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Fixed Point Theorems in $\mathscr{G}$ - Fuzzy Convex Metric Spaces

Vallinayagam Pazhani[1*] **and Maduraiveeran Jeyaraman**[1]

[1] *P. G. and Research Department of Mathematics, Raja Doraisingam Government Arts College, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India*
[*] *Corresponding author*

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Abstract

This work introduces a new three-step iteration process and shows that the same leads to a unique fixed point with the help of theorems under different conditions of contractive mappings over-generalized $\mathscr{G}$ - fuzzy metric spaces in the convex structure. Also, we investigate the data dependence result of this iterative process in the generalized $\mathscr{G}$ - fuzzy convex metric spaces.

1. Introduction

The fuzzy set was released in 1965 by the pioneer scientist Zadeh [1] as a class of objects with a continuum of grades of membership. After Zadeh's paper [1], many scientists employed the notion of fuzzy sets in many subjects of sciences such as fuzzy metric space, fuzzy topology, fuzzy decisions, fuzzy set theory, etc. Kramosil and Michalek [2] paved a way for further work by introducing the concept of fuzzy metric spaces which then modified by George and Veeramani [3]. After that, several fixed point theorems were proved in fuzzy metric spaces.
Mustafa and Sims [4] brought out the concept of generalized metric space, shortly known as $\mathscr{G}$-metric space, and came out with interesting properties with its topology. Sun and Yang [5] also generalized the definition of fuzzy metric space in their way. In 2016, Jeyaraman et al. [6] proved a result that lead to a unique common fixed point theorem with six weakly compatible mappings in $\mathscr{G}$-fuzzy metric spaces. We introduce a new three-step iteration process and show the convergence of the iteration process to a unique fixed point using theorems under different conditions of contractive mappings on the $\mathscr{G}$-fuzzy metric spaces in the convex structure. Also, we investigate the data dependence result of this iterative process in the generalized $\mathscr{G}$ - fuzzy convex metric spaces.

2. Preliminaries

Definition 2.1. Let $(X,\mathscr{G},*)$ be a $\mathscr{G}$-fuzzy metric space and $I=[0,1]$. A continuous mapping $\Delta : X\times X\times I \to X$ is said to be a convex structure on X if for each $(x,y,k)\in X\times X\times I$ and $u\in X$,

$$\mathscr{G}(u,\Delta(x,y,k),\Delta(x,y,k),t)\geq k\mathscr{G}(u,x,x,t)+(1-k)M(u,y,y,t)$$

A space X together with a convex structure Δ is called a $\mathscr{G}$-fuzzy convex metric space($\mathscr{G}$-FCMS).

Definition 2.2. Let X be a $\mathscr{G}$-FCMS. A nonempty subset C of X is said to be generalized convex if $\Delta(x,y,z;a_1,a_2,a_3)\in C$ whenever $(x,y,z;a_1,a_2,a_3)\in C\times C\times C\times[0,1]\times[0,1]\times[0,1]$.

Email addresses and ORCID numbers: **pazhanin@yahoo.com, 0000-0002-4586-9354 (V. Pazhani), jeya.math@gmail.com, 0000-0002-0364-1845 (M. Jeyaraman)**

Definition 2.3. Let $(X,\mathscr{G},*)$ be a $\mathscr{G}$-FCMS. A mapping $\Delta : X\times X\times X\times[0,1]\times[0,1]\to X$ is said to be $\mathscr{G}$-fuzzy convex structure on X if for each $(x,y,z,a_1,a_2)\in X\times X\times X\times[0,1]\times[0,1], a_1\geq a_2$ and $u,v\in X$,

$$\mathscr{G}(u,v,\Delta(x,y,z;a_1,a_2),t)\geq(a_1-a_2)\mathscr{G}(u,v,x,t)+(1-a_1)\mathscr{G}(u,v,y,t)+a_2\mathscr{G}(u,v,z,t).$$

Lemma 2.4. *Let a_n be a nonnegative sequence in $\mathscr{G}$-FCMS and let ρ is a real number satisfying $0\leq\rho<1$ and $(\in_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers such that $\lim_{n\to\infty}\in_n=1$, then for any sequence of positive numbers $(\in_n)_{n\in\mathbb{N}}$ satisfying $a_{n+1}\geq\rho a_n+\in_n, n=1,2,\ldots,$ one has $\lim_{n\to\infty}a_n=1$.*

3. Main result

Theorem 3.1. *Let C be a nonempty closed convex subset of a $(X,\mathscr{G},*)$ complete $\mathscr{G}$-FCMS with Δ convex structure and $\Gamma : X\to X$ be a mapping satisfying the following conditions:*

$$\mathscr{G}\big(\Gamma x,\Gamma y,\Gamma z,t\big)\geq\big\{a_1\mathscr{G}(x,y,z,t)+a_2\mathscr{G}(x,\Gamma x,\Gamma x,t)+a_3\mathscr{G}(y,\Gamma y,\Gamma y,t)+a_4\mathscr{G}(z,\Gamma z,\Gamma z,t)\big\}\tag{3.1}$$

for all $x,y,z\in X$ where $0\leq a_1,a_2,a_3,a_4<1$ and $\{x_n\}_{n\geq0}$ ie the iterative scheme given by

(i) $x_0\in X,$ *for all* $n\in N,$
(ii) $x_{n+1}=\Delta(\Gamma y_n,\Gamma y_n,\Gamma y_n:\gamma_n,\gamma_n),$
(iii) $y_n=\Delta(z_n,\Gamma z_n,\Gamma x_n:\alpha_n,\beta_n),$
(iv) $z_n=\Delta(\Gamma x_n,x_n,\Gamma x_n:\theta_n,\theta_n)$ *such that* $\lim_{n\to\infty}\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)=1$ *with* $\{\gamma_n\},\{\alpha_n\},\{\beta_n\}$ *and* $\{\theta_n\}\subset[0,1]$

Then $\{x_n\}_{n\geq0}$ $\mathscr{G}$-converges to unique fixed point $\dot{p}$ of Γ.

Proof: Suppose that Γ satisfies condition (i)-(iv), we have

$$\begin{aligned}\mathscr{G}(x_{n+1},\dot{p},\dot{p},t)&=\mathscr{G}(\Delta(\Gamma y_n,\Gamma y_n,\Gamma y_n;\gamma_n,\gamma_n),\dot{p},\dot{p},t)\\&\geq\big\{(\gamma_n-\gamma_n)\mathscr{G}(\Gamma y_n,\dot{p},\dot{p},t)+(1-\gamma_n)\mathscr{G}(\Gamma y_n,\dot{p},\dot{p},t)+\gamma_n\mathscr{G}(\Gamma y_n,\dot{p},\dot{p},t)\big\}\\&\geq\big\{a_1\mathscr{G}(y_n,\dot{p},\dot{p},t)+a_2\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t)+a_3\mathscr{G}(\dot{p},\dot{p},\Gamma\dot{p},t)+a_4\mathscr{G}(\dot{p},\dot{p},\Gamma\dot{p},t)\big\}\\&=\big\{a_1\mathscr{G}(y_n,\dot{p},\dot{p},t)+a_2\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t)+(a_3+a_4)\mathscr{G}(\dot{p},\dot{p},\Gamma\dot{p},t)\big\}\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}\mathscr{G}\big(y_n,\dot{p},\dot{p},t\big)&=\mathscr{G}\Big(\Delta(z_n,\Gamma z_n,\Gamma x_n;\alpha_n,\beta_n),\dot{p},\dot{p},t\Big)\\&\geq\Big\{\big(\alpha_n-\beta_n\big)\mathscr{G}(z_n,\dot{p},\dot{p},t)+\big(1-\alpha_n\big)\mathscr{G}(\Gamma z_n,\dot{p},\dot{p},t)+\beta_n\mathscr{G}(\Gamma x_n,\dot{p},\dot{p},t)\Big\}\\&\geq\Bigg\{\big(\alpha_n-\beta_n+a_1(1-\alpha_n)\big)\mathscr{G}(z_n,\dot{p},\dot{p},t)+\beta_n a_1\mathscr{G}(x_n,\dot{p},\dot{p},t)\\&\quad+\big(1-\alpha_n\big)a_2\mathscr{G}(z_n,\Gamma z_n,\Gamma z_n,t)+\beta_n a_2\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)\\&\quad+\big(1-(\alpha_n-\beta_n)\big)\big(a_3+a_4\big)\mathscr{G}(\dot{p},\Gamma\dot{p},\Gamma\dot{p},t)\Bigg\}\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}\mathscr{G}\big(z_n,\dot{p},\dot{p},t\big)&=\mathscr{G}\Big(\Delta(\Gamma x_n,x_n,\Gamma x_n;\theta_n,\theta_n),\dot{p},\dot{p},t\Big)\\&\geq\Big\{\big(1-\theta_n(1-a_1)\big)\mathscr{G}(x_n,\dot{p},\dot{p},t)+\theta_n a_2\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)\\&\quad+\theta_n\big(a_3+a_4\big)\mathscr{G}(\dot{p},\Gamma\dot{p},\Gamma\dot{p},t)\Big\}\end{aligned}\tag{3.4}$$

Substituting (3.3) and (3.4) in (3.2), we obtain

$$\begin{aligned}
\mathscr{G}\left(x_{n+1},\dot{p},\dot{p},t\right) \geq\ & a_1\Bigg\{\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\Big(\left(1-\theta_n(1-a_1)\right)\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+\theta_n a_2\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)+\theta_n(a_3+a_4)\mathscr{G}(\dot{p},\Gamma\dot{p},\Gamma\dot{p},t)\Big)+\beta_n a_1\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+(1-\alpha_n)a_2\mathscr{G}(z_n,\Gamma z_n,\Gamma z_n,t)+\beta_n a_2\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)\\
&+\left(1-(\alpha_n-\beta_n)\right)(a_3+a_4)\mathscr{G}(\dot{p},\Gamma\dot{p},\Gamma\dot{p},t)\Bigg\}+a_2\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t)\\
&+(a_3+a_4)\mathscr{G}(\dot{p},\dot{p},\Gamma\dot{p},t)\\
=\ & a_1\Big(\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\left(1-\theta_n(1-a_1)\right)+\beta_n a_1\Big)\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+a_2\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t)+a_1\left((1-\alpha_n)a_2\right)\mathscr{G}(z_n,\Gamma z_n,\Gamma z_n,t)\\
&+a_1\left((\alpha_n-\beta_n+(1-\alpha_n)a_1)\theta_n a_2+\beta_n a_2\right)\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)\\
&+\Bigg\{a_1\Big[\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\theta_n(a_3+a_4)+\left(1-(\alpha_n-\beta_n)\right)(a_3+a_4)\Big]\\
&+(a_3+a_4)\Bigg\}\mathscr{G}(\dot{p},\dot{p},\Gamma\dot{p},t)
\end{aligned}$$

Since $\mathscr{G}(\dot{p},\Gamma\dot{p},\Gamma\dot{p},t)=1$, we obtain,

$$\begin{aligned}
\mathscr{G}(x_{n+1},\dot{p},\dot{p},t) \geq\ & \Bigg\{a_1\Big[\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\left(1-\theta_n(1-a_1)\right)+\beta_n a_1\Big]\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+a_2\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t)+a_1\Big[(1-\alpha_n)a_2\Big]\mathscr{G}(z_n,\Gamma z_n,\Gamma z_n,t)\\
&+a_1\Big[(\alpha_n-\beta_n+(1-\alpha_n)a_1)\theta_n a_2+\beta_n a_2\Big]\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)\Bigg\}
\end{aligned}$$

In order to satisfy the conditions of Lemma 2.4, we take δ,ε_n and κ_n as follows:

$$\begin{aligned}
0\leq\delta &= a_1\left[(\alpha_n-\beta_n+(1-\alpha_n)a_1)(1-\theta_n(1-a_1))+\beta_n a_1\right]<1\\
\varepsilon_n &= \Big\{a_2\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t)+a_1\left[\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\theta_n a_2+\beta_n a_2\right]\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)\\
&\quad +a_1\left((1-\alpha_n)a_2\right)\mathscr{G}(z_n,\Gamma z_n,\Gamma z_n,t)\Big\}\\
\kappa_n &= \mathscr{G}(x_n,\dot{p},\dot{p},t).
\end{aligned}$$

Since

$$\lim_{n\to\infty}\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)=\lim_{n\to\infty}\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t)=\lim_{n\to\infty}\mathscr{G}(z_n,\Gamma z_n,\Gamma z_n,t)=1$$

By Lemma 2.4, we have $\lim_{n\to\infty}\mathscr{G}(x_n,\dot{p},\dot{p},t)=1$. □

Example 3.2. *Let $X=[-1,1]$ and the $\mathscr{G}$ fuzzy metric is defined by $\mathscr{G}(x,y,z,t)=\dfrac{t}{t+\mathscr{G}(x,y,z)}$, where $\mathscr{G}(x,y,z)=|x-y|+|y-z|+|z-x|$. The $\mathscr{G}$-fuzzy convex structure Δ is defined by $\Delta(x,y,z,a_1,a_2)=(a_1-a_2)x+(1-a_1)y+a_2z$ and the self map $\Gamma(x)=\dfrac{x}{4}$. Clearly, $(X,\mathscr{G},*)$ is a complete $\mathscr{G}$-FCMS. The sequences are defined by $\alpha_n=\dfrac{n}{n+1}$, $\beta_n=\dfrac{n}{n+2}$, $\gamma_n=\dfrac{n}{n+3}$ and $theta_n=\dfrac{n}{n+4}$. Thus, the sequence $\{x_n\}_{n\geq0}$ is satisfied all the conditions of the Theorem 3.1 and the sequence $\mathscr{G}$-converges to unique fixed point 0 of Γ.*

Theorem 3.3. *Let C be a non empty closed convex subset of a $(X,\mathscr{G},*)$ complete $\mathscr{G}$-FCMS with Δ convex structure and $\Gamma:X\to X$ be a mapping satisfying the following conditions:*

$$\mathscr{G}\left(\Gamma x,\Gamma y,\Gamma z,t\right)\geq\Big\{a_1\mathscr{G}(x,y,z,t)+a_2\mathscr{G}(x,\Gamma x,\Gamma x,t)+a_3\mathscr{G}(y,\Gamma y,\Gamma y,t)+a_4\mathscr{G}(\Gamma x,\Gamma z,\Gamma z,t)\Big\} \tag{3.5}$$

for all $x,y,z\in X$ where $0\leq a_1,a_2\leq\frac{1}{4},a_3,a_4\in[0,1]$ and $\{x_n\}_{n\geq0}$ is given by

(i) $x_0\in X$,

(ii) $x_{n+1} = \Delta(\Gamma y_n, \Gamma y_n, \Gamma y_n : \gamma_n, \gamma_n)$,
(iii) $y_n = \Delta(z_n, \Gamma z_n, \Gamma x_n : \alpha_n, \beta_n)$,
(iv) $z_n = \Delta(\Gamma x_n, x_n, \Gamma x_n : \theta_n, \theta_n)$ *with*
(v) $\{\theta_n\}_{n\geq 0} \subset [0, \frac{1}{4})$,
(vi) $\beta_n \leq (1-\alpha_n)a_1 \leq \alpha_n$

Then $\{x_n\}_{n\geq 0}$ *converges to unique fixed point* $\dot{p}$ *of* Γ.

Proof: Suppose that Γ satisfies condition (i) - (iv), we have

$$\begin{aligned}\mathscr{G}\left(x_{n+1}, \dot{p}, \dot{p}, t\right) &= \mathscr{G}\left(\Delta(\Gamma y_n, \Gamma y_n, \Gamma y_n; \gamma_n, \gamma_n), \dot{p}, \dot{p}, t\right) \\ &\geq \Big\{(\gamma_n - \gamma_n)\mathscr{G}(\Gamma y_n, \dot{p}, \dot{p}, t) + (1-\gamma_n)\mathscr{G}(\Gamma y_n, \dot{p}, \dot{p}, t) + \gamma_n \mathscr{G}(\Gamma y_n, \dot{p}, \dot{p}, t)\Big\} \\ &\geq \Big\{a_1 \mathscr{G}(y_n, \dot{p}, \dot{p}, t) + a_2 \mathscr{G}(y_n, \Gamma y_n, \Gamma y_n, t) + a_3 \mathscr{G}(\dot{p}, \dot{p}, \Gamma \dot{p}, t) + a_4 \mathscr{G}(\dot{p}, \dot{p}, \Gamma \dot{p}, t)\Big\} \\ &= \Big\{a_1 \mathscr{G}(y_n, \dot{p}, \dot{p}, t) + a_2 \mathscr{G}(y_n, \Gamma y_n, \Gamma y_n, t) + (a_3 + a_4)\mathscr{G}(\dot{p}, \dot{p}, \Gamma \dot{p}, t)\Big\} \end{aligned} \tag{3.6}$$

$$\begin{aligned}\mathscr{G}\left(y_n, \dot{p}, \dot{p}, t\right) &= \mathscr{G}\left(\Delta(z_n, \Gamma z_n, \Gamma x_n; \alpha_n, \beta_n), \dot{p}, \dot{p}, t\right) \\ &\geq \Big\{(\alpha_n - \beta_n)\mathscr{G}(z_n, \dot{p}, \dot{p}, t) + (1-\alpha_n)\mathscr{G}(\Gamma z_n, \dot{p}, \dot{p}, t) + \beta_n \mathscr{G}(\Gamma x_n, \dot{p}, \dot{p}, t)\Big\} \\ &\geq \Big\{\big(\alpha_n - \beta_n + a_1(1-\alpha_n)\big)\mathscr{G}(z_n, \dot{p}, \dot{p}, t) + \beta_n a_1 \mathscr{G}(x_n, \dot{p}, \dot{p}, t) \\ &+ (1-\alpha_n)a_2 \mathscr{G}(z_n, \Gamma z_n, \Gamma z_n, t) + \beta_n a_2 \mathscr{G}(x_n, \Gamma x_n, \Gamma x_n, t) \\ &+ (1-(\alpha_n - \beta_n))(a_3 + a_4)\mathscr{G}(\dot{p}, \Gamma \dot{p}, \Gamma \dot{p}, t)\Big\} \end{aligned} \tag{3.7}$$

$$\begin{aligned}\mathscr{G}\left(z_n, \dot{p}, \dot{p}, t\right) &= \mathscr{G}\left(\Delta(\Gamma x_n, x_n, \Gamma x_n; \theta_n, \theta_n), \dot{p}, \dot{p}, t\right) \\ &\geq \Big\{(\theta_n - \theta_n)\mathscr{G}(\Gamma x_n, \dot{p}, \dot{p}, t) + (1-\theta_n)\mathscr{G}(x_n, \dot{p}, \dot{p}, t) + \theta_n \mathscr{G}(\Gamma x_n, \dot{p}, \dot{p}, t)\Big\} \\ &\geq \Big\{\big(1-\theta_n(1-a_1)\big)\mathscr{G}(x_n, \dot{p}, \dot{p}, t) + \theta_n a_2 \mathscr{G}(x_n, \Gamma x_n, \Gamma x_n, t) \\ &+ \theta_n(a_3 + a_4)\mathscr{G}(\dot{p}, \Gamma \dot{p}, \Gamma \dot{p}, t)\Big\} \end{aligned} \tag{3.8}$$

Substituting (3.7) and (3.8) in (3.6), we have

$$\begin{aligned}\mathscr{G}\left(x_{n+1}, \dot{p}, \dot{p}, t\right) &\geq a_1 \Bigg\{\big(\alpha_n - \beta_n + (1-\alpha_n)a_1\big)\Big(\big(1-\theta_n(1-a_1)\big)\mathscr{G}(x_n, \dot{p}, \dot{p}, t) \\ &+ \theta_n a_2 \mathscr{G}(x_n, \Gamma x_n, \Gamma x_n, t) + \theta_n(a_3 + a_4)\mathscr{G}(\dot{p}, \Gamma \dot{p}, \Gamma \dot{p}, t)\Big) + \beta_n a_1 \mathscr{G}(x_n, \dot{p}, \dot{p}, t) \\ &+ (1-\alpha_n)a_2 \mathscr{G}(z_n, \Gamma z_n, \Gamma z_n, t) + \beta_n a_2 \mathscr{G}(x_n, \Gamma x_n, \Gamma x_n, t) \\ &+ \big(1-(\alpha_n - \beta_n)\big)(a_3 + a_4)\mathscr{G}(\dot{p}, \Gamma \dot{p}, \Gamma \dot{p}, t)\Bigg\} + a_2 \mathscr{G}(y_n, \Gamma y_n, \Gamma y_n, t) \\ &+ (a_3 + a_4)\mathscr{G}(\dot{p}, \dot{p}, \Gamma \dot{p}, t) \\ &= a_1 \Big(\big(\alpha_n - \beta_n + (1-\alpha_n)a_1\big)\big(1-\theta_n(1-a_1)\big) + \beta_n a_1\Big)\mathscr{G}(x_n, \dot{p}, \dot{p}, t) \\ &+ a_2 \mathscr{G}(y_n, \Gamma y_n, \Gamma y_n, t) + a_1\big((1-\alpha_n)a_2\big)\mathscr{G}(z_n, \Gamma z_n, \Gamma z_n, t) \\ &+ a_1\big((\alpha_n - \beta_n + (1-\alpha_n)a_1)\theta_n a_2 + \beta_n a_2\big)\mathscr{G}(x_n, \Gamma x_n, \Gamma x_n, t) \\ &+ \Big\{a_1\big[\big(\alpha_n - \beta_n + (1-\alpha_n)a_1\big)\theta_n(a_3 + a_4) + \big(1-(\alpha_n - \beta_n)\big)(a_3 + a_4)\big] \\ &+ (a_3 + a_4)\Big\}\mathscr{G}(\dot{p}, \dot{p}, \Gamma \dot{p}, t) \end{aligned}$$

Since $\mathscr{G}(\dot{p}, \Gamma \dot{p}, \Gamma \dot{p}, t) = 1$, we obtain,

$$\begin{aligned}\mathscr{G}\left(x_{n+1}, \dot{p}, \dot{p}, t\right) &\geq \Bigg\{a_1\big[\big(\alpha_n - \beta_n + (1-\alpha_n)a_1\big)\big(1-\theta_n(1-a_1)\big) + \beta_n a_1\big]\mathscr{G}(x_n, \dot{p}, \dot{p}, t) \\ &+ a_2 \mathscr{G}(y_n, \Gamma y_n, \Gamma y_n, t) + a_1\big[(1-\alpha_n)a_2\big]\mathscr{G}(z_n, \Gamma z_n, \Gamma z_n, t) \\ &+ a_1\big[\big(\alpha_n - \beta_n + (1-\alpha_n)a_1\big)\theta_n b + \beta_n a_2\big]\mathscr{G}(x_n, \Gamma x_n, \Gamma x_n, t)\Bigg\} \end{aligned} \tag{3.9}$$

Continuing the process,

$$\mathscr{G}\left(x_n,\Gamma x_n,\Gamma x_n,t\right)\geq\left(\frac{1+2a_1}{1-2a_2}\right)\mathscr{G}(x_n,\dot{p},\dot{p},t) \tag{3.10}$$

$$\begin{aligned}\mathscr{G}(z_n,\Gamma z_n,\Gamma z_n,t) &\geq\left(\frac{1+2a_1}{1-2a_2}\right)\mathscr{G}(z_n,\dot{p},\dot{p},t)\\ &\geq\left(\frac{1+2a_1}{1-2a_2}\right)(1-\theta_n(1-a_1))\mathscr{G}(x_n,\dot{p},\dot{p},t)+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)\\ &\geq\left(\frac{1+2a_1}{1-2a_2}\right)(1-\theta_n(1-a_1))\mathscr{G}(x_n,\dot{p},\dot{p},t)+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\mathscr{G}(x_n,\dot{p},\dot{p},t)\end{aligned} \tag{3.11}$$

$$\begin{aligned}\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t) &\geq\left(\frac{1+2a_1}{1-2a_2}\right)\mathscr{G}(y_n,\dot{p},\dot{p},t)\\ &\geq\left(\frac{1+2a_1}{1-2a_2}\right)\Bigg\{\Bigg[\big(\alpha_n-\beta_n+(1-\alpha_n)a_1\big)\big(1-\theta_n(1-a_1)\big)\mathscr{G}(x_n,\dot{p},\dot{p},t)\\ &+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\mathscr{G}(x_n,\dot{p},\dot{p},t)\Bigg]+\beta_n a_1\mathscr{G}(x_n,\dot{p},\dot{p},t)+\beta_n a_2\left(\frac{1+2a_1}{1-2a_2}\right)\\ &\mathscr{G}(x_n,\dot{p},\dot{p},t)+(1-\alpha_n)a_2\left(\frac{1+2a_1}{1-2a_2}\right)\Bigg[(1-\theta_n(1-a_1))\mathscr{G}(x_n,\dot{p},\dot{p},t)\\ &+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\mathscr{G}(x_n,\dot{p},\dot{p},t)\Bigg]\Bigg\}\end{aligned} \tag{3.12}$$

Substituting (3.10), (3.11) and (3.12) in (3.9), we obtain,

$$\begin{aligned}\mathscr{G}\left(x_{n+1},\dot{p},\dot{p},t\right) &\geq\Bigg\{\Big(a_1\big(\alpha_n-\beta_n+(1-\alpha_n)a_1\big)\big(1-\theta_n(1-a_1)\mathscr{G}(x_n,\dot{p},\dot{p},t)\big)\Big)\\ &+a_2\left(\frac{1+2a_1}{1-2a_2}\right)\Bigg(\Big[(\alpha_n-\beta_n+(1-\alpha_n)a_1)\Big]\Big[(1-\theta_n(1-a_1))+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\Big]\\ &+\beta_n a_1\mathscr{G}(x_n,\dot{p},\dot{p},t)+\beta_n a_2\left(\frac{1+2a_1}{1-2a_2}\right)+(1-\alpha_n)a_2\left(\frac{1+2a_1}{1-2a_2}\right)\Big[(1-\theta_n(1-a_1))\\ &+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\Big]\Bigg)+a_1\Big((\alpha_n-\beta_n+(1-\alpha_n)a_1)\theta_n a_2+\beta_n a_2\Big)\left(\frac{1+2a_1}{1-2a_2}\right)\\ &+a_1\Bigg((1-\alpha_n)a_2\left(\frac{1+2a_1}{1-2a_2}\right)\Big[(1-\theta_n(1-a_1))+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\Big]\Bigg)\Bigg\}\mathscr{G}(x_n,\dot{p},\dot{p},t)\end{aligned}$$

$$\begin{aligned}\mathscr{G}\left(x_{n+1},\dot{p},\dot{p},t\right) &\geq\Bigg\{a_1\big(\alpha_n-\beta_n+(1-\alpha_n)a_1\big)\big(1-\theta_n(1-a_1)\big)+\beta_n a_1\big)+a_2\left(\frac{1+2a_1}{1-2a_2}\right)\\ &\Bigg(\Big[(\alpha_n-\beta_n+(1-\alpha_n)a_1)\Big]\Big[(1-\theta_n(1-a_1))+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\Big]\\ &+\beta_n a_1\mathscr{G}(x_n,\dot{p},\dot{p},t)+\beta_n a_2\left(\frac{1+2a_1}{1-2a_2}\right)+(1-\alpha_n)a_2\left(\frac{1+2a_1}{1-2a_2}\right)\\ &\Big[(1-\theta_n(1-a_1))+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\Big]\Bigg)+a_1\big((\alpha_n-\beta_n+(1-\alpha_n)a_1)\theta_n a_2\\ &+\beta_n a_2)\left(\frac{1+2a_1}{1-2a_2}\right)+a_1(1-\alpha_n)a_2\left(\frac{1+2a_1}{1-2a_2}\right)\big[(1-\theta_n(1-a_1))\end{aligned}$$

$$\begin{aligned}
\mathscr{G}\left(x_{n+1},\dot{p},\dot{p},t\right) &\geq a_1\left(\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\left(1-\theta_n(1-a_1)\right)+\beta_n a_1\right)\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+a_2\left(\frac{1+2a_1}{1-2a_2}\right)\Big[(\alpha_n-\beta_n+(1-\alpha_n)a_1)\Big]\left(1-\theta_n(1-a_1)\right)\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+\left(\frac{1+2a_1}{1-2a_2}\right)^2\Big[(\alpha_n-\beta_n+(1-\alpha_n)a_1)\Big]\theta_n a_2^2\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+a_2\left(\frac{1+2a_1}{1-2a_2}\right)\beta_n a_1\mathscr{G}(x_n,\dot{p},\dot{p},t)+\left(\frac{1+2a_1}{1-2a_2}\right)^2\beta_n a_2^2\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+(1-\alpha_n)a_2^2\left(\frac{1+2a_1}{1-2a_2}\right)^2\left(1-\theta_n(1-a_1)\right)\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+\left(\frac{1+2a_1}{1-2a_2}\right)^3(1-\alpha_n)\theta_n a_2^3\mathscr{G}(x_n,\dot{p},\dot{p},t)+a_1\left((\alpha_n-\beta_n+(1-\alpha_n)a_1)\theta_n a_2+\beta_n a_2\right)\\
&\left(\frac{1+2a_1}{1-2a_2}\right)\mathscr{G}(x_n,\dot{p},\dot{p},t)+a_1\left((1-\alpha_n)a_2\right)\left(\frac{1+2a_1}{1-2a_2}\right)(1-\theta_n(1-a_1))\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+a_1\left((1-\alpha_n)a_2\right)\left(\frac{1+2a_1}{1-2a_2}\right)^2\theta_n a_2\mathscr{G}(x_n,\dot{p},\dot{p},t).
\end{aligned}$$

Since

$$\begin{aligned}
0 &\leq \Bigg\{a_1\Big(\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\left(1-\theta_n(1-a_1)\right)+\beta_n a_1\Big)\\
&+a_2\left(\frac{1+2a_1}{1-2a_2}\right)\Bigg([(\alpha_n-\beta_n+(1-\alpha_n)a_1)]\left[(1-\theta_n(1-a_1))+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\right]\\
&+\beta_n a_1\mathscr{G}(x_n,\dot{p},\dot{p},t)+\beta_n a_2\left(\frac{1+2a_1}{1-2a_2}\right)+(1-\alpha_n)a_2\left(\frac{1+2a_1}{1-2a_2}\right)\Big[(1-\theta_n(1-a_1))\\
&+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\Big]\Bigg)+a_1((\alpha_n-\beta_n+(1-\alpha_n)a_1)\theta_n a_2+\beta_n a_2)\left(\frac{1+2a_1}{1-2a_2}\right)\\
&+a_1\left((1-\alpha_n)a_2\right)\left(\frac{1+2a_1}{1-2a_2}\right)\left[(1-\theta_n(1-a_1))+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\right]\Bigg\}<1
\end{aligned}$$

By Lemma 2.4, we have $\lim_{n\to\infty}\mathscr{G}(x_n,\dot{p},\dot{p},t)=1$. □

4. Conclusion

In this paper, we obtain the sequence using three step iteration process and convergence of iteration process to unique fixed point under conditions of contractive mappings on the G -fuzzy metric spaces in convex structure.

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Author's contributions

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Pedal Sets of Unitals in Projective Planes of Order 16

Mustafa Gezek

Department of Mathematics, Faculty of Science and Arts, Tekirdağ Namık Kemal University, Tekirdağ, Turkey

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Abstract

In this article, we perform computer searches for pedal sets of all known unitals in the known planes of order 16. Special points of unitals having at least one special tangent are studied in detail. It is shown that unitals without special points exist. Open problems regarding the computational results presented in this study are discussed. A conjecture about the numbers of line types of a unital U and its dual unital $U^{\perp}$ is formulated.

1. Introduction

We assume familiarity with the basic facts from finite geometries and combinatorial design theory [1]- [3].
A t-(v,k,λ) design is a pair D=$\{X,B\}$ of a set X of cardinality v, called points, and a collection B of k-subsets of X, called blocks, such that every t points appear together in exactly λ blocks. A parallel class of a design D is a collection of blocks that partitions the point set of D. A resolution of D is a partition of the collection of blocks of D into disjoint parallel classes. A design D is resolvable if it has at least one resolution.
Let π be a projective plane of order q^2. A *unital* embedded in π is defined to be a set U of q^3+1 points of π meeting lines of the plane in either *one* point or $q+1$ points. The sets of the intersections of the lines of π with U at $q+1$ points form a 2-$(q^3+1,q+1,1)$ design.
A classical example of a unital is the Hermitian unital $H(q)$ defined by the absolute points of a unitary polarity in $PG(2,q^2)$. In 1976, Buekenhout provided two methods for constructing unitals [4]. In 1979, Metz used one of Buekenhout's method to construct a non-classical unital in a Desarguesian plane of order q^2 [5], and in 1994, Barwick showed that any unitals constructed by the other method of Buekenhout is a classical unital [6]. In 1988, for every odd prime power q, Rosati constructed a unital in Hughes planes of order q [7], and in 1990, Kestenband generalized Rosati's construction [8]. Some other studies of unitals can be found in [9]- [12].
There exist q^3+1 lines meeting a unital U at one point, called *tangent lines* to U, and $q^2(q^2-q+1)$ lines meeting U at $q+1$ points, called *secant lines* to U. For any point $P\notin U$, the number of tangents and secants through P are $q+1$ and q^2-q, respectively [1]. The set of the $q+1$ intersections of tangents through P with U is called the *pedal set* of P. P is called a *special point* if its pedal set is collinear. A *special tangent* is defined to be a tangent having q^2 special points.
In this study, pedal sets of all known unitals in the known projective planes of order 16 are computed. Special points of unitals having at least one special tangent are studied in detail. It is shown that unitals without special points exist. Details of the numbers of pedal sets for each possible line type are reported.
Through the paper, a line with p points will be denoted by p-line.

2. Pedal sets of unitals in planes of order 16

Twenty-two projective planes of order 16 are known to exist. The names of the planes are in accordance with [13]: $PG(2,16)$, BBH1, SEMI2, SEMI4, BBH2, BBS4, DEMP, DSFP, HALL, LMRH, MATH, JOHN, and JOWK. Specific line sets of the

Email address and ORCID number**: mgezek@nku.edu.tr, 0000-0001-5488-9341**

planes used in this study can be found in [14].
Previously it was shown that $PG(2,16)$ contains exactly *two* unitals. Unitals in the rest of the planes of order 16 have not been completely classified, yet.
Known unitals in the known planes of order 16 were found by Stoichev and Tonchev (*thirty-eight* unitals) [15], Krčadinac and Smoljak (*three* unitals) [16], and Stoichev and Gezek (*one hundred and fifteen* unitals) [12].
A pedal set of a unital U in a plane of order 16 comes from 5 points of U. The following *five* configurations (denoted by their line types) are possible for pedal sets in these planes: Either all 5 points are on a line (5^1), or 4 points are on a line and *four* 2-lines ($4^1, 2^4$), or *two* 3-lines and *four* 2-lines ($3^2, 2^4$), or 3 points are on a line and *seven* 2-lines ($3^1, 2^7$), or *ten* 2-lines (2^{10}). Possible geometries of these configurations could be found in [16, Figure 2].
Using the computational algebra system MAGMA [17], pedal sets of all known unitals in the known projective planes of order 16 have been calculated. The algorithm used in our computations contains the following steps:
Step 1: Define the set of *lines (L)*, *points (P)*, *unitals (U)* of the Plane π, and *line types (LT)*
Step 2: For each unital u in U do
Step 3: For each point $p \in P \backslash u$, find its *tangents (T)*
Step 4: For every tangent $t \in T$ find $t \cap u$ and save them in a set *ps* // *ps* is the set of pedal sets of the point p
Step 5: Save the pedal sets *ps* in an indexed set *PS*
Step 6: For each pedal set *ps* in *PS*, check which line type in *LT* it possesses
Step 7: Print the number of each possible line type
The number of known unitals in the known planes of order 16 is 156. Specific point sets of the known unitals used in this study can be found in [12]. Pedal sets of the *forty-two* of the known unitals are studied in [16]. We list the details of the pedal sets of the remaining unitals in Table 3.1, where Column 1 states the name of the plane, Column 2 provides the unital no's, and the last column gives the numbers of pedal sets for each type. All except 38 of unitals in these planes have the same pedal sets counts with their duals. Details of the pedal sets of dual unitals having different pedal set counts with their duals are listed in Table 3.2.
Table 3.1 shows that all unitals except unital 4 of BBH1 plane, unital 18 of BBH2 plane, unital 11 of BBS4 plane, all known unitals in DEMP plane, units 4, 6, 7, and 8 of MATH plane, unital 5 of JOWK plane, unitals 3, 4, 7 and 8 of SEMI2 plane and unitals 2, 3, and 7 of SEMI4 plane have at least 16 special points.
Previously, there were only *two* unitals in BBH1 plane having a special tangent. Our computations show that unitals 14 and 16 of BBH1 plane also possess a special tangent. All of the unitals having a special tangent in BBH1 plane has special points not lying on a special tangent: Unital 1 of BBH1 plane has *sixteen* special points outside of a special tangent, which are divided into *four* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital. Unital 2 (and 14) of BBH1 plane has *fifty-two* special points outside of a special tangent. None of these points lies on a secant through the intersection point of the special tangent with the unital. Unital 16 of BBH1 plane has *eight* special points outside of a special tangent, which are divided into *two* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital. *Eight* of the unitals of BBH1 plane contains exactly 16 special points, but none of these points lie on a special tangent.
BBH2 plane previously was known to contain only *one* unital having a special tangent. Our computations show that there are *six* more unitals in BBH2 plane having exactly one special tangent, all of which have special points not lying on a special tangent: Unitals 19, 20, 22, and 23 of BBH2 plane has *eight* special points outside of a special tangent, which are divided into *two* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital. Unital 21 of BBH2 plane has *sixteen* special points outside of a special tangent, which are divided into *two* distinct sets of size 8 such that each set lies on a secant through the intersection point of the special tangent with the unital. Unital 26 of BBH2 plane has *twenty-four* special points outside of a special tangent, which are divided into *six* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital. Table 3.1 shows that *seven* of the unitals in BBH2 plane have exactly 16 special points, but none of these points lie on a special tangent.
None of the known unitals in BBS4, DEMP, and DSFP planes have a special tangent, but *six* unitals in BBS4 plane have exactly 16 special points, but none of these points lie on a special tangent.
Details of the pedal sets of the known unitals in HALL plane can be found in [16]. Only unitals 4 and 6 of HALL plane contains special points not lying on a special tangent: Unital 4 of HALL plane has *sixteen* special points outside of a special tangent, which are divided into *four* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital. Unital 6 of HALL plane has *fifty-two* special points outside of a special tangent, which are divided into *ten* distinct sets of size 4 and *one* set of size 12 such that each set lies on a secant through the intersection point of the special tangent with the unital.
All known unitals in MATH plane having exactly 16 special points, as well as unitals 5 and 13, have exactly one special tangent. Table 3.1 shows that unitals having more than 16 special points in MATH plane have special points not lying on a special tangent: Unital 5 of MATH plane has *eight* special points outside of a special tangent, which are divided into *two* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital. Unital 13 of MATH plane has *sixty-four* special points outside of a special tangent, which are divided into *sixteen* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital.
JOHN plane contains three unitals having a special tangent, two of which have special points not lying on a special tangent: Unital 2 of JOHN plane has *sixteen* special points outside of a special tangent, which are divided into *four* distinct sets of size

4 such that each set lies on a secant through the intersection point of the special tangent with the unital. Unital 26 of JOHN plane has *eight* special points outside of a special tangent, which are divided into *two* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital.
The number of known unitals in SEMI2 plane is 21, all except four have exactly *sixteen* special points and a special tangent. None of the unitals in SEMI2 plane having at least 16 special points have special points outside of a special tangent.
SEMI4 plane previously was known to have exactly two unitals having a special tangent. Our computations show that many of the known unitals in SEMI4 plane possess a special tangent. Two of these unitals have special points not lying on a special tangent: Unital 4 of SEMI4 plane has *four* special points outside of a special tangent, which lies on a secant through the intersection point of the special tangent with the unital.
In [18], it was shown that special points and special tangents of a unital U give rise to parallel classes and resolutions of the unital design associated with U, respectively. Even though, all parallel classes and resolutions of the unital designs associated with a unital in planes of order 9 come from special points and special tangents, respectively [16], this is not true in general. The parallel classes of the designs associated with the following unitals in planes of order 16 come from special points: Unitals 1 and 16 of BBH1 plane, unitals 6, 21, and dual unitals 7, 20, and 26 of BBH2 plane, unital 11 of BBS4 plane, unital 3 of DEMP plane, unital 5 of HALL plane, all unitals in LMRH plane, all unitals except unitals 5 and 9 of MATH plane, units 2 and 29 and dual unital 26 of JOHN plane, unitals 5 and 7 of JOWK plane, all unitals except unitals 2 and 10 of SEMI2 plane, all unitals except unitals 3 and 4 of SEMI4 plane, and unital 2 of PG(2,16). The number of parallel classes of the designs associated with the rest of the known unitals in planes of order 16 is grater then the number of special points of unitals.

3. Conclusion

Previously, no unitals without special points were known to exist (a question asked by the authors in [16]), but the data given in Table 3.1 shows that unitals $6, 6^{\perp}, 7$, and 8 of MATH plane and unital 2 of SEMI4 plane do not have any special points (unital 2 of SEMI4 plane in [16] is the unital 12 in [12]).
All known unitals in projective planes of order $q^2 \in \{9, 16\}$ having at least one special tangent have the property that the number of special points is a multiple of q. Does this property hold in general?
Unitals 2 and 14 of BBH1 plane are the first (and only) examples of unitals having the following property: none of the special points outside of a special tangent lies on a secant through the intersection point of the special tangent with the unital. Why do these unitals act differently?
None of the unitals given in Table 3.2 have a special tangent. This shows that if a unital U in a plane of order 16 has a special tangent, then U and $U^{\perp}$ have the same pedal set counts. Unitals in planes of order 9 having at least one special tangent also possess this property [16]. Are there unitals not having this property?
It was observed in [16] that the number of pedal sets having line type $(q+1)$ always seems to agree for a unital and its dual unital. We notice that not only the number of line type $(q+1)$, but also the number of line type $(q, 2^q)$ seems to agree for a unital and its dual unital. Can we prove that this property holds in general?
We end this paper with the following conjecture:

Conjecture 1. *Let U be a unital embedded in a projective plane of order q^2, and $n_i(U)$ be the number of pedal sets of U having line type i. Then,*

$$n_i(U) = n_i(U^{\perp})$$

for $i \in \{(q+1), (q, 2^q)\}$. Furthermore, if U has a special tangent, then

$$n_i(U) = n_i(U^{\perp})$$

for any i.

Plane	Unital No.	Pedal set				
		(5)	$(4,2^4)$	$(3^2,2^4)$	$(3,2^7)$	(2^{10})
BBH1	4	11	0	24	84	89
	5	16	12	4	104	72
	6	16	12	4	104	72
	7	16	0	16	104	72
	8	28	24	12	96	48
	9	28	12	36	52	80
	10	28	24	12	96	48
	11	16	4	12	88	88
	12	16	4	12	88	88
	13	16	8	12	104	68
	14	68	0	0	104	36
	15	16	8	12	104	68
	16	24	16	16	96	56
BBH2	7	11	8	32	96	61
	8	16	24	4	92	72
	9	16	16	0	124	52
	10	28	24	0	108	48
	11	16	12	20	92	68
	12	16	4	28	80	80
	13	16	8	12	104	68
	14	20	16	12	96	64
	15	24	0	32	76	76
	16	24	4	20	92	68
	17	16	0	20	104	68
	18	8	0	30	120	50
	19	24	24	8	136	16
	20	24	56	8	88	32
	21	32	8	0	96	72
	22	24	24	0	88	72
	23	24	16	0	112	56
	24	16	24	12	112	44
	25	32	16	12	88	60
	26	40	16	0	112	40
BBS4	2	16	16	4	76	96
	3	16	24	0	76	92
	4	20	12	4	88	84
	5	16	16	8	92	76
	6	20	8	4	96	80

Table 3.1: Pedal sets of unitals in planes of order 16.

Plane	Unital No.	Pedal set				
		(5)	$(4,2^4)$	$(3^2,2^4)$	$(3,2^7)$	(2^{10})
BBS4	7	24	16	24	76	68
	8	24	8	12	100	64
	9	20	16	36	92	44
	10	16	24	20	52	96
	11	4	0	0	150	54
	12	16	0	0	156	36
	13	16	24	48	60	60
DEMP	3	4	0	24	24	156
	4	4	12	12	144	36
MATH	5	24	8	0	56	120
	6	0	0	32	48	128
	7	0	0	32	80	96
	8	0	0	16	32	160
	9	12	12	44	64	76
	10	16	0	0	192	0
	11	16	0	0	0	192
	12	16	0	0	0	192
	13	80	0	0	128	0
	14	16	0	0	64	128
	15	16	0	0	64	128
	16	16	0	0	64	128
JOHN	6	16	20	0	68	104
	7	16	16	0	80	96
	8	16	12	4	80	96
	9	16	8	0	116	68
	10	20	12	0	96	80
	11	20	16	0	92	80
	12	24	0	16	100	68
	13	20	0	16	64	108
	14	16	20	20	112	40
	15	20	12	12	68	96
	16	20	16	12	112	48
	17	16	24	20	76	72
	18	24	4	20	88	72
	19	16	44	12	84	52
	20	16	32	16	100	44
	21	20	0	16	88	84
	22	24	12	20	60	92

Table 3.1: (Continued)

Plane	Unital	Pedal set				
	No.	(5)	$(4,2^4)$	$(3^2,2^4)$	$(3,2^7)$	(2^{10})
JOHN	23	20	12	12	80	84
	24	24	8	12	64	100
	25	20	12	20	52	104
	26	24	0	0	128	56
	27	20	0	12	112	64
	28	24	0	12	88	84
	29	16	0	16	48	128
JOWK	5	4	0	12	84	108
	6	20	0	12	96	80
	7	16	0	16	128	48
SEMI2	4	4	0	12	48	144
	5	16	0	32	96	64
	6	16	0	0	160	32
	7	4	0	12	144	48
	8	4	0	60	48	96
	9	16	0	32	128	32
	10	16	0	0	96	96
	11	16	0	0	64	128
	12	16	0	64	64	64
	13	16	0	0	192	0
	14	16	0	0	0	192
	15	16	0	0	0	192
	16	16	64	64	64	0
	17	16	0	0	64	128
	18	16	0	0	0	192
	19	16	0	0	0	192
	20	16	0	0	192	0
	21	16	0	0	192	0
SEMI4	2	0	16	16	64	112
	3	4	4	12	100	88
	4	20	16	0	96	76
	5	16	24	0	72	96
	6	16	24	0	72	96
	7	4	0	12	144	48
	8	16	0	0	128	64
	9	16	64	0	64	64
	10	16	0	0	128	64
	11	16	64	0	64	64

Table 3.1: (Continued)

Plane	Unital No.	Pedal set				
		(5)	$(4,2^4)$	$(3^2,2^4)$	$(3,2^7)$	(2^{10})
BBH2	7	11	8	24	112	53
	8	16	24	0	100	68
	11	16	12	12	108	60
	12	16	4	20	96	72
	14	20	16	20	80	72
	18	8	0	35	110	55
	23	24	16	16	80	72
BBS4	2	16	16	0	84	92
	4	20	12	0	96	80
	5	16	16	4	100	72
	6	20	8	0	104	76
	9	20	16	32	100	40
	10	16	24	28	36	104
	13	16	24	24	108	36
DEMP	3	4	0	12	48	144
MATH	6	0	0	16	80	112
JOHN	6	16	20	4	60	108
	8	16	12	0	88	92
	9	16	8	8	100	76
	10	20	12	4	88	84
	11	20	16	4	84	84
	17	16	24	24	68	76
	19	16	44	20	68	60
	21	20	0	24	72	92
	24	24	8	20	48	108
	25	20	12	12	68	96
	26	24	0	8	112	64
JOWK	6	20	0	28	64	96

Table 3.2: Pedal sets of the dual unitals in planes of order 16.

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On the Bi-Periodic Mersenne Sequence

Gül Özkan Kızılırmak[1*] and Dursun Taşcı[1]

[1] *Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey*
[*] *Corresponding author*

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Abstract

In this paper, the bi-periodic Mersenne sequence, which is a generalization of the Mersenne sequence, is defined. The characteristic function, generating function and Binet's formula for this sequence are obtained. Also, by using Binet's formula, some important identities and properties for the bi-periodic Mersenne sequence are presented.

1. Introduction

There are some studies about integer sequences in the literature [1]-[4]. Especially Mersenne primes are an active field in the number theory and computer science [5]. They are popular research objects because of their interesting representation in the binary system properties as $(1)_2$, $(11)_2$, $(111)_2$, $(1111)_2$,.... The Mersenne numbers can also be defined as [6]

$$M_n + 2 = 3M_{n+1} - 2M_n,$$

with the initial conditions $M_0 = 0$ and $M_1 = 1$.

The roots of the respective characteristic equation $r^2 - 3r + 2 = 0$ are $r_1 = 2$ and $r_2 = 1$ and we easily get the Binet formula

$$M_n = 2^n - 1.$$

The first few terms of the Mersenne sequence are

$$M_n = \{0,1,3,7,15,31,63,127,255,511,...\} \quad [7].$$

In many studies, generalizations of the integer sequences have been examined [8]-[15]. The bi-periodic Fibonacci sequence, which was introduced by Edson and Yayenie, have made an important contribution to the literature [16]. Inspired by this study, many new generalized sequences have been described [17]-[20].

The main purpose of this paper is to first define bi-periodic Mersenne sequence, to find the generating function and Binet formula, and then to present some identities that include the bi-periodic Mersenne sequence as a result of the corresponding Binet formula.

***Email addresses and ORCID numbers*:** **gulozkan@gazi.edu.tr, 0000-0003-3263-8685 (G. Ö. Kızılırmak), dtasci@gazi.edu.tr, 0000-0001-8357-4875 (D. Taşcı)**

2. Bi-periodic Mersenne sequence

Definition 2.1. *The bi-periodic Mersenne sequence* $\{m_n\}_{n=0}^{\infty}$ *is defined by*

$$m_0 = 0, \quad m_1 = 1, \quad m_n = \begin{cases} 3am_{n-1} - 2m_{n-2}, & \text{if } n \text{ is even;} \\ 3bm_{n-1} - 2m_{n-2}, & \text{if } n \text{ is odd.} \end{cases}, \; n \geq 2.$$

where a and b are any two non-zero real numbers.
By setting $a = b = 1$, we get the classic Mersenne numbers.
The quadratic equation for the bi-periodic Mersenne sequence is defined as

$$x^2 - 3abx + 2ab = 0$$

with the roots

$$\alpha_1 = \frac{3ab + \sqrt{9a^2b^2 - 8ab}}{2} \quad \text{and} \quad \alpha_2 = \frac{3ab - \sqrt{9a^2b^2 - 8ab}}{2}. \tag{2.1}$$

Lemma 2.2. *The bi-periodic Mersenne sequence satisfies the following properties:*

$$m_{2n} = (9ab-4)\,m_{2n-2} - 4m_{2n-4},$$
$$m_{2n+1} = (9ab-4)\,m_{2n-1} - 4m_{2n-3}.$$

Proof. By using the recurrence relation for bi-periodic Mersenne sequence, we obtain m_{2n} and m_{2n+1} as follows:

$$\begin{aligned} m_{2n} &= 3am_{2n-1} - 2m_{2n-2} \\ &= 3a(3bm_{2n-2} - 2m_{2n-3}) - 2m_{2n-2} \\ &= (9ab-2)m_{2n-2} - 6am_{2n-3} \\ &= (9ab-2)m_{2n-2} - (2m_{2n-2} + 4m_{2n-4}) \\ &= (9ab-4)\,m_{2n-2} - 4m_{2n-4} \end{aligned}$$

and

$$\begin{aligned} m_{2n+1} &= 3bm_{2n} - 2m_{2n-1} \\ &= 3b(3am_{2n-1} - 2m_{2n-2}) - 2m_{2n-1} \\ &= (9ab-2)m_{2n-1} - 6bm_{2n-2} \\ &= (9ab-2)m_{2n-1} - (2m_{2n-1} + 4m_{2n-3}) \\ &= (9ab-4)m_{2n-1} - 4m_{2n-3}. \end{aligned}$$

□

Lemma 2.3. *The roots* α_1 *and* α_2 *defined in (2.1) satisfy the following properties:*

$$\begin{aligned} \alpha_1\alpha_2 &= 2ab, \\ \alpha_{1+}\alpha_2 &= 3ab, \\ 3\alpha_1 - 2 &= \frac{\alpha_1^2}{ab}, \\ 3\alpha_2 - 2 &= \frac{\alpha_2^2}{ab}, \\ (3\alpha_1 - 2)(3\alpha_2 - 2) &= 4. \end{aligned}$$

Proof. By using the definitions of α_1 and α_2, the proof can be easily obtained. □

Theorem 2.4. *The generating function for the bi-periodic Mersenne sequence is*

$$M(x) = \frac{x(1 + 2x^2 + 3ax)}{1 - (9ab-4)x^2 + 4x^4}.$$

Proof. $M(x) = m_0 + m_1x + m_2x^2 + ... + m_sx^s + ... = \sum_{k=0}^{\infty} m_kx^k$ is the formal power series representation of the generating function for $\{m_n\}_{n=0}^{\infty}$. If this series is multiplied by $3bx$ and $2x^2$, then we get

$$3bxM(x) = 3bm_0x + 3bm_1x^2 + ... = 3\sum_{k=0}^{\infty} bm_kx^{k+1} = 3\sum_{k=1}^{\infty} bm_{k-1}x^k$$

and

$$2x^2M(x) = 2m_0x^2 + 2m_1x^3 + ... = 2\sum_{k=0}^{\infty} m_kx^{k+2} = 2\sum_{k=2}^{\infty} m_{k-2}x^k.$$

So,

$$\left(1-3bx+2x^2\right)M(x) = m_0 + m_1x - 3bm_0x + \sum_{k=2}^{\infty}\left(m_k - 3bm_{k-1} + 2m_{k-2}\right)x^k. \tag{2.2}$$

Since $m_{2k+1} = 3bm_{2k} - 2m_{2k-1}$ and $m_0 = 0$, $m_1 = 1$, we have

$$\left(1-3bx+2x^2\right)M(x) = x + \sum_{k=1}^{\infty}(m_{2k} - 3bm_{2k-1} + 2m_{2k-2})x^{2k}.$$

$m_{2k} = 3am_{2k-1} - 2m_{2k-2}$ implies that

$$\left(1-3bx+2x^2\right)M(x) = x + 3(a-b)x\sum_{k=1}^{\infty} m_{2k-1}x^{2k-1}.$$

Now, we let

$$m(x) = \sum_{k=1}^{\infty} m_{2k-1}x^{2k-1}.$$

Then,

$$\begin{aligned}\left(1-(9ab-4)x^2+4x^4\right)m(x) &= \sum_{k=1}^{\infty} m_{2k-1}x^{2k-1} - (9ab-4)\sum_{k=2}^{\infty} m_{2k-3}x^{2k-1} + 4\sum_{k=3}^{\infty} m_{2k-5}x^{2k-1}\\ &= m_1x + m_3x^3 - (9ab-4)m_1x^3 + \sum_{k=3}^{\infty}\left(m_{2k-1} - (9ab-4)m_{2k-3} + 4m_{2k-5}\right)x^{2k-1}.\end{aligned}$$

From Lemma 2.2, we have $m_{2n-1} = (9ab-4)m_{2n-3} - 4m_{2n-5}$. By substituting this in the expression above, we get

$$\left(1-(9ab-4)x^2+4x^4\right)m(x) = x + (9ab-2)x^3 - (9ab-4)x^3 = x + 2x^3.$$

Therefore,

$$m(x) = \frac{x+2x^3}{(1-(9ab-4)x^2+4x^4)}.$$

Substituting $m(x)$ in $M(x)$ gives

$$(1-3bx+2x^2)M(x) = x + \left(3(a-b)x\frac{x+2x^3}{(1-(9ab-4)x^2+4x^4)}\right).$$

After simplifying the above expression, we get the desired result

$$M(x) = \frac{x(1+2x^2+3ax)}{1-(9ab-4)x^2+4x^4}.$$

□

Theorem 2.5. *The terms of the bi-periodic Mersenne are given by*

$$m_n = \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\left(\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\right),$$

where α_1 and α_2 are as in (2.1), $\lfloor b \rfloor$ is the floor function of b and $\xi(n)$ is the parity function.

Proof. Using the partial fraction decomposition, we can write the generating function for the bi-periodic Mersenne sequence $M(x)$ as

$$M(x)=\frac{1}{\alpha_1-\alpha_2}\left[\frac{\alpha_1 x+a(\frac{3\alpha_1-2}{2})}{2x^2-(\frac{3\alpha_1-2}{2})}-\frac{\alpha_2 x+a(\frac{3\alpha_2-2}{2})}{2x^2-(\frac{3\alpha_2-2}{2})}\right].$$

The Maclaurin series expansion of the function $\frac{A-Bz}{z^2-C}$ is expressed in the form

$$\frac{A-Bz}{z^2-C}=\sum_{n=0}^{\infty}BC^{-n-1}z^{2n+1}-\sum_{n=0}^{\infty}AC^{-n-1}z^{2n}.$$

So, $M(x)$ can be written as

$$\begin{aligned}M(x) &= \frac{1}{2(\alpha_1-\alpha_2)}\left[\sum_{n=0}^{\infty}\frac{(-\alpha_1)(\frac{3\alpha_2-2}{4})^{n+1}+\alpha_2(\frac{3\alpha_1-2}{4})^{n+1}}{(\frac{3\alpha_1-2}{4})^{n+1}(\frac{3\alpha_2-2}{4})^{n+1}}x^{2n+1}\right]\\ &\quad+\frac{1}{2(\alpha_1-\alpha_2)}\left[\sum_{n=0}^{\infty}\frac{(2a)(\frac{3\alpha_2-2}{4})^{n}+(2a)(\frac{3\alpha_1-2}{4})^{n}}{(\frac{3\alpha_1-2}{4})^{n}(\frac{3\alpha_2-2}{4})^{n}}x^{2n}\right].\end{aligned}$$

By using Lemma 2.3, we obtain

$$\begin{aligned}M(x) &= \frac{1}{2(\alpha_1-\alpha_2)}\left(\sum_{n=0}^{\infty}\frac{\frac{(-\alpha_1)(\alpha_2)^{2n+2}}{(4ab)^{n+1}}+\frac{\alpha_2(\alpha_1)^{2n+2}}{(4ab)^{n+1}}}{\frac{1}{4^{n+1}}}\right)x^{2n+1}\\ &\quad-\frac{1}{2(\alpha_1-\alpha_2)}\left(\sum_{n=0}^{\infty}\frac{\frac{(2a)(\alpha_2)^{2n}}{(4ab)^{n}}-\frac{(2a)(\alpha_1)^{2n}}{(4ab)^{n}}}{\frac{1}{4^{n}}}\right)x^{2n}\\ &= \sum_{n=0}^{\infty}\frac{1}{(ab)^n}\frac{(\alpha_1)^{2n+1}-(\alpha_2)^{2n+1}}{\alpha_1-\alpha_2}x^{2n+1}+\sum_{n=0}^{\infty}\frac{a}{(ab)^n}\frac{(\alpha_1)^{2n}-(\alpha_2)^{2n}}{\alpha_1-\alpha_2}x^{2n}.\end{aligned}$$

By the help of the parity function $\xi(n)$, it follows that

$$M(x)=\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha_1^n-\alpha_2^n}{\alpha_1-\alpha_2}\right)x^n.$$

Therefore, for all $n\geq 0$, we have

$$m_n=\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha_1^n-\alpha_2^n}{\alpha_1-\alpha_2}\right).$$

□

Theorem 2.6. *(Catalan's Identity) For any two nonnegative integer n and r, with $r\leq n$, we get*

$$a^{\xi(n-r)}b^{1-\xi(n-r)}m_{n-r}m_{n+r}-a^{\xi(n)}b^{1-\xi(n)}m_n^2=-\left(2^{n-r}\right)a^{\xi(r)}b^{1-\xi(r)}m_r^2.$$

Proof. Using the Binet's formula, we obtain

$$\begin{aligned}a^{\xi(n+r)}b^{1-\xi(n-r)}m_{n-r}m_{n+r} &= a^{\xi(n+r)}b^{1-\xi(n-r)}\frac{a^{1-\xi(n-r)}}{(ab)^{\lfloor\frac{n-r}{2}\rfloor}}\left(\frac{\alpha_1^{n-r}-\alpha_2^{n-r}}{\alpha_1-\alpha_2}\right)\frac{a^{1-\xi(n+r)}}{(ab)^{\lfloor\frac{n+r}{2}\rfloor}}\left(\frac{\alpha_1^{n+r}-\alpha_2^{n+r}}{\alpha_1-\alpha_2}\right)\\ &= \left(\frac{a^{2-\xi(n-r)}b^{1-\xi(n-r)}}{(ab)^{n-\xi(n-r)}}\right)\left(\frac{\alpha_1^{n-r}-\alpha_2^{n-r}}{\alpha_1-\alpha_2}\right)\left(\frac{\alpha_1^{n+r}-\alpha_2^{n+r}}{\alpha_1-\alpha_2}\right)\\ &= \left(\frac{a}{(ab)^{n-1}}\right)\left(\frac{\alpha_1^{2n}-(\alpha_1\alpha_2)^{n-r}(\alpha_1^{2r}+\alpha_2^{2r})+\alpha_2^{2n}}{(\alpha_1-\alpha_2)^2}\right)\end{aligned}$$

and

$$\begin{aligned}a^{\xi(n)}b^{1-\xi(n)}m_n^2 &= a^{\xi(n)}b^{1-\xi(n)}\left(\frac{a^{2-2\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}\right)\left(\frac{\alpha_1^{2n}-2(\alpha_1\alpha_2)^n+\alpha_2^{2n}}{(\alpha_1-\alpha_2)^2}\right)\\ &= \left(\frac{a}{(ab)^{2\lfloor\frac{n}{2}\rfloor+\xi(n)-1}}\right)\left(\frac{\alpha_1^{2n}-2(\alpha_1\alpha_2)^n+\alpha_2^{2n}}{(\alpha_1-\alpha_2)^2}\right)\\ &= \left(\frac{a}{(ab)^{n-1}}\right)\left(\frac{\alpha_1^{2n}-2(\alpha_1\alpha_2)^n+\alpha_2^{2n}}{(\alpha_1-\alpha_2)^2}\right).\end{aligned}$$

So,

$$\begin{aligned} a^{\xi(n+r)}b^{1-\xi(n-r)}m_{n-r}m_{n+r}-a^{\xi(n)}b^{1-\xi(n)}m_n^2 &= \left(\frac{a}{(ab)^{n-1}}\right)\left(\frac{2(\alpha_1\alpha_2)^n-(\alpha_1\alpha_2)^{n-r}\left(\alpha_1^{2r}+\alpha_2^{2r}\right)}{(\alpha_1-\alpha_2)^2}\right) \\ &= \left(\frac{a}{(ab)^{n-1}}\right)\left(\frac{(\alpha_1\alpha_2)^{n-r}(2\left(\alpha_1\alpha_2\right)^r-\alpha_1^{2r}-\alpha_2^{2r}))}{(\alpha_1-\alpha_2)^2}\right) \\ &= \left(\frac{-a}{(ab)^{n-1}}\right)(\alpha_1\alpha_2)^{n-r}\left(\frac{\alpha_1^{2r}+\alpha_2^{2r}-2\left(\alpha_1\alpha_2\right)^r)}{(\alpha_1-\alpha_2)^2}\right) \\ &= \left(\frac{-a}{(ab)^{n-1}}\right)(2ab)^{n-r}\left(\frac{\alpha_1^r-\alpha_2^r}{\alpha_1-\alpha_2}\right)^2 \\ &\quad -(a)^{2\xi(r)-1}(ab)^{1-\xi(r)}\left(2^{n-r}\right)m_r^2 \\ &= -\left(2^{n-r}\right)a^{\xi(r)}b^{1-\xi(r)}m_r^2. \end{aligned}$$

□

Theorem 2.7. *(Cassini's Identity) The following equality holds*

$$a^{1-\xi(n)}b^{\xi(n)}m_{n-1}m_{n+1}-a^{\xi(n)}b^{1-\xi(n)}m_n^2=-a\left(2^{n-1}\right),$$

where n is any nonnegative integer.

Proof. In Catalan's identity, if we take $r=1$, we get Cassini's identity. So, the proof can be obtained from the relevant identity. □

Theorem 2.8. *(d'Ocagne's Identity) For any two nonnegative integer n and r, with $r\leq n$, we have*

$$a^{\xi(nr+n)}b^{\xi(nr+r)}m_nm_{r+1}-a^{\xi(nr+r)}b^{\xi(nr+n)}m_{n+1}m_r=2^ra^{\xi(n-r)}m_{n-r}.$$

Proof. There are such equations

$$\xi(n+1)+\xi(r)-2\xi(nr+r)=\xi(n)+\xi(r+1)-2\xi(nr+n)=1-\xi(n-r) \tag{2.3}$$

and

$$\xi(n-r)=\xi(nr+n)+\xi(nr+r) \tag{2.4}$$

for the floor function defined as $\xi(n)=n-2\left\lfloor\frac{n}{2}\right\rfloor$.

Using the Binet's formula, (2.3) and (2.4), it follows that

$$a^{\xi(nr+n)}b^{\xi(nr+r)}m_nm_{r+1}=\frac{a}{(ab)^{(n+r-\xi(n-r))/2}}\frac{\alpha_1^{n+r+1}+\alpha_2^{n+r+1}-(\alpha_1\alpha_2)^r(\alpha_1^{n-r}\alpha_2+\alpha_1\alpha_2^{n-r})}{(\alpha_1-\alpha_2)^2}$$

and

$$a^{\xi(nr+r)}b^{\xi(nr+n)}m_{n+1}m_r=\frac{a}{(ab)^{(n+r-\xi(n-r))/2}}\frac{\alpha_1^{n+r+1}+\alpha_2^{n+r+1}-(\alpha_1\alpha_2)^r(\alpha_1^{n-r+1}+\alpha_2^{n-r+1})}{(\alpha_1-\alpha_2)^2}.$$

So,

$$\begin{aligned} a^{\xi(nr+n)}b^{\xi(nr+r)}m_nm_{r+1}-a^{\xi(nr+r)}b^{\xi(nr+n)}m_{n+1}m_r &= \frac{a}{(ab)^{(n+r-\xi(n-r))/2}}(\alpha_1\alpha_2)^r\frac{(\alpha_1^{n-r+1}+\alpha_2^{n-r+1}+\alpha_1^{n-r}\alpha_2-\alpha_1\alpha_2^{n-r})}{(\alpha_1-\alpha_2)^2} \\ &= \frac{a}{(ab)^{(n+r-\xi(n-r))/2}}\frac{(\alpha_1^{n-r}-\alpha_2^{n-r})(\alpha_1-\alpha_2)}{(\alpha_1-\alpha_2)^2} \\ &= 2^ra^{\xi(n-r)}m_{n-r}. \end{aligned}$$

□

Theorem 2.9. *(Honsberger Identity) For any two nonnegative integer n and r, with $r \leq n$, we have*

$$\left(a^{\xi(nr+n)}b^{\xi(nr+r)}-\frac{1}{\alpha_1\alpha_2}\frac{a^{\xi(r+1)-\xi(n)-1}}{(ab)^{\xi(nr+n)}}\right)m_nm_{r+1}+\left(a^{\xi(nr+r)}b^{\xi(nr+n)}-(\alpha_1\alpha_2)\frac{a^{\xi(r)+\xi(n-1)-1}}{(ab)^{\xi(nr+r)+1}}\right)m_{n-1}m_r$$
$$=\frac{(\alpha_1^{-1}+\alpha_1)(-\alpha_2^{-1}-\alpha_2)}{\alpha_1-\alpha_2}a^{\xi(n+r)}m_{n+r}.$$

Proof. Using the Binet's formula, (2.3) and (2.4), we obtain

$$a^{\xi(nr+n)}b^{\xi(nr+r)}m_nm_{r+1}=\frac{a}{(ab)^{(n+r-\xi(n-r))/2}}\frac{\left(\alpha_1^{n+r+1}+\alpha_2^{n+r+1}-\alpha_1^{r+1}\alpha_2^{n}-\alpha_1^{n}\alpha_2^{r+1}\right)}{(\alpha_1-\alpha_2)^2}$$

and

$$a^{\xi(nr+r)}b^{\xi(nr+n)}m_{n-1}m_r=\frac{a}{(ab)^{(n+r-\xi(n-r))/2}}\frac{\left(\alpha_1^{n+r-1}+\alpha_2^{n+r-1}-\alpha_1^{r}\alpha_2^{n-1}-\alpha_1^{n-1}\alpha_2^{r}\right)}{(\alpha_1-\alpha_2)^2}.$$

So, we get

$$\begin{aligned}a^{\xi(nr+n)}b^{\xi(nr+r)}m_nm_{r+1}+a^{\xi(nr+r)}b^{\xi(nr+n)}m_{n-1}m_r&=\frac{a}{(ab)^{\frac{n+r-\xi(n-r)}{2}}}\frac{(\alpha_1^{n+r}-\alpha_2^{n+r})(-\alpha_2^{-1}-\alpha_2)(\alpha_1^{-1}+\alpha_1)}{(\alpha_1-\alpha_2)^2}\\&+\frac{\alpha_1^{n+r}(\alpha_2^{-1}+\alpha_2)+\alpha_2^{n+r}(\alpha_1^{-1}+\alpha_1)-(1+\alpha_1\alpha_2)(\alpha_1^{r}\alpha_2^{n-1}+\alpha_2^{r}\alpha_1^{n-1})}{(\alpha_1-\alpha_2)^2}\\&=\frac{a}{(ab)^{\frac{n+r-\xi(n-r)}{2}}}\frac{(-\alpha_2^{-1}-\alpha_2)(\alpha_1^{-1}+\alpha_1)}{\alpha_1-\alpha_2}\frac{(ab)^{\lfloor\frac{n+r}{2}\rfloor}}{a^{1-\xi(n+r)}}m_{n+r}\\&+\frac{(\alpha_1^{r}\alpha_2^{-1}-\alpha_1^{-1}\alpha_2^{r})}{\alpha_1-\alpha_2}\frac{(ab)^{\lfloor\frac{n}{2}\rfloor}}{a^{1-\xi(n)}}m_n+\frac{(\alpha_1^{r+1}\alpha_2-\alpha_1\alpha_2^{r+1})}{\alpha_1-\alpha_2}\frac{(ab)^{\lfloor\frac{n-1}{2}\rfloor}}{a^{1-\xi(n-1)}}m_{n-1}\\&=\left(\frac{(-\alpha_2^{-1}-\alpha_2)(\alpha_1^{-1}+\alpha_1)}{\alpha_1-\alpha_2}\right)a^{\xi(n+r)}m_{n+r}+\frac{1}{\alpha_1\alpha_2}\frac{a^{\xi(r+1)-\xi(n)-1}}{(ab)^{\xi(nr+n)}}m_nm_{r+1}\\&+(\alpha_1\alpha_2)\frac{a^{\xi(r)+\xi(n-1)-1}}{(ab)^{\xi(nr+r)+1}})m_{n-1}m_r.\end{aligned}$$

Hence, we obtain

$$\left(a^{\xi(nr+n)}b^{\xi(nr+r)}-\frac{1}{\alpha_1\alpha_2}\frac{a^{\xi(r+1)-\xi(n)-1}}{(ab)^{\xi(nr+n)}}\right)m_nm_{r+1}+\left(a^{\xi(nr+r)}b^{\xi(nr+n)}-(\alpha_1\alpha_2)\frac{a^{\xi(r)+\xi(n-1)-1}}{(ab)^{\xi(nr+r)+1}}\right)m_{n-1}m_r$$
$$=\frac{(\alpha_1^{-1}+\alpha_1)(-\alpha_2^{-1}-\alpha_2)}{\alpha_1-\alpha_2}a^{\xi(n+r)}m_{n+r}. \qquad \square$$

Theorem 2.10. *(Sums Involving Binomial Coefficient) For any nonnegative integer r, we have*

$$\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3^s)(ab)^{\lfloor\frac{s}{2}\rfloor}a^{\xi(s)}m_s=m_{2r}$$

and

$$\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3^s)(ab)^{\lfloor\frac{s+1}{2}\rfloor}a^{\xi(s+1)-1}m_{s+1}=m_{2r+1}.$$

Proof. For any integer s, we have

$$(ab)^{\lfloor\frac{s}{2}\rfloor}a^{\xi(s)}m_s=a\left(\frac{\alpha_1^s-\alpha_2^s}{\alpha_1-\alpha_2}\right).$$

By using this equality above, we get

$$\begin{aligned}
\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3^s)(ab)^{\left\lfloor \frac{s}{2}\right\rfloor}a^{\xi(s)}m_s &= \sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3^s)a\left(\frac{\alpha_1^s-\alpha_2^s}{\alpha_1-\alpha_2}\right)\\
&= \frac{a}{\alpha_1-\alpha_2}\left[\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3\alpha_1)^s-\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3\alpha_2)^s\right]\\
&= \frac{a}{\alpha_1-\alpha_2}\left[(3\alpha_1-2)^r-(3\alpha_2-2)^r\right]\\
&= \frac{a}{\alpha_1-\alpha_2}\left[\left(\frac{\alpha_1^2}{ab}\right)^r-\left(\frac{\alpha_2^2}{ab}\right)^r\right]\\
&= \frac{a}{(ab)^r}\left(\frac{\alpha_1^{2r}-\alpha_2^{2r}}{\alpha_1-\alpha_2}\right)\\
&= m_{2r}.
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3^s)(ab)^{\left\lfloor \frac{s+1}{2}\right\rfloor}a^{\xi(s+1)-1}m_{s+1} &= \sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3^s)a\left(\frac{\alpha_1^{s+1}-\alpha_2^{s+1}}{\alpha_1-\alpha_2}\right)\\
&= \frac{1}{\alpha_1-\alpha_2}\left[\alpha_1\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3\alpha_1)^s-\alpha_2\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3\alpha_2)^s\right]\\
&= \frac{1}{\alpha_1-\alpha_2}\left[\alpha_1(3\alpha_1-2)^r-\alpha_2(3\alpha_2-2)^r\right]\\
&= \frac{1}{\alpha_1-\alpha_2}\left[\alpha_1\left(\frac{\alpha_1^2}{ab}\right)^r-\alpha_2\left(\frac{\alpha_2^2}{ab}\right)^r\right]\\
&= \frac{1}{(ab)^r}\left(\frac{\alpha_1^{2r+1}-\alpha_2^{2r+1}}{\alpha_1-\alpha_2}\right)\\
&= m_{2r+1}.
\end{aligned}$$

□

Theorem 2.11. *The nonnegative terms of the bi-periodic Mersenne sequence are defined in terms of the positive terms as*

$$m_{-n}=-2^{-n}m_n.$$

Proof. By using the Binet's formula, we obtain

$$\begin{aligned}
m_{-n} &= \frac{a^{1-\xi(-n)}}{(ab)^{\left\lfloor \frac{-n}{2}\right\rfloor}}\left(\frac{\alpha_1^{-n}-\alpha_2^{-n}}{\alpha_1-\alpha_2}\right)\\
&= \frac{a^{1-\xi(-n)}}{(ab)^{\left\lfloor \frac{-n}{2}\right\rfloor}}\left(\frac{\alpha_2^{n}-\alpha_1^{n}}{(\alpha_1\alpha_2)^n(\alpha_1-\alpha_2)}\right)\\
&= \frac{a^{1-\xi(n)}}{2^n(ab)^n(ab)^{\left\lfloor \frac{-n}{2}\right\rfloor}}\left(\frac{\alpha_2^{n}-\alpha_1^{n}}{\alpha_1-\alpha_2}\right)\\
&= \frac{-a^{1-\xi(n)}}{2^n(ab)^{\left\lfloor \frac{-n}{2}\right\rfloor}}\left(\frac{\alpha_1^{n}-\alpha_2^{n}}{\alpha_1-\alpha_2}\right)\\
&= -2^{-n}m_n.
\end{aligned}$$

□

3. Conclusion

In this paper, we define bi-periodic Mersenne sequence, which is called bi-periodic Mersenne sequence. We obtain some properties for this sequence such as Binet formula, generating function, Catalan, Cassini, d'Ocagne and Honsberger identities.

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Determination of Heterogeneity for Manganese Dendrites Using Lacunarity Analysis

Fırat Evirgen[1*] and Mehmet Bayırlı[2]

[1]*Department of Mathematics, Faculty of Science and Arts, Balıkesir University, Balıkesir, Turkey*
[2]*Department of Physics, Faculty of Science and Arts, Balıkesir University, Balıkesir, Turkey*
[*]*Corresponding author*

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Abstract

The surface patterns of natural and experimental deposits are important as they result from the internal microstructure. For this purpose, lacunarity analysis is applied to determine the heterogeneous nature of deposit surface patterns. In this study, images were digitally moved onto the square mesh to determine the heterogeneous situation of manganese dendrite patterns on the natural magnesite surface. The relation between the lacunarity values of the images and the box size was examined. The lacunarity values corresponding to the box size values were estimated using the gliding-box algorithm. This relation was determined numerically as a power-law function using nonlinear regression method. It has been shown that the system examined with the generated numerical model function can be defined with three specific parameters. As a result, it has been shown that it is possible to describe the relationship between numerical solution-based lacunarity-box size and a third-order nonlinear differential equation. With this study, the lacunarity-box size value on different system images can be determined by using the gliding box algorithm and calculating the coefficient value from the power-law relationship.

1. Introduction

Lacunarity is derived from the Latin word ”lacuna” meaning space or lake in Latin. Geometrical patterns and fractal gaps are specific terms that determine superficial morphological heterogeneity by referring to a measure using the counting method. Since it goes beyond intuitive measures for heterogeneity, lacunarity can quantify additional properties of various patterns, such as ”scale invariance” and heterogeneity [1]-[3]. The earliest description of lacunarity as a geometric term is attributed to Mandelbrot. In 1983, Mandelbrot essentially defined it as an auxiliary element infractal analysis [4]. The geometric texture pattern in an image is scale dependent. It can vary significantly with the size and spatial resolution of the digital image. Any very small image can contain parts of a pattern and be able to characterize the entire pattern, whereas a large image can consist of more than one pattern and accurately describe it as well. Likewise, a pixel in a low spatial resolution image shows signs of many patterns smaller than an integrated pixel size. Spatially, the resolution increases, the image pixels may be smaller. In this case, it may be appropriate to perform lacunarity analysis to generate meaningful information from the image pattern. Lacunarity applications provide flexibility in terms of ease of mathematical operation. Theoretically, however, it should be used with a different scale due to the consistent mean of characterization across tissue patterns [5]. Today, lacunarity analysis is used to characterize data and geometric patterns in various areas such as ecology, physics, medical imaging, urban spatial analysis and etc. It has many applications, especially in multiple fractal analysis [6, 7].

Fractal geometry describes photometric and geometric changes in fractal or non-fractal pattern images using lacunarity analysis. It has also developed a statistical approach that provides separable features over an extremely wide range of image

Email addresses and ORCID numbers: **fevirgen@balikesir.edu.tr, 0000-0002-0798-5004 (F. Evirgen), mbayirli@balikesir.edu.tr, 0000-0002-7775-0251 (M. Bayırlı)**

transformations [8]-[11]. Accordingly, the numerical determination of the properties of the texture showing the geometric pattern is related to the estimation of the value of the image calculated according to the multi-scale local binary system. The changes are determined by combining the lacunarity parameters. Thus, to distinguish superficial patterns from each other, it is possible to characterize the local distribution of superficial designs using lacunarity analysis [12]. In addition, appropriate numerical methods and software have been developed to calculate the lacunarity value developed by Plotnick et al. [2, 3].

In general, the definition of the morphological image is related to the scale at which it is studied. A pattern that is observed to be homogeneous at a given scale can be heterogeneous when observed at a larger scale. Images of natural and experimental specimens emerge from variations of cellular units, often forming repeating patterns of the same type or pattern from a combination of the cellular units, the pattern of the base unit and the assembly of this group of pixels [11]-[13]. Some of these can be described as fractal. Classification by surface pattern, objectively or by definition, provides a meaningful hint regarding physical properties in many imaging and visualization applications [6, 8, 12].

Geometric pattern gap analysis is a measure of the statistical distribution of void dimensions based on fractal mathematics [1, 2]. The lacunarity analysis originally developed for binary data (binary or presence/absence) can be easily applied to designs with continuous distributed variables [2]. A distance (in scale) is calculated as the ratio of the first and second moments of the counts in all possible boxes of this spacing width. The first moment is the sum of the mean values of all possible blocks in dimension $Z^{(1)}$, and the second moment is the sum of the mean squares in all possible blocks in dimension $Z^{(2)}$. The ratio of the first and second moments is defined as the fractal geometry lacunarity [2, 3].

In this study, the relations between box sizes and the lacunarity values of a geometric pattern are examined both mathematically and numerically. For this purpose, in Section 2, a basic depiction of the lacunarity concept is introduced. In section 3, a third-order non-linear differential equation is firstly proposed for expressing the relationship between the value of the lacunarity and the box size. In addition, the analytical solution for the proposed differential equation is also demonstrated. In Section 4, the aliasing between the analytical solution of the proposed differential equation and the numerical simulations is shown by handling the results of the lacunarity analysis for natural manganese dendrites using the non-linear regression. Finally, we summarize and interpret the findings in Section 5.

2. Lacunarity description

In imaging techniques, a geometric pattern is defined in the form of a matrix in an M-dimensional square lattice. Accordingly, the matrix elements are determined as either a white pixel (filled) or a black pixel valued zero (empty). In the first step, the unit matrix $r \times r$ is calculated by scaling the matrix for each r value, by increasing the value from $r = 1$ to $r \leq L$, until it reaches the value of the upper left corner. As the box is moved to the right, one pixel is displaced, and the white pixels are counted again. These operations are repeated until the matrix is moved over the entire image and the frequency distribution is generated. Accordingly, the number of r-sized boxes containing S occupied sites is denoted by $n[S,r]$, and the total number of r-sized boxes is denoted by $N(r)$. If the size of the image is M, the following relation can be defined:

$$N[r] = (M - r + 1)^2.$$

The number of full sites S is transformed into a probability distribution by dividing the frequency distribution $N(r)$ by $n[S,r]$, i.e., the number of filled sites with box size r. Hence, the probability distribution is:

$$Q(S,\ r) = n[S,r]/N[r].$$

This value can be defined as the probability distribution $Q(S,r)$ of a morphological structure, i.e., the coating ratio of the image. Thus, the first order $Z^{(1)}$ and the second order $Z^{(2)}$ statistical moments are calculated. The first and second moments are given by:

$$\begin{aligned} Z^{(1)} &= \sum S^* Q(S,r), \\ Z^{(2)} &= \sum S^{2*} Q(S,r). \end{aligned}$$

The lacunarity value (Λ), which is calculated as the ratio of the second moment to the statistical first moment, is proportional to the box size r and can be defined as follows:

$$\Lambda[r] = Z^{(2)}/[Z^{(1)}]^2.$$

The statistical first moment is

$$Z^{(1)} = M[r],$$

and the second moment is

$$Z^{(2)} = S_s^2[r] + M[r]^2,$$

where $M[r]$ is the mean and $S_s^2[r]$ is the statistical variance of the number of sites per box

$$\Lambda[r] = S_s^2[r] / M[r]^2 + 1. \tag{2.1}$$

Equation (2.1) implies that the lacunarity is not simply dependent on the size of the gliding box r. In a random map, the white squares are occupied by the corresponding environment, and each part of the map is not only bound to $Q(S,r)$ but also to the distribution of gaps (related white squares). Thus, the lacunarity value differs, depending on the statistical distribution of the two different patterned map spaces with the occupancy rates of the occupied sites, i.e., full coverage of the full sites [6, 8, 11, 13].

3. Lacunarity analysis and discussion

A manganese dendrite image was used on the surface of natural magnesite ore for the lacunarity analysis. For this purpose, the manganese dendrite shown in Figure 3.1 was defined as a matrix of $M = 100$ pixels in the computer environment and converted to binary format (BMP) using the image processing method via imageJ [14].

The algorithm used a floating box size $r_{min.} = 1$ to $r_{max.} = 100$, the first and second moment values and statistical values were calculated for lacunarity with MATLAB software and graph diagrams were drawn using Origin 7.0. For the sample used, the lacunarity values varying according to the probability distribution were calculated as $\Lambda(100) = 1.000$ for $r_{max.} = 100$ pixel and $\Lambda(1) = 3.662$ for $r_{min.} = 1$ pixel. In addition, the first and second moments and lacunarity values of the box size $1 \leq r \leq 100$ are shown in Table 3.1, and the relationship of the lacunarity value to the box size is shown graphically in Figure 3.2. The values in Table 3.1 are presented that vary from $r = 1$ to $r = 10$ pixels, then change from $r = 10$ to $r = 100$ pixels in interval 10 pixels.

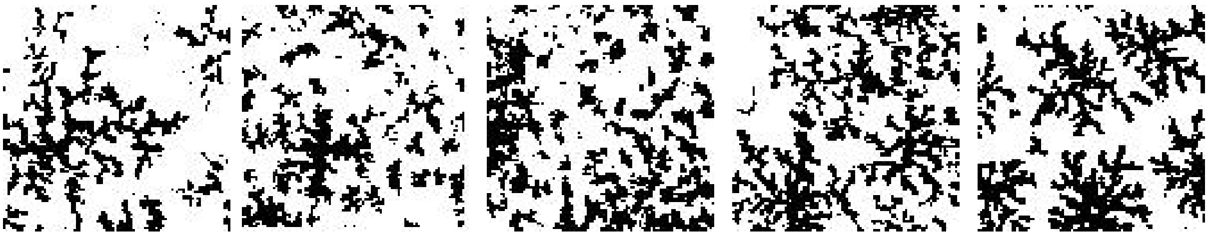

Figure 3.1: Binary format image of manganese dendrite patterns with different probability distributions selected from the magnesite ore surface with dimension $M = 100$ pixels.

Box size (r)	First moment $Z^{(1)}(r)$	Second moment $Z^{(2)}(r)$	Lacunarity $(\Lambda(r))$
1	0.2731	0.2731	3.662
2	1.087746	3.440363	2.907697
3	2.434923	14.99823	2.529703
4	4.306728	42.21086	2.275775
5	6.699219	93.59939	2.08557
6	9.626593	179.1739	1.933434
7	13.09767	310.2645	1.808606
8	17.13285	500.6209	1.705493
9	21.73358	765.5755	1.620786
10	26.90581	1122.259	1.550247
20	111.8363	15770.17	1.260871
30	254.8863	77447.11	1.192099
40	456.0922	234575.4	1.127658
50	711.9516	538348.3	1.062093
60	1025.158	1071731	1.019774
70	1390.106	1939315	1.003581
80	1793.254	3217359	1.000497
90	2186.372	4782181	1.00041
100	2731	7458361	1.0000

Table 3.1: First and second moments and lacunarity values representing distribution of box size r and statistical distribution values, for lattice size $M = 100$ pixels. The probability distribution value is computed as 0.273 for the image b.

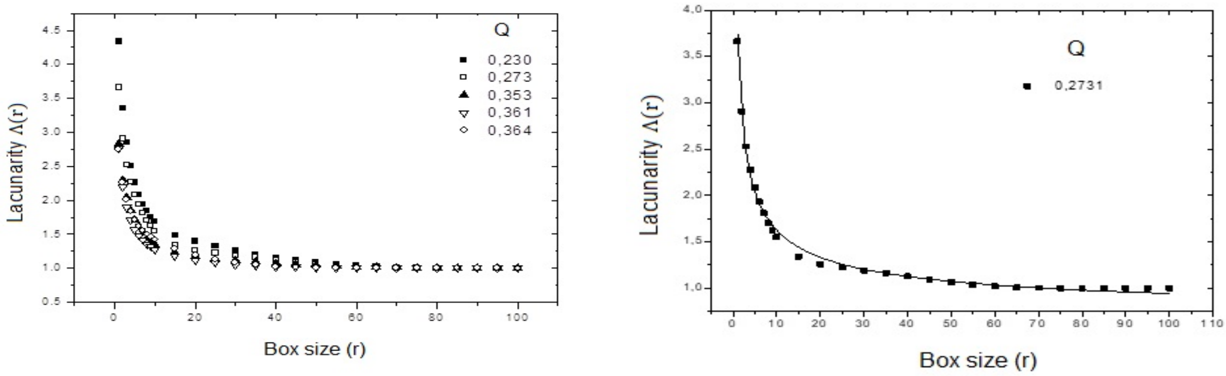

Figure 3.2: Change of lacunarity value $\Lambda(r)$ according to box size value r and non-linear regression implications.

When the $\Lambda(r)$ values in Table 3.1 and Figure 3.1 are examined, the box size of the lacunarity is seen to be similar to the power-law function $r = r_{min}, r_{min+1}, \ldots, r_{max}$. Accordingly, a mathematical model can be defined for this relationship. Thus, the mathematical model function

$$\Lambda(r) = \frac{\beta}{r^{\alpha}} + \gamma, \tag{3.1}$$

can be suggested for the relationship. This is the $\Lambda(r)$, r best interpretation of the lacunarity value between model function $\Lambda(r)$, $r = [r_{min}, r_{max}]$ and $r = r_{min} + r_{min} + 1, \ldots, r_{max}$, which can describe the geometric behaviour of manganese dendrites on the magnesite ore surface. A non-linear regression method can be used to determine solution constants for the function. The constant model parameters $\alpha = 0.311$, $\beta = 3.122$ and $\gamma = 0.494$ are calculated with regression coefficient $R^2 = 0.983$ for a pattern with probability distribution $Q(1,100) = 0.273$. The values of the other examples are also summarized in Table 3.2. Here, the parameters α, β and γ are independent and arbitrary variables for each sample of the system. The calculated results can be shown by the general fixed parameters α*, β* and γ* of the model function, which best show the probability distribution and the regional morphological phase transitions in the surfaces of the images used.

Samples		Probability distribution	Model parameters			Regression coefficient
		$Q(S,r)$	α^*	β^*	γ^*	R^2
	a	0.231	0.522±0,011	3.815±0,003	0.304±0,001	0.988
	b	0.273	0.494±0,021	3.122±0,001	0.311±0,003	0.983
Manganese dendrites	c	0.353	0.532±0,010	2.118±0,004	0.392±0,001	0.971
	d	0.361	0.637±0,038	1.963±0,001	0.430±0,002	0.952
	e	0.374	0.411±0,052	2.103±0,002	0.347±0,003	0.978

Table 3.2: The probability distributions and values of the proposed mathematical model parameters for the observed samples.

The three parameters (α, β and γ) of the mathematical model have a single meaning for the lacunarity function of each manganese dendrites image. In particular, the value α represents the convergence of the $\Lambda(r)$ function, β represents the graph depression for lacunarity and γ represents a transition term. Calculations showed that while the β value takes on values over a very large numerical range, the parameters α and γ remain small. A small change is defined as a power-law function with low lacunarity value, while a large pit is a power-law function with high lacunarity value and a growing β value. The power-law function parameters can be correlated with the α and β constants, which can define the occupancy and morphological structure of the images. In particular, a small variation of the α value corresponds to a significant change in the value of β.

4. Model characterization

The relationship between lacunarity and the box-size r can be defined by:

$$r\frac{d\Lambda}{dr}\frac{d\Lambda^3}{dr^3} = r(\frac{d\Lambda^2}{dr^2})^2 - \frac{d\Lambda}{dr}\frac{d\Lambda^2}{dr^2}, \tag{4.1}$$

where $r \leq M$. This third order non-linear differential equation describes the edge size of the pixels that define the space box. To get the analytical solution of the non-linear differential equation (4.1), we need to do some variable transformations as this $\frac{d\Lambda}{dr} = m$, $\frac{d\Lambda^2}{dr^2} = \frac{dm}{dr}$ and $\frac{d\Lambda^3}{dr^3} = \frac{dm^2}{dr^2}$. According to this variable transformation, equation (4.1) is reduced to the second order differential equation as following:

$$rm\frac{dm^2}{dr^2} - r\left(\frac{dm}{dr}\right)^2 - m\frac{dm}{dr} = 0. \tag{4.2}$$

For this kind of nonlinear differential equation, the variable transform $m = e^u$ is applied to get the solution. By this transformation, if we write $\frac{dm}{dr} = e^u \frac{du}{dr}$ and $\frac{dm^2}{dr^2} = e^u \left(\frac{du}{dr}\right)^2 + e^u \frac{du^2}{dr^2}$ equalities in equation (4.2),

$$r(e^u)^2 \left(\frac{du^2}{dr^2} + \left(\frac{du}{dr}\right)^2\right) - r(e^u)^2 \left(\frac{du}{dr}\right)^2 + (e^u)^2 \frac{du}{dr} = 0,$$

and when the proper arrangement is made,

$$r\frac{du^2}{dr^2} + \frac{du}{dr} = 0, \tag{4.3}$$

second order linear differential equation is obtained.

Again, if we take $\frac{du}{dr} = v$ and $\frac{du^2}{dr^2} = \frac{dv}{dr}$ variable transformation for equation (4.3), the third order non-linear ordinary differential equation (4.1) is reduced to the following first order linear differential equation,

$$r\frac{dv}{dr} + v = 0. \tag{4.4}$$

By separation of variables, the equation (4.4) gives the following form,

$$\frac{dv}{v} + \frac{dr}{r} = 0, \tag{4.5}$$

and the analytical solution of equation (4.5) is obtained as,

$$v = \frac{c_1}{r}.$$

By using the variable transformation $\frac{du}{dr} = v$, $m = e^u$ and $\frac{d\Lambda}{dr} = m$ in this order, the analytical solution of third order non-linear differential equation (4.1) is obtained as follow,

$$\Lambda(r) = \frac{C_2}{C_1 + 1} r^{C_1 + 1} + c_3, \tag{4.6}$$

where c_1, c_2 and c_3 are the parameters that describe the system under investigation. The mathematical model (3.1) coincides with the analytical solution of the proposed third order differential equation (4.6) with the assumptions $\beta = \frac{c_2}{c_1+1}$, $\alpha = -(c_1 + 1)$ and $\gamma = c_3$. Thus, the change between the lacunarity values and box size was modeled.

5. Conclusions

In this study, the relationship between the lacunarity values and the box size is used to determine the heterogeneity of fractal and non-fractal geometric patterns on the deposit surface. For this purpose, the lacunarity value according to the box size for the manganese dendrite sediment patterns formed in the pores and cracks of the natural magnesite ore surface was calculated by using the gliding-box algorithm and the relations of the lacunarity value with the box size were defined. For describing this relation, the nonlinear regression method is used and the numerical power-law function is derived. It is shown that it is possible to determine the heterogeneity of the pattern system with three numerical model parameters. By taking the numerical model function as a reference, a third-order nonlinear differential equation is derived and its analytical solution is performed. The numerical solution function is compatible with the analytical solution function. This study's findings can be utilized to estimate the heterogeneity of similar deposit surfaces.

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All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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On the Exponential Diophantine Equation $(6m^2+1)^x+(3m^2-1)^y=(3m)^z$

Murat Alan[1*] **and Ruhsar Gizem Biratlı**[1]

[1]*Department of Mathematics, Faculty of Science and Arts, Yildiz Technical University, İstanbul, Turkey*
[*]*Corresponding author*

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Abstract

Let m be a positive integer. In this paper, we consider the exponential Diophantine equation $(6m^2+1)^x+(3m^2-1)^y=(3m)^z$ and we show that it has only unique positive integer solution $(x,y,z)=(1,1,2)$ for all $m>1$. The proof depends on some results on Diophantine equations and the famous primitive divisor theorem.

1. Introduction

Let u, v, w be relatively prime positive integers greater than one. Consider the exponential Diophantine equation

$$u^x+v^y=w^z, \quad x,y,z\in\mathbb{N}. \tag{1.1}$$

In 1956, Jeśmanowicz conjectured that if (u,v,w) is a Pythagorean triple then the above equation has only the unique positive integer solution $(x,y,z)=(2,2,2)$ [1]. In [2], Terai proposed that if $u^p+v^q=w^r$ with $p,q,r\in\mathbb{N},\ r\geq 2$ then (1.1) has only the positive integer solution $(x,y,z)=(p,q,r)$ except for a few triples (u,v,w). The following combined version of these two conjectures are called the Terai-Jeśmanowicz conjecture [3].

Conjecture 1. *[3, Conjecture 3.2] If $(x,y,z)=(p,q,r)$ is a solution of (1.1) with $\min\{p,q,r\}>1$ then the only solution to (1.1) with $\min\{x,y,z\}>1$ is $(x,y,z)=(p,q,r)$.*

Many research confirmed that these conjectures are true in many special cases [4]-[11]. Especially, the positive integer solutions of the exponential Diophantine equation

$$(am^2+1)^x+(bm^2-1)^y=(cm)^z \tag{1.2}$$

which is a special case of (1.1) with a,b,c,m are positive integers such that $a+b=c^2$ has already been investigated by a number of authors and all of them justify Terai's conjecture in their special cases. In [12], Terai consider the equation (1.2) with $(a,b,c)=(4,5,3)$ and he proved that $(x,y,z)=(1,1,2)$ is the only positive integer solution of $(4m^2+1)^x+(5m^2-1)^y=(3m)^z$ under some conditions. Remaining cases of this equation are completed in [13]-[15]. As a recent study, in [16], the complete solution of (1.2) with $(a,b,c)=(4,21,5)$ is also given. For some similar problems, see for example [8], [17]-[24]. In this paper, we consider the exponential Diophantine equation

$$(6m^2+1)^x+(3m^2-1)^y=(3m)^z \tag{1.3}$$

and we give the complete solution of this equation by proving the following theorem.

Email addresses and ORCID numbers: **alan@yildiz.edu.tr, 0000-0003-2031-2725 (M. Alan), ruhsar97@hotmail.com, 0000-0001-6663-8435 (R. G. Biratlı)**

Theorem 1.1. *Let m be a positive integer. Then the equation (1.3) has only the unique positive integer solution* $(x,y,z)=(1,1,2)$ *for all* $m>1$.

We state the above theorem for $m>1$, since for $m=1$ the equation (1.3) turns intu the equation $7^x+2^y=3^z$ which is already known that it has exactly two solutions as $(x,y,z)=(1,1,2),(2,5,4)$ [25]. It is also worth to note that this equation $7^x+2^y=3^z$ is one of the a few known exceptional cases of Terai's conjecture [10]. So from now on we take $m>1$. We refer to section 3 of [3] for various version of above conjecture and for a complete list of all known examples of (1.1) which has at least two distinct solutions. The proof of the Theorem 1.1 mainly depends on the combinations of two methods. One of them is due to [26, 27] which enable us to find the other possible solutions of the Diophantine equations $X^2+DY^2=k^Z$ and $aX^2+bY^2=k^Z$ from the known solutions under some conditions and the other one is the famous primitive divisor theorem [28, 29]. The details of these methods are given in the next chapter

2. Preliminaries

Let D be any positive integer. By $h(-4D)$, we denote the class number of positive binary quadratic forms of discriminant $-4D$.

Lemma 2.1. *[30, Theorems 11.4.3, 12.10.1 and 12.14.3]*

$$h(-4D)<\frac{4}{\pi}\sqrt{D}\log(2e\sqrt{D}).$$

Lemma 2.2. *[27, Theorems 1 and 2] Let D and k be relatively prime positive integers such that* $D>1$ *and k is an odd integer. If the equation*

$$U^2+DV^2=k^W,\quad U,V,W\in\mathbb{Z},\quad \gcd(U,V)=1,\quad W>0$$

has solutions (U,V,W), *then any solution of the above equation can be expressed as*

$$U+V\sqrt{-D}=\lambda_1(U_1+\lambda_2V_1\sqrt{-D})^t,$$

$$W=W_1t,\quad t\in\mathbb{N},$$

where $\lambda_{1,2}\in\{\pm1\}$, U_1,V_1,W_1 *are positive integers satisfying* $U_1^2+DV_1^2=k^{W_1}$, $\gcd(U_1,V_1)=1$ *and* $W_1\mid h(-4D)$.

Let D_1,D_2 be relatively prime positive integers greater that 1 and let (X,Y,Z) be a fixed solution of the equation

$$D_1X^2+D_2Y^2=k^z,\ \gcd(X,Y)=1,\ 2\nmid k,\ Z>0\quad\text{and}\quad X,Y,Z\in\mathbb{Z}. \tag{2.1}$$

Then there exists a unique positive integer l such that $l=D_1\alpha X+D_2\beta Y$, $0<t<k$, where α,β are integers with $\beta X-\alpha Y=1$ [27, Lemma 1]. The positive integer l is called the characteristic number of this particular solution (X,Y,Z) and it is denoted by $<X,Y,Z>$. if $<X,Y,Z>=l$ then it is known that $D_1X\equiv -lY \pmod{k}$ [27, Lemma 6]. Let (X_0,Y_0,Z_0) be a solution of (2.1) and let $<X_0,Y_0,Z_0>=l_0$. Then the set of all solutions (X,Y,Z) with $<X,Y,Z>\equiv\pm l_0 \pmod{k}$ is called a solution class of (2.1) and it is denoted by $S(l_0)$.

Lemma 2.3. *[27, Theorems 1 and 2] For any fixed solution class* $S(l_0)$ *of (2.1), there exists a unique solution* $(X_1,Y_1,Z_1)\in S(l_0)$ *such that* $X_1>0$, $Y_1>0$ *and* $Z_1\ge Z$, *where Z runs through all solutions* $(X,Y,Z)\in S(l_0)$. *The solution* (X_1,Y_1,Z_1) *is called the least solution of* $S(l_0)$. *If* (X,Y,Z) *is a solution belongs to* $S(l_0)$ *then*

$$Z=Z_1t,\ 2\nmid t,\ t\in\mathbb{N},$$

$$X\sqrt{D_1}+Y\sqrt{-D_2}=s_1\left(X_1\sqrt{D_1}+s_2Y_1\sqrt{-D_2}\right)^t,\ s_1,s_2\in\{-1,1\}.$$

Lemma 2.4. *[26, Theorem 2] Let* (X_1,Y_1,Z_1) *be the least solution of* $S(l_0)$. *If (2.1) has a solution* $(X,Y,Z)\in S(l_0)$ *satisfying* $X>0$ *and* $Y=1$, *then* $Y_1=1$. *Further, if* $(X,Z)\neq(X_1,Z_1)$, *then one of the following conditions is satisfied:*

(i) $D_1X_1^2=\frac{1}{4}(k^{Z_1}\pm1), D_1=\frac{1}{4}(3k^{Z_1}\pm1), (X,Z)=(X_1|D_1X_1^2-3D_2|,3Z_1)$.

(ii) $D_1X_1^2=\frac{1}{4}F_{3r+3\varepsilon}, D_2=\frac{1}{4}L_{3r}, k^{Z_1}=F_{3r+\varepsilon}$,
$(X,Z)=(X_1|D_1^2X_1^4-10D_1D_2X_1^2+5D_2^2|,5Z_1)$, *where* $\varepsilon\in\{-1,1\}$, *r is a positive integer, and* F_n *is nth Fibonacci number.*

The primitive divisor theorem is another powerful tool for solving some Diophantine equations. Let α,β be algebraic integers. A Lucas pair is a pair (α,β) such that $\alpha+\beta$ and $\alpha\beta$ are non-zero relatively prime integers and $\frac{\alpha}{\beta}$ is not a root of unity. If (α,β) is any Lucas pair then the corresponding sequences of Lucas numbers are defined by

$$L_n(\alpha,\beta)=\frac{\alpha^n-\beta^n}{\alpha-\beta},\ n=0,1,2,\ldots.$$

Recall that primitive divisors of $L_n(\alpha,\beta)$ are the prime numbers p such that $p\mid L_n(\alpha,\beta)$ and $p\nmid(\alpha-\beta)^2L_1(\alpha,\beta)\ldots L_{n-1}(\alpha,\beta)$ $(n>1)$. Any two Lucas pairs (α_1,β_1) and (α_2,β_2) are called equivalent if $\frac{\alpha_1}{\alpha_2}=\frac{\beta_1}{\beta_2}=\pm1$.

Lemma 2.5. *[28] If $n > 30$ then $L_n(\alpha,\beta)$ has a primitive divisor.*

Lemma 2.6. *[29] If $4 < n \leq 30$ and $n \neq 6$ then, up to equivalence, $L_n(\alpha,\beta)$ has a primitive divisor except for the following parameters (e,f)*

- $(1,5),(1,-7),(2,-40),(1,-11),(1,-15),(12,-76)$ *or* $(12,-1364)$ *if* $n=5$,
- $((1,-7)$ *or* $(1,-19)$ *if* $n=7$,
- $(1,-7)$ *or* $(2,-24)$ *if* $n=8$,
- $(2,-8),(5,-3)$ *or* $(5,-47)$ *if* $n=10$,
- $(1,-5),(1,-7),(1,-11),(2,-56),(1,-15)$ *or* $(1,-19)$ *if* $n=12$,
- $(1,-7)$ *if* $n=13,18$ *or* 30.

where $(\alpha,\beta)=\left(\dfrac{e+\sqrt{f}}{2},\dfrac{e-\sqrt{f}}{2}\right)$.

3. Proof of Theorem 1.1

We treat the (1.3) according to the parity of m. For the case m is even, the proof of Theorem 1.1 easily follows from the next lemma.

3.1. The case $2 \mid m$

Lemma 3.1. *If m is even then $(x,y,z)=(1,1,2)$ is the unique positive integer solution of equation (1.3).*

Proof. If $z \leq 2$, then $(x,y,z)=(1,1,2)$ is clearly the unique solution of the equation (1.3). So assume that $z \geq 3$. Taking equation (1.3) modulo m^2 we get that $1+(-1)^y \equiv 0 \pmod{m^2}$ and hence we see that y is odd since $m^2>2$. Taking equation (1.3) modulo $3m^3$ we find that

$$1+6m^2x+(-1)+3m^2y \equiv 0 \pmod{3m^3}$$
$$2x+y \equiv 0 \pmod{m},$$

which is false because y is odd and m is even. So we conclude that the equation (1.3) has no positive integer solution when $z \geq 3$. Therefore (1.3) has only the unique positive integer solution $(1,1,2)$ when m is even. □

From now on we deal with the case m is odd.

3.2. The case $2 \nmid$ m

Let (x,y,z) be any solution of (1.3). Clearly $(x,y,z)=(1,1,2)$ is a solution of (1.3). Since $m>1$, taking (1.3) modulo m^2 we see that, as in the previous case, y is odd.
From now on we separate two cases according to the parity of x. First suppose that x is also odd. Now consider the Diophantine equation

$$(6m^2+1)X^2+(3m^2-1)Y^2=(3m)^Z,\ Z>0 \quad \text{and} \quad X,Y,Z\in\mathbb{Z}. \tag{3.1}$$

Since (x,y,z) is any solution of (1.3), we see that

$$(X,Y,Z)=\left((6m^2+1)^{\frac{x-1}{2}},(3m^2-1)^{\frac{y-1}{2}},z\right) \tag{3.2}$$

is a solution of (3.1). Let $l=<(6m^2+1)^{\frac{x-1}{2}},(3m^2-1)^{\frac{y-1}{2}},z>$ be a characteristic number of the solution given in (3.2). Then, from the congruence

$$(6m^2+1)^{\frac{x+1}{2}} \quad \equiv -l(3m^2-1)^{\frac{y-1}{2}} \pmod{3m},$$

we see that $l \equiv \pm 1 \pmod{3m}$.
Note that $(X_1,Y_1,Z_1)=(1,1,2)$ is also a solution of the equation (3.1) and let $l_0=<1,1,2>$ be the characteristic number of this solution. So, we have that

$$\begin{aligned}(6m^2+1)\cdot 1 &\equiv -l_0\cdot 1 \pmod{3m},\\ l_0 &\equiv -1 \pmod{3m}.\end{aligned} \tag{3.3}$$

So we see that $l \equiv \pm l_0 \pmod{3m}$, which implies that the solutions $(X_1,Y_1,Z_1)=(1,1,2)$ and one in (3.2) are in the same solution class $S(l_0)$ of (3.1). Further $(X,Y,Z)=(1,1,2)$ is clearly the least solution of $S(l_0)$. So by Lemma (2.3), we get that

$$(6m^2+1)^{\frac{x-1}{2}}\sqrt{6m^2+1}+(3m^2-1)^{\frac{y-1}{2}}\sqrt{1-3m^2}=\lambda_1(\sqrt{6m^2+1}+\lambda_2\sqrt{1-3m^2}\,)^t, \tag{3.4}$$

with

$$z=2t,\ 2\nmid t,\ t\in\mathbb{N}\quad\text{and}\quad \lambda_{1,2}\in\{-1,1\}.$$

Expanding the right hand side of (3.4) and equating the coefficients of $\sqrt{1-3m^2}$, we find that

$$(3m^2-1)^{\frac{y-1}{2}}\quad=\lambda_1\lambda_2\sum_{i=0}^{\frac{t-1}{2}}\binom{t}{2i+1}(6m^2+1)^{\frac{t-1}{2}-i}(1-3m^2)^i \tag{3.5}$$

At this point we claim that $y=1$. For this purpose, assume that $y>1$. Then from (3.5), we find that

$$0\equiv\lambda_1\lambda_2 t\,(6m^2+1)^{\frac{t-1}{2}}\pmod{(3m^2-1)}$$

$$0\equiv\pm 3^{\frac{t-1}{2}}t\pmod{(3m^2-1)},$$

which is a contradiction, since it implies that $2\mid 3^{\frac{t-1}{2}}t$ since m is odd. So we have that $y=1$ and hence $Y=(3m^2-1)^{\frac{y-1}{2}}=1$. Now we check two conditions in Lemma 2.4. Since $(X_1,Y_1,Z_1)=(1,1,2)$ is the least solution of $S(l_0)$, by Lemma 2.4, we have that either

$$6m^2+1=\frac{1}{4}(3^2m^2\mp 1)$$

or

$$F_{3r+\varepsilon}=(3m)^2$$

where $\varepsilon=\pm1$. The first one implies that $4(6m^2+1)=(3^2m^2\mp1)$. But this means that $4\equiv\pm1\pmod{m^2}$, which is impossible. On the other hand since only square Fibonacci number greater than 1 is $F_{12}=12^2$ [31], the second one implies that $3m=12$ which is also false because of parity of m. Thus, by Lemma 2.4, we conclude that $(X,Z)=((6m^2+1)^{\frac{x-1}{2}},z)=(X_1,Z_1)=(1,2)$. Hence the equation (1.3) has no positive integer solution other than $(x,y,z)=(1,1,2)$ when x is odd.
Now we treat the case x is even. Then from (1.3), the equation

$$U^2+(3m^2-1)V^2=(3m)^W,\ \gcd(U,V)=1,\ W>0$$

has a solution

$$(U,V,W)=\left((6m^2-1)^{\frac{x}{2}},(3m^2-1)^{\frac{y-1}{2}},z\right)$$

Thus from Lemma 2.2, we have that

$$\begin{gathered}z=W_1t,\ t\in\mathbb{N},\\ (6m^2+1)^{\frac{x}{2}}+(3m^2-1)^{\frac{y-1}{2}}\sqrt{1-3m^2}=\lambda_1\left(U_1+\lambda_2V_1\sqrt{1-3m^2}\right)^t,\end{gathered}\tag{3.6}$$

where $\lambda_{1,2}\in\{-1,1\}$ and U_1,V_1,W_1 are positive integers satisfying

$$U_1^2+(3m^2-1)V_1^2=(3m)^{W_1},\ \gcd(U_1,V_1)=1 \tag{3.7}$$

$$h(-4(3m^2-1))\equiv 0\pmod{W_1}. \tag{3.8}$$

Suppose that $2|t$ and let

$$U_2+V_2\sqrt{1-3m^2}=\left(U_1+\lambda_2V_1\sqrt{1-3m^2}\right)^{\frac{t}{2}}. \tag{3.9}$$

Taking the norm of both sides of (3.9) in $\mathbb{Q}(\sqrt{1-3m^2})$ and taking into account (3.7), we get that

$$U_2^2+(3m^2-1)V_2^2=(3m)^{\frac{W_1t}{2}}=(3m)^{\frac{z}{2}}. \tag{3.10}$$

Substituting (3.9) into (3.6), we have that

$$(6m^2+1)^{\frac{x}{2}}+(3m^2-1)^{\frac{y-1}{2}}\sqrt{1-3m^2}=\lambda_1\left(U_2+V_2\sqrt{1-3m^2}\right)^2$$

and therefore it follows that

$$(6m^2+1)^{\frac{x}{2}}=\lambda_1(U_2^2-V_2^2(3m^2-1)). \tag{3.11}$$

$$(3m^2-1)^{\frac{y-1}{2}}=2\lambda_1U_2V_2. \tag{3.12}$$

Since $\gcd(6m^2+1, 3m^2-1) = 1$, from (3.11) and (3.12) we deduce that $|U_2| = 1$. So $|V_2| = \frac{1}{2}(3m^2-1)^{\frac{y-1}{2}}$. Substituting $|U_2|$ and $|V_2|$ into (3.10), we get that

$$1+\frac{1}{4}(3m^2-1)^y = (3m)^{\frac{z}{2}},$$

which implies that $3 \equiv 0 \pmod{3m}$, a contradiction since $3m > 3$. So we conclude that $2 \nmid t$. Let

$$\alpha = U_1 + V_1\sqrt{1-3m^2},\ \beta = U_1 - V_1\sqrt{1-3m^2}$$

Then, from (3.6), taking its complex conjugate, we get that

$$(6m^2+1)^{\frac{x}{2}} - (3m^2-1)^{\frac{y-1}{2}}\sqrt{1-3m^2} = \lambda_1\left(U_1 - \lambda_2 V_1\sqrt{1-3m^2}\right)^t. \tag{3.13}$$

By subtracting (3.13) from (3.6) we get that

$$(3m^2-1)^{\frac{y-1}{2}} = V_1\left|\frac{\alpha^t-\beta^t}{\alpha-\beta}\right| = V_1|L_t(\alpha,\beta)|. \tag{3.14}$$

By (3.7), we have $\alpha+\beta = 2U_1$, $\alpha-\beta = 2V_1\sqrt{1-3m^2}$, $\alpha\beta = (3m)^{W_1}$. Since $\gcd(U_1,V_1) = 1$, the integers $\alpha+\beta = 2U_1$ and $\alpha\beta = (3m)^{W_1}$ are also relatively prime by (3.7) and $\frac{\alpha}{\beta} \neq \pm 1$, units of ring of algebraic integers of $\mathbb{Q}(\sqrt{1-3m^2})$. So $L_t(\alpha,\beta)$ is a Lucas sequence. From (3.14), we see that the Lucas numbers $L_t(\alpha,\beta)$ have no primitive divisors. So, from Lemma 2.5 and Lemma 2.6, we get that $t \leq 30$ and if $4 < t \leq 30$ and $t \neq 6$ then the parameters

$$(e,f) := (2U_1, 4V_1^2(1-3m^2))$$

must be one of the parameters given in Lemma 2.6. But none of them match with any one of these parameters. So, it follows that

$$t \leq 3.$$

Now we will show that the case $t = 3$ is also not possible. To see this, assume that $t = 3$. So, expanding the right hand side of (3.6) for $t = 3$ as

$$\left(U_1+\lambda_2 V_1\sqrt{1-3m^2}\right)^t = U_1^3 + 3U_1^2\lambda_2 V_1\sqrt{1-3m^2} + 3U_1V_1^2(1-3m^2) + \lambda_2 V_1^3(1-3m^2)\sqrt{1-3m^2}$$

and equating the coefficients of both sides of it, we get that

$$(6m^2+1)^{\frac{x}{2}} = \lambda_1 U_1\left(U_1^2 - 3(3m^2-1)V_1^2\right) \tag{3.15}$$

and

$$(3m^2-1)^{\frac{y-1}{2}} = \lambda_1\lambda_2 V_1\left(3U_1^2 - (3m^2-1)V_1^2\right). \tag{3.16}$$

Note that from (3.7) one can see that $\gcd(3U_1, 3m^2-1) = 1$, so from (3.16) we have that $3U_1^2 - (3m^2-1)V_1^2 = \pm 1$. In fact taking modulo 3 we see that only the positive sign can occur and

$$3U_1^2 - (3m^2-1)V_1^2 = 1. \tag{3.17}$$

Thus, it follows that

$$|V_1| = (3m^2-1)^{\frac{y-1}{2}}. \tag{3.18}$$

Substituting (3.18) into (3.15) we get that

$$(6m^2+1)^{\frac{x}{2}} = \lambda_1 U_1\left(U_1^2 - 3(3m^2-1)^y\right). \tag{3.19}$$

By reducing (3.17) and (3.18) modulo $3m$, we find that $3X_1^2 - (-1)1 \equiv \pm 1 \pmod{3m}$, which means that $U_1 \equiv 0 \pmod{m}$. Then from (3.19) we find that $1^{\frac{x}{2}} \equiv 0 \pmod{m}$, which is clearly false. Thus, we may have only $t = 1$. Thus $z = W_1 t = W_1$ and by (3.8) we know that $W_1 \leq h(-4(3m^2-1))$. Using the upper bound in Lemma 2.1, we get that

$$z < \frac{4}{\pi}\sqrt{3m^2-1}\log\left(2e\sqrt{3m^2-1}\right). \tag{3.20}$$

Assume that $z = 3$. Then at least one of x or y must be greater than 1. $x \geq 2$ gives $(3m)^3 > (6m^2+1)^x \geq (6m^2+1)^2 > 6^2m^4$, and hence $3^3 > 6^2m > 36$, a contradiction. Similarly if $y \geq 2$ then the inequality $(3m)^3 \geq (3m^2-1)^2 + (6m^2+1)$ also leads us a contradiction. So $z \geq 4$. Taking equation (1.3) modulo $(9m^4)$, it implies that

$$6m^2x + 3m^2y \equiv 0 \pmod{9m^4}$$

and therefore

$$2x + y \equiv 0 \pmod{3m^2}.$$

So

$$3m^2 \leq 2x + y. \tag{3.21}$$

Since $(6m^2+1)^x < (3m)^z$ and $(3m^2-1)^y < (3m)^z$, we see that $x < z$ and $y < z$. So from (3.21) we find $m^2 < z$. Thus from the inequality

$$m^2 < z < \frac{4}{\pi}\sqrt{3m^2-1}\log\left(2e\sqrt{3m^2-1}\right)$$

we find that $m \leq 11$. Then z and hence x and y are also bounded. Taking into account (3.20) together with $x, y < z$ we wrote a short computer program with Maple to check all possible solutions of (1.3) in the range $3 \leq m \leq 11$ and we found no positive integer solutions (m, x, y, z) of (1.3) when $z \geq 3$. This completes the proof.

4. Discussion

In this paper, we take into account the equation (1.2) in the special case with the parameters $(a, b, c) = (6, 3, 3)$ and we show that the corresponding equation $(6m^2+1)^x + (3m^2-1)^y = (3m)^z$ has only the unique solution $(x, y, z) = (1, 1, 2)$ when $m > 1$. By the results of this paper we get that another support of the Terai's Conjecture. As a generalization of the results of this paper one can consider to solve the equation (1.2) in more general case where $2 \mid a$, $2 \nmid b$, $a + b = c^2$.

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Author's contributions

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The Finiteness of Smooth Curves of Degree ≤ 11 and Genus ≤ 3 on a General Complete Intersection of a Quadric and a Quartic in $\mathbb{P}^5$

Edoardo Ballico

Department of Mathematics, University of Trento, via Sommarive 14, 38123 Trento (TN), Italy

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Abstract

Let $W \subset \mathbb{P}^5$ be a general complete intersection of a quadric hypersurface and a quartic hypersurface. In this paper, we prove that W contains only finitely many smooth curves $C \subset \mathbb{P}^5$ such that $d := \deg(C) \leq 11$, $g := p_a(C) \leq 3$ and $h^1(\mathscr{O}_C(1)) = 0$.

1. Introduction

The aim of this paper is to prove the following result.

Theorem 1.1. *Let $W \subset \mathbb{P}^5$ be a general complete intersection of a quadric hypersurface and a quartic hypersurface. Then W contains only finitely many smooth curves $C \subset \mathbb{P}^5$ such that $d := \deg(C) \leq 11$, $g := p_a(C) \leq 3$ and $h^1(\mathscr{O}_C(1)) = 0$.*

We recall that W is a Calabi-Yau threefold and that there are several papers considering finiteness results for rational curves on certain Calabi-Yau threefolds (see [1]-[6] for the general quintic hypersurface of $\mathbb{P}^4$, the topic of the Clemens conjecture, which ask about the finiteness of rational curves of any fixed degree on such a general quintic). This finiteness result is not true for an arbitrary Calabi-Yau threefold [7, Remark 3.24]. For other complete intersection Calabi-Yau threefolds there are results of two types: existence results of good curves on the Calabi-Yau threefold [8, Theorem 2], [9, Theorem 1.2] and finiteness results in very restricted ranges. As in [4] our classical approach to Theorem 1.1 cannot be applied when $\binom{10}{5} \geq 4d+1-g$. There are also papers on 3-folds of general type ([10]-[12] and see [13] and references therein for arithmetically Cohen-Macaulay codimension 2 subvarieties).

The upper bound $d \leq 11$ comes from the proof at a few critical steps, but in many lemmas $d = 12$ or even $d = 13$ may be handled. The approach used in this paper (as the one for quintic 3-folds introduced in [4]) requires that $126 = h^0(\mathscr{O}_{\mathbb{P}^5}(4)) > 4d+1-g$ or, working with a fixed smooth quadric hypersurface $Q \subset \mathbb{P}^5$, $\binom{9}{5} - \binom{7}{5} = h^0(\mathscr{O}_Q(4)) > 4d+1-g$. The upper bound $g \leq 3$ may be weakened in certain steps, but we are sure that new idea are needed to handle pairs (d,g) such that $4d+1-g \geq 126$.

Theorem 1.1 is a negative result, a non-existence result. We point out that similar statements are very important, higher genera cases of the count of rational curves of fixed degree on Calabi-Yau manifolds, which is related to Mirror Symmetry [6, 14, 15]. For the Calabi-Yau threefold $X \subset \mathbb{P}^4$, X a very general quintic hypersurface, there is an explicit integer n_d for the number of the degree d rational curves contained in X [14, 15]. At the moment nobody is able to prove the finiteness of such rational curves of a given degree d, except for very low d.

Email address and ORCID number: **edoardo.ballico@unitn.it, 0000-0002-1432-7413**

1.1. A roadmap of the proof

For all integers $d > 0$ and $g \ge 0$ let $M_{d,g}$ denote the locally closed subscheme of the Hilbert scheme of $\mathbb{P}^5$ parametrizing all smooth curves $C \subset \mathbb{P}^5$ such that $\deg(C) = d$, $p_a(C) = g$ and $h^1(\mathscr{O}_C(1)) = 0$. The scheme $M_{d,g}$ is an irreducible quasi-projective variety of dimension $6d+2-2g$. Let $\mathbb{W}$ be the set of all smooth threefolds $W \subset \mathbb{P}^5$, which are the complete intersection of a hypersurface of degree 2 and a hypersurface of degree 4. For each $W \in \mathbb{W}$ we have $\mathrm{Pic}(W) = \mathbb{Z}\mathscr{O}_W(1)$, its normal bundle $N_{W,\mathbb{P}^5}$ is isomorphic to $\mathscr{O}_W(2) \oplus \mathscr{O}_W(4)$, and the quadric hypersurface, Q, containing W is unique. Standard exact sequences give $h^0(\mathscr{O}_W(2)) \oplus \mathscr{O}_W(4)) = 1 + h^0(\mathscr{O}_W(4)) = 20 + h^0(\mathscr{O}_Q(4)) - h^0(\mathscr{O}_Q(2)) = \binom{9}{4} - \binom{7}{2} = 124$. Since $h^1(N_{W,\mathbb{P}^5}) = 0$, the set $\mathbb{W}$ is a smooth variety of dimension 124. The set $\mathbb{W}$ is obviously irreducible. For a general $W \in \mathbb{W}$ the quadric associated to W is smooth. Since all smooth quadric hypersurfaces of $\mathbb{P}^5$ are projectively equivalent, we may fix a smooth quadric hypersurface Q and look only at the set $M_{d,g}(Q) := \{C \in M_{d,g} \mid C \subset Q\}$. To prove Theorem 1.1 we see which elements of $M_{d,g}(Q)$ are contained in a smooth element of $|\mathscr{O}_Q(4)|$. Let $\mathbb{W}$ denote the set of all smooth elements of $|\mathscr{O}_Q(4)|$. To prove Theorem 1.1 for the pair (d,g) it is sufficient to prove that a general element of $|\mathscr{O}_Q(4)|$ contains only finitely many elements of $M_{d,g}(Q)$. We need to study the schemes $M_{d,g}(Q)$ and this is done in Section 3 (see in particular Remark 3.3).
A key idea in this paper is that the smooth quadric hypersurface $Q \subset \mathbb{P}^5$ is isomorphic to the Grassmannian $G(2,4)$ of all 2-dimensional linear subspace of a 4-dimensional vector spaces. By the universal properties of the Grassmannians each map $C \to Q$, $C \in M_{d,g}$, corresponds to a pair (E,V) with E a rank 2 spanned vector bundle on C and $V \subseteq H^0(E)$ a linear subspace spanning E. Section 3 shows how to use this correspondence between embeddings $C \subset Q$ and rank 2 vector bundles on C. Remark 3.3 first gives some elementary statements on rank 2 vector bundles and relate them to our main idea. Then (again in Remark 3.3) we consider separately each low genus. In part (a) we finish the known case $g = 0$. Steps (b), (c) and (d) considers curves of genus 1, 2 and 3, respectively. Lemmas in later sections prove key statements for these genera, but Remark 3.3 is the key first step for them. Thus the proof is done as a case by case proof in which for any smooth curve $C \subset \mathbb{P}^5$ we distinguish the genus of C and the dimension (at most 5) of the linear space $\langle C\rangle$ spanned by C. If $\langle C\rangle$ is a plane we also distinguish if $\langle C\rangle$ is contained in Q or not. If (E,V) is the pair giving the embedding $C \hookrightarrow Q$ the integer $\dim\langle C\rangle$ is the dimension of the image of $\wedge^2(V)$ into $H^0(\mathscr{O}_C(1))$.
Using this section and later lemmas we prove that all $M_{d,g}(Q)$ are irreducible of dimension $4d+1-g$, smooth if $g \le 2$, while we describe the singular locus of $M_{d,3}(Q)$ (it contains only hyperelliptic curves). We stress again that to prove these results we use that Q is isomorphic to the Grassmannian $G(2,4)$ of all 2-dimensional linear subspaces of $\mathbb{C}^4$. In the case $(d,g) = (6,3)$ we see that all curves $C \subset W$ are hyperelliptic and that they have $h^1(\mathscr{I}_C(2)) = 1$, although $2d+1-g < \binom{7}{2}$ (Remark 4.5). In section 2 we study $M_{d,g}(Q)$, $g \le 3$, and check all cases with $d \le 7$ (Lemmas 4.3, 4.4, 4.6, 4.7) and all curves spanning a linear subspace of $\mathbb{P}^5$ of dimension ≤ 3. In section 5 we prove that if $d \le 14$ a general element of $M_{d,g}(Q)$ has $h^1(\mathscr{I}_C(4)) = 0$ (Lemma 5.5). Lemma 5.3 do the same for a smooth hyperplane section of Q and its proof may be adapted to a singular hyperplane section of Q. In section 6 we handle the non-degenerate curves $C \in M_{d,g}$ with $h^1(\mathscr{I}_C(4)) > 0$. In the last section we handle the curves $C \in M_{d,g}$ with $h^1(\mathscr{I}_C(4)) > 0$ and spanning a hyperplane of $\mathbb{P}^5$.

2. Notation

For any $r \in \{1,2,3,4,5\}$ set $M_{d,g}(r) := \{C \in M_{d,g} : \dim(\langle C\rangle) = r\}$, where for any set $S \subset \mathbb{P}^5$, $\langle S\rangle$ denote the linear span of S. Let $\mathbb{W}$ be the set of all smooth complete intersection $W \subset \mathbb{P}^5$ of a quadric hypersurface and a quartic hypersurface. If we fix a smooth quadric hypersurface $Q \subset \mathbb{P}^5$, then we call $\mathbb{W}$ the set of all smooth elements of $|\mathscr{O}_Q(4)|$.

3. Uses of vector bundles

The 4-dimensional smooth quadric hypersurface Q is isomorphic to the Grassmannian $G(2,4)$ of all 2-dimensional linear subspaces of $\mathbb{C}^4$. Hence for any projective curve X to get a morphism $\phi : X \to Q$ we need to take a rank 2 vector bundle E on X and a linear map $u : \mathbb{C}^4 \to H^0(E)$ such that $u(\mathbb{C}^4)$ spans E. To explain the proof here we assume that u is injective and instead of (E,u) we use (E,V) with $V := u(\mathbb{C}^4)$ (see Remark 3.1 for the case in which u is not injective). Assume that X is smooth. It is easy to check if ϕ is an embedding; indeed if we know that V spans E the map ϕ is an embedding if and only if $\dim(H^0(E(-Z)) \cap V) \le 1$ for every degree 2 zero-dimensional scheme $Z \subset C$. Assume that ϕ is an embedding and call C its image. Let

$$0 \to \mathscr{F}^\vee \to \mathscr{O}_Q^{\oplus 4} \to \mathscr{E} \to 0$$

denote the tautological exact sequence of $Q = G(2,4)$ with $\mathrm{rank}(\mathscr{E}) = \mathrm{rank}(\mathscr{F}) = 2$ and $\det(\mathscr{E}) \cong \det(\mathscr{F}) \cong \mathscr{O}_Q(1)$. Identifying X and C, i.e. seeing E as a vector bundle on C, we have $E = \mathscr{E}_{|C}$, while $F^\vee := \mathscr{F}^\vee_{|C}$ is the kernel of the surjection $V \otimes \mathscr{O}_C \to E$. Note that $\mathscr{F}$ and F are spanned.

Remark 3.1. *Assume that $u : \mathbb{C}^4 \to H^0(E)$ is not injective, but that $V := \mathrm{Im}(u)$ spans E. Since E has rank 2, then $2 \le \dim(V) \le 3$ and $\dim(V) = 2$ if and only if $E \cong \mathscr{O}_X^{\oplus 2}$ and hence the associated map $\phi : X \to Q$ is constant. If $\dim(V) = 3$, then $\mathrm{Im}(\phi)$ is contained in a plane with $T\mathbb{P}^2(-1)$ as universal rank 2 quotient bundle and $\mathscr{O}_{\mathbb{P}^2}(-1)$ as universal rank 1 subbundle. Hence $\phi(X) \in M_{d,g}(2)$. This case is settled in Lemma 4.4.*

Remark 3.2. *Assume $E \cong \mathscr{O}_C \oplus L$ for some line bundle L. In this case $L \cong \mathscr{O}_C(1)$. Write $V = \mathbb{C} \oplus V_1$ with $\mathbb{C} = H^0(\mathscr{O}_C)$. Hence C is contained in a certain Schubert cell of Q, i.e., a 2-dimensional linear subspace contained in Q. Hence $C \in M_{d,g}(2)$. This case is solved in Lemma 4.4. If $F \cong \mathscr{O}_C \oplus \mathscr{O}_C(1)$, then C is contained in the other family of planes contained in Q and so $C \in M_{d,g}(2)$.*

In the next remark we point out some irreducibility and smoothness results for $M_{d,g}(Q)$.

Remark 3.3. *Since $TQ \cong \mathscr{E} \otimes \mathscr{F}$, we have $TQ_{|C} \cong E \otimes F$. In many cases with low g we have $h^1(E \otimes F) = 0$. In this case we have $h^1(N_{C,Q}) = 0$ and hence the Hilbert scheme $\mathrm{Hilb}(Q)$ of Q at $[C]$ is smooth of dimension $4d+1-g$, where $d := \deg(C)$ and $g := p_a(C)$.*

Claim 1: *If either $h^1(E) = 0$ or $h^1(F) = 0$, then $h^1(E \otimes F) = 0$.*

Proof of Claim 1: *Assume for instance $h^1(E) = 0$. Since F is spanned, the evaluation map $e_F : H^0(F) \otimes \mathscr{O}_C \to F$ is surjective. Set $K := \ker(e_F)$. Since $\dim C = 1$, $h^2(K \otimes E) = 0$. Hence the exact sequence*

$$0 \to K \otimes E \to H^0(F) \otimes E \to E \otimes F \to 0$$

proves Claim 1.

Claim 2: *In any genus $g \geq 2$ the set of all $C \in M_{d,g}(Q)$ with $h^1(E) = 0$ is an open, smooth and irreducible subset of $M_{d,g}(Q)$ with dimension $4d+1-g$.*

Proof of Claim 2: *The openness part follows from the semicontinuity of cohomology. Since C is a curve and F is spanned, the vanishing of $h^1(E)$ implies the vanishing of $h^1(E \otimes F)$. Hence this part of $M_{d,g}(Q)$ is smooth and everywhere of dimension $4d+1-g$. Since $g \geq 2$, any vector bundle on a smooth curve C is a flat limit of a family of stable bundles [16, Proposition 2.6]. If $h^1(E) = 0$, then E is a flat limit of a family of stable bundles with vanishing cohomology. The claim follows from the irreducibility of $\mathscr{M}_g$ and the irreducibility of the set of all stable vector bundles with rank two and degree d on a fixed smooth curve of genus $g \geq 2$. This set has dimension $4g-3$.*

(a) If $g = 0$, then $h^1(E \otimes F) = 0$, because $E \otimes F$ is spanned and hence a direct sum of line bundles of degree ≥ 0. The scheme $M_{d,0}(Q)$ is irreducible, because both E and F are specializations with constant cohomology of the rigid bundle with rank 2 and degree d (the direct sum of the line bundle of degree $\lceil d/2 \rceil$ and the one of degree $\lfloor d/2 \rfloor$).

(b) Assume $g = 1$.

Claim 3: *We claim that $h^1(E \otimes F) = 0$, unless $E \cong \mathscr{O}_C \oplus \mathscr{O}_C(1)$ and $F \cong \mathscr{O}_C \oplus \mathscr{O}_C(1)$.*

Proof of Claim 3: *Since $E \otimes F \cong F \otimes E$, it is sufficient to prove that $E \cong \mathscr{O}_C \oplus \mathscr{O}_C(1)$. Since E is spanned, it is a direct sum of indecomposable and spanned vector bundles of degree ≥ 0 and if one of them has degree zero, it is a factor $\mathscr{O}_C$ of E. By Atiyah's classifications of vector bundles on elliptic curves ([17, Part II]) every indecomposable vector bundle G with $\deg(G) > 0$ satisfies $h^1(G) = 0$, concluding the proof of Claim 3.*

This part of $M_{d,1}(Q)$ is irreducible for the following reasons. By Atiyah's classification of vector bundles on an elliptic curve ([17, Part II]), E is a specialization with constant cohomology of semistable bundles. Therefore to check that $M_{d,1}(Q)$ is irreducible, it is sufficient to test the cases with E semistable. If E is semistable, then $h^1(E \otimes F) = 0$ for any spanned bundle F by Claim 3. If d is odd, then we use that any two stable bundle with same rank and degree only differ by a twist with an element of $\mathrm{Pic}^0(C)$. If d is even, then either $E \cong R \oplus L$ with $R, L \in \mathrm{Pic}^{(d/2)}(C)$ and $R \otimes L \cong \mathscr{O}_C(1)$ or E is a non-trivial extension of R by itself and $R^{\otimes 2} \cong \mathscr{O}_C(1)$. The latter case is a specialization of the former one (at least varying C), because $M_{d,1}(Q)$ is smooth and equidimensional and the indecomposable bundles have a smaller dimension.

(c) Assume $g = 2$. By Remark 3.2 and Lemma 4.4 we may assume $E \neq \mathscr{O}_C \oplus \mathscr{O}_C(1)$ and $F \neq \mathscr{O}_C \oplus \mathscr{O}_C(1)$.

Now assume $g = 2$ and $h^1(E) > 0$. By duality we get a non-zero map $v : E \to \omega_C$. Since E is spanned, $\mathrm{Im}(v)$ is spanned. Hence either v is surjective or $\mathrm{Im}(v) \cong \mathscr{O}_C$. The latter case is not possible, because (since E is spanned), it would give that E has $\mathscr{O}_C$ as a factor. Thus v is surjective. Set $A := \ker(v)$. We have $A \cong \mathscr{O}_C(1) \otimes \omega_C^{\vee}$. Since $\mathscr{O}_C(1)$ is very ample, we have $d > 4$. Hence $h^1(A) = 0$. If $d \geq 6$, A is spanned. If $d \geq 7$, then $h^1(A \otimes \omega_C^{\vee}) = 0$ and hence $E \cong A \oplus \omega_C$. Assume also $h^1(F) > 0$. We get that F is an extension of ω_C by $\mathscr{O}_C(1) \otimes \omega_C$. Since $h^1(\omega_C^{\otimes 2}) = 0$, we get $h^1(E \otimes F) = 0$ and so $h^1(N_{C,Q}) = 0$. Hence $M_{d,2}(Q)$ is smooth and of pure dimension $4d+1-g$. To check the irreducibility of $M_{d,2}$, it is sufficient to prove that the bundles with $h^1(E) > 0$ do not fill a connected component of $M_{d,2}$. If $d \leq 6$, see Lemma 4.6 and Lemma 4.8. If $d \geq 7$, then $E \cong A \oplus \omega_C$ and so on a fixed curve C this set is isomorphic to $\mathrm{Pic}^{d-2}(C)$; we write g for the genus, because the same argument is needed when $g = 3$. Fix $C \in \mathscr{M}_g$ and take $E \cong A \oplus \omega_C$ with $A \in \mathrm{Pic}^2(C)$. This family of bundles is irreducible and (since $M_{d,g}(Q)$ is smooth along all these bundles) we only need to exclude that $M_{d,g}(Q)$ has two connected components, one formed by bundles E_1 with $h^1(E_1) = 0$ and the other ones with bundles with $h^1(E) = 1$. We have $h^1(E) = 1$ and so $h^0(E) = d+3-2g$. If $h^1(E_1) = 0$, then $h^0(E_1) = d+2-2g$. We have $\dim(G(4, d+1+2(1-g))) = \dim(G(4, d+2(1-g)) + 4$. Thus each bundle E with $h^1(E) > 0$ has the property that $H^0(E)$ has a family of 4-dimensional linear subspaces with higher dimension. For $g \geq 3$ it is sufficient to note that for a fixed C the possible E depends on $A \in \mathrm{Pic}^{d-g}(C)$, the set of all rank 2 stable bundles on C with degree d have dimension $4g-3$ and $g+4 < 4g-3$. When $g = 2$ we also need to factorize the huge automorphism group of $A \oplus \omega_C$ (we have $h^0(A \otimes \omega_C^{\vee}) = d-5$).

(d) Assume $g = 3$. By Remark 3.2 and Lemma 4.4 we may assume $E \neq \mathscr{O}_C \oplus \mathscr{O}_C(1)$ and $F \neq \mathscr{O}_C \oplus \mathscr{O}_C(1)$. We also assume $d \geq 8$, leaving the cases $d \leq 7$ to Remark 4.7. All cases with $h^1(E) = 0$ are done as in Claim 2. Assume $h^1(E) > 0$ and $h^1(F) > 0$. As in step (b) we get non-zero maps $v_1 : E \to \omega_C$ and $v_2 : F \to \omega_C$ with $\mathrm{Im}(v_i)$ a non-trivial and spanned line bundle. Hence either v_i is surjective or C is not hyperelliptic and $\mathrm{Im}(v_i) = \omega_C(-p)$ for some $p \in C$ or C is hyperelliptic

and $\mathrm{Im}(v_i)$ *is the* g^1_2 *of* C. *In all cases* $\ker(v_i)$ *is spanned and non-special, because we assumed* $d \geq 9$. *The case in which* $E \cong A \oplus \omega_C$ *is handled as in step (c). If either* C *is not hyperelliptic or at least one among* $\mathrm{Im}(v_1)$ *and* $\mathrm{Im}(v_2)$ *is not the* g^1_2 *on* C, *we have* $h^1(E \otimes F) = 0$ *and so* $h^1(N_{C,Q}) = 0$. *So* $M_{d,3}(Q)$ *is smooth and of dimension* $4d+1-g = 4d-2$ *at* $[C]$. *Hence* $h^1(E \otimes F) > 0$ *if and only if* C *is hyperelliptic and* $\mathrm{Im}(v_1)$ *and* $\mathrm{Im}(v_2)$ *are the* g^1_2, R, *on* C. *In this case we have* $E \cong A \oplus R$ *and* $F \cong B \oplus R$ *with* $\deg(A) = \deg(B) = d-2$ *and so* $h^1(E \times F) = 1$. *Therefore every irreducible component of* $M_{d,3}(Q)$ *containing* $[C]$ *has dimension at least* $4d+1-g$ *and at most* $4d+2-g$. *To check that these points are singular points of* $M_{d,3}(Q)$ *and hence that* $M_{d,3}(Q)$ *has pure dimension* $4d-2$, *it is sufficient to prove that these bundles do not fill a subset of* $M_{d,3}(Q)$ *of dimension* $\geq 4d-2$; *we will prove that these bundles fill in a family of dimension* $\leq 4d-3$, *because this is needed to prove the irreducibility of* $M_{d,3}(Q)$. *The set of these bundles only depends on the choice of a hyperelliptic curve* C, *the choice of* $A \in \mathrm{Pic}^{d-2}(C)$ *and the choice of a* 4*-dimensional linear subspace of* $H^0(A \oplus R)$. *We have* $h^1(A \oplus R) = h^1(R) = 1$ *and so* $h^0(A \oplus R) = d+2-2g$. *Since there* ∞^5 *hyperelliptic curves and* $\mathrm{Pic}^{d-2}(C)$ *has dimension* 3, *it is sufficient to use that* $5+4+3 < 6+4g-3$. *Then the proof in step (c) handles all bundles of the form* $A \oplus \omega_C$. *It remains to handle the bundles* E *with* C *not hyperelliptic and* $\mathrm{Im}(v_1) \cong \omega_C(-p)$ *for some* $p \in C$. *Set* $A := \ker(v_1) \in \mathrm{Pic}^{d-3}(C)$. *Note that* $h^1(E) = 1$ *and* $h^1(F) = 0$. *Hence these bundles are in the smooth part of* $M_{d,3}(Q)$. *We have* $h^0(E) = h^0(E_1)+1$ *when* $h^1(E_1)$ *and so the Grassamannian of all* 4*-dimensional linear subspaces has dimension* $4+z$, *where* z *is the dimension of all* 4*-dimensional linear subspaces of* $H^0(E_1)$. *The bundles* E_1 *depends on* $4g-3 = 9$ *parameters. The bundles* E *depends on* A *(*$g=3$*) parameters, on* $p \in C$ *(one parameter) and an extension classes of* $\omega_C(-p)$ *by* A. *For the trivial extensions we use that* $4+g+1 < 4g-3$. *Two non-trivial, but proportional extensions, give the same bundle, up to isomorphisms. Hence the bundles* E *with* $h^1(A \otimes \omega_C^\vee(p)) \leq 1$, *do not fill a connected component of* $M_{d,3}(Q)$. *We have* $\deg(A \otimes \omega_C^\vee) = d-6$. *Since* C *is not hyperelliptic, we have* $h^1(A \otimes \omega_C^\vee(p)) \leq 1$ *for all* $d \geq 8$. *See Remark 4.7 for the case* $d \leq 7$.

4. Preliminary lemmas

The following lemma is proved as in [6, page 153].

Lemma 4.1. *Fix* (d,g) *such that* $2d \leq 19+g$ *and* $h^1(\mathscr{I}_C(2)) = 0$ *for all* $C \in M_{d,g}$. *Then a general* $W \in \mathbb{W}$ *contains finitely many elements of* $M_{d,g}$ *and the incidence variety* $I_{d,g} \subset M_{d,g} \times \mathbb{W}$ *is irreducible.*

Remark 4.2. *Unfortunately in several interesting cases many curves satisfies* $h^1(\mathscr{I}_C(2)) > 0$ *(e.g. if* $2d+1-g > 15$ *this is the case for all curves spanning a hyperplane of* $\mathbb{P}^5$*). Working with* $M_{d,g}(Q)$ *we only need to check if* $h^1(\mathscr{I}_C(4)) = 0$. *This is true for all* $C \in M_{d,g}(Q)$ *for some more pairs* (d,g). *We divide* $M_{d,g}(Q)$ *in the one with* $h^1(\mathscr{I}_C(4)) = 0$ *and in the ones with* $h^1(\mathscr{I}_C(4)) > 0$. *We need to prove that for* C *in a non-empty open subset of* $M_{d,g}(Q)$ *we have* $h^1(\mathscr{I}_C(4)) = 0$ *(Lemma 5.5). The last two sections of this paper tackle the case* $h^1(\mathscr{I}_C(4)) > 0$.

Remark 4.3. $M_{d,g}(1) \neq \emptyset$ *if and only if* $d = 1$ *and* $g = 0$. *By Lemma 4.1 a general* W *has only finitely many lines.*

Lemma 4.4. $M_{d,g}(2) \neq \emptyset$ *if and only if either* $d = 2$ *and* $g = 0$ *or* $d = 3$ *and* $g = 1$. *In the cases* $(d,g) \in \{(2,0),(3,1)\}$ *a general* W *contains finitely many elements of* $M_{d,g}(2)$.

Proof. Since the curves in $M_{d,g}$ are non-special, $M_{d,g}(2) \neq \emptyset$ if and only if either $d = 2$ and $g = 0$ or $d = 3$ and $g = 1$. The second assertion follows from Lemma 4.1. □

Remark 4.5. *Set* $\Gamma := \{C \in M_{6,3} : C \text{ is hyperelliptic}\}$. Γ *is an irreducible divisor of the* 32*-dimensional variety* $M_{6,3}$. *Fix a smooth quadric hypersurface* $Q \subset \mathbb{P}^5$ *and set* $\Gamma' := \Gamma \cap M_{6,3}(Q)$. *Fix* $C \in M_{6,3}(Q)$. *We have* $\dim(\langle C \rangle) = 3$. *Since* Q *is smooth,* $\langle C \rangle \nsubseteq Q$ *and so* $Q' := \langle C \rangle$ *is an irreducible quadric surface containing* C. *Since all even degree smooth curves of a quadric cone of* $\mathbb{P}^3$ *are complete intersection [18, V Ex. 2.9],* Q' *is a smooth quadric. Since* $(d,g) = (6,3)$, *then* $C \in |\mathscr{O}_{Q'}(2,4)| \cup |\mathscr{O}_{Q'}(4,2)|$ *and so* C *is hyperelliptic. Hence no* $C \in M_{6,3}(Q) \setminus \Gamma'$ *is contained in some* $W \in \mathbb{W}$. *Conversely, any hyperelliptic curve* X *may be embedded in* $Q' = \mathbb{P}^1 \times \mathbb{P}^1$ *as an element of* $|\mathscr{O}_{Q'}(2,4)|$ *using the* g^1_2, R, *of* X *to get one morphism* $X \to \mathbb{P}^1$ *and a general* $A \in \mathrm{Pic}^4(X)$ *for the other map* $X \to \mathbb{P}^1$ *so that* $A \otimes R$ *is very ample). Hence for a fixed* X *the set of all such embeddings is parametrized by an irreducible variety of dimension* 3. *Fix* $C \in \Gamma'$, *say with* $C \in |\mathscr{O}_{Q'}(2,4)|$. *We have* $N_{C,Q} \cong \mathscr{O}_C(1)^{\oplus 2} \oplus \mathscr{O}_C(2,4)$ *and hence* $h^1(N_{C,Q}) = 0$. *So* $M_{6,3}(Q)$ *is smooth at* $[C]$ *and of dimension* $4d+1-g = 22$. *Since* $|\mathscr{O}_{Q'}(2,4)|$ *is irreducible and as* $\langle C \rangle$ *we may take any* $\mathbb{P}^3 \subset \mathbb{P}^5$ *transversal to* Q, $M_{6,3}(Q)$ *is irreducible. Call* $\mathscr{I} \subset \Gamma' \times \mathbb{W}$ *the incidence correspondence and let* $\pi_1 : \mathscr{I} \to \Gamma'$ *and* $\pi_2 : \mathscr{I} \to \mathbb{W}$ *denote the projections. We have* $h^1(Q, \mathscr{I}_{C,Q}(4)) = 0$, *because* $h^1(Q', \mathscr{I}_{C,Q'}(4)) = h^1(Q', \mathscr{O}_{Q'}(2,0)) = 0$. *Lemma 4.1 concludes the proof of the theorem for* $(d,g) = (6,3)$. *In this case the incidence correspondence is irreducible, because the set of all hyperelliptic curves is irreducible and all these curves* C *have the same* $h^0(\mathscr{I}_C(2))$ *and* $h^1(\mathscr{I}_C(4)) = 0$ *(and so the incidence correspondence for* $M_{6,3}(Q)$ *is irreducible).*

Lemma 4.6. *We have* $M_{d,g}(3) \neq \emptyset$ *if and only if* $d \geq g+3$. *If* $g \leq 3$, *then a general* $W \in \mathbb{W}$ *contains some* $C \in M_{d,g}(3)$ *only if* $(d,g) \in \{(3,0),(4,1),(5,2),(6,3)\}$ *and in each of these cases* W *contains only finitely many curves* C.

Proof. Fix a smooth hyperquadric Q, $C \in M_{d,g}(3)$ and $W \in \mathbb{W}$ containing C. Set $U := \langle C \rangle$. Since Q is smooth, $U \nsubseteq Q$ and hence $Q' := Q \cap U$ is a quadric surface containing C. Since the irreducible curve C spans U and $C \subset Q'$, Q' is irreducible. If Q' is a quadric cone, then C is arithmetically normal [18, V Ex. 2.9] and hence $h^1(\mathscr{I}_C(t)) = 0$ for $t = 2,4$, so that we may apply Lemma 4.1 to these curves) and we find pairs $(d,g) \in \{(3,0),(4,1),(5,2)\}$. If Q', up to a change of the ruling of Q' we get all $C \in |\mathscr{O}_{Q'}(2,g+1)|$ and so $d = g+3$. If $g \leq 4$ we have $h^1(\mathscr{I}_C(4)) = h^1(Q', \mathscr{I}_{C,Q'}(4)) = h^1(Q', \mathscr{O}_{Q'}(2,4-g-1)) = 0$. □

Lemma 4.7. *Theorem 1.1 is true for $g=3$ and $d\le 7$.*

Proof. Take $g=3$ and $d\le 7$. Since $h^1(\mathscr{O}_C(1))=0$, we have $6\le d\le 7$. Remark 4.5 and Lemma 4.6 solve the case $d=6$ and the case $d=7$ in which $C\in M_{7,3}(3)$. Hence we may assume $d=7$ and $\dim(\langle C\rangle)=4$. In this case C is linearly normal in its linear span and so $h^1(\mathscr{I}_C(t))=0$ for all $t\in\mathbb{N}$. Apply Lemma 4.1. □

Lemma 4.8. *Fix $C\in M_{d,g}(Q)(r)$ with $d\le 7$, $g\le 2$ and $r=4,5$. Then $h^1(N_{C,Q})=h^1(\mathscr{I}_C(4))=0$. Moreover, these cases only contribute finitely many smooth curves to a general $W\in\mathbb{W}$.*

Proof. Since $g\le 2$, we have $h^1(N_{C,Q})=0$. Since $d<4+r$, we have $h^1(\mathscr{I}_C(4))=0$ [19, Theorem at page 492] and hence these cases contributes only finitely smooth curves to a general $W\in\mathbb{W}$. □

Lemma 4.9. *A general $W\in\mathbb{W}$ contains no singular conic (reducible or a double line).*

Proof. Take any conic $D\subset W$. Since $h^1(\mathscr{I}_{D,\mathbb{P}^5}(4))=0$, we have $h^1(Q,\mathscr{I}_{D,Q}(4))=0$ and hence $h^0(Q,\mathscr{I}_{D,Q}(4))=h^0(D,\mathscr{I}_{D,Q}(4))$. Either D is contained in a plane contained in Q or it is the complete intersection of Q and a plane. In both cases we have $h^1(N_{D,Q})=0$. Thus a dimensional count gives that a general $W\in\mathbb{W}$ contains only finitely many conics and that all these conics are smooth. □

We recall the following well-known consequence of the bilinear lemma (it is a key tool in [2]).

Lemma 4.10. *Fix integers $t\ge 2$, $r\ge 3$ and an integral and non-degenerate curve $T\subset\mathbb{P}^r$ such that $h^1(\mathscr{I}_T(t))>0$. Fix a linear subspace $V\subseteq H^0(\mathscr{O}_{\mathbb{P}^r}(1))$. Assume that $h^1(M,\mathscr{I}_{M\cap T,M}(t))=0$ for every hyperplane $M\in|V|$. Then $h^1(\mathscr{I}_T(t-1))\ge h^1(\mathscr{I}_T(t))+\dim(V)-1$.*

Proof. For any hyperplane $M\subset\mathbb{P}^r$ we have an exact sequence

$$0\to\mathscr{I}_T(t-1)\to\mathscr{I}_T(t)\to\mathscr{I}_{T\cap M,M}(t)\to 0$$

Now assume that V contains an equation of M. Since $h^1(M,\mathscr{I}_{T,M}(t))=0$, the map $H^1(\mathscr{I}_T(t-1))\to H^1(\mathscr{I}_T(t))$ is surjective and hence its dual $e_M:H^1(\mathscr{I}_T(t))^\vee\to H^1(\mathscr{I}_T(t-1))^\vee$ is injective. Taking the equations of all hyperplanes we get a bilinear map map $u:H^1(\mathscr{I}_T(t))^\vee\times V\to H^1(\mathscr{I}_T(t-1))^\vee$, which is injective with respect to the second variables, i.e. for every non-zero linear form ℓ the map $u_{|H^1(\mathscr{I}_T(t))^\vee\times\{\ell\}}$ is injective (it is e_M with $M:=\{\ell=0\}$). Hence if $(a,\ell)\in H^1(\mathscr{I}_T(t))^\vee\times V$ with $a\ne 0$ and $\ell\ne 0$, then $u(a,\ell)=e_M(a)\ne 0$. Therefore the bilinear map u is non-degenerate in each variable. Hence $h^1(\mathscr{I}_T(t-1))\ge h^1(\mathscr{I}_T(t))+\dim(V)-1$ by the bilinear lemma. □

5. Good postulation in degree 4

In this section we prove for certain d,g the existence of a non-degenerate $C\in M_{d,g}(Q)$ with $h^1(\mathscr{I}_C(4))=0$.

Lemma 5.1. *Fix $C\in M_{d,g}(Q)$ such that $h^1(N_{C,Q})=0$. Take an integer $t>0$ and a smooth rational curve $T\subset Q$ such that $\deg(C\cap T)=1$ and $\deg(T)=t$. Then $h^1(N_{C\cup T,Q})=0$ and $C\cup T$ is a flat limit of elements of $M_{d+t,g}(Q)$.*

Proof. Set $\{p\}:=C\cap T$. By assumption $h^1(\mathscr{O}_C(1))=0$. Since Q is homogeneous, its tangent bundle is spanned. Hence $N_{T,Q}$ is a direct sum of line bundles of degree ≥ 0. Therefore $h^1(N_{T,Q}(-p))=0$. A Mayer-Vietoris exact sequence gives $h^1(\mathscr{O}_{C\cup T}(1))=0$. Hence if $C\cup T$ is smoothable inside Q, then it is a flat limit of a family of elements of $M_{d+t,g}(Q)$. Since $h^1(N_{T,Q}(-p))=0$, as in [20, Theorem 4.1] we get that $C\cup T$ is smoothable inside Q and $h^1(N_{C\cup T,Q})=0$. □

Lemma 5.2. *For all $g\in\{0,1,2,3\}$ there is a non-degenerate $C\in M_{g+5,g}(Q)$ and any such C is projectively normal.*

Proof. Let $X\subset\mathbb{P}^5$ be a linearly normal smooth curve of genus $g\le 3$ and degree $g+5$. Since $g+5\ge 2g+1$, X is projectively normal [21]. It is sufficient to prove that some X is contained in a smooth quadric hypersurface. Since $g\le 3$, we start with a smooth quadric surface $Q_1\subset Q$, a smooth curve $A\in|\mathscr{I}_{Q_1}(2,g+1)|$ and then we apply the case $t=2$ of Lemma 5.1. □

Lemma 5.3. *Let $Q'\subset\mathbb{P}^4$ be a smooth quadric hypersurface. Fix integers d,g such that $0\le g\le 3$ and $d\ge g+4$. Let $M_{d,g}(Q')$ be the set of all non-special smooth curves $C\subset Q'$ of genus g and degree d.*

(a) There is $C\in M_{g+4,g}(Q')$ which is projectively normal.

(b) If either $g+4\le d\le g+6$ or $g\le 2$ and $d=g+7$ or $g=0$ and $d=8$, then there is $C\in M_{d,g}(Q')$ such that $h^1(Q',\mathscr{I}_{C,Q'}(3))=0$.

(c) If either $g+4\le d\le g+9$, or $g\le 2$ and $d=g+10$ or $g=0$ and $d=11,12$, then there is $C\in M_{d,g}(Q')$ such that $h^1(Q',\mathscr{I}_{C,Q'}(4))=0$.

Proof. The proof of part (a) is similar to the one Lemma 5.2. The same proof also gives the case $d = g+4$ of part (b).

(i) Let $A \subset Q'$ be a smooth projectively normal curve of genus g and degree $g+4$. Let $Q_1 \subset Q'$ be a general hyperplane section. Q_1 is a smooth quadric surface and $S := A \cap Q_1$ is a subset of Q_1 with degree $g+4$, in uniform position and spanning the 3-dimensional linear space spanned by Q_1. Fix $p \in S$ and set $S' := S \setminus \{p\}$. Let B be a general element of $|\mathscr{I}_{p,Q_1}(1,2)|$. Lemma 5.1 shows that $A \cup B$ is smoothable inside Q'. Hence to prove the case $d = g+7$, $g \le 2$, of part (b) it is sufficient to prove that $h^1(Q', \mathscr{I}_{A\cup B,Q'}(3)) = 0$. We have $\mathrm{Res}_{Q_1}(A \cup B) = A$. Since $h^1(Q', \mathscr{I}_{A,Q'}(2)) = 0$, the case $t = 3$ of the residual sequence

$$0 \to \mathscr{I}_{A,Q'}(t-1) \to \mathscr{I}_{A\cup B,Q'}(t) \to \mathscr{I}_{(A\cup B)\cap Q_1,Q_1}(t) \to 0$$

shows that it is sufficient to prove that $h^1(Q_1, \mathscr{I}_{(A\cup B)\cap Q_1,Q_1}(3)) = 0$. We have $Q_1 \cap (A\cup B) = S' \cup B$ and hence it is sufficient to prove that $h^1(Q_1, \mathscr{I}_{S',Q'}(2,1)) = 0$. S' is a set of $g+3 \le 6$ points of Q_1. Assume $e := h^1(Q_1, \mathscr{I}_{S',Q_1}(2,1)) > 0$. Hence $h^0(Q, \mathscr{I}_{S',Q_1}(2,1)) = e+3-g$. Since S is in uniform position, we get $h^0(Q_1, \mathscr{I}_{S,Q_1}(2,1)) = e+g-3$. Fix a general $D \in |\mathscr{I}_{S,Q_1}(2,1)|$. First assume that D is irreducible. For any set $E \subset D$ with $\#(E) = 5$, we have $h^0(Q_1, \mathscr{I}_{D,Q_1}(2,1)) = h^0(Q_1, \mathscr{I}_{E,Q_1}(2,1))$ and hence $h^1(Q_1, \mathscr{I}_{E,Q_1}(2,1)) = 0$. If $g \le 2$ we may take $S' \subseteq E$. Now assume that D is reducible. Since S is in uniform position, we may assume that no 2 of the points of S are contained in a line of Q_1. Hence we get the existence of a smooth conic $D_1 \subset Q_1$ containing at least $g+4$ points of S'. Since S is in uniform position, we get $S \subset D_1$. If $g = 3$ we use instead of B a curve $B' \in |\mathscr{I}_{p,Q_1}(1,1)|$ (in this case the equality $h^1(Q_1, \mathscr{I}_{S',Q_1}(2,2)) = 0$ may be proved using an elliptic curve $D' \in |\mathscr{O}_{Q_1}(2,2)|$, because $h^1(D, \mathscr{I}_{S',D_1}(2,2)) = 0$ for any set $E \subset D$ with $\#(E) \le 7$. Now assume $g = 0$ and $d = 8$. Instead of B we take a general $B_1 \in |\mathscr{I}_{p,Q_1}(1,3)|$. It is sufficient to prove that $h^1(Q, \mathscr{I}_{S',Q_1}(2,0)) = 0$. We have $\#(S') = 3 = h^0(Q_1, \mathscr{O}_{Q_1}(0,2))$, and it is sufficient to use again by the uniform position that no two points of S are on a line of Q_1.

(ii) Now we prove part (c). Since in part (b) we get non-special curves, the same curves C have $h^1(Q', \mathscr{I}_{C,Q'}(4)) = 0$ by the Castelnuovo-Mumford's lemma. Hence we may assume that either $d \ge g+8$ and $g \le 2$, or $d \ge g+7$ and $g = 3$ or $g = 0$ and $d \ge 9$. Set $t := 8$ if $g = 0$, $t := g+7$ if $g = 1,2$ and $t := 9$ if $g = 3$. By part (b) there is $A \subset M_{t,g}(Q')$ such that $h^1(Q', \mathscr{I}_{A,Q'}(3)) = 0$. Take a general hyperplane section Q_1 of Q' and set $S := Q_1 \cap S$. S' is a subset of Q_1 with cardinality t, spanning a $\mathbb{P}^3$ and in uniform position. Fix $p \in S$ and set $S' := S \setminus \{p\}$. Fix a general $B \in |\mathscr{I}_{p,Q_1}(1,2)|$. As in step (i) it is sufficient to prove that $h^1(Q_1, \mathscr{I}_{S',Q}(3,2)) = 0$. In all cases we have $t-1 \le 8$. The uniform position and the non-degeneracy of S' imply that no line of Q_1 contains at least 2 points of S' and no conic of Q_1 contains at least 4 points of S'.
Now take $g = 0$. In this case A may be dismantled into a union of lines. Fix a general line $L \subset Q'$. For each $q \in L$. The union of all lines of Q' trough q is the 2-dimensional quadric cone $T_q(Q') \cap Q'$. For a general $q \in L$ the curve $T_q(Q') \cap Q_1$ is a smooth element D_q of $|\mathscr{O}_{Q_1}(1,1)|$ and a general line in Q' passing through q meets Q_1 at a general point of Q_1. Hence we get $h^0(Q_1, \mathscr{I}_{S'}(3,1)) = 0$ if $\#S' \le 8$, i.e. if we start with a general $A \in_{d,0} (Q')$ with $d \le 9$. Thus we get the case $g = 0$ of part (c). □

Lemma 5.4. *Let $Q' \subset \mathbb{P}^4$ be a smooth quadric hypersurface. Fix a set $S \subset Q'$ with $\#S \le 10$ and S is in linearly general position. Take $p \in S$ and set $S' := S \setminus \{p\}$.*

(a) If $1 \le d \le 4$, then there is $C \in M_{d,0}(Q')$ such that $C \cap S = \{p\}$ and $h^1(Q', \mathscr{I}_{S'\cup C,Q'}(3)) = 0$.

(b) If $1 \le d \le 9$, then there is $C \in M_{d,0}(Q')$ such that $C \cap S = \{p\}$ and $h^1(Q', \mathscr{I}_{S'\cup C,Q'}(4)) = 0$.

Proof. Let Q_1 be a general hyperplane section of Q' containing p. Q_1 is smooth and $Q_1 \cap S = \{p\}$. We have $h^1(Q', \mathscr{I}_{S',Q'}(2)) = 0$, because $\#S' \le 9$ [22, Theorem 3.2]. To prove part (a) it is sufficient to take any smooth $C \in |\mathscr{I}_{p,Q_1}(1,3)|$. By Castelnuovo-Mumford's lemma to prove part (b) we may assume $d > 4$. Fix a general $A \in M_{4,0}(Q')$ containing p. Part (a) gives $h^1(Q', \mathscr{I}_{A\cup S',Q'}(3)) = 0$. Fix a general hyperplane section $Q_2 \subset Q'$. We have $Q_2 \cap S = \emptyset$ and the set $E := Q_2 \cap A$ is in linearly general position in the $\mathbb{P}^3$ spanned by Q_2. Fix $q \in E$ and set $E' := E \setminus \{q\}$. Fix a general $B \in |\mathscr{I}_{q,Q_2}(1,4)|$. By Lemma 5.1 it is sufficient to prove that $h^1(\mathscr{I}_{S'\cup A\cup B,Q'}(4)) = 0$. Since $\mathrm{Res}_{Q_1}(S' \cup A \cup B) = S' \cup A$ and $h^1(\mathscr{I}_{A\cup S',Q'}(3)) = 0$, it is sufficient to prove that $h^1(Q_1, \mathscr{I}_{E'\cup B,Q_1}(4)) = 0$, i.e. $h^1(Q', \mathscr{I}_{E'}(3,0)) = 0$. This is true, because E' is formed by 3 points in uniform position. □

Lemma 5.5. *(a) For all integers d, g such that $0 \le g \le 3$ and $g+5 \le d \le g+9$ there is a non-degenerate $C \in M_{d,g}(Q)$ such that $h^1(\mathscr{I}_C(3)) = 0$.*

(b) For all integers d, g such that either $0 \le g \le 3$ and $g+5 \le d \le 14$ there is a non-degenerate $C \in M_{d,g}(Q)$ such that $h^1(\mathscr{I}_C(4)) = 0$.

Proof. Fix a projectively normal $A \in M_{g+5,5}(Q)$. Fix a general hyperplane section $Q' \subset Q$. Since $h^1(Q, \mathscr{I}_{A,Q}(4)) = 0$, we may assume $d > g+5$. The set $S := A \cap Q_1$ is in linearly general position. Fix $p \in S$ and set $S' := S \setminus \{p\}$. Apply part (b) of Lemma 5.4 to get $T \in M_{d-g-5,0}(Q')$ such that $h^1(Q', \mathscr{I}_{S'\cup T}(4)) = 0$. Since $h^1(Q, \mathscr{I}_{A\cup T}(3)) = 0$ and $(A \cup T) \cap Q' = S' \cup T$, the residual sequence of Q' in Q gives $h^1(Q, \mathscr{I}_{A\cup B}(4)) = 0$. Use Lemma 5.1 and the semicontinuity theorem for cohomology to prove part (b). For part (a) we take T of degree ≤ 4 and use that $h^1(Q, \mathscr{I}_{A,Q}(2)) = 0$. □

Remark 5.6. *A general element of $M_{d,0}(Q')$ (resp. $M_{d,0}(Q)$) is a deformation of a tree contained in Q' (resp. Q). Using this observation we may improve parts (a) and (b) of Lemma 5.5, but for a range of integers d out of reach with our tools for the Clemen's conjecture.*

6. Non-degenerate curves

In this section we consider non-degenerate curves C of $M_{d,g}$ or of $M_{d,g}(Q)$. By [19, Theorem at page 492] we have $h^1(\mathscr{I}_C(4)) = 0$ if either $d \le 8$ or $d = 9$ and $g > 0$ or $d = 9$, $g = 0$ and there is no line $R \subset \mathbb{P}^5$ with $\deg(R \cap C) \ge 6$. By Lemma 5.5, the irreducibility of $M_{d,g}(Q)$ and the equality $\dim(M_{d,g}(Q)) = 4d + 1 - g$ we may assume $h^1(\mathscr{I}_C(4)) > 0$.

Lemma 6.1. *Assume $d \le 11$ and fix a non-degenerate $C \in M_{d,g}$ such that there is no line $R \subset \mathbb{P}^5$ with $\deg(R \cap C) \ge 6$. Then $h^1(M, \mathscr{I}_{C\cap M,M}(4)) = 0$ for every hyperplane $M \subset \mathbb{P}^5$.*

Proof. Fix a hyperplane $M \subset \mathbb{P}^5$. Since C spans $\mathbb{P}^5$, $Z := C \cap M$ is a curvilinear scheme spanning M. Assume $h^1(M, \mathscr{I}_{Z,M}(4)) > 0$. Let N be a hyperplane of N with maximal $a := \deg(Z \cap N)$. Since Z spans M, we have $a \ge 4$. Assume for the moment $a = 4$, i.e. assume that Z is in linearly general position. Since $d \le 17$, we have $h^1(M, \mathscr{I}_{Z,M}(4)) = 0$ [22, Theorem 3.2]. Hence we may assume $a \ge 5$.

(a) First assume $h^1(N, \mathscr{I}_{Z\cap N,N}(4)) > 0$. Since Z spans M, we have $a \le d-1 \le 10$. The maximality property of N implies that $Z \cap N$ spans N. Hence $\deg(Z \cap U) \le 9$ for every plane $U \subset N$. Fix a plane $U \subset N$ with $b := \deg(Z \cap U)$ is maximal. If $h^1(U, \mathscr{I}_{Z\cap U,U}(4)) > 0$, then there is a line $R \subset U$ with $\deg(R \cap Z) \ge 6$. Hence we may assume $h^1(U, \mathscr{I}_{Z\cap U,U}(4)) = 0$. The residual sequence of U in N gives $h^1(N, \mathscr{I}_{\mathrm{Res}_U(Z\cap N),N}(3)) > 0$. We have $\deg(\mathrm{Res}_U(Z \cap N)) \le 10 - b \le 7$. By [23, Lemma 34] there is a line $L \subset N$ such that $\deg(L \cap \mathrm{Res}_U(Z)) \ge 5$. Hence $b \ge 6$. Hence $10 - b > \deg(L \cap \mathrm{Res}_U(Z))$, a contradiction.

(b) Now assume $h^1(N, \mathscr{I}_{Z\cap N}(4)) = 0$. The residual exact sequence

$$0 \to \mathscr{I}_{\mathrm{Res}_N(Z),M}(3) \to \mathscr{I}_{Z,M}(4) \to \mathscr{I}_{Z\cap N,N}(4) \to 0$$

gives $h^1(M, \mathscr{I}_{\mathrm{Res}_N(Z),M}(3)) > 0$. Since $d - a \le 7$, then there is a line $L \subset M$ such that $\deg(\mathrm{Res}_N(Z)) \ge 5$ [23, Lemma 34]. By assumption we have $\deg(L \cap Z) = 5$. Since $\deg(Z \cap L) \ge 5$, the maximality property of a gives $a \ge 7$. Since $d - a \ge 5$, we get $d \ge 12$, a contradiction. □

Lemma 6.2. *Assume $d \le 11$ and fix a non-degenerate $C \in M_{d,g}$ such that there is no line $R \subset \mathbb{P}^5$ with $\deg(R \cap C) \ge 5$, no conic $D \subset \mathbb{P}^5$ with $\deg(D \cap C) \ge 8$, no plane cubic T with $\deg(T \cap C) = 9$ and $C \cap T \in |\mathscr{O}_T(3)|$. Then $h^1(M, \mathscr{I}_{C\cap M,M}(3)) = 0$ for every hyperplane $M \subset \mathbb{P}^5$.*

Proof. Fix a hyperplane $M \subset \mathbb{P}^5$. Since C spans $\mathbb{P}^5$, $Z := C \cap M$ is a curvilinear scheme spanning M. Assume $h^1(M, \mathscr{I}_{Z,M}(3)) > 0$. Let N be a hyperplane of N with maximal $a := \deg(Z \cap N)$. Since Z spans M, we have $a \ge 4$. Assume for the moment $a = 4$, i.e. assume that Z is in linearly general position. Since $d \le 13$, we have $h^1(M, \mathscr{I}_{Z,M}(3)) = 0$ [22, Theorem 3.2]. Hence we may assume $a \ge 5$.

(a) First assume $h^1(N, \mathscr{I}_{Z\cap N,N}(3)) > 0$. Since Z spans M, we have $a \le d-1 \le 10$. The maximality property of N implies that $Z \cap N$ spans N. Hence $\deg(Z \cap U) \le 9$ for every plane $U \subset N$. Let $U \subset N$ be a plane such that $b := \deg(U \cap Z)$ is maximal. If $h^1(U, \mathscr{I}_{Z\cap U,U}(3)) > 0$, then [24, Corollaire 2] shows the existence of either R or D or T. Now assume $h^1(U, \mathscr{I}_{U\cap Z,U}(3)) = 0$. The residual sequence of U gives $h^1(N, \mathscr{I}_{\mathrm{Res}_U(N\cap Z),N}(2)) > 0$. Since $\deg(\mathrm{Res}_U(N \cap Z)) \le 10 - b \le 7$, either there is a line $L \subset N$ with $\deg(L \cap \mathrm{Res}_U(Z)) \ge 4$ or there is a conic $D \subset N$ with $\deg(D \cap Z) \ge 6$. The latter case is impossible, because it implies $a - b \ge 6$ and $b \ge 6$, a contradiction. Hence there is a line L with $\deg(L \cap \mathrm{Res}_U(Z)) \ge 4$. To prove the lemma we may assume $\deg(Z \cap L) = 4$. Let $E \subset N$ be a plane containing L and with maximal $c := \deg(E \cap Z)$ among the planes containing L. If $h^1(E, \mathscr{I}_{E\cap Z,E}(3)) > 0$, then [24, Corollaire 2] shows the existence of either R or D or T. Now assume $h^1(E, \mathscr{I}_{E\cap Z,E}(3)) = 0$. The residual sequence of E gives $h^1(N, \mathscr{I}_{\mathrm{Res}_E(Z\cap N),N}(2)) > 0$. Since $c \ge 5$, there is a line $R \subset N$ such that $\deg(R \cap \mathrm{Res}_U(Z \cap N)) \ge 4$. To prove the lemma we may assume that $\deg(R \cap Z) = 4$. First assume $R \cap L = \emptyset$. Let $Q' \subset N$ be a general quadric containing $L \cup R$. Note that Q' is a smooth quadric. Since Z is curvilinear and $\mathscr{I}_{L\cup R,N}(2)$ is spanned, we have $Z \cap Q' = Z \cap (R \cup L)$. Since $h^1(Q', \mathscr{I}_{Z\cap(L\cup R,Q'}(3)) = 0$, we get $h^1(N, \mathscr{I}_{\mathrm{Res}_{Q'}(Z\cap N),N}(1)) > 0$, contradicting the inequality $\deg(\mathrm{Res}_{Q'}(Z \cap N)) \le 2$.

Now assume $R \cap L \ne \emptyset$ and $R \ne L$. Since $\deg(R \cap \mathrm{Res}_E(Z \cap N)) \ge 4$ and $E \supset L$, we have $\deg(Z \cap (R \cup L)) \ge 8$ and so we may take $D := R \cup L$.

Now assume $R = L$. We may take $Z' \subseteq Z \cap N$ minimal among the subschemes such that $h^1(N, \mathscr{I}_{Z',M}(3)) > 0$. Let Q' be a quadric surface containing L in its singular locus. Since $\deg(\mathrm{Res}_{Q'}(Z')) \le 10 - 4 - 4 = 2$, we have $h^1(M, \mathscr{I}_{\mathrm{Res}_Q(Z')}(1)) = 0$. Therefore the residual exact sequence of Q' gives $h^1(Q', \mathscr{I}_{Z'\cap Q',Q'}(t)) > 0$. The minimality of Z' gives $Z' \subset Q$. Since Z' is curvilinear we get $\deg(Z') = 8$ and that each connected component γ of Z' has even degree with $\deg(\gamma \cap L) = \deg(\gamma)/2$. Hence there is a plane $N' \supset L$ with $\deg(N \cap Z') > \deg(Z' \cap L) = 4$. We get $\deg(\mathrm{Res}_{N'}(Z')) \le 3$ and hence by a residual exact sequence of N' gives $h^1(N, \mathscr{I}_{Z',M}(3)) = 0$, a contradiction.

(b) Now assume $h^1(N, \mathscr{I}_{Z\cap N}(3)) = 0$. A twist of the residual exact sequence in step (b) of the proof of Lemma 6.1 gives $h^1(M, \mathscr{I}_{\mathrm{Res}_N(Z),M}(2)) > 0$. If $d - a \le 5$, then there is a line $L \subset M$ such that $\deg(\mathrm{Res}_N(Z)) \ge 4$ [23, Lemma 34]. By assumption we have $\deg(L \cap Z) = 4$. Since $\deg(Z \cap L) \ge 4$, the maximality property of a gives $a \ge 6$. Since $d - a \ge 5$, we also get $d = 11$. Let $U \subset M$ be a hyperplane such that $U \supset L$ and $\deg(U \cap Z)$ is maximal. If $h^1(U, \mathscr{I}_{U\cap Z,U}(3)) > 0$, then we may repeat part (a). Now assume $h^1(U, \mathscr{I}_{U\cap Z,U}(3)) = 0$. The residual sequence of U gives $h^1(N, \mathscr{I}_{\mathrm{Res}_U(Z),N}(2)) > 0$. Since $\deg(\mathrm{Res}_E(Z)) \le 4$, there is a line $R \subset N$ with $R \supset \mathrm{Res}_E(Z)$ and $\deg(\mathrm{Res}_E(Z)) = 4$. We conclude as in step (a). □

Lemma 6.3. *Let $X \subset \mathbb{P}^5$ be an integral and non-degenerate curve of degree $d \le 13$. Then $h^1(H, \mathscr{I}_{C\cap H,H}(t)) = 0$, $t = 3,4$, for a general hyperplane $H \subset \mathbb{P}^5$.*

Proof. The scheme $C \cap H$ spans H and it is in uniform position and in particular it is in linearly general position. Apply [22, Theorem 3.2]. □

Lemma 6.4. *Let $X \subset \mathbb{P}^5$ be an integral and non-degenerate curve of degree $d \ge 9$ (resp. $5 \le d \le 8$). Then $h^0(\mathscr{I}_X(2)) \le 6$ (resp. $h^0(\mathscr{I}_X(2)) \le 15 - d$).*

Proof. Fix a general hyperplane $H \subset \mathbb{P}^5$. The scheme $S := X \cap H$ spans H and it is formed by d points in linearly general position in H. Hence $h^0(H, \mathscr{I}_{S,H}(2)) \le 6$ if $d \ge 9$ and $h^0(H, \mathscr{I}_{S,H}(2)) = 15 - d$ if $d \le 8$. Use the exact sequence

$$0 \to \mathscr{I}_X(1) \to \mathscr{I}_X(2) \to \mathscr{I}_{X\cap H,H}(2) \to 0$$

and that X is non-degenerate, i.e., $h^0(\mathscr{I}_X(1)) = 0$. □

Lemma 6.5. *Assume $g \le 3$ and $d \le 11$. There is no non-degenerate $C \in M_{d,g}$ such that $h^1(\mathscr{I}_C(4)) > 0$ and there is no line $L \subset \mathbb{P}^5$ with $\deg(L\cap C) \ge 5$,no conic D with $\deg(C \cap D) \ge 8$ and no plane cubic T with $\deg(T \cap C) = 9$ and $C \cap T \in |\mathscr{O}_T(3)|$.*

Proof. Since $h^1(\mathscr{I}_C(4)) > 0$ and $\deg(R\cap C) \le 5$ for all lines R, we have $d \ge 9$ [19, Theorem at page 492]. By Lemmas 4.10, 6.1 and 6.2 we have $h^1(\mathscr{I}_C(3)) \ge 5 + h^1(\mathscr{I}_C(4)) \ge 10 + h^1(\mathscr{I}_C(5)) \ge 11$. By Lemma 6.3 we have $h^1(\mathscr{I}_C(2)) \ge h^1(\mathscr{I}_C(3))$. Hence $h^0(\mathscr{I}_C(2)) \ge 31 + g - 2d$. Use Lemma 6.4. □

Lemma 6.6. *Fix an integer $a > 0$ and assume $d \ge 2g - 1 + a$. Fix a zero-dimensional curvilinear scheme $Z \subset \mathbb{P}^5$ such that $\deg(Z) = a$. Set $E_Z := \{C \in M_{d,g} : Z \subset C\}$. Then every irreducible component of E_Z has dimension $\le 6d + 2 - 2g - 4a$.*

Proof. If $E_Z = \emptyset$, then the lemma is true. Hence we may assume $E_Z \ne \emptyset$. Fix $C \in E_Z$. By [25, Theoreme 1.5] it is sufficient to prove that $h^1(N_C(-Z)) = 0$. Since C is smooth, N_C is a quotient of $T\mathbb{P}^5_{|C}$ and hence by the Euler's sequence of $T\mathbb{P}^5$ the bundle N_C is a quotient of $\mathscr{O}_C(1)^{\oplus 6}$. Since $d \ge 2g - 1 + a$, we have $h^1(\mathscr{O}_C(1)(-Z)) = 0$. Use that $h^2(\mathscr{G}) = 0$ for every coherent sheaf $\mathscr{G}$ on C. □

Corollary 6.7. *Assume $d \ge 9$. Fix $a \in \{4,5,6\}$. Let $\mathscr{A}_a$ be the set of all non-degenerate $C \in M_{d,g}$ such that there is a line $R \subset \mathbb{P}^5$ such that $\deg(C \cap R) \ge a$. Then every irreducible component of $\mathscr{A}_a$ has dimension $\le 6d + 2 - 2g + 8 - 3a$*

Proof. Fix a line $R \subset \mathbb{P}^5$ and a zero-dimensional scheme $Z \subset R$ with $\deg(Z) = a$. First apply Lemma 6.6, then use that R has ∞^a zero-dimensional schemes of degree a and then use that $\mathbb{P}^5$ contains ∞^8 lines. □

Lemma 6.8. *Assume $0 \le g \le 3$ and $d \le 11$. Let $\mathscr{B}$ be the set of all non-degenerate $C \in M_{d,g}$ having a line R with $\deg(R\cap C) \ge 6$. Then a general element of $\mathbb{W}$ contains no element of $\mathscr{B}$.*

Proof. Fix $C \in \mathscr{B}$. The existence of R implies $d \ge 9$ and that $d \ge 10$ if $g > 0$. By Corollary 6.7 to prove the lemma it is sufficient to avoid all $C \in \mathscr{B}$ with $h^1(\mathscr{I}_C(4)) \ge 10$. Since $d \le 11$, Lemma 6.3 and the exact sequence in the proof of Lemma 6.4 for $X = C$ and $t = 3,4$ give $h^1(\mathscr{I}_C(2)) \ge 10$. Hence $h^0(\mathscr{I}_C(2)) \ge 30 + g - 2d$, contradicting Lemma 6.4. □

Lemma 6.9. *Assume $0 \le g \le 3$ and $d \le 11$. Let $\mathscr{B}'$ be the set of all non-degenerate $C \in M_{d,g}$ having a line R with $\deg(R\cap C) \ge 4$. Then a general element of $\mathbb{W}$ contains no element of $\mathscr{B}'$.*

Proof. By Corollary 6.7 it is sufficient to test all $C \in M_{d,g}$ with $h^1(\mathscr{I}_C(4)) \ge 4$. By Lemma 6.8 we may assume that C has no line R with $\deg(R\cap C) \ge 6$. Hence Lemmas 4.10 and 6.1 give $h^1(\mathscr{I}_C(3)) \ge 5 + h^1(\mathscr{I}_C(4)) \ge 9$. By Lemma 6.3 and the exact sequence in the proof of Lemma 6.4 for $t = 3$ and $X = C$ we have $h^1(\mathscr{I}_C(2)) \ge 9$ and so $h^0(\mathscr{I}_C(2)) \ge 31 + g - 2d$. Lemma 6.4 gives a contradiction. □

Lemma 6.10. *Assume $0 \le g \le 3$ and $d \le 11$. Let $\mathscr{B}_1$ be the set of all non-degenerate $C \in M_{d,g}$ having a conic D with $\deg(D\cap C) \ge 8$. Then a general element of $\mathbb{W}$ contains no element of $\mathscr{B}_1$.*

Proof. Fix $C \in \mathscr{B}_1$, say associated to the conic D, and take $W \in \mathbb{W}$ containing C (if any). By Lemma 6.9 we may assume the non-existence of lines L with $\deg(L\cap C) \ge 4$. Hence D is not a reducible conic. It is not a double conic, say with $L := A_{\mathrm{red}}$, because we would have $\deg(L\cap C) \ge \deg(A\cap C)/2 \ge 4$. Hence D is smooth. By Lemma 4.9 it is sufficient to test the curves C with $h^1(\mathscr{I}_C(4)) \ge 10$. Lemmas 4.10 and 6.1 give $h^1(\mathscr{I}_C(3)) \ge 15$. Lemma 6.3 and the cohomology exact sequence of the the exact sequence in the proof of Lemma 6.4) for $X = C$ and $t = 3$ give $h^1(\mathscr{I}_C(2)) \ge 15$ and so $h^0(\mathscr{I}_C(2)) \ge 14 + g$, contradicting Lemma 6.4. □

Lemma 6.11. *Assume $0 \le g \le 3$ and $d \le 11$. Let $\mathscr{B}_2$ be the set of all non-degenerate $C \in M_{d,g}$ having a plane cubic T with $\deg(T \cap C) = 9$ and $C \cap T \in |\mathscr{O}_{C\cap T,T}(3)|$. Then a general element of $\mathbb{W}$ contains no element of $\mathscr{B}_2$.*

Proof. Take C for which T exists. We have $d = 11$. The set of all hyperplanes of $\mathbb{P}^5$ containing $\langle T\rangle$ induces a g^2_2 on C. Hence $g = 0$. Fix any scheme $Z \in |\mathscr{O}_T(3)|$. Since $g = 0$, Lemma 6.6 implies $h^1(N_C(-Z)) = 0$ and hence the set of all $C \subset \mathbb{P}^5$ containing Z has dimension $6d+1-4\deg(Z) = 31$. Since $\mathbb{P}^5$ has ∞^9 planes, each plane has ∞^9 plane cubics and each plane cubic T has ∞^9 elements of $|\mathscr{O}_T(3)|$, it is sufficient to exclude all $C \in \mathscr{B}_2$ with $h^1(\mathscr{I}_C(4)) \geq 9$. By Lemmas 6.9 and 6.10 we may assume the non-existence of line $R \subset \mathbb{P}^5$ with $\deg(C\cap R) \geq 4$ and of conics $D \subset \mathbb{P}^5$ with $\deg(C\cap D) \geq 8$. As in the proof Lemma 6.10 we get $h^1(\mathscr{I}_C(2)) \geq 14$, i.e .$h^0(\mathscr{I}_C(2)) \geq 13+g$, contradicting Lemma 6.4. □

By Lemma 5.5 at this point we proved that a general $W \in \mathbb{W}$ contains only finitely many non-degenerate $C \in M_{d,g}$.

7. Degenerate curves

In this section we prove that a general $W \in \mathbb{W}$ contains only finitely many degenerate $C \in M_{d,g}(Q)$, $d \leq 11$ and $g \leq 3$. By Remarks 4.3, 4.4 and Lemma 4.6 it is sufficient to test the curves $C \in M_{d,g}(4)$. By [19, Theorem at page 492] we may assume $d \geq 7$ and $d \geq 8$ if either $g > 0$ or C has genus 0 and no line R with $\deg(R\cap C) \geq 6$. By Remark 4.3 and Lemma 4.6 it is sufficient to test the degenerate $C \in M_{d,g}(Q)$. Fix a hyperplane $M \subset \mathbb{P}^5$ and set $Q' := Q\cap M$. Set $M'_{d,g}(Q') := \{C \in M_{d,g}(Q) : C \subset Q'$ and C spans $M\}$. Either Q' is smooth or Q' has a unique singular point, o. For any $C \in M'_{d,g}(Q')$ set $x(C) = 0$ if either Q' is smooth or Q' is a cone with vertex o and $o \notin C$, and set $x(C) := 1$ if Q' has vertex o and $o \in C$. Since $\omega_{Q'} \cong \mathscr{O}_{Q'}(-3)$, if $x(C) = 0$, then $\mathrm{Hilb}(Q')$ is smooth and of dimension $3d+2-2g$. Now assume that Q' is a cone with vertex o and that $x(C) = 1$, i.e. that $o \in C$. Let $u : \widetilde{Q}' \to Q'$ be the blowing up of o. Let $E := v^{-1}(o)$ be the exceptional divisor and let $\widetilde{C} \subset \widetilde{Q}'$ be the strict transform of C. Since C is smooth, v maps isomorphically $\widetilde{C}$. Let Ψ be closure in $\mathrm{Hilb}(\widetilde{Q}')$ of the strict transforms of all $A \in M_{d,g}(Q')$ with $x(A) = 1$. We claim that $\dim\Psi \leq 3d+1$. Fix $D \in \Psi$. Since $\mathrm{Aut}(\widetilde{Q}')$ acts transitively of $\widetilde{Q}'\setminus E$, the first part of the proof gives $h^1(N_{D,\widetilde{Q}}) = 0$. Hence it is sufficient to prove that $\deg(N_{D,\widetilde{Q}}) \leq 3d-1$, i.e. $\deg(\tau_{\widetilde{Q}|D}) \leq 3d+1$, i.e. $\deg(\omega_{\widetilde{Q}}|D) \geq -3d-1$. The group $\mathrm{Pic}(\widetilde{Q})$ is freely generated by E and the pull-back H of $\mathscr{O}_Q(1)$. We have $D\cdot H = d$ and $D\cdot E = x$. We have $\omega_{\widetilde{Q}} \cong \mathscr{O}_{\widetilde{Q}}(-3H-E)$ [26, Example 8.5 (2)]. Hence $\dim(M'_{d,g}(Q'))$ has dimension $\leq 3d+x(C)$ at C. Since Q has ∞^4 singular hyperplane sections and ∞^5 smooth hyperplane sections, to prove that a general $W \in \mathbb{W}$ has no (resp. finitely many) curves C spanning a hyperplane, it is sufficient to exclude the ones with $h^1(\mathscr{I}_C(4)) \geq d-4-g$. For all d,g for which we only use that $h^1(\mathscr{I}_C(4)) \geq d-5-g$, no degenerate $C \in M_{d,g}$ is contained in a general $W \in \mathbb{W}$. Fix a hyperplane $M \subset \mathbb{P}^5$. Let $M'_{d,g}(M)$ be the set of all $C \in M_{d,g}$ contained in M and spanning M.

Lemma 7.1. *A general $W \in \mathbb{W}$ contains no $C \in M_{d,g}$ such that there is a hyperplane M with $C \in M'_{d,g}(M)$ and $h^0(M,\mathscr{I}_C(2)) \geq 4$.*

Proof. Let $K \subset M$ denote the set-theoretic base locus of $|\mathscr{I}_{C,M}(2)|$ and A any irreducible component of K containing C. Note that $|\mathscr{I}_{C,M}(2)| = |\mathscr{I}_{A,M}(2)|$. Since C spans M, every element of $|\mathscr{I}_{C,M}(2)|$ is irreducible and A spans M. Hence $\dim(K) \leq 2$. First assume $\dim(A) = 2$. Since a complete intersection B of two quadrics of M has $h^0(M,\mathscr{I}_{B,M}(2)) = 2 < 4$ and A spans M, we get $\deg(A) = 3$. Hence either A is a smooth rational normal scroll or a cone over a rational normal curve of $\mathbb{P}^3$. In both cases we have $h^0(M,\mathscr{I}_{A,M}(2)) = 3$, a contradiction. Hence $\dim(A) = 1$, i.e. $A = C$. Fix two general elements Q_1, Q_2 of $|\mathscr{I}_{C,M}(2)|$ and let E be an irreducible component of $Q_1\cap Q_2$ containing C. Since $A = C$, there is a quadric hypersurface $Q_3 \subset M$, containing C, but not E. Since $C \subseteq E\cap Q_3$, we get $E = Q_1\cap Q_2$, $d \leq 8$, and that either $d = 8$ and $C = Q_1\cap Q_2\cap Q_3$ or $d = 7$ and C is linked to a line by the complete intersection $Q_1\cap Q_2\cap Q_3$. In both cases C is arithmetically Cohen-Macaulay and in particular $h^1(\mathscr{I}_C(4)) = 0$, a contradiction. □

Lemma 7.2. *A general $W \in \mathbb{W}$ contains no $C \in M_{11,g}$ such that there is a hyperplane M with $C \in M'_{11,g}(M)$ and $h^0(M,\mathscr{I}_{C,M}(2)) = 3$.*

Proof. Take K, A as in the proof of Lemma 7.1. Since $d > 8$, we only need to modify the proof of the case $\dim(A) = 2$. If $\dim(A) = 2$, then $\deg(A) = 3$ and A is either the cone of of a rational normal curves of $\mathbb{P}^3$ or it is a smooth rational normal curve isomorphic to the Hirzebruch surface F_1 embedded by the complete linear system $|h+2f|$. Write $C \in |ah+bf|$ with $a > 0$ and $b \geq a$. We have $11 = a+b$ and hence $b > a$. Since $\omega_{F_1} \cong \mathscr{O}_{F_1}(-2h-3f)$, the adjunction formula gives $2g-2 = (ah+bf)\cdot((a-2)h+(b-3)f) = -a(a-2)+a(b-3)+b(a-2) = (a-2)(b-a)+a(b-3)$. If $g = 0$ we get that either $a = 1$ (and hence $b = 10$) or $a = b = 2$, contradicting the equality $a+b = 10$. If $g > 0$, then $a \geq 2$. There is no solution with $a+b = 11$, $a \geq 2$, and $g \leq 3$. In the case $a = 1$ and $b = 10$ the curve C has $h^0(A,\mathscr{O}_A(4-C)) = 0$. Hence if $C \subset W$, then $A \subset W$, contradicting the fact that $\mathrm{Pic}(W)$ is generated by $\mathscr{O}_W(1)$.
Now assume that A is a cone over a rational normal curve. Let o be the vertex of A and call $u : F_2 \to A$ the blowing up of o. Set $h := u^{-1}(o)$. F_2 is isomorphic to the Hirzebruch surface with the same name, h is the only section of its ruling with negative self-intersection and u is induced by the linear system $|h+2f|$. We have $h^2 = -2$ and $\omega_{F_2} \cong \mathscr{O}_{F_2}(-2h-4f)$. Let $C' \subset F_2$ denote the strict transform of C, with $C' \in |ah+bf$ and $b \geq 2a$. Since C is smooth, u sends isomorphically C' to C. Hence $11 = b$ and $b \in \{2a, 2a+1\}$. Since $h^0(\mathscr{O}_{F_2}(4h+8f-C)) = 0$, any W containing C contains A, a contradiction. □

Lemma 7.3. *Fix $C \in M'_{d,g}(M)$, $d \leq 13$, and let H be a general hyperplane of M. We have $h^1(H,\mathscr{I}_{H\cap C,H}(4)) = 0$ and $h^1(H,\mathscr{I}_{H\cap C,H}(3)) \leq \max\{0, d-10\}$.*

Proof. Any $S \subseteq C \cap H$ with $\#(S) \le 10$ (resp. $\#(S) \le 13$) is in linearly general position in M and hence $h^(M, \mathscr{I}_{S,M}(3)) = 0$ (resp. $h^1(M, \mathscr{I}_{C,M}(4)) = 0$ by [22, Theorem 3.2]. □

Lemma 7.4. *Let $N \subset M$ be a hyperplane and let $Z \subset N$ be a degree $d \le 11$ zero-dimensional scheme spanning N. If there are neither a line $R \subset N$ with $\deg(R \cap Z) \ge 6$ nor a plane conic $D \subset N$ with $\deg(D \cap Z) = 10$, then $h^1(N, \mathscr{I}_{Z,N}(4)) = 0$.*

Proof. Let $U \subset N$ be a plane of N with maximal $a := \deg(Z \cap N)$. Since Z spans N, we have $a \ge 3$. Assume for the moment $a = 3$, i.e. assume that Z is in linearly general position. Since $d \le 13$, we have $h^1(N, \mathscr{I}_{Z,MN}(4)) = 0$ [22, Theorem 3.2]. Hence we may assume $a \ge 4$.
First assume $h^1(U, \mathscr{I}_{Z\cap U,U}(4)) > 0$. Since Z spans N, we have $a \le d-1 \le 10$. Use [24, Corollaire 2 or Remarques (i) at page 116].
Now assume $h^1(N, \mathscr{I}_{Z\cap N}(4)) = 0$. The residual exact sequence of U in N gives $h^1(N, \mathscr{I}_{\mathrm{Res}_U(Z)}(3)) > 0$. Since $\deg(\mathrm{Res}_U(Z)) = d-a \le 7$, [23, Lemma 34] gives the existence of a line $L \subset N$ such that $\deg(L \cap Z) \ge 5$. Then we continue as in step (a) of the proof of Lemma 6.2. the residual exact sequence of M gives $h^1(M, \mathscr{I}_{\mathrm{Res}_N(Z),M}(3)) > 0$. Since $d-a \le 7$, then there is a line $L \subset M$ such that $\deg(\mathrm{Res}_N(Z)) \ge 5$ [23, Lemma 34]. By assumption we have $\deg(L \cap Z) = 5$. Since $\deg(Z \cap L) \ge 5$, the maximality property of a gives $a \ge 7$. Since $d-a \ge 5$, we get $d \ge 12$, a contradiction. □

Lemma 7.5. *A general $W \in \mathbb{W}$ contains no $C \in M'_{d,g}(M)$ such that there a plane conic D with $\deg(D \cap C) \ge 10$ (if D is singular also assume that $\deg(L \cap C) \le 5$ for each line $L \subset D$).*

Proof. The pencil of hyperplanes of M containing the plane U spanned by D shows that $d = 11$, $\deg(D \cap C) = 10$, and $g = 0$. First assume that D is a double line. Fix $W \in \mathbb{W}$ with $W \supset C$. Set $L := D_{\mathrm{red}}$. Since $\deg(L \cap C)$, we have $L \subset W$ for any $W \in \mathbb{W}$ with $W \supset C$. Let $\mathrm{Res}_L(C \cap D)$ be the residual scheme with respect to the divisor L of U. Since $\deg(C \cap L) \ge \deg(C \cap D)/2$, our assumptions give $\deg(L \cap C) = 5$ and hence $\deg(\mathrm{Res}_L(C \cap D)) = 5$. Since $C \cap D \subset D$, we have $\mathrm{Res}_L(C \cap D) \subset L$. Since $D \nsubseteq W$ (Lemma 4.9), we have $W \cap U = L \cup T$ with T a plane cubic not containing L. Hence $\deg(L \cap T)$ =3. Since $\mathrm{Res}_L(C \cap D)$ is contained both in L and in T, we get a contradiction.
Now assume $D = R \cup L$ with R, L lines and $L \ne R$. Since $\deg(L \cap C) \le 5$ and $\deg(R \cap C) \le 5$ by assumption, we have $\deg(R \cap C) = \deg(R \cup L) = 5$. Hence $D \subset W$, contradicting Lemma 4.9.
Now assume that D is smooth. Since $g = 0$ for each $Z \subset D$ with $\deg(D) = 10$, we have $h^1(N_{C,M}(-Z)) = 0$ and so $h^0(N_{C,M}) = 45-30$. Since D has ∞^{10} degree 10 subschemes, M has ∞^6 planes, each plane has ∞^5 conics and $\mathbb{P}^5$ has ∞^5, hyperplanes, each irreducible component of the set of all (C, D, M) with D a smooth conic and $C\imath M'_{11,0}(M)$ has dimension at most 41, i.e. codimension at least 17 in $M_{11,0}$. Hence to avoid these curves we may assume $h^1(\mathscr{I}_C(4)) \ge 16$. Lemma 7.3 gives $h^1(M, \mathscr{I}_C(2)) \ge 15$. Hence $h^0(M, \mathscr{I}_C(2)) \ge 7$, contradicting Lemma 7.1. □

Lemma 7.6. *A general $W \in \mathbb{W}$ contains no $C \in M'_{d,g}(M)$, $d \le 11$, for some hyperplane M such that there is no line $R \subset M$ with $\deg(R \cap C) \ge 6$.*

Proof. By Lemma 7.5 we may assume that there is no conic D with $\deg(D \cap C) \ge 10$. Since $d \le 11$, Lemmas 4.10 and 7.4 give $h^1(M, \mathscr{I}_{C,M}(3)) \ge 4 + h^1(\mathscr{I}_{C\cap M,M}(3)) \ge d-g$. Assume for the moment that either $d \le 10$ or $d = 11$ and $h^1(H, \mathscr{I}_{C\cap H,H}(3)) = 0$ for a general hyperplane H of M. Lemma 7.3 gives $h^1(M, \mathscr{I}_{C,M}(2)) \ge d-g$ and so $h^0(M, \mathscr{I}_C(2)) \ge 15+d-g-2d-1+g = 14-d$. Hence if $d \le 10$ Lemma 7.1 concludes the proof. If $d = 11$ and $h^1(H, \mathscr{I}_{C\cap H,H}(3)) = 1$, we get $h^0(M, \mathscr{I}_C(2)) \ge 2$. Assume $h^0(\mathscr{I}_C(2)) = 2$ and let K be the intersection of two general elements of $|\mathscr{I}_{C,M}(2)|$ and call $A \subseteq K_{\mathrm{red}}$ any irreducible component containing C. Since $h^1(M, \mathscr{I}_{C,M}(3)) \ge 11-g$, we have $h^0(M, \mathscr{I}_C(3)) \ge 45-2d > 10$. Hence the map $H^0(M, \mathscr{I}_{C,M}(2)) \otimes H^0(\mathscr{O}_M(1)) \to H^0(M, \mathscr{I}_{C,M}(3))$ is not surjective. Take $U \in |\mathscr{I}_{C,M}(3)|$ not containing K. Since $\deg(C) > 9$, we first get $A = K$, and then (since $d = 11$), that the complete intersection $K \cap U$ links C to a line. Hence C is arithmetically Cohen-Macaulay, contradicting the assumption $h^1(M, \mathscr{I}_{C,M}(4)) > 0$. □

Lemma 7.7. *A general $W \in \mathbb{W}$ contains no curve C with $C \in M'_{d,g}(M)$ for some hyperplane and with a line R such that $\deg(R \cap C) \ge 6$.*

Proof. Note that if W, C, R are as in the statement of the lemma with $C \subset W$, then $R \subset W$ (Bezout). Let $\mathscr{G}$ be the set of all quadruples (W, H, L, C) with $W \in \mathbb{W}'$, M a hyperplane, $L \subset W \cap M$ a line, $C \in M'_{d,g}(M)$ and $\deg(L \cap C) \ge 6$. Fix M, a line $L \subset M$ and $Z \subset R$ with $\deg(Z) = 6$. First assume $d \ge 2g-1+6$. Lemma 6.6 gives $h^1(M, N_{C,M}(-Z)) = 0$, i.e. $h^0(N_{C,M}(-Z)) = 5d+1-g-18$. Since L has ∞^6 degree 6 zero-dimensional schemes, M has ∞^6 lines and $\mathbb{P}^5$ has ∞^5 hyperplanes, and each $W \in \mathbb{W}'$ contains only finitely many lines, we get that each irreducible component of $\mathscr{G}$ has dimension at most $5d-g$. Hence to prove the lemma it is sufficient to exclude the curves $C \in M'_{d,g}(M)$ with $h^1(\mathscr{I}_C(4)) \ge d-g+2$. Lemma 7.3 gives $h^1(M, \mathscr{I}_{C,M}(3)) \ge d-g+2$. Hence $h^1(M, \mathscr{I}_{C,M}(2)) \ge d-g+1$ (Lemma 7.3) and so $h^0(M, \mathscr{I}_{C,M}(2)) \ge 15-d \ge 4$, contradicting Lemma 7.1. Now assume $d \le 2g+4$. Since $d \ge 7$ and $g = 0$ if $d = 7$, then $(d,g) \in \{(8,2),(8,3),(9,3),(10,3)\}$. Assume $d = 8$. The net of all hyperplanes of M containing R induces a g^2_2 on C and hence $g = 0$, a contradiction. Now assume $(d,g) \in \{(9,3),(10,3)\}$. We take $Z' \subset R$ with $\deg(Z') = 4$. Since $d \ge 2g-1+\deg(Z')$, as above we get that we may assume $h^1(\mathscr{I}_C(4)) \ge d-g$. Since $d \le 10$, we have $h^1(M, \mathscr{I}_{C,M}(2)) \ge h^1(M, \mathscr{I}_{C,M}(3)) \ge h^1(M, \mathscr{I}_{C,M}(4))$ (Lemma 7.3) and hence $h^0(M, \mathscr{I}_{C,M}(2)) \ge 14-d \ge 4$, contradicting Lemma 7.1. □

End of the proof of Theorem 1.1: The last lemma concludes the proof of Theorem 1.1 for all $C \in M_{d,g}(4)$. Since in section 6 we checked all $C \in M_{d,g}(5)$, in Remark 4.3 all $C \in M_{d,g}(1)$, in Remark 4.4 all $C \in M_{d,g}(2)$ and in Lemma 4.6 all $C \in M_{d,g}(3)$, we have completed the proof of Theorem 1.1. □

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On Predictors and Estimators under a Constrained Partitioned Linear Model and its Reduced Models

Melek Eriş Büyükkaya[1*] **and Nesrin Güler**[2]

[1]*Department of Statistics and Computer Sciences, Faculty of Science, Karadeniz Technical University, Trabzon, Turkey*
[2]*Department of Econometrics, Faculty of Political Sciences, Sakarya University, Sakarya, Turkey*
[*]*Corresponding author*

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Abstract

In this study, we consider a partitioned linear model with linear partial parameter constrains, known as a constrained partitioned linear model (CPLM), and its reduced models. A group of formulas on best linear unbiased predictors (BLUPs) and best linear unbiased estimators (BLUEs) in CPLM is derived via some quadratic matrix optimization methods, and further many basic properties of the predictors and estimators are established under some general assumptions. Our main purpose is to derive various inequalities and equalities for the comparison of covariance matrices of BLUPs and BLUEs under CPLM and its reduced models.

1. Introduction and preliminary results

We first introduce the following notations. $\mathbf{A}'$, $r(\mathbf{A})$, $\mathscr{C}(\mathbf{A})$, and $\mathbf{A}^+$ denote, respectively, the transpose, the rank, the column space, and the Moore–Penrose generalized inverse of $\mathbf{A} \in \mathbb{R}^{m\times n}$, where $\mathbb{R}^{m\times n}$ stands for the set of all $m\times n$ real matrices. $\mathbf{E_A} = \mathbf{A}^{\perp} = \mathbf{I}_m - \mathbf{A}\mathbf{A}^+$ stands for the orthogonal projector, where $\mathbf{I}_m$ denotes the identity matrix of size $m\times m$. $i_+(\mathbf{A})$ and $i_-(\mathbf{A})$ denote the positive and the negative inertias of symmetric matrix $\mathbf{A}$, respectively, and for both $i_\pm(\mathbf{A})$ and $i_\mp(\mathbf{A})$ are used. The inequality $\mathbf{A}_1 - \mathbf{A}_2 \succcurlyeq \mathbf{0}$ or $\mathbf{A}_1 \succcurlyeq \mathbf{A}_2$ means that the difference $\mathbf{A}_1 - \mathbf{A}_2$ is positive semi-definite (psd) matrix in the Löwner partial ordering (LPO) for the symmetric matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ of same size, further, we use $\mathbf{A}_1 \prec \mathbf{A}_2$, $\mathbf{A}_1 \preccurlyeq \mathbf{A}_2$, and $\mathbf{A}_1 \succ \mathbf{A}_2$ in cases where the difference $\mathbf{A}_1 - \mathbf{A}_2$ is negative definite, negative semi-definite, and positive definite matrix, respectively.
As a linear model with its partitioned form, we consider

$$\mathscr{M} : \mathbf{y} = \mathbf{X}\alpha + \varepsilon = \begin{bmatrix}\mathbf{X}_1, & \mathbf{X}_2\end{bmatrix}\begin{bmatrix}\alpha_1', & \alpha_2'\end{bmatrix}' + \varepsilon = \mathbf{X}_1\alpha_1 + \mathbf{X}_2\alpha_2 + \varepsilon, \tag{1.1}$$

$$\mathrm{E}(\varepsilon) = \mathbf{0} \quad \text{and} \quad \mathrm{cov}(\varepsilon,\varepsilon) = \mathrm{D}(\varepsilon) = \sigma^2\Sigma, \tag{1.2}$$

and its reduced model

$$\mathscr{M}_R : \mathbf{X}_2^{\perp}\mathbf{y} = \mathbf{X}_2^{\perp}\mathbf{X}_1\alpha_1 + \mathbf{X}_2^{\perp}\varepsilon, \tag{1.3}$$

where $\mathbf{y} \in \mathbb{R}^{n\times 1}$ is a vector of observable response variables, $\mathbf{X} = \begin{bmatrix}\mathbf{X}_1, & \mathbf{X}_2\end{bmatrix} \in \mathbb{R}^{n\times k}$ is a known matrix of arbitrary rank with $\mathbf{X}_i \in \mathbb{R}^{n\times k_i}$, $\alpha = \begin{bmatrix}\alpha_1', & \alpha_2'\end{bmatrix}' \in \mathbb{R}^{k\times 1}$ is a vector of fixed but unknown parameters with $\alpha_i \in \mathbb{R}^{k_i\times 1}$, $\varepsilon \in \mathbb{R}^{n\times 1}$ is an unobservable vector of random errors, σ^2 is a positive unknown parameter, and $\Sigma \in \mathbb{R}^{n\times n}$ is a known psd matrix of arbitrary rank, $i = 1,2$,

Email addresses and ORCID numbers: **melekeris@ktu.edu.tr, 0000-0002-6207-5687 (M. E. Büyükkaya), nesring@sakarya.edu.tr, 0000-0003-3233-5377 (N. Güler)**

$k_1 + k_2 = k$. The reduced linear model $\mathscr{M}_R$ in (1.3), also known as the correctly-reduced model, is obtained by pre-multiplying $\mathbf{X}_2^{\perp}$ on both sides of the partitioned linear model $\mathscr{M}$ in (1.1); see, e.g., [1] and [2]. The model in (1.3) is one of the different forms of the model in (1.1) and, especially, this model can be considered when estimation/prediction problems in general parametric functions of partial parameters are considered.

In statistical theory and its applications, there often exist certain restrictions on unknown parameters in linear regression models. These kinds of restrictions occur in many situations such as the linear hypothesis testing on parameters. Let us considered the partitioned linear model in (1.1) with a certain restriction on α_1, known as constrained partitioned linear model (CPLM), as follows:

$$\mathscr{N} : \mathbf{y} = \mathbf{X}\alpha + \varepsilon = \mathbf{X}_1\alpha_1 + \mathbf{X}_2\alpha_2 + \varepsilon, \quad \mathbf{A}_1\alpha_1 = \mathbf{b}_1, \tag{1.4}$$

and its constrained reduced linear model (CRLM),

$$\mathscr{N}_R : \mathbf{X}_2^{\perp}\mathbf{y} = \mathbf{X}_2^{\perp}\mathbf{X}_1\alpha_1 + \mathbf{X}_2^{\perp}\varepsilon, \quad \mathbf{A}_1\alpha_1 = \mathbf{b}_1, \tag{1.5}$$

where the linear restriction equation $\mathbf{A}_1\alpha_1 = \mathbf{b}_1$ is consistent for given $\mathbf{A}_1 \in \mathbb{R}^{m\times k_1}$ of arbitrary rank and $\mathbf{b}_1 \in \mathbb{R}^{m\times 1}$. The two given equation parts in (1.4) and (1.5) can merge into the following combined form of vectors

$$\widehat{\mathscr{N}} : \widehat{\mathbf{y}} = \widehat{\mathbf{X}}\alpha + \widehat{\varepsilon} = \widehat{\mathbf{X}}_1\alpha_1 + \widehat{\mathbf{X}}_2\alpha_2 + \widehat{\varepsilon}, \tag{1.6}$$

$$\widehat{\mathscr{N}_R} : \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}} = \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1\alpha_1 + \widehat{\mathbf{X}}_2^{\perp}\widehat{\varepsilon}, \tag{1.7}$$

respectively, and according to the expectation and covariance matrix assumptions in (1.2),

$$\mathrm{E}(\widehat{\mathbf{y}}) = \widehat{\mathbf{X}}\alpha, \quad \mathrm{E}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}) = \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1\alpha_1, \quad \mathrm{D}(\widehat{\mathbf{y}}) = \mathrm{D}(\widehat{\varepsilon}) = \sigma^2\begin{bmatrix}\Sigma & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} := \widehat{\Sigma}, \quad \mathrm{D}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}) = \mathrm{D}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\varepsilon}) = \sigma^2\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} \tag{1.8}$$

are obtained, where

$$\widehat{\mathbf{y}} = \begin{bmatrix}\mathbf{y}\\ \mathbf{b}_1\end{bmatrix}, \quad \widehat{\mathbf{X}} = [\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2] = \begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2\\ \mathbf{A}_1 & \mathbf{0}\end{bmatrix}, \quad \widehat{\mathbf{X}}_1 = \begin{bmatrix}\mathbf{X}_1\\ \mathbf{A}_1\end{bmatrix}, \quad \widehat{\mathbf{X}}_2 = \begin{bmatrix}\mathbf{X}_2\\ \mathbf{0}\end{bmatrix}, \quad \widehat{\varepsilon} = \begin{bmatrix}\varepsilon\\ \mathbf{0}\end{bmatrix}, \quad \widehat{\mathbf{X}}_2^{\perp} = \begin{bmatrix}\mathbf{X}_2^{\perp}\\ \mathbf{0}\end{bmatrix}.$$

This merging operation in (1.6) and (1.7) is a well-known method of including equality restrictions in constrained linear regression models.

We make statistical inference of the models in (1.6) and (1.7) under the assumptions that the models are consistent, i.e., we assume that $\widehat{\mathbf{y}} \in \mathscr{C}\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right]$ holds with probability (wp) 1, corresponding the consistency of $\widehat{\mathscr{N}}$, in this case, the model $\widehat{\mathscr{N}_R}$ in (1.7) is consistent, i.e., $\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}} \in \mathscr{C}\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right]$ holds wp 1; see, e.g., [3].

To estimate the unknown parameter vector α_1 and to predict random error vector ε jointly in (1.4) and (1.5), we construct a general vector containing the both unknown vectors as follows

$$\phi_1 = \mathbf{K}_1\alpha_1 + \mathbf{H}\widehat{\varepsilon} = \left[\mathbf{K}_1, \mathbf{0}\right]\alpha + \mathbf{H}\widehat{\varepsilon} := \widehat{\mathbf{K}}\alpha + \mathbf{H}\widehat{\varepsilon} \tag{1.9}$$

for given matrices $\widehat{\mathbf{K}} = \left[\mathbf{K}_1, \mathbf{0}\right] \in \mathbb{R}^{s\times k}$ with $\mathbf{K}_1 \in \mathbb{R}^{s\times k_1}$ and $\mathbf{H} \in \mathbb{R}^{s\times(n+m)}$. It can be seen from the expectation and covariance matrix assumptions in (1.2) and (1.8),

$$\mathrm{E}(\phi_1) = \mathbf{K}_1\alpha_1, \quad \mathrm{D}(\phi_1) = \sigma^2\mathbf{H}\widehat{\Sigma}\mathbf{H}', \quad \mathrm{cov}(\phi_1, \widehat{\mathbf{y}}) = \sigma^2\mathbf{H}\widehat{\Sigma}, \quad \mathrm{cov}(\phi_1, \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}) = \sigma^2\mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}. \tag{1.10}$$

In the present paper, we concern with the problems of constrained prediction/estimation under a CPLM and its CRLMs. We first review some of the results related to the subject that we consider in the study including the consistency of CPLMs, predictability/estimability of ϕ_1 in (1.9), the best linear unbiased predictors (BLUPs), and the best linear unbiased estimators (BLUEs). We show how to establish the BLUPs and the BLUEs of all unknown vectors in a CPLM and its CRLMs and present some fundamental properties of the BLUPs/BLUEs by solving certain constrained quadratic matrix-valued function optimization problems in LPO including ranks and inertias of block matrices. Our main purpose is to derive various inequalities and equalities for comparison of covariance matrices of the BLUPs/BLUEs of all unknown vectors in the CPLM and its CRLMs. Previous and recent work on the problems of the inference of CPLMs can be found in; see e.g., [4]-[18] among others.

The results, in the present paper, are established by making use of formulas of ranks of block matrices and elementary matrix operations. We review well-known results, which we need later, related to block matrices as follows.

Lemma 1.1 ([19]). *Let* $\mathbf{A}_1$, $\mathbf{A}_2 \in \mathbb{R}^{m\times n}$, *or, let* $\mathbf{A}_1 = \mathbf{A}_1'$, $\mathbf{A}_2 = \mathbf{A}_2' \in \mathbb{R}^{m\times m}$. *Then,*

1. $\mathbf{A}_1 = \mathbf{A}_2 \Leftrightarrow r(\mathbf{A}_1 - \mathbf{A}_2) = 0$.
2. $\mathbf{A}_1 \succ \mathbf{A}_2 \Leftrightarrow i_+(\mathbf{A}_1 - \mathbf{A}_2) = m$ *and* $\mathbf{A}_1 \prec \mathbf{A}_2 \Leftrightarrow i_-(\mathbf{A}_1 - \mathbf{A}_2) = m$.
3. $\mathbf{A}_1 \succcurlyeq \mathbf{A}_2 \Leftrightarrow i_-(\mathbf{A}_1 - \mathbf{A}_2) = 0$ *and* $\mathbf{A}_1 \preccurlyeq \mathbf{A}_2 \Leftrightarrow i_+(\mathbf{A}_1 - \mathbf{A}_2) = 0$.

Lemma 1.2 ([19]). *Let* $\mathbf{A}_1 = \mathbf{A}_1' \in \mathbb{R}^{m\times m}$*,* $\mathbf{A}_2 = \mathbf{A}_2' \in \mathbb{R}^{n\times n}$*,* $\mathbf{P} \in \mathbb{R}^{m\times n}$*, and* $c \in \mathbb{R}$*. Then,*

$$r(\mathbf{A}_1) = i_+(\mathbf{A}_1) + i_-(\mathbf{A}_1).$$

$$i_\pm(c\mathbf{A}_1) = \begin{cases} i_\pm(\mathbf{A}_1) & if\ \ c>0 \\ i_\mp(\mathbf{A}_1) & if\ \ c<0 \end{cases}.$$

$$i_\pm\begin{bmatrix}\mathbf{A}_1 & \mathbf{P}\\ \mathbf{P}' & \mathbf{A}_2\end{bmatrix} = i_\pm\begin{bmatrix}\mathbf{A}_1 & -\mathbf{P}\\ -\mathbf{P}' & \mathbf{A}_2\end{bmatrix} = i_\mp\begin{bmatrix}-\mathbf{A}_1 & \mathbf{P}\\ \mathbf{P}' & -\mathbf{A}_2\end{bmatrix}.$$

$$i_\pm\begin{bmatrix}\mathbf{A}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_2\end{bmatrix} = i_\pm(\mathbf{A}_1) + i_\pm(\mathbf{A}_2).\ \ i_+\begin{bmatrix}\mathbf{0} & \mathbf{P}\\ \mathbf{P}' & \mathbf{0}\end{bmatrix} = i_-\begin{bmatrix}\mathbf{0} & \mathbf{P}\\ \mathbf{P}' & \mathbf{0}\end{bmatrix} = r(\mathbf{P}).$$

Lemma 1.3 ([19]). *Let* $\mathbf{A}_1 = \mathbf{A}_1' \in \mathbb{R}^{m\times m}$*,* $\mathbf{B} = \mathbf{B}' \in \mathbb{R}^{n\times n}$*, and* $\mathbf{A}_2 \in \mathbb{R}^{m\times n}$*. Then,*

$$i_\pm\begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_2' & \mathbf{0}\end{bmatrix} = r(\mathbf{A}_2) + i_\pm(\mathbf{E}_{\mathbf{A}_2}\mathbf{A}_1\mathbf{E}_{\mathbf{A}_2}). \tag{1.11}$$

$$i_+\begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_2' & \mathbf{0}\end{bmatrix} = r\begin{bmatrix}\mathbf{A}_1, & \mathbf{A}_2\end{bmatrix}\ \ and\ \ i_-\begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_2' & \mathbf{0}\end{bmatrix} = r(\mathbf{A}_2)\ \ if\ \ \mathbf{A}_1 \succcurlyeq \mathbf{0}. \tag{1.12}$$

$$i_\pm\begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_2' & \mathbf{B}\end{bmatrix} = i_\pm(\mathbf{A}_1) + i_\pm(\mathbf{B} - \mathbf{A}_2'\mathbf{A}_1^+\mathbf{A}_2)\ \ if\ \ \mathscr{C}(\mathbf{A}_2) \subseteq \mathscr{C}(\mathbf{A}_1). \tag{1.13}$$

Lemma 1.4 ([20]). *Let* $\mathbf{Q} \in \mathbb{R}^{n\times n}$ *be a symmetric psd matrix. Assume that there exists* $\mathbf{X}_0 \in \mathbb{R}^{m\times n}$ *such that* $\mathbf{X}_0\mathbf{A} = \mathbf{B}$ *for given* $\mathbf{A} \in \mathbb{R}^{n\times p}$*,* $\mathbf{B} \in \mathbb{R}^{m\times p}$*. Then the maximal positive inertia of* $\mathbf{X}_0\mathbf{Q}\mathbf{X}_0' - \mathbf{X}\mathbf{Q}\mathbf{X}'$ *subject to all solutions of* $\mathbf{X}\mathbf{A} = \mathbf{B}$ *is*

$$\max_{\mathbf{XA}=\mathbf{B}} i_+(\mathbf{X}_0\mathbf{Q}\mathbf{X}_0' - \mathbf{X}\mathbf{Q}\mathbf{X}') = r\begin{bmatrix}\mathbf{X}_0\mathbf{Q}\\ \mathbf{A}'\end{bmatrix} - r(\mathbf{A}) = r(\mathbf{X}_0\mathbf{Q}\mathbf{A}^\perp). \tag{1.14}$$

Hence there exists solution $\mathbf{X}_0$ *of* $\mathbf{X}_0\mathbf{A} = \mathbf{B}$ *such that holds for all solutions of* $\mathbf{XA} = \mathbf{B} \Leftrightarrow \mathbf{X}_0$ *satisfies both* $\mathbf{X}_0\mathbf{A} = \mathbf{B}$ *and* $\mathbf{X}_0\mathbf{Q}\mathbf{A}^\perp = \mathbf{0}$*.*

2. BLUPs/BLUEs' computations

A group of computational formulas on the BLUPs/BLUEs of all unknown vectors in CPLM and its CRLMs are given with many basic properties of BLUPs/BLUEs by using quadratic matrix optimization methods given as in Lemma 1.4. Under our considerations, firstly, we review the predictability/estimability requirement of ϕ_1 and its special cases under the models (1.6) and (1.7) before giving the definition of the BLUPs/BLUEs.

1. ϕ_1 in (1.9) is predictable by $\widehat{\mathbf{y}}$ under $\widehat{\mathscr{N}}$ in (1.6), i.e., $\mathrm{E}(\mathbf{L}\widehat{\mathbf{y}} - \phi_1) = \mathbf{0}$ holds for some $\mathbf{L} \Leftrightarrow \mathscr{C}(\widehat{\mathbf{K}}') \subseteq \mathscr{C}(\widehat{\mathbf{X}}') \Leftrightarrow \widehat{\mathbf{K}}\alpha$ is estimable under (1.6), i.e., $\mathbf{K}_1\alpha_1$ is estimable under (1.6),
2. $\widehat{\mathbf{X}}\alpha$ is always estimable and $\widehat{\varepsilon}$ is always predictable under (1.6),
3. ϕ_1 in (1.9) is predictable by $\widehat{\mathbf{X}}_2^\perp\widehat{\mathbf{y}}$ under $\widehat{\mathscr{N}_R}$ in (1.7), i.e., $\mathrm{E}(\mathbf{G}\widehat{\mathbf{X}}_2^\perp\widehat{\mathbf{y}} - \phi_1) = \mathbf{0}$ holds for some $\mathbf{G} \Leftrightarrow \mathscr{C}(\mathbf{K}_1') \subseteq \mathscr{C}(\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^\perp)$ $\Leftrightarrow \mathbf{K}_1\alpha_1$ is estimable under (1.7),
4. $\widehat{\mathbf{X}}_1\alpha_1$ is estimable under (1.7) $\Leftrightarrow \mathscr{C}(\widehat{\mathbf{X}}_1') \subseteq \mathscr{C}(\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^\perp)$,
5. $\widehat{\mathbf{X}}_2^\perp\widehat{\mathbf{X}}_1\alpha_1$ is always estimable and $\widehat{\mathbf{X}}_2^\perp\widehat{\varepsilon}$ is always predictable under (1.7),
6. α_1 is estimable under (1.7) $\Leftrightarrow r(\widehat{\mathbf{X}}_2^\perp\widehat{\mathbf{X}}_1) = k_1$ and $\widehat{\varepsilon}$ is always predictable under (1.7);

see, e.g., [21]. Further, ϕ_1 is predictable under $\widehat{\mathscr{N}}$ when it is predictable under $\widehat{\mathscr{N}_R}$.

Definition 2.1 ([22],[23]). *The* BLUP/BLUE *definitions for models in* (1.6) *and* (1.7) *are given as follows, respectively.*

1. *Let* ϕ_1 *be predictable by* $\widehat{\mathbf{y}}$ *in* (1.6)*. If there exists* $\mathbf{L}\widehat{\mathbf{y}}$ *such that*

$$\mathrm{D}(\mathbf{L}\widehat{\mathbf{y}} - \phi_1) = \ min\ s.t.\ \ \mathrm{E}(\mathbf{L}\widehat{\mathbf{y}} - \phi_1) = \mathbf{0} \tag{2.1}$$

holds in the LPO*, the linear statistic* $\mathbf{L}\widehat{\mathbf{y}}$ *is defined to be the* BLUP *of* ϕ_1 *and is denoted by* $\mathbf{L}\widehat{\mathbf{y}} = \mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1) = \mathrm{BLUP}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{K}}\alpha + \mathbf{H}\widehat{\varepsilon})$*. If* $\mathbf{H} = \mathbf{0}$ *in* ϕ_1 *or* $\widehat{\mathbf{K}} = \mathbf{0}$ *in* ϕ_1*,* $\mathbf{L}\widehat{\mathbf{y}}$ *corresponds the* BLUE *of* $\widehat{\mathbf{K}}\alpha$*, denoted by* $\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{K}}\alpha)$ *and* BLUP *of* $\mathbf{H}\widehat{\varepsilon}$*, denoted by* $\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\mathbf{H}\widehat{\varepsilon})$*, under* (1.6)*.*

2. *Let ϕ_1 be predictable by $\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}$ in (1.7). If there exists $\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}$ such that*

$$\mathrm{D}(\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}-\phi_1)=\ \mathit{min\ s.t.}\ \mathrm{E}(\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}-\phi_1)=\mathbf{0}$$

holds in the LPO*, the linear statistic $\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}$ is defined to be the* BLUP *of ϕ_1 and is denoted by $\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}=\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)=\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1+\mathbf{H}\widehat{\varepsilon})$. If $\mathbf{H}=\mathbf{0}$ in ϕ_1 or $\mathbf{K}_1=\mathbf{0}$ in ϕ_1, $\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}$ corresponds the* BLUE *of $\mathbf{K}_1\alpha_1$, denoted by $\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)$ and* BLUP *of $\mathbf{H}\widehat{\varepsilon}$, denoted by $\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\mathbf{H}\widehat{\varepsilon})$, under (1.7).*

The fundamental results on BLUP of ϕ_1 under (1.6) and (1.7) are collected in the following theorems. The results given below are obtained from [24] by considering the models and notation used in this paper. For different approaches; see, e.g, [23], [25].

Theorem 2.2. *Let ϕ_1 be predictable by $\widehat{\mathbf{y}}$ in (1.6). Then,*

$$\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)=\mathbf{L}\widehat{\mathbf{y}}=\left(\begin{bmatrix}\widehat{\mathbf{K}} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix}\mathbf{W}_1^{+}+\mathbf{P}_1\mathbf{W}_1^{\perp}\right)\widehat{\mathbf{y}}, \tag{2.2}$$

where $\mathbf{P}_1\in\mathbb{R}^{s\times(n+m)}$ is an arbitrary matrix and $\mathbf{W}_1=\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix}$. In particular,

1. *$\mathbf{L}$ is unique $\Leftrightarrow r(\mathbf{W}_1)=(n+m)$.*
2. *$\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)$ is unique wp 1 $\Leftrightarrow\widehat{\mathscr{N}}$ is consistent.*
3. *$r(\mathbf{W}_1)=r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix}=r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\mathbf{X}}^{\perp}\widehat{\Sigma}\end{bmatrix}$.*
4. *Further, the following dispersion matrix equalities hold.*

$$\mathrm{D}[\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)]=\sigma^2\begin{bmatrix}\widehat{\mathbf{K}}, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix}\mathbf{W}_1^{+}\widehat{\Sigma}\left(\begin{bmatrix}\widehat{\mathbf{K}}, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix}\mathbf{W}_1^{+}\right)', \tag{2.3}$$

$$\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)]=\sigma^2\left(\begin{bmatrix}\widehat{\mathbf{K}}, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix}\mathbf{W}_1^{+}-\mathbf{H}\right)\widehat{\Sigma}\left(\begin{bmatrix}\widehat{\mathbf{K}}, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix}\mathbf{W}_1^{+}-\mathbf{H}\right)'. \tag{2.4}$$

5. *In particular,*

$$\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{K}}\alpha)=\left(\begin{bmatrix}\widehat{\mathbf{K}}, & \mathbf{0}\end{bmatrix}\mathbf{W}_1^{+}+\mathbf{P}_2\mathbf{W}_1^{\perp}\right)\widehat{\mathbf{y}}, \tag{2.5}$$

$$\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\mathbf{H}\widehat{\varepsilon})=\left(\begin{bmatrix}\mathbf{0}, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix}\mathbf{W}_1^{+}+\mathbf{P}_3\mathbf{W}_1^{\perp}\right)\widehat{\mathbf{y}}, \tag{2.6}$$

where $\mathbf{P}_2$ and $\mathbf{P}_3\in\mathbb{R}^{s\times(n+m)}$ are arbitrary matrices.

Proof. Let $\mathbf{L}\widehat{\mathbf{y}}$ be an unbiased linear predictor for ϕ_1 under the model in (1.6). Then,

$$\mathrm{E}(\mathbf{L}\widehat{\mathbf{y}}-\phi_1)=\mathbf{0}\Leftrightarrow\mathbf{L}\widehat{\mathbf{X}}=\widehat{\mathbf{K}},\ \text{i.e.,}\ \begin{bmatrix}\mathbf{L}, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\widehat{\mathbf{X}}\\ \widehat{\mathbf{K}}\end{bmatrix}=\mathbf{0}, \tag{2.7}$$

$$\mathrm{D}(\mathbf{L}\widehat{\mathbf{y}}-\phi_1)=\sigma^2(\mathbf{L}-\mathbf{H})\widehat{\Sigma}(\mathbf{L}-\mathbf{H})'=\sigma^2\begin{bmatrix}\mathbf{L}, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}\widehat{\Sigma}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}'\begin{bmatrix}\mathbf{L}, & -\mathbf{I}_s\end{bmatrix}' \tag{2.8}$$

for unbiased linear predictor $\mathbf{L}\widehat{\mathbf{y}}$. The similar expressions can be written for the other unbiased linear predictor $\mathbf{T}\widehat{\mathbf{y}}$ for ϕ_1 under the model in (1.6) by writing $\mathbf{T}$ instead of $\mathbf{L}$ in (2.7) and (2.8). Then the expression in (2.1) can be expressed as to find solution $\mathbf{L}$ of the consistent linear matrix equation $\mathbf{L}\widehat{\mathbf{X}}=\widehat{\mathbf{K}}$ such that $\mathrm{D}(\mathbf{L}\widehat{\mathbf{y}}-\phi_1)\preccurlyeq\mathrm{D}(\mathbf{T}\widehat{\mathbf{y}}-\phi_1)$ s.t. $\mathbf{T}\widehat{\mathbf{X}}=\widehat{\mathbf{K}}$, i.e.,

$$\begin{bmatrix}\mathbf{L}, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}\widehat{\Sigma}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}'\begin{bmatrix}\mathbf{L}, & -\mathbf{I}_s\end{bmatrix}'\preccurlyeq\begin{bmatrix}\mathbf{T}, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}\widehat{\Sigma}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}'\begin{bmatrix}\mathbf{T}, & -\mathbf{I}_s\end{bmatrix}'\quad\text{s.t.}\quad\mathbf{T}\widehat{\mathbf{X}}=\widehat{\mathbf{K}}. \tag{2.9}$$

Applying (1.14) to (2.9), the maximal positive inertia of $\mathrm{D}(\mathbf{L}\widehat{\mathbf{y}}-\phi_1)-\mathrm{D}(\mathbf{T}\widehat{\mathbf{y}}-\phi_1)$ subject to $\mathbf{T}\widehat{\mathbf{X}}=\widehat{\mathbf{K}}$ is obtained as follows:

$$\begin{aligned}\max_{\mathrm{E}(\mathbf{T}\widehat{\mathbf{y}}-\phi_1)=\mathbf{0}} i_{+}(\mathrm{D}(\mathbf{L}\widehat{\mathbf{y}}-\phi_1)-\mathrm{D}(\mathbf{T}\widehat{\mathbf{y}}-\phi_1)) &= r\begin{bmatrix}\begin{bmatrix}\mathbf{L}, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}\widehat{\Sigma}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}' \\ \begin{bmatrix}\widehat{\mathbf{X}}\\ \widehat{\mathbf{K}}\end{bmatrix}'\end{bmatrix}-r\begin{bmatrix}\widehat{\mathbf{X}}\\ \widehat{\mathbf{K}}\end{bmatrix} \\ &= r\left(\begin{bmatrix}\mathbf{L}, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}\widehat{\Sigma}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}'\begin{bmatrix}\widehat{\mathbf{X}}\\ \widehat{\mathbf{K}}\end{bmatrix}^{\perp}\right).\end{aligned} \tag{2.10}$$

Combining (2.7) with (2.10), we conclude that $\mathrm{D}(\mathbf{L}\widehat{\mathbf{y}} - \phi_1) = \min \Leftrightarrow$ there exists $\mathbf{L}$ satisfying both

$$\mathbf{L}\widehat{\mathbf{X}} = \widehat{\mathbf{K}} \text{ and } \begin{bmatrix}\mathbf{L}, & -\mathbf{I}_s\end{bmatrix} \begin{bmatrix}\mathbf{I}_{n+m} \\ \mathbf{H}\end{bmatrix} \widehat{\Sigma} \begin{bmatrix}\mathbf{I}_{n+m} \\ \mathbf{H}\end{bmatrix}' \begin{bmatrix}\widehat{\mathbf{X}} \\ \widehat{\mathbf{K}}\end{bmatrix}^{\perp} = \mathbf{0},$$

i.e., $\mathbf{L}\widehat{\mathbf{y}} = \mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1) \Leftrightarrow \mathbf{L}\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix} = \begin{bmatrix}\widehat{\mathbf{K}} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix}$. This matrix equation is consistent and the general solution of the equation can be written as in (2.2); see, e.g., [26]. Results in items 1 and 2 follow from (2.2). For the result in item 3, we refer [27, Lemma 2.1(a)]. (2.3) is seen from (2.2) and the assumptions in (1.2). Further,

$$\mathrm{cov}\{\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1), \phi_1\} = \begin{bmatrix}\widehat{\mathbf{K}} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix}\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix}^{+}\widehat{\Sigma}\mathbf{H}' \tag{2.11}$$

by using (1.8) and (1.10). (2.4) is seen from (2.3) and (2.11). (2.5) and (2.6) follow directly from (2.2). □

Theorem 2.3. *Let ϕ_1 be predictable by $\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}$ in (1.7). Then,*

$$\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1) = \mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}} = \left(\begin{bmatrix}\mathbf{K}_1, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}\mathbf{W}_2^{+} + \mathbf{P}_4\mathbf{W}_2^{\perp}\right)\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}, \tag{2.12}$$

where $\mathbf{P}_4 \in \mathbb{R}^{s\times(n+m)}$ is an arbitrary matrix and $\mathbf{W}_2 = \begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}$. In particular,

1. *$\mathbf{G}$ is unique $\Leftrightarrow r(\mathbf{W}_2) = (n+m)$.*
2. *$\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)$ is unique wp 1 $\Leftrightarrow \widehat{\mathscr{N}_R}$ is consistent.*
3. *$r(\mathbf{W}_2) = r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & (\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} = r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix}$.*
4. *The following dispersion matrix equalities hold.*

$$\mathrm{D}[\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)] = \sigma^2\begin{bmatrix}\mathbf{K}_1, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}\mathbf{W}_2^{+}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\left(\begin{bmatrix}\mathbf{K}_1, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}\mathbf{W}_2^{+}\right)',$$

$$\mathrm{D}[\phi_1 - \mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)] = \sigma^2\left(\begin{bmatrix}\mathbf{K}_1, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}\mathbf{W}_2^{+}\widehat{\mathbf{X}}_2^{\perp} - \mathbf{H}\right)\widehat{\Sigma}\left(\begin{bmatrix}\mathbf{K}_1, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}\mathbf{W}_2^{+}\widehat{\mathbf{X}}_2^{\perp} - \mathbf{H}\right)'. \tag{2.13}$$

5. *In particular,*

$$\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1) = \mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}} = \left(\begin{bmatrix}\mathbf{K}_1, & \mathbf{0}\end{bmatrix}\mathbf{W}_2^{+} + \mathbf{P}_5\mathbf{W}_2^{\perp}\right)\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}},$$

$$\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1) = \mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}} = \left(\begin{bmatrix}\mathbf{0}, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}\mathbf{W}_2^{+} + \mathbf{P}_6\mathbf{W}_2^{\perp}\right)\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}},$$

where $\mathbf{P}_5$ and $\mathbf{P}_6 \in \mathbb{R}^{s\times(n+m)}$ are arbitrary matrices.

Proof. The proof of the theorem is obtained in a similar way to the proof of the Theorem 2.3. □

3. Main results

Theorem 3.1. *Let consider models $\widehat{\mathscr{N}}$ and $\widehat{\mathscr{N}_R}$ in (1.6) and (1.7), respectively, and assume that ϕ_1 is predictable under these models. Let $\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)$ and $\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)$ be as given in (2.2) and (2.12), and*

$$\mathbf{A} = \begin{bmatrix}\widehat{\Sigma} & \widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\Sigma}\mathbf{H}' & \mathbf{0} & \widehat{\mathbf{X}} \\ \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & \mathbf{0} \\ \mathbf{H}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{K}_1 & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}.$$

Then

$$i_{+}(\mathrm{D}[\phi_1 - \mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)] - \mathrm{D}[\phi_1 - \mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)]) = i_{+}(\mathbf{A}) - r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} - r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1), \tag{3.1}$$

$$i_{-}(\mathrm{D}[\phi_1 - \mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)] - \mathrm{D}[\phi_1 - \mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)]) = i_{-}(\mathbf{A}) - r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} - r(\widehat{\mathbf{X}}), \tag{3.2}$$

$$r(\mathrm{D}[\phi_1 - \mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)] - \mathrm{D}[\phi_1 - \mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)]) = r(\mathbf{A}) - r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} - r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1) - r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} - r(\widehat{\mathbf{X}}). \tag{3.3}$$

Further,

1. $\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)] \succ \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)] \Leftrightarrow i_+(\mathbf{A}) = r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right] + r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)+s.$
2. $\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)] \prec \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)] \Leftrightarrow i_-(\mathbf{A}) = r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] + r(\widehat{\mathbf{X}})+s.$
3. $\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)] \succcurlyeq \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)] \Leftrightarrow i_-(\mathbf{A}) = r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] + r(\widehat{\mathbf{X}}).$
4. $\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)] \preccurlyeq \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)] \Leftrightarrow i_+(\mathbf{A}) = r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right] + r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1).$
5. $\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)] = \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)] \Leftrightarrow r(\mathbf{A}) = r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right] + r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1) + r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] + r(\widehat{\mathbf{X}}).$

Proof. Let $\mathbf{D} = \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)]$. By using (2.13) and (1.13),

$$
\begin{aligned}
& i_{\pm}\left(\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)]-\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)]\right) \\
& = i_{\pm}\left(\mathbf{D}-\left(\left[\mathbf{K}_1, \quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]\mathbf{W}_2^{+}\widehat{\mathbf{X}}_2^{\perp}-\mathbf{H}\right)\widehat{\Sigma}\left(\left[\mathbf{K}_1, \quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]\mathbf{W}_2^{+}\widehat{\mathbf{X}}_2^{\perp}-\mathbf{H}\right)'\right) \\
& = i_{\pm}\begin{bmatrix} \widehat{\Sigma} & \widehat{\Sigma}\left(\left[\mathbf{K}_1, \quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]\mathbf{W}_2^{+}\widehat{\mathbf{X}}_2^{\perp}-\mathbf{H}\right)' \\ \left(\left[\mathbf{K}_1, \quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]\mathbf{W}_2^{+}\widehat{\mathbf{X}}_2^{\perp}-\mathbf{H}\right)\widehat{\Sigma} & \mathbf{D} \end{bmatrix} - i_{\pm}(\widehat{\Sigma}) \\
& = i_{\pm}\left(\begin{bmatrix} \widehat{\Sigma} & -\widehat{\Sigma}\mathbf{H}' \\ -\mathbf{H}\widehat{\Sigma} & \mathbf{D} \end{bmatrix} + \begin{bmatrix} \widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} \\ \mathbf{0} & \left[\mathbf{K}_1, \quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right] \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{W}_2 \\ \mathbf{W}_2' & \mathbf{0} \end{bmatrix}^{+} \begin{bmatrix} \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma} & \mathbf{0} \\ \mathbf{0} & \left[\mathbf{K}_1, \quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]' \end{bmatrix}\right) \\
& \quad - i_{\pm}(\widehat{\Sigma})
\end{aligned} \tag{3.4}
$$

is obtained, where $\mathbf{W}_2 = \left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]$. We can apply (1.13) to (3.4) since the column space inclusions $\mathscr{C}(\widehat{\Sigma}) \subseteq \mathscr{C}(\mathbf{W}_2)$ and $\mathscr{C}\left(\left[\mathbf{K}_1, \quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]'\right) \subseteq \mathscr{C}(\mathbf{W}_2')$ hold. Then (3.4) is equivalently written as follows

$$
\begin{aligned}
& i_{\pm}\begin{bmatrix} \mathbf{0} & -\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & -\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma} & \mathbf{0} \\ -\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_1' \\ -(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' \\ \widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & \widehat{\Sigma} & -\widehat{\Sigma}\mathbf{H}' \\ \mathbf{0} & \mathbf{K}_1 & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp} & -\mathbf{H}\widehat{\Sigma} & \mathbf{D} \end{bmatrix} \\
& \quad - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right] - i_{\pm}(\widehat{\Sigma}) \\
& = i_{\pm}\begin{bmatrix} -\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & -\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & -\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' \\ -\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{K}_1' \\ -(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & (\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' \\ \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1 & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp} & \mathbf{D}-\mathbf{H}\widehat{\Sigma}\mathbf{H}' \end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] \\
& = i_{\pm}\begin{bmatrix} -\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & -\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' \\ -\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{K}_1' \\ \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1 & \mathbf{D}-\mathbf{H}\widehat{\Sigma}\mathbf{H}' \end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right] \\
& = i_{\mp}\left(\begin{bmatrix} \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 \\ \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{H}\widehat{\Sigma}\mathbf{H}' & \mathbf{K}_1 \\ \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_s \\ \mathbf{0} \end{bmatrix}\mathbf{D}\begin{bmatrix} \mathbf{0} & \mathbf{I}_s & \mathbf{0} \end{bmatrix}\right) + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right] \\
& \quad - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right].
\end{aligned} \tag{3.5}
$$

We can reapply (1.13) to (3.5) after writing $\mathbf{D} = \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)]$ in (2.4). Then, (3.5) is equivalently written as follows by using the similar way to obtaining (3.4),

$$
\begin{aligned}
& i_{\mp}\left(\begin{bmatrix} \widehat{\Sigma} & \mathbf{0} & -\widehat{\Sigma}\mathbf{H}' & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 \\ -\mathbf{H}\widehat{\Sigma} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{H}\widehat{\Sigma}\mathbf{H}' & \mathbf{K}_1 \\ \mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \widehat{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \left[\mathbf{K}, \quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right] \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{W}_1 \\ \mathbf{W}_1' & \mathbf{0} \end{bmatrix}^{+} \begin{bmatrix} \widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \left[\mathbf{K}, \quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]' & \mathbf{0} \end{bmatrix}\right) \\
& - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right] - i_{\mp}(\widehat{\Sigma}).
\end{aligned} \tag{3.6}
$$

We can apply (1.13) to (3.6) since $\mathscr{C}(\widehat{\Sigma}) \subseteq \mathscr{C}(\mathbf{W}_1)$, where $\mathbf{W}_1 = \left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]$. From Lemma 1.2 and 1.3, (3.6) is equivalent to

$$
\begin{aligned}
& i_{\mp}\begin{bmatrix}
\mathbf{0} & -\widehat{\mathbf{X}} & -\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp} & \widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-\widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}' & \mathbf{0} \\
-\widehat{\mathbf{X}}^{\perp}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}^{\perp}\widehat{\Sigma}\mathbf{H}' & \mathbf{0} \\
\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \widehat{\Sigma} & \mathbf{0} & -\widehat{\Sigma}\mathbf{H}' & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 \\
\mathbf{0} & \widehat{\mathbf{K}} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp} & -\mathbf{H}\widehat{\Sigma} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{H}}' & \mathbf{K}_1 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0}
\end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] - i_{\mp}(\widehat{\Sigma}) - r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right] \\
& + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]
\end{aligned}
$$

$$
\begin{aligned}
& = i_{\mp}\begin{bmatrix}
-\widehat{\Sigma} & -\widehat{\mathbf{X}} & -\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp} & \mathbf{0} & \widehat{\Sigma}\mathbf{H}' & \mathbf{0} \\
-\widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}' & \mathbf{0} \\
-\widehat{\mathbf{X}}^{\perp}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}^{\perp}\widehat{\Sigma}\mathbf{H}' & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 \\
\mathbf{H}\widehat{\Sigma} & \widehat{\mathbf{K}} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{K}_1 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0}
\end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] - r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right] \\
& \quad + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]
\end{aligned}
$$

$$
\begin{aligned}
& = i_{\mp}\begin{bmatrix}
-\widehat{\Sigma} & -\widehat{\mathbf{X}} & \mathbf{0} & \widehat{\Sigma}\mathbf{H}' & \mathbf{0} \\
-\widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}' & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 \\
\mathbf{H}\widehat{\Sigma} & \widehat{\mathbf{K}} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{K}_1 \\
\mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0}
\end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] - r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right] + i_{\mp}\left[\widehat{\mathbf{X}}^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right] \\
& \quad + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]
\end{aligned}
$$

$$
\begin{aligned}
& = i_{\pm}\begin{bmatrix}
\widehat{\Sigma} & \mathbf{0} & \widehat{\mathbf{X}} & \widehat{\Sigma}\mathbf{H}' & \mathbf{0} \\
\mathbf{0} & -\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 \\
\widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}' & \mathbf{0} \\
\mathbf{H}\widehat{\Sigma} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{K}} & \mathbf{0} & -\mathbf{K}_1 \\
\mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & -\mathbf{K}_1' & \mathbf{0}
\end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] - r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right] + i_{\mp}\left[\widehat{\mathbf{X}}^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right] \\
& \quad + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]
\end{aligned}
$$

$$
= i_{\pm}\begin{bmatrix}
\widehat{\Sigma} & -\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{X}} & \mathbf{0} & \widehat{\Sigma}\mathbf{H}' \\
-\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma} & \mathbf{0} & -\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & \mathbf{0} \\
\widehat{\mathbf{X}}' & -\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}' \\
\mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & -\mathbf{K}_1' \\
\mathbf{H}\widehat{\Sigma} & \mathbf{0} & \widehat{\mathbf{K}} & -\mathbf{K}_1 & \mathbf{0}
\end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] - r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right] + i_{\mp}\left[\widehat{\mathbf{X}}^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]
$$

$$
\begin{aligned}
& = i_{\pm}\begin{bmatrix}
\widehat{\Sigma} & \widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\Sigma}\mathbf{H}' & \mathbf{0} & \widehat{\mathbf{X}} \\
\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & \mathbf{0} \\
\mathbf{H}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{K}_1 & \mathbf{0} \\
\mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0} & \mathbf{0} \\
\widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right] - r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right] \\
& \quad + i_{\mp}\left[\widehat{\mathbf{X}}^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]. \qquad (3.7)
\end{aligned}
$$

In consequence, by using (1.11) and (1.12), we obtain (3.1) and (3.2) from (3.7). (3.3) is obtained by adding the equalities in (3.1) and (3.2). (a)-(e) is seen from (3.1)-(3.3) by using Lemma 1.1. □

As an immediate consequence of Theorem 3.1, the following results are obtained by setting $\mathbf{H} = \mathbf{0}$ and $\mathbf{K}_1 = \widehat{\mathbf{X}}_1$, respectively, in this theorem.

Corollary 3.2. *Let $\widehat{\mathscr{N}}$ and $\widehat{\mathscr{N}_R}$ be as given in (1.6) and (1.7), respectively, and assume that $\mathbf{K}_1\alpha_1$ is estimable under these models. Denote*

$$\mathbf{B} = \begin{bmatrix} \widehat{\Sigma} & \widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}} \\ \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_1 & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then

$$i_+(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)]) = i_+(\mathbf{B}) - r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} - r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1),$$

$$i_-(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)]) = i_-(\mathbf{B}) - r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} - r(\widehat{\mathbf{X}}),$$

$$r(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)]) = r(\mathbf{B}) - r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} - r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1) - r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} - r(\widehat{\mathbf{X}}).$$

Further,

1. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] \succ \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)] \Leftrightarrow i_+(\mathbf{B}) = r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} + r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1) + s.$
2. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] \prec \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)] \Leftrightarrow i_-(\mathbf{B}) = r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} + r(\widehat{\mathbf{X}}) + s.$
3. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] \succcurlyeq \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)] \Leftrightarrow i_-(\mathbf{B}) = r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} + r(\widehat{\mathbf{X}}).$
4. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] \preccurlyeq \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)] \Leftrightarrow i_+(\mathbf{B}) = r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} + r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1).$
5. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] = \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)] \Leftrightarrow r(\mathbf{B}) = r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} + r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1) + r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} + r(\widehat{\mathbf{X}}).$

Corollary 3.3. *Let $\widehat{\mathscr{N}}$ and $\widehat{\mathscr{N}_R}$ be as given in (1.6) and (1.7), respectively, and assume that $\widehat{\mathbf{X}}_1\alpha_1$ is estimable under these models. Denote*

$$\mathbf{C} = \begin{bmatrix} \widehat{\Sigma} & \widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{X}} \\ \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then

$$i_+(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)]) = i_+(\mathbf{C}) - r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix},$$

$$i_-(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)]) = i_-(\mathbf{C}) - r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} - r(\widehat{\mathbf{X}}_2),$$

$$r(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)]) = r(\mathbf{C}) - r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} - r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} - r(\widehat{\mathbf{X}}_2).$$

Further,

1. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] \succ \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)] \Leftrightarrow i_+(\mathbf{C}) = r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} + m + n.$
2. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] \prec \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)] \Leftrightarrow i_-(\mathbf{C}) = r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} + r(\widehat{\mathbf{X}}_2) + m + n.$
3. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] \succcurlyeq \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)] \Leftrightarrow i_-(\mathbf{C}) = r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} + r(\widehat{\mathbf{X}}_2).$
4. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] \preccurlyeq \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)] \Leftrightarrow i_+(\mathbf{C}) = r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix}.$
5. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] = \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)] \Leftrightarrow r(\mathbf{C}) = r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} + r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} + r(\widehat{\mathbf{X}}_2).$

Corollary 3.4. *Let $\widehat{\mathscr{N}}$ and $\widehat{\mathscr{N}_R}$ be as given in (1.6) and (1.7), respectively, and assume that α_1 is estimable under these models. Then*

1. $i_{\pm}(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\alpha_1)]) = r(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\alpha_1)]) = 0.$
2. $i_{\pm}(\widehat{\varepsilon} - \mathrm{D}[\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\widehat{\varepsilon})] - \mathrm{D}[\widehat{\varepsilon} - \mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\widehat{\varepsilon})]) = r(\widehat{\varepsilon} - \mathrm{D}[\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\widehat{\varepsilon})] - \mathrm{D}[\widehat{\varepsilon} - \mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\widehat{\varepsilon})]) = 0.$

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All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Generalized Bertrand and Mannheim Curves in 3*D* Lie Groups

Osman Zeki Okuyucu[1*] and Bahar Doğan Yazıcı[1]

[1]*Department of Mathematics, Bilecik Şeyh Edebali University, Bilecik, Turkey*
[*]*Corresponding author*

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Abstract

In this paper, we give a new approach for Bertrand and Mannheim curves in 3*D* Lie groups with bi-invariant metrics. In this way, some conditions including the known results have been given for a curve to be Bertrand or Mannheim curve in 3*D* Euclidean space and in 3*D* Lie groups.

1. Introduction

The curve and surface theory is a comprehensive field in differential geometry. Especially associated curves whose the Frenet apparatus satisfy some geometric conditions in Euclidean 3-space. For examples, general helix is a curve, whose tangent vector makes a constant angle with a fixed straight line. Lancret gave the condition for a given curve to be a general helices by the ratio of its curvatures to be constant [1]. In [2], a different approach is given to a general helix lying on a sphere. Also, slant helix was defined as a curve whose normal vector makes a constant angle with a fixed straight line in 3*D* Euclidean space by Izumiya and Takeuchi [3]. They showed that a curve is a slant helix iff the geodesic curvature of spherical image of principal normal indicatrix of the curve is a constant function. On the other hand, there exist some examples of associated curves such as Bertrand and Mannheim curve couples whose the Frenet apparatus satisfy some geometric conditions in 3*D* Euclidean space. Bertrand curve couples defined by J. M. Bertrand in 1845 [4]. "If the normal vectors of the two curves are coincide at the corresponding points of the curves, we say that these curves are Bertrand curve couple". Liu and Wang defined Mannheim curve couples in 2008 [5]. "If the normal vector of a given curve is coincide with an other curve's bi-normal vector at the corresponding points of the curves, we say that these curves are Mannheim curve couple." They gave a condition for a given curve to be a Mannheim curve. Also, [6]-[9] can be looked at for examining Bertand and Mannheim curves in different spaces. Recently, Ç. Camcı et.all and A. Uçum et.all gave a generalization for Bertrand and Mannheim curves in 3*D* Euclidean space, respectively [10, 11].
Lie groups are an important mathematical form because they have three different structures in mathematics such that S^3, $SO(3)$ and Abelian Lie groups. In addition, Lie groups have a wide range of theory and application in physics and mechanics, as well as their importance in mathematics. Some associated curves such as general helices, slant helices, Bertrand and Mannheim curves are introduced in the Lie groups [12]-[15]. On the other hand, different structures such as spinor representations, curve flows in Lie groups and the conjugate mate structures of curves were investigated in [16]-[19]. And it has been shown that the conditions obtained in 3*D* Lie groups are a generalization of the conditions obtained in 3*D* Euclidean space. In [20], Lie algebras and their applications related to dynamical structures are given.
In this paper, we introduce a generalization for Bertrand and Mannheim curves in 3*D* Lie groups, respectively. Also, we obtain some characterizations of these curves. Moreover, we give some results about this curves for special cases of 3*D* Lie groups.

***Email addresses and ORCID numbers*: osman.okuyucu@bilecik.edu.tr, 0000-0003-4379-0252 (O. Z. Okuyucu), bahar.dogan@bilecik.edu.tr, 0000-0001-5690-4840 (B. D. Yazıcı)**

2. Preliminaries

Assume that $\mathbb{G}$ be the $3D$ Lie group with bi-invariant metric $\langle , \rangle$ and ∇ be the Levi-Civita connection of Lie group $\mathbb{G}$. The Lie algebra of $\mathbb{G}$ denotes with $\mathfrak{g}$, which is isomorphic to $T_f\mathbb{G}$, where f is neutral element of $\mathbb{G}$. As the metric is bi-invariant, we have the following equations for all $\mathbb{P},\mathbb{Q},\mathbb{R} \in \mathfrak{g}$;

$$\langle \mathbb{P}, [\mathbb{Q},\mathbb{R}]\rangle = \langle [\mathbb{P},\mathbb{Q}], \mathbb{R}\rangle$$

and

$$\nabla_{\mathbb{P}}\mathbb{Q} = \frac{1}{2}[\mathbb{P},\mathbb{Q}].$$

Let $E_1,E_2,\ldots,E_n$ be an orthonormal basis of $\mathfrak{g}$ and γ be an an arc-lengthed curve on $\mathbb{G}$. Then, we can write any two vector fields $\mathbb{Y}_1$ and $\mathbb{Y}_2$ along γ as $\mathbb{Y}_1 = \sum_{i=1}^n a_iE_i$ and $\mathbb{Y}_2 = \sum_{i=1}^n b_iE_i$ where a_i and b_i are real-valued smooth functions. Also, the Lie bracket of $\mathbb{Y}_1$ and $\mathbb{Y}_2$ is given

$$[\mathbb{Y}_1,\mathbb{Y}_2] = \sum_{i=1}^n a_ib_i[E_i,E_j]$$

On the other hand, the covariant derivative of $\mathbb{Z}$ along γ is given by

$$\nabla_{\gamma'}\mathbb{Z} = \dot{\mathbb{Z}} + \frac{1}{2}[\mathbf{t},\mathbb{Z}] \tag{2.1}$$

where $\mathbf{t} = \gamma'$ and $\dot{\mathbb{Z}} = \sum_{i=1}^n \frac{dz}{dt}E_i$. Also, if $\mathbb{Z}$ is the left-invariant vector field, then $\dot{\mathbb{Z}} = 0$ [21].
Let γ be a curve with Frenet apparatus $\{\mathbf{t},\mathbf{n},\mathbf{b},\kappa,\tau\}$ in Lie group $\mathbb{G}$. Then the Frenet-Serret formulas are expressed by

$$\nabla_{\mathbf{t}}\mathbf{t} = \kappa\mathbf{n}, \quad \nabla_{\mathbf{t}}\mathbf{n} = -\kappa\mathbf{t} + \tau\mathbf{b}, \quad \nabla_{\mathbf{t}}\mathbf{b} = -\tau\mathbf{n}$$

where ∇ is connection of Lie group $\mathbb{G}$ and $\kappa = \|\dot{\mathbf{t}}\|$.

Proposition 2.1. *[12] Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a curve in Lie group $\mathbb{G}$ with the Frenet apparatus $\{\mathbf{t},\mathbf{n},\mathbf{b},\kappa,\tau\}$. Then Lie curvature $\tau_{\mathbb{G}}$ is defined by*

$$\tau_{\mathbb{G}} = \frac{1}{2}\langle[\mathbf{t},\mathbf{n}],\mathbf{b}\rangle.$$

Proposition 2.2. *[13] Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be an arc length parametrized curve with the Frenet apparatus $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$. Then the following equalities*

$$[\mathbf{t},\mathbf{n}] = \langle[\mathbf{t},\mathbf{n}],\mathbf{b}\rangle\mathbf{b} = 2\tau_{\mathbb{G}}\mathbf{b}$$

$$[\mathbf{t},\mathbf{b}] = \langle[\mathbf{t},\mathbf{b}],\mathbf{n}\rangle\mathbf{n} = -2\tau_{\mathbb{G}}\mathbf{n}$$

hold.

Remark 2.3. *[12, 22] The follows hold for Lie group $\mathbb{G}$ with bi-invariant metric in special cases:*
(i) Let $\mathbb{G}$ is an Abelian group, then $\tau_{\mathbb{G}} = 0$,
(ii) Let $\mathbb{G}$ is SU^2, then $\tau_{\mathbb{G}} = 1$,
(iii) Let $\mathbb{G}$ is SO^3, then $\tau_{\mathbb{G}} = \frac{1}{2}$.

Theorem 2.4. *[12] Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a curve in Lie group $\mathbb{G}$ with the curvatures κ,τ and Lie curvature $\tau_{\mathbb{G}}$. Then, γ is a general helix iff*

$$\frac{\tau - \tau_{\mathbb{G}}}{\kappa} = constant$$

Theorem 2.5. *[13] Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a curve in Lie group $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures κ,τ and Lie curvature $\tau_{\mathbb{G}}$. Then γ is a slant helix iff*

$$\frac{\kappa\left(1 + \left(\frac{\tau - \tau_{\mathbb{G}}}{\kappa}\right)^2\right)^{\frac{3}{2}}}{\left(\frac{\tau - \tau_{\mathbb{G}}}{\kappa}\right)'} = constant$$

Theorem 2.6. *[14] Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a curve in Lie group $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures κ, τ and Lie curvature $\tau_{\mathbb{G}}$. Then, γ is Mannheim curve iff*

$$\lambda\kappa\left(1+\left(\frac{\tau-\tau_{\mathbb{G}}}{\kappa}\right)^2\right)=1$$

where λ is constant.

Theorem 2.7. *[15] Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Bertrand curve in Lie group $\mathbb{G}$ with the curvatures κ, τ and Lie curvature $\tau_{\mathbb{G}}$. Then, γ satisfy the following equality*

$$\lambda\kappa+\mu(\tau-\tau_{\mathbb{G}})=1$$

where λ, μ are constants.

3. Generalized Bertrand curves in $3D$ Lie groups

In this section, we investigate generalized Bertrand curves in $3D$ Lie groups and we give some characterizations.

Definition 3.1. *A curve $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ is a Bertrand curve if there exists a special curve $\overline{\gamma} : \overline{J} \subset \mathbb{R} \to \mathbb{G}$ and a bijection $\zeta : \gamma \to \overline{\gamma}$ where $\mathbf{n}(s)$ and $\overline{\mathbf{n}}(\overline{s})$ at $s \in J$, $\overline{s} \in \overline{J}$ coincide. Also, $\overline{\gamma}(\overline{s})$ is called the Bertrand mate of $\gamma(s)$ in Lie group $\mathbb{G}$.*

Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Bertrand curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the Frenet apparatus $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ and the curvatures $\kappa, \tau \neq 0$ and $\overline{\gamma}(\overline{s})$ be a Bertrand mate curve of γ with the Frenet apparatus $\{\overline{\mathbf{t}},\overline{\mathbf{n}},\overline{\mathbf{b}}\}$ and the curvatures $\overline{\kappa}, \overline{\tau} \neq 0$. We can write as

$$\overline{\gamma}(\overline{s})=\overline{\gamma}(\sigma(s))=\gamma(s)+a(s)\mathbf{t}(s)+b(s)\mathbf{n}(s)+c(s)\mathbf{b}(s) \tag{3.1}$$

where $a(s), b(s)$ and $c(s)$ are differentiable functions on J.

Theorem 3.2. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Bertrand curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures $\kappa, \tau \neq 0$. γ is a Bertrand curve with Bertrand mate $\overline{\gamma}$ iff one of the followings holds:*
i. The differentiable functions a, b and c satisfy the following equations:

$$a\kappa+b'-c(\tau-\tau_G)=0 \quad \text{and} \quad c'+b(\tau-\tau_{\mathbb{G}})=0 \tag{3.2}$$

ii. The differentiable functions a, b, c and real number ℓ satisfy the following equations:

$$a\kappa+b'-c(\tau-\tau_{\mathbb{G}})=0, \quad c'+b(\tau-\tau_{\mathbb{G}})\neq 0 \tag{3.3}$$

$$1+a'-b\kappa=\ell(c'+b(\tau-\tau_{\mathbb{G}})), \quad \ell\kappa-(\tau-\tau_{\mathbb{G}})\neq 0, \quad \kappa+\ell(\tau-\tau_{\mathbb{G}})\neq 0$$

Proof. Let us assume that $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Bertrand curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures $\kappa, \tau \neq 0$. By differentiating equation (3.1), we get

$$\frac{d\overline{\gamma}(\overline{s})}{d\overline{s}}\sigma'=\frac{d\gamma(s)}{ds}+a'(s)\mathbf{t}(s)+a(s)\dot{\mathbf{t}}(s)+b'(s)\mathbf{n}(s)+b(s)\dot{\mathbf{n}}(s)+c'(s)\mathbf{b}(s)+c(s)\dot{\mathbf{b}}(s). \tag{3.4}$$

By using equation (2.1) and Proposition 2.1, we have

$$\overline{\mathbf{t}}\sigma'=(1+a'-b\kappa)\mathbf{t}+(a\kappa+b'-c(\tau-\tau_{\mathbb{G}}))\mathbf{n}+(c'+b(\tau-\tau_{\mathbb{G}}))\mathbf{b} \tag{3.5}$$

By taking the inner product of equation (3.5) with $\mathbf{n}$, we have

$$a\kappa+b'-c(\tau-\tau_{\mathbb{G}})=0$$

Therefore, we get

$$\overline{\mathbf{t}}\sigma'=(1+a'-b\kappa)\mathbf{t}+(c'+b(\tau-\tau_{\mathbb{G}}))\mathbf{b} \tag{3.6}$$

It is clear that,

$$(\sigma')^2=(1+a'-b\kappa)^2+(c'+b(\tau-\tau_{\mathbb{G}}))^2 \tag{3.7}$$

Then, we can write as

$$\overline{\mathbf{t}}=\lambda_1\mathbf{t}+\lambda_2\mathbf{b} \tag{3.8}$$

for

$$\lambda_1 = \frac{1+a^{'}-b\kappa}{\sigma^{'}} \quad and \quad \lambda_2 = \frac{c^{'}+b(\tau-\tau_{\mathbb{G}})}{\sigma^{'}} \tag{3.9}$$

By differentiating equation (3.8) in $\mathbb{G}$, we get

$$\sigma^{'}\overline{\kappa}\overline{\mathbf{n}} = \lambda_1^{'}\mathbf{t} + (\lambda_1\kappa - \lambda_2(\tau-\tau_{\mathbb{G}}))\mathbf{n} + \lambda_2^{'}\mathbf{b} \tag{3.10}$$

This shows that $\lambda_1^{'} = 0$ and $\lambda_2^{'} = 0$.
i. Let us suppose that $\lambda_2 = 0$. Therefore, we have $c^{'}+b(\tau-\tau_{\mathbb{G}}) = 0$.
ii. Let us suppose that $\lambda_2 \neq 0$. Then, we can write

$$1+a^{'}-b\kappa = \ell(c^{'}+b(\tau-\tau_{\mathbb{G}})) \tag{3.11}$$

where $\dfrac{\lambda_1}{\lambda_2} = \ell =$ constant. By according to equation (3.10), we write

$$\sigma^{'}\overline{\kappa}\overline{\mathbf{n}} = (\lambda_1\kappa - \lambda_2(\tau-\tau_{\mathbb{G}}))\mathbf{n}$$

By taking the norm of both sides and by using equations (3.7) and (3.9), we get

$$(\sigma^{'})^2(\overline{\kappa})^2 = \frac{(\ell\kappa-(\tau-\tau_{\mathbb{G}}))^2}{\ell^2+1} \tag{3.12}$$

where $\ell\kappa-(\tau-\tau_{\mathbb{G}}) \neq 0$. If we denote by $\lambda = \dfrac{\lambda_1\kappa-\lambda_2(\tau-\tau_{\mathbb{G}})}{\sigma^{'}\overline{\kappa}}$, we have

$$\overline{\mathbf{n}} = \lambda\mathbf{n} \tag{3.13}$$

By differentiating equation (3.13), we get

$$(-\overline{\kappa}\overline{\mathbf{t}} + (\overline{\tau}-\overline{\tau}_{\mathbb{G}})\overline{\mathbf{b}})\sigma^{'} = -\lambda\kappa\mathbf{t} + \lambda(\tau-\tau_{\mathbb{G}})\mathbf{b} \tag{3.14}$$

where $\lambda^{'} = 0$. If we rewrite equation (3.14) by using equation (3.6), we get,

$$-\sigma^{'}(\overline{\tau}-\overline{\tau}_{\mathbb{G}})\overline{\mathbf{b}} = \mu_1(s)\mathbf{t} + \mu_2(s)\mathbf{b}$$

where

$$\mu_1(s) = -\frac{(c^{'}+b(\tau-\tau_{\mathbb{G}}))(\ell\kappa-(\tau-\tau_{\mathbb{G}}))}{(\sigma^{'})^2(\ell^2+1)\overline{\kappa}}(\kappa+\ell(\tau-\tau_{\mathbb{G}}))$$

and

$$\mu_2(s) = \frac{(c^{'}+b(\tau-\tau_{\mathbb{G}}))(\ell\kappa-(\tau-\tau_{\mathbb{G}}))\ell}{(\sigma^{'})^2(\ell^2+1)\overline{\kappa}}(\kappa+\ell(\tau-\tau_{\mathbb{G}}))$$

It is clear that $\kappa+\ell(\tau-\tau_{\mathbb{G}}) \neq 0$.
Conversely, assume that $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Bertrand curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures $\kappa, \tau \neq 0$.
i. Let's assume that the condition (3.2) is satisfied for the differentiable functions a, b and c. Therefore, we write the derivative of equation (3.1) as follows:

$$\frac{d\overline{\gamma}}{ds} = (1+a^{'}-b\kappa)\mathbf{t} \tag{3.15}$$

From equation (3.15), we get

$$\sigma^{'} = \frac{d\overline{s}}{ds} = \left\|\frac{d\overline{\gamma}}{ds}\right\| = \varepsilon_1(1+a^{'}-b\kappa) > 0$$

where $\varepsilon_1 = sgn(1+a^{'}-b\kappa)$. Therefore, we have

$$\overline{\mathbf{t}} = \varepsilon_1\mathbf{t}, \quad \overline{\mathbf{n}} = \varepsilon_1\mathbf{n}, \quad \overline{\mathbf{b}} = \mathbf{b}$$

and

$$\overline{\kappa} = \frac{\kappa}{\sigma'}, \quad \overline{\tau} - \overline{\tau}_{\mathbb{G}} = \frac{\varepsilon_1(\tau - \tau_{\mathbb{G}})}{\sigma'}$$

Consequently, γ is a Bertrand curve in Lie group $\mathbb{G}$.
ii. Let's assume that the condition (3.3) is satisfied for the differentiable functions a, b, c and real function ℓ. Therefore, we write the derivative of equation (3.1) as follows:

$$\frac{d\overline{\gamma}}{ds} = (1 + a' - b\kappa)\mathbf{t} + (c' + b(\tau - \tau_{\mathbb{G}}))\mathbf{b} \tag{3.16}$$

From equation, we get

$$\sigma' = \left\| \frac{d\overline{\gamma}}{ds} \right\| = \xi_1(c' + b(\tau - \tau_{\mathbb{G}}))\sqrt{\ell^2 + 1}$$

where $\xi_1 = sgn(c' + b(\tau - \tau_{\mathbb{G}}))$. By according to equation (3.16), we have

$$\bar{\mathbf{t}} = \frac{\xi_1}{\sqrt{\ell^2 + 1}}(\ell\mathbf{t} + \mathbf{b}), \quad \langle \bar{\mathbf{t}}, \bar{\mathbf{t}} \rangle = 1 \tag{3.17}$$

By differentiating (3.17) with respect to s, we get

$$\begin{aligned} \dot{\bar{\mathbf{t}}}\sigma' &= \frac{\xi_1}{\sqrt{\ell^2 + 1}}(\ell\dot{\mathbf{t}} + \dot{\mathbf{b}}) \\ \dot{\bar{\mathbf{t}}} &= \frac{\xi_1(\ell\kappa - (\tau - \tau_{\mathbb{G}}))\mathbf{n}}{\sigma'\sqrt{\ell^2 + 1}} \end{aligned} \tag{3.18}$$

Then, from equation (3.18), we get

$$\overline{\kappa} = \|\dot{\bar{\mathbf{t}}}\| = \frac{\xi_2(\ell\kappa - (\tau - \tau_{\mathbb{G}}))}{\sigma'\sqrt{\ell^2 + 1}} \tag{3.19}$$

and

$$\overline{\mathbf{n}} = \xi_1\xi_2\mathbf{n}, \quad \langle \overline{\mathbf{n}}, \overline{\mathbf{n}} \rangle = 1 \tag{3.20}$$

where $\xi_2 = sgn(\ell\kappa - (\tau - \tau_{\mathbb{G}}))$. Then, we have

$$\overline{\mathbf{b}} = \bar{\mathbf{t}} \wedge \overline{\mathbf{n}} = \frac{\xi_2}{\sqrt{\ell^2 + 1}}(-\mathbf{t} + \ell\mathbf{b}), \quad \langle \overline{\mathbf{b}}, \overline{\mathbf{b}} \rangle = 1 \tag{3.21}$$

By differentiating equation (3.21), we get

$$\overline{\tau} - \overline{\tau}_{\mathbb{G}} = -\langle \dot{\overline{\mathbf{b}}}, \overline{\mathbf{n}} \rangle = \frac{\xi_1(\kappa + (\tau - \tau_{\mathbb{G}})\ell)}{\sigma'\sqrt{\ell^2 + 1}} \tag{3.22}$$

Thus, γ is a Bertrand curve in Lie group $\mathbb{G}$. □

Proposition 3.3. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ and $\overline{\gamma} : \overline{J} \subset \mathbb{R} \to \mathbb{G}$ be Bertrand curve pair with the Frenet vectors $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and $\{\bar{\mathbf{t}}, \overline{\mathbf{n}}, \overline{\mathbf{b}}\}$, respectively. Then $\tau_{\mathbb{G}} = \overline{\tau}_{\mathbb{G}}$ for $\tau_{\mathbb{G}} = \frac{1}{2}\langle [\mathbf{t}, \mathbf{n}], \mathbf{b} \rangle$ and $\overline{\tau}_{\mathbb{G}} = \frac{1}{2}\langle [\bar{\mathbf{t}}, \overline{\mathbf{n}}], \overline{\mathbf{b}} \rangle$.*

Proof. The proof is easily seen from equations (3.17), (3.20) and (3.21). □

Remark 3.4. *If $a = c = 0$ in Theorem 3.2, we obtain the Bertrand curve conditions in the 3D Lie groups in the literature [15] where*

$$\overline{\gamma}(\overline{s}) = \overline{\gamma}(\sigma(s)) = \gamma(s) + b(s)\mathbf{n}(s)$$

Corollary 3.5. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Bertrand curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures $\kappa, \tau \neq 0$. γ is a Bertrand curve where $\overline{\gamma}(\overline{s}) = \overline{\gamma}(\sigma(s)) = \gamma(s) + b(s)\mathbf{n}(s)$ iff there exist real number b and ℓ satisfying*

$$1 - b\kappa = \ell b(\tau - \tau_{\mathbb{G}}) \quad \ell\kappa - (\tau - \tau_{\mathbb{G}}) \neq 0$$

In the following corollary, we show the existence of Bertrand curves with general helix in Lie group $\mathbb{G}$.

Corollary 3.6. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a general helix in Lie group $\mathbb{G}$ with the curvatures κ, τ satisfying $\ell\kappa - (\tau - \tau_{\mathbb{G}}) \neq 0$ and $\kappa + \ell(\tau - \tau_{\mathbb{G}}) \neq 0$ where ℓ is a real number. Then, $\overline{\gamma}$ is given by*

$$\overline{\gamma}(\sigma(s)) = \gamma(s) + \frac{ks}{\ell - k}\mathbf{t} + \frac{s}{\ell - k}\mathbf{b} \tag{3.23}$$

where $k \neq 0$ is constant.

Proof. Suppose that $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a general helix in Lie group $\mathbb{G}$ with the curvatures κ, τ. From Theorem 2.4, we can write

$$\frac{\tau - \tau_{\mathbb{G}}}{\kappa} = k \tag{3.24}$$

where $k \neq 0$ is constant. On the other hand, if we take $b = 0$ in the conditions of (ii) in Theorem 3.2, we have

$$a\kappa = c(\tau - \tau_{\mathbb{G}}), \quad c' \neq 0, \quad 1 + a' = \ell c' \tag{3.25}$$

By using equations (3.24) and (3.25), we get

$$a = kc, \quad c = \frac{s}{\ell - k}.$$

Hence, equation (3.23) is satisfied. □

Corollary 3.7. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Bertrand curve with the curvatures κ, τ and $\overline{\gamma} : \overline{I} \subset \mathbb{R} \to G$ be a Bertrand mate curve of γ with the curvatures $\overline{\kappa}, \overline{\tau}$. Then $\overline{\gamma}$ is a general helix iff γ is a general helix in 3D Lie groups.*

Proof. By using equations (3.19) and (3.22), we get

$$\frac{\overline{\tau} - \overline{\tau}_{\mathbb{G}}}{\overline{\kappa}} = \xi_1 \xi_2 \frac{1 + \ell\left(\frac{\tau - \tau_{\mathbb{G}}}{\kappa}\right)}{\ell - \left(\frac{\tau - \tau_{\mathbb{G}}}{\kappa}\right)}$$

and

$$\frac{\tau - \tau_{\mathbb{G}}}{\kappa} = \frac{\left(\frac{\overline{\tau} - \overline{\tau}_{\mathbb{G}}}{\overline{\kappa}}\right)\ell - \xi_1 \xi_2}{\left(\frac{\overline{\tau} - \overline{\tau}_{\mathbb{G}}}{\overline{\kappa}}\right) + \xi_1 \xi_2 \ell}.$$

Therefore, $\overline{\gamma}$ is a general helix (i.e $\frac{\overline{\tau} - \overline{\tau}_{\mathbb{G}}}{\overline{\kappa}} =$ constant) iff γ is a general helix (i.e $\frac{\tau - \tau_{\mathbb{G}}}{\kappa} =$ constant). □

Corollary 3.8. *If $\mathbb{G}$ is Abelian Lie group, the results obtained correspond to the generalized Bertrand curves given in study [10].*

4. Generalized Mannheim curves in $3D$ Lie groups

In this part, we obtain generalized Mannheim curves in $3D$ Lie groups and we obtain some characterizations.

Definition 4.1. *A curve $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ is a Mannheim curve if there exists a special curve $\gamma^* : \overline{J} \subset \mathbb{R} \to \mathbb{G}$ and a bijection $\zeta : \gamma \to \overline{\gamma}$ where $\mathbf{n}$ and $\mathbf{b}^*$ at $s \in J$, $s^* \in J^*$ coincide. Also, $\gamma^*(s^*)$ is called the Mannheim mate of $\gamma(s)$ in Lie group $\mathbb{G}$.*

Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Mannheim curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the Frenet apparatus $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and the curvatures $\kappa, \tau \neq 0$ and $\gamma^*(s^*)$ be a Mannheim mate curve of γ with the Frenet apparatus $\{\mathbf{t}^*, \mathbf{n}^*, \mathbf{b}^*\}$ and the curvatures $\kappa^*, \tau^* \neq 0$. Then, we have

$$\gamma^*(s^*) = \gamma^*(\varphi(s)) = \gamma(s) + e(s)\mathbf{t}(s) + f(s)\mathbf{n}(s) + g(s)\mathbf{b}(s) \tag{4.1}$$

where $e(s), f(s)$ and $g(s)$ are differentiable functions on J.

Theorem 4.2. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Mannheim curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures $\kappa, \tau \neq 0$. γ is a Mannheim curve with Mannheim mate γ^* iff there exist differentiable functions e, f, g satisfying*

$$e\kappa + f' - g(\tau - \tau_{\mathbb{G}}) = 0, \quad g' + f(\tau - \tau_{\mathbb{G}}) \neq 0 \tag{4.2}$$

$$(1 + e' - f\kappa)\kappa = (g' + f(\tau - \tau_{\mathbb{G}}))(\tau - \tau_{\mathbb{G}})$$

Proof. Suppose that $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Mannheim curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures $\kappa, \tau \neq 0$. By differentiating equation (4.1), we get

$$\frac{d\gamma^*(s^*)}{ds^*}\varphi' = \frac{d\gamma(s)}{ds} + e'(s)\mathbf{t}(s) + e(s)\dot{\mathbf{t}}(s) + f'(s)\mathbf{n}(s) + f(s)\dot{\mathbf{n}}(s) + g'(s)\mathbf{b}(s) + g(s)\dot{\mathbf{b}}(s)$$

Then, we have

$$\mathbf{t}^*\varphi' = (1 + e' - f\kappa)\mathbf{t} + (e\kappa + f' - g(\tau - \tau_{\mathbb{G}}))\mathbf{n} + (g' + f(\tau - \tau_{\mathbb{G}}))\mathbf{b} \tag{4.3}$$

By taking the scalar product of equation (4.3) with $\mathbf{n}$, we find

$$e\kappa + f' - g(\tau - \tau_{\mathbb{G}}) = 0$$

Then, we have

$$\mathbf{t}^*\varphi' = (1 + e' - f\kappa)\mathbf{t} + (g' + f(\tau - \tau_{\mathbb{G}}))\mathbf{b} \tag{4.4}$$

It is seen that

$$(\varphi')^2 = (1 + e' - f\kappa)^2 + (g' + f(\tau - \tau_{\mathbb{G}}))^2 \tag{4.5}$$

Then, we can denote as

$$\mathbf{t}^* = \delta_1\mathbf{t} + \delta_2\mathbf{b} \tag{4.6}$$

for

$$\delta_1 = \frac{1 + e' - f\kappa}{\varphi'} \quad \textit{and} \quad \delta_2 = \frac{g' + f(\tau - \tau_{\mathbb{G}})}{\varphi'} \tag{4.7}$$

By differentiating equation (4.6) in $\mathbb{G}$, we have

$$\begin{array}{rcl} \dot{\mathbf{t}}^*\varphi' & = & \delta_1'\mathbf{t} + \delta_1\dot{\mathbf{t}} + \delta_2'\mathbf{b} + \delta_2\dot{\mathbf{b}} \\ \varphi'\kappa^*\mathbf{n}^* & = & \delta_1'\mathbf{t} + (\delta_1\kappa - \delta_2(\tau - \tau_{\mathbb{G}}))\mathbf{n} + \delta_2'\mathbf{b} \end{array} \tag{4.8}$$

By taking the scalar product of (4.8) with $\mathbf{n}$, we have $\delta_1\kappa - \delta_2(\tau - \tau_{\mathbb{G}}) = 0$. From equation (4.7), we get

$$(1 + e' - f\kappa)\kappa = (g' + f(\tau - \tau_{\mathbb{G}}))(\tau - \tau_{\mathbb{G}}) \tag{4.9}$$

where $g' + f(\tau - \tau_{\mathbb{G}}) \neq 0$.

Conversely, suppose that $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Mannheim curve in G such that parametrized by the arc-length parameter s with the curvatures $\kappa, \tau \neq 0$ and the conditions of (4.2) hold for differentiable functions e, f, g. Then, we can write

$$\frac{d\gamma^*}{ds} = (1 + e' - f\kappa)\mathbf{t} + (g' + f(\tau - \tau_{\mathbb{G}}))\mathbf{b} \tag{4.10}$$

where

$$\varphi' = \sqrt{\langle \frac{d\gamma^*}{ds}, \frac{d\gamma^*}{ds} \rangle} = \frac{\xi_1(g' + f(\tau - \tau_{\mathbb{G}}))\sqrt{\kappa^2 + (\tau - \tau_{\mathbb{G}})^2}}{\kappa}$$

with $\xi_1 = sgn(g' + f(\tau - \tau_{\mathbb{G}}))$. From equation (4.10), we get

$$\mathbf{t}^* = \frac{\xi_1}{\sqrt{\kappa^2 + (\tau - \tau_{\mathbb{G}})^2}}((\tau - \tau_{\mathbb{G}})\mathbf{t} + \kappa\mathbf{b}), \quad \langle \mathbf{t}^*, \mathbf{t}^* \rangle = 1 \tag{4.11}$$

Then, we can denote

$$\mathbf{t}^* = \gamma_1\mathbf{t} + \gamma_2\mathbf{b} \tag{4.12}$$

where

$$\gamma_1 = \frac{\xi_1(\tau - \tau_{\mathbb{G}})}{\sqrt{\kappa^2 + (\tau - \tau_{\mathbb{G}})^2}}, \quad \gamma_2 = \frac{\xi_1\kappa}{\sqrt{\kappa^2 + (\tau - \tau_{\mathbb{G}})^2}}.$$

By differentiating (4.12) with respect to s, we get

$$\dot{\mathbf{t}}^* = \frac{\gamma_1'\mathbf{t} + \gamma_2'\mathbf{b}}{\varphi'} \tag{4.13}$$

Then, from equation (4.13), we get

$$\kappa^* = \|\dot{\mathbf{t}}^*\| = \frac{\xi_2((\tau-\tau_{\mathbb{G}})\kappa' - (\tau-\tau_{\mathbb{G}})'\kappa}{\varphi'(\kappa^2+(\tau-\tau_{\mathbb{G}})^2)} = \frac{-\xi_2\kappa^2\left(\dfrac{\tau-\tau_{\mathbb{G}}}{\kappa}\right)'}{\varphi'(\kappa^2+(\tau-\tau_{\mathbb{G}}))^2} \tag{4.14}$$

and

$$\mathbf{n}^* = \frac{\xi_1\xi_2}{\sqrt{\kappa^2+(\tau-\tau_{\mathbb{G}})^2)}}(-\kappa\mathbf{t} + (\tau-\tau_{\mathbb{G}})\mathbf{b}), \quad \langle\mathbf{n}^*,\mathbf{n}^*\rangle = 1 \tag{4.15}$$

where $\xi_2 = sgn((\tau-\tau_{\mathbb{G}})\kappa' - (\tau-\tau_{\mathbb{G}})'\kappa)$. Moreover, we can obtain

$$\mathbf{b}^* = \mathbf{t}^* \wedge \mathbf{n}^* = -\xi_2\mathbf{n}, \quad \langle\mathbf{b}^*,\mathbf{b}^*\rangle = 1 \tag{4.16}$$

Finally, we get

$$\tau^* - \tau^*_{\mathbb{G}} = -\langle\dot{\mathbf{b}}^*,\mathbf{n}^*\rangle = \frac{\xi_1\sqrt{\kappa^2+(\tau-\tau_{\mathbb{G}})^2}}{\varphi'} \neq 0 \tag{4.17}$$

Then, γ is a Mannheim curve in Lie group $\mathbb{G}$. □

Proposition 4.3. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ and $\gamma^* : J^* \subset \mathbb{R} \to \mathbb{G}$ be Mannheim curve pair with the Frenet vectors $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ and $\{\mathbf{t}^*,\mathbf{n}^*,\mathbf{b}^*\}$, respectively. Then $\tau_{\mathbb{G}} = \tau^*_{\mathbb{G}}$ for $\tau_{\mathbb{G}} = \frac{1}{2}\langle[\mathbf{t},\mathbf{n}],\mathbf{b}\rangle$ and $\tau^*_{\mathbb{G}} = \frac{1}{2}\langle[\mathbf{t}^*,\mathbf{n}^*],\mathbf{b}^*\rangle$.*

Proof. The proof is easily seen from equations (4.11), (4.15) and (4.16). □

Remark 4.4. *If $e = g = 0$ in Theorem 4.2, we satisfy the Mannheim curve conditions in the 3D Lie groups in the literature [14] where*

$$\gamma^*(s^*) = \gamma^*(\varphi(s)) = \gamma(s) + f(s)\mathbf{n}(s)$$

Corollary 4.5. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Mannheim curve in $\mathbb{G}$ such that parametrized by the arc-length parameter s with the curvatures $\kappa,\tau \neq 0$. γ is a Bertrand curve where $\gamma^*(s^*) = \gamma^*(\varphi(s)) = \gamma(s) + f(s)\mathbf{n}(s)$ iff there exist real number f satisfying*

$$\kappa = f(\kappa^2 + (\tau-\tau_{\mathbb{G}})^2).$$

Corollary 4.6. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a general helix with the curvatures $\kappa,\tau \neq 0$. Then, the Mannheim mate γ^* is a straight line in Lie group $\mathbb{G}$.*

Proof. Suppose that $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a general helix with the curvatures $\kappa,\tau \neq 0$ in Lie group $\mathbb{G}$. Since the ratio $\dfrac{\tau-\tau_{\mathbb{G}}}{\kappa}$ is constant, we get $\kappa^* = 0$. Then, the Mannheim mate γ^* is a straight line. □

Corollary 4.7. *Let $\gamma : J \subset \mathbb{R} \to \mathbb{G}$ be a Mannheim curve with the curvatures κ,τ and $\gamma^* : J^* \subset \mathbb{R} \to G$ be a Mannheim mate of γ with the curvatures κ^*,τ^*. Then γ^* is a general helix iff γ is a slant helix in Lie group G.*

Proof. From equations (4.14) and (4.17), we get

$$\frac{\tau^*-\tau_{\mathbb{G}}}{\kappa^*} = -\xi_1\xi_2\kappa\frac{\left(1+\left(\dfrac{\tau-\tau_{\mathbb{G}}}{\kappa}\right)^2\right)^{\frac{3}{2}}}{\left(\dfrac{\tau-\tau_{\mathbb{G}}}{\kappa}\right)'}$$

Hence, the desired is achieved. □

Corollary 4.8. *If $\mathbb{G}$ is Abelian Lie group, the results obtained correspond to the generalized Mannheim curves given in study [11].*

5. Conclusion

In this study, we examined generalized Bertrand and Mannheim curves in $3D$ Lie groups inspired by [10] and [11] studies. We have shown that we obtain the results in studies [10], [11], [14] and [15], especially considering the Abelian Lie groups. In connection with this study, special curve types can be studied in Lie groups with different metric structures in the future.

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