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On Some Fixed Point Theorems for $\mathscr{G}(\Sigma,\vartheta,\Xi)-$Contractions in Modular $b-$Metric Spaces

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Abstract

This article aims to specify a new $C-$class function endowed with altering distance and ultra altering distance function via generalized $\Xi-$contraction, which is called the $\mathscr{G}(\Sigma,\vartheta,\Xi)-$contraction in modular $b-$metric spaces. Regarding these new contraction type mappings, the study includes some existence and uniqueness theorems, and to indicate the usability and productivity of these results, some applications related to integral type contractions and an application to the graph structure.

1. Introduction and preliminaries

In this study, the set of all natural and non-negative real numbers will be symbolized by $\mathbf{N}$ and $\mathbf{R}^+$, respectively.
Fixed point theory is an active and popular area for researchers in nonlinear analysis. Especially metric fixed point theory is a cornerstone for this research area. Researchers working in this field are indebted to S. Banach [1]. The focal point of this topic is to achieve the best suitable conditions on mappings to guarantee the existence and the uniqueness of fixed points, mainly the Banach Fixed Point Theorem put forward by Banach in 1922. In particular, extensive progress has been made in improving and expanding these conditions over the past few decades.
However, the metric structure has been generalized in many directions. One of the crucial results defined in different periods by Bakhtin [2] and Czerwik [3, 4] is $b-$metric space, as noted below.

Definition 1.1. [3] *Let $\mathbf{S}$ be a non-void set and $\kappa\geq 1,(\kappa\in\mathbf{R})$. Presume that the function $\eta:\mathbf{S}\times\mathbf{S}\to\mathbf{R}^+$ provides the following terms: for every $\varpi,\xi,\rho\in\mathbf{S}$,*

(η_1) $\eta(\varpi,\xi)=0\Leftrightarrow\varpi=\xi$,
(η_2) $\eta(\varpi,\xi)=\eta(\xi,\varpi)$,
(η_3) $\eta(\varpi,\xi)\leq\kappa[\eta(\varpi,\rho)+\eta(\rho,\xi)]$.

The function η is entitled a $b-$metric on $\mathbf{S}$, and the pair $(\mathbf{S},\eta)$ is a $b-$metric space.

In the case of $\kappa=1$, the concept of $b-$metric and ordinary metric coincide. Also, unlike standard metrics, the $b-$metric is not continuous. Accordingly, the following lemma is valuable and exceptionally significant for a $b-$metric space.

Lemma 1.2. [5] *Let $(\mathbf{S},\eta)$ be a $b-$metric space with $\kappa\geq 1$, the sequences $\{\varpi_q\}$ and $\{\xi_q\}$ be convergent to ϖ and ξ, respectively. So, we have*

$$\frac{1}{\kappa^2}\eta(\varpi,\xi)\leq\liminf_{q\to\infty}\eta(\varpi_q,\xi_q)\leq\limsup_{q\to\infty}\eta(\varpi_q,\xi_q)\leq\kappa^2\eta(\varpi,\xi).$$

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Especially, if $\varpi = \xi$*, then we have* $\lim_{q\to\infty} \eta(\varpi_q, \xi_q) = 0$*. Also, for* $z \in \mathbf{S}$*, we have*

$$\frac{1}{\kappa}\eta(\varpi, z) \leq \liminf_{q\to\infty} \eta(\varpi_q, z) \leq \limsup_{q\to\infty} \eta(\varpi_q, z) \leq \kappa\eta(\varpi, z).$$

In 2008, V. V. Chistyakov [6] proposed a new concept of modular metric space generated by $F-$modular and the theory of this space. Afterward, in 2010, V. V. Chistyakov [7] defined the modular metric space using a modular that identifies an arbitrary set.
Initially, let $\mathbf{S}$ be a non-empty set and $\upsilon : (0,\infty) \times \mathbf{S} \times \mathbf{S} \to [0,\infty]$ be a function. For brevity, we will write:

$$\upsilon_\lambda(\varpi, \xi) = \upsilon(\lambda, \varpi, \xi),$$

for all $\lambda > 0$ and $\varpi, \xi \in \mathbf{S}$.

Definition 1.3. [7] *Let* $\mathbf{S}$ *be a non-empty set and* $\upsilon : (0,\infty) \times \mathbf{S} \times \mathbf{S} \to [0,\infty]$ *be a function that admits the following axioms. Thereupon, we say that* υ *is named a modular metric for all* $\varpi, \xi, \rho \in \mathbf{S}$

(υ_1) $\upsilon_\lambda(\varpi, \xi) = 0$ *for all* $\lambda > 0$ *if and only if* $\varpi = \xi$*,*
(υ_2) $\upsilon_\lambda(\varpi, \xi) = \upsilon_\lambda(\xi, \varpi)$ *for all* $\lambda > 0$*,*
(υ_3) $\upsilon_{\lambda+\mu}(\varpi, \xi) \leq \upsilon_\lambda(\varpi, \rho) + \upsilon_\mu(\rho, \xi)$ *for all* $\lambda, \mu > 0$*.*

If we only exchange the (υ_1) *with*

($\upsilon_1{}'$) $\upsilon_\lambda(\varpi, \varpi) = 0$ *for all* $\lambda > 0$*,*

then υ *is said to be a (metric) pseudomodular on* $\mathbf{S}$*.*

For more detail, it refers to [6]-[10].
In 2018, M. E. Ege and C. Alaca [11] introduced modular $b-$metric spaces by combining the structures of $b-$metric and modular metrics and, besides, established some fixed point theorems in the new space setting.

Definition 1.4. [11] *Let* $\mathbf{S}$ *be a non-empty set and let* $\kappa \geq 1$ ($\kappa \in \mathbf{R}$)*. A map* $\ell : (0,\infty) \times \mathbf{S} \times \mathbf{S} \to [0,\infty]$ *is entitled as modular* $b-$*metric, provided that the following circumstances satisfied for all* $\varpi, \xi, \rho \in \mathbf{S}$*,*

(ℓ_1) $\ell_\lambda(\varpi, \xi) = 0$ *for all* $\lambda > 0$ *if and only if* $\varpi = \xi$*,*
(ℓ_2) $\ell_\lambda(\varpi, \xi) = \ell_\lambda(\xi, \varpi)$ *for all* $\lambda > 0$*,*
(ℓ_3) $\ell_{\lambda+\mu}(\varpi, \xi) \leq \kappa\left[\ell_\lambda(\varpi, \rho) + \ell_\mu(\rho, \xi)\right]$ *for all* $\lambda, \mu > 0$*.*

The pair $(\mathbf{S}, \ell)$ *is a modular* $b-$*metric space expressed in MbMS.*

In fact, for $\kappa = 1$, it can be seen that MbMS is an extension of the modular metric space.

Example 1.5. [11] *Let us consider the space*

$$l_p = \left\{(\varpi_j) \subset \mathbf{R} : \sum_{j=1}^{\infty} \left|\varpi_j\right|^p < \infty\right\} \quad 0 < p < 1.$$

For $\lambda \in (0,\infty)$ *if we define* $\ell_\lambda(\varpi, \xi) = \frac{m(\varpi,\xi)}{\lambda}$ *such that*

$$m(\varpi, \xi) = \left(\sum_{j=1}^{\infty} \left|\varpi_j - \xi_j\right|^p\right)^{\frac{1}{p}}, \quad \varpi = \varpi_j, \xi = \xi_j \in l_p$$

then we see that $(\mathbf{S}, \ell)$ *is an MbMS with* $\kappa = 2^{\frac{1}{p}}$*.*

Example 1.6. [12] *Let* $(\mathbf{S}, \upsilon)$ *be a modular metric space and let* $p \geq 1$ *be a real number. Take* $\ell_\lambda(\varpi, \xi) = (\upsilon_\lambda(\varpi, \xi))^p$*. Due to the fact that the function* $\Gamma(t) = t^p$ *is convex for* $t \geq 0$*, by Jensen inequality, we attain*

$$(\alpha + \beta)^p \leq 2^{p-1}(\alpha^p + \beta^p)$$

for $\alpha, \beta \geq 0$*. Thus,* $(\mathbf{S}, \ell)$ *is an MbMS with* $\kappa = 2^{p-1}$*.*

Definition 1.7. [11] *Let* ℓ *be a modular* $b-$*metric on a set* $\mathbf{S}$*, and a modular set is identified by*

$$\mathbf{S}_\ell = \left\{\xi \in \mathbf{S} : \xi \overset{\ell}{\sim} \varpi\right\},$$

where the $\overset{\ell}{\sim}$ *is a binary relation on* $\mathbf{S}$ *defined by,*

$$\varpi \sim \xi \Leftrightarrow \lim_{\lambda\to\infty} \ell_\lambda(\varpi,\xi) = 0,$$

for $\varpi,\xi \in \mathbf{S}$*. Also, note that the set*

$$\mathbf{S}^*_\ell = \mathbf{S}^*_\ell(\varpi_0) = \{\varpi \in \mathbf{S} : \exists \lambda = \lambda(\varpi) > 0 \text{ such that } \ell_\lambda(\varpi,\varpi_0) < \infty\} \ (\varpi_0 \in \mathbf{S})$$

is mentioned as a modular metric space (around ϖ_0*).*

In what follows, we recollect some basic topological properties of MbMS.

Definition 1.8. [11] *Let* $(\mathbf{S},\ell)$ *be an MbMS and* $(\varpi_q)_{q\in\mathbf{N}}$ *be a sequence in* $\mathbf{S}^*_\ell$*.*

(i) $(\varpi_q)_{q\in\mathbf{N}}$ *is called* $\ell-$*convergent to* $\varpi \in \mathbf{S}^*_\ell$ *if and only if* $\ell_\lambda(\varpi_q,\varpi) \to 0$*, as* $q\to\infty$ *for all* $\lambda > 0$*.*
(ii) $(\varpi_q)_{q\in\mathbf{N}}$ *in* $\mathbf{S}^*_\ell$ *is named* $\ell-$*Cauchy sequence if* $\lim_{q,m\to\infty} \ell_\lambda(\varpi_q,\varpi_m) = 0$ *for all* $\lambda > 0$*.*
(iii) $\mathbf{S}^*_\ell$ *is called* $\ell-$*complete if any* $\ell-$*Cauchy sequence in* $\mathbf{S}^*_\ell$ *is* $\ell-$*convergent to the point of* $\mathbf{S}^*_\ell$*.*

In [13], A.H. Ansari presented a novel class of functions named $C-$class functions. This class has extended many results for metric fixed point theory, which contains almost all types of contractions.

Definition 1.9. [13] *Let* $\mathscr{G} : [0,\infty)\times[0,\infty) \to \mathbf{R}$ *be a function. If for all* $p,q \in [0,\infty)$*, the function* $\mathscr{G}$ *is continuous and satisfies the below circumstances, then we say that* $\mathscr{G}$ *is a* $C-$*class function.*

$(\mathscr{G}_1)$ $\mathscr{G}(p,q) \le p$;
$(\mathscr{G}_2)$ $\mathscr{G}(p,q) = p$ *implies that either* $p = 0$ *or* $q = 0$*.*

The $C-$*class functions are symbolized by* $\mathscr{C}$*.*

Example 1.10. [13] *The following ones from* $\mathscr{G}_1$ *to* $\mathscr{G}_5$ *are examples of* $\mathscr{G} \in \mathscr{C}$*.*

(i) $\mathscr{G}_1(p,q) = p - q$ *for all* $p,q \in [0,\infty)$*,*
(ii) $\mathscr{G}_2(p,q) = mp$ *for all for all* $p,q \in [0,\infty)$ *where* $0 < m < 1$*,*
(iii) $\mathscr{G}_3(p,q) = \frac{p}{(1+q)^r}$ *for all* $p,q \in [0,\infty)$*, where* $r \in (0,\infty)$*,*
(iv) $\mathscr{G}_4(p,q) = p\beta(p)$ *for all* $p,q \in [0,\infty)$*, where* $\beta : [0,\infty) \to [0,\infty)$ *and is continuous,*
(v) $\mathscr{G}_5(p,q) = \sqrt[n]{\ln(1+p^n)}$ *for all* $p,q \in [0,\infty)$*.*

Definition 1.11. [14] *The family* Ω *denotes all function* $\Sigma : [0,\infty) \to [0,\infty)$*, which is named altering distance function, if*

(Σ_1) Σ *is continuous and non-decreasing;*
(Σ_2) $\Sigma(\iota) = 0$ *if and only if* $\iota = 0$*.*

Definition 1.12. [13] *The family* Π *denotes all function* $\vartheta : [0,\infty) \to [0,\infty)$*, which is named ultra altering distance function if*

(ϑ_1) ϑ *is continuous;*
(ϑ_2) $\vartheta(t) > 0$ *for all* $t > 0$*.*

Also, for $C-$class functions, it refers to [15]-[18].
In 2017, Fulga and Proca [19] introduced a new contraction mapping involving the following expression and proved a fixed point theorem on a complete metric space,

$$\Xi(\varpi,\xi) = m(\varpi,\xi) + |m(\varpi,\Gamma\varpi) - m(\xi,\Gamma\xi)|,$$

whenever $(\mathbf{S},m)$ is a complete metric space and $\varpi,\xi \in \mathbf{S}$. Subsequently, it is used as $\Xi-$contraction and appears in many articles, see, [20]-[22].
In [23], Proca specified a new expression of $\Xi-$contraction with the "max operator" and also, in [24] verified a fixed point theorem, as indicated below.

Theorem 1.13. [24] *Let* $\Gamma : \mathbf{S} \to \mathbf{S}$ *be a mapping on a complete metric space* $(\mathbf{S},m)$*.* Γ *admits a unique fixed point in* $\mathbf{S}$ *if there exists* $\alpha \in [0,1)$ *such that for all* $\varpi,\xi \in \mathbf{S}$

$$m(\Gamma\varpi,\Gamma\xi) \le \alpha(M^*(\varpi,\xi)),$$

where

$$\begin{aligned} M^*(\varpi,\xi) = \max\{& m(\varpi,\xi) + |m(\varpi,\Gamma\varpi) - m(\xi,\Gamma\xi)|; m(\varpi,\Gamma\varpi) + |m(\varpi,\xi) - m(\xi,\Gamma\xi)|; \\ & m(\xi,\Gamma\xi) + |m(\varpi,\xi) - m(\varpi,\Gamma\varpi)|; \tfrac{1}{2}[m(\varpi,\Gamma\xi) + m(\xi,\Gamma\varpi) + |m(\varpi,\Gamma\varpi) - m(\xi,\Gamma\xi)|]\}. \end{aligned}$$

Furthermore, in [24], Proca has given an example to explain that $M^*(\varpi,\xi)$ is more general than the value of the maximum of Ciric type contraction [25].
The following notion will be used throughout the study.

Definition 1.14. [26] *Let* $(\mathbf{S},m)$ *be a metric space and* $\Gamma,\Upsilon : \mathbf{S} \to \mathbf{S}$ *be two mappings. Then,* Γ *and* Υ *are said to be weakly compatible if* $\Gamma\varpi = \Upsilon\varpi$ *implies* $\Gamma\Upsilon\varpi = \Upsilon\Gamma\varpi$ *for some* $\varpi \in \mathbf{S}$*.*

2. Main results

Owing to the fact that the concept of modular metrics does not have to be finite, the following requirements are essential to assuring the existence and uniqueness of fixed points of contraction mappings in modular metric and modular $b-$metric spaces.

(M_1) $\ell_\lambda(\varpi,\Gamma\varpi)<\infty$ for all $\lambda>0$ and $\varpi\in\mathbf{S}_\ell^*$,
(M_2) $\ell_\lambda(\varpi,\xi)<\infty$ for all $\lambda>0$ and $\varpi,\xi\in\mathbf{S}_\ell^*$.

In this section, we aim to characterize the concept of $\mathscr{G}(\Sigma,\vartheta,\Xi)-$contractions by considering the $C-$class function endowed with the functions Σ and ϑ, including generalized $\Xi-$contractions for four mappings in the framework of modular $b-$metric spaces. We also put forward some new results derived immediately from the main result.

Definition 2.1. *Let ℓ be a modular $b-$metric with $\kappa\geq 1$ on set $\mathbf{S}_\ell^*$, and let $\Gamma,\Upsilon,J,\zeta:\ \mathbf{S}_\ell^*\to\ \mathbf{S}_\ell^*$ be mappings. The mappings $\Gamma,\Upsilon,J,$ and ζ are called $\mathscr{G}(\Sigma,\vartheta,\Xi)$-contraction, if there exist $\mathscr{G}\in\mathscr{C}$, $\Sigma\in\Omega$, and $\vartheta\in\Pi$ such that*

$$\Sigma\left(\kappa^3\ell_\lambda(\Gamma\varpi,\Upsilon\xi)\right)\leq\mathscr{G}(\Sigma(\Xi(\varpi,\xi)),\vartheta(\Xi(\varpi,\xi))), \tag{2.1}$$

where

$$\begin{aligned}\Xi(\varpi,\xi)=\max\{&\ell_\lambda(\zeta\varpi,J\xi)+|\ell_\lambda(\zeta\varpi,\Gamma\varpi)-\ell_\lambda(J\xi,\Upsilon\xi)|;\ \ell_\lambda(\zeta\varpi,\Gamma\varpi)+|\ell_\lambda(\zeta\varpi,J\xi)-\ell_\lambda(J\xi,\Upsilon\xi)|;\\ &\ell_\lambda(J\xi,\Upsilon\xi)+|\ell_\lambda(\zeta\varpi,J\xi)-\ell_\lambda(\zeta\varpi,\Gamma\varpi)|;\ \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(\zeta\varpi,\Upsilon\xi)+\ell_{2\lambda}(J\xi,\Gamma\varpi)}{\kappa}+|\ell_\lambda(\zeta\varpi,\Gamma\varpi)-\ell_\lambda(J\xi,\Upsilon\xi)|\right]\Big\},\end{aligned}$$

for all distinct $\varpi,\xi\in\mathbf{S}_\ell^*$, and all $\lambda>0$.

Theorem 2.2. *Let $\mathbf{S}_\ell^*$ be an $\ell-$complete MbMS with constant $\kappa\geq 1$. Assume that the following statements are ensured:*

(i) The mappings $\Gamma,\Upsilon,J,$ and ζ are a $\mathscr{G}(\Sigma,\vartheta,\Xi)-$contraction such that $\Gamma(\mathbf{S}_\ell^*)\subset J(\mathbf{S}_\ell^*)$ and $\Upsilon(\mathbf{S}_\ell^*)\subset\zeta(\mathbf{S}_\ell^*)$,
(ii) One of the sets $\Gamma(\mathbf{S}_\ell^*),J(\mathbf{S}_\ell^*),\Upsilon(\mathbf{S}_\ell^*)$ and $\zeta(\mathbf{S}_\ell^*)$ is a closed subset of $\mathbf{S}_\ell^*$,
(iii) The pairs $\{J,\Upsilon\}$ and $\{\zeta,\Gamma\}$ are weakly compatible.

If the condition (M_1) is satisfied, then Γ,Υ,J and ζ admit a common fixed point in $\mathbf{S}_\ell^*$. Moreover, the condition (M_2) is provided, the common fixed point of $\Gamma,\Upsilon,J,$ and ζ is unique.

Proof. Let $\varpi_0\in\mathbf{S}_\ell^*$ be an arbitrary point. If we take into account the condition (i), there is a point $\varpi_1\in\mathbf{S}_\ell^*$ such that $\xi_0=\Gamma\varpi_0=J\varpi_1$. In a similar way, one can find a point $\varpi_2\in\mathbf{S}_\ell^*$ such that $\xi_1=\Upsilon\varpi_1=\zeta\varpi_2$ as $\Upsilon(\mathbf{S}_\ell^*)\subseteq\zeta(\mathbf{S}_\ell^*)$. Following the above process, we acquire a sequence $\{\xi_q\}$ such that

$$\xi_{2q}=\Gamma\varpi_{2q}=J\varpi_{2q+1}\quad\textit{and}\quad\xi_{2q+1}=\Upsilon\varpi_{2q+1}=\zeta\varpi_{2q+2}.$$

Assume that $\xi_{q_0}\neq\xi_{q_0+1}$, because if we accept that $\xi_{q_0}=\xi_{q_0+1}$ for some q_0, the proof is evident. Therefore, we have $\ell_\lambda(\xi_{2q},\xi_{2q+1})>0$ for all $\lambda>0$. From (2.1), we procure

$$\Sigma\left(\kappa^3\ell_\lambda(\Gamma\varpi_{2q},\Upsilon\varpi_{2q+1})\right)\leq\mathscr{G}\left(\Sigma\left(\Xi(\varpi_{2q},\varpi_{2q+1})\right),\vartheta\left(\Xi(\varpi_{2q},\varpi_{2q+1})\right)\right),$$

where

$$\begin{aligned}\Xi(\varpi_{2q},\varpi_{2q+1})=\max\Big\{&\ell_\lambda(\zeta\varpi_{2q},J\varpi_{2q+1})+\left|\ell_\lambda(\zeta\varpi_{2q},\Gamma\varpi_{2q})-\ell_\lambda(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})\right|;\\ &\ell_\lambda(\zeta\varpi_{2q},\Gamma\varpi_{2q})+\left|\ell_\lambda(\zeta\varpi_{2q},J\varpi_{2q+1})-\ell_\lambda(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})\right|;\\ &\ell_\lambda(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})+\left|\ell_\lambda(\zeta\varpi_{2q},J\varpi_{2q+1})-\ell_\lambda(\zeta\varpi_{2q},\Gamma\varpi_{2q})\right|;\\ &\frac{1}{2}\left[\frac{\ell_{2\lambda}(\zeta\varpi_{2q},\Upsilon\varpi_{2q+1})+\ell_{2\lambda}(J\varpi_{2q+1},\Gamma\varpi_{2q})}{\kappa}+\left|\ell_\lambda(\zeta\varpi_{2q},\Gamma\varpi_{2q})-\ell_\lambda(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})\right|\right]\Big\}\\ =\max\Big\{&\ell_\lambda(\xi_{2q-1},\xi_{2q})+\left|\ell_\lambda(\xi_{2q-1},\xi_{2q})-\ell_\lambda(\xi_{2q},\xi_{2q+1})\right|;\\ &\ell_\lambda(\xi_{2q-1},\xi_{2q})+\left|\ell_\lambda(\xi_{2q-1},\xi_{2q})-\ell_\lambda(\xi_{2q},\xi_{2q+1})\right|;\\ &\ell_\lambda(\xi_{2q},\xi_{2q+1})+\left|\ell_\lambda(\xi_{2q-1},\xi_{2q})-\ell_\lambda(\xi_{2q-1},\xi_{2q})\right|;\\ &\frac{1}{2}\left[\frac{\ell_{2\lambda}(\xi_{2q-1},\xi_{2q+1})+\ell_{2\lambda}(\xi_{2q},\xi_{2q})}{\kappa}+\left|\ell_\lambda(\xi_{2q-1},\xi_{2q})-\ell_\lambda(\xi_{2q},\xi_{2q+1})\right|\right]\Big\}.\end{aligned}$$

Now, if we assume that $\sigma_q=\ell_\lambda(\xi_{q-1},\xi_q)$ and use the triangle inequality

$$\ell_{2\lambda}(\xi_{2q-1},\xi_{2q+1})\leq\kappa\left[\ell_\lambda(\xi_{2q-1},\xi_{2q})+\ell_\lambda(\xi_{2q},\xi_{2q+1})\right],$$

we get that

$$\Sigma\left(\kappa^3\sigma_{2q+1}\right)\ \leq\mathscr{G}\left(\Sigma\left(\Xi(\varpi_{2q},\varpi_{2q+1})\right),\vartheta\left(\Xi(\varpi_{2q},\varpi_{2q+1})\right)\right), \tag{2.2}$$

where

$$\Xi\left(\varpi_{2q},\varpi_{2q+1}\right) \leq \max\left\{\sigma_{2q}+\left|\sigma_{2q}-\sigma_{2q+1}\right|;\ \sigma_{2q}+\left|\sigma_{2q}-\sigma_{2q+1}\right|;\ \sigma_{2q+1}+\left|\sigma_{2q}-\sigma_{2q}\right|;\ \frac{1}{2}\left[\frac{\kappa\left(\sigma_{2q}+\sigma_{2q+1}\right)}{\kappa}+\left|\sigma_{2q}-\sigma_{2q+1}\right|\right]\right\}.$$

If $\sigma_{2q+1} \geq \sigma_{2q}$, we achieve

$$\Xi\left(\varpi_{2q},\varpi_{2q+1}\right) \leq \max\left\{\sigma_{2q+1},\frac{\sigma_{2q}+\sigma_{2q+1}+\sigma_{2q+1}-\sigma_{2q}}{2}\right\} = \sigma_{2q+1}.$$

From the above, it is concluded that

$$\Sigma\left(\sigma_{2q+1}\right) \leq \Sigma\left(\kappa^3\sigma_{2q+1}\right) \leq \mathscr{G}\left(\Sigma\left(\sigma_{2q+1}\right),\vartheta\left(\sigma_{2q+1}\right)\right) \leq \Sigma\left(\sigma_{2q+1}\right),$$

which means

$$\mathscr{G}\left(\Sigma\left(\sigma_{2q+1}\right),\vartheta\left(\sigma_{2q+1}\right)\right) = \Sigma\left(\sigma_{2q+1}\right).$$

From $(\mathscr{G}_2)$, either $\Sigma\left(\sigma_{2q+1}\right)=0$ or $\vartheta\left(\sigma_{2q+1}\right)=0$. Nevertheless, a contradictory situation arises in both cases due to our assumption. For $\sigma_{2q+1}<\sigma_{2q}$, we have $\left|\sigma_{2q}-\sigma_{2q+1}\right| = \sigma_{2q}-\sigma_{2q+1}$ and

$$\Xi\left(\varpi_{2q},\varpi_{2q+1}\right) \leq \max\left\{2\sigma_{2q}-\sigma_{2q+1},\sigma_{2q+1}\frac{\sigma_{2q}+\sigma_{2q+1}+\sigma_{2q}-\sigma_{2q+1}}{2}\right\}.$$

As $\left[2\sigma_{2q}-\sigma_{2q+1}>\sigma_{2q}>\sigma_{2q+1}\right]$, we yield that

$$\Xi\left(\varpi_{2q},\varpi_{2q+1}\right) = \max\left\{2\sigma_{2q}-\sigma_{2q+1},\sigma_{2q+1},\sigma_{2q}\right\} = 2\sigma_{2q}-\sigma_{2q+1}. \tag{2.3}$$

Moreover, by repeating similar steps, we acquire that $\sigma_{2q}<\sigma_{2q-1}$. Then, it ensures $\sigma_{q+1}<\sigma_q$. So, we say $\left\{\sigma_q\right\} = \left\{\ell_\lambda\left(\xi_{q-1},\xi_q\right)\right\}$ is a non-increasing sequence of non-negative real numbers. Thereby, there exists $\tau \geq 0$ such that $\lim_{q\to\infty}\sigma_q=\tau$ for all $\lambda>0$. Now, we aim to show $\tau=0$.

By using $(\mathscr{G}_1)$ and (2.3), contemplating the inequality (2.2), we get

$$\Sigma\left(\sigma_{2q+1}\right) \leq \Sigma\left(\kappa^3\sigma_{2q+1}\right) \leq \mathscr{G}\left(\Sigma\left(2\sigma_{2q}-\sigma_{2q+1}\right),\vartheta\left(2\sigma_{2q}-\sigma_{2q+1}\right)\right) \leq \Sigma\left(2\sigma_{2q}-\sigma_{2q+1}\right).$$

If we take the limit in the above inequality, we have

$$\Sigma(\tau) \leq \mathscr{G}\left(\Sigma(\tau),\vartheta(\tau)\right) \leq \Sigma(\tau),$$

and consequently, $\mathscr{G}\left(\Sigma(\tau),\vartheta(\tau)\right)=\Sigma(\tau)$. Then, from $(\mathscr{G}_2)$, either $\Sigma(\tau)=0$ or $\vartheta(\tau)=0$. This implies that $\tau=0$, i.e., for all $\lambda>0$

$$\ell_\lambda\left(\xi_{q-1},\xi_q\right) \to 0, \quad (q\to\infty). \tag{2.4}$$

We need to show that $\left\{\xi_q\right\}$ is an $\ell-$Cauchy sequence. It is adequate to demonstrate that $\left\{\xi_{2q}\right\}$ is an $\ell-$Cauchy sequence. Presume on, by contrast; we will find $\varepsilon>0$ and also form two sequences $\{a_i\}$ and $\{b_i\}$ of positive integers fulfilling $b_i>a_i\geq i$ such that b_i is the smallest index for which

$$\ell_\lambda\left(\xi_{2a_i},\xi_{2b_i}\right) \geq \varepsilon \quad \text{and} \quad \ell_\lambda\left(\xi_{2a_i},\xi_{2b_i-2}\right) < \varepsilon, \quad \text{for all } \lambda>0. \tag{2.5}$$

From (2.5), we gain

$$\varepsilon \leq \ell_\lambda\left(\xi_{2a_i},\xi_{2b_i}\right) \leq \kappa\ell_{\frac{\lambda}{2}}\left(\xi_{2a_i},\xi_{2b_i+1}\right)+\kappa\ell_{\frac{\lambda}{2}}\left(\xi_{2b_i+1},\xi_{2b_i}\right).$$

Taking the limit superior in the above expression as $i\to\infty$ and by utilizing (2.4), we attain

$$\limsup_{q\to\infty}\ell_\lambda\left(\xi_{2a_i},\xi_{2b_i+1}\right) \geq \frac{\varepsilon}{\kappa}, \quad \text{for all } \lambda>0. \tag{2.6}$$

From (ℓ_3), we acquire

$$\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right) \leq \kappa\ell_{\frac{\lambda}{2}}\left(\xi_{2a_i-1},\xi_{2a_i}\right)+\kappa^2\ell_{\frac{\lambda}{2}}\left(\xi_{2a_i},\xi_{2b_i-2}\right)+\kappa^3\ell_{\frac{\lambda}{4}}\left(\xi_{2b_i-2},\xi_{2b_i-1}\right)+\kappa^3\ell_{\frac{\lambda}{4}}\left(\xi_{2b_i-1},\xi_{2b_i}\right).$$

Again, by taking the limit as $i\to\infty$ and taking the expressions (2.4) and (2.5) into account, the above inequality provides that

$$\limsup_{k\to\infty}\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)\leq\kappa^2\varepsilon\,,\quad\text{for all }\lambda>0. \tag{2.7}$$

Thereby, by (2.1), we procure

$$\Sigma\left(\kappa^3\ell_\lambda\left(\Gamma\varpi_{2a_i},\Upsilon\varpi_{2b_i+1}\right)\right)\leq\mathscr{G}\left(\Sigma\left(\Xi\left(\varpi_{2a_i},\varpi_{2b_i+1}\right)\right),\vartheta\left(\Xi\left(\varpi_{2a_i},\varpi_{2b_i+1}\right)\right)\right), \tag{2.8}$$

where

$$\begin{aligned}\Xi\left(\varpi_{2a_i},\varpi_{2b_i+1}\right)=\max\Big\{&\ell_\lambda\left(\zeta\varpi_{2a_i},J\varpi_{2b_i+1}\right)+\left|\ell_\lambda\left(\zeta\varpi_{2a_i},\Gamma\varpi_{2a_i}\right)-\ell_\lambda\left(J\varpi_{2b_i+1},\Upsilon\varpi_{2b_i+1}\right)\right|;\\&\ell_\lambda\left(\zeta\varpi_{2a_i},\Gamma\varpi_{2a_i}\right)+\left|\ell_\lambda\left(\zeta\varpi_{2a_i},J\varpi_{2b_i+1}\right)-\ell_\lambda\left(J\varpi_{2b_i+1},\Upsilon\varpi_{2b_i+1}\right)\right|;\\&\ell_\lambda\left(J\varpi_{2b_i+1},\Upsilon\varpi_{2b_i+1}\right)+\left|\ell_\lambda\left(\zeta\varpi_{2a_i},J\varpi_{2b_i+1}\right)-\ell_\lambda\left(\zeta\varpi_{2a_i},\Gamma\varpi_{2a_i}\right)\right|;\\&\frac{1}{2}\left[\frac{\ell_{2\lambda}\left(\zeta\varpi_{2a_i},\Upsilon\varpi_{2b_i+1}\right)+\ell_{2\lambda}\left(J\varpi_{2b_i+1},\Gamma\varpi_{2a_i}\right)}{\kappa}+\left|\ell_\lambda\left(\zeta\varpi_{2a_i},\Gamma\varpi_{2a_i}\right)-\ell_\lambda\left(J\varpi_{2b_i+1},\Upsilon\varpi_{2b_i+1}\right)\right|\right]\Big\}\\=\max\Big\{&\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)+\left|\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)-\ell_\lambda\left(\xi_{2a_i},\xi_{2b_i+1}\right)\right|;\\&\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)+\left|\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)-\ell_\lambda\left(\xi_{2a_i},\xi_{2b_i+1}\right)\right|;\\&\ell_\lambda\left(\xi_{2a_i},\xi_{2b_i+1}\right)+\left|\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)-\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)\right|;\\&\frac{1}{2}\left[\frac{\ell_{2\lambda}\left(\xi_{2a_i-1},\xi_{2b_i+1}\right)+\ell_{2\lambda}\left(\xi_{2b_i},\xi_{2a_i}\right)}{\kappa}+\left|\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)-\ell_\lambda\left(\xi_{2a_i},\xi_{2b_i+1}\right)\right|\right]\Big\}.\end{aligned}$$

Also, by using (ℓ_3), we derive

$$\begin{aligned}\ell_{2\lambda}\left(\xi_{2a_i-1},\xi_{2b_i+1}\right)&\leq\kappa\left[\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)+\ell_\lambda\left(\xi_{2b_i},\xi_{2b_i+1}\right)\right],\\\ell_{2\lambda}\left(\xi_{2b_i},\xi_{2a_i}\right)&\leq\kappa\left[\ell_\lambda\left(\xi_{2b_i},\xi_{2a_i-1}\right)+\ell_\lambda\left(\xi_{2a_i-1},\xi_{2a_i}\right)\right].\end{aligned} \tag{2.9}$$

Consequently, we combine the inequalities (2.8) and (2.9), we deduce $\Xi\left(\varpi_{2a_i},\varpi_{2b_i+1}\right)\leq\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)$, thence, we get

$$\Sigma\left(\kappa^3\ell_\lambda\left(\xi_{2a_i},\xi_{2b_i+1}\right)\right)\leq\mathscr{G}\left(\Sigma\left(\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)\right),\vartheta\left(\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)\right)\right). \tag{2.10}$$

Now, by employing (2.6), (2.7), and $(\mathscr{G}_1)$, if we take the limit as $(i\to\infty)$ in (2.10), then we achieve

$$\Sigma\left(\kappa^3\frac{\varepsilon}{\kappa}\right)\leq\mathscr{G}\left(\Sigma\left(\kappa^2\varepsilon\right),\vartheta\left(\kappa^2\varepsilon\right)\right)\leq\Sigma\left(\kappa^2\varepsilon\right)$$

which stands for

$$\mathscr{G}\left(\Sigma\left(\kappa^2\varepsilon\right),\vartheta\left(\kappa^2\varepsilon\right)\right)=\Sigma\left(\kappa^2\varepsilon\right),$$

hence, it must be either $\Sigma\left(\kappa^2\varepsilon\right)=0$ or $\vartheta\left(\kappa^2\varepsilon\right)=0$. As $\kappa\geq1$ and $\varepsilon>0$, it is a contradiction, that is, $\left\{\xi_{2q}\right\}$ is an $\ell-$Cauchy sequence. Thus, $\left\{\xi_q\right\}$ is an $\ell-$Cauchy sequence in $\mathbf{S}_\ell^*$. Since $\mathbf{S}_\ell^*$ is an $\ell-$complete MbMS, there exists $c\in\mathbf{S}_\ell^*$ such that

$$\lim_{q\to\infty}\xi_q=c. \tag{2.11}$$

Now, we aim to show that $\Gamma c=\Upsilon c=Jc=\zeta c=c$. Primarily, we prove that $\Gamma c=\zeta c=c$, that is, c is a common fixed point for the maps Γ and ζ. The following statements are obvious.

$$\lim_{q\to\infty}\xi_{2q}=\lim_{q\to\infty}\Gamma\varpi_{2q}=\lim_{q\to\infty}J\varpi_{2q+1}=c,$$

$$\lim_{q\to\infty}\xi_{2q+1}=\lim_{q\to\infty}\Upsilon\varpi_{2q+1}=\lim_{q\to\infty}\zeta\varpi_{2q+2}=c.$$

Considering the hypothesis, let $\zeta\left(\mathbf{S}_\ell^*\right)$ be a closed subset of $\mathbf{S}_\ell^*$, there exists $u\in\mathbf{S}_\ell^*$ such that $c=\zeta u$. We claim that $\Gamma u=c$. Let us replace ϖ and ξ in expression (2.1) with u and ϖ_{2q+1}, respectively.

$$\Sigma\left(\kappa^3\ell_\lambda\left(\Gamma u,\Upsilon\varpi_{2q+1}\right)\right)\leq\mathscr{G}\left(\Sigma\left(\Xi\left(u,\varpi_{2q+1}\right)\right),\vartheta\left(\Xi\left(u,\varpi_{2q+1}\right)\right)\right),$$

where

$$\begin{aligned}\Xi\left(u,\varpi_{2q+1}\right)=\max\Big\{&\ell_\lambda\left(\zeta u,J\varpi_{2q+1}\right)+\left|\ell_\lambda\left(\zeta u,\Gamma u\right)-\ell_\lambda\left(J\varpi_{2q+1},\Upsilon\varpi_{2q+1}\right)\right|;\\&\ell_\lambda\left(\zeta u,\Gamma u\right)+\left|\ell_\lambda\left(\zeta u,J\varpi_{2q+1}\right)-\ell_\lambda\left(J\varpi_{2q+1},\Upsilon\varpi_{2q+1}\right)\right|;\\&\ell_\lambda\left(J\varpi_{2q+1},\Upsilon\varpi_{2q+1}\right)+\left|\ell_\lambda\left(\zeta u,J\varpi_{2q+1}\right)-\ell_\lambda\left(\zeta u,\Gamma u\right)\right|;\\&\frac{1}{2}\left[\frac{\ell_{2\lambda}\left(\zeta u,\Upsilon\varpi_{2q+1}\right)+\ell_{2\lambda}\left(J\varpi_{2q+1},\Gamma u\right)}{\kappa}+\left|\ell_\lambda\left(\zeta u,\Gamma u\right)-\ell_\lambda\left(J\varpi_{2q+1},\Upsilon\varpi_{2q+1}\right)\right|\right]\Big\}\\=\max\Big\{&\ell_\lambda\left(c,\xi_{2q}\right)+\left|\ell_\lambda\left(c,\Gamma u\right)-\ell_\lambda\left(\xi_{2q},\xi_{2q+1}\right)\right|;\\&\ell_\lambda\left(c,\Gamma u\right)+\left|\ell_\lambda\left(c,\xi_{2q}\right)-\ell_\lambda\left(\xi_{2q},\xi_{2q+1}\right)\right|;\\&\ell_\lambda\left(\xi_{2q},\xi_{2q+1}\right)+\left|\ell_\lambda\left(c,\xi_{2q}\right)-\ell_\lambda\left(c,\Gamma u\right)\right|;\\&\frac{1}{2}\left[\frac{\ell_{2\lambda}\left(c,\xi_{2q+1}\right)+\ell_{2\lambda}\left(\xi_{2q},\Gamma u\right)}{\kappa}+\left|\ell_\lambda\left(c,\Gamma u\right)-\ell_\lambda\left(\xi_{2q},\xi_{2q+1}\right)\right|\right]\Big\}.\end{aligned}$$

Also, by using (2.11) and $(\mathscr{G}_1)$, if we take the limit as $q\to\infty$ in the above, and note that

$$\ell_{2\lambda}\left(\xi_{2q},\Gamma u\right)\leq\kappa\left[\ell_\lambda\left(\xi_{2q},\xi_{2q+1}\right)+\ell_\lambda\left(\xi_{2q+1},\Gamma u\right)\right],$$

we conclude $\Xi\left(u,\varpi_{2q+1}\right)\leq\ell_\lambda\left(c,\Gamma u\right)$. Hence, we obtain

$$\Sigma\left(\ell_\lambda\left(\Gamma u,c\right)\right)\leq\Sigma\left(\kappa^3\ell_\lambda\left(\Gamma u,c\right)\right)\leq\mathscr{G}\left(\Sigma\left(\ell_\lambda\left(\Gamma u,c\right)\right),\vartheta\left(\ell_\lambda\left(\Gamma u,c\right)\right)\right)\leq\Sigma\left(\ell_\lambda\left(\Gamma u,c\right)\right),$$

which implies the following

$$\mathscr{G}\left(\Sigma\left(\ell_\lambda\left(\Gamma u,c\right)\right),\vartheta\left(\ell_\lambda\left(\Gamma u,c\right)\right)\right)=\Sigma\left(\ell_\lambda\left(\Gamma u,c\right)\right).$$

Then, from $(\mathscr{G}_2)$, either $\Sigma\left(\ell_\lambda\left(\Gamma u,c\right)\right)=0$ or $\vartheta\left(\ell_\lambda\left(\Gamma u,c\right)\right)=0$, which yields $\ell_\lambda\left(\Gamma u,c\right)=0\Leftrightarrow\Gamma u=c$. Therefore, $\Gamma u=\zeta u=c$. Since the mappings Γ and ζ are weakly compatible, we have $\Gamma c=\Gamma\zeta u=\zeta\Gamma u=\zeta c$. Next, we claim that $\Gamma c=c$. Again, from (2.1), we get

$$\Sigma\left(\kappa^3\ell_\lambda\left(\Gamma c,\Upsilon\varpi_{2q+1}\right)\right)\leq\mathscr{G}\left(\Sigma\left(\Xi\left(c,\varpi_{2q+1}\right)\right),\vartheta\left(\Xi\left(c,\varpi_{2q+1}\right)\right)\right),$$

where

$$\begin{aligned}\Xi\left(c,\varpi_{2q+1}\right)=\max\Big\{&\ell_\lambda\left(\zeta c,J\varpi_{2q+1}\right)+\left|\ell_\lambda\left(\zeta c,\Gamma c\right)-\ell_\lambda\left(J\varpi_{2q+1},\Upsilon\varpi_{2q+1}\right)\right|;\\&\ell_\lambda\left(\zeta c,\Gamma c\right)+\left|\ell_\lambda\left(\zeta c,J\varpi_{2q+1}\right)-\ell_\lambda\left(J\varpi_{2q+1},\Upsilon\varpi_{2q+1}\right)\right|;\\&\ell_\lambda\left(J\varpi_{2q+1},\Upsilon\varpi_{2q+1}\right)+\left|\ell_\lambda\left(\zeta c,J\varpi_{2q+1}\right)-\ell_\lambda\left(\zeta c,\Gamma c\right)\right|;\\&\frac{1}{2}\left[\frac{\ell_{2\lambda}\left(\zeta c,\Upsilon\varpi_{2q+1}\right)+\ell_{2\lambda}\left(J\varpi_{2q+1},\Gamma c\right)}{\kappa}+\left|\ell_\lambda\left(\zeta c,\Gamma c\right)-\ell_\lambda\left(J\varpi_{2q+1},\Upsilon\varpi_{2q+1}\right)\right|\right]\Big\}\\=\max\Big\{&\ell_\lambda\left(\Gamma c,\xi_{2q}\right)+\left|\ell_\lambda\left(\zeta c,\Gamma c\right)-\ell_\lambda\left(\xi_{2q},\xi_{2q+1}\right)\right|;\\&\ell_\lambda\left(\zeta c,\Gamma c\right)+\left|\ell_\lambda\left(\Gamma c,\xi_{2q}\right)-\ell_\lambda\left(\xi_{2q},\xi_{2q+1}\right)\right|;\\&\ell_\lambda\left(\xi_{2q},\xi_{2q+1}\right)+\left|\ell_\lambda\left(\Gamma c,\xi_{2q}\right)-\ell_\lambda\left(\zeta c,\Gamma c\right)\right|;\\&\frac{1}{2}\left[\frac{\ell_{2\lambda}\left(\Gamma c,\xi_{2q+1}\right)+\ell_{2\lambda}\left(\xi_{2q},\Gamma c\right)}{\kappa}+\left|\ell_\lambda\left(\zeta c,\Gamma c\right)-\ell_\lambda\left(\xi_{2q},\xi_{2q+1}\right)\right|\right]\Big\}.\end{aligned}$$

Likewise, by utilizing (2.11), $(\mathscr{G}_1)$ and noting

$$\ell_{2\lambda}\left(\xi_{2q},\Gamma c\right)\leq\kappa\left[\ell_\lambda\left(\xi_{2q},\xi_{2q+1}\right)+\ell_\lambda\left(\xi_{2q+1},\Gamma c\right)\right],$$

taking the limit as $q\to\infty$ in the above, we get

$$\Xi\left(c,\varpi_{2q+1}\right)\leq\ell_\lambda\left(\Gamma c,c\right)$$

and, so

$$\Sigma\left(\ell_\lambda\left(\Gamma c,c\right)\right)\leq\Sigma\left(\kappa^3\ell_\lambda\left(\Gamma c,c\right)\right)\leq\mathscr{G}\left(\Sigma\left(\Xi\left(\Gamma c,c\right)\right),\vartheta\left(\Xi\left(\Gamma c,c\right)\right)\right)\leq\Sigma\left(\Xi\left(\Gamma c,c\right)\right).$$

Thus, we have

$$\mathscr{G}\left(\Sigma\left(\ell_\lambda\left(\Gamma c,c\right)\right),\vartheta\left(\ell_\lambda\left(\Gamma c,c\right)\right)\right)=\Sigma\left(\ell_\lambda\left(\Gamma c,c\right)\right).$$

By $(\mathscr{G}_2)$, either $\Sigma\left(\ell_\lambda\left(\Gamma c,c\right)\right)=0$ or $\vartheta\left(\ell_\lambda\left(\Gamma c,c\right)\right)=0$. This shows that $\Gamma c=c$. The next step is to prove that c is the fixed point of Υ and J. Because $\Gamma\left(\mathbf{S}_\ell^*\right)\subset J\left(\mathbf{S}_\ell^*\right)$, there is an element v in $\mathbf{S}_\ell^*$ such that $\Gamma c=Jv$. Then, $\Gamma c=Jv=\zeta c=c$. We claim that $\Upsilon v=c$. From (2.1)

$$\Sigma\left(\kappa^3\ell_\lambda\left(\Gamma c,\Upsilon v\right)\right)\leq\mathscr{G}\left(\Sigma\left(\Xi\left(c,v\right)\right),\vartheta\left(\Xi\left(c,v\right)\right)\right),$$

where

$$\begin{aligned}\Xi\left(c,v\right)=\max\Big\{&\ell_\lambda\left(\zeta c,Jv\right)+\left|\ell_\lambda\left(\zeta c,\Gamma c\right)-\ell_\lambda\left(Jv,\Upsilon v\right)\right|;\ \ell_\lambda\left(\zeta c,\Gamma c\right)+\left|\ell_\lambda\left(\zeta c,Jv\right)-\ell_\lambda\left(Jv,\Upsilon v\right)\right|;\\&\ell_\lambda\left(Jv,\Upsilon v\right)+\left|\ell_\lambda\left(\zeta c,Jv\right)-\ell_\lambda\left(\zeta c,\Gamma c\right)\right|;\ \frac{1}{2}\left[\frac{\ell_{2\lambda}\left(\zeta c,\Upsilon v\right)+\ell_{2\lambda}\left(Jv,\Gamma c\right)}{\kappa}+\left|\ell_\lambda\left(\zeta c,\Gamma c\right)-\ell_\lambda\left(Jv,\Upsilon v\right)\right|\right]\Big\}\\=\max\Big\{&\ell_\lambda\left(\zeta c,c\right)+\left|\ell_\lambda\left(\zeta c,c\right)-\ell_\lambda\left(c,\Upsilon v\right)\right|;\ \ell_\lambda\left(\zeta c,c\right)+\left|\ell_\lambda\left(\zeta c,c\right)-\ell_\lambda\left(c,\Upsilon v\right)\right|;\\&\ell_\lambda\left(c,\Upsilon v\right)+\left|\ell_\lambda\left(\zeta c,c\right)-\ell_\lambda\left(\zeta c,c\right)\right|;\ \frac{1}{2}\left[\frac{\ell_{2\lambda}\left(c,\Upsilon v\right)+\ell_{2\lambda}\left(c,\Gamma c\right)}{\kappa}+\left|\ell_\lambda\left(c,\Gamma c\right)-\ell_\lambda\left(c,\Upsilon v\right)\right|\right]\Big\}.\end{aligned}$$

Note that $\ell_{2\lambda}\left(c,\Upsilon v\right)\leq\kappa\left[\ell_\lambda\left(c,\xi_{2q}\right)+\ell_\lambda\left(\xi_{2q},\Upsilon v\right)\right]$, we get $\Xi\left(c,v\right)\leq\ell_\lambda\left(c,\Upsilon v\right)$. Then, by using $(\mathscr{G}_1)$, we have

$$\begin{aligned}\Sigma\left(\ell_\lambda\left(c,\Upsilon v\right)\right)\leq\Sigma\left(\kappa^3\ell_\lambda\left(c,\Upsilon v\right)\right)&\leq\mathscr{G}\left(\Sigma\left(\ell_\lambda\left(c,\Upsilon v\right)\right),\vartheta\left(\ell_\lambda\left(c,\Upsilon v\right)\right)\right)\\&\leq\Sigma\left(\ell_\lambda\left(c,\Upsilon v\right)\right),\end{aligned}$$

which implies

$$\mathscr{G}(\Sigma(\ell_\lambda(c,\Upsilon v)),\vartheta(\ell_\lambda(c,\Upsilon v)))=\Sigma(\ell_\lambda(c,\Upsilon v)).$$

So, similar to the above, by using $(\mathscr{G}_2)$, it is clear that $\Upsilon v=c$. By the weak compatibility of the mappings Υ and J, we achieve that $\Upsilon c=\Upsilon Jv=JJv=Jc$.
Finally, we demand that $\Upsilon c=c$. Using from (2.1) we achieve

$$\Sigma\left(\kappa^3\ell_\lambda(\Gamma c,\Upsilon c)\right)\leq\mathscr{G}(\Sigma(\Xi(c,c)),\vartheta(\Xi(c,c))),$$

where

$$\begin{aligned}\Xi(c,c)=\max\{&\ell_\lambda(\zeta c,Jc)+|\ell_\lambda(\zeta c,\Gamma c)-\ell_\lambda(Jc,\Upsilon c)|;\ \ell_\lambda(\zeta c,\Gamma c)+|\ell_\lambda(\zeta c,Jc)-\ell_\lambda(Jc,\Upsilon c)|;\\ &\ell_\lambda(Jc,\Upsilon c)+|\ell_\lambda(\zeta c,Jc)-\ell_\lambda(\zeta c,\Gamma c)|;\ \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(\zeta c,\Upsilon c)+\ell_{2\lambda}(Jc,\Gamma c)}{\kappa}+|\ell_\lambda(\zeta c,\Gamma c)-\ell_\lambda(Jc,\Upsilon c)|\right]\Big\}\\ =\max\{&\ell_\lambda(c,\Upsilon c)+|\ell_\lambda(c,c)-\ell_\lambda(\Upsilon c,\Upsilon c)|;\ \ell_\lambda(\zeta c,c)+|\ell_\lambda(\zeta c,c)-\ell_\lambda(c,\Upsilon c)|;\\ &\ell_\lambda(c,\Upsilon v)+|\ell_\lambda(\zeta c,c)-\ell_\lambda(\zeta c,c)|;\ \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(c,\Upsilon c)+\ell_{2\lambda}(\Upsilon c,c)}{\kappa}+|\ell_\lambda(c,c)-\ell_\lambda(\Upsilon c,\Upsilon c)|\right]\Big\}.\end{aligned}$$

Hence, this implies

$$\Xi(c,c)=\max\left\{\ell_\lambda(\Upsilon c,c),\frac{\ell_{2\lambda}(\Upsilon c,c)}{\kappa}\right\}=\ell_\lambda(\Upsilon c,c)$$

and from $(\mathscr{G}_1)$, we obtain

$$\Sigma(\ell_\lambda(c,\Upsilon c))\leq\Sigma\left(\kappa^3\ell_\lambda(\Upsilon c,c)\right)\leq\mathscr{G}(\Sigma(\ell_\lambda(\Upsilon c,c)),\vartheta(\ell_\lambda(\Upsilon c,c)))\leq\Sigma(\ell_\lambda(\Upsilon c,c)),$$

which yields

$$\mathscr{G}(\Sigma(\ell_\lambda(c,\Upsilon c)),\vartheta(\ell_\lambda(c,\Upsilon c)))=\Sigma(\ell_\lambda(c,\Upsilon c)).$$

Then, from$(\mathscr{G}_2)$, one can conclude $\Gamma c=\Upsilon c=\zeta c=Jc=c$. Since $\Gamma\left(\mathbf{S}_\ell^*\right)\subset J\left(\mathbf{S}_\ell^*\right)$ and $\Upsilon\left(\mathbf{S}_\ell^*\right)\subset\zeta\left(\mathbf{S}_\ell^*\right)$, similar calculations can be done for the case in which $J\left(\mathbf{S}_\ell^*\right)$(or $\Gamma\left(\mathbf{S}_\ell^*\right)$, $\Upsilon\left(\mathbf{S}_\ell^*\right)$) is closed.
In conclusion, for the uniqueness of the common fixed point of Γ,Υ, J, and ζ, suppose that c^* is another common fixed point of our mappings, that is, $c^*=\Gamma c^*=\Upsilon c^*=Jc^*=\zeta c^*$ such that $c\neq c^*$. Then, from (2.1), we have

$$\Sigma\left(\kappa^3\ell_\lambda(\Gamma c,\Upsilon c^*)\right)\leq\mathscr{G}(\Sigma(\Xi(c,c^*)),\vartheta(\Xi(c,c^*))),$$

where

$$\begin{aligned}\Xi(c,c^*)=\max\{&\ell_\lambda(\zeta c,Jc^*)+|\ell_\lambda(\zeta c,\Gamma c)-\ell_\lambda(Jc^*,\Upsilon c^*)|;\ \ell_\lambda(\zeta c,\Gamma c)+|\ell_\lambda(\zeta c,Jc^*)-\ell_\lambda(Jc^*,\Upsilon c^*)|;\\ &\ell_\lambda(Jc^*,\Upsilon c^*)+|\ell_\lambda(\zeta c,Jc^*)-\ell_\lambda(\zeta c,\Gamma c)|;\ \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(\zeta c,\Upsilon c^*)+\ell_{2\lambda}(Jc^*,\Gamma c)}{\kappa}+|\ell_\lambda(\zeta c,\Gamma c)-\ell_\lambda(Jc^*,\Upsilon c^*)|\right]\Big\}\\ =\max\{&\ell_\lambda(c,c^*)+|\ell_\lambda(c,c)-\ell_\lambda(c^*,c^*)|;\ \ell_\lambda(c,c)+|\ell_\lambda(c,c^*)-\ell_\lambda(c^*,c^*)|;\\ &\ell_\lambda(c^*,c^*)+|\ell_\lambda(c,c^*)-\ell_\lambda(c,c)|;\ \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(c,c^*)+\ell_{2\lambda}(c^*,c)}{\kappa}+|\ell_\lambda(c,c)-\ell_\lambda(c^*,c^*)|\right]\Big\}\\ =\max&\left\{\ell_\lambda(c,c^*),\tfrac{\ell_{2\lambda}(c,c^*)}{\kappa}\right\}=\ell_\lambda(c,c^*).\end{aligned}$$

Thus, by $(\mathscr{G}_1)$, we get

$$\Sigma(\ell_\lambda(c,c^*))\leq\Sigma\left(\kappa^3\ell_\lambda(c,c^*)\right)\leq\mathscr{G}(\Sigma(\ell_\lambda(c,c^*)),\vartheta(\ell_\lambda(c,c^*)))\leq\Sigma(\ell_\lambda(c,c^*)),$$

which implies

$$\mathscr{G}(\Sigma(\ell_\lambda(c,c^*)),\vartheta(\ell_\lambda(c,c^*)))=\Sigma(\ell_\lambda(c,c^*)).$$

Thereby, it is easy to show that $c=c^*$, which means that the common fixed point of Γ,Υ, J, and ζ is unique. □

Inferences drawn directly from the main result are presented below.

Corollary 2.3. *Let $\mathbf{S}_\ell^*$ be an ℓ-complete MbMS with constant $\kappa\geq 1$ and let $\Gamma,\Upsilon:\ \mathbf{S}_\ell^*\to\ \mathbf{S}_\ell^*$ be two self-mappings. Presume that the following statements are satisfied:*

(i) There exist $\mathscr{G} \in \mathscr{C}$, $\Sigma \in \Omega$, and $\vartheta \in \Pi$ such that

$$\Sigma\left(\kappa^3 \ell_\lambda\left(\Gamma\varpi, \Upsilon\xi\right)\right) \leq \mathscr{G}\left(\Sigma\left(\Xi\left(\varpi,\xi\right)\right), \vartheta\left(\Xi\left(\varpi,\xi\right)\right)\right),$$

where

$$\begin{aligned}\Xi\left(\varpi,\xi\right) = \max\{&\ell_\lambda\left(\varpi,\xi\right) + \left|\ell_\lambda\left(\varpi,\Gamma\varpi\right) - \ell_\lambda\left(\xi,\Upsilon\xi\right)\right|; \ell_\lambda\left(\varpi,\Gamma\varpi\right) + \left|\ell_\lambda\left(\varpi,\xi\right) - \ell_\lambda\left(\xi,\Upsilon\xi\right)\right|;\\ &\ell_\lambda\left(\xi,\Upsilon\xi\right) + \left|\ell_\lambda\left(\varpi,\xi\right) - \ell_\lambda\left(\varpi,\Gamma\varpi\right)\right|; \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(\varpi,\Upsilon\xi)+\ell_{2\lambda}(\xi,\Gamma\varpi)}{\kappa} + \left|\ell_\lambda\left(\varpi,\Gamma\varpi\right) - \ell_\lambda\left(\xi,\zeta\xi\right)\right|\right]\Big\},\end{aligned}$$

for all distinct $\varpi,\xi \in \mathbf{S}_\ell^*$ and for all $\lambda > 0$.

(ii) The conditions (M_1) and (M_2) are provided.

Then, Γ and Υ admit a common unique fixed point in $\mathbf{S}_\ell^*$.

If we distinguish $\Gamma = \Upsilon$ in Corollary 2.3, we yield the below one.

Corollary 2.4. *Let $\mathbf{S}_\ell^*$ be an ℓ-complete MbMS with coefficient $\kappa \geq 1$ and let $\Gamma: \mathbf{S}_\ell^* \to \mathbf{S}_\ell^*$ be a mapping. Presume that the following circumstances are satisfied:*

(i) There exist $\mathscr{G} \in \mathscr{C}$, $\Sigma \in \Omega$, and $\vartheta \in \Pi$ such that

$$\Sigma\left(\kappa^3 \ell_\lambda\left(\Gamma\varpi, \Gamma\xi\right)\right) \leq \mathscr{G}\left(\Sigma\left(\Xi\left(\varpi,\xi\right)\right), \vartheta\left(\Xi\left(\varpi,\xi\right)\right)\right),$$

where

$$\begin{aligned}\Xi\left(\varpi,\xi\right) = \max\{&\ell_\lambda\left(\varpi,\xi\right) + \left|\ell_\lambda\left(\varpi,\Gamma\varpi\right) - \ell_\lambda\left(\xi,\Gamma\xi\right)\right|; \ell_\lambda\left(\varpi,\Gamma\varpi\right) + \left|\ell_\lambda\left(\varpi,\xi\right) - \ell_\lambda\left(\xi,\Gamma\xi\right)\right|;\\ &\ell_\lambda\left(\xi,\Gamma\xi\right) + \left|\ell_\lambda\left(\varpi,\xi\right) - \ell_\lambda\left(\varpi,\Gamma\varpi\right)\right|; \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(\varpi,\Gamma\xi)+\ell_{2\lambda}(\xi,\Gamma\varpi)}{\kappa} + \left|\ell_\lambda\left(\varpi,\Gamma\varpi\right) - \ell_\lambda\left(\xi,\Gamma\xi\right)\right|\right]\Big\},\end{aligned} \tag{2.12}$$

for all distinct $\varpi,\xi \in \mathbf{S}_\ell^*$ and for all $\lambda > 0$.

(ii) The conditions (M_1) and (M_2) are provided.

Then Γ possesses a unique fixed point in $\mathbf{S}_\ell^*$.

Also, we get the following corollary if we choose $\mathscr{G}(p,q) = p - q$ for all $p,q \in [0,\infty)$ in Corollary 2.4.

Corollary 2.5. *Let $\mathbf{S}_\ell^*$ be an ℓ-complete MbMS with constant $\kappa \geq 1$ and let $\Gamma: \mathbf{S}_\ell^* \to \mathbf{S}_\ell^*$ be a mapping. Presume that the below statements hold:*

(i) There exist $\Sigma \in \Omega$ and $\vartheta \in \Pi$ such that

$$\Sigma\left(\kappa^3 \ell_\lambda\left(\Gamma\varpi, \Gamma\xi\right)\right) \leq \Sigma\left(\Xi\left(\varpi,\xi\right)\right) - \vartheta\left(\Xi\left(\varpi,\xi\right)\right),$$

where $\Xi(\varpi,\xi)$ is as in (2.12), for all distinct $\varpi,\xi \in \mathbf{S}_\ell^*$ and for all $\lambda > 0$.

(ii) The conditions (M_1) and (M_2) are provided.

Then, Γ holds a unique fixed point in $\mathbf{S}_\ell^*$.

If we choose that $\mathscr{G}(p,q) = kp$, $k \in (0,1)$, for all $p \in [0,\infty)$, then we possess the below one.

Corollary 2.6. *Let $\mathbf{S}_\ell^*$ be an ℓ-complete MbMS with $\kappa \geq 1$ and let $\Gamma: \mathbf{S}_\ell^* \to \mathbf{S}_\ell^*$ be a mapping. Presume that the following conditions hold:*

(i) There exist $\Sigma \in \Omega$ and $k \in (0,1)$ such that

$$\Sigma\left(\kappa^3 \ell_\lambda\left(\Gamma\varpi, \Gamma\xi\right)\right) \leq k\Sigma\left(\Xi\left(\varpi,\xi\right)\right),$$

where $\Xi(\varpi,\xi)$ is defined as in (2.12) and for all $\varpi,\xi \in \mathbf{S}_\ell^*$ and $\lambda > 0$.

(ii) The conditions (M_1) and (M_2) are provided.

Thus, Γ admits a unique fixed point in $\mathbf{S}_\ell^*$.

3. An application to graph theory

Let $\mathbf{S}_\ell^*$ be an ℓ-complete MbMS with $\kappa \geq 1$, and let consider: $\Lambda = \{(\varpi,\varpi) : \varpi \in \mathbf{S}_\ell^*\}$, which denotes the diagonal of the Cartesian product $\mathbf{S}_\ell^* \times \mathbf{S}_\ell^*$. Also, let H be a directed graph such that

- $V(H)$: vertices coincide with $\mathbf{S}_\ell^*$,
- $B(H)$: edges contain all loops such that $\Lambda \subseteq B(H)$.

The pair $(V(H), B(H))$ could be displayed as the graph H. The following set

$$B\left(H^{-1}\right) = \{(\varpi,\xi) \in \mathbf{S}_\ell^* \times \mathbf{S}_\ell^* | \quad (\xi,\varpi) \in B(H)\},$$

is identified where H^{-1} is obtained in the graph H by reversing the direction of edges and called conversion of H. The graph H can be called an undirected graph, which denotes $\tilde{H}$, in case of the direction is ignored and so, we get

$$B\left(\tilde{H}\right) = B(H) \cup B\left(H^{-1}\right).$$

Let K be a subgraph of a graph H such that $V(K) \subseteq V(H)$ and $B(K) \subseteq B(H)$. If ϖ and ξ be vertices in a graph H, then a path from ϖ to ξ of length $j \in \mathbf{N}$ is a sequence (ϖ_j), which has $j+1$ distinct vertices such that $\varpi = \varpi_0, \varpi_1, ..., \varpi_j$ and $(\varpi_{i-1}, \varpi_i) \in B(H)$ for $i = 1, \ldots, j$.

Also, H is called a "connected graph" if there is a path between two vertices. Moreover, H is a "weakly connected graph", provided that $\tilde{H}$ is connected. For more detail about the graph theory, see [27]-[29].

Definition 3.1. *Let $\Gamma : \mathbf{S} \to \mathbf{S}$ be a mapping on a metric space $(\mathbf{S}, m)$. Presume that the followings hold:*

(i) $(\varpi,\xi) \in B(H) \Rightarrow (\Gamma\varpi, \Gamma\xi) \in B(H)$, *for* all $\varpi, \xi \in \mathbf{S}$,
(ii) $m(\Gamma\varpi,\Gamma\xi) \leq \mu m(\varpi,\xi)$, *for* all $(\varpi,\xi) \in B(H)$, *and* $\mu \in (0,1)$.

Then, Γ is called a Banach H-contraction mapping on $\mathbf{S}$.

Let $\mathbf{S}_\ell^*$ be an MbMS endowed with a graph H and $\Gamma : \mathbf{S}_\ell^* \to \mathbf{S}_\ell^*$. We set

$$\mathbf{S}_\ell^{*\Gamma} = \{\varpi \in \mathbf{S}_\ell^* | (\varpi, \Gamma\varpi) \in B(H)\}.$$

We present a new concept using the graph structure entitled $\mathscr{G}(\Sigma, \vartheta, \Xi)$-graphic contraction, as noted below.

Definition 3.2. *Let $\mathbf{S}_\ell^*$ be an MbMS endowed with a graph H. Presume that the following statements are provided:*

(i) *Γ preserves edges of H, i.e.,*

$$(\varpi,\xi) \in B(H) \quad \Rightarrow \quad (\Gamma\varpi,\Gamma\xi) \in B(H)$$

for all $\varpi\xi \in \mathbf{S}_\ell^*$.

(ii) *There exists a $\mathscr{G} \in \mathscr{C}$, $\Sigma \in \Omega$, and $\vartheta \in \Pi$ such that*

$$\Sigma\left(\kappa^3 \ell_\lambda(\Gamma\varpi,\Gamma\xi)\right) \leq \mathscr{G}(\Sigma(\Xi(\varpi,\xi)), \vartheta(\Xi(\varpi,\xi))), \tag{3.1}$$

where

$$\begin{aligned}\Xi(\varpi,\xi) = \max\{&\ell_\lambda(\varpi,\xi) + |\ell_\lambda(\varpi,\Gamma\varpi) - \ell_\lambda(\xi,\Gamma\xi)|; \ell_\lambda(\varpi,\Gamma\varpi) + |\ell_\lambda(\varpi,\xi) - \ell_\lambda(\xi,\Gamma\xi)|;\\ &\ell_\lambda(\xi,\Gamma\xi) + |\ell_\lambda(\varpi,\xi) - \ell_\lambda(\varpi,\Gamma\varpi)|; \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(\varpi,\Gamma\xi)+\ell_{2\lambda}(\xi,\Gamma\varpi)}{\kappa} + |\ell_\lambda(\varpi,\Gamma\varpi) - \ell_\lambda(\xi,\Gamma\xi)|\right]\Big\},\end{aligned}$$

for all $\varpi, \xi \in B(H)$, and $\lambda > 0$.

Then, Γ is called a $\mathscr{G}(\Sigma, \vartheta, \Xi)$-graphic contraction on $\mathbf{S}_\ell^*$.

Theorem 3.3. *Let $\mathbf{S}_\ell^*$ be an $\ell-$complete MbMS endowed with a graph H and $\Gamma : \mathbf{S}_\ell^* \to \mathbf{S}_\ell^*$ be a mapping. Assume that the circumstances below hold:*

(i) *There exists $\varpi_0 \in \mathbf{S}_\ell^{*\Gamma}$,*
(ii) *Γ is a $\mathscr{G}(\Sigma, \vartheta, \Xi)-$graphic contraction,*
(iii) *If $\{\varpi_k\}$ is a sequence in $\mathbf{S}_\ell^*$ such that $\lim_{k\to\infty} \ell_\lambda(\varpi_k, \varpi^*) = 0$ and $(\varpi_k, \varpi_{k+1}) \in B(H)$, then there exists a subsequence $\{\varpi_{k_s}\}$ of $\{\varpi_k\}$ such that $(\varpi_{k_s}, \varpi^*) \in B(H)$,*
(iv) *H is a weakly connected graph.*

Then, by the conditions (M_1) and (M_2), Γ holds a unique fixed point in $\mathbf{S}_\ell^*$.

Proof. Define the sequence $\{\varpi_k\}$ in $\mathbf{S}_\ell^*$ by $\varpi_{k+1} = \Gamma\varpi_k$ for all $k \in \mathbf{N}$. From (i), since $\varpi_0 \in \mathbf{S}_\ell^{*\Gamma}$; we have $(\varpi_0, \Gamma\varpi_0) \in B(H)$. If we denote $\varpi_1 = \Gamma\varpi_0$, then

$$(\varpi_0, \Gamma\varpi_0) = (\varpi_0, \varpi_1) \in B(G).$$

Because Γ preserves the edges of H, the following expression is provided:

$$(\varpi_0, \varpi_1) \in B(H) \quad \Rightarrow \quad (\Gamma\varpi_0, \Gamma\varpi_1) \in B(H).$$

Continuing this way, we procure

$$(\varpi_k, \varpi_{k+1}) \in B(H).$$

So, from Corollary 2.4, we get $\{\varpi_k\}$ is an $\ell-$Cauchy sequence in $\mathbf{S}_\ell^*$. Because $\mathbf{S}_\ell^*$ is an $\ell-$complete space, there exists $\varpi^* \in \mathbf{S}_\ell^*$ such that

$$\lim_{k\to\infty} \ell_\lambda(\varpi_k, \varpi^*) = 0. \tag{3.2}$$

Now, we show that ϖ^* is a fixed point of Γ. Using (iii), we have $(\varpi_{k_s}, \varpi^*) \in B(H)$. Then, from (3.1), we obtain

$$\Sigma\left(\kappa^3 \ell_\lambda(\Gamma\varpi_{k_s}, \Gamma\varpi^*)\right) \leq \mathscr{G}\left(\Sigma(B(\varpi_{k_s}, \varpi^*)), \vartheta(B(\varpi_{k_s}, \varpi^*))\right), \tag{3.3}$$

where

$$\begin{aligned}
\Xi(\varpi_{k_s}, \varpi^*) = \max\{&\ell_\lambda(\varpi_{k_s}, \varpi^*) + |\ell_\lambda(\varpi_{k_s}, \Gamma\varpi_{k_s}) - \ell_\lambda(\varpi^*, \Gamma\varpi^*)|;\\
&\ell_\lambda(\varpi_{k_s}, \Gamma\varpi_{k_s}) + |\ell_\lambda(\varpi_{k_s}, \varpi^*) - \ell_\lambda(\varpi^*, \Gamma\varpi^*)|;\\
&\ell_\lambda(\varpi^*, \Gamma\varpi^*) + |\ell_\lambda(\varpi_{k_s}, \varpi^*) - \ell_\lambda(\varpi_{k_s}, \Gamma\varpi_{k_s})|;\\
&\tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(\varpi_{k_s}, \Gamma\varpi^*) + \ell_{2\lambda}(\varpi^*, \Gamma\varpi_{k_s})}{\kappa} + |\ell_\lambda(\varpi_{k_s}, \Gamma\varpi_{k_s}) - \ell_\lambda(\varpi^*, \Gamma\varpi^*)|\right]\Big\}\\
= \max\{&\ell_\lambda(\varpi_{k_s}, \varpi^*) + |\ell_\lambda(\varpi_{k_s}, \varpi_{k_s-1}) - \ell_\lambda(\varpi^*, \Gamma\varpi^*)|;\\
&\ell_\lambda(\varpi_{k_s}, \varpi_{k_s-1}) + |\ell_\lambda(\varpi_{k_s}, \varpi^*) - \ell_\lambda(\varpi^*, \Gamma\varpi^*)|;\\
&\ell_\lambda(\varpi^*, \Gamma\varpi^*) + |\ell_\lambda(\varpi_{k_s}, \varpi^*) - \ell_\lambda(\varpi_{k_s}, \varpi_{k_s-1})|;\\
&\tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(\varpi_{k_s}, \Gamma\varpi^*) + \ell_{2\lambda}(\varpi^*, \varpi_{k_s-1})}{\kappa} + |\ell_\lambda(\varpi_{k_s}, \varpi_{k_s-1}) - \ell_\lambda(\varpi^*, \Gamma\varpi^*)|\right]\Big\}.
\end{aligned} \tag{3.4}$$

Taking the limit as $s \to \infty$ in (3.3) and (3.4) and by employing (3.2) and $(\mathscr{G}_1)$, we deduce that

$$\Sigma(\ell_\lambda(\varpi^*, \Gamma\varpi^*)) \leq \Sigma\left(\kappa^3 \ell_\lambda(\varpi^*, \Gamma\varpi^*)\right) \leq \mathscr{G}\left(\Sigma(\ell_\lambda(\varpi^*, \Gamma\varpi^*)), \vartheta(\ell_\lambda(\varpi^*, \Gamma\varpi^*))\right) \leq \Sigma(\ell_\lambda(\varpi^*, \Gamma\varpi^*)),$$

and as a consequence, the subsequent term is found.

$$\mathscr{G}\left(\Sigma(\ell_\lambda(\varpi^*, \Gamma\varpi^*)), \vartheta(\ell_\lambda(\varpi^*, \Gamma\varpi^*))\right) = \Sigma(\ell_\lambda(\varpi^*, \Gamma\varpi^*)).$$

Using the properties of $\mathscr{G}$, we say that $\varpi^* = \Gamma\varpi^*$, that is, ϖ^* is a fixed point of Γ.
Next, we show that ϖ^* is a unique fixed point of Γ. Let w be another fixed point of Γ, i.e., $w = \Gamma w$, there exists $r \in \mathbf{S}_\ell^*$ such that $(\varpi^*, r) \in B(H)$ and $(r, w) \in B(H)$. Using (iv), we have $(\varpi^*, w) \in B(H)$.
Thence, from (3.2), we achieve

$$\Sigma\left(\kappa^3 \ell_\lambda(\Gamma\varpi^*, \Gamma w)\right) \leq \mathscr{G}\left(\Sigma(\Xi(\varpi^*, w)), \vartheta(\Xi(\varpi^*, w))\right), \tag{3.5}$$

where

$$\begin{aligned}
\Xi(\varpi^*, w) = \max\{&\ell_\lambda(\varpi^*, w) + |\ell_\lambda(\varpi^*, \Gamma\varpi^*) - \ell_\lambda(w, \Gamma w)|; \ell_\lambda(\varpi^*, \Gamma\varpi^*) + |\ell_\lambda(\varpi^*, w) - \ell_\lambda(w, \Gamma w)|;\\
&\ell_\lambda(w, \Gamma w) + |\ell_\lambda(\varpi^*, w) - \ell_\lambda(\varpi^*, \Gamma\varpi^*)|; \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(\varpi^*, \Gamma w) + \ell_{2\lambda}(w, \Gamma\varpi^*)}{\kappa} + |\ell_\lambda(\varpi^*, \Gamma\varpi^*) - \ell_\lambda(w, \Gamma w)|\right]\Big\}.
\end{aligned} \tag{3.6}$$

Together with (3.5) and (3.6), also applying the features of the functions $\mathscr{G}, \Sigma, \vartheta$, we yield that $\varpi^* = w$, hence ϖ^* is a unique fixed point of Γ. □

4. Application to integral type contractions

This section consists of a common fixed point theorem, including integral type contraction and some consequences, which can be obtained by applying particular expressions in the main result.

Definition 4.1. *Let* $\Theta := \{\mu : \mathbf{R}^+ \to \mathbf{R}^+ |\ \mu$ Lebesgue integrable mapping$\}$ *be a class of mappings satisfying the followings:*

(μ_1) μ *is non-negative and summable function;*
(μ_2) *for all* $\varepsilon > 0$

$$\int_0^{\varepsilon} \mu(\rho)\, d\rho > 0.$$

In what follows, Branciari [30] demonstrated a fixed point theorem regarding a contractive condition of integral type.

Theorem 4.2. [30] *Let* Γ *be a self-mapping on a complete metric space* $(\mathbf{S}, m)$*, and there exists* $k \in (0,1)$ *and* $\mu \in \Theta$*. If for* $\varpi, \xi \in \mathbf{S}$

$$\int_0^{m(\Gamma\varpi,\Gamma\xi)} \mu(\rho)\, d\rho \leq k \int_0^{m(\varpi,\xi)} \mu(\rho)\, d\rho$$

is satisfied, then Γ *possesses a unique fixed point in* $(\mathbf{S}, m)$.

Subsequently, numerous studies have been done about the consequence of Branciari with some known properties. In [31], Azadifar et al. verified that a common fixed point theorem was satisfying a contractive condition of integral type in the sense of modular metric spaces.
Now, we construct our main result of this section by defining the $\mathscr{G}(\Sigma, \vartheta, \Xi)-$contraction of integral type, as indicated below.

Definition 4.3. *Let* $\mathbf{S}_\ell^*$ *be an MbMS with the coefficient* $\kappa \geq 1$ *and let* $\Gamma, \Upsilon, J, \zeta : \mathbf{S}_\ell^* \to \mathbf{S}_\ell^*$ *be mappings. The mappings* Γ, Υ, J *and* ζ *are called* $\mathscr{G}(\Sigma, \vartheta, \Xi)$*-contraction of integral type, if there exist* $\mathscr{G} \in \mathscr{C}$*,* $\Sigma \in \Omega$*,* $\vartheta \in \Pi$*, and* $\mu \in \Theta$ *such that*

$$\Sigma\left(\kappa^3 \int_0^{\ell_{\frac{\lambda}{c}}(\Gamma\varpi,\Upsilon\xi)} \mu(\rho)\, d\rho\right) \leq \mathscr{G}\left(\Sigma(\Xi(\varpi,\xi)), \vartheta(\Xi(\varpi,\xi))\right), \tag{4.1}$$

where

$$\begin{aligned}
\Xi(\varpi,\xi) = \max\Bigg\{ & \int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi,J\xi)} \mu(\rho)\, d\rho + \left| \int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi,\Gamma\varpi)} \mu(\rho)\, d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(J\xi,\Upsilon\xi)} \mu(\rho)\, d\rho \right|; \\
& \int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi,\Gamma\varpi)} \mu(\rho)\, d\rho + \left| \int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi,J\xi)} \mu(\rho)\, d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(J\xi,\Upsilon\xi)} \mu(\rho)\, d\rho \right|; \\
& \int_0^{\ell_{\frac{\lambda}{l}}(J\xi,\Upsilon\xi)} \mu(\rho)\, d\rho + \left| \int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi,J\xi)} \mu(\rho)\, d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi,\Gamma\varpi)} \mu(\rho)\, d\rho \right|; \\
& \frac{1}{2}\left[\frac{\int_0^{\ell_{\frac{2\lambda}{l}}(\zeta\varpi,\Upsilon\xi)} \mu(\rho)d\rho + \int_0^{\ell_{\frac{2\lambda}{l}}(J\xi,\Gamma\varpi)} \mu(\rho)d\rho}{\kappa} + \left| \int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi,\Gamma\varpi)} \mu(\rho)\, d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(J\xi,\Upsilon\xi)} \mu(\rho)\, d\rho \right| \right] \Bigg\}
\end{aligned}$$

for all distinct $\varpi, \xi \in \mathbf{S}_\ell^*$*, for* $c, l \in \mathbf{R}^+$*,* $c > l$ *and for all* $\lambda > 0$.

Theorem 4.4. *Let* $\mathbf{S}_\ell^*$ *be an* $\ell-$*complete MbMS with coefficient* $\kappa \geq 1$*. Assume that the following statements hold:*

(i) Γ, Υ, J *and* ζ *be a* $\mathscr{G}(\Sigma, \vartheta, \Xi)-$*contraction such that* $\Gamma(\mathbf{S}_\ell^*) \subset J(\mathbf{S}_\ell^*)$ *and* $\Upsilon(\mathbf{S}_\ell^*) \subset \zeta(\mathbf{S}_\ell^*)$*,*
(ii) One of the sets $\Gamma(\mathbf{S}_\ell^*), J(\mathbf{S}_\ell^*), \Upsilon(\mathbf{S}_\ell^*)$ and $\zeta(\mathbf{S}_\ell^*)$ *is a closed subset of* $\mathbf{S}_\ell^*$*,*
(iii) The pairs $\{J, \Upsilon\}$ *and* $\{\zeta, \Gamma\}$ *are weakly compatible.*

Under the conditions (M_1) *and* (M_2)*, the mappings* Γ*,* Υ*,* J*, and* ζ *admit a unique common fixed point in* $\mathbf{S}_\ell^*$.

Proof. Let $\varpi_0 \in \mathbf{S}_\ell^*$ be an arbitrary point and similar to the proof of Theorem 2.2, we construct a sequence $\{\xi_q\}$ in $\mathbf{S}_\ell^*$ such that

$$\xi_{2q} = \Gamma\varpi_{2q} = J\varpi_{2q+1}, \quad \xi_{2q+1} = \Upsilon\varpi_{2q+1} = \zeta\varpi_{2q+2}.$$

From (4.1), we get

$$\Sigma\left(\kappa^3\int_0^{\ell_{\frac{\lambda}{c}}(\Gamma\varpi_{2q},\Upsilon\varpi_{2q+1})}\mu(\rho)\,d\rho\right)\leq\mathscr{G}\left(\Sigma\left(\Xi\left(\varpi_{2q},\varpi_{2q+1}\right)\right),\vartheta\left(\Xi\left(\varpi_{2q},\varpi_{2q+1}\right)\right)\right),$$

where

$$\Xi\left(\varpi_{2q},\varpi_{2q+1}\right)=\max\left\{\begin{array}{l}\int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi_{2q},J\varpi_{2q+1})}\mu(\rho)\,d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi_{2q},\Gamma\varpi_{2q})}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})}\mu(\rho)\,d\rho\right|;\\ \int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi_{2q},\Gamma\varpi_{2q})}\mu(\rho)\,d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi_{2q},J\varpi_{2q+1})}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})}\mu(\rho)\,d\rho\right|;\\ \int_0^{\ell_{\frac{\lambda}{l}}(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})}\mu(\rho)\,d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi_{2q},J\varpi_{2q+1})}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi_{2q},\Gamma\varpi_{2q})}\mu(\rho)\,d\rho\right|;\\ \frac{1}{2}\left[\frac{\int_0^{\ell_{\frac{2\lambda}{l}}(\zeta\varpi_{2q},\Upsilon\varpi_{2q+1})}\mu(\rho)d\rho+\int_0^{\ell_{\frac{2\lambda}{l}}(J\varpi_{2q+1},\Gamma\varpi_{2q})}\mu(\rho)d\rho}{\kappa}+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi_{2q},\Gamma\varpi_{2q})}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})}\mu(\rho)\,d\rho\right|\right]\end{array}\right\}$$

$$=\max\left\{\begin{array}{l}\int_0^{\ell_{\frac{\lambda}{l}}(\xi_{2q-1},\xi_{2q})}\mu(\rho)\,d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\xi_{2q-1},\xi_{2q})}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(\xi_{2q},\xi_{2q+1})}\mu(\rho)\,d\rho\right|;\\ \int_0^{\ell_{\frac{\lambda}{l}}(\xi_{2q-1},\xi_{2q})}\mu(\rho)\,d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\xi_{2q-1},\xi_{2q})}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(\xi_{2q},\xi_{2q+1})}\mu(\rho)\,d\rho\right|;\\ \int_0^{\ell_{\frac{\lambda}{l}}(\xi_{2q},\xi_{2q+1})}\mu(\rho)\,d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\xi_{2q-1},\xi_{2q})}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(\xi_{2q-1},\xi_{2q})}\mu(\rho)\,d\rho\right|;\\ \frac{1}{2}\left[\frac{\int_0^{\ell_{\frac{2\lambda}{l}}(\xi_{2q-1},\xi_{2q+1})}\mu(\rho)d\rho+\int_0^{\ell_{\frac{2\lambda}{l}}(\xi_{2q},\xi_{2q})}\mu(\rho)d\rho}{\kappa}+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\xi_{2q-1},\xi_{2q})}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(\xi_{2q},\xi_{2q+1})}\mu(\rho)\,d\rho\right|\right]\end{array}\right\}.$$

Now, we accept $\sigma_q=\int_0^{\ell_{\frac{\lambda}{c}}(\xi_{q-1},\xi_q)}\mu(\rho)\,d\rho$, and $\sigma_q^*=\int_0^{\ell_{\frac{\lambda}{l}}(\xi_{q-1},\xi_q)}\mu(\rho)\,d\rho$ with $c>l$ and suppose that $\sigma_{2q+1}^*\geq\sigma_{2q}^*$. Again, similar to the proof of Theorem 2.2 and by using ($\mathscr{G}_1$), we obtain

$$\Sigma\left(\sigma_{2q+1}^*\right)\leq\Sigma\left(\sigma_{2q+1}\right)\leq\Sigma\left(\kappa^3\sigma_{2q+1}\right)\leq\mathscr{G}\left(\Sigma\left(\sigma_{2q+1}^*\right),\vartheta\left(\sigma_{2q+1}^*\right)\right)\leq\Sigma\left(\sigma_{2q+1}^*\right).$$

From the properties of $\mathscr{G}$, we have a contradiction. Then, we get $\sigma_{2q+1}^*<\sigma_{2q}^*$ such that $\Xi\left(\varpi_{2q},\varpi_{2q+1}\right)=2\sigma_{2q}^*-\sigma_{2q+1}^*$. Also, in a similar way, we yield that $\sigma_{2q}^*<\sigma_{2q-1}^*$. So, it ensures that $\sigma_{q+1}^*<\sigma_q^*$ such that

$$\sigma_q^*=\left\{\int_0^{\ell_{\frac{\lambda}{l}}(\xi_{q-1},\xi_q)}\mu(\rho)\,d\rho\right\}$$

is a non-increasing sequence of non-negative real numbers and so the following sequence

$$\left\{\int_0^{\ell_{\frac{\lambda}{l}}(\xi_{q-1},\xi_q)}\mu(\rho)\,d\rho\right\}$$

converges to a non-negative number r. We shall prove that $r=0$. By the same argument, we conclude that

$$\Sigma\left(\sigma_{2q+1}\right)\leq\Sigma\left(\kappa^{3}\sigma_{2q+1}\right)\leq\mathscr{G}\left(\Sigma\left(2\sigma^{*}_{2q}-\sigma^{*}_{2q+1}\right),\vartheta\left(2\sigma^{*}_{2q}-\sigma^{*}_{2q+1}\right)\right)\leq\Sigma\left(2\sigma^{*}_{2q}-\sigma^{*}_{2q+1}\right).$$

If we take the limit above, we gain

$$\Sigma(r)\leq\mathscr{G}(\Sigma(r),\vartheta(r))\leq\Sigma(r),$$

and consequently, $\mathscr{G}(\Sigma(r),\vartheta(r))=\Sigma(r)$. Then, from ($\mathscr{G}_2$), either $\Sigma(r)=0$ or $\vartheta(r)=0$. This implies that $r=0$, that is, $\int_{0}^{\ell_{\lambda}(\xi_{q-1},\xi_{q})}\mu(\rho)\,d\rho\to 0$, as $q\to\infty$. It follows that

$$\ell_{\lambda}\left(\xi_{q-1},\xi_{q}\right)\to 0\,,\quad(q\to\infty).$$

In the next step, we demonstrate that $\left\{\xi_{q}\right\}$ is an $\ell-$Cauchy sequence. It is enough to achieve that $\left\{\xi_{2q}\right\}$ is an $\ell-$Cauchy sequence. If it is not, then we can find $\varepsilon>0$ such that there exist two sequences $\{a_i\}$ and $\{b_i\}$ of positive integers satisfying $b_i>a_i\geq i$ such that b_i is the smallest index for which

$$\ell_{\frac{\lambda}{l}}\left(\xi_{2a_i},\xi_{2b_i}\right)\geq\varepsilon,\quad\text{and}\quad\ell_{\frac{\lambda}{l}}\left(\xi_{2a_i},\xi_{2b_i-2}\right)<\varepsilon,\quad\text{for all }\lambda>0.$$

Note that $\varepsilon\leq\ell_{\frac{\lambda}{l}}\left(\xi_{2a_i},\xi_{2b_i}\right)\leq\ell_{\frac{\lambda}{c}}\left(\xi_{2a_i},\xi_{2b_i}\right)$ and continuing as in the proof of Theorem 2.2, we deduce that

$$\begin{aligned}\Sigma\left(\int_{0}^{\ell_{\frac{\lambda}{c}}\left(\xi_{2a_i},\xi_{2b_i+1}\right)}\mu(\rho)\,d\rho\right)&\leq\Sigma\left(\kappa^{3}\int_{0}^{\ell_{\frac{\lambda}{c}}\left(\xi_{2a_i},\xi_{2b_i+1}\right)}\mu(\rho)\,d\rho\right)\\&\leq\mathscr{G}\left(\Sigma\left(\int_{0}^{\ell_{\frac{\lambda}{l}}\left(\xi_{2a_i-1},\xi_{2b_i}\right)}\mu(\rho)\,d\rho\right),\vartheta\left(\int_{0}^{\ell_{\frac{\lambda}{l}}\left(\xi_{2a_i-1},\xi_{2b_i}\right)}\mu(\rho)\,d\rho\right)\right)\\&\leq\Sigma\left(\int_{0}^{\ell_{\frac{\lambda}{l}}\left(\xi_{2a_i-1},\xi_{2b_i}\right)}\mu(\rho)\,d\rho\right).\end{aligned}$$

Nevertheless, the above inequality causes a contradiction, that is, $\left\{\xi_{2q}\right\}$ is an $\ell-$Cauchy sequence. Thus, $\left\{\xi_{q}\right\}$ is an $\ell-$Cauchy sequence in $\mathbf{S}^{*}_{\ell}$. Since $\mathbf{S}^{*}_{\ell}$ is $\ell-$complete MbMS, there exists $z\in\mathbf{S}^{*}_{\ell}$ such that

$$\lim_{q\to\infty}\xi_{q}=z.$$

Now, we shall prove that $\Gamma z=\Upsilon z=Jz=\zeta z=z$. Indeed, we only need to show that $\Gamma z=\zeta z=z$. Also, similar to Theorem 2.2, it is clear that z is a common fixed point of Υ and J. Assuming that $\zeta\left(\mathbf{S}^{*}_{\ell}\right)$ is a closed subset of $\mathbf{S}^{*}_{\ell}$, there is $u\in\mathbf{S}^{*}_{\ell}$ such that $z=\zeta u$. We claim that $\Gamma u=z$. From (4.1), we have

$$\Sigma\left(\kappa^{3}\int_{0}^{\ell_{\frac{\lambda}{c}}\left(\Gamma u,\Upsilon\varpi_{2q+1}\right)}\mu(\rho)\,d\rho\right)\leq\mathscr{G}\left(\Sigma\left(\Xi\left(u,\varpi_{2q+1}\right)\right),\vartheta\left(\Xi\left(u,\varpi_{2q+1}\right)\right)\right),$$

where

$$\Xi\left(u,\varpi_{2q+1}\right)=\max\left\{\int_{0}^{\ell_{\frac{\lambda}{l}}\left(z,\xi_{2q}\right)}\mu\left(\rho\right)d\rho+\left|\int_{0}^{\ell_{\frac{\lambda}{l}}\left(z,\Gamma u\right)}\mu\left(\rho\right)d\rho-\int_{0}^{\ell_{\frac{\lambda}{l}}\left(\xi_{2q},\xi_{2q+1}\right)}\mu\left(\rho\right)d\rho\right|;\right.$$

$$\int_{0}^{\ell_{\frac{\lambda}{l}}\left(z,\Gamma u\right)}\mu\left(\rho\right)d\rho+\left|\int_{0}^{\ell_{\frac{\lambda}{l}}\left(z,\xi_{2q}\right)}\mu\left(\rho\right)d\rho-\int_{0}^{\ell_{\frac{\lambda}{l}}\left(\xi_{2q},\xi_{2q+1}\right)}\mu\left(\rho\right)d\rho\right|;$$

$$\int_{0}^{\ell_{\frac{\lambda}{l}}\left(\xi_{2q},\xi_{2q+1}\right)}\mu\left(\rho\right)d\rho+\left|\int_{0}^{\ell_{\frac{\lambda}{l}}\left(z,\xi_{2q}\right)}\mu\left(\rho\right)d\rho-\int_{0}^{\ell_{\frac{\lambda}{l}}\left(z,\Gamma u\right)}\mu\left(\rho\right)d\rho\right|;$$

$$\left.\frac{1}{2}\left[\frac{\int_{0}^{\ell_{\frac{2\lambda}{l}}\left(z,\xi_{2q+1}\right)}\mu(\rho)d\rho+\int_{0}^{\ell_{\frac{2\lambda}{l}}\left(\xi_{2q},\Gamma u\right)}\mu(\rho)d\rho}{\kappa}+\left|\int_{0}^{\ell_{\frac{\lambda}{l}}\left(z,\Gamma u\right)}\mu\left(\rho\right)d\rho-\int_{0}^{\ell_{\frac{\lambda}{l}}\left(\xi_{2q},\xi_{2q+1}\right)}\mu\left(\rho\right)d\rho\right|\right]\right\}.$$

Step by step, similar to the proof of Theorem 2.2, we obtain

$$\begin{aligned}\Sigma\left(\int_{0}^{\ell_{\frac{\lambda}{l}}\left(\Gamma u,z\right)}\mu\left(\rho\right)d\rho\right)&\leq\Sigma\left(\int_{0}^{\ell_{\frac{\lambda}{c}}\left(\Gamma u,z\right)}\mu\left(\rho\right)d\rho\right)\leq\Sigma\left(\kappa^{3}\int_{0}^{\ell_{\frac{\lambda}{c}}\left(\Gamma u,z\right)}\mu\left(\rho\right)d\rho\right)\\&\leq\mathscr{G}\left(\Sigma\left(\int_{0}^{\ell_{\frac{\lambda}{l}}\left(\Gamma u,z\right)}\mu\left(\rho\right)d\rho\right),\vartheta\left(\int_{0}^{\ell_{\frac{\lambda}{l}}\left(\Gamma u,z\right)}\mu\left(\rho\right)d\rho\right)\right)\\&\leq\Sigma\left(\int_{0}^{\ell_{\frac{\lambda}{l}}\left(\Gamma u,z\right)}\mu\left(\rho\right)d\rho\right).\end{aligned}$$

Hence, from $(\mathscr{G}_2)$, we achieve that $\Gamma u=\zeta u=z$. Since the maps Γ and ζ are weakly compatible, we have $\Gamma z=\Gamma\zeta u=\zeta\Gamma u=\zeta z$. In another step, we show that $\Gamma z=z$. However, it can be shown similar to the above proof. So $\Gamma z=\zeta z=z$ is procured. Finally, for the uniqueness, we assume that w be another common fixed point, i.e., $\Gamma w=\Upsilon w=Jw=\zeta w$ such that $z\neq w$. Then, from (4.1), we get

$$\Sigma\left(\kappa^{3}\int_{0}^{\ell_{\frac{\lambda}{c}}\left(\Gamma z,\Upsilon w\right)}\mu\left(\rho\right)d\rho\right)\leq\mathscr{G}\left(\Sigma\left(\Xi\left(z,w\right)\right),\vartheta\left(\Xi\left(z,w\right)\right)\right),$$

and it is straightforward that $\Xi(z,w)=\int_{0}^{\ell_{\frac{\lambda}{l}}(z,w)}\mu\left(\rho\right)d\rho$. Thereupon, we get

$$\begin{aligned}\Sigma\left(\int_{0}^{\ell_{\frac{\lambda}{l}}\left(z,w\right)}\mu\left(\rho\right)d\rho\right)&\leq\Sigma\left(\int_{0}^{\ell_{\frac{\lambda}{c}}\left(z,w\right)}\mu\left(\rho\right)d\rho\right)\leq\Sigma\left(\kappa^{3}\int_{0}^{\ell_{\frac{\lambda}{c}}\left(z,w\right)}\mu\left(\rho\right)d\rho\right)\\&\leq\mathscr{G}\left(\Sigma\left(\int_{0}^{\ell_{\frac{\lambda}{l}}\left(z,w\right)}\mu\left(\rho\right)d\rho\right),\vartheta\left(\int_{0}^{\ell_{\frac{\lambda}{l}}\left(z,w\right)}\mu\left(\rho\right)d\rho\right)\right)\\&\leq\Sigma\left(\int_{0}^{\ell_{\frac{\lambda}{l}}\left(z,w\right)}\mu\left(\rho\right)d\rho\right).\end{aligned}$$

Hence, from the properties of the function $\mathscr{G}$, $z=w$ is gained as a unique common fixed point of Γ, Υ, J and ζ. $\square$

Some conclusions can be drawn from the main result of this section are given.

Corollary 4.5. *Let $\mathbf{S}_\ell^*$ be an ℓ-complete MbMS with constant $\kappa \geq 1$ and let Γ and Υ be self mappings in $\mathbf{S}_\ell^*$. The following statements hold:*

(i) *There exist $\mathscr{G} \in \mathscr{C}$, $\Sigma \in \Omega$, $\vartheta \in \Pi$, and $\mu \in \Theta$ such that*

$$\Sigma\left(\kappa^3 \int_0^{\ell_{\frac{\lambda}{c}}(\Gamma\varpi,\Upsilon\xi)} \mu(\rho)\,d\rho\right) \leq \mathscr{G}\left(\Sigma\left(\Xi(\varpi,\xi)\right), \vartheta\left(\Xi(\varpi,\xi)\right)\right),$$

where

$$\Xi(\varpi,\xi) = \max\left\{\begin{array}{l} \int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\xi)} \mu(\rho)\,d\rho + \left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)} \mu(\rho)\,d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Upsilon\xi)} \mu(\rho)\,d\rho\right|; \\ \int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)} \mu(\rho)\,d\rho + \left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\xi)} \mu(\rho)\,d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Upsilon\xi)} \mu(\rho)\,d\rho\right|; \\ \int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Upsilon\xi)} \mu(\rho)\,d\rho + \left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\xi)} \mu(\rho)\,d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)} \mu(\rho)\,d\rho\right|; \\ \frac{1}{2}\left[\frac{\int_0^{\ell_{\frac{2\lambda}{l}}(\varpi,\Upsilon\xi)} \mu(\rho)d\rho + \int_0^{\ell_{\frac{2\lambda}{l}}(\xi,\Gamma\varpi)} \mu(\rho)d\rho}{\kappa} + \left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)} \mu(\rho)\,d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Upsilon\xi)} \mu(\rho)\,d\rho\right|\right] \end{array}\right\},$$

for all distinct $\varpi, \xi \in \mathbf{S}_\ell^*$, for $c, l \in \mathbf{R}^+$, $c > l$ and for all $\lambda > 0$.

(ii) (M_1) *and* (M_2) *are provided.*

The mappings Γ and Υ admit a unique common fixed point in $\mathbf{S}_\ell^*$.

By choosing $\Gamma = \Upsilon$ in Corollary 4.5, we obtain the following one.

Corollary 4.6. *Let $\mathbf{S}_\ell^*$ be an ℓ-complete MbMS with coefficient $\kappa \geq 1$ and let $\Gamma : \mathbf{S}_\ell^* \to \mathbf{S}_\ell^*$ be a mapping. Suppose that the following circumstances are satisfied:*

(i) *There exists $\mathscr{G} \in \mathscr{C}$, $\Sigma \in \Omega$, $\vartheta \in \Pi$ and $\mu \in \Theta$ such that*

$$\Sigma\left(\kappa^3 \int_0^{\ell_{\frac{\lambda}{c}}(\Gamma\varpi,\Gamma\xi)} \mu(\rho)\,d\rho\right) \leq \mathscr{G}\left(\Sigma\left(\Xi(\varpi,\xi)\right), \vartheta\left(\Xi(\varpi,\xi)\right)\right),$$

where

$$\Xi(\varpi,\xi)=\max\left\{\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\xi)}\mu(\rho)\,d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Gamma\xi)}\mu(\rho)\,d\rho\right|;\right.$$
$$\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)}\mu(\rho)\,d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\xi)}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Gamma\xi)}\mu(\rho)\,d\rho\right|;$$
$$\int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Gamma\xi)}\mu(\rho)\,d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\xi)}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)}\mu(\rho)\,d\rho\right|; \tag{4.2}$$
$$\left.\frac{1}{2}\left[\frac{\int_0^{\ell_{\frac{2\lambda}{l}}(\varpi,\Gamma\xi)}\mu(\rho)d\rho+\int_0^{\ell_{\frac{2\lambda}{l}}(\xi,\Gamma\varpi)}\mu(\rho)d\rho}{\kappa}+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Gamma\xi)}\mu(\rho)\,d\rho\right|\right]\right\},$$

for all distinct $\varpi,\xi\in\mathbf{S}_\ell^*$, for $c,l\in\mathbf{R}^+$, $c>l$ and for all $\lambda>0$.

(ii) *The conditions (M_1) and (M_2) are provided.*

The mapping Γ holds a unique fixed point in $\mathbf{S}_\ell^*$.

Besides, we attain the following consequence if we perceive $\mathscr{G}(p,q)=p-q$ for all $p,q\in[0,\infty)$ in Corollary 4.6.

Corollary 4.7. *Let $\mathbf{S}_\ell^*$ be an ℓ-complete MbMS with coefficient $\kappa\geq 1$ and let $\Gamma:\ \mathbf{S}_\ell^*\to\mathbf{S}_\ell^*$ be a mapping. Assume that the following statements hold:*

(i) *There exist $\Sigma\in\Omega$, $\vartheta\in\Pi$ and $\mu\in\Theta$ such that*

$$\Sigma\left(\kappa^3\int_0^{\ell_{\frac{\lambda}{c}}(\Gamma\varpi,\Gamma\xi)}\mu(\rho)\,d\rho\right)\leq\Sigma(\Xi(\varpi,\xi))-\vartheta(\Xi(\varpi,\xi)),$$

where $\Xi(\varpi,\xi)$ is defined as in (4.2) and for all distinct $\varpi,\xi\in\mathbf{S}_\ell^*$, for $c,l\in\mathbf{R}^+$, $c>l$ and for all $\lambda>0$.

(ii) *The condition (M_1) and (M_2) are provided.*

Then, Γ admits a unique fixed point in $\mathbf{S}_\ell^*$.

If we constitute $\mathscr{G}(p,q)=kp,\quad k\in(0,1)$ for all $p\in[0,\infty)$, then we get the below one.

Corollary 4.8. *Let $\mathbf{S}_\ell^*$ be an ℓ-complete MbMS with $\kappa\geq 1$ and let $\Gamma:\ \mathbf{S}_\ell^*\to\mathbf{S}_\ell^*$ be a mapping. Assume that the following ones hold:*

(i) *There exists $\Sigma\in\Omega$ and $\mu\in\Theta$ such that*

$$\Sigma\left(\kappa^3\int_0^{\ell_{\frac{\lambda}{c}}(\Gamma\varpi,\Gamma\xi)}\mu(\rho)\,d\rho\right)\leq k\Sigma(\Xi(\varpi,\xi)),$$

where $\Xi(\varpi,\xi)$ is defined as in (4.2) and for all distinct $\varpi,\xi\in\mathbf{S}_\ell^*$, $c,l\in\mathbf{R}^+$, $c>l$ for all $\lambda>0$.

(ii) *The conditions (M_1) and (M_2) are provided.*

Then, we yield that Γ admits a unique fixed point $\mathbf{S}_\ell^*$.

5. Conclusion

Consequently, we extended the results of Fulga and Proca [19, 20] and [23, 24] to modular $b-$metric space via $C-$class functions for four mappings and examined that our results can be applied to graph structure and integral type contractions. In the meantime, our consequences are still valid in the case of

$$\Xi(\varpi,\xi)=\ell_\lambda(\varpi,\xi)+|\ell_\lambda(\varpi,\Gamma\varpi)-\ell_\lambda(\xi,\Gamma\xi)|.$$

Moreover, by taking $\kappa=1$, the obtained conclusions for MbMS are valid in the setting of modular metric space.

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Author's contributions

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Constacyclic and Negacyclic Codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2$ and their Equivalents over $\mathbb{F}_2$

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Abstract

In this work, we consider the finite ring $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2$, $u^2=1, v^2=0$, $u\cdot v=v\cdot u=0$ which is not Frobenius and chain ring. We studied constacyclic and negacyclic codes in $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2$ with odd length. These codes are compared with codes that had priorly been obtained on the finite field $\mathbb{F}_2$. Moreover, we indicate that the Gray image of a constacyclic and negacyclic code over $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2$ with odd length n is a quasicyclic code of index 4 with length $4n$ in $\mathbb{F}_2$. In particular, the Gray images are applied to two different rings $S_1=\mathbb{F}_2+v\mathbb{F}_2$, $v^2=0$ and $S_2=\mathbb{F}_2+u\mathbb{F}_2$, $u^2=1$ and negacyclic and constacyclic images of these rings are also discussed.

1. Introduction

The fundamental problem in coding theory, such as distance, polynomial representation over codes, weight, etc. were examined in [1]. The Gray images of cyclic and negacyclic codes defined on $\mathbb{Z}_4$ were studied, and their relationships on $\mathbb{Z}_2$ were researched in [2]. In [3], differently in the previously studied the ring $\mathbb{Z}_4$, the images of the $(1+u)-constacyclic$ codes on the finite chain ring $\mathbb{F}_2+u\mathbb{F}_2$ were studied in the case $u^2=0$, and the relationship of cyclic codes between this ring and field $\mathbb{F}_2$ has been mentioned. Moreover, in [4] gray images of $(1+u^2)-constacyclic$ codes on $\mathbb{F}_2+u\mathbb{F}_2+u^2\mathbb{F}_2$ with 8 elements were given on the field $\mathbb{F}_2$ by the same authors in [3]. X. Xiaofang [5] investigated $(1+v)-constacyclic$ codes over $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2, u^2=v^2=0, v\cdot u=u\cdot v=0$, and $(1+v)-constacyclic$ codes in $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2$ of odd length were described through cyclic codes over $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2$.

In this study, unlike in [6], we take the properties of the variables in the ring structure differently. Therefore, a different ring structure emerged. In the next section, we give the primary form of the ring and define the Gray transformations. In the third section, we show that the images of the codes on this ring correspond to codes in the finite rings. Finally, in the last part, we also match the codes found to codes on $\mathbb{F}_2$.

2. Preliminaries

We denote $R=\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2$ as a ring with characteristic 2, where $u^2=1, v^2=0, u\cdot v=v\cdot u=0$. It is clearly see that $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2\cong\mathbb{F}_2[u,v]/\langle u^2=1, v^2=0, u\cdot v=v\cdot u=0\rangle$. Consider $R=\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2=\{0,1,u,1+u,v,1+v,u+v,1+u+v\}$. Thus R is a ring under "$+$" and "." operations. Also, 1 and $1+v$ are units in R, and all the ideals of R can be given by $\{0\}=I_0$, I_u, I_v, I_{u+v}, $I_{1+u}=I_{1+u+v}$, $I_{1+v}=R$. We consider R as a natural extension of $S_1=\mathbb{F}_2+v\mathbb{F}_2$, $v^2=0$. Thus, $S_1\cong\mathbb{F}_2[v]/\langle v^2\rangle$. Then, the elements of S_1 are $0,1,v,1+v$ where the units in S_1 are 1 and $1+v$. We consider R as a natural

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extension of $S_2 = \mathbb{F}_2 + u\mathbb{F}_2$, $u^2 = 1$. Therefore, $S_2 \cong \mathbb{F}_2[u]/\langle u^2\rangle$. The elements of S_2 are $0,1,u$ and $1+u$. Then, the only units in S_2 are 1 and u. Let us take C as a linear code with length n over R (S_1^n or S_2^n). Thus C is a R (S_1^n or S_2^n) submodule of R^n (S_1^n or S_2^n). If D is a linear code with length n in $\mathbb{F}_2$, in this case, D is a $\mathbb{F}_2$ subvector space $\mathbb{F}_2^n$. An element of C and D is called a codeword.

Let Γ_1 denote the Gray map on R (see [6]).

$$\Gamma_1 : R \mapsto S_1^2$$

$$a+ub+vc \mapsto \Gamma_1(a+ub+vc) = \Gamma_1(r+uq) = (v\cdot r, q)$$

where $r = a+vc$ and $q = b+vc$. It can be extended to R^n as shown below:
$\Gamma_1(c_0,c_1,\dots,c_{n-1}) = (v\cdot r_0, v\cdot r_1,\dots, v\cdot r_{n-1}, q_0,q_1,\dots,q_{n-1})$ where $c_i = r_i + u\cdot q_i$ for all $0 \le i \le n-1$. Let the Gray map Ψ_1 on R be defined as indicated below:

$$\Psi_1 : R \mapsto S_2^2$$

$$a+ub+vc \mapsto \Psi_1(a+ub+vc) = \Psi_1(r+vq) = (u\cdot r, q) \tag{2.1}$$

such that $r = a+ub$ and $q = c+ub$. We will extend Ψ_1 to R^n, that is, $\Psi_1(c_0,c_1,\dots,c_{n-1}) = (u\cdot r_0, u\cdot r_1,\dots,u\cdot r_{n-1},q_0,q_1,\dots,q_{n-1})$ where $c_i = r_i + v\cdot q_i$ for all $0\le i\le n-1$.

Let us define the Gray map Γ_2 on S_1 as the following:

$$\Gamma_2 : S_1 \mapsto \mathbb{F}_2^2$$

$$s+vt \mapsto (s, s+t) \tag{2.2}$$

where $s,t\in\mathbb{F}_2$. The extension of Γ_2 to S_1^n is given by

$$\Gamma_2 : S_1^n \mapsto \mathbb{F}_2^{2n}$$

$$(c_0,c_1,\dots,c_{n-1}) \mapsto (s_0,\dots,s_{n-1},s_0+t_0,\dots,s_{n-1}+t_{n-1})$$

where $c_i = s_i + v\cdot t_i$, $s_i,t_i\in\mathbb{F}_2$ for all $0\le i\le n-1$. The Gray map Ψ_2 on S_2 is given by

$$\Psi_2 : S_2 \mapsto \mathbb{F}_2^2$$

$$s+ut \mapsto (s, s+t) \tag{2.3}$$

where $s,t\in\mathbb{F}_2$. The extension of Ψ_2 to S_2^n is given by

$$\Psi_2 : S_2^n \mapsto \mathbb{F}_2^{2n}$$

$$(c_0,c_1,\dots,c_{n-1}) \mapsto (s_0,\dots,s_{n-1},s_0+t_0,\dots,s_{n-1}+t_{n-1})$$

where $c_i = s_i + u\cdot t_i$, $s_i,t_i\in\mathbb{F}_2$ for all $0\le i\le n-1$. For $r\in R$, we define the weight function $w_1(r)$ by

$$w_1(r) = \begin{cases} 0\,;\ r=0 \\ 1\,;\ r=1 \\ 2\,;\ r=v,1+u,1+u+v \\ 3\,;\ r=u,u+v,1+v \end{cases}$$

For $r\in R$, we define the weight function $w_2(r)$ by

$$w_2(r) = \begin{cases} 0\,;\ r=0 \\ 1\,;\ r=1,u,u+v \\ 2\,;\ r=u,1+v,1+u+v \\ 3\,;\ r=1+u \end{cases}$$

Then $w_1(r)$ and $w_2(r)$ extend to a weight function in R^n. If $r=(r_0,r_1,\dots,r_{n-1})\in R^n$, then we write $w_1(r)=\sum_{i=0}^{n-1} w_1(r_i)$ and $w_2(r)=\sum_{i=0}^{n-1} w_2(r_i)$. Let $x,\ y\in R^n$ be any distinct vectors. The distance $d_1(x,y)$ and $d_2(x,y)$ can be defined to be $w_1(x-y)$ and $w_2(x-y)$. The d_1 and d_2 minimum distance of C can be given by $d_1(C)=min\{d_1(x,y)\}$ and $d_2(C)=min\{d_2(x,y)\}$ for any $x,y\in C$ such that $x\neq y$. The weights $w_3(t)$ of $t\in S_1$ and $w_4(t)$ of $t\in S_2$ can be given by

$$w_3(t)=\begin{cases} 0\,;\ t=0 \\ 1\,;\ t=v,1+v \\ 2\,;\ t=1 \end{cases}$$

$$w_4(t)=\begin{cases} 0\,;\ t=0 \\ 1\,;\ t=u,1+u \\ 2\,;\ t=1 \end{cases}$$

These extend to w_3 and w_4 weight functions in S_1^n and S_2^n. If $t=(t_0,t_1,\dots,t_{n-1})\in S_1^n,\ S_2^n$, then we have $w_3(t)=\sum_{i=0}^{n-1} w_3(t_i)$ and $w_4(t)=\sum_{i=0}^{n-1} w_4(t_i)$. Let $x,y\in S_1^n,S_2^n$ be any distinct vectors. The distance $d_3(x,y)$ and $d_4(x,y)$ between x,y can be given by $w_{S_1}(x-y)$ and $w_{S_2}(x-y)$, respectively. Also, the d_3 and d_4 minimum distance of C is defined as $d_3(C)=min\ \{\ d_3\ (x,y)\ \}$ and $d_4(C)=min\{d_4(x,y)\}$ for any $x,y\in C$ such that $x\neq y$. Let D as a code with length n over $\mathbb{F}_2$ and $c=(c_0,c_1,\dots,c_{n-1})$ be a codeword of D. The Hamming weight of D is defined as $w_H(c)=\sum_{i=0}^{n-1} w_H(c_i)$ where $w_H(c_i)=1$ if $c_i=1$ and $w_H(c_i)=0$ if $c_i=0$. In addition, we can define the minimum Hamming distance of D such as $d_H=min\ \{d_H(c,\tilde{c})\}$ for any $c,\tilde{c}\in D$, $c\neq\tilde{c}$. The elements of R as $a+ub+vc=r+vq$ where $r=a+ub$ and $q=c+ub$ are in S_2, we have

$$w_1(a+ub+vc)=w_1(r+vq)=w_4(ur,q)=w_4(b+ua,c+ub)=w_H(b,c,b+a,c+b)$$

Similarly, the elements of R as $a+ub+vc=r+vq$ where $r=a+vc$ and $q=b+vc$ are in S_1 and so we obtain the following

$$w_2(a+ub+vc)=w_2(r+uq)=w_3(vr,q)=w_3(av,b+vc)=w_H(0,b,a,b+c)$$

Definition 2.1. [1] Let C be a linear code over R with length n. A cyclic shift on R^n is a permutation σ such that $\sigma\ (c_0,c_1,\dots,c_{n-1})=(c_{n-1},c_0,\dots,c_{n-1})$. If $\sigma(C)=C$, the code C is said to be cyclic code. A $(1+v)-constacylic$ shift μ act on R^n as $\mu(c_0,c_1,\dots,c_{n-1})=((1+v)c_{n-1},c_0,\dots,c_{n-2})$. The code C is called $(1+v)-constacyclic$ code if $\mu(C)=C$. A *negacylic* shift δ act on R^n as $\delta(c_0,c_1,\dots,c_{n-1})=(-c_{n-1},c_0,\dots,c_{n-2})$. If $\delta(C)=C$, C is said to be *negacylic* code.

Let $P(C)=\left\{\sum_{i=0}^{n-1} r_ix^i:(r_0,r_1,\dots,r_{n-1})\in C\right\}$. $P(C)$ is a polynomial representation of code C with length n over R. Note that C is cyclic if and only if $P(C)$ is an ideal of $R[x]/\langle x^n-1\rangle$ and C is $(1+v)-constacyclic$ if and only if $P(C)$ is an ideal of $R[x]/\langle x^n-(1+v)\rangle$.

Definition 2.2. [1] Let $a\in S_1^{2n}$ with $a=(a_0,a_1,\dots,a_{2n-1})=(a^{(0)}|a^{(1)})$, $a^{(i)}\in S_1^n$ for all $i=0,1$ and σ be the usual cyclic shift.

$$\sigma_1^{*2}:S_1^{2n}\mapsto S_1^{2n}$$
$$a\mapsto\sigma_1^{*2}(a)=(\sigma(a^{(0)})|\sigma(a^{(1)}))$$

A code $\hat{C}$ of length $2n$ in S_1 is called *quasicyclic* code with index 2 if $\sigma_1^{*2}(\hat{C})=\hat{C}$. Let $a\in S_2^{2n}$ with $a=(a_0,a_1,\dots,a_{2n-1})=(a^{(0)}|a^{(1)})$, $a^{(i)}\in S_2^n$ for all $i=0,1$ and σ be the usual cyclic shift.

$$\sigma_2^{*2}:S_2^{2n}\mapsto S_2^{2n}$$
$$a\mapsto\sigma_2^{*2}(a)=(\sigma(a^{(0)})|\sigma(a^{(1)}))$$

A code $\hat{C}$ with length $2n$ in S_2 is called *quasicyclic* code with index 2 if $\sigma_2^{*2}(\hat{C})=\hat{C}$. Take $a\in\mathbb{F}_2^{4n}$ with $a=(a_0,a_1,\dots,a_{4n-1})=(a^{(0)}|a^{(1)}|a^{(2)}|a^{(3)})$, $a^{(i)}\in\mathbb{F}_2^n$ for all $i=0,1,2,3$ and let σ be the usual cyclic shift.

$$\sigma^{*4}:\mathbb{F}_2^{4n}\mapsto\mathbb{F}_2^{4n}$$
$$a\mapsto\sigma^{*4}(a)=(\sigma(a^{(0)})|\sigma(a^{(1)})|\sigma(a^{(2)})|\sigma(a^{(3)}))$$

A code $\hat{D}$ of length $4n$ over $\mathbb{F}_2$ is called *quasicyclic* code with index 4 if $\sigma^{*4}(\hat{D})=\hat{D}$.

3. Negacyclic codes and their gray images

We get *quasicyclic* code of index 2 with even length in S_2 as the Gray image Ψ_1 of *negacyclic* code over R. Therefore, we construct the Gray image Ψ_2 of *quasicyclic* code of index 2 in S_2 with even length.

Proposition 3.1. $\sigma_2^{*2}\Psi_1=\Psi_1\delta$

Proof. Ψ_1, σ_2^{*2} and δ are defined in (2.1) and in [1], respectively. Let $c=(c_0,c_1,\dots,c_{n-1})\in R^n$ such that $c_i=r_i+v\cdot q_i$ for $i=0,1,\dots,n-1$.

$$\Psi_1(c_0,c_1,\dots,c_{n-1})=\Psi_1(r_0+v\cdot q_0,r_1+v\cdot q_1,\dots,r_{n-1}+v\cdot q_{n-1})=(u\cdot r_0,u\cdot r_1,\dots,u\cdot r_{n-1},q_0,q_1,\dots,q_{n-1})$$

By applying σ_2^{*2}, we have

$$\Psi_1(c_0,c_1,\ldots,c_{n-1})=\sigma_2^{*2}(u\cdot r_0,u\cdot r_1,\ldots,u\cdot r_{n-1},q_0,q_1,\ldots,q_{n-1})=(u\cdot r_{n-1},u\cdot r_0,\ldots,u\cdot r_{n-2},q_{n-1},q_0,q_1,\ldots,q_{n-2})$$

Conversely, $\delta(c_0,\ldots,c_{n-1})=(-c_{n-1},c_0,\ldots,c_{n-2})$ where $-c_{n-1}=r_{n-1}+v\cdot q_{n-1}$. Therefore,

$$\Psi_1(\delta(c))=\Psi_1(r_{n-1}+v\cdot q_{n-1},r_0+v\cdot q_0,\ldots,r_{n-2}+v\cdot q_{n-2})=(u\cdot r_{n-1},u\cdot r_0,\ldots,u\cdot r_{n-2},q_{n-1},q_0,\ldots,q_{n-2})$$

Equality is obtained by using the above equations. □

Theorem 3.1 A code C_1 of length n over R is a *negacyclic* code if and only if $\Psi_1(C_1)$ is a *quasicyclic* code of index 2 and length $2n$ over S_2.

Proof. Assume that C_1 is a *negacyclic* code. Then we write $\delta(C_1)=C_1$. By applying Ψ_1, we have $\Psi_1(\delta(C_1))=\Psi_1(C_1)$. By using Proposition 3.1, we have $\sigma_2^{*2}(\Psi_1(C_1))=\Psi_1(\delta(C_1))=\Psi_1(C_1)$. Therefore $\Psi_1(C_1)$ is a quasicyclic code with index 2. On the contrary, if $\Psi_1(C_1)$ is a quasicyclic code with index 2, then $\sigma_2^{*2}(\Psi_1(C_1))=\Psi_1(C_1)$. Again by Proposition 3.1, we have $\sigma_2^{*2}(\Psi_1(C_1))=\Psi_1(\delta(C_1))=\Psi_1(C_1)$. Since $\delta(C_1)=C_1$, C_1 is a *negacyclic* code. □

Proposition 3.2. $\sigma^{*4}\Psi_2=\Psi_2\sigma_2^{*2}$

Proof. Ψ_2, σ_2^{*2} and σ^{*4} are given in (2.3) and in [1], respectively.

$$\begin{aligned}\sigma_2^{*2}(a)&=\sigma_2^{*2}(a_0,a_1,\ldots,a_{2n-1})=(\sigma(a^{(0)})|\sigma(a^{(1)}))\\&=(\sigma(a_0,a_1,\ldots,a_{n-1})|\sigma(a_n,\ldots,a_{2n-1}))\\&=(a_{n-1},a_0,\ldots,a_{n-2},a_{2n-1},a_n,\ldots,a_{2n-2})\end{aligned}$$

where $a_{n-1}=s_{n-1}+u\cdot t_{n-1},a_0=s_0+u\cdot t_0,\ldots,a_{2n-2}=s_{2n-2}+u\cdot t_{2n-2}$. By applying Ψ_2, we have

$$\begin{aligned}&\Psi_2(\sigma_2^{*2}(a))=\Psi_2(a_{n-1},a_0,\ldots,a_{n-2},a_{2n-1},a_n,\ldots,a_{2n-2})\\&=\Psi_2(s_{n-1}+u\cdot t_{n-1},s_0+u\cdot t_0,\ldots,s_{n-2}+u\cdot t_{n-2},s_{2n-1}+u\cdot t_{2n-1},\ldots,s_{2n-2}+u\cdot t_{2n-2})\\&=(s_{n-1},s_0,\ldots,s_{n-2},s_{2n-1},\ldots,s_{2n-2},s_{n-1}+t_{n-1},s_0+t_0,\ldots,s_{n-2}+t_{n-2},s_{2n-1}+t_{2n-1},\ldots,s_{2n-2}+t_{2n-2})\end{aligned}$$

Conversely, $\Psi_2(a)=\Psi_2(a_0,a_1,\ldots,a_{2n-1})=(s_0,s_1,\ldots,s_{2n-1},s_0+t_0,s_1+t_1,\ldots,s_{2n-1}+t_{2n-1})$ where $a_0=s_0+ut_0$,

$a_1=s_1+ut_1$ $,\ldots,a_{2n-1}=s_{2n-1}+ut_{2n-1}$. By applying σ^{*4}, we have

$$\begin{aligned}&\sigma^{*4}(\Psi_2(a))=\sigma^{*4}(\Psi_2(a_0,a_1,\ldots,a_{2n-1}))=\sigma^{*4}(s_0,s_1,\ldots,s_{2n-1},s_0+t_0,s_1+t_1,\ldots,s_{2n-1}+t_{2n-1})\\&=(\sigma(s_0,s_1,\ldots,s_{n-1})|\sigma(s_n,\sigma(s_{n+1},\ldots,s_{2n-1})|\sigma(s_0+t_0,s_1+t_1,\ldots,s_{n-1}+t_{n-1})|\sigma(s_n+t_n,s_{n+1}+t_{n+1},\ldots,s_{2n-1}+t_{2n-1}))\\&=(s_{n-1},s_0,\ldots,s_{n-2},s_{2n-1},s_{n+1},\ldots,s_{2n-2},s_{n-1}+t_{n-1},s_0+t_0,\ldots,s_{n-2}+t_{n-2},s_{2n-1}+t_{2n-1},\ldots,s_{2n-2}+t_{2n-2})\end{aligned}$$

Equality is obtained by using the above equations. □

Theorem 3.2 A code C_2 with length $2n$ over S_2 is a quasicyclic code of index 2 if and only if $\Psi_2(C_2)$ is a *quasicyclic* code with length $4n$ over $\mathbb{F}_2$ and has index 4.

Proof. Assume C_2 is a *quasicyclic* code with index 2. Then $\sigma_2^{*2}(C_2)=C_2$. By applying Ψ_2, we get $\Psi_2(\sigma_2^{*2}(C_2))=\Psi_2(C_2)$. Using Proposition 3.2, we can write $\sigma^{*4}(\Psi_2(C_2))=\Psi_2(\sigma_2^{*2}(C_2))=\Psi_2(C_2)$. So $\Psi_2(C_2)$ is a quasicyclic code with index 4. Conversely, if $\Psi_2(C_2)$ is a quasicyclic code of index 4, then we say that $\sigma^{*4}(\Psi_2(C_2))=\Psi_2(C_2)$. From Proposition 3.2, we have $\sigma^{*4}(\Psi_2(C_2))=\Psi_2(\sigma_2^{*2}(C_2))=\Psi_2(C_2)$. Since Ψ_2 is injective, it follows that $\sigma_2^{*2}(C_2)=C_2$. □

4. Constacyclic codes and their gray images

In this part, we present even length quasicyclic code of index 2 over S_1 as the Gray image Γ_1 of *constacyclic* code over R and we also give the Gray image Γ_2 of *constacyclic* code with index 2 over S_1 with even length.

Proposition 4.1. $\sigma_1^{*2}\Gamma_1=\Gamma_1\delta$

Proof. The proof is given in [5]. □

Theorem 4.1 A code C_3 of length n in R is a *constacyclic* code if and only if $\Gamma_1(C_3)$ is a *quasicyclic* code with length $2n$ over S_1 and has index 2.

Proof. The proof is given in [5]. □

Proposition 4.2. $\sigma^{*4}\Gamma_2=\Gamma_2\sigma_1^{*2}$

Proof. Γ_2, σ_1^{*2} and σ^{*4} are given in (2.2) and in [1], respectively.

$$\begin{aligned}\sigma_1^{*2}(a) &= \sigma_1^{*2}(a_0,a_1,\ldots,a_{2n-1}) = (\sigma(a^{(0)})|\sigma(a^{(1)}))\\ &= (\sigma(a_0,a_1,\ldots,a_{n-1})|\sigma(a_n,\ldots,a_{2n-1}))\\ &= (a_{n-1},a_0,\ldots,a_{n-2},a_{2n-1},a_n,\ldots,a_{2n-2})\end{aligned}$$

where $a_{n-1} = s_{n-1} + v\cdot t_{n-1}, a_0 = s_0 + v\cdot t_0,\ldots,a_{2n-2} = s_{2n-2} + v\cdot t_{2n-2}$. By applying Γ_2, we have
$\Gamma_2(\sigma_1^{*2}(a)) = \Psi_2(a_{n-1},a_0,\ldots,a_{n-2},a_{2n-1},a_n,\ldots,a_{2n-2})$
$= \Psi_2(s_{n-1}+v\cdot t_{n-1},s_0+v\cdot t_0,\ldots,s_{n-2}+v\cdot t_{n-2},s_{2n-1}+v\cdot t_{2n-1},\ldots,s_{2n-2}+v\cdot t_{2n-2})$

$= (s_{n-1},s_0,\ldots,s_{n-2},s_{2n-1},\ldots,s_{2n-2},s_{n-1}+t_{n-1},s_0+t_0,\ldots,s_{n-2}+t_{n-2},s_{2n-1}+t_{2n-1},\ldots,s_{2n-2}+t_{2n-2})$

Conversely, $\Gamma_2(a) = \Gamma_2(a_0,a_1,\ldots,a_{2n-1}) = (s_0,s_1,\ldots,s_{2n-1},s_0+t_0,s_1+t_1,\ldots,s_{2n-1}+t_{2n-1})$ where $a_0 = s_0 + v\cdot t_0$,

$a_1 = s_1 + v\cdot t_1,\ldots,a_{2n-1} = s_{2n-1} + v\cdot t_{2n-1}$. By applying σ^{*4}, we have
$\sigma^{*4}(\Gamma_2(a)) = \sigma^{*4}(\Gamma_2(a_0,a_1,\ldots,a_{2n-1})) = \sigma^{*4}(s_0,s_1,\ldots,s_{2n-1},s_0+t_0,s_1+t_1,\ldots,s_{2n-1}+t_{2n-1})$

$= (\sigma(s_0,s_1,\ldots,s_{n-1})|\sigma(s_n,\sigma(s_{n+1},\ldots,s_{2n-1})|\sigma(s_0+t_0,s_1+t_1,\ldots,s_{n-1}+t_{n-1})|\sigma(s_n+t_n,s_{n+1}+t_{n+1},\ldots,s_{2n-1}+t_{2n-1}))$
$= (s_{n-1},s_0,\ldots,s_{n-2},s_{2n-1},s_{n+1},\ldots,s_{2n-2},s_{n-1}+t_{n-1},s_0+t_0,\ldots,s_{n-2}+t_{n-2},s_{2n-1}+t_{2n-1},\ldots,s_{2n-2}+t_{2n-2})$

Equality is obtained by using the above equations.

□

Theorem 4.2 A code C_4 with length $2n$ over S_1 is a quasicyclic code of index 2 if and only if $\Gamma_2(C_4)$ is a *quasicyclic* code of index 4 over $\mathbb{F}_2$ with length $4n$.

Proof. Assume that C_4 is a *quasicyclic* code with index 2. So $\sigma_1^{*2}(C_4) = C_4$. By applying Γ_2, we have $\Gamma_2(\sigma_1^{*2}(C_4)) = \Gamma_2(C_4)$. From Proposition 4.2, it follows that $\sigma^{*4}(\Gamma_2(C_4)) = \Gamma_2(\sigma_1^{*2}(C_4)) = \Gamma_2(C_4)$. Hence $\Gamma_2(C_4)$ is a quasicyclic code with index 4. Conversely, if $\Gamma_2(C_4)$ is a quasicyclic code of index 4, then $\sigma^{*4}(\Gamma_2(C_4)) = \Gamma_2(C_4)$. By Proposition 4.2, it can be written as $\sigma^{*4}(\Gamma_2(C_4)) = \Gamma_2(\sigma_1^{*2}(C_4)) = \Gamma_2(C_4)$. Since Γ_2 is injective, it follows that $\sigma_1^{*2}(C_4) = C_4$. □

5. Conclusion

We examined the constacyclic and negacyclic codes over $R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2$, $u^2 = 1$, $v^2 = 0$, $u\cdot v = v\cdot u = 0$ which is not Frobenius and chain ring. We compare these codes with the codes over finite field $\mathbb{F}_2$. Besides, we mention the Gray image of constacyclic and negacyclic codes over R with odd length n, and it is a quasicyclic code of index 4 with length $4n$ in $\mathbb{F}_2$.

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Some Results on the p-Weak Approximation Property in Banach Spaces

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Abstract

In this study, some existing results dealing with the weak approximation property of Banach spaces are considered for the p-weak approximation property. Also, an observation on the bounded weak approximation and the p-bounded weak approximation properties is given. Moreover, the proof of the solution of the duality problem for the p-weak approximation property which exists in the literature is given in a shorter way as an alternative.

1. Introduction

The approximation property, which closely related to basis property of Banach spaces, appeared in Banach's book in 1932 [1], and the variants of this property were systematically studied by Grothendieck, in 1955 [2]. A Banach space W has the approximation property (AP) if for every $\varepsilon > 0$ and every compact set M in W, there is a finite-rank operator $R : W \to W$ satisfying $\|Rw - w\| < \varepsilon$, for every $w \in M$ [2]. Let $1 \leq \lambda < \infty$. A Banach space W has the λ-bounded approximation property (λ-BAP) if for every $\varepsilon > 0$ and every compact set M in W, there exists a finite-rank operator $R : W \to W$ satisfying $\|R\| \leq \lambda$ and $\|Rw - w\| < \varepsilon$, for every $w \in M$ (see [3]). W has the bounded approximation property (BAP) if W has the λ-BAP, for some λ (see [3]). W has the metric approximation property (MAP) if W has the 1-BAP (see [3]). Clearly, a Banach space with the BAP has the AP, but the converse is not generally true (see [3]). It is possible to find many studies on the AP and its versions in the literature. For examples, we can mention from [4]-[9].

Grothendieck characterized the concept of compactness in Banach spaces as follows. Let W be a Banach space and let $M \subset W$. M is a relatively compact set if and only if there is a null sequence $(w_n)_n$ in W satisfying $M \subset \{\sum_{n=1}^{\infty} a_n w_n : (a_n)_n \in B_{l_1}\}$ ([2] and see [10, Proposition1.e.2]). Inspired by Grothendieck's characterization, Sinha and Karn [11] introduced the concept of p-compactness in Banach spaces. Let M be a subset of the Banach space W, and let $1 \leq p \leq \infty$. If there exists a p-summable sequence $(w_n)_n$ in W ($\|w_n\| \to 0$ as $p = \infty$) such that $M \subset \{\sum_{n=1}^{\infty} a_n w_n : (a_n)_n \in B_{l_q}\}$ (where $\frac{1}{p} + \frac{1}{q} = 1$), M is said to be relatively p-compact [11]. We remember that the ∞-compact sets are exactly the compact sets, and p-compact sets are r-compact if $1 \leq p < r \leq \infty$ [11].

The concept of a p-compact set leaded to the concept of the p-approximation property (p-AP). Sinha and Karn [11] defined the concept of the p-approximation property by replacing compact sets with p-compact sets in the definition of the AP. In recent years, the plenty of studies which focused on p-compactness, the p-AP, and some versions of the p-AP appeared. Some from these are [12]-[16]. Note that any Banach space has the 2-AP (and thus the p-AP for $1 \leq p \leq 2$) [11, Theorem 6.4].

Inspired from a result characterizing the AP given by Grothendieck, Choi and Kim [4] defined the weak approximation and the bounded weak approximation properties as weaker versions of the AP. Let W be a Banach space. If for every compact

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operator $R : W \to W$, every $\varepsilon > 0$, and every compact subset M of W, there exists a finite-rank operator $R_0 : W \to W$ satisfying $\|Rw - R_0 w\| < \varepsilon$ for all $w \in M$, then W has the weak approximation property (WAP) [4]. Also, W has the bounded weak approximation property (BWAP) if for every compact operator $R : W \to W$, there exists a positive number λ_R such that for every compact subset M of W, and every $\varepsilon > 0$, there is a finite-rank operator $R_0 : W \to W$ satisfying $\|R_0\| \le \lambda_R$ and $\|R_0 w - Rw\| < \varepsilon$ for all $w \in M$ [4]. It is clear that the BWAP implies the WAP. Also, it is showed in [4] that the AP implies the BWAP. In the BWAP, if for every compact operator $R : W \to W$ with $\|R\| \le 1$, $\lambda_R = 1$, then W has the metric weak approximation property (MWAP) [6].

As the weaker versions of the WAP and the BWAP, Li and Fang in [14] introduced the concepts of the p-weak approximation property (p-WAP) and the p-bounded weak approximation property (p-BWAP), respectively. A Banach space W has the p-weak approximation property (p-WAP) if for every compact operator $R : W \to W$, every p-compact subset M of W and, every $\varepsilon > 0$, there exists a finite-rank operator $R_0 : W \to W$ satisfying $\|R_0 w - Rw\| < \varepsilon$ for all $w \in M$ [14]. W has the p-bounded weak approximation property (p-BWAP) if for every compact operator $R : W \to W$, there exists a positive number λ_R such that for every p-compact subset M of W and every $\varepsilon > 0$, there is a finite-rank operator $R_0 : W \to W$ satisfying $\|R_0\| \le \lambda_R$ and $\|R_0 w - Rw\| < \varepsilon$ for all $w \in M$ [14]. It is clear that the WAP implies the p-WAP and the BWAP implies the p-BWAP.

The aim of this study is to obtain for the p-WAP the some results which given on the WAP in [6]-[8], by using the proof techniques in these results. Firstly, through a characterization given on the BWAP in [4, Lemma 3.7], it has been observed that the concepts of the p-BWAP and the BWAP are equivalent to each other. After, as a modification for the p-WAP of [6, Theorem 1.4 (a)], it is shown that the p-WAP of a Banach space W passes to its closed subspace N whenever $N^{\perp}$ is a complemented subspace of the dual space W^* and W^* has the v_p^* density, and also shown that the metric weak* density property in [6, Theorem 1.4 (b)] can be changed with the metric v_p^* density property. The proof of the solution of the duality problem for the p-WAP (respectively, p-BWAP) proved by Li and Fang [14] has been proved in a shorter way as an alternative. Moreover, as modifications of [7, Theorem 3.5] and [8, Theorem 1.3], respectively, it has been observed that the direct sum of two Banach spaces with the p-WAP and the p-AP has the p-WAP, and every ideal in a Banach space W has the p-WAP if and only if W has the p-WAP.

2. Notation and preliminaries

The symbols W and Z will denote Banach spaces. Let K be a subset of W. The symbol I_K represents the identity mapping on K, and for any topology τ on W, $\overline{K}^{\tau}$ denotes the τ-closure of K in W. If the τ is a norm topology, then we write $\overline{K}$. The symbol B_W denotes the closed unit ball of W. For $1 \le p < \infty$, the symbol $l_p(W)$ (respectively, $l_\infty(W)$) denotes the Banach space of all p-summable sequences (respectively, bounded sequences) in W, and $c_0(W)$ denotes the Banach space of all null sequences in W, respectively. $L(W,Z)$ denotes the Banach space of all linear bounded operators from W to Z with usual operator norm $\|,\|$. In this case $F = \mathbb{C}$, we write W^* instead of $L(W,\mathbb{C})$. A linear operator R from W to Z is called compact if $\overline{R(B_W)}$ is a compact subset of Z. The symbols $F(W,Z)$ and $K(W,Z)$ denote subspaces of finite rank and compact operators of $L(W,Z)$, respectively. Let $\lambda > 0$. $K^{\lambda}(W,W)$ (respectively, $F^{\lambda}(W,W)$) denotes the collection of compact (respectively, finite rank) operators $R : W \to W$ with $\|R\| \le \lambda$. $K^{\lambda}_{z^*}(W^*,W^*)$ (respectively, $F^{\lambda}_{z^*}(W^*,W^*)$) denotes the collection of compact (respectively, finite rank) and *weak**-to-*weak** continuous operators $R : W^* \to W^*$ with $\|R\| \le \lambda$. For a set $K \subset W$, the annihilator of K in W^* will be denoted by $K^{\perp}$. That is, $K^{\perp} = \{w^* \in W^* : w^*(w) = 0 \text{ for each } w \in K\}$. The notations τ and τ_p will denote the topologies on $L(W,Z)$, which of uniform convergence on the compact sets and p-compact sets in W, respectively. Through the paper, for p with $1 < p < \infty$, the q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

3. Some results for the p-weak approximation property

In this section, we will give an observation on the p-BWAP, some results on the p-WAP, and an alternative proof of solution of the duality problem for the p-WAP (respectively, p-BWAP). Firstly, we remember that the definitions of the v_p and v_p^* topologies given in [15] as the modifications of the v and *weak** topologies in [5, 6], respectively.

Definition 3.1. *([15], see [5, 6]) Let $1 < p < \infty$. Let X_1 be space of all linear functionals ϑ on $L(W,W)$ as in the form below*

$$\vartheta(S) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \lambda_i^k (w_k^*)(Sw_i)$$

in which $(w_i)_{i=1}^{\infty} \in l_p(W)$, $(w_k^*)_{k=1}^{\infty} \subset W^*$ and $z_k = (\lambda_i^k)_{i=1}^{\infty} \in l_q$ for $\forall k \in \mathbb{N}$ with $\sum_{k=1}^{\infty} \|z_k\|_q \|w_k^*\| < \infty$.

Let X_2 be space of all linear functionals ϕ on $L(W^*,W^*)$ as in the form below

$$\phi(R) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \beta_i^k (Rw_k^*)(w_i)$$

in which $(w_i)_{i=1}^{\infty} \in l_p(W)$, $(w_k^*)_{k=1}^{\infty} \subset W^*$ and $t_k = (\beta_i^k)_{i=1}^{\infty} \in l_q$ for $\forall k \in \mathbb{N}$ with $\sum_{k=1}^{\infty} \|t_k\|_q \|w_k^*\| < \infty$.

Let v_p be the topology induced by X_1 on $L(W,W)$, and let v_p^* be the topology induced by X_2 on $L(W^*,W^*)$. From elementary facts, the v_p and v_p^* are locally convex topologies (see [5, 6, 17, 18]). Also, by using [13, Theorem 2.5], we get $(L(W,W),\tau_p)^* = X_1 = (L(W,W),v_p)^*$.
An operator S and a net $(S_\alpha)_\alpha$ in $L(W,W)$,

$$S_\alpha \xrightarrow{v_p} S \text{ if and only if } \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\lambda_i^k(w_k^*)(S_\alpha w_i) \to \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\lambda_i^k(w_k^*)(Sw_i)$$

for every $(w_i)_{i=1}^{\infty} \in l_p(W)$, $(w_k^*)_{k=1}^{\infty} \subset W^*$ and $z_k = (\lambda_i^k)_{i=1}^{\infty} \in l_q$ for $\forall k \in \mathbb{N}$ with $\sum_{k=1}^{\infty}\|z_k\|_q\|w_k^*\| < \infty$ ([15], see [5, 6]).
Similarly, for an operator R and a net $(R_\alpha)_\alpha$ in $L(W^*,W^*)$,

$$R_\alpha \xrightarrow{v_p^*} R \text{ if and only if } \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(R_\alpha w_k^*)(w_i) \to \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(Rw_k^*)(w_i)$$

for every $(w_i)_{i=1}^{\infty} \in l_p(W)$, $(w_k^*)_{k=1}^{\infty} \subset W^*$ and $z_k = (\beta_i^k)_{i=1}^{\infty} \in l_q$ for $\forall k \in \mathbb{N}$ with $\sum_{k=1}^{\infty}\|z_k\|_q\|w_k^*\| < \infty$ ([15], see [5, 6]).

Remark 3.2. *([15], see [5]) For any $1 < p < \infty$, by [13, Theorem 2.5], we can easily see that the τ_p-topology on the space $L(W,W)$ is stronger than the v_p-topology. Also, the v_p^*-topology on the space $L(W^*,W^*)$ is weaker than the v_p-topology. The v_p and v_p^* topologies coincide if W is a reflexive Banach space. Also, we remember that for an operator S and a net $(S_\alpha)_\alpha$ in $L(W,W)$*

$$S_\alpha \xrightarrow{v_p} S \textit{ if and only if } S_\alpha^* \xrightarrow{v_p^*} S^*.$$

Remark 3.3. *([15], see [5, 6]) We have the following for a Banach space W.*

- *Let $2 < p < \infty$. W has the p-AP if and only if $I_W \in \overline{F(W,W)}^{v_p}$.*
- *Let $1 < p < \infty$. W has the λ-BAP if and only if $I_W \in \overline{F^\lambda(W,W)}^{v_p}$.*
- *Let $2 < p < \infty$. W has the p-WAP if and only if $K(W,W) \subset \overline{F(W,W)}^{v_p}$.*

Now we recall that the properties v_p^*D and Bv_p^*D given in [15] for compact operators on the dual space W^*.

Definition 3.4. *([15], see [5, 6]) Let W be a Banach space and let $1 < p < \infty$.*
(a) *W^* is said to have the v_p^* density (v_p^*D) if $K(W^*,W^*) \subset \overline{K_{z^*}(W^*,W^*)}^{v_p^*}$.*
(b) *W^* is said to have the bounded v_p^* density (Bv_p^*D) if $K^1(W^*,W^*) \subset \overline{K_{z^*}^\lambda(W^*,W^*)}^{v_p^*}$ for some $\lambda > 0$.*

W^* is said to have the metric v_p^* density (Mv_p^*D) if the Bv_p^*D is satisfied for $\lambda = 1$.

Lemma 3.5. *([15], see [10, Lemma 1.e.17], see [4, Lemma 3.11]) For a Banach space W and $1 < p < \infty$, we have the following.*
(a) *$F(W^*,W^*) \subset \overline{F_{z^*}(W^*,W^*)}^{\tau_p} \subset \overline{F_{z^*}(W^*,W^*)}^{v_p^*}$.*
(b) *$F^\lambda(W^*,W^*) \subset \overline{F_{z^*}^\lambda(W^*,W^*)}^{\tau_p} \subset \overline{F_{z^*}^\lambda(W^*,W^*)}^{v_p^*}$ for all $\lambda > 0$.*

Lemma 3.6. *([6, Lemma 3.6]) Let W be a Banach space, let N be a closed subspace of W such that let $N^\perp$ be a complemented subspace in W^*. Then, there exists a linear bounded map $U : N^* \to W^*$ satisfying $(Un^*)(n) = n^*(n)$ for $\forall\, n^* \in N^*$ and $n \in N$.*

3.1. Main results

Now, we give the main results of this paper.

Remark 3.7. *For any $1 < p < \infty$, Li and Fang in [14] defined the p-BWAP as weak version of the BWAP. On the other hand, Choi and Kim in [4, Lemma 3.7] showed that compact sets can be replaced by finite sets in the BWAP. Since every finite set is p-compact and every p-compact set is compact, the [4, Lemma 3.7] will also be correct when its part (a) is replaced with the p-BWAP. So, p-compact sets can be replaced with finite sets in p-BWAP. Thus, the concepts of the p-BWAP and the BWAP are equivalent.*

Using Remark 3.7, the relation definitions and (see [18, Lemma 3.5]), the following characterizations are obtained.

Remark 3.8. *(see [4, Lemma 3.7], and see [18, Lemma 3.5]) Let $1 < p < \infty$. We get the followings.*
(a) *A Banach space W has the BWAP if and only if for every $R \in K(W,W)$, there is a $\lambda_R > 0$ such that $R \in \overline{F^{\lambda_R}(W,W)}^{v_p}$.*
(b) *A Banach space W has the MWAP if and only if $K^1(W,W) \subset \overline{F^1(W,W)}^{v_p}$.*

The part (a) of the following theorem is a modification of [6, Theorem 1.4 (a)] for the p-WAP, and the part (b) shows that a similar result will be obtained when the metric weak* density (MW*D) property are replaced with the Mv_p^*D in [6, Theorem 1.4 (b)].

Theorem 3.9. *(see [6, Theorem 1.4]) Let $2 < p < \infty$. Let W be a Banach space, let N a closed subspace of W such that let $N^{\perp}$ be a complemented subspace in W^*.*

(a) *N has the p-WAP if W has the p-WAP and W^* has the v_p^*D.*
(b) *$K^1(N,N) \subset \overline{F^{\mu}(N,N)}^{\tau_p}$ for some $\mu > 0$ if W has the MWAP and W^* has the Mv_p^*D. In particular, N has the BWAP.*

Proof. **(a)** By using that W has the p-WAP, $K_{z^*}(W^*,W^*) \subset \overline{F_{z^*}(W^*,W^*)}^{v_p^*}$ is obtained. If this inclusion is combined with the property v_p^*D of W^*, then we get $K(W^*,W^*) \subset \overline{F_{z^*}(W^*,W^*)}^{v_p^*}$. Now, let $R \in K(N,N)$. We show that $R \in \overline{F(N,N)}^{v_p}$.
Let $I_N : N \to W$ be the inclusion map, and let the operator $U : N^* \to W^*$ be such as in Lemma 3.6. Since $UR^*I_N^* \in K(W^*,W^*)$, there exists a net $(R_\alpha^*)_\alpha \subset F_{z^*}(W^*,W^*)$ such that $R_\alpha^* \xrightarrow{v_p^*} UR^*I_N^*$. That means,

$$\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\lambda_i^k(R_\alpha^* w_k^*)(w_i) \xrightarrow{\alpha} \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\lambda_i^k(UR^*I_N^* w_k^*)(w_i) \tag{3.1}$$

for every $(w_i)_{i=1}^{\infty} \in l_p(W)$, $(w_k^*)_{k=1}^{\infty} \subset W^*$ and $z_k = (\lambda_i^k)_{i=1}^{\infty} \in l_q$ for each $k \in \mathbb{N}$ satisfying $\sum_{k=1}^{\infty}\|z_k\|_q\|w_k^*\| < \infty$.

Now, we take the sequences $(n_i)_{i=1}^{\infty} \in l_p(N)$, $(n_k^*)_{k=1}^{\infty} \subset N^*$ and, $t_k = (\beta_i^k)_{i=1}^{\infty} \in l_q$ for each $k \in \mathbb{N}$ satisfying $\sum_{k=1}^{\infty}\|t_k\|_q\|n_k^*\| < \infty$. Therefore, we get from (3.1)

$$\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(R_\alpha^* U(n_k^*))(I_N n_i) \xrightarrow{\alpha} \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(UR^*I_N^*U(n_k^*))(I_N n_i) = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(UR^*I_N^*U(n_k^*))(n_i). \tag{3.2}$$

Therefore, by (3.2), and the definition of the operator U, we get

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(I_N^*R_\alpha^*Un_k^*)(n_i) &= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(R_\alpha^*U(n_k^*))(I_N n_i) \\
&\xrightarrow{\alpha} \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(UR^*I_N^*U(n_k^*))(n_i) \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(R^*I_N^*U(n_k^*))(n_i) \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(Un_k^*)(Rn_i) \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(n_k^*)(Rn_i) \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(R^*n_k^*)(n_i).
\end{aligned}$$

Thus, from the definition v_p^*, we have $I_N^*R_\alpha^*U \xrightarrow{v_p^*} R^*$. It follows that $R^* \in \overline{F(N^*,N^*)}^{v_p^*}$. From Lemma 3.5 (a), $R^* \in \overline{F_{z^*}(N^*,N^*)}^{v_p^*}$, and by Remark 3.2, we get $R \in \overline{F(N,N)}^{v_p}$. This proves (a).

(b) Since W has the MWAP, $K^1(W,W) \subset \overline{F^1(W,W)}^{v_p}$. Thus, as in the proof of (a), we get $K_{z^*}^1(W^*,W^*) \subset \overline{F_{z^*}^1(W^*,W^*)}^{v_p^*}$. Now, let R be an operator in $K^1(N,N)$. Then $UR^*I_N^* \in K^{\|U\|}(W^*,W^*)$. Using that W^* has the Mv_p^*D, we get $UR^*I_N^* \in K^{\|U\|}(W^*,W^*) \subset \overline{F_{z^*}^{\|U\|}(W^*,W^*)}^{v_p^*}$. By following similar steps in the proof of the part (a), if Lemma 3.5 (b) is applied, then it is obtained $R^* \in \overline{F_{z^*}^{\|U\|^2}(N^*,N^*)}^{v_p^*}$. By Remark 3.2, $R \in \overline{F^{\|U\|^2}(N,N)}^{v_p}$. Since $(L(W,W),\tau_p)^* = (L(W,W),v_p)^*$, by (see [18, Lemma 3.5]), $R \in \overline{F^{\|U\|^2}(N,N)}^{\tau_p}$, where $\mu =: \|U\|^2$. Thus, the proof is completed. □

Remark 3.10. *Let $2 < p < \infty$. Li and Fang [14] proved that if the W^* has p-WAP (respectively, p-BWAP), then W has the p-WAP (respectively, p-BWAP). The proof of this theorem can be shortened by using Remark 3.2 and Lemma 3.5. Actually, suppose that W^* has the p-WAP, and let $R \in K(W,W)$. It follows from Lemma 3.5 (a) that $R^* \in \overline{F_{z^*}(W^*,W^*)}^{v_p^*}$. Thus, there exists a net $(R_\alpha)_\alpha$ in $F(W,W)$ such that $R_\alpha^* \xrightarrow{v_p^*} R^*$. By Remark 3.2, $R_\alpha \xrightarrow{v_p} T$. Thus, $R \in \overline{F(W,W)}^{v_p}$.This shows that W has the p-WAP. Using Lemma 3.5 (b), the shortened proof for the p-BWAP can be made as similar.*

The following theorem is a modification for the p-WAP of [7, Theorem 3.5]. The proof of theorem is omitted since similar to [7, Theorem 3.5].

Theorem 3.11. *(see [7, Theorem 3.5]) The Banach space $W \oplus Z$ has the p-WAP if W has the p-WAP and Z has the p-AP.*

Li and Fang in [14] proved that the complemented subspaces of a Banach space with the p-WAP have the p-WAP. Combining this result with Theorem 3.11, we obtain the following result.

Corollary 3.12. *Let $2 < p < \infty$. Let a closed subspace N of a Banach space W be complemented in W. Then, we have the following:*
(a) *The space N has the p-WAP if W has the p-WAP, [14].*
(b) *The space W/N has the p-WAP if W has the p-WAP.*
(c) *The space W has the p-WAP if N has the p-WAP and W/N has the p-AP.*

Proof. **(a)** This part is proved by [14].
(b) Since N is a complemented subspace of W, it is well known that there is a closed subspace M of W such that M is complementary of N and the spaces W/N and M are isomorphic (see [17]). From (a), since every complemented subspace of W has the p-WAP, M has the p-WAP. Thus, W/N has the p-WAP.
(c) As in (b), there exists a closed subspace M of W such that the spaces W/N and M are isomorphic. Thus, from the hypothesis, M has the p-AP. Since N has the p-WAP and M has the p-AP, from Theorem 3.11 and [17, p. 65], we get that W has the p-WAP. ☐

By a modification of [8, Theorem 1.3], we obtain the following theorem for the p-WAP of a Banach space W. The proof of this theorem is omitted since similar to [8, Theorem 1.3]. (The locally complemented subspace and ideal concepts in the following theorem can be found in [8].)

Theorem 3.13. *(see [8, Theorem 1.3]) Let $2 < p < \infty$. For a Banach space W the following are equivalent.*
(a) *W has the p-WAP.*
(b) *Every locally complemented subspace of W has the p-WAP.*
(c) *Every ideal in W has the p-WAP.*
(d) *For every closed and separable subspace Z of W, there is a closed and separable subspace $Y \subset W$ containing the subspace Z such that Y has the p-WAP.*

Remark 3.14. *The above theorem also shows that without the property $v_p^* D$ on the space W^* in Theorem 3.9 (a), Theorem 3.9 (a) will still be true (see [19, 20]).*

4. Conclusion

In the paper, it has been observed that the BWAP and the p-BWAP concepts are equivalent to each other. Some results on the p-WAP of Banach spaces have been given. The proof of the solution of the duality problem for the p-WAP (respectively, p-BWAP) which exists in the literature is given in a shorter way as an alternative.

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Error Elimination From Bloom Filters in Computer Networks Represented by Graphs

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Abstract

An undirected mathematical graph, $G=(V,E)$ where V is a set of vertices and $E=V\times V$ is the set of edges, can model a computer network. By this consideration we search for solutions to real computer network problems with a theoretical approach. This approach is based on labelling each edge by a subset of a universal set, and then encoding a path as the union of the labels of its edges. We label each vertex $v\in V$ by using a subset of universal set U, then we present a way to encode shortest paths in the graph G by using a way optimizing the data. By mathematical approach, it is provable that the labelling method we introduced eliminates the errors from the shortest paths in the graph. We aim to obtain the results in a more efficient use of network resources and to reduce network traffic. This shows how our theoretical approach works in real world network systems.

1. Introduction

We consider an undirected and unweighted regular graph $G=(V,E)$ where V is the set of vertices and $E=V\times V$ is the set of edges in the graph. This graph may represent a computer network. Therefore, it may be a reasonable approach that a real-time routing scenario in a computer network can be modeled in a mathematical graph. As a realistic model for a computer network we choose a graph denoted by a king's graph. The king's graph is a graph $G=(V,E)$ with a set of vertices V and a set of edges E where $V=\{(i,j)|i\in[0,M],j\in[0,N],M,N\in\mathbb{Z}\}$. The vertices (i,j) and (p,q) are connected in a king's graph by an edge if and only if $i=p$ and $j=q\pm1$ or $i=p\pm1$ and $j=q$ or $i=p\pm1$ and $j=q\pm1$ (see Figure 1.1).
In literature, some applications of king's graph such as tracking vehicles [1] has been studied by [2, 3]. In order to produce solutions to some network problems encoding the verticies [4] or edges [5] have been suggested in literature.
One of the labelling idea denoted by Bloom filter has been studied by [6]. A Bloom filter is a way to compress the data. Bloom filter has been widely preferred to seek for solutions to network problems [7]-[9]. This is because it saves time and space when querying the element whether in the set or not. Another application of the Bloom filter is to save space in big sized graphs [10]. Using small spaces in some models is an advantage for saving memory or reducing the network traffic [11]. A wide range of research of network applications of Bloom filters has been presented in [12]. Bloom filter is a random data structure, therefore it may produce errors denoted by false positives. These errors can be tolerable in the set, if the probability of false positives is highly small. Therefore, some applications of the Bloom filter aim to reduce the probability of false positives.
Considering routing scenarios in networks the users may face some delivery problems. For network deliveries using shortest paths [13] is an advantage to save time and network resources. However, this approach can cause additional network traffic in practice. In this case, the users may be forced to use any path between two distinct nodes rather than shortest paths. We have introduced encoding methods for shortest paths without false positives in some graphs [14]-[16]. An encoding method for the shortest path in king's graph were considered in [5]. In this paper, we consider the routing scenarios using any path for delivery in a king's graph. We build Bloom filters do not produce false positives and uses less space than the Bloom filter obtain in [5].

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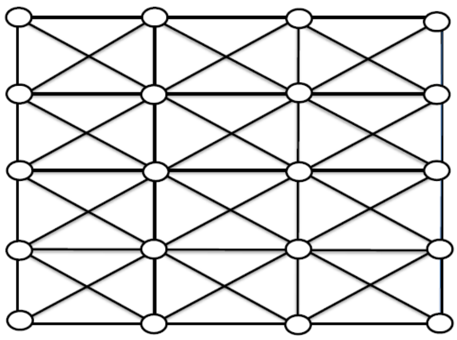

Figure 1.1: A king's graph with computers on each vertex [5].

In this research, we consider to encode any path by using the main idea behind the Bloom filters. This encoding method do not produce false positives. Note that a computer tends to send the message throughout all connections to it, hence the false positives in this model are the adjacent edges to path chosen in advance. The Bloom filter in this model of routing has a role of packet header that is sent with the message between computers. A sender can send messages through any path between two distinct nodes. We assume that the sender chooses a particular delivery path, encodes it as a Bloom filter, and this Bloom filter is sent along together with the message. We introduce a certain encoding method for the paths in this paper for this model to function.

2. Bloom filters for routing models

Bloom filter is a way to represent a subset S with n elements of a universe U [6]. We denote the Bloom filters by β. Each element in the set U is represented by a binary array of length m. The number of the bits 1 in this binary string is k.

The Bloom filter can be obtained by applying the binary operations to the binary strings of the elements in the set S. This structure of the Bloom filter is determined by design of the applications. We use bit-wise OR operation in this research like some other applications of the Bloom filter [17]. Binary OR operation takes the bit 0 and 1, then it produces the bit 1, otherwise it produces the bit 0.

An element x from the set U can be queried whether in the subset S or not by comparing the Bloom filter of the subset S with the binary array of the element x in all bits positions. This property of the Bloom filter provides the users to access the set very quickly.

We may denote the representative binary array of an element by $\beta(x)$ and the Bloom filter of the set S by $\beta(S)$. If $\beta(x) \not\leq \beta(S)$, then it can be concluded that x is definitely not in the set S. However, if $\beta(x) \leq \beta(S)$, then we cannot be certain about the existence of the element x in the set S. Since, the standard implementation of a Bloom filter, the bits 1s are placed in the array of each element randomly and the Bloom filter of the set S is obtained by adding these binary arrays together.

Because of this randomness, one can obtain that $\beta(x) \leq \beta(S)$ for some elements in the universe. Some of these elements may seem like an element of the set S, but they may not belong to the set S. These elements are called false positives. The probability of false positives is obtained after a simple calculation as $(1-e^{\frac{-kn}{m}})^k$ where m is the length of the Bloom filter, n is the number of elements in the set S and k is the number of bit 1 in the Bloom filter of an element [6].

The probability of false positives can be reduced depending on the number of the bits 1 in the Bloom filter of the set S.Therefore, the optimum number of the bits 1s denoted by k in the Bloom filter is computed by the formula $\lceil ln2 \times \frac{m}{n} \rceil$ which is obtained by taking the derivation of the false positives probability formula [12].

3. A way of edge labelling

Suppose $G=(V,E)$ is a king's graph with a set of vertices V and a set of edges E, and U is a universal set of labels, obviously $V<E$ in the king's graph. Consider a labelling such that $U=V$, and for each $e \in E$, $\beta(e)=\{u,v\}$, where u and v are the endpoints of the edge e and $\beta(e)$ is the label of the edge e. That means each vertex in the label of a path is represented by one bit. Therefore, we may denote this labelling method by a bit-per-vertex labelling.

There are $(M+1)(N+1)=MN+M+N+1$ vertices in total in a king's graph of size of $M \times N$. Therefore, the length of the labels in the graph is $MN+M+N+1$. A path is a sequence of the consecutive distinct edges and an edge is denoted by $e=\{v_i,v_j\}$ where v_i and v_j are the end vertices of the edge e. A path $P=v_0,v_1,\ldots,v_n$ where v_i is a vertex and $i \in \{0,1,2,\ldots,n\}$ is represented by a label that is obtained by applying bitwise OR operation to the labels of the vertices belonging to the path P. Binary OR operation takes the bits 0 and 1 as inputs and produces a bit 1, if at least one of the input is 1. Otherwise, it produces a bit 0, when all inputs are 0. The number of the bit 1 in the label of a vertex, an edge and a path are $1,2$ and n, which is the number of veritices in the path, respectively.

4. Properties of the shortest paths

The edges in a king's graph has four orientation of compass directions that are north (or south), east (or west), north-east (or south-west) and north-west (or south-east. For instance, vertical and horizontal edges have the orientation of the way of north or equally south and east or equally west, respectively. The orientations of the diagonal edges are north-east (south-west) and north-west (south-east) in a king's graph.

Lemma 4.1. *If a shortest path between the vertices $u = (x_i, y_j)$ and $v = (x_{(i+m)}, y_{(j+n)})$ consists of horizontal and diagonal edges, with at least one horizontal edge, then the first components of the vertices have a sequence of $x_i, x_{(i+1)}, x_{(i+2)}, \dots, x_{(i+n)}$. If a shortest path between the vertices $u = (x_i, y_j)$ and $v = (x_{(i+m)}, y_{(j+n)})$ consists of vertical and diagonal edges, with at least one vertical edge, then the second components of the vertices have a sequence of $y_i, y_{(i+1)} y_{(i+2)}, \dots, y_{(i+n)}$.*

Proof. Suppose a path P between vertices $u = (x_i, y_j)$ and $v = (x_{(i+m)}, y_{(j+n)})$. The endpoints of the vertical, horizontal and diagonal edges have a form of $\{(x_i, y_j), (x_i, y_{(j+1)})\}$, $\{(x_i, y_j), (x_{(i+1)}, y_j)\}$, and $\{(x_i, y_j), (x_{(i+1)}, y_{(j+1)})\}$, respectively.
There can be found two paths between the vertices (x_i, y_j) and $v = (x_{(i+1), y_(j+1)})$. One path P_1 is the diagonal edge where the sequence of the vertices in the path is $\{(x_i, y_j), (x_{(i+1)}, y_{(j+1)})\}$, and other path P_2 consists of one vertical and one horizontal edge with the sequence of vertices $\{(x_i, y_j), (x_{(i+1)}, y_j), (x_{(i+1)}, y_{(j+1)})\}$ or $\{(x_i, y_j), (x_i, y_{(j+1)}), (x_{(i+1)}, y_{(j+1)})\}$. Therefore, $|P_1| < |P_2|$. This concludes that vertical and horizontal edges do not appear in a shortest path. Besides, if $m > 0$ and $n > 0$ where the path lies between the vertices $u = (x_i, y_j)$ and $v = (x_{(i+m)}, y_{(j+n)})$, then this shortest path contains diagonal edges as many as possible.

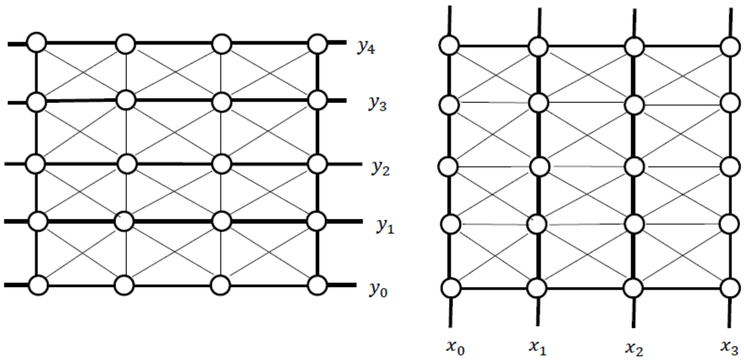

Figure 4.1: Vertical and horizontal lines in the king's graphs

The adjacent vertical edges which have the same first component constitute a vertical line in the graph (see Figure 4.1). The vertical lines in the graph are parallel to the y-coordinate and one can assign each of them with a number of a point from x-coordinate such as $\{x_1, x_2, ..., x_n\}$. Similarly, the adjacent horizontal edges which have the same second component constitute a horizontal line in the graph. The horizontal lines are parallel to the x-coordinate, then the number of the lines are $\{y_1, y_2, ..., y_n\}$. Therefore, each edge in the shortest path consisting of horizontal and diagonal edges takes place between the ordered pair of two consecutive vertical lines and the edges in the shortest path consisting of vertical and diagonal edges takes place between the ordered pair of two consecutive horizontal lines.
Suppose there are two edges from the shortest path consisting of horizontal and diagonal edges between two consecutive vertical lines numbered x_i and x_{i+1} where $i \in \{1, 2, ..., n\}$ and there is one edge from the shortest path between all other consecutive vertical lines. The previous edge of the edges lying between the vertical lines x_i and x_{i+1} has the endpoints between the lines x_{i-1} and x_i . The sequence of the vertical lines containing the endpoints of the these three edges is $x_{i-1}, x_i, x_{i+1}, x_i$. Theoretically, this fragment of the path P contains the edges between the lines x_{i-1} and x_i . Obviously, there can be found a another path between these two lines which is shorter than the path P which also contain another edges between the lines x_i and x_{i+1}. Similarly, we may suppose the shortest path consisting of vertical and diagonal edges contains two edges between two horizontal lines y_i and y_{i+1} and there is one edge from the shortest path between all other consecutive horizontal lines. In a fragment of the path, the endpoints of the adjacent edges belong to the horizontal lines y_{i-1}, y_i, y_{i+1}. There is one edge between the lines y_{i-1} and y_i and two edges between the lines y_i and y_{i+1}. However, there is another path containing the edges between the lines y_{i-1} and y_i and this path is shorter than the path containing two edges between the lines y_i and y_{i+1}.
□

5. Zero false positives zone

We suppose a king's graph represents a network and each node in the graph represents a computer. The shortest paths between distinct nodes are used for the message delivery. The sender computer chooses a shortest path to the receiver and the messages follow this route from the sender to the receiver. The messages are not sent back. Yet, a computer tends to send the message through the computers connected to it. Therefore, the false positives in this model are the adjacent edges to the chosen shortest path.

Theorem 5.1. *If a path P is one of the shortest between the vertices u and v in a king's graph, the label of the path does not produce a false positive.*

Proof. Suppose the edges in a king's graph are labelled by one-bit-vertex labelling. Consider a shortest path P with the sequence of vertices $v_0, v_1, ..., v_n$. The edges in the path P are represented by $e_k = \{v_k, v_{(k+1)}\}$ where $k \in \{0, 1, \dots, n\}$. Suppose that there is a false positive $f = \{v_i, v_j\} \in E$.
Since f is adjacent to the shortest path, either $v_i \in \{v_0, v_1, ..., v_n\}$ or $v_j \in \{v_0, v_1, ..., v_n\}$. Suppose $v_i \in \{v_0, v_1, ..., v_n\}$ and $v_i = \{(p,q)\}$ where $p \in [0, M]$ and $q \in [0, N]$ in a $M \times N$ sized king's graph. Therefore, v_j belongs to $V' = \{(p+1,q),(p-1,q),(p,q+1),(p,q-1),(p+1,q+1),(p-1,q-1),(p+1,q-1),(p-1,q+1)\}$. A vertex in a shortest path is connected with two adjacent vertices which belong to the shortest path. Therefore, two adjacent vertices v_{i-1} and v_{i+1} to the vertex v_i also belong to V'. If $v_j = v_{i-1}$ or $v_j = v_{i+1}$, then we can conclude that f is an edge in the shortest path. According to bit-per-vertex labelling each node is represented by a bit in the Bloom filter of the shortest path. Therefore, it is obtained that $\beta(f) \leq \beta(P)$ and f is not a false positive.
Suppose $v_j \in V' - \{v_{i-1}, v_{i+1}\}$. The vertices v_{i-1}, v_i, v_{i+1} can belong to consecutive vertical lines x_{i-1}, x_i, x_{i+1}. Yet, the other vertices in $V' - \{v_{i-1}, v_{i+1}\}$ also belongs to the lines x_{i-1}, x_i, x_{i+1}.
However, if the shortest path consists of horizontal and diagonal edges, then by the Lemma 4.1 the endpoints of the edges in the shortest path belong to the vertical lines and there is one vertex from the shortest path on each vertical line. Since each vertex is represented by one bit in the shortest path, βP recognizes that the other vertices whether on the path or not. Therefore, when $v_j \in V' - \{v_{i-1}, v_{i+1}\}$, then $\beta(f) \nleq \beta(P)$ and f is not a false positive.
Similarly, if the shortest path consists of vertical and diagonal edges, then by the Lemma 4.1 the endpoints of the edges in the shortest path belong to the horizontal lines and there is one vertex from the shortest path on each horizontal line. Each horizontal line contains one vertex from the shortest path by Lemma 4.1. Therefore, when $v_j \in V' - \{v_{i-1}, v_{i+1}\}$, then $\beta(f) \nleq \beta(P)$ and f is not a false positive.
In conclusion, bit-per-vertex labelling method in king's graph do not produce any false positives for the shortest paths. □

6. Practical performance of bit-per-vertex encoding method

The edges in a graph G can be encoded by one bit. Therefore, in a Bloom filter of a shortest path each edge is represented by one bit 1 and each bit has a particular bit position in the Bloom filter. Therefore, this encoding method also do not produce a false positive.
Obviously, the length of the Bloom filter is $|E|$ and the number of the bits 1s k in the Bloom filter is the number of edges in the path P. The number of edges in a king's graph $|G|$, where the size of the graph is $M \times N$, is $4MN + M + N$. By using bit-per-vertex encoding method, the length of the Bloom filter is obtained as $|V| = MN + M + N + 1$. Obviously, in a king's graph $|V| < |E|$. Hence, the space is saved with the parameters used for coding edges in our method for the king's graphs.
Besides, if the standard Bloom filter is used for encoding the edges with the parameters that we have obtained with bit-per-vertex encoding, then we probably obtain false positives. The probability of false positives is [6] by the formula $(1 - e^{\frac{-kn}{m}})^k$ where m is length of Bloom filter, n is the number of edges in a shortest path between two distinct nodes and k is the number of the bits 1s in the Bloom filters of the edges.
In our model, $m = |V|$ and $k = 2$. If we take n as its maximum value. This is $max(M,N)$ where the size of the king's graph is $M \times N$, when the shortest path is one of the the number shortest path lying between one corner to opposite corner of a king's graph. In order to obtain less probability of false positives, we recalculate k by using the formula $\lceil ln2 \times \frac{m}{n} \rceil$, [11]. By this formula, optimum k is obtained with the maximum number of edges in a shortest path and the length of Bloom filters of the edges.

Size of a king's graph	$m = (M+1)^2$	Optimum k	$n = max(M,M)$	Probability of false positives
2×2	9	≈ 3	2	$\approx 0,115$
10×10	121	≈ 8	10	$\approx 0,002$
18×18	361	≈ 14	18	$\approx 0,000065$
25×25	676	≈ 18	25	$\approx 0,0000022$
30×30	961	≈ 22	30	$\approx 2,07090265e-7$

Table 1: The probabilities of false positives are obtained by using parameters from different sizes of graphs

Therefore k is $\lceil ln2 \times \frac{M^2+2M+1}{M} \rceil$ with the parameters of our model where the size of the graph is $M \times M$, the optimum k depends on the size of the king's graph. This results that the more the size of the graph increases, the more the value of the k rises. For example; $k \approx 2$, when the king's graph has a size of 1×1. Obviously, $k > 2$ in the other sizes of the king's graphs. However, real-world network models have bigger sizes than 1×1 sized king's graph. In the encoding method we introduce k is 2 in the Bloom filters of edges in any size of the king's graph. This is another advantage of bit-per-vertex encoding.

We can conclude that if the edges in a king's graph would have been encoded by standard Bloom filter with optimum parameters of our model, then the probability of the false positives would be calculated with $(1-e^{\frac{-kn}{m}})^k$. We list some examples of probabilities of false positives obtained from different sizes of graphs (see Table 1).
Another encoding method introduced in [5] do not produce false positives for shortest paths in king's graphs. The length of the Bloom filter in that study is $12\times(M\times N)$ where the size of the king's graph is $M\times N$. It can be seen that $12\times(M\times N)\leq MN+M+N+1$ when $M=N=22$. Therefore, one may think the method introduced in [5] which offers $12\times(M\times N)$ length Bloom filter saves more space than bit-per-vertex encoding method in the big size of king's graph where $M>22$ and $N\geq 22$. However, in literature the practicable size of a Bloom filter m has been chosen as 256 [13]. Hence, bit-per-vertex coding works for bigger sizes of king's graph models with more advantages than the other encoding methods, if m is chosen from 256 up to 506.

7. Conclusion

In this research we have chosen a graph with $V<E$ that is a realistic network model and we have built labels for the edges in this graph. The labels have reasonably small length. This property of the labels is an advantages, if the network users have a small space to store the data. Additionally, the bit-per-vertex labelling do not produce false positives for shortest paths. This is another advantage for some routing scenarios taking the mathematical graphs as a network model. We show that if a Bloom filter is built with some assumptions, then it is possible to obtain a model using less space without false positives. The encoding method and routing model we have introduced in this paper can work, when the graph is regular and has an overall shape of a king's graph. For the future work, we aim to generalize this encoding idea to arbitrary graphs. The shape of graph in this paper has been chosen specifically, it is regular and undirected. This graph has its specific geometric structure. Therefore, the coding structure can be changed for other types of graphs. Also, we may extend the work in neutrosophic environment for future studies.

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Author's contributions

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Approximately Near Rings in Proximal Relator Spaces

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Abstract

The motivation of this article is to define approximately near rings, some types of approximately near rings, approximately N-groups, approximately ideals, and approximately near rings of all descriptive approximately cosets. Moreover, some properties of these approximately algebraic structures are given. Furthermore, approximately near-ring homomorphisms are introduced and their some properties are investigated.

1. Introduction

Let X be a nonempty set and $\mathscr{R}_\delta$ be a set of proximity relations on X. Then $(X, \mathscr{R}_\delta)$ is called a proximal relator space. Efremovič proximity, descriptive proximity and Lodato proximity are different types of proximity relations [1]-[3]. Non-abstract points have locations and features. In proximal relator space, the sets consist of these points.
The aim of this work is to obtain algebraic structures in proximal relator spaces using descriptively upper approximations of the subsets of X. In 2017 and 2018, approximately semigroups and approximately ideals, approximately groups, approximately subgroups and approximately rings were introduced by İnan [4]-[7]. Approximately Γ-semigroups were also defined [8]. In these articles some examples of these approximately algebraic structures in digital images endowed with proximity relations were given as in this article. Approximately algebraic structures satisfy a framework for further applied areas such as image analysis or classification problems.
In 1983, Pilz introduced the near-rings as a generalization of rings. In near rings, the addition operation does not need to be commutative as only one distributive law is sufficient [9].
Essentially, the focus of this article is to define approximately near rings, some types of approximately near rings, approximately N-groups, approximately ideals and approximately near rings of all descriptive approximately cosets. Moreover, some properties of these approximately algebraic structures are given. Furthermore, approximately near ring homomorphisms are introduced and their some properties are investigated.

2. Preliminaries

Let X be a nonempty set and $\mathscr{R}$ be a family of relations on X. If $\mathscr{R}$ is a family of proximity relations on X, then $(X, \mathscr{R}_\delta)$ is called proximal relator space, where $\mathscr{R}_\delta$ contains proximity relations, for example Efremovič proximity δ_E [1], Lodato proximity $\delta_{\mathscr{L}}$ [2], Wallman proximity δ_ω or descriptive proximity δ_Φ [3, 10, 11].
Throughout this article, the Efremovič proximity [1] and the descriptive proximity relations are considered.
An Efremovič proximity δ is a relation on $P(X)$ that satisfies the conditions: For $I, J, K \subseteq X$

1^o $I \,\delta\, J \Rightarrow J \,\delta\, I$.

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2^o $I\ \delta\ J \Rightarrow I \neq \emptyset$ and $J \neq \emptyset$.
3^o $I \cap J \neq \emptyset \Rightarrow I\ \delta\ J$.
4^o $I\ \delta\ (J \cup K) \Leftrightarrow I\ \delta\ J$ or $I\ \delta\ K$.
5^o $\{x\}\ \delta\ \{y\} \Leftrightarrow x = y$.
6^o $I\ \underline{\delta}\ J \Rightarrow \exists E \subseteq X$ such that $I\ \underline{\delta}\ E$ and $E^c\ \underline{\delta}\ J$ (*Efremovič Axiom*).

Lodato proximity [2] swaps the *Efremovič Axiom* with:

$$I\ \delta\ J \text{ and } \forall b \in J, \{b\}\ \delta\ K \Rightarrow I\ \delta\ K\ (\textit{Lodato Axiom}).$$

Here, $I\ \delta\ J$ means that I is proximal to J. Also, $I\ \underline{\delta}\ J$ means that I is not proximal to J.
Let X be a set of non-abstract points which has a location and features [12, §3] in $(X, \mathscr{R}_{\delta_\Phi})$. Let $\Phi = \{\phi_1, \ldots, \phi_n\}$ be a set of probe functions that represents features of any $x \in X$.
A *probe function* $\phi_i : X \to \mathbb{R}$ represents features of a sample non-abstract point. Let $\Phi(x) = (\phi_1(x), \ldots, \phi_n(x)), (n \in \mathbb{N})$ be an object description denoting a feature vector of x, which is a description of each $x \in X$. After choosing a set of probe functions, one can obtain a descriptive proximity relation δ_Φ as follows:
[13] Let $I, J \subseteq X$.

$$Q(I) = \{\Phi(a) \mid a \in I\}$$

is a *set description* of $I \subseteq X$. And

$$I \underset{\Phi}{\cap} J = \{x \in I \cup J \mid \Phi(x) \in Q(I) \ \textit{and}\ \Phi(x) \in Q(J)\}\ .$$

is a descriptive intersection of I and J.
[10] If $Q(I) \cap Q(J) \neq \emptyset$, then I is called *descriptively proximal (near)* to J, denoted by $I\delta_\Phi J$.
Throughout the article, $\left(X, \mathscr{R}_{\delta_\Phi}\right)$ or shortly X is considered as descriptive proximal relator space, unless otherwise stated.
[14] Let X be a descriptive proximal relator space and $A \subseteq X$. Let $(A, \circ)$ and $(Q(A), \cdot)$ be groupoids. Consider the object description Φ by means of a function

$$\Phi : A \subseteq X \longrightarrow Q(A) \subset \mathbb{R}^n, x \mapsto \Phi(x), x \in A.$$

The object description Φ of A into $Q(A)$ is an *object descriptive homomorphism* if $\Phi(x \circ y) = \Phi(x) \cdot \Phi(y)$ for all $x, y \in A$.

Definition 2.1. *[5] Let $A \subseteq X$. A descriptively upper approximation of A is defined with*

$$\Phi^* A = \{x \in X \mid x\delta_\Phi A\}.$$

It is clear that $A \subseteq \Phi^* A$ for all $A \subseteq X$.

Lemma 2.2. *[5] Let I, J be subsets of X. Then*
(i) $Q(I \cap J) = Q(I) \cap Q(J)$,
(ii) $Q(I \cup J) = Q(I) \cup Q(J)$.

Definition 2.3. *[5] Let "$\cdot$" be a binary operation on X. $G \subseteq X$ is called an approximately groupoid if $x \cdot y \in \Phi^* G$ for all $x, y \in G$.*

Definition 2.4. *[4] Let "$\cdot$" be a binary operation on X. Then $G \subseteq X$ is called an approximately group if the following conditions are true:*

$(\mathscr{A}G_1)$ *$x \cdot y \in \Phi^* G$ for all $x, y \in G$,*
$(\mathscr{A}G_2)$ *$(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $\Phi^* G$ for all $x, y, z \in G$,*
$(\mathscr{A}G_3)$ *There exists $e \in \Phi^* G$ such that $x \cdot e = e \cdot x = x$ for all $x \in G$ (e is called the approximately identity element of G),*
$(\mathscr{A}G_4)$ *There exists $y \in G$ such that $x \cdot y = y \cdot x = e$ for all $x \in G$ (y is called the inverse of x in G and denoted as x^{-1}).*

A subset S of X is called an *approximately semigroup* if

$(\mathscr{A}S_1)$ $x \cdot y \in \Phi^* S$ for all $x, y \in S$,
$(\mathscr{A}S_2)$ $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $\Phi^* S$ for all $x, y, z \in S$
properties are satisfied.
If an approximately semigroup S has an approximately identity element $e \in \Phi^* S$ such that $x \cdot e = e \cdot x = x$ for all $x \in S$, then S is called an *approximately monoid*.
If $x \cdot y = y \cdot x$ for all $x, y \in S$ holds in $\Phi^* S$, then S is called *commutative approximately groupoid (semigroup, monoid or group)*.

Theorem 2.5. *[4] Let $G \subseteq X$ be an approximately group. Then the followings are true:*
(i) There is one and only one approximately identity element in G.
(ii) There is one and only one inverse of elements in G.
(iii) If either $x \cdot z = y \cdot z$ or $z \cdot x = z \cdot y$, then $x = y$ for all $x, y, z \in G$.

Theorem 2.6. *[4] Let G be an approximately group, H be a nonempty subset of G and $\Phi^* H$ be a groupoid. Then H is an approximately subgroup of G if and only if $x^{-1} \in H$ for all $x \in H$.*

Let G be an approximately groupoid in $\left(X, \mathscr{R}_{\delta_\Phi}\right)$, $x \in G$ and $A, B \subseteq G$. Then the subsets $x \cdot A,\ A \cdot x,\ A \cdot B \subseteq \Phi^* G \subseteq X$ are defined as:

$$\begin{aligned} x \cdot A = xA &= \{xa \mid a \in A\}, \\ A \cdot x = Ax &= \{ax \mid a \in A\}, \\ A \cdot B = AB &= \{ab \mid a \in A, b \in B\}. \end{aligned}$$

Lemma 2.7. *[4] Let $A, B \subseteq X$ and $A, B, Q(A), Q(B)$ be groupoids. If $\Phi : X \longrightarrow \mathbb{R}$ is an object descriptive homomorphism, then*

$$Q(A)Q(B) = Q(AB).$$

Theorem 2.8. *[6] Let G be an approximately group, H be an approximately subgroup of G and $G/_{\rho_l}$ be a set of all descriptive approximately left cosets of G by H. If $(\Phi^* G)/_{\rho_l} \subseteq \Phi^* \left(G/_{\rho_l}\right)$, then $G/_{\rho_l}$ is an approximately group with the binary operation $xH \odot yH = (x \cdot y)H$ for all $x, y \in G$.*

Definition 2.9. *[9] Let N be a nonempty set and "$+$" and "$\cdot$" be binary operations defined on N. Then N is called a (right) near-ring if the following properties are satisfied:*
(N_1) N is a group with "$+$" (need not be commutative),
(N_2) N is a semigroup with "$\cdot$",
(N_3) For all $x, y, z \in N$, $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$.

3. Approximately near rings

Definition 3.1. *Let "$+$" and "$\cdot$" be binary operations on $\left(X, \mathscr{R}_{\delta_\Phi}\right)$. For a subset N of X is called an approximately near ring if the following conditions are satisfied:*

$(\mathscr{A}N_1)$ N is an approximately group with "$+$" (need not be abelian),
$(\mathscr{A}N_2)$ N is an approximately semigroup with "$\cdot$",
$(\mathscr{A}N_3)$ For all $x, y, z \in N$,
$(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ property holds in $\Phi^* N$.

In addition,

$(\mathscr{A}N_4)$ If $x \cdot y = y \cdot x$ for all $x, y \in N$,

then N is a commutative approximately near ring.

$(\mathscr{A}N_5)$ If $\Phi^* N$ contains an element 1_N such that $1_N \cdot x = x \cdot 1_N = x$ for all $x \in N$,

then N is called an approximately near ring with identity.

Since $(\mathscr{A}N_3)$, instead of approximately near-ring it can be used approximately right near ring. Furthermore, if consider the condition $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ for all $x, y, z \in N$ instead of $(\mathscr{A}N_3)$, then it can be named an approximately left near ring. Throughout this study approximately near ring will be used.
In general, the identity element of the approximately group $(N, +)$ is defined as zero of the approximately near ring N. Also, the set of all approximately near rings is shown with the notation $\mathscr{A}_{\mathscr{N}}$.
It should be noted here that, these conditions $(\mathscr{A}N_1) - (\mathscr{A}N_3)$ have to be hold in $\Phi^* N$. Addition or multiplying of finite number of elements in N may not always belong to $\Phi^* N$. Therefore we cannot always say that $kx \in \Phi^* N$ or $x^k \in \Phi^* N$ for all $x \in N$ and some $k \in \mathbb{Z}^+$. If $(\Phi^* N, +)$ and $(\Phi^* N, \cdot)$ are groupoids, then $kx \in \Phi^* N$ for all integer k or $x^k \in \Phi^* N$ for all positive integer k, for all $x \in N$.
An element x in approximately near ring N with identity is called a *left (resp. right) approximately invertible* if there exists $y \in N$ (resp. $z \in N$) such that $y \cdot x = 1_N$ (resp. $x \cdot z = 1_N$). The element y (resp. z) is called *a left (resp. right) approximately inverse* of x. If $x \in R$ is both left and right approximately invertible, then x is called an *approximately invertible* or an *approximately unit*. The set of approximately units in an approximately near ring N with identity forms is an approximately group with multiplication.

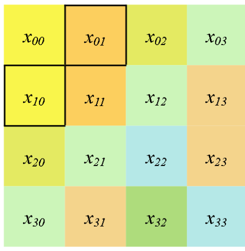

Figure 3.1: Digital Image I

Example 3.2. *Let I be a digital image endowed with δ_Φ. It is composed of* 16 *pixels (image elements) as shown in the Fig. 3.1. An image element x_{ij} is a pixel in the location (i,j). Let ϕ be a probe function that represents RGB (Red, Green, Blue) codes of pixels that are shown in Table 1.*

	x_{00}	x_{01}	x_{02}	x_{03}	x_{10}	x_{11}	x_{12}	x_{13}	x_{20}	x_{21}	x_{22}	x_{23}	x_{30}	x_{31}	x_{32}	x_{33}
Red	249	252	228	204	249	252	204	244	228	204	181	244	204	244	174	181
Green	245	207	234	245	245	207	245	212	234	245	232	212	245	212	220	232
Blue	75	94	98	185	75	94	185	140	98	185	231	140	185	140	124	231

Table 3.1: RGB codes of pixels

Let

$$+ \ : \ \begin{array}{ll} I\times I & \longrightarrow I \\ (x_{ij},x_{kl}) & \longmapsto x_{ij}+x_{kl} \end{array},$$

$$x_{ij}+x_{kl}=x_{mn}, \quad i+k\equiv m \ (\mathrm{mod}\ 2) \text{ and } j+l\equiv n \ (\mathrm{mod}\ 2)$$

be a binary operation on I such that $0\leq i,j,k,l\leq 3$. Let $N=\{x_{01},x_{10}\}\subseteq I$.
From Definition 2.1, descriptively upper approximation of N is $\Phi^*N=\{x_{ij}\in X \mid x_{ij}\delta_\phi N\}$. Hence $\phi(x_{ij})\cap Q(N)\neq\emptyset$ such that $x_{ij}\in I$, $Q(N)=\{\phi(x_{ij}) \mid x_{ij}\in N\}$. From Table 1,

$$\begin{aligned} Q(N) &= \{\phi(x_{01}),\phi(x_{10})\} \\ &= \{(252,207,94),(249,245,75)\}. \end{aligned}$$

Hence we get $\Phi^*N=\{x_{00},x_{01},x_{10},x_{11}\}$ as in Fig. 3.2.

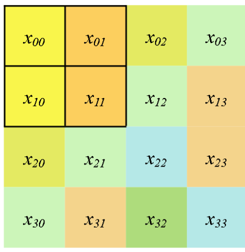

Figure 3.2: Upper Approximation of N

Hence N is an approximately group with "$+$" in $(I,\mathscr{R}_{\delta_\Phi})$ from Definition 2.4. Furthermore, let

$$\begin{array}{ll} \cdot : I\times I & \longrightarrow I \\ (x_{ij},x_{kl}) & \longmapsto x_{ij}\cdot x_{kl}=x_{ij} \end{array}$$

be a binary operation on I. Then it is obvious that N is an approximately semigroup with "$\cdot$" in $(I,\mathscr{R}_{\delta_\Phi})$. Also for all $x_{ij},x_{kl},x_{mn}\in N$,
$(x_{ij}+x_{kl})\cdot x_{mn}=x_{ij}\cdot x_{mn}+x_{kl}\cdot x_{mn}$ property holds in Φ^*N. But since $x_{01}\cdot(x_{01}+x_{01})\neq x_{01}\cdot x_{01}+x_{01}\cdot x_{01}$, so $x_{ij}\cdot(x_{kl}+x_{mn})=x_{ij}\cdot x_{kl}+x_{ij}\cdot x_{mn}$ property does not hold in Φ^*N. Consequently, N is an approximately right near ring.

Example 3.3. *Let I be a digital image endowed with δ_Φ. It is composed of* 25 *pixels (image elements) as shown in the Fig. 3.3. An image element x_{ij} is a pixel in the location (i,j). Let ϕ be a probe function that represents RGB (Red, Green, Blue) codes of pixels that are shown in Table 2.*
Let

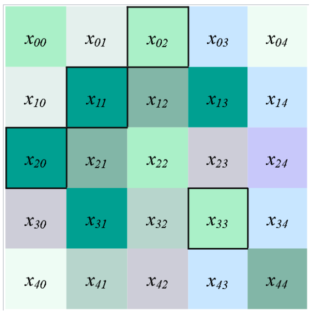

Figure 3.3: Digital Image I

	x_{00}	x_{01}	x_{02}	x_{03}	x_{04}	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{20}	x_{21}	x_{22}	x_{23}	x_{24}
Red	170	228	170	200	238	228	0	130	0	200	0	130	170	205	200
Green	240	240	240	230	252	240	160	182	160	230	160	182	240	205	200
Blue	200	237	200	255	244	237	145	167	145	255	145	167	200	216	250

	x_{30}	x_{31}	x_{32}	x_{33}	x_{34}	x_{40}	x_{41}	x_{42}	x_{43}	x_{44}
Red	205	0	183	170	200	238	183	205	200	130
Green	205	160	213	240	230	252	213	205	230	182
Blue	216	145	204	200	255	244	204	216	255	167

Table 3.2: RGB codes of pixels

$$+\ :\ \begin{array}{ll} I\times I & \longrightarrow I \\ (x_{ij},x_{kl}) & \longmapsto x_{ij}+x_{kl} \end{array},$$

$$x_{ij}+x_{kl}=x_{mn},\quad i+k\equiv m \pmod 4 \text{ and } j+l\equiv n \pmod 4$$

be a binary operation on I such that $0\leq i,j,k,l\leq 4$. *Let* $N=\{x_{02},x_{11},x_{20},x_{33}\}\subseteq I$.
From Definition 2.1, $\Phi^*N=\{x_{ij}\in I \mid x_{ij}\delta_\phi N\}$. *Hence* $\phi(x_{ij})\cap Q(N)\neq\emptyset$ *such that* $x_{ij}\in I$, $Q(N)=\{\phi(x_{ij})\mid x_{ij}\in N\}$. *From Table 2,*

$$\begin{aligned} Q(N) &= \{\phi(x_{02}),\phi(x_{11}),\phi(x_{20}),\phi(x_{33})\} \\ &= \{(170,240,200),(0,160,145)\}. \end{aligned}$$

Hence we get $\Phi^*N=\{x_{00},x_{02},x_{11},x_{13},x_{20},x_{22},x_{31},x_{33}\}$.
And so N is an approximately group with "+" in $\left(I,\mathscr{R}_{\delta_\Phi}\right)$ *from Definition 2.4. Furthermore, let*

$$\begin{array}{ll} \cdot: I\times I & \longrightarrow I \\ (x_{ij},x_{kl}) & \longmapsto x_{ij}\cdot x_{kl}=x_{ij} \end{array}$$

be a binary operation on I. Then it is obivious that N is an approximately semigroup with "·" in $\left(I,\mathscr{R}_{\delta_\Phi}\right)$. *Also for all* $x_{ij},x_{kl},x_{mn}\in N$,
$(x_{ij}+x_{kl})\cdot x_{mn}=x_{ij}\cdot x_{mn}+x_{kl}\cdot x_{mn}$ *property holds in* Φ^*N. *But since* $x_{02}\cdot(x_{02}+x_{02})\neq x_{02}\cdot x_{02}+x_{02}\cdot x_{02}$, *so* $x_{ij}\cdot(x_{kl}+x_{mn})=x_{ij}\cdot x_{kl}+x_{ij}\cdot x_{mn}$ *property does not hold in* Φ^*N. *Consequently, N is an approximately right near ring.*

Theorem 3.4. *All ordinary near rings in proximal relator spaces are approximately near rings.*

Proof. Let $N\subseteq X$ be a near ring. Since $N\subseteq\Phi^*N$, then the properties $(\mathscr{A}N_1)-(\mathscr{A}N_3)$ hold in Φ^*N. Therefore N is an approximately near ring. □

Theorem 3.5. *All approximately rings in descriptive proximal relator space are approximately near rings.*

Proof. Let $N\subseteq X$ be an approximately ring. From definition of approximately ring, it is easily shown that N is an approximately near ring. □

Lemma 3.6. *Let* $N\subseteq X$ *be an approximately near ring and* $0_N\in N$. *If* $0_N\cdot x\in N$ *for all* $x\in N$, *then*
(i) $0_N\cdot x=0_N$,
(ii) $(-x)\cdot y=-(x\cdot y)$
for all $x,y\in N$.

Proof. (i) For all $x \in N$, $0_N \cdot x = (0_N + 0_N) \cdot x = 0_N \cdot x + 0_N \cdot x$.
From Theorem 2.5 (i), since the identity element is unique, $0_N \cdot x = 0_N$.
(ii) From (i), $0_N \cdot y = 0_N$ for all $y \in N$. Then $0_N = 0_N \cdot y = ((-x)+x) \cdot y = (-x) \cdot y + x \cdot y$.
From Theorem 2.5 (ii), since the approximately inverse element is unique, $(-x) \cdot y = -(x \cdot y)$. $\square$

Definition 3.7. *Let N be an approximately near ring. The set*

$$N_0 = \{x \in N \mid x \cdot 0_N = 0_N\}$$

is called zero symmetric part of N and the set

$$N_c = \{x \in N \mid x \cdot 0_N = x\}$$

is called constant part of N.

If $N = N_0$, then N is called a zero symmetric approximately near ring and if $N = N_c$, then N is called constant approximately near ring. The set of all zero symmetric approximately near rings is represented as $\mathcal{N}_0$ and the set of all constant approximately near rings is represented as $\mathcal{N}_c$.
If the condition $d \cdot (x+y) = d \cdot x + d \cdot y$ holds in $\Phi^* N$ for all $x, y \in N$, then d is called distributive element. Also, the set of all approximately near ring with the identity is represented as $\mathcal{N}_1$ and the set of all distributive elements in N is represented as N_d.
If $N = N_d$, then N is called distributive approximately near ring.

Definition 3.8. *Let $(G,+)$ be an approximately group, N be an approximately near ring and*

$$\omega : \Phi^* N \times G \to \Phi^* G, \ \omega((x,g)) = xg.$$

The pair (G,ω) is called an approximately N-group if $(x+y)g = xg + yg$ and $(x \cdot y)g = x(yg)$ properties satisfy in $\Phi^* G$ for all $g \in G$ and all $x, y \in N$. It is denoted by ${}_N G$ and the set of all approximately N-groups is denoted by ${}_N\mathcal{G}$.

Theorem 3.9. *All approximately near-ring $(N,+,\cdot)$ are approximately N-groups.*

Definition 3.10. *Let $N \in \mathcal{N}_1$ and ${}_N G \in {}_N\mathcal{G}$. If $1_N g = g$ property holds in $\Phi^* G$ for all $g \in G$, then ${}_N G$ is called an unitary approximately N-group.*

Lemma 3.11. *Let N be an approximately near ring and G be an approximately N-group. Then*
(i) $0_N g = 0_G$ for all $g \in G$.
(ii) $(-x)g = -xg$ for all $g \in G$ and all $x \in N$.
(iii) $x0_G = 0_G$ for all $x \in N_0$.
(iv) $xg = x0_G$ for all $g \in G$ and all $x \in N_c$.

Proof. (i) For all $g \in G$, $0_N g = (0_N + 0_N) g = 0_N g + 0_N g$. From Theorem 2.5 (i), $0_N g = 0_G$.
(ii) From (i), $0_N g = 0_G$ for all $g \in G$. Then $0_G = 0_N g = ((-x)+x) g = (-x) g + xg$.From Theorem 2.5 (ii), $(-x) g = -xg$.
(iii) Since $x \cdot 0_N = 0_N$ for all $x \in N_0$, $x0_G = x(0_N g) = (x \cdot 0_N) g = 0_N g = 0_G$ by (i).
(iv) Since $x \cdot 0_N = x$ for all $x \in N_c$, $xg = (x \cdot 0_N) g = x(0_N g) = x0_G$ by (i). $\square$

Definition 3.12. *Let N be an approximately near ring and M be an approximately subgroup of $(N,+)$. M is called an approximately subnear ring of N if $M \cdot M \subseteq \Phi^* M$.*

Theorem 3.13. *Let $N \subseteq X$ be an approximately near ring, $M \subseteq N$ and $(\Phi^* M,+)$, $(\Phi^* M,\cdot)$ be groupoids. Then M is an approximately subnear ring of N iff $-x \in M$ for all $x \in M$.*

Proof. $(\Rightarrow)$ Let M is an approximately subnear ring of M. Then $(M,+)$ is an approximately group and hence $-x \in M$ for all $x \in M$.
$(\Leftarrow)$ Let $-x \in M$ for all $x \in M$. Since $(\Phi^* M,+)$ a groupoid, $(M,+)$ is an approximately group from Theorem 2.6. Therefore, since $(\Phi^* M,\cdot)$ is a groupoid and $M \subseteq N$, $x \cdot y \in \Phi^* M$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $\Phi^* M$ for all $x, y, z \in M$. Hence $(M,\cdot)$ is an approximately semigroup. Furthermore, since $(\Phi^* M,+)$ and $(\Phi^* M,\cdot)$ are groupoids and M is an approximately near ring, $(x+y) \cdot z = (x \cdot z) + (y \cdot z)$ property holds in $\Phi^* M$ for all $x, y, z \in M$. Consequently, M is an approximately subnear ring of N. $\square$

Definition 3.14. *Let N be an approximately near ring, G be an approximately N-group and H be an approximately subgroup of $(G,+)$. Then H is called an approximately N-subgroup of G if $N \cdot H \subseteq \Phi^* H$.*

Definition 3.15. *Let N be an approximately near ring and I be an approximately subgroup of $(N,+)$. Then I is called an approximately ideal of N if the following properties are satisfied:*
(1) $I \cdot N \subseteq \Phi^* I$,
(2) $x \cdot (y+a) - x \cdot y \in \Phi^* I$ for all $x, y \in N$ and all $a \in I$.
Furthermore, I is called right approximately ideal of M if only the condition (1) satisfies. Also, I is called left approximately ideal of M if only the condition (2) satisfies.

Definition 3.16. *Let N be an approximately near ring, G be an approximately N-group and H be an approximately N-subgroup of G. Then H is called an approximately ideal of G if $x(g+h)-xg \in N_r(B)^* H$ for all $g \in G$, all $h \in H$ and all $x \in N$.*

Theorem 3.17. *Let $N \subseteq X$ be an approximately near ring, M_1 and M_2 two approximately subnear rings of N and $\Phi^* M_1$, $\Phi^* M_2$ be groupoids with the binary operations "$+$" and "$\cdot$". If*

$$(\Phi^* M_1) \cap (\Phi^* M_2) = \Phi^* (M_1 \cap M_2),$$

then $M_1 \cap M_2$ is an approximately subnear ring of N.

Corollary 3.18. *Let $N \subseteq X$ be an approximately near ring, $\{M_i : i \in \Delta\}$ be a nonempty family of approximately subnear rings of N and $\Phi^* M_i$ be groupoids for all $i \in \Delta$. If*

$$\bigcap_{i\in\Delta} (\Phi^* M_i) = \Phi^* \left(\bigcap_{i\in\Delta} M_i \right),$$

then $\bigcap\limits_{i\in\Delta} M_i$ is an approximately subnear ring of N.

3.1. Approximately near rings of weak cosets

Let N be an approximately near ring and M be an approximately subnear ring of N. The relation "$\sim_r$" defined as

$$a \sim_r b \Leftrightarrow a+(-b) \in M \cup \{0_N\}$$

where $a, b \in N$.

Theorem 3.19. *Let N be an approximately near ring. Then "$\sim_r$" is a right weak equivalence relation on N.*

Proof. Since $(N,+)$ is an approximately group, $-a \in N$ for all $a \in N$. Due to $a+(-a) = 0_N \in M \cup \{0_N\}$, $a \sim_r a$. Let $a \sim_r b$ for all $a, b \in N$. Then $a+(-b) \in M \cup \{0_N\}$, that is $a+(-b) \in M$ or $a+(-b) \in \{0_N\}$. If $a+(-b) \in M$, since $(M,+)$ is an approximately group, then $-(a+(-b)) = b+(-a) \in M$. Hence $b \sim_r a$. Also if $a+(-b) \in \{0_N\}$, then $a+(-b) = 0_N$. Therefore $b+(-a) = -(a+(-b)) = -0_N = 0_N$ and so $b \sim_r a$. Consequently, "$\sim_r$" is a right weak equivalence relation on N. □

A weak class containing the element $a \in N$ according to the relation "$\sim_r$" is defined by

$$\tilde{a}_r = \{m+a | m \in M, a \in N, m+a \in N\} \cup \{a\}.$$

Definition 3.20. *Let N be an approximately near ring. A weak class determined by right weak equivalence relation "$\sim_r$" is called near right weak coset.*

Similarly, the relation "$\sim_\ell$" defined as

$$a \sim_\ell b \Leftrightarrow (-a)+b \in M \cup \{0_N\}$$

where $a, b \in N$.

Theorem 3.21. *Let N be an approximately near ring. Then "$\sim_\ell$" is a left weak equivalence relation on N.*

Proof. Since $(N,+)$ is an approximately group, $-a \in N$ for all $a \in N$. Due to $(-a) + a = 0_N \in M \cup \{0_N\}$, $a \sim_\ell a$. Let $a \sim_\ell b$ for all $a, b \in N$. Then $(-a)+b \in M \cup \{0_N\}$, that is, $(-a)+b \in M$ or $(-a)+b \in \{0_N\}$. If $(-a)+b \in M$, since $(M,+)$ is a an approximately group, then $-((-a)+b) = (-b)+a \in M$. Hence $b \sim_\ell a$. Also if $(-a)+b \in \{0_N\}$, then $(-a)+b = 0_N$. Therefore $(-b)+a = -((-a)+b) = -0_N = 0_N$ and so $b \sim_\ell a$. Consequently, "$\sim_\ell$" is a left weak equivalence relation on N. □

A class that contains the element $a \in N$, determined by relation "$\sim_\ell$" is

$$\tilde{a}_\ell = \{a+m | m \in M, a \in N, a+m \in N\} \cup \{a\}.$$

Definition 3.22. *Let N be an approximately near ring. A class determined by left weak equivalence relation "$\sim_\ell$" is called near left weak coset.*

We can easily show that $\tilde{a}_r = M+a$ and $\tilde{a}_\ell = a+M$. Approximately group $(M,+)$ may not always abelian. If $(M,+)$ is an abelian approximately group, $\tilde{a}_r = \tilde{a}_\ell$. Otherwise $\tilde{a}_r \neq \tilde{a}_\ell$.
Let N be an approximately near ring and M be an approximately subnear ring of N. Then

$$N/_{\sim_\ell} = \{a+M | a \in N\}$$

is a set of all near left weak cosets of N determined by M. If we consider $\Phi^* N$ instead of approximately near ring N

$$(\Phi^* N)/_{\sim_\ell} = \{a+M | a \in \Phi^* N\}.$$

Hence

$$a+M = \{a+m | m \in M, a \in \Phi^* N, a+m \in N\} \cup \{a\}.$$

Definition 3.23. *Let N be an approximately near ring and M be an approximately subnear ring of N. For $a,b \in N$, let $a+M$ and $b+M$ be two near left weak cosets that determined the elements a and b, respectively. Then sum of two near left weak cosets that determined by $a+b \in \Phi^* N$ can be defined as*

$$\{(a+b)+m | m \in M, a+b \in \Phi^* N, (a+b)+m \in N\} \cup \{a+b\}$$

and denoted by

$$(a+M) \oplus (b+M) = (a+b)+M.$$

Definition 3.24. *Let N be an approximately near ring and M be an approximately subnear ring of N. For $a,b \in N$, let $a+M$ and $b+M$ be two near left weak cosets that determined the elements a and b, respectively. Then product of two near left weak cosets that determined by $a \cdot b \in \Phi^* N$ can be defined as*

$$\{(a \cdot b)+m | m \in M, a \cdot b \in \Phi^* N, (a \cdot b)+m \in N\} \cup \{a \cdot b\}$$

and denoted by

$$(a+M) \odot (b+M) = (a \cdot b)+M.$$

Definition 3.25. *Let $N/_{\sim_\ell}$ be a set of all near left weak cosets of N determined by M and $\xi_\Phi(S)$ be a descriptive approximately collection of $S \in P(X)$. Then*

$$\Phi^*\left(N/_{\sim_\ell}\right) = \bigcup_{\xi_\Phi(S) \underset{\Phi}{\cap} N/_{\sim_\ell} \neq \emptyset} \xi_\Phi(S)$$

is called upper approximation of $N/_{\sim_\ell}$.

Theorem 3.26. *Let N be an approximately near ring, M be an approximately subnear ring of N and $N/_{\sim_\ell}$ be a set of all near left weak cosets of N determined by M. If*

$$(\Phi^* N)/_{\sim_\ell} \subseteq \Phi^*\left(N/_{\sim_\ell}\right),$$

then $N/_{\sim_\ell}$ is an approximately near ring with the operations given by

$$(a+M) \oplus (b+M) = (a+b)+M$$

and

$$(a+M) \odot (b+M) = (a \cdot b)+M$$

for all $a,b \in N$.

Proof. $(\mathscr{A}N_1)$ Let $(\Phi^* N)/_{\sim_\ell} \subseteq \Phi^*\left(N/_{\sim_\ell}\right)$. Since N is an approximately near ring, $\left(N/_{\sim_\ell}, \oplus\right)$ is an approximately group of all near left weak cosets of N determined by M from Theorem 2.8.
$(\mathscr{A}N_2)$

$(\mathscr{A}S_1)$ Since $(N,\cdot)$ is an approximately semigroup, $a \cdot b \in \Phi^* N$ for all $a,b \in N$ and $(a+M) \odot (b+M) = (a \cdot b)+M \in (\Phi^* N)/_{\sim_\ell}$ for all $(a+M),(b+M) \in N/_{\sim_\ell}$. From the hypothesis, $(a+M) \odot (b+M) = (a \cdot b)+M \in \Phi^*\left(N/_{\sim_\ell}\right)$ for all $(a+M),(b+M) \in N/_{\sim_\ell}$.

$(\mathscr{A}S_2)$ Since $(N,\cdot)$ is an approximately semigroup, associative property holds in $\Phi^* N$. Hence

$$\begin{aligned}
& ((a+M) \odot (b+M)) \odot (c+M) \\
= & ((a \cdot b)+M) \odot (c+M) \\
= & ((a \cdot b) \cdot c)+M \\
= & (a \cdot (b \cdot c))+M \\
= & (a+M) \odot ((b \cdot c)+M) \\
= & (a+M) \odot ((b+M) \odot (c+M))
\end{aligned}$$

holds in $(\Phi^* N)/_{\sim_\ell}$ for all $(a+M),(b+M),(c+M) \in N/_{\sim_\ell}$. From the hypothesis, associative property holds in $\Phi^*\left(N/_{\sim_\ell}\right)$. So $\left(N/_{\sim_\ell}, \odot\right)$ is an approximately semigroup of all near left weak cosets of N determined by M.

$(\mathscr{A}N_3)$ Since N is an approximately near ring, right distributive property holds in Φ^*N for all $a,b,c \in N$. Then

$$\begin{aligned} & ((a+M)\oplus(b+M))\odot(c+M) \\ = & ((a+b)+M)\odot(c+M) \\ = & ((a+b)\cdot c)+M \\ = & ((a\cdot c)+(b\cdot c))+M \\ = & ((a\cdot c)+M)\oplus((b\cdot c)+M) \\ = & ((a+M)\odot(c+M))\oplus((b+M)\odot(c+M)) \end{aligned}$$

for all $(a+M),(b+M),(c+M) \in N/_{\sim_\ell}$.
Hence right distributive property holds in $\Phi^*\left(N/_{\sim_\ell}\right)$ from the hypothesis.
Consequently, $N/_{\sim_\ell}$ is an approximately near ring. □

Definition 3.27. *Let N be an approximately near ring and M be an approximately subnear ring of N. The approximately near ring $N/_{\sim_\ell}$ is called an approximately near ring of all near left weak cosets of N determined by M and denoted by $N/_wM$.*

3.2. Approximately near ring homomorphisms

Definition 3.28. *Let $N_1, N_2 \subseteq X$ be two approximately near rings and*

$$\psi : \Phi^*N_1 \to \Phi^*N_2$$

be a mapping. If

$$\psi(a+b) = \psi(a)+\psi(b)$$

and

$$\psi(a\cdot b) = \psi(a)\cdot\psi(b)$$

for all $a,b \in N_1$, then ψ is called an approximately near ring homomorphism. Furthermore, N_1 is called approximately homomorphic to N_2 and denoted by $N_1 \simeq_a N_2$.
An approximately near ring homomorphism $\psi : \Phi^*N_1 \to \Phi^*N_2$ is called
(1) an approximately near ring monomorphism if ψ is one-one,
(2) an approximately near ring epimorphism if ψ is onto,
(3) an approximately near ring isomorphism if ψ is one-one and onto.
Set of all approximately near ring homomorphisms from Φ^*N_1 into Φ^*N_2 is denoted by $Hom(\Phi^*N_1, \Phi^*N_2)$.

Theorem 3.29. *Let N_1, N_2 be two approximately near rings and ψ be an approximately near ring homomorphism from Φ^*N_1 into Φ^*N_2. Then*
(i) $\psi(0_{N_1}) = 0_{N_2}$, where $0_{N_2} \in \Phi^*N_2$ is the near zero of N_2.
(ii) $\psi(-a) = -\psi(a)$ for all $a \in N_1$.

Proof. (i) Since $0_{N_1} = 0_{N_1}+0_{N_1}$ and ψ is an approximately near ring homomorphism, $\psi(0_{N_1}) = \psi(0_{N_1}+0_{N_1}) = \psi(0_{N_1})+\psi(0_{N_1})$. Hence $\psi(0_{N_1}) = 0_{N_2}$ as the approximately identity element is unique.
(ii) $a+(-a) = 0_{N_1}$ for all $a \in N_1$. Then $0_{N_2} = \psi(0_{N_1}) = \psi(a+(-a)) = \psi(a)+\psi(-a)$ by (i). Similarly, $0_{N_2} = \psi(-a)+\psi(a)$ for all $a \in N_1$. By Theorem 2.5 (ii), since $\psi(a)$ has a unique approximately inverse, $\psi(-a) = -\psi(a)$ for all $a \in N_1$. □

Definition 3.30. *Let $N_1, N_2 \subseteq X$ be two approximately near rings and $\psi \in Hom(\Phi^*N_1, \Phi^*N_2)$. The set*

$$Ker\psi = \{a \in N_1 \mid \psi(a) = 0_{N_2}\}$$

is called kernel of approximately near ring homomorphism ψ.

Theorem 3.31. *Let $N_1, N_2 \subseteq X$ be two approximately near rings, $\psi \in Hom(\Phi^*N_1, \Phi^*N_2)$ and $(\Phi^*Ker\psi, +)$, $(\Phi^*Ker\psi, \cdot)$ be groupoids. Then $Ker\psi$ is a approximately subnear ring of N_1.*

Proof. Let $a \in Ker\psi$. Then $\psi(a) = 0_{N_2}$. Since $N_1, N_2 \subseteq X$ are two approximately near rings, $0_{N_1} \in \Phi^*N_1$ and $0_{N_2} \in \Phi^*N_2$, $\psi(0_{N_1}) = 0_{N_2}$ by Theorem 3.29 (i). Hence $0_{N_2} = \psi(0_{N_1}) = \psi(a+(-a)) = \psi(a)+\psi(-a)$ and so $\psi(-a) = 0_{N_2}$ from $\psi(a) = 0_{N_2}$. Thus from Definition 3.30, $-a \in Ker\psi$. Therefore $Ker\psi$ is an approximately subnear ring of N_1 from Theorem 3.13. □

Theorem 3.32. *Let $N_1, N_2 \subseteq X$ be two approximately near rings, $\psi \in Hom(\Phi^* N_1, \Phi^* N_2)$ and $(\Phi^* N_1, +)$, $(\Phi^* N_1, \cdot)$ be groupoids. If S is an approximately subnear ring of N_1 and*

$$\psi(\Phi^* S) = \Phi^* \psi(S),$$

then $\psi(S) = \{\psi(a) | a \in S\}$ is an approximately subnear ring of N_2.

Proof. Since $N_1, N_2 \subseteq X$ are two approximately near rings, $0_{N_1} \in \Phi^* N_1$ and $0_{N_2} \in \Phi^* N_2$, $\psi(0_{N_1}) = 0_{N_2}$ by Theorem 3.29 (i). Thus $0_{N_2} = \psi(0_{N_1}) \in \psi(\Phi^* S) = \Phi^* \psi(S)$. This means that $\Phi^* \psi(S) \neq \emptyset$, i.e., $\psi(S) \neq \emptyset$. Since S is an approximately subnear ring of N_1, $-a \in S$ for all $a \in S$ from Theorem 3.13. Therefore $-\psi(a) = \psi(-a) \in \psi(S)$ for all $\psi(a) \in \psi(S)$ by Theorem 3.29 (ii). Consequently, $\psi(S)$ is an approximately subnear ring of N_2 from Theorem 3.13. □

Theorem 3.33. *Let $N_1, N_2 \subseteq X$ be two approximately near rings, $T \subseteq N_2$, $\psi \in Hom(\Phi^* N_1, \Phi^* N_2)$ and $(\Phi^* T, +)$, $(\Phi^* T, \cdot)$ be groupoids. If T is an approximately subnear ring of N_2, then $\psi^{-1}(T) = \{a \in N_1 | \psi(a) \in T\}$ is an approximately subnear ring of N_1.*

Proof. Let $a \in \psi^{-1}(T)$. Then $\psi(a) \in T$. Since T is an approximately subnear ring of N_2, $-\psi(a) \in T$ from Theorem 3.13. Hence $\psi(-a) \in T$ and so $-a \in \psi^{-1}(T)$ by Theorem 3.29 (ii). Consequently, $\psi^{-1}(T)$ is an approximately subnear ring of N_1 from Theorem 3.13. □

Theorem 3.34. *Let N be an approximately near ring and M be an approximately subnear ring of N. Then the mapping $\Pi : \Phi^* N \to \Phi^* (N/_w M)$ defined by $\Pi(a) = a + M$ for all $a \in \Phi^* N$ is an approximately near ring homomorphism.*

Proof. From the definition of Π, Definitions 3.23 and 3.24,

$$\Pi(a+b) = (a+b) + M = (a+M) \oplus (b+M) = \Pi(a) \oplus \Pi(b), \Pi(a \cdot b) = (a \cdot b) + M = (a+M) \odot (b+M) = \Pi(a) \odot \Pi(b)$$

for all $a, b \in N$. Thus Π is an approximately near ring homomorphism from Definition 3.28. □

Definition 3.35. *The approximately near ring homomorphism Π is called a natural approximately near ring homomorphism from $\Phi^* N$ into $\Phi^* (N/_w M)$.*

Definition 3.36. *Let $N_1, N_2 \subseteq X$ be two approximately near rings, $S \subseteq N_1$. Let*

$$\tau : \Phi^* N_1 \longrightarrow \Phi^* N_2$$

be a mapping and

$$\tau_S = {}^{\tau}\big|_S : S \longrightarrow \Phi^* N_2$$

a restricted mapping. If

$$\tau(a+b) = \tau_S(a+b) = \tau_S(a) + \tau_S(b) = \tau(a) + \tau(b)$$

and

$$\tau(a \cdot b) = \tau_S(a \cdot b) = \tau_S(a) \cdot \tau_S(b) = \tau(a) \cdot \tau(b)$$

for all $a, b \in S$, then τ is called a restricted approximately near ring homomorphism and also, N_1 is called restricted approximately homomorphic to N_2, denoted by $N_1 \simeq_{ra} N_2$.

Theorem 3.37. *Let $N_1, N_2 \subseteq X$ be two approximately near rings and $\tau \in Hom(\Phi^* N_1, \Phi^* N_2)$. Let $(\Phi^* Ker\tau, +)$, $(\Phi^* Ker\tau, \cdot)$ be groupoids and $(\Phi^* N_1)/_{\sim_\ell}$ be a set of all approximately left weak cosets of $\Phi^* N_1$ determined by $Ker\tau$. If*

$$(\Phi^* N_1)/_{\sim_\ell} \subseteq \Phi^* \left(N_1/_{\sim_\ell}\right)$$

and

$$\Phi^* \tau(N_1) = \tau(\Phi^* N_1),$$

then

$$N_1/_{\sim_\ell} \simeq_{ra} \tau(N_1).$$

Proof. Since $(\Phi^* Ker\tau, +)$ and $(\Phi^* Ker\tau, \cdot)$ are groupoids, $Ker\tau$ is an approximately subnear ring of N_1 from Theorem 3.31. Since $Ker\tau$ is an approximately subnear ring of N_1 and $(\Phi^* N_1)/_{\sim_\ell} \subseteq \Phi^* \left(N_1/_{\sim_\ell}\right)$, then $N_1/_{\sim_\ell}$ is an approximately near ring of all near left weak cosets of N_1 determined by $Ker\tau$, from Theorem 3.26. Since $\Phi^* \tau(N_1) = \tau(\Phi^* N_1)$, $\tau(N_1)$ is an approximately subnear ring of N_2 from Theorem 3.32. Let

$$\begin{array}{rcl} \sigma:\Phi^*\left(N_1/_{\sim_\ell}\right) & \longrightarrow & \Phi^*\tau(N_1) \\ A & \longmapsto & \sigma(A)=\left\{\begin{array}{ll} \sigma_{N_1/_{\sim_\ell}}(A) & ,A\in(\Phi^*N_1)/_{\sim_\ell} \\ 0_{\tau(N_1)} & ,A\notin(\Phi^*N_1)/_{\sim_\ell} \end{array}\right. \end{array}$$

be a mapping where

$$\begin{array}{rcl} \sigma_{N_1/_{\sim_\ell}}={}^{\sigma}\big|_{N_1/_{\sim_\ell}}:N_1/_{\sim_\ell} & \longrightarrow & \Phi^*\tau(N_1) \\ a+Ker\tau & \longmapsto & \sigma_{N_1/_{\sim_\ell}}(a+Ker\tau)=\tau(a) \end{array}$$

for all $a+Ker\tau\in N_1/_{\sim_\ell}$.
Since

$$\begin{aligned} a+Ker\tau &= \{a+k\mid k\in Ker\tau, a+k\in N_1\}\cup\{a\}, \\ b+Ker\tau &= \left\{b+k'\mid k'\in Ker\tau, b+k'\in N_1\right\}\cup\{b\} \end{aligned}$$

and the mapping τ is an approximately near ring homomorphism,

$$\begin{array}{rl} & a+Ker\tau=b+Ker\tau \\ \Rightarrow & a\in b+Ker\tau \\ \Rightarrow & a\in\{b+k'\mid k'\in Ker\tau, b+k'\in N_1\} \text{ or } a\in\{b\} \\ \Rightarrow & a=b+k',\ k'\in Ker\tau,\ b+k'\in N_1 \text{ or } a=b \\ \Rightarrow & -b+a=(-b+b)+k',\ k'\in Ker\tau \text{ or } \tau(a)=\tau(b) \\ \Rightarrow & -b+a=k',\ k'\in Ker\tau \\ \Rightarrow & -b+a\in Ker\tau \\ \Rightarrow & \tau(-b+a)=0_{\tau(N_1)} \\ \Rightarrow & \tau(-b)+\tau(a)=0_{\tau(N_1)} \\ \Rightarrow & -\tau(b)+\tau(a)=0_{\tau(N_1)} \\ \Rightarrow & \tau(a)=\tau(b) \\ \Rightarrow & \sigma_{N_1/_{\sim_\ell}}(a+Ker\tau)=\sigma_{N_1/_{\sim_\ell}}(b+Ker\tau) \end{array}$$

Therefore $\sigma_{N_1/_{\sim_\ell}}$ is well defined.
For $A,B\in\Phi^*\left(\sigma_{N_1/_{\sim_\ell}}\right)$, we suppose that $A=B$. Since the mapping $\sigma_{N_1/_{\sim_\ell}}$ is well defined,

$$\begin{aligned} \sigma(A) &= \left\{\begin{array}{ll} \sigma_{N_1/_{\sim_\ell}}(A) & ,A\in(\Phi^*N_1)/_{\sim} \\ 0_{\tau(N_1)} & ,A\notin(\Phi^*N_1)/_{\sim} \end{array}\right. \\ &= \left\{\begin{array}{ll} \sigma_{N_1/_{\sim_\ell}}(B) & ,B\in(\Phi^*N_1)/_{\sim} \\ 0_{\tau(N_1)} & ,B\notin(\Phi^*N_1)/_{\sim} \end{array}\right. \\ &= \sigma(B). \end{aligned}$$

Consequently σ is well defined.
For all $a+Ker\tau, b+Ker\tau\in N_1/_{\sim_\ell}\subset\Phi^*\left(N_1/_{\sim_\ell}\right)$,

$$\begin{array}{rl} & \sigma((a+Ker\tau)\oplus(b+Ker\tau)) \\ = & \sigma((a+b)+Ker\tau) \\ = & \sigma_{N_1/_{\sim_\ell}}((a+b)+Ker\tau) \\ = & \tau(a+b) \\ = & \tau(a)+\tau(b) \\ = & \sigma_{N_1/_{\sim_\ell}}(a+Ker\tau)+\sigma_{N_1/_{\sim_\ell}}(b+Ker\tau) \\ = & \sigma(a+Ker\tau)+\sigma(b+Ker\tau) \end{array}$$

and

$$\begin{array}{rl} & \sigma((a+Ker\tau)\odot(b+Ker\tau)) \\ = & \sigma((a\cdot b)+Ker\tau) \\ = & \sigma_{N_1/_{\sim_\ell}}((a\cdot b)+Ker\tau) \\ = & \tau(a\cdot b) \\ = & \tau(a)\cdot\tau(b) \\ = & \sigma_{N_1/_{\sim_\ell}}(a+Ker\tau)\cdot\sigma_{N_1/_{\sim_\ell}}(b+Ker\tau) \\ = & \sigma(a+Ker\tau)\cdot\sigma(b+Ker\tau). \end{array}$$

Therefore σ is a restricted approximately near ring homomorphism by Definition 3.36. Consequently, $N_1/_{\sim_\ell}\simeq_{ra}\tau(N_1)$. $\square$

4. Conclusion

Algebraic structures provide a consistent theoretical background for all mathematical research. The theoretical background is very important in all problems of processing digital images. In this study, approximately near-rings as an algebraic structure on digital images were investigated. We hope this research will inspire the investigations in both some theoretical and applied sciences.

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Author's contributions

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*Author [Ayşegül Kocamaz]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%40)

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Employing the $\exp(-\varphi(z))$-Expansion Method to Find Analytical Solutions for a (2+1)-dimensional Combined KdV-mKdV Equation

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Abstract

In this paper, we obtain exact solutions of the (2+1)-dimensional combined KdV-mKdV equation by using a symbol calculation approach. First, we give some background on the equation. Second, the $\exp(-\varphi(z))$-expansion method will be introduced to solve the equation. After, using the $\exp(-\varphi(z))$-expansion method to solve the equation, we can get four types of exact solutions, which are hyperbolic, trigonometric, exponential, and rational function solutions. Finally, we can observe the characteristics of the exact solutions via computer simulation more easily.

1. Introduction

Seeking the exact solutions of nonlinear partial differential equations (NLPDEs) is a hotspot in nonlinear science research and the related theory has developed rapidly in recent decades. Because many nonlinear phenomena existing in nature and various fields can be described as NLPDEs. More importantly, the solutions of NLPDEs can account for these complex phenomena as well as applying in these fields [1]-[12], such as atmosphere, optical fiber communications and fluid mechanics. There is a series of NLPDEs, for example, the KdV equation, the KP equation and the Schrödinger equation. Also, there are many effective methods to search exact solutions of NLPDEs, such as Lie symmetry [13], the Hirota bilinear method [14, 15], the extended complex metho d[16], and the $\exp(-\varphi(z))$-expansion method [17]. Particularly, the $\exp(-\varphi(z))$-expansion method first proposed by Zhao and Li [17] can be used to attain analytical traveling wave solutions of numerous NLPDEs, such as the combined KdV-mKdV equation[18], the (1+1)-dimensional classical Boussinesq equations [19] and the Caudrey-Dodd-Gibbon-Sawada-Kotera equation [20].
As we all know, the KdV equation becoming a kind of classical nonlinear partial differential equations can be used to describe small amplitude shallow water waves, stratified internal waves, ion acoustic waves and its model has great practical value in many fields [21]-[23], such as plasma physics, solid state physics and fluid mechanics. With the development of soliton theory and the in-depth research in the KdV equation, we fully understand the properties of it and its abundant solutions. Meanwhile, various extensions of KdV equations are derived. More recently, Wang and Kara [24] built the new (2+1)-dimensional KdV and mKdV equations as

$$u_t - 6uu_x + 6uu_y - u_{xxx} + u_{yyy} + 3u_{xxy} - 3u_{xyy} = 0 \tag{1.1}$$

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$$u_t - 6u^2u_x + 6u^2u_y - u_{xxx} + u_{yyy} + 3u_{xxy} - 3u_{xyy} = 0 \tag{1.2}$$

Then Malik et al. [25] proposed the (2+1)-dimensional combined KdV-mKdV equation by combining them, which is given by:

$$u_t - a_1uu_x + a_1uu_y - a_2u^2u_x + a_2u^2u_y - a_3u_{xxx} + a_3u_{yyy} + a_4u_{xxy} - a_4u_{xyy} = 0 \tag{1.3}$$

By considering $a_1 = 6, a_2 = 0, a_3 = 1, a_4 = 3$ and $a_1 = 0, a_2 = 6, a_3 = 1, a_4 = 3$, Eq.(1.3) reduces to Eqs.(1.1) and (1.2) respectively. Additionally, although the authors have obtained the analytical solutions to the combined KdV-mKdV equation in [18], the (2+1)-dimensional combined KdV-mKdV equation in [25] has more dimensions and the mixed partial derivatives in contrast to the former, which means a much broader researching space for scholars. Therefore, it makes sense to research the the (2+1)-dimensional combined KdV-mKdV equation deeply. The integrability of the equation and some forms of its solutions are illuminated in Sandeep Malik's paper. In this article, we use the $\exp(-\varphi(z))$-expansion method to attain exact solutions to the (2+1)-dimensional combined KdV-mKdV equation and observe the characteristics of them by computer simulation, which can obtain more abundant solutions to the equation and indicate the validity of the $\exp(-\varphi(z))$-expansion method. The results and simulations are gained by using Maple.

2. The $\exp(-\varphi(z))$-expansion method

Considering the following nonlinear PDE:

$$F(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, \cdots) = 0, \tag{2.1}$$

in which F is a polynomial of the unknown function $u(x,y,t)$ and its partial derivatives, and it also involves nonlinear terms.

Step 1. Insert traveling wave transform

$$u(x,y,t) = u(z), \quad z = \kappa x + \lambda y + \omega t,$$

into Eq.(2.1) to reduce it into the ODE,

$$P(u, u', u'', u''', \cdots) = 0, \tag{2.2}$$

in which P is a polynomial of u and its derivatives, while $' := \frac{d}{dz}$.

Step 2. Assume that the exact solutions of Eq.(2.2) have the following form:

$$u(z) = \sum_{\upsilon=0}^{n} C_\upsilon (\exp(-\varphi(z)))^\upsilon, \tag{2.3}$$

in which C_υ, $(0 \le \upsilon \le n)$ are constants to be determined later, such that $C_n \ne 0$ and $\varphi = \varphi(z)$ satisfies the ODE as follows:

$$\varphi'(z) = b_1 + \exp(-\varphi(z)) + b_2 \exp(\varphi(z)). \tag{2.4}$$

where b_1 and b_2 are constants and the solutions of Eq.(2.4) are given as below:
When $b_1^2 - 4b_2 > 0$, $b_2 \ne 0$,

$$\varphi(z) = \ln\left(\frac{-\sqrt{(b_1^2-4b_2)}\tanh(\frac{\sqrt{b_1^2-4b_2}}{2}(z+\varsigma)) - b_1}{2b_2}\right), \tag{2.5}$$

$$\varphi(z) = \ln\left(\frac{-\sqrt{(b_1^2-4b_2)}\coth(\frac{\sqrt{b_1^2-4b_2}}{2}(z+\varsigma)) - b_1}{2b_2}\right). \tag{2.6}$$

When $b_1^2 - 4b_2 < 0$, $b_2 \ne 0$,

$$\varphi(z) = \ln\left(\frac{\sqrt{(4b_2-b_1^2)}\tan(\frac{\sqrt{(4b_2-b_1^2)}}{2}(z+\varsigma)) - b_1}{2b_2}\right), \tag{2.7}$$

$$\varphi(z)=\ln\left(\frac{\sqrt{(4b_2-b_1^2)}\cot(\frac{\sqrt{(4b_2-b_1^2)}}{2}(z+\varsigma))-b_1}{2b_2}\right) \tag{2.8}$$

When $b_1^2-4b_2>0$, $b_1\neq 0$, $b_2=0$,

$$\varphi(z)=-\ln\left(\frac{b_1}{\exp(b_1(z+\varsigma))-1}\right). \tag{2.9}$$

When $b_1^2-4b_2=0$, $b_1\neq 0$, $b_2\neq 0$,

$$\varphi(z)=\ln\left(-\frac{2(b_1(z+\varsigma)+2)}{b_1^2(z+\varsigma)}\right). \tag{2.10}$$

When $b_1^2-4b_2=0$, $b_1=0$, $b_2=0$,

$$\varphi(z)=\ln(z+\varsigma). \tag{2.11}$$

in which ς is an arbitrary constant and $C_n\neq 0, b_1, b_2$ are constants in Eqs.(2.5)-(2.11). Considering the homogeneous balance between the highest order derivatives and nonlinear terms of Eq.(2.2), we define the degree of $u(z)$ as $D(u(z))=n$ and the positive integer n can be ascertained by the following expressions

$$D\left(\frac{d^\alpha u}{dz^\alpha}\right)=n+\alpha, D\left(u^\beta\left(\frac{d^\alpha u}{dz^\alpha}\right)^s\right)=n\beta+s(n+\alpha). \tag{2.12}$$

Step 3. Plugging Eq.(2.3) into Eq.(2.2), we obtain a polynomial of $\exp(-\varphi(z))$. Then collect all terms with the same power about $\exp(-\varphi(z))$ and let the coefficients of them equal zero respectively. After that, we get a set of algebraic equations and by solving them we confirm the values of $C_n\neq 0, b_1, b_2$. Finally, we substitute the obtained values into Eq.(2.3) as well as Eqs.(2.5)-(2.11) to achieve the determination of the exact solutions for the original PDE.

3. Exact solutions of the (2+1)-dimensional combined KdV-mKdV equation

Substituting traveling wave transform

$$u(x,y,t)=u(z), \quad z=\kappa x+\lambda y+\omega t,$$

into Eq.(1.3) and then integrating it, we obtain

$$u+(\frac{1}{2}a_1\lambda-\frac{1}{2}a_1\kappa)u^2+(\frac{1}{3}a_2\lambda-\frac{1}{3}a_2\kappa)u^3+(a_3\lambda^3-a_3\kappa^3+a_4\kappa^2\lambda-a_4\kappa\lambda^2)u''+\delta=0. \tag{3.1}$$

where δ is the integration constant.

Taking the homogeneous balance between u^3 and u'' of Eq.(3.1) according to Eqs.(2.12), we can yield $n=1$ and hence

$$u(z)=C_0+C_1\exp(-\varphi(z)), \tag{3.2}$$

where $C_1\neq 0$, C_0 are constants.

Plugging u, u^2, u^3, u'' into Eq.(3.1) and equating the coefficients with the same order of $\exp(-\varphi(z))$ to zero, we obtain

$$\begin{aligned}
e^{0(-\varphi(z))}:\ &\omega C_0+1/2a_1\lambda C_0{}^2-1/2a_1\kappa C_0{}^2+1/3a_2\lambda C_0{}^3\\
&-1/3a_2\kappa C_0{}^3+\delta-C_1\kappa^3a_3b_1b_2+C_1\lambda^3a_3b_1b_2\\
&-C_1a_4\kappa\lambda^2b_1b_2+C_1a_4\kappa^2\lambda b_1b_2=0
\end{aligned}$$

$$\begin{aligned}
e^{1(-\varphi(z))}:\ &-C_1a_3b_1{}^2\kappa^3+C_1a_3b_1{}^2\lambda^3+C_1a_4b_1{}^2\kappa^2\lambda\\
&-C_1a_4b_1{}^2\kappa\lambda^2-2a_3\kappa^3b_2C_1+2a_3\lambda^3b_2C_1\\
&+2C_1\lambda a_4b_2\kappa^2-2C_1\lambda^2a_4b_2\kappa-C_0{}^2C_1a_2\kappa\\
&+C_0{}^2C_1a_2\lambda-C_0C_1a_1\kappa+C_0C_1a_1\lambda\\
&+C_1\omega=0
\end{aligned}$$

$$\begin{aligned}
e^{2(-\varphi(z))}:\ &-3C_1a_4\kappa\lambda^2b_1+3C_1a_4\kappa^2\lambda b_1+1/2a_1\lambda C_1{}^2\\
&-1/2a_1\kappa C_1{}^2+a_2\lambda C_0C_1{}^2-a_2\kappa C_0C_1{}^2\\
&-3C_1a_3\kappa^3b_1+3C_1a_3\lambda^3b_1=0
\end{aligned}$$

$$
\begin{aligned}
e^{3(-\varphi(z))}: & -2C_1\,a_3\,\kappa^3+2C_1\,a_3\,\lambda^3+1/3\,a_2\,\lambda\,{C_1}^3 \\
& -1/3\,a_2\,\kappa {C_1}^3-2C_1\,a_4\,\kappa\,\lambda^2+2C_1\,a_4\,\kappa^2\lambda=0
\end{aligned}
$$

Having solved the above algebraic equations,we get two different cases:

Case 1.

$$
\begin{aligned}
C_0 &= \frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1-a_1}{2a_2}, \\
C_1 &= \frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}}{a_2}, \\
\delta &= \frac{1}{24a_2^2}\Big(-3\,b_1\,\sqrt{-(a_3\,\lambda^2+\kappa\,(a_3-a_4)\,\lambda+a_3\,\kappa^2)\,a_2}\,\big(\big(2\,\big({b_1}^2-4\,b_2\big) \\
&\quad (\kappa-\lambda)\,\big(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda\big)+4\,\omega\big)\,a_2 \\
&\quad +{a_1}^2\,(\kappa-\lambda)\big)\,\sqrt{6}+2\,a_1\,\big(6\,\omega\,a_2+{a_1}^2\,(\kappa-\lambda)\big)\Big),
\end{aligned}
\tag{3.3}
$$

where b_1 and b_2 are arbitrary.
Plugging Eqs.(3.3) into Eq.(3.2), we can obtain

$$
\begin{aligned}
u(z) = & \frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1-a_1}{2a_2} \\
& +\frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}}{a_2}\exp(-\varphi(z)).
\end{aligned}
\tag{3.4}
$$

Employing Eqs.(2.5) to (2.11) into Eq.(3.4) respectively, attains the exact solutions to the (2+1)-dimensional combined KdV-mKdV equation in the following.

Case 1.1. When $b_1^2-4b_2>0$, $b_2\neq 0$,

$$
\begin{aligned}
u_{11}(z) = & \frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1-a_1}{2a_2} \\
& -\frac{2b_2\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}}{a_2(\sqrt{(b_1^2-4b_2)}\tanh(\frac{\sqrt{b_1^2-4b_2}}{2}(z+\varsigma))+b_1)}, \\
u_{12}(z) = & \frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1-a_1}{2a_2} \\
& -\frac{2b_2\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}}{a_2(\sqrt{(b_1^2-4b_2)}\coth(\frac{\sqrt{b_1^2-4b_2}}{2}(z+\varsigma))+b_1)}.
\end{aligned}
$$

Case 1.2. When $b_1^2-4b_2<0$, $b_2\neq 0$,

$$
\begin{aligned}
u_{13}(z) = & \frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1-a_1}{2a_2} \\
& +\frac{2b_2\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}}{a_2(\sqrt{(4b_2-b_1^2)}\tan(\frac{\sqrt{4b_2-b_1^2}}{2}(z+\varsigma))-b_1)}, \\
u_{14}(z) = & \frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1-a_1}{2a_2} \\
& +\frac{2b_2\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}}{a_2(\sqrt{(4b_2-b_1^2)}\cot(\frac{\sqrt{4b_2-b_1^2}}{2}(z+\varsigma))-b_1)}.
\end{aligned}
$$

Case 1.3. When $b_1^2-4b_2>0$, $b_1\neq 0$, $b_2=0$,

$$
\begin{aligned}
u_{15}(z) = & \frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1-a_1}{2a_2} \\
& +\frac{\sqrt{-6a_2\,(a_3\,\kappa^2+a_3\,\kappa\,\lambda+a_3\,\lambda^2-a_4\,\kappa\,\lambda)}b_1}{a_2(\exp(b_1(z+\varsigma))-1)}.
\end{aligned}
$$

Case 1.4. When $b_1^2-4b_2=0$, $b_1\neq 0$, $b_2\neq 0$,

$$u_{16}(z)=\frac{\sqrt{-6a_2\left(a_3\kappa^2+a_3\kappa\lambda+a_3\lambda^2-a_4\kappa\lambda\right)}b_1-a_1}{2a_2} - \frac{b_1^2(z+\varsigma)\sqrt{-6a_2\left(a_3\kappa^2+a_3\kappa\lambda+a_3\lambda^2-a_4\kappa\lambda\right)}}{2a_2(b_1(z+\varsigma)+2)}.$$

Case 1.5. When $b_1^2-4b_2=0$, $b_1=0$, $b_2=0$,

$$u_{17}(z)=\frac{\sqrt{-6a_2\left(a_3\kappa^2+a_3\kappa\lambda+a_3\lambda^2-a_4\kappa\lambda\right)}b_1-a_1}{2a_2} + \frac{\sqrt{-6a_2\left(a_3\kappa^2+a_3\kappa\lambda+a_3\lambda^2-a_4\kappa\lambda\right)}}{a_2(z+\varsigma)}.$$

Case 2.

$$C_0=-\frac{\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}b_1+a_1}{2a_2},$$

$$C_1=-\frac{\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}}{a_2},$$

$$\delta=\frac{1}{24a_2^2}\Big(3b_1\sqrt{-\left(a_3\lambda^2+\kappa\left(a_3-a_4\right)\lambda+a_3\kappa^2\right)a_2}\,\big(\big(2\left({b_1}^2-4b_2\right)\left(\kappa-\lambda\right)\left(a_3\kappa^2+a_3\kappa\lambda+a_3\lambda^2-a_4\kappa\lambda\right)+4\omega\big)a_2 +{a_1}^2\left(\kappa-\lambda\right)\big)\sqrt{6}+2a_1\left(6\omega a_2+{a_1}^2\left(\kappa-\lambda\right)\right)\Big), \tag{3.5}$$

where b_1 and b_2 are arbitrary constants.
Plugging Eqs.(3.5) into Eq.(3.2), we can obtain

$$u(z)=-\frac{\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}b_1+a_1}{2a_2} - \frac{\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}}{a_2}\exp(-\varphi(z)). \tag{3.6}$$

Employing Eqs.(2.5) to (2.11) into Eq.(3.6) respectively, attains the exact solutions to the (2+1)-dimensional combined KdV-mKdV equation in the following.

Case 2.1. When $b_1^2-4b_2>0$, $b_2\neq 0$,

$$u_{21}(z)=-\frac{\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}b_1+a_1}{2a_2} + \frac{2b_2\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}}{a_2\left(\sqrt{(b_1^2-4b_2)}\tanh\left(\frac{\sqrt{b_1^2-4b_2}}{2}(z+\varsigma)\right)+b_1\right)},$$

$$u_{22}(z)=-\frac{\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}b_1+a_1}{2a_2} + \frac{2b_2\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}}{a_2\left(\sqrt{(b_1^2-4b_2)}\coth\left(\frac{\sqrt{b_1^2-4b_2}}{2}(z+\varsigma)\right)+b_1\right)}.$$

Case 2.2. When $b_1^2-4b_2<0$, $b_2\neq 0$,

$$u_{23}(z)=-\frac{\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}b_1+a_1}{2a_2} - \frac{2b_2\sqrt{-6a_2\left(a_3\lambda^2+a_3\kappa\lambda+a_3\kappa^2-a_4\kappa\lambda\right)}}{a_2\left(\sqrt{(4b_2-b_1^2)}\tan\left(\frac{\sqrt{4b_2-b_1^2}}{2}(z+\varsigma)\right)-b_1\right)},$$

$$u_{24}(z) = -\frac{\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}b_1 + a_1}{2a_2} - \frac{2b_2\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}}{a_2(\sqrt{(4b_2 - b_1^2)}\cot(\frac{\sqrt{4b_2-b_1^2}}{2}(z+\varsigma)) - b_1)}.$$

Case 2.3. When $b_1^2 - 4b_2 > 0$, $b_1 \neq 0$, $b_2 = 0$,

$$u_{25}(z) = -\frac{\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}b_1 + a_1}{2a_2} - \frac{\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}b_1}{a_2(\exp(b_1(z+\varsigma)) - 1)}.$$

Case 2.4. When $b_1^2 - 4b_2 = 0$, $b_1 \neq 0$, $b_2 \neq 0$,

$$u_{26}(z) = -\frac{\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}b_1 + a_1}{2a_2} + \frac{b_1^2(z+\varsigma)\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}}{2a_2(b_1(z+\varsigma)+2)}.$$

Case 2.5. When $b_1^2 - 4b_2 = 0$, $b_1 = 0$, $b_2 = 0$,

$$u_{27}(z) = -\frac{\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}b_1 + a_1}{2a_2} - \frac{\sqrt{-6a_2\,(a_3\,\lambda^2 + a_3\,\kappa\lambda + a_3\,\kappa^2 - a_4\,\kappa\lambda)}}{a_2(z+\varsigma)}.$$

4. Computer simulations

In this section, the results are illustrated by computer simulations respectively.

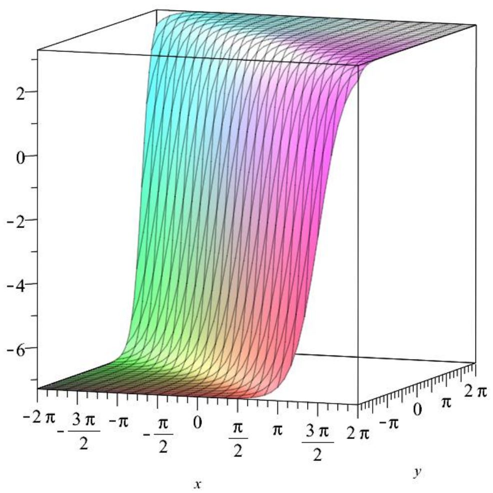

Figure 4.1: 3D profile of $u_{11}(z)$ for $a_1 = 1.2$, $a_2 = 0.3$, $a_3 = -0.2$, $a_4 = 0.8$, $\kappa = 1$, $\lambda = 1$, $t = 1$, $\omega = 2$, $\varsigma = -1$, $b_1 = 4$, and $b_2 = 3$.

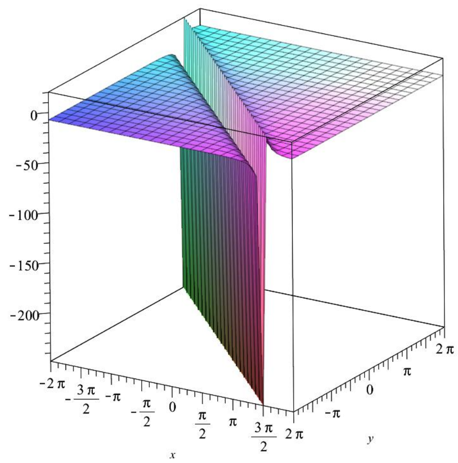

Figure 4.2: 3D profile of $u_{12}(z)$ for $a_1 = 1.2$, $a_2 = 0.3$, $a_3 = -0.2$, $a_4 = 0.8$, $\kappa = 1$, $\lambda = 1$, $t = 1$, $\omega = 2$, $\varsigma = -1$, $b_1 = 4$, and $b_2 = 3$.

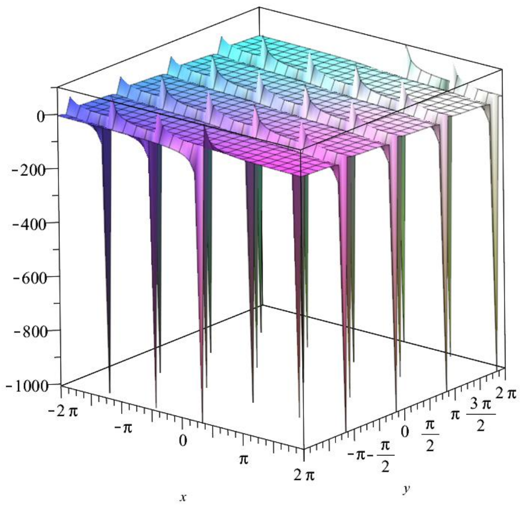

Figure 4.3: 3D profile of $u_{13}(z)$ for $a_1 = 1.2$, $a_2 = 0.3$, $a_3 = -0.2$, $a_4 = 0.8$, $\kappa = \frac{1}{10}$, $\lambda = 1$, $t = 1$, $\omega = 2$, $\varsigma = -1$, $b_1 = 4$, and $b_2 = 5$.

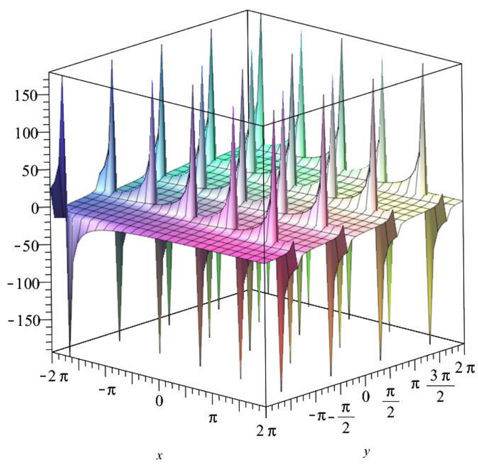

Figure 4.4: 3D profile of $u_{14}(z)$ for $a_1 = 1.2$, $a_2 = 0.3$, $a_3 = -0.2$, $a_4 = 0.8$, $\kappa = \frac{1}{10}$, $\lambda = 1$, $t = 1$, $\omega = 2$, $\varsigma = -1$, $b_1 = 4$, and $b_2 = 5$.

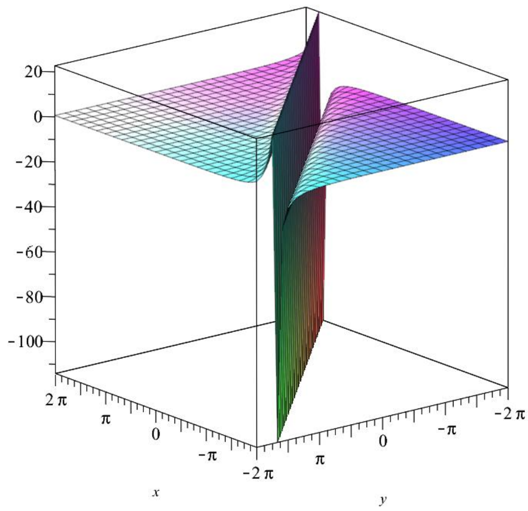

Figure 4.5: 3D profile of $u_{15}(z)$ for $a_1 = 1.2$, $a_2 = 0.3$, $a_3 = -0.2$, $a_4 = 0.8$, $\kappa = 1$, $\lambda = 1$, $t = 1$, $\omega = 2$, $\varsigma = -1$, and $b_1 = 1$.

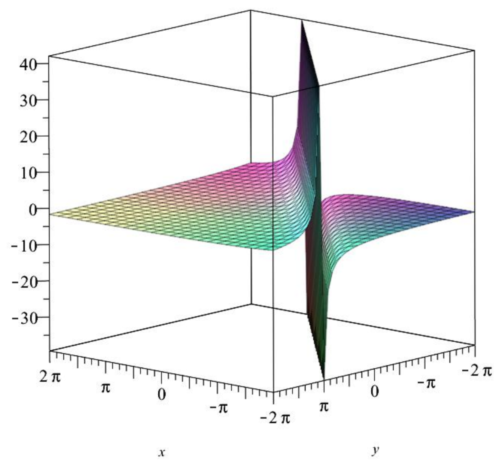

Figure 4.6: 3D profile of $u_{16}(z)$ for $a_1 = 1.2$, $a_2 = 0.3$, $a_3 = -0.2$, $a_4 = 0.8$, $\kappa = 1$, $\lambda = 1$, $t = 1$, $\omega = 2$, $\varsigma = -1$, and $b_1 = 1$.

5. Conclusion

In this study, we use the $\exp(-\varphi(z))$-expansion method to obtain abundant new exact solutions to the (2+1)-dimensional combined KdV-mKdV equation. Except the types of hyperbolic and exponential function solutions which are the same as those of Sandeep Malik's paper [25], we also get new types of function solutions including trigonometric and rational solutions. Additionally, the results indicate that utilizing the $\exp(-\varphi(z))$-expansion method to the combined KdV-mKdV equation and the (2+1)-dimensional combined KdV-mKdV equation can get the same forms of solutions, while the solutions to the (2+1)-dimensional combined KdV-mKdV equation have one more case. These solutions can widely stimulate mathematicians and physicists' interest and have potential value to be applied in mathematics and physics. Meanwhile, the effectiveness of the $\exp(-\varphi(z))$-expansion method to seek exact solutions for nonlinear differential equations can be seen from the obtained results.

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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A Note on Trigonometric Approximations of Bessel Functions of the First Kind, and Trigonometric Power Sums

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Abstract

I reconsider the approximation of Bessel functions with finite sums of trigonometric functions, in the light of recent evaluations of Neumann-Bessel series with trigonometric coefficients. A proper choice of angle allows for an efficient choice of the trigonometric sum. Based on these series, I also obtain straightforward non-standard evaluations of new parametric sums with powers of cosine and sine functions.

1. Introduction

The aim of this paper is twofold: to investigate trigonometric approximations of Bessel functions via Bessel-Neumann series whose sums are finite trigonometric sums, and use the same series to provide new sums of powers of sines with cosines.
Bessel functions are among the most useful and studied special functions. Analytic expansions exist for different regimes [1], and numerical algorithms for their precise evaluation [2]-[5]. Their simplest approximations are polynomials [6]-[8] and finite trigonometric sums, that can be advantageous in applications [9].
Let's consider J_0. Several trigonometric sums appeared in the decades, sometimes being rediscovered. These very simple ones

$$J_0(x) \simeq \tfrac{1}{4}[1+\cos x+2\cos(\tfrac{\sqrt{2}}{2}x)] \tag{1.1}$$

$$J_0(x) \simeq \tfrac{1}{6}\left[1+\cos x+2\cos(\tfrac{1}{2}x)+2\cos(\tfrac{\sqrt{3}}{2}x)\right] \tag{1.2}$$

have errors $\varepsilon = J_0 - J_0^{\text{approx}}$ with power series (the marvel of Mathematica)

$$\varepsilon(x) = -\tfrac{x^8}{2^8\cdot 20160}(1-\tfrac{x^2}{36}+\ldots), \qquad \varepsilon(x) = -\tfrac{x^{12}}{2^{12}\cdot 239500800}(1-\tfrac{x^2}{52}+\ldots).$$

In practice, an error less than 0.001 is achieved for $x \le 3$ or $x \le 5.9$. These approximations were obtained by Fettis with the Poisson formula [10]. Rehwald [11] and later Waldron [12], Blachman and Mousavinezhad [13] and [14] used the strategy of truncating to the first term Neumann-Bessel series like this one

$$J_0(x)+2J_8(x)+2J_{16}(x)+\ldots = \tfrac{1}{4}\cos[1+\cos x+2\cos(\tfrac{\sqrt{2}}{2}x)],$$

that can be obtained from the Bessel generating function. The examples (1.1), (1.2) correspond to $n = 4, 6$ of eq.19 in [15]:

$$J_0(x)+2\sum_{k=1}^{\infty}(-1)^{kn}J_{2kn}(x) = \frac{1}{n}\sum_{\ell=0}^{n-1}\cos(x\cos\tfrac{\pi}{n}\ell) \tag{1.3}$$

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where only J_0 is kept, and the errors reflect the behaviour $J_{2n}(x) \approx \frac{(x/2)^{2n}}{(2n)!}$ of the first neglected term. The truncation yields J_0 as a sum of cosines that corresponds to the evaluation of the Bessel integral $J_0(x) = \int_0^\pi \frac{d\theta}{\pi} \cos(x\cos\theta)$ with the trapezoidal rule with n nodes [16]-[18]. Increasing n increases accuracy: $n = 15$ is a formula by Fettis [10] with 8 cosines (instead of 15, symmetries of the roots of unity reduce the number of terms):

$$J_0(x) \simeq \tfrac{1}{15}\cos x + \tfrac{2}{15}\sum_{k=1}^{7} \cos(x\cos\tfrac{k\pi}{15}). \tag{1.4}$$

The error now is order $x^{30} \times 10^{-42}$ and less than 10^{-6} for $x < 15$.
I reconsider the approximations for J_0 in the light of new Neumann-Bessel trigonometric series in [19]. They extend the series (1.3) by including an angular parameter, that is chosen to kill the term with J_{2n}, so that the truncation involves the next-to-next term J_{4n} of the series. This is presented in Section 2. The same strategy is then used in Section 3 to approximate Bessel functions J_n of low order by appropriate series. The quality of the approximations is the same as earlier ones with same number of terms, but the terms are different and the source of error is more clear.
In Section 4, I show that the same Neumann series give in very simple way some parametric sums of powers of sines and cosines. Some are found in the recent literature [20], while the following ones, to my knowledge, are new:

$$\sum_{\ell=0}^{n} \sin^p(\tfrac{\theta+2\pi\ell}{n})\{\begin{matrix}\sin\\ \cos\end{matrix}\}(q\tfrac{\theta+2\pi\ell}{n}) \qquad (p,q = 0,1,\dots).$$

Many other trigonometric sums are available in the literature. For example, sums like $\sum_{k=1}^{m-1} \sin\frac{2kq\pi}{m}\cot^n\frac{k\pi}{m}$ and variants are studied in [21, 22]. Ratios of powers of sines and cosines are evaluated in [23], and many results are given in the remarkable paper [24].

2. The Bessel function J_0

Consider the Neumann trigonometric series eq.11 in [19]:

$$J_0(x) + 2\sum_{k=1}^{\infty}(-1)^{kn}J_{2kn}(x)\cos(2kn\theta) = \frac{1}{n}\sum_{\ell=0}^{n-1}\cos[x\cos(\theta + \tfrac{\pi}{n}\ell)]. \tag{2.1}$$

The approximations (1.1), (1.2) and (1.4) are obtained with $\theta = 0$, $n = 4$, 6, 15, and neglecting functions J_8, J_{12}, J_{30} and higher orders. However they are not optimal. The advantage of eq.(2.1) is the possibility to choose the angle $\theta = \pi/4n$ to kill all terms J_{2n}, J_{6n}, etc. Then:

$$J_0(x) - 2J_{4n}(x) + 2J_{8n}(x) - \ldots = \frac{1}{n}\sum_{\ell=0}^{n-1}\cos(z\cos\tfrac{1+4\ell}{4n}\pi).$$

An expansion for J_0 results, again, by neglecting the other terms.

Some examples:

• $n = 2$. It is $J_0(x) = \frac{1}{2}[\cos(x\cos\frac{\pi}{8}) + \cos(x\sin\frac{\pi}{8})] + \varepsilon_2(x)$. If we neglect the error, the first zero occurs at $\pi\sqrt{2-\sqrt{2}} = 2.4045$ ($j_{0,1} = 2.4048$).
• $n = 3$. The approximation has three cosines:

$$\begin{aligned} J_0(x) &= \tfrac{1}{3}[\cos(x\tfrac{1}{\sqrt{2}}) + \cos(x\tfrac{\sqrt{3}-1}{2\sqrt{2}}) + \cos(x\tfrac{\sqrt{3}+1}{2\sqrt{2}})] + \varepsilon_3(x) \\ \varepsilon_3(x) &= \tfrac{x^{12}}{2^{12}\cdot 239500800}\left[1 - \tfrac{x^2}{52} + \tfrac{x^4}{52\cdot 112} - \tfrac{x^6}{52\cdot 112\cdot 180} + \ldots\right]. \end{aligned} \tag{2.2}$$

Remarkably, the first powers of the error are opposite of those for the expansion eq.(1.2), that would involve 6 terms if not for the degeneracy of the roots of unity. The half-sum of (1.2) and (2.2),

$$J_0(x) \simeq \tfrac{1}{12}\left[1 + \cos x + 2\cos(\tfrac{1}{2}x) + 2\cos(\tfrac{\sqrt{3}}{2}x) + 2\cos(x\tfrac{1}{\sqrt{2}}) + 2\cos(x\tfrac{\sqrt{3}-1}{2\sqrt{2}}) + 2\cos(x\tfrac{\sqrt{3}+1}{2\sqrt{2}})\right] \tag{2.3}$$

has error $\varepsilon(x) = -\frac{x^{24}}{5.2047} \times 10^{-30}[1 - \frac{x^2}{100} + \frac{x^4}{20800} - \ldots]$.
• $n = 6$ gives a precision similar to the sum (2.3):

$$J_0(z) = \tfrac{1}{6}\left[\cos(x\cos\tfrac{\pi}{24}) + \cos(x\cos\tfrac{3\pi}{24}) + \cos(x\cos\tfrac{5\pi}{24}) + \cos(x\sin\tfrac{\pi}{24}) + \cos(x\sin\tfrac{3\pi}{24}) + \cos(x\sin\tfrac{5\pi}{24})\right] + \varepsilon_6(x). \tag{2.4}$$

The error has power expansion $\varepsilon_6(x) = \frac{x^{24}}{5.2047} \times 10^{-30}[1 - \frac{x^2}{100} + \ldots]$.
• $n = 8$ is a sum of 8 cosines and compares with the formula (1.4) by Fettis. The two approximations are different but with the same number of terms (because $\theta = 0$ produces degenerate terms) and similar precision.

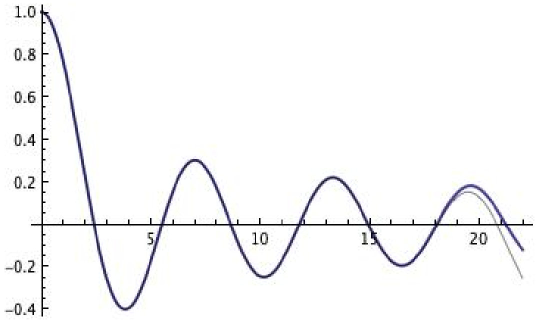

Figure 2.1: The Bessel function J_0 (thick) and the trigonometric expansion (2.4). The difference increases with x; it is less than 10^{-9} for $x<8$ and 10^{-3} for $x<15$.

3. Bessel functions J_n.

• $\mathbf{J_1}$ is evaluated via $J_1=-J_0'$. Eq. (2.3) gives:

$$J_1(x)\simeq\tfrac{1}{12}\left[\sin x+\sin(\tfrac12 x)+\sqrt3\sin(\tfrac{\sqrt3}{2}x)+\sqrt2\sin(x\tfrac{1}{\sqrt2})+(\tfrac{\sqrt3-1}{\sqrt2})\sin(x\tfrac{\sqrt3-1}{2\sqrt2})+(\tfrac{\sqrt3+1}{\sqrt2})\sin(x\tfrac{\sqrt3+1}{2\sqrt2})\right]$$

with error $\varepsilon(x)\simeq(x/20)^{23}\times3.87\times[1-\frac{13}{1200}x^2+\dots]$.

• $\mathbf{J_2},\mathbf{J_4}$ can be evaluated with the following identity (the real part of eq.(5) in [19]):

$$J_p(x)+\sum_{k=1}^{\infty}[J_{kn+p}(x)+(-1)^{kn+p}J_{kn-p}(x)]\cos(kn\theta)=\tfrac1n\sum_{\ell=0}^{n-1}\cos[x\sin(\theta+\tfrac{2\pi\ell}{n})+p(\theta+\tfrac{2\pi\ell}{n})]. \tag{3.1}$$

Because of the term J_{n-p}, we take $2p<n$. With $y=\frac{\pi}{2n}$:

$$J_p(x)-(-1)^pJ_{2n-p}(x)+\ldots=\tfrac1n\sum_{\ell=0}^{n-1}\cos[x\sin(\tfrac{1+4\ell}{2n}\pi)+p\tfrac{1+4\ell}{2n}\pi].$$

If only J_p is kept, the approximation depends on the parity of p:

$$J_p(x)\simeq\begin{cases}\cos(p\frac{1+4\ell}{2n}\pi)\times\frac1n\sum_{\ell=0}^{n-1}\cos[x\sin(\frac{1+4\ell}{2n}\pi)] & p \text{ even}\\ -\sin(p\frac{1+4\ell}{2n}\pi)\times\frac1n\sum_{\ell=0}^{n-1}\sin[x\sin(\frac{1+4\ell}{2n}\pi)] & p\text{ odd}\end{cases}$$

$p=2$, $n=6$, give the short formula

$$J_2(x)\simeq\tfrac{1}{2\sqrt3}\left[\cos(x\sin\tfrac{\pi}{12})-\cos(x\cos\tfrac{\pi}{12})\right]$$

with error $\varepsilon(x)=2.69114\times(x/10)^{-10}[1-\frac{x^2}{44}+\ldots]$. The first zero is evaluated $\frac23\pi\sqrt6\simeq5.1302$ ($j_{2,1}=5.13562$). A better approximation is $n=8$, $y=\frac{\pi}{16}$:

$$J_2(x)\simeq\tfrac14\cos(\tfrac{\pi}{8})[\cos(x\sin\tfrac{\pi}{16})-\cos(x\cos\tfrac{\pi}{16})]+\tfrac14\sin(\tfrac{\pi}{8})[\cos(x\cos\tfrac{5\pi}{16})-\cos(x\sin\tfrac{5\pi}{16})]$$

with error $\varepsilon(x)=7.00119\times10^{-16}x^{14}[1-\frac{x^2}{60}+\ldots]$; $\varepsilon(5)=3\times10^{-6}$, $\varepsilon(8)=0.0010$.
For J_4 we select $p=4$, $n=8$, $\theta=\frac{\pi}{16}$. Now the lowest neglected term is J_{12}:

$$J_4(x)\simeq\tfrac{\sqrt2}{8}\left[\cos(x\sin\tfrac{\pi}{16})+\cos(x\cos\tfrac{\pi}{16})-\cos(x\sin\tfrac{5\pi}{16})-\cos(x\cos\tfrac{5\pi}{16})\right].$$

The error is less that 10^{-3} at $x<6.3$.

• $\mathbf{J_3},\mathbf{J_5}$. A useful sum for odd-order Bessel functions is eq.(17) in [19]:

$$\sum_{k=0}^{\infty}(-1)^{n+k}J_{(2n+1)(2k+1)}(x)\cos[(2k+1)\theta]=\sum_{\ell=0}^{2n}\frac{\sin[x\cos(\frac{\theta+2\pi\ell}{2n+1})]}{2(2n+1)}. \tag{3.2}$$

The angle $\theta=\frac{\pi}{6}$ cancels J_{6n+3}, J_{14n+7} etc. and gives the approximation

$$J_{2n+1}(x)\simeq\frac{(-1)^n}{\sqrt3}\sum_{\ell=0}^{2n}\frac{\sin[x\cos\frac{1+12\ell}{12n+6}\pi]}{2n+1}$$

that neglects J_{10n+5} etc. With $n=1$ and $n=2$ we obtain:

$$J_3(x)\simeq-\tfrac{1}{3\sqrt3}[\sin(x\cos\tfrac{\pi}{18})-\sin(x\sin\tfrac{2\pi}{9})-\sin(x\sin\tfrac{\pi}{9})]$$

$$J_5(x)\simeq\tfrac{1}{5\sqrt3}\left[\sin(x\cos\tfrac{\pi}{30})+\sin(x\sin\tfrac{\pi}{15})-\sin(\tfrac{\sqrt3}{2}x)-\sin(x\sin\tfrac{4\pi}{15})+\sin(x\cos\tfrac{2\pi}{15})\right].$$

The expansion for J_3 has error $\varepsilon=2.33373\times10^{-17}x^{15}[1-\frac{x^2}{64}+\ldots]$. The second one has error $\varepsilon=1.92134x^{25}\times10^{-33}\times[1-\frac{x^2}{104}+\ldots]$.

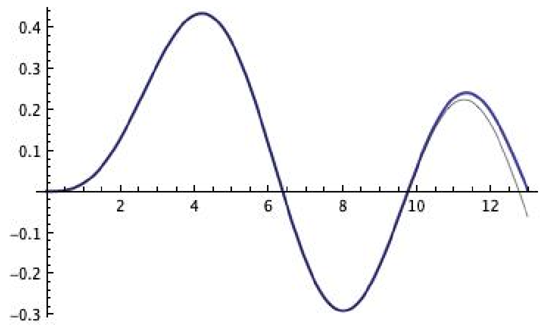

Figure 3.1: The Bessel function J_3 (thick) and the approximation (3.3). The difference is $\varepsilon(6) = 6\times 10^{-6}$, $\varepsilon(8) = .0003$, $\varepsilon(10) = .0045$.

4. Trigonometric identities

The Neumann-Bessel series here used provide sums of powers of sines and cosines. They arise by expanding in powers of x the Bessel functions in the series,

$$J_n(x) = \sum_{k=0}^{\infty}(-1)^k \frac{(x/2)^{n+2k}}{k!(k+n)!}$$

and the trigonometric functions in the sum of the series.

• Consider the series eq.(2.1). At threshold powers x^{2n}, x^{4n} etc. new Bessel functions $(-1)^n 2J_{2n}\cos(2n\theta)$, $2J_{4n}\cos(4n\theta)$ etc. enter a term in the sum of cosines.

$$\frac{1}{n}\sum_{\ell=0}^{n-1}\left[\cos\frac{\theta+\ell\pi}{n}\right]^{2k} = \begin{cases} \frac{1}{4^k}\binom{2k}{k} & 0\le k<n \\ \frac{1}{4^k}\left[\binom{2k}{k}+2\binom{2k}{k-n}\cos(2\theta)\right] & n\le k<2n \\ \frac{1}{4^k}\left[\binom{2k}{k}+2\binom{2k}{k-n}\cos(2\theta)+2\binom{2k}{k-2n}\cos(4\theta)\right] & 2n\le k<3n \\ \dots & \dots \end{cases}.$$

By replacing θ with $\theta+n\pi$ we obtain:

$$\frac{1}{n}\sum_{\ell=0}^{n-1}\left[\sin\frac{\theta+\ell\pi}{n}\right]^{2k} = \begin{cases} \frac{1}{4^k}\binom{2k}{k} & 0\le k<n \\ \frac{1}{4^k}\left[\binom{2k}{k}+(-1)^n 2\binom{2k}{k-n}\cos(2\theta)\right] & n\le k<2n \\ \frac{1}{4^k}\left[\binom{2k}{k}+(-1)^n 2\binom{2k}{k-n}\cos(2\theta)+2\binom{2k}{k-2n}\cos(4\theta)\right] & 2n\le k<3n \\ \dots & \dots \end{cases}.$$

Examples:

$\frac{1}{9}\sum_{\ell=0}^{8}\left[\sin\frac{\theta+\ell\pi}{9}\right]^{20} = \frac{1}{4^{10}}\left[\binom{20}{10}-2\binom{20}{1}\cos(2\theta)\right].$
$\frac{1}{n}\sum_{\ell=0}^{n-1}\left[\cos(\frac{1+6\ell}{6n}\pi)\right]^{2n} = \frac{1}{4^n}\left[\binom{2n}{n}+1\right],$
$\frac{1}{n}\sum_{\ell=0}^{n-1}\left[\cos(\frac{1+4\ell}{4n}\pi)\right]^{4n} = \frac{1}{16^n}\left[\binom{4n}{2n}-2\right].$

For $\theta=0$ and $\theta=\frac{\pi}{2}$ these identities are eqs. 4.4.2 in [25], 2.1 and 2.2 (together with several other non-parametric sums) in [26]. The series had also been studied in [27]. Parametric averages on the full circle were recently evaluated by Jelitto [20], with a different method.

• With the Neumann series (3.2) we obtain:

$$\frac{1}{2n+1}\sum_{\ell=0}^{2n}\left[\cos\frac{\theta+2\pi\ell}{2n+1}\right]^{2k+1} = \begin{cases} 0 & 1\le 2k+1<2n+1 \\ \frac{1}{4^k}\binom{2k+1}{k-n}\cos\theta & 2n+1\le 2k+1<3(2n+1) \\ \frac{1}{4^k}\left[\binom{2k+1}{k-n}\cos\theta+\binom{2k+1}{k-3n-1}\cos(3\theta)\right] & 3(2n+1)\le 2k+1<5(2n+1) \\ \dots & \dots \end{cases}.$$

The sums of even powers of cosines are obtained from the series eq.16 in [19]:

$$J_0(x)+2\sum_{k=1}^{\infty}(-1)^k J_{(4n+2)k}(x)\cos(2k\theta) = \sum_{\ell=0}^{2n}\frac{\cos[x\cos\frac{\theta+2\pi\ell}{2n+1}]}{2n+1}$$

$$\frac{1}{2n+1}\sum_{\ell=0}^{2n}\left[\cos\frac{\theta+2\pi\ell}{2n+1}\right]^{2k}=\begin{cases}\frac{1}{4^k}\binom{2k}{k} & 0\le k<2n+1\\ \frac{1}{4^k}\left[\binom{2k}{k}+2\binom{2k}{k-2n-1}\cos(2\theta)\right] & 2n+1\le k<4n+2\\ \frac{1}{4^k}\left[\binom{2k}{k}+2\binom{2k}{k-2n-1}\cos(2\theta)+2\binom{2k}{k-4n-2}\cos(4\theta)\right] & 4n+2\le k<6n+3\\ \dots & \dots\end{cases}.$$

Example: $(\cos\frac{\theta}{3})^{12}+(\cos\frac{\theta+\pi}{3})^{12}+(\cos\frac{\theta+2\pi}{3})^{12}=\frac{3}{4^6}[\binom{12}{6}+2\binom{12}{3}\cos(2\theta)+2\cos(4\theta)]$.

The formulae with sines are obtained by shifting the parameter θ.

• Now let's consider the sum eq.(3.1) with $p<n-p$.
The equations are new and are easier to state if we distinguish the parity of n and of p.

Case $\mathbf{n=2m}$ and $\mathbf{p=2q}$. Eq.(3.1) now is:

$$\frac{1}{2m}\sum_{\ell=0}^{2m-1}\cos[x\sin(\tfrac{\theta+\pi\ell}{m})]\cos[2q\tfrac{\theta+\pi\ell}{m}]=J_{2q}(x)+[J_{2m-2q}(x)+J_{2m+2q}(x)]\cos(2\theta)$$
$$+[J_{4m-2q}(x)+J_{4m+2q}(x)]\cos(4\theta)+\dots$$

Separation of even and odd parts in x, and expansion in x give:

$$\frac{1}{2m}\sum_{\ell=0}^{2m-1}[\sin\tfrac{\theta+\pi\ell}{m}]^{2k+1}\sin(2q\tfrac{\theta+\pi\ell}{m})=0,\qquad k=0,1,2,\dots$$

This result is obvious as the sum from 0 to $m-1$ is opposite of the rest of the sum. The symmetry is used also in the other result:

$$\frac{1}{m}\sum_{\ell=0}^{m-1}[\sin\tfrac{\theta+\pi\ell}{m}]^{2k}\cos(2q\tfrac{\theta+\pi\ell}{m})=$$
$$=\frac{(-1)^q}{4^k}\begin{cases}0 & k<q\\ \binom{2k}{k-q} & q\le k<m-q\\ \binom{2k}{k-q}+(-1)^m\binom{2k}{k-m+q}\cos(2\theta) & m-q\le k<m+q\\ \binom{2k}{k-q}+(-1)^m\left[\binom{2k}{k-m+q}+\binom{2k}{k-m-q}\right]\cos(2\theta) & m+q\le k<2m-q\\ \dots & \dots\end{cases}$$

Case $\mathbf{n=2m}$, $\mathbf{p=2q+1}$. Eq.(3.1) becomes:

$$-\frac{1}{2m}\sum_{\ell=0}^{2m-1}\sin[x\sin(\tfrac{\theta+\pi\ell}{m})]\sin[(2q+1)\tfrac{\theta+\pi\ell}{m}]=J_{2q+1}(x)+$$
$$+[-J_{2m-2q-1}(x)+J_{2m+2q+1}(x)]\cos(2\theta)+[J_{4m-2q-1}(x)+J_{4m+2q+1}(x)]\cos(4\theta)+\dots$$

The non trivial result is:

$$\frac{1}{m}\sum_{\ell=0}^{m-1}[\sin\tfrac{\theta+\pi\ell}{m}]^{2k+1}\sin[(2q+1)\tfrac{\theta+\pi\ell}{m}]=$$
$$=\frac{(-1)^q}{2^{2k+1}}\begin{cases}0 & k<q\\ \binom{2k+1}{k-q} & q\le k<m-q-1\\ \binom{2k+1}{k-q}+(-1)^m\binom{2k+1}{k+m-q}\cos(2\theta) & m-q-1\le k<m+q\\ \binom{2k+1}{k-q}+(-1)^m\left[\binom{2k+1}{k+m-q}+\binom{2k+1}{k-m-q}\right]\cos(2\theta) & m+q\le k<2m-q-1\\ \dots & \dots\end{cases}$$

Example: $\frac{1}{5}\sum_{\ell=0}^{4}\sin^{13}(\frac{\pi\ell}{5})\sin(\frac{3\pi\ell}{5})=-\frac{1}{2^{13}}\left[\binom{13}{5}-\binom{13}{10}-\binom{13}{0}\right]=-\frac{125}{1024}$.

Case $\mathbf{n=2m+1}$ and $\mathbf{p=2q}$:

$$\frac{1}{2m+1}\sum_{\ell=0}^{2m}\cos[x\sin(\tfrac{\theta+2\pi\ell}{2m+1})]\cos(2q\tfrac{\theta+2\pi\ell}{2m+1})=J_{2q}(x)+[J_{4m+2-2q}(x)+J_{4m+2+2q}(x)]\cos(2\theta)+\dots$$
$$\frac{1}{2m+1}\sum_{\ell=0}^{2m}\sin[x\sin(\tfrac{\theta+2\pi\ell}{2m+1})]\sin(2q\tfrac{\theta+2\pi\ell}{2m+1})=[J_{2m+1-2q}(x)-J_{2m+1+2q}(x)]\cos\theta+\dots$$

$$\frac{1}{2m+1}\sum_{\ell=0}^{2m}[\sin\tfrac{\theta+2\pi\ell}{2m+1}]^{2k}\cos(2q\tfrac{\theta+2\pi\ell}{2m+1})=$$
$$=\frac{(-1)^q}{4^k}\begin{cases}0 & k<q\\ \binom{2k}{k-q} & q\le k<2m+1-q\\ \binom{2k}{k-q}-\binom{2k}{k+q-1-2m}\cos(2\theta) & 2m+1-q\le k<2m+1+q\\ \binom{2k}{k-q}-\left[\binom{2k}{k+q-1-2m}+\binom{2k}{k-q-1-2m}\right]\cos(2\theta) & 2m+1+q\le k<4m+2-q\\ \dots & \dots\end{cases}$$

$$\frac{1}{2m+1}\sum_{\ell=0}^{2m}[\sin\tfrac{\theta+2\pi\ell}{2m+1}]^{2k+1}\sin(2q\tfrac{\theta+2\pi\ell}{2m+1})=\frac{(-1)^{q+m+1}}{2^{2k+1}}\begin{cases}0 & k<m-q\\ \binom{2k+1}{k-m+q}\cos\theta & m-q\le k<m+q\\ \left[\binom{2k+1}{k-m+q}-\binom{2k+1}{k-m-q}\right]\cos\theta & m+q\le k<3m+1-q\\ \dots & \dots\end{cases}$$

Case $\mathbf{n=2m+1}$ and $\mathbf{p=2q+1}$:

$$\frac{1}{2m+1}\sum_{\ell=0}^{2m}\cos[x\sin\tfrac{\theta+2\pi\ell}{2m+1}]\cos[(2q+1)\tfrac{\theta+2\pi\ell}{2m+1}]=[J_{2m-2q}(x)+J_{2m+2q+2}(x)]\cos\theta+\dots$$
$$-\frac{1}{2m+1}\sum_{\ell=0}^{2m}\sin[x\sin\tfrac{\theta+2\pi\ell}{2m+1}]\sin[(2q+1)\tfrac{\theta+2\pi\ell}{2m+1}]=J_{2q+1}(x)+[-J_{4m-2q+1}(x)+J_{4m+2q+3}(x)]\cos(2\theta)+\dots$$

$$\frac{1}{2m+1}\sum_{\ell=0}^{2m}\sin[\tfrac{\theta+2\pi\ell}{2m+1}]^{2k}\cos[(2q+1)\tfrac{\theta+2\pi\ell}{2m+1}]=$$
$$=\frac{(-1)^{m+q}}{4^k}\begin{cases}0 & k<m-q\\ \binom{2k}{k-m+q}\cos\theta & m-q\le k<m+q+1\\ \left[\binom{2k}{k-m+q}-\binom{2k}{k-m-q-1}\right]\cos\theta & m+q+1\le k<3m+q+2\\ \dots & \dots\end{cases}$$

$$\frac{1}{2m+1}\sum_{\ell=0}^{2m}\sin[\tfrac{\theta+2\pi\ell}{2m+1}]^{2k+1}\sin[(2q+1)\tfrac{\theta+2\pi\ell}{2m+1}]=$$
$$=\frac{(-1)^q}{2^{2k+1}}\begin{cases}0 & k<q\\ \binom{2k+1}{k-q} & q\le k<2m-q\\ \binom{2k+1}{k-q}-\binom{2k+1}{k+q-2m}\cos(2\theta) & 2m-q\le k<2m+q+1\\ \binom{2k+1}{k-q}-\left[\binom{2k+1}{k+q-2m}+\binom{2k+1}{k-q-1-2m}\right]\cos(2\theta) & 2m+q+1\le k<4m-q+1\\ \dots & \dots\end{cases}.$$

Examples:

$\frac{1}{13}\sum_{\ell=0}^{12}\sin^9(\frac{\theta+2\pi\ell}{13})\sin(5\frac{\theta+2\pi\ell}{13})=\frac{1}{2^9}\binom{9}{2}=\frac{9}{128}$ for all θ,
$\frac{1}{13}\sum_{\ell=0}^{12}\sin^{31}(\frac{\theta+2\pi\ell}{13})\sin(5\frac{\theta+2\pi\ell}{13})=\frac{1}{2^{31}}\left[\binom{31}{13}-\left(\binom{31}{5}+1\right)\cos(2\theta)\right].$

Conclusion

Bessel functions of the first kind may be well approximated on an interval containing the origin, by the trapezoidal rule applied the Bessel integral. The result is a finite trigonometric sum. Here I show that comparable accuracy is obtained by exploiting some exact results for Neumann series of Bessel functions and cosines, as finite trigonometric sums. At appropriate angles, the second term of the Neumann series cancels and, by the rapid decay of third and next terms, the trigonometric sum well represents the first Bessel term in the Neumann series.
The same Neumann series allow for the evaluation of new trigonometric sums of powers of sine and cosine functions. They extend recent results by Jelitto [20], and are not included in the extensive paper by Al Jarrah et al. [15].

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Characterization of Intuitionistic Fuzzy Collineations in Intuitionistic Fuzzy Projective Planes

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Abstract

In this paper, the intuitionistic fuzzy counterparts of the collineations defined in classical projective planes are defined in intuitionistic fuzzy projective planes. The properties of the intuitionistic fuzzy projective plane left invariant under the intuitionistic fuzzy collineations are characterized depending on the base point, base line, membership degrees, and the non-membership degrees of the intuitionistic fuzzy projective plane.

1. Introduction

The fuzzy concept was first proposed by Zadeh in 1965 [1], and many scientists have contributed to this field. Projective planes have been fuzzified by Kuijken et al., see [2]. Also a fuzzy group corresponding to the fuzzy projective geometry was created, so that through these fuzzy projective geometries a relationship between fuzzy vector spaces and fuzzy groups was obtained by Kuijken, Maldeghem and Kerre in 1999 [3]. The fuzzy projective plane collinations were described by Kuijken and Maldeghem in 2003 [4].
As a generalization of Zadeh's Fuzzy Sets, Intuitionistic Fuzzy Set which is characterized by a membership function and a nonmembership function was proposed by Atannassov [5]. In 2009, a new model of intuitionistic fuzzy projective geometry was constructed by Ghassan [6] and it is seen that this new intuitionistic fuzzy projective plane is closely related to the fibered projective geometry. And also in different algebraic structures many theories were introduced. By Sharma, a relation relating to the Intuitionistic fuzzy subgroup of a group with its homomorphic image by using the properties of their (α,β)–cut sets was determined [7]. By developing and holding to some properties of Atanassov's intuitionistic fuzzy relations, some connections of their properties with lattice operations were introduced in 2012 [8]. Pradhan and Pal introduced the set of all linear transformations L(V) defined over an intuitionistic fuzzy vector space V not form an vector space and the concept of the inverse of an intuitionistic fuzzy linear transformation (IFLT) in 2012 [9]. In 2015, Bayar and Ekmekci showed that intuitionistic fuzzy versions of some classical configurations in projective plane are valid in the intuitionistic fuzzy projective plane with base Desarguesian or Pappian plane [10]. In intuitionistic fuzzy projective plane, the conditions to the intuitionistic fuzzy versions of Menelaus and Ceva 6-figures have been determined by Akca et. al. [11]. In 2021, by constructing a homomorphism between intuitionistic fuzzy abstract algebras, intuitionistic fuzzy congruence relations were examined and also some isomorphism theorems on intuitionistic abstract algebras were introduced by Cuvalcioglu and Tarsuslu [12]. In the fuzzy and intuitionistic fuzzy projective planes, Altintas and Bayar introduced the fuzzy counterparts and the intuitionistic fuzzy counterparts of the central collineations defined in the classical projective planes and showed some properties of central fuzzy and intuitionistic fuzzy collineations [13]. This paper is an extention of examinations by Altintas et. al. [14] on the

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Fuzzy Collineations of Fuzzy Projective Planes to intuitionistic fuzzy projective planes. In [14], we introduced fuzzy versions of some classical properties related to collineations of projective plane by collineations in fuzzy projective planes by using the base point, the base line and the membership degrees of fuzzy projective plane.
The aim of this study is to define the intuitionistic fuzzy equivalents of collineations defined in classical projective planes in intuitionistic fuzzy projective planes such that every point and every line in the base plane possess the degree of membership and the degree of non-membership and to prove the properties that are invariant under the intuitionistic fuzzy collineations in intuitionistic fuzzy projective planes.

Definition 1.1. *A fuzzy set λ of a set X is a function $\lambda : X \to [0,1] : x \to \lambda(x)$. The number $\lambda(x)$is called the degree of membership of the point x in λ. The intersection $\lambda \wedge \mu$ of the two fuzzy sets λ and μ on X is given by the fuzzy set*
$\lambda \wedge \mu : X \to [0,1] : \lambda(x) \wedge \mu(x)$,
where $\wedge$ denotes the minimum operator and also $\vee$ denotes the maximum operator [1].

Definition 1.2. *[5] Let X be a nonempty fixed set. An intuitionistic fuzzy set A on X is an object having the form $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ where the function $\lambda : X \to I$ and $\mu : X \to I$ denote the degree of membership (namely, $\lambda(x)$) and the degree of nonmembership (namely, $\mu(x)$) of each element $x \in X$ to the set A, respectively $0 \leq \lambda(x) + \mu(x) \leq 1$ for each $x \in X$. An intuitionistic fuzzy set $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ can be written in the $A = \{\langle x, \lambda, \mu\rangle : x \in X\}$, or simply $A = \langle \lambda, \mu\rangle$. Let $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ and $B = \{\langle x, \delta(x), \gamma(x)\rangle : x \in X\}$ be an intuitionistic fuzzy sets on X. Then,*

(a) *$\bar{A} = \{\langle x, \mu(x), \lambda(x)\rangle : x \in X\}$ (the complement of A).*
(b) *$A \cap B = \{\langle x, \lambda(x) \wedge \delta(x), \mu(x) \vee \gamma(x)\rangle : x \in X\}$ (the meet of A and B).*
(c) *$A \cup B = \{\langle x, \lambda(x) \vee \delta(x), \mu(x) \wedge \gamma(x)\rangle : x \in X\}$ (the join of A and B).*
(d) *$A \subseteq B \Leftrightarrow \lambda(x) \leq \delta(x)$ and $\mu(x) \geq \gamma(x)$ for each $x \in X$.*
(e) *$A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.*
(f) *$\tilde{1} = \{\langle x, 1, 0\rangle : x \in X\}, \tilde{0} = \{\langle x, 0, 1\rangle : x \in X\}$.*

Definition 1.3. *Let $A = (\lambda, \mu)$ be an intuitionistic fuzzy set of a classical vector space V over F. For any $x, y \in V$ and $\alpha, \beta \in F$, if it satisfy $\lambda_A(\alpha x + \beta y) \geq min\{\lambda_A(x), \lambda_A(y)\}$ and $\mu_A(\alpha x + \beta y) \leq max\{\mu_A(x), \mu_A(y)\}$, then A is called an intuitionistic fuzzy subspace of V. Let V_n denotes the set of all $n-$tuples $(\langle x_{1\lambda}, x_{1\mu}\rangle, \langle x_{2\lambda}, x_{2\mu}\rangle, ..., \langle x_{n\lambda}, x_{n\mu}\rangle)$ over F. An element of V_n is called an intuitionistic fuzzy vector (IFV) of dimension n, where $x_{i\lambda}$ and $x_{i\mu}$ are the membership and non-membership values of the component x_i [9].*

Definition 1.4. *An intuitionistic fuzzy set $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ on $n-$dimensional projective space S is an intuitionistic fuzzy $n-$dimensional projective space on S if $\lambda(p) \geq \lambda(q) \wedge \lambda(r)$ and $\mu(p) \leq \mu(q) \vee \mu(r)$,for any three collinear points p, q, r of A we denoted $[A, S]$. The projective space S is called the base projective space of $[A, S]$ if $[A, S]$ is an intuitionistic fuzzy point, line, plane, ... , we use base point, base line, base plane, ..., respectively [6].*

Definition 1.5. *Let $\langle \lambda, \mu\rangle$ be an intuitionistic fuzzy projective space and let U be a subspace of $\mathscr{P}$. Then (λ_U, U) is called a fuzzy subspace of $(\lambda, \mathscr{P})$ if $\lambda_U(x) \leq \lambda(x)$ and $\mu_U(x) \geq \mu(x)$ for $x \in U$, and $\lambda_U(x) = 0, \mu_U(x) = 1$ for $x \notin U$.*

Definition 1.6. *Let (λ, μ) be an intuitionistic fuzzy projective space of dimension n. Then there are constants $a_i, b_i \in [0,1]$, $i = 0, 1, ..., n$, with $a_i + b_i \leq 1$, and a chain of subspaces $(U_i)_{0 \leq i \leq n}$ with $U_i \leq U_{i+1}$ and $dimU_i = i$, such that*

$$\begin{array}{rcccl} (\lambda, \mu) & : & \mathscr{P} & \to & [0,1] \times [0,1] \\ & & \bar{u} & \to & (a_0, b_0) \quad for\, \bar{u} \in U_0 \\ & & \bar{u} & \to & (a_i, b_i) \quad for\ \bar{u} \in U_i \setminus U_{i-1}, i = 1, 2, ..., n \end{array}$$

Definition 1.7. *[10] Consider the projective plane $\mathscr{P} = (\mathscr{N}, \mathscr{D}, \circ)$. Suppose $a \in \mathscr{N}$ and $\alpha, \beta \in [0,1]$. The IF-point (a, α, β) is the following intuitionistic fuzzy set on the point set $\mathscr{N}$ of $\mathscr{P}$:*

$$(a, \alpha, \beta) : \mathscr{N} \to [0,1] : \begin{array}{l} a \to \alpha, \ a \to \beta \\ x \to 0, \ x \in \mathscr{N} \setminus \{a\} \end{array}$$

The point $a \in \mathscr{N}$ is called the base point of the IF-point (a, α, β). An IF-line (L, α, β) with base line L is defined in a similar way. The IF- lines (L, α, β) and (M, σ, ω) intersect in the unique IF-point $(L \cap M, \alpha \wedge \sigma, \beta \vee \omega)$. The IF-points (a, α, β) and(b, σ, ω) span the unique IF-line $(\langle a, b\rangle, \alpha \wedge \sigma, \beta \vee \omega)$.

Definition 1.8. *Suppose $\mathscr{P}$ is a projective plane $\mathscr{P} = (\mathscr{N}, \mathscr{D}, \circ)$. The intuitionistic fuzzy set $Z = \langle \lambda, \mu\rangle$ on $\mathscr{N} \cup \mathscr{D}$ is an intuitionistic fuzzy projective plane on $\mathscr{P}$ if :*

$$\textbf{(1)}\ \lambda(L) \geq \lambda(p) \wedge \lambda(q) \text{ and } \mu(L) \leq \mu(p) \vee \mu(q); \forall p, q : \langle p, q\rangle = L$$

$$\textbf{(2)} \lambda(p) \geq \lambda(L) \wedge \lambda(M) \text{ and } \mu(p) \leq \mu(L) \vee \mu(M); \forall L, M : L \cap M = p.$$

The intuitionistic fuzzy projective plane can be considered as an ordinary projective plane, where to every point (and only to points) one (and only one) degrees of membership and nonmembership are assigned [6].

Now here, the intuitionistic fuzzy counterparts of the theorems and proofs related to the fuzzy linear maps in Abdulhalikov's works [15] are given by using the intuitionistic fuzzy linear maps definition.

Definition 1.9. *Let V and W be two vector spaces over the same field F and T be is a linear map from V to W. Suppose that (V,λ_V,μ_V) and (W,λ_W,μ_W) be intuitionistic fuzzy vector spaces on F. For all $x\in V$, if*

$$\lambda_W(T(x))\geq\lambda_V(x) \text{ and } \mu_W(T(x))\leq\mu_V(x)$$

is satisfied such that $0\leq\lambda_V+\mu_V\leq 1$ and $0\leq\lambda_W+\mu_W\leq 1$, T is called as an intuitionistic fuzzy linear maps from the intuitionistic fuzzy vector space (V,λ_V,μ_V) to the intuitionistic fuzzy vector space (W,λ_W,μ_W).

Definition 1.10. *If T which is a zero linear map from V to W, is an intuitionistic fuzzy linear map defined between the intuitionistic fuzzy vector spaces (V,λ_V,μ_V) and (W,λ_W,μ_W), then T is called as an intuitionistic fuzzy zero linear map.*

2. Intuitionistic collineations of intuitionistic fuzzy projective plane

In this paper, our aim is to investigate intuitionistic collineations of intuitionistic fuzzy projective planes. Compared to isomorphisms, collineations of projective plane and fuzzy isomorphisms, fuzzy collineations of fuzzy projective plane have the advantages and properties. In projective planes, a collineation is a point-to-point and line-to-line transformation that preserves the relation of incidence. Thus it transforms ranges into ranges, pencils into pencils, quadrangles into quadrangles, and so on. Clearly, it is a self-dual concept, the inverse of a collineation, and the product of two collineations is again a collineation [4]. Our aim is now to define the intuitionistic fuzzy counterparts of homomorphism and isomorphism defined in vector spaces in intuitionistic fuzzy projective planes and to apply theorems about properties of collineations in projective plane to intuitionistic fuzzy projective plane. Furthermore, we will show that each collineation can be uniquely extended to a fuzzy projective collineation.
The definitions of homomorphism, isomorphism and collineation in projective planes can be adopted to fuzzy projective planes as follows:

Definition 2.1. *Let $[\mathscr{P},\lambda_{\mathscr{P}},\mu_{\mathscr{P}}]$ and $[\mathscr{P}',\lambda_{\mathscr{P}'},\mu_{\mathscr{P}'}]$ be two intuitionistic fuzzy projective planes with base planes $\mathscr{P}=(\mathscr{N},\mathscr{D},\circ)$, $\mathscr{P}'=\left(\mathscr{N}',\mathscr{D}',\circ'\right)$, respectively. Suppose that f be a homomorphism of a projective plane $\mathscr{P}$ into a projective plane P'P'.$\bar{f}$ is called as* **intuitionistic fuzzy homomorphism** *from $[\mathscr{P},\lambda_{\mathscr{P}},\mu_{\mathscr{P}}]$ into $[\mathscr{P}',\lambda_{\mathscr{P}'},\mu_{\mathscr{P}'}]$ if $\bar{f}(p,\alpha,\beta)=(f(p),\alpha',\beta')$ for all $(p,\alpha,\beta)\in[\mathscr{P},\lambda_{\mathscr{P}},\mu_{\mathscr{P}}]$ where $\lambda_{\mathscr{P}}(p)=\alpha$, $\mu_{\mathscr{P}}(p)=\beta$, $\lambda_{\mathscr{P}'}(f(p))=\alpha'$, $\mu_{\mathscr{P}'}(f(p))=\beta'$ and $\alpha\leq\alpha'$, $\beta\geq\beta'$. If f is an isomorphism of $\mathscr{P}$ into $\mathscr{P}'$ and $\alpha=\alpha',\beta=\beta'$, then $\bar{f}$ is called as* **intuitionistic fuzzy isomorphism** *between the intuitionistic fuzzy projective planes $[\mathscr{P},\lambda_{\mathscr{P}},\mu_{\mathscr{P}}]$ and $[\mathscr{P}',\lambda_{\mathscr{P}'},\mu_{\mathscr{P}'}]$. Also if $\mathscr{P}=\mathscr{P}'$, this $\bar{f}$ intuitionistic fuzzy isomorphism is called as* **intuitionistic fuzzy collineation**.

Theorem 2.2. *Let $\bar{f}:[\mathscr{P},\lambda,\mu]\to[\mathscr{P}',\lambda',\mu']$ is intuitionistic fuzzy isomorphism, the following holds:*
(i) *For any pair of intuitionistic fuzzy points (p_1,α_1,β_1) and (p_2,α_2,β_2), $p_1\neq p_2$ in $[\mathscr{P},\lambda,\mu]$,*

$$\bar{f}(\langle(p_1,\alpha_1,\beta_1),(p_2,\alpha_2,\beta_2)\rangle)=\langle\bar{f}(p_1,\alpha_1,\beta_1),\bar{f}(p_2,\alpha_2,\beta_2)\rangle.$$

(ii) *For any pair of intuitionistic fuzzy lines (L,γ_1,σ_1) and (M,γ_2,σ_2), $L\neq M$ in $[\mathscr{P},\lambda,\mu]$,*

$$\bar{f}((L,\gamma_1,\sigma_1)\cap(M,\gamma_2,\sigma_2))=\bar{f}(L,\gamma_1,\sigma_1)\cap\bar{f}(M,\gamma_2,\sigma_2).$$

(iii) *For any intuitionistic fuzzy point (p,α,β) and intuitionistic fuzzy line (L,γ,σ) in $[\mathscr{P},\lambda,\mu]$, if p is not on L, then the intuitionistic fuzzy point $\bar{f}(p,\alpha,\beta)$ is not on $\bar{f}(L,\gamma,\sigma)$ in $[\mathscr{P}',\lambda',\mu']$.*

Proof **(i)** Let $\bar{f}$ be an intuitionistic fuzzy isomorphism between $[\mathscr{P},\lambda,\mu]$ and $[\mathscr{P}',\lambda',\mu']$. The intuitionistic fuzzy line spanned by the intuitionistic fuzzy points (p_1,α_1,β_1) and (p_2,α_2,β_2) with distinct base points $p_1,\ p_2$ is $\langle(p_1,\alpha_1,\beta_1),(p_2,\alpha_2,\beta_2)\rangle=(\langle p_1,p_2\rangle,\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)$. Since f is isomorphism between the base projective planes $\mathscr{P}$ and $\mathscr{P}'$, $f(p_1)\neq f(p_2)$. So $\bar{f}(p_1,\alpha_1,\beta_1)\neq\bar{f}(p_2,\alpha_2,\beta_2)$. Using the definitions of $\bar{f}$ and f,

$$\begin{aligned}\bar{f}(\langle(p_1,\alpha_1,\beta_1),(p_2,\alpha_2,\beta_2)\rangle)&=(f(\langle p_1,p_2\rangle),\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)=(\langle f(p_1),f(p_2)\rangle,\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)\\&=\langle(f(p_1),\alpha_1,\beta_1),(f(p_2),\alpha_2,\beta_2)\rangle=\langle\bar{f}(p_1,\alpha_1,\beta_1),\bar{f}(p_2,\alpha_2,\beta_2)\rangle\end{aligned}$$

is obtained.

(ii) Let $\bar{f}$ be an intuitionistic fuzzy isomorphism between $[\mathscr{P},\lambda,\mu]$ and $[\mathscr{P}',\lambda',\mu']$. The intersection point of the intuitionistic fuzzy lines (L,γ_1,σ_1) and (M,γ_2,σ_2) with distinct base lines $L,\ M$ is $(L,\gamma_1,\sigma_1)\cap(M,\gamma_2,\sigma_2)=(L\cap M,\gamma_1\wedge\gamma_2,\sigma_1\vee\sigma_2)$. Since f is isomorphism between the projective planes $\mathscr{P}$ and $\mathscr{P}'$, $f(L)\neq f(M)$. So $\bar{f}(L,\gamma_1,\sigma_1)\neq\bar{f}(M,\gamma_2,\sigma_2)$. Using the definition of $\bar{f}$ and f

$$\begin{aligned}\bar{f}((L,\gamma_1,\sigma_1)\cap(M,\gamma_2,\sigma_2))&=(f(L\cap M),\gamma_1\wedge\gamma_2,\sigma_1\vee\sigma_2)=(f(L)\cap f(M),\gamma_1\wedge\gamma_2,\sigma_1\vee\sigma_2)\\&=((f(L),\gamma_1,\sigma_1)\cap(f(M),\gamma_2,\sigma_2))=\bar{f}(L,\gamma_1,\sigma_1)\cap\bar{f}(M,\gamma_2,\sigma_2)\end{aligned}$$

(iii) Suppose that the intuitionistic fuzzy point $\bar{f}((p,\alpha,\beta))$ is on the intuitionistic fuzzy line $\bar{f}((L,\gamma,\sigma))$ when the base point p is not on the base line L. Then the intuitionistic fuzzy point (p,α,β) is not on the intuitionistic fuzzy line (L,γ,σ). From definitions of f and $\bar{f}$, $\bar{f}((p,\alpha,\beta)) = (f(p),\alpha,\beta)$ and $\bar{f}((L,\gamma,\sigma)) = (f(L),\gamma,\sigma)$. Since the intuitionistic fuzzy point $\bar{f}((p,\alpha,\beta))$ is on the intuitionistic fuzzy line $\bar{f}((L,\gamma,\sigma))$ and f is isomorphism, $f(p)\circ f(L)$ and $p\circ L$ are obtained. This contradicts the hypothesis.
From now on, we considered the intuitionistic fuzzy projective plane $[\mathscr{P},\lambda,\mu]$ with base plane $\mathscr{P}$ and (λ,μ) in the following form:

$$\begin{array}{rcccl} (\lambda,\mu) & : & PG(V) & \rightarrow & [0,1]\times[0,1] \\ & & q & \rightarrow & (a_0,b_0) \\ & & p & \rightarrow & (a_1,b_1),\, p\in L\setminus\{q\} \\ & & p & \rightarrow & (a_2,b_2),\, p\in \mathscr{P}\setminus\{L\} \end{array}$$

where L is a projective line of $\mathscr{P}$ contains q and $a_0 \geq a_1 \geq a_2$, $b_0 \leq b_1 \leq b_2$, $0 \leq a_i + b_i \leq 1,\ i = 0,\ 1,\ 2.$

The intuitionistic fuzzy point (q,a_0,b_0) and the intuitionistic fuzzy line (L,a_1,b_1) are called as the base point, the base line of the intuitionistic fuzzy projective plane $[\mathscr{P},\lambda,\mu]$, respectively. The invariant properties under any intuitionistic fuzzy collineation in $[\mathscr{P},\lambda,\mu]$ depending on the base line, the base point, the membership degrees and nonmembership degrees of $[\mathscr{P},\lambda,\mu]$ are investigated in detail with the following theorems.

Theorem 2.3. *Suppose that $\bar{f}$ is an intuitionistic fuzzy collineation of $[\mathscr{P},\lambda,\mu]$ defined by the collineation f of the base plane $\mathscr{P}$. Then,*

(i) *If $a_0 \neq a_1 \neq a_2$, then the intuitionistic fuzzy collineation $\bar{f}$ leaves invariant the base point and the base line of $[\mathscr{P},\lambda,\mu]$.*

(ii) *If $a_0 \neq a_1 = a_2$, then the base point is invariant and the base line turns into a line passing through the base point under the intuitionistic fuzzy collineation $\bar{f}$.*

Proof **(i)** Let $a_0 \neq a_1 \neq a_2$.
The image of the base point (q,a_0,b_0) is $\bar{f}(q,a_0,b_0) = (f(q),a_0,b_0)$. Since there is no other point which has membership degree (a_0,b_0) in $[\mathscr{P},\lambda,\mu]$, $(f(q),a_0,b_0)$ must be the base point. So $f(q) = q$, $\bar{f}(q,a_0,b_0) = (q,a_0,b_0)$.
Since $(L,a_1,b_1) = \langle (q,a_0,b_0),(p,a_1,b_1)\rangle \ni p\circ L$, $p \neq q$ and from Theorem 2. 2. i), the base line is

$$\begin{aligned} \bar{f}((L,a_1,b_1)) &= \left\langle \bar{f}(q,a_0,b_0),\bar{f}(p,a_1,b_1)\right\rangle = \langle (f(q),a_0,b_0),(f(p),a_1,b_1)\rangle,\,(f(q)=q) \\ &= \langle (q,a_0,b_0),(f(p),a_1,b_1)\rangle = (\langle q,f(p)\rangle,a_0\wedge a_1,b_0\vee b_1) = (\langle q,f(p)\rangle,a_1,b_1). \end{aligned}$$

Since there is no other line with (a_1,b_1) membership degree, $\bar{f}((L,a_1,b_1)) = (\langle q,f(p_i)\rangle,a_1,b_1) = (L,a_1,b_1)$ is obtained. So the base point and the base line are invariant under the intuitionistic fuzzy collineation $\bar{f}$.
The converse of this proposition is not true. While the base point and the base line are invariant, the membership degrees can be different or equal.

(ii) Let $a_0 \neq a_1 = a_2$.
The image of the base point (q,a_0,b_0) is $\bar{f}(q,a_0,b_0) = (f(q),a_0,b_0)$. Since there is no other line with (a_0,b_0) membership degree in $[\mathscr{P},\lambda,\mu]$, $(f(q),a_0ib_0)$ must be base point. So $f(q) = q$ and $\bar{f}(q,a_0,b_0) = (f(q),a_0,b_0) = (q,a_0,b_0)$. Since $\bar{f}$ is intuitionistic fuzzy isomorphism, $\bar{f}(q,a_0,b_0)\circ\bar{f}(L,a_1,b_1)$. Hence, the base point (q,a_0,b_0) is on $(f(L),a_1,b_1)$.

$$\begin{aligned} \bar{f}(L,a_1,b_1) &= \bar{f}(\langle (q,a_0,b_0),(p,a_1,b_1)\rangle) = \left\langle \bar{f}(q,a_0,b_0),\bar{f}(p,a_1,b_1)\right\rangle = \langle (f(q),a_0,b_0),(f(p),a_1,b_1)\rangle \\ &= \langle (q,a_0,b_0),f(p,a_1,b_1)\rangle = (\langle q,f(p)\rangle,a_0\wedge a_1,b_0\vee b_1) = (\langle q,f(p)\rangle,a_1,b_1) \end{aligned}$$

So the base line $f(L) = \langle q,f(p)\rangle$ turns into the line through the base point.
The following theorem states the properties of $\bar{f}$ intuitionistic fuzzy collineation while the base point is invariant.

Theorem 2.4. *Suppose that $\bar{f}$ is an intuitionistic fuzzy collineation of $[\mathscr{P},\lambda,\mu]$ defined by the collineation f of the base plane $\mathscr{P}$ and the base point (q,a_0,b_0) be invariant under the intuitionistic fuzzy collineation $\bar{f}$.*

(i) *If the base line (L,a_1,b_1) is invariant under $\bar{f}$, $[\mathscr{P},\lambda,\mu]$ has at most three membership degrees.*

(ii) *If the base line (L,a_1,b_1) turns into a line other than itself passing through the base point (q,a_0,b_0), there are at most two membership degrees in $[\mathscr{P},\lambda,\mu]$ such that $a_0 \geq a_1 = a_2$ and $b_0 \leq b_1 = b_2$.*

(iii) *The base line (L,a_1,b_1) does not turn into an intuitionistic fuzzy line that does not pass through the base point (q,a_0,b_0) under $\bar{f}$ in $[\mathscr{P},\lambda,\mu]$.*

Proof **(i)** Let the base point (q,a_0,b_0) and the base line (L,a_1,b_1) be invariant under the intuitionistic fuzzy collineation $\bar{f}$. Then $\bar{f}(q,a_0,b_0) = (q,a_0,b_0)$. The image point $\bar{f}(p,a_1,b_1)$ of the intuitionistic fuzzy point (p,a_1,b_1) on the base line (L,a_1,b_1) is $(f(p),a_1,b_1)$ and is on the base line L.

If $a_0 \neq a_1 \neq a_2$ is taken, there are at most three membership degrees in $[\mathscr{P},\lambda,\mu]$.

(ii) Let the base point (q,a_0,b_0) be invariant and the base line turns into a line other than the base line passing through the base point $[\mathscr{P},\lambda,\mu]$. Since the base point (q,a_0,b_0) on (L,a_1,b_1) and $\bar{f}$ is an intuitionistic fuzzy isomorphism, the image of the base point (q,a_0,b_0) is also on the image of the base line $(f(L),a_1,b_1)$. Also $L \neq f(L)$ and the line $f(L)$ passes through points of degree of membership (a_2,b_2) not on the base line, the membership degree of $f(L)$ is (a_2,b_2). So, $a_1 = a_2$ and $b_1 = b_2$ are obtained. Consequently, $[\mathscr{P},\lambda,\mu]$ has at most two membership degrees.

(iii) Since the base point is on the base line, its image is on the image of the base line. However, the being invariant of the base point gives rise to that the image line has to pass through the base point.

Theorem 2.5. *Suppose that $\bar{f}$ is an intuitionistic fuzzy collineation of $[\mathscr{P},\lambda,\mu]$ defined by the collineation f of the base plane $\mathscr{P}$ and the base point is not invariant and turns into an intuitionistic fuzzy point on the base line under the intuitionistic fuzzy collineation $\bar{f}$.*

(i) *If the base line is invariant under the intuitionistic fuzzy collineation $\bar{f}$ of $[\mathscr{P},\lambda,\mu]$, among the membership degrees $0 \leq a_i + b_i \leq 1,\ i = 0,1,2$, there is a relationship $a_0 = a_1 \geq a_2$ and $b_0 = b_1 \leq b_2$.*

(ii) *If the base point turns into an intuitionistic fuzzy point on the base line other than itself under the intuitionistic fuzzy collineation $\bar{f}$, then there is one membership degree in $[\mathscr{P},\lambda,\mu]$.*

(iii) *If the base point of (q,a_0,b_0) turns into any point not on the base line under the collineation f in $\mathscr{P}$, then there is only one membership degree in $[\mathscr{P},\lambda,\mu]$.*

Proof **(i)** Let the base point q of the intuitionistic fuzzy point (q,a_0,b_0) be not invariant and turn into another point on the base line L, $\bar{f}(q,a_0,b_0) = (f(q),a_1,b_1)$.
Suppose that the base line L is invariant under the collineation f. It is clear that the intuitionistic fuzzy point (q,a_0,b_0) turns into the intuitionistic fuzzy point (p,a_1,b_1) with $p \circ L$, $q \neq p$. Since $\bar{f}$ is intuitionistic fuzzy isomorphism, $a_0 = a_1$ and $b_0 = b_1$. Hence, there are at most two membership degree in $[\mathscr{P},\lambda,\mu]$.

(ii) Since the intuitionistic fuzzy point (q,a_0,b_0) turns into the intuitionistic fuzzy point (p,a_1,b_1), $p \neq q$ on (L,a_1,b_1). It is clear that $a_0 = a_1$ and $b_0 = b_1$. Next suppose that $f(L) \neq L$. So any intuitionistic fuzzy point different from the base point on the base line L with membership degree (a_1,b_1) turns into any other intuitionistic fuzzy point with membership degree (a_2,b_2). Since $\bar{f}$ is intuitionistic fuzzy isomorphism, then $a_1 = a_2$ and $b_1 = b_2$. Hence, $a_0 = a_1 = a_2$ and $b_0 = b_1 = b_2$.

(iii) Since the base point turns into a point not on the base line, the image of (q,a_0,b_0) under $\bar{f}$ is (p,a_2,b_2) with $p \not\circ L$. It is clearly $a_0 = a_2$ and $b_0 = b_2$. If we use this equality and the conditions $a_0 \geq a_1 \geq a_2$ and $b_0 \leq b_1 \leq b_2$ among the membership degrees in $[\mathscr{P},\lambda,\mu]$, $a_0 = a_1 = a_2$ and $b_0 = b_1 = b_2$ are obtained.

Corollary If the base point of (q,a_0,b_0) turns into any point not on the base line L of (L,a_1,b_1) under the collineation f, the base line (L,a_1,b_1) with base line L of $[\mathscr{P},\lambda,\mu]$ is not invariant under the intuitionistic fuzzy collineation $\bar{f}$.

Proof Let the base point q turns into any point not on the base line under the collineation f of $\mathscr{P}$. Since the intuitionistic fuzzy point (q,a_0,b_0) such that the base point q is on the base line L turns into $(f(q),a_2,b_2)$ such that $f(q)$ is not on the base line L, then the base line $L = \langle q,p \rangle$ spanning by the points p and q turns into $f(L) = \langle f(q),f(p) \rangle \neq L$ under the intuitionistic fuzzy collineation $\bar{f}$. Hence, (L,a_1,b_1) is not invariant.

Theorem 2.6. *Suppose that $\bar{f}$ is an intuitionistic fuzzy collineation of $[\mathscr{P},\lambda,\mu]$ defined by the collineation f of the base plane $\mathscr{P}$.*

(i) *If two distinct points p_1 and p_2 in the base plane $\mathscr{P}$ are invariant under the collineation f of $\mathscr{P}$, the intuitionistic fuzzy line spanned by fuzzy points (p_1,α_1,β_1) and (p_2,α_2,β_2) is invariant under the intuitionistic fuzzy collineation $\bar{f}$ of $[\mathscr{P},\lambda,\mu]$.*

(ii) *If two distinct lines L_1 and L_2 in the base plane $\mathscr{P}$ are invariant under the collineation f of $\mathscr{P}$ and the intersection point of L_1 and L_2 is not on the base line L in $\mathscr{P}$, then the intersection point of the intuitionistic fuzzy lines (L_1,α_1,β_1) and (L_2,α_2,β_2) is invariant under the intuitionistic fuzzy collineation $\bar{f}$ in $[\mathscr{P},\lambda,\mu]$.*

(iii) *Suppose that two distinct lines L_1 and L_2 different from the base line L in the base plane $\mathscr{P}$ are invariant under the collineation f of $\mathscr{P}$ and the intersection point of these lines is on the base line L in $\mathscr{P}$. If the intersection point of the intuitionistic fuzzy lines (L_1,α_1,β_1) and (L_2,α_2,β_2) is invariant under the intuitionistic fuzzy collineation $\bar{f}$ in $[\mathscr{P},\lambda,\mu]$. There is a relationship $a_0 = a_1 = a_2,\ b_0 = b_1 = b_2$ or $a_1 = a_2,\ b_1 = b_2$ among the membership degrees in $[\mathscr{P},\lambda,\mu]$.*

Proof

(i) Let the base points p_1 and p_2 in $\mathscr{P}$ of (p_1,α_1,β_1) and (p_2,α_2,β_2) in $[\mathscr{P},\lambda,\mu]$ be invariant under the collineation f of $\mathscr{P}$. Then by the definition of intuitionistic fuzzy collineation $\bar{f}$ in $[\mathscr{P},\lambda,\mu]$, $\bar{f}(p_1,\alpha_1,\beta_1) = (f(p_1),\alpha_1,\beta_1) = (p_1,\alpha_1,\beta_1)$ and $\bar{f}(p_2,\alpha_2,\beta_2) = (f(p_2),\alpha_2,\beta_2) = (p_2,\alpha_2,\beta_2)$.
For any pair $((p_1,\alpha_1,\beta_1),(p_2,\alpha_2,\beta_2))$ of fuzzy points, $p_1 \neq p_2$, the fuzzy line $(\langle p_1,p_2 \rangle,\alpha_1 \wedge \alpha_2,\beta_1 \vee \beta_2)$ spanned by them, also belongs to the intuitionistic fuzzy projective plane $[\mathscr{P},\lambda,\mu]$. By using the definition of $\bar{f}$ of $[\mathscr{P},\lambda,\mu]$ and the remaining invariant of the points p_1 and p_2 under the collineation f in $\mathscr{P}$, the image of the intuitionistic fuzzy line $(\langle (p_1,\alpha_1,\beta_1),(p_2,\alpha_2,\beta_2) \rangle)$ under the intuitionistic fuzzy collineation $\bar{f}$ is

$$(f(\langle p_1,p_2\rangle),\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)=(\langle f(p_1),f(p_2)\rangle,\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)=(\langle p_1,p_2\rangle,\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2).$$

Hence, the intuitionistic fuzzy line $(\langle p_1,p_2\rangle,\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)$ is invariant under the intuitionistic fuzzy collineation $\bar{f}$.

(ii) Let the base lines L_1 and L_2 in $\mathscr{P}$ of (L_1,α_1,β_1) and (L_2,α_2,β_2) in $[\mathscr{P},\lambda,\mu]$ be invariant under the collineation f of $\mathscr{P}$. Since $L_1\neq L_2\neq L$, the membership degrees $\alpha_i=a_2$, and $\beta_i=b_2$ $i=1,2$. By the definition of $\bar{f}$ and being invariant of the lines L_1 and L_2 under the collineation f in $\mathscr{P}$, $\bar{f}(L_1,a_2,b_2)=(L_1,a_2,b_2)$ and $\bar{f}(L_2,a_2,b_2)=(L_2,a_2,b_2)$. The image of the intersection intuitionistic fuzzy point $(L_1\cap L_2,a_2,b_2)$ under $\bar{f}$ is

$$(f(L_1\cap L_2),a_2,b_2)=(f(L_1)\cap f(L_2),a_2,b_2)=(L_1\cap L_2,a_2,b_2).$$

It is implies that the intuitionistic fuzzy point $(L_1\cap L_2,a_2,b_2)$ remains invariant under the intuitionistic fuzzy collineation $\bar{f}$.

(iii) Let different base lines L_1 and L_2 of (L_1,α_1,β_1) and (L_2,α_2,β_2) be invariant under the collineation f in $\mathscr{P}$. Since $L_1\neq L_2\neq L$, $\alpha_i=a_2$, and $\beta_i=b_2$ $i=1,2$. The intersection point of (L_1,a_2,b_2) and (L_2,a_2,b_2) is the base point (q,a_0,b_0) or any intuitionistic fuzzy point (p,a_1,b_1) on the base line (L,a_1,b_1) of $[\mathscr{P},\lambda,\mu]$. If the intersection point is (q,a_0,b_0), then $a_0=a_1=a_2$ and $b_0=b_1=b_2$. If the intersection point is (p,a_1,b_1), then $a_1=a_2$ and $b_1=b_2$ are obtained.

Theorem 2.7. *Suppose that $\bar{f}$ is any intuitionistic fuzzy collineation of $[\mathscr{P},\lambda,\mu]$ defined by the collineation f of the base plane $\mathscr{P}$. In this case,*

(i) *If M is a pointwise invariant line under the collineation f in the base projective plane $\mathscr{P}$, then the corresponding intuitionistic fuzzy line (M,γ,σ) is also pointwise invariant under the intuitionistic fuzzy collineation $\bar{f}$ in $[\mathscr{P},\lambda,\mu]$.*

(ii) *If two distinct lines L_1 and L_2 are pointwise invariant under the collineation f of the base plane $\mathscr{P}$, then the intersection point of the intuitionistic fuzzy lines (L_1,γ_1,σ_1) and (L_2,γ_2,σ_2) is invariant under the intuitionistic fuzzy collineation $\bar{f}$.*

(iii) *If the base line L and L_1, $L_1\neq L$ are pointwise invariant lines under the collineation f of the base plane $\mathscr{P}$, then the intuitionistic fuzzy collineation $\bar{f}$ is identity collineation in $[\mathscr{P},\lambda,\mu]$.*

Proof **(i)** Let the base line M of the intuitionistic fuzzy line (M,γ,σ) in $[\mathscr{P},\lambda,\mu]$ be pointwise invariant under the collineation f of $\mathscr{P}$. From the definition of $\bar{f}$ and being pointwise invariant of M under collineation f,$\bar{f}(p,\alpha,\beta)=(f(p),\alpha,\beta)=(p,\alpha,\beta)$ for every fuzzy point (p,α,β) on (M,γ,σ). Hence the fuzzy line (M,γ,σ) is pointwise invariant in $[\mathscr{P},\lambda,\mu]$.

(ii) Let the base lines L_1 and L_2 of the intuitionistic fuzzy lines (L_1,γ_1,σ_1) and (L_2,γ_2,σ_2) in $[\mathscr{P},\lambda,\mu]$ be pointwise invariant under the collineation f of the base plane $\mathscr{P}$, respectively. From (i), the fuzzy lines (L_1,γ_1,σ_1) and (L_2,γ_2,σ_2) are pointwise invariant under the intuitionistic fuzzy collineation $\bar{f}$ of $[\mathscr{P},\lambda,\mu]$. Since (L_i,γ_i,σ_i) are pointwise invariant and (p,α,β) is on (L_i,γ_i,σ_i), $i=1,2$, hence the intersection point of (L_1,γ_1,σ_1) and (L_2,γ_2,σ_2) is invariant.

(iii) Let the base line L and L_1, $L_1\neq L$ be pointwise invariant under the collineation f of $\mathscr{P}$. So $\gamma_1=a_2$ and $\sigma_1=b_2$ are obtained. It is well-known that if there are two distinct pointwise lines under a collineation of projective plane $\mathscr{P}$, then the collineation f is identity collineation. From *i)* (L,a_1,b_1) and (L_1,a_2,b_2), are pointwise invariant. The image of (p,a_2,b_2) such that $p\,\phi L$ and $p\,\phi L_1$ is $\bar{f}(p,a_2,b_2)=(f(p),a_2,b_2)=(p,a_2,b_2)$ under $\bar{f}$. Hence every intuitionistic fuzzy point in $[\mathscr{P},\lambda,\mu]$ are invariant, and this means that $\bar{f}$ is the identity collineation of $[\mathscr{P},\lambda,\mu]$.

Corollary If f is identity collineation of $\mathscr{P}$, $\bar{f}$ is identity collineation of $[\mathscr{P},\lambda,\mu]$.

3. Conclusion

The concepts of intuitionistic fuzzy isomorphism and intuitionistic fuzzy collineations between two intuitionistic fuzzy projective planes are introduced and then some important results are obtained. It is seen that the intuitionistic fuzzy collineations of intuitionistic fuzzy projective planes cannot hold the intuitionistic fuzzy versions of some classical properties related to collineations of projective plane. The properties of intuitionistic fuzzy projective plane left invariant under the intuitionistic fuzzy collineations are characterized depending on the base point, base line and the membership degrees of intuitionistic fuzzy projective plane. Consequently, these obtained results on intuitionistic fuzzy isomorphism and intuitionistic fuzzy collineation have an important effect on enriching the theory of intuitionistic fuzzy geometries.

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