http://dergipark.gov.tr/cams

Communications in Advanced Mathematical Sciences

VOLUME V
ISSUE I

ISSN 2651-4001

VOLUME 5 ISSUE 1
ISSN 2651-4001

March 2022
https://dergipark.org.tr/en/pub/cams

COMMUNICATIONS IN ADVANCED MATHEMATICAL SCIENCES

Editors in Chief

Managing Editor

Editorial Board

Contents

Communications in Advanced Mathematical Sciences
Vol. 5, No. 1, 1-7, 2022
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1022686

Gap Between Operator Norm and Spectral Radius for the Square of Antidiagonal Block Operator Matrices

Elif Otkun Çevik[1]*

Abstract
In this work, the gap between operator norm and spectral radius for the square of antidiagonal block operator matrices in the direct sum of Banach spaces has been investigated, and also the gap between operator norm and numerical radius for the square of same matrices in the direct sum of Hilbert spaces has been studied.

Keywords: Antidiagonal operator matrix, Numerical radius, Operator norm, Spectral radius
2010 AMS: 47A12, 47A30, 15A60

[1] *Department of Mathematics, Faculty of Engineering and Architecture, Avrasya University, Trabzon, Turkey, ORCID: 0000-0001-8506-1889*
*Corresponding author: elifotkuncevik@gmail.com.tr
Received: 12 November 2021, **Accepted:** 29 December 2021, **Available online:** 19 March 2022

1. Introduction

As it is known from the mathematical literature that one of the fundamental problems of the spectral theory of linear operators is to obtain the spectrum set, numerical range set and calculate spectral and numerical radii of a given operator. In many cases, serious theoretical and technical difficulties are encountered in finding the spectrum set and numerical range of non-selfadjoint linear bounded operators. Note that there is one formula for the calculation of the spectral radius $r(A)$ of the linear bounded operator in any Banach space $r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$ [1]. On the other hand, it is also known that $r(A) \le w(A) \le \|A\|$ and $\frac{1}{2}\|A\| \le w(A) \le \|A\|$ for $A \in L(H)$.

In addition for the linear normal bounded operator A in Hilbert space we have the following relations $r(A) = w(A) = \|A\|$.

It is beneficial to recall that for the spectrum set $\sigma(A)$ and numerical range $W(A)$ of any linear bounded operator A the following spectral inclusion holds $\sigma(A) \subset \overline{W(A)}$ (See [1, 2] for more information).

In [3] some spectral radius inequalities for 2×2 block operator matrix, sum, product, and commutators of two linear bounded Hilbert space operators have been examined. In [4] some estimates for numerical and spectral radii of the Frobenius companion matrix has been obtained.

Some upper and lower bounds for the numerical radius in Hilbert space operators have been obtained in [5].

In [6] some estimates for spectral and numerical radii have been obtained for the product, sum, commutator, anticommutator of two Hilbert spaces operators.

In [7] several numerical radius inequalities for $n\times n$ block operator matrices in the direct sum of Hilbert spaces have been proved. The numerical radius inequalities for $n\times n$ accretive matrices have also been given in [8].

Several new norms and numerical radius inequalities for 2×2 block operator matrices have been researched in [9].

Recently, several new $\mathbb{A}$-numerical radius inequalities for many type $n\times n$ block operator matrices have been offered in [9] in the direct sum of Hilbert spaces.

Subadditivity of the spectral radius of commutative two operators in Banach spaces has been investigated in [10]. By the same author the subadditivity and submultiplicativity properties of local spectral radius of bounded positive operators have been researched in Banach spaces [11]. The same properties of local spectral radius in partially ordered Banach spaces have been established in [12]. In Banach space ordered by a normal and generating core, several inequalities for the spectral radius of a positive commutator of positive operators have been surveyed in [13].

The numerical range and numerical radius of some Volterra integral operator in Hilbert Lebesgue spaces at finite interval have been considered in [14, 15].

Demuth's open problem in 2015 had a great impact on the emergence and shaping of the subject examined in this article (see [16]).

This paper is organized as fallows: In section 2 one important result will be contrived. The evaluations of gaps between operator norm with spectral and numerical radii will be given in Section 3 and Section 4, respectively. And also, throughout this paper, we will use the notations as fallows:

$$(\cdot,\cdot)_H := (\cdot,\cdot),\ \|\cdot\|_H := \|\cdot\|,\ (\cdot,\cdot)_{H_m} := (\cdot,\cdot)_m,\ \|\cdot\|_{H_m} := \|\cdot\|_m,\ 1 \le m \le n.$$

2. Auxiliary important result

It will be proved the following elementary result.

Theorem 2.1. *For each $n \in \mathbb{N}$ and numbers $a_1, a_2, ..., a_n \in \mathbb{R}$ and $b_1, b_2, ..., b_n \in \mathbb{R}$*

$$\min_{1\le m\le n}(a_m - b_m) \le \max_{1\le m\le n} a_m - \max_{1\le m\le n} b_m \le \max_{1\le m\le n}(a_m - b_m)$$

are true.

Proof. In proof, we will use by mathematical induction method.
For $n = 2$, it is clear that

$$\begin{aligned}
\max\{a_1,a_2\} - \max\{b_1,b_2\} &= \frac{1}{2}[(a_1+a_2+|a_1-a_2|) - (b_1+b_2+|b_1-b_2|)] \\
&= \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) + (|a_1-a_2| - |b_1-b_2|)] \\
&= \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) - (|b_1-b_2| - |a_1-a_2|)] \\
&\ge \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) - ||b_1-b_2| - |a_1-a_2||] \\
&\ge \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) - |(b_1-b_2) - (a_1-a_2)|] \\
&= \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) - |(a_1-b_1) - (a_2-b_2)|] \\
&= \min\{a_1-b_1, a_2-b_2\}.
\end{aligned}$$

Now assume that

$$\max_{1\le m\le n-1} a_m - \max_{1\le m\le n-1} b_m \ge \min_{1\le m\le n-1}(a_m - b_m)$$

for any $n \in \mathbb{N}$, $n > 2$. Then one can easily have that

$$\begin{aligned}
\max\{a_1,...,a_n\} - \max\{b_1,...,b_n\} &= \max\{\max_{1\le m\le n-1} a_m, a_n\} - \max\{\max_{1\le m\le n-1} b_m, b_n\} \\
&\ge \min\{\max_{1\le m\le n-1} a_m - \max_{1\le m\le n-1} b_m, a_n - b_n\} \\
&\ge \min\{\min_{1\le m\le n-1}(a_m - b_m), a_n - b_n\} \\
&= \min_{1\le m\le n}(a_m - b_m).
\end{aligned}$$

From this and by mathematical induction method, for any $n \in \mathbb{N}$

$$\min_{1\le m\le n}(a_m - b_m) \le \max_{1\le m\le n} a_m - \max_{1\le m\le n} b_m$$

holds. Similarly, for $n = 2$ by simple calculations we again have that

$$\begin{aligned}
\max\{a_1,a_2\} - \max\{b_1,b_2\} &= \frac{1}{2}[(a_1+a_2+|a_1-a_2|) - (b_1+b_2+|b_1-b_2|)] \\
&= \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) + (|a_1-a_2| - |b_1-b_2|)] \\
&= \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) - (|a_1-a_2| - |b_1-b_2|)] \\
&\leq \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) - ||a_1-a_2| - |b_1-b_2||] \\
&\leq \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) - |(a_1-b_1)-(a_2-b_2)|] \\
&= \max\{a_1-b_1, a_2-b_2\}.
\end{aligned}$$

Now assume that for $n \in \mathbb{N}$, $n > 2$

$$\max_{1\leq m\leq n-1} a_m - \max_{1\leq m\leq n-1} b_m \leq \max_{1\leq m\leq n-1} (a_m - b_m).$$

From this assumption one can have that

$$\begin{aligned}
\max_{1\leq m\leq n} a_m - \max_{1\leq m\leq n} b_m &= \max\{\max_{1\leq m\leq n-1} a_m, a_n\} - \max\{\max_{1\leq m\leq n-1} b_m, b_n\} \\
&\leq \max\{\max_{1\leq m\leq n-1} a_m - \max_{1\leq m\leq n-1} b_m, a_n - b_n\} \\
&\leq \max\{\min_{1\leq m\leq n-1} (a_m - b_m), a_n - b_n\} \\
&= \max_{1\leq m\leq n} (a_m - b_m).
\end{aligned}$$

Consequently, by mathematical induction method it is obtained that,

$$\max_{1\leq m\leq n} a_m - \max_{1\leq m\leq n} b_m \leq \max_{1\leq m\leq n} (a_m - b_m)$$

holds for any $n \in \mathbb{N}$. This completes the proof of theorem. □

3. Gap between operator norm and spectral radius for the square of antidiagonal block operator matrices

Remember that the traditional direct sum of Banach spaces $\mathfrak{X}_m, 1 \leq m \leq n$ in the sense of $\ell_p, 1 \leq p < \infty$ and the direct sum of linear densely defined closed operator A_m in $\mathfrak{X}_m, 1 \leq m \leq n$ are defined as

$$\mathfrak{X} = \left(\bigoplus_{m=1}^{n} \mathfrak{X}_m\right)_p = \left\{ x = (x_1,\ldots,x_n) : x_m \in \mathfrak{X}_m,\ 1 \leq m \leq n,\ \|x\|_p = \left(\sum_{m=1}^{n} \|x_m\|^p_{\mathfrak{X}_m}\right)^{1/p} < +\infty \right\}$$

and

$$\begin{aligned}
A &= \bigoplus_{m=1}^{n} A_m,\ A : D(A) \subset \mathfrak{X} \to \mathfrak{X}, \\
D(A) &= \{x = (x_1,\ldots,x_n) \in \mathfrak{X} : x_m \in D(A_m),\ 1 \leq m \leq n,\ Ax = (A_1x_1, A_2x_2, \ldots, A_nx_n) \in \mathfrak{X}\},
\end{aligned}$$

respectively [17].

Definition 3.1. *[16] For any operator $A \in L(\mathscr{X})$ in any Banach space $\mathscr{X}$, $gap(A)$ denotes as fallows*

$$gap(A) = \|A\| - r(A).$$

Theorem 3.2. *For any $1 \leq m \leq n$ $\mathfrak{X}_m$ be a Banach space and $A_m \in L(\mathfrak{X}_m)$, $\mathfrak{X} = (\bigoplus_{m=1}^{n} \mathfrak{X}_m)_p$ and*

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & A_1 \\ 0 & 0 & \cdots & A_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n & 0 & \cdots & 0 & 0 \end{pmatrix} : \mathfrak{X} \to \mathfrak{X}.$$

Then for the operator $A \in L(\mathfrak{X})$

$$\min_{1\le m\le n} gap(A_mA_{n-m+1}) \le gap(A^2) \le \max_{1\le m\le n} gap(A_mA_{n-m+1}).$$

Proof. It is clear that

$$A^2 = \begin{pmatrix} A_1A_n & & & \\ & A_2A_{n-1} & & 0 \\ 0 & & \ddots & \\ & & & A_nA_1 \end{pmatrix} : \mathfrak{X} \to \mathfrak{X}.$$

Then from this we obtain

$$\|A^2\| = \max_{1\le m\le n} \|A_mA_{n-m+1}\| \text{ [18]}$$

and

$$\sigma(A^2) = \bigcup_{m=1}^{n} \sigma(A_mA_{n-m+1}) \text{ [19]}.$$

In this case

$$r(A^2) = \max_{1\le m\le n} r(A_mA_{n-m+1}).$$

Consequently by Theorem 2.1 it implies that

$$\begin{aligned} \min_{1\le m\le n} gap(A_mA_{n-m+1}) &\le gap(A^2) \\ &= \|A^2\| - r(A^2) \\ &= \max_{1\le m\le n} \|A_mA_{n-m+1}\| - \max_{1\le m\le n} r(A_mA_{n-m+1}) \\ &\le \max_{1\le m\le n} gap(A_mA_{n-m+1}). \end{aligned}$$

□

Corollary 3.3. *It is known that by spectral mapping theorem $r(A^2) = r^2(A)$ [3]. Then from the inequality*

$$\|A^2\| - r(A^2) \le \|A\|^2 - r^2(A)$$

and last theorem it implies that

$$\frac{\min_{1\le m\le n} gap(A_mA_{n-m+1})}{\|A\| + r(A)} \le gap(A)$$

for $A \neq 0$. From this, additionally for $A \neq 0$ we have

$$\frac{1}{2\|A\|} \min_{1\le m\le n} gap(A_mA_{n-m+1}) \le gap(A).$$

Example 3.4. *Let us*

$$\mathfrak{X}_1 = \mathfrak{X}_2 = L_2(0,1),\ \mathfrak{X} = (\mathfrak{X}_1 \oplus \mathfrak{X}_2)_2,\ A_1, A_2 : L_2(0,1) \to L_2(0,1),$$

$$A_1 f_1 = \alpha_1 V f_1 = \alpha_1 \int_0^t f_1(s)ds,\ f_1 \in L_2(0,1),\ \alpha_1 > 0,$$

$$A_2 f_2 = \alpha_2 V f_2 = \alpha_2 \int_0^t f_2(s)ds,\ f_2 \in L_2(0,1),\ \alpha_2 > 0,$$

$$A : \mathfrak{X} \to \mathfrak{X},\ \begin{pmatrix} 0 & \alpha_1 V \\ \alpha_2 V & 0 \end{pmatrix}.$$

In this case,

$$\|A_1\| = \frac{2\alpha_1}{\pi},\ r(A_1) = 0,\ gap(A_1) = \frac{2\alpha_1}{\pi},$$
$$\|A_2\| = \frac{2\alpha_2}{\pi},\ r(A_2) = 0,\ gap(A_2) = \frac{2\alpha_2}{\pi}$$

and

$$r(A_1A_2) = r(A_2A_1) = \{0\}\ [3].$$

Then by Theorem 3.2

$$\frac{2}{\pi} min\{\alpha_1, \alpha_2\} < gap(A^2) < \frac{2}{\pi} max\{\alpha_1, \alpha_2\},$$

where $A^2 = \begin{pmatrix} \alpha_1\alpha_2 V^2 & 0 \\ 0 & \alpha_1\alpha_2 V^2 \end{pmatrix}$.

4. Gap between operator norm and numerical radius for the square of antidiagonal block operator matrices

Let us for each $1 \le m \le n < \infty$ H_m be a Hilbert space with inner product $(\cdot,\cdot)_m$ and $H = \bigoplus_{m=1}^{n} H_m, 1 \le m \le n$ be direct sum of Hilbert spaces $H_m, 1 \le m \le n$ with inner product

$$(x,y) = \sum_{m=1}^{n} (x_m, y_m)_m,\ x = (x_1, ..., x_m),\ y = (y_1, ..., y_m) \in H.$$

In this section it will be investigated gap between operator norm and numerical radius of block antidiagonal operator matrix in the form

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & A_1 \\ 0 & 0 & \cdots & A_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n & 0 & \cdots & 0 & 0 \end{pmatrix} : H \to H,$$

in case when $A_m \in L(H_{n-m+1}, H_m),\ 1 \le m \le n$. Then $A \in L(H)$.

Definition 4.1. *[2] The numerical range and numerical radius of an operator T in any Hilbert space $(\mathscr{H}, (\cdot,\cdot))$ are*

$$W(T) = \{(Tx,x) : x \in \mathscr{H}, \|x\| = 1\},\ w(T) = sup\{|\lambda| : \lambda \in W(T)\},$$

respectively. And also along of this section the nonnegative number $wgap(T) = \|T\| - w(T)$ will be called the numerical gap of the operator T.

Note that $\frac{1}{2}\|T\| \le w(T) \le \|T\|$ and $w(T) \le \|T\| \le 2w(T)$ for any $T \in L(\mathscr{H})$ (see [1] and [2]).

Theorem 4.2. *For the block antidiagonal operator matrix A in Hilbert space H*

$$\min_{1\le m\le n} wgap(A_mA_{n-m+1}) \le wgap(A^2) \le \max_{1\le m\le n} wgap(A_mA_{n-m+1})$$

are true.

Proof. In this case the simple calculations shown that

$$A^2 = \begin{pmatrix} A_1A_n & & & 0 \\ & A_2A_{n-1} & & \\ & & \ddots & \\ 0 & & & A_nA_1 \end{pmatrix} : H \to H.$$

In this case from [20] it is clear that

$$\|A^2\| - w(A^2) = \max_{1\le m\le n} \|A_mA_{n-m+1}\| - \max_{1\le m\le n} w(A_mA_{n-m+1}).$$

□

From this relation and proof of Theorem 2.1 it is clear that

$$\min_{1\leq m\leq n} wgap(A_mA_{n-m+1}) \leq wgap(A^2) \leq \max_{1\leq m\leq n} wgap(A_mA_{n-m+1}).$$

Acknowledgement

The author thank the authors which listed in References for many useful support.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] P. R. Halmos, *A Hilbert Space Problem Book*, Springer-Verlag, New York, 1982.

[2] K. E. Gustafson, D. K. M. Rao, *Numerical Range: The field of values of linear operators and matrices*, Springer-Verlag, New York, 1982.

[3] F. Kittaneh, *Spectral radius inequalities for Hilbert space operators*, Proc. Am. Math. Soc., **134**(2) (2006), 385-390.

[4] A. Abu-Omar, F. Kittaneh, *Estimates for the numerical radius and spectral radius of the Frobenius companion matrix and bounds for the zeros of polynomials*, Ann. Func. Anal., **5**(1) (2014), 56-62.

[5] A. Abu-Omar, F. Kittaneh, *Upper and lower bounds for the numerical radius with an application to involution operators*, Rocky Mt. J. Math., **45**(4) (2015), 1055-1065.

[6] A. Abu-Omar, F. Kittaneh, *Notes on some spectral radius and numerical radius inequalities*, Studia Math., **227**(2) (2015), 97-109.

[7] H. Guelfen, F. Kittaneh, *On numerical radius inequalities for operator matrices*, Numer. Funct. Anal. Optim., **40**(11) (2019), 1231-1241.

[8] Y. Bedrani, F. Kittaneh, M. Sabbaheh, *Numerical radii of accretive matrices*, Linear Multilinear Algebra, **69**(5) (2021), 957-970.

[9] W. Bani-Domi, F. Kittaneh, *Norm and numerical radius inequalities for Hilbert space operators*, Linear Multilinear Algebra, **69**(5) (2021), 934-945.

[10] M. Zima, *A theorem on the spectral-radius of the sum of* 2 *operators and its application*, Bull. Aust. Math. Soc., **48**(3) (1993), 427-434.

[11] M. Zima, *On the local spectral radius of positive operators*, Proc. Am. Math. Soc., **131**(3) (2003), 845-850.

[12] M. Zima, *On the local spectral radius in partially ordered Banach spaces*, Czechoslov. Math. J., **49**(4) (1999), 835-841.

[13] M. Zima, *Spectral radius inequalities for positive commutators*, Czechoslov. Math. J., **64**(1) (2014), 1-10.

[14] M. Gurdal, M. T. Garayev, S. Saltan, *Some concrete operators and their properties*, Turk. J. Math., **39**(6) (2015), 970-989.

[15] M. T. Karaev, N. S. Iskenderov, *Numerical range and numerical radius for some operators*, Linear Algebra Appl., **432**(12) (2010), 3149-3158.

[16] M. Demuth, *Mathematical aspect of physics with non-selfadjoint operators*, List of open problem (American Institute of Mathematics Workshop, 8-12 June), 2015.

[17] J. Lindenstrauss , L. Tzafriri, *Classical Banach Spaces*, First ed., Springer-Verlag, 1977.

[18] M. A. Naimark, S. V. Fomin, *Continuous direct sums of Hilbert spaces and some of their applications*, Uspehi. Mat. Nauk., **10** (1955), 111-142.

[19] E. O. Çevik, Z. I. Ismailov, *Spectrum of the direct sum of operators*, Electron. J. Differ. Equ., **210** (2012), 1-8.

[20] E. O. Çevik, *Some numerical characteristics of direct sum of operators*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **69**2 (2020), 1221-1227.

Communications in Advanced Mathematical Sciences
Vol. 5, No. 1, 8-11, 2022
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.962877

On Fuzzy Differential Equations with Finite Delay via ψ-type Riemann-Liouville Fractional Derivative

Devaraj Vivek[1], Elsayed M. Elsayed[2]*, Kuppusamy Kanagarajan[3]

Abstract
In the article, the existence of a solution for a class of boundary value problem for a fuzzy differential equation with finite delay is discussed. By applying the contraction mapping principle, we gain an existence of a solution

Keywords: Existence, Fuzzy differential equations, Fuzzy solution, ψ-fractional derivative
2010 AMS: 34A12, 26E50, 45G10

[1] Department of Mathematics, PSG College of Arts & Science, Coimbatore-641014, India, ORCID: 0000-0003-0951-8060
[2] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, ORCID: 0000-0003-0894-8472
[3] Department of Mathematics, Faculty of Science, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore-641020, India, ORCID: 0000-0001-5556-2658
***Corresponding author**: emmelsayed@yahoo.com
Received: 5 July 2021, **Accepted:** 21 December 2021, **Available online:** 19 March 2022

1. Introduction

The idea of this paper is to look into the existence of fuzzy solution for three-point boundary value problem for ψ-type fractional differential equation:

$$\begin{cases} \mathscr{D}^{\alpha;\psi}x(t) = f\left(t, x_t, \mathscr{D}^{\beta;\psi}x(t)\right), & t \in J := [0,1],\ 1<\alpha<2, \\ x(t) = \phi(t), & t \in [-r,0], \\ x(\zeta) = x(1), \end{cases} \tag{1.1}$$

where $\mathscr{D}^{\alpha;\psi}$, $\mathscr{D}^{\beta;\psi}$ are ψ-type Riemann-Liouville (R-L) fractional derivatives, $\alpha-\beta \geq 1$, $\varepsilon \in [0,1)$, $f: J\times C_0 \times E^n \to E^n$ is a fuzzy function, $\phi \in C_0$, $\phi(0)=\widehat{0}$, and $C_0 = C\left([-r,0],E^n\right)$. For any function x defined on $[-r,1]$ and any $t\in J$. We denote by x_t the element of C_0 defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-r,0]$.

To the best of our information, even if various results for fuzzy differential equations (FDEs) have been established until now, results for FDEs with fractional order are rarely seen, papers [1, 2, 3] related to it only. The plan of the present paper is to establish some simple criteria for the existence and uniqueness of solution of the problem (1.1). The paper is structured as follows. In Section 2, we present some preliminaries and lemmas. In Section 3, we discuss the existence of solution for problem (1.1).

2. Prerequisites

Let $P_k(\mathbb{R})$ be the family of all nonempty compact convex subsets of $\mathbb{R}^n$. For $A,B \in P_k(\mathbb{R}^n)$, the Hausdorff metric is defined by

$$d_H(A,B) = \max\left\{ \sup_{a\in A}\inf_{b\in B}\|a-b\|, \quad \sup_{b\in B}\inf_{a\in A}\|a-b\| \right\}.$$

A fuzzy set in $\mathbb{R}^n$ is a function with domain $\mathbb{R}^n$ and values in $[0,1]$, that is, an element of $[0,1]^{\mathbb{R}^n}$.

Let $u \in [0,1]^{\mathbb{R}^n}$, the α-level set is

$$[u]^\alpha = \{x \in \mathbb{R}^n | u(x) \geq \alpha\}, \quad \alpha \in (0,1].$$

By E^n, we denote the all upper-semi continuous, normal fuzzy convex sets with $[u]^0 = cl\{x \in \mathbb{R} | u(x) > 0\}$ is compact. Let $d : E^n \times E^n \to [0,+\infty)$ be defined by

$$d(u,v) = \sup\{d_H([u]^\alpha,[v]^\alpha) | \alpha \in [0,1]\}.$$

Then, (E^n,d) is a complete metric space. We define $\widehat{0} \in E^n$ as $\widehat{0}(0) = 1$ if $x = 0$ and $\widehat{0} = 0$ if $x \neq 0$.

Definition 2.1. *[4] The ψ-type R-L fractional integral of order α for a function $f : [a,b] \to E^n$ is defined by*

$$I_{a^+}^{\alpha;\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^b \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} f(s) ds, \quad \alpha > 0.$$

When $a = 0$, we write $I^{\alpha;\psi} f(t)$.

Definition 2.2. *[4] For a function $f : [a,b] \to E^n$, the ψ-type R-L derivative of fractional order $\alpha > 0$ is defined by*

$$\mathscr{D}^{\alpha;\psi} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} f(s) ds, \ n = [\alpha] + 1.$$

Denote by $C(J,E^n)$ the set of all continuous mapping from J to E^n the metric on $C(J,E^n)$ is defined by $H(u,v) = \sup_{t \in J} d(u(t),v(t))$. And we metricize C_0 by setting

$$H_0(x,y) = \max\{d(x,y), y(t) | t \in [-r,0]\},$$

for all $x,y \in C_0$. Set $X = \{x | x \in C([-r,1],E^n), \mathscr{D}^\beta x \in C(J,E^n), \text{ and } x(t) = \phi(t), \ t \in [-r,0]\}$, the metric on X will be defined later.

3. Main Result

Theorem 3.1. *Assume that $f : J \times D_r \times E^n \to E^n$ and there exist positive constant K,L such that*

$$d\left(f(t,u,\mathscr{D}^{\beta;\psi}u), f(t,v,\mathscr{D}^{\beta;\psi}v)\right) \leq KH_0(u,v) + Ld\left(\mathscr{D}^{\beta;\psi}u, \mathscr{D}^{\beta;\psi}v\right)$$

for all $t \in J$ and all $u,v \in C_0$. Then

$$\left(\frac{K}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha-\beta+1)}\right)\left(1 + \frac{1+(\psi(\zeta))^\alpha}{1-(\psi(\zeta))^{\alpha-1}}\right) < 1,$$

implies that the problem (1.1) has a unique fuzzy solution on $[-r,1]$.

Proof. The metric H on X is defined by

$$H(u,v) = K \max_{t \in [-r,1]} d(u(t),v(t)) + L \max_{t \in [0,1]} d(\mathscr{D}^{\beta;\psi}u, \mathscr{D}^{\beta;\psi}v),$$

$u,v \in X$. Then (X,H) is a complete metric space.

Transform the problem into a fixed point problem. It is clear that the solutions of problem (1.1) are fixed points of the problem $F : X \to X$ defined by

$$F(x)(t) = \begin{cases} \phi(t), \quad t \in [-r,0], \\ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\psi(t) - \psi(s))^{\alpha-1} f(s,x_s,\mathscr{D}^{\beta;\psi}x(s)) ds \\ + \frac{1}{\Gamma(\alpha)(1-(\psi(\zeta))^{\alpha-1})} \int_0^\zeta \psi'(\psi(\zeta) - \psi(s))^{\alpha-1} f(s,x_s,\mathscr{D}^{\beta;\psi}x(s)) ds \\ - \frac{1}{\Gamma(\alpha)(1-(\psi(\zeta))^{\alpha-1})} \int_0^1 \psi'(\psi(1) - \psi(s))^{\alpha-1} f(s,x_s,\mathscr{D}^{\beta;\psi}x(s)) ds, \ t \in (0,1). \end{cases}$$

For $u,v \in X$, then

$$d(Fu(t),Fv(t)) = 0, \quad t \in [-r,0], \tag{3.1}$$

and for $t \in J$, we have

$$
\begin{aligned}
& d\left(Fu(t), Fv(t)\right) \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1} d\left(f(s,u_s,\mathscr{D}^{\beta;\psi}u(s)), f(s,v_s,\mathscr{D}^{\beta;\psi}v(s))\right) ds \\
& + \frac{(\psi(t))^{\alpha-1}}{\Gamma(\alpha)(1-(\psi(\zeta))^{\alpha-1})} \int_0^\zeta \psi^{'}(s)(\psi(\zeta)-\psi(s))^{\alpha-1} d\left(f(s,u_s,\mathscr{D}^{\beta;\psi}u(s)), f(s,v_s,\mathscr{D}^{\beta;\psi}v(s))\right) ds \\
& + \frac{(\psi(t))^{\alpha-1}}{\Gamma(\alpha)(1-(\psi(\zeta))^{\alpha-1})} \int_0^1 \psi^{'}(s)(\psi(1)-\psi(s))^{\alpha-1} d\left(f(s,u_s,\mathscr{D}^{\beta;\psi}u(s)), f(s,v_s,\mathscr{D}^{\beta;\psi}v(s))\right) ds \\
& \leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1} \left[Kd(u_s(\theta),v_s(\theta)) + Ld(\mathscr{D}^{\beta;\psi}u(s),\mathscr{D}^{\beta;\psi}v(s))\right] ds\right. \\
& + \frac{(\psi(t))^{\alpha-1}}{(1-(\psi(\zeta))^{\alpha-1})} \int_0^\zeta \psi^{'}(s)(\psi(\zeta)-\psi(s))^{\alpha-1} \left[Kd(u_s(\theta),v_s(\theta)) + Ld(\mathscr{D}^{\beta;\psi}u(s),\mathscr{D}^{\beta;\psi}v(s))\right] ds \\
& \left.+ \frac{(\psi(t))^{\alpha-1}}{(1-(\psi(\zeta))^{\alpha-1})} \int_0^1 \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1} \left[Kd(u_s(\theta),v_s(\theta)) + Ld(\mathscr{D}^{\beta;\psi}u(s),\mathscr{D}^{\beta;\psi}v(s))\right] ds\right) \\
& \leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1} ds + \frac{(\psi(t))^{\alpha-1}}{(1-(\psi(\zeta))^{\alpha-1})} \int_0^\zeta \psi^{'}(s)(\psi(\zeta)-\psi(s))^{\alpha-1} ds\right. \\
& \left.+ \frac{(\psi(t))^{\alpha-1}}{(1-(\psi(1))^{\alpha-1})} \int_0^1 \psi^{'}(s)(\psi(\zeta)-\psi(s))^{\alpha-1} ds\right) H(u,v) \\
& \leq \frac{1}{\Gamma(\alpha+1)} \sup_{t\in J} \left((\psi(t))^{\alpha} + \frac{(\psi(\zeta))^{\alpha}(\psi(t))^{\alpha-1}}{1-(\psi(\zeta))^{\alpha-1}} + \frac{(\psi(t))^{\alpha-1}}{1-(\psi(\zeta))^{\alpha-1}}\right) H(u,v)
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
& d\left(\mathscr{D}^{\beta;\psi}Fu(t), \mathscr{D}^{\beta;\psi}Fv(t)\right) \\
& \leq \frac{1}{\Gamma(\alpha-\beta)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-\beta-1} d\left(f(s,u_s,\mathscr{D}^{\beta;\psi}u(s)), f(s,v_s,\mathscr{D}^{\beta;\psi}v(s))\right) ds \\
& + \frac{(\psi(t))^{\alpha-\beta-1}}{(1-(\psi(\zeta))^{\alpha-1})} \int_0^\zeta \psi^{'}(s)(\psi(\zeta)-\psi(s))^{\alpha-\beta-1} d\left(f(s,u_s,\mathscr{D}^{\beta;\psi}u(s)), f(s,v_s,\mathscr{D}^{\beta;\psi}v(s))\right) ds \\
& + \frac{(\psi(t))^{\alpha-\beta-1}}{(1-(\psi(\zeta))^{\alpha-1})} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-\beta-1} d\left(f(s,u_s,\mathscr{D}^{\beta;\psi}u(s)), f(s,v_s,\mathscr{D}^{\beta;\psi}v(s))\right) ds \\
& \leq \frac{1}{\Gamma(\alpha-\beta+1)} \sup_{t\in J} \left((\psi(t))^{\alpha-\beta} + \frac{(\psi(\zeta))^{\alpha}(\psi(t))^{\alpha-\beta-1}}{(1-(\psi(\zeta))^{\alpha-1})} + \frac{(\psi(t))^{\alpha-\beta-1}}{(1-(\psi(\zeta))^{\alpha-1})}\right) H(u,v).
\end{aligned}
$$

Therefore, for each $t \in J$, we have

$$
H(Fu,Fv) \leq \left(\frac{K}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha-\beta+1)}\right)\left(1 + \frac{1+(\psi(\zeta))^{\alpha}}{1-(\psi(\zeta))^{\alpha-1}}\right) H(u,v).
$$

So, F is contraction and thus F has a unique fixed point x on X, then $x(t)$ is the unique solution to problem (1.1) on $[-r,1]$. □

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] R. P. Agarwal, V. Lakshmikantham, J. J. Nieto, *On the concept of solution for fractional differential equations with uncertainty*, Nonlinear Anal., **72**(2010), 2859-2862.

[2] J. U. Jeong, *Existence results for fractional order Fuzzy differential equations with infinite delay*, Int. Math. Forum, **5** (2010), 3221-3230.

[3] H. Wang, Y. Liu, *Existence results for fractional Fuzzy differential equations with finite delay*, Int. Math. Forum, **6** (2011), 2535-2538.

[4] R. Almeida, *A Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci. Numer. Simul., **44** (2017), 460-481.

Communications in Advanced Mathematical Sciences
Vol. 5, No. 1, 12-24, 2022
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1021878

How to Find a Bézier Curve in $\mathbf{E}^3$

Şeyda Kılıçoğlu[1], Süleyman Şenyurt[2]*

Abstract
"How to find any n^{th} order Bézier curve if we know its first, second, and third derivatives?" Hence we have examined the way to find the Bézier curve based on the control points with matrix form, while derivatives are given in $\mathbf{E}^3$. Further, we examined the control points of a cubic Bézier curve with given derivatives as an example. In this study first we have examined how to find any n^{th} order Bezier curve with known its first, second and third derivatives, which are inherently, the $(n-1)^{th}$ order, the $(n-2)^{th}$ and the $(n-3)^{th}$ Bezier curves in respective order. There is a lot of the number of Bézier curves with known the derivatives with control points. Hence to find a Bézier curve we have to choose any control point of any derivationİn this study we have chosen two special points which are the initial point P_0 and the endpoint P_n.

Keywords: Bézier curves, Cubic Bezier curves, Derivatives of Bezier curve
2010 AMS: 53A04, 53A05

[1] *Department of Mathematics, Faculty of Science and Arts, Başkent University, Ankara, Turkey, ORCID: 0000-0003-0252-1574*
[2] *Department of Mathematics, Faculty of Science and Arts, Ordu University, Ordu, Turkey, ORCID: 0000-0003-1097-5541*
***Corresponding author**: ssenyurt@odu.edu.tr
Received: 10 November 2021, **Accepted:** 17 January 2022, **Available online:** 19 March 2022

1. Introduction and Preliminaries

A Bézier curve is frequently used in computer graphics and related fields, in vector graphics and in animations as a tool to control motions. Especially, in animation applications, such as Adobe Flash and Synfig, Bézier curves are used to outline object's behaviors. Users sketch the desired path in Bézier curves, and the application creates the required frames for an object moving along in that given path. For 3D animation, Bézier curves are often used to define 3D paths as well as 2D curves by key-frame interpolation. We have been motivated by the following studies. First, Bézier-curves with curvature and torsion continuity has been examined in [1]. In [2, 3], Bézier curves are outlined for Computer-Aided Geometric Design. Bézier curves and surfaces have been discussed deeply in [4, 5]. Frenet apparatus of both the n^{th} degree Bézier curves have been examined in $\mathbf{E}^3$, in [6]. The Bishop frame and the alternative frame have been associated with the Bézier curves in [7] and [8], respectively. The matrix forms of the cubic Bézier curve and its involute have been examined in [9] and [10], respectively. Cubic Bézier like curves have been studied with different basis in [11]. 5^{th} order Bézier curve and its, first, second, and third derivatives are examined based on the control points of 5^{th} order Bézier Curve in $\mathbf{E}^3$ by [12]. Further, the Bertrand and the Mannheim partner of a cubic Bézier curve based on the control points with matrix form according to Frenet apparatus have been examined in [13, 14]. Some other couples of Bézier curves have been studied in [15].

Generally, a Bézier curve of an n^{th} degree can be defined by $n+1$ control points $P_0, P_1, \ldots, P_n$ by the following parametrization:

$$\mathbf{B}(t) = \sum_{i=0}^{n} B_i^n(t)\,[P_i]$$

where $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ is known to be Bernstein polynomials and $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ are the binomial coefficients. Bézier curves have some specific properties inherited by Bernstein polynomials. Since $B_i^n(t) \geq 0$ for $t \in [0,1]$ and $i = 0, \ldots, n$, and the polynomials have the property of partition of unity that is $\sum_{i=0}^{n} B_i^n(t) = 1$, the Bézier curves are invariant to affine transformations. Moreover, the curve lies in the convex hull of its control points by two of these properties. The end point interpolation property ensures that any Bézier curve has the first and the last of control points on it, whereas none of others do not touch the curve, necessarily. Moreover, the recursiveness property and the derivatives of the polynomials lead this study, intrinsically.

2. How to find a Bézier curve with known derivatives

Before responding the main question of this paper, we suggest readers to see [9] and [10], where another question that "How to find the control points of a given Bézier curve?" was studied. To solve the latter, we have referred the matrix form of Bézier curves as it is relatively the simplest representation. Further, it is advised to check the matrix representation of 5^{th} and n^{th} order Bézier Curve and derivatives provided in [12] and [16], respectively. Now, let us consider the main argument "How to find a Bézier curve if we know its first derivative ?" with the background of a knowledge on finding the control points of a given Bézier curve.

Theorem 2.1. *For $t \in [0,1]$, $i \in \mathbf{N_0}$ and $P_i \in \mathbf{E}^3$, a Bézier curve of n^{th} order defined by $\mathbf{B}(t) = \sum_{i=0}^{n} B_i^n(t)[P_i]$ has the following control points by means of the given its first derivative and the initial point P_0*

$$P_i = P_0 + \frac{Q_0 + Q_1 + Q_2 + \ldots + Q_{i-1}}{n}, \; 1 \leq i \leq n$$
$$P_k = P_{k-1} + \frac{Q_{k-1}}{n}.$$

Proof. The derivative of the any Bézier curve $\mathbf{B}(t)$ is

$$\mathbf{B}'(t) = \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1-t)^{n-i-1} Q_i$$

where Q_0, $Q_1, \ldots, Q_{n-1}$ are the control points. The first derivative of a n^{th} order Bézier curve has the following matrix representation

$$\alpha'(t) = \begin{bmatrix} t^{n-1} \\ . \\ . \\ . \\ t \\ 1 \end{bmatrix}^T [B'] \begin{bmatrix} Q_0 \\ Q_1 \\ . \\ . \\ . \\ Q_{n-1} \end{bmatrix}$$

where $[B']$ is the coefficient matrix of the $(n-1)^{th}$ order Bezier curve which is the derivative of the n^{th} order Bezier curve and the control points $Q_0, Q_1, \ldots, Q_{n-1}$ are

$$Q_0 = n(P_1 - P_0)$$
$$Q_1 = n(P_2 - P_1)$$
$$\ldots$$
$$Q_{n-1} = n(P_n - P_{n-1}).$$

For more detail see in [16] .There are a lot of number Bézier curves with the first dervatives have these control points. Then we have to choose any initial point. In this study we choose first two special points which are the initial point P_0 and the end point P_n, Let the n^{th} order Bézier curve pass through from a given the initial point P_0, then

$$\begin{aligned}
P_1 &= P_0 + \frac{Q_0}{n}\\
P_2 &= P_1 + \frac{Q_1}{n}\\
P_3 &= P_2 + \frac{Q_2}{n}\\
&\dots\\
P_n &= P_{n-1} + \frac{Q_{n-1}}{n}
\end{aligned}$$

if replace we get all the control points based on the Q_i,

$$\begin{aligned}
P_1 &= P_0 + \frac{Q_0}{n}\\
P_2 &= P_0 + \frac{Q_0}{n} + \frac{Q_1}{n}\\
P_3 &= P_0 + \frac{Q_0}{n} + \frac{Q_1}{n} + \frac{Q_2}{n}\\
&\dots.\\
P_n &= P_0 + \frac{Q_0}{n} + \frac{Q_1}{n} + \frac{Q_2}{n} + \dots + \frac{Q_{n-1}}{n}.
\end{aligned}$$

This complete the proof. □

Corollary 2.2. *The derivative of n^{th} order Bézier curve can not has the origin $(0,0,0)$ as a control point.*

Proof. Let first derivative of n^{th} order Bézier curve has the origin $Q_i = (0,0,0)$

$$\begin{aligned}
Q_i &= n(P_{i+1} - P_i) = (0,0,0)\\
P_{i+1} &= P_i.
\end{aligned}$$

Hence Bézier curve has n control points and cant be n^{th} order Bézier curve. So derivative of n^{th} order Bézier curve cannot has the origin $Q_i(0,0,0)$ as a control point. □

Corollary 2.3. *If the first derivative of n^{th} order Bézier curve with given control points $Q_i, 0 < i < n-1$, is given and n^{th} order Bézier curve has initial point $P_0 = (0,0,0)$,has the following control points*

$$P_i = \frac{Q_0 + Q_1 + Q_2 + \dots + Q_{i-1}}{n},\ 1 \le i \le n.$$

Proof. Since $P_i = P_0 + \frac{Q_0+Q_1+Q_2+\dots+Q_{i-1}}{n}$, $1 \le i \le n$, and $P_0 = (0,0,0)$, it is clear that

$$P_i = \frac{Q_0 + Q_1 + Q_2 + \dots + Q_{i-1}}{n}, 1 \le i \le n.$$

□

Corollary 2.4. *n^{th} order Bézier curve with given the first derivative and the initial point P_0, under the condition $P_0 = Q_0$, has the following control points*

$$\begin{aligned}
P_i &= \frac{(n+1)P_0}{n} + \frac{Q_1 + Q_2 + \dots + Q_{i-1}}{n},\ 1 \le i \le n\\
P_i &= P_{i-1} + \frac{Q_{i-1}}{n}.
\end{aligned}$$

Theorem 2.5. *The Bézier curve based on the control points a n^{th} order Bézier curve with given the first derivative and the end point P_n, has the following control points as in the following ways*

$$
\begin{aligned}
P_{i-1} &= P_n - \frac{Q_0 + Q_1 + Q_2 + ... + Q_{i-1}}{n}, \ 1 \leq i \leq n \\
P_{i-1} &= P_n - \frac{Q_{i-1}}{n}.
\end{aligned}
$$

Proof. If the first derivative of n^{th} order Bézier curve is given,

$$
\alpha'(t) = \begin{bmatrix} t^{n-1} \\ . \\ . \\ . \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} B' \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ . \\ . \\ . \\ Q_{n-1} \end{bmatrix}
$$

then the control points Q_0, $Q_1, ..., Q_{n-1}$ are given , where

$$
\begin{aligned}
Q_0 &= n(P_1 - P_0) \\
Q_1 &= n(P_2 - P_1) \\
&\dots \\
Q_{n-1} &= n(P_n - P_{n-1}).
\end{aligned}
$$

Let the n^{th} order Bézier curve passing through the end point P_n, then

$$
\begin{aligned}
P_n & \text{ is given} \\
P_{n-1} &= P_n - \frac{Q_{n-1}}{n} \\
P_{n-2} &= P_{n-1} - \frac{Q_{n-2}}{n} \\
P_{n-3} &= P_{n-2} - \frac{Q_{n-3}}{n} \\
&\dots \\
P_2 &= P_3 - \frac{Q_2}{n} \\
P_1 &= P_0 - \frac{Q_1}{n} \\
P_0 &= P_1 - \frac{Q_0}{n}
\end{aligned}
$$

if replace we get all the control points based on the Q_i,

$$
\begin{aligned}
&P_n \text{ is given}\\
P_{n-1} &= P_n - \frac{Q_{n-1}}{n}\\
P_{n-2} &= P_n - \frac{Q_{n-1}}{n} - \frac{Q_{n-2}}{n}\\
P_{n-3} &= P_n - \frac{Q_{n-1}}{n} - \frac{Q_{n-2}}{n} - \frac{Q_{n-3}}{n}\\
&.\\
&.\\
&.\\
P_2 &= P_n - \frac{Q_{n-1}}{n} - \frac{Q_{n-2}}{n} - \frac{Q_{n-3}}{n} - \ldots - \frac{Q_2}{n}\\
P_1 &= P_n - \frac{Q_{n-1}}{n} - \frac{Q_{n-2}}{n} - \frac{Q_{n-3}}{n} - \ldots - \frac{Q_2}{n} - \frac{Q_1}{n}\\
P_0 &= P_n - \frac{Q_{n-1}}{n} - \frac{Q_{n-2}}{n} - \frac{Q_{n-3}}{n} - \ldots - \frac{Q_2}{n} - \frac{Q_1}{n} - \frac{Q_0}{n}
\end{aligned}
$$

This completes the proof. □

Corollary 2.6. *The n^{th} order Bézier curve with given the first derivative and the end point $P_n = (0,0,0)$ has the following control points as in the following ways*

$$P_{i-1} = -\frac{Q_0 + Q_1 + Q_2 + \ldots + Q_{i-1}}{n}, 1 \le i \le n.$$

Corollary 2.7. *The Bézier curve based on the control points a n^{th} order Bézier curve with given the first derivative and the end point P_n, under the condition $P_n = Q_0$, has the following control points as in the following ways*

$$
\begin{aligned}
P_{i-1} &= \frac{(n-1)}{n} P_n - \frac{Q_1 + Q_2 + \ldots + Q_{i-1}}{n},\ 1 \le i \le n\\
P_{k-1} &= P_n - \frac{Q_{k-1}}{n}.
\end{aligned}
$$

Theorem 2.8. *The n^{th} order Bézier curve with given the first derivative and any point P_k ,$0 < k < n$, is given, has the following control points $P_{k+1}, P_{k+2}, \ldots, P_n$ and $P_0, P_1, \ldots, P_{k-1}$*

$$
\begin{aligned}
P_{k+1} &= P_k + \frac{Q_k}{n}\\
P_{k+2} &= P_k + \frac{Q_k}{n} + \frac{Q_{k+1}}{n}\\
&\ldots\\
P_n &= P_k + \frac{Q_k}{n} + \frac{Q_{k+1}}{n} + \ldots \frac{Q_{n-1}}{n}
\end{aligned}
$$

$$
\begin{aligned}
P_{k-1} &= P_k - \frac{Q_{k-1}}{n}\\
P_{k-2} &= P_{k-} - \frac{Q_{k-1}}{n} - \frac{Q_{k-2}}{n}\\
&\ldots\\
P_0 &= P_k - \frac{Q_{k-1}}{n} - \frac{Q_{k-2}}{n} - \frac{Q_2 + Q_1 + Q_0}{n}.
\end{aligned}
$$

Second, lets find the answer of ”How to find a Bézier curve if we know the second derivative ? ”

Theorem 2.9. *The n^{th} order Bézier curve with given the initial point P_0, the initial point Q_0 of the first derivative and the control points $R_0, R_1, ..., R_{n-2}$ of the second derivation, has the following control points as in the following ways*

$$P_1 = P_0 + \frac{Q_0}{n}$$
$$P_i = P_0 + i\frac{Q_0}{n} + \frac{(i-1)R_0}{n(n-1)} + \frac{(i-2)R_1}{n(n-1)} + \frac{(i-3)R_2}{n(n-1)} + ... + 1\frac{R_{i-2}}{n(n-1)},\ 2 \leq i \leq n.$$

Proof. The second derivative of n^{th} order Bézier curve by using matrix representation is

$$\alpha''(t) = \begin{bmatrix} t^{n-2} \\ . \\ . \\ . \\ t \\ 1 \end{bmatrix}^T [B''] \begin{bmatrix} R_0 \\ R_1 \\ . \\ . \\ . \\ R_{n-2} \end{bmatrix}$$
$$\alpha''(t) = \begin{bmatrix} t^{n-2} \\ . \\ . \\ . \\ t \\ 1 \end{bmatrix}^T [B''] \begin{bmatrix} (n-1)(Q_1 - Q_0) \\ (n-1)(Q_2 - Q_1) \\ . \\ . \\ . \\ (n-1)(Q_{n-1} - Q_{n-2}) \end{bmatrix}$$

Control points $R_0, R_1, ..., R_{n-2}$, and Q_0 are given, we can easily find the $Q_1, Q_2, ..., Q_{n-1}$.

Q_0 is given

$$Q_1 = Q_0 + \frac{R_0}{n-1}$$
$$Q_2 = Q_0 + \frac{R_0}{n-1} + \frac{R_1}{n-1}$$
$$Q_3 = Q_0 + \frac{R_0}{n-1} + \frac{R_1}{n-1} + \frac{R_2}{n-1}$$
$$\vdots$$
$$Q_{n-1} = Q_0 + \frac{R_0}{n-1} + \frac{R_1}{n-1} + ...\frac{R_{n-2}}{n-1}.$$

Also if the initial control point P_0 is given we can find easly control points of n^{th} order Bézier curve

P_0 and Q_0 are given

$$P_1 = P_0 + \frac{Q_0}{n}$$
$$P_2 = P_0 + \frac{2Q_0}{n} + \frac{R_0}{n(n-1)}$$
$$P_3 = P_0 + \frac{3Q_0}{n} + \frac{2R_0}{n(n-1)} + \frac{R_1}{n(n-1)}$$
$$P_4 = P_0 + \frac{4Q_0}{n} + \frac{3R_0}{n(n-1)} + \frac{2R_1}{n(n-1)} + \frac{1R_2}{n(n-1)}$$
$$...$$
$$P_i = P_0 + \frac{iQ_0}{n} + \frac{(i-1)R_0}{n(n-1)} + \frac{(i-2)R_1}{n(n-1)} + \frac{(i-3)R_2}{n(n-1)} + ... + \frac{R_{n-2}}{n(n-1)}.$$

□

Corollary 2.10. *The n^{th} order Bézier curve with given the initial point P_0, the initial point Q_0 of the first derivative and the control points $R_0, R_1, ..., R_{n-2}$ of the second derivation, under the condition $P_0 = Q_0 = R_0$, has the following control points as in the following ways*

$$P_i = \frac{(in + n(n-1) - 1)}{n(n-1)} P_0 + \frac{(i-2)R_1}{n(n-1)} + \frac{(i-3)R_2}{n(n-1)} + ... + 1\frac{R_{i-2}}{n(n-1)}, 2 \leq i \leq n.$$

Proof. Since

$$P_i = P_0 + i\frac{P_0}{n} + \frac{(i-1)P_0}{n(n-1)} + \frac{(i-2)R_1}{n(n-1)} + \frac{(i-3)R_2}{n(n-1)} + ... + 1\frac{R_{i-2}}{n(n-1)},$$
$$P_i = \frac{n(n-1)P_0 + i(n-1)P_0 + (i-1)P_0}{n(n-1)} + \frac{(i-2)R_1}{n(n-1)} + \frac{(i-3)R_2}{n(n-1)} + ... + 1\frac{R_{i-2}}{n(n-1)}$$
$$P_i = \frac{n(n-1) + i(n-1) + (i-1)}{n(n-1)} P_0 + \frac{(i-2)R_1}{n(n-1)} + \frac{(i-3)R_2}{n(n-1)} + ... + 1\frac{R_{i-2}}{n(n-1)}$$

it is clear. □

Now, let us find the answer to "How to find a Bézier curve if we know its third derivative ? "

Theorem 2.11. *The n^{th} order Bézier curve with given the initial point P_0 , the initial point Q_0 of the first derivative , the initial point R_0 of the second derivative and the control points $S_0, S_1, ..., S_{n-3}$ of the third derivative has the following control points as in the following ways*

$$P_i = P_0 + i\frac{Q_0}{n} + \frac{((i-1) + ... + 3 + 2 + 1)R_0}{n(n-1)} + \frac{((i-2) + ... + 3 + 2 + 1)S_0}{n(n-1)(n-2)}$$
$$+ \frac{((i-3) + ... + 3 + 2 + 1)S_1}{n(n-1)(n-2)} + ... + \frac{3S_{n-4}}{n(n-1)(n-2)} + \frac{1S_{n-3}}{n(n-1)(n-2)}.$$

Proof. The third derivative of n^{th} order Bézier curve by using matrix representation is

$$\alpha'''(t) = \begin{bmatrix} t^{n-3} \\ . \\ . \\ . \\ t \\ 1 \end{bmatrix}^T \left[B'''\right] \begin{bmatrix} S_0 \\ S_1 \\ . \\ . \\ . \\ S_{n-3} \end{bmatrix},$$
$$= \begin{bmatrix} t^{n-3} \\ . \\ . \\ . \\ t \\ 1 \end{bmatrix}^T \left[B'''\right] \begin{bmatrix} (n-2)(R_1 - R_0) \\ (n-2)(R_2 - R_1) \\ . \\ . \\ . \\ (n-2)(R_{n-2} - R_{n-3}) \end{bmatrix}$$

Control points $S_0, S_1, ..., S_{n-3}$, and R_0 are given, hence solving the following system

R_0 is given,

$$R_1 = R_0 + \frac{S_0}{(n-2)},$$
$$R_2 = R_0 + \frac{S_0}{(n-2)} + \frac{S_1}{(n-2)},$$
$$R_3 = R_0 + \frac{S_0}{(n-2)} + \frac{S_1}{(n-2)} + \frac{S_2}{(n-2)},$$
$$...$$
$$R_{n-2} = R_0 + \frac{S_0}{(n-2)} + \frac{S_1}{(n-2)} + ... + \frac{S_{n-3}}{(n-2)}.$$

We can easily find the $R_1, R_2, ..., R_{n-2}$. Also if the initial control point Q_0 of first derivative is given we can find easily Q_i control points of n^{th} order Bézier curve

$$\begin{aligned}
&Q_0 \text{ is given,}\\
&Q_1 = Q_0 + \frac{R_0}{n-1},\\
&Q_2 = Q_0 + \frac{R_0}{n-1} + \frac{R_1}{n-1},\\
&Q_3 = Q_0 + \frac{R_0}{n-1} + \frac{R_1}{n-1} + \frac{R_2}{n-1},\\
&Q_{n-1} = Q_0 + \frac{R_0}{n-1} + \frac{R_1}{n-1} + ... \frac{R_{n-2}}{n-1}.
\end{aligned}$$

$$\begin{aligned}
&Q_0, R_0 \text{ are given,}\\
&Q_1 = Q_0 + \frac{R_0}{(n-1)},\\
&Q_2 = Q_0 + 2\frac{R_0}{(n-1)} + 1\frac{S_0}{(n-1)(n-2)},\\
&Q_3 = Q_0 + 3\frac{R_0}{n-1} + \frac{2S_0}{(n-1)(n-2)} + \frac{S_1}{(n-1)(n-2)},\\
&Q_4 = Q_0 + \frac{4R_0}{n-1} + \frac{3S_0}{(n-1)(n-2)} + \frac{2S_1}{(n-1)(n-2)} + 1\frac{S_2}{(n-1)(n-2)},\\
&\quad ...\\
&Q_{i-1} = Q_0 + \frac{(i-1)R_0}{n-1} + \frac{(i-2)S_0}{(n-1)(n-2)} + \frac{(i-3)S_1}{(n-1)(n-2)}\\
&\qquad + ... + \frac{2S_{i-4}}{(n-1)(n-2)} + \frac{1S_{i-3}}{(n-1)(n-2)}
\end{aligned}$$

the n^{th} order Bézier curve pass through from a given initial point P_0, then

$$\begin{aligned}
&P_1 = P_0 + \frac{Q_0}{n},\\
&P_2 = P_0 + \frac{2Q_0}{n} + \frac{R_0}{n(n-1)},\\
&P_3 = P_0 + \frac{3Q_0}{n} + \frac{3R_0}{n(n-1)} + \frac{S_0}{n(n-1)(n-2)},\\
&P_4 = P_0 + \frac{4Q_0}{n} + \frac{6R_0}{n(n-1)} + \frac{3S_0}{n(n-1)(n-2)} + \frac{S_1}{n(n-1)(n-2)},\\
&\quad ...\\
&P_n = P_0 + n\frac{Q_0}{n} + \frac{((i-1)+...+3+2+1)R_0}{n(n-1)} + \frac{((i-2)+...+3+2+1)S_0}{n(n-1)(n-2)} + ...\\
&\qquad + \frac{3S_{n-4}}{n(n-1)(n-2)} + 1\frac{S_{n-3}}{n(n-1)(n-2)}.
\end{aligned}$$

□

3. How to find a cubic Bézier curve with known derivatives

In this section as an application we will study on cubic Bézier curves which are defined in $\mathbf{E}^3$. For more detail see in [3] .

Definition 3.1. *A cubic Bézier curve is a special Bézier curve has only four control points P_0, P_1, P_2 and P_3, with the parametrization*

$$\alpha(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t) P_2 + t^3 P_3$$

and matrix form of its the cubic Bézier curve with control points P_0, P_1, P_2, P_3, is

$$\alpha(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

Also using the derivatives of a cubic Bézier curve Frenet apparatus $\{T, N, B, \kappa, \tau\}$ have already been given as in the [9]. The first derivative of a cubic Bézier curve by using matrix representation is given by

$$\alpha'(t) = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix}$$

where $Q_0 = 3(P_1 - P_0), Q_1 = 3(P_2 - P_1), Q_2 = 3(P_3 - P_2)$ are control points. The second derivative of a cubic Bézier curve in matrix representation is

$$\alpha''(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \end{bmatrix}$$

where $R_0 = 6(P_2 - 2P_1 + P_0)$, $R_1 = 6(P_3 - 2P_2 + P_1)$ are control points.

Theorem 3.2. *The cubic Bézier curve with given the first derivative and the initial point P_0, has the following control points*

$$P_1 = P_0 + \frac{Q_0}{3},$$
$$P_2 = P_0 + \frac{Q_0 + Q_1}{3},$$
$$P_3 = P_0 + \frac{Q_0 + Q_1 + Q_2}{3}.$$

Corollary 3.3. *The cubic Bézier curve with given the first derivative and the initial point P_0,under the condition $P_0 = Q_0$,has the following control points*

$$P_1 = P_0 + \frac{P_0}{3} = \frac{4P_0}{3},$$
$$P_2 = P_0 + \frac{P_0 + Q_1}{3} = \frac{4P_0}{3} + \frac{Q_1}{3},$$
$$P_3 = P_0 + \frac{P_0 + Q_1 + Q_2}{3} = \frac{4P_0}{3} + \frac{Q_1 + Q_2}{3}.$$

Theorem 3.4. *The cubic Bézier curve with given the first derivative and the end point P_3, has the following control points*

$$P_2 = P_3 - \frac{Q_2}{3},$$
$$P_1 = P_3 - \frac{Q_2}{3} - \frac{Q_1}{3},$$
$$P_0 = P_3 - \frac{Q_2 + Q_1 + Q_0}{3}.$$

Corollary 3.5. *The cubic Bézier curve with given the first derivative and the end point P_3, under the condition $P_3 = Q_2$ has the following control points*

$$P_2 = P_3 - \frac{P_3}{3} = \frac{2P_3}{3},$$
$$P_1 = \frac{2P_3}{3} - \frac{Q_1}{3},$$
$$P_0 = \frac{2P_3}{3} - \frac{Q_1 + Q_0}{3}.$$

For an example, let us consider the cubic Bézier curve $\alpha(t) = (3t^3 - 6t^2 + 3t, -3t^3 + 3t^2, t^3)$ with the control points $P_0 = (0,0,0), P_1 = (1,0,0), P_2 = (0,1,0), P_3 = (0,0,1)$. (See, Figure 3.1)

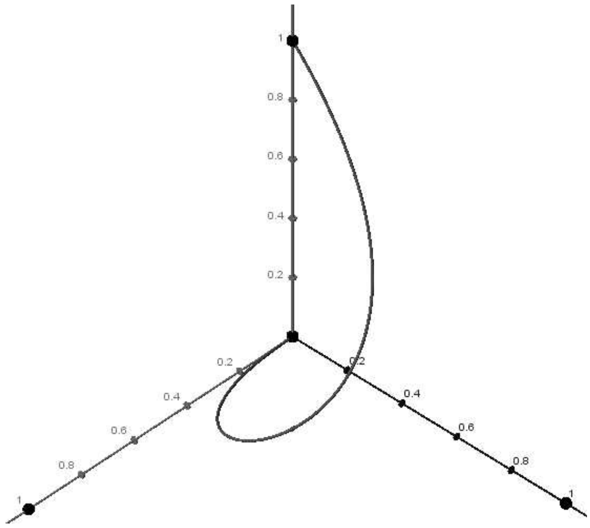

Figure 3.1.
The 3^{rd} order cubic Bézier curve $\alpha(t) = (3t^3 - 6t^2 + 3t, -3t^3 + 3t^2, t^3)$

Example 3.6. *If the first derivative of the cubic Bézier curve is $\alpha'(t) = (9t^2 - 12t + 3, -9t^2 + 6t, 3t^2)$ given. It's matrix representation is*

$$\alpha'(t) = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix}.$$

We can find the control points Q_i, $0 < i < 2$ as in the following way easily

$$\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 9 & -9 & 3 \\ -12 & 6 & 0 \\ 3 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 3 & 0 \\ 0 & -3 & 3 \end{bmatrix}.$$

There are a lot of number Bézier curves with the first dervatives have these control points. Then we have to choose any initial point. To make the correction our example, let the initial point be $P_0 = (0,0,0)$ with $Q_0 = (3,0,0), Q_1 = (-3,3,0), Q_2 = (0,-3,3)$. Since

$$P_1 = P_0 + \frac{Q_0}{3},$$
$$P_2 = P_0 + \frac{Q_0 + Q_1}{3},$$
$$P_3 = P_0 + \frac{Q_0 + Q_1 + Q_2}{3},$$

we get

$$P_1 = (0,0,0) + \frac{(3,0,0)}{3} = (1,0,0),$$
$$P_2 = P_0 + \frac{(3,0,0) + (-3,3,0)}{3} = (0,1,0),$$
$$P_3 = P_0 + \frac{(3,0,0) + (-3,3,0) + (0,-3,3)}{3} = (0,0,1).$$

Now we can write the cubic Bézier curve

$$\alpha(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$
$$= \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3t^3 - 6t^2 + 3t & 3t^2 - 3t^3 & t^3 \end{bmatrix}.$$

Let the end point be $P_3 = (0,0,1)$ with $Q_0 = (3,0,0), Q_1 = (-3,3,0), Q_2 = (0,-3,3)$. Since

$$P_2 = P_3 - \frac{Q_2}{3},$$
$$P_1 = P_3 - \frac{Q_2}{3} - \frac{Q_1}{3},$$
$$P_0 = P_3 - \frac{Q_2 + Q_1 + Q_0}{3},$$

$$P_2 = (0,0,1) - \frac{(0,-3,3)}{3} = (0,1,0),$$
$$P_1 = (0,0,1) - \frac{(0,-3,3)}{3} - \frac{(-3,3,0)}{3} = (1,0,0),$$
$$P_0 = (0,0,1) - \frac{(0,-3,3) + (-3,3,0) + (3,0,0)}{3} = (0,0,0).$$

Let the any point except the initial or the end point be $P_2 = (0,1,0)$ with $Q_0 = (3,0,0), Q_1 = (-3,3,0), Q_2 = (0,-3,3)$ are given. Since

$$P_3 = P_2 + \frac{Q_2}{3},$$
P_2 is given,
$$P_1 = P_2 + \frac{Q_2}{3} - \frac{Q_2}{3} - \frac{Q_1}{3}$$
$$= P_2 - \frac{Q_1}{3},$$
$$P_0 = P_2 + \frac{Q_2}{3} - \frac{Q_2 + Q_1 + Q_0}{3}$$
$$= P_2 - \frac{Q_1 + Q_0}{3},$$

$$P_3 = (0,1,0) + \frac{(0,-3,3)}{3} = (0,0,1),$$
P_2 is given,
$$P_1 = P_2 + \frac{Q_2}{3} - \frac{Q_2}{3} - \frac{Q_1}{3}$$
$$= (0,1,0) - \frac{(-3,3,0)}{3} = (1,0,0),$$
$$P_0 = P_2 + \frac{Q_2}{3} - \frac{Q_2 + Q_1 + Q_0}{3},$$
$$= (0,1,0) - \frac{(-3,3,0) + (3,0,0)}{3} = (0,0,0)$$

To find cubic Bézier curve with given second derivative we have the following theorem;

Theorem 3.7. *The cubic Bézier curve with given the second derivative,the initial point Q_0 and the initial point P_0, has the following control points*

$$P_0 \text{ and } Q_0 \text{ are given},$$
$$P_1 = P_0 + \frac{Q_0}{3},$$
$$P_2 = P_0 + 2\frac{Q_0}{3} + \frac{R_0}{6},$$
$$P_3 = P_0 + 3\frac{Q_0}{3} + 2\frac{R_0}{6} + \frac{R_1}{6}.$$

Example 3.8. *The second derivative $\alpha''(t) = (18t-12, -18t+6, 6t)$ of a cubic Bézier curve in matrix representation is*

$$\alpha''(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}^T \begin{bmatrix} 18 & -18 & 6 \\ -12 & 6 & 0 \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \end{bmatrix}$$

where $R_0 = \begin{pmatrix} -12, & 6, & 0 \end{pmatrix}$ and $R_1 = \begin{pmatrix} 6, & -12, & 6 \end{pmatrix}$ are the control points

$$P_0 = (0,0,0) \text{ and } Q_0 = (3,0,0)\text{are given},$$
$$P_1 = P_0 + \frac{Q_0}{3},$$
$$P_2 = P_0 + 2\frac{Q_0}{3} + \frac{R_0}{6},$$
$$P_3 = P_0 + Q_0 + \frac{R_0}{3} + \frac{R_1}{6},$$

$$P_0 = (0,0,0) \text{ and } Q_0 = (3,0,0) \text{ are given},$$
$$P_1 = (0,0,0) + \frac{(3,0,0)}{3} = (1,0,0),$$
$$P_2 = (0,0,0) + 2\frac{(3,0,0)}{3} + \frac{(-12,6,0)}{6} = (0,1,0),$$
$$P_3 = (0,0,0) + (3,0,0) + \frac{(-12,6,0)}{3} + \frac{(6,-12,6)}{6} = (0,0,1).$$

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] H. Hagen, *Bézier-curves with curvature and torsion continuity*, Rocky Mountain J. Math., **16**(3), (1986), 629-638.

[2] D. Marsh, Applied Geometry for Computer Graphics and CAD. Springer Science and Business Media., 2006.

[3] G. Farin, *Curves and Surfaces for Computer-Aided Geometric Design*, Academic Press, 1996.

[4] H. Zhang, F. Jieqing, *Bézier Curves and Surfaces (2)*, State Key Lab of CAD&CG Zhejiang University, 2006.

[5] S. Michael, *Bézier Curves and Surfaces*, Lecture 8, Floater Oslo Oct., 2003.

[6] E. Erkan, S. Yüce, *Serret-Frenet frame and curvatures of Bézier curves*, Mathematics, **6** (12) (2018), 321.

[7] H. K. Samanci, S. Celik, M. Incesu, *The Bishop frame of Bézier curves*, Life Sci. J, **12**(6) 2015, 175-180.

[8] H. K. Samanci, M. Incesu, *Investigating a quadratic Bezier curve due to NCW and N-Bishop frames*, Turk. J. Math. Compu. Sci., **12**(2) (2020), 120-127.

[9] Ş. Kılıçoğlu, S. Şenyurt, *On the cubic bezier curves in E^3*, Ordu Uni. J. Sci. Techno., **9**(2) (2019), 83-97.

[10] Ş. Kılıçoğlu, S. Şenyurt, *On the involute of the cubic Bézier curve by using matrix representation in E^3*, European J. Pure App. Math., **13** (2020), 216-226.

[11] A. Levent, B. Sahin, *Cubic bezier-like transition curves with new basis function*, Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, **44**(2) (2008) , 222-228.

[12] Ş. Kılıçoğlu, S. Şenyurt, *On the matrix representation of 5th order Bézier curve and derivatives*, Comm. Fac. Sci. Uni. Ankara Series A1 Math. Stat., 71(1) (2022), 133-152.

[13] Ş. Kılıçoğlu, S. Şenyurt, *On the Bertrand mate of a cubic Bézier curve by using matrix representation in* $\mathbf{E}^3$, 18^{th} International Geometry Sym. 2021.

[14] Ş. Kılıçoğlu, S. Şenyurt, *On the Mannheim partner of a cubic Bézier curve in* $\mathbf{E}^3$, 10th International Eurasian Conference on Mathematical Sciences and Applications, 2021.

[15] A. Y. Ceylan, *Curve couples of Bezier curves in Euclidean 2-space*, Fundamental J. Math. App., **4**(4) (2021), 245-250.

[16] Ş. Kılıçoğlu, S. Şenyurt, *On the matrix representation of Bezier curves and derivatives in E^3*, Sigma J. Engineering and Natural Sci., in Press 2021.

Communications in Advanced Mathematical Sciences
Vol. 5, No. 1, 25-34, 2022
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1023022

Set Invariant Means and Set Fixed Point Properties

Moslem Amini Nia[1]*

Abstract
In this paper, we introduce a concept of fixed point property for a semigroup S called A-fixed point property, where A is a non-empty subset of S. Also, the relationship between A-amenability and A-fixed point property is investigated.

Keywords: A-amenability, A-fixed point, Semigroup
2010 AMS: 43A07

[1] *Department of Mathematics, Payame Noor University, Tehran, Iran, ORCID: 0000-0001-8879-0921*
***Corresponding author**: amininia.math@yahoo.com
Received: 13 November 2021, **Accepted:** 14 February 2022, **Available online:** 19 March 2022

1. Introduction

Let S be a semitopological semigroup, i.e. S is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \to as$ and $s \to sa$ from S into S are continuous. Let $\ell^\infty(S)$ denotes the C^*-algebra of bounded real-valued functions on S with the supremum norm and pointwise multiplication. For each $a \in S$ and $f \in \ell^\infty(S)$, let $_af$ and f_a denote, respectively, the left and right translations of S by a, i.e. $(_af)(s) = (f \cdot a)(s) = f(as)$ and $(f_a)(s) = (a \cdot f)(s) = f(sa)$, $s \in S$. Let X be a closed subspace of $\ell^\infty(S)$ containing the constant functions and being invariant under translations. A linear functional $m \in X^*$ is called a mean if $\|m\| = m(1) = 1$; where 1 denotes the constant function on S with value 1. Then m is called a left invariant mean if

$$m(_sf) = m(f),$$

for all $s \in S$ and $f \in X$. If X is a subalgebra of $\ell^\infty(S)$, then m is multiplicative if $m(fg) = m(f)m(g)$ for all $f, g \in X$.

Recently, a new version of amenability of discrete semigroups, namely set amenability is defined by authors in [1] as follows:

Definition 1.1. *Let S be a semigroup and $\emptyset \neq A \subseteq S$. We say that a mean m on $\ell^\infty(S)$ is an A-invariant mean if for all $a \in A$ and $f \in \ell^\infty(S)$ we have*

$$m(_af) = m(f).$$

A semigroup S which admits A-invariant means is called left A-amenable. If for every pure subset A of S, S is left A-amenable, then we say that S is left set-amenable. The right A-amenability may be defined similarly. A semigroup S which is both left and right A-amenable is called A-amenable. It follows immediately that every amenable semigroup is A-amenable for all subsets A of S, but the converse is not true in general, see the examples are given in [1] for more details.

A semitopological semigroup S is said to be act on a topological space X from the left if there is a map $S \times X \to X$ denoted by $(s, x) \to s \cdot x$ for each $(s, x) \in S \times X$ such that $(st) \cdot x = s \cdot (t \cdot x)$ for all $s, t \in S$ and $x \in X$. The action is separately continuous if the mapping is continuous in each of the variables when the other is kept fixed. Moreover, the action is jointly continuous if the mapping is continuous when $S \times X$ has the product topology.

When C is a convex subset of a linear topological space X, we say that an action of S on C is affine if for each $s \in S$, the mapping from $C \to C$ defined by $x \mapsto s \cdot x \ (x \in C)$ is affine, i.e. it satisfies $s \cdot (\lambda x + (1-\lambda)y) = \lambda(s \cdot x) + (1-\lambda)(s \cdot y)$, for all $s \in S$, $x, y \in C$ and $0 \leq \lambda \leq 1$.

Let $C_b(S)$ be the Banach space of all continuous bounded real-valued functions on S with the supremum norm topology, $A(K)$ be the closed subspace of $C_b(K)$ consisting of all real valued continuous affine functions on a compact Hausdorff space K and $LUC(S)$ be the space of left uniformly continuous functions on S, that is, all $f \in C_b(S)$ such that the mappings $s \mapsto {}_s f$ from S into $C_b(S)$ are continuous when $C_b(S)$ has the supremum norm topology. Then $LUC(S)$ is a C^*-subalgebra of $C_b(S)$ invariant under translations and contains the constant functions.

A function $f \in C_b(S)$ is strongly almost periodic if $\{{}_a f : a \in S\}$ is relatively compact in the supremum norm topology of $C_b(S)$ and the set of all strongly almost periodic functions is denoted by $AP(S)$. Also, it is weakly almost periodic if $\{{}_a f : a \in S\}$ is relatively compact in the weak topology of $C_b(S)$ and the set of all weakly almost periodic functions is denoted by $WAP(S)$.

In this paper, we investigate a new version of the fixed point property for semitopological semigroups that we call A-fixed point property, where A is a non-empty subset of S. In the next section, we introduce and study the concept of set-reversibility of semitopological semigroups that is a generalization of reversibility that is defined for discrete semigroups in [1]. Section 3, introduce the notion of set-fixed point theory for semitopological semigroups and gives some relations between set-amenability and set-fixed point property for them.

Finally, in section 4, we give some examples for clarifying this new version of fixed point property for semitopological semigroups that they show that set-fixed point property is weaker than fixed point property.

2. Set-reversiblity for semigroups

For discrete semigroups, set reversibility is defined in [1, Definition 4.7], now we start off with the following definition for semitopological semigroups:

Definition 2.1. *Let S be a semigroup and $\emptyset \neq A \subseteq S$. We say that S is left A-reversible if $\overline{aS} \cap \overline{bS} \neq \emptyset$ for all $a, b \in A$. If S for every pure subset A is A-reversible, then we call it set-reversible.*

Clearly, every reversible semigroup is set-reversible, but, the converse is not true [1, Example 4.8]. For discrete semigroup S, if it is left A-amenable, then S is left A-reversible [1, Lemma 4.9]. However, a general semitopological semigroup S needs not be left A-reversible even when $C_b(S)$ has a left A-invariant mean unless S is normal (see Proposition 2.5).

Proposition 2.2. *Let S be a compact semitopological semigroup with minimal right ideal I, then S is left I-reversible.*

Proof. Since I is a right ideal, aS and bS are closed right ideals of S contained in I for each $a, b \in I$. Furthermore, since I is minimal, $aS = bS = I$. Thus

$$aS \cap bS = I \neq \emptyset.$$

□

Proposition 2.3. *Let S be a compact semitopological semigroup with subset A containig of a minimal right ideal. If S is left A-reversible, then A consists a unique minimal right ideal of S.*

Proof. Let I_1 and I_2 be two distinct minimal right ideals in A. It is easy to verify that they are closed and disjoint. Now, if we consider $a_1 \in I_1$ and $a_2 \in I_2$, then we may write

$$\overline{a_1 S} \subseteq I_1 \ \text{ and } \ \overline{a_2 S} \subseteq I_2.$$

By minimility of I_1 and I_2, we obtain that $\overline{a_1 S} = I_1$ and $\overline{a_2 S} = I_2$. Therefore

$$\overline{a_1 S} \cap \overline{a_2 S} = I_1 \cap I_2 = \emptyset.$$

This is a contradiction. □

The following Proposition is a set-reversibility version of [2, Lemma 3.1] that its proof is similar and we omit it.

Proposition 2.4. *Let S be a semitopological semigroup with non-empty subset A and X be a left translation invariant subspace of $C_b(S)$ containing constants. If X which separates closed subsets of S and has a left A-invariant mean, then S is left A-reversible.*

Above proposition implies immediately the following result:

Proposition 2.5. *Let S be a normal semitopological semigroup and $C_b(S)$ has a left A-invariant mean, then S is left A-reversible.*

In light [1, Lemma 4.9] of the above proposition we have the following result:

Proposition 2.6. *Let S be a discrete semigroup left A-amenable, then S is left A-reversible.*

If S is not a normal semitopological semigroup, then Proposition 2.5 does not hold. For example, if S is a left zero semigroup, that is a semigroup whose multiplication is defined by $st = s$ for all $s,t \in S$ and is the topological space which is regular and Hausdorff such that $C_b(S)$ consists of constant functions only. For a fixed $a \in S$ define $m(f) = f(a)$, for all $f \in C_b(S)$. Then for each subset A of S, contains more than one element, m is a left A-invariant mean on $C_b(S)$, but S is not left A-reversible.

By the following result, we show that set-reversibility can be transfered by a continuouse and onto semigroup homomorphism.

Proposition 2.7. *Let S and T be two semitopological semigroups and φ be a continuouse homomorphism of S onto T. If S is left A-reversible, then T is left $\varphi(A)$-reversible.*

Proof. For each $b_1, b_2 \in \varphi(A)$, there exist $a_1, a_2 \in A$ such that $\varphi(a_1) = b_1$ and $\varphi(a_2) = b_2$. Since S is left A-reversible, $\overline{a_1 S} \cap \overline{a_2 S} \neq \emptyset$. Let $x_0 \in \overline{a_1 S} \cap \overline{a_2 S}$, then there are nets $(a_1 s_\alpha) \subset a_1 S$ and $(a_2 t_\beta) \subset a_2 S$ such that $a_1 s_\alpha \longrightarrow x_0$ and $a_2 t_\beta \longrightarrow x_0$. Continuity of φ implies

$$b_1 \varphi(s_\alpha) = \varphi(a_1 s_\alpha) \longrightarrow \varphi(x_0) \text{ and } b_2 \varphi(t_\beta) = \varphi(a_2 t_\beta) \longrightarrow \varphi(x_0).$$

It follows that there exists $\varphi(x_0) \in \overline{b_1 T} \cap \overline{b_2 T}$, hence T is left $\varphi(A)$-reversible. □

Recall that if S is a semigroup, the intersection of all the two-sided ideals of S is called the kernel of S and denoted by $K(S)$. If $K(S)$ is non-empty, it is clearly the smallest two-sided ideal of S (see [3]).

Similar to [4, Lemma 2.8], we have the following result:

Proposition 2.8. *Let S be a compact semitopological semigroup with unit and let A be a subset of S that consists a minimal right ideal of S. Then the following statements are equivalent:*

(a) A consists a unique minimal right ideal of S.

(b) $C_b(S)$ has a left A-invariant mean.

Proof. $(b) \Rightarrow (a)$. Let m be a left A-invariant mean on $C_b(S)$ and I_1 and I_2 be two distinct minimal right ideals in A. It is obvious that I_1 and I_2 are closed and disjoint. Now, we define $f \in C_b(S)$ by

$$f(s) = \begin{cases} 0 & s \in I_1 \\ 1 & s \in I_2. \end{cases}$$

Then for any $a \in I_1$ and $b \in I_2$, we have ${}_a f = 0$ and ${}_b f = 1$. But, by the definition of m,

$$1 = m({}_a f) = m(f) = m({}_b f) = 0,$$

which this is a contradiction.

$(a) \Rightarrow (b)$. By [4, Corollary 2.4], $K(S)$ is a union of compact semitopological groups that are left ideals. Normalized Haar measure on any one of these will be a left A-invariant mean for $C_b(S)$. □

By the following result, we rewrite [4, Lemma 2.10] as follows that its proof is similar to that mentioned Lemma and for clarify we state its proof.

Proposition 2.9. *Let S and T be semitopological semigroups with T be compact and $\varphi : S \to T$ be a continuous homomorphism with $\varphi(S)$ dense in T. Let $\tilde{\varphi} : C_b(T) \longrightarrow C_b(S)$ be the dual map taking f into $f \circ \varphi$, then $C_b(T)$ has a left $\varphi(A)$-invariant mean if and only if $\tilde{\varphi}(C_b(T))$ has a left A-invariant mean.*

Proof. Suppose that there is a left $\varphi(A)$-invariant mean m on $C_b(T)$. For each $f \in C_b(T)$, define n by

$$n(\tilde{\varphi} f) = m(f).$$

Furthermore, it is clear that for any f in $C_b(T)$,

$$_s(\tilde{\varphi} f) = \tilde{\varphi}(_{\varphi(s)} f) \text{ for all } s \in S.$$

Now, for all $a \in A$ and $f \in C_b(T)$, we have

$$n(_a(\tilde{\varphi} f)) = n(\tilde{\varphi}(_{\varphi(a)} f)) = m(_{\varphi(a)} f) = m(f) = n(\tilde{\varphi} f).$$

This means that n is a left A-invariant mean on $\tilde{\varphi}(C_b(T))$.

On the other hand, let n be a left A-invariant mean on $\tilde{\varphi}(C_b(T))$, we can define a mean m on $C_b(T)$ by

$$m(f) = n(\tilde{\varphi} f) \text{ for all } f \in C_b(T).$$

Since m satisfies

$$m(_{\varphi(a)} f) = n(\tilde{\varphi}(_{\varphi(a)} f)) = n(_a(\tilde{\varphi} f)) = n(\tilde{\varphi} f) = m(f),$$

for all $a \in A$, m is a left $\varphi(A)$-invariant mean on $C_b(T)$. □

3. Common set-fixed point

Fixed point property for semigroups is one of the interesting concepts related to the semigroups theory that investigated by many authors, see [5]-[9]. Set-fixed point property for discrete semigroups is defined in [1, Definition 4.5]. Now, we define it for semitopological semigroups as follows:

Definition 3.1. *Let X be a non-empty Hausdorff topological space and S is a semigroup acting on X from the left with $\emptyset \neq A \subseteq S$. A point $x \in X$ is called a common A-fixed point of S in X if $a \cdot x = x$ for each $a \in A$. If S for every pure subset A has a common A-fixed point, then we say that it has a common set-fixed point.*

It follows immediately that every common fixed point of S in X is a common A-fixed point. But the converse is not true in general (see the Example 4.2). In this section, we rewrite some well-known results related to fixed point properties of semitopological semigroup for the set fixed point properties.

Before stating the following result, recall that when K is convex subset of a Banach space, a mapping $T : K \longmapsto K$ is called non-expansive self-maps if $\|Tx - Ty\| \leq \|x - y\|$, for each $x, y \in K$.

Theorem 3.2. *Let S be a left I-reversible semigroup of non-expansive self-maps on a non-empty compact convex subset K of a Banach space with ideal I, then K contains a common I-fixed point.*

Proof. By using Zorn's Lemma, there is a minimal I-invariant non-empty compact convex set $X \subseteq K$. By using Zorn's Lemma again, we can find a minimal I-invariant nonempty compact set $M \subseteq X$. Since S is left I-reversible, if $\{a_1, a_2, \dots, a_n\}$ is any finite subset of I, there is a finite subset $\{s_1, s_2, \dots, s_n\}$ of S such that $a_1 s_1 = a_2 s_2 = \cdots = a_n s_n$. Hence

$$\bigcap_{i=1}^{n} a_i M \supseteq \bigcap_{i=1}^{n} a_i(s_i a_1 M) = a_1 s_1 a_1 M \neq \emptyset.$$

Thus the family $\{aM : a \in I\}$ has the finite intersection property. By compactness of M, $F = \bigcap_{a \in I} aM$ is non-empty. Assume that $x \in F$. For each pair $a, b \in I$, there exist $c, d \in S$ such that $ac = bd$. Since $F \subseteq caM$, $x = cay$ for some $y \in M$. Furthermore,

$$ax = a(cay) = b(day) \in bM.$$

Moreover, $aF \subseteq F$, for all $a \in I$. By minimality of M, we have $F = M$. Therefore $M = aM$, for all $a \in I$.

If we assume that M contains more than one point, there is an element u in the closed convex hull of M such that $\rho = \sup\{\|u - x\| : x \in M\} < \delta(M)$, where $\delta(M)$ is the diameter of M. Define

$$X_0 = \bigcap_{x \in M} \{y \in X : \|x - y\| \leq \rho\},$$

then X_0 is a proper non-empty compact convex subset of X such that it is I-invariant, which contradicts the minimality of X. Hence M is a singleton set, which proves the theorem. □

Note that in the above Theorem if we replace I by S we have the following result:

Corollary 3.3. *Let S be a left reversible semigroup of non-expansive self-maps on a non-empty compact convex subset K of a Banach space, then K contains a common fixed point of S.*

Definition 3.4. *An action S on a convex subset K of a linear topological space X is A-affine if $a\cdot(\lambda x+(1-\lambda)y)=\lambda(a\cdot x)+(1-\lambda)(a\cdot y)$, for all $a\in A$, $x,y\in K$ and $0\leq\lambda\leq 1$.*

Clearly every affine action is an A-affine, but the converse is not true in general. For example, let S be a semigroup of real-valued functions on $\mathbb{R}$ with function composition operation and $A=\{f:\mathbb{R}\to\mathbb{R}\mid f(x)=kx\}$. Define the action S on $\mathbb{R}$ by $f\cdot x=f(x)$. It is easy to see that this action is A-affine, but it does not act affinely on $\mathbb{R}$.

There is a strong connection between left A-amenability and A-fixed point properties. By a similar method for discrete semigroups in [1, Theorem 4.6] and [9, Theorem 2.1], we have the following result:

Theorem 3.5 (Day's Fixed Point Theorem)**.** *Let S be a semigroup. Then the following statements are equivalent:*

(a) S is left A-amenable.

(b) Whenever S acts A-affinely on a non-empty compact convex subset K of a locally convex space, there is a common A-fixed point of S in K.

Proof. According to [1, Theorem 4.6], it suffices that we prove $(b)\Rightarrow(a)$.

$(b)\Rightarrow(a)$. As well-known that $M(S)$ the set of all means on $\ell^\infty(S)$ is a w^*-compact convex subset of $\ell^\infty(S)^*$. For $s\in S$ and $m\in M(S)$, we can define the action S on the left $\ell^\infty(S)^*$ by $(s\cdot m)(f)=m(f\cdot s)$ for each $f\in\ell^\infty(S)$. It is easy to verify that the map $m\mapsto s\cdot m$ is w^*-w^*-continuous and S acts A-affinely on $(M(S),w^*)$. Hence, by the assumption there is a common A-fixed point of S in $M(S)$ which is a left A-invariant mean on $\ell^\infty(S)$. □

Theorem 3.6. *Let S be a semitopological semigroup. Then the following statements are equivalent:*

(a) $LUC(S)$ has a multiplicative left A-invariant mean.

(b) Whenever action of S on a compact Hausdorff space X is jointly continuous, then X contains a common A-fixed point of S.

Proof. $(a)\Rightarrow(b)$. From [8, Theorem 1], for each $x\in X$ and each $f\in C_b(X)$, we have $f_x\in LUC(S)$, where f_x is defined by $f_x(s)=f(s\cdot x)$. Now, for $x\in X$ we consider $T_x:C(X)\to LUC(S)$ by $T_x(f)=f_x$ for each $f\in C(X)$. Let $T_x^*:LUC(S)^*\to C(X)^*$ be the adjoint map of T_x. Thus, if m is a multiplicative left A-invariant mean on $LUC(S)$, then there exists a point $x_0\in X$ such that $f(x_0)=(T_x^*m)(f)=m(T_xf)$ for all $f\in C(X)$.

For each $s\in S$, define $\theta_s:C(X)\to C(X)$ by $(\theta_sf)(x)=f(s\cdot x)$ for all $f\in C(X)$, $x\in X$. Hence, for each $t\in S$, we have

$$(T_x(\theta_sf))(t)=(\theta_s)(t\cdot x)=f(s\cdot(t\cdot x))=f(st\cdot x)=f_x(st)=(T_xf)(st)={}_s(T_xf)(t).$$

Therefore, $T_x(\theta_sf)={}_s(T_xf)$. Thus it follows that for all $f\in C(X)$ and $a\in A$,

$$\begin{aligned} f(a\cdot x_0) &= (\theta_af)(x_0)\\ &= m(T_x(\theta_af))\\ &= m({}_a(T_xf))\\ &= m(T_xf)\\ &= f(x_0). \end{aligned}$$

But $C(X)$ separates points of X and this implies that x_0 is the A-fixed point.

$(b)\Rightarrow(a)$. Assume that X is the compact Hausdorff space of the set of all multiplicative means on $LUC(S)$, where X is given by the w^*-topology of $LUC(S)^*$. By a simillar method in [8, Theorem 1] and use the notations there in, one can show the action of S on X its jointly continuous. Thus, (b) implies that there exists $m_0\in LUC(S)^*$ such that $m_0({}_af)=m_0(f)$, for all $f\in LUC(S)$ and $a\in A$. □

Theorem 3.7. *Let S be a semitopological semigroup. Then the following properties are equivalent:*

(a) $LUC(S)$ has a left A-invariant mean.

(b) *Whenever action of S on a nonempty compact convex subset X of a locally convex linear topological space is A-affine, then X contains a common A-fixed point of S.*

Proof. $(a) \Rightarrow (b)$. For each $x \in X$ and each $f \in C_b(X)$, the proof of Theorem 3.6 yields $f_x \in LUC(S)$, where f_x is defined by $f_x(s) = f(s \cdot x)$. For a specific $x \in X$ we consider $T_x : A(X) \to LUC(S)$ by $T_x(f) = f_x$ for each $f \in A(X)$. Now, by simmilar method in Theorem 3.6 we can obtain (b).

$(b) \Rightarrow (a)$. Assume that X is the compact convex set of the space of all means on $LUC(S)$, with the w^*-topology of $LUC(S)^*$. Define the A-affine action of S on X by $s \cdot m = l_s^* m$, for each $s \in S$ and $m \in X$. Now, the argument used in the proof of Theorem 3.6 can be used to show that the action is jointly continuous, hence by (b), there exists an A-fixed point of S on X. □

An action of S on a compact convex subset K of locally convex space X is equicontinuous if for each neighborhood U of 0, there exists a neighborhood V of 0 in X such that $x, y \in K$ and $x - y \in V$ imply $s \cdot x - s \cdot y \in U$ for each $s \in S$.

In the following Theorem, we state a relation between the existence of left A-invariant mean on $AP(S)$, the space of continuous almost periodic functions on S and A-fixed point properties of S acting on certain subsets of a locally convex space. In light of [10, Theorem 3.2], we have the following result which its proof is similar to the mentioned result and for clarifying we write its proof completely.

Theorem 3.8. *Let S be a semitopological semigroup. Then the following statements are equivalent:*

(a) *$AP(S)$ is left A-amenable.*

(b) *Whenever action of S on a compact convex subset K of a separated locally convex space is separately continuous, equicontinuous and A-affine, then there exists a common A-fixed point of S in K.*

Proof. $(a) \Rightarrow (b)$. Suppose that m is a left A-invariant mean on $AP(S)$. Since the finite means are w^*-dense in the set of means, we can find a net of finite means $\varphi_\alpha = \sum_{i_\alpha=1}^n \lambda_{i_\alpha} \delta_{s_{i_\alpha}}$, $\lambda_{i_\alpha} > 0$ and $\sum_{i_\alpha=1}^n \lambda_{i_\alpha} = 1$ such that w^*-converges to m in $AP(S)^*$. Let $x \in K$ be fixed and x_0 be a cluster point of the net $(\sum_{i_\alpha=1}^n \lambda_{i_\alpha} s_{i_\alpha} \cdot x)_\alpha$ in K. Now by [10, Lemma 3.1], for each $f \in A(K)$, we have $f_x \in AP(S)$ and hence

$$\begin{aligned} f(a \cdot x_0) &= f(a \cdot \lim_\alpha \sum_{i_\alpha=1}^n \lambda_{i_\alpha} s_{i_\alpha} \cdot x) = f(\lim_\alpha \sum_{i_\alpha=1}^n \lambda_{i_\alpha} a s_{i_\alpha} \cdot x) \\ &= \lim_\alpha f(\sum_{i_\alpha=1}^n \lambda_{i_\alpha} a s_{i_\alpha} \cdot x) = \lim_\alpha (\sum_{i_\alpha=1}^n \lambda_{i_\alpha} f(a s_{i_\alpha} \cdot x)) \\ &= \lim_\alpha \sum_{i_\alpha=1}^n \lambda_{i_\alpha} \delta_{s_{i_\alpha}} ({}_a(f_x)) = \lim_\alpha \varphi_\alpha ({}_a(f_x)) \\ &= m({}_a(f_x)) = m(f_x) \\ &= f(x_0), \end{aligned}$$

for all $a \in A$. Since $A(K)$ separates points, this shows that x_0 is an A-fixed point for S.

$(b) \Rightarrow (a)$. Let the compact convex set K be the space of all means on $AP(S)$, where K has the w^*-topology of $AP(S)^*$. Let the A-affine action of S on K be given by $s \cdot m = l_s^* m$, for each $s \in S$ and $m \in K$. By the similar method in [10, Theorem 3.2], the action of S on (K, w^*) is both separately continuous and equicontinuous. Consequently, any A-fixed point in K under this action is a left A-invariant mean on $AP(S)$. □

Recall that the right translation operators r_a on the Banach space $AP(S)$, clearly, form an almost periodic semigroup of operators. In fact, the strong operator closure of this semigroup is a compact semitopological semigroup, having jointly continuous multiplication, in the strong (or equivalently weak) operator topology. It will be denoted by S^a and called the almost periodic compactification of S.

Corollary 3.9. *Let S be a semitopological semigroup with subset A containig of minimal right ideal. If S is left A-reversible, then $AP(S)$ is left A-amenable.*

Proof. Assume that S is left A-reversible. In light of [4, Theorem 6.1], the homomorphism $r : S \mapsto \overline{S}^a$ defined by $r(a) = r_a$ is continuous. This implies that $\overline{S}^a$ is also $r(A)$-reversible. By Proposition 2.3, $\overline{S}^a$ has a unique minimal right ideal in $r(A)$. Hence, by Theorem 2.8, $C_b(\overline{S}^a)$ has a left $r(A)$-invariant mean. Consequently, again it follows from [4, Theorem 6.1] and Proposition 2.9, $AP(S)$ has a left A-invariant mean. □

Form the above Corollary and Proposition 2.5, we immediately have the following result:

Corollary 3.10. *Let S be a normal semitopological semigroup and $C_b(S)$ has a left A-invariant mean, then $AP(S)$ has a left A-invariant mean.*

Note that the converse of Corollary 3.9 is false in general, since there exist the exampels of topological semigroups such as S such that they are not left A-reversible, but $AP(S)$ (or even $C_b(S)$) has a left A-invariant mean (see Exampel 4.3).

Definition 3.11. *Let Q be a (fixed) family of continuous semi-norms on a separated locally convex space X which determines the topology of X. Then an action of S on a subset $K \subseteq X$ is Q_A-nonexpansive if $\rho(a \cdot x - a \cdot y) \leq \rho(x - y)$ for all $a \in A$, $x, y \in K$ and $\rho \in Q$.*

Theorem 3.12. *Let S be a semitopological semigroup. Then the following statements are equivalent:*

(a) $AP(S)$ is left A-amenable.

(b) Whenever S is a separately continuous and Q_A-non-expansive action on a compact convex subset K of a separately locally convex space, there is a common A-fixed point of S in K.

Proof. $(a) \Rightarrow (b)$. Assume that m is a left A-invariant mean on $AP(S)$. An application of Zorn's Lemma shows that there exists a minimal non-empty compact convex $X \subseteq K$, that is invariant under A. In particular, If X is not a singleton, apply Zorn's Lemma for the second time to get a minimal non-empty compact $F \subseteq X$, that is invariant under A.

Let $x \in X$ be a fixed. By using [10, Lemma 3.1], we may define a mean μ on $C(F)$ by $\mu(f) = m(f_x)$ for all $f \in C(F)$. Since $\mu(f) \geq 0$ whenever $f \geq 0$, and $\mu(1) = 1$ and,

$$\mu({}_a f) = m(({}_a f)_x) = m({}_a(f_x)) = m(f_x) = \mu(f).$$

It is easy to see that μ is a left A-invariant mean on $C(F)$. From Riesz representation Theorem, μ can be viewed as a regular probability measure on F and it satisfies $\mu(B) = \mu(a^{-1}B)$ for each Borel set $B \subseteq F$ and $a \in A$, where as usual, $a^{-1}B = \{x \in F : a \cdot x \in B\}$. Let $\Gamma = \{B \subseteq F : \text{B is closed subset}, \ \mu(B) = 1\}$. Set $F_0 = \bigcap_{B \in \Gamma} B$. Then by finite intersection property F_0 is a non-empty compact subset of F. Since for each $B \in \Gamma$ and $a \in A$, we have $a^{-1}B \in \Gamma$ then $a^{-1}F_0 \supseteq F_0$ or $F_0 \supseteq aF_0$. Hence $F = F_0$ by the minimality of F. Since

$$\mu(aF) = \mu(a^{-1}(aF)) = \mu(F) = 1,$$

$aF \in \Gamma$ for all $a \in A$. Consequently, $F \supseteq aF \supseteq F_0 = F$. This means that $aF = F$ for all $a \in A$.

Now, if F is a singleton we are done, otherwise, there exists a continuous seminorm ρ in Q such that $r = \sup\{\rho(x - y) : \ x, y \in F\} > 0$. Then, by De Marr's Lemma [6], there exists an element u in the closed convex hull of F such that $r_0 = \sup\{\rho(u - x) : \ x \in F\} < r$. Consider

$$X_0 = \bigcap_{x \in F} \{y \in X : \rho(y - x) \leq r_0 \}.$$

Then $u \in X_0$ and X_0 is a nonempty closed convex proper subset of X. From $aF = F$ for each $a \in A$ and Q_A-nonexpansiveness of S on X, we can write

$$\rho(a \cdot x_0 - a \cdot y) \leq \rho(x_0 - y) \leq r_0,$$

for each $x_0 \in X_0$ and $y \in F$. This leads to $aX_0 \subseteq X_0$ for all $a \in A$, contradicting the minimality of X. Consequently, F contains only one point, which, in fact, is a common A-fixed point for S.

$(b) \Rightarrow (a)$. We can prove it by the same argument in Theorem 3.8.

□

Finally, in this section, by the similar method in [7, Theorem 3.4], we have the following result:

Theorem 3.13. *Let S be a semitopological semigroup with separable ideal I. Then the following statements are equivalent:*

(a) $WAP(S)$ is left I-amenable.

(b) Whenever S acts on a weakly compact convex subset K of a separated locally convex space and the action is weakly separately continuous, weakly quasi-equicontinuous and Q_I-non-expansive, there is a common I-fixed point of S in K.

4. Examples

Example 4.1. *Consider the semigroup $S = \{a,b,c,d\}$ defined as follows:*

	a	b	c	d
a	a	a	c	c
b	b	b	d	d
c	a	a	c	c
d	b	b	d	d

(i) Let $A = \{a,b\}$. The sets $aS = \{a,c\}$ and $bS = \{b,d\}$ are disjoint minimal right ideals. Thus S is not left A-reversible.

Letting $f = \chi_{\{a,c\}}$ be the characteristic function. It is easy to see that ${}_af = 1$ and ${}_bf = 0$. Now, if we take m a left A-invariant mean on $AP(S) = WP(S) = C_b(S)$, then

$$1 = m({}_af) = m(f) = m({}_bf) = 0,$$

which is impossible.

(ii) Let $A = \{a,c\}$. Since $aS \cap cS = \{a,c\}$, S is left A-reversible. Now, if we for a fix $a \in S$ define $m(f) = f(a)$ for all $f \in AP(S)$. It is evident that ${}_af = {}_cf$. Also, we have

$$m({}_af) = {}_af(a) = f(aa) = f(a) = m(f).$$

This means that m is a left A-invariant mean on $AP(S)$.

In the following, we denote the cardinal number of a set A by $|A|$.

Example 4.2. *Let $K = [0,1]$ and consider a semigroup $S = \{h_s : s \in K\}$ with functional composition operation. Define the action S on K by $h_s(x) = s$ for each $x \in K$. Then for any subset A of S we have:*

(i) If $|A| = 1$, then there is an A-fixed point of S in K.

(ii) If $|A| \geq 2$, then there is no common A-fixed point of S in K.

Example 4.3. *Consider the partially bicyclic semigroups $S_2 = \langle e,a,b,c|ab = e, ac = e\rangle$ and $S_{1,1} = \langle e,a,b,c,d|ac = e, bd = e\rangle$. For $A_1 = \{b,c\}$ and $A_2 = \{b,d\}$, since*

$$bS_2 \cap cS_2 = \emptyset \text{ and } cS_{1,1} \cap dS_{1,1} = \emptyset$$

S_2 and $S_{1,1}$ are not left A_i-reversible $(i = 1,2)$, respectively. Of course, it is worth to mention that both $AP(S_2)$ and $AP(S_{1,1})$ have an invariant mean [7, Proposition 4.6].

We consider for a semitopological semigroup S the following A-fixed point property:

(F_A): Every jointly continuous action of S on a non-empty compact convex set K of a separated locally convex topological vector space has a common A-fixed point.

Proposition 4.4. *If a semitopological semigroup S has the common A-fixed point property (F_A), then $LUC(S)$ has a left A-invariant mean.*

Proof. Suppose that $M(S)$ is the set of all means on $LUC(S)$, where $M(S)$ is given the w^*-topology of $LUC(S)^*$. Then, $M(S)$ is w^*-compact convex subset of $LUC(S)^*$. Define an action of S on X by $s \cdot m = l_s^* m$ for each $s \in S$ and each $m \in M(S)$. This action is jointly continuous on $M(S)$. Therefore, the common A-fixed point of this action gives a left A-invariant mean on $LUC(S)$. □

Corollary 4.5. *If S is a discrete semigroup with the common A-fixed point property (F_A), then S is left A-amenable.*

In the following, by A_i's we mean the sets in Example 4.3.

Example 4.6. *We know that the partially bicyclic semigroups S_2 and $S_{1,1}$ are not left A_i-amenable $(i = 1,2)$, respectively. Hence, by Corollary 4.5, they do not have the common A_i-fixed point property (F_A).*

Proposition 4.7. *Let S and T be two semigroups and φ be a homomorphism of S onto T. If S has the common A-fixed point property (F_A), then T has the common $\varphi(A)$-fixed point property (F_A).*

Proposition 4.8. *Let S and T be semigroups such that $S \times T$ has the common $(A \times B)$-fixed point property (F_A).Then both semigroups S and T have the common A-fixed point and common B-fixed point property (F_A), respectively.*

Proof. Consider the projection homomorphisms $\pi_S : S \times T \longrightarrow S$ and $\pi_T : S \times T \longrightarrow T$ defined by $\pi_S(s,t) = s$ and $\pi_T(s,t) = t$, respectively. Let $S \times T$ has the common $(A \times B)$-fixed point property (F_A). Then by $\pi_S(A \times B) = A$ and Proposition 4.7, we obtain that S has the common A-fixed point property (F_A). Similarly, we conclude that T has the common B-fixed point property (F_A). □

In the following, we show that the converse of the Proposition 4.8, is not true in general.

Example 4.9. *Commutative free semigroup on two generators does not have (F_A).*

First recall from the well-known Schauder's Fixed Point Theorem that every free commutative discrete semigroup on one generator has the fixed point property (F_A). Let N_0 denote the additive semigroup of non-negative integers, which is the free commutative semigroup on one generator. Hence, it has the common A-fixed point property (F_A).

We know from [11] that there are two continuous functions f and g mapping the unit interval $[0,1]$ into itself which commute under the function composition but do not have any common fixed point in $[0,1]$.

Consider set $A = \{(0,0),(1,0),(0,1)\}$ and define the action of $N_0 \times N_0$ on $[0,1]$ by

$$(0,0) \cdot x = x, \ (1,0) \cdot x = f(x) \ \text{ and } \ (0,1) \cdot x = g(x).$$

Then, $N_0 \times N_0$ has no common A-fixed point on $[0,1]$. Therefore, this semigroup does not has (F_A), since it is isomorphic to $N_0 \times N_0$.

5. Conclusion

In this paper, we investigate a new version of the fixed point property for semitopological semigroups and also we introduce and study the concept of set-reversibility of semitopological semigroups that is a generalization of reversibility that is defined for discrete semigroups. Finally, some examples are given to illustrate the theoretical results.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] M. Amini Nia, A. Ebadian, A. Jabbari, *Set amenability for semigroups*, Semigroup Forum, **95**(3)(2017), 555-568.

[2] A.T-M. Lau, W. Takahashi, *Invariant means and fixed point properties for nonexpansive representations of topological semigroups*, Topol. Methods Nonlinear Anal., **5**(1995), 39-57.

[3] A. H. Clifford, *Semigroups containing minimal ideals*, Amer. J. Math., **70**(1948), 521-526.

[4] K. Deleeuw, I. Glicksberg, *Applications of almost periodic comcompactications*, Acta Math., **105**(8) (1961), 63-97.

[5] M. M. Day, *Amenable semigroups*, Illinois. J. Math., **1**(1957), 509-544.

[6] R. E. De Marr, *Common fixed points for commuting contraction mappings*, Pacific J. Math., **13**(1963) 1139-1141.

[7] A. T-M. Lau, Y. Zhang, *Fixed point properties of semigroups of non-expansive mappings*, J. Funct. Anal., **254**(2008), 2534-2554.

[8] T. Mitchell, *Topological semigroups and fixed points*, Illinois J. Math., **14**(1970), 630-641.

[9] I. Namioka, *Kakutani-type fixed point theorems: A survey*, J. Fixed Point Theory and Applications, **9**(2011), 1-23.

[10] A. T-M. Lau, *Invariant means on almost periodic functions and fixed point properties*, Rocky Mountain J. Math., **3**(1973), 69-76.

[11] W. M. Boyce, *Commuting functions with no common fixed point*, Trans. Amer. Math. Soc., **137**(1969), 77-92.

Communications in Advanced Mathematical Sciences
Vol. 5, No. 1, 35-45, 2022
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1025928

On Triple Difference Sequences of Real Numbers in Neutrosophic Normed Spaces

Ömer Kişi[1]*, Verda Gürdal[2]

Abstract
The aim of this article is to investigate triple Δ-statistical convergent sequences in a neutrosophic normed space (NNS). Also, we examine the notions of Δ-statistical limit points and Δ-statistical cluster points and prove their important features.

Keywords: Difference sequence, Lacunary sequence, Neutrosophic normed space, Triple sequence
2020 AMS: 40A05, 40C05

[1] *Faculty of Science, Department of Mathematics, Bartın University, Bartın, Turkey, ORCID: 0000-0001-6844-3092*
[2] *Department of Mathematics, Süleyman Demirel University, Isparta, Turkey, ORCID: 0000-0001-5130-7844*
***Corresponding author**: okisi@bartin.edu.tr
Received: 19 November 2021, **Accepted:** 21 December 2021, **Available online:** 19 March 2022

1. Introduction

The initial work on fuzzy sets was established by Zadeh [1]. Then, several authors have advanced the theory of fuzzy set. Park [2] defined intuitionistic fuzzy metric space and also intuitionistic fuzzy normed space was examined by Lael and Nourouzi [3]. Some beneficial results on this topic can be found in [4]-[7].

The neutrosophic set (NS) was worked by F. Smarandache [8] who defined the degree of indeterminacy (i) as indepedent component. In [9], neutrosophic logic was firstly examined. It is a logic where each proposition is identified to have a degree of truth (T), falsity (F), and indeterminacy (I). A Neutrosophic set (NS) is specified as a set where each component of the universe has a degree of T, F and I. Kirişçi and Şimşek [10] discussed neutrosophic metric space (NMS) with continuous t-norms and continuous t-conorms. The theory of NNS and statistical convergence in NNS were first developed by Kirişci and Şimşek [11]. Neutrosophic set and neutrosophic logic has utilized by applied sciences and theoretical science for instance summability theory, decision making, robotics. Some remarkable results on this topic can be reviewed in [12]-[15].

The concept of statistical convergence was investigated under the name almost convergence by Zygmund [16]. It was formally introduced by Fast [17]. Later the idea was associated with summability theory by Fridy [18], and many others (see [19]-[22]). The studies of triple sequences have seen rapid growth. The initial work on the statistical convergence of triple sequences was establised by Şahiner et al. [23] and the other researches continued by [24, 25]. The idea of difference sequences was given by Kızmaz [26] where $\Delta x = (\Delta x_k) = x_k - x_{k+1}$. Başarır [27] investigated the Δ -statistical convergence of sequences. Also, the generalized difference sequence spaces were worked by various authors [28]-[30].

Since sequence convergence plays a very significant role in the fundamental theory of mathematics, there are many convergence notions in summability theory, in approximation theory, in classical measure theory, in probability theory, and the relationships between them are discussed. The interested reader may consult Hazarika et al. [31], the monographs [32] and [33] for the background on the sequence spaces and related topics. Inspired by this, in this study, a further investigation into the mathematical features of triple sequences will be thought. Section 2 recalls some definitions in summability theory and NNS.

In Section 3, we study triple Δ-statistical convergent sequences in a NNS. Also, we examine the notions of Δ-statistical limit point and Δ-statistical cluster point and prove their important features.

2. Definitions and preliminaries

Now, we remember essential definitions required in this study.

Triangular norms (t-norms) (TN) were considered by Menger [34]. Triangular conorms (t-conorms) (TC) recognized as dual operations of TNs. TNs and TCs are significant for fuzzy operations.

Definition 2.1. *([34]) Let $*:[0,1]\times[0,1]\to[0,1]$ be an operation. If $*$ provides subsequent cases, it is named continuous TN. Take $a,b,c,d\in[0,1]$,*

(a) $a*1=a$,
(b) *If* $a\le c$ *and* $b\le d$, *then* $a*b\le c*d$,
(c) $*$ *is continuous,*
(d) $*$ *associative and commutative.*

Definition 2.2. *([34]) Let $\diamondsuit:[0,1]\times[0,1]\to[0,1]$ be an operation. If $\diamondsuit$ provides subsequent cases, it is named to be continuous TC.*

(a) $a\diamondsuit 0=a$,
(b) *If* $a\le c$ *and* $b\le d$, *then* $a\diamondsuit b\le c\diamondsuit d$,
(c) $\diamondsuit$ *is continuous,*
(d) $\diamondsuit$ *associative and commutative.*

Definition 2.3. *([11]) Let F be a vector space, $\mathscr{N}=\{\langle\varpi,\mathscr{G}(\varpi),\mathscr{B}(\varpi),\mathscr{Y}(\varpi)\rangle:\varpi\in F\}$ be a normed space (NS) such that $\mathscr{N}:F\times\mathbb{R}^+\to[0,1]$. While subsequent situations hold, $V=(F,\mathscr{N},*,\diamondsuit)$ is called to be NNS. For each $\varpi,\kappa\in F$ and $\lambda,\mu>0$ and for all $\sigma\neq 0$,*

(a) $0\le\mathscr{G}(\varpi,\lambda)\le 1$, $0\le\mathscr{B}(\varpi,\lambda)\le 1$, $0\le\mathscr{Y}(\varpi,\lambda)\le 1$ $\forall\lambda\in\mathbb{R}^+$,
(b) $\mathscr{G}(\varpi,\lambda)+\mathscr{B}(\varpi,\lambda)+\mathscr{Y}(\varpi,\lambda)\le 3$ (*for* $\lambda\in\mathbb{R}^+$),
(c) $\mathscr{G}(\varpi,\lambda)=1$ (*for* $\lambda>0$) *iff* $\varpi=0$,
(d) $\mathscr{G}(\sigma\varpi,\lambda)=\mathscr{G}\left(\varpi,\frac{\lambda}{|\sigma|}\right)$,
(e) $\mathscr{G}(\varpi,\mu)*\mathscr{G}(\kappa,\lambda)\le\mathscr{G}(\varpi+\kappa,\mu+\lambda)$,
(f) $\mathscr{G}(\varpi,.)$ *is non-decreasing continuous function,*
(g) $\lim_{\lambda\to\infty}\mathscr{G}(\varpi,\lambda)=1$,
(h) $\mathscr{B}(\varpi,\lambda)=0$ (*for* $\lambda>0$) *iff* $\varpi=0$,
(i) $\mathscr{B}(\sigma\varpi,\lambda)=\mathscr{B}\left(\varpi,\frac{\lambda}{|\sigma|}\right)$,
(j) $\mathscr{B}(\varpi,\mu)\diamondsuit\mathscr{B}(\kappa,\lambda)\ge\mathscr{B}(\varpi+\kappa,\mu+\lambda)$,
(k) $\mathscr{B}(\varpi,.)$ *is non-decreasing continuous function,*
(l) $\lim_{\lambda\to\infty}\mathscr{B}(\varpi,\lambda)=0$,
(m) $\mathscr{Y}(\varpi,\lambda)=0$ (*for* $\lambda>0$) *iff* $\varpi=0$,
(n) $\mathscr{Y}(\sigma\varpi,\lambda)=\mathscr{Y}\left(\varpi,\frac{\lambda}{|\sigma|}\right)$,
(o) $\mathscr{Y}(\varpi,\mu)\diamondsuit\mathscr{Y}(\kappa,\lambda)\ge\mathscr{Y}(\varpi+\kappa,\mu+\lambda)$,
(p) $\mathscr{Y}(\varpi,.)$ *is non-decreasing continuous function,*
(r) $\lim_{\lambda\to\infty}\mathscr{Y}(\varpi,\lambda)=0$,
(s) *If* $\lambda\le 0$, *then* $\mathscr{G}(\varpi,\lambda)=0,\mathscr{B}(\varpi,\lambda)=1$ *and* $\mathscr{Y}(\varpi,\lambda)=1$.
Then $\mathscr{N}=(\mathscr{G},\mathscr{B},\mathscr{Y})$ *is called Neutrosophic norm (NN).*

We recall the notions of convergence, statistical convergence, lacunary statistical convergence for single sequences in a NNS.

Definition 2.4. *([11]) Take V as an NNS. Let $\varepsilon\in(0,1)$ and $\lambda>0$. Then, a sequence (x_k) is converges to $L\in F$ iff there is $N\in\mathbb{N}$ such that $\mathscr{G}(x_k-L,\lambda)>1-\varepsilon$, $\mathscr{B}(x_k-L,\lambda)<\varepsilon$, $\mathscr{Y}(x_k-L,\lambda)<\varepsilon$. That is,*

$$\lim_{k\to\infty}\mathscr{G}(x_k-L,\lambda)=1,\ \lim_{k\to\infty}\mathscr{B}(x_k-L,\lambda)=0 \text{ and } \lim_{k\to\infty}\mathscr{Y}(x_k-L,\lambda)=0$$

as $\lambda>0$. The convergent in NNS is signified by $\mathscr{N}-\lim x_k=L$.

Definition 2.5. *([11]) A sequence (x_k) is named to be statistically convergent to $L \in F$ with regards to NN (SC-NN), provided that, for each $\lambda > 0$ and $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : \mathscr{G}(x_k - L, \lambda) \le 1-\varepsilon \text{ or } \mathscr{B}(x_k - L, \lambda) \ge \varepsilon,\ \mathscr{Y}(x_k - L, \lambda) \ge \varepsilon\}\right| = 0.$$

It is demonstrated by $S_{\mathscr{N}}$-$\lim x_k = L$.

Now we give the following notion.

Definition 2.6. *([23]) A subset K of $\mathbb{N}^3$ is said to have natural density $\delta_3(K)$ if*

$$\delta_3(K) = P - \lim_{n,l,k\to\infty} \frac{|K_{nlk}|}{nlk}$$

exists, where the vertical bars denote the number of (n,l,k) in K such that $p \le n$, $q \le l$, $r \le k$. Then, a real triple sequence $x = (x_{pqr})$ is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\delta_3\left(\left\{(n,l,k) \in \mathbb{N}^3 : p \le n, q \le l, r \le k,\ \left|x_{pqr} - L\right| \ge \varepsilon\right\}\right) = 0.$$

3. Main results

Definition 3.1. *A triple sequence $w = (w_{nlk})$ in V is named to be Δ-convergent to $\zeta \in F$ with regards to (w.r.t in short) NN $(\mathscr{G}, \mathscr{B}, \mathscr{Y})$ on condition that for every $\lambda > 0$ and $\varepsilon \in (0,1)$, there is a positive integer k_0 such that*

$$\mathscr{G}(\Delta w_{nlk} - \zeta, \lambda) > 1-\varepsilon \text{ and } \mathscr{B}(\Delta w_{nlk} - \zeta, \lambda) < \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \zeta, \lambda) < \varepsilon$$

for every $n \ge k_0, l \ge k_0, k \ge k_0$ where $n,l,k \in \mathbb{N}$ and $\Delta w_{nlk} = w_{nlk} - w_{n,l+1,k} - w_{n,l,k+1} + w_{n,l+1,k+1} - w_{n+1,l,k} + w_{n+1,l+1,k} + w_{n+1,l,k+1} - w_{n+1,l+1,k+1}$. We indicate $(\mathscr{G}, \mathscr{B}, \mathscr{Y}) - \lim \Delta w = \zeta$ or $\Delta w \to \zeta\ ((\mathscr{G}, \mathscr{B}, \mathscr{Y}))$ as $n,l,k \to \infty$.

Definition 3.2. *A triple sequence $w = (w_{nlk})$ is named to be Δ-Cauchy in V w.r.t NN $(\mathscr{G}, \mathscr{B}, \mathscr{Y})$ if for each $\varepsilon \in (0,1)$ and $\lambda > 0$, there are positive integers t_0, t_1, t_2 such that $\mathscr{G}(\Delta w_{nlk} - \Delta w_{pqr}, \lambda) > 1-\varepsilon$ and $\mathscr{B}(\Delta w_{nlk} - \Delta w_{pqr}, \lambda) < \varepsilon$, $\mathscr{Y}(\Delta w_{nlk} - \Delta w_{pqr}, \lambda) < \varepsilon$, whenever $n,p \ge t_0$, $l,q \ge t_1$, $k,r \ge t_2$.*

Definition 3.3. *A triple sequence $w = (w_{nlk})$ is named to be Δ-statistical convergent to ζ in V w.r.t NN $(\mathscr{G}, \mathscr{B}, \mathscr{Y})$ if for each $\varepsilon \in (0,1)$ and $\lambda > 0$,*

$$\delta_3\left(\left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \zeta, \lambda) \le 1-\varepsilon \text{ or } \mathscr{B}(\Delta w_{nlk} - \zeta, \lambda) \ge \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \zeta, \lambda) \ge \varepsilon\right\}\right) = 0.$$

In this case, we denote $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta$.

Definition 3.4. *A triple sequence $w = (w_{nlk})$ is named to be Δ-statistically Cauchy in V w.r.t NN $(\mathscr{G}, \mathscr{B}, \mathscr{Y})$ if for each $\varepsilon \in (0,1)$ and $\lambda > 0$, there are positive integers U, V and Y such that*

$$\delta_3\left(\left\{\begin{array}{c} (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \Delta w_{pqr}, \lambda) \le 1-\varepsilon \text{ or} \\ \mathscr{B}(\Delta w_{nlk} - \Delta w_{pqr}, \lambda) \ge \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \Delta w_{pqr}, \lambda) \ge \varepsilon \end{array}\right\}\right) = 0$$

for all $n,p \ge U$, $l,q \ge V$, $k,r \ge Y$.

Lemma 3.5. *For each $\varepsilon \in (0,1)$ and $\lambda > 0$, the subsequent cases are equivalent.*

(a) $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta$.

(b)

$$\delta_3\left(\left\{\begin{array}{c} (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \le 1-\varepsilon \text{ or} \\ \mathscr{B}(\Delta w_{nlk} - \xi, \lambda) \ge \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \xi, \lambda) \ge \varepsilon \end{array}\right\}\right) = 0,$$

(c)

$$\delta_3\left(\left\{\begin{array}{c} (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) > 1-\varepsilon \text{ and} \\ \mathscr{B}(\Delta w_{nlk} - \xi, \lambda) < \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \xi, \lambda) < \varepsilon \end{array}\right\}\right) = 1,$$

(d)

$$\begin{array}{l} st^3_{N(\Delta)} - \lim \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) = 1 \text{ and} \\ st^3_{N(\Delta)} - \lim \mathscr{B}(\Delta w_{nlk} - \xi, \lambda) = 0, \\ st^3_{N(\Delta)} - \lim \mathscr{Y}(\Delta w_{nlk} - \xi, \lambda) = 0. \end{array}$$

Proof. $(a) \Rightarrow (b)$ Presume that $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta$. Then, we get for each $\varepsilon \in (0,1)$ and $\lambda > 0$,

$$\delta_3\left(\left\{\begin{array}{c}(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \zeta, \lambda) \leq 1-\varepsilon \text{ or} \\ \mathscr{B}(\Delta w_{nlk} - \zeta, \lambda) \geq \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \zeta, \lambda) \geq \varepsilon\end{array}\right\}\right) = 0.$$

$(b) \Rightarrow (c)$ Take $\varepsilon \in (0,1)$ and $\lambda > 0$. Then, we acquire

$$\begin{aligned}&\delta_3\left(\left\{\begin{array}{c}(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) > 1-\varepsilon \text{ and} \\ \mathscr{B}(\Delta w_{nlk} - \xi, \lambda) < \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \xi, \lambda) < \varepsilon\end{array}\right\}\right) \\ &= 1 - \delta_3\left(\left\{\begin{array}{c}(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \leq 1-\varepsilon \text{ or} \\ \mathscr{B}(\Delta w_{nlk} - \xi, \lambda) \geq \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \xi, \lambda) \geq \varepsilon\end{array}\right\}\right) = 1.\end{aligned}$$

$(c) \Rightarrow (d)$ Take $\varepsilon \in (0,1)$ and $\lambda > 0$. Then, we obtain

$$\begin{aligned}&\left\{(n,l,k) \in \mathbb{N}^3 : |\mathscr{G}(\Delta w_{nlk} - \xi, \lambda) - 1| \geq \varepsilon\right\} \\ &= \left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \leq 1-\varepsilon\right\} \\ &\cup \left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \geq 1+\varepsilon\right\}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\delta_3\left(\left\{(n,l,k) \in \mathbb{N}^3 : |\mathscr{G}(\Delta w_{nlk} - \xi, \lambda) - 1| \geq \varepsilon\right\}\right) \\ &= \delta_3\left(\left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \leq 1-\varepsilon\right\}\right) \\ &+ \delta_3\left(\left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \geq 1+\varepsilon\right\}\right).\end{aligned}$$

Since,

$$\begin{aligned}&\delta_3\left(\left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \leq 1-\varepsilon\right\}\right) = 0 \text{ and} \\ &\delta_3\left(\left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \geq 1+\varepsilon\right\}\right) = 0\end{aligned}$$

we get

$$\delta_3\left(\left\{(n,l,k) \in \mathbb{N}^3 : |\mathscr{G}(\Delta w_{nlk} - \xi, \lambda) - 1| \geq \varepsilon\right\}\right) = 0.$$

So $st^3_{N(\Delta)} - \lim \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) = 1$. Similarly, we obtain $st^3_{N(\Delta)} - \lim \mathscr{B}(\Delta w_{nlk} - \xi, \lambda) = 0$, $st^3_{N(\Delta)} - \lim \mathscr{Y}(\Delta w_{nlk} - \xi, \lambda) = 0$. □

Theorem 3.6. *If $w = (w_{nlk})$ is Δ-statistically convergent to ζ in V w.r.t NN $(\mathscr{G}, \mathscr{B}, \mathscr{Y})$, then $st^3_{N(\Delta)} - \lim w_{nlk}$ is determined unique.*

Proof. Let $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta_1$ and $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta_2$, where $\zeta_1 \neq \zeta_2$. For a given $\varepsilon \in (0,1)$ select $\nu \in (0,1)$ such that $(1-\nu) * (1-\nu) > 1-\varepsilon$ and $\nu \Diamond \nu < \varepsilon$. For any $\lambda > 0$, we identify the subsequent sets:

$$\begin{aligned}F_{\mathscr{G},1}(\nu,\lambda) &= \left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi_1, \lambda) \leq 1-\nu\right\} \\ F_{\mathscr{G},2}(\nu,\lambda) &= \left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi_2, \lambda) \leq 1-\nu\right\} \\ F_{\mathscr{B},1}(\nu,\lambda) &= \left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{B}(\Delta w_{nlk} - \xi_1, \lambda) \geq \nu\right\} \\ F_{\mathscr{B},2}(\nu,\lambda) &= \left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{B}(\Delta w_{nlk} - \xi_2, \lambda) \geq \nu\right\} \\ F_{\mathscr{Y},1}(\nu,\lambda) &= \left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{Y}(\Delta w_{nlk} - \xi_1, \lambda) \geq \nu\right\} \\ F_{\mathscr{Y},2}(\nu,\lambda) &= \left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{Y}(\Delta w_{nlk} - \xi_2, \lambda) \geq \nu\right\}\end{aligned}$$

Since $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta_1$, we say

$$\delta_3\left(F_{\mathscr{G},1}(\nu,\lambda)\right) = \delta_3\left(F_{\mathscr{B},1}(\nu,\lambda)\right) = \delta_3\left(F_{\mathscr{Y},1}(\nu,\lambda)\right) = 0$$

for all $\lambda > 0$. In addition, utilizing $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta_2$, we acquire

$$\delta_3\left(F_{\mathscr{G},2}(\nu,\lambda)\right) = \delta_3\left(F_{\mathscr{B},2}(\nu,\lambda)\right) = \delta_3\left(F_{\mathscr{Y},2}(\nu,\lambda)\right) = 0$$

for all $\lambda > 0$.

Now, take

$$F_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\nu,\lambda) := \left(F_{\mathscr{G},1}(\nu,\lambda)\cup F_{\mathscr{G},2}(\nu,\lambda)\right)\cap\left(F_{\mathscr{B},1}(\nu,\lambda)\cup F_{\mathscr{B},2}(\nu,\lambda)\right)\cap\left(F_{\mathscr{Y},1}(\nu,\lambda)\cup F_{\mathscr{Y},2}(\nu,\lambda)\right).$$

Then, contemplate that $\delta_3\left(F_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\nu,\lambda)\right)=0$ that implies $\delta_3\left(\mathbb{N}^3\diagdown F_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\nu,\lambda)\right)=1$. If $(n,l,k)\in\mathbb{N}^3\diagdown F_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\nu,\lambda)$, then we get three possible situations. The former is the situation of $(n,l,k)\in\mathbb{N}^3\diagdown\left(F_{\mathscr{G},1}(\nu,\lambda)\cup F_{\mathscr{G},2}(\nu,\lambda)\right)$, the second is $(n,l,k)\in\mathbb{N}^3\diagdown\left(F_{\mathscr{B},1}(\nu,\lambda)\cup F_{\mathscr{B},2}(\nu,\lambda)\right)$ and the third is $(n,l,k)\in\mathbb{N}^3\diagdown\left(F_{\mathscr{Y},1}(\nu,\lambda)\cup F_{\mathscr{Y},2}(\nu,\lambda)\right)$. First think that $(n,l,k)\in\mathbb{N}^3\diagdown\left(F_{\mathscr{G},1}(\nu,\lambda)\cup F_{\mathscr{G},2}(\nu,\lambda)\right)$. Then, we acquire

$$\mathscr{G}(\xi_1-\xi_2,\lambda)\geq\mathscr{G}\left(\Delta w_{nlk}-\xi_1,\frac{\lambda}{2}\right)*\mathscr{G}\left(\Delta w_{nlk}-\xi_2,\frac{\lambda}{2}\right)>(1-\nu)*(1-\nu).$$

Since $(1-\nu)*(1-\nu)>1-\varepsilon$, we have $\mathscr{G}(\xi_1-\xi_1,\lambda)>1-\varepsilon$. Since $\varepsilon\in(0,1)$ was arbitrary, we get $\mathscr{G}(\xi_1-\xi_2,\lambda)=1$ for all $\lambda>0$ which means that $\xi_1=\xi_2$. At the same time, if $(n,l,k)\in\mathbb{N}^3\diagdown\left(F_{\mathscr{B},1}(\nu,\lambda)\cup F_{\mathscr{B},2}(\nu,\lambda)\right)$, we can see

$$\mathscr{B}(\xi_1-\xi_2,\lambda)<\mathscr{B}\left(\Delta w_{nlk}-\xi_1,\frac{\lambda}{2}\right)\Diamond\mathscr{B}\left(\Delta w_{nlk}-\xi_2,\frac{\lambda}{2}\right)<\nu\Diamond\nu.$$

Since $\nu\Diamond\nu<\varepsilon$, we get $\mathscr{B}(\xi_1-\xi_2,\lambda)<\varepsilon$. Since $\varepsilon\in(0,1)$ was arbitrary, we acquire $\mathscr{B}(\xi_1-\xi_2,\lambda)=0$ for all $\lambda>0$ which means that $\xi_1=\xi_2$. If we observe the third case, we see that $\xi_1=\xi_2$. Hence, in all conditions, we obtain $st^3_{N(\Delta)}-\lim w_{nlk}$ is determined unique. □

Theorem 3.7. *Let $w=(w_{nlk})$ be a sequence in V. If $\Delta w\to\zeta\ ((\mathscr{G},\mathscr{B},\mathscr{Y}))$, then $st^3_{N(\Delta)}-\lim w=\zeta$.*

Proof. By supposition, for every $\lambda>0$ and $\varepsilon\in(0,1)$, there is a $k_0\in\mathbb{N}$ such that

$$\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\varepsilon \text{ and } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon$$

for every $n\geq k_0, l\geq k_0, k\geq k_0$. This assurances that the set

$$\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)\leq 1-\varepsilon \text{ or } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)\geq\varepsilon\right\}$$

has at most finitely many terms. Every finite subset of the $\mathbb{N}$ has density zero, so we acquire

$$\delta_3\left(\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)\leq 1-\varepsilon \text{ or } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)\geq\varepsilon\right\}\right)=0$$

gives the result. Hence, $st^3_{N(\Delta)}-\lim w_{nlk}=\zeta$. □

The subsequent example indicates that the converse of Theorem 3.7 is not valid.

Example 3.8. *Let $(F,\|.\|)$ be a NS. For each $a,b\in[0,1]$, select the TN $a*b=ab$ and the TC $a\Diamond b=\min\{a+b,1\}$. For every $w=(w_{nlk})\in F$ and each $\lambda>0$, we contemplate $\mathscr{G}(w,\lambda)=\frac{\lambda}{\lambda+\|w\|}$, $\mathscr{B}(w,\lambda)=\frac{\|w\|}{\lambda+\|w\|}$ and $\mathscr{Y}(w,\lambda)=\frac{\|w\|}{\lambda}$. Then V is an NNS. We identify a sequence (w_{nlk}) by*

$$w_{nlk}=\begin{cases}1, & n=k^2, l=v^2, k=t^2\ (k,v,t\in\mathbb{N})\\ 0, & otherwise.\end{cases}$$

Consider

$$A_{pqr}(\varepsilon,\lambda)=\left\{\begin{array}{c} n\leq p, l\leq q, k\leq r:\mathscr{G}(w_{nlk}-\xi,\lambda)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}(w_{nlk}-\xi,\lambda)\geq\varepsilon,\ \mathscr{Y}(w_{nlk}-\xi,\lambda)\geq\varepsilon\end{array}\right\}$$

for every $\varepsilon\in(0,1)$ and for any $\lambda>0$. Then we acquire

$$\begin{aligned}A_{pqr}(\varepsilon,\lambda) &=\left\{n\leq p,l\leq q,k\leq r:\tfrac{\lambda}{\lambda+\|w_{nlk}\|}\leq 1-\varepsilon \text{ or } \tfrac{\|w_{nlk}\|}{\lambda+\|w_{nlk}\|}\geq\varepsilon,\ \tfrac{\|w_{nlk}\|}{\lambda}\geq\varepsilon\right\}\\ &=\left\{n\leq p,l\leq q,k\leq r:\|w_{nlk}\|\geq\tfrac{\lambda\varepsilon}{1-\varepsilon},\text{ or } \|w_{nlk}\|\geq\lambda\varepsilon\right\}\\ &=\{n\leq p,l\leq q,k\leq r:\|w_{nlk}\|=1\}\\ &=\left\{n\leq p,l\leq q,k\leq r:n=k^2,l=v^2,k=t^2\ (k,v,t\in\mathbb{N})\right\}\end{aligned}$$

we get

$$\frac{1}{pqr}\left|A_{pqr}(\varepsilon,\lambda)\right| = \frac{1}{pqr}\left|\left\{n \le p, l \le q, k \le r : n = k^2, l = v^2, k = t^2 \ (k,v,t \in \mathbb{N})\right\}\right| \le \frac{\sqrt{pqr}}{pqr}$$

which means that $\lim_{pqr\to\infty} \frac{1}{pqr}\left|A_{pqr}(\varepsilon,\lambda)\right| = 0$*. Hence, we have* $st^3_{N(\Delta)} - \lim w_{nlk} = 0$*. However, the sequence* $w = (w_{nlk})$ *is not* Δ*-convergent in the space* $(F, \|.\|)$*.*

Theorem 3.9. *Take NNS as* V*. Then,* $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta$ *iff there is a subset*

$$K = \left\{(n,l,k) \in \mathbb{N}^3 : n,l,k = 1,2,3,...\right\} \subset \mathbb{N}^3$$

such that $\delta_3(K) = 1$ *and* $(\mathscr{G},\mathscr{B},\mathscr{Y}) - \lim_{(n,l,k)\in K, n,l,k\to\infty} \Delta w_{nlk} = \zeta$*.*

Proof. Presume that $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta$. Then, for every $\lambda > 0$ and $j \ge 1$,

$$K(j,\lambda) = \left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \zeta, \lambda) > 1 - \frac{1}{j} \text{ and } \mathscr{B}(\Delta w_{nlk} - \zeta, \lambda) < \frac{1}{j},\ \mathscr{Y}(\Delta w_{nlk} - \zeta, \lambda) < \frac{1}{j}\right\}$$

and

$$M(j,\lambda) = \left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \zeta, \lambda) \le 1 - \frac{1}{j} \text{ or } \mathscr{B}(\Delta w_{nlk} - \zeta, \lambda) \ge \frac{1}{j},\ \mathscr{Y}(\Delta w_{nlk} - \zeta, \lambda) \ge \frac{1}{j}\right\}.$$

Then $\delta_3(M(j,\lambda)) = 0$ since

$$K(j,\lambda) \supset K(j+1,\lambda) \tag{3.1}$$

and

$$\delta_3(K(j,\lambda)) = 1 \tag{3.2}$$

for $\lambda > 0$ and $j \ge 1$. Now we need to show that for $(n,l,k) \in K(j,\lambda)$ the triple sequence $w = (w_{nlk})$ is Δ-convergent to $\zeta \in F$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$. Suppose $w = (w_{nlk})$ be not Δ-convergent to $\zeta \in F$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$. Therefore, there are $\beta > 0$ and $k_0 > 0$ such that $\mathscr{G}(\Delta w_{nlk} - \zeta, \lambda) \le 1 - \beta$ or $\mathscr{B}(\Delta w_{nlk} - \zeta, \lambda) \ge \beta$, $\mathscr{Y}(\Delta w_{nlk} - \zeta, \lambda) \ge \beta$ for all $n \ge k_0, l \ge k_0, k \ge k_0$. Let $\beta > \frac{1}{j}$ and

$$K(\beta,\lambda) = \left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \zeta, \lambda) > 1 - \beta \text{ and } \mathscr{B}(\Delta w_{nlk} - \zeta, \lambda) < \beta,\ \mathscr{Y}(\Delta w_{nlk} - \zeta, \lambda) < \beta\right\}.$$

Then, we have $\delta_3(K(\beta,\lambda)) = 0$. Since $\beta > \frac{1}{j}$, by (3.1) we get $\delta_3(K(j,\lambda)) = 0$, which contradicts by (3.2). Therefore, $w = (w_{nlk})$ is Δ-convergent to $\zeta \in F$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$.

Conversely presume that there is a subset $K = \left\{(n,l,k) \in \mathbb{N}^3 : n,l,k = 1,2,3,...\right\} \subset \mathbb{N}^3$ such that $\delta_3(K) = 1$ and $(\mathscr{G},\mathscr{B},\mathscr{Y}) - \lim_{(n,l,k)\in K, n,l,k\to\infty} \Delta w_{nlk} = L$. Then for every $\lambda > 0$ and $\varepsilon \in (0,1)$, there is $k_0 \in \mathbb{N}$ such that $\mathscr{G}(\Delta w_{nlk} - \zeta, \lambda) > 1 - \varepsilon$ and $\mathscr{B}(\Delta w_{nlk} - \zeta, \lambda) < \varepsilon$, $\mathscr{Y}(\Delta w_{nlk} - \zeta, \lambda) < \varepsilon$ for all $n \ge k_0, l \ge k_0, k \ge k_0$. Let

$$\begin{aligned} M(\varepsilon,\lambda) \quad &:= \left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \zeta, \lambda) \le 1 - \varepsilon \text{ or } \mathscr{B}(\Delta w_{nlk} - \zeta, \lambda) \ge \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \zeta, \lambda) \ge \varepsilon\right\} \\ &\subseteq \mathbb{N}^3 - \left\{\left(n_{k_0+1}, l_{k_0+1} k_{k_0+1}\right), \left(n_{k_0+2}, l_{k_0+2} k_{k_0+2}\right), ...\right\} \end{aligned}$$

and as a consequence $\delta_3(M(\varepsilon,\lambda)) \le 1 - 1 = 0$. Hence $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta$. Then, the desired result has been acquired. □

Definition 3.10. *Let* V *be an NNS, then* ζ *is named a* Δ*-limit point of the sequence* $w = (w_{nlk})$ *w.r.t NN* $(\mathscr{G},\mathscr{B},\mathscr{Y})$ *on condition that there is a subsequence of the sequence* w *which* Δ*-converges to* ζ *w.r.t NN* $(\mathscr{G},\mathscr{B},\mathscr{Y})$*. Let* $L^3_{(\mathscr{G},\mathscr{B},\mathscr{Y})(\Delta)}(w)$*, indicate the set of all limit points of the sequence* w *w.r.t NN* $(\mathscr{G},\mathscr{B},\mathscr{Y})$*. Let* $\left\{\left(w_{n(j_1)l(j_2)k(j_3)}\right)\right\}$ *be a subsequence of* $w = (w_{nlk})$ *and* $P = \left\{(n(j_1), l(j_2), k(j_3)) \in \mathbb{N}^3,\ j_1, j_2, j_3 \in \mathbb{N}\right\}$*, then we contract* $\left\{\left(w_{n(j_1)l(j_2)k(j_3)}\right)\right\}$ *by* $\{w\}_P$*, which in case* $\delta_3(P) = 0$*,* $\{w\}_P$ *is named a thin subsequence or subsequence of density zero. At the same time,* $\{w\}_P$ *is a non-thin subsequence of* w *if* P *does not have density zero.*

Definition 3.11. *Let* V *be an NNS. Then,* ζ *is named a* Δ*-statistical limit point of the sequence* $w = (w_{nlk})$ *w.r.t NN* $(\mathscr{G},\mathscr{B},\mathscr{Y})$ *on condition that there is a non-thin subsequence of* w *that* Δ*-converges to* $\zeta \in V$ *w.r.t NN* $(\mathscr{G},\mathscr{B},\mathscr{Y})$*. In that case, we say* ζ *is* $st_{N(\Delta)}$*-limit point of sequence* w*. Throughout* $\Lambda^3_{N(\Delta)}(w)$ *demonstrates the set of all* $st^3_{N(\Delta)}$*-limit point of sequence* w*.*

Definition 3.12. *Let V be an NNS. Then, ζ is named a Δ-statistical cluster point of the sequence $w=(w_{nlk})$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$ on condition that for every $\lambda>0$ and $\varepsilon\in(0,1)$,*

$$\overline{\delta_3}\left(\left\{\begin{array}{c}(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon\end{array}\right\}\right)>0$$

where $\overline{\delta_3}=\limsup\delta_3$. In that case, we say that ζ is $st^3_{N(\Delta)}$-cluster point of sequence w. Throughout $Cl^3_{N(\Delta)}(w)$ indicates the set of all $st^3_{N(\Delta)}$-cluster point of sequence w.

Definition 3.13. *A NNS V is called to be Δ-complete if every Δ-Cauchy sequence is Δ-convergent in V w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$.*

Theorem 3.14. *Let V be an NNS. Then, for any sequence $w=(w_{nlk})\in V$, $\Lambda^3_{N(\Delta)}(w)\subset Cl^3_{N(\Delta)}(w)$.*

Proof. Let $\zeta\in\Lambda_{N(\Delta)}(w)$, then there is a non-thin subsequence $\left(w_{n(j_1)l(j_2)k(j_3)}\right)$ of w that Δ-converges to $\zeta\in V$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$, i.e.

$$\delta_3\left(\left\{\begin{array}{c}(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)>1-\varepsilon\\ \text{and } \mathscr{B}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)<\varepsilon,\ \mathscr{Y}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)<\varepsilon\end{array}\right\}\right)=d>0.$$

Since

$$\begin{array}{l}\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\varepsilon \text{ and } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon\right\}\\ \supseteq\left\{\begin{array}{c}(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)>1-\varepsilon\\ \text{and } \mathscr{B}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)<\varepsilon,\ \mathscr{Y}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)<\varepsilon\end{array}\right\}.\end{array}$$

For every $\lambda>0$ and $\varepsilon\in(0,1)$, we obtain

$$\begin{array}{l}\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\varepsilon \text{ and } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon\right\}\\ \supseteq\left\{(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:j_1,j_2,j_3\in\mathbb{N}\right\}\\ \setminus\left\{\begin{array}{c}(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\le 1-\varepsilon\\ \text{or } \mathscr{B}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\ge\varepsilon,\ \mathscr{Y}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\ge\varepsilon\end{array}\right\}.\end{array}$$

Since $\left\{\left(w_{n(j_i)l(j_2)k(j_3)}\right)\right\}$ is Δ-convergent to ζ w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$, the set

$$\left\{\begin{array}{c}(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\le 1-\varepsilon\\ \text{or } \mathscr{B}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\ge\varepsilon,\ \mathscr{Y}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\ge\varepsilon\end{array}\right\}$$

is finite, for any $\varepsilon\in(0,1)$, so

$$\begin{array}{l}\overline{\delta_3}\left(\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\varepsilon \text{ and } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon\right\}\right)\\ \ge\overline{\delta_3}\left(\left\{(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:j_1,j_2,j_3\in\mathbb{N}\right\}\right)\\ -\overline{\delta_3}\left(\left\{\begin{array}{c}(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\le 1-\varepsilon\\ \text{or } \mathscr{B}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\ge\varepsilon,\ \mathscr{Y}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\ge\varepsilon\end{array}\right\}\right).\end{array}$$

Hence

$$\overline{\delta_3}\left(\left\{\begin{array}{c}(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon\end{array}\right\}\right)>0,$$

which gives $\zeta\in Cl^3_{N(\Delta)}(w)$. Therefore, we acquire $\Lambda^3_{N(\Delta)}(w)\subset Cl^3_{N(\Delta)}(w)$. ☐

Theorem 3.15. *For any sequence $w=(w_{nlk})\in V$, $Cl^3_{N(\Delta)}(w)\subset L^3_{(\mathscr{G},\mathscr{B},\mathscr{Y})(\Delta)}(w)$.*

Proof. Let $\zeta\in Cl^3_{N(\Delta)}(w)$, then

$$\delta_3\left(\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nlk}-\zeta,\lambda)>1-\varepsilon \text{ and } \mathscr{B}(\Delta x_{nlk}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nlk}-\zeta,\lambda)<\varepsilon\right\}\right)>0$$

for every $\lambda>0$ and $\varepsilon\in(0,1)$. Let $\{w\}_P$ be a non-thin subsequence of w such that

$$P=\left\{\begin{array}{c}(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)>1-\varepsilon \text{ and}\\ \mathscr{B}\left(\Delta x_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)<\varepsilon,\mathscr{Y}\left(\Delta x_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)<\varepsilon\end{array}\right\}$$

for each $\varepsilon\in(0,1)$ and $\delta_3(P)\neq 0$. Since there are infinitely many elements in P, $\zeta\in L^3_{(\mathscr{G},\mathscr{B},\mathscr{Y})(\Delta)}(w)$. Therefore, we obtain $Cl^3_{N(\Delta)}(w)\subset L^3_{(\mathscr{G},\mathscr{B},\mathscr{Y})(\Delta)}(w)$. ☐

Theorem 3.16. *For any sequence $w=(w_{nlk})\in V$, $st^3_{N(\Delta)}-\lim w_{nlk}=\zeta$, gives $\Lambda^3_{N(\Delta)}(w)=Cl^3_{N(\Delta)}(w)=\{\zeta\}$.*

Proof. First we denote that $\Lambda^3_{N(\Delta)}(w)=\{\zeta\}$. Presume that $\Lambda^3_{N(\Delta)}(w)=\{\zeta,\eta\}$ such that $\zeta\neq\eta$. In that case, there are two non-thin subsequences $\{(w_{n(j_1)l(j_2)k(j_3)})\}$ and $\{(w_{p(j_1)q(j_2)r(j_3)})\}$ of $w=(w_{nlk})$ those Δ-converge to ζ and η respectively w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$. Since $\{(w_{p(j_1)q(j_2)r(j_3)})\}$ is Δ-convergent to η w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$, so for every $\lambda>0$ and $\varepsilon\in(0,1)$,

$$P=\left\{\begin{array}{c}(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)\geq\varepsilon,\ \mathscr{Y}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)\geq\varepsilon\end{array}\right\}$$

is a finite set and so $\delta_3(P)=0$. Then, we observe that

$$\begin{aligned}&\{(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3\in\mathbb{N}^3,\ j_1,j_2,j_3\in\mathbb{N}\}\\ &=\left\{\begin{array}{c}(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)>1-\varepsilon \text{ and}\\ \mathscr{B}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon,\ \mathscr{Y}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon\end{array}\right\}\\ &\cup\left\{\begin{array}{c}(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)\geq\varepsilon,\ \mathscr{Y}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)\geq\varepsilon\end{array}\right\}\end{aligned}$$

which gives that

$$\delta_3\left(\left\{\begin{array}{c}(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)>1-\varepsilon \text{ and}\\ \mathscr{B}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon,\ \mathscr{Y}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon\end{array}\right\}\right)\neq 0. \tag{3.3}$$

Since $st^3_{N(\Delta)}-\lim w_{nlk}=\zeta$, we get

$$\delta_3\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon\end{array}\right\}\right)=0, \tag{3.4}$$

for every $\lambda>0$ and $\varepsilon\in(0,1)$. Therefore, we can write

$$\delta_3\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)<\varepsilon\end{array}\right\}\right)\neq 0.$$

For every $\zeta\neq\eta$, we get

$$\begin{aligned}&\left\{\begin{array}{c}(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)>1-\varepsilon \text{ and}\\ \mathscr{B}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon,\ \mathscr{Y}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon\end{array}\right\}\\ &\cap\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)<\varepsilon\end{array}\right\}\right)=\emptyset.\end{aligned}$$

Hence

$$\begin{aligned}&\left\{\begin{array}{c}(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)>1-\varepsilon \text{ and}\\ \mathscr{B}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon,\ \mathscr{Y}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon\end{array}\right\}\\ &\subset\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon\end{array}\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\overline{\delta_3}\left(\left\{\begin{array}{c}(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)>1-\varepsilon \text{ and}\\ \mathscr{B}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon,\ \mathscr{Y}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon\end{array}\right\}\right)\\ &\leq\overline{\delta_3}\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon\end{array}\right\}\right)=0.\end{aligned}$$

This contradicts (3.3). Hence $\Lambda^3_{N(\Delta)}(w)=\{\zeta\}$.

Next we demonstrate that $Cl^3_{N(\Delta)}(w)=\{\zeta\}$. Presume that $Cl^3_{N(\Delta)}(w)=\{\zeta,\gamma\}$ such that $\zeta\neq\gamma$. Then

$$\overline{\delta_3}\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\gamma,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon\end{array}\right\}\right)\neq 0. \tag{3.5}$$

Since

$$\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)<\varepsilon\end{array}\right\}$$
$$\cap\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\gamma,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon\end{array}\right\}=\emptyset$$

for every $\zeta\neq\gamma$, so

$$\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon\end{array}\right\}$$
$$\supseteq\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\gamma,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon\end{array}\right\}.$$

Therefore

$$\begin{aligned}&\overline{\delta_3}\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon\end{array}\right\}\right)\\ &\geq\overline{\delta_3}\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\gamma,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon\end{array}\right\}\right).\end{aligned}\tag{3.6}$$

From (3.5), the right hand side of (3.6) is greater than zero and from (3.4) the left hand side of (3.6) equals zero. This causes a contradiction. Hence $Cl^3_{N(\Delta)}(w)=\{\zeta\}$. □

Theorem 3.17. *The set $Cl^3_{N(\Delta)}$ is closed in V for each $w=(w_{nlk})$ of elements of V.*

Proof. Let $q\in\overline{Cl^3_{N(\Delta)}(w)}$. Let $r\in(0,1)$ and $\lambda>0$, there is $\sigma\in Cl^3_{N(\Delta)}(w)\cap B(q,r,\lambda)$ such that

$$B(q,r,\lambda)=\{s\in V:\mathscr{G}(q-s,\lambda)>1-r \text{ and } \mathscr{B}(q-s,\lambda)<r,\ \mathscr{Y}(q-s,\lambda)<r\}$$

Select $\xi>0$ such that $B(\xi,\sigma,\lambda)\subset B(q,r,\lambda)$. Then, we get

$$\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(q-\Delta w_{nkl},\lambda)>1-r \text{ and}\\ \mathscr{B}(q-\Delta w_{nkl},\lambda)<r,\ \mathscr{Y}(q-\Delta w_{nkl},\lambda)<r\end{array}\right\}$$
$$\supset\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\sigma-\Delta w_{nkl},\lambda)>1-\xi \text{ and}\\ \mathscr{B}(\sigma-\Delta w_{nkl},\lambda)<\xi,\ \mathscr{Y}(\sigma-\Delta w_{nkl},\lambda)<\xi\end{array}\right\}.$$

Since $\sigma\in Cl^3_{N(\Delta)}(w)$ so

$$\overline{\delta_3}\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\sigma-\Delta w_{nkl},\lambda)>1-\xi \text{ and}\\ \mathscr{B}(\sigma-\Delta w_{nkl},\lambda)<\xi,\ \mathscr{Y}(\sigma-\Delta w_{nkl},\lambda)<\xi\end{array}\right\}\right)>0.$$

Hence

$$\overline{\delta_3}\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(q-\Delta x_{nkl},\lambda)>1-r \text{ and}\\ \mathscr{B}(q-\Delta x_{nkl},\lambda)<r,\ \mathscr{Y}(q-\Delta x_{nkl},\lambda)<r\end{array}\right\}\right)>0.$$

Thus $q\in Cl^3_{N(\Delta)}(w)$. This ends the proof. □

Acknowledgement

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

Not applicable.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] L.A. Zadeh, *Fuzzy sets*, Inform. Control, **8** (1965), 338-353.

[2] J. H. Park, *Intuitionistic fuzzy metric spaces*, Chaos Solitons Fractals, **22** (2004), 1039-1046.

[3] F. Lael, K. Nourouzi, *Some results on the IF-normed spaces*, Chaos Solitons Fractals, **37** (2008), 931-939.

[4] S. Karakuş, K. Demirci, O. Duman, *Statistical convergence on intuitionistic fuzzy normed spaces*, Chaos Solitons Fractals, **35** (2008), 763-769.

[5] S. Karakuş, K. Demirci, Ş. Yardımcı, *Statistical limit points of sequences on intuitionistic fuzzy normed spaces,* J. Concr. Appl. Math., **6**(4) (2008), 375-386.

[6] P. Debnath, *A generalised statistical convergence in intuitionistic fuzzy n-normed linear spaces*, Ann. Fuzzy Math. Inform., **12**(4) (2016) 559-572.

[7] E. Savaş, M. Gürdal, *A generalized statistical convergence in intuitionistic fuzzy normed spaces*, Science Asia, **41** (2015), 289-294.

[8] F. Smarandache, *Neutrosophic set, a generalisation of the intuitionistic fuzzy sets*, Int. J. Pure Appl. Math., **24**(3) (2005), 287-297.

[9] F. Smarandache, *Neutrosophy, Neutrosophic Probability, Set and Logic*, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 1998.

[10] M. Kirisci, N. Şimşek, *Neutrosophic metric spaces*, Math. Sci., **14** (2020), 241-248.

[11] M. Kirisci, N. Şimşek, *Neutrosophic normed spaces and statistical convergence*, J. Anal., **28** (2020), 1059-1073.

[12] M. Kirisci, N. Şimşek, M. Akyiğit, *Fixed point results for a new metric space*, Math. Methods Appl. Sci., **44**(9) (2020), 7416-7422.

[13] Ö. Kişi, *Lacunary statistical convergence of sequences in neutrosophic normed spaces*, 4th International Conference on Mathematics: An Istanbul Meeting for World Mathematicians, Istanbul, 345-354, 2020.

[14] Ö. Kişi, *Ideal convergence of sequences in neutrosophic normed spaces*, J. Intell. Fuzzy Syst., **41**(2) (2021), 2581-2590.

[15] V. A. Khan, M. D. Khan, M. Ahmad, *Some new type of lacunary statistically convergent sequences in neutrosophic normed space*, Neutrosophic Sets Syst., **42** (2021), 239-252.

[16] A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge, UK, 1979.

[17] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241-244.

[18] J. A. Fridy, *On statistical convergence*, Analysis, **5** (1985), 301-313.

[19] A. A. Nabiev, E. Savaş, M. Gürdal, *Statistically localized sequences in metric spaces*, J. Appl. Anal. Comput., **9**(2) (2019), 739-746.

[20] E. Savaş, M. Gürdal, *Generalized statistically convergent sequences of functions in fuzzy* 2*-normed spaces*, J. Intell. Fuzzy Systems., **27**(4) (2014), 2067-2075.

[21] M. Mursaleen, O. H. H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl., **288** (2003), 223-231.

[22] B. Altay, F. Başar, *Some new spaces of double sequences*, J. Math. Anal. Appl., **309**(1) (2005), 70-90.

[23] A. Şahiner, M. Gürdal, F. K. Düden, *Triple sequences and their statistical convergence*, Selçuk J. Appl. Math., **8**(2) (2007), 49-55.

[24] A. Esi, *Statistical convergence of triple sequences in topological groups*, Annals Univ. Craiova. math. Comput. Sci. Ser., **10**(1) (2013), 29-33.

[25] B. C. Tripathy, R. Goswami, *On triple difference sequences of real numbers in propobabilistic normed space*, Proyecciones J. Math., **33**(2) (2014), 157-174.

[26] H. Kızmaz, *On certain sequence spaces*, Canad. Math. Bull., **24** (1981), 169-176.

[27] M. Başarır, *On the statistical convergence of sequences*, Fırat Univ. Turk. J. Sci. Technol., **2** (1995), 1-6.

[28] R. Çolak, H. Altınok, M. Et, *Generalized difference sequences of fuzzy numbers*, Chaos Solitons Fractals, **40**(3) (2009), 1106-1117.

[29] Y. Altın, M. Başarır, M. Et, *On some generalized difference sequences of fuzzy number*s, Kuwait J. Sci., **34**(1A) (2007), 1-14.

[30] S. Altundağ, E. Kamber, *Lacunary Δ-statistical convergence in intuitionistic fuzzy n-normed space*, J. Inequal. Appl., **2014**(40) (2014), 1-12.

[31] B. Hazarika, A. Alotaibi, S.A. Mohiudine, *Statistical convergence in measure for double sequences of fuzzy-valued functions*, Soft Comput., **24**(9) (2020), 6613-6622.

[32] F. Başar, *Summability theory and its applications*, Bentham Science Publishers, Istanbul, 2012.

[33] M. Mursaleen, F. Başar, *Sequence Spaces: Topics in Modern Summability Theory,* CRC Press, Taylor & Francis Group, Series: Mathematics and Its Applications, Boca Raton London New York, 2020.

[34] K. Menger, *Statistical metrics*, Proc. Natl. Acad. Sci. USA, **28**(12) (1942), 535-537.

Communications in Advanced Mathematical Sciences
Vol. 5, No. 1, 46-47, 2022
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1034541

Erratum to "A New Pre-Order Relation for Set Optimization using ℓ-difference" [Communications in Advanced Mathematical Sciences, 4(3) (2021), 163-170]

Emrah Karaman[1]*

Abstract
In this work, an erratum for a proposition in the paper "A New Pre-Order Relation for Set Optimization using ℓ-difference" is outlined. It was pointed out by Stefan Rocktäschel and Ernest Quintana that the proof of Proposition 3.11 is wrong in [1]. A small detail in the proof of Proposition 3.11 has been overlooked. A new proposition, which is closely related to Proposition 3.11 in [1], is presented. The main results of the paper are not affected by this erratum.

Keywords: Order relation, pre-order relation, set optimization

[1] *Eskişehir Technical University, Faculty of Science, Department of Mathematics, 26470 Eskişehir, Turkey, ORCID: 0000-0002-0466-3827*
Corresponding author: e.karaman42@gmail.com
Received: 9 December 2021, **Accepted:** 25 January 2022, **Available online:** 19 March 2022

1. A new order relation for set approach

There is a subtle error in the proof of Proposition 3.11 in [1]. So, Proposition 3.11 in [1] needs to be restate. The main results of the paper are not effected from this erratum. The following example can be given as a counter example for Proposition 3.11 (i) in [1]:

Example 1.1. *Let $Y = \mathbb{R}^2$, $K = \{(x,y) \in \mathbb{R}^2 \mid y = x \text{ and } x \geq 0\}$, $A = \{(x,y) \in \mathbb{R}^2 \mid y = x\}$ and $B = \{0_{\mathbb{R}^2}\}$. Then, we can find $a \in A$ and $b \in B$ such that $b \leq_K a$. But, $A \preceq^{\ell_1} B$ is not satisfied.*

I want to put the following proposition instead of Proposition 3.11 in [1]:

Proposition 1.2. *Let $A, B \in \mathscr{P}(Y)$. If $b \leq_K a$ for all $a \in A$ and all $b \in B$, then $A \preceq^{\ell_1} B$.*

Proof. Assume that $b \leq_K a$ for all $a \in A$ and all $b \in B$. By contradiction, suppose that $A \not\preceq^{l_1} B$. Then, $(B \ominus_\ell A) \cap K = \emptyset$, and we have $k + A \not\subset B + K$ for all $k \in K$. By setting $k = 0_Y \in K$, we have $A \not\subset B + K$. Hence, there exists $a \in A$ with $a \notin B + K$. Consequently, it holds $a \notin b + K$ for all $b \in B$ and therefore, $b \not\leq_K a$ for all $b \in B$, which is contradict. □

Besides of all them, we can easily show that the order relation $\preceq^3$ implies the order relation $\preceq^{\ell_1}$, where the order of sets should be changed. That is, if $B \preceq^3 A$ (or $A \subset B + K$) for any $A, B \in \mathscr{P}(Y)$, then $A \preceq^{\ell_1} B$. But, the inverse inclusion may not be true. For example, let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = (-1, 0)$ and $B = (0, 0)$, where $\mathbb{R}^2_+$ is nonnegative orthant of the space. Although $A \preceq^{\ell_1} B$, we have $B \not\preceq^3 A$.

Acknowledgements

I would like thank to Stefan Rocktäschel and Ernest Quintana for spotting the error in the original paper. Also, the author put a sincere thanks to the reviewers and editors for their valuable comments to enhance the paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] E. Karaman, *A new pre-order relation for set optimization using l-difference*, Comm. Adv. Math. Sci., **4**(3) (2021), 163-170.