, the concept of gravity has no meaning. Therefore, unlike the Riemannian case, Eq. (1.2) describes only spacelike surfaces with prescribed angle between $\mathbf{n}$ and the $z-$axis. Point out that H is non-vanishing in Eq. (1.2) if $\alpha \neq 0$ because $\langle \mathbf{n}, (0,0,1)\rangle_L \neq 0$ for timelike vectors $\mathbf{n}$ and $(0,0,1)$ and so we can not adapt Eq. (1.2) to our study as is. For this reason, we modify the concept as follows:

Definition 1.2. *Let $\mathbf{r}$ be a smooth immersion of an oriented timelike surface M^2 in $\mathbb{R}^3_1$ and $\mathbf{n}$ unit normal vector field on M^2 and H the mean curvature. Let $\mathbf{v} \in \mathbb{R}^3_1$, $\mathbf{v} \neq \mathbf{0}$, a spacelike vector non-parallel to $\mathbf{n}$ such that $\mathbf{n}$ and $\mathbf{v}$ span a spacelike 2-space. Then M^2 is called singular minimal surface with respect to $\mathbf{v}$ if satisfies*

$$2H = \alpha\frac{\langle \mathbf{n}, \mathbf{v}\rangle_L}{\langle \mathbf{r}, \mathbf{v}\rangle_L}, \ \alpha \in \mathbb{R}, \ \alpha \neq 0. \tag{1.3}$$

With Definition 1.2, we may view the singular minimal surface M^2 as a timelike surface in $\mathbb{R}^3_1$ with prescribed Lorentz spacelike angle between $\mathbf{n}$ and $\mathbf{v}$. If M^2 is minimal, it follows from Eq. (1.3) that $\langle \mathbf{n}, \mathbf{v}\rangle_L = 0$, namely the angle is $\frac{\pi}{2}$, and, as in Riemannian case, M^2 becomes a timelike constant angle surface which has to be a plane (see [21, Theorem 3.1]), yielding the following trivial outcome.

Proposition 1.2. *Let M^2 be a singular minimal surface in $\mathbb{R}^3_1$ with respect to a spacelike vector $\mathbf{v}$. If M^2 is minimal, then it is a plane parallel to $\mathbf{v}$.*

In Section 4, we also state non-trivial results in $\mathbb{R}^3_1$ for singular minimal surfaces which are minimal with respect to the connections ∇ and D given by Eqs. (4.1) and (4.19), respectively.

2. Preliminaries

Most of following notions can be found [8,40,46].
Let $(\bar{M},\bar{g})$ be a semi-Riemannian manifold and $\bar{\nabla}$ a linear connection on $\bar{M}$. The *torsion tensor field* T of $\bar{\nabla}$ is defined by

$$T(\bar{\mathbf{x}},\bar{\mathbf{y}})=\bar{\nabla}_{\bar{\mathbf{x}}}\bar{\mathbf{y}}-\bar{\nabla}_{\bar{\mathbf{x}}}\bar{\mathbf{y}}-[\bar{\mathbf{x}},\bar{\mathbf{y}}],$$

where $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are vector fields on $\bar{M}$. A linear connection is called a *semi-symmetric* (resp. *non-*) *metric connection* if there exist a $1-$form π such that

$$T(\bar{\mathbf{x}},\bar{\mathbf{y}})=\pi(\bar{\mathbf{y}})\bar{\mathbf{x}}-\pi(\bar{\mathbf{x}})\bar{\mathbf{y}},\ \bar{\nabla}\bar{g}=0\ (\text{resp. } \bar{\nabla}\bar{g}\neq 0).$$

The linear connection $\bar{\nabla}$ is called *Levi-Civita connection* if $T=0$ and $\bar{\nabla}\bar{g}=0$. We denote the Levi-Civita connection by $\bar{\nabla}^L$.
Let M be a semi-Riemannian submanifold of $\bar{M}$ and ∇^L and g the induced Levi-Civita connection and metric tensor, respectively. Then the *Gauss formula* follows

$$\bar{\nabla}^L_{\mathbf{x}}\mathbf{y}=\nabla^L_{\mathbf{x}}\mathbf{y}+h(\mathbf{x},\mathbf{y}),$$

where h is so-called *second fundamental form* of M and $\mathbf{x}$ and $\mathbf{y}$ tangent vector fields to M. Let $\{\mathbf{f}_1,...,\mathbf{f}_n\}$ be an orthonormal frame on M at any point $p\in M$. Then the *mean curvature vector* $\mathbf{H}(p)$ at p is defined by

$$\mathbf{H}(p)=\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i h(\mathbf{f}_i,\mathbf{f}_i),$$

where $\varepsilon_i=g(\mathbf{f}_i,\mathbf{f}_i)$ and $n=\dim M$. The length of mean curvature vector is called *mean curvature*. A semi-Riemannian submanifold is called *minimal* if its mean curvature vanishes identically.
Let $\bar{M}=\mathbb{R}^3_1$ be the Lorentz-Minkowski $3-$space and $\bar{g}=\langle\cdot,\cdot\rangle_L=dx^2+dy^2-dz^2$. A vector field $\mathbf{x}$ on $\mathbb{R}^3_1$ is said to be *spacelike* (resp. *timelike*) if $\mathbf{x}=0$ or $\langle\mathbf{x},\mathbf{x}\rangle_L>0$ (resp. $\langle\mathbf{x},\mathbf{x}\rangle_L<0$). A vector field $\mathbf{x}$ is said to be *null* if $\langle\mathbf{x},\mathbf{x}\rangle_L=0$ and $\mathbf{x}\neq 0$. A timelike vector $\mathbf{x}=(a,b,c)$ is said to be *future pointing* (resp. *past pointing*) if $c>0$ (resp. $c<0$). A *Lorentz timelike angle* θ between two future (past) pointing timelike vectors $\mathbf{x}$ and $\mathbf{y}$ is associated with [17]

$$|\langle\mathbf{x},\mathbf{y}\rangle_L|=\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle_L|}\sqrt{|\langle\mathbf{y},\mathbf{y}\rangle_L|}\cosh\theta.$$

A *Lorentz spacelike angle* θ between two spacelike vectors $\mathbf{x}$ and $\mathbf{y}$ spanning a spacelike vector subspace ($\mathbb{R}^3_1$ induces a Riemannian metric on it) is associated with [17]

$$|\langle\mathbf{x},\mathbf{y}\rangle_L|=\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle_L|}\sqrt{|\langle\mathbf{y},\mathbf{y}\rangle_L|}\cos\theta.$$

Let M^2 be an immersed surface into $\mathbb{R}^3_1$. The surface M^2 is said to be *spacelike* (resp. *timelike*) if all tangent planes of M^2 are spacelike (resp. timelike). For such a spacelike (resp. timelike) surface, we have the decomposition $\mathbb{R}^3_1=T_pM^2\oplus\left(T_pM^2\right)^{\perp}$, where T_pM^2 is the tangent plane of M^2 at the point p. Notice that $\left(T_pM^2\right)^{\perp}$ is a timelike (resp. spacelike) $1-$space of $\mathbb{R}^3_1$. A *Gauss map* $\mathbf{n}$ of M^2 is a smooth map $\mathbf{n}:M^2\to\mathbb{R}^3_1$, $|\langle\mathbf{n},\mathbf{n}\rangle_L|=1$.
We finish this section remarking that a spacelike (resp. timelike) surface in $\mathbb{R}^3_1$ is locally a graph of a smooth function $u(x,y)$ (resp. $u(x,z)$ or $u(y,z)$) [28, Proposition 3.3].

3. Singular minimal surfaces in $\mathbb{R}^3$

3.1. $\nabla-$Singular minimal surfaces

Let ∇^L be the Levi-Civita connection on $\mathbb{R}^3$ and $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ the standard basis on $\mathbb{R}^3$ and $\mathbf{x},\mathbf{y}$ tangent vector fields to $\mathbb{R}^3$. Consider the following semi-symmetric metric connection on $\mathbb{R}^3$ [44]

$$\nabla_{\mathbf{x}}\mathbf{y}=\nabla^L_{\mathbf{x}}\mathbf{y}+\langle\mathbf{y},\mathbf{e}_3\rangle\mathbf{x}-\langle\mathbf{x},\mathbf{y}\rangle\mathbf{e}_3. \tag{3.1}$$

The nonzero derivatives are

$$\nabla_{\mathbf{e}_1}\mathbf{e}_1=-\mathbf{e}_3,\ \nabla_{\mathbf{e}_1}\mathbf{e}_3=\mathbf{e}_1,\ \nabla_{\mathbf{e}_2}\mathbf{e}_2=-\mathbf{e}_3,\ \nabla_{\mathbf{e}_2}\mathbf{e}_3=\mathbf{e}_2.$$

Definition 3.1. *Let $\mathbf{r}$ be a smooth immersion of an oriented surface M^2 into $\mathbb{R}^3$ and $\mathbf{n}$ unit normal vector field on M^2 and H^{∇} the mean curvature with respect to ∇. Let $\mathbf{v}\in\mathbb{R}^3$, $\mathbf{v}\neq\mathbf{0}$, a unit fixed vector non-parallel to $\mathbf{n}$. The surface M^2 is called $\nabla-$singular minimal surface with respect to $\mathbf{v}$ if holds*

$$2H^{\nabla}=\alpha\frac{\langle\mathbf{n},\mathbf{v}\rangle}{\langle\mathbf{r},\mathbf{v}\rangle},\ \alpha\in\mathbb{R},\ \alpha\neq 0. \tag{3.2}$$

In particular, the surface M^2 is said to be $\nabla-$*minimal* if $H^{\nabla}=0$. With Definition 3.1, we first observe the $\nabla-$singular minimal surfaces of type $z=u(x,y)$ which are $\nabla-$minimal.

Theorem 3.1. *Let M^2 be a $\nabla-$singular minimal surface in $\mathbb{R}^3$ of type $z=u(x,y)$ with respect to a unit vector $\mathbf{v}=(a,b,c)$, $a^2+b^2\neq 0$. If M^2 is $\nabla-$minimal, then one of the following happens*

1. $\mathbf{v} = (0, b \neq 0, c)$ *and*

$$u(x,y) = \frac{c}{b}y + \frac{1}{2b^2}\ln\left[\cos\left(2bx + \lambda_1\right)\right] + \lambda_2;$$

2. $\mathbf{v} = (a \neq 0, 0, c)$ *and*

$$u(x,y) = \frac{c}{a}x + \frac{1}{2a^2}\ln\left[\cos\left(2ay + \lambda_3\right)\right] + \lambda_4;$$

3. $\mathbf{v} = (a,b,c)$*,* $ab \neq 0$*, and*

$$u(x,y) = \frac{c}{a}x - \frac{1}{2\left(a^2+b^2\right)}\ln\left[\cos\left(-2|a|\left(y - \frac{b}{a}x\right) + \lambda_5\right)\right] + \frac{bc}{a^2+b^2}\left(y - \frac{b}{a}x\right) + \lambda_6,$$

where $\lambda_1, ..., \lambda_6 \in \mathbb{R}$.

Proof. The unit normal vector field on M^2 is

$$\mathbf{n} = \frac{-u_x\mathbf{e}_1 - u_y\mathbf{e}_2 + \mathbf{e}_3}{\sqrt{1 + (u_x)^2 + (u_y)^2}},$$

where $u_x = \frac{\partial u}{\partial x}$ and so. Suppose that M^2 is $\nabla-$minimal. Due to $\alpha \neq 0$, Eq. (3.2) gives $\langle \mathbf{n}, \mathbf{v} \rangle = 0$ and

$$au_x + bu_y = c. \tag{3.3}$$

The condition of $\nabla-$minimality yields

$$\left[1 + (u_y)^2\right]u_{xx} - 2u_xu_yu_{xy} + \left[1 + (u_x)^2\right]u_{yy} - 2\left[1 + (u_x)^2 + (u_y)^2\right] = 0. \tag{3.4}$$

We distinguish several cases: the first case is that $a = 0$. Then Eq. (3.3) follows $u(x,y) = f(x) + \frac{c}{b}y$, for an arbitrary smooth function f. Considering this into Eq. (3.4) leads to

$$\frac{bf''}{1 + (bf')^2} = 2b, \tag{3.5}$$

where $f' = \frac{df}{dx}$ and so. The first statement of the theorem is obtained by integrating Eq. (3.5). The roles of x and y in Eq. (3.4) are symmetric and hence we may conclude the second statement of the theorem by similar steps when $a \neq 0$ and $b = 0$. The last case is that $ab \neq 0$. Then the solution to Eq. (3.3) is given by

$$u(x,y) = \frac{c}{a}x + g\left(y - \frac{b}{a}x\right), \tag{3.6}$$

for a smooth function g. Substituting Eq. (3.6) into Eq. (3.4) follows

$$g'' - 2\left[a^2 + (c - bg')^2 + (ag')^2\right] = 0, \tag{3.7}$$

for $g' = \frac{dg}{d\tilde{y}}$, $g'' = \frac{d^2g}{d\tilde{y}^2}$, $\tilde{y} = y - \frac{b}{a}x$. Eq. (3.7) can be rewritten as

$$\frac{\left(a^2+b^2\right)g''}{a^2 + \left(bc - \left(a^2+b^2\right)g'\right)^2} = 2. \tag{3.8}$$

The proof is completed by integrating Eq. (3.8). □

Remark 3.1. *The surface given in the first statement of Theorem 3.1 is a generalized cylinder (see [20, p. 439]) and may be written parametrically*

$$\mathbf{r}(x,y) = \left(x, 0, \frac{1}{2b^2}\ln\left[\cos\left(2bx + \lambda_1\right)\right] + \lambda_2\right) + y\left(0, 1, \frac{c}{b}\right).$$

This is a $\nabla-$*minimal translation surface of type* $z = p(x) + q(y)$ *which was already found by Wang [44]. The same may be concluded for the above second statement. However, the surface described in the last statement of Theorem 3.1 is the generalized cylinder parametrically written by*

$$\mathbf{r}(x,\tilde{y}) = x\left(1, \frac{b}{a}, \frac{c}{a}\right) + \left(0, \tilde{y}, g(\tilde{y})\right),$$

where $\tilde{y} = y - \frac{b}{a}x$*. Due to* $b \neq 0$*, it belongs to the class of affine translation surfaces and a new example of* $\nabla-$*minimal surfaces.*

In the following we classify $\nabla-$singular minimal surfaces in $\mathbb{R}^3$ of type $y = u(x,z)$ which are $\nabla-$minimal.

Theorem 3.2. *Let* M^2 *be a* $\nabla-$*singular minimal surface in* $\mathbb{R}^3$ *of type* $y = u(x,z)$ *with respect to a unit vector* $\mathbf{v} = (a,b,c)$, $a^2 + c^2 \neq 0$*. If* M^2 *is* $\nabla-$*minimal, then one of the following happens*

1. *M^2 is a plane parallel to the vector* $(0,0,1)$*;*
2. $\mathbf{v}=(0,b,c)$, $bc\neq 0$, *and*
$$u(x,z)=\frac{b}{c}z+\frac{1}{2bc}\ln\left[\cos\left(2bx+\lambda_1\right)\right]+\lambda_2;$$
3. $\mathbf{v}=(a,b,0)$, $a\neq 0$, *and*
$$u(x,z)=\frac{b}{a}x\pm\frac{1}{2|a|}\arctan\left(\frac{1}{|a\lambda_2|}\sqrt{e^{4z}-a^2}\right)+\lambda_3;$$
4. $\mathbf{v}=(a,0,c)$, $ac\neq 0$, *and*
$$u(x,z)=\pm\frac{1}{2|a|}\arctan\left(\frac{1}{|\lambda_4|}\sqrt{e^{4a^2\left(z-\frac{c}{a}x\right)}-\lambda_4^2}\right)+\lambda_5,\ \lambda_4\neq 0;$$
5. $\mathbf{v}=(a,b,c)$, $ac\neq 0$, *and*
$$u(x,z)=\frac{b}{a}x+h\left(z-\frac{c}{a}x\right),$$
where h is a smooth function satisfying
$$z-\tfrac{c}{a}x=\tfrac{1}{2|a|(a^2+c^2)(a^2+b^2c^2)}\left\{bc\left(2|a|h+\lambda_6\right)-\right.$$
$$\left.-|a|\ln\left[bc\cos\left(2|a|h+\lambda_6\right)-|a|\sin\left(2|a|h+\lambda_6\right)\right]\right\}+\lambda_7,$$
for $\lambda_1,\ldots,\lambda_7\in\mathbb{R}$.

Proof. Let M^2 be locally given by
$$(x,z)\longmapsto\mathbf{r}(x,z)=(x,u(x,z),z),$$
for a smooth function $u=u(x,z)$. The normal vector field on M^2 is
$$\mathbf{n}=\frac{u_x\mathbf{e}_1-\mathbf{e}_2+u_z\mathbf{e}_3}{\sqrt{1+(u_x)^2+(u_z)^2}}. \tag{3.9}$$
Because M^2 is $\nabla-$singular minimal, we get Eq. (3.2). Assume that M^2 is $\nabla-$minimal. Due to $\alpha\neq 0$, Eqs. (3.2) and (3.9) follow $\langle\mathbf{n},\mathbf{v}\rangle=0$ and
$$au_x+cu_z=b. \tag{3.10}$$
Remark also that we may write $\mathbf{v}=a\mathbf{r}_x+c\mathbf{r}_z$, which means that the tangent plane of M at any point is parallel to $\mathbf{v}$. The condition of $\nabla-$minimality leads to
$$\left[1+(u_z)^2\right]u_{xx}-2u_xu_zu_{xz}+\left[1+(u_x)^2\right]u_{zz}+2\left[1+(u_x)^2+(u_z)^2\right]u_z=0. \tag{3.11}$$
We distinguish several cases:

1. $a=0$, $c\neq 0$. Then Eq. (3.10) gives $u_z=\frac{b}{c}$ and so Eq. (3.11) turns M^2 to a plane parallel to $\mathbf{v}$ if $b=0$. Otherwise, $b\neq 0$, the solution to Eq. (3.10) is given by $u(x,z)=\frac{b}{c}z+f(x)$, for an arbitrary smooth function f. Hence Eq. (3.11) reduces to
$$\frac{cf''}{1+(cf')^2}=-2b, \tag{3.12}$$
where $f'=\frac{df}{dx}$, etc. The second statement of the theorem is obtained by integrating Eq. (3.12).
2. $a\neq 0$, $c=0$. Then Eq. (3.10) gives $u(x,z)=\frac{b}{a}x+g(z)$ for an arbitrary smooth function g and so Eq. (3.11) may be written as
$$\frac{g''}{g'}-\frac{a^2g'g''}{1+(ag')^2}=-2, \tag{3.13}$$
where $g'=\frac{dg}{dz}$, etc. Integrating Eq. (3.13), we obtain the third statement of the theorem.
3. $ac\neq 0$. The solution to Eq. (3.10) is
$$u(x,z)=\frac{b}{a}x+h\left(z-\frac{c}{a}x\right), \tag{3.14}$$
for an arbitrary smooth function h. By plugging the partial derivatives of Eq. (3.14) into Eq. (3.11), we write
$$h''+2\left[a^2+\left(b-ch'\right)^2+\left(ah'\right)^2\right]h'=0, \tag{3.15}$$
where $h'=\frac{dh}{d\tilde{z}}$, $h''=\frac{d^2h}{d\tilde{z}^2}$, $\tilde{z}=z-\frac{c}{a}x$. We have two subcases: the first subcase is that $b=0$. Then Eq. (3.15) may be rewritten as
$$\frac{h''}{h'}-\frac{h'h''}{a^2+(h')^2}=-2a^2. \tag{3.16}$$

The fourth statement of the theorem is proved by integrating Eq. (3.16). The second subcase is $b \neq 0$. Hence, we may write Eq. (3.15) as

$$\frac{-\left(a^2+c^2\right)h''}{a^2+\left(bc-\left(a^2+c^2\right)h'\right)^2}=2h'. \tag{3.17}$$

A first integration of Eq. (3.17) yields

$$\frac{\left(a^2+c^2\right)dh}{-|a|\tan\left(2|a|h+\lambda\right)+bc}=d\tilde{z}, \tag{3.18}$$

for $\lambda \in \mathbb{R}$. By a first integration of Eq. (3.18), we finish the proof.

□

Remark 3.2. *The surfaces given in the second and third statements of Theorem 3.2 are $\nabla-$minimal generalized cylinders and are examples of $\nabla-$minimal translation surfaces of type $y=p(x)+q(z)$, which was found by Wang [44]. However, the surfaces given in the last two statements of Theorem 3.2 are a $\nabla-$minimal affine translation surface.*

Lastly, we deal with a surface M^2 of type $x=u(y,z)$. The unit normal vector field on M^2 is

$$\mathbf{n}=\frac{\mathbf{e}_1-u_y\mathbf{e}_2-u_z\mathbf{e}_3}{\sqrt{1+\left(u_y\right)^2+\left(u_z\right)^2}}.$$

Suppose that M^2 is $\nabla-$singular minimal with respect to the vector $\mathbf{v}=(a,b,c)$. The mean curvature is same as that of the surface of type $y=u(x,z)$. If M^2 is also $\nabla-$minimal, then Eq. (1.3) gives

$$bu_y+cu_z=a,$$

where $b^2+c^2 \neq 0$. Therefore, without giving a proof, we may state a similar result for those surfaces of type $x=u(y,z)$ to Theorem 3.2 by replacing x with y and a with b.

3.2. $D-$Singular minimal surfaces

Let D be the semi-symmetric non-metric connection on $\mathbb{R}^3$ given by [44]

$$D_{\mathbf{x}}\mathbf{y}=\nabla^L_{\mathbf{x}}\mathbf{y}+\langle\mathbf{y},\mathbf{e}_3\rangle\mathbf{x}, \tag{3.19}$$

where $\mathbf{x},\mathbf{y}$ are tangent vector fields to $\mathbb{R}^3$. The nonzero derivatives are

$$D_{\mathbf{e}_1}\mathbf{e}_3=\mathbf{e}_1,\; D_{\mathbf{e}_2}\mathbf{e}_3=\mathbf{e}_2,\; D_{\mathbf{e}_3}\mathbf{e}_3=\mathbf{e}_3.$$

Definition 3.2. *Let $\mathbf{r}$ be a smooth immersion of an oriented surface M^2 into $\mathbb{R}^3$ and $\mathbf{n}$ unit normal vector field on M^2 and H^D denote the mean curvature with respect to D. Let $\mathbf{v}\in\mathbb{R}^3$, $\mathbf{v}\neq\mathbf{0}$, a unit fixed vector non-parallel to $\mathbf{n}$. The surface M^2 is called $D-$singular minimal surface with respect to $\mathbf{v}$ if holds*

$$2H^D=\alpha\frac{\langle\mathbf{n},\mathbf{v}\rangle}{\langle\mathbf{r},\mathbf{v}\rangle},\; \alpha\in\mathbb{R},\; \alpha\neq 0. \tag{3.20}$$

In particular, the surface M^2 is said to be *$D-$minimal* if $H^D=0$. We first consider the $D-$singular minimal surfaces of type $z=u(x,y)$ which are $D-$minimal. Hence Eq. (3.20) gives $\langle\mathbf{n},\mathbf{v}\rangle=0$ and

$$au_x+bu_y=c, \tag{3.21}$$

where $\mathbf{v}=(a,b,c)$ and $a^2+b^2\neq 0$. Morever the condition of $D-$minimality yields

$$\left[1+\left(u_y\right)^2\right]u_{xx}-2u_xu_yu_{xy}+\left[1+\left(u_x\right)^2\right]u_{yy}=0, \tag{3.22}$$

where the roles of x and y are symmetric. If $a=0$, then Eq. (3.21) follows $u(x,y)=f(x)+\frac{c}{b}y$, for an arbitrary smooth function f. Considering this into Eq. (3.22) yields $\frac{1}{b^2}\frac{d^2f}{dx^2}=0$, which leads M^2 to be a plane parallel to $\mathbf{v}$. By symmetry, we may obtain same obtain when $a\neq 0$ and $b=0$. Let $ab\neq 0$. Then the solution to Eq. (3.21) is $u(x,y)=\frac{c}{a}x+g\left(y-\frac{b}{a}x\right)$, for an arbitrary smooth function f. After substituting its partial derivatives into Eq. (3.22), we conclude $\frac{1}{a^2}\frac{d^2g}{d\tilde{y}^2}=0$, $\tilde{y}=y-\frac{b}{a}x$, yielding that M is a plane parallel to $\mathbf{v}$.
Therefore we state the following

Theorem 3.3. *Let M^2 be a $D-$singular minimal surface in $\mathbb{R}^3$ of type $z=u(x,y)$ with respect to a unit vector $\mathbf{v}=(a,b,c)$, $a^2+b^2\neq 0$. If M^2 is $D-$minimal, then it is a plane parallel to $\mathbf{v}$.*

When we take surfaces of type $y=u(x,z)$ and $x=u(y,z)$, we get similar equations to Eqs. (3.21) and (3.22) and thus the above result remains true for those surfaces as well.

4. Singular minimal surfaces in $\mathbb{R}_1^3$

4.1. $\nabla-$Singular minimal surfaces

Let ∇^L be the Levi-Civita connection $\mathbb{R}_1^3$ and $\mathbf{x},\mathbf{y}$ tangent vector fields to $\mathbb{R}_1^3$. Consider the following semi-symmetric metric connection on $\mathbb{R}_1^3$ [44]

$$\nabla_{\mathbf{x}}\mathbf{y}=\nabla_{\mathbf{x}}^L\mathbf{y}+\langle\mathbf{y},\mathbf{e}_3\rangle_L\mathbf{x}-\langle\mathbf{x},\mathbf{y}\rangle_L\mathbf{e}_3. \tag{4.1}$$

The nonzero derivatives are

$$\nabla_{\mathbf{e}_1}\mathbf{e}_1=-\mathbf{e}_3,\ \nabla_{\mathbf{e}_1}\mathbf{e}_3=-\mathbf{e}_1,\ \nabla_{\mathbf{e}_2}\mathbf{e}_2=-\mathbf{e}_3,\ \nabla_{\mathbf{e}_2}\mathbf{e}_3=-\mathbf{e}_2.$$

Definition 4.1. *Let $\mathbf{r}$ be a smooth immersion of an oriented timelike surface M^2 in $\mathbb{R}_1^3$ and $\mathbf{n}$ unit normal vector field on M^2 and H^∇ the mean curvature of M^2 with respect to ∇. Let $\mathbf{v}\neq\mathbf{0}\in\mathbb{R}_1^3$ a unit fixed spacelike vector non-parallel to $\mathbf{n}$ such that $\mathbf{n}$ and $\mathbf{v}$ span a spacelike 2-space. M^2 is called $\nabla-$singular minimal surface with respect to $\mathbf{v}$ if satisfies*

$$2H^\nabla=\alpha\frac{\langle\mathbf{n},\mathbf{v}\rangle_L}{\langle\mathbf{r},\mathbf{v}\rangle_L},\ \alpha\in\mathbb{R},\ \alpha\neq0. \tag{4.2}$$

The surface M^2 is called $\nabla-$*minimal* if $H^\nabla=0$. With Definition 4.1, we classify the $\nabla-$singular minimal surfaces of type $z=u(x,y)$, which are $\nabla-$minimal.

Theorem 4.1. *Let M^2 be a $\nabla-$singular minimal surface in $\mathbb{R}_1^3$ of type $z=u(x,y)$ with respect to a unit spacelike vector $\mathbf{v}=(a,b,c)$. If M^2 is $\nabla-$minimal, then one of the following happens*

1. $\mathbf{v}=(0,b\neq0,c)$ *and*
$$u(x,y)=\frac{c}{b}y+\frac{1}{2b^2}\ln\left[\cosh\left(2bx+\lambda_1\right)\right]+\lambda_2;$$
2. $\mathbf{v}=(a\neq0,0,c)$ *and*
$$u(x,y)=\frac{c}{a}x+\frac{1}{2a^2}\ln\left[\cosh\left(2ay+\lambda_3\right)\right]+\lambda_4;$$
3. $\mathbf{v}=(a,b,c)$, $ab\neq0$, *and*
$$u(x,y)=\frac{c}{a}x+\frac{bc}{a^2+b^2}\left(y-\frac{b}{a}x\right)+\frac{1}{2\left(a^2+b^2\right)}\ln\left[\cosh\left(-2\left|a\right|\left\{y-\frac{b}{a}x\right\}+\lambda_5\right)\right]+\lambda_6,$$
where $\lambda_1,...,\lambda_6\in\mathbb{R}$, $\lambda_5\neq0$.

Proof. The unit spacelike normal vector field on M^2 is

$$\mathbf{n}=\frac{-u_x\mathbf{e}_1-u_y\mathbf{e}_2-\mathbf{e}_3}{\sqrt{-1+(u_x)^2+(u_y)^2}},$$

Suppose that M^2 is $\nabla-$minimal. Due to $\alpha\neq0$, Eq. (4.2) gives $\langle\mathbf{n},\mathbf{v}\rangle=0$ and

$$au_x+bu_y=c. \tag{4.3}$$

The condition of $\nabla-$minimality yields

$$\left[1-(u_y)^2\right]u_{xx}+2u_xu_yu_{xy}+\left[1-(u_x)^2\right]u_{yy}+2\left[-1+(u_x)^2+(u_y)^2\right]=0. \tag{4.4}$$

We distinguish several cases: the first case is that $a=0$. Then Eq. (4.3) follows $u(x,y)=f(x)+\frac{c}{b}y$, for an arbitrary smooth function f. Considering this into Eq. (4.4) yields

$$\frac{bf''}{1-(bf')^2}=2b, \tag{4.5}$$

where $f'=\frac{df}{dx}$ and so. The first statement of the theorem is derived by integrating Eq. (4.5). The roles of x and y in Eq. (4.4) are symmetric and hence we may conclude the second statement of the theorem by similar steps when $a\neq0$ and $b=0$. The last case is that $ab\neq0$. Then the solution to Eq. (4.3) is given by

$$u(x,y)=\frac{c}{a}x+g\left(y-\frac{b}{a}x\right), \tag{4.6}$$

for a smooth function g. Substituting Eq. (4.6) into Eq. (4.4) follows

$$g''+2\left[-a^2+\left(c-bg'\right)^2+\left(ag'\right)^2\right]=0, \tag{4.7}$$

where $g'=\frac{dg}{d\tilde{y}}$, $g''=\frac{d^2g}{d\tilde{y}^2}$, $\tilde{y}=y-\frac{b}{a}x$. Eq. (4.7) may be rewritten as

$$\frac{-\left(a^2+b^2\right)g''}{a^2-\left[bc-\left(a^2+b^2\right)g'\right]^2}=-2. \tag{4.8}$$

The proof is completed by integrating Eq. (4.8). □

Remark 4.1. *The last statement of Theorem 4.1 is a new example in $\mathbb{R}^3_1$ of $\nabla-$minimal surfaces while the first two statements are $\nabla-$minimal translation surfaces, introduced by Wang [44].*

In the following we classify $\nabla-$singular minimal surfaces in $\mathbb{R}^3_1$ of type $y=u(x,z)$ which are $\nabla-$minimal.

Theorem 4.2. *Let M^2 be a $\nabla-$singular minimal surface in $\mathbb{R}^3_1$ of type $y=u(x,z)$ with respect to a unit spacelike vector $\mathbf{v}=(a,b,c)$, $a^2+c^2\neq 0$. If M^2 is $\nabla-$minimal, then one of the following happens*

1. *M^2 is a plane parallel to $\mathbf{v}=(a,b,0)$, $a\neq 0$;*
2. *$\mathbf{v}=(0,b,c)$, $bc\neq 0$ and*

$$u(x,y)=\frac{b}{c}z+\frac{1}{2bc}\ln\left[\cosh\left(2bx+\lambda_1\right)\right]+\lambda_2;$$

3. *$\mathbf{v}=(a,b,0)$, $a\neq 0$, and*

$$u(x,z)=\frac{b}{a}x\pm\frac{1}{2|a|}\sinh^{-1}\left(\lambda_3 e^{2z}\right)+\lambda_4,\ \lambda_3\neq 0;$$

4. *$\mathbf{v}=(a,0,c)$, $ac\neq 0$, and*

$$u(x,z)=\pm\frac{1}{2|a|}\sinh^{-1}\left[|\lambda_5|\,e^{2a^2\left(z-\frac{c}{a}x\right)}\right]+\lambda_6,\ \lambda_5\neq 0;$$

5. *$\mathbf{v}=(a,\pm 1,c)$, $a=\pm c$, $c\neq 0$, and*

$$u(x,z)=\frac{\pm 1}{c}x\pm\frac{1}{4c}\ln\left[1\pm 2\lambda_7 e^{2\left(1+c^2\right)(z\pm x)}\right]+\lambda_8,\ \lambda_7\neq 0;$$

6. *$\mathbf{v}=(a,b,c)$, $abc\neq 0$, and*

$$u(x,z)=\frac{b}{a}x+h\left(z-\frac{c}{a}x\right),$$

where h is a smooth function satisfying

$$z-\frac{c}{a}x=\frac{-bc\left(c^2-a^2\right)}{2|a|\left(a^2-b^2c^2\right)}\left(2|a|h+\lambda_9\right)-$$
$$-\frac{c^2-a^2}{2\left(a^2-b^2c^2\right)}\ln\left[bc\cosh\left(2|a|h+\lambda_9\right)-|a|\sinh\left(2|a|h+\lambda_9\right)\right]+\lambda_{10},$$

for $\lambda_1,\ldots,\lambda_{10}\in\mathbb{R}$.

Proof. Let M^2 be locally given by

$$(x,z)\longmapsto \mathbf{r}(x,z)=(x,u(x,z),z),$$

for a smooth function $u=u(x,z)$. The normal vector field on M^2 is

$$\mathbf{n}=\frac{u_x\mathbf{e}_1-\mathbf{e}_2-u_z\mathbf{e}_3}{\sqrt{1+(u_x)^2-(u_z)^2}}.$$

Because M^2 is $\nabla-$singular minimal, we get Eq. (4.1). Assume that M^2 is $\nabla-$minimal. Due to $\alpha\neq 0$, Eq. (4.1) gives $\langle\mathbf{n},\mathbf{v}\rangle=0$ and

$$au_x+cu_z=b. \tag{4.9}$$

Remark also that we may write $\mathbf{v}=a\mathbf{r}_x+c\mathbf{r}_z$, implying the tangent plane of M at any point is parallel to $\mathbf{v}$. The condition of $\nabla-$minimality yields

$$\left[(u_z)^2-1\right]u_{xx}-2u_xu_zu_{xz}+\left[1+(u_x)^2\right]u_{zz}-2\left[1+(u_x)^2-(u_z)^2\right]u_z=0. \tag{4.10}$$

We distinguish several cases:

1. $a=0$, $c\neq 0$. Then $b\neq 0$ because $\mathbf{v}$ is spacelike. The solution to Eq. (4.9) is given by $u(x,z)=\frac{b}{c}z+f(x)$, for an arbitrary smooth function f. Hence Eq. (4.10) turns to

$$\frac{cf''}{1-(cf')^2}=-2b, \tag{4.11}$$

where $f'=\frac{df}{dx}$, etc. Because M^2 is non-degenerate, $1-(cf')^2\neq 0$. Therefore the second statement of the theorem is proved by integrating Eq. (4.11).

2. $a \neq 0$, $c = 0$. Then Eq. (4.9) gives $u(x,z) = \frac{b}{a}x + g(z)$ for an arbitrary smooth function g and so Eq. (4.10) leads to

$$g'' - 2\left[1 - \left(ag'\right)^2\right] g' = 0, \tag{4.12}$$

where $g' = \frac{dg}{dz}$, etc. That $g' = 0$ is a trivial solution to Eq. (4.12), implying the first statement of the theorem. Otherwise, $g' \neq 0$, Eq. (4.12) may be rewritten as

$$\frac{g''}{g'} + \frac{a}{2}\left(\frac{g''}{1 - ag'} - \frac{g''}{1 + ag'}\right) = 2. \tag{4.13}$$

The third statement of the theorem is obtained by integrating Eq. (4.13).

3. $ac \neq 0$. The solution to Eq. (4.9) is

$$u(x,z) = \frac{b}{a}x + h\left(z - \frac{c}{a}x\right),$$

for an arbitrary smooth function h. Therefore Eq. (4.10) reduces to

$$h'' - 2\left[a^2 + \left(b - ch'\right)^2 - \left(ah'\right)^2\right] h' = 0, \tag{4.14}$$

where $h' = \frac{dh}{d\tilde{z}}$, $h' = \frac{d^2h}{d\tilde{z}^2}$, $\tilde{z} = z - \frac{c}{a}x$. We have three subcases: the first one is that $b = 0$. Then Eq. (4.14) may be rewritten as

$$\frac{h''}{h'} + \frac{h''}{2(a - h')} - \frac{h''}{2(a + h')} = 2a^2. \tag{4.15}$$

Integrating Eq. (4.15) gives the fourth statement of the theorem. The second subcase is that $a^2 = c^2$ and $b = \pm 1$. Then Eq. (4.14) may be rewritten as

$$\frac{\pm 2ch''}{1 + c^2 \mp 2ch'} + \frac{h''}{h'} = 2\left(1 + c^2\right). \tag{4.16}$$

After integrating Eq. (4.16), we obtain the fifth statement of the theorem. The third subcase is that $a^2 \neq c^2$. Then Eq. (4.14) may be rewritten as

$$\frac{-\left(c^2 - a^2\right) h''}{a^2 - \left[bc - \left(c^2 - a^2\right) h'\right]^2} = 2h'. \tag{4.17}$$

A first integration of Eq. (4.17) yields

$$\frac{\left(c^2 - a^2\right) dh}{-|a| \tanh\left(2|a| h + \lambda\right) + bc} = d\tilde{z}, \tag{4.18}$$

for $\lambda \in \mathbb{R}$. The proof is completed by a first integration of Eq. (4.18).

□

Remark 4.2. *The last three statements of Theorem 4.1 are new examples in $\mathbb{R}^3_1$ of $\nabla-$minimal surfaces while the second and third statements are $\nabla-$minimal translation surfaces, introduced by Wang [44].*

Let M^2 be a timelike surface $\mathbb{R}^3_1$ of type $x = u(y,z)$. The spacelike unit normal vector field on M^2 is

$$\mathbf{n} = \frac{\mathbf{e}_1 - u_y\mathbf{e}_2 + u_z\mathbf{e}_3}{\sqrt{1 + \left(u_y\right)^2 - \left(u_z\right)^2}}.$$

Suppose that M^2 is $\nabla-$singular minimal with respect to the vector $\mathbf{v} = (a,b,c)$. If M^2 is also $\nabla-$minimal, then Eq. (4.2) gives

$$bu_y + cu_z = a,$$

where $b^2 + c^2 \neq 0$. Notice that the mean curvature is same as that of the surface of type $y = u(x,z)$. Therefore, without giving a proof, we may state a similar result for those surfaces of type $x = u(y,z)$ to Theorem 4.1 by replacing x with y and a with b.

4.2. $D-$Singular minimal surfaces

Consider the following semi-symmetric non-metric connection on $\mathbb{R}^3_1$ [44]

$$D_{\mathbf{x}}\mathbf{y} = \nabla^L_{\mathbf{x}}\mathbf{y} + \langle \mathbf{y}, \mathbf{e}_3 \rangle \mathbf{x}, \tag{4.19}$$

where $\mathbf{x}, \mathbf{y}$ are tangent vector fields to $\mathbb{R}^3$. The nonzero derivatives are

$$D_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1,\ D_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2,\ D_{\mathbf{e}_3}\mathbf{e}_3 = \mathbf{e}_3.$$

Definition 4.2. *Let $\mathbf{r}$ be a smooth immersion of an oriented timelike surface M^2 in $\mathbb{R}^3_1$ and $\mathbf{n}$ unit normal vector field on M^2 and H^D the mean curvature of M^2 with respect to D. Let $\mathbf{v} \neq \mathbf{0} \in \mathbb{R}^3_1$ a unit fixed spacelike vector non-parallel to $\mathbf{n}$ such that $\mathbf{n}$ and $\mathbf{v}$ span a spacelike 2-space. M^2 is called $D-$singular minimal surface with respect to $\mathbf{v}$ if satisfies*

$$2H^D = \alpha \frac{\langle \mathbf{n}, \mathbf{v} \rangle_L}{\langle \mathbf{r}, \mathbf{v} \rangle_L},\ \alpha \in \mathbb{R},\ \alpha \neq 0. \tag{4.20}$$

The surface M^2 is called $D-$*minimal* if $H^D = 0$. With Definition 4.2, we first observe the $D-$singular minimal surfaces of type $z = u(x,y)$ which are $D-$minimal. Hence Eq. (4.20) gives $\langle \mathbf{n}, \mathbf{v} \rangle = 0$ and

$$au_x + bu_y = c, \tag{4.21}$$

where $\mathbf{v} = (a,b,c)$. The condition of $D-$minimality yields

$$\left[1 - (u_y)^2\right] u_{xx} + 2u_x u_y u_{xy} + \left[1 - (u_x)^2\right] u_{yy} = 0, \tag{4.22}$$

where the roles of x and y are symmetric. If $a = 0$, then Eq. (4.21) follows $u(x,y) = f(x) + \frac{c}{b}y$, for an arbitrary smooth function f. Considering this into Eq. (4.22) yields $\frac{1}{b^2}\frac{d^2 f}{dx^2} = 0$, which leads M^2 to be a plane parallel to $\mathbf{v}$. By symmetry, we can obtain same result when $a \neq 0$ and $b = 0$. Let $ab \neq 0$. Then the solution to Eq. (4.21) is $u(x,y) = \frac{c}{a}x + g\left(y - \frac{b}{a}x\right)$, for a smooth function g. After substituting its partial derivatives into Eq. (4.22), we conclude $\frac{1}{a^2}\frac{d^2 g}{d\tilde{y}^2} = 0$, $\tilde{y} = y - \frac{c}{a}x$, yielding that M is a plane parallel to $\mathbf{v}$.
Therefore, we state the following

Theorem 4.3. *Let M^2 be a $D-$singular minimal surface in $\mathbb{R}^3_1$ of type $z = u(x,y)$ with respect to a unit spacelike vector $\mathbf{v}$. If M^2 is $D-$minimal, then it is a plane parallel to $\mathbf{v}$.*

When we take the surfaces of type $y = u(x,z)$ or $x = u(y,z)$, we may state a similar result to Theorem 4.3.

5. Conclusions and further remarks

In this study, we discussed the singular minimal surfaces in $\mathbb{R}^3$ (resp. $\mathbb{R}^3_1$) which are minimal and expressed a trivial outcome, Proposition 1.1 (resp. Proposition 1.2). Nevertheless, the non-trivial outcomes, Theorems 3.1 and 3.2 (resp. Theorems 4.1 and 4.2), were obtained by using the modified version, Definition 3.1 (resp. Definition 4.1), of singular minimality. With this definition, we observed that the singular minimal surfaces which are minimal are a generalized cylinder. Since the generalized cylinders belong to a subcase of translation surfaces, the $\nabla-$minimal translation surfaces introduced by Wang [44] were presented by some of our results. Still, we also exhibited new examples of $\nabla-$minimal surfaces, as explained in Remarks 3.1 and 3.2 (resp. Remarks 4.1 and 4.2). Morever, a trivial outcome, Theorem 3.3 (resp. Theorem 4.3), was found by using the semi-symmetric non-metric connection D given by Eq. (3.19) (resp. Eq. (4.19)).
On the other hand, let M^2 be locally a graph surface in $\mathbb{R}^3$ of a smooth function $u(x,y)$ and H and H^∇ denote the mean curvatures with respect to the Levi-Civita connection and the semi-symmetric metric connection ∇ given by Eq. (3.1), respectively. Then, the following relation occurs

$$H^\nabla = H - \langle \mathbf{n}, (0,0,1) \rangle, \tag{5.1}$$

where $\mathbf{n}$ is the unit normal vector field on M^2. Notice also that Eq. (5.1) remains true for a graph of the forms $u(x,z)$ or $u(y,z)$ up to a sign. Therefore, $\nabla-$minimal graph surfaces turn to the *translating solitons* whose the mean curvature satisfies

$$H = \langle \mathbf{n}, (0,0,1) \rangle \tag{5.2}$$

Eq. (5.2) appears in the theories of mean curvature flow and manifolds with density, for details see ([23, 24, 29–31]). Eventually, the above discussion imply that $\nabla-$singular minimal surfaces which are $\nabla-$minimal are a cylindrical translating soliton. Such surfaces were considered in [23, 31]. Nevertheless, this paper provides a novel approach.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Universal Journal of Mathematics and Applications, 4 (4) (2021) 147-153
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.983509

Geometric Structure of the Set of Pairs of Matrices under Simultaneous Similarity

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Article Info

Keywords: *Bundles, simultaneous similarity, connection, curvature*
2010 AMS: *15A18, 57S25, 15A22*
Received: *16 August 2021*
Accepted: *19 November 2021*
Available online: *30 December 2021*

Abstract

In this paper pairs of matrices under similarity are considered because of their scientific applications, especially pairs of matrices being simultaneously diagonalizable. For example, a problem in quantum mechanics is the position and momentum operators, because they do not have a shared base representing the system's states. They do not commute, and that is why switching operators form a crucial element in quantum physics.
A study of the set of linear operators consisting of pairs of simultaneously diagonalizable matrices is done using geometric constructions such as the principal bundles. The main goal of this work is to construct connections that allow us to establish a relationship between the local geometry around a point with the local geometry around another point. The connections give us a way to help distinguish bundle sections along tangent vectors.

1. Introduction

Let $\mathfrak{M}$ be the manifold of pairs of n-order real matrices $T = (X_1, X_2)$. A frequent question, in both mathematics and physics, is to ask if it is possible to find a base in the space $\mathbb{R}^n$ in which both matrices diagonalize, that is, to ask if they diagonalize simultaneously. Concretely, the simultaneous diagonalization of pairs of symmetric matrices has a particular interest, (see [5], [7] and [8], for example), due to its applications in sciences. For example, they appear when we must give the "instanton"solution of Yang-Mills field presented in an octonion form, and it can be represented by triples of traceless matrices [1], [6], [13]. Another application of simultaneous diagonalization is found when studying, for example, thermal transmissivity, whose study is different depending on whether the interaction matrices diagonalize simultaneously [12].
In order to formalize the simultaneous diagonalization problem, it is necessary to start by defining an equivalence relation called similarity, which allows establishing criteria for simultaneous diagonalization.
It is well known that, in the space of n-square real matrices, the subset of diagonalizable matrices is generic. Then, any non-diagonalizable matrix is arbitrarily close to a diagonalizable matrix and reduced to a diagonal form by a small perturbation of its entries. This property cannot be generalized to the case of simultaneous diagonalization of a pair (or m-tuple) of n-order real square matrices. Necessary or sufficient conditions for simultaneous diagonalization have been studied. These studies have been realized under different points of view, for example, analysing the spectra of families of pairs of matrices [8] computing versal deformations [2].
A good tool for distinguishing one subset from another within a differentiable variety could be by trying to identify it from the zeros of bundle sections built on the variety, then, the characteristic classes allow to identify its obstructions. In this particular setup, the interest is about the set of the m-tuples of simultaneously diagonalizable real matrices. Some results about families of pairs of matrices that are simultaneously diagonalizable can be found in [7], [8].
Principal bundles [10] have significant applications in different mathematical areas as topology and differential geometry, in special bundles given by a Lie group action. The first attempts to apply the theory of fiber bundles in the field of physics were made by E. Lubkin [11], who pointed out that the caliber fields had a fiber bundle structure. Further, they form part of the basic framework of gauge theories describing the interaction of forces by differentiating connections [14], and quantum theory [3].
An important object in principal bundles theory is that of connection. Visually, a connection gives us a way to move through the fibers of a principal bundle through isomorphisms between them, which leads us to curvature invariants. In this article, a connection on a specific main bundle is defined as well as the curvature derived from the connection.

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2. Preliminaries

2.1. Simultaneous equivalence of pairs of matrices

The purpose of this section is to give necessary and sufficient conditions that for two pairs of matrices,$T=(X_1,X_2),T'=(Y_1,Y_2)$ are simultaneously diagonalizable. First of all, we define the simultaneous similarity equivalence relating the elements of $\mathfrak{M}$.

Definition 2.1. *Let $T=(X_1,X_2),T'=(Y_1,Y_2)$ be two pairs of matrices in $\mathfrak{M}$. Then, T is simultaneously similar to T' if and only if there exists $P\in\mathfrak{G}=Gl(n;\mathbb{R})$ such that*

$$T'=(Y_1,Y_2)=(PX_1P^{-1},PX_2P^{-1})=PTP^{-1}. \tag{2.1}$$

A particular case of pairs of matrices is that those that are similar to a pair of matrices which are both diagonal, that is, they diagonalize simultaneously.

Definition 2.2. *The pair of matrices $T=(X_1,X_2)$ is simultaneously diagonalizable if and only if there exists an equivalent pair formed by diagonal matrices.*

Necessary conditions for simultaneous diagonalizable pairs can be found in the following propositions(see [7], [9]):

Proposition 2.3. *Let $T=(X_1,X_2)$ be a simultaneously diagonalizable pair. Then both matrices X_i must be diagonalizable. (The converse is false).*

Proposition 2.4. *Let $T=(X_1,X_2)$ be a simultaneously diagonalizable pair. Then, the Lie bracket $[X_1,X_2]=0$.*

Regarding sufficient conditions, we have the following results.

Theorem 2.5. *Let $T=(X_1,X_2)$ be a pair of commuting n-order square matrices and suppose that the matrix X_i, for some $i=1,2$, is diagonalizable with simple eigenvalues ($\lambda_j\neq\lambda_k$ for all $j\neq k$; $k,j=1,\dots n$). Then T is a pair of simultaneously diagonalizable matrices.*

Proof. Without loss of generality, we can assume that X_1 is diagonalizable.
Let $v_1,\dots,v_n$ be a basis such that $X_1(v_i)=\lambda_iv_i$ for $i=1,\dots,n$.
Then, $X_1(X_2v_i)=X_2(X_1v_i)=\lambda_iX_2v_i$.
So, if $X_2v_i\neq 0$, it is an eigenvector of X_1 of eigenvalue λ_i, but condition $\lambda_k\neq\lambda_\ell$ implies that $\dim\operatorname{Ker}(X_1-\lambda_jI)=1$, then, $X_2v_i=\mu_2v_i$, that is to say v_i is an eigenvector for X_2 of eigenvalue μ_2. If $X_2v_i=0$ the vector v_i is an eigenvector of X_2 of eigenvalue zero. That is to say, $\{v_1,\dots,v_n\}$ is a basis of eigenvectors for X_2, and T is a pair of simultaneous diagonalizable matrices with $P=\begin{pmatrix}v_1^t & \dots & v_n^t\end{pmatrix}^{-1}$. □

Remark 2.6. *The matrix X_2 does not necessarily have simple eigenvalues.*

Theorem 2.7. *Let X_1,X_2 be two n-order square matrices. If X_1 and X_2 are commuting and diagonalizable matrices, then, $T=(X_1,X_2)$ is simultaneously diagonalizable.*

Proof. Let P_1 be an invertible matrix such that $D_1=P_1X_1P_1^{-1}=\begin{pmatrix}D_1^1 & & \\ & \ddots & \\ & & D_r^1\end{pmatrix}$ with $D_i^1=\lambda_i^1I\in M_{n_i}(\mathbb{C})$, $1\leq i\leq r$ and $n_1+\ldots+n_r=n$.
Let us consider $v_{1_1},\dots,v_{n_1},\dots,v_{1_r},\dots,v_{n_r}$ the vector columns of P_1^{-1}, then

$$\begin{aligned}X_2X_1v_{i_\ell}&=X_2\lambda_\ell v_{i_\ell}=\lambda_\ell X_2v_{i_\ell}\\ X_2X_1v_{i_\ell}&=X_1X_2v_{i_\ell}\end{aligned}$$

Consequently $X_2v_{i_\ell}$ is an eigenvector of X_1 of eigenvalue λ_ℓ or $X_2v_{i_\ell}=0$, in any case we have that $X_2v_{i_\ell}\in[v_{1_\ell},\dots,v_{n_\ell}]=F_\ell$, consequently, $X_2F_\ell\subset F_\ell$. That is, the subspace F_ℓ is X_2 invariant for all $1\leq\ell\leq r$, and $P_1X_2P_1^{-1}$ is block-diagonal matrix

$$P_1X_2P_1^{-1}=\begin{pmatrix}Y_1^2 & & \\ & \ddots & \\ & & Y_r^2\end{pmatrix}$$

where the size of each block Y_j^2 is the same of the corresponding block D_j^1 in the matrix $P_1X_aP_1^{-1}$.
If all matrices Y_k^2 are diagonal the proof is concluded. If any submatrix Y_k^2 is not diagonal, then taking into account that the matrix X_2 diagonalizes, all submatrices Y_k^2 diagonalize.

Consider $P_2=\begin{pmatrix}P_2^1 & & \\ & \ddots & \\ & & P_2^r\end{pmatrix}$ where P_2^j diagonalizes Y_2^2 for $1\leq j\leq r$.
Clearly P_2 diagonalizes D_1:

$$\begin{pmatrix}P_2^1 & & \\ & \ddots & \\ & & P_2^r\end{pmatrix}\begin{pmatrix}D_1^1 & & \\ & \ddots & \\ & & D_r^1\end{pmatrix}\begin{pmatrix}P_2^1 & & \\ & \ddots & \\ & & P_2^r\end{pmatrix}^{-1}=\begin{pmatrix}P_2^1D_1^1(P_2^1)^{-1} & & \\ & \ddots & \\ & & P_2^rD_r^1(P_2^r)^{-1}\end{pmatrix}=\begin{pmatrix}P_2^1\lambda_1^1I(P_2^1)^{-1} & & \\ & \ddots & \\ & & P_2^r\lambda_r^1I(P_2^r)^{-1}\end{pmatrix}=\begin{pmatrix}D_1^1 & & \\ & \ddots & \\ & & D_r^1\end{pmatrix}$$

Then P_2P_1 diagonalizes X_1 and X_2.

□

2.1.1. Invariant polynomials associated to a pair of matrices under similarity

We are going to construct polynomials $\mathscr{P}(T)$ with $2n^2$ variables x^1_{ij}, x^2_{ij} $1 \leq i, j \leq n$, corresponding to the elements of the pair of matrices $T = (X_1, X_2) = ((x^1_{ij}), (x^2_{ij}))$.

Example 2.8. *Let $T = ((x^1_{ij}), (x^2_{ij}))$ be a pair of matrices. We can define the polynomial*

$$\mathscr{P}(T) = trace\, X_1 + trace\, X_2 = x^1_{11} + \ldots + x^1_{nn} + x^2_{11} + \ldots + x^2_{nn}.$$

We are interested in those which will be invariant under simultaneous similarity in the following sense.

Definition 2.9. *Let $T \in \mathfrak{M}$. A polynomial $\mathscr{P}(T)$ is called invariant under similarity, if $\mathscr{P}(T) = \mathscr{P}(PTP^{-1})$ for all $P \in GL(n;\mathbb{R})$.*

For this study, we will use the characteristic polynomials associated with each matrix of the pair.
Given the pair of matrices $T = (X_1, X_2)$, we can associate it with the following polynomial:

$$\sigma_T(t) = \det(tI - X_1) \cdot \det(tI - X_2) = \prod_{j=1}^{2} \det(tI - X_j) \tag{2.2}$$

Proposition 2.10. *The polynomial (2.2) is invariant under simultaneous similarity.*

Proof.

$$\sigma_{PTP^{-1}}(t) = \prod_{j=1}^{2} \det(tI - PX_jP^{-1}) = (\det P)^2 (\det P^{-1})^2 \prod_{j=1}^{2} \det(tI - X_j) = \sigma_T(t)$$

□

The polynomial $\sigma_T(t)$ can be written in the following manner

$$\sigma_T(t) = \prod_{j=1}^{2} \left(\Sigma_{i=0}^{n} \sigma_i^j(X_j) t^i \right) = \sum_{i=0}^{2n} \sigma_i(T) t^i$$

where, clearly, $\sigma_0(T) = \prod_{j=1}^{2} \det X_j$ and $\sigma_{2n}(T) = 1$.
For the set of variables x^1_{ij}, x^2_{ij} $1 \leq i, j \leq n$, we consider the corresponding pair $T = ((x^1_{ij}), (x^2_{ij}))$. Then, the polynomials $\sigma_i(T)$ are homogeneous polynomials in the given variables.

Proposition 2.11. *Each polynomial $\sigma_i(T)$ is an invariant polynomial.*

Proof. It suffices to note that $\sigma_i^j(X_j)$ is invariant. □

Let be now $T = (X_1, X_2)$, a simultaneously diagonalizable pair, then and taking into account the invariance of the characteristic polynomial we have that $\sigma_i(T)$ are expressed in terms of the eigenvalues of X_i, $i = 1, 2$:

$$\prod_{j=1}^{2} \det(tI - X_j) = \prod_{j=1}^{2} \prod_{k=1}^{n} (t - \lambda_k^j) = \Sigma_{i=0}^{2n} \sigma_i(\lambda_1^1, \ldots, \lambda_n^1, \lambda_1^2, \ldots, \lambda_n^2) t^i$$

Then, in this class of pairs of matrices, these polynomials can be expressed with $2n$ variables instead of the $2n^2$ variables intervening in the general case.

2.2. Fiber Bundles

Following Husmoller [10], a fiber bundle is a structure (E, B, π, F), where E, B, and F are topological spaces called the total space, base space of the bundle, and the fiber, respectively, and $\pi : E \to B$ is a continuous surjection called the bundle projection, satisfying the following local triviality condition: for every $x \in E$, there is an open neighborhood $U \subset B$ of $\pi(x)$ (called a trivializing neighborhood) such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ in such a way that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\pi_1"] \\ U & \end{tikzcd}$$

where $\pi_1 : U \times F \to U$ is the natural projection. The set of all $\{(U_i, \varphi_i)\}$ is called a local trivialization of the bundle.
Thus for any $p \in B$, $\pi^{-1}(\{p\})$ is homeomorphic to F and is called the fiber over p.
A trivial example of a bundle is the one given by

$$(B \times F, \pi, B, F)$$

where $\pi : B \times F \longrightarrow B$ is the projection on the first factor, in this case, the fibers are $\{p\} \times F$ for all $p \in B$.
A fiber bundle (E', B', π', F') is a subbundle of (E, B, π, F) provided E' is a subspace of E, B' is a subspace of B, and π' is the restriction of π to E', $\pi' = \pi_{E'} : E' \longrightarrow B'$,

In the special case where the fiber is a group G, the fiber bundle is called the principal bundle. In this case, any fiber $\pi^{-1}(b)$ is a space isomorphic to G. More specifically, G acts freely without fixed points on the fibers.
In the case where E, B and F are smooth manifolds and all the functions above are smooth maps, the fiber bundle is called a smooth fiber bundle.
It is possible to induce bundles in the following manner.
Let $\pi : E \longrightarrow B$ be a fiber bundle with fiber F and let $f : B' \longrightarrow B$ be a continuous map. Then, a fiber bundle over B' can be deduced as follows:

$$f^*E = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\} \subseteq B' \times E$$

and equip it with the subspace topology and the projection map $\pi' : f^*E \longrightarrow B'$ defined as the projection onto the first factor:

$$\pi'(b', e) = b'$$

Defining f' so that the following diagram is commutative

$$\begin{tikzcd} f^*E \arrow[r, "f'"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

we have that (f^*E, B', π') is a fiber bundle so that the fibers on $b \in B$ correspond to the fibers on $f^{-1}(b)$.
An important concept on fiber bundles is the cross-section notion.

Definition 2.12. *A cross section of a bundle (E, B, π, F) is a map $s : B \longrightarrow E$ such that $\pi s = I_B$. In other words, a cross section is a map $s : B \longrightarrow E$ such that $s(b) \in \pi^{-1}(b)$, the fibre over b, for each $b \in B$.*

Let (E', B, π', F') be a subbundle of (E, B, π, F), and let s be a cross section of (E, B, π, F). Then s is a cross section of (E', B, π', F') if and only if $s(b) \in E'$ for each $b \in B$.
One of the main goals of studying cross sections is to account for the existence or non-existence of global sections. When there are some problems with constructing a global section, one says that there are some obstructions.

3. Bundle of pairs of matrices

3.1. Lie group actions

The simultaneous equivalence relation defined in (2.1), can be seen as the action of a Lie group $\mathfrak{G}$ over $\mathfrak{M}$ in the following manner:
Let us consider the following map:

$$\begin{aligned} \alpha : \mathfrak{G} \times \mathfrak{M} &\longrightarrow \mathfrak{M} \\ (P, T) &\longrightarrow PTP^{-1} = (PX_1P^{-1}, PX_2P^{-1}) \end{aligned}$$

that verifies

i) If $I \in \mathfrak{G}$ is the identity element, then $\alpha(I, T) = T$ for all $T \in \mathfrak{M}$.
ii) If P_1 and P_2 are in $\mathfrak{G}$, then $\alpha(P_1, \alpha(P_2, T)) = \alpha(P_1P_2, T)$ for all $T \in \mathfrak{M}$.
Indeed: $\alpha(P_1, \alpha(P_2, T)) = \alpha(P_1, P_2TP_2^{-1}) = P_1P_2TP_2^{-1}P_1^{-1} = (P_1P_2)T(P_1P_2)^{-1} = \alpha(P_1P_2, T)$

So, the map α defines an action of $\mathfrak{G}$ over the differentiable manifold $\mathfrak{M}$ that allows seeing the equivalent classes as differentiable manifolds providing a hard link between geometry and algebra.
Analogously we can define an action of $\mathfrak{G}$ over $\mathfrak{G} \times \mathfrak{M}$ in the following manner:

$$\begin{aligned} \beta : \mathfrak{G} \times (\mathfrak{G} \times \mathfrak{M}) &\longrightarrow \mathfrak{G} \times \mathfrak{M} \\ (Q, (P, T)) &\longrightarrow (PQ^{-1}, \alpha(Q^{-1}, T)). \end{aligned}$$

It will be denoted by β_Q the restriction of β to the set $\{Q\} \times (\mathfrak{G} \times \mathfrak{M})$ and by $\beta_{(P,T)}$ the restriction of β to the set $\mathfrak{G} \times \{(P, T)\}$.

Proposition 3.1. *The $\mathfrak{G}$-action β is free, transitive and its orbits are diffeomorphic to $\mathfrak{G}$*

Proof. Suppose that $\beta(Q, (P, T)) = (P, T)$, so

$$\begin{aligned} \beta(Q, (P, T)) &= (PQ^{-1}, \alpha(Q^{-1}, T)) \\ &= (PQ^{-1}, Q^{-1}TQ) \\ &= (P, T) \end{aligned}$$

then, $PQ^{-1} = P$ and $Q^{-1}TQ = T$ and $Q = I$.

$$\begin{aligned}\beta(R,\beta(Q,(P,T))) &= \beta(R,(PQ^{-1},\alpha(Q,T)))\\ &= \beta(R,(PQ^{-1},QTQ^{-1}))\\ &= (PQ^{-1}R^{-1},\alpha(R,QTQ^{-1}))\\ &= (PQ^{-1}R^{-1},RQTQ^{-1}R^{-1})\\ &= (P(RQ)^{-1},\alpha(RQ,T))\\ &= \beta(RQ,(P,T)),\end{aligned}$$

Let us denote by $\mathscr{O}(P,T)$ the orbit of T under $\mathfrak{G}$-action $\mathscr{O}(P,T)=\{(\overline{P},\overline{T})=\beta(Q,(P,T)),\forall Q\in\mathfrak{G}\}$

$$\begin{aligned}\varphi:\mathfrak{G} &\longrightarrow \mathscr{O}(P,T)\\ Q &\longrightarrow (\overline{P},\overline{T})=\beta(Q,(P,T))\end{aligned}$$

φ is a diffeomorphism:
If $\varphi(Q)=\varphi(\overline{Q})$, then $PQ=P\overline{Q}$ consequently $Q=\overline{Q}$
And, for $(\overline{P},\overline{T})\in\mathscr{O}(P,T)$, there exists $Q\in\mathfrak{G}$ with $(\overline{P},\overline{T})=(PQ^{-1},QTQ^{-1})$, so $\varphi(Q)=(\overline{P},\overline{T})$. □

The set $\mathfrak{M}$ is identified as the set of orbits class $\mathfrak{G}\times\mathfrak{M}/\beta$.

Proposition 3.2. *There exists a bijection between $\mathfrak{M}$ and $\mathfrak{G}\times\mathfrak{M}/\beta$.*

Proof. We define f as

$$\begin{aligned}\mathfrak{G}\times\mathfrak{M}/\beta &\longrightarrow \mathfrak{M}\\ (P,T)\circ\mathfrak{G} &\longrightarrow T'\end{aligned}$$

where T' is in such a way that there exists $Q\in\mathfrak{G}$ such that $\beta(Q,(P,T))=(I,T')$

1) It suffices to take $Q=P$ to obtain $T'=P^{-1}TP$.
2) f is well-defined because of unicity of T':
   Let $(I,T')\sim(I,T'')$, then, there exists Q such that

$$\beta(Q,(I,T'))=(IQ^{-1},\alpha(Q^{-1},T'))=(I,T'').$$

   So, $IQ^{-1}=I$ and $Q^{-1}=I=Q$ and $IQ^{-1}\alpha(Q^{-1},T')=\alpha(I,T')=T'$.
3) f is bijective:
   If $f(I,T')\circ\mathfrak{G}=f(I,T'')\circ\mathfrak{G}$, then $T'=T''$ and $f(I,T')\circ\mathfrak{G}=f(I,T'')\circ\mathfrak{G}$, so f is injective.
   And, clearly, for all $T\in\mathfrak{M}$, $f(I,T)\circ\mathfrak{G}=T$ and f is surjective.

□

Proposition 3.3. *The $\mathfrak{G}$-action preserves the fibers $F_T=\alpha^{-1}(T)$ of $\alpha:\mathfrak{G}\times\mathfrak{M}\longrightarrow\mathfrak{M}$.*

Proof. Let $(P,\overline{T})\in\alpha^{-1}(T)$, then

$$\alpha(Q,(P,\overline{T}))=(PQ^{-1},QTQ^{-1})=PQ^{-1}Q\overline{T}Q^{-1}QP^{-1}=P\overline{T}P^{-1}=T.$$

So, $(PQ^{-1},QTQ^{-1})\in\alpha^{-1}(T)$. □

From propositions 3.1 and 3.3 we can deduce the following result.

Proposition 3.4. $(\mathfrak{G}\times\mathfrak{M},\mathfrak{M},\alpha,\mathfrak{G})$ *is a principal fiber bundle.*

Clearly, we observe that F_T is diffeomorphic to $\mathfrak{G}$:

$$\begin{aligned}\psi:F_T &\longrightarrow \mathfrak{G}\\ (Q,\overline{T}) &\longrightarrow Q\end{aligned}$$

If $\psi(Q,\overline{T})=(\overline{Q},\overline{\overline{Q}})$, then $Q=\overline{Q}$ and $Q\overline{T}Q^{-1}=\overline{Q}\overline{\overline{T}}\overline{Q}^{-1}=Q\overline{\overline{T}}Q^{-1}$, so $\overline{T}=\overline{\overline{T}}$ and the map ψ is injective.
On the other hand, for all $Q\in\mathfrak{G}$, there exists $(Q,Q^{-1}TQ)\in F_T$ such that $\psi(Q,Q^{-1}TQ)=T$, so, the map ψ is surjective.

4. Connections and curvature

A connection is a mathematical object defined over a differentiable manifold that allows the local geometry around a point to be related to local geometry around another point in the manifold. The connection is an object that shows us how to derive local sections and thus compare the fibers on different points of the base space [4].

Curvature is useful to obtain characteristic classes that are global invariants that measure the deviation of the local product structure from a global product structure. The theory of characteristic classes generalizes the idea of obstructions to construct cross-sections of fiber bundles.

Let us use the notation $T_I\mathfrak{G}$ for the tangent space to the manifold $\mathfrak{G}$ at the unit element I. Since $\mathfrak{G}$ is an open subset of $M_n(\mathbb{R})$, we have that $T_I(\mathfrak{G}) = M_n(\mathbb{R})$, and, since $\mathfrak{M}$ is a linear space, $T_T(\mathfrak{M}) = \mathfrak{M}$, then $T_{(I,T)}(\mathfrak{G}\times\mathfrak{M}) = M_n \times \mathfrak{M}$.

The action β of $\mathfrak{G}$ over $\mathfrak{G}\times\mathfrak{M}$ permits us to construct a vertical subspace $T_{(P,T)}\mathscr{O}(P,T) \subset T(\mathfrak{G}\times\mathfrak{M})$.

$$T_{(P,T)}\mathscr{O}(P,T) = \mathscr{I}mg\, d\beta_{(P,T)} = \{(-PQ,[T,Q]) \mid \forall Q \in T\mathfrak{G}\}$$

where $[T,Q] = ([X_1,Q],[X_2,Q])$.

(To describe $\mathscr{I}mg\, d\beta_{(P,T)}$ it suffices to compute the linear approximation of $\beta_{(P,T)}(I+\varepsilon Q)^{-1} \sim \beta_{(P,T)}(I-\varepsilon Q)$).

The subspace $T_{(P,T)}\mathscr{O}(P,T)$ is generated by $d\beta(A_{ij})$ with $\{A_{ij}\}$ a basis for $T\mathfrak{G} = M_n(\mathbb{R})$.

Consider the Euclidean scalar product in the space $T_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ defined as:

$$\langle (P,T_1),(Q,T_2\rangle = \langle (P,(X_1,X_2)),(Q,(Y_1,Y_2))\rangle = \operatorname{tr} P\overline{Q}^t + \operatorname{tr} X_1\overline{Y}_1^t + \operatorname{tr} X_2\overline{Y}_2^t = \operatorname{tr} P\overline{Q}^t + \operatorname{tr} T_1\overline{T}_2^t.$$

An orthogonal element $(X,Y) \in T_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ to $T_{(P,T)}\mathscr{O}(P,T)$ is a solution of the equation:

$$\langle (-PQ,[T,Q]),(X,Y)\rangle = \operatorname{tr}(-PQ\overline{X}^t) + \operatorname{tr}([T,Q]\overline{Y}^t) = 0.$$

It is possible to construct a horizontal subspace

$$\begin{aligned} T_{(P,T)}\mathscr{O}(P,T)^{\perp} &= \{(A,B) \in T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \mid -\overline{A}^t P + \overline{B}^t T - T\overline{B}^t = 0\} \\ &= \{(A,B) \in T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \mid -\overline{A}^t P + [\overline{B}^t, T] = 0\}, \end{aligned}$$

where $[\overline{B}^t, T]$ denotes $[\overline{B}_1^t, X_1] + [\overline{B}_2^t, X_2]$.

Definition 4.1. *Given a principal bundle $(\mathfrak{G}\times\mathfrak{M},\mathfrak{M},\alpha,\mathfrak{G})$, a differentiable distribution $\mathscr{H}$ of fields over $\mathfrak{G}\times\mathfrak{M}$ such that for each point $(P,T)\in\mathfrak{G}\times\mathfrak{M}$, the subspace $H_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \subset T_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ is called connection if it verifies:*

a) $T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) = V_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \oplus H_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$

b) For each $(P,T)\in\mathfrak{G}\times\mathfrak{M}$ and for each $Q\in\mathfrak{G}$, for the translation $\beta_Q(P,T) = (PQ^{-1},Q^{-1}TQ)$, the space $H_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ fulfills $H_{\beta_Q(P,T)}(\mathfrak{G}\times\mathfrak{M}) = H_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$

So, the connections allow us to decompose the vectors into a vertical part $V_u(E)$ and a horizontal part $H_u(E)$, which we will call the vertical and horizontal subspace, respectively, of $T_u(E)$ concerning the connection $\mathscr{H}$.

Proposition 4.2. *The subgroup $T_{(P,(X_1,X_2))}\mathscr{O}(P,(X_1,X_2))^{\perp}$ verifies the conditions of definition 4.1.*

Proof. Let $(C,D) \in T_{\beta_Q(P,T)}\mathscr{O}(\beta_Q(P,T))^{\perp}$, then,

$$-\overline{C}^t PQ + \overline{D}_1^t Q^{-1}X_1Q - Q^{-1}X_1Q\overline{D}_1^t + \overline{D}_2^t Q^{-1}X_2Q - Q^{-1}X_2Q\overline{D}_2^t = 0$$

or, equivalently:

$$-Q\overline{C}^t P + Q\overline{D}_1^t Q^{-1}X_1 - X_1Q\overline{D}_1^t Q^{-1} + Q\overline{D}_2^t Q^{-1}X_2 - X_2Q\overline{D}_2^t Q^{-1} = 0$$

Then, setting $\overline{E}^t = Q\overline{C}^t$, $\overline{F}_1^t = Q\overline{D}_1^t Q^{-1}$ and $\overline{F}_2^t = Q\overline{D}_2^t Q^{-1}$, we have that equivalently $(E,F) \in T_{(P,T)}\mathscr{O}(P,T)^{\perp}$. □

The bijectivity of the map

$$\begin{aligned} d\beta_{(P,T)}(\mathfrak{G}) &\longrightarrow T_{(P,T)}\mathscr{O}(P,T) \\ Q &\longrightarrow (-PQ, X_1Q - QX_1, X_2Q - QX_2) \end{aligned}$$

permit us to define a 1-form ω over $\mathfrak{G}\times\mathfrak{M}$ with values in the Lie algebra $\mathfrak{G}$ as follows: Given $(A,B) \in T_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ there exists a unique element $Q\in T\mathfrak{G}$ such that the vertical component of (A,B) is $A_v = d\beta_{(P,T)}Q$, thus we define $\omega(A,B) = Q$. It is clear that $\omega(A,B) = 0$ if and only if (A,B) is horizontal.

From this 1-form, it is possible to build a 2-form Ω in the following maner:

$$\begin{aligned} \Omega : T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \times T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) &\longrightarrow T_I\mathfrak{G} \\ (X,Y) = ((P_1,T_1),(P_2,T_2)) &\longrightarrow \Omega((P_1,T_1),(P_2,T_2)) = \Omega(X,Y) \end{aligned}$$

verifying:

$$\Omega(X,Y) = d\omega(X,Y) + \frac{1}{2}[\omega(X),\omega(Y)] \tag{4.1}$$

and (4.1) is called curvature of the connection.

5. Conclusion

In this work, we build bundles to obtain algebraic objects providing more geometric information about the space of pairs of matrices. With them, we consider an operator called connection, and we define the curvature associated with it. These are ingredients to obtain invariants that measure in a certain way how the local product structure of the bundle separates from a global product structure.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Universal Journal of Mathematics and Applications, 4 (4) (2021) 154-163
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.978875

Multi-Parametric Families of Real and Non Singular Solutions of the Kadomtsev-Petviasvili I Equation

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Article Info

Keywords: *Kadomtsev-Petviashvili equation, rational solutions*
2010 AMS: *33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd*
Received: *4 August 2021*
Accepted: *16 December 2021*
Available online: *30 December 2021*

Abstract

Multi-parametric solutions to the Kadomtsev-Petviashvili equation (KPI) in terms of Fredholm determinants are constructed in function of exponentials. A representation of these solutions as a quotient of wronskians of order $2N$ in terms of trigonometric functions is deduced. All these solutions depend on $2N-1$ real parameters.
A third representation in terms of a quotient of two real polynomials depending on $2N-2$ real parameters is given; the numerator is a polynomial of degree $2N(N+1)-2$ in x, y and t and the denominator is a polynomial of degree $2N(N+1)$ in x, y and t. The maximum absolute value is equal to $2(2N+1)^2-2$.
We explicitly construct the expressions for the first third orders and we study the patterns of their absolute value in the plane (x,y) and their evolution according to time and parameters.
It is relevant to emphasize that all these families of solutions are real and non singular.

1. Introduction

We consider the Kadomtsev-Petviashvili I equation (KPI)

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \tag{1.1}$$

where subscripts x, y and t denote partial derivatives.
This equation was introduced by Kadomtsev and Petviashvili [1] in 1970. It is considered as a model in hydrodynamic for surface and internal water waves [2] or in nonlinear optics [3]. Dryuma showed in 1974 how the KP equation could be written in Lax form [4].
Manakov, Zakharov, Bordag and Matveev first constructed rational solutions in 1977 [5] and two month later Krichever published other solutions [6].
In the frame of algebraic geometry, Krichever constructed for the first time in 1976 [7] the solutions to KPI equation in terms of Riemann theta functions and a little later, it was done by Dubrovin [8].
Others rational solutions of the KPI equation were obtained. For example, one can quote of the studies of Krichever in 1978 [9], Satsuma and Ablowitz in 1979 [10], Matveev in 1979 [11], Freeman and Nimmo in 1983 [12, 13], Matveev in 1987 [14], Pelinovsky and Stepanyants in 1993 [15], Pelinovsky in 1994 [16], Ablowitz, Villarroel, Chakravarty, Trubatch [17–19] in 1997-2000, Biondini and Kodama [20–22] in 2003-2007.

We give in the following three types of representations of the solutions to the KPI equation : first, in terms of Fredholm determinants of order $2N$ depending on $2N-1$ real parameters in function of exponentials, then in terms of wronskians of order $2N$ with $2N-1$ real parameters in function of some trigonometric functions.
In a third representation, real rational solutions of order N depending on $2N-2$ real parameters are constructed and they can be written as a ratio of two polynomials; the numerator is a polynomial in x, y and t of degree $2N(N+1)-2$ and the denominator a polynomial in x, y and t of degree $2N(N+1)$.
So we get rational real and non singular solutions to the KPI equation at each order N depending on $2N-2$ real parameters. We present explicit rational solutions and the representations of their absolute value in the plane of the coordinates (x,y) according to the $2N-2$ real parameters a_i and b_i $(1 \leq i \leq N-1)$ and time t for the first three orders.

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2. Families of solutions of order N depending on $2N-1$ real parameters in terms of Fredholm determinants to the KPI equation

We define the numbers λ_ν, κ_ν, δ_ν, γ_ν, $x_{r,\nu}$, e_ν depending on a real number ε by

$$\begin{array}{l}
\lambda_j = 1-2\varepsilon^2 j^2,\ \lambda_{N+j} = -\lambda_j, \quad \kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j = \kappa_j\lambda_j, \quad \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}},\\
x_{r,j} = (r-1)\ln\frac{\gamma_j - i}{\gamma_j + i},\ r = 1,3, \quad \tau_j = -12i\lambda_j^2\sqrt{1-\lambda_j^2} - 4i(1-\lambda_j^2)\sqrt{1-\lambda_j^2},\\
\kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = \gamma_j^{-1}, \quad x_{r,N+j} = -x_{r,j},, \quad \tau_{N+j} = \tau_j\\
e_j = 2i\left(\sum_{k=1}^{1/2M-1} a_k (je)^{2k+1} - i\sum_{k=1}^{1/2M-1} b_K (je)^{2k+1}\right),\\
e_{N+j} = 2i\left(\sum_{k=1}^{1/2M-1} a_k (je)^{2k+1} + i\sum_{k=1}^{1/2M-1} b_k (je)^{2k+1}\right),\\
a_j, b_j \text{ real numbers }, \quad \varepsilon_j = 1, \quad \varepsilon_{N+j} = 0, \quad \varphi \text{ a real number}\\
1 \le j \le N.
\end{array} \tag{2.1}$$

Then we have the following statement:

Theorem 2.1. *Let $v(x,y,t)$ be the expression defined by*

$$v(x,y,t) = \frac{\det(I + D_3(x,y,t))}{\det(I + D_1(x,y,t))}, \tag{2.2}$$

with I the unit matrix and $D_r = (d_{jk}^{(r)})_{1\le j,k\le 2N}$ the matrix

$$d_{\nu\mu}^{(r)} = (-1)^{\varepsilon_\nu}\prod_{\eta\neq\mu}\left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu}\right)\exp(i\kappa_\nu x - 2\delta_\nu y + \tau_\nu t + x_{r,\nu} + e_\nu). \tag{2.3}$$

Then the function defined by

$$u(x,y,t) = -2(|v(x+3t,y,t)|^2 - 1) \tag{2.4}$$

is a solution to the KPI equation (1.1) depending on $2N-1$ real parameters a_k, b_k, $1 \le k \le N-1$ and ε.

Proof. We have proven in [23] that the function w defined by (2.5)

$$w(x,y) = \frac{\det(I + D_3(x,y,0))}{\det(I + D_1(x,y,0))}\exp(2iy - i\varphi) \tag{2.5}$$

is a solution to the nonlinear Schrödinger equation (2.6)

$$iw_y + w_{xx} + 2|w|^2 w = 0. \tag{2.6}$$

It can then be similarly proven that the function v defined by

$$\tilde{v}(x,y,t) = v(x,y,t)\times\exp(2iy - i\varphi) = \frac{\det(I + D_3(x,y,t))}{\det(I + D_1(x,y,t))}\times\exp(2iy - i\varphi)$$

is a solution to the NLS equation (2.6) by considering t as a parameter. We can then deduce that the function u defined by (3)

$$u(x,t) = -2(|v(x+3t,y,t)|^2 - 1) = -2(|\tilde{v}(x+3t,y,t)|^2 - 1)$$

is a solution to the KPI equation, which proves the result. ☐

3. Families of solutions of order N depending on $2N-1$ real parameters in terms of wronskians to the KPI equation

We denote $W_r(w)$ the wronskian of the functions $\phi_{r,1}, \ldots, \phi_{r,2N}$ defined by

$$W_r(w) = \det[(\partial_w^{\mu-1}\phi_{r,\nu})_{\nu,\mu\in[1,\ldots,2N]}]. \tag{3.1}$$

We consider the matrix $D_r = (d_{\nu\mu}^{(r)})_{\nu,\mu\in[1,\ldots,2N]}$ defined in (2.3).
We consider the real parameters a_k, b_k $1 \le k \le N-1$ and ε, and κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν defined in the previous section.
Then we have the following statement

Theorem 3.1. *Let $\Phi_{r,\nu}$ be the functions defined by*

$$\begin{aligned}
\phi_{r,\nu}(x,y,t,w) &= \sin\left(\frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{r,\nu}}{2} - i\frac{\tau_\nu}{2}t + \gamma_\nu w - i\frac{e_\nu}{2}\right), \quad 1 \leq \nu \leq N, \\
\phi_{r,\nu}(x,y,t,w) &= \cos\left(\frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{r,\nu}}{2} - i\frac{\tau_\nu}{2}t + \gamma_\nu w - i\frac{e_\nu}{2}\right), \quad N+1 \leq \nu \leq 2N, \quad r = 1,3,
\end{aligned}$$

Let v be the expression defined by

$$v(x,y,t) = \frac{W_3(\phi_{3,1},\ldots,\phi_{3,2N})(x,y,t,0)}{W_1(\phi_{1,1},\ldots,\phi_{1,2N})(x,y,t,0)} \tag{3.2}$$

Then the function defined by

$$u(x,y,t) = -2(|v(x+3t,y,t)|^2 - 1) \tag{3.3}$$

is a solution to the KPI equation (1.1) depending on $2N-1$ real parameters a_k, b_k, $1 \leq k \leq N-1$ and ε.

Proof. We have proven in [24] that the function v defined by (3.4)

$$w(x,y) = \frac{W_3(\phi_{3,1},\ldots,\phi_{3,2N})(x,y,0,0)}{W_1(\phi_{1,1},\ldots,\phi_{1,2N})(x,y,0,0)}\exp(2iy - i\varphi) \tag{3.4}$$

is a solution to the nonlinear Schrödinger equation (3.5)

$$iw_y + w_{xx} + 2|w|^2 w = 0. \tag{3.5}$$

We can similarly prove that the function v defined by

$$\tilde{v}(x,y,t) = v(x,y,t) \times \exp(2iy - i\varphi)$$

is a solution of the NLS equation by considering t as a parameter. We can then deduce that the function u defined by

$$u(x,t) = -2(|v(x+3t,y,t)|^2 - 1) = -2(|\tilde{v}(x+3t,y,t)|^2 - 1)$$

is a solution to the KPI equation which proves the result. □

4. Real and non singular rational solutions to the KPI equation of order N depending on $2N-2$ real parameters

We construct in this section rational solutions to the KPI equation as a quotient of two determinants.
We define functions of the following arguments:

$$X_\nu = \frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{3,\nu}}{2} - i\frac{\tau_\nu}{2}t - i\frac{e_\nu}{2}, \tag{4.1}$$

$$Y_\nu = \frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{1,\nu}}{2} - i\frac{\tau_\nu}{2}t - i\frac{e_\nu}{2}, \tag{4.2}$$

for $1 \leq \nu \leq 2N$, with κ_ν, δ_ν, $x_{r,\nu}$ defined in the first section.
We consider the following functions:

$$\begin{array}{l}
\varphi_{4j+1,k} = \gamma_k^{4j-1}\sin X_k, \quad \varphi_{4j+2,k} = \gamma_k^{4j}\cos X_k, \\
\varphi_{4j+3,k} = -\gamma_k^{4j+1}\sin X_k, \quad \varphi_{4j+4,k} = -\gamma_k^{4j+2}\cos X_k, \\
\varphi_{4j+1,N+k} = \gamma_k^{2N-4j-2}\cos X_{N+k}, \quad \varphi_{4j+2,N+k} = -\gamma_k^{2N-4j-3}\sin X_{N+k}, \\
\varphi_{4j+3,N+k} = -\gamma_k^{2N-4j-4}\cos X_{N+k}, \quad \varphi_{4j+4,N+k} = \gamma_k^{2N-4j-5}\sin X_{N+k}, \\
\psi_{4j+1,k} = \gamma_k^{4j-1}\sin Y_k, \quad \psi_{4j+2,k} = \gamma_k^{4j}\cos Y_k, \\
\psi_{4j+3,k} = -\gamma_k^{4j+1}\sin Y_k, \quad \psi_{4j+4,k} = -\gamma_k^{4j+2}\cos Y_k, \\
\psi_{4j+1,N+k} = \gamma_k^{2N-4j-2}\cos Y_{N+k}, \quad \psi_{4j+2,N+k} = -\gamma_k^{2N-4j-3}\sin Y_{N+k}, \\
\psi_{4j+3,N+k} = -\gamma_k^{2N-4j-4}\cos Y_{N+k}, \quad \psi_{4j+4,N+k} = \gamma_k^{2N-4j-5}\sin Y_{N+k}, \\
1 \leq k \leq N
\end{array} \tag{4.3}$$

Then we get the following result

Theorem 4.1. *Let v be the expression defined by*

$$v(x,y,t) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} \tag{4.4}$$

where

$$
\begin{aligned}
&n_{j1} = \varphi_{j,1}(x,y,t,0), \quad n_{jk} = \frac{\partial^{2k-2}\varphi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&n_{jN+1} = \varphi_{j,N+1}(x,y,t,0), \quad n_{jN+k} = \frac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&d_{j1} = \psi_{j,1}(x,y,t,0), \quad d_{jk} = \frac{\partial^{2k-2}\psi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&d_{jN+1} = \psi_{j,N+1}(x,y,t,0), \quad d_{jN+k} = \frac{\partial^{2k-2}\psi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&2 \le k \le N, \quad 1 \le j \le 2N,
\end{aligned}
\tag{4.5}
$$

the functions φ and ψ being defined in (4.3).
Then the function defined by

$$u(x,t) = -2(|v(x+3t,y,t)|^2 - 1) \tag{4.6}$$

is a solution to the KPI equation (1.1) depending on $2N-2$ parameters a_k, b_k, $1 \le k \le N-1$.

Proof. It is still a consequence of our previous works. Precisely, we have proven in [25] that the function w defined by

$$w(x,y) = \frac{\det(n_{jk})_{j,k\in[1,2N]_{t=0}}}{\det(d_{jk})_{j,k\in[1,2N]_{t=0}}} \times \exp(2iy - i\varphi) \tag{4.7}$$

is a solution to the nonlinear Schrödinger equation (4.8)

$$iw_y + w_{xx} + 2|w|^2 w = 0. \tag{4.8}$$

We can prove in the same way that the function v defined by

$$\tilde{v}(x,y,t) = v(x,y,t) \times \exp(2iy - i\varphi)$$

is a solution of the NLS equation by considering t as a parameter. Then, we can deduce that the function u defined by

$$u(x,t) = -2(|v(x+3t,y,t)|^2 - 1) = -2(|\tilde{v}(x+3t,y,t)|^2 - 1)$$

is a solution to the KPI equation which proves the result. □

5. The structure of the solutions to the KPI equation

The structure of the rational solutions to the KPI equation is given by the following result

Theorem 5.1. *Let u the function defined by*

$$u(x,t) = -2(|v(x+3t,y,t)|^2 - 1) = \frac{n(x,y,t)}{d(x,y,t)},$$

with

$$v(x,y,t) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})}$$

$$
\begin{aligned}
&n_{j1} = \varphi_{j,1}(x,y,t,0), \quad n_{jk} = \frac{\partial^{2k-2}\varphi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&n_{jN+1} = \varphi_{j,N+1}(x,y,t,0), \quad n_{jN+k} = \frac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&d_{j1} = \psi_{j,1}(x,y,t,0), \quad d_{jk} = \frac{\partial^{2k-2}\psi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&d_{jN+1} = \psi_{j,N+1}(x,y,t,0), \quad d_{jN+k} = \frac{\partial^{2k-2}\psi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&2 \le k \le N, \quad 1 \le j \le 2N,
\end{aligned}
$$

Then the function v is a rational solution to the KPI equation (1.1) quotient of two polynomials $n(x,y,t)$ and $d(x,y,t)$ depending on $2N-2$ real parameters a_j and b_j, $1 \le j \le N-1$.
n is a polynomial of degree $2N(N+1)-2$ in x, y and t.
d is a polynomial of degree $2N(N+1)$ in x, y and t.

Proof. It is already proven in the previous section that this function is a solution to the KPI equation.
The proof of the structure of the solution is similar to this given in [26]. The difference in this present case is due to the reduction of the fraction which cancel the terms in $x^{2N(N+1)}$, $y^{2N(N+1)}$, $t^{2N(N+1)}$ in the numerator, these terms having the same maximal power in numerator and denominator, and the fact that the elevation by the power 2 makes that the succeeding terms in x, y and t are to the power $2N(N+1)-2$. □

Theorem 5.2. *Let u the function defined by*

$$u(x,t) = -2(|v(x+3t,y,t)|^2 - 1) = \frac{n(x,y,t)}{d(x,y,t)},$$

with

$$v(x,y,t) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})}$$

$$\begin{array}{l} n_{j1} = \varphi_{j,1}(x,y,t,0), \quad n_{jk} = \dfrac{\partial^{2k-2}\varphi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0), \\ n_{jN+1} = \varphi_{j,N+1}(x,y,t,0), \quad n_{jN+k} = \dfrac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0), \\ d_{j1} = \psi_{j,1}(x,y,t,0), \quad d_{jk} = \dfrac{\partial^{2k-2}\psi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0), \\ d_{jN+1} = \psi_{j,N+1}(x,y,t,0), \quad d_{jN+k} = \dfrac{\partial^{2k-2}\psi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0), \\ 2 \le k \le N, \quad 1 \le j \le 2N, \end{array}$$

Then the function v_0 defined by

$$v_0(x,y,t) = v(x,y,t)_{(a_j=b_j=0,\, 1\le j\le N-1)} \tag{5.1}$$

is the solution of order N solution to the KPI equation (1.1) whose highest amplitude in modulus is equal to $2(2N+1)^2-2$.

Proof. The proof of this result is similar to this given in [26]. We do not give more details. The reader can do by himself the rewriting of this proof. □

6. Explicit expressions and patterns of the rational solutions to the KPI equation in function of parameters and time

We have explicitly constructed rational solutions to the KPI equation of order N depending on $2N-2$ parameters for $1 \le N \le 3$.
In the following, we only give patterns of the modulus of the solutions in the plane (x,y) of coordinates in function of the parameters a_i, and b_i, for $1 \le i \le N-1$ for $2 \le N \le 3$, and time t.
We present the solutions using the following notations $X = 2x$, $Y = 4y$, $T = 2t$,

$$u_N(X,Y,T) = 1 - \frac{G_N(X,Y,T)}{Q_N(X,Y,T)}$$

with

$$\begin{array}{l} G_N(X,Y,T) = \sum_{k=0}^{2N(N+1)} g_k(Y,T)X^k, \\ Q_N(X,Y,T) = \sum_{k=0}^{2N(N+1)} q_k(Y,T)X^k. \end{array}$$

By construction, all these solutions constructed in this study are real. Moreover, we know from the study of the NLS equation that the solutions constructed by ourself were non singular. From the construction, the denominators of the solutions to the KPI equation being the square of those of the solutions of the NLS equation, we get the non singularity of all these families of solutions to the KPI equation.

6.1. Case $N = 1$

The polynomials Q_1 and G_1 are given by
$\mathbf{q}_4 = 1$, $\mathbf{q}_3 = -12T$, $\mathbf{q}_2 = 54T^2+2Y^2+2$, $\mathbf{q}_1 = -108T^3+(-12Y^2-12)T$, $\mathbf{q}_0 = 81T^4+Y^4+\left(18Y^2+18\right)T^2+2Y^2+1$
$\mathbf{g}_4 = 1$, $\mathbf{g}_3 = -12T$, $\mathbf{g}_2 = 54T^2+2Y^2-14$, $\mathbf{g}_1 = -108T^3+(-12Y^2+84)T$, $\mathbf{g}_0 = 81T^4+Y^4+\left(18Y^2-126\right)T^2+18Y^2+17$
This type of solution to the KPI equation is different from our previous works.
In our previous works [26–31], we constructed solution of order 1 to KPI equation and got
$\tilde{v}_1(X,Y,T)$
$= -2\,\frac{9-6X^2+72XT+X^4+1296T^4+216X^2T^2-216T^2+10Y^2+Y^4-24XTY^2-24X^3T+2X^2Y^2-864XT^3+72T^2Y^2}{(X^2-12XT+36T^2+Y^2+1)^2}.$
The solution of order 1 obtained in this paper can be rewritten as

$$v_1(X,Y,T) = 16\,\frac{-1+X^2-6XT+9T^2-Y^2}{X^2-6XT+9T^2+Y^2+1)^2} = 16\,\frac{-1+(X-3T)^2-Y^2}{(1+(X-3T)^2+Y^2)^2}.$$

It can be easily seen in this example that these two solutions are different and non singular. Moreover, we can verify that the maximum of the absolute value of v_1 is equal to $2(2N+1)^2-2 = 16$ obtained when $X = Y = T = 0$.

In the case $N = 1$, one obtains a peak which the height decreases very quickly as t increases.

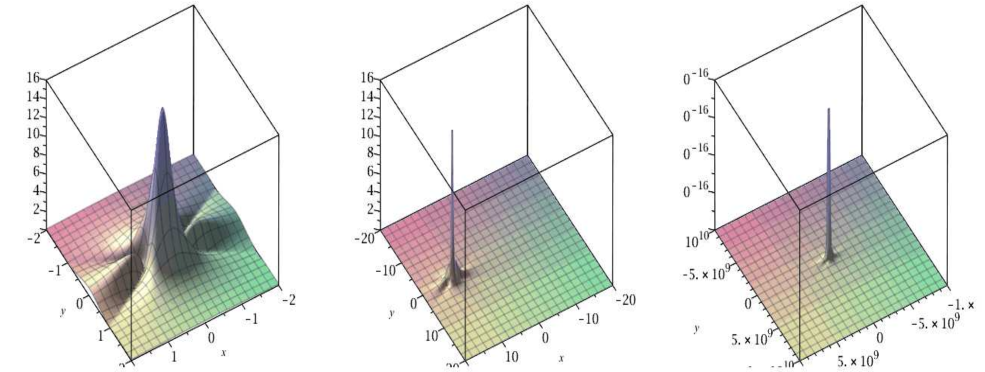

Figure 1. Solution of order 1 to the KPI equation, on the left for $t = 0$; in the center for $t = 4$; on the right for $t = 10^8$.

6.2. Case $N = 2$

In the case $N = 2$, the polynomials G_2 and G_2 more complex are given by

$\mathbf{q}_{12} =,1$ $\mathbf{f}_{11} = -36T$, $\mathbf{q}_{10} = 594T^2+6Y^2+6$, $\mathbf{q}_9 = -5940T^3+(-180Y^2+12)T-12a_1$, $\mathbf{q}_8 = 40095T^4+6Y^4+(2430Y^2-2754)T^2+324Ta_1-36Y^2+36Yb_1+(3Y^2+3)^2+54$, $\mathbf{q}_7 = -192456T^5+(-19440Y^2+42768)T^3-3888T^2a_1+36Y^2a_1+(-144Y^4+288Y^2-864Yb_1-1872+2(-36Y^2+60)(3Y^2+3))T+36a_1-12(3Y^2+3)a_1$, $\mathbf{q}_6 = 673596T^6+2Y^6+(102060Y^2-333396)T^4+27216T^3a_1+54Y^4-12Y^3b_1+(1512Y^4+3024Y^2+9072Yb_1+30312+2(162Y^2-702)(3Y^2+3)+(-36Y^2+60)^2)T^2+198Y^2-108Yb_1+54a_1^2+18b_1^2+(-756Y^2a_1-1332a_1-12(-36Y^2+60)a_1+108(3Y^2+3)a_1)T+2(3Y^4-18Y^2+18Yb_1+27)(3Y^2+3)+18$, $\mathbf{q}_5 = -1732104T^7+(-367416Y^2+1592136)T^5-122472T^4a_1+(-9072Y^4-54432Y^2-54432Yb_1-273456+2(162Y^2-702)(-36Y^2+60)+2(-324Y^2+2268)(3Y^2+3))T^3+(6804Y^2a_1+17172a_1-324(3Y^2+3)a_1-12(162Y^2-702)a_1+108(-36Y^2+60)a_1)T^2+(-36Y^6-972Y^4+216Y^3b_1-3564Y^2+1944Yb_1-972a_1^2-324b_1^2+2(-18Y^4-180Y^2-108Yb_1-450)(3Y^2+3)-324+2(3Y^4-18Y^2+18Yb_1+27)(-36Y^2+60))T-12(3Y^4-18Y^2+18Yb_1+27)a_1+2(18Y^2a_1+18a_1)(3Y^2+3)$, $\mathbf{q}_4 = 3247695T^8+(918540Y^2-4960116)T^6+367416T^5a_1+(34020Y^4+340200Y^2+204120Yb_1+1472580+(162Y^2-702)^2+2(243Y^2-2349)(3Y^2+3)+2(-324Y^2+2268)(-36Y^2+60))T^4+(-34020Y^2a_1-12(-324Y^2+2268)a_1-111780a_1-324(-36Y^2+60)a_1+108(162Y^2-702)a_1+324(3Y^2+3)a_1)T^3+(270Y^6+7290Y^4-1620Y^3b_1+26730Y^2-14580Yb_1+7290a_1^2+2430b_1^2+2430+2(27Y^4+702Y^2+162Yb_1+3411)(3Y^2+3)+2(-18Y^4-180Y^2-108Yb_1-450)(-36Y^2+60)+2(3Y^4-18Y^2+18Yb_1+27)(162Y^2-702))T^2+(-12(-18Y^4-180Y^2-108Yb_1-450)a_1+108(3Y^4-18Y^2+18Yb_1+27)a_1+2(-54Y^2a_1-342a_1)(3Y^2+3)+2(18Y^2a_1+18a_1)(-36Y^2+60))T-12(18Y^2a_1+18a_1)a_1+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)(3Y^2+3)+(3Y^4-18Y^2+18Yb_1+27)^2$, $\mathbf{q}_3 = -4330260T^9+(-1574640Y^2+10182672)T^7-734832T^6a_1+(-81648Y^4-1143072Y^2-489888Yb_1-4856112+2(243Y^2-2349)(-36Y^2+60)+2(-324Y^2+2268)(162Y^2-702))T^5+(102060Y^2a_1+413100a_1-324(162Y^2-702)a_1+108(-324Y^2+2268)a_1-12(243Y^2-2349)a_1+324(-36Y^2+60)a_1)T^4+(-1080Y^6-29160Y^4+6480Y^3b_1-106920Y^2+58320Yb_1-29160a_1^2-9720b_1^2-9720+2(27Y^4+702Y^2+162Yb_1+3411)(-36Y^2+60)+2(-18Y^4-180Y^2-108Yb_1-450)(162Y^2-702)+2(-324Y^2+2268)(3Y^4-18Y^2+18Yb_1+27))T^3+(-12(27Y^4+702Y^2+162Yb_1+3411)a_1+108(-18Y^4-180Y^2-108Yb_1-450)a_1-324(3Y^4-18Y^2+18Yb_1+27)a_1+2(-54Y^2a_1-342a_1)(-36Y^2+60)+2(18Y^2a_1+18a_1)(162Y^2-702))T^2+(-12(-54Y^2a_1-342a_1)a_1+108(18Y^2a_1+18a_1)a_1+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)(-36Y^2+60)+2(-18Y^4-180Y^2-108Yb_1-450)(3Y^4-18Y^2+18Yb_1+27))T-12(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)a_1+2(18Y^2a_1+18a_1)(3Y^4-18Y^2+18Yb_1+27)$, $\mathbf{q}_2 = 3897234T^{10}+(1771470Y^2-13345074)T^8+944784T^7a_1+(122472Y^4+2204496Y^2+734832Yb_1+(-324Y^2+2268)^2+9640296+2(243Y^2-2349)(162Y^2-702))T^6+(-183708Y^2a_1-883548a_1+324(162Y^2-702)a_1+108(243Y^2-2349)a_1-324(-324Y^2+2268)a_1)T^5+(2430Y^6+65610Y^4-14580Y^3b_1+240570Y^2-131220Yb_1+65610a_1^2+21870b_1^2+21870+2(27Y^4+702Y^2+162Yb_1+3411)(162Y^2-702)+2(243Y^2-2349)(3Y^4-18Y^2+18Yb_1+27)+2(-18Y^4-180Y^2-108Yb_1-450)(-324Y^2+2268))T^4+(324(3Y^4-18Y^2+18Yb_1+27)a_1-324(-18Y^4-180Y^2-108Yb_1-450)a_1+108(27Y^4+702Y^2+162Yb_1+3411)a_1+2(-54Y^2a_1-342a_1)(162Y^2-702)+2(18Y^2a_1+18a_1)(-324Y^2+2268))T^3+(-324(18Y^2a_1+18a_1)a_1+108(-54Y^2a_1-342a_1)a_1+(-18Y^4-180Y^2-108Yb_1-450)^2+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)(162Y^2-702)+2(27Y^4+702Y^2+162Yb_1+3411)(3Y^4-18Y^2+18Yb_1+27))T^2+(108(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)a_1+2(-54Y^2a_1-342a_1)(3Y^4-18Y^2+18Yb_1+27)+2(18Y^2a_1+18a_1)(-18Y^4-180Y^2-108Yb_1-450))T+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)(3Y^4-18Y^2+18Yb_1+27)+(18Y^2a_1+18a_1)^2$, $\mathbf{q}_1 = -2125764T^{11}+(-1180980Y^2+10156428)T^9-708588T^8a_1+(-104976Y^4-2309472Y^2-629856Yb_1+2(243Y^2-2349)(-324Y^2+2268)-10602576)T^7+(183708Y^2a_1+324(-324Y^2+2268)a_1-324(243Y^2-2349)a_1+1023516a_1)T^6+(-2916Y^6-78732Y^4+17496Y^3b_1-288684Y^2+157464Yb_1-78732a_1^2-26244b_1^2+2(243Y^2-2349)(-18Y^4-180Y^2-108Yb_1-450)+2(27Y^4+702Y^2+162Yb_1+3411)(-324Y^2+2268)-26244)T^5+(-324(27Y^4+702Y^2+162Yb_1+3411)a_1+324(-18Y^4-180Y^2-108Yb_1-450)a_1+2(-54Y^2a_1-342a_1)(-324Y^2+2268)+2(243Y^2-2349)(18Y^2a_1+18a_1))T^4+(-324(-54Y^2a_1-342a_1)a_1+324(18Y^2a_1+18a_1)a_1+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)(-324Y^2+2268)+2(27Y^4+702Y^2+162Yb_1+3411)(-18Y^4-180Y^2-108Yb_1-450))T^3+(-324(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)a_1+2(-54Y^2a_1-342a_1)(-18Y^4-180Y^2-108Yb_1-450)+2(27Y^4+702Y^2+162Yb_1+3411)(18Y^2a_1+18a_1))T^2+(2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)(-18Y^4-180Y^2-108Yb_1-450)+2(-54Y^2a_1-342a_1)(18Y^2a_1+18a_1))T+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)(18Y^2a_1+18a_1)$, $\mathbf{q}_0 = 531441T^{12}+Y^{12}+(354294Y^2-3424842)T^{10}+236196T^9a_1+54Y^{10}-12Y^9b_1+(98415Y^4-118098Y^2+236196Yb_1+10491039)T^8-1259712T^7a_1+927Y^8-432Y^7b_1+18Y^6a_1^2+54Y^6b_1^2+(14580Y^6+253692Y^4+69984Y^3b_1-1495908Y^2-839808Yb_1+39366a_1^2+13122b_1^2-16011756)T^6+5364Y^6-4104Y^5b_1+486Y^4a_1^2+1134Y^4b_1^2-108Y^3a_1^2b_1-108Y^3b_1^3+(-17496Y^4a_1+314928Y^2a_1+52488Ya_1b_1+2711880a_1)T^5+(1215Y^8+46332Y^6+5832Y^5b_1+598266Y^4+229392Y^3b_1-13122Y^2a_1^2+30618Y^2b_1^2+4328316Y^2+1358856Yb_1-153090a_1^2-42282b_1^2+11592639)T^4+10287Y^4-10800Y^3b_1+1782Y^2a_1^2+4698Y^2b_1^2-972Ya_1^2b_1-972Yb_1^3+81a_1^4+162a_1^2b_1^2+81b_1^4+(-2592Y^6a_1-85536Y^4a_1-19440Y^3a_1b_1-816480Y^2a_1-128304Ya_1b_1+$

$$2916a_1{}^3+2916a_1b_1{}^2-2330208a_1)T^3+(54Y^{10}+2862Y^8+50076Y^6-2592Y^5b_1+3402Y^4a_1{}^2-1458Y^4b_1{}^2+323676Y^4-84672Y^3b_1+49572Y^2a_1{}^2-4860Y^2b_1{}^2+2916Ya_1{}^2b_1+2916Yb_1{}^3+688014Y^2-365472Yb_1+178362a_1{}^2+61398b_1{}^2+61398)T^2+1782Y^2-972Yb_1+162a_1{}^2+162b_1{}^2+(-108Y^8a_1-3600Y^6a_1+648Y^5a_1b_1-29160Y^4a_1+9936Y^3a_1b_1-972Y^2a_1{}^3-972Y^2a_1b_1{}^2-68688Y^2a_1+36936Ya_1b_1-6156a_1{}^3-6156a_1b_1{}^2-6156a_1)T+81$$

$$\mathbf{g}_{12}=1,\quad \mathbf{g}_{12}=1,\quad \mathbf{g}_{11}=-36T,\quad \mathbf{g}_{10}=594T^2+6Y^2-42,\quad \mathbf{g}_9=-5940T^3+(-180Y^2+1452)T-12a_1,\quad \mathbf{g}_8=40095T^4+15Y^4+(2430Y^2-22194)T^2+324Ta_1-162Y^2+36Yb_1-81,\quad \mathbf{g}_7=-192456T^5+(-19440Y^2+198288)T^3-3888T^2a_1+(-360Y^4+3888Y^2-864Yb_1+1944)T,\quad \mathbf{g}_6=673596T^6+20Y^6+(102060Y^2-1149876)T^4+27216T^3a_1-132Y^4+96Y^3b_1+(3780Y^4-40824Y^2+9072Yb_1-6588)T^2-1728Ta_1-1476Y^2+1152Yb_1+54a_1{}^2+18b_1{}^2+1620,\quad \mathbf{g}_5=-1732104T^7+(-367416Y^2+4531464)T^5-122472T^4a_1+72Y^4a_1+(-22680Y^4+244944Y^2-54432Yb_1-126360)T^3+31104T^2a_1+3888Y^2a_1-216Ya_1b_1+(-360Y^6+1224Y^4-1728Y^3b_1-35640Y^2-17280Yb_1-972a_1{}^2-324b_1{}^2-81000)T+3240a_1,\ \mathit{quad}\mathbf{g}_4=3247695T^8+15Y^8+(918540Y^2-12308436)T^6+367416T^5a_1+156Y^6+72Y^5b_1+(85050Y^4-918540Y^2+204120Yb_1+1406970)T^4-233280T^3a_1+8442Y^4-3312Y^3b_1-162Y^2a_1{}^2+378Y^2b_1{}^2+(2700Y^6-540Y^4+12960Y^3b_1+692388Y^2+103680Yb_1+7290a_1{}^2+2430b_1{}^2+1286604)T^2+23004Y^2-13176Yb_1+1134a_1{}^2+2214b_1{}^2+(-1080Y^4a_1-53136Y^2a_1+3240Ya_1b_1-84888a_1)T+1647,\quad \mathbf{g}_3=-4330260T^9+(-1574640Y^2+22779792)T^7-734832T^6a_1+96Y^6a_1+(-204120Y^4+2204496Y^2-489888Yb_1-6362712)T^5+933120T^4a_1+1440Y^4a_1+720Y^3a_1b_1+(-10800Y^6-32400Y^4-51840Y^3b_1-4304016Y^2-311040Yb_1-29160a_1{}^2-9720b_1{}^2-8581680)T^3-864Y^2a_1+4752Ya_1b_1-108a_1{}^3-108a_1b_1{}^2+(6480Y^4a_1+287712Y^2a_1-19440Ya_1b_1+644112a_1)T^2+(-180Y^8-3408Y^6-864Y^5b_1-124344Y^4+28224Y^3b_1+1944Y^2a_1{}^2-4536Y^2b_1{}^2-262224Y^2+82080Yb_1-8424a_1{}^2-24840b_1{}^2-144180)T+7776a_1,\quad \mathbf{g}_2=3897234T^{10}+6Y^{10}+(1771470Y^2-27516834)T^8+944784T^7a_1+270Y^8+(306180Y^4-3306744Y^2+734832Yb_1+15142788)T^6-2099520T^5a_1+9468Y^6-2592Y^5b_1+378Y^4a_1{}^2-162Y^4b_1{}^2+(24300Y^6+150660Y^4+116640Y^3b_1+12763332Y^2+466560Yb_1+65610a_1{}^2+21870b_1{}^2+27660204)T^4+26460Y^4-1728Y^3b_1+2916Y^2a_1{}^2-4860Y^2b_1{}^2+324Ya_1{}^2b_1+324Yb_1{}^3+(-19440Y^4a_1-769824Y^2a_1+58320Ya_1b_1-2087856a_1)T^3+(810Y^8+22248Y^6+3888Y^5b_1+759996Y^4-75168Y^3b_1-8748Y^2a_1{}^2+20412Y^2b_1{}^2+1864296Y^2+55728Yb_1+14580a_1{}^2+104004b_1{}^2+2079594)T^2+51678Y^2-33696Yb_1+3402a_1{}^2+3078b_1{}^2+(-864Y^6a_1-25056Y^4a_1-6480Y^3a_1b_1-85536Y^2a_1-53136Ya_1b_1+972a_1{}^3+972a_1b_1{}^2-178848a_1)T-8586,\quad \mathbf{g}_1=-2125764T^{11}+(-1180980Y^2+19604268)T^9-708588T^8a_1+36Y^8a_1+(-262440Y^4+2834352Y^2-629856Yb_1-18738216)T^7+2519424T^6a_1-144Y^6a_1-216Y^5a_1b_1+(-29160Y^6-274104Y^4-139968Y^3b_1-18563256Y^2-279936Yb_1-78732a_1{}^2-26244b_1{}^2-42766056)T^5+1080Y^4a_1-5616Y^3a_1b_1+324Y^2a_1{}^3+324Y^2a_1b_1{}^2+(29160Y^4a_1+1014768Y^2a_1-87480Ya_1b_1+2991816a_1)T^4+(-1620Y^8-58320Y^6-7776Y^5b_1-2114424Y^4+46656Y^3b_1+17496Y^2a_1{}^2-40824Y^2b_1{}^2-7917264Y^2-1127520Yb_1+17496a_1{}^2-192456b_1{}^2+527148)T^3-27216Y^2a_1+8424Ya_1b_1-2268a_1{}^3-2268a_1b_1{}^2+(2592Y^6a_1+111456Y^4a_1+19440Y^3a_1b_1+785376Y^2a_1+190512Ya_1b_1-2916a_1{}^3-2916a_1b_1{}^2-878688a_1)T^2+(-36Y^{10}-2196Y^8-54504Y^6+19008Y^5b_1-2268Y^4a_1{}^2+972Y^4b_1{}^2-176040Y^4+100224Y^3b_1-33048Y^2a_1{}^2+23976Y^2b_1{}^2-1944Ya_1{}^2b_1-1944Yb_1{}^3+125388Y^2+67392Yb_1+88452a_1{}^2+17820b_1{}^2+212220)T-10044a_1,\quad \mathbf{g}_0=,531441T^{12}+Y^{12}+(354294Y^2-6259194)T^{10}+236196T^9a_1+102Y^{10}-12Y^9b_1+(98415Y^4-1062882Y^2+236196Yb_1+9546255)T^8-1259712T^7a_1+1935Y^8-432Y^7b_1+18Y^6a_1{}^2+54Y^6b_1{}^2+(14580Y^6+183708Y^4+69984Y^3b_1+10681308Y^2+39366a_1{}^2+13122b_1{}^2+25348788)T^6+2772Y^6+2808Y^5b_1-1674Y^4a_1{}^2-162Y^4b_1{}^2-108Y^3a_1{}^2b_1-108Y^3b_1{}^3+(-17496Y^4a_1-524880Y^2a_1+52488Ya_1b_1-1487160a_1)T^5+(1215Y^8+54108Y^6+5832Y^5b_1+2176794Y^4+42768Y^3b_1-13122Y^2a_1{}^2+30618Y^2b_1{}^2+12189852Y^2+1732104Yb_1-48114a_1{}^2+132678b_1{}^2-11229921)T^4-6129Y^4+30672Y^3b_1-5994Y^2a_1{}^2-13446Y^2b_1{}^2+1620Ya_1{}^2b_1+1620Yb_1{}^3+81a_1{}^4+162a_1{}^2b_1{}^2+81b_1{}^4+(-2592Y^6a_1-147744Y^4a_1-19440Y^3a_1b_1-1562976Y^2a_1-221616Ya_1b_1+2916a_1{}^3+2916a_1b_1{}^2+2708640a_1)T^3+(54Y^{10}+4158Y^8+82908Y^6-33696Y^5b_1+3402Y^4a_1{}^2-1458Y^4b_1{}^2-138564Y^4-312768Y^3b_1+72900Y^2a_1{}^2-28188Y^2b_1{}^2+2916Ya_1{}^2b_1+2916Yb_1{}^3-2375730Y^2+515808Yb_1-171558a_1{}^2-39690b_1{}^2-186138)T^2+41958Y^2-21708Yb_1+1458a_1{}^2+4050b_1{}^2+(-108Y^8a_1-144Y^6a_1+648Y^5a_1b_1+50328Y^4a_1+20304Y^3a_1b_1-972Y^2a_1{}^3-972Y^2a_1b_1{}^2+273456Y^2a_1-77112Ya_1b_1+1620a_1{}^3+1620a_1b_1{}^2-16524a_1)T+3969$$

For $N=2$, the formation of three peaks is obtained when the parameters a_1 or b_1 are not equal to 0.

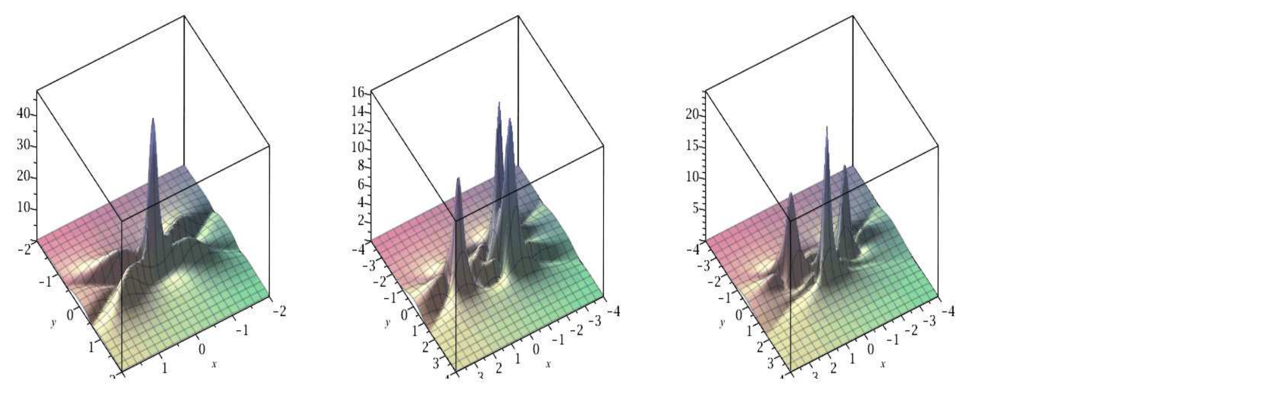

Figure 2. Solution of order 2 to the KPI equation for $t=0$, on the left $a_1=0, b_1=0,$; in the center $a_1=10, b_1=0,$; on the right $a_1=10, b_1=10$.

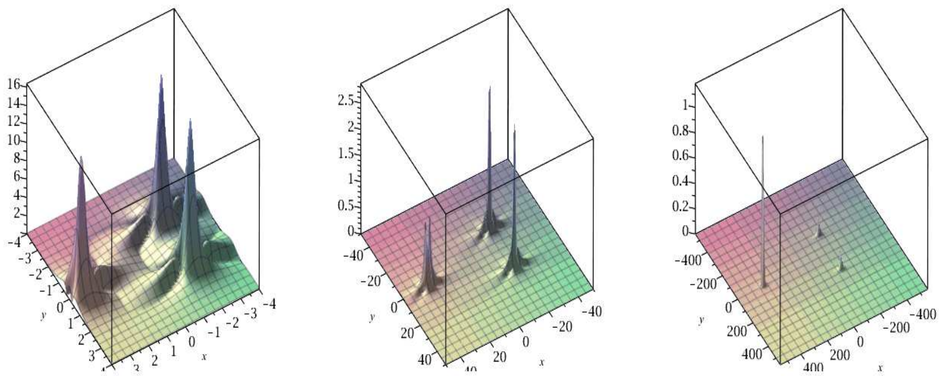

Figure 3. Solution of order 2 to the KPI equation for $t = 0$, on the left $a_1 = 10^2$, $b_1 = 0$; in the center $a_1 = 10^4$, $b_1 = 0$; on the right for $t = 10$, $a_1 = 10^8$, $b_1 = 0$.

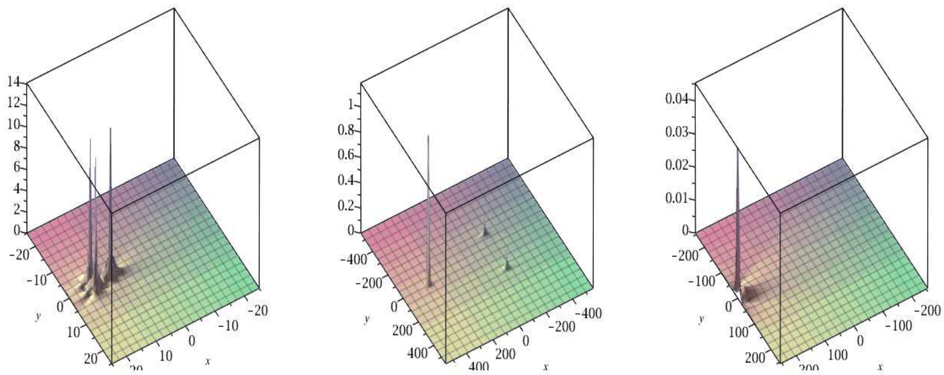

Figure 4. Solution of order 2 to the KPI equation, on the left for $t = 5$, $a_1 = 0$, $b_1 = 0$; in the center for $t = 10$, $a_1 = 0$, $b_1 = 0$; on the right for $t = 100$, $a_1 = 0$, $b_1 = 0$.

6.3. Case $N = 3$

In this case, polynomials G_3 and Q_3 depending on 4 parameters being too complex, we cannot give their explicit expressions. Even without parameters, due to the length of the solution, the explicit expression cannot be given here.

In the case $N = 3$, for $a_1 \neq 0$ or $b_1 \neq 0$ and the other parameters equal to zero, we obtain a triangle with 6 peaks; for $a_2 \neq 0$ or $b_2 \neq 0$, and other parameters equal to zero, we obtain a concentric rings of 5 peaks with a peak in the center.

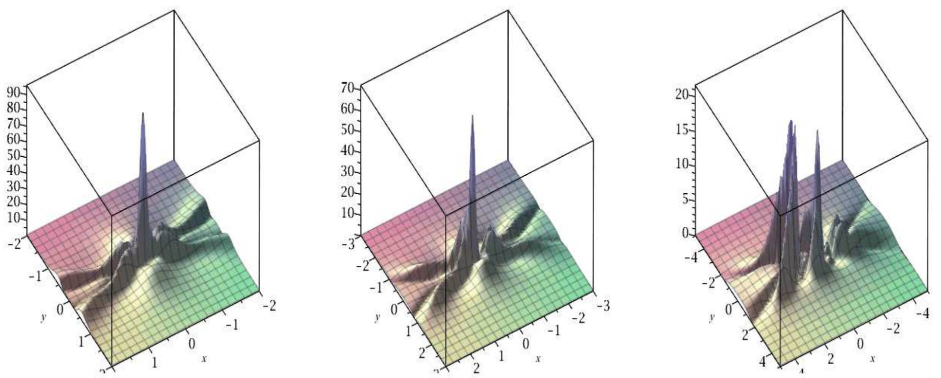

Figure 5. Solution of order 3 to the KPI equation, on the left for $t = 0$; in the center for $t = 0,01$; on the right for $t = 0,1$; all the parameters are equal to 0.

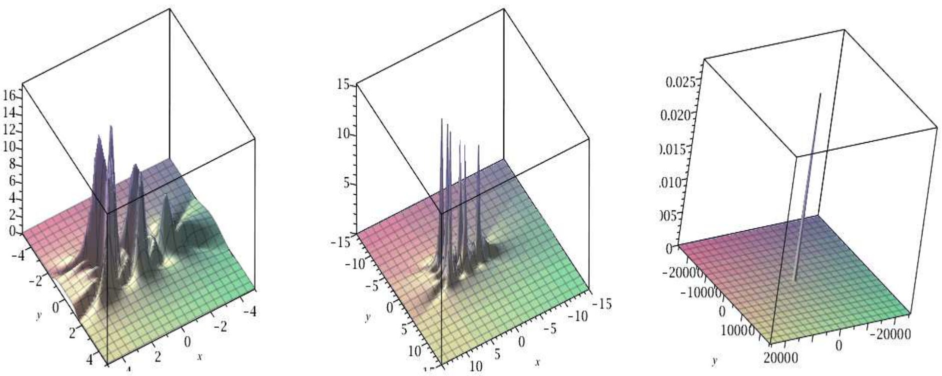

Figure 6. Solution of order 3 to the KPI equation, on the left for $t = 0,2$; in the center for $t = 10^0$; on the right for $t = 10^1$; all the parameters are equal to 0.

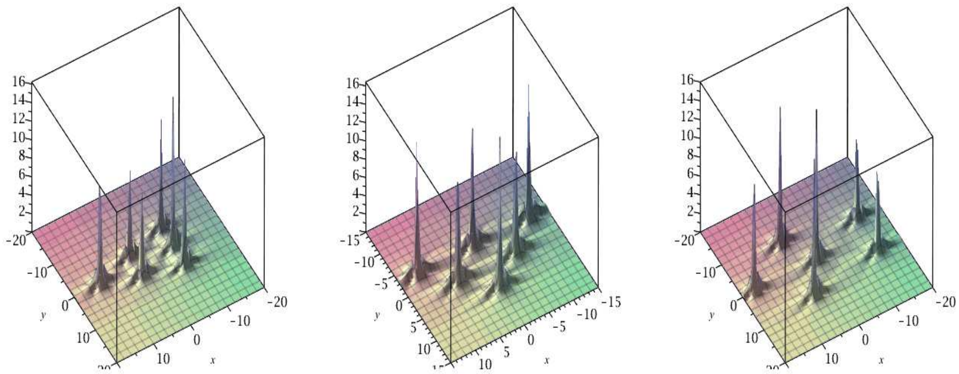

Figure 7. Solution of order 3 to the KPI equation, on the left for $a_1 = 10^3$; in the center for $b_1 = 10^3$; on the right for $a_2 = 10^6$; here $t = 0$.

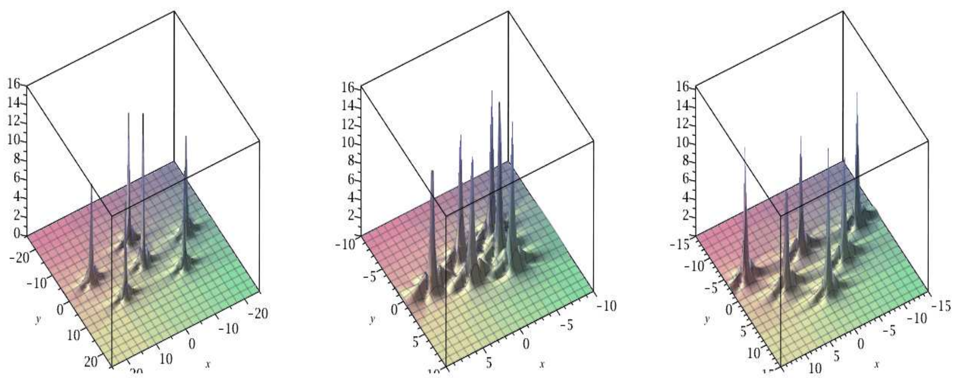

Figure 8. Solution of order 3 to the KPI equation, on the left for $t = 0$, $b_2 = 10^6$; in the center for $t = 0,01$, $a_1 = 10^3$ all the other parameters are equal to 0; on the right for $t = 0,1$, $b_1 = 10^3$ all the parameters are equal to 0.

7. Conclusion

We have given three representations of the solutions to the KPI equation: in terms of Fredholm determinants of order $2N$ depending on $2N-1$ real parameters in function of exponentials; in terms of wronskians of order $2N$ depending on $2N-1$ real parameters by means of trigonometric functions; in terms of real rational solutions as a quotient of two polynomials $n(x,y,t)$ and $d(x,y,t)$ of degrees $2N(N+1)-2$ in x, y, t and $2N(N+1)$ in x, y, t respectively and depending on $2N-2$ real parameters a_j and b_j, $1 \leq j \leq N-1$.
The maximum of the modulus of those solutions is equal to $2(2N+1)^2-2$. That gives a new approach to find explicit solutions for higher orders and try to describe the structure of those rational solutions.
In the (x,y) plane of coordinates, different structures appear.
All the solutions described in this study are different from those constructed in previous works [26–31] .
It will be relevant to go on this study for higher orders to try to understand the structure of those rational solutions.

Acknowledgements

The author would like to thank the referees for their suggestions and comments which improve the original version of the paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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