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In this issue, we publish 4 more valuable papers written with pleasure by our authors, carefully reviewed by our referees, despite all their busy time.

We thank our authors, reviewers, editors, and editing team for their contribution to thisVolume.

We expect support from you, valuable researchers and writers, for our journal, which will be published in January 2022.

We wish you a successful scientific life.

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THEORY OF GENERALIZED SEPARATION AXIOMS IN GENERALIZED TOPOLOGICAL SPACES

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Abstract. In this paper, a new class of generalized separation axioms (briefly, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*separation axioms*) whose elements are called $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{K}}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{F}}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{H}}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{R}}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{N}}$-axioms is defined in terms of generalized sets (briefly, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*sets*) in generalized topological spaces (briefly, $\mathcal{T}_{\mathfrak{g}}$-*spaces*) and the properties and characterizations of a $\mathcal{T}_{\mathfrak{g}}$-space endowed with each such $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{K}}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{F}}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{H}}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{R}}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{N}}$-axioms are discussed. The study shows that $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{F}}$-axiom implies $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{K}}$-axiom, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{H}}$-axiom implies $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{F}}$-axiom, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{R}}$-axiom implies $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{H}}$-axiom, and $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{N}}$-axiom implies $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{R}}$-axiom. Considering the $\mathrm{T}_{\mathfrak{g},\mathrm{K}}$, $\mathrm{T}_{\mathfrak{g},\mathrm{F}}$, $\mathrm{T}_{\mathfrak{g},\mathrm{H}}$, $\mathrm{T}_{\mathfrak{g},\mathrm{R}}$, $\mathrm{T}_{\mathfrak{g},\mathrm{N}}$-axioms as their analogues but defined in terms of corresponding elements belonging to the class of open, closed, semi-open, semi-closed, preopen, preclosed, semi-preopen, and semi-preclosed sets, the study also shows that the statement $\mathrm{T}_{\mathfrak{g},\alpha}$-axiom implies $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axiom holds for each $\alpha \in \{\mathrm{K},\mathrm{F},\mathrm{H},\mathrm{R},\mathrm{N}\}$. Diagrams expose the various implications amongst the classes presented here and in the literature, and a nice application supports the overall theory.

1. Introduction

Whether it concerns the theory of $\mathcal{T}$-spaces or $\mathcal{T}_{\mathfrak{g}}$-spaces, the idea of adding a T_{α} or a $\mathfrak{g}$-T_{α}-axiom[1] (with $\alpha = 0, 1, 2, \ldots$) to the axioms for a $\mathcal{T}$-space $\mathfrak{T} = (\Omega, \mathcal{T})$ to obtain a $\mathcal{T}^{(\alpha)}$-space $\mathfrak{T}^{(\alpha)} = \left(\Omega, \mathcal{T}^{(\alpha)}\right)$ or a $\mathfrak{g}$-$\mathcal{T}^{(\alpha)}$-space $\mathfrak{g}$-$\mathfrak{T}^{(\alpha)} = \left(\Omega, \mathfrak{g}\text{-}\mathcal{T}^{(\alpha)}\right)$ or, the idea of adding a $\mathrm{T}_{\mathfrak{g},\alpha}$ or a $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axiom (with $\alpha = 0, 1, 2, \ldots$) to the axioms for a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ to obtain a $\mathcal{T}_{\mathfrak{g}}^{(\alpha)}$-space $\mathfrak{T}_{\mathfrak{g}}^{(\alpha)} = \left(\Omega, \mathcal{T}_{\mathfrak{g}}^{(\alpha)}\right)$ or a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\alpha)}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\alpha)} = \left(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\alpha)}\right)$ has never played little role in Generalized Topology and Abstract Analysis [1, 2, 3]. Because the defining attributes of a $\mathcal{T}$-space in terms of a collection of $\mathcal{T}$-open or $\mathfrak{g}$-$\mathcal{T}$-open sets or a $\mathcal{T}_{\mathfrak{g}}$-space in terms of a collection

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[1]Notes to the reader: The notations T_{α}-axiom and $\mathfrak{g}$-T_{α}-axiom (with $\alpha = 0, 1, 2, \ldots$), founded upon the notions of $\mathcal{T}$-open and $\mathfrak{g}$-$\mathcal{T}$-open sets, respectively, designate an ordinary and a generalized separation axioms for a $\mathcal{T}$-space $\mathfrak{T} = (\Omega, \mathcal{T})$; the notations $\mathrm{T}_{\mathfrak{g},\alpha}$-axiom and $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axiom (with $\alpha = 0, 1, 2, \ldots$), founded upon $\mathcal{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open sets, respectively, designate an ordinary and a generalized separation axioms for a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$.

of $\mathcal{T}_{\mathfrak{g}}$-open or $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open sets, respectively, does little to guarantee that the points in the $\mathcal{T}$-space $\mathfrak{T}$ or the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ are somehow distinct or far apart. The more types and categories of T_{α} or $\mathfrak{g}$-T_{α}-axioms or $\mathfrak{g}$-T_{α} or $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms (with $\alpha = 0$, 1, 2, ...) are added to the axioms for a $\mathcal{T}$-space or a $\mathcal{T}_{\mathfrak{g}}$-space, respectively, the greater the role they will play in any topological endeavours [4, 5, 6, 7, 8]. For instance, for a sequence $\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}, \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\beta}, \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\gamma} \rangle$ (with α, β, $\gamma = 0$, 1, 2, ...), the $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\beta}$ and $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\gamma}$-axioms can be arranged in increasing order of strength in the sense that $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\gamma}$ implies $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\beta}$ and the latter implies $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$.

In the literature of $\mathcal{T}$-spaces and $\mathcal{T}_{\mathfrak{g}}$-spaces, respectively, several classes of T_{α}, $\mathfrak{g}$-T_{α}-axioms, founded upon the concepts of $\mathcal{T}$, $\mathfrak{g}$-$\mathcal{T}$-open sets, and $\mathrm{T}_{\mathfrak{g},\alpha}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms (with $\alpha = 0$, 1, 2, ...), founded upon the concepts of $\mathcal{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open sets, have been introduced and studied [9, 10, 11, 12, 13, 14, 15, 16]. The T_{α}-axioms called $\mathrm{T}_{\mathrm{Kolmogorov}}$, $\mathrm{T}_{\mathrm{Frchet}}$, $\mathrm{T}_{\mathrm{Hausdorff}}$, $\mathrm{T}_{\mathrm{Regular}}$, and $\mathrm{T}_{\mathrm{Normal}}$-axioms (shortly, T_{K}, T_{F}, T_{H}, T_{R}, and T_{N}), founded upon the concepts of $\mathcal{T}$-open, closed sets, are four classical examples, among others, which have gained extensive studies [17]. The $\mathfrak{g}$-T_{α}-axioms called generalized T_{α}, S_{β}-axioms (with $\alpha = 0$, 1, 2; $\beta = 1$, 2), founded upon $\mathfrak{g}$-$\mathcal{T}$-open sets instead of $\mathcal{T}$-open sets, are five examples of generalized T_{α}-axioms which have been discussed in the paper of [18]; the $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms called generalized $\mathrm{T}_{\frac{\alpha}{8}}$-axioms (with $\alpha = 2$, 3, 4), founded upon $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open sets instead of $\mathcal{T}_{\mathfrak{g}}$-open sets, are three examples of generalized $\mathrm{T}_{\mathfrak{g},\alpha}$-axioms which have been introduced and studied by [19]. Several other classes of T_{α}, $\mathfrak{g}$-T_{α}-axioms for $\mathcal{T}$-spaces and $\mathrm{T}_{\mathfrak{g},\alpha}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms for $\mathcal{T}_{\mathfrak{g}}$-spaces (with $\alpha = 0, 1, 2, \ldots$) have also been introduced and discussed in many papers [20, 21, 15, 3, 22, 23].

In view of the above references, we remark that the quintuple sequence $\langle \mathrm{T}_{\alpha} \rangle_{\alpha \in \Lambda}$, where $\Lambda = \{\mathrm{K}, \mathrm{F}, \mathrm{H}, \mathrm{R}, \mathrm{N}\}$, is based on the notions of $\mathcal{T}$-open, closed sets. From this remark and the conclusion drawn by [18], it is no error to state that $\langle \mathfrak{g}\text{-}\mathrm{T}_{\alpha} \rangle_{\alpha \in \Lambda}$ are based on the notions of $\mathfrak{g}$-$\mathcal{T}$-open, closed sets; $\langle \mathrm{T}_{\mathfrak{g},\alpha} \rangle_{\alpha \in \Lambda}$ on the notions of $\mathcal{T}_{\mathfrak{g}}$-open, closed sets, and $\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha} \rangle_{\alpha \in \Lambda}$ on the notions of $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open, closed sets. Thus, the idea of adding a quintuple sequence $\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha} \rangle_{\alpha \in \Lambda}$ of $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms (with $\Lambda = \{\mathrm{K}, \mathrm{F}, \mathrm{H}, \mathrm{R}, \mathrm{N}\}$), founded upon a new class of $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open, closed sets, to the axioms for a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ to obtain a corresponding sequence $\langle \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\alpha)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\alpha)}) \rangle_{\alpha \in \Lambda}$ of $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\alpha)}$-spaces might be an interesting subject of inquiry.

Hitherto, the introduction of several types of T_{α} and $\mathfrak{g}$-T_{α}-axioms in $\mathcal{T}$-spaces and $\mathrm{T}_{\mathfrak{g},\alpha}$ and $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms (with $\alpha = 0$, 1, 2, ...) in $\mathcal{T}_{\mathfrak{g}}$-spaces have contributed extensively to the geometrical specifications of $\mathcal{T}$-spaces and $\mathcal{T}_{\mathfrak{g}}$-spaces. However, despite these contributions not a single work has been devoted to the generalization of the sequence $\langle \mathrm{T}_{\alpha} \rangle_{\alpha \in \Lambda}$ in terms of the notions of $\mathfrak{g}$-$\mathcal{T}$-open, closed sets. With this view in mind, the idea therefore suggests itself, of introducing the generalized versions of the Kolmogorov, Fréchet, Hausdorff, Regular and Normal separation axioms in terms of the notions of $\mathfrak{g}$-$\mathcal{T}$-open, closed sets in a $\mathcal{T}_{\mathfrak{g}}$-space, adequate for the obtention of $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-spaces in this direction.

In this paper, we attempt to make a contribution to such a development by introducing a new theory, called *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Separation Axioms*, in which it is presented the generalized version of the sequence $\langle \mathrm{T}_{\alpha} \rangle_{\alpha \in \Lambda}$ in terms of the notions of $\mathfrak{g}$-$\mathcal{T}$-open, closed sets, discussing the fundamental properties and giving characterizations of its elements, on this ground and with respect to existing works [?, 14].

The paper is organised as follows: In Sect. 2, preliminary notions are described in Sect. 2.1 and the main results of the $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms in a $\mathcal{T}_{\mathfrak{g}}$-space are reported in Sect. 3. In Sect. 4, the establishment of the various relationships between these $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms are discussed in Sect. 4.1. To support the work, a nice application of the $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms in a $\mathcal{T}_{\mathfrak{g}}$-space is presented in Sect. 4.2. Finally, Sect. 5 provides concluding remarks and future directions of the $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms in a $\mathcal{T}_{\mathfrak{g}}$-space.

2. Theory

2.1. **Preliminaries.** Though foreign terms are neatly defined in [30], we thought it necessary to recall some basic definitions and notations of most essential concepts presented in [30].

The set $\mathfrak{U}$ represents the universe of discourse, fixed within the framework of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation axioms and containing as elements all sets ($\mathcal{T}$, $\mathfrak{g}$-$\mathcal{T}$, $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-sets; $\mathcal{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$, $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets) considered in this theory, and $I_n^0 \stackrel{\text{def}}{=} \{\nu \in \mathbb{N}^0 : \ \nu \leq n\}$; index sets I_∞^0, I_n^*, I_∞^* are defined in an analogous way. Granted $\Omega \subset \mathfrak{U}$, $\mathcal{P}(\Omega) \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g},\nu} : \ \mathcal{O}_{\mathfrak{g},\nu} \subseteq \Omega\}$ denotes the family of all subsets $\mathcal{O}_{\mathfrak{g},1}$, $\mathcal{O}_{\mathfrak{g},2}$, …, of Ω. A one-valued map of the type $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ satisfying $\mathcal{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}) \subseteq \mathcal{O}_{\mathfrak{g}}$, and $\mathcal{T}_{\mathfrak{g}}\big(\bigcup_{\nu \in I_\infty^*} \mathcal{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu \in I_\infty^*} \mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$ is called a $\mathfrak{g}$-topology on Ω, and the structure $\mathfrak{T}_{\mathfrak{g}} \stackrel{\text{def}}{=} (\Omega, \mathcal{T}_{\mathfrak{g}})$ is called a $\mathcal{T}_{\mathfrak{g}}$-space, on which a quintuple sequence $\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha} \rangle_{\alpha \in \Lambda}$ of $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms (with $\Lambda = \{\mathrm{K}, \mathrm{F}, \mathrm{H}, \mathrm{R}, \mathrm{N}\}$) will be discussed [24, 25, 26].

The operator $\mathrm{cl}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ carrying each $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ into its closure $\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \mathfrak{T}_{\mathfrak{g}} - \mathrm{int}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}}$ is termed a $\mathfrak{g}$-closure operator and the operator $\mathrm{int}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ carrying each $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ into its interior $\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \mathfrak{T}_{\mathfrak{g}} - \mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}}$ is called a $\mathfrak{g}$-interior operator. Let $\complement : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ denotes the absolute complement with respect to the underlying set $\Omega \subset \mathfrak{U}$, and let $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set. The classes

$$
\begin{aligned}
\mathcal{T}_{\mathfrak{g}} &\stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \ \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}\}, \\
\neg\mathcal{T}_{\mathfrak{g}} &\stackrel{\text{def}}{=} \{\mathcal{K}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \ \complement_{\Lambda}(\mathcal{K}_{\mathfrak{g}}) \in \mathcal{T}_{\mathfrak{g}}\},
\end{aligned} \tag{2.1}
$$

respectively, denote the classes of all $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets relative to the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}$, and the classes

$$
\begin{aligned}
\mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}} : \ \mathcal{O}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}\}, \\
\mathrm{C}^{\mathrm{sup}}_{\neg\mathcal{T}_{\mathfrak{g},}}[\mathcal{S}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{\mathcal{K}_{\mathfrak{g}} \in \neg\mathcal{T}_{\mathfrak{g}} : \ \mathcal{K}_{\mathfrak{g}} \supseteq \mathcal{S}_{\mathfrak{g}}\},
\end{aligned} \tag{2.2}
$$

respectively, denote the classes of $\mathcal{T}_{\mathfrak{g}}$-open subsets and $\mathcal{T}_{\mathfrak{g}}$-closed supersets (complements of the $\mathcal{T}_{\mathfrak{g}}$-open subsets) of the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ relative to the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}$. To this end, the $\mathfrak{g}$-closure and the $\mathfrak{g}$-interior of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathcal{T}_{\mathfrak{g}}$-space [27] define themselves as

$$
\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcup_{\mathcal{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}]} \mathcal{O}_{\mathfrak{g}}, \quad \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcap_{\mathcal{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}]} \mathcal{K}_{\mathfrak{g}}. \tag{2.3}
$$

Throughout this paper, the composition operators $\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}(\cdot)$, $\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}(\cdot)$, and $\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}(\cdot)$, respectively, stand for the functionals $\mathrm{cl}_{\mathfrak{g}}(\mathrm{int}_{\mathfrak{g}}(\cdot))$, $\mathrm{int}_{\mathfrak{g}}(\mathrm{cl}_{\mathfrak{g}}(\cdot))$, and $\mathrm{cl}_{\mathfrak{g}}(\mathrm{int}_{\mathfrak{g}}(\mathrm{cl}_{\mathfrak{g}}(\cdot)))$; other composition operators are defined similarly. Furthermore, the backslash $\mathfrak{T}_{\mathfrak{g}}\setminus\mathcal{S}_{\mathfrak{g}}$ refers to the set-theoretic difference $\mathfrak{T}_{\mathfrak{g}}-\mathcal{S}_{\mathfrak{g}}$. The mapping $\mathrm{op}_{\mathfrak{g}}:\mathcal{P}(\Omega)\longrightarrow\mathcal{P}(\Omega)$ is called a $\mathfrak{g}$-operation on $\mathcal{P}(\Omega)$ if the following statements hold:

$$\begin{aligned}&\forall\mathcal{S}_{\mathfrak{g}}\in\mathcal{P}(\Omega)\setminus\{\emptyset\},\ \exists(\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}})\in\mathcal{T}_{\mathfrak{g}}\setminus\{\emptyset\}\times\neg\mathcal{T}_{\mathfrak{g}}\setminus\{\emptyset\}:\\ &\big(\mathrm{op}_{\mathfrak{g}}(\emptyset)=\emptyset\big)\vee\big(\neg\,\mathrm{op}_{\mathfrak{g}}(\emptyset)=\emptyset\big),\ \big(\mathcal{S}_{\mathfrak{g}}\subseteq\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\big)\vee\big(\mathcal{S}_{\mathfrak{g}}\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\big),\end{aligned}\tag{2.4}$$

where $\neg\,\mathrm{op}_{\mathfrak{g}}:\mathcal{P}(\Omega)\longrightarrow\mathcal{P}(\Omega)$ is called the "complementary $\mathfrak{g}$-operation" on $\mathcal{P}(\Omega)$ and, for all $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathcal{S}_{\mathfrak{g}}$, $\mathcal{S}_{\mathfrak{g},\nu}$, $\mathcal{S}_{\mathfrak{g},\mu}\in\mathcal{P}(\Omega)\setminus\{\emptyset\}$, the following axioms are satisfied:

- Ax. I. $\big(\mathcal{S}_{\mathfrak{g}}\subseteq\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\big)\vee\big(\mathcal{S}_{\mathfrak{g}}\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\big)$,
- Ax. II. $\big(\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\subseteq\mathrm{op}_{\mathfrak{g}}\circ\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\big)\vee\big(\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}\circ\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\big)$,
- Ax. III. $\big(\mathcal{S}_{\mathfrak{g},\nu}\subseteq\mathcal{S}_{\mathfrak{g},\mu}\ \longrightarrow\ \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})\subseteq\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\mu})\big)\vee\big(\mathcal{S}_{\mathfrak{g},\mu}\subseteq\mathcal{S}_{\mathfrak{g},\nu}\ \longleftarrow\ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\mu})\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu})\big)$,
- Ax. IV. $\big(\mathrm{op}_{\mathfrak{g}}\big(\bigcup_{\sigma=\nu,\mu}\mathcal{S}_{\mathfrak{g},\sigma}\big)\subseteq\bigcup_{\sigma=\nu,\mu}\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\sigma})\big)\vee\big(\neg\,\mathrm{op}_{\mathfrak{g}}\big(\bigcup_{\sigma=\nu,\mu}\mathcal{S}_{\mathfrak{g},\sigma}\big)\supseteq\bigcup_{\sigma=\nu,\mu}\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\sigma})\big)$,

for some $\mathcal{T}_{\mathfrak{g}}$-open sets $\mathcal{O}_{\mathfrak{g}}$, $\mathcal{O}_{\mathfrak{g},\nu}$, $\mathcal{O}_{\mathfrak{g},\mu}\in\mathcal{T}_{\mathfrak{g}}\setminus\{\emptyset\}$ and $\mathcal{T}_{\mathfrak{g}}$-closed sets $\mathcal{K}_{\mathfrak{g}}$, $\mathcal{K}_{\mathfrak{g},\nu}$, $\mathcal{K}_{\mathfrak{g},\mu}\in\neg\mathcal{T}_{\mathfrak{g}}$ [28, 29]. The class $\mathcal{L}_{\mathfrak{g}}[\Omega]=\mathcal{L}_{\mathfrak{g}}^{\omega}[\Omega]\times\mathcal{L}_{\mathfrak{g}}^{\kappa}[\Omega]$, where

$$\mathcal{L}_{\mathfrak{g}}[\Omega]\stackrel{\mathrm{def}}{=}\big\{\mathbf{op}_{\mathfrak{g},\nu\mu}(\cdot)=\big(\mathrm{op}_{\mathfrak{g},\nu}(\cdot),\neg\,\mathrm{op}_{\mathfrak{g},\mu}(\cdot)\big):\ (\nu,\mu)\in I_3^0\times I_3^0\big\}\tag{2.5}$$

in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, stands for the class of all possible $\mathfrak{g}$-operators and their complementary $\mathfrak{g}$-operators in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Its elements are defined as:

$$\begin{aligned}\mathrm{op}_{\mathfrak{g}}(\cdot)\ &\in\ \mathcal{L}_{\mathfrak{g}}^{\omega}[\Omega]\stackrel{\mathrm{def}}{=}\big\{\mathrm{op}_{\mathfrak{g},0}(\cdot),\ \mathrm{op}_{\mathfrak{g},1}(\cdot),\ \mathrm{op}_{\mathfrak{g},2}(\cdot),\ \mathrm{op}_{\mathfrak{g},3}(\cdot)\big\}\\ &=\ \big\{\mathrm{int}_{\mathfrak{g}}(\cdot),\ \mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}(\cdot),\ \mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}(\cdot),\ \mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}(\cdot)\big\};\\ \neg\,\mathrm{op}_{\mathfrak{g}}(\cdot)\ &\in\ \mathcal{L}_{\mathfrak{g}}^{\kappa}[\Omega]\stackrel{\mathrm{def}}{=}\big\{\neg\,\mathrm{op}_{\mathfrak{g},0}(\cdot),\ \neg\,\mathrm{op}_{\mathfrak{g},1}(\cdot),\ \neg\,\mathrm{op}_{\mathfrak{g},2}(\cdot),\ \neg\,\mathrm{op}_{\mathfrak{g},3}(\cdot)\big\}\\ &=\ \big\{\mathrm{cl}_{\mathfrak{g}}(\cdot),\ \mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}(\cdot),\ \mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}(\cdot),\ \mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}(\cdot)\big\}.\end{aligned}\tag{2.6}$$

A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ in a $\mathcal{T}_{\mathfrak{g}}$-space is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set if and only if there exist a pair $(\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}})\in\mathcal{T}_{\mathfrak{g}}\times\neg\mathcal{T}_{\mathfrak{g}}$ of $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets, and a $\mathfrak{g}$-operator $\mathbf{op}_{\mathfrak{g}}(\cdot)\in\mathcal{L}_{\mathfrak{g}}[\Omega]$ such that the following statement holds:

$$(\exists\xi)\big[(\xi\in\mathcal{S}_{\mathfrak{g}})\wedge\big(\big(\mathcal{S}_{\mathfrak{g}}\subseteq\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\big)\vee\big(\mathcal{S}_{\mathfrak{g}}\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\big)\big)\big].\tag{2.7}$$

The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ is said to be of category ν if and only if it belongs to the following class of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-sets:

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]\stackrel{\mathrm{def}}{=}\big\{\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}:\ &\big(\exists\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}},\mathbf{op}_{\mathfrak{g},\nu}(\cdot)\big)\\ &\big[\big(\mathcal{S}_{\mathfrak{g}}\subseteq\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g}})\big)\vee\big(\mathcal{S}_{\mathfrak{g}}\supseteq\neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g}})\big)\big]\big\}.\end{aligned}\tag{2.8}$$

It is called a $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open set if it satisfies the first property in $\mathfrak{g}$-ν-$\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ and a $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed set if it satisfies the second property in $\mathfrak{g}$-ν-$\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$. The classes of

$\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, respectively, are defined by

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] &\stackrel{\text{def}}{=} \big\{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}: \ \big(\exists \mathcal{O}_{\mathfrak{g}}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)\big)\ \big[\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g}})\big]\big\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] &\stackrel{\text{def}}{=} \big\{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}: \ \big(\exists \mathcal{K}_{\mathfrak{g}}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)\big)\ \big[\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g}})\big]\big\}. \end{aligned} \tag{2.9}$$

From these classes, the following relation holds:

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g}}\big] &= \textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \\ &= \textstyle\bigcup_{\nu\in I_3^0} \big(\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cup \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big) \\ &= \big(\textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big) \cup \big(\textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big) \\ &= \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cup \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]. \end{aligned} \tag{2.10}$$

When the subscript $\mathfrak{g}$ are omitted in almost all symbols of the above definitions, very similar definitions are derived but in a $\mathcal{T}$-space.

A $\mathfrak{T}$-set $\mathcal{S} \subset \mathfrak{T}$ in a $\mathcal{T}$-space is called a $\mathfrak{g}$-$\mathfrak{T}$-set if and only if there exists a pair $(\mathcal{O},\mathcal{K}) \in \mathcal{T} \times \neg\mathcal{T}$ of $\mathcal{T}$-open and $\mathcal{T}$-closed sets, and an operator $\mathbf{op}(\cdot) \in \mathcal{L}\big[\Omega\big]$ such that the following statement holds:

$$(\exists\xi)\ \big[(\xi \in \mathcal{S}) \wedge \big((\mathcal{S} \subseteq \mathrm{op}(\mathcal{O})) \vee (\mathcal{S} \supseteq \neg\,\mathrm{op}(\mathcal{K}))\big)\big]. \tag{2.11}$$

The $\mathfrak{g}$-$\mathfrak{T}$-set $\mathcal{S} \subset \mathfrak{T}$ is said to be of category ν if and only if it belongs to the following class of $\mathfrak{g}$-ν-$\mathcal{T}$-sets:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\big[\mathfrak{T}\big] \stackrel{\text{def}}{=} \big\{\mathcal{S} \subset \mathfrak{T}: \ &(\exists \mathcal{O}, \mathcal{K}, \mathbf{op}_{\nu}(\cdot)) \\ &\big[(\mathcal{S} \subseteq \mathrm{op}_{\nu}(\mathcal{O})) \vee (\mathcal{S} \supseteq \neg\,\mathrm{op}_{\nu}(\mathcal{K}))\big]\big\}. \end{aligned} \tag{2.12}$$

It is called a $\mathfrak{g}$-ν-$\mathfrak{T}$-open set if it satisfies the first property in $\mathfrak{g}$-ν-$\mathrm{S}\big[\mathfrak{T}\big]$ and a $\mathfrak{g}$-ν-$\mathfrak{T}$-closed set if it satisfies the second property in $\mathfrak{g}$-ν-$\mathrm{S}\big[\mathfrak{T}\big]$. The classes of $\mathfrak{g}$-ν-$\mathfrak{T}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}$-closed sets, respectively, are defined by

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}\big] &\stackrel{\text{def}}{=} \big\{\mathcal{S} \subset \mathfrak{T}: \ (\exists \mathcal{O}, \mathbf{op}_{\nu}(\cdot))\ \big[\mathcal{S} \subseteq \mathrm{op}_{\nu}(\mathcal{O})\big]\big\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}\big] &\stackrel{\text{def}}{=} \big\{\mathcal{S} \subset \mathfrak{T}: \ (\exists \mathcal{K}, \mathbf{op}_{\nu}(\cdot))\ \big[\mathcal{S} \supseteq \neg\,\mathrm{op}_{\nu}(\mathcal{K})\big]\big\}. \end{aligned} \tag{2.13}$$

The following relations are immediate consequences of the above definitions:

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}\big] &= \textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\big[\mathfrak{T}\big] \\ &= \textstyle\bigcup_{\nu\in I_3^0} \big(\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}\big] \cup \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}\big]\big) \\ &= \big(\textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}\big]\big) \cup \big(\textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}\big]\big) \\ &= \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}\big] \cup \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}\big]. \end{aligned} \tag{2.14}$$

The classes $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ denote the families of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets, respectively, in $\mathfrak{T}_{\mathfrak{g}}$, with $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$; the classes $\mathrm{O}[\mathfrak{T}]$ and $\mathrm{K}[\mathfrak{T}]$ denote the families of $\mathfrak{T}$-open and $\mathfrak{T}$-closed sets, respectively, in $\mathfrak{T}$, with $\mathrm{S}[\mathfrak{T}] = \mathrm{O}[\mathfrak{T}] \cup \mathrm{K}[\mathfrak{T}]$.

In regard to the above descriptions, by a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open set and a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed set are meant a $\mathcal{T}_{\mathfrak{g}}$-open set $\mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}$ and a $\mathcal{T}_{\mathfrak{g}}$-closed set $\mathcal{K}_{\mathfrak{g}} \in \neg\mathcal{T}_{\mathfrak{g}}$ satisfying $\mathcal{O}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$ and $\mathcal{K}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})$, respectively. Likewise, by a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open set of category ν and a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed set of category ν are meant a $\mathcal{T}_{\mathfrak{g}}$-open set $\mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}$ and a $\mathcal{T}_{\mathfrak{g}}$-closed set $\mathcal{K}_{\mathfrak{g}} \in \neg\mathcal{T}_{\mathfrak{g}}$ satisfying $\mathcal{O}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g}})$ and $\mathcal{K}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g}})$, respectively; $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-sets of category ν will be called $\mathfrak{g}$-ν-$\mathcal{T}_{\mathfrak{g}}$-sets. We are now in a position to present a carefully chosen set of terms used in the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation axioms in $\mathcal{T}_{\mathfrak{g}}$-spaces.

Agreed to let $\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\rangle_{\alpha\in\Lambda}$ denote a sequence of $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms, indexed by the set $\Lambda \stackrel{\text{def}}{=} \{\mathrm{K},\mathrm{F},\mathrm{H},\mathrm{R},\mathrm{N}\}$, throughout the present paper, the sequence $\langle \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\alpha)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\alpha)})\rangle_{\alpha\in\Lambda}$ will stand for the resulting sequence of $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\alpha)}$-spaces, obtained after endowing a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ with $\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\rangle_{\alpha\in\Lambda}$. Hence, the definition follows.

Definition 2.1 ($\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\alpha)}$-Space)**.** A $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ endowed with a $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axiom is called a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\alpha)}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\alpha)} \stackrel{\text{def}}{=} (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\alpha)})$.

The elements of $\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\rangle_{\alpha\in\Lambda}$ concern the separation of points, points from $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open sets, and $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open sets from each other. They are nicely discussed through the notions of pairwise disjoint points and $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-sets in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. We let $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\xi}) \in \mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ denote a pair of $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets containing the point $\xi \in \mathfrak{T}_{\mathfrak{g}}$ and let $(\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}, \mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}) \in \mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ denote either a pair of $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed subsets or supersets of the set $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{T}_{\mathfrak{g}}$, and consider the following definition.

Definition 2.2 (ξ, $\mathcal{S}_{\mathfrak{g}}$-Pairwise Disjoint)**.** Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ be a $\mathcal{T}_{\mathfrak{g}}$-space. For some $\sigma \geq 0$ and $\mathcal{S}_{\mathfrak{g}} \supseteq \emptyset$, the families

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma] &\stackrel{\text{def}}{=} \{(\xi,\zeta) \in \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}} :\ \mathcal{N}_{\mathfrak{g}}(\xi,\zeta) \geq \sigma\},\\
\mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{O}}[\mathcal{S}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathcal{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} :\ \textstyle\bigcap_{\lambda=\xi,\zeta} \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\lambda}) \subseteq \mathcal{S}_{\mathfrak{g}}\},\\
\mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{K}}[\mathcal{S}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{(\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta}) \in \neg\mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}} :\ \mathcal{S}_{\mathfrak{g}} \supseteq \textstyle\bigcap_{\lambda=\xi,\zeta} \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\lambda})\},
\end{aligned} \tag{2.15}$$

respectively, denote the collections of pairwise points, and $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed sets of category ν in $\mathfrak{T}_{\mathfrak{g}}$. They form the collections of pairwise distinct points, and pairwise disjoint $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed sets of category ν whenever $\sigma > 0$ and $\mathcal{S}_{\mathfrak{g}} = \emptyset$, respectively.

Granted $\mathfrak{g}$-$\mathrm{F}_{\mathrm{P}}[\sigma]$, $\mathfrak{g}$-ν-$\mathrm{F}_{\mathrm{O}}[\mathcal{S}_{\mathfrak{g}}]$, and $\mathfrak{g}$-ν-$\mathrm{F}_{\mathrm{K}}[\mathcal{S}_{\mathfrak{g}}]$, the elements of $\langle \mathfrak{g}\text{-}\nu\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\rangle_{\alpha\in\Lambda}$ may well be stated as thus:

Definition 2.3 ($\langle \mathfrak{g}\text{-}\nu\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\rangle_{\alpha\in\Lambda}$-Axioms)**.** Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ be a $\mathcal{T}_{\mathfrak{g}}$-space and suppose $\mathfrak{g}$-$\mathrm{F}_{\mathrm{P}}[\sigma]$, and $\mathfrak{g}$-ν-$\mathrm{F}_{\mathrm{O}}[\mathcal{S}_{\mathfrak{g}}] \subseteq \mathcal{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}}$ and $\mathfrak{g}$-ν-$\mathrm{F}_{\mathrm{K}}[\mathcal{S}_{\mathfrak{g}}] \subseteq \neg\mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ be given, where $\sigma \geq 0$ and $\mathcal{S}_{\mathfrak{g}} \supseteq \emptyset$. Then:

- I. $\mathfrak{g}$-ν-$\mathrm{T}_{\mathfrak{g},\mathrm{K}}$-Axiom: For every $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists a pair $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{O}}[\mathcal{S}_{\mathfrak{g}} \supset \emptyset]$ such that:

$$\begin{aligned}
&\big[(\xi \in \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\xi})) \wedge (\zeta \notin \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\xi}))\big] \vee \big[(\xi \notin \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\zeta}))\\
&\qquad \wedge (\zeta \in \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\zeta}))\big].
\end{aligned} \tag{2.16}$$

- II. $\mathfrak{g}$-ν-$\mathrm{T}_{\mathfrak{g},\mathrm{F}}$-Axiom: For every $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists a pair $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{O}}[\mathcal{S}_{\mathfrak{g}} \supset \emptyset]$ such that:

$$\big[(\xi,\zeta) \in {}_{\lambda=\xi,\zeta}\, \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\lambda})\big] \wedge \big[(\xi,\zeta) \notin {}_{\lambda=\zeta,\xi}\, \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\lambda})\big]. \tag{2.17}$$

- III. $\mathfrak{g}$-ν-$\mathrm{T}_{\mathfrak{g},\mathrm{H}}$-Axiom: For every $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists a pair $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{O}}\big[\emptyset\big]$ such that:

$$\big[\xi \in \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\xi})\big] \wedge \big[\zeta \in \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\zeta})\big]. \tag{2.18}$$

- IV. $\mathfrak{g}$-ν-$\mathrm{T}_{\mathfrak{g},\mathrm{R}}$-Axiom: For every $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$ and $(\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{K}}\big[\emptyset\big]$ such that $(\zeta,\xi) \notin (\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta})$, there exists a pair $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{O}}\big[\emptyset\big]$ such that:

$$\begin{aligned}&\big[\big(\neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\xi}) \subset \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\xi})\big) \wedge \big(\zeta \in \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\zeta})\big)\big] \vee \big[\big(\neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\zeta}) \\ &\qquad \subset \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\zeta})\big) \wedge \big(\xi \in \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\xi})\big)\big].\end{aligned} \tag{2.19}$$

- V. $\mathfrak{g}$-ν-$\mathrm{T}_{\mathfrak{g},\mathrm{N}}$-Axiom: For every $(\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{K}}[\emptyset]$, there exists a pair $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{O}}[\emptyset]$ such that:

$$\big[\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\xi}) \supset \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\xi})\big] \wedge \big[\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\zeta}) \supset \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\zeta})\big]. \tag{2.20}$$

Granted $\big\langle \mathfrak{g}\text{-}\nu\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\big\rangle_{\alpha\in\Lambda}$, we form $\big\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\big\rangle_{\alpha\in\Lambda} \stackrel{\mathrm{def}}{=} \big\langle \bigvee_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\big\rangle_{\alpha\in\Lambda}$, and define the $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{K}}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{F}}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{H}}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{R}}$, and $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{N}}$-axioms as thus.

Definition 2.4 ($\big\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\big\rangle_{\alpha\in\Lambda}$-Axioms)**.** Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ be a $\mathcal{T}_{\mathfrak{g}}$-space and let $\mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma]$, $\mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}[\mathcal{S}_{\mathfrak{g}}] = \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{O}}[\mathcal{S}_{\mathfrak{g}}] \subseteq \mathcal{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{F}_{\mathrm{K}}[\mathcal{S}_{\mathfrak{g}}] = \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{K}}[\mathcal{S}_{\mathfrak{g}}] \subseteq \neg\mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ be given, where $\sigma \geq 0$ and $\mathcal{S}_{\mathfrak{g}} \supseteq \emptyset$. Then:

- I. $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{K}}$-Axiom: For every $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\mathcal{S}_{\mathfrak{g}} \supset \emptyset\big]$ such that:

$$\begin{aligned}&\big[\big(\xi \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big) \wedge \big(\zeta \notin \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big)\big] \vee \big[\big(\xi \notin \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big) \\ &\qquad \wedge \big(\zeta \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big)\big].\end{aligned} \tag{2.21}$$

- II. $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{F}}$-Axiom: For every $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists a pair $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\mathcal{S}_{\mathfrak{g}} \supset \emptyset\big]$ such that:

$$\big[(\xi,\zeta) \in {}_{\lambda=\xi,\zeta}\,\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\big] \wedge \big[(\xi,\zeta) \notin {}_{\lambda=\zeta,\xi}\,\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\big]. \tag{2.22}$$

- III. $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{H}}$-Axiom: For every $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists a pair $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\emptyset\big]$ such that:

$$\big[\xi \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big] \wedge \big[\zeta \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big]. \tag{2.23}$$

- IV. $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{R}}$-Axiom: For every $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$ and $(\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{K}}\big[\emptyset\big]$ such that $(\zeta,\xi) \notin (\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta})$, there exists $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\emptyset\big]$ such that:

$$\begin{aligned}&\big[\big(\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi}) \subset \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big) \wedge \big(\zeta \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big)\big] \vee \big[\big(\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\zeta}) \\ &\qquad \subset \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big) \wedge \big(\xi \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big)\big].\end{aligned} \tag{2.24}$$

- V. $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{N}}$-Axiom: For every $(\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{K}}[\emptyset]$, there exists a pair $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}[\emptyset]$ such that:

$$\big[\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi}) \supset \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi})\big] \wedge \big[\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta}) \supset \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\zeta})\big]. \tag{2.25}$$

In the following sections, the main results of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-maps are presented.

3. Main Results

A necessary and sufficient condition for a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = \left(\Omega, \mathcal{T}_{\mathfrak{g}}\right)$ to be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{K})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{K})} = \left(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{K})}\right)$ may be given in terms of the complementary $\mathfrak{g}$-operator $\neg\operatorname{op}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ and any pairs $(\{\xi\}, \{\zeta\}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ of unit sets.

Theorem 3.1. *A $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = \left(\Omega, \mathcal{T}_{\mathfrak{g}}\right)$ is said to be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{K})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{K})} = \left(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{K})}\right)$ if and only if the following condition holds:*

$$\neg\operatorname{op}_{\mathfrak{g}}(\{\xi\}) \neq \neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\}) \quad \forall (\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]. \tag{3.1}$$

Proof. Necessity. Let the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{K})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{K})}$. Then, for every $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\left[\mathcal{S}_{\mathfrak{g}} \supset \emptyset\right]$ such that:

$$\begin{aligned}\left[\left(\xi \in \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\right) \wedge \left(\zeta \notin \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\right)\right] \vee \big[&\left(\xi \notin \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\right) \\ &\wedge \left(\zeta \in \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\right)\big].\end{aligned}$$

Consequently,

$$\begin{aligned}\left[\left(\xi \notin \mathtt{C}\left(\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\right)\right) \wedge \left(\zeta \in \mathtt{C}\left(\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\right)\right)\right] \vee \big[&\left(\xi \in \mathtt{C}\left(\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\right)\right) \\ &\wedge \left(\zeta \notin \mathtt{C}\left(\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\right)\right)\big],\end{aligned}$$

implying $\mathtt{C}\left(\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\right), \mathtt{C}\left(\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\right) \in \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, respectively, are $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closed sets containing $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ and $\xi \in \mathfrak{T}_{\mathfrak{g}}$. Thus, there exists $(\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta}) \in \neg\mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ such that $\mathtt{C}\left(\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\right) \supseteq \neg\operatorname{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\zeta})$ and $\mathtt{C}\left(\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\right) \supseteq \neg\operatorname{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi})$. But, for every $\lambda \in \{\xi,\zeta\}$, $\mathtt{C}(\{\lambda\}) \supseteq \mathtt{C}\left(\operatorname{op}_{\mathfrak{g}}(\{\lambda\})\right) \supseteq \mathtt{C}\left(\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\right)$ and $\neg\operatorname{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\lambda}) \supseteq \neg\operatorname{op}_{\mathfrak{g}}(\{\lambda\})$. Therefore, $\mathtt{C}(\{\xi\}) \supseteq \neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\})$ and $\mathtt{C}(\{\zeta\}) \supseteq \neg\operatorname{op}_{\mathfrak{g}}(\{\xi\})$. Since, $\mathtt{C}(\{\xi\}) \neq \mathtt{C}(\{\zeta\})$, it follows that $\neg\operatorname{op}_{\mathfrak{g}}(\{\xi\}) \neq \neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\})$.

Sufficiency. Conversely, suppose the condition $\neg\operatorname{op}_{\mathfrak{g}}(\{\xi\}) \neq \neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\})$ holds for every $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$. Then there exists $\eta \in \mathfrak{T}_{\mathfrak{g}}$ such that

$$\begin{aligned}\left[\left(\eta \in \neg\operatorname{op}_{\mathfrak{g}}(\{\xi\})\right) \wedge \left(\eta \notin \neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\})\right)\right] \vee \big[&\left(\eta \notin \neg\operatorname{op}_{\mathfrak{g}}(\{\xi\})\right) \\ &\wedge \left(\eta \in \neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\})\right)\big].\end{aligned}$$

If $\left[\xi \in \neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\})\right] \wedge \left[\zeta \in \neg\operatorname{op}_{\mathfrak{g}}(\{\xi\})\right]$, then

$$\left[\neg\operatorname{op}_{\mathfrak{g}}(\{\xi\}) \subseteq \neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\})\right] \wedge \left[\neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\}) \subseteq \neg\operatorname{op}_{\mathfrak{g}}(\{\xi\})\right].$$

Consequently,

$$\begin{aligned}\left[\left(\eta \in \neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\})\right) \wedge \left(\eta \notin \neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\})\right)\right] \vee \big[&\left(\eta \notin \neg\operatorname{op}_{\mathfrak{g}}(\{\xi\})\right) \\ &\wedge \left(\eta \in \neg\operatorname{op}_{\mathfrak{g}}(\{\xi\})\right)\big].\end{aligned}$$

This is a contradiction; hence, $\left[\xi \notin \neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\})\right] \wedge \left[\zeta \notin \neg\operatorname{op}_{\mathfrak{g}}(\{\xi\})\right]$, implying

$$\left[\xi \in \mathtt{C}\left(\neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\})\right)\right] \wedge \left[\zeta \in \mathtt{C}\left(\neg\operatorname{op}_{\mathfrak{g}}(\{\xi\})\right)\right].$$

Since $\mathtt{C}\left(\neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\})\right)$, $\mathtt{C}\left(\neg\operatorname{op}_{\mathfrak{g}}(\{\xi\})\right) \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, respectively, are $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open sets containing $\xi \in \mathfrak{T}_{\mathfrak{g}}$ and $\zeta \in \mathfrak{T}_{\mathfrak{g}}$, there exists $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathcal{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}}$ such that $\mathtt{C}\left(\neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\})\right) \subseteq \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})$ and $\mathtt{C}\left(\neg\operatorname{op}_{\mathfrak{g}}(\{\xi\})\right) \subseteq \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})$. Hence, for every $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\left[\mathcal{S}_{\mathfrak{g}} \supset \emptyset\right]$ such that:

$$\begin{aligned}\left[\left(\xi \in \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\right) \wedge \left(\zeta \notin \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\right)\right] \vee \big[&\left(\xi \notin \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\right) \\ &\wedge \left(\zeta \in \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\right)\big].\end{aligned}$$

Therefore, $\mathfrak{T}_{\mathfrak{g}} = \big(\Omega, \mathcal{T}_{\mathfrak{g}}\big)$ is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{K})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{K})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{K})}\big)$; this completes the proof of the theorem. □

A necessary and sufficient condition for a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = \big(\Omega, \mathcal{T}_{\mathfrak{g}}\big)$ to be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{F})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}\big)$ may be given in terms of the complementary $\mathfrak{g}$-operator $\neg\operatorname{op}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ and a unit set $\{\xi\} \subset \mathfrak{T}_{\mathfrak{g}}$.

Theorem 3.2. *A $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = \big(\Omega, \mathcal{T}_{\mathfrak{g}}\big)$ is said to be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{F})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}\big)$ if and only if the following condition holds:*

$$\{\xi\} \supseteq \neg\operatorname{op}_{\mathfrak{g}}(\{\xi\}) \quad \forall \xi \in \mathfrak{T}_{\mathfrak{g}}. \tag{3.2}$$

Proof. Necessity. Let the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{F})}$. Then, for every $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\mathcal{S}_{\mathfrak{g}} \supset \emptyset\big]$ such that:

$$\big[(\xi,\zeta) \in {}_{\lambda=\xi,\zeta}\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\big] \wedge \big[(\xi,\zeta) \notin {}_{\lambda=\zeta,\xi}\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\big].$$

Consequently,

$$\big[(\xi,\zeta) \notin {}_{\lambda=\xi,\zeta}\complement\big(\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\big)\big] \wedge \big[(\xi,\zeta) \in {}_{\lambda=\zeta,\xi}\complement\big(\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\big)\big].$$

Since, for every $\lambda \in \{\xi,\zeta\}$, $\complement\big(\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\big) \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ is a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closed set, there exists $(\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta}) \in \neg\mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ such that $\complement\big(\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big) \supseteq \neg\operatorname{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\zeta})$ and $\complement\big(\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big) \supseteq \neg\operatorname{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi})$. But, for every $\lambda \in \{\xi,\zeta\}$, $\neg\operatorname{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\lambda}\big) \supseteq \neg\operatorname{op}_{\mathfrak{g}}(\{\lambda\})$. Therefore

$$\big[(\xi,\zeta) \in {}_{\lambda=\xi,\zeta}\neg\operatorname{op}_{\mathfrak{g}}(\{\lambda\})\big] \wedge \big[(\xi,\zeta) \notin {}_{\lambda=\zeta,\xi}\neg\operatorname{op}_{\mathfrak{g}}(\{\lambda\})\big].$$

Hence, for every $\xi \in \mathfrak{T}_{\mathfrak{g}}$, $\{\xi\} \supseteq \neg\operatorname{op}_{\mathfrak{g}}(\{\xi\})$.

Sufficiency. Conversely, suppose the condition $\{\xi\} \supseteq \neg\operatorname{op}_{\mathfrak{g}}(\{\xi\})$ holds for every $\xi \in \mathfrak{T}_{\mathfrak{g}}$. Let $(\xi,\zeta) \in \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ such that $\xi \neq \zeta$. Then

$$\big[(\xi,\zeta) \notin {}_{\lambda=\xi,\zeta}\complement\big(\{\lambda\}\big)\big] \wedge \big[(\xi,\zeta) \in {}_{\lambda=\zeta,\xi}\complement\big(\{\lambda\}\big)\big].$$

But, for every $\lambda \in \{\xi,\zeta\}$, $\complement\big(\{\lambda\}\big) \subseteq \complement\big(\neg\operatorname{op}_{\mathfrak{g}}(\{\lambda\})\big)$, and $\complement\big(\neg\operatorname{op}_{\mathfrak{g}}(\{\lambda\})\big) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ is a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open set. Thus, there exists $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathcal{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}}$ such that $\complement\big(\neg\operatorname{op}_{\mathfrak{g}}(\{\xi\})\big) \subseteq \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})$ and $\complement\big(\neg\operatorname{op}_{\mathfrak{g}}(\{\zeta\})\big) \subseteq \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})$. By substitution, it thus follows that, for every $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\mathcal{S}_{\mathfrak{g}} \supset \emptyset\big]$ such that:

$$\big[(\xi,\zeta) \in {}_{\lambda=\xi,\zeta}\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\big] \wedge \big[(\xi,\zeta) \notin {}_{\lambda=\zeta,\xi}\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\big].$$

Therefore, $\mathfrak{T}_{\mathfrak{g}} = \big(\Omega, \mathcal{T}_{\mathfrak{g}}\big)$ is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{F})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}\big)$; this completes the proof of the theorem. □

Proposition 1. If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{F})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}\big)$, then it is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{K})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{K})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{K})}\big)$.

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{F})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}\big)$. Then, for every $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists a pair $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\mathcal{S}_{\mathfrak{g}} \supset \emptyset\big]$ such that:

$$\big[(\xi,\zeta) \in {}_{\lambda=\xi,\zeta}\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\big] \wedge \big[(\xi,\zeta) \notin {}_{\lambda=\zeta,\xi}\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\big].$$

Set $\mathrm{P}(\xi,\zeta) = \big(\xi \in \operatorname{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\xi}\big)\big) \wedge \big(\zeta \notin \operatorname{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\xi}\big)\big)$. Then, the above logical statement is equivalent to $\mathrm{P}(\xi,\zeta) \wedge \mathrm{P}(\zeta,\xi)$. But, logically,

$$\mathrm{P}(\xi,\zeta) \vee \mathrm{P}(\zeta,\xi) = \mathrm{P}(\xi,\zeta) \vee \mathrm{P}(\zeta,\xi) \vee \big(\mathrm{P}(\xi,\zeta) \wedge \mathrm{P}(\zeta,\xi)\big).$$

Consequently, $\mathrm{P}(\xi,\zeta)\vee\mathrm{P}(\zeta,\xi)\longleftarrow\mathrm{P}(\xi,\zeta)\wedge\mathrm{P}(\zeta,\xi)$, from which it then follows that, if $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathcal{T}_{\mathfrak{g}})$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{F})}=\big(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}\big)$, then for every $(\xi,\zeta)\in$ $\mathfrak{g}$-$\mathrm{F}_{\mathrm{P}}[\sigma>0]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi},\mathcal{O}_{\mathfrak{g},\zeta})\in\mathfrak{g}$-$\mathrm{F}_{\mathrm{O}}\big[\mathcal{S}_{\mathfrak{g}}\supset\emptyset\big]$ such that:

$$\big[\big(\xi\in\mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\xi}\big)\big)\wedge\big(\zeta\notin\mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\xi}\big)\big)\big]\vee\big[\big(\xi\notin\mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\zeta}\big)\big)\wedge\big(\zeta\in\mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\zeta}\big)\big)\big],$$

the logical statement characterising $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathcal{T}_{\mathfrak{g}})$ as a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{K})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{K})}=\big(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{K})}\big)$. ☐

A necessary and sufficient condition for a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=\big(\Omega,\mathcal{T}_{\mathfrak{g}}\big)$ to be a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}=\big(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$ may be given in terms of the $\mathfrak{g}$-operator $\mathrm{op}_{\mathfrak{g}}:\mathcal{P}(\Omega)\longrightarrow\mathcal{P}(\Omega)$, a unit set $\{\xi\}\subset\mathfrak{T}_{\mathfrak{g}}$, and $\mathcal{T}_{\mathfrak{g}}$-closed sets.

Theorem 3.3. *A $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=\big(\Omega,\mathcal{T}_{\mathfrak{g}}\big)$ is said to be a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}=\big(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$ if and only if the following conditions hold:*

$$\{\xi\}=\bigcap_{\mathcal{K}_{\mathfrak{g},\zeta}\in\neg\mathcal{T}_{\mathfrak{g}}}\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\zeta}\right)\quad\forall\xi\in\mathfrak{T}_{\mathfrak{g}}.\tag{3.3}$$

Proof. Necessity. Let the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$. Then, for every $(\xi,\zeta)\in\mathfrak{g}$-$\mathrm{F}_{\mathrm{P}}[\sigma>0]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi},\mathcal{O}_{\mathfrak{g},\zeta})\in\mathfrak{g}$-$\mathrm{F}_{\mathrm{O}}\big[\emptyset\big]$ such that:

$$\big[\xi\in\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)\big]\wedge\big[\zeta\in\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\zeta}\right)\big].$$

Consequently,

$$\big[\xi\notin\complement\big(\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)\big)\big]\wedge\big[\zeta\notin\complement\big(\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\zeta}\right)\big)\big].$$

But, for every $\lambda\in\{\xi,\zeta\}$, $\complement\big(\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\lambda}\right)\big)\in\mathfrak{g}$-$\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set. Consequently, there exists a pair $(\mathcal{K}_{\mathfrak{g},\xi},\mathcal{K}_{\mathfrak{g},\zeta})\in\neg\mathcal{T}_{\mathfrak{g}}\times\neg\mathcal{T}_{\mathfrak{g}}$ such that the relations $\complement\big(\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)\big)\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\zeta}\right)$ and $\complement\big(\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\zeta}\right)\big)\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\xi}\right)$ hold true. Therefore, the relations $\xi\in\complement\big(\{\zeta\}\big)\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}\big(\{\xi\}\big)$ and $\zeta\in\complement\big(\{\xi\}\big)\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}\big(\{\zeta\}\big)$ are true for all $(\xi,\zeta)\in\mathfrak{T}_{\mathfrak{g}}\times\mathfrak{T}_{\mathfrak{g}}$. But, $\complement\big(\{\xi\}\big)=\bigcup_{\mathcal{O}_{\mathfrak{g},\zeta}\in\mathcal{T}_{\mathfrak{g}}}\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\zeta}\right)$ and hence,

$$\{\xi\}=\complement\big(\complement\big(\{\xi\}\big)\big)=\bigcap_{\mathcal{O}_{\mathfrak{g},\zeta}\in\mathcal{T}_{\mathfrak{g}}}\complement\big(\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\zeta}\right)\big)=\bigcap_{\mathcal{K}_{\mathfrak{g},\zeta}\in\neg\mathcal{T}_{\mathfrak{g}}}\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\zeta}\right)\quad\forall\xi\in\mathfrak{T}_{\mathfrak{g}}.$$

Sufficiency. Conversely, suppose $\{\xi\}=\bigcap_{\mathcal{K}_{\mathfrak{g},\zeta}\in\neg\mathcal{T}_{\mathfrak{g}}}\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\zeta}\right)$ holds for all $\xi\in\mathfrak{T}_{\mathfrak{g}}$. Then, there exists a $\mathcal{K}_{\mathfrak{g},\xi}\in\neg\mathcal{T}_{\mathfrak{g}}$ such that $\zeta\notin\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\xi}\right)$. Since $\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\xi}\right)\in\mathfrak{g}$-$\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set $\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)\in\mathfrak{g}$-$\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that $\xi\in\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)\subseteq\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\xi}\right)$. But, since $\complement\big(\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\xi}\right)\big)\in\mathfrak{g}$-$\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set containing $\zeta\in\mathfrak{T}_{\mathfrak{g}}$, it follows that $\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)$, $\complement\big(\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\xi}\right)\big)\in\mathfrak{g}$-$\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ are disjoint $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets. Thus, for every $(\xi,\zeta)\in\mathfrak{g}$-$\mathrm{F}_{\mathrm{P}}[\sigma>0]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi},\mathcal{O}_{\mathfrak{g},\zeta})\in\mathfrak{g}$-$\mathrm{F}_{\mathrm{O}}\big[\emptyset\big]$ such that:

$$\big[\xi\in\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)\big]\wedge\big[\zeta\in\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\zeta}\right)\big].$$

Therefore, $\mathfrak{T}_{\mathfrak{g}}=\big(\Omega,\mathcal{T}_{\mathfrak{g}}\big)$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}=\big(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$; this completes the proof of the theorem. ☐

Proposition 2. If $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathcal{T}_{\mathfrak{g}})$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}=\big(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$, then it is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{F})}=\big(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}\big)$.

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$. Then, for every $(\xi, \zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists a pair $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\emptyset\big]$ such that $\big[\xi \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big] \wedge \big[\zeta \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big]$. But since $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\emptyset\big] \subset \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\mathcal{S}_{\mathfrak{g}} \supset \emptyset\big]$ and

$$\begin{aligned} & \big[\xi \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big] \wedge \big[\zeta \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big] \\ \Leftrightarrow \quad & \big[\big(\xi \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big) \wedge \big(\zeta \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big)\big] \wedge \big[\big(\xi \notin \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big) \\ & \qquad \wedge \big(\zeta \notin \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big)\big], \end{aligned}$$

it follows that, if $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$, then for every $(\xi, \zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists a pair $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\mathcal{S}_{\mathfrak{g}} \supset \emptyset\big]$ such that:

$$\big[(\xi, \zeta) \in {}_{\lambda=\xi,\zeta}\, \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\big] \wedge \big[(\xi, \zeta) \notin {}_{\lambda=\zeta,\xi}\, \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\big].$$

Hence, if $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$, then it is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{F})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}\big)$. ☐

A necessary and sufficient condition for a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = \big(\Omega, \mathcal{T}_{\mathfrak{g}}\big)$ to be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{R})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}\big)$ may be given in terms of the $\mathfrak{g}$-operator $\mathrm{op}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$, a $\mathfrak{T}_{\mathfrak{g}}$-closed set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, and $\mathcal{T}_{\mathfrak{g}}$-closed neighbourhood sets.

Theorem 3.4. *A $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = \big(\Omega, \mathcal{T}_{\mathfrak{g}}\big)$ is said to be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{R})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}\big)$ if and only if the following condition holds:*

$$\mathcal{S}_{\mathfrak{g}} = \bigcap_{\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}} \in \neg\mathcal{T}_{\mathfrak{g}}} \neg\, \mathrm{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big) \quad \forall \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]. \tag{3.4}$$

Proof. Necessity. Let the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{R})}$. Then, for every $(\xi, \zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$ and $(\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{K}}\big[\emptyset\big]$ such that $(\zeta, \xi) \notin (\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta})$, there exists $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\emptyset\big]$ such that:

$$\begin{aligned} & \big[\big(\neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi}) \subset \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big) \wedge \big(\zeta \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big)\big] \vee \big[\big(\neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\zeta}) \\ & \qquad \subset \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big) \wedge \big(\xi \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big)\big]. \end{aligned}$$

Consequently,

$$\begin{aligned} & \big[\big(\neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi}) \subset \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big) \wedge \big(\zeta \notin \complement\big(\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big)\big)\big] \vee \big[\big(\neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\zeta}) \\ & \qquad \subset \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big) \wedge \big(\xi \notin \complement\big(\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big)\big)\big]. \end{aligned}$$

But, for every $\lambda \in \{\xi, \zeta\}$, $\complement\big(\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})\big) \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ is a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closed set. Consequently, there exists $\big(\mathcal{K}_{\mathfrak{g},\mathcal{R}_{\mathfrak{g}}}, \mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big) \in \neg\mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ such that $\complement\big(\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big) \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\mathcal{R}_{\mathfrak{g}}}\big)$ and $\complement\big(\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big) \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)$. Therefore

$$\begin{aligned} & \big[\big(\neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi}) \subset \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big) \wedge \big(\zeta \notin \neg\, \mathrm{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big)\big] \vee \big[\big(\neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\zeta}) \\ & \qquad \subset \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big) \wedge \big(\xi \notin \neg\, \mathrm{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\mathcal{R}_{\mathfrak{g}}}\big)\big)\big]. \end{aligned}$$

By virtue of this logical statement, it consequently follows that

$$\begin{aligned} & \big[\big(\neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi}) \subset \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big) \wedge \big(\neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi}) \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big)\big] \\ \vee & \big[\big(\neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\zeta}) \subset \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big) \wedge \big(\neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi}) \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\mathcal{R}_{\mathfrak{g}}}\big)\big)\big], \end{aligned}$$

and, consequently,

$$\big[\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\right)\subseteq\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\xi}\right)\subset\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)\big]$$
$$\vee\big[\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\mathcal{R}_{\mathfrak{g}}}\right)\subseteq\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\zeta}\right)\subset\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\zeta}\right)\big],$$

But, for any $\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$, $\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\right)\supseteq\mathcal{S}_{\mathfrak{g}}$. Consequently,

$$\big[\mathcal{S}_{\mathfrak{g}}\subseteq\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\right)\subseteq\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\xi}\right)\subset\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)\big]$$
$$\vee\big[\mathcal{R}_{\mathfrak{g}}\subseteq\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\mathcal{R}_{\mathfrak{g}}}\right)\subseteq\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\zeta}\right)\subset\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\zeta}\right)\big],$$

Hence, $\mathcal{S}_{\mathfrak{g}}=\bigcap_{\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\in\neg\mathcal{T}_{\mathfrak{g}}}\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\right)$ for all $\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$.

Sufficiency. Conversely, suppose $\mathcal{S}_{\mathfrak{g}}=\bigcap_{\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\in\neg\mathcal{T}_{\mathfrak{g}}}\neg\operatorname{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)$ holds for all $\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$, let $\xi\notin\mathcal{S}_{\mathfrak{g}}$. Then, $\mathcal{S}_{\mathfrak{g}}\subseteq\neg\operatorname{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)$ for every $\mathcal{T}_{\mathfrak{g}}$-closed neighbourhood set $\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\in\neg\mathcal{T}_{\mathfrak{g}}$ satisfying $\mathcal{S}_{\mathfrak{g}}\subseteq\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}$. Therefore, there exists a $\mathcal{T}_{\mathfrak{g}}$-closed neighbourhood set $\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\in\neg\mathcal{T}_{\mathfrak{g}}$ such that $\xi\notin\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}$. But, since $\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\in\neg\mathcal{T}_{\mathfrak{g}}$ is a $\mathcal{T}_{\mathfrak{g}}$-closed neighbourhood set, there exists a $\mathcal{T}_{\mathfrak{g}}$-open set $\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\in\mathcal{T}_{\mathfrak{g}}$ such that $\mathcal{S}_{\mathfrak{g}}\subset\operatorname{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\subset\neg\operatorname{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)$, and $\complement(\mathcal{S}_{\mathfrak{g}})\supset\complement\big(\operatorname{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big)\supset\complement\big(\neg\operatorname{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big)$. Because $\xi\in\complement\big(\neg\operatorname{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big)$ and $\xi\in\complement\big(\operatorname{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big)$, it follows that $\operatorname{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\cap\complement\big(\neg\operatorname{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big)=\emptyset$ and $\complement\big(\operatorname{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big)\cap\neg\operatorname{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)=\emptyset$, respectively. In other words, for every $(\xi,\zeta)\in\mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}\left[\sigma>0\right]$ and $(\mathcal{K}_{\mathfrak{g},\xi},\mathcal{K}_{\mathfrak{g},\zeta})\in\mathfrak{g}\text{-}\mathrm{F}_{\mathrm{K}}\big[\emptyset\big]$ such that $(\zeta,\xi)\notin(\mathcal{K}_{\mathfrak{g},\xi},\mathcal{K}_{\mathfrak{g},\zeta})$, there exists $(\mathcal{O}_{\mathfrak{g},\xi},\mathcal{O}_{\mathfrak{g},\zeta})\in\mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\emptyset\big]$ such that:

$$\big[\big(\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\xi}\right)\subset\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)\big)\wedge\big(\zeta\in\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\zeta}\right)\big)\big]\vee\big[\big(\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\zeta}\right)$$
$$\subset\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\zeta}\right)\big)\wedge\big(\xi\in\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)\big)\big].$$

Therefore, $\mathfrak{T}_{\mathfrak{g}}=\big(\Omega,\mathcal{T}_{\mathfrak{g}}\big)$ is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{R})}=\big(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}\big)$; this completes the proof of the theorem. □

Proposition 3. If $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathcal{T}_{\mathfrak{g}})$ is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{R})}=\big(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}\big)$, then it is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}=\big(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$.

Proof. Let $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathcal{T}_{\mathfrak{g}})$ be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{R})}=\big(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}\big)$. Then, for every $(\xi,\zeta)\in\mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}\left[\sigma>0\right]$ and $(\mathcal{K}_{\mathfrak{g},\xi},\mathcal{K}_{\mathfrak{g},\zeta})\in\mathfrak{g}\text{-}\mathrm{F}_{\mathrm{K}}\big[\emptyset\big]$ such that $(\zeta,\xi)\notin(\mathcal{K}_{\mathfrak{g},\xi},\mathcal{K}_{\mathfrak{g},\zeta})$, there exists $(\mathcal{O}_{\mathfrak{g},\xi},\mathcal{O}_{\mathfrak{g},\zeta})\in\mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\emptyset\big]$ such that:

$$\big[\big(\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\xi}\right)\subset\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)\big)\wedge\big(\zeta\in\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\zeta}\right)\big)\big]\vee\big[\big(\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\zeta}\right)$$
$$\subset\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\zeta}\right)\big)\wedge\big(\xi\in\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)\big)\big].$$

Set $\mathrm{Q}\left(\xi\right)=\big(\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\xi}\right)\subset\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)\big)$, $\mathrm{R}\left(\zeta\right)=\big(\zeta\in\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\zeta}\right)\big)$, and $\mathrm{P}\left(\xi,\zeta\right)=\mathrm{Q}\left(\xi\right)\wedge\mathrm{R}\left(\zeta\right)$. Then, the above logical statement is equivalent to $\mathrm{P}\left(\xi,\zeta\right)\vee\mathrm{P}\left(\zeta,\xi\right)$. But since $\lambda\in\neg\operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\lambda}\right)$ for every $\lambda\left\{\xi,\zeta\right\}$, it consequently follows that $\mathrm{R}\left(\lambda\right)\longleftarrow\mathrm{Q}\left(\lambda\right)$ for every $\lambda\left\{\xi,\zeta\right\}$. Therefore $\mathrm{R}\left(\xi\right)\wedge\mathrm{R}\left(\zeta\right)\longleftarrow\mathrm{P}\left(\xi,\zeta\right)$. Because associativity with respect to $\wedge$ holds, it then follows that

$$\mathrm{P}\left(\xi,\zeta\right)\vee\mathrm{P}\left(\zeta,\xi\right)\longrightarrow\left[\mathrm{R}\left(\xi\right)\wedge\mathrm{R}\left(\zeta\right)\right]\vee\left[\mathrm{R}\left(\zeta\right)\wedge\mathrm{R}\left(\xi\right)\right]=\mathrm{R}\left(\xi\right)\wedge\mathrm{R}\left(\zeta\right).$$

Hence, for every $(\xi,\zeta)\in\mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}\left[\sigma>0\right]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi},\mathcal{O}_{\mathfrak{g},\zeta})\in\mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\emptyset\big]$ such that:

$$\big[\xi\in\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\xi}\right)\big]\wedge\big[\zeta\in\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\zeta}\right)\big].$$

This proves that, if $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathcal{T}_{\mathfrak{g}})$ is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{R})}=\big(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}\big)$, then it is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}=\big(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$. □

A necessary and sufficient condition for a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = \big(\Omega, \mathcal{T}_{\mathfrak{g}}\big)$ to be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{N})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}\big)$ may be given in terms of the $\mathfrak{g}$-operator $\mathrm{op}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$, a $\mathfrak{T}_{\mathfrak{g}}$-closed set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, $\mathcal{T}_{\mathfrak{g}}$-open sets, and a $\mathcal{T}_{\mathfrak{g}}$-closed set.

Theorem 3.5. *A $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = \big(\Omega, \mathcal{T}_{\mathfrak{g}}\big)$ is said to be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{N})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}\big)$ if and only if the following condition holds:*

$$\mathcal{S}_{\mathfrak{g}} \subset \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{O}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big) \subset \neg\, \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{K}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big) \subset \mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big) \quad \forall \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]. \tag{3.5}$$

Proof. Necessity. Let the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{N})}$ and, let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ and $\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}} \in \mathcal{T}_{\mathfrak{g}}$, respectively, be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closed set and a $\mathcal{T}_{\mathfrak{g}}$-open neighbourhood set of $\mathcal{S}_{\mathfrak{g}}$. Then, $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{N})}$ implies that, for every $(\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{K}}[\emptyset]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}[\emptyset]$ such that:

$$\big[\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi}) \supset \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi})\big] \wedge \big[\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta}) \supset \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\zeta})\big].$$

Clearly, $\complement\big(\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big) \cap \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi}) = \emptyset$ for any $\xi \in \mathfrak{T}_{\mathfrak{g}}$. The relation $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ implies that there exists a $\hat{\mathcal{K}}_{\mathfrak{g},\xi} \in \neg\mathcal{T}_{\mathfrak{g}}$ such that $\mathcal{S}_{\mathfrak{g}} \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{K}}_{\mathfrak{g},\xi}\big)$ and, $\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}} \in \mathcal{T}_{\mathfrak{g}}$ is a $\mathcal{T}_{\mathfrak{g}}$-open neighbourhood set of $\mathcal{S}_{\mathfrak{g}}$ implies that there exists $\big(\hat{\mathcal{O}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}, \hat{\mathcal{K}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big) \in \mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ such that $\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}} \subseteq \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{O}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big) \subset \neg\, \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{K}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)$. But, $\mathcal{S}_{\mathfrak{g}} \subset \mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}$ and, for some $\hat{\mathcal{O}}_{\mathfrak{g},\xi} \in \neg\mathcal{T}_{\mathfrak{g}}$, the relation $\neg\, \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{K}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big) \subset \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{O}}_{\mathfrak{g},\xi}\big)$ holds in a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{N})}$. Hence,

$$\neg\, \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{K}}_{\mathfrak{g},\xi}\big) \subseteq \mathcal{S}_{\mathfrak{g}} \subset \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{O}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big) \subset \neg\, \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{K}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big) \subset \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{O}}_{\mathfrak{g},\xi}\big)$$

for all $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. At this stage, it suffices to set $\hat{\mathcal{O}}_{\mathfrak{g},\xi} \subseteq \mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}$ and the result follows.

Sufficiency. Conversely, suppose the following relation holds:

$$\mathcal{S}_{\mathfrak{g}} \subset \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{O}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big) \subset \neg\, \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{K}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big) \subset \mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big) \quad \forall \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big].$$

Then, its complementary reads

$$\complement\big(\mathcal{S}_{\mathfrak{g}}\big) \supset \complement\big(\mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{O}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big) \supset \complement\big(\neg\, \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{K}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big) \supset \complement\big(\mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big),$$

where $\complement\big(\mathcal{S}_{\mathfrak{g}}\big)$, $\complement\big(\neg\, \mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{K}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ are $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open sets and, $\complement\big(\mathrm{op}_{\mathfrak{g}}\big(\hat{\mathcal{O}}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big)$, $\complement\big(\mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big) \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ are $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closed sets. Thus, $\mathcal{S}_{\mathfrak{g}} \cap \complement\big(\mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big) = \emptyset$ for any $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. But since the relation $\mathcal{S}_{\mathfrak{g}} \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi})$ holds for some $\mathcal{K}_{\mathfrak{g},\xi} \in \neg\mathcal{T}_{\mathfrak{g}}$, it consequently follows that $\complement\big(\mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big)\big) \subset \complement\big(\mathcal{S}_{\mathfrak{g}}\big) \subseteq \complement\big(\neg\, \mathrm{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\xi}\big)\big)$ which, in turn, implies $\mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\mathcal{S}_{\mathfrak{g}}}\big) \supset \mathcal{S}_{\mathfrak{g}} \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}\big(\mathcal{K}_{\mathfrak{g},\xi}\big)$. Thus, for every $(\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{K}}[\emptyset]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}[\emptyset]$ such that:

$$\big[\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi}) \supset \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi})\big] \wedge \big[\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta}) \supset \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\zeta})\big].$$

Therefore, $\mathfrak{T}_{\mathfrak{g}} = \big(\Omega, \mathcal{T}_{\mathfrak{g}}\big)$ is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{N})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}\big)$; this completes the proof of the theorem. □

Proposition 4. If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{N})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}\big)$, then it is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{R})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}\big)$.

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{N})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}\big)$. Then, for every $(\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{K}}[\emptyset]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}[\emptyset]$ such that:

$$\big[\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi}) \supset \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi})\big] \wedge \big[\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta}) \supset \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\zeta})\big].$$

Set $\mathrm{Q}(\xi) = \big(\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi}) \subset \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big)$ and $\mathrm{R}(\xi) = \big(\xi \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big)$ so that the above logical statement now reads $\mathrm{Q}(\xi) \wedge \mathrm{Q}(\zeta)$. Then, since $(\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{K}}[\emptyset]$, $\mathrm{Q}(\xi) \wedge \mathrm{R}(\zeta) \longleftarrow \mathrm{Q}(\xi)$ and $\mathrm{Q}(\zeta) \wedge \mathrm{R}(\xi) \longleftarrow \mathrm{Q}(\zeta)$ hold. Consequently,

$$\big[\mathrm{Q}(\xi) \wedge \mathrm{R}(\zeta)\big] \wedge \big[\mathrm{Q}(\zeta) \wedge \mathrm{R}(\xi)\big] \longleftarrow \mathrm{Q}(\xi) \wedge \mathrm{Q}(\zeta).$$

But

$$\big[\mathrm{Q}(\xi) \wedge \mathrm{R}(\zeta)\big] \vee \big[\mathrm{Q}(\zeta) \wedge \mathrm{R}(\xi)\big] \longleftarrow \big[\mathrm{Q}(\xi) \wedge \mathrm{R}(\zeta)\big] \wedge \big[\mathrm{Q}(\zeta) \wedge \mathrm{R}(\xi)\big],$$

and, therefore,

$$\big[\mathrm{Q}(\xi) \wedge \mathrm{R}(\zeta)\big] \vee \big[\mathrm{Q}(\zeta) \wedge \mathrm{R}(\xi)\big] \longleftarrow \mathrm{Q}(\xi) \wedge \mathrm{Q}(\zeta).$$

Thus, for every $(\xi, \zeta) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0]$ and $(\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{K}}\big[\emptyset\big]$ such that $(\zeta, \xi) \notin (\mathcal{K}_{\mathfrak{g},\xi}, \mathcal{K}_{\mathfrak{g},\zeta})$, there exists $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{O}}\big[\emptyset\big]$ such that:

$$\begin{aligned}\big[\big(\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\xi}) \subset \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big) \wedge \big(\zeta \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big)\big] \vee \big[\big(\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\zeta}) \\ \subset \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})\big) \wedge \big(\xi \in \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})\big)\big].\end{aligned}$$

This proves that, if $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ is a $\mathfrak{g}\text{-}\mathcal{T}^{(\mathrm{N})}_{\mathfrak{g}}$-space $\mathfrak{g}\text{-}\mathfrak{T}^{(\mathrm{N})}_{\mathfrak{g}} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}^{(\mathrm{N})}_{\mathfrak{g}}\big)$, then it is a $\mathfrak{g}\text{-}\mathcal{T}^{(\mathrm{R})}_{\mathfrak{g}}$-space $\mathfrak{g}\text{-}\mathfrak{T}^{(\mathrm{R})}_{\mathfrak{g}} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}^{(\mathrm{R})}_{\mathfrak{g}}\big)$. □

By virtue of the above propositions, every $\mathfrak{g}\text{-}\mathcal{T}^{(\mathrm{H})}_{\mathfrak{g}}$-space is a $\mathfrak{g}\text{-}\mathcal{T}^{(\mathrm{F})}_{\mathfrak{g}}$-space, and hence, a $\mathfrak{g}\text{-}\mathcal{T}^{(\mathrm{K})}_{\mathfrak{g}}$-space. Also, every $\mathfrak{g}\text{-}\mathcal{T}^{(\mathrm{N})}_{\mathfrak{g}}$-space is a $\mathfrak{g}\text{-}\mathcal{T}^{(\mathrm{R})}_{\mathfrak{g}}$-space, and hence, a $\mathfrak{g}\text{-}\mathcal{T}^{(\mathrm{H})}_{\mathfrak{g}}$-space. But, the converse of both statements are untrue, and thus, the corollary follows. If $\big\langle \mathfrak{g}\text{-}\mathfrak{T}^{(\alpha)}_{\mathfrak{g}} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}^{(\alpha)}_{\mathfrak{g}}\big)\big\rangle_{\alpha\in\Lambda}$, $\Lambda = \big\{\mathrm{K}, \mathrm{F}, \mathrm{H}, \mathrm{R}, \mathrm{N}\big\}$, denotes a sequence of $\mathfrak{g}\text{-}\mathcal{T}^{(\alpha)}_{\mathfrak{g}}$-spaces, obtained after endowing a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = \big(\Omega, \mathcal{T}_{\mathfrak{g}}\big)$ with the sequence of $\mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}$-axioms $\big\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\big\rangle_{\alpha\in\Lambda}$, then the following relations hold:

- I. $\mathfrak{T}^{(\mathrm{K})}_{\mathfrak{g}} \subseteq \mathfrak{T}^{(\mathrm{F})}_{\mathfrak{g}} \subseteq \mathfrak{T}^{(\mathrm{H})}_{\mathfrak{g}} \subseteq \mathfrak{T}^{(\mathrm{R})}_{\mathfrak{g}} \subseteq \mathfrak{T}^{(\mathrm{N})}_{\mathfrak{g}}$.
- II. $\mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\mathrm{N}} \Rightarrow \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\mathrm{R}} \Rightarrow \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\mathrm{H}} \Rightarrow \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\mathrm{F}} \Rightarrow \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\mathrm{K}}$.

4. Discussion

4.1. **Categorical Classifications.** Having adopted a categorical approach in the classifications of the $\mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}$-axioms, $\alpha \in \Lambda = \big\{\mathrm{K}, \mathrm{F}, \mathrm{H}, \mathrm{R}, \mathrm{N}\big\}$, in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, the aims here are, to establish the various relationships amongst the elements of the sequence $\big\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\big\rangle_{\alpha\in\Lambda}$ and, to illustrate them through diagrams.

We have seen that, both the $\mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\mathrm{N}}$, $\mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\mathrm{R}}$-axioms imply the $\mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\mathrm{K}}$, $\mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\mathrm{F}}$-axioms and, on the other hand, the $\mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\mathrm{N}}$-axiom implies the $\mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\mathrm{R}}$-axiom and the $\mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\mathrm{H}}$-axiom implies the $\mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\mathrm{F}}$-axioms. The *separation axioms diagram* presented in Figs 1 illustrates these implications.

We called the elements of the sequence $\big\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\big\rangle_{\alpha\in\Lambda}$ $\mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}$*-axioms*. To this end it does make sense to call those of $\big\langle \mathrm{T}_{\mathfrak{g},\alpha}\big\rangle_{\alpha\in\Lambda}$ $\mathrm{T}_{\mathfrak{g},\alpha}$*-axioms*. Thus, in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$, $\big\langle \mathrm{T}_{\mathfrak{g},\alpha}\big\rangle_{\alpha\in\Lambda}$ stands for a sequence of separation axioms in the *ordinary sense* while $\big\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\big\rangle_{\alpha\in\Lambda}$ stands for its analogue but in the *generalized sense*, just as, in a $\mathcal{T}$-space $\mathfrak{T} = (\Omega, \mathcal{T})$, $\big\langle \mathrm{T}_{\alpha}\big\rangle_{\alpha\in\Lambda}$ stands for a sequence of separation axioms in the *ordinary sense* while $\big\langle \mathfrak{g}\text{-}\mathrm{T}_{\alpha}\big\rangle_{\alpha\in\Lambda}$ stands for its analogue but in the *generalized*

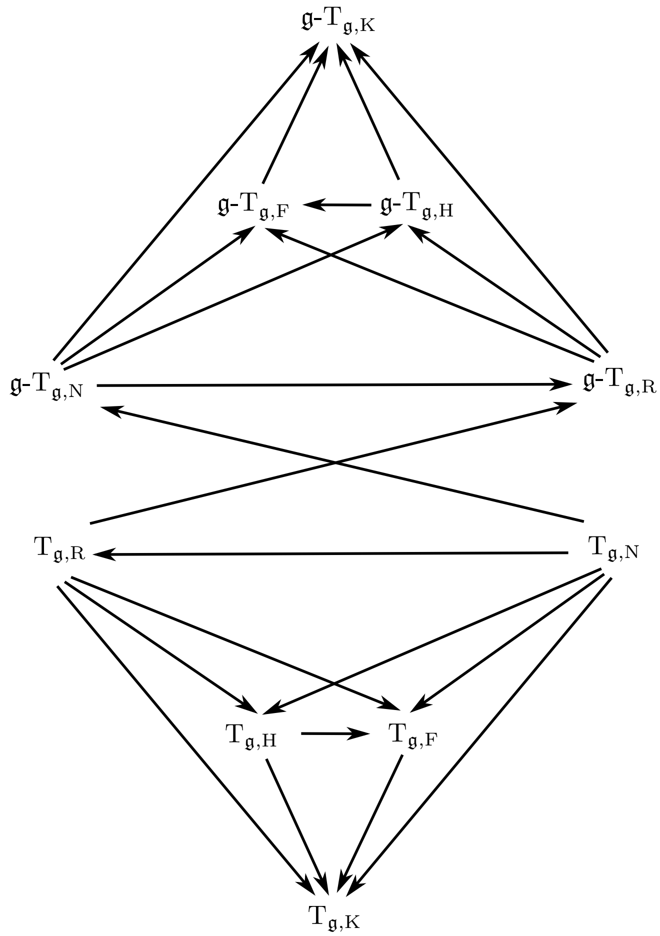

FIGURE 1. Relationships: Separation axioms diagram.

sense. Let $\mathrm{F}_{\mathrm{P}}[\sigma] = \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma]$ and set

$$\begin{aligned} \mathrm{F}_{\mathrm{O}}[\mathcal{S}_{\mathfrak{g}}] &\stackrel{\mathrm{def}}{=} \big\{(\mathcal{O}_{\mathfrak{g},\xi},\mathcal{O}_{\mathfrak{g},\zeta}) \in \mathcal{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} :\ \textstyle\bigcap_{\lambda=\xi,\zeta}\mathcal{O}_{\mathfrak{g},\lambda} \subseteq \mathcal{S}_{\mathfrak{g}}\big\}, \\ \mathrm{F}_{\mathrm{K}}[\mathcal{S}_{\mathfrak{g}}] &\stackrel{\mathrm{def}}{=} \big\{(\mathcal{K}_{\mathfrak{g},\xi},\mathcal{K}_{\mathfrak{g},\zeta}) \in \neg\mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}} :\ \mathcal{S}_{\mathfrak{g}} \supseteq \textstyle\bigcap_{\lambda=\xi,\zeta}\mathcal{K}_{\mathfrak{g},\lambda}\big\}, \end{aligned} \tag{4.1}$$

where $\sigma \geq 0$ and $\mathcal{S}_{\mathfrak{g}} \supseteq \emptyset$. Then, the notions of $\mathrm{T}_{\mathfrak{g},\mathrm{K}}$, $\mathrm{T}_{\mathfrak{g},\mathrm{F}}$, $\mathrm{T}_{\mathfrak{g},\mathrm{H}}$, $\mathrm{T}_{\mathfrak{g},\mathrm{R}}$, and $\mathrm{T}_{\mathfrak{g},\mathrm{N}}$-axioms in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ may well be defined as follows:

- I. $\mathrm{T}_{\mathfrak{g},\mathrm{K}}$-AXIOM: For every $(\xi,\zeta) \in \mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi},\mathcal{O}_{\mathfrak{g},\zeta}) \in \mathrm{F}_{\mathrm{O}}\big[\mathcal{S}_{\mathfrak{g}} \supset \emptyset\big]$ such that:

$$\big[(\xi \in \mathcal{O}_{\mathfrak{g},\xi}) \wedge (\zeta \notin \mathcal{O}_{\mathfrak{g},\xi})\big] \vee \big[(\xi \notin \mathcal{O}_{\mathfrak{g},\zeta}) \wedge (\zeta \in \mathcal{O}_{\mathfrak{g},\zeta})\big] \tag{4.2}$$

- II. $\mathrm{T}_{\mathfrak{g},\mathrm{F}}$-AXIOM: For every $(\xi,\zeta) \in \mathrm{F}_{\mathrm{P}}[\sigma > 0]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi},\mathcal{O}_{\mathfrak{g},\zeta}) \in \mathrm{F}_{\mathrm{O}}\big[\mathcal{S}_{\mathfrak{g}} \supset \emptyset\big]$ such that:

$$\big[(\xi,\zeta) \in {}_{\lambda=\xi,\zeta}\mathcal{O}_{\mathfrak{g},\lambda}\big] \wedge \big[(\xi,\zeta) \notin {}_{\lambda=\zeta,\xi}\mathcal{O}_{\mathfrak{g},\lambda}\big]. \tag{4.3}$$

- III. $\mathrm{T}_{\mathfrak{g},\mathrm{H}}$-AXIOM: For every $(\xi,\zeta)\in\mathrm{F}_{\mathrm{P}}[\sigma>0]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi},\mathcal{O}_{\mathfrak{g},\zeta})\in\mathrm{F}_{\mathrm{O}}[\emptyset]$ such that:

$$\left[\xi\in\mathcal{O}_{\mathfrak{g},\xi}\right]\wedge\left[\zeta\in\mathcal{O}_{\mathfrak{g},\zeta}\right]. \tag{4.4}$$

- IV. $\mathrm{T}_{\mathfrak{g},\mathrm{R}}$-AXIOM: For every $(\xi,\zeta)\in\mathrm{F}_{\mathrm{P}}[\sigma>0]$ and $(\mathcal{K}_{\mathfrak{g},\xi},\mathcal{K}_{\mathfrak{g},\zeta})\in\mathrm{F}_{\mathrm{K}}[\emptyset]$ such that $(\zeta,\xi)\notin(\mathcal{K}_{\mathfrak{g},\xi},\mathcal{K}_{\mathfrak{g},\zeta})$, there exists $(\mathcal{O}_{\mathfrak{g},\xi},\mathcal{O}_{\mathfrak{g},\zeta})\in\mathrm{F}_{\mathrm{O}}[\emptyset]$ such that:

$$\left[\left(\mathcal{K}_{\mathfrak{g},\xi}\subset\mathcal{O}_{\mathfrak{g},\xi}\right)\wedge\left(\zeta\in\mathcal{O}_{\mathfrak{g},\zeta}\right)\right]\vee\left[\left(\mathcal{K}_{\mathfrak{g},\zeta}\subset\mathcal{O}_{\mathfrak{g},\zeta}\right)\wedge\left(\xi\in\mathcal{O}_{\mathfrak{g},\xi}\right)\right]. \tag{4.5}$$

- V. $\mathrm{T}_{\mathfrak{g},\mathrm{N}}$-AXIOM: For every $(\mathcal{K}_{\mathfrak{g},\xi},\mathcal{K}_{\mathfrak{g},\zeta})\in\mathrm{F}_{\mathrm{K}}[\emptyset]$, there exists $(\mathcal{O}_{\mathfrak{g},\xi},\mathcal{O}_{\mathfrak{g},\zeta})\in\mathrm{F}_{\mathrm{O}}[\emptyset]$ such that:

$$\left[\mathcal{O}_{\mathfrak{g},\xi}\supset\mathcal{K}_{\mathfrak{g},\xi}\right]\wedge\left[\mathcal{O}_{\mathfrak{g},\zeta}\supset\mathcal{K}_{\mathfrak{g},\zeta}\right]. \tag{4.6}$$

By virtue of the relations $\mathcal{O}_{\mathfrak{g},\lambda}\subseteq\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\lambda})$ and $\mathcal{K}_{\mathfrak{g},\lambda}\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\lambda})$ for every $(\mathcal{O}_{\mathfrak{g},\lambda},\mathcal{K}_{\mathfrak{g},\lambda},\lambda)\in\mathcal{T}_{\mathfrak{g}}\times\neg\mathcal{T}_{\mathfrak{g}}\times\{\xi,\zeta\}$, these implications follow: $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{K}}\longleftarrow\mathrm{T}_{\mathfrak{g},\mathrm{K}}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{F}}\longleftarrow\mathrm{T}_{\mathfrak{g},\mathrm{F}}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{H}}\longleftarrow\mathrm{T}_{\mathfrak{g},\mathrm{H}}$, $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{R}}\longleftarrow\mathrm{T}_{\mathfrak{g},\mathrm{R}}$, and $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{N}}\longleftarrow\mathrm{T}_{\mathfrak{g},\mathrm{N}}$. When the statements preceding the above definitions are taken into account, another separation axioms diagram is obtained. In FIG. 2, we have illustrated the various relationships amongst the elements of $\langle\mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\rangle_{\alpha\in\Lambda}$ and $\langle\mathrm{T}_{\mathfrak{g},\alpha}\rangle_{\alpha\in\Lambda}$. It is interesting to present a third separation axioms diagram illustrating both the implications and the categorical classifications of the elements of $\langle\mathfrak{g}\text{-}\nu\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\rangle_{\alpha\in\Lambda}$, where $\nu\in I_3^0$.

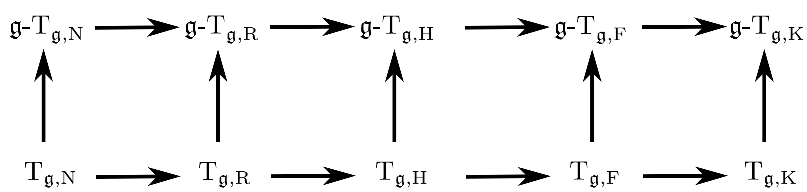

FIGURE 2. Relationships: Separation axioms diagram.

For every fixed $\nu\in I_3^0$, it is immediate that the implications $\mathfrak{g}$-ν-$\mathrm{T}_{\mathfrak{g},\mathrm{K}}\longleftarrow\mathfrak{g}$-$\nu$-$\mathrm{T}_{\mathfrak{g},\mathrm{F}}$, $\mathfrak{g}$-ν-$\mathrm{T}_{\mathfrak{g},\mathrm{F}}\longleftarrow\mathfrak{g}$-$\nu$-$\mathrm{T}_{\mathfrak{g},\mathrm{H}}$, $\mathfrak{g}$-ν-$\mathrm{T}_{\mathfrak{g},\mathrm{H}}\longleftarrow\mathfrak{g}$-$\nu$-$\mathrm{T}_{\mathfrak{g},\mathrm{R}}$, and $\mathfrak{g}$-ν-$\mathrm{T}_{\mathfrak{g},\mathrm{R}}\longleftarrow\mathfrak{g}$-$\nu$-$\mathrm{T}_{\mathfrak{g},\mathrm{N}}$ hold. On the other hand, we saw in the first part of our works, on the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, that

$$\begin{aligned}\mathrm{op}_{\mathfrak{g},0}(\mathcal{S}_{\mathfrak{g}})\subseteq\mathrm{op}_{\mathfrak{g},1}(\mathcal{S}_{\mathfrak{g}})\subseteq\mathrm{op}_{\mathfrak{g},3}(\mathcal{S}_{\mathfrak{g}})\supseteq\mathrm{op}_{\mathfrak{g},2}(\mathcal{S}_{\mathfrak{g}})\ \ \forall\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}},\\ \neg\,\mathrm{op}_{\mathfrak{g},0}(\mathcal{S}_{\mathfrak{g}})\supseteq\neg\,\mathrm{op}_{\mathfrak{g},1}(\mathcal{S}_{\mathfrak{g}})\supseteq\neg\,\mathrm{op}_{\mathfrak{g},3}(\mathcal{S}_{\mathfrak{g}})\subseteq\neg\,\mathrm{op}_{\mathfrak{g},2}(\mathcal{S}_{\mathfrak{g}}),\end{aligned} \tag{4.7}$$

as a consequence of the definitions of the $\mathfrak{g}$-operators $\mathrm{op}_{\mathfrak{g},\nu},\ \neg\,\mathrm{op}_{\mathfrak{g},\nu}:\mathcal{P}(\Omega)\longrightarrow\mathcal{P}(\Omega)$. Hence, it results that, for every $\alpha\in\Lambda$, $\mathfrak{g}$-0-$\mathrm{T}_{\mathfrak{g},\alpha}\longrightarrow\mathfrak{g}$-1-$\mathrm{T}_{\mathfrak{g},\alpha}\longrightarrow\mathfrak{g}$-3-$\mathrm{T}_{\mathfrak{g},\alpha}$ and $\mathfrak{g}$-3-$\mathrm{T}_{\mathfrak{g},\alpha}\longleftarrow\mathfrak{g}$-2-$\mathrm{T}_{\mathfrak{g},\alpha}$. When these properties are taken into consideration, the resulting separation axioms diagram so obtained is that presented in FIG. 3. It is reasonable to call them $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-*axioms of type* α *and of category* ν, where $(\alpha,\nu)\in\Lambda\times I_3^0$.

In order to exemplify the concept of $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axiom of type α and of category ν, where $(\alpha,\nu)\in\Lambda\times I_3^0$, a nice application is presented in the following section.

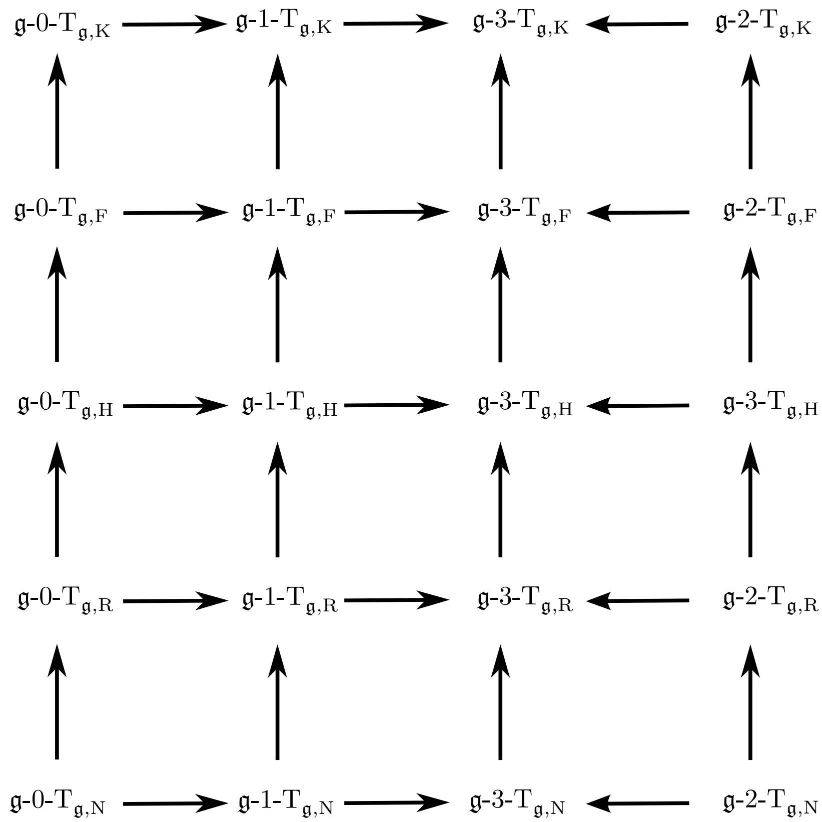

FIGURE 3. Relationships: Separation axiom diagram.

4.2. **A Nice Application.** Focusing on the fundamental notions of the sequence $\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}\rangle_{\alpha\in\Lambda}$ of $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms, $\Lambda = \{\mathrm{K},\mathrm{F},\mathrm{H},\mathrm{R},\mathrm{N}\}$, in a $\mathcal{T}_{\mathfrak{g}}$-space, founded upon the class of $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open sets, we shall now present a nice application. Let $\Omega = \{\xi_\nu : \nu \in I_3^*\}$ denotes the underlying set and consider the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$, where

$$\begin{aligned} \mathcal{T}_{\mathfrak{g}}(\Omega) &= \{\emptyset, \{\xi_1\}, \{\xi_2\}, \{\xi_3\}, \{\xi_1,\xi_2\}, \{\xi_1,\xi_3\}, \{\xi_2,\xi_3\}, \Omega\} \\ &= \{\mathcal{O}_{\mathfrak{g},1}, \mathcal{O}_{\mathfrak{g},2}, \mathcal{O}_{\mathfrak{g},3}, \mathcal{O}_{\mathfrak{g},4}, \mathcal{O}_{\mathfrak{g},5}, \mathcal{O}_{\mathfrak{g},6}, \mathcal{O}_{\mathfrak{g},7}, \mathcal{O}_{\mathfrak{g},8}\}, \qquad (4.8) \\ \neg\mathcal{T}_{\mathfrak{g}}(\Omega) &= \{\Omega, \{\xi_2,\xi_3\}, \{\xi_1,\xi_3\}, \{\xi_1,\xi_2\}, \{\xi_3\}, \{\xi_2\}, \{\xi_1\}, \emptyset\} \\ &= \{\mathcal{K}_{\mathfrak{g},1}, \mathcal{K}_{\mathfrak{g},2}, \mathcal{K}_{\mathfrak{g},3}, \mathcal{K}_{\mathfrak{g},4}, \mathcal{K}_{\mathfrak{g},5}, \mathcal{K}_{\mathfrak{g},6}, \mathcal{K}_{\mathfrak{g},7}, \mathcal{K}_{\mathfrak{g},8}\}, \qquad (4.9) \end{aligned}$$

respectively, stand for the classes of $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets. In both settings, the $\mathcal{T}_{\mathfrak{g}}$-open, closed sets occupying the ν^{th} position corresponds to $\mathcal{O}_{\mathfrak{g},\nu}$, $\mathcal{K}_{\mathfrak{g},\nu}$, $\nu \in I_8^*$, respectively, as is easily understood. Since conditions $\mathcal{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}) \subseteq \mathcal{O}_{\mathfrak{g},\nu}$ for every $\nu \in I_8^*$, and $\mathcal{T}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_8^*} \mathcal{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu\in I_8^*} \mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$ are satisfied, it is clear that the one-valued map $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}\big(\{\xi_\nu : \ \nu \in I_8^*\}\big)$ is a $\mathfrak{g}$-topology. After computing the elements of the set $\{\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\xi_\alpha}) : (\alpha,\nu) \in I_8^* \times I_3^0\}$, called $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open sets, we obtain:

$$\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\xi_\alpha}) \in \begin{cases} \{\mathcal{O}_{\mathfrak{g},2}, \mathcal{O}_{\mathfrak{g},5}, \mathcal{O}_{\mathfrak{g},6}, \mathcal{O}_{\mathfrak{g},8}\} & \forall(\alpha,\nu) \in \{1\} \times \{0,2\}, \\ \{\mathcal{O}_{\mathfrak{g},3}, \mathcal{O}_{\mathfrak{g},5}, \mathcal{O}_{\mathfrak{g},7}, \mathcal{O}_{\mathfrak{g},8}\} & \forall(\alpha,\nu) \in \{2\} \times \{0,2\}, \\ \{\mathcal{O}_{\mathfrak{g},4}, \mathcal{O}_{\mathfrak{g},6}, \mathcal{O}_{\mathfrak{g},7}, \mathcal{O}_{\mathfrak{g},8}\} & \forall(\alpha,\nu) \in \{3\} \times \{0,2\}, \\ \{\mathcal{K}_{\mathfrak{g},1}, \mathcal{K}_{\mathfrak{g},3}, \mathcal{K}_{\mathfrak{g},4}, \mathcal{K}_{\mathfrak{g},7}\} & \forall(\alpha,\nu) \in \{1\} \times \{1,3\}, \\ \{\mathcal{K}_{\mathfrak{g},1}, \mathcal{K}_{\mathfrak{g},2}, \mathcal{K}_{\mathfrak{g},4}, \mathcal{K}_{\mathfrak{g},6}\} & \forall(\alpha,\nu) \in \{2\} \times \{1,3\}, \\ \{\mathcal{K}_{\mathfrak{g},1}, \mathcal{K}_{\mathfrak{g},2}, \mathcal{K}_{\mathfrak{g},3}, \mathcal{K}_{\mathfrak{g},5}\} & \forall(\alpha,\nu) \in \{3\} \times \{1,3\}. \end{cases}$$

(4.10)

Similarly, the elements of $\{\neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\xi_\alpha}) : (\alpha,\nu) \in I_8^* \times I_3^0\}$, called $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed sets, are:

$$\neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\xi_\alpha}) \in \begin{cases} \{\mathcal{K}_{\mathfrak{g},1}, \mathcal{K}_{\mathfrak{g},3}, \mathcal{K}_{\mathfrak{g},4}, \mathcal{K}_{\mathfrak{g},7}\} & \forall(\alpha,\nu) \in \{1\} \times \{0,2\}, \\ \{\mathcal{K}_{\mathfrak{g},1}, \mathcal{K}_{\mathfrak{g},2}, \mathcal{K}_{\mathfrak{g},4}, \mathcal{K}_{\mathfrak{g},6}\} & \forall(\alpha,\nu) \in \{2\} \times \{0,2\}, \\ \{\mathcal{K}_{\mathfrak{g},1}, \mathcal{K}_{\mathfrak{g},2}, \mathcal{K}_{\mathfrak{g},3}, \mathcal{K}_{\mathfrak{g},5}\} & \forall(\alpha,\nu) \in \{3\} \times \{0,2\}, \\ \{\mathcal{O}_{\mathfrak{g},2}, \mathcal{O}_{\mathfrak{g},5}, \mathcal{O}_{\mathfrak{g},6}, \mathcal{O}_{\mathfrak{g},8}\} & \forall(\alpha,\nu) \in \{1\} \times \{1,3\}, \\ \{\mathcal{O}_{\mathfrak{g},3}, \mathcal{O}_{\mathfrak{g},5}, \mathcal{O}_{\mathfrak{g},7}, \mathcal{O}_{\mathfrak{g},8}\} & \forall(\alpha,\nu) \in \{2\} \times \{1,3\}, \\ \{\mathcal{O}_{\mathfrak{g},4}, \mathcal{O}_{\mathfrak{g},6}, \mathcal{O}_{\mathfrak{g},7}, \mathcal{O}_{\mathfrak{g},8}\} & \forall(\alpha,\nu) \in \{3\} \times \{1,3\}. \end{cases}$$

(4.11)

First, for every $\nu \in I_3^0$, set $\mathcal{I}^{\mathrm{op}}_{\mathfrak{g},(\xi_\alpha,\xi_\beta)} = \bigcap_{\lambda=\alpha,\beta} \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\xi_\lambda})$ and $\mathcal{I}^{\mathrm{cl}}_{\mathfrak{g},(\xi_\alpha,\xi_\beta)} = \bigcap_{\lambda=\alpha,\beta} \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\xi_\lambda})$. Next, for all $(\alpha,\beta,\nu) \in I_3^* \times I_3^* \times I_3^0$, calculate $\mathcal{I}^{\mathrm{op}}_{\mathfrak{g},(\xi_\alpha,\xi_\beta)}$, $\mathcal{I}^{\mathrm{cl}}_{\mathfrak{g},(\xi_\alpha,\xi_\beta)}$. Finally, for every $(r,s) \in I_8^* \times I_8^*$, set $\mathcal{O}_{\mathfrak{g},(r,s)} = (\mathcal{O}_{\mathfrak{g},r}, \mathcal{O}_{\mathfrak{g},s})$ and $\mathcal{K}_{\mathfrak{g},(r,s)} = (\mathcal{K}_{\mathfrak{g},r}, \mathcal{K}_{\mathfrak{g},s})$. These procedures yield:

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}[\sigma > 0] &= \bigcup_{\alpha \in I_3^*} \{(\xi_\alpha, \xi_\beta) : \beta \in I_3^* \setminus \{\alpha\}\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{O}}[\emptyset] &= \{\mathcal{O}_{\mathfrak{g},(3,2)}, \mathcal{O}_{\mathfrak{g},(3,6)}, \mathcal{O}_{\mathfrak{g},(4,2)}, \mathcal{O}_{\mathfrak{g},(4,3)}, \\ &\quad\ \mathcal{O}_{\mathfrak{g},(4,5)}, \mathcal{O}_{\mathfrak{g},(6,3)}, \mathcal{O}_{\mathfrak{g},(7,2)}\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{K}}[\emptyset] &= \{\mathcal{K}_{\mathfrak{g},(2,7)}, \mathcal{K}_{\mathfrak{g},(3,6)}, \mathcal{K}_{\mathfrak{g},(5,4)}, \mathcal{K}_{\mathfrak{g},(5,6)}, \\ &\quad\ \mathcal{K}_{\mathfrak{g},(5,7)}, \mathcal{K}_{\mathfrak{g},(6,3)}, \mathcal{K}_{\mathfrak{g},(6,7)}\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{O}}[\mathcal{S}_{\mathfrak{g}} \supset \emptyset] &= \{\mathcal{O}_{\mathfrak{g},(r,s)} : (r,s) \in I_8^* \times I_8^*\} \supset \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{O}}[\emptyset] \quad \forall \nu \in I_3^0, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{K}}[\mathcal{S}_{\mathfrak{g}} \supset \emptyset] &= \{\mathcal{K}_{\mathfrak{g},(r,s)} : (r,s) \in I_8^* \times I_8^*\} \supset \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{K}}[\emptyset]. \end{aligned}$$

(4.12)

We are now in a position to discuss the $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms, $\Lambda = \{\mathrm{K}, \mathrm{F}, \mathrm{H}, \mathrm{R}, \mathrm{N}\}$.

Let $\mathcal{O}_{\mathfrak{g},(p,q)} \supset \mathcal{K}_{\mathfrak{g},(r,s)}$ stand for the relations $\mathcal{O}_{\mathfrak{g},p} \supset \mathcal{K}_{\mathfrak{g},r}$ and $\mathcal{O}_{\mathfrak{g},q} \supset \mathcal{K}_{\mathfrak{g},s}$, where $\mathcal{O}_{\mathfrak{g},(p,q)} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{O}}[\emptyset]$ and $\mathcal{K}_{\mathfrak{g},(r,s)} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{K}}[\emptyset]$. Further, for every $\nu \in I_3^0$, let $\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},(p,q)}) \supset \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},(r,s)})$ stand for $\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},p}) \supset \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},r})$, $\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},q}) \supset \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},s})$. Then, the following relations are easily checked:

$\mathscr{O}_{\mathfrak{g},(7,2)} \supset \mathscr{K}_{\mathfrak{g},(2,7)}$; $\mathscr{O}_{\mathfrak{g},(6,3)} \supset \mathscr{K}_{\mathfrak{g},(3,6)}$; $\mathscr{O}_{\mathfrak{g},(4,5)} \supset \mathscr{K}_{\mathfrak{g},(5,4)}$; $\mathscr{O}_{\mathfrak{g},(4,3)} \supset \mathscr{K}_{\mathfrak{g},(5,6)}$; $\mathscr{O}_{\mathfrak{g},(4,2)} \supset \mathscr{K}_{\mathfrak{g},(5,7)}$; $\mathscr{O}_{\mathfrak{g},(3,6)} \supset \mathscr{K}_{\mathfrak{g},(6,3)}$ and $\mathscr{O}_{\mathfrak{g},(3,2)} \supset \mathscr{K}_{\mathfrak{g},(6,7)}$. But, for every $\nu \in I_3^0$, the relations $\mathscr{O}_{\mathfrak{g},(p,q)} \subseteq \mathrm{op}_{\mathfrak{g},\nu}\big(\mathscr{O}_{\mathfrak{g},(p,q)}\big)$ and $\mathscr{K}_{\mathfrak{g},(r,s)} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}\big(\mathscr{K}_{\mathfrak{g},(r,s)}\big)$ hold for all $(p,q) = (3,2)$, $(3,6)$, $(4,2)$, $(4,3)$, $(4,5)$, $(6,3)$, $(7,2)$ and all $(r,s) = (6,7)$, $(6,3)$, $(5,7)$, $(5,6)$, $(5,4)$, $(3,6)$, $(2,7)$. Combining these last two relations with $\mathscr{O}_{\mathfrak{g},(p,q)} \supset \mathscr{K}_{\mathfrak{g},(r,s)}$, it follows that $\mathrm{op}_{\mathfrak{g},\nu}\big(\mathscr{O}_{\mathfrak{g},(p,q)}\big) \supset \neg\,\mathrm{op}_{\mathfrak{g},\nu}\big(\mathscr{K}_{\mathfrak{g},(r,s)}\big)$. Hence, for every $\mathscr{K}_{\mathfrak{g},(r,s)} \in \mathfrak{g}$-$\nu$-$\mathrm{F_K}[\emptyset]$, there exists $\mathscr{O}_{\mathfrak{g},(p,q)} \in \mathfrak{g}$-$\nu$-$\mathrm{F_O}[\emptyset]$ such that:

$$\big[\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},p}) \supset \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g},r})\big] \wedge \big[\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},q}) \supset \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g},s})\big].$$

This shows that $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{N})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{N})}\big)$.

Let $(\xi_i,\xi_j) \in \mathscr{K}_{\mathfrak{g},(r,s)}$ mean $\xi_i \in \mathscr{K}_{\mathfrak{g},r}$ and $\xi_j \in \mathscr{K}_{\mathfrak{g},s}$, where $\mathscr{K}_{\mathfrak{g},(r,s)} \in \mathfrak{g}$-$\nu$-$\mathrm{F_K}\big[\emptyset\big]$. Then, the following results are easily checked: $(\xi_2,\xi_1) \in \mathscr{K}_{\mathfrak{g},(2,7)}$, $\mathscr{K}_{\mathfrak{g},(6,3)}$, $\mathscr{K}_{\mathfrak{g},(6,7)}$ and $(\xi_1,\xi_2) \notin \mathscr{K}_{\mathfrak{g},(2,7)}$, $\mathscr{K}_{\mathfrak{g},(6,3)}$, $\mathscr{K}_{\mathfrak{g},(6,7)}$; $(\xi_3,\xi_1) \in \mathscr{K}_{\mathfrak{g},(2,7)}$, $\mathscr{K}_{\mathfrak{g},(5,4)}$, $\mathscr{K}_{\mathfrak{g},(5,7)}$ and $(\xi_1,\xi_3) \notin \mathscr{K}_{\mathfrak{g},(2,7)}, \mathscr{K}_{\mathfrak{g},(5,4)}, \mathscr{K}_{\mathfrak{g},(5,7)}$; $(\xi_3,\xi_2) \in \mathscr{K}_{\mathfrak{g},(3,6)}, \mathscr{K}_{\mathfrak{g},(5,4)}, \mathscr{K}_{\mathfrak{g},(5,6)}$ and $(\xi_2,\xi_3) \notin \mathscr{K}_{\mathfrak{g},(3,6)}$, $\mathscr{K}_{\mathfrak{g},(5,4)}$, $\mathscr{K}_{\mathfrak{g},(5,6)}$. But, $\mathscr{O}_{\mathfrak{g},(7,2)} \supset \mathscr{K}_{\mathfrak{g},(2,7)}$; $\mathscr{O}_{\mathfrak{g},(6,3)} \supset \mathscr{K}_{\mathfrak{g},(3,6)}$; $\mathscr{O}_{\mathfrak{g},(4,5)} \supset \mathscr{K}_{\mathfrak{g},(5,4)}$; $\mathscr{O}_{\mathfrak{g},(4,3)} \supset \mathscr{K}_{\mathfrak{g},(5,6)}$; $\mathscr{O}_{\mathfrak{g},(4,2)} \supset \mathscr{K}_{\mathfrak{g},(5,7)}$; $\mathscr{O}_{\mathfrak{g},(3,6)} \supset \mathscr{K}_{\mathfrak{g},(6,3)}$ and $\mathscr{O}_{\mathfrak{g},(3,2)} \supset \mathscr{K}_{\mathfrak{g},(6,7)}$. Furthermore, for every $\nu \in I_3^0$, $\mathscr{O}_{\mathfrak{g},(p,q)} \subseteq \mathrm{op}_{\mathfrak{g},\nu}\big(\mathscr{O}_{\mathfrak{g},(p,q)}\big)$ for all $(p,q) = (3,2)$, $(3,6)$, $(4,2)$, $(4,3)$, $(4,5)$, $(6,3)$, $(7,2)$ and $\mathscr{K}_{\mathfrak{g},(r,s)} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}\big(\mathscr{K}_{\mathfrak{g},(r,s)}\big)$ for all $(r,s) = (2,7)$, $(3,6)$, $(5,4)$, $(5,6)$, $(5,7)$, $(6,3)$, $(6,7)$. Thus, for every $(\xi_i,\xi_j) \in \mathfrak{g}$-$\mathrm{F_P}[\sigma > 0]$ and $\mathscr{K}_{\mathfrak{g},(r,s)} \in \mathfrak{g}$-$\nu$-$\mathrm{F_K}\big[\emptyset\big]$ such that $(\xi_j,\xi_i) \notin \mathscr{K}_{\mathfrak{g},(r,s)}$, there exists $\mathscr{O}_{\mathfrak{g},(p,q)} \in \mathfrak{g}$-$\nu$-$\mathrm{F_O}\big[\emptyset\big]$ such that:

$$\Big[\big(\neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g},r}) \subset \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},p})\big) \wedge \big(\zeta \in \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},q})\big)\Big] \vee \Big[\big(\neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g},s}) \subset \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},q})\big) \wedge \big(\xi \in \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},p})\big)\Big].$$

This shows that $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{R})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{R})}\big)$.

Let $(\xi_i,\xi_j) \in \mathscr{O}_{\mathfrak{g},(p,q)}$ mean $\xi_i \in \mathscr{O}_{\mathfrak{g},p}$ and $\xi_j \in \mathscr{O}_{\mathfrak{g},q}$, where $\mathscr{O}_{\mathfrak{g},(p,q)} \in \mathfrak{g}$-$\nu$-$\mathrm{F_O}\big[\emptyset\big]$. Then, the following relations are easily verified: $(\xi_2,\xi_1) \in \mathscr{O}_{\mathfrak{g},(3,2)}$, $\mathscr{O}_{\mathfrak{g},(3,6)}$, $\mathscr{O}_{\mathfrak{g},(7,2)}$ and $(\xi_1,\xi_2) \notin \mathscr{O}_{\mathfrak{g},(3,2)}$, $\mathscr{O}_{\mathfrak{g},(3,6)}$, $\mathscr{O}_{\mathfrak{g},(7,2)}$; $(\xi_3,\xi_1) \in \mathscr{O}_{\mathfrak{g},(4,2)}$, $\mathscr{O}_{\mathfrak{g},(4,5)}$, $\mathscr{O}_{\mathfrak{g},(7,2)}$ and $(\xi_1,\xi_3) \notin \mathscr{O}_{\mathfrak{g},(4,2)}, \mathscr{O}_{\mathfrak{g},(4,5)}, \mathscr{O}_{\mathfrak{g},(7,2)}$; $(\xi_3,\xi_2) \in \mathscr{O}_{\mathfrak{g},(4,3)}, \mathscr{O}_{\mathfrak{g},(4,5)}, \mathscr{O}_{\mathfrak{g},(6,3)}$ and $(\xi_2,\xi_3) \notin \mathscr{O}_{\mathfrak{g},(4,3)}$, $\mathscr{O}_{\mathfrak{g},(4,5)}$, $\mathscr{O}_{\mathfrak{g},(6,3)}$. But, for every $\nu \in I_3^0$, $\mathscr{O}_{\mathfrak{g},(p,q)} \subseteq \mathrm{op}_{\mathfrak{g},\nu}\big(\mathscr{O}_{\mathfrak{g},(p,q)}\big)$ and $\mathscr{O}_{\mathfrak{g},(p,q)} \in \mathfrak{g}$-$\nu$-$\mathrm{F_O}[\emptyset]$ for all $(p,q) = (3,2)$, $(3,6)$, $(4,2)$, $(4,3)$, $(4,5)$, $(6,3)$, $(7,2)$. Thus, for every $(\xi_i,\xi_j) \in \mathfrak{g}$-$\mathrm{F_P}[\sigma > 0]$, there exists $\mathscr{O}_{\mathfrak{g},(p,q)} \in \mathfrak{g}$-$\nu$-$\mathrm{F_O}\big[\emptyset\big]$ such that:

$$\big[\xi_i \in \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},p})\big] \wedge \big[\xi_j \in \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},q})\big].$$

This shows that $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$.

Let $(\xi_i,\xi_j) \in \mathscr{O}_{\mathfrak{g},(p,q)}$ mean $\xi_i \in \mathscr{O}_{\mathfrak{g},p}$, $\xi_j \in \mathscr{O}_{\mathfrak{g},q}$, and $(\xi_j,\xi_i) \notin \mathscr{O}_{\mathfrak{g},(p,q)}$, where $\mathscr{O}_{\mathfrak{g},(p,q)} \in \mathfrak{g}$-$\nu$-$\mathrm{F_O}\big[\mathscr{S}_{\mathfrak{g}} \supset \emptyset\big]$. Then, the following relations are easily verified: $(\xi_1,\xi_2) \in \mathscr{O}_{\mathfrak{g},(p,q)}$ and $(\xi_2,\xi_1) \notin \mathscr{O}_{\mathfrak{g},(p,q)}$ for all $(p,q) = (2,3)$, $(2,7)$, $(6,3)$, $(6,7)$; $(\xi_1,\xi_3) \in \mathscr{O}_{\mathfrak{g},(p,q)}$ and $(\xi_3,\xi_1) \notin \mathscr{O}_{\mathfrak{g},(p,q)}$ for all $(p,q) = (2,4)$, $(2,7)$, $(5,4)$, $(5,7)$; $(\xi_2,\xi_3) \in \mathscr{O}_{\mathfrak{g},(p,q)}$ and $(\xi_3,\xi_1) \notin \mathscr{O}_{\mathfrak{g},(p,q)}$ for all $(p,q) = (3,4)$, $(3,6)$, $(5,4)$, $(5,6)$. But, $\mathscr{O}_{\mathfrak{g},(p,q)} \subseteq \mathrm{op}_{\mathfrak{g},\nu}\big(\mathscr{O}_{\mathfrak{g},(p,q)}\big)$ for all $(p,q) = (2,3)$, $(2,4)$, $(2,7)$, $(3,4)$, $(3,6)$, $(5,4)$, $(5,6)$, $(5,7)$, $(6,3)$, $(6,7)$. Hence, for every $(\xi_i,\xi_j) \in \mathfrak{g}$-$\mathrm{F_P}[\sigma > 0]$, there exists $\mathscr{O}_{\mathfrak{g},(p,q)} \in \mathfrak{g}$-$\nu$-$\mathrm{F_O}\big[\mathscr{S}_{\mathfrak{g}} \supset \emptyset\big]$ such that:

$$\big[(\xi_i,\xi_j) \in {}_{\lambda=p,q}\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},\lambda})\big] \wedge \big[(\xi_i,\xi_j) \notin {}_{\lambda=q,p}\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},\lambda})\big].$$

This shows that $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{F})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{F})}\big)$.

Let $(\xi_i, \xi_j) \in \mathcal{O}_{\mathfrak{g},(p,q)}$ mean $\xi_i \in \mathcal{O}_{\mathfrak{g},p}$ and $\xi_j \notin \mathcal{O}_{\mathfrak{g},p}$, or $\xi_i \notin \mathcal{O}_{\mathfrak{g},p}$ and $\xi_j \in \mathcal{O}_{\mathfrak{g},q}$, where $\mathcal{O}_{\mathfrak{g},(p,q)} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{O}}\big[\mathcal{S}_{\mathfrak{g}} \supset \emptyset\big]$. Then, the following relations are easily verified: $(\xi_1, \xi_2) \in \mathcal{O}_{\mathfrak{g},(2,3)}$, $\mathcal{O}_{\mathfrak{g},(2,7)}$, $\mathcal{O}_{\mathfrak{g},(6,3)}$, $\mathcal{O}_{\mathfrak{g},(6,7)}$; $(\xi_1, \xi_3) \in \mathcal{O}_{\mathfrak{g},(2,4)}$, $\mathcal{O}_{\mathfrak{g},(2,7)}$, $\mathcal{O}_{\mathfrak{g},(5,4)}$, $\mathcal{O}_{\mathfrak{g},(5,7)}$; $(\xi_2, \xi_3) \in \mathcal{O}_{\mathfrak{g},(3,4)}$, $\mathcal{O}_{\mathfrak{g},(3,6)}$, $\mathcal{O}_{\mathfrak{g},(5,4)}$, $\mathcal{O}_{\mathfrak{g},(5,6)}$. But, $\mathcal{O}_{\mathfrak{g},(p,q)} \subseteq \mathrm{op}_{\mathfrak{g},\nu}\big(\mathcal{O}_{\mathfrak{g},(p,q)}\big)$ for all $(p,q) = (2,3)$, $(2,4)$, $(2,7)$, $(3,4)$, $(3,6)$, $(5,4)$, $(5,6)$, $(5,7)$, $(6,3)$, $(6,7)$. Hence, for every $(\xi_i, \xi_j) \in \mathfrak{g}\text{-}\mathrm{F}_{\mathrm{P}}\,[\sigma > 0]$, there exists $\mathcal{O}_{\mathfrak{g},(p,q)} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{F}_{\mathrm{O}}\big[\mathcal{S}_{\mathfrak{g}} \supset \emptyset\big]$ such that:

$$\big[\big(\xi_i \in \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},p})\big) \wedge \big(\xi_j \notin \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},p})\big)\big] \vee \big[\big(\xi_i \notin \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},q})\big) \wedge \big(\xi_j \in \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},q})\big)\big].$$

This shows that $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{K})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{K})} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{K})}\big)$.

The elements discussed in the preceding sections can be easily checked from this nice application. In the next section, we provide concluding remarks and future directions of the theory of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-separation axioms developed in the earlier sections.

5. Conclusion

In this paper, we developed a new theory, called *Theory of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-Separation Axioms*. The theory is based on the *Theory of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-Sets* but not on the *Theory of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-Maps*. In its own rights, the proposed theory has several advantages. The very first advantage is that the theory holds equally well when $(\Omega, \mathcal{T}_{\mathfrak{g}}) = (\Omega, \mathcal{T})$ and other characteristics adapted on this ground, in which case it might be called *Theory of* $\mathfrak{g}\text{-}\mathfrak{T}$*-Separation Axioms*.

Thus, in a $\mathcal{T}_{\mathfrak{g}}$-space the proposed theoretical framework categorises each element of the quintuple sequence $\big\langle \mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha} \big\rangle_{\alpha\in\Lambda}$ as $\mathfrak{g}\text{-}\mathrm{T}_{\mathfrak{g},\alpha}$-axioms of type α and of categories ν, where $(\alpha,\nu) \in \Lambda \times I_3^0$ and $\Lambda = \big\{\mathrm{K}, \mathrm{F}, \mathrm{H}, \mathrm{R}, \mathrm{N}\big\}$ and theorises the concepts in a unified way; in a $\mathcal{T}$-space it categorises each element of the quintuple sequence $\big\langle \mathfrak{g}\text{-}\mathrm{T}_{\alpha} \big\rangle_{\alpha\in\Lambda}$ as $\mathfrak{g}\text{-}\mathrm{T}_{\alpha}$-axioms of type α and of categories ν, where $(\alpha,\nu) \in \Lambda \times I_3^0$ and $\Lambda = \big\{\mathrm{K}, \mathrm{F}, \mathrm{H}, \mathrm{R}, \mathrm{N}\big\}$ and theorises the concepts in a unified way.

Since the theory of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-separation Axioms has been based solely on theory of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-sets, as pointed out above, it is an interesting topic for future research either to develop the theory of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-separation axioms of mixed categories based on the aforementioned theory or to develop it but based on the theory of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-maps. More precisely, either for some pair $(\nu,\mu) \in I_3^0 \times I_3^0$ such that $\nu \neq \mu$, to develop the theory of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-separation axioms based on the theory of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open sets belonging to the class $\big\{\mathcal{O}_{\mathfrak{g}} = \mathcal{O}_{\mathfrak{g},\nu} \cup \mathcal{O}_{\mathfrak{g},\mu} :\ (\mathcal{O}_{\mathfrak{g},\nu}, \mathcal{O}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big\}$ and the theory of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closed sets belonging to the class $\big\{\mathcal{K}_{\mathfrak{g}} = \mathcal{K}_{\mathfrak{g},\nu} \cup \mathcal{K}_{\mathfrak{g},\mu} :\ (\mathcal{K}_{\mathfrak{g},\nu}, \mathcal{K}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big\}$ in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ or, to develop the theory of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-separation axioms based on the theory of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-maps, called $\mathfrak{g}\text{-}(\mathfrak{T}_{\Lambda}, \mathfrak{T}_{\Theta})$-continuous maps, $\mathfrak{g}\text{-}(\mathfrak{T}_{\Lambda}, \mathfrak{T}_{\Theta})$-irresolute maps and $\mathfrak{g}\text{-}(\mathfrak{T}_{\Lambda}, \mathfrak{T}_{\Theta})$-homeomorphism maps, where $\Lambda,\ \Theta \in \big\{\Omega, \Sigma, \Upsilon\big\}$, between any two of such $\mathcal{T}_{\mathfrak{g}}$-spaces $\mathfrak{T}_{\mathfrak{g},\Omega}$, $\mathfrak{T}_{\mathfrak{g},\Sigma}$, and $\mathfrak{T}_{\mathfrak{g},\Upsilon}$. Such two theories are what we thought would certainly be worth considering, and the discussion of this paper ends here.

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The Declaration of Research and Publication Ethics
The authors declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the authors declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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PERFECT CODES OVER HURWITZ INTEGERS INDUCED BY CIRCULANT GRAPHS

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Abstract. In this paper, a new family of $t-$error correcting perfect codes over Hurwitz integers is presented. To obtain these perfect codes, the perfect $t-$dominating sets over the circulant graphs are used. The codewords of such perfect codes are generated by the elements of a subgroup of the considered group.

1. Introduction

If a code satisfy the sphere-packing bound in any given metric, then the code called perfect code. Perfect codes are important since perfect codes plays an importance role in coding theory. The first perfect codes which were subspaces of $\mathbb{Z}_2^n$ were defined by Hamming in [4]. The first non-linear perfect $1-$error correcting binary code was constructed by Vasil'ev in [15]. Vasil'ev's construction was generalized to $q-$ary case by Lindström and independently Schönheim in [10, 14]. Group and non-group perfect codes which were not equal to any linear code were given by Heden in [5]. Besides, perfect codes have been investigated with respect to some other metrics such as the Lee metric, the Mannheim metric, the Lipschitz metric. Some perfect codes with respect to the Lee metric introduced in [9]. Huber defined Mannheim metric, and presented perfect $1-$Mannheim error correcting codes (shortly OMEC) in the metric in [8]. The dimension of OMEC codes not only $n-1$, but also $n-k$ $(k>1)$ were constructed by Güzeltepe and Heden in [3]. The Lipschitz metric was presented and some perfect codes over Lipschitz integers were introduced with respect to the Lipschitz metric in [11, 12]. A generalization of perfect Lee-error-correcting codes and perfect $1-$error correcting Lipschitz weight codes were presented by Heden and Güzeltepe in [6, 7].

The Hurwitz metric was introduced in [1, 3]. Besides, Güzeltepe constructed linear codes over Hurwitz integers with respect to the Hurwitz metric for a Hurwitz prime in [1]. These linear codes were not perfect.

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On the other hand, the common trait of the papers [1, 2, 3, 6] is that the perfect codes were obtained by using a chosen prime over relevant structures. Unlike these articles, we use only odd Hurwitz integers being product of distinct primes to construct perfect codes over Hurwitz integers. The main idea in the presented paper is inspired by the article [13]. In that paper [13], a method for defining new metrics over two-dimension signal spaces and families of perfect codes of length one over lattice constellations obtained by Gaussian integers and Eisenstein Jacobi integers were presented by Martínez *et al.* They mainly considered QAM-like signal spaces and defined a new distance over QAM-like constellations imported from degree-four circulant graphs whose nodes were labeled with Gaussian integers. By means of these graphs, they constructed perfect $t-$correcting codes over Gaussian integers with length one.

The rest of the paper is organized as follows. In Section 2, basic definitions and theorems in Hurwitz integers are given. A connection between Circulant graph and Hurwitz integers is obtained in Section 3. Perfect $t-$dominating sets is defined in Section 4. Using these sets, perfect codes over Hurwitz integers are constructed in that section. In terms of average energy and bandwidth occupancy, a simple comparison between these perfect codes and some perfect codes given in literature is presented in the last section.

2. On Hurwitz integers

In this section, we give some basic definitions and theorems which we need throughout our study.

Definition 2.1 (see [3]). Hamilton Quaternions $\mathcal{H}(\mathbb{R})$ is the free $\mathbb{R}-$module over the symbols $1, i, j, k$ and the set of Hamilton Quaternions is defined as following:

$$\mathcal{H}(\mathbb{R}) = \{a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

Here, 1 is the multiplicative identity. Morever,

(1) $i^2 = j^2 = k^2 = -1$ and
(2) $ij = -ji = k$; $jk = -kj = i$; $ki = -ik = j$.
(3) If $q = a_0 + a_1 i + a_2 j + a_3 k$ is a quaternion, then its conjugate is denoted by q^* and $q* = a_0 - (a_1 i + a_2 j + a_3 k)$.
(4) The norm $N(q)$ of $q \in \mathcal{H}(\mathbb{R})$ is $N(q) = qq^* = a_0^2 + a_1^2 + a_2^2 + a_3^2$ and $N(q_1 q_2) = N(q_1) N(q_2)$, that is, the norm N is a multiplicative norm.

Definition 2.2 (see [3]). The Lipschitz integers $\mathcal{H}(\mathbb{Z})$ is a subset of $\mathcal{H}(\mathbb{R})$ and is defined as

$$\mathcal{H}(\mathbb{Z}) = \{a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{Z}\}.$$

Definition 2.3 (see [2]). The set of Hurwitz integers is the set $\mathcal{H} = \mathcal{H}(\mathbb{Z}) \cup \mathcal{H}\left(\mathbb{Z} + \frac{1}{2}\right)$, that is,

$$\begin{aligned} \mathcal{H} = & \left\{a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{Z} \ \vee \ a_0, a_1, a_2, a_3 \in \mathbb{Z} + \tfrac{1}{2}\right\} \\ = & \left\{\tfrac{a_0 + a_1 i + a_2 j + a_3 k}{2} : a_0, a_1, a_2, a_3 \in \mathbb{Z},\ a_0 \equiv a_1 \equiv a_2 \equiv a_3 \pmod 2\right\}. \end{aligned}$$

Definition 2.4 (see [3]). If the norm of a Hurwitz integer q is an odd integer, then the element q is called an odd Hurwitz integer. Similarly, if the norm of a Hurwitz integer α is a prime integer, then the element α is called prime Hurwitz integer.

In this study, we use only odd Hurwitz integers to construct perfect codes over Hurwitz integers.

Definition 2.5. Let q_1, q_2 be two elements of Hurwitz integers $\mathcal{H}$ and let α be an odd Hurwitz integer. If there exists $\beta \in \mathcal{H}$ such that $q_1 - q_2 = \alpha\beta$, then $q_1, q_2 \in \mathcal{H}$ are left congruent modulo α and it is denoted as $q_1 \equiv_l q_2 \pmod{\alpha}$.

Let $\mathcal{H}_\alpha$ denotes the complete set of left coset representatives. In this situation, the elements of $\mathcal{H}_\alpha$ are not left congruent each other modulo α. Right congruent can be defined like left congruent.

Theorem 2.6 (see [2])**.** *If α is an odd Hurwitz prime, then the size of $\mathcal{H}_\alpha$ is equal to $N(\alpha)^2$.*

Corollary 2.7. Let $0 \neq \alpha$ and β be in $\mathcal{H}$ and let β be a left-divisor of α. Then the subgroup generated by the element β is denoted by $\langle\beta\rangle$ and the number of the elements of the subgroup $\langle\beta\rangle$ is equal to ${N(\alpha)^2}/{N(\beta)^2}$.

Proof. $\mathcal{H}_\alpha$ is an additive group and $\langle\beta\rangle$ is a subgroup of $\mathcal{H}_\alpha$. So, the proof is clear from the Lagrange Theorem. □

3. Circulant graph and Hurwitz integers

In this section, a connection between circulant graph $C_N(j_1, \ldots, j_m)$ and $\mathcal{H}_\alpha$ is given.

Definition 3.1. The distance $\beta, \gamma \in \mathcal{H}_\alpha$ is defined as

$$d_\alpha(\beta, \gamma) = N(\delta),$$

where $\delta = a_0 + a_1 i + a_2 j + a_3 k$ denotes an element in the coset $\beta - \gamma$ in $\mathcal{H}_\alpha$ with $|a_0| + |a_1| + |a_2| + |a_3|$ minimum. We also define the weight of $\beta \in \mathcal{H}_\alpha$ as

$$w_\alpha = d_\alpha(\beta, 0).$$

There are 24 elements of weight one in Hurwitz integers $\mathcal{H}$. These elements are $\pm1, \pm i, \pm j, \pm k$ and $\pm\frac{1}{2} \pm \frac{i}{2} \pm \frac{j}{2} \pm \frac{k}{2}$. From now on ε denotes the following set:

$$\left\{\pm1, \pm i, \pm j, \pm k, \pm\frac{1}{2} \pm \frac{i}{2} \pm \frac{j}{2} \pm \frac{k}{2}\right\}.$$

By adding the elements of the set ε one by one to $\gamma \in \mathcal{H}_\alpha$, we determine the elements at distance one from exactly γ.

Definition 3.2. Let $0 \neq \alpha \in \mathcal{H}$ be an odd Hurwitz integer. If we take

(1) $V = \mathcal{H}_\alpha$ is the set of vertices (nodes) and
(2) $E = \{(\beta, \gamma) \in V \times V : d_\alpha(\beta, \gamma) = 1\}$ is the set of edges,

then $G_\alpha(V, E)$ defines a graph generated by α.

Definition 3.3. (see [13]) A circulant graph with N vertices and jumps $\{j_1, j_2, \cdots, j_m\}$, where $m < N/2$, is an undirected graph in which each vertex n, $0 \leq n \leq N-1$, is adjacent to all the vertices $n \pm j_i$, with $1 \leq i \leq m$. We denote this graph as $C_N(j_1, j_2, \cdots, j_m)$.

Theorem 3.4. *Let $e_1 \in \{i, j, k\}$ and let $\alpha = a_0 + a_1 i + a_2 j + a_3 k = a_0 + a_1 i + (a_2 + a_3 i)e_1 \in \mathcal{H}$ be an odd Hurwitz integer. Then $C_{N(\alpha)^2}(j_1, \ldots, j_{12})$ and G_α are isomorphic graphs.*

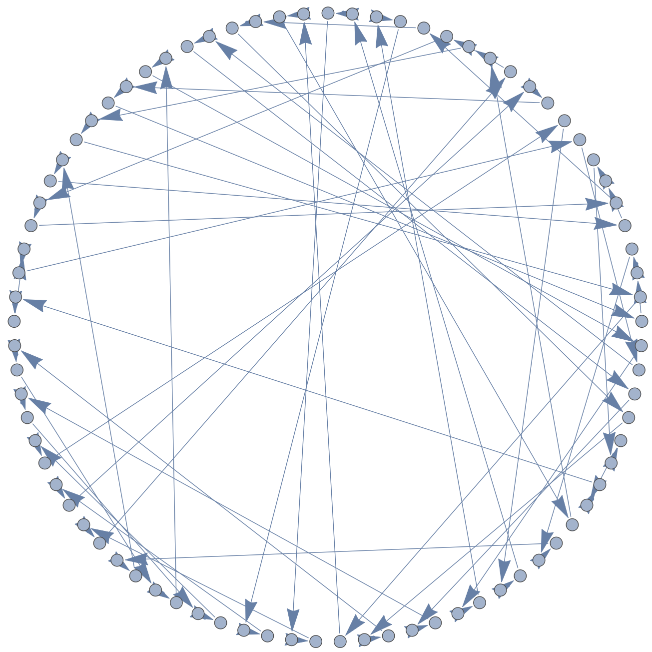
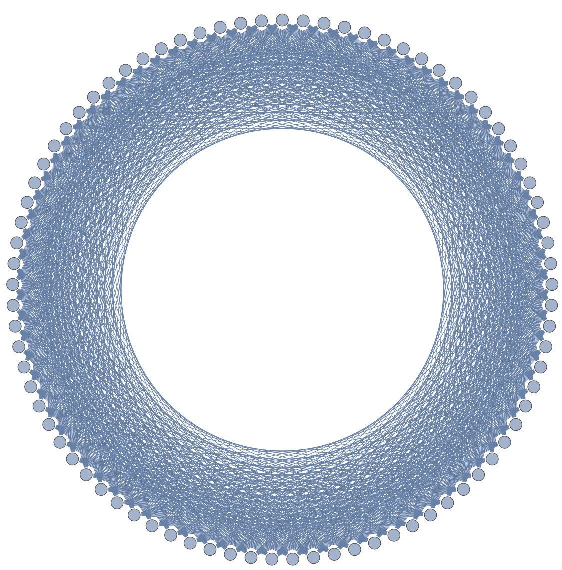


FIGURE 1. The graph $G_{-1+2i+2j}$

```
In[4]:= A = CirculantGraph@81, 813, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24<D
        EdgeCount@AD
        GraphDiameter@AD
```

Out[4]=



```
Out[5]= 972
Out[6]= 3
```

FIGURE 2. The graph $C_{81}(13,\ldots,24)$

Proof. The edges of G_α and the edges of $C_{N(\alpha)^2}(j_1,\dots,j_{12})$ are chosen as the elements of $\mathcal{H}_\alpha$ and the elements of $\mathbb{Z}_{N(\alpha)}\times\mathbb{Z}_{N(\alpha)}$, respectively. Therefore, it is sufficient to prove that $\mathcal{H}_\alpha$ and $\mathbb{Z}_{N(\alpha)}\times\mathbb{Z}_{N(\alpha)}$ are isomorphic groups. We now consider the function

$$\begin{array}{rl} \psi : \mathbb{Z}_{N(\alpha)}\times\mathbb{Z}_{N(\alpha)} \to \mathcal{H}_\alpha & \\ (q_1,q_2) & \mapsto (x_1+y_1)\,i+(x_2+y_2)\,e_1 \pmod{\alpha}, \end{array}$$

where $x_1,y_1,x_2,y_2\in\mathbb{Z}_{N(\alpha)}$, $q_1=a_0x_1+a_1y_1,\ q_2=a_2x_2+a_3y_2 \pmod{N(\alpha)}$. The function ψ is a bijective function. The bases of these groups $\mathcal{H}_\alpha$ and $\mathbb{Z}_{N(\alpha)}\times\mathbb{Z}_{N(\alpha)}$ are $e_2,e_3\in\{1,i,j,k\}$ and $\{(1,0),(0,1)\}$, respectively. Hence we get $\psi((1,0))=e_2,\ \psi((0,1))=e_3,$ where $\psi((x,y))=\beta_1e_2+\beta_2e_3 \pmod{\alpha}$, $x,y\in\mathbb{Z}_{N(\alpha)}$ and $\beta_1,\beta_2\in\mathcal{H}_\alpha$. Hence, the proof is completed.

□

Example 3.5. Let $\alpha=-1+2i+2j$. Fig. 1 shows the graph $\mathcal{H}_\alpha$ and Fig. 2 shows the graph $C_{81}(13,\dots,24)$. The vertexes given in Fig. 1 shows one twelfth of all vertexes. The diameter of these graphs is 3. This shows that the distance between 0 and the elements of $\mathcal{H}_\alpha$ is less than or equal to 3, that is, $N(q)\le 3$ for all $q\in\mathcal{H}_\alpha$.

4. Perfect $t-$Dominating Sets and Perfect Codes Over the Hurwitz Integers

In this paper, we study on arbitrary parameter t, give conditions for the existence of perfect $t-$dominating sets.

Proposition 1. If α is a Hurwitz integer and $\rho_1,\rho_2\in\varepsilon$ then the norm $N(\alpha)$ is equal to the norm $N(\rho_1\alpha\rho_2)$.

Proof. Recall that the norm N is a multiplicative norm and $N(\rho_1)=N(\rho_2)=1$ since $\rho_1,\rho_2\in\varepsilon$. Hence, we have

$$N(\rho_1\alpha\rho_2)=N(\rho_1)\,N(\alpha)\,N(\rho_2)=N(\alpha)\,.$$

□

It is clear that if $\alpha_1,\cdots,\alpha_r$ are odd Hurwitz integers then $\alpha_1\dots\alpha_r$ is an odd Hurwitz integer.

Proposition 2. Let α be an odd Hurwitz integer and let $\beta_1,\beta_2\in\mathcal{H}$. If

$$\beta_1=\beta_2 \pmod{\alpha},$$

then

$$\rho_1\beta_1\rho_2=\rho_1\beta_2\rho_2 \pmod{\rho_1\alpha\rho_2}.$$

Proof. If $\beta_1=\beta_2 \pmod{\alpha}$, then we get

$$\beta_2=\beta_1+\alpha\delta, \delta\in\mathcal{H}.$$

Multiplying left sides of the above equation by ρ_1 and right sides by ρ_2, we obtain

$$\begin{aligned} \rho_1\beta_2\rho_2 &= \rho_1(\beta_1+\alpha\delta)\rho_2=\rho_1\beta_1\rho_2+\rho_1(\alpha\delta)\rho_2 \\ &= \rho_1\beta_1\rho_2+\rho_1\left(\alpha\left(\rho_2\rho_2^{-1}\right)\delta\right)\rho_2 \\ &= \rho_1\beta_1\rho_2+\rho_1(\alpha\rho_2)\left(\rho_2^{-1}\delta\rho_2\right)=\rho_1\beta_1\rho_2+(\rho_1\alpha\rho_2)\,\delta_1. \end{aligned}$$

This shows that

$$\rho_1\beta_1\rho_2=\rho_1\beta_2\rho_2 \pmod{\rho_1\alpha\rho_2}.$$

□

The proof of next proposition is straightforward from the proof of Prop. 2.

Proposition 3. Let α be an odd Hurwitz integer. If the set $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$ is a partition of $\mathcal{H}_\alpha$ then the set $\{\rho_1\varepsilon_1\rho_2\,, \rho_1\varepsilon_2\rho_2, \cdots, \rho_1\varepsilon_n\rho_2\}$ becomes a partition of $\mathcal{H}_{\rho_1\alpha\rho_2}$.

The proof of next lemma is straightforward from Prop. 3.

Lemma 4.1. *$G_{\alpha_1} \cong G_{\alpha_2}$ if there exist $\rho_1, \rho_2 \in \varepsilon$ such that $\alpha_1 = \rho_1\alpha_2\rho_2$, where $\alpha_1, \alpha_2 \in \mathcal{H}$.*

Definition 4.2. Let α be a Hurwitz integer. A sphere (ball) centered at γ with radius t in G_α is defined as

$$B_t(\gamma) = \{\beta \in \mathcal{H}_\alpha : d_\alpha(\beta, \gamma) \le t\},$$

where $t \in \mathbb{N}$. If $q \in \mathcal{H}_\alpha$ is in the $B_t(\gamma)$, then it is said that the vertex q is $t-$dominated by the vertex γ.

We give the following definition as in [13].

Definition 4.3. Let a vertex subset $S \subset G_\alpha$ and $t \in \mathbb{Z}^+$. If every vertex of G_α is $t-$dominated by a unique vertex in S, then S is called a perfect $t-$dominating set.

Example 4.4. For $\alpha = 1+3i+2j+k$ and $\gamma = -2j-k$, the set of $B_1(-2j-k) = \{\beta \in \mathcal{H}_{1+3i+2j+k} : d_{1+3i+2j+k}(\beta, -2j-k) \le 1\}$ is a sphere centered at $-2j-k$ with radius 1 in $\mathcal{H}_{1+3i+2j+k}$.

If $t=0$, then $d_{1+3i+2j+k}(\beta, -2j-k) = 0$. Hence, we get $\beta = -2j-k$ and $-2j-k \in B_1(-2j-k)$.

If $t=1$, then $d_{1+3i+2j+k}(\beta, -2j-k) = 1$. So, we add all of the elemets of weight one to $\gamma = -2j-k$ in an effort to determine $1-$dominating set of $\gamma = -2j-k$.

For -1: $\beta = (-2j-k)-1 = -1-2j-k \equiv \frac{1}{2} - \frac{i}{2} + \frac{j}{2} + \frac{3k}{2} \pmod{1+3i+2j+k}$. Then we get $\frac{1}{2} - \frac{i}{2} + \frac{j}{2} + \frac{3k}{2} \in B_1(-2j-k)$.

For 1: $\beta = (-2j-k)+1 = 1-2j-k \equiv -\frac{1}{2} + \frac{i}{2} + \frac{3j}{2} - \frac{k}{2} \pmod{1+3i+2j+k}$. Then we get $-\frac{1}{2} + \frac{i}{2} + \frac{3j}{2} - \frac{k}{2} \in B_1(-2j-k)$.

By processing similar technique for 24 elements of weight one, we obtain $1-$dominating set of $\gamma = -2j-k \in \mathcal{H}_\alpha$ as

$$\begin{aligned} B_1(-2j-k) = \quad & \Big\{-2j-k, -\tfrac{1}{2}+\tfrac{i}{2}+\tfrac{3j}{2}-\tfrac{k}{2}, \tfrac{1}{2}-\tfrac{i}{2}+\tfrac{j}{2}+\tfrac{3k}{2}, \tfrac{3}{2}+\tfrac{i}{2}+\tfrac{j}{2}+\tfrac{3k}{2}, 1+2i, \\ & -j-k, -\tfrac{3}{2}+\tfrac{i}{2}+\tfrac{j}{2}-\tfrac{k}{2}, -2j, \tfrac{3}{2}-\tfrac{i}{2}+\tfrac{j}{2}+\tfrac{k}{2}, \tfrac{1}{2}+\tfrac{i}{2}-\tfrac{3j}{2}-\tfrac{k}{2}, \\ & -\tfrac{1}{2}+\tfrac{i}{2}-\tfrac{3j}{2}-\tfrac{k}{2}, \tfrac{1}{2}-\tfrac{i}{2}-\tfrac{3j}{2}-\tfrac{k}{2}, -1+i+j, \tfrac{1}{2}+\tfrac{i}{2}-\tfrac{3j}{2}-\tfrac{3k}{2}, \\ & -\tfrac{1}{2}-\tfrac{i}{2}-\tfrac{3j}{2}-\tfrac{k}{2}, \tfrac{1}{2}-\tfrac{3i}{2}+\tfrac{j}{2}-\tfrac{3k}{2}, 1+j+k, -1+j, \tfrac{1}{2}-\tfrac{i}{2}-\tfrac{3j}{2}-\tfrac{3k}{2}, \\ & -1+i+j-k, -2+j, 1-i+j+k, 1+k, -1+j-k, 1-i+k\Big\}. \end{aligned}$$

Theorem 4.5. (1) *If $0 \ne \beta \in \mathcal{H}_\alpha$, $N(\beta) = 5$ and $\beta \,|\alpha$, then the set of the subgroup $\langle\beta\rangle$ generated by β is a perfect $1-$dominating set in G_α.*

(2) *If $0 \ne \beta \in \mathcal{H}_\alpha$, $N(\beta) = 7$ and $\beta \,|\alpha$, then the set of the subgroup $\langle\beta\rangle$ generated by β is a perfect $2-$dominating set in G_α.*

Proof. 1. Let $N(\beta) = 5$ and $\beta \,|\alpha$ We prove that $d_\alpha(\sigma, \tau) \ge 3$ for all $\sigma, \tau \in \langle\beta\rangle, \sigma \ne \tau$. Since σ and τ are the elements of $\langle\beta\rangle$, there are δ_1, δ_2 in $\mathcal{H}_\alpha$ such that $\sigma = \beta\delta_1$ and $\tau = \beta\delta_2$. Thus, we have

$$d_\alpha(\sigma, \tau) = d_\alpha(\sigma - \tau, 0) = d_\alpha(\beta\delta_1 - \beta\delta_2, 0) = d_\alpha(\beta\gamma, 0),$$

where $\gamma = \delta_1 - \delta_2 (\mathrm{mod} \alpha)$. Let us assume that

$$d_\alpha(\sigma, \tau) = d_\alpha(\beta\gamma, 0) < 3.$$

In this situation, there is an element q in $\mathcal{H}_\alpha$ such that $\beta\gamma$ is equal to q modulo α, that is, $q = \beta\gamma \ (\mathrm{mod} \alpha)$. According to Def. 6, we get

$$d_\alpha(\sigma, \tau) = d_\alpha(\sigma - \tau, 0) = d_\alpha(\beta\gamma, 0) = N(q) < 3.$$

Since $q = \beta\gamma \ (\mathrm{mod} \alpha)$ and β is a left divisor of α, we get

$$\beta\gamma = q + \alpha\gamma_1, \ \ \alpha = \beta\gamma_2,$$

respectively, for some $\gamma, \gamma_2 \in \mathcal{H}$. Thus, we obtain

$$\beta\gamma = q + \alpha\gamma_1 = q + (\beta\gamma_2)\gamma_1 = q + \beta(\gamma_2\gamma_1)$$
$$\Rightarrow \ q = \beta(\gamma - \gamma_2\gamma_1)$$

and

$$N(\gamma - \gamma_2\gamma_1) = \frac{N(q)}{N(\beta)}.$$

But, this is contradict to the definition of the norm since $N(\beta) = 5, \ N(q) < 3$, so, $N(\gamma - \gamma_2\gamma_1) = \frac{N(q)}{N(\beta)} \notin \mathbb{Z}$. Hence, the proof is completed.

□

Here, note that $\sigma \neq \tau$, so, $N(q) \neq 0$. A similar proof can be obtained for the case 2.

Theorem 4.6. *Let us assume that there is a Hurwitz integer β and let $N(\beta)^2$ denotes the number of Hurwitz integers which the norm of these integers is less than or equal to t. In this situation, there exists a perfect $t-$dominating set in $\mathcal{H}_\alpha$ if there exists a prime p such that the norm of β is equal to p.*

It is well known that there is a natural way of defining perfect error-correcting group codes with length one by means of perfect dominating sets over known graphs. Some examples associated with this topic can be seen in [13].

We use Mathematica software program to determine the codes given in this paper. As an illustration, we give the following algorithm in Fig. 5 for getting perfect 1−dominating set. In that algorithm, we take $\alpha = 1 + 3i + 2j + k$ and $\beta = 2 + i$. The first column of the table "K", one can see the table "K" when the program runs, denotes the elements of $\mathcal{H}_\alpha$, the second column denotes the elements which dominated by the elements of $\langle\beta\rangle$, the set "SW1" denotes the set $\mathcal{E} \cup \{0\}$, the set "B1γ" denotes Hurwitz integers which the norm between $\gamma \in \langle\beta\rangle$ and these elements is 1. Note that any Hurwitz integer $a_0 + a_1 i + a_2 j + a_3 k$ is shown as $Quaternion[a_0, a_1, a_2, a_3]$ in Mathematica. We don't show the outputs since they takes up too much space in the paper.

Fig. 4 shows the graph $C_{225}(13, 14, \ldots, 24)$ which it is isomorphic to the graph given in Fig. 3. The diameter of the graph $C_{225}(13, 14, \ldots, 24)$ is 5. Using the technic presented in this paper, one can construct a code that the minimum distance of the code is less than or equal to 5. The graph immediately gives the minimum distance of a code presented in this paper.

Fig. 3 shows representation of the Hurwitz graph generated by $\alpha = 1 + 3i + 2j + k$. In the figure, points labeled red denotes the set $\langle\beta\rangle$. Note all vertexes are not given in the figure. The vertexes given in the figure shows one twelfth of all vertexes.

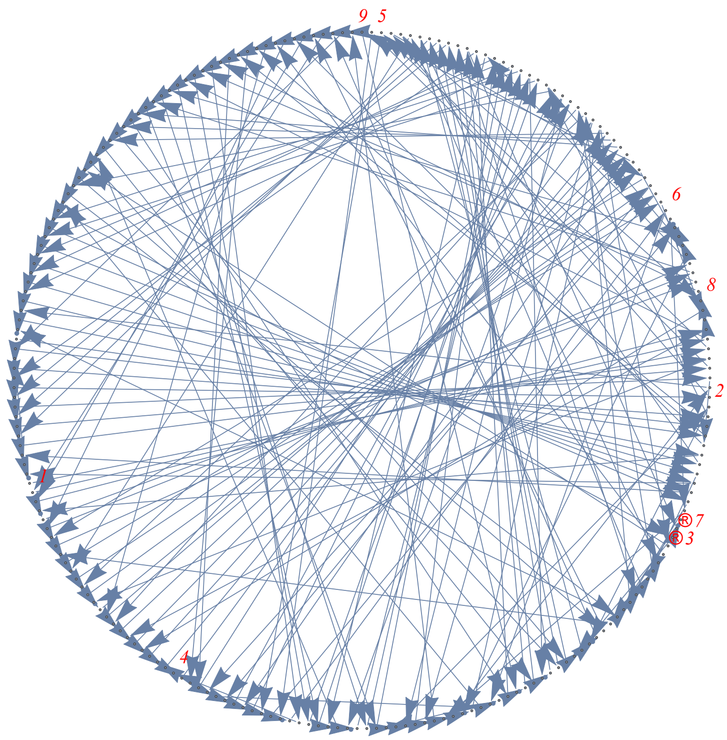


FIGURE 3. A representation of the Hurwitz graph generated by $\alpha = 1 + 3i + 2j + k$

Table I: Some perfect code parameters

α	β	t
$1 + 2i - 3j + k$	$i - 2k$	1
$-\frac{5}{2} + \frac{5i}{2} + \frac{3j}{2} + \frac{k}{2}$	$\frac{3}{2} + \frac{i}{2} + \frac{3j}{2} + \frac{k}{2}$	1
$-\frac{3}{2} + \frac{i}{2} - \frac{5j}{2} + \frac{7k}{2}$	$\frac{1}{2} + \frac{i}{2} + \frac{j}{2} + \frac{5k}{2}$	2
$\frac{3}{2} + \frac{5i}{2} + \frac{9j}{2} + \frac{5k}{2}$	$\frac{3}{2} + \frac{3i}{2} + \frac{3j}{2} + \frac{k}{2}$	2
$3 + 4i + 3j + k$	$2 + i + j + k$	2
$-4 + 5i + 5j + 5k$	$1 + 2i + 2j + 2k$	4

Also, we give some perfect $t-$dominating sets, which directly are perfect codes at Table I.

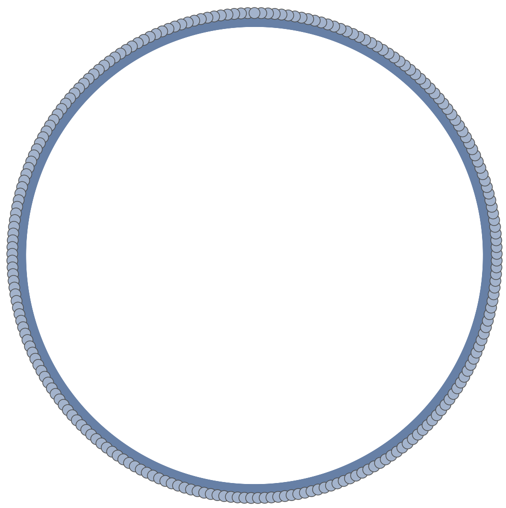
5. Some Comparisons

In this section, we compare codes given in the present paper and some codes given in literature in terms of average energy and bandwidth occupancy. Firstly, we give a comparison associated with average energy. The average energy calculated as:

$$E_{avg} = \frac{1}{M} \sum_{r=0}^{M-1} |q_r|^2,$$

```
In[1]:= A = CirculantGraph@5^2, 813, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24<D
        EdgeCount@AD
        GraphDiameter@AD
```

Out[1]=



```
Out[2]= 2700

Out[3]= 5
```

FIGURE 4. The graph $C_{225}(13, 14, \ldots, 24)$

.

where q_r is in signal space and it has a magnitude (distance from the origin) of $|q_r| = \sqrt{q_{r,0}^2 + q_{r,1}^2 + q_{r,2}^2 + q_{r,3}^2}$ and M denotes the number of the constellation.

Table II: Average Energy Comparison

α	$N(\alpha)$	Base group	Number of constellation	E_{avg}
$2+i$	5	$\mathbb{Z}[i]_{2+i}$	5	0.8
$2+i$	5	$\mathcal{H}_{2+i}$	25	0.96
$3+4i$	25	$\mathbb{Z}[i]_{3+4i}$	25	4.16
15	$1+3i+2j+k$	$\mathcal{H}_{1+3i+2j+k}$	225	3.27
$3+4i$	25	$\mathcal{H}_{3+4i}$	625	5.30
$3+4i$	25	$\mathbb{Z}[i]_{3+4i}^2$	625	8.32

Table II shows that the average energy of codes over Hurwitz integers is better than the average energy of codes over Gaussian integers.

Secondly, we compare codes in terms of bandwidth occupancy. One of the most important parameter of analog/digital communication systems is bandwidth. So far, various modulation and coding techniques are developed to provide bandwidth efficiency. As we know from the communication theory, to attain the equal channel capacity required bandwidth must be higher when the codeword number increases [16, 17]. Bandwidth occupancy BW is calculates as

$$BW = \frac{C_a}{\log(1 + SNR)},$$

where C_a and SNR denote the channel capacity and signal-to-noise ratio, respectively. The BW of codes over Hurwitz integers is better than the BW of codes over

```
<< Quaternions`
α = Quaternion[1, 3, 2, 1]; k = Norm[α]; A = Table[1, {k}];
Do[A[[n]] = n, {n, 1, k}]; B = Quaternion[0, 0, 1, 0] * A;
GG = Table[1, {k^2}];
Do[Do[GG[[n + k * (m - 1)]] = A[[n]] + B[[m]], {n, 1, k}], {m, 1, k}];
Hα = Table[1, {k^2}]; (*Hα denotes the Hα*)
Do[Hα[[tt]] = Mod[GG[[tt]], α], {tt, 1, k^2}];
β = Quaternion[2, 1, 0, 0]; Dimensions[Hα]; B = Table[1, {k^2}];
Do[B[[t]] = Mod[β ** Hα[[t]], α], {t, 1, k^2}]; Bβ = Union[B];
Dimensions[Bβ]; Bβ1 = Table[0, {Dimensions[Bβ][[1]]}];
Do[Bβ1[[t]] = If[Bβ[[t]] == Mod[Bβ[[t]] + (α ** Quaternion[1/2, 1/2, 1/2, 1/2]), α],
    Bβ[[t]]], {t, 1, Dimensions[Bβ][[1]]}];
MatrixForm[Bβ1];

       ( Quaternion[2, 1, 0, 0]   )
       ( Quaternion[0, 0, -2, -1] )
       ( Quaternion[0, 0, -1, 2]  )
       ( Quaternion[-1, 2, 0, 0]  )
Bβ2 =  ( Quaternion[0, 0, 0, 0]   ) ;
       ( Quaternion[0, 0, 2, 1]   )
       ( Quaternion[1, -2, 0, 0]  )
       ( Quaternion[0, 0, 1, -2]  )
       ( Quaternion[-2, -1, 0, 0] )

(*Bβ2 shows the set generated by β. To obtain the set Bβ2,
firstly Bβ is obtained. secondly Bβ1 is checked. Using the commend MemberQ,
the equivalent elements  are elected. *)
(*The set SW1 denotes the elements of weight 1. *)
SW1 = Table[1, {25}];
Do[Do[SW1[[m + 5 * (n - 1)]] = Mod[Quaternion[m, 1, 0, 0] + Quaternion[1, 0, n, 0],
    Quaternion[2, 1, 0, 0]], {m, 1, 5}], {n, 1, 5}];
B1γ = Table[1, {k^2}]; Do[Do[B1γ[[m + 25 * (n - 1)]] = SW1[[m]] + Bβ2[[n]],
  {m, 1, 25}], {n, 1, Dimensions[Bβ2][[1]]}]; MatrixForm[Hα];
Dimensions[B1γ]; K = Table[1, {225}, {2}]; Do[K[[t, 1]] = Hα[[t]];
 K[[t, 2]] = Mod[B1γ[[t]], α], {t, 1, 225}]; Dimensions[Union[Hα]];
MatrixForm[K]
BB = Table[1, {k^2}, {4}]; Do[BB[[t, 1]] = Hα[[t, 1]]; BB[[t, 2]] = Hα[[t, 2]];
 BB[[t, 3]] = Hα[[t, 3]]; BB[[t, 4]] = Hα[[t, 4]];, {t, 1, k^2}];
KK = Table[1, {k^2 - 1}];
Do[KK[[t]] = BB[[t]] → BB[[t + 1]], {t, 1, k^2 - 1}];
KKK = Union[KK, {{0, 0, 0, 0} → {1, 0, 1, 0}}];
Graph[KKK, VertexLabels → {{0, 0, 0, 0} → "1", {0, 0, -2, -1} → "2",
   {0, 0, -1, 2} → "→3", {-1, 2, 0, 0} → "4", {2, 1, 0, 0} → "  5 ", {0, 0, 2, 1} → "6",
   {1, -2, 0, 0} → "→7", {0, 0, 1, -2} → "8", {-2, -1, 0, 0} → "9"},
 VertexLabelStyle → Directive[Red, Italic, 11], VertexSize → 0.2]
```

Gaussian integers since the number of codewords in a code over Hurwitz integers is equal to square of the number of the codewords in a code over Gaussian integers for the same integer $N(\alpha)$ and the same length n. For example, a code C has 625 codewords in Hurwitz integers while a code C has 25 codewords in Gaussian integers for the same integer $N(\alpha) = 25$, $\alpha = 3 + 4i$, and the same length $n = 1$.

6. Conclusion

The paper devotes a new family of $t-$error correcting perfect codes over Hurwitz integers. Using perfect $t-$dominating sets over the circulant graphs, these perfect codes are constructed. Codes given in the present paper and some codes given in literature in terms of average energy and bandwidth occupancy are compared. It is shows that the average energy of codes over Hurwitz integers is better than the average energy of codes over Gaussian integers.

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This study does not be necessary ethical committee permission or any special permission.

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The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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A NEW VIEW ON FIXED POINT

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Abstract. In this paper, we examine a view on fixed point with near soft mapping. First, we study the relationship between. soft mapping and almost smooth mapping. Also, the notion of near soft point, near soft mappings, a different approach to the study of near soft topological spaces. Shows how a near soft fixed point is derived from near soft topological spaces. Finally, many cases such as conservation of near soft compact topological spaces under near soft continuous mapping have been obtained.

1. Introduction

Near sets is a concept given by Peters [4] who deals with the proximity of objects. Here it causes the sample objects to be divided by the feature selection. The nearness of sets foundation on object definitions can be seen by introducing the near approximation space and finding nearby sets there.

The soft set concept, another concept proposed by Molodtsov [5], has been studied by many scientists [2, 6, 7, 8, 9, 10]. The soft sets and soft topological spaces and some of their related concepts have studied by Shabir and Hussain in [1, 2]. Wardowski [14], studied on a fixed points of soft mapping. The notion of near soft set emerges by considering the soft sets approximation and the near set theory as a common concept. Tasbozan [3] introduce the soft topology and sets based on a nearness approximation space. And many studies have been conducted on this subject [13, 11, 12]. The aim of this article is to create different concepts on nearness approximation space. In this study, we create the near soft point notion of near soft set and near soft mapping. These new concepts are explained with examples. The notions of near soft point, near soft topological space are described and their basic properties are explored with the help of examples. New definitions and theorems about near soft continuous mapping and near soft compactness have been obtained. Also, discuss the contrasting image and properties of an image in near soft mapping, based on the presented near soft element concept. In the last part

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of this study, a near soft compact Hausdorff topological space, near soft mapping, and a new fixed point result were created.

2. Near Soft Sets and Near Soft Topology

Definition 2.1. Let $(\mathcal{O}, \mathcal{F}, \sim_{Br}, N_r, \upsilon_{N_r})$ be a nearness approximation space(NAS) and $\sigma = (F, B)$ be a soft set(SS) over $\mathcal{O}$.

$N_r*((F,B)) = (N_r*(F(\phi) = \cup\{x \in \mathcal{O} : [x]_{Br} \subseteq F(\phi)\}, B))$ and $N_r^*((F,B)) = (N_r^*(F(\phi) = \cup\{x \in \mathcal{O} : [x]_{Br} \cap F(\phi) \neq \emptyset\}, B))$ are lower and upper near approximation operators. The SS $N_r((F,B))$ with $Bnd_{N_r(B)}((F,B)) \geqslant 0$ called a near soft set(NSS) [3].

Definition 2.2. Let $\mathcal{O}$ be an initial universe set, E be the universe set of parameters and $A, B \subseteq E$

(1) (F, A) is called a relative null NSS if $F(\phi) = \emptyset$, $\forall c \in A$.
(2) (G, B) is called a relative whole NSS if $G(\phi) = \mathcal{O}$, $\forall \phi \in B$. [3]

Definition 2.3. $(F, A)^c = (F^c, A)$ NSS is a complement of (F, A) if $F^c(\phi) = \mathcal{O} - F(\phi)$ $\forall \phi \in A$ [3].

Definition 2.4. Let (F, B) be a NSS over $\mathcal{O}$ and τ be the collection of near soft subsets NSs of $\mathcal{O}$, if if the following are provided

i): $(\emptyset, B)$, $(\mathcal{O}, B) \in \tau$
ii): $(F_1, B), (F_2, B) \in \tau$ then $(F_1, B) \cap (F_2, B) \in \tau$
iii): $(F_i, B), \forall \phi \in B$ then $\underset{i}{\cup}(F_i, B) \in \tau$

Then $(\mathcal{O}, \tau, B)$ is a near soft topological space($NSTS$) [3].

Definition 2.5. Let $(\mathcal{O}, \tau, B)$ be a $NSTS$ over $\mathcal{O}$, then the members of τ are said to be near soft open sets ($NSOS$) in $\mathcal{O}$. If its complement is open and a member of τ then a NSs of $(\mathcal{O}, \tau, B)$ is called near soft closed(NSC).

Definition 2.6. Let (F, B) be a NSS over $\mathcal{O}$. If for the element $\phi \in B$, $F(\phi) = \{x\}$ and $F(\phi^{'}) = \emptyset$, $\forall \phi^{'} \in B - \{\phi\}$ then NSS (F, B) is a near soft point (NSP), denoted by (x, ϕ).

Proposition 1. Let $(\mathcal{O}, \tau, B)$ be a $NSTS$ over $\mathcal{O}$, then the collection $\tau_\phi = \{F(\phi) : (F, B) \in \tau\}$ for each $\phi \in B$, defines a topology on $\mathcal{O}$.

Definition 2.7. Let $(\mathcal{O}, \tau, B)$ be a $NSTS$ over $\mathcal{O}$ and (F, B) be a NSS over $\mathcal{O}$. Then the near soft closure $(F, B)^c$ is the intersection of all NSC super sets of (F, B).

Definition 2.8. Let $(\mathcal{O}, \tau, B)$ be a $NSTS$ over $\mathcal{O}$ and (F, B) be a NSS over $\mathcal{O}$. Then the near soft interior $(F, B)^\circ$ is the collection of all $NSOS$ of (F, B).

Example 2.9. $\mathcal{O} = \{x_1, x_2, x_3\}, B = \{\phi_1, \phi_2\} \subseteq \mathcal{F}$ be denote a set of objects and a set of parameters respectively. Let (F, B) be a SS defined by $(F, B) = \{(\phi_2, x_2)\}$. Then $\sigma = (F, B)$ is a NSS with $r = 1$.

$$\begin{aligned} [x_1]_{\phi_1} &= \{x_1, x_2\}, [x_2]_{\phi_2} = \{x_2\} \\ [x_3]_{\phi_1} &= \{x_3\}, [x_1]_{\phi_2} = \{x_1, x_3\} \end{aligned}$$

Then $N_*(\sigma) = N_*(F(\phi), B) = (F_*(\phi), B) = \{(\phi_2, \{x_2\})\}$,$N^*(\sigma) = N^*(F(\phi), B) = (F, B)$ and $Bnd_N(\sigma) \geq 0$.Thus (F, B) is a NSS.

Then $\sigma = (F, B)$ is a NSS with $r = 2$.

$$[x_1]_{\phi_1,\phi_2} = \{x_1\}, [x_2]_{\phi_1,\phi_2} = \{x_2\}, [x_3]_{\phi_1,\phi_2} = \{x_3\}$$

$N^*(\sigma) = N_*(\sigma) = (F, B)$.Thus (F, B) is a NSS. Also $\phi_2 \in B, F(\phi_2) = \{x_2\}$ and $\phi_2' \in B - \{\phi_2\}, F(\phi_2') = \emptyset$. Thus (F, B) is a NSP and denote (x_2, ϕ_2) or $(x_2)_{\phi_2}$.

Definition 2.10. Let $(\mathcal{O}, \tau, B)$ be a $NSTS$ over $\mathcal{O}$. If there exists a $NSOS$ (G, B) such that $(x_\phi, B) \in (G, B) \subset (F, B)$ then a NSS (F, B) in $(\mathcal{O}, \tau, B)$ is a near soft neighbourhood of the NSP $(x_\phi, B) \in (F, B)$.

3. Near Soft Compactness, Near Soft Mapping and Its Fixed Points

In this section, we will give some definitions using NSP.

Definition 3.1. Let $(\mathcal{O}, \tau, B)$ be a $NSTS$ and $x, y \in \mathcal{O}$ such that $x \neq y$. $(\mathcal{O}, \tau, B)$ is a near soft Hausdorff space($NSHS$) if for each $NSOS$ $(F, D), (G, C) \in (\mathcal{O}, B)$ such that $x \in (F, D)$, $y \in (G, C)$ and $(F, D) \cap (G, C) = \emptyset$. Similarly for each NSP $(x_\phi, B), (y_{\phi'}, B) \in (\mathcal{O}, B)$ such that $(x_\phi, B) \neq (y_{\phi'}, B)$ there are $NSOS$ $(F_1, B), (F_2, B) \subset (\mathcal{O}, B)$ so that $(x_\phi, B) \in (F_1, B), (y_{\phi'}, B) \in (F_2, B)$ and $(F_1, B) \cap (F_2, B) = (\emptyset, B)$.

Definition 3.2. Two NSS (F, B) and (G, B) in $(\mathcal{O}, B)$ are near soft disjoint denoted by $(F, B) \cap (G, B) = (\emptyset, B)$, if $F(\phi) \cap G(\phi) = \emptyset, \forall \phi \in B$.

Definition 3.3. Two NSP (x_ϕ, B) and $(y_{\phi'}, B)$ over a common universe $\mathcal{O}$ are distinct, written $(x, \phi) \neq (y, \phi^{'})$ if there corresponding NSS (F, B) and (G, B) are disjoint.

Definition 3.4. Let $(\mathcal{O}, \tau, B)$ be a $NSTS$ and a NSS $(V, B) \subseteq (\mathcal{O}, B)$ is near soft open $\Leftrightarrow$ for each a NSS $(W, B) \in \tau$ which $\alpha \in (W, B) \subseteq (V, B)$.

Definition 3.5. Let $(\mathcal{O}, \tau, B)$ be a $NSTS$ and $G \subseteq \mathcal{O}$. The near soft topology (NST) on (G, B) incited by the NST τ is the family τ_G of the NSs of G of the shape $\tau_G = \{V \cap G : V \in \tau\}$. Thus (G, τ_G, B) is a near soft topological subspace of $(\mathcal{O}, \tau, B)$.

Definition 3.6. Let $(\mathcal{O}, \tau, B)$ be a $NSTS$ and $C \subseteq \mathcal{O}$. If $(C, B) \subseteq \cup_{i \in I} (V_i, B)$ then $\{V_i\}_{i \in I} \subseteq \tau$ is a NSO cover of (C, B).

Definition 3.7. If for each NSO cover $\{V_i\}_{i \in I}$ of (C, B) there exists $i_1, i_2, ..., i_k \in I$, $k \in N$ such that $(C, B) \subseteq \cup_{n=1}^{k} (V_{in}, B)$ then (C, τ, B) is a near soft compact space ($NSCoS$).

Definition 3.8. Let $(\mathcal{O}, \tau, B)$ be a $NSTS$ and $C \subseteq \mathcal{O}$. If the (G, τ_G, B) is $NSCoS$ then the (G, B) NSS is compact in $(\mathcal{O}, \tau, B)$.

Definition 3.9. Let $(\mathcal{O}, \tau, B)$ be a $NSHS$. Then every $NSCo$ set in $(\mathcal{O}, \tau, B)$ is NSC in $(\mathcal{O}, \tau, B)$.

Proof. Let (C, B) be a $NSCo$ set in $(\mathcal{O}, \tau, B)$ and $x \in C^{'}$. For every $y \in C$ let $x, y \in \mathcal{O}$ and $x \in (F, D), y \in (G, P), (F, D), (G, P) \in (\mathcal{O}, B)$ such that $(F, D) \cap (G, P) = \emptyset$. From the near soft compactness of (C, B) there exists $y_1, y_2, ..., y_k \in (C, B)$ such that $(C, B) \subseteq (G_1, P) \cup ... \cup (G_k, C)$. Denote $(F, D) = (F_1, D) \cup ... \cup (F_k, D)$ and $(G_1, P) \cup ... \cup (G_k, C) = (G, P)$ then $F \cap G = \emptyset$ and thus $F \cap C = \emptyset$, which $x \in F \subseteq C^{'}$ thus (C, B) is NSC. $\square$

Definition 3.10. $NS(\mathcal{O},B)$ denotes the family of all NSS over $(\mathcal{O},B)$. Let $(F,A),(G,C) \in NS(\mathcal{O},B), A, C \subseteq B$. The near soft cartesian product $(F,A) \times (G,C)$ is a NSS on $(\mathcal{O},B) \times (\mathcal{O},B)$ such that $(F,A) \times (G,C) = \{((\phi_1,\phi_2), F(\phi_1) \times G(\phi_2)) : \phi_1, \phi_2 \in B\}$

Definition 3.11. A near soft relation(NSR) from (F,A) to $(G,C)$is a NSS (R,B) , $R \subseteq (F,A) \times (G,C)$ with

$$(R,B) = \{((\phi_1,\phi_2), \mathcal{O}(\phi_1) \times \mathcal{O}(\phi_2)) : \phi_1, \phi_2 \in B, \mathcal{O}(\phi_1) \subseteq F(\phi_1), \mathcal{O}(\phi_2) \subseteq G(\phi_1)\}$$

If $((\phi_1,\phi_2), \mathcal{O}(\phi_1) \times \mathcal{O}(\phi_2)) \in (R,B)$ then $(\phi_1, \mathcal{O}(\phi_1) R(\phi_2, \mathcal{O}(\phi_2))$.

Definition 3.12. Let $(F,A),(G,C) \in NS(\mathcal{O},B)$. If the following conditions satisfied then a NSR $f \subseteq (F,A) \times (G,C)$ is a NSM denoted by $f : (F,A) \to (G,C)$;

(1) For each NSP $\alpha = (x_e, A) \in (F,A)$ there exists only one NSP $\beta = (f(x)_e, A) \in (G,C)$ such that $f(\alpha) = \beta$ or $\alpha f \beta$.
(2) For each empty NSP $\alpha \in (F,A), f(\alpha)$ is an empty NSP of (G,C) .

Definition 3.13. Let $(F,A),(G,C) \in NS(\mathcal{O},B)$ and $f : (F,A) \to (G,C)$ be a NSM.

(1) The view of $X \subseteq F$ under (NSM) f is the NSS of $(f(X),C) = (\cup_{\alpha \in X} f(\alpha), C)$ and for each NSM $(f(\emptyset),B) = (\emptyset,B)$.
(2) The inverse of $Y \subseteq G$ under NSM f is the NSS of $(f^{-1}(Y),A) = (\cup\{\{\alpha\} : \alpha \in (F,A), f(\alpha) \in (Y,C)\}, B)$.

Definition 3.14. Let $(F,B),(G,B) \in NS(\mathcal{O},B)$. $(W,B),(W_1,B),(W_2,B) \subseteq (F,B)$,$(Z,B),(Z_1,B),(Z_2,B) \subseteq (G,B)$ and let $f : (F,B) \to (G,B)$ be a (NSM). Then the following hold:

(1) $W_1 \subseteq W_2 \Rightarrow f(W_1) \subseteq f(W_2)$
(2) $Z_1 \subseteq Z_2 \Rightarrow f^{-1}(Z_1) \subseteq f^{-1}(Z_2)$
(3) $W \subseteq f^{-1}(f(W))$
(4) $f(f^{-1}(Z)) \subseteq Z$
(5) $f(W_1 \cup W_2) = f(W_1) \cup f(W_2)$
(6) $f(W_1 \cap W_2) \subseteq f(W_1) \cap f(W_2)$
(7) $f^{-1}(Z_1 \cup Z_2) = f^{-1}(Z_1) \cup f^{-1}(Z_2)$
(8) $f^{-1}(Z_1 \cap Z_2) = f^{-1}(Z_1) \cap f^{-1}(Z_2)$

Definition 3.15. Let $(F,\tau,B),(G,\upsilon,B)$ be a $NSTS$ and $f : (F,B) \to (G,B)$ be a NSM. If $\forall V \in \upsilon, f^{-1}(V) \in \tau$ then f is a near soft continuous mapping and denoted by $NSCM$.

Definition 3.16. Let $(g,h) : (F,B) \to (G,B')$ be a NSM. A NSM (g,h) is an injective, surjective and bijective if g,h are both injective, surjective and bijective, respectively.

Definition 3.17. Let $(\mathcal{O}_k,\tau,B)$ and $(\mathcal{O}_l,\tau,B)$ be two $NSTS$. $f : (\mathcal{O}_k,\tau,B) \to (\mathcal{O}_l,\tau,B)$ be a mapping. For each near soft neighbourhood (H,B) of $(f(x)_\phi,B)$, if there exists a near soft neighbourhood $f((F,B)) \subset (H,B)$ then f is a $NSCM$ (x_ϕ,B). If f is $NSCM$ for all (x_ϕ,B), then f is called $NSCM$.

Definition 3.18. Let $(\mathcal{O}_1,\tau,B)$ and $(\mathcal{O}_2,\tau,B)$ be two $NSTS$. $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a mapping. $\mathcal{O}_1$ is near soft homeomorphic to $\mathcal{O}_2$ if f is a bijection, NSC and f^{-1} is a near soft homeomorphism.

Example 3.19. $\mathcal{O} = \{x_1, x_2, x_3\}, B = \{\phi_1, \phi_2\} \subseteq \mathcal{F}$. Let (F, B) be a NSS defined by $(F, B) = \{(\phi_2, \{x_1, x_2\}), (\phi_2, \{x_2, x_3\})\}$. Then $\sigma = (F, B)$ is a NSS with

$$\begin{aligned}[x_1]_{\phi_1} &= \{x_1, x_2\}, [x_2]_{\phi_2} = \{x_2\} \\ [x_3]_{\phi_1} &= \{x_3\}, [x_1]_{\phi_2} = \{x_1, x_3\}\end{aligned}$$

. And think τ of $NSSs$ of (F, B);

$$\tau = \{\emptyset, (\phi_2, \{x_2\}), (F, B), \{(\phi_1, \{x_1, x_2\}), (\phi_2, \{x_2\})\}\}$$

Then (F, τ) is a $NSTS$. Now taking a NSM $f : (F, B) \to (F, B)$ as follows:

$$\begin{aligned}f(\phi_1, \emptyset) &= (\phi_2, \emptyset), f(\phi_2, \{x_2\}) = (\phi_2, \{x_2\}) \\ f(\phi_2, \emptyset) &= (\phi_1, \emptyset), f(\phi_1, \{x_1, x_2\}) = (\phi_1, \{x_1, x_2\}) \\ f(\phi_3, \emptyset) &= (\phi_3, \emptyset), f(F, B) = (F, B)\end{aligned}$$

Then $f^{-1}(v) \in \tau, \forall v \in \tau$ then f is a $NSCM$.

Proposition 2. Let (C, τ, B) be a $NSCoTS$ and let $f : (C, B) \to (C, B)$ $(NSCM)$. Then $f(C)$ is a $(NSCo)$ in (C, τ, B).

Proof. Let $\{V_i\}_{i \in I} \subseteq \tau$ which $f(C) \subseteq \cup_{i \in I} V_i$. From the near soft continuity of $f, \{f^{-1}(V_i)\}_{i \in I}$ is a family of $(NSOS)$. Then $C \subseteq f^{-1}(f(C)) \subseteq f^{-1}(\cup_{i \in I} V_i) = \cup_{i \in I} f^{-1}(V_i)$ and from $NSCo$ of C there exists $i_1, i_2, ..., i_k \in I, k \in N$ which

$$\begin{aligned}C &\subseteq f^{-1}(V_{i1}) \cup f^{-1}(V_{i2}) \cup ... \cup f^{-1}(V_{ik}) \\ f(C) &\subseteq (V_{i1}) \cup (V_{i2}) \cup ... \cup (V_{ik})\end{aligned}$$

Thus $f(C)$ is a $NSCo$. □

Definition 3.20. Let (F, B) be a NSS and $f : (F, B) \to (F, B)$ be a NSM. If $f(\alpha) = \alpha$ then a NSP $\alpha \in (F, B)$ is a fixed point of f.

Theorem 3.21. *Let (C, τ) be a $NSCoHTS$ and let $f : (C, B) \to (C, B)$ be a $NSCM$ such that:*

(1) for each nonempty NSP $\alpha \in (C, B)$, $f(\alpha)$ is a nonempty NSP of (C, B),
(2) If $f(X, B) = (X, B)$then only one nonempty NSP $\alpha \in (C, B)$ which $f(\alpha) = \alpha$, for each NSC set $(X, B) \subseteq (C, B)$.

Example 3.22. Let $f : (F, B) \to (F, B)$ be a NSM defined in example 35. Then the NSP $(\phi_2, \{x_2\}), (\phi_3, \emptyset), (\phi_1, \{x_1, x_2\})$ are fixed points of f.

4. Conclusions

In this study, we describe the notion of NSM and its fixed point. In the near soft topological space, we tried to create a fixed point structure with near soft mapping, which we created based on the concept of near soft point. Expressions explaining these concepts and showing the necessity of some assumptions are presented. With a different approach to the near soft cluster, it will facilitate the solution of many problems and will help new studies.

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VERSIONS OF FUGLEDE-PUTNAM THEOREM ON p-w-HYPONORMAL OPERATORS

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Abstract. The aim of the article is the presentation of certain extensions of the famous Fuglede-Putnam Theorem on the class of p-w-hyponormal operators, which generalize some results proved by authors in [10].

1. Introduction and Preliminaries

Throughout this work, $B(H)$ denotes the Banach algebra of bounded linear operators on a complex separable Hilbert space H. By $\ker(T)$ and $ran(T)$ respectively, we mean the null space and the range of an operator $T \in B(H)$. Given $T, S \in B(H)$, the generalized derivation $\delta_{T,S}$ induced by T and S is defined for all $X \in B(H)$ by $\delta_{T,S}(X) = TX - XS$. Recall that T is said to be normal if T commutes with its adjoint $T^\star$. The well-known Fuglede-Putnam Theorem states that $\ker(\delta_{T,S}) \subset \ker(\delta_{T^\star,S^\star})$ whenever T and S are normal operators, see [5, 6, 7] and [15] where several generalizations of this result are given for operators T and S belonging to some classes of non normal operators. For $0 < p \leq 1$, an operator $T \in B(H)$ is said to be p-hyponormal if $|T|^{2p} - |T^\star|^{2p} \geq 0$, where $|T| = (T^\star T)^{\frac{1}{2}}$ is the module of T. A 1-hyponormal operator is hyponormal and $\frac{1}{2}$-hyponormal is semi-hyponormal. Reader can find many interesting spectral properties of this class in [1, 2, 12, 15]. In [1], it is defined the Aluthge transform of an operator $T = U|T|$ by $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$, and in [2], it is shown that if T is p-hyponormal, then $\widetilde{T}$ is $(p + \frac{1}{2})$-hyponormal for $0 < p \leq \frac{1}{2}$ and hyponormal for $\frac{1}{2} \leq p \leq 1$. Also, T is said to be log-hyponormal if T is invertible and $\log(T^\star T) \geq \log(TT^\star)$. The operator $T \in B(H)$ is said to be dominant if $ran(T - \lambda) \subset ran(T - \lambda)^\star$ for each λ in the spectrum $\sigma(T)$ of T. Also, if there exists $M > 0$ such that $(T - \lambda)(T - \lambda)^\star \leq M(T - \lambda)^\star(T - \lambda)$ for each $\lambda \in \sigma(T)$, then T is said to be M-hyponormal. Clearly,

$$\text{Hyponormal} \subset M\text{-hyponormal} \subset \text{dominant}$$

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In [15], it is presented an example of an M-hyponormal operator that is not hyponormal. An operator T is said to be w-hyponormal if $\left|\widetilde{T}\right| \geq |T| \geq \left|\widetilde{T^\star}\right|$ [2, 4, 8]. Useful results of the class of w-hyponormal operators are presented in [2, 8, 9], and it was proved that it contains the class of p-hyponormal operators. The following inclusions hold

$$\text{Hyponormal} \subset p\text{-hyponormal} \subset w\text{-hyponormal}$$

The operator T is said to be p-w-hyponormal for certain $0 < p \leq 1$, if

$$\left|\widetilde{T}\right|^p \geq |T|^p \geq \left|\widetilde{T^\star}\right|^p$$

[10, 16]. A 1-w-hyponormal is w-hyponormal, and w-hyponormal operators are evidently p-w-hyponormal. In this article, we'll extend the Fuglede-Putnam theorem for p-w-hyponormal with p-hyponormal operators or with log-hyponormal operators. Other spectral related results are also added.

2. Known Results

The following known results will be needed for the rest of the paper.

Lemma 2.1. [13] *Let T be in $B(H)$ and S be in $B(K)$. The following assertions are equivalent*

1. *The pair (T, S) satisfies the Fuglede-Putnam theorem.*
2. *If $TX = XS$ for some X in $B(K, H)$, then $\overline{ran(X)}$ reduces T, $(\ker(X))^\perp$ reduces S, and restrictions $T\left|\overline{ran(X)}\right.$, $S\left|(\ker(X))^\perp\right.$ are unitarily equivalent normal operators.*

Lemma 2.2. [2] *Let $T \in B(H)$ be a w-hyponormal operator and let $M \subset H$ be an invariant subspace under T. Then $T\,|M$ is w-hyponormal.*

Lemma 2.3. [2] *Let $T \in B(H)$ be a w-hyponormal operator. Then $\widetilde{T}$ is semi-hyponormal.*

Lemma 2.4. [1] *If T is a p-hyponormal operator, then $\widetilde{T}$ is $(p + \frac{1}{2})$-hyponormal for $0 < p \leq \frac{1}{2}$ and hyponormal for $\frac{1}{2} \leq p \leq 1$.*

Lemma 2.5. [16] *Let $T \in B(H)$ be p-w-hyponormal, and let $M \subset H$ be a T-invariant subspace. Then $T\,|M$ is p-w-hyponormal.*

Lemma 2.6. [16] *Let $T \in B(H)$ be a p-w-hyponormal operator. Then $\widetilde{T}$ is $\frac{p}{2}$-hyponormal.*

3. Main results

The familiar Fuglede-Putnam Theorem asserts that for normal operators T and S on H, equation $\delta_{T,S}(X) = 0$ implies $\delta_{T^\star,S^\star}(X) = 0$ for all X in $B(H)$. Extensions of this result for certain classes of non normal operators are presented in many papers, see [5, 6] and [7]. Authors in [11] showed that this result remains true for an M-hyponormal operator T and a dominant operator S.

The following result gives an extension of the Fuglede-Putnam property for M-hyponormal and p-hyponormal operators.

Proposition 1. For an M-hyponormal operator T and for a p-hyponormal operator $S^\star$ in $B(H)$, $\ker(\delta_{T,S}) \subset \ker(\delta_{T^\star,S^\star})$.

Proof. Due to [7] and since an M-hyponormal operator is dominant, the pair (T, S) satisfies the Fuglede-Putnam property. □

Theorem 3.1. *Let T be M-hyponormal and let $S^\star$ be w-hyponormal operators in $B(H)$. Then, $\delta_{T,S}(X) = 0$ entails $\delta_{T^\star,S^\star}(X) = 0$ for all X in $B(H)$. Moreover, $\overline{ran(X)}$ reduces T, $(\ker(X))^\perp$ reduces S and restrictions $T\left|\overline{ran(X)}\right.$, $S\left|(\ker(X))^\perp\right.$ are unitarily equivalent normal operators.*

Proof. Subspaces $\overline{ran(X)}$ and $(\ker(X))^\perp$ are invariant for T and S respectively since $\delta_{T,S}(X) = 0$. Then, we can write

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}, \ S = \begin{pmatrix} S_1 & 0 \\ S_2 & S_3 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : H_2 \longrightarrow H_1$$

under the decompositions

$$\begin{aligned} H &= H_1 = \overline{ran(X)} \oplus ran(X)^\perp \\ H &= H_2 = (\ker X)^\perp \oplus \ker X \end{aligned}$$

From $\delta_{T,S}(X) = 0$ we get

$$T_1X_1 = X_1S_1 \tag{3.1}$$

where T_1 is M-hyponormal, and S_1 is w-hyponormal by Lemma 2.2. Let $S_1 = U|S_1|$ be the polar decomposition of T_1. Since $U|S_1| = |S_1^*|U$, equality (3.1) can be writen

$$T_1X_1 = X_1|S_1^\star|U \tag{3.2}$$

Multiplying the two sides of (3.2) at right by $|S_1^\star|^{\frac{1}{2}}$, we obtain

$$T_1(X_1|S_1^\star|^{\frac{1}{2}}) = X_1|S_1^\star|U|S_1^\star|^{\frac{1}{2}} = (X_1|S_1^\star|^{\frac{1}{2}})\widetilde{S_1^\star}$$

The Aluthge transform $\widetilde{S_1^\star}$ of $S_1^\star$ is semi-hyponormal by Lemma 2.3. Hence, the pair $(T_1, \widetilde{S_1^\star})$ satisfies the Fuglede-Putnam property by Proposition 1. Thus, restrictions $T_1\Big|_{\overline{ran(X_1|S_1^\star|^{\frac{1}{2}})}}$ and $\widetilde{S_1^\star}\Big|_{(\ker((X_1|S_1^\star|^{\frac{1}{2}})^\perp}$ are equivalent normal operators by Lemma 2.1. Since X_1 is quasiaffinity, i.e., one-to-one with dense range, and $|S_1^\star|^{\frac{1}{2}}$ is injective,

$$\overline{ran(X_1|S_1^\star|^{\frac{1}{2}})} = \overline{ranX_1} = \overline{ranX}$$

and

$$\ker(X_1|S_1^\star|^{\frac{1}{2}}) = \ker X_1 = \ker X$$

Thus, $\widetilde{S_1^\star}$ is normal and then S_1 is normal by [15]. The operator $S^\star$ is M-hyponormal and its restriction $S_1^\star$ on $(\ker X)^\perp$ is normal. Consequently, $\ker X$ reduces $S^\star$. Hence $S_2 = 0$.

Similarly, T is M hyponormal, and its restriction T_1 on $\overline{ranX}$ is normal. Then, $\overline{ranX}$ reduces T. Thus $T_2 = 0$. Since the pair (T_1, S_1) satisfies the Fuglede-Putnam theorem, $T_1^\star X_1 = X_1S_1^\star$. Finally $T^\star X = XS^\star$. □

Theorem 3.2. *Let T be a p-w-hyponormal operator in $B(H)$. If $|T|$ is invertible, then for all $\lambda \notin \sigma(T)$*

i. $\left\| \left|\widetilde{\widetilde{T}}\right|^{\frac{1}{2}} \left|\widetilde{T}\right|^{\frac{1}{2}} |T|^{\frac{1}{2}} (T-\lambda)^{-1} |T|^{-\frac{1}{2}} \left|\widetilde{T}\right|^{-\frac{1}{2}} \left|\widetilde{\widetilde{T}}\right|^{-\frac{1}{2}} \right\| \leq \frac{1}{dist(\lambda,\sigma(T))}$

ii. $\left\|T^{-1}\right\| \leq \frac{1}{\min(|\lambda|,\lambda\in\sigma(T))}$

Proof. i. $\widetilde{T}$ is $\frac{p}{2}$-hyponormal by Lemma 2.6, and $0 < \frac{p}{2} \leq \frac{1}{2}$. Since $\sigma(T) = \sigma(\widetilde{T})$ by [3],

$$\left\| \left|\widetilde{\widetilde{T}}\right|^{\frac{1}{2}} \left|\widetilde{T}\right|^{\frac{1}{2}} (\widetilde{T}-\lambda)^{-1} \left|\widetilde{T}\right|^{-\frac{1}{2}} \left|\widetilde{\widetilde{T}}\right|^{-\frac{1}{2}} \right\| \leq \frac{1}{dist(\lambda,\sigma(T))}$$

for $\lambda \notin \sigma(T)$ by [1]. The proof derives then from the fact that

$$(\widetilde{T}-\lambda)^{-1} = |T|^{\frac{1}{2}} (T-\lambda)^{-1} |T|^{-\frac{1}{2}}$$

ii. Since $\left\|\widetilde{T}\right\| \leq \|T\|$ for an arbitary operator T in $B(H)$,

$$\left\|T^{-1}\right\| \leq \left\|\widetilde{T}^{-1}\right\| = \frac{1}{\min(|\lambda|\,,\lambda\in\sigma(\widetilde{T}))} = \frac{1}{\min(|\lambda|\,,\lambda\in\sigma(T))}$$

$\square$

As a consequence of the previous result, and since the Aluthge transform of a log-hyponormal operator is semi-hyponormal [14], we can then state the following generalization of the Fuglede-Putnam's Theorem for p-w-hyponormal with log-hyponormal operators as follows

Theorem 3.3. *The Fuglede-Putnam Theorem holds for a p-w-hyponormal operator $T \in B(H)$ with $\ker T \subset \ker T^\star$, and a p-hyponormal operator $S^\star \in B(H)$.*

Proof. Let

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix},\quad S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

according to the decompositions

$$\begin{aligned} H &= H_1 = (\ker T)^{\perp} \oplus (\ker T) \\ H &= H_2 = (\ker S^\star)^{\perp} \oplus (\ker S^\star) \end{aligned}$$

From equation $TX = XS$, we get

$$T_1X_1 = X_1S_1 \tag{3.3}$$

and $T_1X_2 = X_3S_1 = 0$. Since T_1 and S_1 are one-to-one, $X_2 = X_3 = 0$. T_1 is a one-to-one p-w-hyponormal operator by Lemma 2.3, and $S_1^\star$ is p-hyponormal. Let $T_1 = U|T_1|$ be the polar decomposition of T_1. Equation (3.3) can be written

$$U|T_1|X_1 = X_1S_1 \tag{3.4}$$

Multiplying the two sides of (3.4) on the left by $|T_1|^{\frac{1}{2}}$ we get

$$|T_1|^{\frac{1}{2}} U |T_1|^{\frac{1}{2}} |T_1|^{\frac{1}{2}} X_1 = |T_1|^{\frac{1}{2}} X_1S_1$$

So $\widetilde{T_1}(|T_1|^{\frac{1}{2}} X_1) = (|T_1|^{\frac{1}{2}} X_1)S_1$. The Aluthge transform $\widetilde{T_1}$ of T_1 is $\frac{p}{2}$-hyponormal by Lemma 2.6, and $S_1^\star$ is p-hyponormal. By [5], the pair $(\widetilde{T_1}, S_1)$ satisfies the Fuglede-Putnam Theorem. Thus,

$$\widetilde{T_1}^*(|T_1|^{\frac{1}{2}} X_1) = (|T_1|^{\frac{1}{2}} X_1)S_1^\star$$

Consequently, restrictions $\widetilde{T_1}\Big|_{\overline{ran(|T_1|^{\frac{1}{2}} X_1)}}$ and $S_1\Big|_{(\ker(|T_1|^{\frac{1}{2}} X_1)^\perp}$ are unitarily equivalent normal operators by Lemma 2.1. Since the operator $|T_1|^{\frac{1}{2}}$ and X_1 are one-to-one, the operator $|T_1|^{\frac{1}{2}} X_1$ so is. Thus

$$(\ker(|T_1|^{\frac{1}{2}} X_1))^\perp = \{0\}^\perp = (\ker X_1)^\perp = (\ker X)^\perp$$

And

$$\overline{ran(\left|\widetilde{T_1}\right|^{\frac{1}{2}} X_1)} = (\ker |T_1|^{\frac{1}{2}} X_1)^\perp = \{0\}^\perp = \overline{ran(X_1)} = \overline{ran(X)}$$

Thus, $\widetilde{T_1}$ is a normal operator. The operator T_1 so is by [15]. Therefore, $\overline{ran(X)}$ reduces T_1 by Lemma 2.1, and $(\ker X_1)^\perp$ reduces $S_1^\star$ by [17]. Since T_1 is normal, and $S_1^\star$ is p-hyponormal, the Fuglede-Putnam property holds for the pair (T_1, S_1). Thus, $T_1^\star X_1 = X_1 S_1^\star$ and then, $T^\star X = XS^\star$. □

Corollary 3.4. The pair (T, S) satisfies the Fuglede-Putnam Theorem if T is a p-hyponormal operator and $S^\star$ is a p-w-hyponormal with $\ker S \subset \ker S^\star$.

Proof. $TX = XS$ for some X in $B(H)$. Put $A = S^\star$, $B = T^\star$ and $C = X^\star$. Then, $B^\star C^\star = C^\star A^\star$. Hence, $AC = CB$, where A is an injective p-w-hyponormal or a p-w-hyponormal with $\ker A \subset \ker A^\star$, and $B^\star$ is p-hyponormal. By the previous result, $A^\star C = CB^*$. Thus, $SX^\star = X^\star T$. Consequently, $T^\star X = XS^\star$. □

Theorem 3.5. *$\delta_{(T,S)} \subset \delta_{(T^\star,S^\star)}$ for a p-w-hyponormal operator T with $\ker T \subset \ker T^\star$, and a* log*-hyponormal operator $S^\star$.*

We need the following property of log-hyponormal operators for the proof.

Lemma 3.6. [15] *Let $T \in B(H)$ be a* log*-hyponormal operator and let $M \subset H$ be a T-invariant closed subspace. Then, the restriction $T\,|M$ is* log*-hyponormal.*

Proof. (of Theorem 3.5) Let's consider the decompositions

$$\begin{aligned} H &= H_1 = (\ker T)^\perp \oplus (\ker T) \\ H &= H_2 = (\ker S^\star)^\perp \oplus (\ker S^\star) \end{aligned}$$

Then

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

From equation $\delta_{T,S}(X) = 0$, we get

$$\delta_{T_1,S_1}(X_1) = 0 \tag{3.5}$$

and $T_1X_2 = X_3S_1 = 0$. Since T_1 and S_1 are one-to-one, $X_2 = X_3 = 0$. T_1 is a one-to-one p-w-hyponormal operator by Lemma 2.6, and $S_1^\star$ is an injective log-hyponormal by Lemma 3.6. Let $S_1 = U\,|S_1|$ be the polar decomposition of S_1. Since $S_1 = |S_1^\star|\,U$, equation (3.5) can be written

$$T_1X_1 = X_1\,|S_1^\star|\,U \tag{3.6}$$

Multiplying the two sides of (3.6) at right by $|S_1^\star|^{\frac{1}{2}}$ we get

$$T_1(X_1\,|S_1^\star|^{\frac{1}{2}}) = (X_1\,|S_1^\star|^{\frac{1}{2}})\,|S_1^\star|^{\frac{1}{2}}\,U\,|S_1^\star|^{\frac{1}{2}} = (X_1\,|S_1^\star|^{\frac{1}{2}})\widetilde{S_1^\star}$$

T_1 is p-w-hyponormal, and the Aluthge transform $\widetilde{S_1^\star}$ of $S_1^\star$ is $\frac{1}{2}$-hyponormal by [14]. By Theorem 3.3, the Fuglede-Putnam's Theorem holds for the pair $(T_1, \widetilde{S_1^\star})$. Hence, $T_1^\star(X_1\,|S_1^\star|^{\frac{1}{2}}) = (X_1\,|S_1^\star|^{\frac{1}{2}})\widetilde{S_1^\star}^\star$. Furthermore, and by Lemma 2.1, $T_1\Big|_{\overline{ran(X_1|S_1^\star|^{\frac{1}{2}})}}$ and $\widetilde{S_1^\star}\Big|_{(\ker(X_1|S_1^\star|^{\frac{1}{2}})^\perp}$ are unitarily equivalent normal operators. Since the operator $|S_1^\star|^{\frac{1}{2}}$ and X_1 are one-to-one, the operator $X_1\,|S_1^\star|^{\frac{1}{2}}$ so is. The rest of proof is similar to Theorem 3.1. □

Corollary 3.7. Let $T \in B(H)$ be a pure log-hyponormal operator, and let $S^* \in B(H)$ be a p-w-hyponormal with $\ker S \subset \ker S^*$. Then, equation $TX = XS$ implies $X = 0$.

Proof. By Theorem 3.3, equations $TX = XS$ and $T^\star X = XS^\star$ hold. Hence, restriction $T\Big|(\overline{ran(X)}$ is a normal operator by Lemma 2.1, which contradicts the hypotheses that T is pure. Thus, $X = 0$. □

Corollary 3.8. An invertible p-w-hyponormal operator $T \in B(H)$ is normal if and only if it is log-hyponormal.

Proof. Put $T = X = S$ in the previous Theorem. □

In [9, Lemma 7], it is shown that if T is w-hyponormal with $\ker T \subset \ker T^\star$ and S is normal, and if $X \in B(H)$ has dense range such that $TX = XS$, then T is normal. We give now, an extension of this result for a p-w-hyponormal operator as follows

Lemma 3.9. *Let $T \in B(H)$ be a p-w-hyponormal operator with $\ker T \subset \ker T^\star$, and let S be normal. If $TX = XS$ for some $X \in B(H)$ with dense range, then T is normal.*

Proof. The pair (T, S) verifies the Fuglede-Putnam property by Theorem 3.2. Then, by Lemma 2.1, the restriction $T\Big|(\overline{ran(X)}$ is a normal operator. This achieves the proof since $\overline{ran(X)} = H$. □

Corollary 3.10. Let $T, S^\star \in B(H)$ be p-w-hyponormal operators with $\ker T \subset \ker T^\star$, and $\ker S \subset \ker S^\star$. If $TX = XS$ and $SY = YT$ for certain $X, Y \in B(H)$ with dense ranges, then T and S are normal.

4. Conclusion

In this paper, are shown some versions of Fuglede-Putnam Theorem on classes of p-w-hyponormal operators with log-hyponormal and with p-hyponormal opertaors. Some spectral results in [16] on w-hyponormal operators are also extended to p-w-hyponormal operators.

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