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Research Article

Equilibria for abstract economies in Hausdorff topological vector spaces

Dedicated to Professor Anthony To-Ming Lau with much admiration.

DONAL O'REGAN*

ABSTRACT. In this paper using new fixed point results of the author, we establish a variety of existence results for equilibria for abstract economies.

Keywords: Fixed points, equilibria, abstract economies.

2020 Mathematics Subject Classification: 47H04, 47H10, 47N10, 90A14.

1. INTRODUCTION

Using strategy sets with constraint and preference correspondences defined on subsets of Hausdorff topological vector spaces, we present in this paper a variety of equilibrium results for abstract economies. These equilibrium results are deduced from recent fixed point results in the literature (see [8, 9, 10]) and our theory improves and generalizes corresponding results in the literature (see [1, 4, 5, 6, 11, 12] and the references therein).

Now, we recall some fixed point results [8, 9, 10] in the literature. First, we recall the following notions from the literature. For a subset K of a topological space X, we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given two maps $F, G : X \to 2^Y$ (here 2^Y denotes the family of nonempty subsets of Y) and $\alpha \in Cov(Y)$, F and G are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$.

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov(Y)$ and any pair (X, K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$ there exists a continuous function $f : X \to Y$ such that $f|_K$ is α–close to f_0.

Let V be a subset of a Hausdorff topological vector space E. Then, we say V is Schauder admissible if for every compact subset K of V and every covering $\alpha \in Cov_V(K)$ there exists a continuous functions $\pi_\alpha : K \to V$ such that

(i). π_α and $i : K \to V$ are α–close,
(ii). $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in AES$ (compact).

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X is said to be q– Schauder admissible if any nonempty compact convex subset Ω of X is Schauder admissible.

An upper semicontinuous map $\phi : X \to CK(Y)$ is said to Kakutani (and we write $\phi \in Kak(X, Y)$); here $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y.

Theorem 1.1. *Let I be an index set and $\{X_i\}_{i\in I}$ be a family of sets each in a Hausdorff topological vector space E_i. For each $i \in I$, let K_i be a nonempty compact subset of X_i and suppose $F_i : X \equiv \prod_{i\in I} X_i \to K_i$ is upper semicontinuous with nonempty convex compact values (i.e. $F_i \in Kak(X, K_i)$). Also assume $K \equiv \prod_{i\in I} K_i$ is a Schauder admissible subset of the Hausdorff topological vector space $E \equiv \prod_{i\in I} E_i$. Then, there exists a $x \in K$ with $x_i \in F_i(x)$ for $i \in I$ (here x_i is the projection of x on X_i).*

Remark 1.1. *One could repace K a Schauder admissible subset of E in Theorem 1.1 (and the other results in this paper) with other admissible subsets of E described in* [7].

Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and G a multifunction. We say $G \in DKT(Z, W)$ [2] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ is open (in Z) for each $w \in W$.

Theorem 1.2. *Let I be an index set and $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and $F_i \in DKT(X, X_i)$. In addition assume for each $i \in I$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Also suppose X is a q–Schauder admissible subset of the Hausdorff topological vector space $E = \prod_{i\in I} E_i$. Then, there exists a $x \in X$ with $x_i \in F_i(x)$ for $i \in I$.*

Remark 1.2. *If I is a finite set, then the assumption that "X is a q–Schauder admissible subset of the Hausdorff topological vector space E" can be removed. In fact we have: Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, ..., N\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$ and $F_i \in DKT(X, X_i)$. In addition assume for each $i \in \{1, ..., N\}$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Then, there exists a $x \in X$ with $x_i \in F_i(x)$ for $i \in \{1, ..., N\}$.*

Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and F a multifunction. We say $F \in HLPY(Z, W)$ [3, 4] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup\{\, int\, S^{-1}(w) : \; w \in W\}$; here $S^{-1}(w) = \{z \in Z : w \in S(z)\}$.

Theorem 1.3. *Let I be an index set and $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and $F_i \in HLPY(X, X_i)$. In addition assume for each $i \in I$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Also suppose X is a q–Schauder admissible subset of the Hausdorff topological vector space $E = \prod_{i\in I} E_i$. Then, there exists a $x \in X$ with $x_i \in F_i(x)$ for $i \in I$.*

Remark 1.3. *If I is a finite set, then the assumption that "X is a q–Schauder admissible subset of the Hausdorff topological vector space E" can be removed. In fact we have: Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, ..., N\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$ and $F_i \in HLPY(X, X_i)$. In addition assume for each $i \in \{1, ..., N\}$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Then, there exists a $x \in X$ with $x_i \in F_i(x)$ for $i \in \{1, ..., N\}$.*

We now state a result from the literature [11] which will be used in Section 2.

Theorem 1.4. *Let X and Y be two topological spaces and A an open subset of X. Suppose $F_1 : X \to 2^Y$, $F_2 : A \to 2^Y$ (here 2^Y denotes the family of nonempty subsets of Y) are upper semicontinuous such that $F_2(x) \subset F_1(x)$ for all $x \in A$. Then, the map $F : X \to 2^Y$ defined by*

$$F(x) = \begin{cases} F_1(x), & x \notin A \\ F_2(x), & x \in A \end{cases}$$

is upper semicontinuous.

2. Abstract economy results

Let I be the set of agents and we describe the abstract economy as $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$, where $A_i,\, B_i : X \equiv \prod_{i\in I} X_i \to 2^{E_i}$ are constraint correspondences, $P_i : X \to 2^{E_i}$ is a preference correspondence and X_i is a choice (or strategy) set which is a subset of a Hausdorff topological vector space E_i. We are interested in finding an equilibrium point for Γ i.e. a point $x \in X$ with $x_i \in \overline{B_i}(x)$ and $co\, A_i(x) \cap co\, P_i(x) = \emptyset$ (or $x_i \in B_i(x)$ and $A_i(x) \cap P_i(x) = \emptyset$) for $i \in I$.

Theorem 2.5. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ be an abstract economy with $\{X_i\}_{i\in I}$ a family of nonempty sets each in a Hausdorff topological vector space E_i (here I is an index set). For each $i \in I$, let $A_i, B_i, P_i : X \equiv \prod_{i\in I} X_i \to 2^{E_i}$ and assume the following conditions are satisfied:*

$$U_i = \{x \in X :\ co\, A_i(x) \cap co\, P_i(x) \neq \emptyset\} \text{ is paracompact and open in } X \tag{2.1}$$

$$cl\, B_i (\equiv \overline{B_i}) : X \to CK(E_i) \text{ is upper semicontinuous} \tag{2.2}$$

$$\begin{cases} \text{there exists a nonempty compact subset } K_i \text{ of } X_i \text{ with } \overline{B_i} : X \to CK(K_i) \\ \text{and } K \equiv \prod_{i\in I} K_i \text{ is a Schauder admissible subset of } E \equiv \prod_{i\in I} E_i \end{cases} \tag{2.3}$$

and

$$x_i \notin co\, A_i(x) \cap co\, P_i(x) \text{ if } x \in U_i;\ \text{here } x_i \text{ is the projection of } x \text{ on } E_i. \tag{2.4}$$

For $i \in I$ and $x \in X$, let $H_i(x) = co\, A_i(x) \cap co\, P_i(x)$ and suppose

$$H_i(x) \subseteq \overline{B_i}(x) \text{ for } x \in U_i \tag{2.5}$$

and

$$\begin{cases} \text{there exists a } S_i : U_i \to 2^{E_i} \text{ with } co\, S_i(x) \subseteq H_i(x) \text{ for } x \in U_i \\ \text{and } S_i^{-1}(y) \text{ is open (in } U_i) \text{ for each } y \in E_i. \end{cases} \tag{2.6}$$

Then there exists a $x \in X$ with for each $i \in I$, we have $x_i \in \overline{B_i}(x)$ and $co\, A_i(x) \cap co\, P_i(x) = \emptyset$.

Proof. Note for each $i \in I$ from (2.6), we have $H_i \in DKT(U_i, E_i)$ so from [2] there exists a continuous (single valued) selection $f_i : U_i \to E_i$ of H_i with $f_i(x) \in co\,(S_i(x)) \subseteq H_i(x)$ for $x \in U_i$. For each $i \in I$, let

$$G_i(x) = \begin{cases} f_i(x), & x \in U_i \\ \overline{B_i}(x), & x \notin U_i \end{cases}.$$

Note for each $i \in I$ that $\{f_i(x)\} \subseteq co\,(S_i(x)) \subseteq H_i(x) \subseteq \overline{B_i}(x)$ (see (2.5)) if $x \in U_i$, so Theorem 1.4 guarantees that $G_i : X \to CK(E_i)$ is upper semicontinuous. Also for each $i \in I$, we have $G_i(x) \subseteq \overline{B_i}(x) \subseteq K_i$ for $x \in X$ so $G_i \in Kak(X, K_i)$. Now, Theorem 1.1 guarantees a $x \in K$ with $x_i \in G_i(x)$ for $i \in I$. If $x \in U_i$ for some $i \in I$, then $x_i = f_i(x) \in H_i(x) = co\, A_i(x) \cap co\, P_i(x)$, which contradicts (2.4). Thus for each $i \in I$, we must have $x \notin U_i$ and then we have $x_i \in \overline{B_i}(x)$ and $co\, A_i(x) \cap co\, P_i(x) = \emptyset$. $\square$

Remark 2.4.

(i). *If $i \in I$ and $H_i^{-1}(y)$ is open (in X) for each $y \in E_i$, then U_i in (2.1) is automatically open in X. This is immediate once one notices that $U_i = \cup_{y \in E_i} H_i^{-1}(y)$.*

(ii). *Of course there are other obvious analogues of Theorem 2.5 if the assumptions on $co\, A_i \cap co\, P_i$ are replaced by assumptions on $co\, A_i \cap P_i$ or $\overline{co}\, A_i \cap P_i$ or $\overline{co}\, A_i \cap \overline{co}\, P_i$ or $\overline{co}\, A_i \cap co\, P_i$ or $A_i \cap co\, P_i$ or $A_i \cap \overline{co}\, P_i$ or $A_i \cap P_i$ or $co\, A_i \cap \overline{co}\, P_i$ and the assumptions on $\overline{B_i}$ are replaced by assumptions on B_i.*

Remark 2.5. *For each $i \in I$ suppose there exists a map $S_i : X \to E_i$ (which may have empty values) with $co\, S_i(x) \subseteq H_i(x)$ for $x \in X$, the fibres $S_i^{-1}(y)$ are open (in X) for each $y \in E_i$ and also assume if $x \in U_i$, then $S_i(x) \neq \emptyset$. Then, (2.6) holds with S_i replaced by $S_i|_{U_i}$. Let $S_i^\star$ denote $S_i|_{U_i}$. For $i \in I$ note $S_i^\star : U_i \to 2^{E_i}$, $co\, S_i^\star(x) \subseteq H_i(x)$ for $x \in U_i$ and for $y \in E_i$ note*

$$(S_i^\star)^{-1}(y) = \{x \in U_i : y \in S_i^\star(x)\} = \{x \in X : y \in S_i(x)\} \cap U_i = S_i^{-1}(y) \cap U_i,$$

so $(S_i^\star)^{-1}(y)$ which is open in U_i.

Theorem 2.6. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy with $\{X_i\}_{i \in I}$ a family of nonempty sets each in a Hausdorff topological vector space E_i (here I is an index set). For each $i \in I$, let $A_i, B_i, P_i : X \equiv \prod_{i \in I} X_i \to 2^{E_i}$ and assume (2.1), (2.2), (2.3) and (2.4) hold. For $i \in I$ and $x \in X$, let $H_i(x) = co\, A_i(x) \cap co\, P_i(x)$ and suppose (2.5) holds. In addition for each $i \in I$ assume*

$$\text{(2.7)} \qquad \begin{cases} \text{there exists a } \; S_i : U_i \to 2^{E_i} \; \text{ with } \; co\, S_i(x) \subseteq H_i(x) \; \text{ for } \; x \in U_i \\ \text{and } \; U_i = \bigcup \{ int_{U_i}\, S_i^{-1}(w) : \; w \in E_i\} \end{cases} .$$

Then there exists a $x \in X$ with for each $i \in I$ we have $x_i \in \overline{B_i}(x)$ and $co\, A_i(x) \cap co\, P_i(x) = \emptyset$.

Proof. Note for each $i \in I$ from (2.7), we have $H_i \in HLPY(U_i, E_i)$ so from [4] there exists a continuous (single valued) selection $f_i : U_i \to E_i$ of H_i with $f_i(x) \in co(S_i(x)) \subseteq H_i(x)$ for $x \in U_i$. Let G_i for $i \in I$ be as in Theorem 2.5 and the same reasoning guarantees a $x \in K$ with $x_i \in G_i(x)$ for $i \in I$. □

Remark 2.6. *For each $i \in I$ suppose there exists a map $S_i : X \to E_i$ (which may have empty values) with $co\, S_i(x) \subseteq H_i(x)$ for $x \in X$, $X = \bigcup \{ int_X\, S_i^{-1}(w) : \; w \in E_i\}$ and also assume if $x \in U_i$, then $S_i(x) \neq \emptyset$. Then, (2.7) holds with S_i replaced by $S_i|_{U_i}$. Let $S_i^\star$ denote $S_i|_{U_i}$. For $i \in I$ note $S_i^\star : U_i \to 2^{E_i}$, $co\, S_i^\star(x) \subseteq H_i(x)$ for $x \in U_i$ and now we show $U_i = \bigcup \{ int_{U_i}\, (S_i^\star)^{-1}(w) : \; w \in E_i\}$. To see this notice*

$$U_i = U_i \cap X = U_i \cap \left(\bigcup \{ int_X\, S_i^{-1}(w) : \; w \in E_i\} \right) = \bigcup \{ U_i \cap int_X\, S_i^{-1}(w) : \; w \in E_i\},$$

so $U_i \subseteq \bigcup \{ int_{U_i}\, (S_i^\star)^{-1}(w) : \; w \in E_i\}$ since for each $w \in E_i$, we have that $U_i \cap int_X\, S_i^{-1}(w)$ is open in U_i. On the other hand clearly $\bigcup \{ int_{U_i}\, (S_i^\star)^{-1}(w) : \; w \in E_i\} \subseteq U_i$ so as a result $U_i = \bigcup \{ int_{U_i}\, (S_i^\star)^{-1}(w) : \; w \in E_i\}$.

Theorem 2.7. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy with $\{X_i\}_{i \in I}$ a family of nonempty convex sets each in a Hausdorff topological vector space E_i (here I is an index set). For each $i \in I$, let $A_i, B_i, P_i : X \equiv \prod_{i \in I} X_i \to 2^{E_i}$ and assume the following conditions are satisfied:*

$$\text{(2.8)} \qquad co(A_i(x)) \subseteq B_i(x) \;\; \text{for} \;\; x \in X,$$

$$\text{(2.9)} \qquad x_i \notin B_i(x) \cap co\, P_i(x) \;\; \text{if} \;\; x \in X \;\; \text{and} \;\; A_i(x) \cap P_i(x) \neq \emptyset,$$

$$\text{(2.10)} \qquad \begin{cases} \text{there exists a nonempty convex compact subset } \; K_i \; \text{ of } \; X_i \\ \text{with } \; B_i(X) \subseteq K_i \subseteq X_i \end{cases}$$

and

$$(2.11)\qquad \left\{\begin{array}{l} \textit{for each } \ y_i\in X_i \ \textit{ the set } \ \big[(co\,P_i)^{-1}(y_i)\cup M_i\big]\cap A_i^{-1}(y_i) \\ \textit{is open in } \ X \ \ (\textit{here } \ M_i=\{x\in X: \ A_i(x)\cap P_i(x)=\emptyset\}) \end{array}\right. .$$

Finally, assume X is a q–Schauder admissible subset of $E=\prod_{i\in I} E_i$. Then there exists a $x\in X$ with for each $i\in I$, we have $x_i\in B_i(x)$ and $A_i(x)\cap P_i(x)=\emptyset$.

Proof. For each $i\in I$, let $N_i=\{x\in X: A_i(x)\cap P_i(x)\neq\emptyset\}$ and for each $x\in X$ let

$$I(x)=\{i\in I: A_i(x)\cap P_i(x)\neq\emptyset\}.$$

For each $i\in I$, let $F_i, G_i: X\to 2^{X_i}$ be given by

$$F_i(x)=\left\{\begin{array}{ll} A_i(x)\cap co\,P_i(x), & i\in I(x)\\ A_i(x), & i\notin I(x)\end{array}\right.$$

and

$$G_i(x)=\left\{\begin{array}{ll} B_i(x)\cap co\,P_i(x) & ,i\in I(x)\\ B_i(x) & ,i\notin I(x)\end{array}\right. .$$

Fix $i\in I$. Note from (2.8) that $co\,F_i(x)\subseteq G_i(x)$ for $x\in X$ (and note $F_i(x)\neq\emptyset$ for $x\in X$). Also note for each $y_i\in X_i$, we have

$$\begin{aligned} F_i^{-1}(y_i)&=\{x\in X: \ y_i\in F_i(x)\}\\ &=\{x\in N_i: \ y_i\in A_i(x)\cap co\,P_i(x)\}\,\cup\,\{x\in M_i: \ y_i\in A_i(x)\}\\ &=\big\{\big[(co\,P_i)^{-1}(y_i)\cap A_i^{-1}(y_i)\big]\cap N_i\big\}\cup\big\{A_i^{-1}(y_i)\cap M_i\big\}\\ &=\big[(co\,P_i)^{-1}(y_i)\cap A_i^{-1}(y_i)\big]\cup\big[A_i^{-1}(y_i)\cap M_i\big]\\ &=\big[(co\,P_i)^{-1}(y_i)\cup M_i\big]\cap A_i^{-1}(y_i)\end{aligned}$$

which (see (2.11)) is open in X. Thus for each $i\in I$, we have $G_i\in DKT(X,X_i)$ and also from (2.10) note $G_i(X)\subseteq K_i\subseteq X_i$. Now, Theorem 1.2 guarantees a $x\in K$ with $x_i\in G_i(x)$ for $i\in I$. Note if $i\in I(x)$ for some $i\in I$ then $A_i(x)\cap P_i(x)\neq\emptyset$ and $x_i\in B_i(x)\cap co\,P_i(x)$, which contradicts (2.9). Thus $i\notin I(x)$ for all $i\in I$. Consequently, $x_i\in B_i(x)$ and $A_i(x)\cap P_i(x)=\emptyset$ for all $i\in I$. □

Remark 2.7. *In Theorem 2.7 if I is a finite set, then the assumption that "X is a q–Schauder admissible subset of the Hausdorff topological vector space E" can be removed (see Remark 1.2).*

Theorem 2.8. *Let $\Gamma=(X_i,A_i,B_i,P_i)_{i\in I}$ be an abstract economy with $\{X_i\}_{i\in I}$ a family of nonempty convex sets each in a Hausdorff topological vector space E_i (here I is an index set). For each $i\in I$, let $A_i,B_i,P_i: X\equiv\prod_{i\in I}X_i\to 2^{E_i}$ and assume (2.8), (2.9) and (2.10) hold. Also suppose X is a q–Schauder admissible subset of $E=\prod_{i\in I}E_i$. For each $x\in X$, let $I(x)=\{i\in I: \ A_i(x)\cap P_i(x)\neq\emptyset\}$ and for each $i\in I$, let*

$$F_i(x)=\left\{\begin{array}{ll} A_i(x)\cap co\,P_i(x), & i\in I(x)\\ A_i(x), & i\notin I(x)\end{array}\right.$$

and assume that

$$(2.12)\qquad X=\cup\{\,int\,F_i^{-1}(w): \ w\in X_i\}.$$

Then there exists a $x\in X$ with for each $i\in I$, we have $x_i\in B_i(x)$ and $A_i(x)\cap P_i(x)=\emptyset$.

Proof. Let N_i and G_i be as in Theorem 2.7. For $i\in I$ note $F_i(x)\neq\emptyset$ and $co\,F_i(x)\subseteq G_i(x)$ for $x\in X$ and $X=\cup\{\,int\,F_i^{-1}(w): \ w\in X_i\}$. Thus for each $i\in I$, we have $G_i\in HLPY(X,X_i)$ and also from (2.10) note $G_i(X)\subseteq K_i\subseteq X_i$. Now, Theorem 1.3 guarantees a $x\in K$ with $x_i\in G_i(x)$ for $i\in I$ and the reasoning in Theorem 2.7 guarantees the result. □

Remark 2.8. *In Theorem 2.8 if I is a finite set, then the assumption that "X is a q–Schauder admissible subset of the Hausdorff topological vector space E" can be removed (see Remark 1.3).*

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Research Article

Generalized eigenvectors of linear operators and biorthogonal systems

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ABSTRACT. The notions of a set of generalized eigenvalues and a set of generalized eigenvectors of a linear operator in Euclidean space are introduced. In addition, we provide a method to find a biorthogonal system of a subsystem of eigenvectors of some linear operators in a Hilbert space whose systems of canonical eigenvectors are over-complete. Related to our problem, we will show an example of a linear differential operator that is formally adjoint to Bessel-type differential operators. We also investigate the basic properties (completeness, minimality, basicity) of the systems of generalized eigenvectors of this differential operator.

Keywords: Linear operator, generalized eigenvector, Bessel function, complete system, minimal system, biorthogonal system.

2020 Mathematics Subject Classification: 33C10, 34B30, 34L10.

1. INTRODUCTION

Let $\mathcal{H}$ be an Euclidean space with inner product $\langle\cdot;\cdot\rangle : \mathcal{H}\times\mathcal{H}\to\mathbb{C}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $m\in\mathbb{N}_0$, $\overline{n;m}=[n;m]\cap\mathbb{N}_0$ and $\overline{n;m}=\emptyset$ if $n>m$. Suppose that a certain linear operator $A:\mathcal{H}\to\mathcal{H}$ has a countable set of simple eigenvalues $\{\lambda_k : k\in\mathbb{N}\}$ and a corresponding system of eigenvectors $\{\psi_k : k\in\mathbb{N}\}$ that is complete and minimal after removing, for example, the first $m\in\mathbb{N}$ members, or the adjoint operator of A has no eigenvalues. Such operators arise naturally in the study of some boundary value problems (see, for example, [3, 4, 10, 14, 16] and the reference therein), for instance, in the study of boundary value problems for Bessel's equation (see [8, 12, 13, 18, 19, 25, 26]). The problem is how to find a biorthogonal system $(U_n : n\in\mathbb{N}\backslash\overline{1;m})$. Such a biorthogonal system will be found if we can find the vectors U_n such that $\langle\psi_k;U_n\rangle=0$ for all $k\in\mathbb{N}\backslash\overline{1;m}$ and $n\in\mathbb{N}\backslash\overline{1;m}$.

Finding such biorthogonal systems often faces certain difficulties (see [3, 4, 8, 12, 13, 18, 19, 25, 26]). Sometimes, in the case of simple eigenvalues, such vectors U_n can be found by using a notion of a set of generalized eigenvectors which we propose in this paper (see Section 2). There are different methods to introduce the generalized eigenvectors with access to a wider space (for details, see [2, 3, 4, 5, 9]). The peculiarity of our interpretation of a set of generalized eigenvectors of a linear operator $B:\mathcal{H}\to\mathcal{H}$ with domain $\mathcal{D}(B)$ is that the generalized eigenvectors belong to $\mathcal{H}$ and the difference of eigenvectors belong to $\mathcal{D}(B)$. We show an example of a linear differential operator $B_\nu:\mathcal{H}^\nu\to\mathcal{H}^\nu$ in some Hilbert space $\mathcal{H}^\nu$ that has no eigenvectors, but has the generalized eigenvectors (see Section 3). In Sections 4 and 5, we will prove that this operator, B_ν, is formally adjoint to Bessel-type differential operators $\widetilde{A}_\nu:\mathcal{H}^\nu\to\mathcal{H}^\nu$ and

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$A_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$ whose systems of canonical eigenvectors are over-complete. We also investigate the basic properties (completeness, minimality, basicity) of the systems of generalized eigenvectors of an operator B_ν.

The introduced notions of the sets of generalized eigenvalues and eigenvectors probably are of interest in some sense for spectral theory.

2. Generalized eigenvectors

Let $\Omega \subseteq \mathbb{N}$ be some non-empty set.

Definition 2.1. *The set $\mathfrak{M}(B) = \{\mu_j : j \in \Omega\}$ is called a set of generalized eigenvalues of a linear operator $B : \mathcal{H} \to \mathcal{H}$ with domain $\mathcal{D}(B)$ in a vector (linear) space $\mathcal{H}$ if there exists a set $\mathfrak{U}(B) = \{U_j : j \in \Omega\}$ of nonzero elements $U_j \in \mathcal{H}$ such that $U_n - U_k \in \mathcal{D}(B)$ and $B(U_n - U_k) = \mu_n U_n - \mu_k U_k$ for every $n \in \Omega$ and $k \in \Omega$. In this case, the set $\mathfrak{U}(B)$ is called a set of generalized eigenvectors of an operator B.*

We say that an operator $B : \mathcal{H} \to \mathcal{H}$ is a *formally adjoint* of an operator $A : \mathcal{H} \to \mathcal{H}$ in a Euclidean space $\mathcal{H}$ with inner product $\langle \cdot ; \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, if $\langle A\psi; u\rangle = \langle \psi; Bu\rangle$ for all $\psi \in \mathcal{D}(A)$ and $u \in \mathcal{D}(B)$.

Theorem 2.1. *Suppose that $A : \mathcal{H} \to \mathcal{H}$ be a linear operator with domain $\mathcal{D}(A)$ in a Euclidean space $\mathcal{H}$ with inner product $\langle \cdot ; \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ having a set of eigenvalues $\{\lambda_j : j \in \Omega\}$ and a set of eigenvectors $\{\psi_j : j \in \Omega\}$. Let each $\mu_j = \overline{\lambda_j}$ be a generalized eigenvalue of an operator $B : \mathcal{H} \to \mathcal{H}$ that is a formally adjoint of A, and let $\{U_j : j \in \Omega\}$ be a set of generalized eigenvectors of B. Then $\langle \psi_k; U_n\rangle = 0$ if $\lambda_k \neq \lambda_n$.*

Proof. Indeed,

$$\begin{aligned}\lambda_k \langle \psi_k; U_n\rangle &= \lambda_k \langle \psi_k; U_n - U_k\rangle + \lambda_k \langle \psi_k; U_k\rangle = \langle A\psi_k; U_n - U_k\rangle + \lambda_k \langle \psi_k; U_k\rangle \\ &= \langle \psi_k; B(U_n - U_k)\rangle + \lambda_k \langle \psi_k; U_k\rangle = \langle \psi_k; \mu_n U_n - \mu_k U_k\rangle + \lambda_k \langle \psi_k; U_k\rangle \\ &= \langle \psi_k; \mu_n U_n\rangle - \langle \psi_k; \mu_k U_k\rangle + \lambda_k \langle \psi_k; U_k\rangle = \langle \psi_k; \mu_n U_n\rangle \\ &= \lambda_n \langle \psi_k; U_n\rangle,\end{aligned}$$

whence the theorem follows. Theorem 2.1 is proved. □

A linear operator can has several sets of generalized eigenvalues. The union of two such sets may not be a set of generalized eigenvalues. Every set of eigenvalues is a set of generalized eigenvalues. If for some $b \in \mathcal{H}$ and each $j \in \Omega$, and the numbers μ_j, the equation $B(u) = \mu_j u + b$ has a nonzero solution $u_j \in \mathcal{D}(B)$, then the set $\mathfrak{M}(B) = \{\mu_j : j \in \Omega\}$ is a set of generalized eigenvalues of an operator $B : \mathcal{H} \to \mathcal{H}$. If $\mathcal{D}(B) = \mathcal{H}$ and the set $\mathfrak{M}(B) = \{\mu_j : j \in \Omega\}$ is a set of generalized eigenvalues of an operator $B : \mathcal{H} \to \mathcal{H}$, then there exists $b \in \mathcal{H}$ such that for every $k \in \Omega$ the equation $B(u) = \mu_k u + b$ has a nonzero solution $u_k \in \mathcal{H}$. In this case, $b = B(U_n) - \mu_n U_n$ and $n \in \Omega$ is arbitrary. If $\mathcal{D}(B) \neq \mathcal{H}$, then a linear operator $B : \mathcal{H} \to \mathcal{H}$ can has generalized eigenvectors of other kinds.

Definition 2.2. *Let $m \in \mathbb{N}_0$ and $\Omega = \mathbb{N} \backslash \overline{1;m}$. The set $\mathfrak{M}(B) = \{\mu_j : j \in \Omega\}$ of generalized eigenvalues of a linear operator $B : \mathcal{H} \to \mathcal{H}$ is called a set of generalized eigenvalues of width m (with respect to an operator $\widehat{B}$) if there exists a vector space $\widehat{\mathcal{H}}$ and a linear operator $\widehat{B} : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ with domain $\mathcal{D}(\widehat{B})$ that has a countable set of eigenvalues $\{\mu_k : k \in \mathbb{N}\}$ and a set of eigenvectors $\{\widehat{u}_k : k \in \mathbb{N}\}$ such that $\widehat{\mathcal{H}} \cap \mathcal{H} \neq \emptyset$, $U_n - U_k \in \mathcal{D}(\widehat{B})$, $B(U_n - U_k) = \widehat{B}(U_n - U_k)$ for any $n \in \Omega$ and $k \in \Omega$, and*

$$U_s := \widehat{u}_s + \sum_{i \in \overline{1;m}} \omega_{i,s} \widehat{u}_i \in \mathcal{H}, \quad \omega_{i,s} := (\mu_s - \mu_i)^{-1},\ s \in \Omega.$$

In this case, the set $\mathfrak{U}(B) = \{U_j : j \in \Omega\}$ is called a set of generalized eigenvectors of width m.

Theorem 2.2. *Assume that $m \in \mathbb{N}_0$, $\Omega = \mathbb{N}\backslash\overline{1;m}$, $B : \mathcal{H} \to \mathcal{H}$ be a linear operator in a vector space $\mathcal{H}$, and $\{U_j : j \in \Omega\}$ be some set of nonzero elements of the space $\mathcal{H}$. Let there exist a vector space $\widehat{\mathcal{H}}$ and a linear operator $\widehat{B} : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ with a countable set of eigenvalues $\{\mu_k : k \in \mathbb{N}\}$ and a set of eigenvectors $\{\widehat{u}_k : k \in \mathbb{N}\}$ satisfying $\widehat{\mathcal{H}} \cap \mathcal{H} \neq \emptyset$,*

$$U_s := \widehat{u}_s + \sum_{i \in \overline{1;m}} \frac{1}{\mu_s - \mu_i} \widehat{u}_i \in \mathcal{H}, \quad s \in \Omega,$$

and $U_n - U_k \in \mathcal{D}(\widehat{B})$, $B(U_n - U_k) = \widehat{B}(U_n - U_k)$ for every $n \in \Omega$ and $k \in \Omega$. Then $\mathfrak{M}(B) = \{\mu_j : j \in \Omega\}$ is a set of generalized eigenvalues of width m of an operator B, and $\mathfrak{U}(B) = \{U_j : j \in \Omega\}$ is a set of generalized eigenvectors of width m.

Proof. Indeed, we have

$$\begin{aligned}
B(U_n - U_k) &= \widehat{B}(U_n - U_k) \\
&= \mu_n \widehat{u}_n + \sum_{i \in \overline{1;m}} \frac{\mu_i}{\mu_n - \mu_i} \widehat{u}_i - \mu_k \widehat{u}_k - \sum_{i \in \overline{1;m}} \frac{\mu_i}{\mu_k - \mu_i} \widehat{u}_i \\
&= \mu_n \left(\widehat{u}_n + \sum_{i \in \overline{1;m}} \frac{1}{\mu_n - \mu_i} \widehat{u}_i \right) - \mu_k \left(\widehat{u}_k + \sum_{i \in \overline{1;m}} \frac{1}{\mu_k - \mu_i} \widehat{u}_i \right) \\
&+ \sum_{i \in \overline{1;m}} \frac{\mu_k - \mu_i}{\mu_k - \mu_i} \widehat{u}_i + \sum_{i \in \overline{1;m}} \frac{\mu_i - \mu_n}{\mu_n - \mu_i} \widehat{u}_i \\
&= \mu_n U_n - \mu_k U_k.
\end{aligned}$$

Theorem 2.2 is proved. □

Remark 2.1. *Due to Theorem 2.2, if U_k and U_n are the generalized eigenvectors of width m of an operator $B : \mathcal{H} \to \mathcal{H}$, then*

$$\sum_{i \in \overline{1;m}} \left((\omega_{i,n} - \omega_{i,k}) \mu_i \widehat{u}_i - (\omega_{i,n} \mu_n - \omega_{i,k} \mu_k) \widehat{u}_i \right) = 0$$

for every $k \in \Omega$ and $n \in \Omega$, because

$$\begin{aligned}
B(U_n - U_k) &= \widehat{B}(U_n - U_k) \\
&= \widehat{B} \left(\widehat{u}_n + \sum_{i \in \overline{1;m}} \omega_{i,n} \widehat{u}_i - \widehat{u}_k - \sum_{i \in \overline{1;m}} \omega_{i,k} \widehat{u}_i \right) \\
&= \mu_n \widehat{u}_n - \mu_k \widehat{u}_k + \sum_{i \in \overline{1;m}} (\omega_{i,n} - \omega_{i,k}) \mu_i \widehat{u}_i,
\end{aligned}$$

$$\mu_n U_n - \mu_k U_k = \mu_n \left(\widehat{u}_n + \sum_{i \in \overline{1;m}} \omega_{i,n} \widehat{u}_i \right) - \mu_k \left(\widehat{u}_k + \sum_{i \in \overline{1;m}} \omega_{i,k} \widehat{u}_i \right).$$

Theorems 2.1 and 2.2 indicate the method of finding a biorthogonal system that can be used in certain cases. In this paper, for illustrative purposes, we shall prove that there exists an operator $B_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$ in some Hilbert space $\mathcal{H}^\nu$ that has no eigenvalues, but has generalized eigenvalues and corresponding eigenvectors of width $m \in \{0; 1; 2\}$ (see Theorem 3.3). We also study the properties of this operator B_ν (see Theorems 4.4 and 5.5).

To prove Theorems 3.3, 4.4 and 5.5, we need some preliminaries.

3. Operator B_ν

Let $C(\Delta)$ be a vector space of continuous functions $f : \Delta \to \mathbb{C}$ on the interval $\Delta \subset \mathbb{C}$, and $C^{(k)}(\Delta)$ be a set of functions $f \in C(\Delta)$ with $f^{(k)} \in C(\Delta)$. Let $\alpha \in \mathbb{R}$ and $L^2((0;1); x^\alpha dx)$ be the space of measurable functions $f : (0;1) \to \mathbb{C}$ such that $t^{\alpha/2} f(t) \in L^2(0;1)$; the inner product and the norm in $L^2((0;1); x^\alpha dx)$ are given by $\langle f_1; f_2 \rangle = \int_0^1 t^\alpha f_1(t)\overline{f_2(t)}\, dt$ and $\|f\| = \sqrt{\int_0^1 t^\alpha |f(t)|^2\, dt}$, respectively. Let also

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}$$

be a Bessel function of the first kind of index $\nu \in \mathbb{R}$, where Γ is the gamma function. The function J_ν is a solution (see, for instance, [1, 17, 27]) of the equation $y'' + y'/x + (1 - \nu^2/x^2)y = 0$, the function $y(x) = J_\nu(xs)$ is a solution of the equation $-y'' - y'/x + y\nu^2/x^2 = s^2 y$, and the functions $y(x) = \sqrt{xs} J_{\pm\nu}(xs)$ satisfy the equation

$$-y'' + \frac{\nu^2 - 1/4}{x^2} y = s^2 y.$$

For $\nu > -1$, the function J_ν has (see [1, p. 59], [17, p. 350], [27, p. 483]) an infinite set $\{\widetilde{s}_k : k \in \mathbb{Z}\}$ of real zeros, among them $\widetilde{s}_k$, $k \in \mathbb{N}$, are the positive zeros and $\widetilde{s}_{-k} := -\widetilde{s}_k$, $k \in \mathbb{N}$, are the negative zeros. All zeros are simple except, perhaps, the zero $\widetilde{s}_0 = 0$. For $\nu > 1$, the function $J_{-\nu}$ has (see [1, p. 59], [27, p. 483]) an infinity of real zeros and also $2[\nu]$ pairwise conjugate complex zeros, among them two pure imaginary zeros when $[\nu]$ is an odd integer. Let s_k, $k \in \mathbb{N}$, be the zeros of the function $J_{-\nu}$ for which $\operatorname{Im} s_k > 0$ if $s_k \in \mathbb{C}$ or $s_k > 0$ if $s_k \in \mathbb{R}$.

Let $\nu = l + 1/2$ with $l \in \mathbb{N}$, $\mathcal{H}^\nu := L^2((0;1); x^{2\nu-1} dx)$ and B_ν is the operator generated by the formal differential operator

$$\ell_\nu^*(u) := -u'' - 2(2\nu - 1)\frac{1}{x}u' - 3((\nu-1)^2 - 1/4)\frac{1}{x^2}u$$

with domain $\mathcal{D}(B_\nu)$ consisting of all functions $u \in C^{(2)}(0;1]$ satisfying the boundary conditions

$$u(x) = O(x^{-\nu+5/2}), \quad x \to 0+, \tag{3.1}$$

$$u(1) = 0, \tag{3.2}$$

and the asymptotic equality (3.1) can be twice differentiated termwise. Then $\ell_\nu^*(u) = O(x^{-\nu+1/2})$ as $x \to 0+$, and $B_\nu(u) \in \mathcal{H}^\nu$ if $u \in \mathcal{D}(B_\nu)$. Let also $\widehat{\mathcal{H}} = C(0;1]$ and $\widehat{B}_\nu$ is the operator generated by the formal differential operator $\ell_\nu^*(u)$ with domain $\mathcal{D}(\widehat{B}_\nu)$ consisting of all functions $u \in C^{(2)}(0;1]$ satisfying the boundary condition (3.2). Then $\widehat{B}_\nu(u) \in \widehat{\mathcal{H}}$ if $u \in \mathcal{D}(\widehat{B}_\nu)$.

In this section, we shall prove the following theorem.

Theorem 3.3. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then the operator B_ν has no eigenvalues. In this case, $\widetilde{\mathfrak{M}}(B_\nu) = \{\widetilde{\mu}_k : k \in \mathbb{N}\}$, $\widetilde{\mu}_k = \widetilde{s}_k^2$, where $\widetilde{s}_k$ are the zeros of J_ν, is the set of generalized eigenvalues of width $m = 0$ of an operator B_ν that corresponds to the operator $\widehat{B}_\nu$, and*

$$\widetilde{U}_{k,\nu}(x) := \frac{\sqrt{x\widetilde{s}_k} J_\nu(x\widetilde{s}_k)}{\widetilde{s}_k^{\nu+1/2} x^{2\nu-1}}, \quad k \in \mathbb{N}$$

are the generalized eigenfunctions of width $m = 0$ of the operator B_ν. Besides, the set $\mathfrak{M}(B_{3/2}) = \{\mu_k : k \in \mathbb{N} \setminus \{1\}\}$, $\mu_k = s_k^2$, where s_k are the zeros of $J_{-\nu}$, is a set of generalized eigenvalues of width $m = 1$ of the operator $B_{3/2}$ which correspond to the operator $\widehat{B}_{3/2}$, and

$$U_{k,3/2}(x) := \frac{s_k\sqrt{xs_k}J_{-3/2}(xs_k) - s_1\sqrt{xs_1}J_{-3/2}(xs_1)}{x^2(s_1^2 - s_k^2)}, \quad k \in \mathbb{N} \setminus \{1\}$$

are the generalized eigenfunctions of width $m = 1$ of $B_{3/2}$. In addition, the set $\mathfrak{M}(B_{5/2}) = \{\mu_k : k \in \mathbb{N} \setminus \{1; 2\}\}$, $\mu_k = s_k^2$, is a set of generalized eigenvalues of width $m = 2$ of an operator $B_{5/2}$ that corresponds to the operator $\widehat{B}_{5/2}$, and

$$\begin{aligned} U_{k,5/2}(x) :=& \frac{s_k^2\sqrt{xs_k}J_{-5/2}(xs_k) - s_1^2\sqrt{xs_1}J_{-5/2}(xs_1)}{x^4(s_k^2 - s_1^2)} \\ &- \frac{s_k^2\sqrt{xs_k}J_{-5/2}(xs_k) - s_2^2\sqrt{xs_2}J_{-5/2}(xs_2)}{x^4(s_k^2 - s_2^2)}, \quad k \in \mathbb{N} \setminus \{1; 2\} \end{aligned}$$

are the generalized eigenfunctions of width $m = 2$ of the operator $B_{5/2}$.

To prove Theorem 3.3, we need some auxiliary lemmas.

Lemma 3.1. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then the operator B_ν has no eigenvalues.*

Proof. In fact, in the case $s = 0$, the functions $u_1(x) = x^{-\nu+3/2}$ and $u_2(x) = x^{-3\nu+3/2}$ are the linearly independent solutions of the equation $u'' + 2(2\nu - 1)x^{-1}u' + 3((\nu - 1)^2 - 1/4)x^{-2}u = -s^2u$. In the case $s \neq 0$, the linearly independent solutions of this equation are the functions $v_1(x) = x^{-2\nu+1}\sqrt{xs}J_\nu(xs)$ and $v_2(x) = x^{-2\nu+1}\sqrt{xs}J_{-\nu}(xs)$. Using relation (see [15, p. 226], [17, p. 346], [27, p. 43])

$$J_\nu(x) = \frac{x^\nu}{2^\nu\Gamma(\nu + 1)} + O(x^{\nu+2}), \quad x \to 0,$$

we obtain

$$\frac{\sqrt{xs}J_\nu(xs)}{x^{2\nu-1}} = \frac{s^{\nu+1/2}}{2^\nu\Gamma(\nu + 1)}x^{-\nu+3/2} + O(x^{-\nu+7/2}), \quad x \to 0+, \tag{3.3}$$

$$\frac{\sqrt{xs}J_{-\nu}(xs)}{x^{2\nu-1}} = \sum_{k\in\overline{0;\nu}} \frac{(-1)^k s^{-\nu+2k+1/2}x^{-3\nu+2k+3/2}}{2^{-\nu+2k}k!\Gamma(-\nu + k + 1)} + O(x^{-3\nu+2[\nu]+7/2}), \quad x \to 0+. \tag{3.4}$$

In view of this, every nonzero linear combination of these functions cannot satisfy (3.1), and hence this operator has no eigenfunctions. Lemma 3.1 is proved. □

Let $l \in \mathbb{N}$, $\nu = l + 1/2$, $\widehat{\mathcal{H}} = C(0; 1]$ and $\widehat{B}_\nu$ is the operator generated by the formal differential operator $\ell_\nu^*(u)$ with domain $\mathcal{D}(\widehat{B}_\nu)$ consisting of all functions $u \in C^{(2)}(0; 1]$ satisfying the boundary condition (3.2). Then $\widehat{B}_\nu(u) \in \widehat{\mathcal{H}}$ if $u \in \mathcal{D}(\widehat{B}_\nu)$.

Lemma 3.2. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then $\widetilde{\mathfrak{M}}(B_\nu) = \{\widetilde{\mu}_k : k \in \mathbb{N}\}$, $\widetilde{\mu}_k = \widetilde{s}_k^2$, where $\widetilde{s}_k$ are the zeros of J_ν, is the set of generalized eigenvalues of width $m = 0$ of an operator B_ν which correspond to the operator $\widehat{B}_\nu$, and $\widetilde{U}_{k,\nu}(x)$, $k \in \mathbb{N}$, are the generalized eigenfunctions of width $m = 0$ of B_ν.*

Proof. Indeed, the numbers $\widetilde{\mu}_k = \widetilde{s}_k^2$ are the eigenvalues of the operator $\widehat{B}_\nu$, and $\widehat{u}_{k,\nu}(x) = \widetilde{U}_{k,\nu}(x) = \widetilde{s}_k^{-\nu-1/2}x^{-2\nu+1}\sqrt{x\widetilde{s}_k}J_\nu(x\widetilde{s}_k)$ are the eigenfunctions of this operator. Further, $\widetilde{U}_{k,\nu} \in \mathcal{H}^\nu$, by using (3.3)

$$\widetilde{U}_{k,\nu}(x) = \frac{1}{2^\nu\Gamma(\nu + 1)}x^{-\nu+3/2} + O(x^{-\nu+7/2}), \quad x \to 0+.$$

Besides,

$$\widetilde{U}_{k,\nu}(x) - \widetilde{U}_{n,\nu}(x) = O(x^{-\nu+7/2}) = O(x^{-\nu+5/2}), \quad x \to 0+.$$

Therefore, $\widetilde{U}_{k,\nu} - \widetilde{U}_{n,\nu} \in \mathcal{D}(B_\nu)$ and $\widetilde{U}_{k,\nu} - \widetilde{U}_{n,\nu} \in \mathcal{D}(\widehat{B}_\nu)$. In addition,

$$B_\nu(\widetilde{U}_{k,\nu} - \widetilde{U}_{n,\nu}) = \widehat{B}_\nu(\widetilde{U}_{k,\nu} - \widetilde{U}_{n,\nu}) = \ell_\nu^*(\widehat{u}_{k,\nu} - \widehat{u}_{n,\nu}) = \widetilde{s}_k^2\widehat{u}_{k,\nu} - \widetilde{s}_n^2\widehat{u}_{n,\nu} = \widetilde{s}_k^2\widetilde{U}_{k,\nu} - \widetilde{s}_n^2\widetilde{U}_{n,\nu}.$$

Hence, $\widetilde{\mathfrak{M}}(B_\nu) = \{\widetilde{\mu}_k : k \in \mathbb{N}\}$ is a set of generalized eigenvalues of width $m = 0$ of the operator B_ν, and $\widetilde{\mathfrak{U}}(B_\nu) = \{\widetilde{U}_{k,\nu} : k \in \mathbb{N}\}$ is the set of generalized eigenfunctions of width $m = 0$. Lemma 3.2 is proved. □

Lemma 3.3. *Let s_k, $k \in \mathbb{N}$ be the zeros of the function $J_{-\nu}$. Then $\mathfrak{M}(B_{3/2}) = \{\mu_k : k \in \mathbb{N} \setminus \{1\}\}$, $\mu_k = s_k^2$, is a set of generalized eigenvalues of width $m = 1$ of the operator $B_{3/2}$ which correspond to the operator $\widehat{B}_{3/2}$, and $U_{k,3/2}(x)$, $k \in \mathbb{N} \setminus \{1\}$, are the generalized eigenfunctions of width $m = 1$ of $B_{3/2}$.*

Proof. Indeed, the numbers $\mu_k = s_k^2$ are the eigenvalues of the operator $\widehat{B}_{3/2}$, and the functions $\widehat{u}_{k,3/2}(x) = x^{-2}(s_1^2 - s_k^2)^{-1}s_k\sqrt{xs_k}J_{-3/2}(xs_k)$, $k \neq 1$, and $\widehat{u}_{1,3/2}(x) = x^{-2}s_1\sqrt{xs_1}J_{-3/2}(xs_1)$ are their corresponding eigenfunctions. Moreover, $U_{k,3/2}(x) = \widehat{u}_{k,3/2}(x) + \omega_{1,k}\widehat{u}_{1,3/2}(x)$ if $\omega_{1,k} = (s_k^2 - s_1^2)^{-1}$. Using (3.4), we obtain

$$U_{k,3/2}(x) = \frac{1}{\sqrt{2\pi x}} + O(x), \quad x \to 0+.$$

Therefore, $U_{k,3/2} \in \mathcal{H}^{3/2}$. Besides, $U_{k,3/2}(x) - U_{n,3/2}(x) = O(x)$ as $x \to 0+$. Hence, $U_{k,3/2} - U_{n,3/2} \in \mathcal{D}(B_{3/2})$, $U_{k,3/2} - U_{n,3/2} \in \mathcal{D}(\widehat{B}_{3/2})$, and

$$\begin{aligned}
B_{3/2}(U_{k,3/2} - U_{n,3/2}) &= \widehat{B}_{3/2}(U_{k,3/2} - U_{n,3/2}) \\
&= \ell_{3/2}^*(U_{k,3/2} - U_{n,3/2}) \\
&= \ell_{3/2}^*(\widehat{u}_{k,3/2} + \omega_{1,k}\widehat{u}_{1,3/2} - \widehat{u}_{n,3/2} - \omega_{1,n}\widehat{u}_{1,3/2}) \\
&= s_k^2\widehat{u}_{k,3/2} + \omega_{1,k}s_1^2\widehat{u}_{1,3/2} - s_n^2\widehat{u}_{n,3/2} - \omega_{1,n}s_1^2\widehat{u}_{1,3/2} \\
&= s_k^2(\widehat{u}_{k,3/2} + \omega_{1,k}\widehat{u}_{1,3/2}) - s_n^2(\widehat{u}_{n,3/2} + \omega_{1,n}\widehat{u}_{1,3/2}) \\
&+ (s_1^2(\omega_{1,k} - \omega_{1,n}) - (\omega_{1,k}s_k^2 - \omega_{1,n}s_n^2))\widehat{u}_{1,3/2} \\
&= s_k^2U_{k,3/2} - s_n^2U_{n,3/2}.
\end{aligned}$$

Thus, $\mathfrak{M}(B_{3/2}) = \{\mu_k : k \in \mathbb{N} \setminus \{1\}\}$ is the set of generalized eigenvalues of the operator $B_{3/2}$, and $\mathfrak{U}(B_{3/2}) = \{U_{k,3/2} : k \in \mathbb{N} \setminus \{1\}\}$ is a set of generalized eigenfunctions of width $m = 1$. Lemma 3.3 is proved. □

Lemma 3.4. *Let s_k, $k \in \mathbb{N}$, be the zeros of the function $J_{-\nu}$. Then $\mathfrak{M}(B_{5/2}) = \{\mu_k : k \in \mathbb{N} \setminus \{1; 2\}\}$, $\mu_k = s_k^2$, is a set of generalized eigenvalues of width $m = 2$ of an operator $B_{5/2}$ which corresponds to the operator $\widehat{B}_{5/2}$, and $U_{k,5/2}(x)$, $k \in \mathbb{N} \setminus \{1; 2\}$, are the generalized eigenfunctions of width $m = 2$ of $B_{5/2}$.*

Proof. In fact, the numbers $\mu_k = s_k^2$ are the eigenvalues of the operator $\widehat{B}_{5/2}$, and the functions

$$\widehat{u}_{k,5/2}(x) = \frac{s_k^2(s_1^2 - s_2^2)\sqrt{xs_k}J_{-5/2}(xs_k)}{x^4(s_k^2 - s_1^2)(s_k^2 - s_2^2)}, \quad k \in \mathbb{N} \setminus \{1; 2\},$$

$\widehat{u}_{1,5/2}(x) = -x^{-4}s_1^2\sqrt{xs_1}J_{-5/2}(xs_1)$ and $\widehat{u}_{2,5/2}(x) = x^{-4}s_2^2\sqrt{xs_2}J_{-5/2}(xs_2)$ are their corresponding eigenfunctions. Moreover, $U_{k,5/2}(x) = \widehat{u}_{k,5/2}(x) + \omega_{1,k}\widehat{u}_{1,5/2}(x) + \omega_{2,k}\widehat{u}_{2,5/2}(x)$ if

$\omega_{i,k} = (s_k^2 - s_i^2)^{-1}$, $i \in \{1;2\}$. Using (3.4), we get

$$U_{k,5/2}(x) = \frac{s_1^2 - s_2^2}{4\sqrt{2\pi}x^2} + O(1), \quad x \to 0+.$$

Therefore, $U_k \in \mathcal{H}^{5/2}$. Furthermore, $U_{k,5/2}(x) - U_{n,5/2}(x) = O(1)$ as $x \to 0+$. Hence, $U_{k,5/2} - U_{n,5/2} \in \mathcal{D}(B_{5/2})$, $U_{k,5/2} - U_{n,5/2} \in \mathcal{D}(\widehat{B}_{5/2})$ and

$$\begin{aligned}
B_{5/2}(U_{k,5/2} - U_{n,5/2}) &= \widehat{B}_{5/2}(U_{k,5/2} - U_{n,5/2}) \\
&= \ell^*_{5/2}(U_{k,5/2} - U_{n,5/2}) \\
&= \ell^*_{5/2}(\widehat{u}_{k,5/2} + \omega_{1,k}\widehat{u}_{1,5/2} + \omega_{2,k}\widehat{u}_{2,5/2} - \widehat{u}_{n,5/2} - \omega_{1,n}\widehat{u}_{1,5/2} - \omega_{2,n}\widehat{u}_{2,5/2}) \\
&= s_k^2\widehat{u}_{k,5/2} + \omega_{1,k}s_1^2\widehat{u}_{1,5/2} + \omega_{2,k}s_2^2\widehat{u}_{2,5/2} - s_n^2\widehat{u}_{n,5/2} - \omega_{1,n}s_1^2\widehat{u}_{1,5/2} - \omega_{2,n}s_2^2\widehat{u}_{2,5/2} \\
&= s_k^2(\widehat{u}_{k,5/2} + \omega_{1,k}\widehat{u}_{1,5/2} + \omega_{2,k}\widehat{u}_{2,5/2}) - s_n^2(\widehat{u}_{n,5/2} + \omega_{1,n}\widehat{u}_{1,5/2} + \omega_{2,n}\widehat{u}_{2,5/2}) \\
&+ (s_1^2(\omega_{1,k} - \omega_{1,n}) - (\omega_{1,k}s_k^2 - \omega_{1,n}s_n^2))\widehat{u}_{1,5/2} + (s_2^2(\omega_{2,k} - \omega_{2,n}) \\
&- (\omega_{2,k}s_k^2 - \omega_{2,n}s_n^2))\widehat{u}_{2,5/2} \\
&= s_k^2U_{k,5/2} - s_n^2U_{n,5/2}.
\end{aligned}$$

Thus, $\mathfrak{M}(B_{5/2})$ is the set of generalized eigenvalues of an operator $B_{5/2}$, and $U_{k,5/2}$ are the generalized eigenfunctions of width $m = 2$. Lemma 3.4 is proved. □

Remark 3.2. *$U_{k,\nu} - \widetilde{U}_{n,\nu} \notin \mathcal{D}(B_\nu)$ if $\nu = 3/2$ or $\nu = 5/2$. Lemmas 3.2–3.4 are leaving aside the existence of other sets of generalized eigenvalues. We have not been able to extend Lemma 3.4 to an arbitrary $\nu = l + 1/2$ with $l \in \mathbb{N}$.*

Theorem 3.3 is an immediate consequence of Lemmas 3.1–3.4.

4. Operator $\widetilde{A}_\nu$ and Approximation Properties of the System $(\widetilde{U}_k : k \in \mathbb{N})$

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}^*$ its dual space, i.e., the space of linear continuous functionals on $\mathcal{H}$. The system of elements $(e_k : k \in \mathbb{N})$ is called *complete* ([11, p. 4258]) in $\mathcal{H}$ if $\overline{\operatorname{span}}\,(e_k : k \in \mathbb{N}) = \mathcal{H}$. The system of elements $(e_k : k \in \mathbb{N})$ is said to be *minimal* ([11, p. 4258]) in $\mathcal{H}$ if $e_{k_0} \notin \overline{\operatorname{span}}\,(e_k : k \in \mathbb{N} \setminus \{k_0\})$ for each $k_0 \in \mathbb{N}$. The system $(e_k : k \in \mathbb{N})$ is called ([11, p. 4258]) a *basis* for the space $\mathcal{H}$ if, for every $f \in \mathcal{H}$, there exists a unique series with respect to the system $(e_k : k \in \mathbb{N})$ which converges to f (in $\mathcal{H}$): $f = \sum_{k=1}^{\infty} d_k e_k$, $d_k \in \mathbb{C}$. Minimality of the system $(e_k : k \in \mathbb{N})$ in $\mathcal{H}$ is equivalent (see [11, p. 4258]) to the existence of the system of conjugate functionals $(f_k : k \in \mathbb{N}) \in \mathcal{H}^*$, i.e., $f_k(e_n) = \delta_{kn}$, where δ_{kn} is the Kronecker delta. The system $(f_k : k \in \mathbb{N})$ is also called a *biorthogonal system* with respect to the system $(e_k : k \in \mathbb{N})$. A system $(e_k : k \in \mathbb{N})$ is said to be *uniformly minimal* ([11, p. 4258]) in $\mathcal{H}$ if there exists $\delta > 0$ such that for every $n \in \mathbb{N}$ the distance of e_n to the closure of the linear span of the system $(e_k : k \in \mathbb{N} \setminus \{n\})$ is greater than $\delta\|u_n\|$. A complete system $(e_k : k \in \mathbb{N})$ that has a biorthogonal system $(f_k : k \in \mathbb{N})$ is uniformly minimal if and only if (see [11, p. 4258])

$$\limsup_{k\to\infty} \|e_k\|^2\|f_k\|^2 < +\infty.$$

Every basis is uniformly minimal system (see [11, p. 4258]).

Let $\nu = l + 1/2$ with $l \in \mathbb{N}$, $\mathcal{H}^\nu = L^2((0;1); x^{2\nu-1}dx)$, and $\widetilde{A}_\nu$ is the operator generated by the formal differential operator $\ell_\nu(\psi) := -\psi'' + (\nu^2 - 1/4)x^{-2}\psi$ with domain $\mathcal{D}(\widetilde{A}_\nu)$ consisting of those functions $\psi \in C^{(2)}[0;1]$ which satisfy the boundary conditions $\psi(0) = \psi(1) = 0$. Then $\ell_\nu(\psi) = O(x^{-1})$ as $x \to 0+$, and $\widetilde{A}_\nu(\psi) \in \mathcal{H}^\nu$ if $\psi \in \mathcal{D}(\widetilde{A}_\nu)$.

In this section, we prove that the operator $B_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$ is formally adjoint in H^ν of an operator $\widetilde{A}_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$. We also investigate completeness, minimality and basicity of the system $(\widetilde{U}_k : k \in \mathbb{N})$ of generalized eigenfunctions of width $m = 0$ of a operator B_ν.

Lemma 4.5. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then, the operator B_ν is formally adjoint in H^ν of an operator $\widetilde{A}_\nu$.*

Proof. Let $a_0 = (\nu^2 - 1/4)x^{2\nu-3}$, $a_2 = -x^{2\nu-1}$ and $\widetilde{\ell}_\nu(\psi) = a_2\psi'' + a_0\psi$. Then $\widetilde{\ell}^*_\nu(u) = (\overline{a}_2 u)'' + \overline{a}_0 u$ is formally adjoint operator of the operator $\widetilde{\ell}_\nu(\psi)$ (see [9, p. 97]). Moreover, $\widetilde{\ell}_\nu(\psi) = -x^{2\nu-1}\psi'' + (\nu^2 - 1/4)x^{2\nu-3}\psi = x^{2\nu-1}\ell_\nu(\psi)$ and $\widetilde{\ell}^*_\nu(u) = \overline{a}_2 u'' + 2\overline{a}'_2 u' + (\overline{a}_0 + \overline{a}''_2)u = -x^{2\nu-1}u'' - 2(2\nu-1)x^{2\nu-2}u' - 3((\nu-1)^2 - 1/4)x^{2\nu-3}u = x^{2\nu-1}\ell^*_\nu(u)$. Furthermore, according to the Lagrange identity (see [9, p. 97]), for every $\psi \in \mathcal{D}(\widetilde{A}_\nu)$ and $u \in \mathcal{D}(B_\nu)$,

$$
\begin{aligned}
x^{2\nu-1}(\ell_\nu(\psi)\overline{u} - \psi\overline{\ell^*_\nu(u)}) &= \widetilde{\ell}_\nu(\psi)\overline{u} - \psi\overline{\widetilde{\ell}^*_\nu(u)} \\
&= \frac{d}{dx}((a_2\psi' - \psi a'_2)\overline{u} - \psi a_2\overline{u}') \\
&= \frac{d}{dx}((-x\psi' + (2\nu-1)\psi)x^{2\nu-2}\overline{u} + x^{2\nu-1}\psi\overline{u}').
\end{aligned} \tag{4.5}
$$

Hence,

$$
\int_0^1 x^{2\nu-1}\ell_\nu(\psi)\overline{u}\,dx = \int_0^1 x^{2\nu-1}\psi\overline{\ell^*_\nu(u)}\,dx.
$$

Lemma 4.5 is proved. □

Lemma 4.6 ([21, 6, 7])**.** *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then the system $(\widetilde{U}_{k,\nu} : k \in \mathbb{N})$, $\widetilde{U}_{k,\nu}(x) = \widetilde{s}_k^{-\nu-1/2}x^{-2\nu+1}\sqrt{x\widetilde{s}_k}J_\nu(x\widetilde{s}_k)$ is complete in the space $\widetilde{\mathcal{H}}^\nu := L^2((0;1); x^{4\nu-4}dx)$.*

Lemma 4.7. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then, the system $(\widetilde{U}_{k,\nu} : k \in \mathbb{N})$ in the space $\mathcal{H}^\nu$ has a biorthogonal system $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ that is formed by the functions*

$$
\widetilde{\gamma}_{k,\nu}(x) := \frac{2\widetilde{s}_k^{\nu-1/2}}{J^2_{\nu+1}(\widetilde{s}_k)}\sqrt{x\widetilde{s}_k}J_\nu(x\widetilde{s}_k).
$$

The system $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ is a system of eigenfunctions of an operator $\widetilde{A}_\nu$ which correspond to their eigenvalues $\widetilde{\mu}_k = \widetilde{s}_k^2$, where $\widetilde{s}_k$ are the zeros of J_ν.

Proof. Since (see [17, p. 347], [27, p. 482])

$$
\int_0^1 xJ_\nu(x\widetilde{s}_k)J_\nu(x\widetilde{s}_n)\,dx = \begin{cases} \frac{1}{2}J^2_{\nu+1}(\widetilde{s}_n), & k = n, \\ 0, & k \neq n, \end{cases}
$$

it follows that

$$
\begin{aligned}
\int_0^1 x^{2\nu-1}\widetilde{U}_{k,\nu}(x)\overline{\widetilde{\gamma}_{n,\nu}(x)}\,dx &= \frac{2\sqrt{\widetilde{s}_k\widetilde{s}_n}\,\widetilde{s}_n^{\nu-1/2}}{\widetilde{s}_k^{\nu+1/2}J^2_{\nu+1}(\widetilde{s}_n)}\int_0^1 xJ_\nu(x\widetilde{s}_k)J_\nu(x\widetilde{s}_n)\,dx \\
&= \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}
\end{aligned}
$$

Furthermore, since $\ell_\nu(\widetilde{\gamma}_{k,\nu}) = \widetilde{s}_k^2 \widetilde{\gamma}_{k,\nu}$ and $J_\nu(x) = O(x^\nu)$ as $x \to 0$, we conclude that the numbers $\widetilde{\mu}_k = \widetilde{s}_k^2$ are the eigenvalues of an operator $\widetilde{A}_\nu$, and $\widetilde{\gamma}_{k,\nu}(x)$, $k \in \mathbb{N}$ are the corresponding eigenfunctions of this operator. Lemma 4.7 is proved. □

Lemma 4.8. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then, the system $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ is complete in the space $\mathcal{H}^\nu$.*

Proof. Assume the contrary. Then, according to the Hahn-Banach theorem ([11, p. 4258]), there exists a nonzero function $h \in \mathcal{H}^\nu$ such that

$$\frac{2\widetilde{s}_k^{\nu-1/2}}{J_{\nu+1}^2(\widetilde{s}_k)} \int_0^1 x^{2\nu-1} \sqrt{x\widetilde{s}_k} J_\nu(x\widetilde{s}_k) h(x)\, dx = 0, \quad k \in \mathbb{N}.$$

Let $q(x) = x^{2\nu-1}h(x)$. Then $q \in L^2(0;1)$ and, therefore, the system $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ is incomplete in the space $L^2(0;1)$. We have a contradiction, because it is well known that the system $(\sqrt{x}J_\nu(x\widetilde{s}_k) : k \in \mathbb{N})$ is complete in $L^2(0;1)$ (see [15, p. 223], [17, p. 357]). Thus, the system $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ is complete in $\mathcal{H}^\nu$. Lemma 4.8 is proved. □

We remark that Lemma 4.7 also follows from Lemmas 4.5, 4.8, 3.2 and Theorem 2.1.

Lemma 4.9. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then, the system $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ is not a basis in the space $\mathcal{H}^\nu$.*

Proof. Using relations (see [15, p. 226], [17, pp. 346, 352], [27, pp. 43, 618])

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O\left(x^{-3/2}\right), \quad x \to \infty,$$

$$J_\nu(x) = O(x^\nu), x \to 0, \widetilde{s}_k = \pi k + \frac{\pi\nu}{2} - \frac{\pi}{4} + O(k^{-1})$$

and

$$|\sqrt{\widetilde{s}_k} J_{\nu+1}(\widetilde{s}_k)| = \sqrt{2/\pi}(1 + O(k^{-1})) \text{ as } k \to \infty,$$

we get

$$\begin{aligned} \|\widetilde{U}_{k,\nu}\|_{\mathcal{H}^\nu}^2 \|\widetilde{\gamma}_{k,\nu}\|_{\mathcal{H}^\nu}^2 &= \frac{4}{J_{\nu+1}^4(\widetilde{s}_k)} \int_0^1 x|J_\nu(x\widetilde{s}_k)|^2\, dx \int_0^1 x^{2\nu}|J_\nu(x\widetilde{s}_k)|^2\, dx \\ &= \frac{O(\widetilde{s}_k^{4\nu})}{J_{\nu+1}^4(\widetilde{s}_k)} \\ &= O(\widetilde{s}_k^{4\nu+2}) \longrightarrow +\infty, \quad k \to \infty. \end{aligned}$$

Hence, the system $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ is not uniformly minimal in the space $\mathcal{H}^\nu$ and therefore is not a basis in this space. □

From Lemmas 4.5–4.9, we obtain the following assertion.

Theorem 4.4. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then, the system $(\widetilde{U}_{k,\nu} : k \in \mathbb{N})$ of the generalized eigenfunctions of width $m = 0$ of an operator B_ν is complete in the space $\widetilde{\mathcal{H}}^\nu$ and minimal in $\mathcal{H}^\nu$. Moreover, the operator B_ν is formally adjoint in $\mathcal{H}^\nu$ of an operator $\widetilde{A}_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$ which has a complete and minimal system of eigenfunctions $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ such that is not a basis in $\mathcal{H}^\nu$.*

Remark 4.3. *Basis properties (completeness, minimality, basicity) of more general systems $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ with $\Theta_{k,\nu,p}(x) = x^{-p-1}\sqrt{x\widetilde{s}_k} J_\nu(x\widetilde{s}_k)$ in the space $L^2((0;1); x^{2p}dx)$, where $\nu \geq 1/2$, $p \in \mathbb{R}$ and $(\widetilde{s}_k)_{k\in\mathbb{N}}$ is a sequence of distinct nonzero complex numbers, have been studied in* [6, 7, 20, 21, 22, 23, 24].

5. OPERATOR A_ν

Let $\nu = l + 1/2$ with $l \in \mathbb{N}$, $\mathcal{H}^\nu = L^2((0;1); x^{2\nu-1}dx)$, and A_ν is the operator generated by the formal differential operator $\ell_\nu(\psi)$ and the boundary conditions

$$\psi(1) = 0, \tag{5.6}$$

$$\psi(x) = \sum_{j\in\overline{0;\nu}} c_j x^{-\nu+2j+1/2} + o(x^{\nu+1/2}), \quad x \to 0+ \tag{5.7}$$

for some constants $c_j \in \mathbb{C}$, $j \in \overline{0;\nu}$. Suppose that the domain $\mathcal{D}(A_\nu)$ consists of those functions $\psi \in C^{(2)}(0;1]$ that satisfy these boundary conditions and the asymptotic equality (5.7) can be twice differentiated termwise. Then $\ell_\nu(\psi) = 4c_1(-1+\nu)x^{-\nu+1/2} + o(x^{-\nu+1/2}) + o(x^{\nu-3/2}) = O(x^{-\nu+1/2})$ as $x \to 0+$, and $A_\nu(\psi) \in \mathcal{H}^\nu$ if $\psi \in \mathcal{D}(A_\nu)$.

In this section, we show that the operator $B_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$ is formally adjoint in $\mathcal{H}^\nu$ of an operator $A_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$ whose systems of canonical eigenfunctions are over-complete. We also remark about basis properties of the systems of generalized eigenfunctions of width $m \in \{1;2\}$ of an operator B_ν.

Lemma 5.10 ([26]). *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. The operator A_ν has a finite set $\{\mu_k : k \in \mathbb{N}\}$ of eigenvalues, where $\mu_k = s_k^2$ and s_k are the zeros of the function $J_{-\nu}$. Moreover, the functions $\psi_{k,\nu}(x) := s_k^{\nu-1/2}\sqrt{xs_k}J_{-\nu}(xs_k)$, $k \in \mathbb{N}$, are the eigenfunctions of this operator.*

Lemma 5.11 ([13]). *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then, the system $(\psi_{k,\nu} : k \in \mathbb{N} \setminus \{1;2;\ldots;l\})$ is complete in $\mathcal{H}^\nu$.*

Lemma 5.12. *Let $l \in \mathbb{N}$ and $\nu = l+1/2$. The operator B_ν is formally adjoint in $\mathcal{H}^\nu$ of an operator A_ν.*

Proof. Using relations (3.1), (3.2), (5.6), (5.7) and

$$\psi'(x) = \sum_{j\in\overline{0;\nu}} c_j(-\nu+2j+1/2)x^{-\nu+2j-1/2} + o(x^{\nu-1/2}), \quad x \to 0+,$$

from (4.5), it follows that

$$\int_0^1 x^{2\nu-1}\ell_\nu(\psi)\overline{u}\,dx = \int_0^1 x^{2\nu-1}\psi\overline{\ell_\nu^*(u)}\,dx.$$

Lemma 5.12 is proved. □

Remark 5.4. *From Lemmas 3.3, 5.10, 5.12 and Theorem 2.1, it follows that $\langle\psi_{k,3/2}; U_{n,3/2}\rangle = 0$, if $k \neq n$, $k \in \mathbb{N}\setminus\{1\}$ and $n \in \mathbb{N}\setminus\{1\}$. By direct calculations, we get $\langle\psi_{n,3/2}; U_{n,3/2}\rangle = 1$ (see also [18, 19, 25]). Lemma 5.11 implies that the system $(\psi_{k,3/2} : k \in \mathbb{N} \setminus \{1\})$ is complete in the space $\mathcal{H}^{3/2}$. Moreover, in [25, 26] the authors proved that this system is minimal and is not a basis in $\mathcal{H}^{3/2}$. Furthermore, the biorthogonal system is formed by the functions $g_{k,3/2}(x) = \pi s_k^{-4}(1+s_k^2)(s_1^2 - s_k^2)U_{k,3/2}(x)$. In [19], it was shown that the system $(g_{k,3/2} : k \in \mathbb{N} \setminus \{1\})$ is also complete in $\mathcal{H}^{3/2}$. In addition, in [19] it has been established that the system $(U_{k,3/2} : k \in \mathbb{N} \setminus \{1\})$ has in the space $\mathcal{H}^{3/2}$ a biorthogonal system $(\gamma_{k,3/2} : k \in \mathbb{N} \setminus \{1\})$ that is formed by the functions $\gamma_{k,3/2}(x) = \pi s_k^{-4}(1+s_k^2)(s_1^2 - s_k^2)\psi_{k,3/2}(x)$.*

Since $s_k = \pi k - \frac{1}{\pi k} + o(k^{-3})$ *as* $k \to \infty$ *(see [1, 27]), and*

$$\begin{aligned}
&\|U_{k,3/2}\|^2_{\mathcal{H}^{3/2}}\|\gamma_{k,3/2}\|^2_{\mathcal{H}^{3/2}} \\
=&\frac{\pi^2(1+s_k^2)^2}{s_k^9}\int_0^{s_k}|t\sqrt{t}J_{-3/2}(t)|^2\,dt\int_0^1\frac{|s_k\sqrt{ts_k}J_{-3/2}(ts_k)-s_1\sqrt{ts_1}J_{-3/2}(ts_1)|^2}{t^2}\,dt \\
=&\frac{\pi(1+s_k^2)^2}{9s_k^3}(1+o(1))\longrightarrow+\infty,\quad k\to\infty,
\end{aligned}$$

the system $(U_{k,3/2} : k \in \mathbb{N}\setminus\{1\})$ *is not uniformly minimal in* $\mathcal{H}^{3/2}$ *and, hence, is not a basis in this space. Lemma 5.11 implies that the system* $(\psi_{k,5/2} : k \in \mathbb{N}\setminus\{1;2\})$ *is complete in the space* $\mathcal{H}^{5/2}$*. From Lemmas 3.4, 5.10, 5.12 and Theorem 2.1, it follows that* $\langle\psi_{k,5/2}; U_{n,5/2}\rangle = 0$ *if* $k \neq n$*,* $k \in \mathbb{N}\setminus\{1;2\}$ *and* $n \in \mathbb{N}\setminus\{1;2\}$*. In [12], it was proven by some other method that the system* $(\psi_{k,5/2} : k \in \mathbb{N})$ *has in* $\mathcal{H}^{5/2}$ *a biorthogonal system* $(U_{k,5/2} : k \in \mathbb{N}\setminus\{1;2\})$*. However, the problem of finding a biorthogonal system* $(U_{k,\nu} : k \in \mathbb{N}\setminus\{1;2;\ldots;l\})$ *to the system* $(\psi_{k,\nu} : k \in \mathbb{N}\setminus\{1;2;\ldots;l\})$ *for an arbitrary* $\nu = l+1/2$ *with* $l \in \mathbb{N}\setminus\{1;2\}$ *remains open.*

From Lemmas 5.10–5.12 and Remark 5.4, we obtain the following statement.

Theorem 5.5. *Let* $l \in \mathbb{N}$ *and* $\nu = l+1/2$*. Then the system* $(U_{k,3/2} : k \in \mathbb{N}\setminus\{1\})$ *of the generalized eigenfunctions of width* $m=1$ *of an operator* $B_{3/2}$ *is complete, minimal and is not a basis in the space* $\mathcal{H}^{3/2}$*. The biorthogonal system* $(\gamma_{k,3/2} : k \in \mathbb{N}\setminus\{1\})$ *also is complete in* $\mathcal{H}^{3/2}$*. Furthermore, the system* $(U_{k,5/2} : k \in \mathbb{N}\setminus\{1;2\})$ *of the generalized eigenfunctions of width* $m=2$ *of an operator* $B_{5/2}$ *is minimal in the space* $\mathcal{H}^{5/2}$*, and its biorthogonal system* $(\psi_{k,5/2} : k \in \mathbb{N}\setminus\{1;2\})$ *is complete in* $\mathcal{H}^{5/2}$*. Moreover, the operator* B_ν *is also formally adjoint in* $\mathcal{H}^\nu$ *of an operator* $A_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$ *whose system of eigenfunctions* $(\psi_{k,\nu} : k \in \mathbb{N})$ *is complete after removing a finite number of eigenfunctions, i.e., the system* $(\psi_{k,\nu} : k \in \mathbb{N}\setminus\{1;2;\ldots;l\})$ *is complete in* $\mathcal{H}^\nu$*.*

Remark 5.5. *Let* $f \in \mathcal{H}^{3/2}$ *and* $d_k = \int_0^1 t^2 f(t)\overline{g_{k,3/2}(t)}\,dt$*. Since the system* $(g_{k,3/2} : k \in \mathbb{N}\setminus\{1\})$ *is complete in the space* $\mathcal{H}^{3/2}$*, the numbers* d_k *determine the function* $f \in \mathcal{H}^{3/2}$ *uniquely. But, the series*

$$\sum_{k=2}^{\infty} d_k\psi_{k,3/2}(x),\quad \psi_{k,3/2}(x) = s_k\sqrt{xs_k}J_{-3/2}(xs_k)$$

does not converge for each function $f \in \mathcal{H}^{3/2}$ *in* $\mathcal{H}^{3/2}$ *to the function* f*. We do not know whether it converges in some sense, for example, whether a given series converges in* $\mathcal{H}^{3/2}$ *to* f *in the sense of Cesàro. Similar questions arise for the other series that can be constructed by using the above considered biorthogonal systems.*

6. Concluding remarks

In this paper, the notions of a set of generalized eigenvalues and a set of generalized eigenvectors of a linear operator in an Euclidean space are introduced. A method is described to find a biorthogonal system of a subsystem of eigenvectors of linear operators in a Hilbert space whose systems of canonical eigenvectors are over-complete. This is illustrated by an example of a linear differential operator that is formally adjoint to Bessel-type differential operators. Also, basic properties of the systems of generalized eigenvectors of those differential operators are studied. Those results can be used for the investigations in spectral theory and nonharmonic analysis.

Remark that there are other points of view on how to study similar problems (see, for example, [2, 3, 4, 5, 9, 10, 16] and the bibliography in them).

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Research Article

Approximating sums by integrals only: multiple sums and sums over lattice polytopes

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ABSTRACT. The Euler–Maclaurin (EM) summation formula is used in many theoretical studies and numerical calculations. It approximates the sum $\sum_{k=0}^{n-1} f(k)$ of values of a function f by a linear combination of a corresponding integral of f and values of its higher-order derivatives $f^{(j)}$. An alternative (Alt) summation formula was presented by the author, which approximates the sum by a linear combination of integrals only, without using derivatives of f. It was shown that the Alt formula will in most cases outperform the EM formula. In the present paper, a multiple-sum/multi-index-sum extension of the Alt formula is given, with applications to summing possibly divergent multi-index series and to sums over the integral points of integral lattice polytopes.

Keywords: Euler–Maclaurin summation formula, alternative summation formula, multiple sums, multi-index series, approximation, lattice polytopes.

2020 Mathematics Subject Classification: 41A35, 52B20, 26D15.

1. INTRODUCTION

The Euler–Maclaurin (EM) summation formula can be written as follows (see e.g. [16]):

$$\sum_{k=0}^{n-1} f(k) \approx \int_0^{n-1} \mathrm{d}x\, f(x) + \frac{f(n-1)+f(0)}{2} + \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!}[f^{(2j-1)}(n-1) - f^{(2j-1)}(0)], \tag{1.1}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth enough function, B_j is the j-th Bernoulli number, and n and m are natural numbers. The EM approximation is exact when f is a polynomial of degree $< 2m+1$.

The EM formula has been used in a large number of theoretical studies and numerical calculations.

Clearly, to use the EM formula in a theoretical or computational study, one will usually need to have an antiderivative F of f and the derivatives $f^{(2j-1)}$ for $j = 1, \dots, m$ in tractable or, respectively, computable form.

In [19], an alternative summation formula (Alt) was offered, which approximates the sum $\sum_{k=0}^{n-1} f(k)$ by a linear combination of values of an antiderivative F of f only, without using values of any derivatives of f:

$$\sum_{k=0}^{n-1} f(k) \approx \sum_{j=1-m}^{m-1} \tau_{m,1+|j|} \int_{j/2-1/2}^{n-1/2-j/2} \mathrm{d}x\, f(x), \tag{1.2}$$

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where f is again a smooth enough function, the coefficients $\tau_{m,r}$ are certain rational numbers not depending on f and such that $\sum_{j=1-m}^{m-1} \tau_{m,1+|j|} = 1$, and n and m are natural numbers. Similarly to the case of the EM formula, the Alt approximation is exact when f is a polynomial of degree $< 2m$.

It was shown in [19] that the Alt formula should be usually expected to outperform the EM one.

Extensions of the EM formula to the multiple sums, including sums over the integral points of integral lattice polytopes, have been of significant interest; see e.g. [20, 8, 7, 13, 21, 14, 6, 3, 10, 22, 18, 4]. In the present paper, a multiple-sum/multi-index-sum extension of the Alt formula will be given. The main result of this paper, Theorem 2.1, is then extended to sums over the integral points of integral lattice polytopes as well.

The rest of this paper is organized as follows.

In Section 2, the multi-index Alt formula is stated, with discussion.

In Section 3, an application of the multi-index Alt formula to summing possibly divergent multi-index series is given. A shift trick then allows one to make the remainder in the Alt formula arbitrarily small.

In Section 4, the mentioned extension to sums over the integral points of integral lattice polytopes is presented.

The necessary proofs are deferred to Section 5.

At the end of this introduction, let us fix notation to be used in the rest of the paper:

Suppose that p and m are natural numbers and $f\colon \mathbb{R}^p \to \mathbb{R}$ is a $2m$-times continuously differentiable function, with partial derivatives $f^{(\boldsymbol{\alpha})}$, where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p) \in \mathbb{Z}_+^p$ and $\mathbb{Z}_+ := \mathbb{Z} \cap [0, \infty)$.

Generally, boldface letters will denote vectors in $\mathbb{R}^p$, in $\mathbb{Z}^p$, or in $\mathbb{Z}_+^p$, with the coordinates denoted by the corresponding non-boldface letters with the indices: $\mathbf{x} = (x_1, \dots, x_p) \in \mathbb{R}^p$, $\mathbf{y} = (y_1, \dots, y_p) \in \mathbb{R}^p$, $\mathbf{u} = (u_1, \dots, u_p) \in \mathbb{R}^p$, $\mathbf{v} = (v_1, \dots, v_p) \in \mathbb{R}^p$, $\mathbf{n} = (n_1, \dots, n_p) \in \mathbb{Z}_+^p$, $\mathbf{k} = (k_1, \dots, k_p) \in \mathbb{Z}_+^p$, $\mathbf{j} = (j_1, \dots, j_p) \in \mathbb{Z}_+^p$, $\mathbf{i} = (i_1, \dots, i_p) \in \mathbb{Z}_+^p$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p) \in \mathbb{Z}_+^p$, and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p) \in \mathbb{Z}^p$. Let $\mathrm{I}\{A\}$ denote the indicator of an assertion A; that is, $\mathrm{I}\{A\} := 1$ if A is true and $\mathrm{I}\{A\} := 0$ if A is false. Let $\|\boldsymbol{\alpha}\| := \|\boldsymbol{\alpha}\|_1 = \alpha_1 + \dots + \alpha_p$; $\boldsymbol{\alpha}! := \alpha_1! \cdots \alpha_p!$; $\mathbf{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \cdots x_p^{\alpha_p}$; $|\boldsymbol{\beta}| := (|\beta_1|, \dots, |\beta_p|)$; $\mathbf{1} := (1, \dots, 1) \in \mathbb{Z}_+^p$; $\mathbf{0} := 0\mathbf{1}$; $\mathbf{j}\mathbf{v} := (j_1 v_1, \dots, j_p v_p)$;

$$\mathbf{j} \geq \mathbf{i} \overset{\text{def}}{\iff} \mathbf{i} \leq \mathbf{j} \overset{\text{def}}{\iff} i_r \leq j_r \text{ for all } r \in [p] := \{1, \dots, p\};$$

$$[\mathbf{u}, \mathbf{v}] := \prod_{r=1}^{p} [u_r, v_r]; \wedge\mathbf{x} := x_1 \wedge \dots \wedge x_p; \vee\mathbf{x} := x_1 \vee \dots \vee x_p;$$

$$\mathbf{u} \wedge \mathbf{v} := (u_1 \wedge v_1, \dots, u_p \wedge v_p); \quad \mathbf{u} \vee \mathbf{v} := (u_1 \vee v_1, \dots, u_p \vee v_p);$$

$$\sum_{\mathbf{i}=\mathbf{j}}^{\mathbf{k}} := \sum_{\mathbf{i} \in \mathbb{Z}_+^p \colon \mathbf{j} \leq \mathbf{i} \leq \mathbf{k}}; \qquad \int_{\mathbf{u}}^{\mathbf{v}} \mathrm{d}\mathbf{x}\, h(\mathbf{x}) := (-1)^{\sum_{r=1}^{p} \mathrm{I}\{u_r > v_r\}} \int_{[\mathbf{u} \wedge \mathbf{v}, \mathbf{u} \vee \mathbf{v}]} \mathrm{d}\mathbf{x}\, h(\mathbf{x});$$

$$\int_{\mathbf{u}}^{\mathbf{v}} := \int_{\mathbf{u}}^{\mathbf{v}} \mathrm{d}\mathbf{x}\, f(\mathbf{x}).$$

Let $\mathbb{R}_+^p := [0, \infty)^p$.

2. A multi-index alternative (Alt) to the EM formula

The following extension of [19, Theorem 3.1] to multiple sums is the main result of this paper:

Theorem 2.1. *One has*

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f(\mathbf{k})\Big[=\sum_{k_1=0}^{n_1-1}\cdots\sum_{k_p=0}^{n_p-1} f(k_1,\dots,k_p)\Big]=A_m-R_m, \tag{2.3}$$

where

$$A_m := \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}\int_{\mathbf{i}-\mathbf{j}/2}^{\mathbf{n}-\mathbf{1}+\mathbf{j}/2-\mathbf{i}} = \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}\int_{-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}^{\mathbf{n}-\mathbf{1}+\mathbf{j}/2-\mathbf{i}} \tag{2.4}$$

$$= \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}}\tau_{m,\mathbf{1}+|\boldsymbol{\beta}|}\int_{\boldsymbol{\beta}/2-\mathbf{1}/2}^{\mathbf{n}-\mathbf{1}/2-\boldsymbol{\beta}/2} = \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}}\tau_{m,\mathbf{1}+|\boldsymbol{\beta}|}\int_{-\mathbf{1}/2-\boldsymbol{\beta}/2}^{\mathbf{n}-\mathbf{1}/2-\boldsymbol{\beta}/2} \tag{2.5}$$

$$= \sum_{\boldsymbol{\alpha}=\mathbf{0}}^{(m-1)\mathbf{1}}\tau_{m,\mathbf{1}+\boldsymbol{\alpha}}\sum_{\boldsymbol{\beta}\colon|\boldsymbol{\beta}|=\boldsymbol{\alpha}}\int_{\boldsymbol{\beta}/2-\mathbf{1}/2}^{\mathbf{n}-\mathbf{1}/2-\boldsymbol{\beta}/2} = \sum_{\boldsymbol{\alpha}=\mathbf{0}}^{(m-1)\mathbf{1}}\tau_{m,\mathbf{1}+\boldsymbol{\alpha}}\sum_{\boldsymbol{\beta}\colon|\boldsymbol{\beta}|=\boldsymbol{\alpha}}\int_{-\mathbf{1}/2-\boldsymbol{\beta}/2}^{\mathbf{n}-\mathbf{1}/2-\boldsymbol{\beta}/2} \tag{2.6}$$

is the integral approximation to the sum $\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f(\mathbf{k})$,

$$\gamma_{m,\mathbf{j}} := \prod_{r=1}^{p}\gamma_{m,j_r}, \quad \gamma_{m,j} := (-1)^{j-1}\,\frac{2}{j}\binom{2m}{m+j}\Big/\binom{2m}{m}, \tag{2.7}$$

$$\tau_{m,\mathbf{j}} := \prod_{r=1}^{p}\tau_{m,j_r}, \quad \tau_{m,j} := \sum_{\beta=0}^{\lfloor m/2-j/2\rfloor}\gamma_{m,j+2\beta} = \sum_{\beta=0}^{\infty}\gamma_{m,j+2\beta}, \tag{2.8}$$

and R_m *is the remainder given by the formula*

$$R_m := \frac{m}{2^{2m+p-1}} \times \sum_{\|\boldsymbol{\alpha}\|=2m}\frac{1}{\boldsymbol{\alpha}!}\int_0^1 \mathrm{d}s\,(1-s)^{2m-1}\int_{-\mathbf{1}}^{\mathbf{1}}\mathrm{d}\mathbf{v}\,\mathbf{v}^{\boldsymbol{\alpha}}\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\,\mathbf{j}^{\boldsymbol{\alpha}+\mathbf{1}}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f^{(\boldsymbol{\alpha})}(\mathbf{k}+s\mathbf{j}\mathbf{v}/2). \tag{2.9}$$

The sum of all the coefficients of the integrals in each of the expressions (2.4), (2.5), *and* (2.6) *of* A_m *is*

$$\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}1 = \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\,\mathbf{j}^{\mathbf{1}} = \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}}\tau_{m,\mathbf{1}+|\boldsymbol{\beta}|} = 1. \tag{2.10}$$

If M_{2m} *is a real number such that*

$$\Big|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f^{(\boldsymbol{\alpha})}(\mathbf{k}+\mathbf{u})\Big| \le M_{2m} \quad \text{for all } \boldsymbol{\alpha} \text{ with } \|\boldsymbol{\alpha}\|=2m \text{ and all } \mathbf{u}\in(-m\mathbf{1}/2, m\mathbf{1}/2], \tag{2.11}$$

then the remainder R_m *can be bounded as follows:*

$$|R_m| \le \frac{M_{2m}}{2^{2m}}\sum_{\|\boldsymbol{\alpha}\|=2m}\frac{1}{(\boldsymbol{\alpha}+\mathbf{1})!}\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}|\gamma_{m,\mathbf{j}}|\,\mathbf{j}^{\boldsymbol{\alpha}+\mathbf{1}} \tag{2.12}$$

$$\le M_{2m}\,\frac{1.0331(\pi m)^{(p+1)/2}}{(2m+1)!}\,(\kappa p m)^{2m}, \tag{2.13}$$

where

$$\kappa := \sqrt{\frac{\Lambda_*}{4}} = 0.27754\dots \tag{2.14}$$

and

$$\Lambda_* := \max_{0<t<1} \Lambda(t) = 0.3081\ldots, \quad \Lambda(t) := (1-t)^{t-1}(1+t)^{-1-t}t^2.$$

If $m \geq 2$, then the factor 1.0331 in (2.13) can be replaced by 1.001.

Recall the convention that the sum of an empty family is 0. In particular, if $\wedge\mathbf{n} = 0$, then $\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f(\mathbf{k}) = 0 = A_m = R_m$.

Also, it is clear that $R_m = 0$ if the function f is any polynomial of degree at most $2m-1$.

One may note here that, in each of the formulas (2.4), (2.5), and (2.6), the first expression is a linear combination of integrals of the form $\int_{-\boldsymbol{\lambda}}^{\mathbf{n}-\mathbf{1}+\boldsymbol{\lambda}}$ for some $\boldsymbol{\lambda} \in \mathbb{R}^p$ with $|\boldsymbol{\lambda}| \leq (m-2)\mathbf{1}/2$. So, provided that $\mathbf{n} \geq (m-1)\mathbf{1}$, each of these integrals equals the Lebesgue integral of the function f over the p-dimensional interval $[-\boldsymbol{\lambda}, \mathbf{n}-\mathbf{1}+\boldsymbol{\lambda}]$, symmetric about the point $(\mathbf{n}-\mathbf{1})/2$.

In contrast, the second expression in each of the formulas (2.4), (2.5), and (2.6) is a linear combination of integrals of the form $\int_{\boldsymbol{\lambda}}^{\mathbf{n}+\boldsymbol{\lambda}}$ for some $\boldsymbol{\lambda} \in \mathbb{R}^p$; so, each of these integrals equals the Lebesgue integral of the function f over the p-dimensional interval $[\boldsymbol{\lambda}, \mathbf{n}+\boldsymbol{\lambda}]$, whose endpoints differ by the vector $\mathbf{n}$. This observation holds whether the condition $\mathbf{n} \geq (m-1)\mathbf{1}$ holds or not.

Remark 2.1. *As in [19] in the special case of ordinary sums, here, instead of assuming that the function f is real-valued, one may assume, more generally, that f takes values in any normed space. In particular, one may allow f to take values in the q-dimensional complex space $\mathbb{C}^q$, for any natural q. An advantage of dealing with a vector-valued function (rather than separately with each of its coordinates) is that this way one has to compute the coefficients – say $\tau_{m,\boldsymbol{\beta}}$ in (2.6) – only once, for all the components of the vector function.* $\square$

3. Application to summing (possibly divergent) multi-index series

Let us say that a function $F\colon \mathbb{R}^p \to \mathbb{R}$ is an antiderivative of the function f if

$$F^{(\mathbf{1})} = f;$$

that is, if F is differentiated once with respect to every one of the p arguments of the function F, then the result of this p-fold partial differentiation is the function f. It is assumed that this result does not depend on the order of the arguments with respect to which the partial derivatives are taken. Here and elsewhere in the paper, f and p are as set in Section 1. In particular, it follows that the function f is continuous. Clearly, this notion of an antiderivative is a generalization of the corresponding notion for functions on $\mathbb{R}$.

For each set $J \subseteq [p]$, let $|J|$ denote the cardinality of J, and also let

$$\mathbf{1}_J := (\mathrm{I}\{1 \in J\}, \ldots, \mathrm{I}\{p \in J\}).$$

In particular, $\mathbf{1}_{[p]} = \mathbf{1}$ and $\mathbf{1}_{\emptyset} = \mathbf{0}$.

Remark 3.2. *A function F on $\mathbb{R}^p$ is an antiderivative of the function f if and only if one has a representation of the form*

$$F(\mathbf{x}) = \int_{\mathbf{0}}^{\mathbf{x}} \mathrm{d}\mathbf{y}\, f(\mathbf{y}) + \sum_{j=1}^{p} c_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p)$$

for all $\mathbf{x} = (x_1, \ldots, x_p) \in \mathbb{R}^p$, where $c_1, \ldots, c_p$ are functions on $\mathbb{R}^{p-1}$ such that, for each $j \in \{1, \ldots, p\}$ and all $(x_1, \ldots, x_p) \in \mathbb{R}^p$, the mixed partial derivative $\dfrac{\partial^{p-1} c_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p)}{\partial x_1 \cdots \partial x_{j-1} \partial x_{j+1} \cdots \partial x_p}$ exists and does not depend on the order of the arguments $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p$ with respect to which the partial derivatives are taken.

The "if" part of the above statement is obvious. The "only if" part of it follows from the multidimensional version of the fundamental theorem of calculus to be given by Lemma 5.1 in Section 5 (take there $\mathbf{0}$ *and* $\mathbf{x}$*, respectively, in place of* $\mathbf{u}$ *and* $\mathbf{v}$ *in Lemma 5.1, and note that then* $F(\mathbf{v}_{[p]}) = F(\mathbf{v}) = F(\mathbf{x})$*).*

In particular, the function F *on* $\mathbb{R}^p$ *given by the condition* $F(\mathbf{x}) = \int_{\mathbf{0}}^{\mathbf{x}} \mathrm{d}\mathbf{y}\, f(\mathbf{y})$ *for all* $\mathbf{x} \in \mathbb{R}^p$ *is clearly an antiderivative of* f*; thus, there always exists an antiderivative of the function* f *– still assuming, of course, that* f *is* $2m$*-times continuously differentiable for some natural* m*; in fact, just the continuity of* f *would be enough for the existence of an antiderivative of* f*.* □

The alternative summation formula presented in Theorem 2.1 can be used for summing (possibly divergent) multi-index series, as follows.

Theorem 3.2. *Let* m_0 *be a natural number, and suppose that* $m \geq m_0$*. Let* F *be any antiderivative of* f*. Suppose that*

$$F^{(\boldsymbol{\alpha})}(x) \underset{\vee\mathbf{x}\to\infty}{\longrightarrow} 0 \text{ for each } \boldsymbol{\alpha} \in \mathbb{Z}_+^p \text{ with } \|\boldsymbol{\alpha}\| = 2m_0 \tag{3.15}$$

and the series

$$\sum_{\mathbf{k}=\mathbf{0}}^{\infty\mathbf{1}} f^{(\boldsymbol{\alpha})}(\mathbf{k}+\mathbf{u}) \text{ converges uniformly in } \mathbf{u} \in [-m\mathbf{1}/2, m\mathbf{1}/2] \quad \text{for each } \boldsymbol{\alpha} \in \mathbb{Z}_+^p \text{ with } \|\boldsymbol{\alpha}\| = 2m, \tag{3.16}$$

in the sense that $\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f^{(\boldsymbol{\alpha})}(\mathbf{k}+\mathbf{u})$ *converges uniformly as* $\wedge\mathbf{n} \to \infty$*. Then*

$$\sum_{\mathbf{k}\geq\mathbf{0}}^{\mathsf{Alt}} f(\mathbf{k}) := \lim_{\wedge\mathbf{n}\to\infty} \Big(\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f(\mathbf{k}) - \tilde{A}_{m_0,F}(\mathbf{n}) \Big) = (-1)^p A^{\emptyset}_{m,F}(\mathbf{0}) - R_{m,f}(\infty), \tag{3.17}$$

where (cf. (2.4), (2.5), and (2.6))

$$\tilde{A}_{m,F}(\mathbf{n}) := \sum_{\emptyset \neq J \subseteq [p]} (-1)^{p-|J|} A^J_{m,F}(\mathbf{n}), \tag{3.18}$$

$$A^J_{m,F}(\mathbf{n}) := \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}} \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}} F(\mathbf{n}\mathbf{1}_J - \mathbf{1} + \mathbf{j}/2 - \mathbf{i}) \tag{3.19}$$

$$= \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}} \tau_{m,\mathbf{1}+|\boldsymbol{\beta}|}\, F(\mathbf{n}\mathbf{1}_J - \mathbf{1}/2 - \boldsymbol{\beta}/2) \tag{3.20}$$

$$= \sum_{\boldsymbol{\alpha}=\mathbf{0}}^{(m-1)\mathbf{1}} \tau_{m,\mathbf{1}+\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta}:\ |\boldsymbol{\beta}|=\boldsymbol{\alpha}} F(\mathbf{n}\mathbf{1}_J - \mathbf{1}/2 - \boldsymbol{\beta}/2), \tag{3.21}$$

and (cf. (2.9))

$$R_{m,f}(\infty) := \frac{m}{2^{2m+p-1}} \times \sum_{\|\boldsymbol{\alpha}\|=2m} \frac{1}{\boldsymbol{\alpha}!} \int_0^1 \mathrm{d}s\,(1-s)^{2m-1} \int_{-\mathbf{1}}^{\mathbf{1}} \mathrm{d}\mathbf{v}\, \mathbf{v}^{\boldsymbol{\alpha}} \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}}\, \mathbf{j}^{\boldsymbol{\alpha}+\mathbf{1}} \sum_{\mathbf{k}=\mathbf{0}}^{\infty\mathbf{1}} f^{(\boldsymbol{\alpha})}(\mathbf{k}+s\mathbf{j}\mathbf{v}/2).$$

If condition (2.11) holds for all $\mathbf{n} \in \mathbb{Z}_+^p$*, then one can replace* R_m *in (2.12)–(2.13) by* $R_{m,f}(\infty)$*, so that*

$$|R_{m,f}(\infty)| \leq M_{2m} \frac{1.0331(\pi m)^{(p+1)/2}}{(2m+1)!} (\kappa p m)^{2m}. \tag{3.22}$$

Looking, say, at the expression of $A^J_{m,F}(\mathbf{n})$ in (3.21), one may note that

$$A^{\emptyset}_{m,F}(\mathbf{0}) = A^{\emptyset}_{m,F}(\mathbf{n}) = A^J_{m,F}(\mathbf{0}) = \sum_{\boldsymbol{\alpha}=\mathbf{0}}^{(m-1)\mathbf{1}} \tau_{m,\mathbf{1}+\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta}\colon |\boldsymbol{\beta}|=\boldsymbol{\alpha}} F(\boldsymbol{\beta}/2 - \mathbf{1}/2) \tag{3.23}$$

for all $\mathbf{n} \in \mathbb{Z}^p_+$ and $J \subseteq [p]$.

The limit $\sum^{\mathsf{Alt}}_{\mathbf{k}\geq\mathbf{0}} f(\mathbf{k})$ in (3.17) may be referred to as the (generalized) sum of the possibly divergent multi-index series $\sum_{\mathbf{k}=\mathbf{0}}^{\infty\mathbf{1}} f(\mathbf{k})$ by means of the Alt formula (2.3).

Theorem 3.2 is a multi-index extension of Proposition 5.1 in [19].

To compute the generalized sum $\sum^{\mathsf{Alt}}_{\mathbf{k}\geq\mathbf{0}} f(\mathbf{k})$ effectively, one has to ensure that the remainder $R_{m,f}(\infty)$ can be made arbitrarily small. This can be done as follows.

For any function $h\colon \mathbb{R}^p \to \mathbb{R}$ and any $\mathbf{c} \in \mathbb{R}^p$, let $h_{\mathbf{c}}$ denote the $\mathbf{c}$-shift of h defined by the formula

$$h_{\mathbf{c}}(\mathbf{x}) := h(\mathbf{x}+\mathbf{c}) \tag{3.24}$$

for all $\mathbf{x} \in \mathbb{R}^p$. Note that, if F is an antiderivative of f, then $F_{\mathbf{c}}$ is an antiderivative of $f_{\mathbf{c}}$.

Theorem 3.3. *Suppose that the conditions of Theorem 3.2 hold. Take any $\mathbf{c} \in \mathbb{Z}^p_+$. Then*

$$\sum^{\mathsf{Alt}}_{\mathbf{k}\geq\mathbf{0}} f(\mathbf{k}) = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{c}-\mathbf{1}} f(\mathbf{k}) - \tilde{A}_{m,F}(\mathbf{c}) - R_{m,f,\mathbf{c}}(\infty), \tag{3.25}$$

where

$$R_{m,f,\mathbf{c}}(\infty) := -\sum_{\emptyset\neq J\subseteq[p]} (-1)^{p-|J|} R_{m,f_{\mathbf{c}\mathbf{1}_J}}(\infty) \tag{3.26}$$

(cf. (3.18)).

Under the conditions of Theorem 3.2, the remainder $R_{m,f,\mathbf{c}}(\infty)$ can be made arbitrarily small by making $\wedge\mathbf{c}$ large enough. The price to pay for this will be the need to compute a possibly large partial sum $\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{c}-\mathbf{1}} f(\mathbf{k})$ of the series.

Theorem 3.3 is a multi-index extension of Corollary 5.6 in [19].

Example 3.1. *In Theorem 3.3, let $p = 2$ and take any 4-times continuously differentiable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that*

$$f(x,y) = (x+y+2)\ln(x+y+2)$$

for real $x, y \geq 0$. Such a function f exists, by Whitney's theorem [24]; however, only the values of f on $[0,\infty)^2$ will matter for the purposes of this example. Then it is straightforward to check by direct differentiation that for an antiderivative F of f and all real $x, y \geq 0$ one will have

$$F(x,y) = \tfrac{1}{6}\,(x+y+2)^3\,\ln(x+y+2) - \tfrac{5}{12}\,(x+1)(y+1)(x+y+2).$$

It is also straightforward to verify conditions (3.15) and (3.16) of Theorem 3.2 with $m_0 = m = 2$.

It also follows that, for $\mathbf{n} = (n,n)$, the term $\tilde{A}_{m,F}(\mathbf{n}) = \tilde{A}_{2,F}\big((n,n)\big)$ (defined in (3.18)) is expressed as a linear combination of certain terms of the form $P(n)\ln(a+bn)$ or $P(n)$, where P is polynomial with real coefficients, a is a nonnegative real number, and b is a positive real number. Replacing, in that expression for $\tilde{A}_{m,F}(\mathbf{n})$, every instance of $\ln(a+bn)$ by its large-n asymptotics $\ln n + \ln b + \frac{a}{bn} - \frac{a^2}{2b^2n^2} + \frac{a^3}{3b^3n^3} + O(\frac{1}{n^4})$, after some rather heavy algebra we find

$$\tilde{A}_{m,F}(\mathbf{n}) = S_n + \delta_n,$$

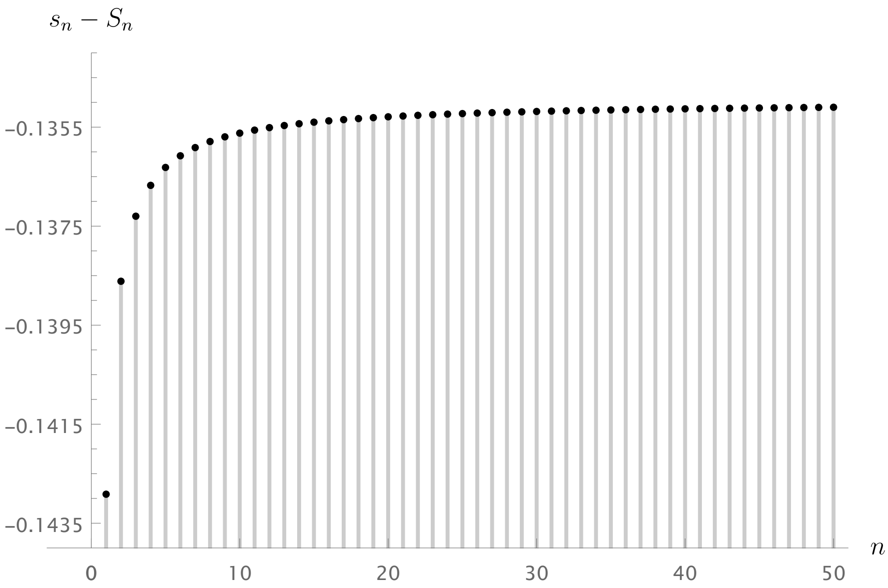

FIGURE 1. Graph $\{(n, s_n - S_n)\colon n \in \{1, \dots, 50\}\}$

where

$$S_n := n^3 \ln \frac{2^{4/3}\, n}{e^{5/6}} + n^2 \ln \frac{4n}{\sqrt{e}} + \frac{5}{6} n \ln 2 - \frac{1}{12} \ln \frac{en}{2} \tag{3.27}$$

and

$$\delta_n = O(1/n^2). \tag{3.28}$$

Thus, by Theorem 3.3,

$$s_n := \sum_{k=1}^{n} \sum_{l=1}^{n} (k+l) \ln(k+l) = S_n + L + r_n,$$

where

$$L := \sum_{\mathbf{k} \geq \mathbf{0}}^{\mathsf{Alt}} f(\mathbf{k}) = \lim_{n\to\infty} (s_n - S_n) \in \mathbb{R}$$

and

$$r_n := \delta_n + R_{2,f,(n,n)}(\infty) = O(1/n), \tag{3.29}$$

in view of (3.28), (3.26), (3.24), (3.22), (2.11), and (2.14); the universal positive real constant factor in $O(1/n)$ in (3.29) can be given explicitly. Note that the bound $O(1/n)$ on the error term r_n in (3.29) can be improved to $O(1/n^{m-1})$ by choosing the "approximation order" m in formula (3.25) to be any natural number greater than 2; of course, then the expression for S_n in (3.27) will have to be replaced by a more complicated expression.

The convergence of $s_n - S_n$ to the limit L is illustrated in Figure 1, which shows the discrete graph $\{(n, s_n - S_n)\colon n \in \{1, \dots, 50\}\}$. □

4. Application to sums over the integral points of integral lattice polytopes

Let P be an integral polytope in $\mathbb{R}^p$, that is, the convex hull of a finite subset of $\mathbb{Z}^p$.

Suppose that P is of full dimension, p. Let V denote the set of all vertices (that is, extreme points) of P.

By the main result of Haase [11], for each $\mathbf{v} \in V$ there exist a finite set $I_\mathbf{v}$, a map $I_\mathbf{v} \ni i \mapsto t_{\mathbf{v},i} \in \{0,1\}$, a map $I_\mathbf{v} \ni i \mapsto A_{\mathbf{v},i}$ into the set of all nonsingular $p \times p$ matrices over $\mathbb{Z}$, and a map $I_\mathbf{v} \ni i \mapsto J_{\mathbf{v},i}$ into the set of all subsets of the set $[p] = \{1, \dots, p\}$ such that

$$\llbracket P \rrbracket = \sum_{\mathbf{v}\in V} \sum_{i \in I_\mathbf{v}} (-1)^{t_{\mathbf{v},i}} \llbracket C_{\mathbf{v},i} \rrbracket, \tag{4.30}$$

where $\llbracket \cdot \rrbracket$ denotes the indicator/characteristic function,

$$C_{\mathbf{v},i} := \mathbf{v} + A_{\mathbf{v},i} \mathbb{R}^+_{J_{\mathbf{v},i}} = \{\mathbf{v} + A_{\mathbf{v},i}\mathbf{x} \colon \mathbf{x} \in \mathbb{R}^+_{J_{\mathbf{v},i}}\}, \tag{4.31}$$

$$\mathbb{R}^+_J := \prod_{j\in[p]} \mathbb{R}^+_{1-\llbracket J \rrbracket(j)} \quad \text{for} \quad J \subseteq [p],$$

and

$$\mathbb{R}^+_\varepsilon := \begin{cases} (0,\infty) & \text{if } \varepsilon = 0, \\ [0,\infty) & \text{if } \varepsilon = 1 \end{cases}$$

(so that the closure of $C_{\mathbf{v},i}$ is a polyhedral cone, for each pair $(\mathbf{v}, i)$). In the case when the polytope P is simple, decomposition (4.30) was obtained earlier by Lawrence [17]. To extend Lawrence's result, Haase used virtual infinitesimal deformations of vertices of P, identified with regular triangulations of the normal cones at the vertices.

Proposition 4.1. *Let A be any nonsingular $p \times p$ matrix over $\mathbb{Z}$, and let J be any subset of the set $[p]$. Then there exist a finite set I, a map $I \ni i \mapsto A_i$ into the set of all* unimodular *$p \times p$ matrices over $\mathbb{Z}$, and a map $I \ni i \mapsto J_i$ into the set of all subsets of the set $[p]$ such that*

$$\llbracket A\mathbb{R}^+_J \rrbracket = \sum_{i\in I} \llbracket A_i \mathbb{R}^+_{J_i} \rrbracket. \tag{4.32}$$

(Recall that a matrix is called unimodular if its determinant is 1 or -1.)

Thus, one can strengthen the statement on the decomposition (4.30) as follows:

Corollary 4.1. *One may assume that all the matrices $A_{\mathbf{v},i}$ in (4.30)–(4.31) are unimodular.*

A similar decomposition, but with polyhedral cones of lower dimensions, was obtained in [5].

The following corollary is almost immediate from Theorem 2.1 and Corollary 4.1.

Corollary 4.2. *Suppose that the function f is compactly supported. Then*

$$\sum_{\mathbf{k}\in P\cap\mathbb{Z}^p} f(\mathbf{k}) = A_m(f,P) - R_m(f,P), \tag{4.33}$$

where

$$A_m(f,P) := \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}} \tau_{m,\mathbf{1}+|\boldsymbol{\beta}|} \sum_{\mathbf{v}\in V} (-1)^{t_\mathbf{v}} \sum_{i\in I_\mathbf{v}} \int_{C_{\mathbf{v},i}+A_{\mathbf{v},i}(\mathbf{1}_{J_{\mathbf{v},i}}-(\mathbf{1}+\boldsymbol{\beta})/2)} \mathrm{d}\mathbf{x} f(\mathbf{x}) \tag{4.34}$$

is the integral approximation to the sum $\sum\limits_{\mathbf{k}\in P\cap\mathbb{Z}^p} f(\mathbf{k})$ and $R_m(f,P)$ is the remainder given by the formula

$$R_m(f,P) := \frac{m}{2^{2m+p-1}} \sum_{\|\boldsymbol{\alpha}\|=2m} \frac{1}{\boldsymbol{\alpha}!} \int_0^1 \mathrm{d}s\,(1-s)^{2m-1} \int_{-\mathbf{1}}^{\mathbf{1}} \mathrm{d}\mathbf{u}\,\mathbf{u}^{\boldsymbol{\alpha}}\,\Sigma_m(s\mathbf{u})$$

with

$$\Sigma_m(\mathbf{w}) := \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}}\,\mathbf{j}^{\boldsymbol{\alpha}+\mathbf{1}} \sum_{\mathbf{v}\in V} \sum_{i\in I_{\mathbf{v}}} (-1)^{t_{\mathbf{v}}} \sum_{\mathbf{k}\geq\mathbf{0}} g_{\mathbf{v},i}^{(\boldsymbol{\alpha})}(\mathbf{k}+\mathbf{1}_{J_{\mathbf{v},i}}+\mathbf{j}\mathbf{w}/2)$$

and

$$g_{\mathbf{v},i}(\mathbf{y}) := f(\mathbf{v}+A_{\mathbf{v},i}\mathbf{y})$$

for $\mathbf{y}\in\mathbb{R}^p$*. If* M_{2m} *is a real number such that*

$$\Big|\sum_{\mathbf{v}\in V}\sum_{i\in I_{\mathbf{v}}}(-1)^{t_{\mathbf{v}}}\sum_{\mathbf{k}\geq\mathbf{0}} g_{\mathbf{v},i}^{(\boldsymbol{\alpha})}(\mathbf{k}+\mathbf{u})\Big| \leq M_{2m} \quad \textit{whenever} \quad \|\boldsymbol{\alpha}\|=2m \quad \textit{and} \quad |\mathbf{u}|\leq(\tfrac{m}{2}+1)\mathbf{1},$$

then

$$|R_m(f,P)| \leq M_{2m}\,\frac{1.0331(\pi m)^{(p+1)/2}}{(2m+1)!}\,(\kappa p m)^{2m},$$

where κ *is as in* (2.14)*.*

Indeed, for $J\subseteq[p]$, let

$$\mathbb{Z}_J^+ := \mathbb{Z}^p\cap\mathbb{R}_J^+ = \mathbb{Z}_+^p+\mathbf{1}_J,$$

where $\mathbb{Z}_+ := \mathbb{Z}\cap[0,\infty)$. Note that $A\mathbb{Z}^p=\mathbb{Z}^p$ for any unimodular matrix A over $\mathbb{Z}$. Now Corollary 4.2 follows by Corollary 4.1 and Theorem 2.1 because

$$\sum_{\mathbf{k}\in C_{\mathbf{v},i}\cap\mathbb{Z}^p} f(\mathbf{k}) = \sum_{\mathbf{q}\in\mathbb{Z}^+_{J_{\mathbf{v},i}}} f(\mathbf{v}+A_{\mathbf{v},i}\mathbf{q}) = \sum_{\mathbf{q}\geq\mathbf{0}} f(\mathbf{v}+A_{\mathbf{v},i}(\mathbf{q}+\mathbf{1}_{J_{\mathbf{v},i}})) = \sum_{\mathbf{q}\geq\mathbf{0}} g_{\mathbf{v},i}(\mathbf{q}+\mathbf{1}_{J_{\mathbf{v},i}})$$

and

$$\int_{[-\mathbf{1}/2-\boldsymbol{\beta}/2,\,\infty\mathbf{1})} \mathrm{d}\mathbf{y}\,g_{\mathbf{v},i}(\mathbf{y}+\mathbf{1}_{J_{\mathbf{v},i}}) = \int_{C_{\mathbf{v},i}+A_{\mathbf{v},i}(\mathbf{1}_{J_{\mathbf{v},i}}-(\mathbf{1}+\boldsymbol{\beta})/2)} \mathrm{d}\mathbf{x}\,f(\mathbf{x}).$$

The expression for $A_m(f,P)$ in (4.34) is based on the second expression for A_m in (2.5); of course, one can quite similarly use any one of the other 5 expressions in (2.4)–(2.6).

Notable differences between the Alt formula in Corollary 4.2 and the EM formula that is the main result of [14] (Theorem 2 therein) include the following: (i) in [14, Theorem 2], the summation is over all faces of the polytope P, whereas in (4.34) the corresponding summation is only over the vertices of P and (ii) instead of the plain summation $\sum_{\mathbf{k}\in P\cap\mathbb{Z}^p} f(\mathbf{k})$ in (4.33), in the corresponding sum in [14] the summands $f(\mathbf{k})$ are weighted (in accordance with the dimension of the relative interior of the face given that $\mathbf{k}$ belongs to that relative interior).

Note also that [14, Theorem 2] is obtained for simple polytopes. In [3], this result was extended to allow more general weights, and then further generalized to non-simple polytopes in [4].

The version of the EM formula for polytopes in [6] is given for polynomial functions f in terms of differential operators of infinite order, with the summation over all faces of the polytope.

It should be possible to extend Corollary 4.2 to the case when the function f is a so-called symbol in the sense of Hörmander [12] – cf. [14, Theorem 3], as well as conditions (3.15) and (3.16). (Recall that a function $f\in C^\infty(\mathbb{R}^p)$ is called a symbol of order N if for every $\boldsymbol{\alpha}\in\mathbb{Z}_+^p$ there is a real constant $C_{\boldsymbol{\alpha}}$ such that $|f^{(\boldsymbol{\alpha})}(\mathbf{x})|\leq C_{\boldsymbol{\alpha}}(1+\|\mathbf{x}\|)^{N-\|\boldsymbol{\alpha}\|}$ for all $\mathbf{x}\in\mathbb{R}^p$; here, as before, $\|\cdot\|:=\|\cdot\|_1$.) One way to attack this goal could be to show that, for any $\boldsymbol{\alpha}\in\mathbb{Z}_+^p$ such that $\boldsymbol{\alpha} \le (m-1)\mathbf{1}$, the essential support (except possibly for a set of Lebesgue measure 0) of the function

$$\sum_{\boldsymbol{\beta}\colon\, |\boldsymbol{\beta}|=\boldsymbol{\alpha}} \sum_{\mathbf{v}\in V} \sum_{i\in I_{\mathbf{v}}} (-1)^{t_{\mathbf{v},i}} \, [\![C_{\mathbf{v},i} + A_{\mathbf{v},i}(\mathbf{1}_{J_{\mathbf{v},i}} - (\mathbf{1}+\boldsymbol{\beta})/2)]\!]$$

is bounded, presumably being just a perturbed version of the indicator of the polytope P; cf. (4.34) and the equality in [14, formula (89)].

Moreover, in view of the results of Section 3, it appears not unlikely that Corollary 4.2 could be extended to general polyhedral sets.

5. Proofs

Proof of Theorem 2.1. Take any $\mathbf{k}$ (in $\mathbb{Z}_+^p$) such that $\mathbf{k} \le \mathbf{n}-\mathbf{1}$ and consider the Taylor expansion

$$f(\mathbf{x}) = \sum_{\|\boldsymbol{\alpha}\|\le 2m-1} \frac{f^{(\boldsymbol{\alpha})}(\mathbf{k})}{\boldsymbol{\alpha}!}\,\mathbf{u}^{\boldsymbol{\alpha}} + \sum_{\|\boldsymbol{\alpha}\|=2m} \frac{2m}{\boldsymbol{\alpha}!}\,\mathbf{u}^{\boldsymbol{\alpha}} \int_0^1 \mathrm{d}s\,(1-s)^{2m-1} f^{(\boldsymbol{\alpha})}(\mathbf{k}+s\mathbf{u}) \tag{5.35}$$

for all $\mathbf{x} \in (\mathbf{k} - m\mathbf{1}/2,\, \mathbf{k}+m\mathbf{1}/2]$, where $\mathbf{u} := \mathbf{x}-\mathbf{k}$. Integrating both sides of this identity in $\mathbf{x} \in (\mathbf{k}-\mathbf{j}/2,\, \mathbf{k}+\mathbf{j}/2]$ (or, equivalently, in $\mathbf{u} \in (-\mathbf{j}/2,\, \mathbf{j}/2]$) for each $\mathbf{j}$ (in $\mathbb{Z}_+^p$) such that $\mathbf{j} \le m\mathbf{1}$, then multiplying by $\gamma_{m,\mathbf{j}}$, and then summing in $\mathbf{j}$, one has

$$A_{m,\mathbf{k}} = S_{m,\mathbf{k}} + R_{m,\mathbf{k}}, \tag{5.36}$$

where

$$A_{m,\mathbf{k}} := \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}} \int_{\mathbf{k}-\mathbf{j}/2}^{\mathbf{k}+\mathbf{j}/2} \mathrm{d}\mathbf{x}\, f(\mathbf{x}), \tag{5.37}$$

$$S_{m,\mathbf{k}} := \sum_{\|\boldsymbol{\alpha}\|\le m-1} \frac{f^{(2\alpha)}(\mathbf{k})}{(2\boldsymbol{\alpha}+1)!\,2^{2\|\boldsymbol{\alpha}\|}} \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}}\,\mathbf{j}^{2\boldsymbol{\alpha}+\mathbf{1}}, \tag{5.38}$$

$$\begin{aligned} R_{m,\mathbf{k}} &:= \sum_{\|\boldsymbol{\alpha}\|=2m} \frac{2m}{\boldsymbol{\alpha}!} \int_0^1 \mathrm{d}s\,(1-s)^{2m-1} \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}} \int_{-\mathbf{j}/2}^{\mathbf{j}/2} \mathrm{d}\mathbf{u}\;\mathbf{u}^{\boldsymbol{\alpha}}\, f^{(\boldsymbol{\alpha})}(\mathbf{k}+s\mathbf{u}) \\ &= \sum_{\|\boldsymbol{\alpha}\|=2m} \frac{2m}{\boldsymbol{\alpha}!} \int_0^1 \mathrm{d}s\,(1-s)^{2m-1} \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}}\,(\mathbf{j}/2)^{\boldsymbol{\alpha}+\mathbf{1}} \int_{-\mathbf{1}}^{\mathbf{1}} \mathrm{d}\mathbf{v}\;\mathbf{v}^{\boldsymbol{\alpha}}\, f^{(\boldsymbol{\alpha})}(\mathbf{k}+s\mathbf{j}\mathbf{v}/2); \end{aligned} \tag{5.39}$$

the latter equality is obtained by the change of variables $\mathbf{u} = \mathbf{j}\mathbf{v}$.

As noted before, in the special case $p=1$ Theorem 2.1 turns into Theorem 3.1 of [19]. So, without loss of generality (w.l.o.g.) $p \ge 2$. Write

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} \int_{\mathbf{k}-\mathbf{j}/2}^{\mathbf{k}+\mathbf{j}/2} \mathrm{d}\mathbf{x}\, f(\mathbf{x}) = \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_p=0}^{n_p-1} \int_{k_p-j_p/2}^{k_p+j_p/2} \mathrm{d}x_p \cdots \int_{k_1-j_1/2}^{k_1+j_1/2} \mathrm{d}x_1\, f(\mathbf{x}). \tag{5.40}$$

In view of the multi-line display next after formula (7.7) in [19] (note, in particular, the penultimate expression there), the right-hand side of (5.40) can be rewritten as

$$\begin{aligned}
&\sum_{k_1=0}^{n_1-1}\cdots\sum_{k_{p-1}=0}^{n_{p-1}-1}\sum_{i_p=0}^{j_p-1}\int_{i_p-j_p/2}^{n_p-1+j_p/2-i_p}\mathrm{d}x_p\int_{k_{p-1}-j_{p-1}/2}^{k_{p-1}+j_{p-1}/2}\mathrm{d}x_{p-1}\cdots\int_{k_1-j_1/2}^{k_1+j_1/2}\mathrm{d}x_1\, f(\mathbf{x})\\
&=\sum_{i_p=0}^{j_p-1}\int_{i_p-j_p/2}^{n_p-1+j_p/2-i_p}\mathrm{d}x_p\sum_{k_1=0}^{n_1-1}\cdots\sum_{k_{p-1}=0}^{n_{p-1}-1}\int_{k_{p-1}-j_{p-1}/2}^{k_{p-1}+j_{p-1}/2}\mathrm{d}x_{p-1}\cdots\int_{k_1-j_1/2}^{k_1+j_1/2}\mathrm{d}x_1\, f(\mathbf{x})\\
&\ \vdots\\
&=\sum_{i_p=0}^{j_p-1}\int_{i_p-j_p/2}^{n_p-1+j_p/2-i_p}\mathrm{d}x_p\cdots\sum_{i_1=0}^{j_1-1}\int_{i_1-j_1/2}^{n_1-1+j_1/2-i_1}\mathrm{d}x_1\, f(\mathbf{x}).
\end{aligned}$$

So,

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}}\int_{\mathbf{k}-\mathbf{j}/2}^{\mathbf{k}+\mathbf{j}/2}\mathrm{d}\mathbf{x}\, f(\mathbf{x})=\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}\int_{\mathbf{i}-\mathbf{j}/2}^{\mathbf{n}-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}\mathrm{d}\mathbf{x}\, f(\mathbf{x})$$

and hence, by (5.37),

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}}A_{m,\mathbf{k}}=\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}}\int_{\mathbf{k}-\mathbf{j}/2}^{\mathbf{k}+\mathbf{j}/2}\mathrm{d}\mathbf{x}\, f(\mathbf{x})=\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}\int_{\mathbf{i}-\mathbf{j}/2}^{\mathbf{n}-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}\mathrm{d}\mathbf{x}\, f(\mathbf{x})=A_m. \tag{5.41}$$

Similarly, but using the last expression in the mentioned multi-line display next after formula (7.7) in [19] rather than the penultimate expression there, we have

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}}A_{m,\mathbf{k}}=\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}\int_{-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}^{\mathbf{n}-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}\mathrm{d}\mathbf{x}\, f(\mathbf{x}).$$

In particular, it follows that the two double sums in (2.4) are the same.

Suppose now that some $\mathbf{i}$ and $\mathbf{j}$ in $\mathbb{Z}_+^p$ and some $\boldsymbol{\beta}\in\mathbb{Z}^p$ are related by the condition $\boldsymbol{\beta}=2\mathbf{i}-\mathbf{j}+\mathbf{1}$. Then the condition $\mathbf{1}\le\mathbf{j}\le m\mathbf{1}\ \&\ \mathbf{0}\le\mathbf{i}\le\mathbf{j}-\mathbf{1}$ is equivalent to the condition

$$(1-m)\mathbf{1}\le\boldsymbol{\beta}\le(m-1)\mathbf{1}\ \&\ \mathbf{1}+|\boldsymbol{\beta}|\le\mathbf{j}\le m\mathbf{1}\ \&\ (\mathbf{j}-\mathbf{1}-|\boldsymbol{\beta}|)/2\in\mathbb{Z}_+^p.$$

So,

$$\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}\int_{\mathbf{i}-\mathbf{j}/2}^{\mathbf{n}-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}=\sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}}\tilde{\tau}_{m,\mathbf{1}+|\boldsymbol{\beta}|}\int_{\boldsymbol{\beta}/2-\mathbf{1}/2}^{\mathbf{n}-\mathbf{1}/2-\boldsymbol{\beta}/2} \tag{5.42}$$

and

$$\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}\int_{-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}^{\mathbf{n}-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}=\sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}}\tilde{\tau}_{m,\mathbf{1}+|\boldsymbol{\beta}|}\int_{-\mathbf{1}/2-\boldsymbol{\beta}/2}^{\mathbf{n}-\mathbf{1}/2-\boldsymbol{\beta}/2}, \tag{5.43}$$

where

$$
\begin{aligned}
\tilde{\tau}_{m,\mathbf{1}+|\boldsymbol{\beta}|} &:= \sum_{\mathbf{j}=\mathbf{1}+|\boldsymbol{\beta}|}^{m\mathbf{1}} \gamma_{m,\mathbf{j}} \operatorname{I}\big\{(\mathbf{j}-\mathbf{1}-|\boldsymbol{\beta}|)/2 \in \mathbb{Z}_+^p\big\} \\
&= \sum_{j_1=1+|\beta_1|}^{m} \cdots \sum_{j_p=1+|\beta_p|}^{m} \prod_{r=1}^{p} \big(\gamma_{m,j_r} \operatorname{I}\big\{(j_r-1-|\beta_r|)/2 \in \mathbb{Z}_+\big\}\big) \\
&= \prod_{r=1}^{p} \sum_{j_r=1+|\beta_r|}^{m} \big(\gamma_{m,j_r} \operatorname{I}\big\{(j_r-1-|\beta_r|)/2 \in \mathbb{Z}_+\big\}\big) \\
&= \prod_{r=1}^{p} \tau_{m,1+|\beta_r|} = \tau_{m,\mathbf{1}+|\boldsymbol{\beta}|},
\end{aligned}
$$

in view of (2.7) and (2.8).

Thus, by (5.42) and (5.43), the first double sum in (2.4) equals the first sum in (2.5), and the second double sum in (2.4) equals the second sum in (2.5).

Also, it is obvious that the first sum in (2.6) equals the first sum in (2.5), and the second sum in (2.6) equals the second sum in (2.5).

Next, for any $\boldsymbol{\alpha}$ (in $\mathbb{Z}_+^p$) with $\|\boldsymbol{\alpha}\| \le m-1$,

$$
\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}}\, \mathbf{j}^{2\boldsymbol{\alpha}+\mathbf{1}} = \sum_{j_1=1}^{m} \cdots \sum_{j_p=1}^{m} \prod_{r=1}^{p} \big(\gamma_{m,j_r} j_r^{2\alpha_r+1}\big) = \prod_{r=1}^{p} \sum_{j=1}^{m} \gamma_{m,j} j^{2\alpha_r+1} = \operatorname{I}\{\boldsymbol{\alpha}=\mathbf{0}\} \tag{5.44}
$$

by formula (7.6) in [19]. So, by (5.38),

$$
S_{m,\mathbf{k}} = f(\mathbf{k}). \tag{5.45}
$$

Also, the case $\boldsymbol{\alpha}=\mathbf{0}$ in (5.44) shows that the first two sums in (2.10), involving the $\gamma_{m,\mathbf{j}}$'s, are equal to 1. The second equality in (2.10) follows from the equality of the first sums in (2.4) and (2.5) to each other by taking there $\mathbf{n} = m\mathbf{1}$ and $f(\mathbf{x}) \equiv \operatorname{I}\{(m/2-1)\mathbf{1} \le \mathbf{x} \le m\mathbf{1}/2\}$; then each of the integrals in (2.4)–(2.6) equals 1.

By (5.39) and (2.9),

$$
\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} R_{m,\mathbf{k}} = R_m.
$$

So, (2.3) follows immediately from (5.36), (5.41), and (5.45).

In view of (2.9) and (2.11),

$$
|R_m| \le \tilde{R}_m := M_{2m}\, \frac{m}{2^{2m+p-1}} \sum_{\|\boldsymbol{\alpha}\|=2m} \frac{1}{\boldsymbol{\alpha}!} \int_0^1 \mathrm{d}s\, (1-s)^{2m-1} \int_{-\mathbf{1}}^{\mathbf{1}} \mathrm{d}\mathbf{v}\, |\mathbf{v}|^{\boldsymbol{\alpha}} \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} |\gamma_{m,\mathbf{j}}|\, \mathbf{j}^{\boldsymbol{\alpha}+\mathbf{1}}.
$$

Computing the integrals here, it is easy to check that $\tilde{R}_m$ equals the upper bound in (2.12). On the other hand, using the multinomial formula, the definition of $\gamma_{m,\mathbf{j}}$ in (2.7), and the Hölder inequality $\left(\sum_{r=1}^p |v_r j_r|\right)^{2m} \le p^{2m-1}\sum_{r=1}^p |v_r j_r|^{2m}$, we see that

$$\begin{aligned}\tilde{R}_m =&\frac{M_{2m}}{2^{2m}(2m)!}\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}|\gamma_{m,\mathbf{j}}|\,\mathbf{j}^{\mathbf{1}}\int_{\mathbf{0}}^{\mathbf{1}}\mathrm{d}\mathbf{v}\sum_{\|\boldsymbol{\alpha}\|=2m}\frac{(2m)!}{\boldsymbol{\alpha}!}(\mathbf{v}\mathbf{j})^{\boldsymbol{\alpha}}\\ =&\frac{M_{2m}}{2^{2m}(2m)!}\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}|\gamma_{m,\mathbf{j}}|\,\mathbf{j}^{\mathbf{1}}\int_{\mathbf{0}}^{\mathbf{1}}\mathrm{d}\mathbf{v}\left(\sum_{r=1}^{p}v_r j_r\right)^{2m}\\ \le&\frac{M_{2m}p^{2m-1}}{2^{2m}(2m)!}\left(\sum_{j_1=1}^{m}\cdots\sum_{j_p=1}^{m}|\gamma_{m,j_1}|j_1\dots|\gamma_{m,j_p}|j_p\right)\sum_{r=1}^{p}j_r^{2m}\int_{\mathbf{0}}^{\mathbf{1}}v_r^{2m}\,\mathrm{d}\mathbf{v}\\ =&\frac{M_{2m}p^{2m}}{2^{2m}(2m+1)!}\sum_{j=1}^{m}|\gamma_{m,j}|j^{2m+1}\left(\sum_{j=1}^{m}|\gamma_{m,j}|j\right)^{p-1}.\end{aligned} \tag{5.46}$$

By Proposition 4.4 in [19],

$$\sum_{j=1}^{m}|\gamma_{m,j}|j^{2m+1}\le 1.0331\pi\Lambda_*^m m^{2m+1}, \tag{5.47}$$

and for $m\ge 2$ the factor 1.0331 can be replaced by 1.001.

It follows from [23] that $\Gamma(x+1)/\Gamma(x+1/2)>\sqrt{x+1/\pi}$ for real $x>0$. For $x=m\in\mathbb{N}$, this inequality can be rewritten as $2^{2m}\Big/\binom{2m}{m}<\sqrt{\pi m+1}$. So, in view of (2.7),

$$\sum_{j=1}^{m}|\gamma_{m,j}|j=2^{2m}\Big/\binom{2m}{m}-1<\sqrt{\pi m}. \tag{5.48}$$

Collecting (5.46), (5.47), (5.48), and (2.14), we obtain (2.13).

Theorem 2.1 is now completely proved. □

To prove Theorem 3.2, we shall need the following multidimensional generalization of the fundamental theorem of calculus (FTC).

Lemma 5.1. (Multidimensional FTC) *Let F be any antiderivative of f. Take any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^p$. Then*

$$\int_{\mathbf{u}}^{\mathbf{v}}\mathrm{d}\mathbf{x}f(\mathbf{x})=\sum_{J\subseteq[p]}(-1)^{p-|J|}F(\mathbf{v}_J), \tag{5.49}$$

where $\mathbf{v}_J:=\mathbf{u}\mathbf{1}_{[p]\setminus J}+\mathbf{v}\mathbf{1}_J=\mathbf{u}+(\mathbf{v}-\mathbf{u})\mathbf{1}_J$.

For $p=2$ and $\mathbf{u}\le\mathbf{v}$, formula (5.49) appears in the proof of Lemma 6.2 [9]; a version of it for general p seems to be implicit on page 515 in [15]. Related formulas were given in [2, (III.1)] and [1, Lemma 1]. The following simple proof – which is essentially just a p-fold application of the one-dimensional FTC, plus some organizing – will be given here for readers' convenience.

Proof of Lemma 5.1. This will be done by induction in p. For $p=1$, (5.49) is the usual, one-dimensional FTC. Suppose that $p\ge 2$ and that (5.49) holds with $p-1$ in place of p.

Introduce some notation, as follows. For $\mathbf{x}=(x_1,\dots,x_{p-1},x_p)\in\mathbb{R}^p$, let $\tilde{\mathbf{x}}:=(x_1,\dots,x_{p-1})$, and similarly define $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$. Also, for any $J\subseteq[p-1]$, define $\tilde{\mathbf{v}}_J$ similarly to $\mathbf{v}_J$, but based on $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ rather than on $\mathbf{u}$ and $\mathbf{v}$. For any function $h\colon\mathbb{R}^p\to\mathbb{R}$ and any real x_p, let h_{x_p} denote the "cross-section" function from $\mathbb{R}^{p-1}$ to $\mathbb{R}$ defined by the formula $h_{x_p}(\tilde{\mathbf{x}}):=h(\mathbf{x})$, again for $\mathbf{x} = (x_1, \dots, x_{p-1}, x_p) \in \mathbb{R}^p$. Note that, for each real x_p, the function $\big(F^{(\mathbf{1}_{\{p\}})}\big)_{x_p}$ is an antiderivative of the function f_{x_p}.

For real u and v, let $\Delta_{u,v} := \delta_v - \delta_u$, where δ_x is the Dirac measure at x. Consider the signed product measures

$$\Delta_{\mathbf{u},\mathbf{v}} := \Delta_{u_1,v_1} \otimes \cdots \otimes \Delta_{u_p,v_p} = \sum_{J\subseteq[p]} (-1)^{p-|J|}\delta_{\mathbf{v}_J}$$

and $\tilde{\Delta}_{\mathbf{u},\mathbf{v}} := \Delta_{u_1,v_1} \otimes \cdots \otimes \Delta_{u_{p-1},v_{p-1}}$, so that $\Delta_{\mathbf{u},\mathbf{v}} = \tilde{\Delta}_{\mathbf{u},\mathbf{v}} \otimes \Delta_{u_p,v_p}$.

Now, appropriately rewriting the right-hand side of (5.49) and then using the Fubini theorem and the induction hypothesis, we have

$$\begin{aligned}
\sum_{J\subseteq[p]} (-1)^{p-|J|} F(\mathbf{v}_J) &= \int_{\mathbb{R}^p} \mathrm{d}\Delta_{\mathbf{u},\mathbf{v}}\, F && \text{(rewriting)}\\
&= \int_{\mathbb{R}} \Delta_{u_p,v_p}(\mathrm{d}x_p) \int_{\mathbb{R}^{p-1}} \mathrm{d}\tilde{\Delta}_{\mathbf{u},\mathbf{v}}\, F_{x_p} && \text{(Fubini)}\\
&= \int_{\mathbb{R}} \Delta_{u_p,v_p}(\mathrm{d}x_p) \sum_{J\subseteq[p-1]} (-1)^{p-1-|J|} F_{x_p}(\tilde{\mathbf{v}}_J) && \text{(similar rewriting)}\\
&= \sum_{J\subseteq[p-1]} (-1)^{p-1-|J|} \int_{\mathbb{R}} \Delta_{u_p,v_p}(\mathrm{d}x_p)\, F_{x_p}(\tilde{\mathbf{v}}_J)\\
&= \sum_{J\subseteq[p-1]} (-1)^{p-1-|J|} \int_{u_p}^{v_p} \mathrm{d}x_p\, \frac{\mathrm{d}}{\mathrm{d}x_p} F_{x_p}(\tilde{\mathbf{v}}_J) && \text{(one-dimensional FTC)}\\
&= \int_{u_p}^{v_p} \mathrm{d}x_p \sum_{J\subseteq[p-1]} (-1)^{p-1-|J|} \frac{\mathrm{d}}{\mathrm{d}x_p} F_{x_p}(\tilde{\mathbf{v}}_J)\\
&= \int_{u_p}^{v_p} \mathrm{d}x_p \sum_{J\subseteq[p-1]} (-1)^{p-1-|J|} \big(F^{(\mathbf{1}_{\{p\}})}\big)_{x_p}(\tilde{\mathbf{v}}_J)\\
&= \int_{u_p}^{v_p} \mathrm{d}x_p \int_{\tilde{\mathbf{u}}}^{\tilde{\mathbf{v}}} \mathrm{d}\tilde{\mathbf{x}}\, f_{x_p}(\tilde{\mathbf{x}}) && \text{(induction)}\\
&= \int_{\mathbf{u}}^{\mathbf{v}} \mathrm{d}\mathbf{x} f(\mathbf{x}). && \text{(Fubini)}
\end{aligned}$$

This completes the proof of Lemma 5.1. □

Proof of Theorem 3.2. Let

$$R_{m,f}(\mathbf{n}) := R_m,$$

with R_m as defined in (2.9). Then, by (3.16),

$$R_{m,f}(\mathbf{n}) \underset{\wedge\mathbf{n}\to\infty}{\longrightarrow} R_{m,f}(\infty). \tag{5.50}$$

Let

$$A_{m,F}(\mathbf{n}) := \sum_{J\subseteq[p]} (-1)^{p-|J|} A^J_{m,F}(\mathbf{n}) = \tilde{A}_{m,F}(\mathbf{n}) + (-1)^p A^{\emptyset}_{m,F}(\mathbf{n}), \tag{5.51}$$

in view of (3.18). By (2.3)–(2.4), Lemma 5.1, (5.51), and (3.19),

$$\begin{aligned}
&\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f(\mathbf{k}) - A_{m_0,F}(\mathbf{n}) + R_{m,f}(\mathbf{n}) \\
&= A_{m,F}(\mathbf{n}) - A_{m_0,F}(\mathbf{n}) \\
&= \sum_{J\subseteq[p]} (-1)^{p-|J|} \big(A^J_{m,F}(\mathbf{n}) - A^J_{m_0,F}(\mathbf{n})\big) \\
&= \sum_{J\subseteq[p]} (-1)^{p-|J|} \big(A^J_{m,T_J}(\mathbf{n}) - A^J_{m_0,T_J}(\mathbf{n}) + A^J_{m,F-T_J}(\mathbf{n}) - A^J_{m_0,F-T_J}(\mathbf{n})\big),
\end{aligned} \tag{5.52}$$

where $T_J = T_{J,\mathbf{n},m_0,F}$ is the Taylor polynomial of order $2m_0 - 1$ for the function F at the point $\mathbf{n1}_J - \mathbf{1}$, so that

$$T_J(\mathbf{x}) = \sum_{\|\boldsymbol{\alpha}\|\le 2m_0-1} \frac{F^{(\boldsymbol{\alpha})}(\mathbf{n1}_J - \mathbf{1})}{\boldsymbol{\alpha}!} (\mathbf{x} - \mathbf{n1}_J + \mathbf{1})^{\boldsymbol{\alpha}}$$

for $\mathbf{x} \in \mathbb{R}^p$.

Consider the monomial $P(\mathbf{x}) = \mathbf{x}^{\boldsymbol{\alpha}}$ of degree $\|\boldsymbol{\alpha}\| \le 2m_0 - 1$, so that $P(\mathbf{x}) = \prod_{r=1}^p P_r(x)$, where $P_r(x) := x^{\alpha_r}$.

Take any $r = 1, \dots, p$ and any $J \subseteq [p]$, and let $n_{r,J} := n_r \operatorname{I}\{r \in J\}$. Following the lines of the proof of Proposition 5.1 in [19] for the case when $f = P_r'$ and $F = P_r$, so that the polynomial T therein coincides with $F = P_r$, we see from [19, (5.5) and (7.19)] that

$$\begin{aligned}
\sum_{\beta=1-m}^{m-1} \tau_{m,1+|\beta|}\, P_r(n - 1/2 - \beta/2) = G_{m,P_r}(n) = G_{m_0,P_r}(n) \\
= \sum_{\beta=1-m_0}^{m_0-1} \tau_{m_0,1+|\beta|}\, P_r(n - 1/2 - \beta/2)
\end{aligned}$$

for any $n \in \mathbb{Z}_+$. So, by (3.20) and (2.8),

$$\begin{aligned}
A^J_{m,P}(\mathbf{n}) &= \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}} \tau_{m,\mathbf{1}+|\boldsymbol{\beta}|}\, P(\mathbf{n1}_J - \mathbf{1}/2 - \boldsymbol{\beta}/2) \\
&= \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}} \prod_{r=1}^{p} \big(\tau_{m,1+|\beta_r|}\, P_r(n_{r,J} - 1/2 - \beta_r/2)\big) \\
&= \prod_{r=1}^{p} \sum_{\beta=1-m}^{m-1} \big(\tau_{m,1+|\beta|}\, P_r(n_{r,J} - 1/2 - \beta/2)\big) \\
&= \prod_{r=1}^{p} \sum_{\beta=1-m_0}^{m_0-1} \big(\tau_{m_0,1+|\beta|}\, P_r(n_{r,J} - 1/2 - \beta/2)\big) = A^J_{m_0,P}(\mathbf{n}).
\end{aligned}$$

Since T_J is a polynomial of degree $\le 2m_0 - 1$ and $A^J_{m,F}(\mathbf{n})$ is linear in F, we conclude that

$$A^J_{m,T_J}(\mathbf{n}) - A^J_{m_0,T_J}(\mathbf{n}) = 0 \quad \text{for all } J \subseteq [p]. \tag{5.53}$$

Further, the remainder $(F-T_J)(\mathbf{n}\mathbf{1}_J-\mathbf{1}+\mathbf{u})$ at point $\mathbf{n}\mathbf{1}_J-\mathbf{1}+\mathbf{u}$ of the Taylor approximation T_J of F at $\mathbf{n}\mathbf{1}_J-\mathbf{1}$ equals (cf. (5.35))

$$\sum_{\|\boldsymbol{\alpha}\|=2m_0}\frac{2m_0}{\boldsymbol{\alpha}!}\,\mathbf{u}^{\boldsymbol{\alpha}}\int_0^1 \mathrm{d}s\,(1-s)^{2m_0-1}F^{(\boldsymbol{\alpha})}(\mathbf{n}\mathbf{1}_J-\mathbf{1}+s\mathbf{u}),$$

which, by (3.15), goes to 0 as $\wedge\mathbf{n}\to\infty$ unless $J=\emptyset$. So, by (3.19),

$$A^J_{m,F-T_J}(\mathbf{n})\underset{\wedge\mathbf{n}\to\infty}{\longrightarrow}0\quad\text{and}\quad A^J_{m_0,F-T_J}(\mathbf{n})\underset{\wedge\mathbf{n}\to\infty}{\longrightarrow}0\quad\text{unless } J=\emptyset.$$

It follows now by (5.53), (3.23), the linearity of $A^J_{m,F}$ in F, and (again) (5.53) that the limit of the last expression in (5.52) as $\wedge\mathbf{n}\to\infty$ equals

$$\begin{aligned}
&(-1)^p\big(A^{\emptyset}_{m,F-T_\emptyset}(\mathbf{0})-A^{\emptyset}_{m_0,F-T_\emptyset}(\mathbf{0})\big)\\
=&(-1)^p\big(A^{\emptyset}_{m,F}(\mathbf{0})-A^{\emptyset}_{m_0,F}(\mathbf{0})\big)-(-1)^p\big(A^{\emptyset}_{m,T_\emptyset}(\mathbf{0})-A^{\emptyset}_{m_0,T_\emptyset}(\mathbf{0})\big)\\
=&(-1)^p\big(A^{\emptyset}_{m,F}(\mathbf{0})-A^{\emptyset}_{m_0,F}(\mathbf{0})\big).
\end{aligned}$$

Now (3.17) follows, in view of (5.50) and (the second equality in) (5.51).

Inequality (3.22) follows immediately from (2.11) and (2.12)–(2.13).

Formula (3.23) follows immediately from (3.21).

Theorem 3.2 is completely proved. □

Proof of Theorem 3.3. Note that

$$\begin{aligned}
\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{c}-\mathbf{1}}f(\mathbf{k})&=\sum_{\mathbf{k}\ge\mathbf{0}}f(\mathbf{k})\,\mathrm{I}\{\mathbf{k}\le\mathbf{c}-\mathbf{1}\}\\
&=\sum_{\mathbf{k}\ge\mathbf{0}}f(\mathbf{k})\prod_{r=1}^{p}\big(\mathrm{I}\{k_r\le n_r+c_r-1\}-\mathrm{I}\{c_r\le k_r\le n_r+c_r-1\}\big)\\
&=\sum_{\mathbf{k}\ge\mathbf{0}}f(\mathbf{k})\sum_{J\subseteq[p]}(-1)^{|J|}\,\mathrm{I}\{k_r\le n_r+c_r-1\ \forall r\in[p]\setminus J,\\
&\qquad\qquad\qquad c_r\le k_r\le n_r+c_r-1\ \forall r\in J\}\\
&=\sum_{J\subseteq[p]}(-1)^{|J|}\sum_{\mathbf{k}=\mathbf{c}\mathbf{1}_J}^{\mathbf{n}+\mathbf{c}-\mathbf{1}}f(\mathbf{k})=\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}+\mathbf{c}-\mathbf{1}}f(\mathbf{k})+\sum_{\emptyset\ne J\subseteq[p]}(-1)^{|J|}\sum_{\mathbf{k}=\mathbf{c}\mathbf{1}_J}^{\mathbf{n}+\mathbf{c}-\mathbf{1}}f(\mathbf{k}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}+\mathbf{c}-\mathbf{1}}f(\mathbf{k})-\tilde{A}_{m_0,F}(\mathbf{n}+\mathbf{c})\\
&=\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{c}-\mathbf{1}}f(\mathbf{k})-\sum_{\emptyset\ne J\subseteq[p]}(-1)^{|J|}\sum_{\mathbf{k}=\mathbf{c}\mathbf{1}_J}^{\mathbf{n}+\mathbf{c}-\mathbf{1}}f(\mathbf{k})-\tilde{A}_{m_0,F}(\mathbf{n}+\mathbf{c})\\
&=\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{c}-\mathbf{1}}f(\mathbf{k})-\sum_{\emptyset\ne J\subseteq[p]}(-1)^{|J|}\Big(\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}+\mathbf{c}-\mathbf{c}\mathbf{1}_J-\mathbf{1}}f_{\mathbf{c}\mathbf{1}_J}(\mathbf{k})-\tilde{A}_{m_0,F_{\mathbf{c}\mathbf{1}_J}}(\mathbf{n}+\mathbf{c}-\mathbf{c}\mathbf{1}_J)\Big)+\mathcal{R},
\end{aligned}\tag{5.54}$$

where

$$\mathcal{R}:=-\sum_{\emptyset\ne J\subseteq[p]}(-1)^{|J|}\tilde{A}_{m_0,F_{\mathbf{c}\mathbf{1}_J}}(\mathbf{n}+\mathbf{c}-\mathbf{c}\mathbf{1}_J)-\tilde{A}_{m_0,F}(\mathbf{n}+\mathbf{c}).$$

By Lemma 5.1 with $F = 1$ (and $f = 0$),

$$\sum_{J\subseteq[p]}(-1)^{|J|} = 0 \quad\text{and hence}\quad \sum_{\emptyset\ne J\subseteq[p]}(-1)^{|J|} = -1. \tag{5.55}$$

Therefore and in view of (3.18) and (3.20),

$$\mathcal{R} = \sum_{\emptyset\ne J\subseteq[p]}(-1)^{|J|}\mathcal{R}_J,$$

where

$$\begin{aligned}\mathcal{R}_J :=& \tilde{A}_{m_0,F}(\mathbf{n}+\mathbf{c}) - \tilde{A}_{m_0,F_{\mathbf{c}\mathbf{1}_J}}(\mathbf{n}+\mathbf{c}-\mathbf{c}\mathbf{1}_J) = \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}}\tau_{m,\mathbf{1}+|\boldsymbol{\beta}|}\,\mathcal{R}_{J,\boldsymbol{\beta}},\\ \mathcal{R}_{J,\boldsymbol{\beta}} :=& \sum_{\emptyset\ne K\subseteq[p]}(-1)^{p-|K|}\big[H\big((\mathbf{n}+\mathbf{c})\mathbf{1}_K\big) - H\big(\mathbf{c}\mathbf{1}_J + (\mathbf{n}+\mathbf{c}-\mathbf{c}\mathbf{1}_J)\mathbf{1}_K\big)\big],\end{aligned}$$

and $H(\mathbf{x}) := F(\mathbf{x} - \mathbf{1}/2 - \boldsymbol{\beta}/2)$. Thus,

$$\mathcal{R} = \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}}\tau_{m,\mathbf{1}+|\boldsymbol{\beta}|}\sum_{\emptyset\ne K\subseteq[p]}(-1)^{p-|K|}\mathcal{R}_{\boldsymbol{\beta},K}, \tag{5.56}$$

where

$$\begin{aligned}\mathcal{R}_{\boldsymbol{\beta},K} :=& \sum_{\emptyset\ne J\subseteq[p]}(-1)^{|J|}\big[H\big((\mathbf{n}+\mathbf{c})\mathbf{1}_K\big) - H\big(\mathbf{c}\mathbf{1}_J + (\mathbf{n}+\mathbf{c}-\mathbf{c}\mathbf{1}_J)\mathbf{1}_K\big)\big]\\ =& \sum_{\emptyset\ne J\subseteq[p]}(-1)^{|J|}\big[H\big((\mathbf{n}+\mathbf{c})\mathbf{1}_K\big) - H\big((\mathbf{n}+\mathbf{c})\mathbf{1}_K + \mathbf{c}\mathbf{1}_{J\setminus K}\big)\big]\\ =& \sum_{L\in\mathcal{L}_K}\big[H\big((\mathbf{n}+\mathbf{c})\mathbf{1}_K\big) - H\big((\mathbf{n}+\mathbf{c})\mathbf{1}_K + \mathbf{c}\mathbf{1}_L\big)\big]\sum_{J\in\mathcal{J}_{K,L}}(-1)^{|J|},\end{aligned} \tag{5.57}$$

$$\mathcal{L}_K := \{L\colon L\subseteq[p],\ L\ne\emptyset,\ L\cap K=\emptyset\},\quad \mathcal{J}_{K,L} := \{J\colon \emptyset\ne J\subseteq[p],\ J\setminus K = L\}.$$

For any $K\subseteq[p]$ and any $L\in\mathcal{L}_K$, the map $J\mapsto I_J := J\cap K$ is a bijection of the set $\mathcal{J}_{K,L}$ onto the set $\{I\colon I\subseteq K\}$, and for any $J\in\mathcal{J}_{K,L}$ the set J is the disjoint union of the sets I_J and L, so that $|J| = |I_J| + |L|$. It follows by (5.55) that for any $K\subseteq[p]$ and any $L\in\mathcal{L}_K$ one has $\sum_{J\in\mathcal{J}_{K,L}}(-1)^{|J|} = \sum_{I\subseteq K}(-1)^{|I|}(-1)^{|L|} = 0$. Looking back at (5.57) and (5.56), we see that $\mathcal{R} = 0$.

Letting now $\wedge\mathbf{n} \to \infty$ and recalling (5.54), (3.17), the definition (3.26) of $R_{m,f,\mathbf{c}}(\infty)$, and formulas (3.23), (3.21), and (3.18), we have

$$\begin{aligned}
&\sum_{\mathbf{k}\geq\mathbf{0}}^{\mathsf{Alt}} f(\mathbf{k}) - \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{c}-\mathbf{1}} f(\mathbf{k}) \\
&= -\sum_{\emptyset\neq J\subseteq[p]} (-1)^{|J|} \sum_{\mathbf{k}\geq\mathbf{0}}^{\mathsf{Alt}} f_{\mathbf{c}\mathbf{1}_J}(\mathbf{k}) \\
&= -\sum_{\emptyset\neq J\subseteq[p]} (-1)^{|J|} \big[(-1)^p A^{\emptyset}_{m,F_{\mathbf{c}\mathbf{1}_J}}(\mathbf{0}) - R_{m,f_{\mathbf{c}\mathbf{1}_J}}(\infty)\big] \\
&= -R_{m,f,\mathbf{c}}(\infty) - \sum_{\emptyset\neq J\subseteq[p]} (-1)^{p-|J|} \sum_{\boldsymbol{\alpha}=\mathbf{0}}^{(m-1)\mathbf{1}} \tau_{m,\mathbf{1}+\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta}:\ |\boldsymbol{\beta}|=\boldsymbol{\alpha}} F(\mathbf{c}\mathbf{1}_J + \boldsymbol{\beta}/2 - \mathbf{1}/2) \\
&= -R_{m,f,\mathbf{c}}(\infty) - \tilde{A}_{m,F}(\mathbf{c}),
\end{aligned}$$

which completes the proof of Theorem 3.3. □

Proof of Proposition 4.1. Let $\mathbf{a}_1, \dots, \mathbf{a}_p$ denote the columns of the matrix A, so that $\mathbf{a}_i \in \mathbb{Z}^p$ for each $i \in [p]$ and

$$C := A\mathbb{R}^+_J = \sum_{i\in[p]} \mathbb{R}^+_{\varepsilon_i}\mathbf{a}_i, \quad \text{where} \quad \varepsilon_i := 1 - [\![J]\!](i).$$

If the matrix A is unimodular, there is nothing to prove. So, w.l.o.g., $|\det A| \geq 2$. Then there is a vector $\mathbf{w} \in \mathbb{Z}^p \setminus \{\mathbf{0}\}$ such that

$$\mathbf{w} = w_1\mathbf{a}_1 + \cdots + w_p\mathbf{a}_p \tag{5.58}$$

for some real numbers $w_1, \dots, w_p$ in the interval $[0, 1)$ (in fact, there are exactly $|\det A| - 1$ such vectors $\mathbf{w}$). Thus, w.l.o.g. for some $k \in [p]$ one has

$$w_j > 0 \text{ for } j \in [k] \quad \text{and} \quad w_j = 0 \text{ for } j \in [p] \setminus [k]. \tag{5.59}$$

For each $i \in [k]$, let A_i be the (integral) matrix obtained from the matrix A by replacing its i-th column, $\mathbf{a}_i$, by $\mathbf{w}$; then $\det A_i = w_i \det A$ and hence

$$0 < |\det A_i| < |\det A|. \tag{5.60}$$

We shall see that (4.32) holds with $I = [k]$, the matrices A_i just defined, and some subsets $J_1, \dots, J_k$ of the set $[p]$.

Then, repeating the step described in the last paragraph – for each of the matrices $A_1, \dots, A_k$ in place of A, in view of (5.60) we shall eventually obtain (4.32) with unimodular $p \times p$ matrices A_i over $\mathbb{Z}$, as required. This step relies mainly on the following combinatorial lemma.

Lemma 5.2. *Let $\mathbf{a}_1, \dots, \mathbf{a}_p$, C, $\mathbf{w}$, and k be as described above. For each $i \in [k]$, let*

$$C_i := \mathbb{R}^+_{\varepsilon_{ii}}\mathbf{w} + \sum_{j\in[p]\setminus\{i\}} \mathbb{R}^+_{\varepsilon_{ij}}\mathbf{a}_j, \tag{5.61}$$

where the ε_{ij}'s are any numbers in the set $\{0, 1\}$ satisfying the following conditions:

(i) $\varepsilon_{ij} = \varepsilon_j$ for $i \in [k]$ and $j \in [p] \setminus [k]$;
(ii) $\varepsilon_{ii} = \varepsilon_i$ for $i \in [k]$;
(iii) $\varepsilon_{ij} + \varepsilon_{ji} = 1$ for any distinct i and j in $[k]$;
(iv) for each $i \in [k]$, the condition $\varepsilon_i = 1$ implies $\varepsilon_{ij} \leq \varepsilon_j$ for all $j \in [k]$;

(v) *for each nonempty subset J of the set $[k]$, there is some $i \in J$ such that for all $j \in J \setminus \{i\}$ one has $\varepsilon_{ij} = 1$.*

Then

$$\llbracket C \rrbracket = \sum_{i\in[k]} \llbracket C_i \rrbracket. \tag{5.62}$$

We also have

Lemma 5.3. *Take any $\varepsilon_1, \dots, \varepsilon_p$ in $\{0, 1\}$ and any $k \in [p]$. Then there exist numbers ε_{ij} in the set $\{0, 1\}$ satisfying all the conditions (i)–(v) in Lemma 5.2.*

We shall prove these two lemmas in a moment.

Letting now $J_i = \{j \in [p] : \varepsilon_{ij} = 0\}$ for each $i \in [k]$ (so that $\varepsilon_{ij} = 1 - \llbracket J_i \rrbracket(j)$ for all $i \in [k]$ and $j \in [p]$), we will have $C_i = A_i\mathbb{R}^+_{J_i}$ for $i \in [k]$, which will complete the step described in the paragraph containing formulas (5.58)–(5.60). Thus, to complete the entire proof of Proposition 4.1, it remains to prove Lemmas 5.2 and 5.3.

Proof of Lemma 5.2. Take any $\mathbf{x} \in \mathbb{R}^p$. Let $(y_1, \dots, y_p) = (y_1(\mathbf{x}), \dots, y_p(\mathbf{x}))$ denote the p-tuple of the coordinates of the vector $\mathbf{x}$ in the basis $(\mathbf{a}_1, \dots, \mathbf{a}_p)$ of $\mathbb{R}^p$, so that

$$\mathbf{x} = \sum_{j\in[p]} y_j \mathbf{a}_j. \tag{5.63}$$

Also, for each $i \in [k]$, let $(y_{i1}, \dots, y_{ip}) = (y_{i1}(\mathbf{x}), \dots, y_{ip}(\mathbf{x}))$ denote the p-tuple of the coordinates of the vector $\mathbf{x}$ in the basis $(\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{w}, \mathbf{a}_{i+1}, \dots, \mathbf{a}_p)$ of $\mathbb{R}^p$, so that

$$\mathbf{x} = y_{ii}\mathbf{w} + \sum_{j\in[p]\setminus\{i\}} y_{ij}\mathbf{a}_j = y_{ii}w_i\mathbf{a}_i + \sum_{j\in[p]\setminus\{i\}} (y_{ii}w_j + y_{ij})\mathbf{a}_j.$$

In view of (5.58) and (5.59),

$$y_{ij} = y_j \quad \text{for } i \in [k],\ j \in [p] \setminus [k]. \tag{5.64}$$

As for i and j in $[k]$, we have $y_i = y_{ii}w_i$ and $y_j = y_{ii}w_j + y_{ij} = \frac{y_i}{w_i}\, w_j + y_{ij}$ if $j \neq i$, which can be rewritten as

$$\forall (i, j) \in [k] \times [k] \ \Big(y_{ii}w_i = y_i \quad \text{and} \quad j \neq i \implies \frac{y_{ij}}{w_j} = r_j - r_i \Big), \tag{5.65}$$

where

$$r_j := r_j(\mathbf{x}) := \frac{y_j}{w_j}.$$

Note that (5.62) means precisely that C is the disjoint union of the C_i's. Thus, the proof of Lemma 5.2 will be completed in the following three steps.

Step 1: checking that $C_i \subseteq C$ for each $i \in [k]$. Take indeed any $i \in [k]$, and then take any $\mathbf{x} \in C_i$, so that, by (5.61), $y_{ij} \in \mathbb{R}^+_{\varepsilon_{ij}}$ for all $j \in [p]$. Then $y_{ii} \in \mathbb{R}^+_{\varepsilon_{ii}} = \mathbb{R}^+_{\varepsilon_i}$ by condition (ii) of Lemma 5.2 and hence $y_i = y_{ii}w_i \in \mathbb{R}^+_{\varepsilon_i}$. Also, by (5.64) and condition (i) of Lemma 5.2, $y_j = y_{ij} \in \mathbb{R}^+_{\varepsilon_{ij}} = \mathbb{R}^+_{\varepsilon_j}$ for $j \in [p] \setminus [k]$.

If $y_i > 0$, then $y_j = \frac{y_i}{w_i}\, w_j + y_{ij} > y_{ij} \geq 0$ for all $j \in [k] \setminus \{i\}$, whence $y_j > 0$ for all $j \in [k]$. So, by (5.63), $\mathbf{x} \in \sum_{j\in[k]} \mathbb{R}^+_0\mathbf{a}_j + \sum_{j\in[p]\setminus[k]} \mathbb{R}^+_{\varepsilon_j}\mathbf{a}_j \subseteq \sum_{j\in[p]} \mathbb{R}^+_{\varepsilon_j}\mathbf{a}_j = C$.

If now $y_i = 0$, then the mentioned condition $y_i \in \mathbb{R}^+_{\varepsilon_i}$ implies $\varepsilon_i = 1$. So, by condition (iv) of Lemma 5.2, for all $j \in [k]$ we have $\varepsilon_{ij} \leq \varepsilon_j$ and hence $\mathbb{R}^+_{\varepsilon_{ij}} \subseteq \mathbb{R}^+_{\varepsilon_j}$, which yields $y_j = \frac{y_i}{w_i}\, w_j + y_{ij} = y_{ij} \in \mathbb{R}^+_{\varepsilon_{ij}} \subseteq \mathbb{R}^+_{\varepsilon_j}$. So, in this case as well, $\mathbf{x} \in C$.

Step 2: checking that the C_i's are disjoint. Take any distinct i and j in $[k]$, and then take any $\mathbf{x} \in C_i \cap C_j$. Then $y_{ij} \in \mathbb{R}^+_{\varepsilon_{ij}}$, whence, by (5.65), $r_j - r_i = y_{ij}/w_j \in \mathbb{R}^+_{\varepsilon_{ij}}$. Similarly, $r_i - r_j \in \mathbb{R}^+_{\varepsilon_{ji}}$, that is, $r_j - r_i \in -\mathbb{R}^+_{\varepsilon_{ji}} = \mathbb{R} \setminus \mathbb{R}^+_{\varepsilon_{ij}}$, by condition (iii) of Lemma 5.2. Thus, $r_j - r_i \in \mathbb{R}^+_{\varepsilon_{ij}} \cap \big(\mathbb{R} \setminus \mathbb{R}^+_{\varepsilon_{ij}}\big) = \emptyset$, which is a contradiction.

Step 3: checking that $C \subseteq \bigcup_{i\in[k]} C_i$. Take any $\mathbf{x} \in C$, so that $y_j \in \mathbb{R}^+_{\varepsilon_j}$ for all $j \in [p]$. Let

$$J_{\mathbf{x}} := \{i \in [k] \colon r_i(\mathbf{x}) \le r_j(\mathbf{x})\ \forall j \in [k]\}.$$

Then, by (5.65), $y_{ij} \ge 0$ for all $i \in J_{\mathbf{x}}$ and $j \in [k]$. Moreover, $r_j(\mathbf{x}) > r_i(\mathbf{x})$ for all $i \in J_{\mathbf{x}}$ and $j \in [k] \setminus J_{\mathbf{x}}$.

So, again by (5.65), for all $i \in J_{\mathbf{x}}$ and $j \in [k] \setminus J_{\mathbf{x}}$ we have $y_{ij} > 0$, so that $y_{ij} \in \mathbb{R}^+_0 \subseteq \mathbb{R}^+_{\varepsilon_{ij}}$. Note that $J_{\mathbf{x}} \ne \emptyset$. So, by condition (v) of Lemma 5.2, there is some $i_{\mathbf{x}} \in J_{\mathbf{x}}$ such that for all $j \in J_{\mathbf{x}} \setminus \{i_{\mathbf{x}}\}$ one has $\varepsilon_{i_{\mathbf{x}}j} = 1$, so that $y_{i_{\mathbf{x}}j} \in \mathbb{R}^+_{\varepsilon_{i_{\mathbf{x}}j}}$. Thus, $y_{i_{\mathbf{x}}j} \in \mathbb{R}^+_{\varepsilon_{i_{\mathbf{x}}j}}$ for all $j \in [k] \setminus \{i_{\mathbf{x}}\}$. Also, $y_{i_{\mathbf{x}}i_{\mathbf{x}}} \in \mathbb{R}^+_{\varepsilon_{i_{\mathbf{x}}i_{\mathbf{x}}}}$ – in view of the first equality in (5.65), the condition $y_i \in \mathbb{R}^+_{\varepsilon_i}$ for all $i \in [p]$, and condition (ii) of Lemma 5.2. Moreover, $y_{i_{\mathbf{x}}j} \in \mathbb{R}^+_{\varepsilon_{i_{\mathbf{x}}j}}$ for all $j \in [p] \setminus [k]$ – in view of (5.64), the condition $y_j \in \mathbb{R}^+_{\varepsilon_j}$ for all $j \in [p]$, and condition (i) of Lemma 5.2. We conclude that $y_{i_{\mathbf{x}}j} \in \mathbb{R}^+_{\varepsilon_{i_{\mathbf{x}}j}}$ for all $j \in [p]$, that is, $\mathbf{x} \in C_{i_{\mathbf{x}}} \subseteq \bigcup_{i\in[k]} C_i$.

Lemma 5.2 is now proved. □

Proof of Lemma 5.3. For $i \in [k]$ and $j \in [p] \setminus [k]$, let $\varepsilon_{ij} := \varepsilon_j$, in accordance with condition (i) of Lemma 5.2.

Similarly, let $\varepsilon_{ii} := \varepsilon_i$ for $i \in [k]$, in accordance with condition (ii) of Lemma 5.2.

Next, w.l.o.g. ε_j is nondecreasing in $j \in [k]$. Let then $\varepsilon_{ij} := 1$ and $\varepsilon_{ji} := 0$ for all i and j in $[k]$ with $i < j$.

It is now straightforward to check that all the conditions (i)–(v) in Lemma 5.2 hold. In particular, concerning condition (iv), note that, if $\varepsilon_i = 1$ and $\varepsilon_{ij} = 1$ for some distinct i and j in $[k]$, then $i < j$ and hence $1 = \varepsilon_i \le \varepsilon_j$, so that $\varepsilon_j = 1$. Concerning condition (v), for each nonempty subset J of the set $[k]$, let $i := \min J$; then for all $j \in J \setminus \{i\}$ one has $i < j$ and hence $\varepsilon_{ij} = 1$. Lemma 5.3 is now proved. □

The entire proof of Proposition 4.1 is thus complete. □

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Research Article

On the singular values of the incomplete Beta function

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ABSTRACT. A new definition of the incomplete beta function as a distribution-valued meromorphic function is given and the finite parts of it and of its partial derivatives at the singular values are calculated and compared with formulas in the literature.

Keywords: Beta function, distribution theory, finite parts.

2020 Mathematics Subject Classification: 33B20, 46F10.

1. INTRODUCTION AND NOTATION

This paper originated when one of the authors (N.O.) came across the article [3]. The explicit formulas in [3] were interesting, but we could not concur with the overall framework in which they had been derived. The calculations in [3] are based on van der Corput's "neutrix calculus", see [1], a way of evaluating divergent integrals, which was inspired by Hadamard's method. This "technique of neglecting appropriately defined infinite quantities", see [12, p. 984] , produces numbers, not distributions. Accordingly, the results in [3] represent the incomplete beta function only on the open interval $(0, 1)$ and do not furnish a distribution on $\mathbf{R}$. So we thought that it might be reasonable to reconsider the calculations in [3] from the nowadays generally adopted viewpoint of distribution theory.

Let us mention that regularizations in Hadamard's sense but employing L. Schwartz' theory of distributions were investigated in [9, pp. 15–19], for three kinds of distributions.

Classically, the incomplete beta function is defined by the integral

$$B_{\lambda,\mu}(x) = \int_0^x t^{\lambda-1}(1-t)^{\mu-1}\,\mathrm{d}t, \quad 0 \le x \le 1,\ \mathrm{Re}\,\lambda > 0,\ \mathrm{Re}\,\mu > 0,$$

see [4, Equ. 8.931]. The goal of the article [3] as well as of this paper consists in defining and evaluating $B_{\lambda,\mu}$ and its partial derivatives with respect to λ and μ at the "singular values", i.e., if $\lambda \in -\mathbf{N}_0$ or $\mu \in -\mathbf{N}_0$.

In Section 2, we define $B_{\lambda,\mu}$ as distributions depending analytically on $(\lambda, \mu) \in \mathbf{C}^2$. At the poles, e.g. if $\lambda = -k \in -\mathbf{N}_0$, we set $B_{-k,\mu} = \mathrm{Pf}_{\lambda=-k}\, B_{\lambda,\mu}$, i.e., $B_{-k,\mu}$ is defined as the finite part of the Laurent series of $B_{\lambda,\mu}$ about $\lambda = -k$. The procedure of embedding a function into a family of distributions which depend analytically on a parameter goes back to M. Riesz, see [14, pp. 31, 32], L. Schwartz, see [15, p. 39], and J. Dieudonné, see [2, pp. 260–262]. With respect to distribution-valued analytic or meromorphic functions, we refer the reader also to [10].

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In Section 3, we collect some algebraic reduction formulas, which show that our task can be reduced to evaluating $B, \partial_\lambda B, \partial_\mu B$ if λ or μ are 1. This is eventually done for B in Section 4 and for $\partial_\lambda B, \partial_\mu B$ in Section 5, respectively.

Let us introduce some notation. As usual, an empty series, as, e.g., in $\sum_{j=1}^{0} c_j$, sums to zero. $\mathbf{N}$ and $\mathbf{N}_0$ denote the sets of positive and of non-negative integers, respectively. We employ the standard notation for the distribution spaces $\mathcal{D}'$, $\mathcal{E}'$, the dual spaces of the spaces $\mathcal{D}$, $\mathcal{E}$ of "test functions" and of C^∞ functions, respectively, see [15, 6, 11]. For the evaluation of a distribution T on a test function ϕ, we use angle brackets, i.e., $\langle \phi, T \rangle$. In this paper, all distributions are on the real axis $\mathbf{R}$, i.e., they belong to $\mathcal{D}'(\mathbf{R})$, but usually depend meromorphically on the complex variables λ, μ. Differentiation with respect to x is denoted by the apostrophe, differentiation with respect to λ, μ by $\partial_\lambda, \partial_\mu$ or $\partial/\partial\lambda, \partial/\partial\mu$ or ∂_1, ∂_2.

The Heaviside function is denoted by Y, see [15, p. 36]. We write δ for the delta distribution with support in 0, i.e., $\delta = Y'$, and δ_1 for the delta distribution with support in 1, i.e., $\delta_1 = Y(x-1)'$. The letter ψ denotes the logarithmic derivative Γ'/Γ of the gamma function and $\mathcal{L}_2$ denotes the dilogarithm, i.e., $\mathcal{L}_2(0) = 0$ and

$$\mathcal{L}_2(x) = \oint_0^1 \frac{\log t}{t - x^{-1}}\, \mathrm{d}t, \qquad x \in \mathbf{R} \setminus \{0\},$$

see [5, Section 323].

2. Definition of the incomplete beta function

Let us first recall some facts concerning the distribution $x_+^\lambda = Y(x)x^\lambda$, see [6, Section 3.2, p. 68], , [11, Exs. 1.3.9, 1.4.8, pp. 32, 49]. If $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > -1$, then x_+^λ is a locally integrable function on $\mathbf{R}$ and hence belongs to $\mathcal{D}'(\mathbf{R})$. The function

$$\{\lambda \in \mathbf{C};\ \operatorname{Re}\lambda > -1\} \longrightarrow \mathcal{D}'(\mathbf{R}) : \lambda \longmapsto x_+^\lambda$$

is analytic and can analytically be extended to $\mathbf{C} \setminus (-\mathbf{N})$. This extension, which is also denoted by x_+^λ, is meromorphic on $\mathbf{C}$ and has simple poles on $-\mathbf{N}$ with the residues

$$\operatorname*{Res}_{\lambda=-k-1} x_+^\lambda = (-1)^k \delta^{(k)}/k!$$

for $k \in \mathbf{N}_0$. For abbreviation, we also set

$$x_+^{-k} = \operatorname*{Pf}_{\lambda=-k} x_+^\lambda \ \text{ if } k \in \mathbf{N}.$$

In [13, pp. 11, 12], the distributions x_+^λ are called *Hadamard kernels*.

Note that $x \cdot x_+^\lambda = x_+^{\lambda+1}$ holds for each $\lambda \in \mathbf{C}$. In contrast, the differentiation formula $(x_+^\lambda)' = \lambda x_+^{\lambda-1}$ is valid for $\lambda \in \mathbf{C} \setminus (-\mathbf{N}_0)$ by analytic continuation, but at $\lambda = -k$, $k \in \mathbf{N}_0$, we obtain

$$\begin{aligned}
(x_+^{-k})' = \operatorname*{Pf}_{\lambda=-k} (x_+^\lambda)' &= \operatorname*{Pf}_{\lambda=-k} \lambda x_+^{\lambda-1} \\
&= \operatorname*{Pf}_{\lambda=-k} [(\lambda+k)x_+^{\lambda-1} - k x_+^{\lambda-1}] \\
&= \lim_{\lambda\to-k} (\lambda+k)x_+^{\lambda-1} - k x_+^{-k-1} \\
&= \operatorname*{Res}_{\lambda=-k} x_+^{\lambda-1} - k x_+^{-k-1} \\
&= \frac{(-1)^k \delta^{(k)}}{k!} - k x_+^{-k-1},
\end{aligned}$$

see also [15, Equ. (II, 2; 28), p. 42], [7, p. 151, Remark], [6, Equ. (3.2.2)$''$, p. 69], , [11, p. 50].

By differentiation with respect to λ, we obtain the distribution-valued function $\lambda \mapsto \partial_\lambda(x_+^\lambda) = x_+^\lambda \log x$, which is meromorphic in λ with double poles on $-\mathbf{N}$. As above we define $x_+^{-k}\log x := \operatorname{Pf}_{\lambda=-k} x_+^\lambda \log x$ for $k \in -\mathbf{N}$ and similarly for the higher derivatives with respect to λ. Hence the Laurent series of x_+^λ about the pole $\lambda = -k$, $k \in \mathbf{N}$, is given by

$$x_+^\lambda = \frac{(-1)^{k-1}\delta^{(k-1)}}{(k-1)!(\lambda+k)} + \sum_{j=0}^{\infty} \frac{x_+^{-k}\log^j x}{j!}(\lambda+k)^j, \quad 0 < |\lambda+k| < 1. \tag{2.1}$$

(In fact, $\operatorname{Pf}_{\lambda=-k} \partial_\lambda^j x_+^\lambda = \operatorname{Pf}_{\lambda=-k} x_+^\lambda \log^j x = x_+^{-k}\log^j x$ for $j \in \mathbf{N}_0$.)

Now we are prepared for giving a distributional definition of the incomplete beta function.

Definition 2.1. *For $\lambda, \mu \in \mathbf{C}$, we call $S_{\lambda,\mu} = x_+^{\lambda-1}\cdot(1-x)_+^{\mu-1} \in \mathcal{E}'(\mathbf{R})$ the M. Riesz kernels of the incomplete beta function and $B_{\lambda,\mu} = Y * S_{\lambda,\mu} \in \mathcal{D}'(\mathbf{R})$ the incomplete (Eulerian) beta function.*

Note that the multiplication of the two distributional factors $x_+^{\lambda-1}$ and $(1-x)_+^{\mu-1}$ of $S_{\lambda,\mu}$ is well-defined since their respective singular supports $\{0\}$ and $\{1\}$ are disjoint, see [6, Thm. 8.2.10, p. 267]. We also observe that $B_{\lambda,\mu}$ is uniquely determined by the two conditions

$$(i)\ B'_{\lambda,\mu} = S_{\lambda,\mu} \qquad \text{and} \qquad (ii)\ \operatorname{supp} B_{\lambda,\mu} \subset [0,\infty).$$

According to the above, the function $(\lambda,\mu) \mapsto S_{\lambda,\mu}$ is analytic for $\lambda, \mu \in \mathbf{C}\setminus(-\mathbf{N}_0)$. Therefore the same holds true for $B_{\lambda,\mu}$ and its derivatives $(\partial_1 B)_{\lambda,\mu} = \partial B_{\lambda,\mu}/\partial\lambda$ and $(\partial_2 B)_{\lambda,\mu} = \partial B_{\lambda,\mu}/\partial\mu$. As before, we abbreviate

$$(\partial_1 B)_{-k,\mu} := \operatorname*{Pf}_{\lambda=-k}(\partial_1 B)_{\lambda,\mu}$$

and

$$(\partial_1 B)_{-k,-l} := \operatorname*{Pf}_{\lambda=-k}\operatorname*{Pf}_{\mu=-l}(\partial_1 B)_{\lambda,\mu} \text{ if } k,l \in \mathbf{N}_0, \mu \in \mathbf{C}\setminus(-\mathbf{N}_0),$$

and similarly for $\partial_2 B$. As related in Section 1, we aim at calculating explicitly $B_{k,l}, (\partial_1 B)_{k,l}, (\partial_2 B)_{k,l}$ for the singular values, i.e., if $k,l \in \mathbf{Z}$ and $[k \in -\mathbf{N}_0$ or $l \in -\mathbf{N}_0]$.

3. Algebraic reduction formulas

The trivial identity

$$S_{\lambda,\mu} = 1\cdot S_{\lambda,\mu} = (x+1-x)\cdot S_{\lambda,\mu} = S_{\lambda+1,\mu} + S_{\lambda,\mu+1}$$

leads to representations of $S_{k,l}$, $k,l \in \mathbf{Z}$, by $S_{j,1}$ and $S_{1,j}$, $j \in \mathbf{Z}$. By convolution with Y and by differentiation with respect to λ and μ, we obtain similar representation formulas for $B_{k,l}, (\partial_1 B)_{k,l}$ and $(\partial_2 B)_{k,l}$, respectively.

Lemma 3.1. *Let $\lambda, \mu \in \mathbf{C}$ and $k, l \in \mathbf{N}_0$. Then the following holds:*

$$S_{\lambda,\mu+l} = \sum_{j=0}^{l}\binom{l}{j}(-1)^j S_{\lambda+j,\mu}; \tag{3.2}$$

$$S_{\lambda-k,\mu-l} = \sum_{j=0}^{k}\binom{k+l-j}{l}S_{\lambda-j,\mu+1} + \sum_{j=0}^{l}\binom{k+l-j}{k}S_{\lambda+1,\mu-j} \tag{3.3}$$

and for $k < l$ we have

$$S_{\lambda-k,\mu+l} = \sum_{j=0}^{k}\binom{l-1}{j}(-1)^j S_{\lambda-k+j,\mu+1} + (-1)^{k+1}\sum_{j=1}^{l-k-1}\binom{l-j-1}{k}S_{\lambda+1,\mu+j}. \tag{3.4}$$

The corresponding formulas hold likewise if S is replaced throughout by $B = Y * S$, by $\partial_1 B$, or by $\partial_2 B$, respectively.

Proof. Equation (3.2) follows directly from the binomial formula:

$$\begin{aligned} S_{\lambda,\mu+l} = x_+^{\lambda-1}(1-x)_+^{\mu+l-1} &= S_{\lambda,\mu} \cdot (1-x)^l \\ &= S_{\lambda,\mu} \cdot \sum_{j=0}^{l} \binom{l}{j} (-1)^j x^j \\ &= \sum_{j=0}^{l} \binom{l}{j} (-1)^j S_{\lambda+j,\mu}. \end{aligned}$$

Formula (3.3) follows similarly by using the polynomial identity

$$1 = \sum_{j=0}^{k} \binom{k+l-j}{l} x^{k-j} (1-x)^{l+1} + \sum_{j=0}^{l} \binom{k+l-j}{k} x^{k+1} (1-x)^{l-j}. \tag{3.5}$$

For completeness, let us indicate shortly how the identity (3.5) is derived from a Mittag-Leffler expansion. In fact, in the representation

$$z^{-k-1}(1-z)^{-l-1} = \sum_{j=0}^{k} c_j z^{-j-1} + \sum_{j=0}^{l} d_j (1-z)^{-j-1}, \quad z \in \mathbf{C} \setminus \{0,1\},$$

the coefficients c_j can be determined from the Laurent expansion

$$z^{-k-1}(1-z)^{-l-1} = \sum_{n=0}^{\infty} \binom{-l-1}{n} (-1)^n z^{n-k-1}, \quad 0 < |z| < 1,$$

i.e.,

$$n = k-j \quad \text{and} \quad c_j = \binom{-l-1}{k-j} (-1)^{k-j} = \binom{k+l-j}{l}, \quad j = 0, \dots, k,$$

and similarly for d_j, $j = 0, \dots, l$.

Equation (3.4) follows in the same way by using the polynomial identity

$$(1-x)^{l-1} = \sum_{j=0}^{k} \binom{l-1}{j} (-1)^j x^j + (-1)^{k+1} \sum_{j=1}^{l-k-1} \binom{l-j-1}{k} x^{k+1} (1-x)^{j-1}.$$

This can be shown by first replacing x by $1-x$ and then employing the Mittag-Leffler expansion of $z^{l-1}(1-z)^{-k-1}$ with respect to the poles 0 and ∞. $\square$

Remark 3.1. *Let us illustrate how the formulas (3.2), (3.3) and (3.4) are applied in order to reduce the singular values $B_{k,l}$ to $B_{j,1}$ and $B_{1,j}$, $j,k,l \in \mathbf{Z}$. E.g., setting $\lambda = \mu = k = l = 0$ in formula (3.3) yields the equation $B_{0,0} = B_{0,1} + B_{1,0}$. Instead, if $l \in \mathbf{N}$ and if we set $\lambda = 0$, $\mu = 1$ and replace l by $l-1$, then formula (3.2) implies*

$$B_{0,l} = \sum_{j=0}^{l-1} \binom{l-1}{j} (-1)^j B_{j,1}, \quad l \in \mathbf{N}. \tag{3.6}$$

Note that formula (3.4) leads to a different representation by setting $\lambda = k = \mu = 0$:

$$B_{0,l} = B_{0,1} - \sum_{j=1}^{l-1} B_{1,j}, \quad l \in \mathbf{N}. \tag{3.7}$$

The formulas (3.6) and (3.7) coincide in the cases $l = 1$ and $l = 2$, but yield different representations for $l \geq 3$. E.g.,

$$B_{0,3} = B_{0,1} - 2B_{1,1} + B_{2,1} = B_{0,1} - B_{1,1} - B_{1,2}.$$

(The last equation amounts to $B_{2,1} = B_{1,1} - B_{1,2}$.)

Let us finally investigate how $B_{\lambda,\mu}$ and $B_{\mu,\lambda}$ are connected. For this we extend the definition of the *complete beta function* or, as it is also called, the *Eulerian integral of the first kind*

$$B(\lambda, \mu) = \int_0^1 x^{\lambda-1}(1-x)^{\mu-1}\,\mathrm{d}x, \qquad \lambda, \mu \in \mathbf{C},\ \operatorname{Re}\lambda > 0,\ \operatorname{Re}\mu > 0,$$

first, as usual, to $[\mathbf{C} \setminus (-\mathbf{N}_0)]^2$ by analytic continuation, i.e.,

$$B(\lambda, \mu) = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}, \qquad \lambda, \mu \in \mathbf{C} \setminus (-\mathbf{N}_0),$$

and then to the singular values in $-\mathbf{N}_0$ by taking the finite part with respect to λ and μ. This implies that $B(\lambda, \mu) = \langle 1, S_{\lambda,\mu} \rangle$ and $B_{\lambda,\mu}(x) = B(\lambda, \mu)$ hold for $x > 1$ and for each $(\lambda, \mu) \in \mathbf{C}^2$.

Lemma 3.2. *For $\lambda, \mu \in \mathbf{C}$, we have $B_{\mu,\lambda}(x) = B(\lambda, \mu) - B_{\lambda,\mu}(1-x)$.*

Proof. If $f, g \in \mathcal{D}(\mathbf{R})$, then

$$\begin{aligned} f(-x) * g(1-x) &= \int f(-t)g(1-(x-t))\,\mathrm{d}t \\ &= \int f(s)g(1-x-s)\,\mathrm{d}s \\ &= (f * g)(1-x) \end{aligned}$$

and this formula holds by density whenever two distributions are convolvable. Hence

$$\begin{aligned} B_{\mu,\lambda} = Y * S_{\mu,\lambda} &= (1 - Y(-x)) * S_{\mu,\lambda} \\ &= \langle 1, S_{\mu,\lambda} \rangle - Y(-x) * S_{\lambda,\mu}(1-x) \\ &= B(\mu, \lambda) - (Y * S_{\lambda,\mu})(1-x) \\ &= B(\lambda, \mu) - B_{\lambda,\mu}(1-x). \end{aligned}$$

□

Let us yet give formulas for the finite parts of the complete beta function $B(\lambda, \mu)$ at the singular points.

Lemma 3.3. *For $k, l \in \mathbf{N}_0$ and $\mu \in \mathbf{C} \setminus \mathbf{Z}$, we have*

$$B(-k, \mu) = (-1)^k \binom{\mu-1}{k} \big[\psi(k+1) - \psi(\mu-k)\big]; \tag{3.8}$$

$$B(-k, l) = \begin{cases} (-1)^k \dbinom{l-1}{k} \Big[\displaystyle\sum_{j=1}^{k} \frac{1}{j} - \sum_{j=1}^{l-k-1} \frac{1}{j}\Big] & : l > k, \\ \dfrac{(-1)^l}{l} \cdot \dbinom{k}{l}^{-1} & : 1 \leq l \leq k; \end{cases} \tag{3.9}$$

$$B(-k, -l) = -\binom{k+l}{k} \Big[\sum_{j=k+1}^{k+l} \frac{1}{j} + \sum_{j=l+1}^{k+l} \frac{1}{j}\Big]. \tag{3.10}$$

Proof. We first calculate

$$\operatorname*{Res}_{\lambda=-k}\Gamma(\lambda)=\operatorname*{Res}_{\lambda=-k}\frac{\Gamma(\lambda+k+1)}{\lambda(\lambda+1)\cdots(\lambda+k)}=\frac{(-1)^k}{k!},\tag{3.11}$$

see [8, Section 13.1.4, p. 156], and

$$\begin{aligned}\operatorname*{Pf}_{\lambda=-k}\Gamma(\lambda)&=\operatorname*{Pf}_{\lambda=-k}\frac{\Gamma(\lambda+k+1)}{\lambda(\lambda+1)\cdots(\lambda+k)}\\&=\partial_\lambda\Big(\frac{\Gamma(\lambda+k+1)}{\lambda(\lambda+1)\cdots(\lambda+k-1)}\Big)\Big|_{\lambda=-k}\\&=\frac{(-1)^k}{k!}\Big(\psi(1)+\sum_{j=1}^k\frac{1}{j}\Big)\\&=\frac{(-1)^k\psi(k+1)}{k!},\end{aligned}\tag{3.12}$$

see [10, p. 65]. This furnishes

$$\begin{aligned}B(-k,\mu)&=\operatorname*{Pf}_{\lambda=-k}\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}\\&=\operatorname*{Pf}_{\lambda=-k}\Gamma(\lambda)\cdot\frac{\Gamma(\mu)}{\Gamma(\mu-k)}+\operatorname*{Res}_{\lambda=-k}\Gamma(\lambda)\cdot\partial_\lambda\Big(\frac{\Gamma(\mu)}{\Gamma(\lambda+\mu)}\Big)\Big|_{\lambda=-k}\\&=(-1)^k\binom{\mu-1}{k}\big[\psi(k+1)-\psi(\mu-k)\big]\end{aligned}$$

and hence formula (3.8).

If $l>k$ and if we set $\mu=l$ in formula (3.8), then we immediately obtain the first equation in (3.9) due to $\psi(n+1)=\psi(1)+\sum_{j=1}^n j^{-1}$ for $n\in\mathbf{N}_0$, see [4, Equ. 8.365.3]. On the other hand, if $1\le l\le k$, then

$$\psi(\mu-k)=\psi(\mu-l+1)-\sum_{j=l}^k\frac{1}{\mu-j},$$

see [4, Equ. 8.365.3], and this implies

$$\begin{aligned}B(-k,l)&=\lim_{\mu\to l}(-1)^k\binom{\mu-1}{k}\big[\psi(k+1)-\psi(\mu-k)\big]\\&=(-1)^l\frac{(l-1)!(k-l)!}{k!}\\&=\frac{(-1)^l}{l}\binom{k}{l}^{-1},\end{aligned}$$

i.e., the second equation in formula (3.9).

Finally, we obtain

$$
\begin{aligned}
B(-k,-l) &= (-1)^k \operatorname*{Pf}_{\mu=-l} \binom{\mu-1}{k} \Big[\psi(k+1) - \psi(\mu+l+1) + \sum_{j=0}^{k+l} \frac{1}{\mu-k+j}\Big] \\
&= (-1)^k \binom{-l-1}{k} \Big[\psi(k+1) - \psi(1) + \sum_{j=0}^{k+l-1} \frac{1}{-k-l+j}\Big] \\
&+ (-1)^k \partial_\mu \binom{\mu-1}{k}\Big|_{\mu=-l} \\
&= -\binom{k+l}{k}\Big[\sum_{j=k+1}^{k+l} \frac{1}{j} + \sum_{j=l+1}^{k+l} \frac{1}{j}\Big].
\end{aligned}
$$

□

4. The singular values of the incomplete Beta function

As explained in Section 3, we can reduce the general case of calculating $B_{k,l}$, $k,l \in \mathbf{Z}$, to the particular cases of $B_{j,1}$ and $B_{1,j}$, $j \in \mathbf{Z}$.

Proposition 4.1. *For $\lambda, \mu \in \mathbf{C} \setminus (-\mathbf{N}_0)$ and $j \in \mathbf{N}$, the following holds:*

$$
B_{\lambda,1} = \frac{1}{\lambda}\big[Y(1-x)x_+^\lambda + Y(x-1)\big], \quad B_{1,\mu} = \frac{Y(x)}{\mu}\big[1-(1-x)_+^\mu\big]; \tag{4.13}
$$

$$
B_{0,1} = Y(x)Y(1-x)\log x, \quad B_{1,0} = -Y(x)Y(1-x)\log(1-x); \tag{4.14}
$$

$$
B_{-j,1} = -\frac{1}{j}\big[Y(1-x)x_+^{-j} + Y(x-1)\big] + \frac{(-1)^j\delta^{(j-1)}}{j\cdot j!}; \tag{4.15}
$$

$$
B_{1,-j} = \frac{Y(x)}{j}\big[(1-x)_+^{-j} - 1\big] + \frac{\delta_1^{(j-1)}}{j\cdot j!}. \tag{4.16}
$$

Proof. For $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > 0$, we have

$$
B_{\lambda,1}(x) = Y(x)\int_0^x Y(1-t)t^{\lambda-1}\,\mathrm{d}t = \frac{1}{\lambda}\big[Y(1-x)x_+^\lambda + Y(x-1)\big].
$$

By analytic continuation, the last expression represents $B_{\lambda,1}$ for all $\lambda \in \mathbf{C} \setminus (-\mathbf{N}_0)$.

For the remaining cases, we use the following formula, which is familiar in the context of complex analysis:

$$
\operatorname*{Pf}_{\lambda=\lambda_0}(f_\lambda \cdot T_\lambda) = \operatorname*{Res}_{\lambda=\lambda_0} f_\lambda \cdot \operatorname*{Pf}_{\lambda=\lambda_0} \partial_\lambda T_\lambda + \operatorname*{Pf}_{\lambda=\lambda_0} f_\lambda \cdot \operatorname*{Pf}_{\lambda=\lambda_0} T_\lambda + \operatorname*{Pf}_{\lambda=\lambda_0} \partial_\lambda f_\lambda \cdot \operatorname*{Res}_{\lambda=\lambda_0} T_\lambda. \tag{4.17}
$$

Here f_λ is an analytic $\mathcal{C}^\infty(\mathbf{R})$-valued function for $0 < |\lambda-\lambda_0| < \epsilon$ and T_λ is an analytic $\mathcal{D}'(\mathbf{R})$-valued function for $0 < |\lambda-\lambda_0| < \epsilon$, $\epsilon > 0$, and both f_λ and T_λ have at most a simple pole in λ_0, see [10, Prop. 1.6.3, p. 28].

Hence

$$
\begin{aligned}
B_{0,1} &= \operatorname*{Pf}_{\lambda=0} \frac{1}{\lambda}\big[Y(1-x)x_+^\lambda + Y(x-1)\big] \\
&= \frac{\partial}{\partial\lambda}\big[Y(1-x)x_+^\lambda + Y(x-1)\big]\big|_{\lambda=0} \\
&= Y(x)Y(1-x)\log x
\end{aligned}
$$

and

$$B_{-j,1} = -\frac{1}{j} \operatorname*{Pf}_{\lambda=-j} \big[Y(1-x)x_+^\lambda + Y(x-1)\big] - \frac{1}{j^2} Y(1-x) \operatorname*{Res}_{\lambda=-j} x_+^\lambda$$
$$= -\frac{1}{j}\big[Y(1-x)x_+^{-j} + Y(x-1)\big] + \frac{(-1)^j \delta^{(j-1)}}{j \cdot j!}.$$

The formulas for $B_{1,\mu}$, $B_{1,0}$ and $B_{1,-j}$ then follow from Lemma 3.2. □

Example 4.1. *Let us calculate here $B_{0,n}$ for $n \in \mathbf{Z}$. If $n = l \in \mathbf{N}$, then we use formula (3.7) and obtain from Proposition 4.1 that*

$$B_{0,l} = B_{0,1} - \sum_{j=1}^{l-1} B_{1,j} = Y(x)Y(1-x)\log x - \sum_{j=1}^{l-1} \frac{Y(x)}{j}\big[1-(1-x)_+^j\big].$$

If $n = -l \in -\mathbf{N}_0$, we set $\lambda = k = \mu = 0$ in formula (3.3) and conclude from Equations (4.14) and (4.16) in Proposition 4.1 that

$$\begin{aligned} B_{0,-l} &= B_{0,1} + \sum_{j=0}^{l} B_{1,-j} \\ &= Y(x)Y(1-x)\log\Big(\frac{x}{1-x}\Big) + \sum_{j=1}^{l}\Big\{\frac{Y(x)}{j}\big[(1-x)_+^{-j} - 1\big] + \frac{\delta_1^{(j-1)}}{j \cdot j!}\Big\}, \qquad l \in \mathbf{N}_0. \end{aligned} \tag{4.18}$$

In the open interval $(0,1)$, Equation (4.18) coincides with the expression given in Thm. 2.1 in [3, p. 5]. *Note that the calculation in this paper is based on van der Corput's neutrix method, which does not produce a distribution but rather represents $B_{0,-l}$ as a function outside its singular support. Similarly, formulas (1), (2), (3) in* [3, pp. 4, 5], *also follow from Lemma 3.1 and Proposition 4.1 or from the above by Lemma 3.2.*

More generally, formula (3.3) yields a representation of $B_{-k,-l}$, $k, l \in \mathbf{N}_0$, which, on the basis of van der Corput's method, is considered in [12, p. 990].

5. On the singular values of the partial derivatives of the incomplete beta function

As indicated above, we denote $\partial B_{\lambda,\mu}/\partial\lambda$ by $\partial_1 B$ and similarly for $\partial_2 B$. Motivated by the calculations in [3], let us derive formulas for $(\partial_1 B)_{1,j}$ and $(\partial_1 B)_{j,1}$, $j \in \mathbf{Z}$. Lemma 3.1 then immediately yields representations of $\partial_1 B$ at the singular values $(k,l) \in \mathbf{Z}^2$, $k \le 0$ or $l \le 0$. Furthermore, we conclude from Lemma 3.2 that

$$\begin{aligned} (\partial_2 B)_{\lambda,\mu} &= \frac{\partial B_{\lambda,\mu}}{\partial\mu} \\ &= \frac{\partial B(\lambda,\mu)}{\partial\mu} - \frac{\partial B_{\mu,\lambda}(1-x)}{\partial\mu} \\ &= \frac{\partial B(\lambda,\mu)}{\partial\mu} - (\partial_1 B)_{\mu,\lambda}(1-x), \end{aligned} \tag{5.19}$$

and hence the derivative $\partial_2 B$ can be expressed by $\partial_1 B$.

Proposition 5.2. *For $\lambda, \mu \in \mathbf{C} \setminus (-\mathbf{N}_0)$ and $k, l \in \mathbf{N}$, the following holds:*

$$(\partial_1 B)_{\lambda,1} = \lambda^{-1} Y(1-x)x_+^\lambda \log x - \lambda^{-2}\big[Y(1-x)x_+^\lambda + Y(x-1)\big]; \tag{5.20}$$

$$(\partial_1 B)_{0,1} = \tfrac{1}{2} Y(x)Y(1-x)\log^2 x; \tag{5.21}$$

$$(\partial_1 B)_{-k,1} = -\tfrac{Y(1-x)}{k} x_+^{-k} \log x - \tfrac{x_+^{-k} Y(1-x) + Y(x-1)}{k^2} + \tfrac{(-1)^k \delta^{(k-1)}}{k^2 \cdot k!}; \tag{5.22}$$

$$(\partial_1 B)_{1,\mu} = -\mu^{-1} Y(x) \log x \cdot (1-x)_+^{\mu} + \mu^{-1} B_{0,\mu+1}; \tag{5.23}$$

$$\begin{aligned}(\partial_1 B)_{1,0} &= -Y(x)Y(1-x)\big[\log x \log(1-x) + \mathcal{L}_2(x)\big] - Y(x-1)\frac{\pi^2}{6} \\ &= Y(x)\Big[Y(1-x)\mathcal{L}_2(1-x) - \frac{\pi^2}{6}\Big].\end{aligned} \tag{5.24}$$

$$\begin{aligned}l(\partial_1 B)_{1,-l} &= Y(x) \log x \cdot (1-x)_+^{-l} - Y(x)Y(1-x) \log\Big(\frac{x}{1-x}\Big) \\ &\quad - \frac{1}{l} Y(x-1) - \sum_{j=1}^{l-1} \frac{Y(x)}{j} \Big\{\big[(1-x)_+^{-j} - 1\big] + \frac{l\, \delta_1^{(j-1)}}{(l-j) \cdot j!}\Big\}.\end{aligned} \tag{5.25}$$

Proof. Formula (5.20) follows immediately from the first equation in formula (4.13) by differentiation with respect to λ.

By taking the finite part at $\lambda = 0$, we infer

$$\begin{aligned}(\partial_1 B)_{0,1} &= \operatorname*{Pf}_{\lambda=0} \frac{1}{\lambda} Y(1-x) x_+^{\lambda} \log x - \operatorname*{Pf}_{\lambda=0} \frac{1}{\lambda^2} Y(1-x) x_+^{\lambda} \\ &= \frac{\partial}{\partial \lambda}\big[Y(1-x) x_+^{\lambda} \log x\big]\big|_{\lambda=0} - \frac{1}{2} \frac{\partial^2}{\partial \lambda^2}\big[Y(1-x) x_+^{\lambda}\big]\big|_{\lambda=0} \\ &= \frac{1}{2} Y(x) Y(1-x) \log^2 x\end{aligned}$$

and hence we obtain formula (5.21).

In order to calculate the finite part of $(\partial_1 B)_{\lambda,1}$ at $\lambda = -k \in -\mathbf{N}$, let us first derive the Laurent series of $x_+^{\lambda} \log x$ about $\lambda = -k$ from that of x_+^{λ}, i.e. formula (2.1), by differentiation with respect to λ :

$$x_+^{\lambda} \log x = \frac{(-1)^k \delta^{(k-1)}}{(k-1)!(\lambda+k)^2} + \sum_{j=0}^{\infty} \frac{x_+^{-k} \log^{j+1} x}{j!} (\lambda+k)^j, \quad 0 < |\lambda + k| < 1.$$

Hence $\operatorname{Res}_{\lambda=-k} x_+^{\lambda} \log x = 0$ and we conclude that

$$\begin{aligned}(\partial_1 B)_{-k,1} &= \operatorname*{Pf}_{\lambda=-k} \Big\{\frac{1}{\lambda} Y(1-x) x_+^{\lambda} \log x - \frac{1}{\lambda^2}\big[Y(1-x) x_+^{\lambda} + Y(x-1)\big]\Big\} \\ &= -\frac{1}{k} Y(1-x) x_+^{-k} \log x - \frac{1}{k^2}\big[Y(1-x) x_+^{-k} + Y(x-1)\big] \\ &\quad + \frac{1}{2} \frac{\partial^2 \lambda^{-1}}{\partial \lambda^2}\Big|_{\lambda=-k} \cdot \frac{(-1)^k \delta^{(k-1)}}{(k-1)!} - \frac{\partial \lambda^{-2}}{\partial \lambda}\Big|_{\lambda=-k} \cdot \operatorname*{Res}_{\lambda=-k} Y(1-x) x_+^{\lambda} \\ &= -\frac{1}{k} Y(1-x) x_+^{-k} \log x - \frac{1}{k^2}\big[Y(1-x) x_+^{-k} + Y(x-1)\big] + \frac{(-1)^k \delta^{(k-1)}}{k^2 \cdot k!}.\end{aligned}$$

This furnishes formula (5.22).

Since $\mu \in \mathbf{C} \setminus (-\mathbf{N}_0)$, we have

$$-\frac{1}{\mu} \frac{\mathrm{d}}{\mathrm{d}x} (1-x)_+^{\mu} = (1-x)_+^{\mu-1}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[-\frac{1}{\mu} Y(x) \log x \cdot (1-x)_+^{\mu}\Big] = Y(x) \log x \cdot (1-x)_+^{\mu-1} - \frac{1}{\mu} x_+^{-1} (1-x)_+^{\mu}.$$

Thus $(\partial_1 S)_{1,\mu} = Y(x)\log x \cdot (1-x)^{\mu-1}$ is the derivative of the distribution $-\mu^{-1}Y(x)\log x \cdot (1-x)_+^{\mu} + \mu^{-1}B_{0,\mu+1}$ and this distribution has its support in the positive half-axis $[0,\infty)$ and coincides therefore with $(\partial_1 B)_{1,\mu}$. This implies formula (5.23).

Evaluating the finite part of $(\partial_1 B)_{1,\mu}$ at $\mu = 0$ in formula (5.23) yields

$$\begin{aligned}(\partial_1 B)_{1,0} &= \operatorname*{Pf}_{\mu=0}(\partial_1 B)_{1,\mu} = -\frac{\partial}{\partial\mu}\, Y(x)\log x \cdot (1-x)_+^{\mu}\big|_{\mu=0} + \frac{\partial B_{0,\mu+1}}{\partial \mu}\bigg|_{\mu=0}\\ &= -Y(x)Y(1-x)\log x \log(1-x) + Y(x)\int_0^x Y(1-t)\log(1-t)\frac{\mathrm{d}t}{t}\\ &= -Y(x)Y(1-x)\big[\log x\log(1-x) + \mathcal{L}_2(x)\big] - Y(x-1)\mathcal{L}_2(1),\end{aligned}$$

see [5, Equ. 323.3a]. Due to $\mathcal{L}_2(1) = \frac{\pi^2}{6}$, this gives the first equation in formula (5.24). On the other hand, a direct calculation yields the following:

$$\begin{aligned}(\partial_1 B)_{1,0} &= Y(x)\int_0^x Y(1-t)(1-t)^{-1}\log t\,\mathrm{d}t\\ &= Y(x)\int_{1-x}^1 Y(t)\log(1-t)\frac{\mathrm{d}t}{t}\\ &= Y(x)\big[Y(1-x)\mathcal{L}_2(1-x) - \mathcal{L}_2(1)\big].\end{aligned}$$

Of course, these two representations of $(\partial_1 B)_{1,0}$ must and do coincide as can be seen from [5, Equ. 323.3g].

Let us finally calculate $(\partial_1 B)_{1,-l}$ for $l \in \mathbf{N}$. From formula (5.23), we obtain

$$\begin{aligned}(\partial_1 B)_{1,-l} &= \operatorname*{Pf}_{\mu=-l}(\partial_1 B)_{1,\mu}\\ &= Y(x)l^{-1}\log x\cdot(1-x)_+^{-l} + Y(x)l^{-2}\log x \cdot \operatorname*{Res}_{\mu=-l}(1-x)_+^{\mu} - l^{-1}B_{0,1-l} - l^{-2}\operatorname*{Res}_{\mu=-l} B_{0,\mu+1}.\end{aligned}$$

Furthermore,

$$\operatorname*{Res}_{\mu=-l}(1-x)_+^{\mu} = \Big(\operatorname*{Res}_{\mu=-l} x_+^{\mu}\Big)(1-x) = \frac{(-1)^{l-1}\delta^{(l-1)}(1-x)}{(l-1)!} = \frac{\delta_1^{(l-1)}}{(l-1)!},$$

and, for a function f which is differentiable at 1 and $m \in \mathbf{N}_0$, we have

$$f \cdot \delta_1^{(m)} = \sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}f^{(m-j)}(1)\,\delta_1^{(j)}$$

and hence

$$(\log x)\cdot \operatorname*{Res}_{\mu=-l}(1-x)_+^{\mu} = -\sum_{j=0}^{l-2}\frac{\delta_1^{(j)}}{(l-j-1)\cdot j!}.$$

From formula (4.18), we infer that

$$B_{0,1-l} = Y(x)Y(1-x)\log\Big(\frac{x}{1-x}\Big) + \sum_{j=1}^{l-1}\Big\{\frac{Y(x)}{j}\big[(1-x)_+^{-j} - 1\big] + \frac{\delta_1^{(j-1)}}{j\cdot j!}\Big\}.$$

In order to evaluate the residue $\operatorname{Res}_{\mu=-l} B_{0,\mu+1}$, we note that

$$\operatorname*{Res}_{\mu=-l} S_{0,\mu+1} = \operatorname*{Res}_{\mu=-l} x_+^{-1}(1-x)_+^{\mu} = x^{-1}\cdot\frac{\delta_1^{(l-1)}}{(l-1)!} = \sum_{j=0}^{l-1}\frac{\delta_1^{(j)}}{j!}$$

and thus

$$\operatorname*{Res}_{\mu=-l} B_{0,\mu+1} = Y * \operatorname*{Res}_{\mu=-l} S_{0,\mu+1} = Y(x-1) + \sum_{j=0}^{l-2} \frac{\delta_1^{(j)}}{(j+1)!}.$$

Collecting terms we arrive at formula (5.25). The proof is complete. □

Remark 5.2. *From formula (5.25) in Proposition 5.2, we conclude that*

$$(\partial_1 B)_{1,-l}(x) = -\frac{1}{l^2} + \frac{1}{l}\sum_{j=1}^{l-1}\frac{1}{j}, \quad l \in \mathbf{N},\ x > 1. \tag{5.26}$$

Let us check this equation by replacing $\log x$ *by its Taylor series about* 1*. If* $l \in \mathbf{N}$ *and* $x > 1$*, then*

$$\begin{aligned}(\partial_1 B)_{1,-l}(x) &= \langle 1, (\partial_1 S)_{1,-l}\rangle \\ &= \langle 1, Y(x)\log x \cdot (1-x)_+^{-l-1}\rangle \\ &= \langle 1, -\sum_{j=1}^{\infty} j^{-1} Y(x)(1-x)_+^{j-l-1}\rangle \\ &= -\langle 1, \sum_{j=1}^{\infty} j^{-1} S_{1,j-l}\rangle.\end{aligned} \tag{5.27}$$

(In fact, these series converge in $\mathcal{E}'(\mathbf{R})$*.) For Re* $\mu > 0$*, we have*

$$\langle 1, S_{1,\mu}\rangle = \langle 1, S_{\mu,1}\rangle = \int_0^1 x^{\mu-1}\,dx = \frac{1}{\mu}$$

and hence

$$\langle 1, S_{1,0}\rangle = 0 \text{ and } \langle 1, S_{1,l}\rangle = l^{-1} \text{ for } l \in \mathbf{Z}\setminus\{0\}$$

by analytic continuation and taking finite parts. Therefore Equation (5.27) implies

$$\begin{aligned}(\partial_1 B)_{1,-l}(x) &= -\sum_{j=1,\, j\neq l}^{\infty} \frac{1}{j(j-l)} \\ &= -\frac{1}{l}\sum_{j=1,\, j\neq l}^{\infty}\Big(\frac{1}{j-l} - \frac{1}{j}\Big) \\ &= -\frac{1}{l}\Big(\frac{1}{l} - \sum_{j=1}^{l-1}\frac{1}{j}\Big), \quad l \in \mathbf{N},\ x > 1.\end{aligned}$$

in accordance with the result in formula (5.26).

Remark 5.3. *In the open interval* $(0,1)$*, the representation of* $(\partial_1 B)_{1,-l}$ *in formula (5.25) coincides with* [3, Thm. 2.2, p. 6]*. Similarly, the formulas for* $(\partial_2 B)_{-k,1}$ *and for* $(\partial_2 B)_{-k,l}$*,* $k, l \in \mathbf{N}$*, in* [3, Thms. 2.3, 2.4, pp. 6, 7]*, follow from Equation (5.19), Lemma 3.1 and Proposition 5.2.*

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Research Article

Powers of Dirichlet kernels and approximation by discrete linear operators I: direct results

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ABSTRACT. The second and third powers of the Dirichlet kernel are used to construct discrete linear operators for the approximation of continuous periodic functions. An estimate of the rate of convergence is given. Approximation of non-periodic functions are also considered.

Keywords: Discrete linear operators, rate of convergence, direct results, Dirichlet kernel.

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1. INTRODUCTION

Let $\mathbb{T}_n$ be the family of all trigonometric polynomial of degree non greater than n and $C_{2\pi}$ the space of 2π-periodic continuous functions f with the norm $\|f\| = \sup\{|f(x)| : x \in [-\pi, \pi]\}$. We denote by $C_{2\pi}^r$ the space of r-times continuously differentiable functions. For $f \in C_{2\pi}^r$ we set $D^r f = f^{(r)}$.

For $f \in C_{2\pi}$, $r \in \mathbb{N}$ and $t > 0$, the modulus of smoothness of order r is defined by

$$\omega_r(f,t) = \sup_{0<h\leq t} \|\Delta_h^r f\|, \quad \text{where} \quad \Delta_h^r f(x) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x+kh).$$

For the approximation of continuous periodic functions several convolution operators have been used. From the computational point of view, it is more useful to work with operators defined discretely (they are given in terms of a finite family of values of the functions). Some authors have employed Riemann sums to replace the integrals in the convolution by discrete sums (see [1]).

For $r \in \mathbb{N}$ and $k \in \mathbb{Z}$, throughout the paper we set

$$x_{r,k} = \frac{2k\pi}{(r+1)}.$$

The Dirichlet kernel is given by (see [3, p. 42])

$$D_n(x) = 1 + 2\sum_{k=1}^{n} \cos(kx) = \frac{\sin((2n+1)x/2)}{\sin(x/2)}, \qquad x \neq 2j\pi, j \in \mathbb{Z}, \tag{1.1}$$

and $D_n(x) = 2n+1$, $x = 2j\pi$, $j \in \mathbb{Z}$. We also set

$$\mathcal{D}_n(x) = \frac{1}{2n+1} D_n(x)$$

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for the normalized Dirichlet kernel. It follows from (1.1) that $|D_n(x)| \leq 2n+1$ and equality holds if $x=0$. That is the reason why we prefer the normalization given by $\mathcal{D}_n(x)$.

For $f \in C_{2\pi}$ the interpolating polynomial of degree n at the equidistant points $x_{2n,k}$ can be written as

$$L_n(f,x) = \sum_{k=0}^{2n} \mathcal{D}_n(x - x_{2n,k}) f(x_{2n,k}). \tag{1.2}$$

The operator L_n is a Riemann sum approximation of the partial sum of the Fourier series of f given by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) D_n(x-t)dx.$$

Notice that for $0 \leq j < k \leq 2n$

$$\mathcal{D}_n(x_{2n,j} - x_{2n,k}) = \frac{1}{2n+1}\frac{\sin((j-k)\pi)}{\sin((j-k)\pi/(2n+1))} = 0.$$

Since for every $i \in \mathbb{N}$, $\mathcal{D}_n^i(0) = 1$, each operator

$$L_{n,i}(f,x) = \sum_{k=0}^{2n} \mathcal{D}_n^i(x - x_{2n,k}) f(x_{2n,k}),$$

interpolates the function f at the points $x_{2n,k}$. It is clear that the new polynomials are of degree non greater than ni. Moreover, if the real numbers $a_1, a_2, \ldots, a_m$ satisfy $\sum\limits_{i=1}^{m} a_i = 1$, then the linear combination

$$\mathcal{M}_{nm}(f,x) = \sum_{i=1}^{m} a_i L_{n,i}(f,x) \tag{1.3}$$

provides an interpolation process. The operators $\mathcal{M}_{nm}$ are useful when we want to approximate properties better than the one provided by $L_{n,1}$.

For instance, Kis and Vértesi studied in [9] the operators

$$K_{4n}(f,x) = 4L_{2n,3}(f,x) - 3L_{2n,4}(f,x),$$

while the arguments given by Saxena and Srivastava in [7] can be used to consider the operators

$$S_{6n}(f,x) = \frac{25}{3}L_{2n,4}(f,x) - \frac{32}{3}L_{2n,5}(f,x) + \frac{10}{3}L_{2n,6}(f,x).$$

In [7] only a modification to non-periodic was included. Notice that, in both cases, the sum of the coefficients is one. Thus, they are interpolating operators of the form (1.3).

It was proved in [9] that there exists a constant C such that, for each $f \in C_{2\pi}$ and $n \in \mathbb{N}$,

$$\|f - K_{4n}(f)\| \leq C\omega\Big(f, \frac{1}{n}\Big). \tag{1.4}$$

Another approach to improve the rate of convergence of a linear approximation process considers iterative combinations. For instance, for a linear operator $L : C_{2\pi} \to \mathbb{T}_n$, we construct the new operator

$$\widetilde{L}(f) = 2L(f) - L^2(f),$$

where $L^2(f) = L(L(f))$. But, for linear interpolation operators this approach is not useful. In particular, if L_n is given by (1.2), then $L_n^2(f) = L_n(f)$. We can avoid this inconvenience by using other Riemann sums in the discretization of a convolution operator.

For $n, m \in \mathbb{N}$ and $f \in C_{2\pi}$, in this paper we study the polynomial operators defined by

$$M_{mn,2}(f,x) = \frac{1}{(2n+1)(mn+1)} \sum_{k=0}^{mn} f(x_{mn,k}) D_n^2(x - x_{mn,k}), \tag{1.5}$$

$$M_{mn,3}(f,x) = \frac{1}{(3n^2+3n+1)} \frac{1}{(mn+1)} \sum_{k=0}^{mn} f(x_{mn,k}) D_n^3(x - x_{mn,k}), \tag{1.6}$$

and

$$Q_{3n}(f,x) = C_n \sum_{k=0}^{4n} f(x_{4n,k}) \Big(\mathcal{D}_n^2(x - x_{4n,k}) + \mathcal{D}_n^3(x - x_{4n,k}) \Big), \tag{1.7}$$

where

$$C_n = \frac{(2n+1)^3}{(7n^2+7n+2)(4n+1)}.$$

We will prove in Section 5 that

$$\|Q_{3n}^2(f) - 2Q_{3n}(f) + f\| \le 14\omega_2\Big(f, \frac{2\pi}{n+1}\Big). \tag{1.8}$$

There are some differences between (1.4) and (1.8). Our polynomials are of a lower degree and the rate of convergence is given in terms of the second order modulus of smoothness, but we need more nodes.

Since, for $m \in \mathbb{N}$, D_n^m is an even trigonometric polynomial of degree nm, there are unique real numbers $\varrho_{n,m}(i)$, $0 \le i \le mn$, such that

$$D_n^m(x) = \sum_{i=0}^{mn} \varrho_{n,m}(i) \cos(ix). \tag{1.9}$$

In particular, for $1 \le i \le mn$,

$$\varrho_{n,m}(i) = \frac{1}{\pi} \int_{-\pi}^{\pi} D_n^m(x) \cos(ix) dx. \tag{1.10}$$

For our approach we need explicit expressions of the coefficients $\varrho_{n,2}(i)$ and $\varrho_{n,3}(i)$, but only for $0 \le i \le n$. This will be accomplished in Section 3. In Section 4 we study the behavior of the operators (1.5) and (1.6) for polynomials of lower degree. The main results are presented in Section 5. Finally, in the last section we investigate the case of approximation of non-periodic functions.

A strong converse result, as well as the saturation class, will be given in the second part of the paper.

2. Auxiliary results

Recall that the Fejér kernel is defined by (see [3, p. 43])

$$F_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(x) = 1 + 2 \sum_{k=1}^{n} \Big(1 - \frac{k}{n+1}\Big) \cos(kx).$$

If $\sin(x/2) \neq 0$, then

$$F_n(x) = \frac{1}{(n+1)} \Big(\frac{\sin((n+1)x/2)}{\sin(x/2)} \Big)^2. \tag{2.11}$$

For $f \in C_{2\pi}$ the associated Fejér operator is defined by

$$\sigma_n(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)F_n(t)dt.$$

Lemma 2.1. *If $g \in C^1_{2\pi}$ and $n \in \mathbb{N}$, then $D(\sigma_n(f)) = \sigma_n(Df)$.*

Proof. It is known that (see [3, Proposition 1.1.14]) if $g \in C_{2\pi}$ and $f \in C^1_{2\pi}$, then $f * g \in C^1_{2\pi}$ and $D(f*g) = (g * D(f))$. □

The following quadrature formula is known.

Proposition 2.1. ([5, p. 20]) *If $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $T \in \mathbb{T}_n$, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} T(t)dt = \frac{1}{n+1}\sum_{k=0}^{n} T\Big(x + \frac{2k\pi}{n+1}\Big).$$

If

$$T(x) = a_0 + \sum_{j=1}^{n}(a_j\cos(jx) + b_j\sin(jx)) = \sum_{j=0}^{n} A_j(T,x), \tag{2.12}$$

the conjugate of T is given by $\widetilde{T}(x) = \sum_{j=1}^{n}(-b_j\cos(jx) + a_j\sin(jx))$. Simple equations related with the conjugate polynomials are presented in Lemma 2.2.

Lemma 2.2. *If $T \in \mathbb{T}_n$ is given by (2.12) and $W = D\widetilde{T}$, then*

$$D\widetilde{T} = \sum_{j=1}^{n} jA_j(T), \qquad D^2T = -\sum_{j=1}^{n} j^2A_j(T),$$

$$D\widetilde{W} = -D^2(T) \qquad \text{and} \qquad D(\widetilde{D^2T}) = D^3\widetilde{T}.$$

Lemma 2.3. *If $n \in \mathbb{N}$, σ_n is the Fejér operator and $T \in \mathbb{T}_n$, then*

$$(I - \sigma_n)T = \frac{1}{(n+1)}D\widetilde{T} \qquad \text{and} \qquad D^3\widetilde{T} = (n+1)(I-\sigma_n)(D^2T).$$

Proof. The first equation is well known (for instance see [2]). For the second one we write

$$(I-\sigma_n)(D^2T) = \frac{1}{(n+1)}D(\widetilde{D^2T}) = \frac{1}{(n+1)}D^3\widetilde{T},$$

where we use Lemma 2.2. □

Theorem 2.1 (Stechkin, [8]). *If $r, n \in \mathbb{N}$ and $T \in \mathbb{T}_n$, then*

$$\|D^rT\| \le \Big(\frac{n}{2\sin(nh/2)}\Big)^r \|\Delta_h^r T\| \tag{2.13}$$

for any $h \in (0, 2\pi/n)$.

We will use the Stechkin theorem in a more convenient form for our purposes.

Proposition 2.2. *If $r, n \in \mathbb{N}$, $f \in C_{2\pi}$, and $T \in \mathbb{T}_n$, then*

$$\frac{1}{n^r}\|D^rT\| \le \frac{1}{2^r}\omega_r\Big(f, \frac{\pi}{n}\Big) + \|f - T_n\|. \tag{2.14}$$

Proof. It follows directly from Theorem 2.1 with $h = \pi/n$ and the inequality $\|\Delta_h^r T\| \le 2^r\|f - T\| + \|\Delta_h^r f\|$. □

We will use Proposition 2.2 in the case when T is the polynomial of the best approximation for f in $\mathbb{T}_n$. It is known that, for every $f \in C_{2\pi}$ and $n \in \mathbb{N}_0$, there exists an unique polynomial $T \in \mathbb{T}_n$ (called the polynomial of the best approximation) such that

$$E_n(f) = \inf_{T_n \in \mathbb{T}_n} \|T_n - f\| = \|T - f\|.$$

Proposition 2.3. *If $f \in C_{2\pi}$, $T \in \mathbb{T}_n$ and $E_n(f) = \|T - f\|$, then*

$$\|D^2 T\| \leq n^2 \Big(\frac{1}{4} \omega_2 \Big(f, \frac{\pi}{n} \Big) + E_n(f) \Big),$$
$$\|D^4 T\| \leq n^4 \Big(\frac{1}{4} \omega_2 \Big(f, \frac{\pi}{n} \Big) + E_n(f) \Big),$$

and

$$\|D^3 \widetilde{T}\| \leq 2n^2(n+1) \Big(\frac{1}{4} \omega_2 \Big(f, \frac{\pi}{n} \Big) + E_n(f) \Big).$$

Proof. It follows from Proposition 2.2 that

$$\|D^2 T\| \leq n^2 \Big(\frac{1}{4} \omega_2 \Big(f, \frac{\pi}{n} \Big) + E_n(f) \Big)$$
$$\|D^4 T\| \leq n^4 \Big(\frac{1}{2^4} \omega_4 \Big(f, \frac{\pi}{n} \Big) + E_n(f) \Big) \leq n^4 \Big(\frac{1}{4} \omega_2 \Big(f, \frac{\pi}{n} \Big) + E_n(f) \Big),$$

because $\omega_4(f,t) \leq 4\omega_2(f,t)$. The last inequality is a consequence of Lemma 2.3. In fact

$$\|D^3 \widetilde{T}\| = (n+1)\|(I - \sigma_n)(D^2 T)\| \leq 2(n+1)\|D^2 T\|.$$

$\square$

3. Expansion of Dirichlet kernels

Proposition 3.4. *For each $n \in \mathbb{N}$, one has*

$$D_n^2(x) = 2n + 1 + 2 \sum_{k=1}^{2n} (2n + 1 - k) \cos(kx).$$

That is, $\varrho_{n,2}(0) = 2n+1$ and $\varrho_{n,2}(j) = 2(2n+1-j)$, for $1 \leq j \leq 2n$ (see (1.9)).

Proof. The computation of D_n^2 is simple, because taking into account (1.1) and (2.11) one has (for $\sin(x/2) \neq 0$)

$$\frac{D_n^2(x)}{2n+1} = \frac{\sin^2((2n+1)x/2)}{(2n+1)\sin^2(x/2)} = F_{2n}(x) = 1 + 2\sum_{k=1}^{n} \Big(1 - \frac{k}{2n+1}\Big) \cos(kx).$$

$\square$

For D_n^3 we need some preparatory computations.

Lemma 3.4. *For each $n, k \in \mathbb{N}$,*

$$\cos(kx) D_n(x) = \begin{cases} \sum\limits_{i=1}^{n+k} \cos(ix) + \sum\limits_{i=0}^{n-k} \cos(ix), & \text{if} \quad 1 \leq k \leq n \\ \sum\limits_{i=k-n}^{n+k} \cos(ix), & \text{if} \quad k > n. \end{cases}$$

Proof. If $k \le n$,

$$\begin{aligned}
\cos(kx)D_n(x) &= \cos(kx) + 2\sum_{j=1}^{n} \cos(kx)\cos(jx) \\
&= \cos(kx) + \sum_{j=1}^{n} (\cos((k+j)x) + \cos((k-j)x)) \\
&= \cos(kx) + \sum_{i=k+1}^{n+k} \cos(ix) + \sum_{i=1}^{k-1} \cos(ix) + \sum_{i=0}^{n-k} \cos(ix) \\
&= \sum_{i=1}^{n+k} \cos(ix) + \sum_{i=0}^{n-k} \cos(ix).
\end{aligned}$$

If $k > n$, then

$$\begin{aligned}
\cos(kx)D_n(x) &= \sum_{i=k}^{n+k} \cos(ix) + \sum_{j=1}^{n} \cos((k-j)x) \\
&= \sum_{i=k}^{n+k} \cos(ix) + \sum_{i=k-n}^{k-1} \cos(ix) = \sum_{i=k-n}^{n+k} \cos(ix).
\end{aligned}$$

□

Proposition 3.5. *If $n \in \mathbb{N}$, $n \ge 3$, and D_n^3 is given as in (1.9), then*

$$\varrho_{n,3}(0) = 3n^2 + 3n + 1,$$

and

$$\varrho_{n,3}(i) = 2(3n^2 + 3n + 1 - i^2), \quad \textit{for} \quad 1 \le i \le n.$$

Proof. Let $\Pi_n : \mathbb{T}_{3n} \to \mathbb{T}_n$ be the projection given by (see (2.12))

$$\Pi_n(T) = \Pi_n\Big(\sum_{j=0}^{3n} A_j(T,x)\Big) = \sum_{j=0}^{n} A_j(T,x).$$

In this proof (for a fixed n) we denote $\varrho(k) = \varrho_{n,2}(k)$ and consider the expansion of D_n^2 given in Proposition 3.4. Hence

$$\begin{aligned}
D_n^3(x) = (D_n^2(x))D_n(x) &= \Big(\sum_{k=0}^{2n} \varrho(k)\cos(kx)\Big) D_n(x) \\
&= \varrho(0)D_n(x) + D_n(x)\sum_{k=1}^{n} \varrho(k)\cos(kx) + D_n(x)\sum_{k=n+1}^{2n} \varrho(k)\cos(kx) \\
&= A_1(x) + A_2(x) + A_3(x).
\end{aligned}$$

For $A_2(x)$ one has

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} A_2(x)dx &= \frac{1}{2\pi}\sum_{k=1}^{n} \varrho(k) \int_{-\pi}^{\pi} D_n(x)\cos(kx)dx \\
&= \frac{1}{2\pi}\sum_{k=1}^{n} \varrho(k) \int_{-\pi}^{\pi} \Big(\cos(kx) + 2\sum_{i=1}^{n}\cos(ix)\cos(kx)\Big)dx = \sum_{k=1}^{n} \varrho(k),
\end{aligned}$$

and, for $1 \le j \le n$, taking into account Lemma 3.4,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} A_2(x)\cos(jx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{n}\varrho(k)\Big(\sum_{i=1}^{n+k}\cos(ix)+\sum_{i=0}^{n-k}\cos(ix)\Big)\cos(jx)dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(\cos^2(jx)\sum_{k=1}^{n}\varrho(k)+\cos(jx)\sum_{i=0}^{n-1}\cos(ix)\Big(\sum_{k=1}^{n-i}\varrho(k)\Big)dx.$$

Hence

$$\Pi_n(A_2)(x) = \sum_{k=1}^{n}\varrho(k)+\sum_{j=1}^{n-1}\Big(\sum_{k=1}^{n}\varrho(k)+\sum_{k=1}^{n-j}\varrho(k)\Big)\cos(jx)+\cos(nx)\sum_{k=1}^{n}\varrho(k).$$

For $j=0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} A_3(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)\sum_{k=n+1}^{2n}\varrho(k)\cos(kx)dx = 0,$$

and, for $1 \le j \le n$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} A_3(x)\cos(jx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(D_n(x)\sum_{k=n+1}^{2n}\varrho(k)\cos(kx)\Big)\cos(jx)dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(\sum_{k=n+1}^{2n}\varrho(k)\Big(\sum_{i=k-n}^{n+k}\cos(ix)\Big)\Big)\cos(jx)dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(\sum_{i=1}^{n}\cos(ix)\Big(\sum_{k=n+1}^{n+i}\varrho(k)\Big)\Big)\cos(jx)dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(\sum_{k=n+1}^{n+j}\varrho(k)\Big)\cos^2(jx)dx = \sum_{k=n+1}^{n+j}\varrho(k).$$

Hence

$$\Pi_n(A_3)(x) = \sum_{j=1}^{n}\Big(\sum_{k=n+1}^{n+j}\varrho(k)\Big)\cos(jx).$$

Therefore

$$\Pi_n(D_n^3)(x) = \sum_{k=0}^{n}\varrho(k)+\sum_{j=1}^{n-1}\Big(\sum_{k=0}^{n}\varrho(k)+\sum_{k=0}^{n-j}\varrho(k)+\sum_{k=n+1}^{n+j}\varrho(k)\Big)\cos(jx)$$
$$+\Big(2\varrho(0)+\sum_{k=1}^{n}\varrho(k)+\sum_{k=n+1}^{2n}\varrho(k)\Big)\cos(nx)$$
$$= 3n^2+3n+1+2\sum_{j=1}^{n}(3n^2+3n+1-j^2)\cos(jx).$$

□

4. The operators $M_{mn,2}$, $M_{mn,3}$ and polynomials of lower degree

In order to proof the estimate announced in (1.8) we follow a method used in [2]. In particular, for $T \in \mathbb{T}_n$, in Proposition 5.9 we will find a representation of $Q_{3n}(T)$ in terms of the some derivatives of the polynomials.

As Proposition 4.6 shows, the operators $M_{mn,2}(f)$ reproduce the constant functions. But, unfortunately, they are not uniformly bounded. Moreover, if we increase the number of points of interpolation the result does not change. That is the reason why we consider only $m = 3$ for the operators Q_{3n}.

Proposition 4.6. *If $m > 2$, $T \in \mathbb{T}_n$ and $M_{mn,2}$ is defined by (1.5), then*

$$M_{mn,2}(T, x) = T(x) - \frac{1}{(2n+1)} D\widetilde{T}(x).$$

Proof. If $T_n \in \mathbb{T}_n$, then $T_n D_n^2 \in \mathbb{T}_{3n}$ and, taking into account Proposition 2.1, one has

$$\sum_{k=0}^{mn} \frac{T(x_{mn,k})}{(mn+1)} D_n^2(x - x_{mn,k}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_n(t) D_n^2(x-t)dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_n(x+t) D_n^2(t)dt.$$

If T is written as in (2.12), then

$$\begin{aligned}
&\frac{1}{(2n+1)(mn+1)} \sum_{k=0}^{mn} T(x_{mn,k}) D_n^2(x - x_{mn,k}) \\
=&\frac{a_0}{(2n+1)} \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n^2(t)dt + \sum_{j=1}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{A_j(T,x)}{(2n+1)} \cos(jt) D_n^2(t)dt \\
=&a_0 + \sum_{j=1}^{n} \frac{1}{2\pi} \frac{A_j(T,x)}{(2n+1)} \int_{-\pi}^{\pi} 2(2n+1-j)\cos^2(jt)dt \\
=&a_0 + \frac{1}{(2n+1)} \sum_{j=1}^{n} A_j(T,x)(2n+1-j) \\
=&T(x) - \frac{1}{(2n+1)} \sum_{j=1}^{n} jA_j(T,x) \\
=&T(x) - \frac{1}{(2n+1)} D\widetilde{T}(x),
\end{aligned}$$

where Proposition 3.4 and Lemma 2.2 were used. □

Proposition 4.7. *If $m > 3$, $T \in \mathbb{T}_n$, and $M_{mn,3}$ is defined by (1.6), then*

$$M_{mn,3}(T, x) = T(x) + \frac{1}{(3n^2+3n+1)} D^2T(x).$$

Proof. Set $u(n) = 3n^2 + 3n + 1$. As before, if $T_n \in \mathbb{T}_n$, then $T_n D_n^3 \in \mathbb{T}_{4n}$ and, taking into account Proposition 2.1, one has

$$\frac{1}{(mn+1)} \sum_{k=0}^{mn} T(x_{mn,k}) D_n^3(x - x_{mn,k}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_n(x+t) D_n^2(t)dt.$$

If T is written as in (2.12), then

$$\frac{1}{(mn+1)}\sum_{k=0}^{mn} T(x_{mn,k})D_n^3(x-x_{mn,k})$$
$$=a_0\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n^3(t)dt+\sum_{j=1}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{A_j(T,x)}{(2n+1)}\cos(jt)D_n^3(t)dt.$$

Taking into account Proposition 3.5

$$\frac{1}{u(n)(mn+1)}\sum_{k=0}^{mn} T(x_{mn,k})D_n^3(x-x_{mn,k})$$
$$=a_0+\frac{1}{u(n)}\sum_{j=1}^{n}\frac{A_j(T,x)(3n^2+3n+1-j^2)}{2\pi}\int_{-\pi}^{\pi} 2\cos^2(jt)dt$$
$$=a_0+\frac{1}{u(n)}\sum_{j=1}^{n}A_j(T,x)(3n^2+3n+1-j^2)$$
$$=T(x)-\frac{1}{u(n)}\sum_{j=1}^{n}j^2A_j(T,x)=T(x)+\frac{1}{u(n)}D^2T(x),$$

here we use Lemma 2.2. □

5. Main results

In the first result of this section we estimate the norms of the operators.

Proposition 5.8. *If $n\in\mathbb{N}$, Q_{3n} is defined by (1.7) and $f\in C_{2\pi}$, then*

$$\|Q_{3n}(f)\|\leq\|f\|$$

and

$$\|Q_{3n}^2(f)-2Q_{3n}(f)+f\|\leq 4\|f\|.$$

Proof. Since $|\mathcal{D}_n(x)|\leq 1$, $1+\mathcal{D}_n(x)\geq 0$. Therefore

$$\mathcal{D}_n^2(x)+\mathcal{D}_n^3(x)=\mathcal{D}_n^2(x)(1+\mathcal{D}_n(x))\geq 0.$$

It is sufficient to verify that Q_{3n} is a positive operator. Moreover

$$Q_{3n}(f,x)=C_n\sum_{k=0}^{4n} f(x_{4n,k})(\mathcal{D}_n^2(x_{4n,k})+\mathcal{D}_n^3(x-x_{4n,k}))$$
$$=\frac{(2n+1)}{(7n^2+7n+2)(4n+1)}\sum_{k=0}^{4n} f(x_{4n,k})D_n^2(x-x_{4n,k})$$
$$+\frac{1}{(7n^2+7n+2)(4n+1)}\sum_{k=0}^{4n} f(x_{4n,k})D_n^3(x-x_{4n,k}))$$
$$=\frac{(2n+1)^2}{(7n^2+7n+2)}M_{4n,2}(f,x)+\frac{(3n^2+3n+1)}{(7n^2+7n+2)}M_{4n,3}(f,x).$$

It follows from Propositions 4.6 and 4.7 that

$$Q_{3n}(1,x)=\frac{(2n+1)^2}{(7n^2+7n+2)}+\frac{(3n^2+3n+1)}{(7n^2+7n+2)}=1.$$

If $f \in C_{2\pi}$ and $x \in [-\pi, \pi)$, then

$$|Q_{3n}(f,x)| \leq \|f\| Q_{3n}(1,x) = \|f\|.$$

The second assertion is a simple consequence of the first one. □

Proposition 5.9. *If $n \in \mathbb{N}$, Q_{3n} is defined by (1.7) and $T \in \mathbb{T}_n$, then*

$$Q_{3n}^2 T - 2Q_{3n}T + T = \frac{-(2n+1)^2 D^2 T - 2(2n+1) D^3 \widetilde{T} + D^4 T}{(7n^2+7n+2)^2}.$$

Proof. It follows from Propositions 4.6 and 4.7 that (we set $u(n) = 3n^2+3n+1$ $v(n) = 7n^2+7n+2$ and $W = D\widetilde{T}$)

$$\begin{aligned} Q_{3n}T &= \frac{(2n+1)^2}{v(n)} M_{4n,2}T + \frac{(3n^2+3n+1)}{v(n)} M_{4n,3}T \\ &= \frac{(2n+1)^2}{v(n)}\Big(T - \frac{1}{(2n+1)} D\widetilde{T}\Big) + \frac{(3n^2+3n+1)}{v(n)}\Big(T + \frac{1}{u(n)} D^2 T\Big) \\ &= T + \frac{1}{v(n)}\Big(D^2 T - (2n+1)W\Big). \end{aligned} \tag{5.15}$$

Hence

$$\begin{aligned} Q_{3n}^2 T &= \frac{(2n+1)^2}{v(n)} M_{4n,2}\Big(T + \frac{D^2T - (2n+1)W}{v(n)}\Big) + \frac{u(n)}{v(n)} M_{4n,3}\Big(T + \frac{D^2T - (2n+1)W}{v(n)}\Big) \\ &= Q_{3n}T + \frac{(2n+1)^2}{v^2(n)} M_{4n,2}\Big(D^2T - (2n+1)W\Big) + \frac{u(n)}{v^2(n)} M_{4n,3}\Big(D^2T - (2n+1)W\Big) \\ &= Q_{3n}T + \frac{(2n+1)^2}{v^2(n)}\Big(D^2T - \frac{D(\widetilde{D^2T})}{(2n+1)} - (2n+1)W + D\widetilde{W}\Big) \\ &+ \frac{u(n)}{v^2(n)}\Big(D^2T + \frac{D^4T}{u(n)} - (2n+1)W - \frac{(2n+1)D^2W}{u(n)}\Big) \end{aligned}$$

(recall $D(\widetilde{D^2T}) = D^3\widetilde{T}$ and $D\widetilde{W} = -D^2T$)

$$\begin{aligned} &= Q_{3n}T + \frac{1}{v(n)} D^2T - \frac{(2n+1)}{v^2(n)} D^3\widetilde{T} - \frac{(2n+1)}{v(n)} W - \frac{(2n+1)^2}{v^2(n)} D^2T \\ &+ \frac{1}{v^2(n)} D^4T - \frac{(2n+1)}{v^2(n)} D^3\widetilde{T} \\ &= Q_{3n}T + \frac{D^2T}{v(n)} - \frac{(2n+1)D\widetilde{T}}{v(n)} - \frac{(2n+1)^2 D^2T}{v^2(n)} - \frac{2(2n+1)}{v^2(n)} D^3\widetilde{T} + \frac{D^4T}{v^2(n)}. \end{aligned}$$

Taking into account (5.15) we conclude that

$$\begin{aligned} Q_{3n}^2(T) - 2Q_{3n}(T) + T &= T - Q_{3n}(T) + \frac{D^2T}{v(n)} - \frac{(2n+1)}{v(n)}D\widetilde{T} \\ &\quad - \frac{(2n+1)^2D^2T}{v^2(n)} - \frac{2(2n+1)}{v^2(n)}D^3\widetilde{T} + \frac{1}{v^2(n)}D^4T \\ &= -\frac{1}{v(n)}\Big(D^2T - (2n+1)D\widetilde{T}\Big) + \frac{D^2T}{v(n)} - \frac{(2n+1)}{v(n)}D\widetilde{T} \\ &\quad - \frac{(2n+1)^2D^2T}{v^2(n)} - \frac{2(2n+1)}{v^2(n)}D^3\widetilde{T} + \frac{1}{v^2(n)}D^4T \\ &= -\frac{(2n+1)^2D^2T}{v^2(n)} - \frac{2(2n+1)}{v^2(n)}D^3\widetilde{T} + \frac{1}{v^2(n)}D^4T. \end{aligned}$$

□

Theorem 5.2. *If $n \in \mathbb{N}$ $(n \geq 3)$, Q_{3n} is defined by (1.7), and $f \in C_{2\pi}$, then*

$$\|Q_{3n}^2(f) - 2Q_{3n}(f) + f\| \leq 5E_n(f) + \omega_2\Big(f, \frac{\pi}{n}\Big).$$

Proof. Fix $f \in C_{2\pi}$ and, for each $n \in \mathbb{N}$, let $T_n \in \mathbb{T}_n$ be the polynomial of the best approximation for f in $\mathbb{T}_n$.

If we set $M_n(f) = Q_{3n}^2(f) - 2Q_{3n}(f)$ and $v(n) = 7n^2+7n+2$, taking into account Propositions 5.8, 5.9, and 2.3 one has

$$\begin{aligned} \|M_n(f) + f\| &= \|M_n(f - T_n) + f - T_n + M_n(T_n) + T_n\| \\ &\leq 4\|f - T_n\| + \|M_n(T_n) + T_n\| \\ &\leq 4E_n(f) + \frac{\|D^4T\| + 2(2n+1)\|D^3\widetilde{T}\| + (2n+1)^2\|D^2T\|}{v^2(n)} \\ &\leq 4E_n(f) + \frac{n^4 + 4n^2(n+1)(2n+1) + n^2(2n+1)^2}{v^2(n)}\Big(\frac{1}{4}\omega_2\Big(f, \frac{\pi}{n}\Big) + E_n(f)\Big) \\ &= 4E_n(f) + \frac{n^2(13n^2+16n+5)}{v^2(n)}\Big(\frac{1}{4}\omega_2\Big(f, \frac{\pi}{n}\Big) + E_n(f)\Big) \\ &\leq 4E_n(f) + \frac{n^2(14n^2+14n+4)}{v^2(n)}\Big(\frac{1}{4}\omega_2\Big(f, \frac{\pi}{n}\Big) + E_n(f)\Big) \\ &= 4E_n(f) + \frac{2n^2}{v(n)}\Big(\frac{1}{4}\omega_2\Big(f, \frac{\pi}{n}\Big) + E_n(f)\Big) \\ &\leq 4E_n(f) + \frac{2}{7}\Big(\frac{1}{4}\omega_2\Big(f, \frac{\pi}{n}\Big) + E_n(f)\Big) \leq 5E_n(f) + \omega_2\Big(f, \frac{\pi}{n}\Big). \end{aligned}$$

□

Remark 5.1. *The term $E_n(f)$ in Theorem 5.2 can be estimate as (see* [6, Theorem 2.5]*)*

$$E_n(f) \leq \frac{5}{2}\omega_2\Big(f, \frac{2\pi}{n+1}\Big).$$

Therefore

$$\|Q_{3n}^2(f) - 2Q_{3n}(f) + f\| \leq \frac{25}{2}\omega_2\Big(f, \frac{2\pi}{n+1}\Big) + \omega_2\Big(f, \frac{\pi}{n}\Big) \leq 14\omega_2\Big(f, \frac{2\pi}{n+1}\Big). \tag{5.16}$$

6. Approximation of non-periodic functions

Let $C[-1,1]$ the space of continuous functions $f:[-1,1]\to\mathbb{R}$ provided with the sup norm $\|f\|_\infty=\sup\{|f(x)|:\ x\in[-1,1]\}$. In this section we follow a known procedure to pass from approximation by trigonometric polynomials to approximation by algebraic polynomials (see Proposition 6.10 below).

For $f\in C[-1,1]$ and $x,h\in[-1,1]$ define

$$(\tau_h f)(x)=\frac{1}{2}\Big(f\Big(xh+\sqrt{(1-x^2)(1-h^2)}\Big)+f\Big(xh-\sqrt{(1-x^2)(1-h^2)}\Big)\Big)$$

and

$$\omega^T(f,t)=\sup_{t\le h\le 1}\|f-\tau_h f\|.$$

We also set

$$E_n(f)_\infty=\inf_{P\in\mathbb{P}_n}\|f-P\|_\infty,$$

where $\mathbb{P}_n$ be the family of all algebraic polynomial of degree not greater than n.

We introduce operators similar to Q_{3n} by setting

$$R_{3n}(f,x)=C_n\sum_{k=0}^{4n}f(\cos x_{4n,k})(\mathcal{D}_n^2(\arccos x-x_{4n,k})+\mathcal{D}_n^3(\arccos x-x_{4n,k}))$$

for $f\in C[-1,1]$ and $x\in[-1,1]$. Notice that $D_n(\arccos x-x_{4n,k})$ can be written in terms of the Chebyshev polynomials. Hence $R_{3n}(f,x)$ is an algebraic polynomial of degree not greater than $3n$ (see Proposition 6.10 below).

Theorem 6.3. *If $n\in\mathbb{N}$ $(n\ge 3)$ and $f\in C[-1,1]$, then*

$$\|R_{3n}^2(f)-2R_{3n}(f)+f\|\le 14\omega^T\Big(f,\cos\frac{2\pi}{n+1}\Big).$$

Proof. Fix $f\in C[-1,1]$ and set $F(t)=f(\cos t)$. It is known that (see [4, Lemma 3]), for $t\in[-1,1]$,

$$\omega^T(f,t)=\omega_2(F,\arccos t). \tag{6.17}$$

If $x\in[-1,1]$ and $x=\cos t$ $(0\le t\le\pi)$, it follows from Theorem 5.2 and (6.17) that

$$\begin{aligned}|R_{3n}^2(f,x)-2R_{3n}(f,x)+f(x)|&=|R_{3n}^2(f,\cos t)-2R_{3n}(f,\cos t)+f(\cos t)|\\&=\big|Q_{3n}^2(F,t)-2Q_{3n}(F,t)+F(t)\big|\\&\le 14\omega_2\Big(F,\frac{2\pi}{n+1}\Big)\\&=14\omega^T\Big(f,\cos\frac{2\pi}{n+1}\Big).\end{aligned}$$

□

Remark 6.2. *Here we only consider estimates in norm, pointwise estimates require another approach.*

Remark 6.3. *Let $X^1[-1,1]$ be the family of $f\in C[-1,1]$ for which there exists $g\in C[-1,1]$ such that*

$$\lim_{h\to 1-}\Big\|\frac{\tau_h f-f}{1-h}-g\Big\|_\infty=0.$$

If $f\in X^1[-1,1]$, then $\omega^T(f,t)\le C(1-t)$ (see [4, Lemma 6]). Hence, for $f\in X^1[-1,1]$,

$$\|R_{3n}^2(f)-2R_{3n}(f)+f\|\le C\Big(1-\cos\frac{2\pi}{n+1}\Big)\le\frac{2C\pi^2}{(n+1)^2}.$$

The following result is known, but we include a proof for the benefit of the reader.

Proposition 6.10. *For each $n, m \in \mathbb{N}$, $f \in C[-1,1]$ and $x \in [-1,1]$, the function*

$$\sum_{k=0}^{4n} f(\cos x_{4n,k}) D_n^m(\arccos x - x_{4n,k})$$

is an algebraic polynomial of degree not greater than mn.

Proof. For $k \in \mathbb{N}_0$, let $T_k(x) = \cos(k \arccos x)$ be the Chebyshev polynomial of degree k.

Since

$$D_n(\arccos x) = 1 + 2\sum_{k=1}^{n} \cos(k \arccos x) = 1 + 2\sum_{k=1}^{n} T_k(x),$$

one has $f(1)D_n^m(\arccos x)$ is an algebraic polynomial.

For $1 \le j, k \le 2n$, we consider the trigonometric identities

$$\cos(j x_{4n,4n+1-k}) = \cos\frac{2j(4n+1-k)\pi}{4n+1} = \cos\frac{2kj\pi}{4n+1} = \cos(x_{4n,jk}),$$
$$\sin(j x_{4n,4n+1-k}) = -\sin\frac{2jk\pi}{4n+1} = -\sin x_{4n,jk}$$

and

$$\begin{aligned}
&\cos j(\arccos x - x_{4n,k}) + \cos j(\arccos x - x_{4n,4n+1-k}) \\
=&T_j(x)\Big(\cos(j x_{4n,k}) + \cos(j x_{4n+1-k,k})\Big) + \sin(j\arccos x)\Big(\sin(j x_{4n,k}) + \sin(j x_{4n+1-k,k})\Big) \\
=&2\cos(j x_{4n,k})\, T_j(x),
\end{aligned}$$

to obtain

$$\begin{aligned}
&\sum_{k=1}^{4n} f(\cos x_{4n,k}) D_n^m(\arccos x - x_{4n,k}) \\
=&\sum_{k=1}^{2n} f(\cos x_{4n,k})\Big(D_n^m(\arccos x - x_{4n,k}) + D_n^m(\arccos x - x_{4n,4n+1-k})\Big) \\
=&\sum_{k=1}^{2n} f(\cos x_{4n,k}) \sum_{j=0}^{mn} \varrho_{n,m}(j)\Big(\cos(j(\arccos x - x_{4n,k})) \\
&+\cos(j(\arccos x - x_{4n,4n+1-k}))\Big) \\
=&2\sum_{k=1}^{2n} f(\cos x_{4n,k}) \sum_{j=0}^{mn} \varrho_{n,m}(j)\cos(j x_{4n,k})\, T_j(x).
\end{aligned}$$

$\square$

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