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Near Soft Topological Groups Based on Near Soft Element

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Abstract. In this article, we introduce the concept of the near soft element and define the near soft group using near soft element with binary operation in the whole set non-empty near soft elements of a dedicated near soft set. In addition, the concepts of topology and continuity of the near soft group are mentioned. The concept of near soft topological group was created with the help of continuous transformations defined on the near soft group. Finally, an example is given for the concept of near soft topological group.

1. Introduction

The concept of near sets Peters [1, 2] and the concept of soft theory is given by Molodtsov [3]. It was later examined by many scientists with [3–11]. The binary soft element definition given by Wardowski [12] is operating on all non-empty soft elements of a dedicated soft set. Next, J.Ghosh [13, 14] defines the soft groupoid according to the soft elements set. Many other researchers have created the topological version of soft set theory and soft algebra. Following these articles, Wardowski [12] gave his ideas about soft topological groups with the help of soft elements. Starting from this definition, scientists studied the soft transformation and continuity of soft mapping [15]. on the other hand Feng and Li [5] came up with the notion of rough soft sets by combining soft sets with rough sets. Similar algebraic studies have been done on rough sets. Later, Tasbozan et al. [16–18] combined the near-set approach with the soft-set. Also, the concept of the near soft set was presented. Next, the concept of a near soft element is defined and the near soft groupoid is defined using a near soft element by binary operation on the set of all nonempty near soft elements of a dedicated near soft set [19].

In this study, first a near soft group definition was made, then a topology structure was created on this near soft group, and the concept of a near soft topological group was defined with the help of the topology defined on this near soft group and continuous mapping. In addition, examples were given for the new concepts defined.

2. Preliminary

In this part, the notions of near approximation(NA), nearness approximation space (NAS) and other definitions of this concept are given. Then we define a binary composition on near soft sets and this form is called near soft group over near soft set.

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Definition 2.1. *[18, 20] Let* $(\mathcal{O}, \mathcal{F}, \sim_{Br}, N_r, \upsilon_{N_r})$ *be a NAS and* $\sigma = (F, B)$ *be a soft set over* $\mathcal{O}$.

$$N_r*((F,B)) = (N_r*(F(k) = \cup\{x \in \mathcal{O} : [x]_{Br} \subseteq F(k)\}, B))$$

and

$$N_r^*((F,B)) = (N_r^*(F(k) = \cup\{x \in \mathcal{O} : [x]_{Br} \cap F(k) \neq \emptyset\}, B))$$

are lower and upper near approximation operators. The SS $N_r((F,B))$ *with* $Bnd_{N_r(B)}((F,B)) \geqslant 0$ *called a near soft set(NSS).*

The collection of all *NSS* on $\mathcal{O}$ will be denoted $NSS(\mathcal{O})$.

Definition 2.2. *[18] Let* $\mathcal{O}$ *be an universe set, E be the parameters and* $B \subseteq E$. *For NSS* (F,B) *over* $\mathcal{O}$*, the set*

$$Supp(F,B) = \{\phi \in B : F(\phi) \neq \emptyset\}$$

is called the support of the NSS (F,B).

1. *A NSS* (F,B) *is called non-null NSS (with respect to the parameters of B) if* $Supp(F,B) \neq \emptyset$. *Otherwise* (F,B) *is called null NSS.*
2. *A near soft set* (F,B) *is called full null NSS if* $Supp(F,B) = B$. *A collection of all full NSS on* $\mathcal{O}$ *will be denoted by* $NS_f(\mathcal{O})$.

Definition 2.3. *[21] Let* $\mathcal{O}$ *be an initial set, E be the parameters and* $B \subseteq E$ *and* $(F,B) \in NSS(\mathcal{O})$. *We say that* $(\phi, \{x_k\})$ *is a nonempty near soft element (NSE) of* (F,B) *if* $\phi \in B$ *and* $x_k \in F(\phi)$. *The pair* $(\phi, \emptyset)$*, where* $\phi \in B$ *is an empty NSE of* (F,B)*.Then* $(\phi, \{x_k\}) = \{x_k\}_\phi$ *is a NSE of* (F,B) *and denoted by* F_B.

3. Near Soft Group

Definition 3.1. *[19] Let* $(\mathcal{F}, \circ)$ *and* $(\mathcal{O}, *)$ *be two groupoids and* $B \subseteq \mathcal{F}$. *Also let* $(F,B) \in NS_f(\mathcal{O})$, $\forall\ \phi \in B$, $\ni \exists$ *nonempty NSE of* (F,B). *We define a binary composition* $*$ *on* (F,B) *by*

$$(\phi_i, \{x_a\}) * (\phi_j, \{x_b\}) = (\phi_i \circ \phi_j, \{x_a * x_b\})$$

for all $(\phi_i, \{x_a\}), (\phi_j, \{x_b\}) \in (F,B)$. (F,B) *is said to be closed under the binary composition* $*$ *if and only if* $(\phi_i \circ \phi_j, \{x_a * x_b\}) \in (F,B)$, $\phi_i \circ \phi_j \in B$ *and* $x_a * x_b \in F(\phi_i \circ \phi_j)$ *for all* $(\phi_i, \{x_a\}), (\phi_j, \{x_b\}) \in (F,B)$. *Then* $(F_B, *)$ *is a near soft groupoid (NSG) over* $(F, \mathcal{O})$.

Theorem 3.2. *[19] Let* $(F,B) \in NS_f(\mathcal{O})$*, then* $((F,B), *)$ *forms a NSG over* $(F, \mathcal{O})$ *if and only if*

1. B is a subgroupoid of F, i.e., $\phi_i \circ \phi_j \in B$ for all $\phi_i, \phi_j \in B$,
2. for $\phi_i, \phi_j \in B$, $x_a \in F(\phi_i)$, $x_b \in F(\phi_j)$ then $x_a * x_b \in F(\phi_i \circ \phi_j)$.

Definition 3.3. *[19] Let* $(F_B, *)$ *be a NSG over* $(F, \mathcal{O})$. *Then* $*$ *binary composition said to be*

1. commutative if $(\phi_i, \{x_a\}) * (\phi_j, \{x_b\}) = (\phi_j, \{x_b\}) * (\phi_i, \{x_a\})$,
2. associative if
$$[(\phi_i, \{x_a\}) * (\phi_j, \{x_b\})] * (\phi_k, \{x_c\}) = (\phi_i, \{x_a\}) * [(\phi_j, \{x_b\}) * (\phi_k, \{x_c\})]$$
for all $(\phi_i, \{x_a\}), (\phi_j, \{x_b\}), (\phi_k, \{x_c\}) \in F_B$.

Definition 3.4. *A (NSE)* $(\phi, \{x\}) \in F_B$ *is a near soft identity element in a NSG* $(F_B, *)$ *if for all* $(\phi_i, \{x_a\}) \in F_B$

$$(\phi, \{x\}) * (\phi_i, \{x_a\}) = (\phi_i, \{x_a\}) = (\phi_i, \{x_a\}) * (\phi, \{x\}).$$

Definition 3.5. *Let $(F_B, *)$ be a NSG with near soft identity element $(\phi, \{x\})$. A NSE $(\phi_i, \{x_a\}) \in F_B$ is an invertible if there exists a (NSE) $(\phi'_i, \{x'_a\}) \in F_B$ such that*

$$(\phi_i, \{x_a\}) * (\phi'_i, \{x'_a\}) = (\phi, \{x\}) = (\phi'_i, \{x'_a\}) * (\phi_i, \{x_a\})$$

Then $(\phi'_i, \{x'_a\})$ is a near soft inverse of $(\phi_i, \{x_a\})$ and denoted by $(\phi_i, \{x_a\})^{-1}$.

Definition 3.6. *Let $B \subseteq \mathcal{F}$ and $(F, B) \in NS(O)$. We say that $(\phi, \{x\})$ is a nonempty NSE of (F, B)if $\phi \in B$ and $x \in F(\phi)$. The pair $(\phi, \emptyset)$, where $\phi \in \mathcal{F}$, is called an empty NSE of (F, B).*

Definition 3.7. *Let $(\mathcal{F}, \circ)$ and $(O, *)$ be two groups, $A, B \subseteq \mathcal{F}$ and $(F, B) \in NS(O)$. (NSG) $(F_B, *)$ is a near soft group (NSGp) over $(O, \mathcal{F})$ if,*

1. $*$ is associative,
2. there exist a *NSE* $(\phi, \{x\})$ such that

   $$(\phi, \{x\}) * (\phi_i, \{x_a\}) = (\phi_i, \{x_a\}) = (\phi_i, \{x_a\}) * (\phi, \{x\})$$

   for all $(\phi_i, \{x_a\}) \in (F, B)$,
3. for each $(\phi_i, \{x_a\}) \in (F, B)$ there exist a *NSE* $(\phi'_i, \{x_a\}')$ such that

   $$(\phi_i, \{x_a\}) * (\phi'_i, \{x_a\}') = (\phi, \{x\}) = (\phi'_i, \{x_a\}') * (\phi_i, \{x_a\}).$$

Example 3.8. *Let $(O, *)$ be a group with $*$ operation being multiplication modula 8 on the set $\{1, 3, 5, 7\}$ and $(B, \circ)$ be a group with $\circ$ operation. The composition table of $\circ$ on B is given by Table 1.*

Table 1: The composition table of $\circ$ on B.

$\circ$	ϕ_1	ϕ_2
ϕ_1	ϕ_1	ϕ_2
ϕ_2	ϕ_2	ϕ_1

and define the (NSS) $\sigma = (F, B) = \{(\phi_1, \{1, 3\}), (\phi_2, \{5, 7\})\}$. For $r = 1$

$$\begin{aligned} [1]_{\phi_1} &= \{1, 3, 7\}, [5]_{\phi_1} = \{5\} \\ [1]_{\phi_2} &= \{1, 3\}, [5]_{\phi_2} = \{5, 7\} \end{aligned}$$

$N_*(\sigma) = \{(\phi_2, \{5, 7\})\}, N^*(\sigma) = \{(\phi_1, \{1, 3\}), (\phi_2, \{5, 7\})\}, Bnd(\sigma) \geq 0.$

For $r = 2$; $N_*(\sigma) = \{(\phi_1, \{1, 3\}), (\phi_2, \{5, 7\})\} = N^*(\sigma), Bnd(\sigma) \geq 0$. Then σ is a *NSS*. Hence all the *NSE* of σ are;

$$F_{B.} = (\phi_1, \{1\}), (\phi_1, \{3\}), (\phi_2, \{5\}), (\phi_2, \{7\}).$$

The table of operation $*$ on $F_{B.}$ is given in Table 2.

Table 2: The table of operation $*$ on $F_{B.}$.

$*$	$(\phi_1, \{1\})$	$(\phi_1, \{3\})$	$(\phi_2, \{5\})$	$(\phi_2, \{7\})$
$(\phi_1, \{1\})$	$(\phi_1, \{1\})$	$(\phi_1, \{3\})$	$(\phi_2, \{5\})$	$(\phi_2, \{7\})$
$(\phi_1, \{3\})$	$(\phi_1, \{3\})$	$(\phi_1, \{1\})$	$(\phi_2, \{7\})$	$(\phi_2, \{5\})$
$(\phi_2, \{5\})$	$(\phi_2, \{5\})$	$(\phi_2, \{7\})$	$(\phi_1, \{1\})$	$(\phi_1, \{3\})$
$(\phi_2, \{7\})$	$(\phi_2, \{7\})$	$(\phi_2, \{5\})$	$(\phi_1, \{3\})$	$(\phi_1, \{1\})$

$(F_{B.}, *)$ is commutative *NSGp* with near soft identity $(\phi_1, \{1\})$.

Definition 3.9. *Suppose that* (O, B) *is a group then* $(F, B) \in NS(O)$ *is called a NSGp (resp. near normal soft group) over* $(O, B) \Leftrightarrow (F(\phi), B)$ *is a subgroup (resp. normal subgroup) of* (O, B), $\forall \phi \in B$.

Definition 3.10. *Suppose that* (F, B) *is a NSGp over* (O, B). *Then* $(G, A) \in NS(O)$ *is called a near soft subgroup* (*NSsGp*) *(resp. near soft normal subgroup* (*NSNsGp*) *of* (F, B) *if and only if* $A \subseteq B$ *and* $(G(\phi), A)$ *is a subgroup (resp. a normal subgroup) of* $(F(\phi), B)$, $\forall \phi \in B$.

Definition 3.11. *Let* (F_B*) *be a NSGp over* $(O, \mathcal{F})$ *and* (G, A) *be* (*NSsGp*) *of* (F, B). *If for all* $(\phi_j, \{x_b\}) \in (F, B)$

$$(\phi_i, \{x_a\}) * (G, A) * (\phi_i, \{x_a\})^{-1} = (G, A)$$

then (G, A)*is called NSNsGp of* (F, B).

Definition 3.12. *Let* (F, B) *be a NSS over* O *and* τ *be the collection of NSs of* O, *if if the following are provided*

i) $(\emptyset, B), (O, B) \in \tau$

ii) $(F_1, B), (F_2, B) \in \tau$ then $(F_1, B) \cap (F_2, B) \in \tau$

iii) $(F_i, B), \forall \phi \in B$ then $\bigcup_i (F_i, B) \in \tau$

Then (O, τ, B) is a near soft topological space(*NSTS*) [18].

Definition 3.13. (O, τ, B) *be a NSTS and* $G \subseteq O$. *The near soft topology* (*NST*) *on* (G, B) *induced by the NST* τ *is the family* τ_G *of the near soft subsets of G of the form* $\tau_G = \{V \cap G : V \in \tau\}$. *Thus* (G, τ_G, B) *is a near soft topological subspace of* (O, τ, B).

Definition 3.14. *NS*(O, B) *denotes the family of all NSS over* (O, B).*Let* $(F, A), (G, C) \in NS(O, B), A, C \subseteq B$. *the near soft cartesian product of* $(F, A), (G, C)$ *denoted by* $(F, A) \times (G, C)$ *is a NSS on* $(O, B) \times (O, B)$ *such that* $(F, A) \times (G, C) = \{((\phi_1, \phi_2), F(\phi_1) \times G(\phi_2)) : \phi_1, \phi_2 \in B\}$

Definition 3.15. *Let* (O, τ, B) *be a NSTS over* O . *A, NSS* (F, B) *in* (O, τ, B) *is called a near soft neighbourhood of the NSP* $(x_e, B) \in (F, B)$ *if there exists a NSOS* (G, B) *such that* $(x_e, B) \in (G, B) \subset (F, B)$.

Definition 3.16. *Let* $(F, A), (G, C) \in NS(O, B)$ *and* $f : (F, A) \to (G, C)$ *a near soft mapping NSM then the following hold:*

1. *The image of* $X \subseteq F$ *under* (*NSM*) f *is the near soft set of the form* $(f(X), C) = (\cup_{\alpha \in X} f(\alpha), C)$ *and for each NSM* $(f(\emptyset), B) = (\emptyset, B)$.
2. *The invese of* $Y \subseteq G$ *under NSM* f *is the NSS of the form* $(f^{-1}(Y), A) = (\cup\{\{\alpha\} : \alpha \in (F, A), f(\alpha) \in (Y, C)\}, B)$.

Definition 3.17. *Let* $(F, \tau, B), (G, \upsilon, B)$ *be a NSTS and* $f : (F, B) \to (G, B)$ *be a NSM. If for each* $V \in \upsilon, f^{-1}(V) \in \tau$ *then* f *is a near soft continuous mapping and denoted by NSCM.*

Definition 3.18. *Let* (O_1, τ, B) *and* (O_2, τ, B) *be two NSTS.* $f : (O_1, \tau, B) \to (O_2, \tau, B)$ *be a mapping. For each near soft neighbourhood* (H, B) *of* $(f(x)_\phi, B)$, *if there exists a near soft neighbourhood* $f((F, B)) \subset (H, B)$ *then* f *is a NSCM* (x_ϕ, B). *If* f *is* (*NSCM*) *for all* (x_ϕ, B), *then* f *is a NSCM.*

Definition 3.19. *Let* (O_1, τ, B) *and* (O_2, τ, B) *be two NSTS.* $f : O_1 \to O_2$ *be a mapping.* O_1 *is near soft homeomorphic to* O_2 *if* f *is a bijection, NSC and* f^{-1} *is a near soft homeomorfizm.*

Example 3.20. *Let $\sigma = (F, B)$ be a NSS given in example 10. Then all near soft subsets of $\sigma = (F, B)$ are;*

$$\begin{aligned}(F_1, B) &= \{(\phi_1, \{1, 3\})\}\\(F_2, B) &= \{(\phi_2, \{5, 7\})\}\\(F_3, B) &= \{(\phi_1, \{1, 3\}), (\phi_2, \{5\})\}\\(F_4, B) &= \{(\phi_1, \{1, 3\}), (\phi_2, \{7\})\}\\(F_5, B) &= \{(\phi_1, \{1\}), (\phi_2, \{5, 7\})\}\\(F_6, B) &= \{(\phi_1, \{3\}), (\phi_2, \{5, 7\})\}\\(F_7, B) &= \{(\phi_1, \{1, 3\}), (\phi_2, \{5, 7\})\}\\&\ldots\end{aligned}$$

$\tau = \{(F_1, B), (F_2, B), (F_7, B), (\emptyset, B)\}$ is a NST on (F, B). $(F_{B.}, *)$ is a NSGp with a topology τ. Then $\imath : F_{B.} \to F_{B.}$ which defined by

$$\begin{aligned}\imath((\phi_i, \{x_a\})) &= ((\phi_i, \{x_a\}))^{-1}\\\imath((\phi_1, \{1\})) &= ((\phi_1, \{1\}))^{-1} = (\phi_1, \{1\})\\\imath((\phi_1, \{3\})) &= ((\phi_1, \{3\}))^{-1} = (\phi_1, \{3\})\\\imath((\phi_2, \{5\})) &= ((\phi_2, \{5\}))^{-1} = (\phi_2, \{5\})\\\imath((\phi_2, \{7\})) &= ((\phi_2, \{7\}))^{-1} = (\phi_2, \{7\})\end{aligned}$$

is continuous.

Definition 3.21. *Let (O, τ) be a NSTS of NSE and $(\phi_j, \{x_a\}) \in F_B$. If $(\phi_j, \{x_a\}) \in H_C \subseteq G_A$ is an open set then a near soft subset G_A of F_B is a near soft neighborhood of NSE $(\phi_j, \{x_a\})$. The collection of all near soft neighborhoods of the NSE $(\phi_j, \{x_a\})$ is denoted $N_{(\phi_j, \{x_a\})}$.*

Definition 3.22. *Let (O_F, τ) be a NSTS over (F, B).A NSS $(G, A) \subseteq (F, B)$ is near soft open $\Leftrightarrow$ for each NSE $\gamma \in (G, A)$ there exist a NSS $(H, C) \in \tau$ such that $\gamma \in (H, C) \subseteq (G, A)$.*

Definition 3.23. *Let (O_F, τ_1) and (O_G, τ_2) be a NSTS over (F, B) and (G, B) respectively and $\lambda = \{O_F \times O_G : F \in \tau_1$ and $G \in \tau_2\}$. The collection τ of all arbitrary union of elements of λ is a near soft product topology over $O_F \times O_G$.*

4. Near Soft Topological Group Based on Near Soft Element

Definition 4.1. *A near soft group $(F_B, *)$ with a topology τ on F_B is called a near soft semi- topological group (NSsTGp) if for each near soft neighborhood F_A of $(\phi_i, \{x_a\}) * (\phi'_i, \{x_a\}')$, there exists a near soft neighborhood F_C of $(\phi_i, \{x_a\})$ and a near soft neighborhood F_D of $(\phi'_i, \{x_a\}')$ such that $F_C * F_D \subseteq F_A$.*

Example 4.2. *Let $\sigma = (F, B)$ be a NSs given in example 10. $\tau = \{(F_1, B), (F_2, B), (F_7, B), (\emptyset, B)\}$ is a NSTS where*

$$\begin{aligned}(F_1, B) &= \{(\phi_1, \{1, 3\})\}\\(F_2, B) &= \{(\phi_2, \{5, 7\})\}\end{aligned}$$

.Then $(F_{B.}, *, \tau)$ is a NSsTGp.

Definition 4.3. *A near soft group $(F_B, *)$ with a topology τ on F_B is a near soft topological group (NSTGp) if the following hold:*

1. $f : F_B \times F_B \to F_B$ which defined by

$$f((\phi_i, \{x_a\}), (\phi'_i, \{x_a\}')) = (\phi_i, \{x_a\}) * (\phi'_i, \{x_a\}')$$

   is continuous with respect to a product topollogy on $F_B \times F_B$.

2. $\imath : F_B \to F_B$ which defined by $\imath((\phi_i, \{x_a\})) = ((\phi_i, \{x_a\}))^{-1}$ is continuous.

Definition 4.4. *Let τ be a topology defined on a additive group G. Let (F, B) be a non-null near soft set defined over G. Then, the triplet (F, B, τ) is a NSTGp over G if*
i. $F(\phi)$ is a subgroup of G for all $\phi \in B$
ii. the mapping $(x_1, x_2) \to x_1 - x_2$ of the topological space $F(\phi) \times F(\phi)$ onto $F(\phi)$ is continuous for all $\phi \in B$.

Definition 4.5. *Suppose that O is an additive group and τ be a near soft topology it. Then the NSTS (O, τ, B) is called a NSTGp if the mapping $(x_1, x_2) \to x_1 - x_2$ is a NSCM from $(O \times O, \tau \times \tau)$ to (O, τ, B).*

Example 4.6. *Let $\sigma = (F, B)$ be a NSS given in example 10. $\tau = \{(F_1, B), (F_2, B), (F_7, B), (\emptyset, B)\}$ is a near soft topology on (F, B). $(F_{B.}, *)$ is a NSGp with a topology τ. $(F_{B.}, *, \tau)$ is a NSTGp;*

1. $f : F_B \times F_B \to F_B$ which defined by

$$f((\phi_i, \{x_a\}), (\phi_i', \{x_a\}')) = (\phi_i, \{x_a\}) * (\phi_i', \{x_a\}')$$

is continuous with respect to a product topollogy on $F_B \times F_B$.

2. $\imath : F_B \to F_B$ which defined by $\imath((\phi_i, \{x_a\})) = ((\phi_i, \{x_a\}))^{-1}$ is continuous from example 25.

5. Conclusion

Soft set theory and near set theory, which have been successfully studied by many researchers to date, have a very good resource for applications. To contribute to the applications in this article, we introduce near soft topological groups with the help of the near soft element. These results provide an environment for studying applications on Near soft topological algebraic structures.

References

[1] Peters JF. Near sets: Special Theory about nearness of objects. Fund. Informaticae. 75, (2007), 407-433.
[2] Peters JF. Near sets: General Theory about nearness of objects. App.Math. Sci. 1, (2007), 2609-2629.
[3] Molodtsov D. Soft set theory-first results. Comput. Math.appl. 37, (1999), 19-31.
[4] Maji PK, Biswas R, Roy AR. Soft set theory. Comput. Math. appl. 45, (2003), 555-562.
[5] Feng F, Li C, Davvaz B, Ali MI. Soft sets combined with fuzzy sets and rough sets. Soft comput. 14, (2010), 899-911.
[6] Cagman N, Karatas S, Enginouglu S, Ali MI. Soft Topology. Computers and Mathematics with Applications. 62, (2011), 351-358.
[7] Aktas H, Cagman N. Soft sets and soft groups. Inform. Sci. 177, (2007), 2726-2735.
[8] Oguz G, Gursoy MH, Icen I. On Soft Topological Categories. Hacettepe Journal of Mathematics and Statistics. 48, (2019), 1675-1681.
[9] Oguz G, Icen I, Gursoy MH. Actions of Soft Groups. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 68, (2019), 1163-1174.
[10] Oguz G. Soft Topological Transformation Groups. Mathematics. 8, (2020), 1545.
[11] Oguz G, Davvaz B. Soft Topological Hyperstructure. Journal of Intelligent Fuzzy Systems. 40, (2021), 8755-8764.
[12] Wardowski D. On a soft mapping and its fixed points. Fixed point theory and Applications. 182, (2013), 1-11.
[13] Ghosh J, Mandal D, Samanta TK. Soft group based on soft element. Jordan journal of mathematics and Statistics. 9, (2016), 141-159.
[14] Ghosh J, Samanta TK. Soft set functions and soft set groups. Pure and Applied Mathematics Letters. (2016), 8-14.
[15] Polat CN, Yaylali G, Tanay B. A new approach for soft semi-topological groups based on soft element. Filomat. 32, (2018), 5743-5751.
[16] Bagirmaz N, Tasbozan H. The left right rough approximations in a group. JP Journal of Algebra. Number Theory and Applications. 42, (2019), 77-83.
[17] Tasbozan H, Icen I. The Upper and Lower Approximations in Rough Subgroupoid of a Groupoid. Moroccan Journal of Pure and Applied Analysis. 4, (2018), 85-93.
[18] Tasbozan H, Icen I, Bagirmaz N, Ozcan AF. Soft sets and soft topology on nearness approximation spaces. Filomat. 13, (2017), 4117-4125.
[19] Tasbozan H. Near soft groupoid. Ikonion Journal of Mathematics. 2, (2020), 35-39.
[20] Tasbozan H. Near Approximations in Vector Spaces. Journal of Universal Mathematics. 3, (2020), 114-120.
[21] Tasbozan H. Near Soft Connectedness. Afyon Kocatepe University Journal of Science and Engineering. 20, (2020), 815-818.

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On the modified orthogonal frames of the non-unit speed curves in Euclidean 3-Space $\mathbb{E}^3$

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Abstract. In this study, the modified frames with both the non-zero curvature and the torsion of the non-unit speed curves in Euclidean 3-space $\mathbb{E}^3$ are examined. The relationships between the derivative vectors of the modified frames and the Frenet vectors or the vectors of the modified frames of the curve are given. Besides, the Darboux vectors obtained from the modified orthogonal frames with both the curvature and torsion of the curve and the unit vectors in the direction of these Darboux vectors are investigated. Finally, all these results are shown on the example curves.

1. INTRODUCTION

The Frenet frame, which is a moving frame at a given point on any regular curve in Euclidean 3-space $\mathbb{E}^3$, is one of the most important tools used to analyze the curve. The Frenet frame is an orthonormal frame consisting of the tangent vector, the principal normal vector and binormal vector of the curve. The curvature and the torsion functions can be defined on the curve using this frame. Studies on the Frenet frame of a regular curve in $\mathbb{E}^3$ are available in various sources, such as [5, 9–11, 16, 19, 20]. According to the fundamental theorem of regular curves, a regular curve is a curve with functions $\kappa > 0$ (curvature) and τ (torsion) that can be differentiated at every point of the curve, [6]. However, it is possible for the curvature function to be zero at certain points on the analytical curves. The principal normal and binormal vectors of these curves are generally discontinuous at the zero point of the curvature, that is, the curvature function is not always differentiable. In this case, the Frenet derivative equations of an analytical curve causes ambiguity at a point where the curvature vanishes. Hord and Sasai pondered this problem and discussed another frame that works fine on these points, [12, 17]. In a simple but useful approach, an orthogonal frame was introduced for unit speed analytical curves by Sasai, [18]. Although the vectors of this modified orthogonal frame are obtained by multiplying each Frenet vector by the curvature function κ, they allow the use of a new formula corresponding to the Frenet derivative equations for the above-mentioned case. It is also a useful tool for investigating analytical curves with singular points. Then, Bükcü and Karacan have developed the Sasai's study and they have obtained the newly modified frame through the coefficient of torsion τ by the Frenet vectors and them spherical curves, [3, 4]. There are many studies on the modified orthogonal frame of a curve in Euclidean or Lorentzian 3-space, [1, 2, 7, 8, 14, 15, 21]. Also, the Darboux

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vector of a space curve is the areal velocity vector of the moving frame of the curve. The direction of the Darboux vector is the direction of the instantaneous rotation axis. The Darboux vector can be expressed in terms of the apparatus of the moving frame. The Darboux vector can also be studied in a modified way for space curves with singular points, [13, 22].

In this study, the modified frames with both the non-zero curvature and the torsion of the non-unit speed curves in Euclidean 3-space $\mathbb{E}^3$ are examined. The derivatives of the vectors belong to these modified frames are calculated. The relationships between these derivative vectors and the Frenet vectors of the curve or the vectors of the modified frame are given. Besides, the Darboux vectors obtained from the modified orthogonal frames with both the curvature and torsion of the curve and the unit vectors in the direction of these Darboux vectors are investigated. Finally, these all results are investigated on the example curves. The aim of this study is to generalize the modified frame formulas, created for unit speed curves by Sasai, for non-unit speed curves. And thus, it provides ease of operation for the solution of the problem at singular points on a non-unit speed analytic curve, that is, there is no need to transform the curve into a unit speed curve every time.

2. PRELIMINARIES

Let the curve $\alpha(t)$ be a differentiable space curve in $\mathbb{E}^3$. The Frenet vectors, the curvature and the torsion of the curve $\alpha(t)$ are given as follows:

$$T(t) = \frac{\alpha'(t)}{v(t)}, \quad B(t) = \frac{\alpha'(t) \wedge \alpha''(t)}{\|\alpha'(t) \wedge \alpha''(t)\|}, \quad N(t) = B(t) \wedge T(t), \tag{1}$$

$$\kappa(t) = \frac{\|\alpha'(t) \wedge \alpha''(t)\|}{v^3(t)} \quad \text{and} \quad \tau(t) = \frac{\det(\alpha'(t), \alpha''(t), \alpha'''(t))}{\|\alpha'(t) \wedge \alpha''(t)\|^2}, \tag{2}$$

where $v(t) = \|\alpha'(t)\|$, respectively. The Frenet derivative formulas of this curve are as follows:

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & v(t)\kappa(t) & 0 \\ -v(t)\kappa(t) & 0 & v(t)\tau(t) \\ 0 & -v(t)\tau(t) & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}, \tag{3}$$

[11], [16]. The Darboux vector $W(t)$ of the non-unit speed curve $\alpha(t)$ is as follows:

$$W(t) = N(t) \wedge N'(t) = v(t)(\tau(t)T(t) + \kappa(t)B(t)), \tag{4}$$

where,

$$T'(t) = W(t) \wedge T(t), \quad N'(t) = W(t) \wedge N(t), \quad B'(t) = W(t) \wedge B(t). \tag{5}$$

The unit vector $C(t)$ in direction of the Darboux vector of the non-unit speed curve $\alpha(t)$ is

$$C(t) = \frac{W(t)}{\|W(t)\|} = \frac{v(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}}(\tau(t)T(t) + \kappa(t)B(t)), \tag{6}$$

or if the angle between of the binormal vector $B(t)$ and the Darboux vector $W(t)$ of the curve $\alpha(t)$ is $\varphi(t)$, then the unit vector is

$$C(t) = \sin\varphi T(t) + \cos\varphi B(t), \tag{7}$$

[16]. If $v(t) = 1$, then the curve $\alpha(t)$ is called unit speed curve. Let's define a orthogonal frame $\{E_1(t), E_2(t), E_3(t)\}$ for the unit speed curve $\alpha(t)$ as follows:

$$E_1(t) = \alpha'(t), \quad E_2(t) = E_1'(t), \quad E_3(t) = E_1(t) \wedge E_2(t). \tag{8}$$

If the curvature $\kappa(t)$ of the curve $\alpha(t)$ is non-zero, then there are the following relationships between the vectors $E_1(t), E_2(t), E_3(t)$ and the Frenet vectors of the curve:

$$E_1(t) = T(t), \quad E_2(t) = \kappa(t) N(t), \quad E_3(t) = \kappa(t) B(t), \tag{9}$$

where,

$$\langle E_1(t), E_2(t)\rangle = \langle E_2(t), E_3(t)\rangle = \langle E_1(t), E_3(t)\rangle = 0, \tag{10}$$

and

$$\langle E_1(t), E_1(t)\rangle = 1, \quad \langle E_2(t), E_2(t)\rangle = \langle E_3(t), E_3(t)\rangle = \kappa^2(t). \tag{11}$$

The orthogonal frame $\{E_1(t), E_2(t), E_3(t)\}$ is called the modified frame with the curvature $\kappa(t)$ of the unit speed curve $\alpha(t)$. It is noted that the modified orthogonal frame coincides with Frenet frame for $\kappa = 1$. There are the following relationships between the vectors $E_1(t), E_2(t), E_3(t)$ and them derivative vectors:

$$\begin{bmatrix} E_1'(t) \\ E_2'(t) \\ E_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2(t) & \dfrac{\kappa'(t)}{\kappa(t)} & \tau(t) \\ 0 & -\tau(t) & \dfrac{\kappa'(t)}{\kappa(t)} \end{bmatrix} \begin{bmatrix} E_1(t) \\ E_2(t) \\ E_3(t) \end{bmatrix}, \tag{12}$$

here, $\tau(t) = \frac{\det(\alpha'(t), \alpha''(t), \alpha'''(t))}{\kappa^2(t)}$ is the torsion of the curve (α), we know that any zero point of $\kappa^2(t)$ is a removable singularity of $\tau(t)$, [17]. Or, let the torsion $\tau(t)$ of the curve $\alpha(t)$ be non-zero. Then let's define the following orthogonal frame $\{A_1(t), A_2(t), A_3(t)\}$ for the unit speed curve $\alpha(t)$ as follows:

$$A_1(t) = T(t), \quad A_2(t) = \tau(t) N(t), \quad A_3(t) = \tau(t) B(t), \tag{13}$$

where,

$$\langle A_1(t), A_2(t)\rangle = \langle A_2(t), A_3(t)\rangle = \langle A_1(t), A_3(t)\rangle = 0, \tag{14}$$

and

$$\langle A_1(t), A_1(t)\rangle = 1, \quad \langle A_2(t), A_2(t)\rangle = \langle A_3(t), A_3(t)\rangle = \tau^2(t). \tag{15}$$

The orthogonal frame $\{A_1(t), A_2(t), A_3(t)\}$ is called the modified frame with the torsion $\tau(t)$ of the unit speed curve $\alpha(t)$. And, there are the following relationships between the vectors $A_1(t), A_2(t), A_3(t)$ and them derivative vectors:

$$\begin{bmatrix} A_1'(t) \\ A_2'(t) \\ A_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\kappa(t)}{\tau(t)} & 0 \\ -\kappa(t)\tau(t) & \dfrac{\tau'(t)}{\tau(t)} & \tau(t) \\ 0 & -\tau(t) & \dfrac{\tau'(t)}{\tau(t)} \end{bmatrix} \begin{bmatrix} A_1(t) \\ A_2(t) \\ A_3(t) \end{bmatrix}. \tag{16}$$

The Darboux vector $D(t)$ obtained from the modified orthogonal frame with the curvature $\kappa(t)$ of a unit speed curve $\alpha(t)$ is obtained as follows:

$$D(t) = \tau(t) E_1(t) + E_3(t) \tag{17}$$

and

$$E_2(t) \wedge E_2'(t) = \kappa^2(t) D(t). \tag{18}$$

If the angle between of the Darboux vector $D(t)$ and the vector $E_3(t)$ is $\varphi(t)$, the unit vector in direction of the Darboux vector is

$$G(t) = \sin\varphi E_1(t) + \frac{\cos\varphi}{\kappa(t)} E_3(t). \tag{19}$$

3. The Modified Orthogonal Frames of the Non-Unit Speed Curve in $\mathbb{E}^3$

Let the Frenet frame, the curvature and the torsion of a non-unit speed curve $\alpha(t)$ be $\{T(t), N(t), B(t)\}, \kappa(t)$ and $\tau(t)$, respectively.

3.1. The Modified Orthogonal Frame With the Curvature $\kappa(t)$ of a Non-Unit Speed Curve in $\mathbb{E}^3$

Theorem 3.1. *Let the Frenet frame and the curvature be $\{T(t), N(t), B(t)\}$ and $\kappa(t)$ of the non-unit speed space curve $\alpha(t)$, respectively. The modified orthogonal frame $\{E_1(t), E_2(t), E_3(t)\}$ with the curvature $\kappa(t)$ of the curve $\alpha(t)$ is as follows:*

$$\begin{cases} E_1(t) = v(t)\,T(t), \\ E_2(t) = v^2(t)\,\kappa(t)\,N(t), \\ E_3(t) = v^3(t)\,\kappa(t)\,B(t). \end{cases} \tag{20}$$

Proof. Let's create the vectors $E_1(t), E_2(t), E_3(t)$ using the Gram-Schmidt orthogonalization procedure as follows:

$$\begin{cases} E_1(t) = \alpha'(t), \\ E_2(t) = E_1'(t) - \dfrac{\left\langle E_1'(t), E_1(t)\right\rangle}{\langle E_1(t), E_1(t)\rangle} E_1(t), \\ E_3(t) = E_1(t) \wedge E_2(t). \end{cases} \tag{21}$$

Here, since the curve $\alpha(t)$ is not an unit speed curve, we can't use the expression (8). From the expression (1), the vector $E_1(t)$ is obtained as follows:

$$E_1(t) = v(t)\,T(t). \tag{22}$$

From the expressions (3) and (22), the following equation is gotten:

$$\frac{\left\langle E_1'(t), E_1(t)\right\rangle}{\langle E_1(t), E_1(t)\rangle} = \frac{v'(t)}{v(t)}. \tag{23}$$

From the expressions (3), (21) and (23), the vectors $E_2(t)$ and $E_3(t)$ are obtained as follows:

$$E_2(t) = v^2(t)\,\kappa(t)\,N(t), \tag{24}$$

$$E_3(t) = v^3(t)\,\kappa(t)\,B(t). \tag{25}$$

The proof is completed from the expressions (22), (24) and (25). □

Corollary 3.2. *As a result of Theorem 3.1, the following equations are obtained for the vectors $E_1(t), E_2(t), E_3(t)$:*

$$\langle E_1(t), E_2(t)\rangle = \langle E_2(t), E_3(t)\rangle = \langle E_1(t), E_3(t)\rangle = 0, \tag{26}$$

$$\begin{cases} \langle E_1(t), E_1(t)\rangle = v^2(t), \\ \langle E_2(t), E_2(t)\rangle = v^4(t)\,\kappa^2(t), \\ \langle E_3(t), E_3(t)\rangle = v^6(t)\,\kappa^2(t), \end{cases} \tag{27}$$

and

$$\begin{cases} E_1(t) \wedge E_2(t) = E_3(t), \\ E_2(t) \wedge E_3(t) = v^4(t)\kappa^2(t)E_1(t), \\ E_3(t) \wedge E_1(t) = v^2(t)E_2(t). \end{cases} \tag{28}$$

Remark 3.3. *The modified frame* $\{E_1(t), E_2(t), E_3(t)\}$ *with the curvature* $\kappa(t)$ *of the non-unit speed curve* $\alpha(t)$ *is indeed orthogonal (from the expression (26)), but is not orthonormal (from the expression (ref20)), because the vectors* $E_1(t), E_2(t), E_3(t)$ *are not unit vectors (if not* $v(t) = \kappa(t) = 1$ *at the same time). If* $v(t) = \kappa(t) = 1$ *at the same time, then the modified frame* $\{E_1(t), E_2(t), E_3(t)\}$ *becomes an orthonormal frame.*

Theorem 3.4. *Let the Frenet frame, the curvature and the torsion be* $\{T(t), N(t), B(t)\}$, $\kappa(t)$ *and* $\kappa(t)$ *of the non-unit speed space curve* $\alpha(t)$, *respectively. And let the modified orthogonal frame with the curvature* $\kappa(t)$ *of the curve* $\alpha(t)$ *be* $\{E_1(t), E_2(t), E_3(t)\}$. *There are the following equations between the Frenet vectors and the derivative vectors* $E_1'(t), E_2'(t), E_3'(t)$:

$$\begin{bmatrix} E_1'(t) \\ E_2'(t) \\ E_3'(t) \end{bmatrix} = \begin{bmatrix} v'(t) & v^2(t)\kappa(t) & 0 \\ -v^3(t)\kappa^2(t) & 2v(t)v'(t)\kappa(t) + v^2(t)\kappa'(t) & v^3(t)\kappa(t)\tau(t) \\ 0 & -v^4(t)\kappa(t)\tau(t) & 3v^2(t)v'(t)\kappa(t) + v^3(t)\kappa'(t) \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}. \quad (29)$$

Proof. By using the expression (3), from the expression (20), we obtain the derivative vectors $E_1'(t)$, $E_2'(t)$, $E_3'(t)$ in terms of the Frenet vectors $T(t), N(t), B(t)$ as follows:

$$\begin{aligned} E_1'(t) &= (v(t)T(t))', \\ E_1'(t) &= v'(t)T(t) + v^2(t)\kappa(t)N(t), \end{aligned} \quad (30)$$

$$\begin{aligned} E_2'(t) &= \left(v^2(t)\kappa(t)N(t)\right)', \\ E_2'(t) &= -v^3(t)\kappa^2(t)T(t) + \left(2v(t)v'(t)\kappa(t) + v^2(t)\kappa'(t)\right)N(t) + v^3(t)\kappa(t)\tau(t)B(t), \end{aligned} \quad (31)$$

$$\begin{aligned} E_3'(t) &= \left(v^3(t)\kappa(t)B(t)\right)' \\ E_3'(t) &= -v^4(t)\kappa(t)\tau(t)N(t) + \left(3v^2(t)v'(t)\kappa(t) + v^3(t)\kappa'(t)\right)B(t). \end{aligned} \quad (32)$$

The proof is completed from the expressions (30), (31) and (32). □

Theorem 3.5. *Let the Frenet frame, the curvature and the torsion be* $\{T(t), N(t), B(t)\}$, $\kappa(t)$ *and* $\tau(t)$ *of the non-unit speed space curve* $\alpha(t)$, *respectively. And let the modified orthogonal frame with the curvature* $\kappa(t)$ *of the curve* $\alpha(t)$ *be* $\{E_1(t), E_2(t), E_3(t)\}$. *There are the following equations between their derivatives and the vectors* $E_1(t), E_2(t), E_3(t)$:

$$\begin{bmatrix} E_1'(t) \\ E_2'(t) \\ E_3'(t) \end{bmatrix} = \begin{bmatrix} \frac{v'(t)}{v(t)} & 1 & 0 \\ -v^2(t)\kappa^2(t) & \frac{2v'(t)}{v(t)} + \frac{\kappa'(t)}{\kappa(t)} & \tau(t) \\ 0 & -v^2(t)\tau(t) & \frac{3v'(t)}{v(t)} + \frac{\kappa'(t)}{\kappa(t)} \end{bmatrix} \begin{bmatrix} E_1(t) \\ E_2(t) \\ E_3(t) \end{bmatrix}. \quad (33)$$

Proof. From the expression (20), the Frenet vectors $T(t), N(t), B(t)$ are written in terms of the vectors $E_1(t), E_2(t), E_3(t)$ as follows:

$$T(t) = \frac{E_1(t)}{v(t)}, \quad N(t) = \frac{E_2(t)}{v^2(t)\kappa(t)}, \quad B(t) = \frac{E_3(t)}{v^3(t)\kappa(t)}. \quad (34)$$

If the expression (34) is substituted in the expressions (30), (31) and (32), respectively, we get

$$E_1'(t) = \frac{v'(t)}{v(t)} E_1(t) + E_2(t), \tag{35}$$

$$E_2'(t) = -v^2(t)\kappa^2(t) E_1(t) + \left(\frac{2v'(t)}{v(t)} + \frac{\kappa'(t)}{\kappa(t)}\right) E_2(t) + \tau(t) E_3(t), \tag{36}$$

$$E_3'(t) = -v^2(t)\tau(t) E_2(t) + \left(\frac{3v'(t)}{v(t)} + \frac{\kappa'(t)}{\kappa(t)}\right) E_3(t). \tag{37}$$

The proof is completed from the expressions (35), (36) and (37). □

Special Case: If the curve $\alpha(t)$ is an unit speed curve, $v(t) = 1$. In this case, from the expressions (20) and (27), the expressions (9) and (11) are obtained, respectively. And the expression (28) would be as follows:

$$E_1(t) \wedge E_2(t) = E_3(t), \quad E_2(t) \wedge E_3(t) = \kappa^2(t) E_1(t), \quad E_3(t) \wedge E_1(t) = E_2(t). \tag{38}$$

Also, from the expression (33), the following equations are obtained:

$$\begin{bmatrix} E_1'(t) \\ E_2'(t) \\ E_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa^2(t) & \kappa'(t) & \kappa(t)\tau(t) \\ 0 & -\kappa(t)\tau(t) & \kappa'(t) \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}. \tag{39}$$

Finally, we obtain the equalities between of the modified orthogonal frame and the derivative vectors of the modified orthogonal frame with the curvature $\kappa(t)$ of the unit speed curve $\alpha(t)$, like the expression (12).

3.2. The Modified Orthogonal Frame With the Torsion $\tau(t)$ of a Non-Unit Speed Curve in $\mathbb{E}^3$

Theorem 3.6. *Let the Frenet frame and the curvature be $\{T(t), N(t), B(t)\}$ and $\kappa(t)$ of the non-unit speed space curve $\alpha(t)$, respectively. The modified orthogonal frame $\{A_1(t), A_2(t), A_3(t)\}$ with the torsion $\tau(t)$ of the curve $\alpha(t)$ is as follows:*

$$\begin{cases} A_1(t) = v(t) T(t), \\ A_2(t) = v^2(t)\tau(t) N(t), \\ A_3(t) = v^3(t)\tau(t) B(t). \end{cases} \tag{40}$$

Proof. Let's create the vectors $A_1(t), A_2(t), A_3(t)$ using the Gram-Schmidt orthogonalization procedure as follows:

$$\begin{cases} A_1(t) = \alpha'(t), \\ A_2(t) = \dfrac{\tau(t)}{\kappa(t)} \left(A_1'(t) - \dfrac{\langle A_1'(t), A_1(t) \rangle}{\langle A_1(t), A_1(t) \rangle} A_1(t) \right), \\ A_3(t) = A_1(t) \wedge A_2(t). \end{cases} \tag{41}$$

Here, since the curve $\alpha(t)$ is not an unit speed curve, we can't use the expression (13). From the expression (1), the vector $A_1(t)$ is obtained as follows:

$$A_1(t) = v(t) T(t). \tag{42}$$

From the expression (22), we see that the vectors $E_1(t)$ and $A_1(t)$ are equal. So from the expression (23), we get

$$\frac{\left\langle A_1'(t), A_1(t)\right\rangle}{\langle A_1(t), A_1(t)\rangle} = \frac{v'(t)}{v(t)}. \tag{43}$$

From the expressions (3), (42) and (43), the vectors $A_2(t)$ and $A_3(t)$ are obtained as follows:

$$A_2(t) = v^2(t)\,\tau(t)\,N(t), \tag{44}$$

$$A_3(t) = v^3(t)\,\tau(t)\,B(t). \tag{45}$$

The proof is completed from the expressions (42), (44) and (45). □

Corollary 3.7. *As a result of Theorem 3.6, the following equations are obtained for the vectors* $A_1(t), A_2(t), A_3(t)$*:*

$$\langle A_1(t), A_2(t)\rangle = \langle A_2(t), A_3(t)\rangle = \langle A_1(t), A_3(t)\rangle = 0, \tag{46}$$

$$\begin{cases} \langle A_1(t), A_1(t)\rangle = v^2(t), \\ \langle A_2(t), A_2(t)\rangle = v^4(t)\,\tau^2(t), \\ \langle A_3(t), A_3(t)\rangle = v^6(t)\,\tau^2(t), \end{cases} \tag{47}$$

and

$$\begin{cases} A_1(t) \wedge A_2(t) = A_3(t), \\ A_2(t) \wedge A_3(t) = v^4(t)\tau^2(t)A_1(t), \\ A_3(t) \wedge A_1(t) = v^2(t)A_2(t). \end{cases} \tag{48}$$

Remark 3.8. *The modified frame* $\{A_1(t), A_2(t), A_3(t)\}$ *with the torsion* $\tau(t)$ *of the non-unit speed curve* $\alpha(t)$ *is indeed orthogonal (from the expression (46)), but is not orthonormal (from the expression (47)), because the vectors* $A_1(t), A_2(t), A_3(t)$ *are not unit vectors (if not* $v(t) = \tau^2(t) = 1$ *at the same time). If* $v(t) = \tau^2(t) = 1$ *at the same time, then the modified frame* $\{A_1(t), A_2(t), A_3(t)\}$ *becomes an orthonormal frame.*

Theorem 3.9. *Let the Frenet frame, the curvature and the torsion be* $\{T(t), N(t), B(t)\}$*,* $\kappa(t)$ *and* $\tau(t)$ *of the non-unit speed space curve* $\alpha(t)$*, respectively. And let the modified orthogonal frame with the torsion* $\tau(t)$ *of the curve* $\alpha(t)$ *be* $\{A_1(t), A_2(t), A_3(t)\}$*. There are the following equations between the Frenet vectors and the derivative vectors* $A_1'(t), A_2'(t), A_3'(t)$*:*

$$\begin{bmatrix} A_1'(t) \\ A_2'(t) \\ A_3'(t) \end{bmatrix} = \begin{bmatrix} v'(t) & v^2(t)\,\kappa(t) & 0 \\ -v^3(t)\,\kappa(t)\,\tau(t) & 2v(t)\,v'(t)\,\tau(t) + v^2(t)\,\tau'(t) & v^3(t)\,\tau^2(t) \\ 0 & -v^4(t)\,\tau^2(t) & 3v^2(t)\,v'(t)\,\tau(t) + v^3(t)\,\tau'(t) \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}. \tag{49}$$

Proof. By using the expression (3), from the expression (40), we obtain the derivative vectors $A_1'(t)$, $A_2'(t)$,

$A_3'(t)$ in terms of the Frenet vectors $T(t)$, $N(t)$, $B(t)$ as follows:

$$\begin{aligned} A_1'(t) &= (v(t)T(t))', \\ A_1'(t) &= v'(t)T(t) + v^2(t)\kappa(t)N(t), \end{aligned} \tag{50}$$

$$\begin{aligned} A_2'(t) &= \left(v^2(t)\tau(t)N(t)\right)' \\ A_2'(t) &= \left(-v^3(t)\kappa(t)\tau(t)\right)T(t) + \left(2v(t)v'(t)\tau(t) + v^2(t)\tau'(t)\right)N(t) + v^3(t)\tau^2(t)B(t), \end{aligned} \tag{51}$$

$$\begin{aligned} A_3'(t) &= \left(v^3(t)\tau(t)B(t)\right)' \\ A_3'(t) &= -v^4(t)\tau^2(t)N(t) + \left(3v^2(t)v'(t)\tau(t) + v^3(t)\tau'(t)\right)B(t). \end{aligned} \tag{52}$$

The proof is completed from the expressions (50), (51) and (52). □

Theorem 3.10. *Let the Frenet frame, the curvature and the torsion be $\{T(t), N(t), B(t)\}$, $\kappa(t)$ and $\tau(t)$ of the non-unit speed space curve $\alpha(t)$, respectively. And let the modified orthogonal frame with the torsion $\tau(t)$ of the curve $\alpha(t)$ be $\{A_1(t), A_2(t), A_3(t)\}$. There are the following equations between their derivatives and the vectors $A_1(t), A_2(t), A_3(t)$:*

$$\begin{bmatrix} A_1'(t) \\ A_2'(t) \\ A_3'(t) \end{bmatrix} = \begin{bmatrix} \frac{v'(t)}{v(t)} & \frac{\kappa(t)}{\tau(t)} & 0 \\ -v^2(t)\kappa(t)\tau(t) & \frac{2v'(t)}{v(t)} + \frac{\tau'(t)}{\tau(t)} & \tau(t) \\ 0 & -v^2(t)\tau(t) & \frac{3v'(t)}{v(t)} + \frac{\tau'(t)}{\tau(t)} \end{bmatrix} \begin{bmatrix} A_1(t) \\ A_2(t) \\ A_3(t) \end{bmatrix}. \tag{53}$$

Proof. From the expression (40), the Frenet vectors $T(t), N(t), B(t)$ are written in terms of the vectors $A_1(t), A_2(t), A_3(t)$ as follows:

$$T(t) = \frac{A_1(t)}{v(t)}, \quad N(t) = \frac{A_2(t)}{v^2(t)\tau(t)}, \quad B(t) = \frac{A_3(t)}{v^3(t)\tau(t)}. \tag{54}$$

If the expression (54) is substituted in the expressions (50), (51) and (52), respectively, we get

$$A_1'(t) = \frac{v'(t)}{v(t)}A_1(t) + \frac{\kappa(t)}{\tau(t)}A_2(t), \tag{55}$$

$$A_2'(t) = -v^2(t)\kappa(t)\tau(t)A_1(t) + \left(\frac{2v'(t)}{v(t)} + \frac{\tau'(t)}{\tau(t)}\right)A_2(t) + \tau(t)A_3(t), \tag{56}$$

$$A_3'(t) = -v^2(t)\tau(t)A_2(t) + \left(\frac{3v'(t)}{v(t)} + \frac{\tau'(t)}{\tau(t)}\right)A_3(t). \tag{57}$$

The proof is completed from the expressions (55), (56) and (57). □

Special Case: If the curve $\alpha(t)$ is an unit speed curve, $v(t) = 1$. In this case, the expressions (40) and (47), the expressions (13) and (15) are obtained, respectively. And the expression (48) would be as follows:

$$A_1(t) \wedge A_2(t) = A_3(t), \quad A_2(t) \wedge A_3(t) = \tau^2(t)A_1(t), \quad A_3(t) \wedge A_1(t) = A_2(t). \tag{58}$$

Also, from the expression (49), the following equations are obtained:

$$\begin{bmatrix} A_1'(t) \\ A_2'(t) \\ A_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t)\tau(t) & \tau'(t) & \tau^2(t) \\ 0 & -\tau^2(t) & \tau'(t) \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}.$$

Finally, we obtain the equalities between of the modified orthogonal frame and the derivative vectors of the modified orthogonal frame with the torsion $\tau(t)$ of the unit speed curve $\alpha(t)$, like the expression (16).

4. The Darboux vectors obtained from the modified orthogonal frames the non-unit speed curves in Euclidean 3-space $\mathbb{E}^3$

In this section, we will calculate the equivalent of the Darboux vector $W(t)$ (or the unit vector in the direction of the Darboux vector $C(t)$) obtained from the Frenet frame in terms of the vectors of the modified frame of a non-unit speed curve $\alpha(t)$. But to avoid confusion, we will denote the Darboux vector (the unit vector in the direction of the Darboux vector) obtained from the modified frame with $D(t)$ (or $G(t)$).

4.1. *The Darboux vector obtained from the modified orthogonal frame with the curvature $\kappa(t)$ of a non-unit speed curve in Euclidean 3-space $\mathbb{E}^3$*

Theorem 4.1. *Let the modified orthogonal frame with the curvature $\kappa(t)$ of the non-unit speed space curve $\alpha(t)$ be $\{E_1(t), E_2(t), E_3(t)\}$. The Darboux vector $D(t)$ obtained from this frame is as follows:*

$$D(t) = \tau(t)E_1(t) + \frac{E_3(t)}{v^2(t)}, \tag{59}$$

here,

$$\left(\frac{E_1(t)}{v(t)}\right)' = D(t) \wedge \frac{E_1(t)}{v(t)} = \frac{E_2(t)}{v(t)}, \tag{60}$$

$$\left(\frac{E_2(t)}{v^2(t)\kappa(t)}\right)' = D(t) \wedge \frac{E_2(t)}{v^2(t)\kappa(t)} = -\kappa(t)E_1(t) + \frac{\tau(t)}{v^2(t)\kappa(t)}E_3(t), \tag{61}$$

$$\left(\frac{E_3(t)}{v^3(t)\kappa(t)}\right)' = D(t) \wedge \frac{E_3(t)}{v^3(t)\kappa(t)} = -\frac{\tau(t)}{v(t)\kappa(t)}E_2(t). \tag{62}$$

Proof. From the expression (4) and (34), the Darboux vector $D(t)$ is obtained as the expression (59). Also, from the expression (33), we can write the following equations:

$$\left(\frac{E_1(t)}{v(t)}\right)' = \frac{1}{v(t)}E_1'(t) - \frac{v'(t)}{v^2(t)}E_1(t), \tag{63}$$

$$\left(\frac{E_1(t)}{v(t)}\right)' = \frac{E_2(t)}{v(t)}. \tag{64}$$

On the other hand, from the expression (28), we get

$$D(t) \wedge \frac{E_1(t)}{v(t)} = \left(\tau(t)E_1(t) + \frac{E_3(t)}{v^2(t)}\right) \wedge \frac{E_1(t)}{v(t)},$$

$$D(t) \wedge \frac{E_1(t)}{v(t)} = \frac{E_2(t)}{v(t)}. \tag{65}$$

From the equality of the expressions (64) and (65), the expression (60) is gotten. If similar operations are applied for $E_2(t)$ and $E_3(t)$ vectors, the equations (61) and (62) are obtained. □

Corollary 4.2. *As a result of Theorem 4.1, the following equations are obtained:*

$$\begin{cases} E_1'(t) = D(t) \wedge E_1(t) + \dfrac{v'(t)}{v(t)} E_1(t), \\ \\ E_2'(t) = D(t) \wedge E_2(t) + \left(\dfrac{2v'(t)}{v(t)} + \dfrac{\kappa'(t)}{\kappa(t)} \right) E_2(t), \\ \\ E_3'(t) = D(t) \wedge E_3(t) + \left(\dfrac{3v'(t)}{v(t)} + \dfrac{\kappa'(t)}{\kappa(t)} \right) E_3(t). \end{cases} \tag{66}$$

Corollary 4.3. *As a result of Corollary 4.2, the following equations are obtained:*

$$\begin{cases} E_1(t) \wedge E_1'(t) = E_3(t), \\ \\ E_2(t) \wedge E_2'(t) = v^4(t)\,\kappa^2(t)\,D(t), \\ \\ E_3(t) \wedge E_3'(t) = v^6(t)\,\kappa^2(t)\,\tau(t)\,E_1(t). \end{cases} \tag{67}$$

Corollary 4.4. *From Corollary 4.3, the following equation are gotten:*

$$D(t) = \frac{E_2(t) \wedge E_2'(t)}{v^4(t)\,\kappa^2(t)}. \tag{68}$$

Remark 4.5. *Here, we have actually expressed the Darboux vector $W(t)$ of the Frenet frame of the non-unit speed curve $\alpha(t)$ in terms of the modified frame $\{E_1(t), E_2(t), E_3(t)\}$ with the curvature $\kappa(t)$ of the curve, with the vector $D(t)$ in the expression (68).*

Special Case: If the curve $\alpha(t)$ is an unit speed curve, $v(t) = 1$. In this case, the expression (59) is obtained as in expression (17). And the expressions (66) and (67) would be as following:

$$\begin{cases} E_1'(t) = D(t) \wedge E_1(t), \\ \\ E_2'(t) = D(t) \wedge E_2(t) + \dfrac{\kappa'(t)}{\kappa(t)} E_2(t), \\ \\ E_3'(t) = D(t) \wedge E_3(t) + \dfrac{\kappa'(t)}{\kappa(t)} E_3(t) \end{cases}$$

and

$$E_1(t) \wedge E_1'(t) = E_3(t), \quad E_2(t) \wedge E_2'(t) = \kappa^2(t)\,D(t), \quad E_3(t) \wedge E_3'(t) = \kappa^2(t)\,\tau(t)\,E_1(t).$$

Finally, the expression (68) is obtained as in the expression (18).

Theorem 4.6. *Let the modified orthogonal frame with the curvature $\kappa(t)$ of the non-unit speed space curve $\alpha(t)$ be $\{E_1(t), E_2(t), E_3(t)\}$. And let the angle between the Darboux vector $D(t)$ and the vector $E_3(t)$ be $\varphi(t)$. The unit vector in the direction of the Darboux vector $G(t)$ is as follows:*

$$G(t) = \frac{\sin\varphi}{v(t)} E_1(t) + \frac{\cos\varphi}{v^3(t)\kappa(t)} E_3(t). \tag{69}$$

Proof. From the expressions (27) and (59), we get

$$\|D(t)\| = \sqrt{\langle E_1(t), E_1(t)\rangle\,\tau^2(t) + \frac{\langle E_3(t), E_3(t)\rangle}{v^4(t)}},$$

$$\|D(t)\| = v(t)\sqrt{\kappa^2(t) + \tau^2(t)}. \tag{70}$$

From the expressions (59) and (70), the unit vector in the direction of the Darboux vector $G(t)$ is gotten as follows:

$$G(t) = \frac{\tau(t)}{v(t)\sqrt{\kappa^2(t)+\tau^2(t)}}E_1(t) + \frac{1}{v^3(t)\sqrt{\kappa^2(t)+\tau^2(t)}}E_3(t). \tag{71}$$

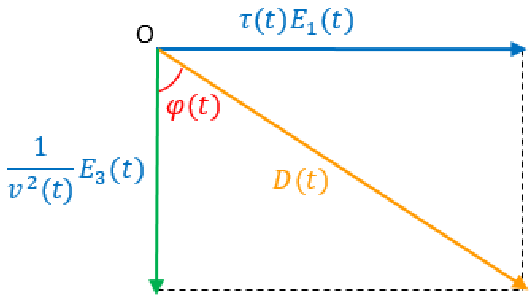

Figure 1: The Darboux vector obtained from the modified orthogonal frame with the curvature $\kappa(t)$

If the angle between of the Darboux vector $D(t)$ and the vector $E_3(t)$ is $\varphi(t)$, from the Figure 1 and the expressions (27) and (70), we write

$$\cos\varphi = \frac{\kappa(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}} \quad \text{and} \quad \sin\varphi = \frac{\tau(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}}. \tag{72}$$

From the expressions (71) and (72), the expression (69) is obtained. □

Special Case: If the curve $\alpha(t)$ is an unit speed curve, $v(t) = 1$. In this case, the expression (69) is obtained as in expression (19).

4.2. *The Darboux vector obtained from the modified orthogonal frame with the torsion* $\tau(t)$ *of a non-unit speed curve in Euclidean 3-space* $\mathbb{E}^3$

Theorem 4.7. *Let the modified orthogonal frame with the torsion $\tau(t)$ of the non-unit speed space curve $\alpha(t)$ be $\{A_1(t), A_2(t), A_3(t)\}$. The Darboux vector $\overline{D}(t)$ obtained from this frame is as follows:*

$$\overline{D}(t) = \tau(t)A_1(t) + \frac{\kappa(t)}{v^2(t)\tau(t)}A_3(t), \tag{73}$$

here,

$$\left(\frac{A_1(t)}{v(t)}\right)' = \overline{D}(t) \wedge \frac{A_1(t)}{v(t)} = \frac{\kappa(t)}{v(t)\tau(t)}A_2(t), \tag{74}$$

$$\left(\frac{A_2(t)}{v^2(t)\tau(t)}\right)' = \overline{D}(t) \wedge \frac{A_2(t)}{v^2(t)\tau(t)} = -\kappa(t)A_1(t) + \frac{A_3(t)}{v^2(t)}, \tag{75}$$

$$\left(\frac{A_3(t)}{v^3(t)\tau(t)}\right)' = \overline{D}(t) \wedge \frac{A_3(t)}{v^3(t)\tau(t)} = -\frac{A_2(t)}{v(t)}. \tag{76}$$

Proof. From the expressions (4) and (54), the Darboux vector $\overline{D}(t)$ is obtained as the expression (73). Also, from the expression (53), we can write the following equations:

$$\left(\frac{A_1(t)}{v(t)}\right)' = \frac{1}{v(t)}A_1'(t) - \frac{v'(t)}{v^2(t)}A_1(t), \tag{77}$$

$$\left(\frac{A_1(t)}{v(t)}\right)' = \frac{\kappa(t)}{v(t)\tau(t)}A_2(t). \tag{78}$$

On the other hand, from the expression (48), we get

$$\begin{aligned}
\overline{D}(t) \wedge \frac{A_1(t)}{v(t)} &= \left(\tau(t) A_1(t) + \frac{\kappa(t)}{v^2(t)\,\tau(t)} A_3(t)\right) \wedge \frac{A_1(t)}{v(t)}, \\
\overline{D}(t) \wedge \frac{A_1(t)}{v(t)} &= \frac{\kappa(t)}{v(t)\,\tau(t)} A_2(t). \qquad (79)
\end{aligned}$$

From the equality of the expressions (78) and (79), the expression (74) is gotten. If similar operations are applied for $A_2(t)$ and $A_3(t)$ vectors, the equations (75) and (76) are obtained. □

Corollary 4.8. *As a result of Theorem 4.7, the following equations are obtained:*

$$\begin{cases}
A_1'(t) = \overline{D}(t) \wedge A_1(t) + \dfrac{v'(t)}{v(t)} A_1(t), \\
A_2'(t) = \overline{D}(t) \wedge A_2(t) + \left(\dfrac{2v'(t)}{v(t)} + \dfrac{\tau'(t)}{\tau(t)}\right) A_2(t), \\
A_3'(t) = \overline{D}(t) \wedge A_3(t) + \left(\dfrac{3v'(t)}{v(t)} + \dfrac{\tau'(t)}{\tau(t)}\right) A_3(t).
\end{cases} \qquad (80)$$

Corollary 4.9. *As a result of Corollary 4.8, the following equations are obtained:*

$$\begin{cases}
A_1(t) \wedge A_1'(t) = \dfrac{\kappa(t)}{\tau(t)} A_3(t), \\
A_2(t) \wedge A_2'(t) = v^4(t)\,\tau^2(t)\,\overline{D}(t), \\
A_3(t) \wedge A_3'(t) = v^6(t)\,\tau^3(t)\,A_1(t).
\end{cases} \qquad (81)$$

Corollary 4.10. *From Corollary 4.9, the following equation is gotten:*

$$\overline{D}(t) = \frac{A_2(t) \wedge A_2'(t)}{v^4(t)\,\tau^2(t)}. \qquad (82)$$

Remark 4.11. *Here, we have actually expressed the Darboux vector $W(t)$ of the Frenet frame of the non-unit speed curve $\alpha(t)$ in terms of the modified frame $\{A_1(t), A_2(t), A_3(t)\}$ with the torsion $\tau(t)$ of the curve, with the vector $\overline{D}(t)$ in the expression (82).*

Special Case: If the curve $\alpha(t)$ is an unit speed curve, $v(t) = 1$. In this case, the expressions (73), (80), (81) and (82) would be as following:

$$\overline{D}(t) = \tau(t) A_1(t) + \frac{\kappa(t)}{\tau(t)} A_3(t),$$

$$\begin{cases}
A_1'(t) = \overline{D}(t) \wedge A_1(t), \\
A_2'(t) = \overline{D}(t) \wedge A_2(t) + \dfrac{\tau'(t)}{\tau(t)} A_2(t), \\
A_3'(t) = \overline{D}(t) \wedge A_3(t) + \dfrac{\tau'(t)}{\tau(t)} A_3(t),
\end{cases}$$

$$A_1(t) \wedge A_1'(t) = \frac{\kappa(t)}{\tau(t)} A_3(t), \quad A_2(t) \wedge A_2'(t) = \tau^2(t)\overline{D}(t), \quad A_3(t) \wedge A_3'(t) = \tau^3(t) A_1(t)$$

and

$$\overline{D}(t) = \frac{A_2(t) \wedge A_2'(t)}{\tau^2(t)}.$$

Theorem 4.12. *Let the modified orthogonal frame with the torsion $\tau(t)$ of the non-unit speed space curve $\alpha(t)$ be $\{A_1(t), A_2(t), A_3(t)\}$. And let the angle between the Darboux vector $\overline{D}(t)$ and the vector $A_3(t)$ be $\overline{\varphi}(t)$. The unit vector in the direction of the Darboux vector $\overline{G}(t)$ is as follows:*

$$\overline{G}(t) = \frac{\sin\overline{\varphi}}{v(t)} A_1(t) + \frac{\cos\overline{\varphi}}{v^3(t)\tau(t)} A_3(t). \tag{83}$$

Proof. From the expressions (47) and (73), we get

$$\begin{aligned} \left\|\overline{D}(t)\right\| &= \sqrt{\langle A_1(t), A_1(t)\rangle \tau^2(t) + \frac{\langle A_3(t), A_3(t)\rangle \kappa^2(t)}{v^4(t)\tau^2(t)}}, \\ \left\|\overline{D}(t)\right\| &= v(t)\sqrt{\kappa^2(t) + \tau^2(t)}. \end{aligned} \tag{84}$$

From the expressions (73) and (84), the unit vector in the direction of the Darboux vector $\overline{G}(t)$ is gotten as follows:

$$\overline{G}(t) = \frac{\tau(t)}{v(t)\sqrt{\kappa^2(t) + \tau^2(t)}} A_1(t) + \frac{\kappa(t)}{v^3(t)\tau(t)\sqrt{\kappa^2(t) + \tau^2(t)}} A_3(t). \tag{85}$$

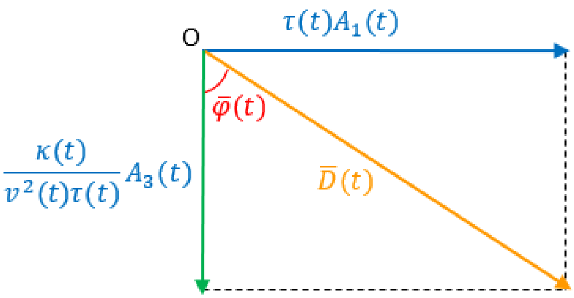

Figure 2: The Darboux vector obtained from the modified orthogonal frame with the torsion $\tau(t)$

If the angle between of the Darboux vector $\overline{D}(t)$ and the vector $A_3(t)$ is $\overline{\varphi}(t)$, from the Figure 2 and the expressions (47) and (84), we write

$$\cos\overline{\varphi} = \frac{\kappa(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}} \quad \text{and} \quad \sin\overline{\varphi} = \frac{\tau(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}}. \tag{86}$$

From the expressions (85) and (86), the expression (83) is obtained. □

Special Case: If the curve $\alpha(t)$ is an unit speed curve, $v(t) = 1$. In this case, the expression (83) is obtained as follows:

$$\overline{G}(t) = \sin\overline{\varphi} A_1(t) + \frac{\cos\overline{\varphi}}{\tau(t)} A_3(t). \tag{87}$$

Example 4.13. *Let's consider the non-unit speed Euler spiral (clothoid or Cornu spiral)*

$$\alpha(t)=\left(a\oint_0^t\cos\left(\frac{\pi x^2}{2}\right)dx,\ a\oint_0^t\sin\left(\frac{\pi x^2}{2}\right)dx,\ at\right),$$

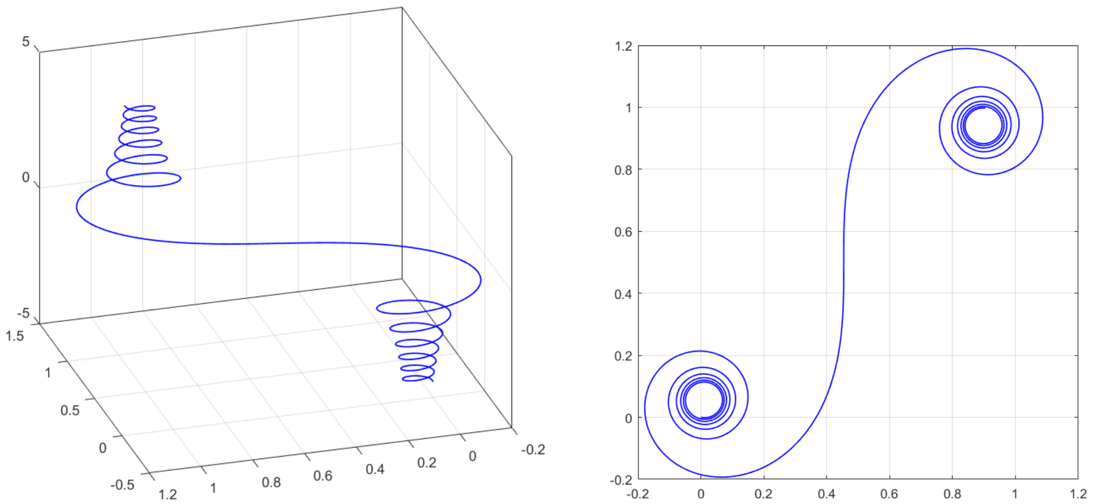

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Figure 3: Euler spiral

Figure 3, [9]. Here the components $\oint_0^t\cos\left(\frac{\pi x^2}{2}\right)dx$ *and* $\oint_0^t\sin\left(\frac{\pi x^2}{2}\right)dx$ *are called Fresnel integrals. Then the first, second and third derivative vectors of* $\alpha(t)$ *are as follows:*

$$\alpha'(t)=\left(a\cos\left(\frac{\pi t^2}{2}\right),\ a\sin\left(\frac{\pi t^2}{2}\right),\ a\right),$$

$$\alpha''(t)=\left(-a\pi t\sin\left(\frac{\pi t^2}{2}\right),\ a\pi t\cos\left(\frac{\pi t^2}{2}\right),\ 0\right),$$

$$\alpha'''(t)=\left(-a\pi^2t^2\cos\left(\frac{\pi t^2}{2}\right),-a\pi^2t^2\sin\left(\frac{\pi t^2}{2}\right),\ 0\right).$$

Here, $\|\alpha'(t)\|=v(t)=\sqrt{2}|a|$, *if* $a\neq\pm\frac{1}{\sqrt{2}}$, *the curve be* $\alpha(t)$ *is not an unit speed curve. So the Frenet vectors, the curvature and the torsion are obtained as follows:*

$$\begin{aligned}
T(t)&=\left(\frac{a}{\sqrt{2}|a|}\cos\left(\frac{\pi t^2}{2}\right),\frac{a}{\sqrt{2}|a|}\sin\left(\frac{\pi t^2}{2}\right),\frac{a}{\sqrt{2}|a|}\right),\\
N(t)&=\left(-\frac{at}{|a||t|}\sin\left(\frac{\pi t^2}{2}\right),\frac{at}{|a||t|}\cos\left(\frac{\pi t^2}{2}\right),0\right),\\
B(t)&=\left(-\frac{t}{\sqrt{2}|t|}\cos\left(\frac{\pi t^2}{2}\right),-\frac{t}{\sqrt{2}|t|}\sin\left(\frac{\pi t^2}{2}\right),\frac{t}{\sqrt{2}|t|}\right),
\end{aligned}$$

$$\kappa(t)=\frac{\pi|t|}{2|a|}\quad,\quad\tau(t)=\frac{\pi t}{2a}.$$

Now, let's examine the left and right limits of the vectors $N(t)$ and $B(t)$:

$$\lim_{t\to 0^+} N(t) = \left(0, \frac{a}{|a|}, 0\right) \quad \text{and} \quad \lim_{t\to 0^-} N(t) = \left(0, -\frac{a}{|a|}, 0\right),$$

$$\lim_{t\to 0^+} B(t) = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \quad \text{and} \quad \lim_{t\to 0^-} B(t) = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right).$$

Since $\lim_{t\to 0^+} N(t) \neq \lim_{t\to 0^-} N(t)$ and $\lim_{t\to 0^+} B(t) \neq \lim_{t\to 0^-} B(t)$, there is no limit at $t = 0$. So the normal vector and binormal vector are discontinuous at $t = 0$. And it is clear that, the curvature function is not differentiable at $t = 0$. Then, to prevent the occurrence of two reverse oriented principal normal vector and binormal vector, it is helpful to use the following modified orthogonal frame with the curvature $\kappa(t)$, obtained from Frenet vectors:

$$\begin{aligned}
E_1(t) &= \left(a\cos\left(\frac{\pi t^2}{2}\right),\ a\sin\left(\frac{\pi t^2}{2}\right),\ a\right),\\
E_2(t) &= \left(-a\pi t\sin\left(\frac{\pi t^2}{2}\right),\ a\pi t\cos\left(\frac{\pi t^2}{2}\right),\ 0\right),\\
E_3(t) &= \left(-a^2\pi t\cos\left(\frac{\pi t^2}{2}\right),\ -a^2\pi t\sin\left(\frac{\pi t^2}{2}\right),\ a^2\pi t\right),
\end{aligned}$$

$$\kappa^2(t) = \frac{\pi^2 t^2}{4a^2} \quad , \quad \tau(t) = \frac{\pi t}{2a}.$$

Thus, the problem of not being able to differentiate of the curvature at the point $t = 0$ is eliminated.

Example 4.14. *Let a non-unit speed space curve be*

$$\beta(t) = \left(t\cos t - \sin t, \cos t + t\sin t, \frac{t^2}{2}\right),$$

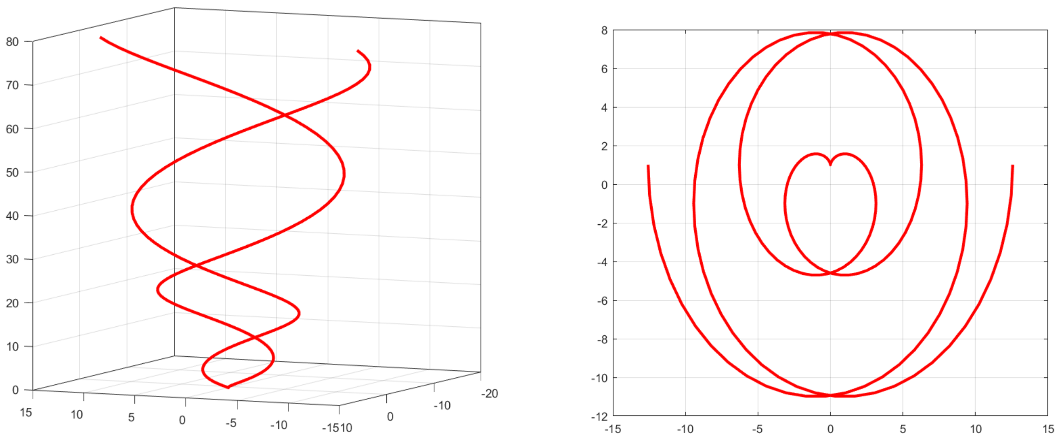

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Figure 4: The curve $\alpha(t) = (t\cos t - \sin t, \cos t + t\sin t, t)$

Figure 4. Then the first, second and third derivative vectors of $\beta(t)$ are as follows:

$$\beta'(t) = (-t\sin t, t\cos t, t),$$

$$\beta''(t) = (-t\cos t - \sin t, \cos t - t\sin t, 1),$$

$$\beta'''(t) = (-2\cos t + t\sin t, -t\cos t - 2\sin t, 0),$$

Since $\left\|\beta'(t)\right\| = v(t) = \sqrt{2}\,|t|$, *the curve be* $\beta(t)$ *is not an unit speed curve. So the Frenet vectors, the curvature, the torsion, the Darboux vector, the unit vector in direction of the Darboux vector of the curve* $\beta(t)$ *are obtained as follows:*

$$\begin{aligned} T(t) &= \left(-\frac{t}{\sqrt{2}\,|t|}\sin t, \frac{t}{\sqrt{2}\,|t|}\cos t, \frac{t}{\sqrt{2}\,|t|}\right), \\ N(t) &= \left(\frac{t}{|t|}\cos t, \frac{t}{|t|}\sin t, 0\right), \\ B(t) &= \left(\frac{1}{\sqrt{2}}\sin t, -\frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\right), \end{aligned}$$

$$\kappa(t) = \frac{1}{\sqrt{2}\,|t|} \quad , \quad \tau(t) = \frac{1}{2t},$$

$$W(t) = \left(\frac{t-t^3}{\sqrt{2}\,|t|}\sin t, \frac{t^3-t}{\sqrt{2}\,|t|}\cos t, \frac{t+t^3}{\sqrt{2}\,|t|}\right),$$

$$C(t) = \left(\frac{t-t^3}{\sqrt{2}\,|t|\,(1+t^4)}\sin t, \frac{t^3-t}{\sqrt{2}\,|t|\,(1+t^4)}\cos t, \frac{t+t^3}{\sqrt{2}\,|t|\,(1+t^4)}\right).$$

Now, let's examine the left and right limits of the vectors $T(t)$ *and* $N(t)$*. Since* $\lim_{t\to 0^+} T(t) \neq \lim_{t\to 0^-} T(t)$, $\lim_{t\to 0^+} N(t) \neq \lim_{t\to 0^-} N(t)$ *and* $\lim_{t\to 0^+} W(t) \neq \lim_{t\to 0^-} W(t)$, *there is no limit at* $t = 0$*. So the tangent vector and principal normal vector are discontinuous at* $t = 0$*. And, it is clear that, the curvature is not differentiable at* $t = 0$*. So, to solve the problem at* $t = 0$*, let's create the modified orthogonal frame of the curve. The modified orthogonal frame with the curvature* $\kappa(t)$ *of the non-unit speed curve* $\beta(t)$ *is gotten as follows:*

$$E_1(t) = (-t\sin t, t\cos t, t),$$

$$E_2(t) = \left(\sqrt{2}t\cos t, \sqrt{2}t\sin t, 0\right),$$

$$E_3(t) = \left(\sqrt{2}t^2\sin t, -\sqrt{2}t^2\cos t, \sqrt{2}t^2\right),$$

where,

$$\langle E_1(t), E_2(t)\rangle = \langle E_2(t), E_3(t)\rangle = \langle E_1(t), E_3(t)\rangle = 0,$$

$$\langle E_1(t), E_1(t)\rangle = \langle E_2(t), E_2(t)\rangle = 2t^2, \qquad \langle E_3(t), E_3(t)\rangle = 4t^4$$

and

$$E_1(t) \wedge E_2(t) = E_3(t),$$

$$E_2(t) \wedge E_3(t) = \left(-2t^3\sin t, 2t^3\cos t, 2t^3\right) = v^4(t)\kappa^2(t)E_1(t),$$

$$E_3(t) \wedge E_1(t) = \left(2\sqrt{2}t^3\cos t, 2\sqrt{2}t^3\sin t, 0\right) = v^2(t)E_2(t).$$

Also, the curvature, the torsion, the Darboux vector and the unit vector in direction of the Darboux vector of the modified orthogonal frame with the curvature $\kappa(t)$ *of the non-unit speed curve* $\beta(t)$ *are as following:*

$$\kappa^2(t) = \frac{1}{2t^2} \quad , \quad \tau(t) = \frac{1}{2t},$$

$$D(t) = \left(\frac{1-t^2}{\sqrt{2}}\sin t, \frac{t^2-1}{\sqrt{2}}\cos t, \frac{1+t^2}{\sqrt{2}}\right),$$

$$G(t) = \left(\frac{1-t^2}{\sqrt{2}(1+t^4)} \sin t, \frac{t^2-1}{\sqrt{2}(1+t^4)} \cos t, \frac{1+t^2}{\sqrt{2}(1+t^4)} \right).$$

Thus, instead of the Frenet frame of the curve, which causes problem at $t = 0$*, the properties of the curve at this point can be examined with this modified orthogonal frame.*

5. Conclusions

At singular points (or sharp points) on the analytical curves (or the discontinuous curves), one or more of the Frenet vectors, or the curvature and torsion functions, cannot be differentiated because their right and left limits are not the same. At these points, the use of the modified orthogonal frame instead of the Frenet frame is sufficient and necessary to solve the problem at that point. Apart from this, there is no harm in creating a modified orthogonal frame of any regular curve. Just like the Frenet frame, the characteristic features of the curve can also be examined with the modified orthogonal frame. Since the Frenet frame provides more ease of operation, the modified orthogonal frame is not preferred much in studies on regular curves. In this study, we have shown that the modified orthogonal frame works well not only for unit speed curves, but also for non-unit speed curves.

References

[1] Arıkan M, Nurkan SK. Adjoint Curve According to Modified Orthogonal Frame with Torsion in 3-Space. Usak University Journal of Science and Natural Sciences. 2, 2020, 54 – 64.
[2] Azak AZ. Involute-Evolute Curves According to Modified Orthogonal Frame. Journal of Science and Arts. 21(2), 2021, 385 – 394.
[3] Bükcü B, Karacan MK. On the modified orthogonal frame with curvature and torsion in 3-space. Mathematical Sciences and Applications E-Notes. 4(1), 2016, 184 – 188.
[4] Bükcü B, Karacan MK. Spherical curves with modified orthogonal frame. Journal of New Results in Science. 10, 2016, 60 – 68.
[5] Clelland JN. From Frenet to Cartan: the method of moving frames. American Mathematical Soc. 178, 2017.
[6] Do Carmo MP. Differential geometry of curves and surfaces: revised and updated second edition. Courier Dover Publications, 2016.
[7] Elsayied HK, Altaha AA, Elsharkawy A. On Some Special Curves According to the Modified Orthogonal Frame in Minkowski 3-space E_1^3. Kasmera, 49, 2021, 2-15.
[8] Eren K, Kosal HH. Evolution of space curves and the special ruled surfaces with modified orthogonal frame. AIMS Mathematics. 5(3), 2020, 2027 – 2039.
[9] Gray A. Modern Differential Geometry of Curves and Surfaces with Mathematica. CRC New York. 1998.
[10] Gür S, Şenyurt S. Frenet Vectors and Geodesic Curvatures of Spheric Indicators of Salkowski Curve in $\mathbb{E}^3$. Hadronic Journal, 33(5), 2010, 485 – 512.
[11] Hacısalihoğlu HH. Differantial Geometry. Ankara University Faculty of Science Press, 2000.
[12] Hord AM. Torsion at an inflection point of a space curve. Am. Math. Mon. 79, 1972, 371 – 374.
[13] Izumiya S, Takeuchi N. New special curves and developable surfaces. Turkish Journal of Mathematics, 28(2), 2004), 153 – 164.
[14] Lone MS, Es H, Karacan MK, Bükcü B. On some curves with modified orthogonal frame in Euclidean 3-Space. Iranian Journal of Science and Technology, Transactions A: Science. 43(4), 2019, 1905 – 1916.
[15] Lone MS, Es H, Karacan MK, Bükcü B. Mannheim curves with modified orthogonal frame in Euclidean 3-space. Turkish Journal of Mathematics, 43(2), 2019, 648 – 663.
[16] Özdemir M. Diferansiyel Geometri. Altin Nokta Yayinevi, 2020.
[17] Sasai T. The fundamental theorem of analytic space curves and apparent singularities of Fuchsian differential equations. Tohoku Math Journal. 36, 1984, 17 – 24.
[18] Sasai T. Geometry of analytic space curves with singularities and regular singularities of differential equations. Funkcial. Ekvac. 30, 1987, 283 – 303.
[19] Şenyurt S, Çalışkan A. Smarandache curves of Mannheim curve couple according to Frenet frame. Mathematical Sciences and Applications E-Notes, 5(1), 2017, 122 – 136.
[20] Şenyurt S, Şardağ H, Çakır O, On Vectorial Moment of the Darboux Vector. Konuralp Journal of Mathematics, 8(1), 2020, 144 – 151.
[21] Uddin S, Stankovic MS, Iqbal M, Yadav SK, Aslam M. Slant helices in Minkowski 3-space E_1^3 with Sasai's modified frame fields. Filomat 36, 2022, 151 – 164.
[22] Yaylı Y, Gök I, Hacısalihoğlu HH. Extended rectifying curves as new kind of modified Darboux vectors. TWMS J Pure Appl Math. 9, 2018, 18 – 31.

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Tubular Hypersurfaces According to Extended Darboux Frame Field of First Kind in E^4

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Abstract. In this paper, we study tubular hypersurfaces according to one of the extended Darboux frame field in Euclidean 4-space. We obtain the Gaussian and mean curvatures of tubular hypersurfaces according to extended Darboux frame field of first kind and give some results for them. Also, we prove a theorem about linear Weingarten tubular hypersurface and construct an example.

1. INTRODUCTION

A canal surface is formed by the envelope of the spheres whose centers lie on a curve and radii vary depending on this curve [18]. In case of a constant radius function, the envelope is called tubular or pipe surface [19]. Also for a canal surface, if the center curve is a straight line, then it becomes a revolution surface. Canal surfaces (especially tubular surfaces) have been applied to many fields, such as the solid and the surface modeling for CAD/CAM, construction of blending surfaces, shape re-construction and so on. In this context, canal and tubular (hyper)surfaces have been studied by many geometers in Euclidean, Minkowskian, Galilean or pseudo-Galilean spaces (see [7], [14], [20]-[24], [28]-[30], [32], [34]-[37], and etc).

On the other hand, Frenet frame has been used in lots of studies about curves and surfaces, but sometimes scienticists have needed alternative frames because Frenet frame cannot be identified at the points where the curvature is zero. Therefore, new alternative frames to the Frenet frame such as Bishop frame, Darboux frame or extended Darboux frame have been defined by geometers and the theories of curves and surfaces have been started to handle according to these alternative frames (see [2], [3], [9]-[13], [25], [27], [33], and etc).

After recalling some basic notions about one type of extended Darboux frame field and the curvatures of hypersurfaces in E^4 in the second section of this paper, we deal with tubular hypersurfaces according to extended Darboux frame field of first kind in E^4 in the third section. We obtain the Gaussian and mean curvatures of tubular hypersurface according to extended Darboux frame field of first kind and give some results when the curve which constructs the tubular hypersurfaces is (unit speed) asymptotic or line of curvature on tubular hypersurface. Finally, we prove a theorem that states the tubular hypersurface according to extended Darboux frame field of first kind in E^4 is a linear Weingarten hypersurface.

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2. PRELIMINARIES

Let $\{e_1, e_2, e_3, e_4\}$ be the standart basis of Euclidean 4-space E^4. If $\overrightarrow{s} = (s_1, s_2, s_3, s_4)$, $\overrightarrow{t} = (t_1, t_2, t_3, t_4)$ and $\overrightarrow{v} = (v_1, v_2, v_3, v_4)$ are three vectors in E^4, then the inner product and vector product are given by

$$\left\langle \overrightarrow{s}, \overrightarrow{t} \right\rangle = s_1t_1 + s_2t_2 + s_3t_3 + s_4t_4$$

and

$$\overrightarrow{s} \times \overrightarrow{t} \times \overrightarrow{v} = \det \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ s_1 & s_2 & s_3 & s_4 \\ t_1 & t_2 & t_3 & t_4 \\ v_1 & v_2 & v_3 & v_4 \end{bmatrix},$$

respectively. Also, the norm of the vector $\overrightarrow{s}$ is $\left\| \overrightarrow{s} \right\| = \sqrt{\left\langle \overrightarrow{s}, \overrightarrow{s} \right\rangle}$. Let $M \subset E^4$ denote a regular hypersurface and $\alpha : I \subset R \longrightarrow M$ be a unit speed curve. If $\{T, n, b_1, b_2\}$ is the moving Frenet frame along α, then the Frenet formulas are given by [15]

$$\begin{bmatrix} T' \\ n' \\ b_1' \\ b_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ n \\ b_1 \\ b_2 \end{bmatrix},$$

where T, n, b_1 and b_2 denote the unit tangent, the principal normal, the first binormal and the second binormal vector fields; k_1, k_2 and k_3 are the curvature functions of the curve α.

Here, we will recall the extended Darboux frame field of first kind (for simplicity, we'll call it *ED^1-frame field* throughout this paper) and for details about the construction of extended Darboux frame fields, we refer to [13].

We consider an embedding $\Psi : U \subset E^3 \longrightarrow E^4$, where U is an open subset of E^3. Now, we denote $M = \Psi(U)$ and identify M and U through the embedding Ψ. Let $\bar{\alpha} : I \longrightarrow U$ be a regular curve and we have a curve $\alpha : I \longrightarrow M \subset E^4$ defined by $\alpha(s) = \Psi(\bar{\alpha}(s))$ and so, the curve α is on the hypersurface M. If M is an orientable hypersurface oriented by the unit normal vector field $\mathcal{N}$ in E^4 and α is a Frenet curve of class $C^n (n \geq 4)$ with arc-length parameter s lying on M, then we denote the unit tangent vector field of the curve by T and denote the hypersurface unit normal vector field restricted to the curve by N, i.e.

$$T(s) = \alpha'(s) \text{ and } N(s) = \mathcal{N}(\alpha(s)).$$

The differential equations of ED-frame fields of first kind $\{T, E, D, N\}$ of the curve α on M in E^4 by matrix notation can be given as

$$\begin{bmatrix} T' \\ E' \\ D' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^1 & 0 & \kappa_n \\ -\kappa_g^1 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & \tau_g^2 \\ -\kappa_n & -\tau_g^1 & -\tau_g^2 & 0 \end{bmatrix} \begin{bmatrix} T \\ E \\ D \\ N \end{bmatrix}, \tag{1}$$

where $\langle E', N \rangle = \tau_g^1$, $\langle D', N \rangle = \tau_g^2$, $\langle T', E \rangle = \kappa_g^1$, $\langle E', D \rangle = \kappa_g^2$ and τ_g^i and κ_g^i are called the geodesic torsions and geodesic curvatures of order i, respectively. Also, $\langle T', N \rangle = \kappa_n$ is the normal curvature of the hypersurface in the direction of the tangent vector T [13].

Now, the relation matrix may be expressed as [13]

$$\begin{bmatrix} T \\ n \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_1 & \cos\phi_2 & \cos\phi_3 \\ 0 & \cos\psi_1 & \cos\psi_2 & \cos\psi_3 \\ 0 & \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \end{bmatrix} \begin{bmatrix} T \\ E \\ D \\ N \end{bmatrix} \tag{2}$$

and

$$\begin{bmatrix} T \\ E \\ D \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_1 & \cos\psi_1 & \cos\theta_1 \\ 0 & \cos\phi_2 & \cos\psi_2 & \cos\theta_2 \\ 0 & \cos\phi_3 & \cos\psi_3 & \cos\theta_3 \end{bmatrix} \begin{bmatrix} T \\ n \\ b_1 \\ b_2 \end{bmatrix}. \tag{3}$$

Also, we have

$$\left.\begin{array}{l} \kappa_g^1 = \langle T', E\rangle = k_1 \cos\phi_1, \ \kappa_n = \langle T', N\rangle = k_1\cos\phi_3, \\ \tau_g^1 = -\phi_1' \sin\phi_1\cos\phi_3 - \psi_1'\sin\psi_1\cos\psi_3 - \theta_1'\sin\theta_1\cos\theta_3 \\ \quad + k_2(\cos\phi_1\cos\psi_3 - \cos\psi_1\cos\phi_3) + k_3(\cos\psi_1\cos\theta_3 - \cos\theta_1\cos\psi_3), \\ \tau_g^2 = -\phi_2'\sin\phi_2\cos\phi_3 - \psi_2'\sin\psi_2\cos\psi_3 - \theta_2'\sin\theta_2\cos\theta_3 \\ \quad + k_2(\cos\phi_2\cos\psi_3 - \cos\psi_2\cos\phi_3) + k_3(\cos\psi_2\cos\theta_3 - \cos\theta_2\cos\psi_3), \\ \kappa_g^2 = -\phi_1'\sin\phi_1\cos\phi_2 - \psi_1'\sin\psi_1\cos\psi_2 - \theta_1'\sin\theta_1\cos\theta_2 \\ \quad + k_2(\cos\phi_1\cos\psi_2 - \cos\psi_1\cos\phi_2) + k_3(\cos\psi_1\cos\theta_2 - \cos\theta_1\cos\psi_2). \end{array}\right\} \tag{4}$$

Furthermore, the differential geometry of different types of (hyper)surfaces in 4-dimensional spaces has been a popular topic for geometers, recently ([1], [4], [5], [6], [8], [16], [17], [26], and etc). If

$$\begin{aligned} \Psi : U \subset E^3 &\longrightarrow E^4 \\ (s,t,v) &\longrightarrow \Psi(s,t,v) = (\Psi_1(s,t,v), \Psi_2(s,t,v), \Psi_3(s,t,v), \Psi_4(s,t,v)) \end{aligned} \tag{5}$$

is a hypersurface in E^4, then the unit normal vector field, the matrix forms of the first and second fundamental forms are

$$\mathcal{N}_\Psi = \frac{\Psi_s \times \Psi_t \times \Psi_v}{\|\Psi_s \times \Psi_t \times \Psi_v\|}, \tag{6}$$

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{7}$$

and

$$[h_{ij}] = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}, \tag{8}$$

respectively. Here $g_{ij} = \left\langle \Psi_{v_i}, \Psi_{v_j} \right\rangle$, $h_{ij} = \left\langle \Psi_{v_i v_j}, \mathcal{N}_\Psi \right\rangle$, $\Psi_{v_i} = \frac{\partial\Psi(v_1,v_2,v_3)}{\partial v_i}$, $\Psi_{v_i v_j} = \frac{\partial^2\Psi(v_1,v_2,v_3)}{\partial v_i v_j}$, $i, j \in \{1,2,3\}$. Also, the shape operator of the hypersurface (5) is

$$S = [a_{ij}] = [g^{ij}].[h_{ij}], \tag{9}$$

where $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$.

With the aid of (6)-(9), the Gaussian and mean curvatures of a hypersurface in E^4 are given by

$$K = \det(S) = \frac{\det[h_{ij}]}{\det[g_{ij}]} \tag{10}$$

and

$$H = \frac{1}{3} tr(S), \tag{11}$$

respectively [31]. We say that a hypersurface is flat or minimal, if it has zero Gaussian curvature or zero mean curvature, respectively.

3. TUBULAR HYPERSURFACES ACCORDING TO ED^1-FRAME FIELD IN EUCLIDEAN 4-SPACE

In this section, we obtain the Gaussian and mean curvatures of tubular hypersurfaces according to ED^1-frame field in Euclidean 4-space E^4 and give some results for these curvatures when the curve α which constructs the tubular hypersurface is an asymptotic curve, a unit-speed asymptotic curve and a line of curvature lying on M.

Let $\alpha : I \longrightarrow M$ be a unit speed curve lying on a regular hypersurface M and we consider the tubular hypersurface $\mathcal{T}$ according to ED^1-frame field of α in E^4 given by

$$\mathcal{T}(s,t,v) = \alpha(s) \pm \rho\left[(\cos t\cos v)\,E(s) + (\sin t\cos v)\,D(s) + (\sin v)\,N(s)\right], \tag{12}$$

where $\alpha(s)$ is the center curve of tubular hypersurface $\mathcal{T}$, $\rho \in \mathbb{R}$ is constant radius, $s \in [0,l]$ and $t,v \in [0,2\pi)$. From now on, we state $\alpha = \alpha(s)$, $T = T(s)$, $E = E(s)$, $D = D(s)$, $N = N(s)$ and we will consider the "±" as "+".

Firstly, from (1) and (12) the first derivatives of the tubular hypersurface (12) are obtained as

$$\left.\begin{aligned}
\mathcal{T}_s &= \left(1-\rho(\kappa_g^1\cos t\cos v+\kappa_n\sin v)\right)T-\rho\left(\kappa_g^2\cos v\sin t+\tau_g^1\sin v\right)E\\
&\quad+\rho\left(\kappa_g^2\cos t\cos v-\tau_g^2\sin v\right)D+\rho\cos v\left(\tau_g^1\cos t+\tau_g^2\sin t\right)N,\\
\mathcal{T}_t &= -\left(\rho\sin t\cos v\right)E+\left(\rho\cos t\cos v\right)D,\\
\mathcal{T}_v &= -\left(\rho\cos t\sin v\right)E-\left(\rho\sin t\sin v\right)D+\left(\rho\cos v\right)N.
\end{aligned}\right\} \tag{13}$$

From (6) and (13), the unit normal vector field of $\mathcal{T}$ in E^4 is

$$\mathcal{N} = (\cos t\cos v)\,E + (\sin t\cos v)\,D + (\sin v)\,N. \tag{14}$$

Also, the coefficients of the first fundamental form are

$$\left.\begin{aligned}
g_{11} &= \left(\rho\left(\kappa_g^2\cos v\sin t+\tau_g^1\sin v\right)\right)^2+\left(\rho\cos v\left(\tau_g^1\cos t+\tau_g^2\sin t\right)\right)^2\\
&\quad+\left(\rho\left(\kappa_g^2\cos t\cos v-\tau_g^2\sin v\right)\right)^2+\left(-1+\rho\kappa_g^1\cos t\cos v+\rho\kappa_n\sin v\right)^2,\\
g_{12} &= g_{21} = \rho^2\cos v\left(\kappa_g^2\cos v+\sin v(\tau_g^1\sin t-\tau_g^2\cos t)\right),\\
g_{13} &= g_{31} = \rho^2\left(\tau_g^2\sin t+\tau_g^1\cos t\right),\\
g_{22} &= \rho^2\cos^2 v,\ g_{23} = g_{32} = 0,\ g_{33} = \rho^2
\end{aligned}\right\} \tag{15}$$

and it follows that

$$\det[g_{ij}] = \rho^4(-1+\rho\kappa_g^1\cos t\cos v+\rho\kappa_n\sin v)^2\cos^2 v. \tag{16}$$

Now, for obtaining the coefficients of the second fundamental form, we give the second derivatives

$\mathcal{T}_{v_iv_j} = \frac{\partial^2 \mathcal{T}}{\partial v_i v_j}$ of the tubular hypersurface (12):

$$\left.\begin{aligned}
\mathcal{T}_{ss} &= \mathcal{T}_{ss}^1 T + \mathcal{T}_{ss}^2 E + \mathcal{T}_{ss}^3 D + \mathcal{T}_{ss}^4 N,\\
\mathcal{T}_{st} &= \mathcal{T}_{ts} = \left(\rho\kappa_g^1 \sin t \cos v\right) T - \left(\rho\kappa_g^2 \cos t \cos v\right) E - \left(\rho\kappa_g^2 \sin t \cos v\right) D\\
&\quad - \left(\rho(\tau_g^1 \sin t - \tau_g^2 \cos t)\cos v\right) N,\\
\mathcal{T}_{sv} &= \mathcal{T}_{vs} = \left(\rho\left(\kappa_g^1 \sin v \cos t - \kappa_n \cos v\right)\right) T + \left(\rho\left(\kappa_g^2 \sin v \sin t - \tau_g^1 \cos v\right)\right) E\\
&\quad - \left(\rho\left(\kappa_g^2 \sin v \cos t + \tau_g^2 \cos v\right)\right) D - \left(\rho\left(\tau_g^1 \cos t + \tau_g^2 \sin t\right)\sin v\right) N,\\
\mathcal{T}_{tt} &= -\left(\rho \cos t \cos v\right) E - \left(\rho \sin t \cos v\right) D,\\
\mathcal{T}_{tv} &= \mathcal{T}_{vt} = \left(\rho \sin t \sin v\right) E - \left(\rho \cos t \sin v\right) D,\\
\mathcal{T}_{vv} &= -\left(\rho \cos t \cos v\right) E - \left(\rho \sin t \cos v\right) D - \left(\rho \sin v\right) N,
\end{aligned}\right\} \tag{17}$$

where

$$\mathcal{T}_{ss}^1 = \rho \begin{pmatrix} \kappa_g^1\left(\kappa_g^2 \cos v \sin t + \tau_g^1 \sin v\right) - (\kappa_n)' \sin v \\ -\left(\tau_g^1 \kappa_n \cos t + \tau_g^2 \kappa_n \sin t + (\kappa_g^1)' \cos t\right)\cos v \end{pmatrix},$$

$$\begin{aligned}
\mathcal{T}_{ss}^2 &= -\rho\left((\kappa_g^1)^2 + (\kappa_g^2)^2 + (\tau_g^1)^2\right)\cos t \cos v + \kappa_g^1(1 - \rho\kappa_n \sin v)\\
&\quad - \rho((-\kappa_g^2\tau_g^2 + (\tau_g^1)')\sin v + (\tau_g^2\tau_g^1 + (\kappa_g^2)')\sin t \cos v),
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_{ss}^3 &= -\rho\left((\kappa_g^2)^2 + (\tau_g^2)^2\right)\sin t \cos v + (\tau_g^2\tau_g^1 - (\kappa_g^2)')\cos t \cos v\\
&\quad + (\kappa_g^2\tau_g^1 + (\tau_g^2)')\sin v),
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_{ss}^4 &= -\rho\tau_g^1(\kappa_g^2 \sin t \cos v + \tau_g^1 \sin v) + \rho\tau_g^2(\kappa_g^2 \cos t \cos v - \tau_g^2 \sin v)\\
&\quad - \kappa_n\left(-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v\right) + \rho\left((\tau_g^1)' \cos t + (\tau_g^2)' \sin t\right)\cos v.
\end{aligned}$$

Thus, from (8), (14) and (17), the coefficients of the second fundamental form are

$$\left.\begin{aligned}
h_{11} &= -\rho\left(\left((\kappa_g^1)^2 + (\tau_g^1)^2\right)\cos^2 t + (\kappa_g^2)^2 + 2\tau_g^1\tau_g^2 \sin t \cos t + (\tau_g^2)^2 \sin^2 t\right)\cos^2 v\\
&\quad - \rho\left((\tau_g^1)^2 + (\tau_g^2)^2 + (\kappa_n)^2\right)\sin^2 v - \kappa_g^1\left(-1 + 2\rho\kappa_n \sin v\right)\cos t \cos v\\
&\quad - \rho\kappa_g^2\left(\tau_g^1 \sin t - \tau_g^2 \cos t\right)\sin(2v) + \kappa_n \sin v,\\
h_{12} &= h_{21} = -\rho\left(\kappa_g^2 \cos v + \sin v(\tau_g^1 \sin t - \tau_g^2 \cos t)\right)\cos v,\\
h_{13} &= h_{31} = -\rho\left(\tau_g^1 \cos t + \tau_g^2 \sin t\right),\\
h_{22} &= -\rho \cos^2 v,\ h_{23} = h_{32} = 0,\ h_{33} = -\rho
\end{aligned}\right\} \tag{18}$$

and it implies that

$$\det[h_{ij}] = -\rho^2 \cos^2 v\left(\kappa_g^1 \cos t \cos v + \kappa_n \sin v\right)\left(-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v\right). \tag{19}$$

So, from (10), (16) and (19), we have

Proposition 3.1. *The Gaussian curvature of the tubular hypersurfaces (12) in E^4 is*

$$K = -\frac{\kappa_g^1 \cos t \cos v + \kappa_n \sin v}{\rho^2(-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v)}. \tag{20}$$

Corollary 3.2. *The Gaussian curvature of the tubular hypersurfaces (12) in E^4 does not depend on the geodesic curvature of order 2 and geodesic torsions of order 1 and order 2.*

Corollary 3.3. *The tubular hypersurfaces (12) in E^4 is flat if and only if*

$$\kappa_g^1 \cos t \cos v = -\kappa_n \sin v$$

holds.

Corollary 3.4. *If $\kappa_g^1 = \kappa_n = 0$, then the tubular hypersurfaces (12) in E^4 is flat.*

Also, after finding the inverse of the matrix of the first fundamental form and using this and (18) in (9), the shape operator of the tubular hypersurface (12) is obtained by

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}, \tag{21}$$

where

$$S_{11} = -\frac{\kappa_g^1 \cos t \cos v + \kappa_n \sin v}{-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v}, \ S_{12} = S_{13} = 0,$$

$$S_{21} = \frac{\sec v \left(\kappa_g^2 + \tan v \left(\tau_g^1 \sin t - \tau_g^2 \cos t\right)\right)}{\rho\left(-\sec v + \rho\kappa_g^1 \cos t + \rho\kappa_n \tan v\right)}, \ S_{22} = -\frac{1}{\rho}, \ S_{23} = 0,$$

$$S_{31} = \frac{\tau_g^1 \cos t + \tau_g^2 \sin t}{\rho\left(-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v\right)}, \ S_{32} = 0, S_{33} = -\frac{1}{\rho}.$$

Hence from (11) and (21), we get

Proposition 3.5. *The mean curvature of the tubular hypersurfaces (12) in E^4 is*

$$H = \frac{2 - 3\rho\left(\kappa_g^1 \cos t \cos v + \kappa_n \sin v\right)}{3\rho\left(-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v\right)}. \tag{22}$$

Corollary 3.6. *The mean curvature of the tubular hypersurfaces (12) in E^4 does not depend on the geodesic curvature of order 2 and geodesic torsions of order 1 and order 2.*

Corollary 3.7. *The tubular hypersurfaces (12) in E^4 is minimal if and only if*

$$\kappa_g^1 \cos t \cos v + \kappa_n \sin v = \frac{2}{3\rho} \tag{23}$$

holds.

Corollary 3.8. *If $\kappa_g^1 = \kappa_n = 0$, then the tubular hypersurface (12) in E^4 has negative constant mean curvature with $\frac{-2}{3\rho}$.*

Here, from (20) and (22), we can state the following theorem which gives an important relation between Gaussian and mean curvatures:

Theorem 3.9. *The Gaussian curvature K and the mean curvature H of tubular hypersurfaces (12) in E^4 satisfy*

$$3H = \rho^2 K - \frac{2}{\rho}. \tag{24}$$

Also, from (21) we have

$$\det(S - \lambda I_3) = -\frac{(1+\lambda\rho)^2\left(-\lambda + (1+\lambda\rho)\,\kappa_g^1 \cos t \cos v + (1+\lambda\rho)\,\kappa_n \sin v\right)}{\rho^2\left(-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v\right)}. \tag{25}$$

By solving the equation $\det(S - \lambda I_3) = 0$ from (25), we obtain the principal curvatures of the tubular hypersurfaces (12) in E^4 as follows:

Proposition 3.10. *The principal curvatures of the tubular hypersurfaces (12) in E^4 are*

$$\lambda_1 = \lambda_2 = -\frac{1}{\rho} \text{ and } \lambda_3 = -\frac{\kappa_g^1 \cos t \cos v + \kappa_n \sin v}{-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v}. \tag{26}$$

Furthermore, if a curve α is a unit-speed asymptotic curve parametrized by arc-length on an oriented hypersurface M in E^4, then we have

$$\kappa_n = 0,\ \kappa_g^1 = k_1,\ \kappa_g^2 = k_2 \cos\varphi,\ \tau_g^1 = -k_2 \sin\varphi,\ \tau_g^2 = k_3 + \frac{d\varphi}{ds}, \tag{27}$$

where φ denotes the angle between D and B_1 [13]. Thus using (27), we have

Corollary 3.11. *If the curve α is a unit-speed asymptotic curve lying on M, then the Gaussian and mean curvatures of tubular hypersurface (12) in E^4 are*

$$K = -\frac{k_1 \cos t \cos v}{\rho^2\left(-1 + \rho k_1 \cos t \cos v\right)} \tag{28}$$

and

$$H = \frac{2 - 3\rho k_1 \cos t \cos v}{3\rho\left(-1 + \rho k_1 \cos t \cos v\right)}, \tag{29}$$

respectively.

Corollary 3.12. *If the curve α is a unit-speed asymptotic curve lying on M, then the Gaussian and mean curvatures of tubular hypersurface (12) in E^4 are independent of the angle φ.*

Also in [24], the authors have studied on canal and tubular hypersurfaces according to the Frenet frame in E^4 and they have obtained the Gaussian and mean curvatures of tubular hypersurface

$$\mathcal{T}(s,t,v) = \alpha(s) + \rho\left[(\cos t \cos v)\, n(s) + (\sin t \cos v)\, b_1(s) + (\sin v)\, b_2(s)\right] \tag{30}$$

as (28) and (29). Therefore

Theorem 3.13. *If the curve α is a unit-speed asymptotic curve lying on M, then the Gaussian and mean curvatures of tubular hypersurfaces (12) according to ED^1-frame field and (30) according to Frenet frame coincide.*

On the other hand, the curve α lying on M is a line of curvature if and only if $\tau_g^1 = \tau_g^2 = 0$ [13]. So, we have

Corollary 3.14. *If the curve α is line of curvature lying on M, then the Gaussian and mean curvatures of tubular hypersurface (12) in E^4 are (20) and (22) respectively.*

Finally, we will give a theorem about linear Weingarten tubular hypersurface according to ED[1]-frame field of unit speed curve α lying on M in E^4. We know that, a hypersurface is called a linear Weingarten hypersurface, if it satisfies

$$aH + bK = c, \tag{31}$$

where a, b, c are not all zero constants. Thus, we have

Theorem 3.15. *The tubular hypersurface (12) in E^4 is a linear Weingarten hypersurface.*

Proof. We know that, the relation between the mean and Gaussian curvatures of the tubular hypersurface (12) in E^4 is given by (24). So, if we take $a = 3,\ b = -\rho^2$ and $c = \frac{-2}{\rho}$ in (31), the proof completes. □

Example 3.16. *We take the unit speed curve*

$$\alpha(s) = \left(\sin\left(\frac{3s}{5}\right), \cos\left(\frac{3s}{5}\right), \sin\left(\frac{4s}{5}\right), \cos\left(\frac{4s}{5}\right)\right) \tag{32}$$

on the hypersphere $M...x^2 + y^2 + z^2 + t^2 = 2$ in E^4. The Frenet apparatus of this curve is

$$\left.\begin{aligned} T &= \tfrac{1}{5}\left(3\cos\left(\tfrac{3s}{5}\right), -3\sin\left(\tfrac{3s}{5}\right), 4\cos\left(\tfrac{4s}{5}\right), -4\sin\left(\tfrac{4s}{5}\right)\right), \\ n &= -\tfrac{1}{\sqrt{337}}\left(9\sin\left(\tfrac{3s}{5}\right), 9\cos\left(\tfrac{3s}{5}\right), 16\sin\left(\tfrac{4s}{5}\right), 16\cos\left(\tfrac{4s}{5}\right)\right), \\ b_1 &= \tfrac{1}{5}\left(4\cos\left(\tfrac{3s}{5}\right), -4\sin\left(\tfrac{3s}{5}\right), -3\cos\left(\tfrac{4s}{5}\right), 3\sin\left(\tfrac{4s}{5}\right)\right), \\ b_2 &= -\tfrac{1}{\sqrt{337}}\left(16\sin\left(\tfrac{3s}{5}\right), 16\cos\left(\tfrac{3s}{5}\right), -9\sin\left(\tfrac{4s}{5}\right), -9\cos\left(\tfrac{4s}{5}\right)\right) \end{aligned}\right\} \tag{33}$$

and

$$k_1 = \frac{\sqrt{337}}{25},\ k_2 = \frac{84}{25\sqrt{337}},\ k_3 = \frac{12}{\sqrt{337}}. \tag{34}$$

Also, we have the ED[1]-frame fields of unit speed curve α as

$$\left.\begin{aligned} T &= \tfrac{1}{5}\left(3\cos\left(\tfrac{3s}{5}\right), -3\sin\left(\tfrac{3s}{5}\right), 4\cos\left(\tfrac{4s}{5}\right), -4\sin\left(\tfrac{4s}{5}\right)\right), \\ E &= \tfrac{1}{\sqrt{2}}\left(\sin\left(\tfrac{3s}{5}\right), \cos\left(\tfrac{3s}{5}\right), -\sin\left(\tfrac{4s}{5}\right), -\cos\left(\tfrac{4s}{5}\right)\right), \\ D &= \tfrac{1}{5}\left(-4\cos\left(\tfrac{3s}{5}\right), 4\sin\left(\tfrac{3s}{5}\right), 3\cos(\tfrac{4s}{5}), -3\sin(\tfrac{4s}{5})\right), \\ N &= \tfrac{1}{\sqrt{2}}\left(\sin\left(\tfrac{3s}{5}\right), \cos\left(\tfrac{3s}{5}\right), \sin\left(\tfrac{4s}{5}\right), \cos\left(\tfrac{4s}{5}\right)\right) \end{aligned}\right\} \tag{35}$$

and the normal curvature, geodesic curvatures and geodesic torsions of order 1 *and* 2 *are obtained as*

$$\kappa_n = -\frac{1}{\sqrt{2}},\ \kappa_g^1 = \frac{7}{25\sqrt{2}},\ \kappa_g^2 = -\frac{12\sqrt{2}}{25},\ \tau_g^1 = 0,\ \tau_g^2 = 0, \tag{36}$$

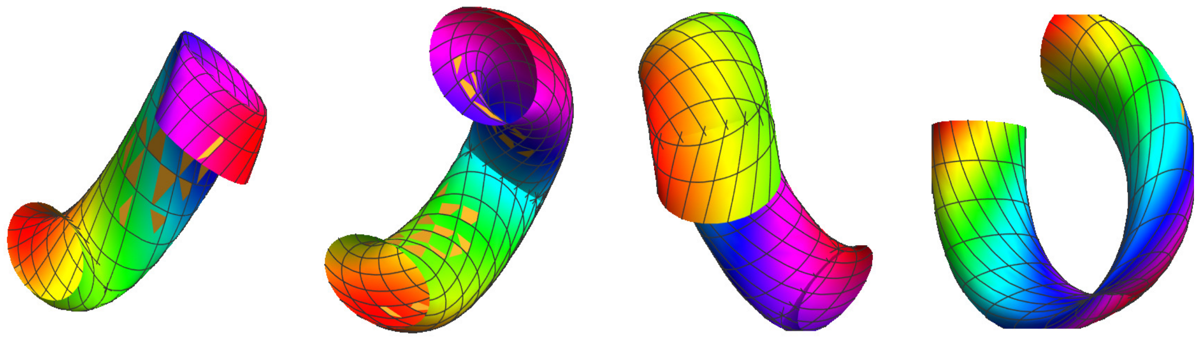
respectively. Hence using (35) in (12), we get the tubular hypersurface according to ED[1]-frame field in E^4 as

$$\mathcal{T}(s,t,v) = \begin{pmatrix} -\frac{4}{5}\rho\cos(\frac{3s}{5})\cos v\sin t + \frac{1}{2}\sin(\frac{3s}{5})\left(2 + \sqrt{2}\rho\left(\cos v\sin t + \sin v\right)\right), \\ \frac{4}{5}\rho\sin(\frac{3s}{5})\cos v\sin t + \frac{1}{2}\cos(\frac{3s}{5})\left(2 + \sqrt{2}\rho\left(\cos v\sin t + \sin v\right)\right), \\ \frac{3}{5}\rho\cos(\frac{4s}{5})\cos v\sin t + \frac{1}{2}\sin(\frac{4s}{5})\left(2 + \sqrt{2}\rho\left(-\cos v\sin t + \sin v\right)\right), \\ -\frac{3}{5}\rho\sin(\frac{4s}{5})\cos v\sin t + \frac{1}{2}\cos(\frac{4s}{5})\left(2 + \sqrt{2}\rho\left(-\cos v\sin t + \sin v\right)\right) \end{pmatrix} \tag{37}$$

and from (20), (22) and (36), we obtain the Gaussian and mean curvatures of the tubular hypersurface (37) as

$$K = -\frac{7\cos t\cos v - 25\sin v}{\rho^2\left(7\rho\cos t\cos v - 25\left(\sqrt{2}+\rho\sin v\right)\right)} \text{ and } H = \frac{100 - 3\sqrt{2}\rho\,(7\cos t\cos v - 25\sin v)}{3\rho\left(-50+\sqrt{2}\rho\,(7\cos t\cos v - 25\sin v)\right)}, \quad (38)$$

respectively. In the following figures, one can see the projections of the tubular hypersurface (37) for $v = \frac{\pi}{3}$ *and* $\rho = 3$ *into* $x_1x_2x_3$ *(A),* $x_1x_2x_4$ *(B),* $x_1x_3x_4$ *(C) and* $x_2x_3x_4$*-spaces (D).*



(a) (b) (c) (d)

Figure 1

References

[1] M. Altın; *Rotational hypersurfaces generated by weighted mean curvature in E41 with density*, Thermal Science, 26(4A), (2022), 3043-3053.
[2] H.S. Abdel-Aziz and M.K. Saad; *Computation of Smarandache curves according to Darboux frame in Minkowski 3-space*, Journal of the Egyptian Mathematical Society, 25, (2017), 382-390.
[3] M. Akyiğit, K. Eren and H.H. Kosal; Tubular Surfaces with Modified Orthogonal Frame in Euclidean 3-Space, Honam Mathematical J., 43(3), (2021), 453–463.
[4] M. Altın, A. Kazan and H.B. Karadağ; Monge Hypersurfaces in Euclidean 4-Space with Density, Journal of Polytechnic, 23(1), (2020), 207-214.
[5] M. Altın, A. Kazan, and D.W. Yoon; 2-Ruled hypersurfaces in Euclidean 4-space, Journal of Geometry and Physics, 166, (2021), 1-13.
[6] M. Altın and A. Kazan; Rotational Hypersurfaces in Lorentz-Minkowski 4-Space, Hacettepe Journal of Mathematics and Statistics, (2021), 1-25.
[7] S. Aslan and Y. Yaylı; *Canal Surfaces with Quaternions*, Adv. Appl. Clifford Algebr., 26, (2016), 31-38.
[8] M.E. Aydın and I. Mihai; On certain surfaces in the isotropic 4-space, Mathematical Communications, 22(1), (2017), 41-51.
[9] R.L. Bishop; *There is more than one way to frame a curve*, Amer. Math. Monthly, 82(3), (1975), 246-251.
[10] M.P.D. Carmo; Differential Geometry of Curves and Surfaces, Prentice Hall, Englewood Cliffs, NJ, 1976.
[11] F. Casorati; *Mesure de la courbure des surfaces suivant l'idée commune*, Acta Mathematica, 14, (1890), 95-110.
[12] F. Doğan and Y. Yaylı; *Tubes with Darboux Frame*, Int. J. Contemp.Math. Sci., 7(16), (2012), 751-758.
[13] M. Düldül, B.U. Düldül, N. Kuruoğlu and E. Özdamar; *Extension of the Darboux frame into Euclidean 4-space and its invariants*, Turk J Math., 41, (2017), 1628-1639.
[14] R. Garcia, J. Llibre and J. Sotomayor; *Lines of Principal Curvature on Canal Surfaces in* $\mathbb{R}^3$, An. Acad. Brasil. Cienc., 78(3), (2006), 405-415.
[15] H. Gluck; *Higher curvatures of curves in Euclidean space*, Amer Math Monthly, 73, (1966), 699-704.
[16] E. Güler, Helical Hypersurfaces in Minkowski Geometry E14, Symmetry, 12, (2020), 1206.
[17] E. Güler, H.H. Hacısalihoğlu and Y.H. Kim, The Gauss map and the third Laplace-Beltrami operator of the rotational hypersurface in 4-Space, Symmetry, 10(9), (2018), 1-11.
[18] A. Gray; Modern Differential Geometry of Curves and Surfaces with Mathematica, 2nd edn. CRC Press, Boca Raton, 1999.
[19] E. Hartman; Geometry and Algorithms for Computer Aided Design, Dept. of Math. Darmstadt Univ. of Technology, 2003.
[20] S. Izumiya and M .Takahashi; *On caustics of submanifolds and canal hypersurfaces in Euclidean space*, Topology Appl., 159, (2012), 501-508.
[21] M.K. Karacan, H. Es and Y. Yaylı; *Singular Points of Tubular Surfaces in Minkowski 3-Space*, Sarajevo J. Math., 2(14), (2006), 73-82.
[22] M.K. Karacan and Y. Tuncer; *Tubular Surfaces of Weingarten Types in Galilean and Pseudo-Galilean*, Bull. Math. Anal. Appl., 5(2), (2013), 87-100.

[23] M.K. Karacan, D.W. Yoon and Y. Tuncer; *Tubular Surfaces of Weingarten Types in Minkowski 3-Space,* Gen. Math. Notes, 22(1), (2014), 44-56.
[24] A.Kazan, M. Altın and D.W. Yoon; Geometric Characterizations of Canal Hypersurfaces in Euclidean Spaces, arXiv:2111.04448v1 [math.DG], (2021).
[25] A. Kazan and H.B. Karadağ; *Magnetic Curves According to Bishop Frame and Type-2 Bishop Frame in Euclidean 3-Space,* British Journal of Mathematics & Computer Science, 22(4), (2017), 1-18.
[26] A.Kazan; *Canal Hypersurfaces according to one of the Extended Darboux Frame Field in Euclidean 4-Space*, Thermal Science, 26(4A), (2022), 3029-3041.
[27] S. Kızıltuğ, M. Dede and C. Ekici; *Tubular Surfaces with Darboux Frame in Galilean 3-Space*, Facta Universitatis Ser. Math. Inform., 34(2), (2019), 253-260.
[28] Y.H. Kim, H. Liu and J. Qian; *Some Characterizations of Canal Surfaces*, Bull. Korean Math. Soc., 53(2), (2016), 461-477.
[29] S.N. Krivoshapko and C.A.B. Hyeng; *Classification of Cyclic Surfaces and Geometrical Research of Canal Surfaces*, International Journal of Research and Reviews in Applied Sciences, 12(3), (2012), 360-374.
[30] Z. Küçükarslan Yüzbaşı and D.W. Yoon; *Tubular Surfaces with Galilean Darboux Frame in* G_3, Journal of Mathematical Physics, Analysis, Geometry, 15(2), (2019), 278-287.
[31] J.M. Lee; Riemannian Manifolds-An Introduction to Curvature, Springer-Verlag New York, Inc, 1997.
[32] T. Maekawa, N.M. Patrikalakis, T. Sakkalis and G. Yu; *Analysis and Applications of Pipe Surfaces*, Comput. Aided Geom. Design, 15, (1998), 437-458.
[33] B. O'Neil; Semi-Riemannian Geometry with Applications to Relativity, Academic Press, London, 1983.
[34] M. Peternell and H. Pottmann; *Computing Rational Parametrizations of Canal Surfaces*, J. Symbolic Comput., 23, (1997), 255-266.
[35] J.S. Ro and D.W. Yoon; *Tubes of Weingarten Type in a Euclidean 3-Space*, Journal of the Chungcheong Mathematical Society, 22(3), (2009), 359-366.
[36] A. Uçum and K. İlarslan; *New Types of Canal Surfaces in Minkowski 3-Space,* Adv. Appl. Clifford Algebr., 26, (2016), 449-468.
[37] Z. Xu, R. Feng and J-G. Sun; *Analytic and Algebraic Properties of Canal Surfaces*, J. Comput. Appl. Math., 195, (2006), 220-228.

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A New Generalization of Szász-Kantorovich Operators on Weighted Space

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Abstract. The purpose of this article is to define a new generalization of Szász-Kantorovich operators. First, by using the Korovkin theorem on the new operator we define, its convergence properties and rates are examined. Then, the Voronovskaja-type theorem for the new operator is proven. Additionally, with the help of the modulus of continuity in the weighted space, rate of convergence the new operator is examined, and a theorem is proven for the operator we define by using functions that satisfy the Lipschitz condition. Finally, the convergence is demonstrated more clearly by numerical examples and plots.

1. INTRODUCTION

Linear positive operators have been studied by several mathematicians in the context of many fields of mathematics from the past to the present. In 1885, Weierstrass [18] proved the existence of a polynomial for any function in a finite interval that converges to this function within the same finite interval. However, he did not provide information on the properties of such a polynomial. In 1912, Russian mathematician Bernstein [3] defined the following operator as proof of the concept defined by Weierstrass.

$$B_b\left(j\left(i\right);s\right) =: B_b\left(j;s\right) = \sum_{l=0}^{b} j\left(\frac{l}{b}\right)\binom{b}{l}s^l\left(1-s\right)^{b-l} \tag{1}$$

here $j \in C[0,1]$, $s \in [0,1]$ and $b \in \mathbb{N}$.

After Bernstein's study, in different places and at different times, Bohman [4] in 1952 and Korovkin [11] in 1953 presented important theorems that proved the possibility of this convergence by providing only three conditions and pioneered this field regarding the convergence of positive operators to a function that is an element of $C[a,b]$ in a finite range. These theorems are generally known as the Korovkin conditions.

Afterward, studies in the field of convergence theory gained momentum, and several mathematicians [1, 2, 7, 9, 10, 12–14] conducted studies in this field.

L. V. Kantorovich [8] in 1930 and O. Szász [15] in 1950 completed their generalizations in which they defined the Bernstein operator in different spaces. These operators they defined are known by their names.

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Later, in 1985, the operator called the Szász-Kantorovich operator was studied by many mathematicians [6, 16]. The classical Szász-Kantorovich operator is as follows: Let $S_{b,f}(s) = e^{-bs}\frac{(bs)^f}{f!}$

$$W_u^j(s) = ue^{-us}\sum_{v=0}^{\infty}\left[\int_{\frac{v}{u}}^{\frac{v+1}{u}} j(s)\,ds\right]\frac{(us)^v}{v!}\ ;\ \ u>0 \tag{2}$$

In this study, we defined an operator as a generalization of classical Szász-Kantorovich operators as follows:

$$Z_b(j\,;s) = \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\int_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{(f+1)}{b}\frac{(b+c)}{(b+e)}} j(i)\,di\ ;0\le i<\infty \tag{3}$$

here $c, e \in \mathbb{R}$ and $0 \le c \le e$.

Remark 1.1. *Szász-Kantorovich type operators defined by* (3) *are linear and positive.*

2. PRELIMINARIES

This section presents the examination of the convergence characteristics of our operator (3) which is a novel generalization of classical Szász Kantorovich operators (2). Additionally, for an arbitrary $A > 0$, the uniform convergences of the operator are examined for continuous functions and functions bounded on the entire real axis in the closed compact interval $[0, A]$. The convergences and convergence rates are calculated in intervals diverging to $[0,\infty)$ and weighted spaces. The Voronovskaja-type theorem for functions that are differentiable in $[0,\infty)$ whose derivatives are in $C_p[0,\infty)$ is calculated.

To make these calculations and demonstrate that our operator satisfies the Korovkin conditions, let us calculate the $1, i, i^2, i^3$, and i^4 values of our operator. After this, with the help of these values, let us calculate its central moments.

Theorem 2.1. *The operator* (3) *satisfies the following equations for* $\forall s \in [0, A]$

$$Z_b(1\,;s) = 1$$

$$Z_b(i\,;s) = s + \frac{(c-e)}{(b+e)}s + \frac{1}{2}\frac{(b+c)}{b(b+e)}$$

$$Z_b\left(i^2\,;s\right) = s^2 + \frac{(2bc+c-2be-e)}{(b+e)^2}s^2 + 2\frac{(b+c)^2}{b(b+e)^2}s + \frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}$$

$$\begin{aligned} Z_b\left(i^3\,;s\right) &= s^3 + \frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(e+b)^3}s^3 \\ &\quad +\frac{9}{2}\frac{(b+c)^3}{b(b+e)^3}s^2 + \frac{7}{2}\frac{(b+c)^3}{b^2(b+e)^3}s + \frac{1}{4}\frac{(b+c)^3}{b^3(b+e)^3} \end{aligned}$$

$$\begin{aligned} Z_b\left(i^4\,;s\right) &= s^4 + \frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)}{(b+e)^4}s^4 \\ &\quad +8\frac{(b+c)^4}{b(b+e)^4}s^3 + 15\frac{(b+c)^4}{b^2(b+e)^4}s^2 + 6\frac{(b+c)^4}{b^3(b+e)^4}s + \frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4} \end{aligned}$$

Proof.

$$\begin{aligned} Z_b(1\,;s) &= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\int\limits_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} 1di \\ &= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\left(\frac{f+1}{b}\frac{(b+c)}{(b+e)}-\frac{f}{b}\frac{(b+c)}{(b+e)}\right) \\ &= \frac{b(b+e)}{(b+c)}.\frac{(b+c)}{b(b+e)}(f+1-f) \\ &= 1 \end{aligned}$$

$$\begin{aligned} Z_b(i\,;s) &= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\int\limits_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} idi \\ &= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\frac{1}{2}\frac{(b+c)^2}{b^2(b+e)^2}\left(\frac{(f+1)^2}{(b)^2}\frac{(b+c)^2}{(b+e)^2}-\frac{(f)^2}{(b)^2}\frac{(b+c)^2}{(b+e)^2}\right) \\ &= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\frac{1}{2}\frac{(b+c)^2}{b^2(b+e)^2}\left(f^2+2f+1-f^2\right) \\ &= \frac{b(b+e)}{(b+c)}\frac{(b+c)^2}{b^2(b+e)^2}\frac{1}{2}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}(2f+1) \\ &= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=1}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs)\frac{1}{2}2f+\frac{(b+c)}{b(b+e)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\frac{1}{2}1 \\ &= \frac{(b+c)}{b(b+e)}(bs)+\frac{1}{2}\frac{(b+c)}{b(b+e)} \\ &= \frac{(b+c)}{(b+e)}s+\frac{1}{2}\frac{(b+c)}{b(b+e)} \\ &= s+\frac{(c-e)}{(b+e)}s+\frac{1}{2}\frac{(b+c)}{b(b+e)} \end{aligned}$$

$$\begin{aligned} Z_b\left(i^2\,;s\right) &= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\int\limits_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} i^2di \\ &= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\frac{1}{3}\left(\frac{(f+1)^3}{(b)^3}\frac{(b+c)^3}{(b+e)^3}-\frac{(f)^3}{(b)^3}\frac{(b+c)^3}{(b+e)^3}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \frac{1}{3} \frac{(b+c)^3}{b^3 (b+e)^3} (f^3 + 3f^2 + 3f + 1 - f^3) \\
&= \frac{1}{3} \frac{(b+c)^2}{b^2 (b+e)^2} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} (3f^2 + 3f + 1) \qquad ; f^2 = f(f-1) + f \\
&= \frac{1}{3} \frac{(b+c)^2}{b^2 (b+e)^2} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} 3f^2 \\
&\quad + \frac{1}{3} 3 \frac{(b+c)^2}{b^2 (b+e)^2} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} f + \frac{1}{3} \frac{(b+c)^2}{b^2 (b+e)^2} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} 1 \\
&= \frac{(b+c)^2}{b^2 (b+e)^2} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} (f(f-1) + f) \\
&\quad + \frac{(b+c)^2}{b^2 (b+e)^2} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^{f-1}}{f(f-1)!} (bs) f + \frac{1}{3} \frac{(b+c)^2}{b^2 (b+e)^2} \\
&= \frac{(b+c)^2}{b^2 (b+e)^2} e^{-bs} \sum_{f=2}^{\infty} \frac{(bs)^{f-2}}{f(f-1)(f-2)!} (bs)^2 f(f-1) \\
&\quad + \frac{(b+c)^2}{b^2 (b+e)^2} e^{-bs} \sum_{f=1}^{\infty} \frac{(bs)^{f-1}}{f(f-1)!} (bs) f + \frac{(b+c)^2}{b^2 (b+e)^2} s + \frac{1}{3} \frac{(b+c)^2}{b^2 (b+e)^2} \\
&= \frac{(b+c)^2}{(b+e)^2} s^2 + \frac{(b+c)^2}{b(b+e)^2} s + \frac{(b+c)^2}{b(b+e)^2} s + \frac{1}{3} \frac{(b+c)^2}{b^2 (b+e)^2} \\
&= \frac{(b+c)^2}{(b+e)^2} s^2 + 2 \frac{(b+c)^2}{b(b+e)^2} s + \frac{1}{3} \frac{(b+c)^2}{b^2 (b+e)^2} \\
&= s^2 + \frac{(2bc + c - 2be - e)}{(b+e)^2} s^2 + 2 \frac{(b+c)^2}{b(b+e)^2} s + \frac{1}{3} \frac{(b+c)^2}{b^2 (b+e)^2}
\end{aligned}
$$

$$
\begin{aligned}
Z_b\left(i^3 ; s\right) &= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \int_{\frac{f}{b} \frac{(b+c)}{(b+e)}}^{\frac{f+1}{b} \frac{(b+c)}{(b+e)}} i^3 di \\
&= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \frac{1}{4} \left(\frac{(f+1)^4}{(b)^4} \frac{(b+c)^4}{(b+e)^4} - \frac{(f)^4}{(b)^4} \frac{(b+c)^4}{(b+e)^4} \right) \\
&= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \frac{1}{4} \frac{(b+c)^4}{b^4 (b+e)^4} ((f+1)^4 - f^4) \\
&= \frac{(b+c)^3}{b^3 (b+e)^3} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \frac{1}{4} (f^4 + 4f^3 + 6f^2 + 4f + 1 - f^4) \\
&= \frac{(b+c)^3}{b^3 (b+e)^3} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \frac{1}{4} 4f^3 + \frac{(b+c)^3}{b^3 (b+e)^3} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \frac{1}{4} 6f^2 \\
&\quad + \frac{(b+c)^3}{b^3 (b+e)^3} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \frac{1}{4} 4f + \frac{1}{4} \frac{(b+c)^3}{b^3 (b+e)^3} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} 1
\end{aligned}
$$

If the equation $f^3 = f(f-1)(f-2) + 3f(f-1) + 3f - 2f$ is substituted into the last equation, the following is obtained:

$$
\begin{aligned}
&= \frac{(b+c)^3}{b^3(b+e)^3} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \left(f(f-1)(f-2) + 3f(f-1) + 3f - 2f\right) \\
&\quad + \frac{(b+c)^3}{b^3(b+e)^3} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \frac{3}{2} \left(f(f-1) + f\right) \\
&\quad + \frac{(b+c)^3}{b^3(b+e)^3} e^{-bs} \sum_{f=1}^{\infty} \frac{(bs)^{f-1}}{f(f-1)!} (bs) f + \frac{1}{4} \frac{(b+c)^3}{b^3(b+e)^3} \\
&= \frac{(b+c)^3}{b^3(b+e)^3} e^{-bs} \sum_{f=3}^{\infty} \frac{(bs)^{f-3}}{f(f-1)(f-2)(f-3)!} (bs)^3 f(f-1)(f-2) \\
&\quad + 3\frac{(b+c)^3}{b^3(b+e)^3} e^{-bs} \sum_{f=2}^{\infty} \frac{(bs)^{f-2}}{f(f-1)(f-2)!} (bs)^2 f(f-1) \\
&\quad + 3\frac{(b+c)^3}{b^3(b+e)^3} e^{-bs} \sum_{f=1}^{\infty} \frac{(bs)^{f-1}}{f(f-1)!} (bs) f - 2\frac{(b+c)^3}{b^3(b+e)^3} e^{-bs} \sum_{f=1}^{\infty} \frac{(bs)^{f-1}}{f(f-1)!} (bs) f \\
&\quad + \frac{3}{2} \frac{(b+c)^3}{b^3(b+e)^3} e^{-bs} \sum_{f=1}^{\infty} \frac{(bs)^{f-1}}{f(f-1)!} (bs) f \\
&\quad + \frac{3}{2} \frac{(b+c)^3}{b^3(b+e)^3} e^{-bs} \sum_{f=2}^{\infty} \frac{(bs)^{f-2}}{f(f-1)(f-2)!} (bs)^2 f(f-1) \\
&\quad + \frac{(b+c)^3}{b^3(b+e)^3} s + \frac{1}{4} \frac{(b+c)^3}{b^3(b+e)^3} \\
&= \frac{(b+c)^3}{(b+e)^3} s^3 + \frac{9}{2} \frac{(b+c)^3}{b(b+e)^3} s^2 + \frac{7}{2} \frac{(b+c)^3}{b^2(b+e)^3} s + \frac{1}{4} \frac{(b+c)^3}{b^3(b+e)^3} \\
&= s^3 + \frac{\left(3bc^2 + 3b^2c + c^3 - 3be^2 - 3b^2e - e^3\right)}{(n+b)^3} s^3 \\
&\quad + \frac{9}{2} \frac{(b+c)^3}{b(b+e)^3} s^2 + \frac{7}{2} \frac{(b+c)^3}{b^2(b+e)^3} s + \frac{1}{4} \frac{(b+c)^3}{b^3(b+e)^3}
\end{aligned}
$$

$$
\begin{aligned}
Z_b\left(i^4; s\right) &= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \int_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} i^4 di \\
&= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \frac{1}{5} \left(\frac{(f+1)^5}{(b)^5} \frac{(b+c)^5}{(b+e)^5} - \frac{(f)^5}{(b)^5} \frac{(b+c)^5}{(b+e)^5} \right) \\
&= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \frac{1}{5} \frac{(b+c)^5}{b^5(b+e)^5} (f^5 + 5f^4 + 5f + 10f^2 + 10f^3 + 1 - f^5)
\end{aligned}
$$

If the equations

$$f^3 = f(f-1)(f-2) + 3f(f-1) + 3f - 2f$$

and

$$f^4 = f(f-1)(f-2)(f-3) + 6f^3 + 11f^2 + 6f$$

are entered into the last equation, the following is obtained:

$$\begin{aligned} = \ & \frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\left(5\left(f(f-1)(f-2)(f-3)+6f^3+11f^2+6f\right)\right) \\ & +10\left(f(f-1)(f-2)+3f(f-1)+3f-2f\right)+10(f(f-1)+f)+5f+1 \end{aligned}$$

Here, the sum can be calculated in two parts for easy operation:

$$\begin{aligned} = \ & \frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\left(5\left(f(f-1)(f-2)(f-3)+6f^3+11f^2+6f\right)\right) \\ & +10\left(f(f-1)(f-2)+3f(f-1)+3f-2f\right)+10(f(f-1)+f)+5f+1 \end{aligned}$$

$$\begin{aligned} I_1 = \ & \frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=4}^{\infty}\frac{(bs)^{f-4}}{f(f-1)(f-2)(f-3)(f-4)!} \\ & f(f-1)(f-2)(f-3)(bs)^4 \\ & +6\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=3}^{\infty}\frac{(bs)^{f-3}}{f(f-1)(f-2)(f-3)!}f(f-1)(f-2)(bs)^3 \\ & +18\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=2}^{\infty}\frac{(bs)^{f-3}}{f(f-1)(f-2)!}f(f-1)(bs)^2 \\ & +18\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=3}^{\infty}\frac{(bs)^{f-3}}{f(f-1)!}f(bs) - 12\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=3}^{\infty}\frac{(bs)^{f-3}}{f(f-1)!}f(bs) \\ & +11\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=2}^{\infty}\frac{(bs)^{f-3}}{f(f-1)(f-2)!}f(f-1)(bs)^2 \\ & +11\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=3}^{\infty}\frac{(bs)^{f-3}}{f(f-1)!}f(bs) + 6\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=3}^{\infty}\frac{(bs)^{f-3}}{f(f-1)!}f(bs) \\ = \ & \frac{(b+c)^4}{(b+e)^4}s^2 + 6\frac{(b+c)^4}{b(b+e)^4}s^3 + 18\frac{(b+c)^4}{b^2(b+e)^4}s^2 + 18\frac{(b+c)^4}{b^3(b+e)^4}s \\ & -12\frac{(b+c)^4}{b^3(b+e)^4}s + 11\frac{(b+c)^4}{b^2(b+e)^4}s^2 + 11\frac{(b+c)^4}{b^3(b+e)^4}s + 6\frac{(b+c)^4}{b^3(b+e)^4}s \\ = \ & \frac{(b+c)^4}{(b+e)^4}s^2 + 6\frac{(b+c)^4}{b(b+e)^4}s^3 + 29\frac{(b+c)^4}{b^2(b+e)^4}s^2 + 23\frac{(b+c)^4}{b^3(b+e)^4}s \end{aligned}$$

and

$$\begin{aligned}
I_2 &= \frac{10}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=3}^{\infty}\frac{(bs)^{f-3}}{f(f-1)(f-2)(f-3)!}(bs)^3 f(f-1)(f-2) \\
&+\frac{30}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=2}^{\infty}\frac{(bs)^{f-2}}{f(f-1)(f-2)!}(bs)^2 f(f-1) \\
&+\frac{30}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=1}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs) f \\
&-\frac{20}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=1}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs) f \\
&+\frac{10}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=2}^{\infty}\frac{(bs)^{f-2}}{f(f-1)(f-2)!}(bs)^2 f(f-1) \\
&+\frac{10}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=1}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs) f \\
&+\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=1}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs) f+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^{f}}{f!} \\
&= 2\frac{(b+c)^4}{b(b+e)^4}s^3+6\frac{(b+c)^4}{b^2(b+e)^4}s^2+6\frac{(b+c)^4}{b^3(b+e)^4}s-4\frac{(b+c)^4}{b^3(b+e)^4}s \\
&+2\frac{(b+c)^4}{b^2(b+e)^4}s^2+2\frac{(b+c)^4}{b^3(b+e)^4}s+\frac{(b+c)^4}{b^3(b+e)^4}s+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4} \\
&= 2\frac{(b+c)^4}{b(b+e)^4}s^3+8\frac{(b+c)^4}{b^2(b+e)^4}s^2+5\frac{(b+c)^4}{b^3(b+e)^4}s+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}
\end{aligned}$$

The following is obtained from I_1 and I_2

$$\begin{aligned}
&= \frac{(b+c)^4}{(b+e)^4}s^4+8\frac{(b+c)^4}{b(b+e)^4}s^3+15\frac{(b+c)^4}{b^2(b+e)^4}s^2+6\frac{(b+c)^4}{b^3(b+e)^4}s+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4} \\
Z_b\left(i^4\,;s\right) &= s^4+\frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)}{(n+b)^4}s^4 \\
&+8\frac{(b+c)^4}{b(b+e)^4}s^3+15\frac{(b+c)^4}{b^2(b+e)^4}s^2+6\frac{(b+c)^4}{b^3(b+e)^4}x+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}
\end{aligned}$$

QED. □

Theorem 2.2. *Some of the central moments for our new Szász-Kantorovich operator* (3) *are as follows:*

$$Z_b\left((i-s)^0\,;s\right)=1$$

$$Z_b\left((i-s)\ ;\ s\right)=\frac{(c-e)}{(b+e)}s+\frac{1}{2b}\frac{(b+c)}{(b+e)}$$

$$
\begin{aligned}
Z_b\left((i-s)^2\ ;\ s\right) &= \left(\frac{(2bc+c-2be-e)}{(b+e)^2}-2\frac{(c-e)}{(b+e)}\right)s^2 \\
&+\left(2\frac{(b+c)^2}{b\,(b+e)^2}-\frac{(b+c)}{b\,(b+e)}\right)s+\frac{1}{3}\frac{(b+c)^2}{b^2\,(b+e)^2}
\end{aligned}
$$

$$
\begin{aligned}
Z_b\left((i-s)^3\ ;\ s\right) &= \Bigg(\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(b+e)^3} \\
&-3\frac{(2bc+c-2be-e)}{(b+e)^2}+3\frac{(c-e)}{(b+e)}\Bigg)s^3 \\
&+\left(4\frac{(b+c)^3}{b\,(b+e)^3}-6\frac{(b+c)^2}{b\,(b+e)^2}+\frac{3}{2}\frac{(b+c)}{b\,(b+e)}\right)s^2+ \\
&\left(\frac{7}{2}\frac{(b+c)^3}{b^2\,(b+e)^3}-\frac{(b+c)^2}{b^2\,(b+e)^2}\right)s+\frac{1}{4}\frac{(b+c)^3}{b^3\,(b+e)^3}
\end{aligned}
$$

$$
\begin{aligned}
Z_n\left((t-x)^4\ ;\ x\right) &= \Bigg(\frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)}{(b+e)^4} \\
&-\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(n+b)^3} \\
&+6\frac{(2bc+c-2be-e)}{(b+e)^2}-4\frac{(c-e)}{(b+e)}\Bigg)s^4 \\
&+\left(8\frac{(b+c)^4}{b\,(b+e)^4}-16\frac{(b+c)^3}{b\,(b+e)^3}+12\frac{(b+c)^2}{b\,(b+e)^2}-2\frac{(b+c)}{b\,(b+e)}\right)s^3 \\
&+\left(15\frac{(b+c)^4}{b^2\,(b+e)^4}-14\frac{(b+c)^3}{b^2\,(b+e)^3}+2\frac{(b+c)^2}{b^2\,(b+e)^2}\right)s^2 \\
&+\left(6\frac{(b+c)^4}{b^3\,(b+e)^4}-\frac{(b+c)^3}{b^3\,(b+e)^3}\right)s+\frac{1}{5}\frac{(b+c)^4}{b^4\,(b+e)^4}
\end{aligned}
$$

Proof.

$$
Z_b\left((i-s)^0\,;s\right)=Z_b\,(1;s)=1
$$

it is clear that.

$$
\begin{aligned}
Z_b\,((i-s)\ ;\ s) &= Z_b\,(i;s)-sZ_b\,(1;s) \\
&= s+\frac{(c-e)}{(b+e)}s+\frac{1}{2}\frac{(b+c)}{b\,(b+e)}-s.1 \\
&= \frac{(c-e)}{(b+e)}s+\frac{1}{2b}\frac{(b+c)}{(b+e)}
\end{aligned}
$$

$$\begin{aligned}
Z_b\left((i-s)^2\ ;\ s\right) &= Z_b\left(i^2-2si+s^2\ ;s\right)\\
&= Z_b\left(i^2;s\right)-2sZ_b\left(i;s\right)+s^2Z_b\left(1;s\right)\\
&= s^2+\frac{(2bc+c-2be-e)}{(b+e)^2}s^2+2\frac{(b+c)^2}{b\left(b+e\right)^2}s\\
&\quad +\frac{1}{3}\frac{(b+c)^2}{b^2\left(b+e\right)^2}-2s\left(s+\frac{(c-e)}{(b+e)}s+\frac{1}{2b}\frac{(b+c)}{(b+e)}\right)+s^2.1\\
&= \left(2+\frac{(2bc+c-2be-e)}{(b+e)^2}-2\frac{(c-e)}{(b+e)}-2\right)s^2\\
&\quad +\left(2\frac{(b+c)^2}{b\left(b+e\right)^2}-\frac{(b+c)}{b\left(b+e\right)}\right)s+\frac{1}{3}\frac{(b+c)^2}{b^2\left(b+e\right)^2}\\
&= \left(\frac{(2bc+c-2be-e)}{(b+e)^2}-2\frac{(c-e)}{(b+e)}\right)s^2\\
&\quad +\left(2\frac{(b+c)^2}{b\left(b+e\right)^2}-\frac{(b+c)}{b\left(b+e\right)}\right)s+\frac{1}{3}\ \frac{(b+c)^2}{b^2\left(b+e\right)^2}
\end{aligned}$$

$$\begin{aligned}
Z_b\left((i-s)^3\ ;\ s\right) &= Z_b\left(i^3-3si^2+3is^2-s^3;s\right)\\
&= Z_b\left(i^3;s\right)-3s\ Z_b\left(i^2;s\right)+3s^2Z_b\left(i;s\right)-s^3Z_b\left(1;s\right)\\
&= s^3+\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(b+e)^3}s^3\\
&\quad +\frac{9}{2}\ \frac{(b+c)^3}{b\left(b+e\right)^3}s^2+\frac{7}{2}\frac{(b+c)^3}{b^2\left(b+e\right)^3}s+\frac{1}{4}\frac{(b+c)^3}{b^3\left(b+e\right)^3}\\
&\quad -3s\left(s^2+\frac{(2bc+c-2be-e)}{(b+e)^2}s^2+2\frac{(b+c)^2}{b\left(b+e\right)^2}s+\frac{1}{3}\frac{(b+c)^2}{b^2\left(b+e\right)^2}\right)\\
&\quad +3s^2\left(s+\frac{(c-e)}{(b+e)}s+\frac{1}{2b}\frac{(b+c)}{(b+e)}\right)-s^3\\
&= s^3+\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(b+e)^3}s^3\\
&\quad +\frac{9}{2}\ \frac{(b+c)^3}{b\left(b+e\right)^3}s^2+\frac{7}{2}\frac{(b+c)^3}{b^2\left(b+e\right)^3}s+\frac{1}{4}\frac{(b+c)^3}{b^3\left(b+e\right)^3}\\
&\quad -3s^3-3\frac{(2bc+c-2be-e)}{(b+e)^2}s^3-6\frac{(b+c)^2}{b\left(b+e\right)^2}s^2\\
&\quad -\frac{(b+c)^2}{b^2\left(b+e\right)^2}s+3s^3+3\frac{(c-e)}{(b+e)}s^3+\frac{3}{2}\frac{(b+c)}{b\left(b+e\right)}s^2-s^3\\
&= \left(\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(b+e)^3}\right.\\
&\quad \left.-3\frac{(2bc+c-2be-e)}{(b+e)^2}+\ 3\frac{(c-e)}{(b+e)}\right)\ s^3
\end{aligned}$$

$$+\left(\frac{9}{2}\frac{(b+c)^3}{b(b+e)^3}-6\frac{(b+c)^2}{b(b+e)^2}+\frac{3}{2}\frac{(b+c)}{b(b+e)}\right)s^2$$
$$+\left(\frac{7}{2}\frac{(b+c)^3}{b^2(b+e)^3}-\frac{(b+c)^2}{b^2(b+e)^2}\right)s+\frac{1}{4}\frac{(b+c)^3}{b^3(b+e)^3}$$

$$\begin{aligned}
Z_b\left((i-s)^4\ ;\ s\right) &= Z_b\left(i^4-4si^3+6s^2i^2-4s^3i+s^4\ ;s\right)\\
&= Z_b\left(i^4\ ;s\right)-4sZ_b\left(i^3\ ;s\right)+6s^2Z_b\left(i^2\ ;e\right)-4s^3Z_b\left(i\ ;s\right)+s^4Z_b\left(1\ ;s\right)\\
&= s^4+\frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)}{(n+b)^4}s^4\\
&\quad +8\frac{(b+c)^4}{b(b+e)^4}s^3+15\frac{(b+c)^4}{b^2(b+e)^4}s^2+6\frac{(b+c)^4}{b^3(b+e)^4}s+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}\\
&\quad -4s\left(s^3+\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(n+b)^3}s^3\right.\\
&\quad \left.\frac{9}{2}\ \frac{(b+c)^3}{b(b+e)^3}s^2+\frac{7}{2}\frac{(b+c)^3}{b^2(b+e)^3}s+\frac{1}{4}\frac{(b+c)^3}{b^3(b+e)^3}\right)\\
&\quad +6s^2\left(s^2+\frac{(2bc+c-2be-e)}{(b+e)^2}s^2+2\frac{(b+c)^2}{b(b+e)^2}s+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\right)\\
&\quad -4s^3\left(s+\frac{(c-e)}{(b+e)}s+\frac{1}{2b}\frac{(b+c)}{(b+e)}\right)+s^4\\
&= s^4+\frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)}{(n+b)^4}s^4\\
&\quad +8\frac{(b+c)^4}{b(b+e)^4}s^3+15\frac{(b+c)^4}{b^2(b+e)^4}s^2+6\frac{(b+c)^4}{b^3(b+e)^4}s+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}\\
&\quad -4s^4-4\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(n+b)^3}s^4-18\frac{(b+c)^3}{b(b+e)^3}s^3\\
&\quad -14\frac{(b+c)^3}{b^2(b+e)^3}s^2-\frac{(b+c)^3}{b^3(b+e)^3}s+6s^4+6\frac{(2bc+c-2be-e)}{(b+e)^2}s^4\\
&\quad +12\frac{(b+c)^2}{b(b+e)^2}s^3+2\frac{(b+c)^2}{b^2(b+e)^2}s^2-4s^4-4\frac{(c-e)}{(b+e)}s^4-2\frac{(b+c)}{b(b+e)}s^3+s^4\\
&= \left(\frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)}{(b+e)^4}\right.\\
&\quad -\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(n+b)^3}\\
&\quad \left.+6\frac{(2bc+c-2be-e)}{(b+e)^2}-\ 4\frac{(c-e)}{(b+e)}\right)s^4\\
&\quad +\left(8\frac{(b+c)^4}{b(b+e)^4}-16\frac{(b+c)^3}{b(b+e)^3}+12\frac{(b+c)^2}{b(b+e)^2}-2\frac{(b+c)}{b(b+e)}\right)s^3
\end{aligned}$$

$$+\left(15\frac{(b+c)^4}{b^2(b+e)^4}-14\frac{(b+c)^3}{b^2(b+e)^3}+2\frac{(b+c)^2}{b^2(b+e)^2}\right)s^2$$
$$+\left(6\frac{(b+c)^4}{b^3(b+e)^4}-\frac{(b+c)^3}{b^3(b+e)^3}\right)s \ +\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}$$

QED. □

Theorem 2.3. *Let $A>0$, and $j\in C[0,A]$ be bounded on the entire real axis. In this case, the following is obtained:*

$$\lim_{b\to\infty}\left\|Z_b j-j\right\|_{C[0,A]}=0$$

Proof. If we use the Korovkin theorem, for $b\to\infty$, it is sufficient to demonstrate the following:

i) $Z_b(1\,;s)\rightrightarrows 1$

ii) $Z_b(i\,;s)\rightrightarrows s$

iii) $Z_b\left(i^2\,;s\right)\rightrightarrows s^2$

The following is clear, $\|Z_b(1\,;s)-1\|_{C[0,A]}=0$. Thus,$Z_b(1\,;s)\rightrightarrows 1$ is obtained.

$$\begin{aligned}\|Z_b(i\,;s)-s\|_{C[0,A]} &= \max_{0\le s\le A}|Z_n(i\,;s)-s| \\ &= \max_{0\le s\le A}\left|s+\frac{(c-e)}{(b+e)}s+\frac{1}{2b}\frac{(b+c)}{(b+e)}-s\right| \\ &= \left|\frac{(c-e)}{(b+e)}A+\frac{1}{2b}\frac{(b+c)}{(b+e)}\right| \ \textit{buradan} \\ &= \left|\frac{2b(c-e)}{2b(b+e)}A+\frac{(b+c)}{2b(b+e)}\right|\end{aligned}$$

We obtain $\lim_{b\to\infty}\left|\frac{2b(c-e)}{2b(b+e)}A+\frac{(b+c)}{2b(b+e)}\right|\to 0$, which shows that $Z_b(i\,;s)\rightrightarrows s$.

$$\begin{aligned}\left\|Z_b\left(i^2\,;s\right)-s^2\right\|_{C[0,A]} &= \max_{0\le s\le A}\left|Z_b\left(i^2\,;s\right)-s^2\right| \\ &= \max_{0\le s\le A}\left|s^2+\frac{(2bc+c-2be-e)}{(b+e)^2}s^2+2\frac{(b+c)^2}{b(b+e)^2}s+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}-s^2\right| \\ &= \max_{0\le s\le A}\left|\frac{(2bc+c-2be-e)}{(b+e)^2}s^2+2\frac{(b+c)^2}{b(b+e)^2}s+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\right| \\ &\le \left|\frac{(2bc+c-2be-e)}{(b+e)^2}A^2+2\frac{(b+c)^2}{b(b+e)^2}A+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\right|\end{aligned}$$

Since it is $\lim_{n\to\infty}\left|\frac{(2bc+c-2be-e)}{(b+e)^2}A^2+\frac{(b+c)^2}{(b+e)^2}\left(\frac{2}{b}A+\frac{1}{3}\frac{1}{b^2}\right)\right|\to 0$, $Z_b\left(i^2\,;s\right)\rightrightarrows s^2$ is obtained. QED. □

Let us now examine convergence in weighted spaces.

Theorem 2.4. *If $j\in C_\rho^0[0,\infty)$, then*

$$\lim_{b\to\infty}\left\|Z_b j-j\right\|_{\rho,[0,\infty)}=0.$$

Proof. Because

$$\left\|Z_b j - j\right\|_{\rho,[0,\infty)} = \sup_{s\in[0,\infty)} \frac{\left|Z_b\left(j\,;s\right) - j\left(s\right)\right|}{1+s^2}$$

and based on the properties of the modulus of continuity in weighted spaces, we obtain the following:

$$\lim_{b\to\infty} \|Z_b 1 - 1\|_{\rho,[0,\infty)} = \lim_{b\to\infty} \sup_{s\in[0,\infty)} \frac{|Z_b\left(1\,;s\right)-1|}{1+s^2} = 0$$

Using the results we obtained in the central moments and the norm definition in weighted spaces, the following can be expressed

$$\begin{aligned}
\|Z_b i - s\|_{\rho,[0,\infty)} &= \sup_{s\in[0,\infty)} \frac{\left|s + \frac{(c-e)}{(b+e)}s + \frac{1}{2b}\frac{(b+c)}{(b+e)} - s\right|}{1+x^2} \\
&\leq \frac{2b\left(c-e\right)}{2b\left(b+e\right)} + \frac{\left(b+c\right)}{2b\left(b+e\right)} \\
\lim_{b\to\infty} \|Z_b i - s\|_{\rho,[0,\infty)} &\leq \lim_{b\to\infty}\left(\frac{2b\left(c-e\right)}{2b\left(b+e\right)} + \frac{\left(b+c\right)}{2b\left(b+e\right)}\right) \to \infty.
\end{aligned}$$

Likewise, because

$$\begin{aligned}
\left\|Z_b i^2 - s^2\right\|_{\rho,[0,\infty)} &= \sup_{s\in[0,\infty)} \frac{\left|s^2 + \frac{(2bc+c-2be-e)}{(b+e)^2}s^2 + 2\frac{(b+c)^2}{b(b+e)^2}s + \frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2} - s^2\right|}{1+s^2} \\
&= \sup_{s\in[0,\infty)} \frac{\left|\frac{3b^2(2bc+c-2be-e)+6b(b+c)^2+(b+c)^2}{3b^2(b+e)^2}\right|}{1+s^2}
\end{aligned}$$

If $0 \leq \frac{s^2}{1+s^2} \leq 1$ based on the above, the following is obtained:

$$\begin{aligned}
&\leq \frac{3b^2\left(2bc+c-2be-e\right)+6b\left(b+c\right)^2+\left(b+c\right)^2}{3b^2\left(b+e\right)^2} \\
\lim_{b\to\infty}\left\|Z_b i^2 - s^2\right\|_{\rho,[0,\infty)} &= \lim_{b\to\infty}\left(\frac{3b^2\left(2bc+c-2be-e\right)+6b\left(b+c\right)^2+\left(b+c\right)^2}{3b^2\left(b+e\right)^2}\right) \to \infty
\end{aligned}$$

Therefore,

$$\lim_{b\to\infty}\left\|Z_b j - j\right\|_{\rho} = 0$$

QED. □

Let as examine rate of convergence the operator in weighted spaces.

Theorem 2.5. *If $j \in C_\rho^0\left[0,\infty\right)$, then*

$$\left\|Z_b j - j\right\|_{\rho,[0,\infty)} \leq M\Omega\left(j\,;\sqrt{\frac{1}{b}}\right).$$

Here, $M = 808$.

Proof. Because our operator is linear and monotone, the following can be written:

$$\left|Z_b\left(j\,;s\right) - j\left(s\right)\right| \leq Z_b\left(\left|j\left(i\right) - j\left(s\right)\right|\,;s\right)$$

Using the properties of the modulus of continuity in weighted spaces, the following is obtained:

$$\begin{aligned}\left|j(i)-j(s)\right| &\leq 2\left(1+\eta_b^2\right)\left(1+s^2\right)\left(1+\frac{|i-s|}{\eta_b}\right)\left(1+(i-s)^2\right)\Omega\left(j\,;\eta_b\right)\\ \left|j(i)-j(s)\right| &\leq 2\left(1+\eta_b^2\right)\left(1+s^2\right)\Omega\left(j\,;\eta_b\right).S_b\left(i\,;s\right)\\ S_b\left(i\,;s\right) &= \left(1+\frac{|i-s|}{\eta_b}\right)\left(1+(i-s)^2\right)\end{aligned}$$

From here,

$$S_b\left(i\,;s\right)\leq\left\{\begin{array}{ll}2\left(1+\eta_b^2\right) & |i-s|<\eta_b\\ 2\left(1+\eta_b^2\right)\frac{|i-s|^4}{\eta_b^4} & ;\ |i-s|\geq\eta_b\end{array}\right\}$$

and thus, the following can be written:

$$S_b\left(i\,;s\right)\leq 2\left(1+\eta_b^2\right)\left[1+\frac{|i-s|^4}{\eta_b^4}\right]$$

Based on the above, the following is obtained:

$$\begin{aligned}\left|Z_b\left(j\,;s\right)-j(s)\right| &\leq Z_b\left(\left|j(i)-j(s)\right|\,;s\right)\\ &\leq 2\left(1+\eta_b^2\right)\left(1+s^2\right)\Omega\left(j\,;\eta_b\right)Z_b\left(S_b\left(i\,;s\right);s\right)\\ &\leq 4\left(1+\eta_b^2\right)\Omega\left(j\,;\eta_b\right)\left(1+s^2\right)\left[1+\frac{1}{\eta_b^4}Z_b\left(i-s\right)^4;s\right]\\ &\leq \left(\frac{\left(4na^3+4n^3a+6n^2a^2+a^4-4nb^3-4n^3b+6n^2b^2+b^4\right)}{(n+b)^4}\right.\\ &\quad\left.+6\frac{(2na+a-2nb-b)}{(n+b)^2}\right)s^4+\left(8\frac{(n+a)^4}{n\left(n+b\right)^4}+12\frac{(n+a)^2}{n\left(n+b\right)^2}\right)s^3\\ &\quad+\left(37\frac{(n+a)^4}{n^2\left(n+b\right)^4}+2\frac{(n+a)^2}{n^2\left(n+b\right)^2}\right)s^2\\ &\quad+\left(28\frac{(n+a)^4}{n^3\left(n+b\right)^4}\right)x+\left(\frac{(n+a)^4}{5n^4\left(n+b\right)^4}\right)\end{aligned}$$

Consequently, we get

$$\begin{aligned}Z_b\left((i-s)^4;s\right) &\leq \left(\frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)+6\left(b+e\right)^2\left(2bc+c-2be-e\right)}{(n+b)^4}\right)s^4\\ &+\left(\frac{8\left(b+c\right)^4+12\left(b+e\right)^2\left(b+c\right)^2}{b\left(b+e\right)^4}\right)s^3+\left(\frac{15\left(b+c\right)^4+2\left(b+e\right)^2\left(b+c\right)^2}{b^2\left(b+e\right)^4}\right)s^2\\ &+\left(\frac{6\left(b+c\right)^4}{b^3\left(b+e\right)^4}\right)s+\left(\frac{(b+c)^4}{5b^4\left(b+e\right)^4}\right)\end{aligned}$$

If the suprema of both sides in $[0,\infty)$ are taken, and if the negative operations are removed,

$$\begin{aligned}\sup_{s\in[0,\infty)} Z_b\left((i-s)^4;s\right) &\leq \frac{\left(5b^4\left(4bc^3+4b^3c+6b^2c^2+c^4+6b^2e^2+e^4\right)+30b^4(b+e)^2(2bc+c)\right)}{5b^4(b+e)^4}\\ &+ \frac{\left(5b^4\left(40c^3(b+c)^4+60(b+e)^2(b+c)^2+185b^2(b+c)^4\right)\right)}{5b^4(b+e)^4}\\ &+ \frac{\left(5b^4\left(^2(b+e)^2(b+c)^2+140b(b+c)^4+(b+c)^4\right)\right)}{5b^4(b+e)^4}\end{aligned}$$

Because $c \leq e$ if c is replaced with e, the following is found:

$$\leq \quad 100\frac{b^3}{(b+e)^4} \leq 100\frac{1}{b}$$

Hence,

$$\sup_{s\in[0,\infty)} \frac{\left|Z_b(j;s)-j(s)\right|}{1+s^2} \leq 4\left(1+\eta_b^2\right)\Omega(j;\eta_b)\left[1+\frac{100}{\eta_b^4}\frac{1}{b}\right]$$

Is obtained. If $\eta_b = \sqrt{\frac{1}{b}}$ and because $\eta_b \to 0$,$\eta_b < 1$ is found after a certain b. Therefore, when $M = 808$, the following is obtained:

$$\left\|Z_b j - j\right\|_{\rho,[0,\infty)} \leq M\Omega\left(j;\sqrt{\frac{1}{b}}\right)$$

QED. □

Theorem 2.6. *Let j be a function that is differentiable in $[0,\infty)$ and $j' \in C_\rho^0[0,\infty)$, the following inequality is provided:*

$$\left\|Z_b j - j\right\|_{\rho,[0,\infty)} \leq L\sqrt{\frac{2}{b}}\Omega\left(j';\sqrt{\frac{2}{b}}\right).$$

Proof. Because j is a function that is differentiable in $[0,\infty)$ and $j' \in C_\rho^0[0,\infty)$, based on the mean value theorem, there is a u between i and s such that

$$j'(u) = \frac{j(i)-j(s)}{i-s}$$

As the equation will not change if we add $-j'(s)+j'(s)$ to the left-hand side, the following can be written:

$$j(i)-j(s) = (i-s)j'(u) + (i-s)\left[j'(u)-j'(s)\right] \tag{1}$$

When $|u-s| \leq |i-s|$, $\Omega(j;|u-s|) \leq \Omega(j;|i-s|)$ Therefore, the following inequality is true:

$$\begin{aligned}\left|j'(u)-j'(s)\right| &\leq 2\left(1+\eta_b^2\right)\left(1+s^2\right)\left(1+\frac{|u-s|}{\eta_b}\right)\left(1+(u-s)^2\right)\Omega(j';\eta_b)\\ &\leq 2\left(1+\eta_b^2\right)\left(1+s^2\right)\left(1+\frac{|i-s|}{\eta_b}\right)\left(1+(i-s)^2\right)\Omega(j';\eta_b)\\ &= 2\left(1+\eta_b^2\right)\left(1+s^2\right)\left[1+\frac{|i-s|}{\eta_b}\right.\\ &\quad\left.+(i-s)^2+\frac{|i-s|(i-s)^2}{\eta_b}\right]\Omega(j';\eta_b)\end{aligned}$$

Let us calculate the multiplication below in equation (1).

The following result is obtained:

$$\begin{aligned} |i-s| \; \left|j'(u)-j'(s)\right| &\leq 2\left(1+\eta_b^2\right)\left(1+s^2\right)[|i-s| \\ &+\frac{(i-s)^2}{\eta_b}+|i-s|(i-s)^2+\frac{(i-s)^4}{\eta_b}\bigg]\Omega\left(j';\eta_b\right) \end{aligned} \tag{2}$$

If we apply the operator to equation (1),

$$Z_b(j;s)-j(s)=Z_b((i-s);s)\,j'(s)+Z_b((i-s)[j'(u)-j'(s)];s)$$

From here, the following is obtained:

$$\left|Z_b(j;s)-j(s)\right| \leq Z_b(|i-s|;s)\left|j'(s)\right|+Z_b(|i-s|[j'(u)-j'(s)];s)$$

Here, let $Z_b(|i-s|;s)\left|j'(s)\right| = I_1$ and $Z_b(|i-s|[j'(u)-j'(s)];s) = I_2$.

Then,

$$\left|Z_b(j;s)-j(s)\right| \leq I_1+I_2$$

Using the Cauchy-Schwarz inequality, the following can be written:

$$I_1 = Z_b(|i-s|;s)\left|j'(s)\right| \leq \sqrt{Z_b\left((i-s)^2;s\right)}\left|j'(s)\right| \leq \sqrt{A_b(b)}M_{j'}\left(1+s^2\right)$$

Here, $M_{j'}$ is a constant dependent on $j'-ne$ and

$$\begin{aligned} A_b(s) &= \left(\left(\frac{(2bc+c-2be-e)}{(b+e)^2}-2\frac{(c-e)}{(b+e)}\right)s^2\right. \\ &\left.+\left(2\frac{(b+c)^2}{b(b+e)^2}-\frac{(b+e)}{b(b+e)}\right)x+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\right). \end{aligned}$$

From (2), the following can be written:

$$\begin{aligned} I_2 &= Z_b\left(|i-s| \; \left|f'(u)-j'(s)\right|;s\right) \\ &\leq 2\left(1+\eta_b^2\right)\left(1+s^2\right)\left(\sqrt{Z_b\left((i-s)^2;s\right)}+\frac{1}{\eta_b}Z_b\left((i-s)^2;s\right)\right. \\ &\left.+\sqrt{Z_b\left((i-s)^2;s\right)}\sqrt{Z_b\left((i-s)^4;s\right)}+\frac{1}{\eta_b}Z_b((i-s)4;s)\right)\Omega\left(j';\eta_b\right) \end{aligned}$$

Here, let $Z_b\left((i-s)^4;s\right)=B_b(s)$; then consequently,

$$
\begin{aligned}
B_b(s) &= \Bigg(\frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)}{(b+e)^4} \\
&-\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(n+b)^3} \\
&+6\frac{(2bc+c-2be-e)}{(b+e)^2}-4\frac{(c-e)}{(b+e)}\Bigg)s^4 \\
&+\left(8\frac{(b+c)^4}{b(b+e)^4}-16\frac{(b+c)^3}{b(b+e)^3}+12\frac{(b+c)^2}{b(b+e)^2}-2\frac{(b+c)}{b(b+e)}\right)s^3 \\
&+\left(15\frac{(b+c)^4}{b^2(b+e)^4}-14\frac{(b+c)^3}{b^2(b+e)^3}+2\frac{(b+c)^2}{b^2(b+e)^2}\right)s^2 \\
&+\left(6\frac{(b+c)^4}{b^3(b+e)^4}-\frac{(b+c)^3}{b^3(b+e)^3}\right)s+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}
\end{aligned}
$$

because

$$
\begin{aligned}
&I_1+I_2 \leq \\
&\sqrt{A_b(s)}M_{j'}\left(1+s^2\right)+2\left(1+\eta_b^2\right)\left(1+s^2\right)\left[\sqrt{A_b(s)}+\frac{1}{\eta_b}A_b(s)+\sqrt{A_b(s)}\sqrt{B_b(s)}+\frac{1}{\eta_b}B_b(s)\right]\Omega\left(j';\eta_b\right)
\end{aligned}
$$

The following is written:

$$
\begin{aligned}
\frac{\left|Z_b(j;s)-j(s)\right|}{1+s^2} &\leq \sqrt{A_b(s)}M_{j'}+2\left(1+\eta_b^2\right)\Bigg[\sqrt{A_b(s)}+\frac{1}{\eta_b}A_b(s) \\
&+\sqrt{A_b(s)}\sqrt{B_b(s)}+\frac{1}{\eta_b}B_b(s)\Bigg]\Omega\left(j';\eta_b\right)
\end{aligned}
$$

If the supremum of each side in $[0,\infty)$ is taken, the following inequality is obtained:

$$
\begin{aligned}
\sup_{s\in[0,\infty)}\frac{\left|Z_b(j;x)-j(s)\right|}{1+s^2} &\leq \sup_{s\in[0,\infty)}\sqrt{A_b(s)}M_{j'}+2\left(1+\eta_b^2\right)\Bigg[\sqrt{A_b(s)} \\
&+\frac{1}{\eta_b}A_b(s)+\sqrt{A_b(s)}\sqrt{B_b(s)}+\frac{1}{\eta_b}B_b(s)\Bigg]\Omega\left(j';\eta_b\right)
\end{aligned}
$$

Here, we obtain

$$
\begin{aligned}
\sup_{s\in[0,\infty)}A_b(s) &= \sup_{s\in[0,\infty)}\left(\frac{(2bc+c-2be-e)}{(b+e)^2}-2\frac{(c-e)}{(b+e)}\right)s^2 \\
&+\left(2\frac{(b+c)^2}{b(b+e)^2}-\frac{(b+c)}{b(b+e)}\right)s+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2} \\
&\leq \frac{(2bc+c-2be-e)}{(b+e)^2}+2\frac{(b+c)^2}{b(b+e)^2}+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2} \\
&\leq \frac{6b(b+e)^2+(b+e)^2}{3b^2(b+e)^2}
\end{aligned}
$$

and the following result is found:

$$\sup_{s\in[0,\infty)} A_b(s) \leq \frac{2}{b}$$

From the result we obtained for Theorem 2.5 $\sup_{s\in[0,\infty)} B_b(s) \leq 100\frac{1}{b}$ is written. Hence,

$$\begin{aligned}\sup_{s\in[0,\infty)} \frac{\left|Z_b(j;s)-j(s)\right|}{1+s^2} &\leq \left(\sqrt{\frac{2}{b}}M_{j'}+2\left(1+\eta_b^2\right)\left[\sqrt{\frac{2}{b}}+\frac{1}{\eta_b}\frac{2}{b}+\sqrt{\frac{2}{b}}\sqrt{100\frac{1}{b}}+\frac{1}{\eta_b}100\frac{1}{b}\right]\Omega(j';\eta_b)\right)\\ &= \sqrt{\frac{2}{b}}\left(M_{j'}+2\left(1+\eta_b^2\right)\left[1+\frac{1}{\eta_b}\sqrt{\frac{2}{b}}+\frac{10}{b}+\frac{1}{\eta_b}100\frac{1}{\sqrt{b}}\right]\Omega(j';\eta_b)\right)\end{aligned}$$

Here, if $\eta_b = \sqrt{\frac{2}{b}}$ is selected, the following is obtained:

$$\begin{aligned}\sup_{s\in[0,\infty)} \frac{\left|Z_b(j;s)-j(s)\right|}{1+s^2} &\leq \sqrt{\frac{2}{b}}\left(M_{j'}+4\left[1+1+10+100\right]\Omega(j';\eta_b)\right)\\ &\leq L\sqrt{\frac{2}{b}}\Omega\left(j';\sqrt{\frac{2}{b}}\right)\end{aligned}$$

Here $L = M_{j'} + 448$. QED. □

Theorem 2.7. *If j is a function providing the Lipschitz condition and $0 < \alpha \leq 1$, the following equation is true:*

$$\left\|Z_b(j;s)-j(s)\right\|_{C[0,A]} = 0\left(\left(\frac{4}{b}\right)^{\alpha/2}\right).$$

Proof. When $Z_b(1;s) = 1$, and because the operator is linear, the following can be written:

$$\begin{aligned}\left|Z_b(j;s)-j(s)\right| &= \left|Z_b(j;s)-j(s)Z_b(1;s)\right|\\ &= \left|Z_b(j;s)-Z_b(j(s);s)\right|\\ \left|Z_b(j;s)-j(s)\right| &\leq \left(Z_b\left|j(i)-j(s)\right|;s\right).\end{aligned}$$

Because j satisfies the Lipschitz condition and $\left|j(i)-j(s)\right| \leq M|i-s|^\alpha$,the following result is obtained:

$$\begin{aligned}\left|Z_b(j;s)-j(s)\right| &\leq \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\int_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} M|i-s|^\alpha\,di\\ &\leq M\frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\int_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} |i-s|^\alpha\,di\\ &= M\frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\int_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} |i-s|^\alpha\,di\\ &= MZ_b\left(|i-s|^\alpha;s\right)\end{aligned}$$

From Hölder's inequality, we obtain the following:

$$Z_b\left(|i-s|^{\alpha};s\right)\le Z_b\left((i-s)^2;s\right)^{\alpha/2}$$

Therefore,

$$Z_b\left(|i-s|^{\alpha};s\right)\le M\left(\left(\left(\frac{(2bc+c-2be-e)}{(b+e)^2}-2\frac{(c-e)}{(b+e)}\right)s^2+\left(2\frac{(b+c)^2}{b(b+e)^2}-\frac{(b+c)}{b(b+e)}\right)s+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\right)\right)^{\alpha/2}$$

If the maximum of the inside expression is taken,

$$\max_{0\le s\le A}\frac{(2bc+c-2be-e)}{(b+e)^2}s^2+2\frac{(b+c)^2}{b(b+e)^2}s+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}$$

$$\begin{aligned}
&= \left(\frac{3b^2(2bc+c-2be-e)}{3b^2(nb+e)^2}+\frac{6b(b+c)^2}{3b^2(b+e)^2}+\frac{(b+c)^2}{3b^2(b+e)^2}\right)\\
&= \frac{3b^2(2bc+c-2be-e)+6b(b+c)^2+(b+c)^2}{3b^2(b+e)^2}
\end{aligned}$$

is obtained. If $a=b$ is selected here,

$$\begin{aligned}
&\le \frac{(b+e)^2(6b+1)}{3b^2(b+e)^2}\\
&\le \frac{6(b+1)}{3b^2}\\
&\le \frac{2(b+1)}{b^2}\\
&\le 2\left(\frac{1}{b}+\frac{1}{b^2}\right)\\
&\le \frac{4}{b}
\end{aligned}$$

Then,

$$\left|Z_b(j;s)-j(s)\right|\le M\left(\frac{4}{b}\right)^{\alpha/2}$$

and thus,

$$\left\|Z_b(j;s)-j(s)\right\|_{C[0,A]}\le M\left(\frac{4}{b}\right)^{\alpha/2}$$

QED. □

Theorem 2.8. *Let the functions $j\in[0,A]$ and j, j', j'' be bounded functions in $[0,A]$, the following limit*

$$\lim_{b\to\infty}(b+e)\left(Z_b(j;s)-j(s)\right)=(c+1-e)\,j'(c)+xj''(s).$$

Proof. The Taylor series expansion of the function j at point s and the form of this expansion in the operator are as follows:

$$j(i)=j(s)+\frac{1}{1!}j'(s)(i-s)+\frac{1}{2!}j''(s)(i-s)^2+\frac{1}{3!}j'''(s)(i-s)^3+\frac{1}{4!}j^4(s)(i-s)^4+\ldots$$

$$j(i)-j(s)=\frac{1}{1!}j'(s)(i-s)+\frac{1}{2!}j''(s)(i-s)^2+(i-s)^2\mu(i-s)$$

$$\mu(i-s)=\left(\frac{1}{3!}j'''(s)(i-s)^3+\frac{1}{4!}j^4(s)(i-s)^4+...\right)$$

$$Z_b(j;s)-j(s)=Z_b((i-s);s)j'(s)+\frac{1}{2}Z_b\left((i-s)^2;s\right)j''(s)+Z_b\left((i-s)^2\mu(i-s);s\right)$$

If we substitute in the central moments in the last equation, we obtain the following:

$$\begin{aligned}Z_b(j;s)-j(s) &= \left(\frac{(c-e)}{(b+e)}s+\frac{1}{2b}\frac{(b+c)}{(b+e)}\right)j'(s)\\ &+\frac{1}{2}\left(\left(\frac{(2bc+c-2be-e)}{(b+e)^2}-2\frac{(c-e)}{(b+e)}\right)s^2\right.\\ &+\left(2\frac{(b+c)^2}{b(b+e)^2}-\frac{(b+c)}{b(b+n)}\right)s\\ &\left.+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\quad j''(s)+Z_b\left((i-s)^2\mu(i-s);s\right)\right)\end{aligned}$$

If both sides of the equation are multiplied by $(b+e)$, we obtain the following result:

$$\begin{aligned}(b+e)(Z_b(j;s)-j(s)) &= (b+e)\left(\frac{(c-e)}{(b+e)}s+\frac{1}{2b}\frac{(b+c)}{(b+e)}\right)j'(s)+\frac{1}{2}(b+e)\\ &\quad\left(\left(\frac{(2bc+c-2be-e)}{(b+e)^2}-2\frac{(c-e)}{(b+e)}\right)s^2\right.\\ &+\quad\left(2\frac{(b+c)^2}{b(b+e)^2}-\frac{(b+c)}{b(b+n)}\right)s\\ &+\quad\left.\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}j''(s)+(b+e)Z_b\left((i-s)^2\mu(i-s);s\right)\right)\end{aligned}$$

Because $\lim_{i\to s}\mu(i-xi)=0$ is bounded, and thus, the following equation can be written:

$$(b+e)Z_b\left((i-s)^2\mu(i-s);s\right)\le\sqrt{(b+e)Z_b\left((i-s)^4;s\right)}\sqrt{(b+e)Z_b\left(\mu(i-s)^2;s\right)}$$

As $\lim_{b\to\infty}(b+e)Z_b\left((i-s)^4;s\right)=0$ when the limit of the last equation is taken, we obtain the following:

$$\lim_{b\to\infty}(b+e)(Z_b(j;s)-j(s))=(c+1-e)j'(s)+sj''(s)$$

QED. □

3. MAIN RESULTS

This section will review the main results. That is, we will exemplify the results of our operator's convergence with graphs and numerical values.

Let us now present the plots that we drew on the Maple program showing the convergence of our operator $Z_b(j;s)$ to the function $h(s)=\sin\left(\frac{\pi s}{2}\right)\sqrt{s}$ for different b values.

Figure 1: Convergence of the operators S and Z to the function $h(s)$ for $b = 10$

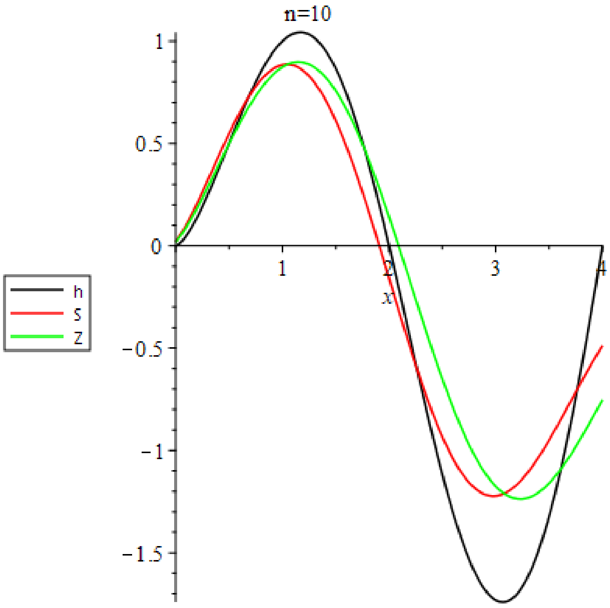


Figure 2: Convergence of the operators S and Z to the function $h(s)$ for $b = 20$

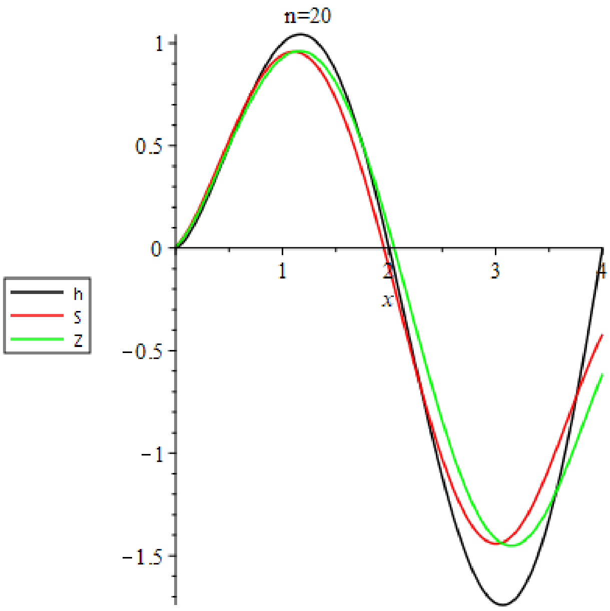
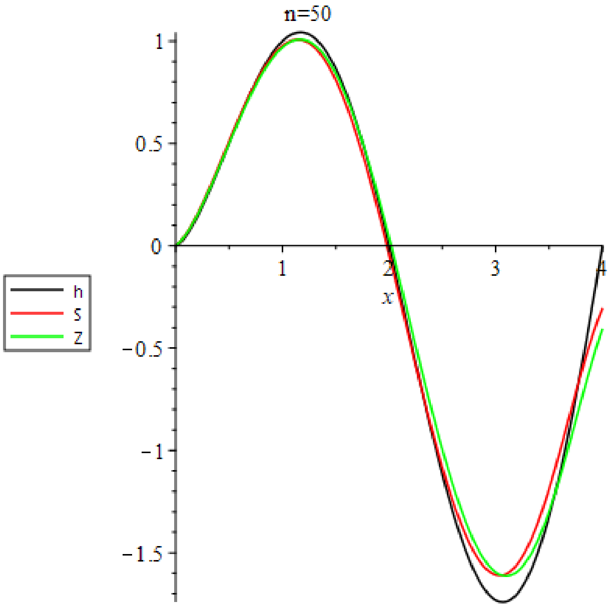


Figure 3: Convergence of the operators S and Z to the function $h(s)$ for $b = 50$



Then, we also demonstrate the convergence numerically with the numerical value table including the error margins. In the plots and the table, S refers to the classical Szász operator, while Z is the operator that we defined.

As seen here, as the values of b increase, the convergence becomes clearer, and our operator Z shows a better convergence than the operator S. Now, let us express the convergence rats of the operators S and Z with numerical values.

When

$$N(s) = \left| \frac{W_u^j(h;s) - h(s)}{Z_b(h;s) - h(s)} \right|$$

for different b and s values in $N(s)$, the following table of numerical values can be given.

$b - s$	$0,1$	$1,5$	$2,5$	4
10	1,02755	0,99899	1,284817	0,92587
100	1,01513	0,96156	1,25418	0,97556
300	1,01415	0,95843	1,25139	0,98571
500	1,01397	0,9578	1,25085	0,98889
700	1,01392	0,95752	1,25072	0,99059
1000	1,01403	0,95732	1,25061	0,99208

Table 1: $N(s)$ rate results for different b and s values.

The values given in the table show better convergence by S when they are greater than 1 and better convergence by Z when they are smaller than 1. Because the values are approximately 1 in general, it may be stated that these to operators are indeed equivalent.

References

[1] Acar, T., Acu, A.M. and Manav, N.(2018). Approximation of functions by genuine Bernstein Durrmeyer type operators. J. Math. Inequal, **12** (4), 975 − 987, .

[2] Acu,A. M., Acar,T., Radu.V. A.(2019).Approximation by modified U_n^{ρ} operators. Rev. R. Acad. Ciene. Exactas Fis. Nat. Ser. A Math.i 113-2715-2729.

[3] Bernstein,S. N.(1913). Démonstration du théoréme de Weierstrass fondée sur le calcul des probabilités. Commun. Kharkov Math. Soc., **13**, 1-2.

[4] Bohman,H.(1952-54). On approximation of continuous and analytic functions. Ark. Math.,**2**, 43-46.

[5] Chlodovsky,I.(1937). Sur le developpement des fonctions definies dans un intervalle infini en series de polynomes de M.S. Bernstein. Compos Math,**4**, 380-393.

[6] Gupta,V. , Vasishtha,V. (2002). Rate of convergene of the Szasz-Kantorovich-Bezier operators for bounded variation function. Publ. Inst. Math.,**72**,137-143.

[7] Izgı,A.(2012). Approximation by a Class of New Type Bernstein Polynomials of one and two Variables. Global Journal of Pure and Applied Mathematics, **8(5)**, 55–71.

[8] Kantorovich ,L. V.(1930). Sur cestain developpemenets suivant les polynomes de la forme de S.Berntein. C.R. Acad.,595-600.

[9] Kajla,A., Miclăuş,D. (2018). Blending Type Approximation by GBS Operators of Generalized Bernstein-Durrmeyer Type. Results Math., **73** Article number: 1 .

[10] Khan,K., Lobiyal,D. K., Kilicman,A. (2019). Bèzier curves and surfaces based on modified Bernstein polynomials. Ajerbaijan Journal of Mathematics, **9** (1), 3-21.

[11] Korovkin, P.P.(1930). On Convergence of Linear Positive Operators in the Space of Continuous Functions. Dokl. Akad. Nauk, **90**, 961-964.

[12] Lenze,B.(1988). On Lipschitz-type maximal functions and their smoothness spaces.Indagationes Mathematicae, 91:1 53–63.

[13] Lorentz,G. G.(2013).Bernstein polynomials, American Mathematical Soc.

[14] Schurer,F. (1962). Linear positive operators in approximation theory. Math. Inst. Techn. Univ. Delft Report.

[15] Szasz,O.(1950). Genelalization of S. Bernstein's polynomials to the infinite interval. J. Research Nat. Bur. Standars, **45**, 239-245.

[16] Totik,V.(1983). Approximation by Szasz-Mirakjan-Kantorovich operators in Lp space. Analysis Mathematica, **9(2)**, 147-167.

[17] Walzack,Z.(2000). On certain Moddified Szasz-Mirakjan operators for functions of two variables. Demonstratio Math ., **31(1)**, 91-100.

[18] Weirestrass,K.(1885). Über die Analytische Darstellbarkeit Sogenannter Willkürlicher Funktionen Einer Reelen Veränderlichen. Sitzungberichte der Akademie zu Berlin, 789-805.

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On the Stability of Finite Difference Scheme for the Schrödinger Equation Including Momentum Operator

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Abstract. In this paper, we apply the finite difference method to a Schrödinger equation which contains a momentum operator. For this, we constitute a difference scheme. A priori estimate for the solution of difference scheme is obtained. By using this estimate, we prove that the difference scheme is unconditionally stable.

1. INTRODUCTION

Schrödinger equation,

$$i\hbar\frac{\partial u}{\partial t}(\varsigma,t)=\left[-\frac{\hbar^2}{2m}\nabla^2+V(\varsigma,t)\right]u(\varsigma,t)=(T+V)\,u(\varsigma,t)$$

is a partial differential equation, where $i^2=-1$, ς and t are the variables of space and time, respectively, $u(\varsigma,t)$ is a wave function; $\hbar=\frac{h}{2m}$ is the reduced Planck's constant; h is the Planck's constant; m is the mass of particle; $T=\frac{p^2}{2m}$ is the kinetic energy operator; $p=-i\hbar\nabla$ is the momentum operator; $V=V(\varsigma,t)$ is the potential energy operator; ∇ is the gradient operator; ∇^2 is the Laplace operator.

As seen, the left hand side (l.h.s.) of above-mentioned equation describes the ratio of change of wave function u according to time, namely; Schrödinger equation is a equation describing how the energy of a quantum mechanical system evolves in time. It is a very sophisticated model applicable to many disciplines in engineering and applied sciences.

Many researchers analyzed the solutions of different versions of Schrödinger equation by using various methods (exactly, approximately or numerically). For example, Khuri and Sadighi et al. applied the Adomian decomposition method to Schrödinger equation [18, 25]; Biazar et al., He, Mousaa et al. studied the linear and nonlinear Schrödinger equations by Homotopy perturbation method [4, 12, 22]; Alomari et al., Ghanbari examined the linear and nonlinear Schrödinger equations by Homotopy analysis method [2, 11]; Hosseinzadeh, Wazwaz analyzed the linear and nonlinear Schrödinger equations by Variational iteration method [13, 29]; Iskenderov et.al., Mahmudov, Yagub et al., Yıldırım Aksoy examined the solvability of

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Schrödinger equations by Galerkin's method [15, 16, 21, 31–33]. Besides, there is a great variety of solution procedure for Schrödinger equation.

In this work, we apply the finite difference method to a linear Schrödinger equation. In studies [3, 7, 8, 10, 16, 27], the solutions of linear Schrödinger equations is examined by finite difference method and, in that studies, generally, the stability and convergence of difference scheme are shown. Also, in studies [5, 9, 14, 17, 23, 24, 26, 28, 30] the finite difference method is applied to the boundary value problems for nonlinear Schrödinger equations and in most of them, the stability, error and convergence of method are analyzed.

In the most of studies mentioned above, Schrödinger equations do not include the momentum operator. Especially, [27], the numerical solution of linear Schrödinger equation including a momentum operator is investigated. For this, the finite difference method is applied to the considered problem and the conditionally stability of method is proved. As distinct from the earlier studies in literature, in this work, we examine a boundary value problem for the linear Schrödinger equation including a momentum operator and apply the finite difference method to it. We analyze the difference scheme and prove that scheme is unconditionally stable.

Consider the following problem for linear Schrödinger equation including a momentum operator;

$$i\frac{\partial u}{\partial t}+a_0\frac{\partial^2 u}{\partial \varsigma^2}+ia_1\frac{\partial u}{\partial \varsigma}-a_2(\varsigma)u+a_3(\varsigma)u=g(\varsigma,t),\ (\varsigma,t)\in\Omega \tag{1}$$

$$u(\varsigma,0)=f(\varsigma),\ \varsigma\in I \tag{2}$$

$$u(0,t)=u(X,t)=0,\ t\in(0,T) \tag{3}$$

where $I=(0,X)$, $\Omega=I\times(0,T)$, $a_0,a_1>0$ are real numbers; $a_2(\varsigma)$ and $a_3(\varsigma)$ are real valued functions such that

$$0 < a_2(\varsigma)\le\mu_0 \text{ almost everywhere (a.e.) in } I,\ \mu_0=const.>0\,, \tag{4}$$

$$a_3 \in L_2(I),\ |a_3(\varsigma)|\le b_0 \text{ a.e. in } I, \tag{5}$$

$b_0>0$ is a given number; $f\in \mathring{W}_2^2(I)$, $g\in W_2^{0,1}(\Omega)$.

Here, $L_\infty(I)$ is the space of all functions that are essentially bounded on I equipped with the norm $\|u\|_{L_\infty(I)}=ess\sup_I|u|$;

$$W_p^r(\Omega)\equiv\left\{\begin{array}{l} u\in L_p(\Omega): D^\gamma u\in L_p(\Omega) \text{ for every multi-index } \gamma \text{ with } |\gamma|\le r,\\ \text{where } D^\gamma u \text{ is the weak(or distributional) partial derivative}\end{array}\right\}$$

and

$$\mathring{W}_p^r(\Omega)\equiv \text{the closure of } C_0^\infty(\Omega) \text{ in the space } W_p^r(\Omega)$$

[1].

In [19], it was shown that the following theorem is valid:

Theorem 1.1. *Assume that (4) and (5) are satisfied and* $f\in\mathring{W}_2^2(I)$, $g\in W_2^{0,1}(\Omega)$. *Then there exists a unique solution* $u\in\mathring{W}_2^{2,1}(\Omega)$ *of the problem (1)-(3) and the following estimate holds*

$$\|u(.,t)\|_{\mathring{W}_2^{2,1}(\Omega)}\le c_0(\|f\|_{\mathring{W}_2^2(I)}+\|g\|_{W_2^{0,1}(\Omega)}) \tag{6}$$

where $c_0>0$ *is a constant independent of* f,g.

2. NOTATIONS AND DIFFERENCE SCHEME

In this section, we will denote the notations used in the paper and discretize the problem (1)-(3). Later, we will express some lemmas and inequalities used in the paper.

Let α, β be any positive integers, $h = \frac{X}{\alpha-1}$, $\tau = \frac{T}{\beta}$,

$$\begin{aligned}
\Omega_h &= \left\{ \varsigma_k : \varsigma_k = kh - \frac{h}{2},\ k = \overline{1, \alpha-1},\ \varsigma_1 - \frac{h}{2} = 0,\ \varsigma_{\alpha-1} + \frac{h}{2} = X, \right\}, \\
\Omega_\tau &= \left\{ t_l : t_l = l\tau,\ l = \overline{0, \beta} \right\}, \\
\Omega_h^\tau &= \Omega_h \times \Omega_\tau .
\end{aligned}$$

Let $u_{kl},\ k = \overline{0, \alpha}, l = \overline{0, \beta}$ be the numerical approximation of $u(\varsigma, t)$ at the point (ς_k, t_l) on Ω_h^τ.

Introduce the following notations:

$$\begin{aligned}
\delta_{\bar{t}} u_{kl} &= \frac{u_{kl} - u_{kl-1}}{\tau},\ \delta_{\bar{\varsigma}} u_{kl} = \frac{u_{kl} - u_{k-1l}}{h}, \\
\delta_{\varsigma} u_{kl} &= \frac{u_{k+1l} - u_{kl}}{h},\ \delta_{\varsigma\bar{\varsigma}} u_{kl} = \frac{\delta_\varsigma u_{kl} - \delta_{\bar{\varsigma}} u_{kl}}{h} = \frac{u_{k+1l} - 2u_{kl} + u_{k-1l}}{h^2}, \\
(v, w) &= h \sum_{k=1}^{\alpha-1} v_k \overline{w}_k,\ \|v\|_p = \sqrt[p]{h \sum_{k=1}^{\alpha-1} |v_k|^p},\ \|v\|_\infty = \max_{1 \le k \le \alpha-1} |v_k|,\ \left\|\delta_\varsigma v\right\|_2 = \sqrt{h \sum_{k=1}^{\alpha-1} \left|\delta_\varsigma v_k\right|^2}
\end{aligned}$$

where $v, w \in V_h = \{v : v = (v_1, v_2, ..., v_{\alpha-1})\}$ are discrete grid functions on Ω_h. We denote by $\|.\|_2$, $\|.\|_\infty$, $(.,.)$ the discrete norms on spaces $L_2(I)$, $L_\infty(I)$ and discrete inner product on $L_2(I)$, respectively. Also, throughout this paper, we denote by $c_k = 1, 2, ..., 5$ the positive constants independent from τ, h and m.

Now, we present finite difference scheme of problem (1)-(3) as follows:

$$i\delta_{\bar{t}} u_{kl} + a_0 \delta_{\varsigma\bar{\varsigma}} u_{kl} + ia_1 \delta_{\bar{\varsigma}} u_{kl} - a_{2k} u_{kl} + a_{3k} u_{kl} = g_{kl},\ k = \overline{1, \alpha-1}, l = \overline{1, \beta}, \quad (7)$$

$$u_{k0} = f_k,\ \ k = \overline{0, \alpha}, \quad (8)$$

$$u_{0l} = u_{\alpha l} = 0,\ \ l = \overline{1, \beta}, \quad (9)$$

where the grid functions a_{2k}, a_{3k}, g_{kl} and f_k are Steklov averages of the functions $a_2(\varsigma)$, $a_3(\varsigma)$, $g(\varsigma, t)$ and $f(\varsigma)$ respectively, defined by

$$\begin{aligned}
a_{2k} &= \frac{1}{h} \int_{\varsigma_k - h/2}^{\varsigma_k + h/2} a_2(\varsigma) d\varsigma,\ \ k = \overline{1, \alpha-1} \\
a_{3k} &= \frac{1}{h} \int_{\varsigma_k - h/2}^{\varsigma_k + h/2} a_3(\varsigma) d\varsigma,\ \ k = \overline{1, \alpha-1} \\
g_{kl} &= \frac{1}{\tau h} \int_{t_{l-1}}^{t_l} \int_{\varsigma_k - h/2}^{\varsigma_k + h/2} g(\varsigma, t) d\varsigma dt,\ \ k = \overline{1, \alpha-1},\ \ l = \overline{1, \beta} \\
f_k &= \frac{1}{h} \int_{\varsigma_k - h/2}^{\varsigma_k + h/2} f(\varsigma) d\varsigma,\ \ k = \overline{1, \alpha-1},\ \ f_0 = f_\alpha = 0
\end{aligned}$$

[6]. Also, from conditions (4) and (5), the inequalities

$$0 \le a_{2k} \le \mu_0,\ k = \overline{1, \alpha-1}, \quad (10)$$

$$0 \le |a_{3k}| \le b_0,\ k = \overline{1, \alpha-1} \quad (11)$$

is written.

In the paper, the lemmas and inequalities we need are as follows:

Lemma 2.1. *(Discrete Gronwall's Inequality [9]): Assume that the nonnegative grid functions* $\{w(z), y(z),\ z = 1, 2, ..., \beta,\ \beta\tau = T\}$ *satisfy the inequality*

$$w(z) \le y(z) + \tau \sum_{\iota=1}^{z} B_\iota w(\iota),$$

where B_ι $(\iota = 1, 2, ..., \beta)$ *are nonnegative constant. Then, for any* $0 \le z \le \beta$*, there is*

$$w(z) \le y(z) \exp\left(z\tau \sum_{\iota=1}^{z} B_\iota\right).$$

Lemma 2.2. *(Summation by Parts Formula): For any two grid functions*

$v, w \in V_h = \{v : v = (v_0, v_1, v_2, ..., v_{\alpha-1}, v_\alpha),\ v_0 = v_\alpha = 0\}$, we have

$$h \sum_{k=1}^{\alpha-1} \left(\delta_{\varsigma\bar{\varsigma}} v_k\right) \overline{w}_k = -h \sum_{k=1}^{\alpha} \left(\delta_{\bar{\varsigma}} v_k\right)\left(\delta_{\bar{\varsigma}} \overline{w}_k\right).$$

Lemma 2.3. *($\in$ −Cauchy's inequality [20]): For any* $\in > 0$ *and arbitrary* a *and* b*, the inequality*

$$ab \le \frac{\in}{2} a^2 + \frac{1}{2\in} b^2$$

is valid.

Lemma 2.4. *(Young's İnequality): Let* $a, b \ge 0$*. Then,*

$$ab \le \frac{1}{p} a^p + \frac{1}{q} b^q$$

when $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p \in (1, +\infty)$.

3. THE STABILITY OF DIFFERENCE SCHEME

In this section, firstly, we obtain an estimate for solution of scheme (7)-(9). Later, using this estimate we prove the stability of scheme.

Theorem 3.1. *Assume that (4) and (5) are satisfied and* $f \in \mathring{W}_2^2(I)$*,* $g \in W_2^{0,1}(\Omega)$*. Then, the solution* u_{km} *of scheme (7)-(9) for any* $m \in \{1, 2, ..., \beta\}$ *satisfies the estimate*

$$h \sum_{k=1}^{\alpha-1} |u_{km}|^2 + 2h \sum_{l=1}^{m} \sum_{k=1}^{\alpha-1} |u_{kl} - u_{kl-1}|^2 + 2a_1\tau \sum_{l=1}^{m} |u_{\alpha-1l}|^2 + 2a_1\tau \sum_{l=1}^{m} \sum_{k=1}^{\alpha-1} |u_{kl} - u_{k-1l}|^2 \le$$
$$c_1\left(h \sum_{k=1}^{\alpha-1} \left|f_k\right|^2 + \tau h \sum_{l=1}^{\beta} \sum_{k=1}^{\alpha-1} \left|g_{kl}\right|^2\right). \tag{12}$$

Proof. For any grid function ξ_{kl} defined on Ω_h^τ with conditions $\xi_{0l} = \xi_{\alpha l} = 0$ for $l = \overline{1, \beta}$, scheme (7)-(9) is equivalent to the summation identity

$$ih \sum_{k=1}^{\alpha-1} \delta_{\bar{t}} u_{kl} \overline{\xi}_{kl} + a_0 h \sum_{k=1}^{\alpha-1} \delta_{\varsigma\bar{\varsigma}} u_{kl} \overline{\xi}_{kl} + ia_1 h \sum_{k=1}^{\alpha-1} \delta_{\bar{\varsigma}} u_{kl} \overline{\xi}_{kl} -$$
$$h \sum_{k=1}^{\alpha-1} a_{2k} u_{kl} \overline{\xi}_{kl} + h \sum_{k=1}^{\alpha-1} a_{3k} u_{kl} \overline{\xi}_{kl} = h \sum_{k=1}^{\alpha-1} g_{kl} \overline{\xi}_{kl}, \tag{13}$$

where $\overline{\xi}_{kl}$ is the conjugate of ξ_{kl}. If we substitute $\tau\overline{u}_{kl}$ for $\overline{\xi}_{kl}$ in (13) and apply the formula of summation by parts, we get

$$ih\tau\sum_{k=1}^{\alpha-1}\delta_{\bar{t}}u_{kl}\overline{u}_{kl}-a_0h\tau\sum_{k=1}^{\alpha-1}\left|\delta_{\bar{\varsigma}}u_{kl}\right|^2+ia_1h\tau\sum_{k=1}^{\alpha-1}\delta_{\bar{\varsigma}}u_{kl}\overline{u}_{kl}-$$
$$h\tau\sum_{k=1}^{\alpha-1}a_{2k}\left|u_{kl}\right|^2+h\tau\sum_{k=1}^{\alpha-1}a_{3k}\left|u_{kl}\right|^2=h\tau\sum_{k=1}^{\alpha-1}g_{kl}\overline{u}_{kl}. \tag{14}$$

If we extract its complex conjugate from (14) and then, use the relations

$$\tau\left(\delta_{\bar{t}}u_{kl}\overline{u}_{kl}+\delta_{\bar{t}}\overline{u}_{kl}u_{kl}\right) = \left|u_{kl}\right|^2-\left|u_{kl-1}\right|^2+\left|u_{kl}-u_{kl-1}\right|^2 \tag{15}$$
$$h\left(\delta_{\bar{\varsigma}}u_{kl}\overline{u}_{kl}+\delta_{\bar{\varsigma}}\overline{u}_{kl}u_{kl}\right) = \left|u_{kl}\right|^2-\left|u_{k-1l}\right|^2+\left|u_{kl}-u_{k-1l}\right|^2 \tag{16}$$

we get

$$h\sum_{k=1}^{\alpha-1}\left(\left|u_{kl}\right|^2-\left|u_{kl-1}\right|^2+\left|u_{kl}-u_{kl-1}\right|^2\right)+a_1\tau\sum_{k=1}^{\alpha-1}\left(\left|u_{kl}\right|^2-\left|u_{k-1l}\right|^2+\left|u_{kl}-u_{k-1l}\right|^2\right)=$$
$$2h\tau\sum_{k=1}^{\alpha-1}Im\left(g_{kl}\overline{u}_{kl}\right)\ for\ l=\overline{1,\beta}. \tag{17}$$

If we sum all equalities in (17) in l from 1 to $m\leq\beta$ and consider

$$\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}\left(\left|u_{kl}\right|^2-\left|u_{kl-1}\right|^2\right) = \sum_{k=1}^{\alpha-1}\left(\left|u_{km}\right|^2-\left|u_{k0}\right|^2\right)=\sum_{k=1}^{\alpha-1}\left|u_{km}\right|^2-\sum_{k=1}^{\alpha-1}\left|f_k\right|^2$$
$$\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}\left(\left|u_{kl}\right|^2-\left|u_{k-1l}\right|^2\right) = \sum_{l=1}^{m}\left(\left|u_{\alpha-1l}\right|^2-\left|u_{0l}\right|^2\right)=\sum_{l=1}^{m}\left|u_{\alpha-1l}\right|^2$$

by (8) and (9), we obtain from (17) the inequality

$$h\sum_{k=1}^{\alpha-1}\left|u_{km}\right|^2+h\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}\left|u_{kl}-u_{kl-1}\right|^2+a_1\tau\sum_{l=1}^{m}\left|u_{\alpha-1l}\right|^2+$$
$$a_1\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}\left|u_{kl}-u_{k-1l}\right|^2\leq 2h\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}\left|g_{kl}\right|\left|u_{kl}\right|+h\sum_{k=1}^{\alpha-1}\left|f_k\right|^2.$$

Let's distinguish m-th term from first summation in the right-hand side (r.h.s.) of above inequality and apply $\epsilon-Cauchy's$ inequality to distinguished term. Then, if we take $\epsilon=2\tau$ and use Young's inequality we get

$$h\sum_{k=1}^{\alpha-1}\left|u_{km}\right|^2+2h\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}\left|u_{kl}-u_{kl-1}\right|^2+2a_1\tau\sum_{l=1}^{m}\left|u_{\alpha-1l}\right|^2+2a_1\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}\left|u_{kl}-u_{k-1l}\right|^2\leq$$
$$2h\tau\sum_{l=1}^{m-1}\sum_{k=1}^{\alpha-1}\left|g_{kl}\right|^2+4T\tau h\sum_{k=1}^{\alpha-1}\left|g_{km}\right|^2+2h\tau\sum_{l=1}^{m-1}\sum_{k=1}^{\alpha-1}\left|u_{kl}\right|^2+2h\sum_{k=1}^{\alpha-1}\left|f_k\right|^2$$

which is equal to

$$h\sum_{k=1}^{\alpha-1}|u_{km}|^2+2h\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|u_{kl}-u_{kl-1}|^2+2a_1\tau\sum_{l=1}^{m}|u_{\alpha-1l}|^2+2a_1\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|u_{kl}-u_{k-1l}|^2\leq$$
$$4Th\tau\sum_{l=1}^{\beta}\sum_{k=1}^{\alpha-1}\left|g_{kl}\right|^2+2h\tau\sum_{l=1}^{m-1}\sum_{k=1}^{\alpha-1}|u_{kl}|^2+2h\sum_{k=1}^{\alpha-1}\left|f_k\right|^2 \tag{18}$$

for *any* $m\in\{1,2,...,\beta\}$. Since all terms in the l.h.s. of (18) are non-negative, it is written that

$$h\sum_{k=1}^{\alpha-1}|u_{km}|^2\leq 4Th\tau\sum_{l=1}^{\beta}\sum_{k=1}^{\alpha-1}\left|g_{kl}\right|^2+2h\tau\sum_{l=1}^{m-1}\sum_{k=1}^{\alpha-1}|u_{kl}|^2+2h\sum_{k=1}^{\alpha-1}\left|f_k\right|^2. \tag{19}$$

In (19), using discrete Gronwall's Inequality, we obtain

$$h\sum_{k=1}^{\alpha-1}|u_{km}|^2\leq c_2\left(h\sum_{k=1}^{\alpha-1}\left|f_k\right|^2+\tau h\sum_{l=1}^{\beta}\sum_{k=1}^{\alpha-1}\left|g_{kl}\right|^2\right)\text{ for } any\ m\in\{1,2,...,\beta\}. \tag{20}$$

If we use the inequality (20) in (18), we get for *any* $m\in\{1,2,...,\beta\}$

$$h\sum_{k=1}^{\alpha-1}|u_{km}|^2+2h\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|u_{kl}-u_{kl-1}|^2+2a_1\tau\sum_{l=1}^{m}|u_{\alpha-1l}|^2+2a_1\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|u_{kl}-u_{k-1l}|^2\leq$$
$$c_3\left(h\sum_{k=1}^{\alpha-1}\left|f_k\right|^2+\tau h\sum_{l=1}^{\beta}\sum_{k=1}^{\alpha-1}\left|g_{kl}\right|^2\right) \tag{21}$$

which shows the hypothesis of theorem 3.1 is valid. □

Theorem 3.2. *Suppose that* u^1_{kl}, *is a solution corresponding to the initial value* f^1_k *and the right side* g^1_{kl} *of scheme (7)-(9) and* u^2_{kl} *is a solution corresponding to the initial value* f^2_k *and the right side* g^2_{kl} *of scheme (7)-(9). Assume that the conditions of theorem 3.1 are fulfilled. Let* $\Phi_{kl}=u^1_{kl}-u^2_{kl}$. *Then, for any* $m\in\{1,2,...,\beta\}$ *and* $h,\tau>0$

$$h\sum_{k=1}^{\alpha-1}|\Phi_{km}|^2\leq c_4\left(h\sum_{k=1}^{\alpha-1}\left|f^1_k-f^2_k\right|^2+h\tau\sum_{l=1}^{\beta-1}\sum_{k=1}^{\alpha-1}\left|g^1_{kl}-g^2_{kl}\right|^2\right).$$

Hence, the difference scheme (7)-(9) is unconditionally stable.

Proof. It is clear that Φ_{kl} satisfies the scheme

$$i\delta_{\bar t}\Phi_{kl}+a_0\delta_{\varsigma\bar\varsigma}\Phi_{kl}+ia_1\delta_{\bar\varsigma}\Phi_{kl}-a_{2k}\Phi_{kl}+a_{3k}\Phi_{kl}=g^1_{kl}-g^2_{kl},\ \ k=\overline{1,\alpha-1},l=\overline{1,\beta}$$
$$\Phi_{k0}=f^1_k-f^2_k,\ \ k=\overline{0,\alpha}$$
$$\Phi_{0l}=\Phi_{\alpha l}=0,\ \ l=\overline{1,\beta}$$

which is equivalent to

$$ih\sum_{k=1}^{\alpha-1}\delta_{\bar t}\Phi_{kl}\overline{\Theta}_{kl}+a_0h\sum_{k=1}^{\alpha-1}\delta_{\varsigma\bar\varsigma}\Phi_{kl}\overline{\Theta}_{kl}+ia_1h\sum_{k=1}^{\alpha-1}\delta_{\bar\varsigma}\Phi_{kl}\overline{\Theta}_{kl}-$$
$$h\sum_{k=1}^{\alpha-1}a_{2k}\Phi_{kl}\overline{\Theta}_{kl}+h\sum_{k=1}^{\alpha-1}a_{3k}\Phi_{kl}\overline{\Theta}_{kl}=h\sum_{k=1}^{\alpha-1}\left(g^1_{kl}-g^2_{kl}\right)\overline{\Theta}_{kl}, \tag{22}$$

for any grid function $\overline{\Theta}_{kl}$, where $\overline{\Theta}_{kl}$ is the conjugate of Θ_{kl} defined on Ω_h^τ such that $\Theta_{0l} = \Theta_{\alpha l} = 0$ for $l = \overline{1,\beta}$. From (22) for $\overline{\Theta}_{kl} = \tau\overline{\Phi}_{kl}$ it is written that

$$\begin{aligned} &ih\tau\sum_{k=1}^{\alpha-1}\delta_{\bar t}\Phi_{kl}\overline{\Phi}_{kl} - a_0h\tau\sum_{k=1}^{\alpha-1}\left|\delta_{\bar\varsigma}\Phi_{kl}\right|^2 + ia_1h\tau\sum_{k=1}^{\alpha-1}\delta_{\bar\varsigma}\Phi_{kl}\overline{\Phi}_{kl} - \\ &h\tau\sum_{k=1}^{\alpha-1}a_{2k}\left|\Phi_{kl}\right|^2 + h\tau\sum_{k=1}^{\alpha-1}a_{3k}\left|\Phi_{kl}\right|^2 = h\tau\sum_{k=1}^{\alpha-1}\left(g_{kl}^1 - g_{kl}^2\right)\overline{\Phi}_{kl}. \end{aligned} \tag{23}$$

with summation by parts. Extracting its complex conjugate from (23) and using (15) and (16) for Φ_{kl}, we obtain

$$\begin{aligned} &h\sum_{k=1}^{\alpha-1}\left(|\Phi_{kl}|^2 - |\Phi_{kl-1}|^2 + |\Phi_{kl} - \Phi_{kl-1}|^2\right) + a_1\tau\sum_{k=1}^{\alpha-1}\left(|\Phi_{kl}|^2 - |\Phi_{k-1l}|^2 + |\Phi_{kl} - \Phi_{k-1l}|^2\right) = \\ &2h\tau\sum_{k=1}^{\alpha-1}Im\left(\left(g_{kl}^1 - g_{kl}^2\right)\overline{\Phi}_{kl}\right)\ for\ l = \overline{1,\beta}. \end{aligned} \tag{24}$$

Summing all equalities in (24) in l from 1 to $m \le \beta$ and using $\Phi_{k0} = f_k^1 - f_k^2$ for $k = \overline{0,\alpha}$, $\Phi_{0l} = 0$ for $l = \overline{1,\beta}$, we have

$$\begin{aligned} &h\sum_{k=1}^{\alpha-1}|\Phi_{km}|^2 + h\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|\Phi_{kl} - \Phi_{kl-1}|^2 + a_1\tau\sum_{l=1}^{m}|\Phi_{\alpha-1l}|^2 + \\ &a_1\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|\Phi_{kl} - \Phi_{k-1l}|^2 \le 2h\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}\left|g_{kl}^1 - g_{kl}^2\right||\Phi_{kl}| + h\sum_{k=1}^{\alpha-1}\left|f_k^1 - f_k^2\right|^2. \end{aligned}$$

which is equal to

$$\begin{aligned} &h\sum_{k=1}^{\alpha-1}|\Phi_{km}|^2 + h\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|\Phi_{kl} - \Phi_{kl-1}|^2 + a_1\tau\sum_{l=1}^{m}|\Phi_{\alpha-1l}|^2 + \\ &a_1\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|\Phi_{kl} - \Phi_{k-1l}|^2 \le 2h\tau\sum_{k=1}^{\alpha-1}\left|g_{km}^1 - g_{km}^2\right||\Phi_{km}| + \\ &2h\tau\sum_{l=1}^{m-1}\sum_{k=1}^{\alpha-1}\left|g_{kl}^1 - g_{kl}^2\right||\Phi_{kl}| + h\sum_{k=1}^{\alpha-1}\left|f_k^1 - f_k^2\right|^2. \end{aligned} \tag{25}$$

Applying $\epsilon - Cauchy's$ and Young's inequalities to (25), we get

$$\begin{aligned} &h\sum_{k=1}^{\alpha-1}|\Phi_{km}|^2 + 2h\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|\Phi_{kl} - \Phi_{kl-1}|^2 + 2a_1\tau\sum_{l=1}^{m}|\Phi_{\alpha-1l}|^2 + 2a_1\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|\Phi_{kl} - \Phi_{k-1l}|^2 \le \\ &4Th\tau\sum_{l=1}^{\beta}\sum_{k=1}^{\alpha-1}\left|g_{kl}^1 - g_{kl}^2\right|^2 + 2h\tau\sum_{l=1}^{m-1}\sum_{k=1}^{\alpha-1}|\Phi_{kl}|^2 + h\sum_{k=1}^{\alpha-1}\left|f_k^1 - f_k^2\right|^2 \end{aligned} \tag{26}$$

by $\epsilon = 2\tau$. It is clear that all terms in the l.h.s. of (26) are non-negative. So, we write that

$$h\sum_{k=1}^{\alpha-1}|\Phi_{km}|^2 \le 4Th\tau\sum_{l=1}^{\beta}\sum_{k=1}^{\alpha-1}\left|g_{kl}^1 - g_{kl}^2\right|^2 + 2h\tau\sum_{l=1}^{m-1}\sum_{k=1}^{\alpha-1}|\Phi_{kl}|^2 + h\sum_{k=1}^{\alpha-1}\left|f_k^1 - f_k^2\right|^2. \tag{27}$$

Thus, applying discrete Gronwall's inequality to (27), we obtain

$$h\sum_{k=1}^{\alpha-1}|\Phi_{km}|^2 \le c_5\left(h\tau\sum_{l=1}^{\beta}\sum_{k=1}^{\alpha-1}\left|g_{kl}^1-g_{kl}^2\right|^2+h\sum_{k=1}^{\alpha-1}\left|f_k^1-f_k^2\right|^2\right) \text{ for } any\ m\in\{1,2,...,\beta\}$$

which this complete the proof. □

4. Conclusion

In this paper, a finite difference scheme for the Schrödinger type equation has been introduced and analyzed. We have obtained a priori estimate for solution of scheme. We have also proved that the proposed scheme is unconditionally stable, without any restriction on both time and spatial step sizes.

References

[1] Adams RA. Sobolev spaces. Academic Press, New York, 1975.
[2] Alomari AK, Noorani MSM, Nazar R. Explicit series solutions of some linear and nonlinear Schrödinger equations via the homotopy analysis method. Communications in Nonlinear Science and Numerical Simulation. 14(4), 2009, 1196-1207.
[3] Becerril R, Guzman FS, Rendon-Romero A, Valdez-Alvarado S. Solving the time-dependent Schrödinger equation using finite difference methods. Revista Mexicana de Fisica E. 54(2), 2008, 120-132.
[4] Biazar J, Ghazvini H. Exact solutions for non-linear Schrödinger equations by He's homotopy perturbation method. Physics Letters A. 366(1-2), 2007, 79-84.
[5] Cavalcanti MM, Corrêa WJ, Sepúlveda CMA, Asem RV. Finite difference scheme for a high order nonlinear Schrödinger equation with localized damping. Studia Universitatis Babeş-Bolyai Mathematica. 64(2), 2019, 161-172.
[6] Chagas CQ, Diehl NML, Guidolin PL. Some properties for the Steklov averages.2017; Available at arXiv:1707.06368.
[7] Chan TF, Lee D, Shen L. Stable explicit schemes for equations of the Schrödinger type. SIAM Journal on Numerical Analysis. 23(2), 1986, 274-281.
[8] Chan TF, Shen L. Stability analysis of difference schemes for variable coefficient Schrödinger type equations. SIAM Journal on Numerical Analysis. 24(2), 1987, 336-349.
[9] Chen J, Zhang LM. Numerical approximation of solution for the coupled nonlinear Schrödinger equations. Acta Mathematicae Applicatae Sinica. English Series 33(2), 2017, 435-450.
[10] Dai W. An unconditionally stable three-level explicit difference scheme for the Schrödinger equation with a variable coefficient. SIAM Journal on Numerical Analysis. 29(1), 1992, 174-181.
[11] Ghanbari B. An analytical study for (2+1)-dimensional Schrödinger equation. The Scientific World Journal . Vol.2014, 2014, 1-5.
[12] He JH. Application of homotopy perturbation method to nonlinear wave equations. Chaos, Solitons and Fractals. 26(3), 2005, 695-700.
[13] Hosseinzadeh Kh. An analytic approximation to the solution of Schrodinger equation by VIM. Applied Mathematical Sciences. 11(16), 2017, 813-818.
[14] Hu H, Hu H. Maximum norm error estimates of fourth-order compact difference scheme for the nonlinear Schrödinger equation involving a quintic term. Journal of inequalities and applications. 2018:180, 2018.
[15] Iskenderov AD, Yagubov GY. Optimal control problem with unbounded potential for multidimensional, nonlinear and nonstationary Schrödinger equation. Proceedings of the Lankaran State University, Natural Sciences Series, 2007, 3-56.
[16] Iskenderov AD, Yagubov GY, Musayeva MA. Identification of Quantum potentials. Casıoglu, Baku, Azerbaijan, 2012.
[17] Ivanauskas F, Radziunas M. On convergence and stability of the explicit difference method for solution of nonlinear Schrödinger equations. SIAM Journal on Numerical Analysis. 36(5), 1999, 1466-1481.
[18] Khuri SA. A new approach to the cubic Schrödinger eguation: An application of the decomposition technique. Applied Mathematics and Computation. 97(2-3), 1998, 251-254.
[19] Kücük GD, Yagub G, Celik E. On the existence and uniqueness of the solution of an optimal control problem for Schrödinger equation. Discrete Contin Dyn Syst-S. 12(3), 2019, 503-512.
[20] Ladyzhenskaya OA, Solonnikov VA, Ural'tseva NN. Linear and Quasilinear Equations of Parabolic Type. Amer. Math. Soc. (Enligsh transl.), Providence, R.I.,1968.
[21] Mahmudov NM. Solvability of boundary value problems for a Schrödinger equation with püre ımaginary coefficient in the nonlinear part of this equation. Proceedings of IMM of NAS of Azerbaijan. 27(35), 2007, 25-36.
[22] Mousaa MM, Ragab SF. Application of the homotopy perturbation method to linear and nonlinear Schrödinger equations. Zeitschrift für Naturforschung A. 63a, 2008, 140-144.
[23] Potapov MM, Razgulin AV. Difference methods in problems of the optimal control problems of the steady selfaction of light beams. USSR Comput. Math. and Math. Phys. 30(4), 1990, 134-142.
[24] Radziūnas M. On convergence and stability of difference schemes for nonlinear Schrödinger type equations. Lithuanian Mathematical Journal. 36(2), 1996, 178-194.
[25] Sadighi A, Ganji DD. Analytic treatment of linear and nonlinear Schrödinger equations: A study with homotopy-perturbation and Adomian decomposition methods. Physics Letters A. 372(4), 2008, 465-469.

[26] Sun ZZ, Zhao DD. On the L∞ convergence of a difference scheme for coupled nonlinear Schrödinger equations. Computers and Mathematics with Applications. 59, 2010, 3286-3300.

[27] Toyoglu F, Yagubov G. Numerical solution of an optimal control problem governed by two dimensional Schrodinger equation. Applied and Computational Mathematics. 4(2), 2015, 30-38.

[28] Wang B, Liang D. The finite difference scheme for nonlinear Schrödinger equations on unbounded domain by artificial boundary conditions. Applied Numerical Mathematics. 128, 2018, 183-204.

[29] Wazwaz AM. A study on linear and nonlinear Schrodinger equations by the variational iteration method. Chaos, Solitons and Fractals. 37(4), 2008, 1136-1142.

[30] Xie SS, Li GX, Yi S. Compact finite difference schemes with high accuracy for one-dimensional nonlinear Schrödinger equation. Computer Methods in Applied Mechanics and Engineering. 198, 2009, 1052-1060.

[31] Yagub G, Aksoy E. The solvability of initial boundary value problem for three-dimensional nonlinear Schrödinger equation with a special gradient term. AIP Conference Proceedings. 1833, 2017, 020042.

[32] Yagub G, Ibrahimov NS, Zengin M. The Solvability of the Initial-Boundary Value Problems for a Nonlinear Schrödinger Equation with a Special Gradient Term. Journal of Mathematical Physics, Analysis, Geometry. 14(2), 2018, 214-232.

[33] Yıldırım Aksoy N. Solution of a nonlinear Schrödinger equation with Galerkin's method. Iğdır Universtiy Journal of the Institute of Science and Technology. 7(2), 2017, 225-239.

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Some Characterizations of PS-Statistical Manifolds

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Abstract. In the present study, firstly we state symmetry properties for curvatures of a statistical manifold and give some relations between the Riemannian curvature $\widehat{R}$ and the curvatures R, R^* and R^S. After, by defining the notion of para-Sasakian statistical manifold, we give the necessary and sufficient conditions for a structure (D, h, Ψ, w, ζ) to be a para-Sasakian structure when (D, h) is a statistical structure and (Ψ, w, ζ, h) is an almost paracontact Riemannian manifold. Also, we give some results for curvatures R, R^*, R^S and Ricci tensor of these curvatures on a para-Sasakian statistical manifold. We construct an example of para-Sasakian statistical manifold of dimension 3. Finally, we examined the Einsteinian of para-Sasakian statistical manifolds according to certain conditions.

1. INTRODUCTION

The theory of statistical manifolds, (at the same time it is called information geometry), has started with a study in 1945, where a statistical model was considered as a Riemannian manifold with the tensor given by the Fisher information matrix [15]. After that, the information geometry, which is typically deals with the study of various geometric structures on a statistical manifold, has begun as a study of the geometric structures possessed by a statistical model of probability distributions.

The notion of dual connection, which is also called conjugate connection in affine geometry, has been first introduced into statistics by Amari in 1985 [2]. A statistical model equipped with a Riemannian metric together with a pair of dual affine connections is called a *statistical manifold*. For details about statistical manifolds and information geometry, one can see [3], [5], [6], [10], [11], [12], [13], [14], [19] and etc.

Also, if Ψ is a tensor field of type $(1, 1)$, w is a 1-form and ζ is a vector field on a $(2n + 1)$-dimensional differentiable manifold M, then almost contact structure (Ψ, w, ζ) which is related to almost complex structures and satisfies the conditions $\Psi^2 = -I + w \otimes \zeta$, $w(\zeta) = 1$ has been determined by Sasaki in 1960 [16]. After in 1976, on an n-dimensional differentiable manifold M, almost paracontact structure which is a similar structure with almost contact structure, related to almost product structures and satisfies the conditions $\Psi^2 = I - w \otimes \zeta$, $w(\zeta) = 1$ has been determined by Sato [17]. With the aid of these definitions, different types of manifolds have been defined and studied by many mathematicians.

According to these notions, nowadays lots of studies have been started to be done by scientists. For example, in [22] the authors have defined the concept of quaternionic Kähler-like statistical manifold and derived the main properties of quaternionic Kähler-like statistical submersions, extending in a new setting some previous results obtained by K. Takano concerning statistical manifolds endowed with almost complex

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(in [20]) and almost contact structures (in [21]). In [7], the authors have introduced the notion of Sasakian statistical structure and obtained the condition for a real hypersurface in a holomorphic statistical manifold to admit such a structure. Also in [8], the notion of a Kenmotsu statistical manifold is introduced, which is locally obtained as the warped product of a holomorphic statistical manifold and a line by authors. And they have showed that, a Kenmotsu statistical manifold of constant Ψ-sectional curvature is constructed from a special Kahler manifold, which is an important example of holomorphic statistical manifold.

In this paper, after giving some basic notions about statistical structures and para-Sasakian manifolds in Preliminaries, in Section 3 we give symmetry properties of curvatures R and R^* which are the curvatures of the connections D and D^*, respectively and R^S which is statistical curvature of a statistical manifold and obtain some results for relations between the Riemannian curvature $\widehat{R}$ and the curvatures R, R^* and R^S. In fourth section, we define the notion of para-Sasakian statistical manifold and give the necessary and sufficient coditions for a structure (D, h, Ψ, w, ζ) to be a para-Sasakian structure when (D, h) is a statistical structure and (Ψ, w, ζ, h) is an almost paracontact Riemannian manifold. Also, we give some results about the curvatures R, R^*, R^S and Ricci tensor of these curvatures on a para-Sasakian statistical manifold. We construct an example of 3-dimensional para-Sasakian statistical manifold and give its all of connections and components of curvature tensors. And in the fifth section, we study on Ricci semi-symmetric and Ricci pseudo-symmetric para-Sasakian statistical manifolds and after we give some characterizations for ζ-projectively flat, projectively flat and Ψ-projectively semi-symmetric para-Sasakian statistical manifolds.

2. PRELIMINARIES

In this section, we recall some notions about statistical structures and para-Sasakian manifolds, respectively. Througout this paper, we suppose that M is an n-dimensional manifold, h is a Riemannian metric and $\Gamma(TM^{(p,q)})$ means the set of tensor fields of type (p, q) on M.

On M, a parametric family of torsion-free connections $D^{(\alpha)}$ indexed by $\alpha \in \mathbb{R}$ can be defined by

$$D^{(\alpha)} = \frac{1+\alpha}{2} D - \frac{1-\alpha}{2} D^*, \tag{1}$$

with

$$D^{(1)} = D,\ D^{(-1)} = D^*,\ D^{(0)} = \frac{1}{2}(D + D^*) := \hat{D}. \tag{2}$$

Here $\hat{D}$ denotes the Levi-Civita (L-C) connection associated with h.

Also, a pair (D, h) is called a *statistical structure* on M, if D is torsion-free and

$$(D_{\Omega_1} h)(\Omega_2, \Omega_3) = (D_{\Omega_2} h)(\Omega_1, \Omega_3),\ \forall \Omega_1, \Omega_2, \Omega_3 \in \Gamma(TM^{(1,0)}) \tag{3}$$

holds, where the equation (3) is generally called *Codazzi equation*. In this case (M, D, h) is called a *statistical manifold.*

If (D, h) is a statistical structure on M, then the connection D^* which is given by

$$\Omega_1 h(\Omega_2, \Omega_3) = h(D_{\Omega_1}\Omega_2, \Omega_3) + h(\Omega_2, D^*_{\Omega_1}\Omega_3) \tag{4}$$

is called *conjugate or dual connection* of D with respect to h. If (D, h) is a statistical structure on M, then (D^*, h) is a statistical structure on M, too.

For a statistical structure (D, h), the *difference tensor field* $\kappa \in \Gamma(TM^{(1,2)})$ can be defined as

$$\kappa(\Omega_1, \Omega_2) = D_{\Omega_1}\Omega_2 - \hat{D}_{\Omega_1}\Omega_2,\ \forall \Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)}). \tag{5}$$

Moreover, κ satisfies

$$\kappa(\Omega_1, \Omega_2) = \kappa(\Omega_2, \Omega_1), \tag{6}$$

$$\tilde{\kappa}(\Omega_1, \Omega_2, \Omega_3) = h(\kappa(\Omega_1, \Omega_2), \Omega_3) = h(\Omega_2, \kappa(\Omega_1, \Omega_3)) = \tilde{\kappa}(\Omega_1, \Omega_3, \Omega_2), \tag{7}$$

where $\tilde{\kappa} \in \Gamma(TM^{(0,3)})$. Furthermore, we have

$$\kappa = \hat{D} - D^* = \frac{1}{2}(D - D^*). \tag{8}$$

For a more detailed treatment, we refer to [6], [7] and [23].

Now, let us recall some fundamental informations about para-Sasakian manifolds.

A differentiable manifold M is said to admit an *almost paracontact Riemannian structure* (Ψ, w, ζ, h), where Ψ is a tensor field of type (1,1), ζ is a vector field, w is a 1-form and h is a Riemannian metric on M such that

$$\begin{aligned} &\Psi\zeta = 0,\ w(\zeta) = 1,\ h(\zeta, \Omega_1) = w(\Omega_1), \\ &\Psi^2\Omega_1 = \Omega_1 - w(\Omega_1)\zeta, \\ &h(\Psi\Omega_1, \Psi\Omega_2) = h(\Omega_1, \Omega_2) - w(\Omega_1)w(\Omega_2), \end{aligned} \tag{9}$$

for any vector fields Ω_1, Ω_2 on M. In addition, if (Ψ, w, ζ, h) satisfy the equations

$$d\eta = 0,\ \hat{D}_{\Omega_1}\zeta = \Psi\Omega_1, \tag{10}$$

$$(\hat{D}_{\Omega_1}\Psi)\Omega_2 = -h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2w(\Omega_1)w(\Omega_2)\zeta, \tag{11}$$

then M is called a *para-Sasakian (PS) manifold*. On a PS-manifold, for $\forall\Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)})$ we have the following equations:

$$\widehat{Ric}(\Omega_1, \zeta) = (1 - n)w(\Omega_1), \tag{12}$$

$$\widehat{Q}\zeta = (1 - n)\zeta, \tag{13}$$

$$\widehat{R}(\Omega_1, \Omega_2)\zeta = w(\Omega_1)\Omega_2 - w(\Omega_2)\Omega_1, \tag{14}$$

$$\widehat{R}(\zeta, \Omega_1)\Omega_2 = w(\Omega_2)\Omega_1 - h(\Omega_1, \Omega_2)\zeta, \tag{15}$$

$$\widehat{R}(\zeta, \Omega_1)\zeta = \Omega_1 - w(\Omega_1)\zeta, \tag{16}$$

$$w(\widehat{R}(\Omega_1, \Omega_2)\Omega_3) = h(\Omega_1, \Omega_3)w(\Omega_2) - h(\Omega_2, \Omega_3)w(\Omega_1), \tag{17}$$

$$\widehat{Ric}(\Psi\Omega_1, \Psi\Omega_2) = \widehat{Ric}(\Omega_1, \Omega_2) - (1 - n)w(\Omega_1)w(\Omega_2), \tag{18}$$

where $\widehat{R}$, $\widehat{Ric}$ and $\widehat{Q}$ denotes the Riemannian curvature tensor, Ricci tensor and Ricci operator of L-C connection $\hat{D}$, respectively (for detail, see [1], [17] and [18]).

3. R, R^* AND R^S CURVATURES OF STATISTICAL MANIFOLDS

In this section, firstly we recall symmetry properties of curvatures R, R^* and give these properties for R^S. After, we give some results for relations between the Riemannian curvature $\widehat{R}$ and the curvatures R, R^* and R^S.

Lemma 3.1. *Let (M, D, h) be a statistical manifold. Then, the curvatures R and R^* satisfy the following symmetry properties:*

i) $R(\Omega_1, \Omega_2)\Omega_3 + R(\Omega_2, \Omega_3)\Omega_1 + R(\Omega_3, \Omega_1)\Omega_2 = 0,$
$R^*(\Omega_1, \Omega_2)\Omega_3 + R^*(\Omega_2, \Omega_3)\Omega_1 + R^*(\Omega_3, \Omega_1)\Omega_2 = 0;$

ii) $\mathcal{R}(\Omega_1, \Omega_2, \Omega_3, \Omega_4) + \mathcal{R}(\Omega_1, \Omega_2, \Omega_4, \Omega_3) = 2h((\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_4) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_4), \Omega_3),$
$\mathcal{R}^*(\Omega_1, \Omega_2, \Omega_3, \Omega_4) + \mathcal{R}^*(\Omega_1, \Omega_2, \Omega_4, \Omega_3) = 2h((\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_4) - (\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_4), \Omega_3);$

iii) $\begin{array}{l} \mathcal{R}(\Omega_1, \Omega_2, \Omega_3, \Omega_4) - \mathcal{R}(\Omega_3, \Omega_4, \Omega_1, \Omega_2) = 0, \\ \mathcal{R}^*(\Omega_1, \Omega_2, \Omega_3, \Omega_4) - \mathcal{R}^*(\Omega_3, \Omega_4, \Omega_1, \Omega_2) = 0 \end{array}$ *if* $(\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_4) = (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_4),$

where $\mathcal{R}$ and $\mathcal{R}^*$ $\in\Gamma(TM^{(0,4)})$ are Riemannian-Christoffel curvature tensors of R and R^*, respectively and they are defined by $h(R(\Omega_1, \Omega_2)\Omega_3, \Omega_4) = \mathcal{R}(\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ and $h(R^*(\Omega_1, \Omega_2)\Omega_3, \Omega_4) = \mathcal{R}^*(\Omega_1, \Omega_2, \Omega_3, \Omega_4)$, for $\forall\Omega_1, \Omega_2, \Omega_3, \Omega_4 \in \Gamma(TM^{(1,0)})$.

Proof. The proof can be found in [9]. □

In [7], the authors have defined a curvature tensor field $S \in \Gamma(TM^{(1,3)})$ as

$$S(\Omega_1, \Omega_2)\Omega_3 = \frac{1}{2}\{R(\Omega_1, \Omega_2)\Omega_3 + R^*(\Omega_1, \Omega_2)\Omega_3\}, \tag{19}$$

for $\forall \Omega_1, \Omega_2, \Omega_3 \in \Gamma(TM^{(1,0)})$ and they have called it *statistical curvature tensor field* of (D, h). Hereafter, in our results we'll denote the statistical curvature tensor field S by R^S. So, let us give the following Theorem which gives the symmetry properties of R^S:

Theorem 3.2. *Let (M, D, h) be a statistical manifold. Then, the statistical curvature tensor field R^S satisfies the following symmetry properties:*
i) $R^S(\Omega_1, \Omega_2)\Omega_3 + R^S(\Omega_2, \Omega_3)\Omega_1 + R^S(\Omega_3, \Omega_1)\Omega_2 = 0$,
ii) $\mathcal{R}^S(\Omega_1, \Omega_2, \Omega_3, \Omega_4) + \mathcal{R}^S(\Omega_1, \Omega_2, \Omega_4, \Omega_3) = 0$,
iii) $\mathcal{R}^S(\Omega_1, \Omega_2, \Omega_3, \Omega_4) - \mathcal{R}^S(\Omega_3, \Omega_4, \Omega_1, \Omega_2) = 0$,
where $\mathcal{R}^S \in \Gamma(TM^{(0,4)})$ is Riemannian-Christoffel curvature tensor of R^S and it is defined by $h(R^S(\Omega_1, \Omega_2)\Omega_3, \Omega_4) = \mathcal{R}^S(\Omega_1, \Omega_2, \Omega_3, \Omega_4)$*, for* $\forall \Omega_1, \Omega_2, \Omega_3, \Omega_4 \in \Gamma(TM^{(1,0)})$.

Proof. Using Lemma 3.1-(i) in (19), we get (i). Using Lemma 3.1-(ii) in (19), we reach (ii). And finally, from (i) and (ii), we have (iii). □

Also, we can give the following relations, which have been stated in [9] too, between Riemannian curvature $\widehat{R}$ and the curvatures R, R^* when (M, D, h) is a statistical manifold and we give these relations for R^S.

Using $D = \hat{D} + \kappa$ in $R(\Omega_1, \Omega_2)\Omega_3 = D_{\Omega_1} D_{\Omega_2}\Omega_3 - D_{\Omega_2} D_{\Omega_1}\Omega_3 - D_{[\Omega_1,\Omega_2]}\Omega_3$, we have

$$R(\Omega_1, \Omega_2)\Omega_3 = \widehat{R}(\Omega_1, \Omega_2)\Omega_3 + (D_{\Omega_1}\kappa)(\Omega_2, \Omega_3) - (D_{\Omega_2}\kappa)(\Omega_1, \Omega_3) - \kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) + \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3)). \tag{20}$$

Again using $D = \hat{D} + \kappa$ in (20), we get

$$R(\Omega_1, \Omega_2)\Omega_3 = \widehat{R}(\Omega_1, \Omega_2)\Omega_3 + (\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_3) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_3) + \kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) - \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3)). \tag{21}$$

Thus, from (20) and (21) we can write

$$(D_{\Omega_1}\kappa)(\Omega_2, \Omega_3) - (D_{\Omega_2}\kappa)(\Omega_1, \Omega_3) - (\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_3) + (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_3) = 2\{\kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) - \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3))\}. \tag{22}$$

Similarly, using $D^* = \hat{D} - \kappa$ in $R^*(\Omega_1, \Omega_2)\Omega_3 = D^*_{\Omega_1} D^*_{\Omega_2}\Omega_3 - D^*_{\Omega_2} D^*_{\Omega_1}\Omega_3 - D^*_{[\Omega_1,\Omega_2]}\Omega_3$, we have

$$R^*(\Omega_1, \Omega_2)\Omega_3 = \widehat{R}(\Omega_1, \Omega_2)\Omega_3 - (D^*_{\Omega_1}\kappa)(\Omega_2, \Omega_3) + (D^*_{\Omega_2}\kappa)(\Omega_1, \Omega_3) - \kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) + \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3)) \tag{23}$$

and again using $D^* = \hat{D} - \kappa$ in (23), we get

$$R^*(\Omega_1, \Omega_2)\Omega_3 = \widehat{R}(\Omega_1, \Omega_2)\Omega_3 - (\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_3) + (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_3) + \kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) - \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3)). \tag{24}$$

So, from (23) and (24) we can write

$$(D^*_{\Omega_1}\kappa)(\Omega_2, \Omega_3) - (D^*_{\Omega_2}\kappa)(\Omega_1, \Omega_3) - (\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_3) + (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_3) = -2\{\kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) - \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3))\}. \tag{25}$$

And finally, using (21) and (24) in (19), we reach that

$$R^S(\Omega_1, \Omega_2)\Omega_3 = \widehat{R}(\Omega_1, \Omega_2)\Omega_3 + \kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) - \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3)). \tag{26}$$

4. PARA-SASAKIAN (PS) STATISTICAL MANIFOLDS

In this section, firstly we define the notion of para-Sasakian (PS) statistical manifold and give the necessary and sufficient coditions for a structure (D, h, Ψ, w, ζ) to be a PS-structure when (D, h) is a statistical structure and (Ψ, w, ζ, h) is an almost paracontact Riemannian manifold. After that, we give some results about the curvatures R, R^* and R^S and Ricci tensor of these curvatures on a PS-statistical manifold.

Let Ω be the fundamental 2-form of a PS-manifold (M, Ψ, w, ζ, h) defined by

$$\Omega(\Omega_1, \Omega_2) = h(\Omega_1, \Psi\Omega_2), \tag{27}$$

for $\forall \Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)})$. Then, we have

Lemma 4.1. *Let (D, h) be a statistical structure and (Ψ, w, ζ, h) be an almost paracontact Riemannian structure on M. Then, we have*
i) $(D_{\Omega_1}\Omega)(\Omega_2, \Omega_3) = h(\Omega_2, D^*_{\Omega_1}\Psi\Omega_3 - \Psi D_{\Omega_1}\Omega_3)$,
ii) $(D_{\Omega_1}\Omega)(\Omega_2, \Omega_3) - (D^*_{\Omega_1}\Omega)(\Omega_2, \Omega_3) = -2g(\Omega_2, \kappa(\Omega_1, \Psi\Omega_3) + \Psi\kappa(\Omega_1, \Omega_3))$,
iii) $D_{\Omega_1}\Psi\Omega_2 - \Psi D^*_{\Omega_1}\Omega_2 = (\hat{D}_{\Omega_1}\Psi)\Omega_2 + \kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2)$,
for $\forall \Omega_1, \Omega_2, \Omega_3 \in \Gamma(TM^{(1,0)})$.

Proof. **i)** From (4) and (27), we get

$$\begin{aligned}(D_{\Omega_1}\Omega)(\Omega_2, \Omega_3) &= D_{\Omega_1}\Omega(\Omega_2, \Omega_3) - \Omega(D_{\Omega_1}\Omega_2, \Omega_3) - \Omega(\Omega_2, D_{\Omega_1}\Omega_3)\\ &= \Omega_1 h(\Omega_2, \Psi\Omega_3) - h(D_{\Omega_1}\Omega_2, \Psi\Omega_3) - h(\Omega_2, \Psi D_{\Omega_1}\Omega_3)\\ &= h(D_{\Omega_1}\Omega_2, \Psi\Omega_3) + h(\Omega_2, D^*_{\Omega_1}\Psi\Omega_3) - h(D_{\Omega_1}\Omega_2, \Psi\Omega_3) - h(\Omega_2, \Psi D_{\Omega_1}\Omega_3)\\ &= h(\Omega_2, D^*_{\Omega_1}\Psi\Omega_3 - \Psi D_{\Omega_1}\Omega_3).\end{aligned} \tag{28}$$

ii) Substracting the dual of equation (28) from (28) and using (8), the proof completes.
iii) From (5) and (8), we have (iii). □

Definition 4.2. *(D, h, Ψ, w, ζ) is a PS-statistical structure on M, if*
i) *(D, h) is a statistical structure,*
ii) *(Ψ, w, ζ, h) is a PS-structure,*
iii) *for $\forall \Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)})$, the equation*

$$\kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2) = 0 \tag{29}$$

is satisfied.

Thus, we can prove the following Theorem:

Theorem 4.3. *Let (D, h) be a statistical structure and (Ψ, w, ζ, h) be an almost paracontact Riemannian structure on M. Then, (D, h, Ψ, w, ζ) is a PS-statistical structure on M iff the equations*

$$D_{\Omega_1}\Psi\Omega_2 - \Psi D^*_{\Omega_1}\Omega_2 = -h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2\eta(\Omega_1)w(\Omega_2)\zeta \tag{30}$$

and

$$D_{\Omega_1}\zeta = \Psi\Omega_1 + w(D_{\Omega_1}\zeta)\zeta \tag{31}$$

hold for $\forall \Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)})$.

Proof. Let (D, h) be a statistical structure and (Ψ, w, ζ, h) be a PS-Riemannian structure on M. Then, from (11) and Lemma 4.1-(iii), we get

$$D_{\Omega_1}\Psi\Omega_2 - \Psi D^*_{\Omega_1}\Omega_2 = -h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2\eta(\Omega_1)w(\Omega_2)\zeta + \kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2).$$

So, if (D, h, Ψ, w, ζ) is a PS-structure on M, then from (29) we have (30). Also, putting $\Omega_2 = \zeta$ in the expression of the dual of (30), from (9) we obtain (31).

Conversely, let us assume that the equations (30) and (31) hold for $\forall\Omega_1, \Omega_2 \in \Gamma(TM)$. Taking $\Psi\Omega_2$ instead of Ω_2 in the equation (30) and applying Ψ to the resulting equation, from (9) and (31) we have

$$D^*_{\Omega_1}\Psi\Omega_2 - \Psi D_{\Omega_1}\Omega_2 = -h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2\eta(\Omega_1)w(\Omega_2)\zeta$$

and this is the dual of (30). Finally we have to see that, (Ψ, w, ζ, h) is a PS-structure and the equation (29) holds. Using (5) and (8) in the equation which is in the Lemma 4.1-(iii), we have

$$D_{\Omega_1}\Psi\Omega_2 - \Psi D^*_{\Omega_1}\Omega_2 = \frac{1}{2}\{(D_{\Omega_1}\Psi)\Omega_2 + (D^*_{\Omega_1}\Psi)\Omega_2\} + \frac{1}{2}\{D_{\Omega_1}\Psi\Omega_2 - D^*_{\Omega_1}\Psi\Omega_2\} + \frac{1}{2}\Psi\{D_{\Omega_1}\Omega_2 - D^*_{\Omega_1}\Omega_2\}.$$

Taking the dual of the last equation, we get

$$D^*_{\Omega_1}\Psi\Omega_2 - \Psi D_{\Omega_1}\Omega_2 = (\hat{D}_{\Omega_1}\Psi)\Omega_2 - \kappa(\Omega_1, \Psi\Omega_2) - \Psi\kappa(\Omega_1, \Omega_2). \tag{32}$$

The dual of the equation (30), i.e. $(30)^*$, is

$$D^*_{\Omega_1}\Psi\Omega_2 - \Psi D_{\Omega_1}\Omega_2 = -h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2\eta(\Omega_1)w(\Omega_2)\zeta.$$

Thus, from (32) and $(30)^*$ we obtain that

$$-(\hat{D}_{\Omega_1}\Psi)\Omega_2 - h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2\eta(\Omega_1)w(\Omega_2)\zeta = -\kappa(\Omega_1, \Psi\Omega_2) - \Psi\kappa(\Omega_1, \Omega_2). \tag{33}$$

Also, from Lemma 4.1-(iii) and (30), we get

$$-(\hat{D}_{\Omega_1}\Psi)\Omega_2 - h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2\eta(\Omega_1)w(\Omega_2)\zeta = \kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2). \tag{34}$$

So, from (33) and (34) we can reach that, $\kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2) = 0$ holds and also we have $(\hat{D}_{\Omega_1}\Psi)\Omega_2 = -h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2\eta(\Omega_1)w(\Omega_2)\zeta$. Thus, (Ψ, w, ζ, h) is a PS-structure and this completes the proof. □

Example 4.4. *Let (Ψ, w, ζ, h) be a PS-Riemannian structure on M. Set the connection $\check{D}$ as*

$$\check{D}_{\Omega_1}\Omega_2 = \hat{D}_{\Omega_1}\Omega_2 + w(\Omega_1)\Omega_2 + w(\Omega_2)\Omega_1 + h(\Omega_1, \Omega_2)\zeta, \tag{35}$$

for any $\Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)})$. Then, $\check{D}$ is torsion-free and satisfies the Codazzi equation (3). So, $(\check{D}, h)$ is a statistical structure on the PS-Riemannian manifold (M, Ψ, w, ζ, h).

Also, from (5) and (35) we have $\kappa(\Omega_1, \Omega_2) = w(\Omega_1)\Omega_2 + w(\Omega_2)\Omega_1 + h(\Omega_1, \Omega_2)\zeta$. So, for this structure we have

$$\kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2) = 2\eta(\Omega_1)\Psi\Omega_2 + w(\Omega_2)\Psi\Omega_1 + h(\Omega_1, \Psi\Omega_2)\zeta.$$

Now, let us suppose that $\kappa(\Omega_1, \Psi\Omega_2) + \Phi\kappa(\Omega_1, \Omega_2) = 0$ is satisfied for this structure. Then, we have

$$2\eta(\Omega_1)\Psi\Omega_2 + w(\Omega_2)\Psi\Omega_1 + h(\Omega_1, \Psi\Omega_2)\zeta = 0.$$

Applying w to the last equation, we get $h(\Omega_1, \Psi\Omega_2) = 0 \Rightarrow \Psi = 0$ and this is a contradiction. So, $\kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2)$ cannot be zero. Hence, $(\check{D}, h)$ is a statistical structure on the PS-Riemannian manifold (M, Ψ, w, ζ, h) but it isn't a p-S statistical structure.

Example 4.5. *Let (Ψ, w, ζ, h) be a PS-Riemannian structure on M. Set the connection $\tilde{D}$ as*

$$\tilde{D}_{\Omega_1}\Omega_2 = \hat{D}_{\Omega_1}\Omega_2 + w(\Omega_1)w(\Omega_2)\zeta, \tag{36}$$

for any $\Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)})$. Then, $\tilde{D}$ is torsion-free and satisfies the Codazzi equation (3). So, $(\tilde{D}, h)$ is a statistical structure on the PS-Riemannian manifold (M, Ψ, w, ζ, h).

Also, from (5) and (36) we have $\kappa(\Omega_1, \Omega_2) = w(\Omega_1)w(\Omega_2)\zeta$. So, $\kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2) = 0$ is satisfied for the connection $\tilde{D}$. Hence $(\tilde{D}, h, \Psi, w, \zeta)$ is a PS-statistical structure on M.

Here, we obtain some results about the curvatures R, R^* and R^S. For this, we'll give some results for a PS-statistical manifold (M, Ψ, w, ζ, h).

Taking $\Omega_2 = \zeta$ in (30) and (30)* and using (9), we have

$$D^*_{\Omega_1}\zeta = \Psi\Omega_1 + w(D^*_{\Omega_1}\zeta)\zeta \tag{37}$$

and

$$D_{\Omega_1}\zeta = \Psi\Omega_1 + w(D_{\Omega_1}\zeta)\zeta, \tag{38}$$

respectively. Also, from (5), (8), (37) and 38) we have

$$\kappa(\Omega_1, \zeta) = D_{\Omega_1}\zeta - \hat{D}_{\Omega_1}\zeta = w(D_{\Omega_1}\zeta)\zeta \tag{39}$$

and

$$\kappa(\Omega_1, \zeta) = \hat{D}_{\Omega_1}\zeta - D^*_{\Omega_1}\zeta = -w(D^*_{\Omega_1}\zeta)\zeta, \tag{40}$$

respectively. Thus, from (37)-(40) we get

$$D^*_{\Omega_1}\zeta = \Psi\Omega_1 - \kappa(\Omega_1, \zeta) \tag{41}$$

$$D_{\Omega_1}\zeta = \Psi\Omega_1 + \kappa(\Omega_1, \zeta) \tag{42}$$

and

$$\Psi\kappa(\Omega_1, \zeta) = 0. \tag{43}$$

Furthermore, from (5), (10), (39) and (42) we have

$$\kappa(\Omega_1, \kappa(\Omega_2, \zeta)) = w(D_{\Omega_1}\zeta)w(D_{\Omega_2}\zeta)\zeta \tag{44}$$

and so, we get

$$\kappa(\Omega_1, \kappa(\Omega_2, \zeta)) = \kappa(\Omega_2, \kappa(\Omega_1, \zeta)). \tag{45}$$

Now, we can give some results about the curvatures R, R^* and R^S.

Using (14), (15) and (45) in (21), we have

$$R(\Omega_1, \Omega_2)\zeta = w(\Omega_1)\Omega_2 - w(\Omega_2)\Omega_1 + (\hat{D}_{\Omega_1}\kappa)(\Omega_2, \zeta) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \zeta) \tag{46}$$

and

$$R(\zeta, \Omega_1)\Omega_2 = w(\Omega_2)\Omega_1 - h(\Omega_1, \Omega_2)\zeta + (\hat{D}_{\zeta}\kappa)(\Omega_1, \Omega_2) - (\hat{D}_{\Omega_1}\kappa)(\zeta, \Omega_2) + \kappa(\zeta, \kappa(\Omega_1, \Omega_2)) - \kappa(\Omega_1, \kappa(\zeta, \Omega_2)). \tag{47}$$

From Lemma 3.1-(iv) and (46), we get

$$w(R(\Omega_1, \Omega_2)\Omega_3) = -w(\Omega_1)h(\Omega_2, \Omega_3) + w(\Omega_2)h(\Omega_1, \Omega_3) + h((\hat{D}_{\Omega_1}\kappa)(\Omega_2, \zeta) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \zeta), \Omega_3). \tag{48}$$

Also, from (21), we have

$$\begin{aligned} w(R(\Omega_1, \Omega_2)\Omega_3) &= -w(\Omega_1)h(\Omega_2, \Omega_3) + w(\Omega_2)h(\Omega_1, \Omega_3) \\ &+ w((\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_3) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_3) + \kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) - \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3))). \end{aligned} \tag{49}$$

Thus from (48) and (49), we obtain that

$$h((\hat{D}_{\Omega_1}\kappa)(\Omega_2, \zeta) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \zeta), \Omega_3) = w((\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_3) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_3) + \kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) - \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3))). \tag{50}$$

Similarly, using (14), (15) and (45) in (24), we have

$$R^*(\Omega_1, \Omega_2)\zeta = w(\Omega_1)\Omega_2 - w(\Omega_2)\Omega_1 - (\hat{D}_{\Omega_1}\kappa)(\Omega_2, \zeta) + (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \zeta) \tag{51}$$

and

$$R^*(\zeta,\Omega_1)\Omega_2 = w(\Omega_2)\Omega_1 - h(\Omega_1,\Omega_2)\zeta - (\hat{D}_\zeta\kappa)(\Omega_1,\Omega_2) + (\hat{D}_{\Omega_1}\kappa)(\zeta,\Omega_2) + \kappa(\zeta,\kappa(\Omega_1,\Omega_2)) - \kappa(\Omega_1,\kappa(\zeta,\Omega_2)). \quad (52)$$

From Lemma 3.1-(iv) and (51), we get

$$w(R^*(\Omega_1,\Omega_2)\Omega_3) = -w(\Omega_1)h(\Omega_2,\Omega_3) + w(\Omega_2)h(\Omega_1,\Omega_3) - h((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\zeta) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\zeta),\Omega_3). \quad (53)$$

Also, from (24), we have

$$\begin{aligned} w(R^*(\Omega_1,\Omega_2)\Omega_3) &= -w(\Omega_1)h(\Omega_2,\Omega_3) + w(\Omega_2)h(\Omega_1,\Omega_3) \\ &\quad - w((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\Omega_3) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\Omega_3) - \kappa(\Omega_1,\kappa(\Omega_2,\Omega_3)) + \kappa(\Omega_2,\kappa(\Omega_1,\Omega_3))). \end{aligned} \quad (54)$$

Thus from (53) and (54), we obtain that

$$h((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\zeta)-(\hat{D}_{\Omega_2}\kappa)(\Omega_1,\zeta),\Omega_3) = w((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\Omega_3)-(\hat{D}_{\Omega_2}\kappa)(\Omega_1,\Omega_3)-\kappa(\Omega_1,\kappa(\Omega_2,\Omega_3))+\kappa(\Omega_2,\kappa(\Omega_1,\Omega_3))). \quad (55)$$

Furthermore, from (50) and (55) we have

$$w(\kappa(\Omega_1,\kappa(\Omega_2,\Omega_3))) = w(\kappa(\Omega_2,\kappa(\Omega_1,\Omega_3))). \quad (56)$$

Hence, the equations (49) and (54) reduces to

$$w(R(\Omega_1,\Omega_2)\Omega_3) = -w(\Omega_1)h(\Omega_2,\Omega_3) + w(\Omega_2)h(\Omega_1,\Omega_3) + w((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\Omega_3) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\Omega_3)) \quad (57)$$

and

$$w(R^*(\Omega_1,\Omega_2)\Omega_3) = -w(\Omega_1)h(\Omega_2,\Omega_3) + w(\Omega_2)h(\Omega_1,\Omega_3) - w((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\Omega_3) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\Omega_3)), \quad (58)$$

respectively. Also, the equations (50) and (55) reduces to

$$h((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\zeta) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\zeta),\Omega_3) = w((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\Omega_3) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\Omega_3)). \quad (59)$$

Likewise, let us obtain some equations about the statistical curvature of a PS-statistical manifold (M,Ψ,w,ζ,h).

From (14), (26) and (45) (or from (19), (46) and (51)), we get

$$R^S(\Omega_1,\Omega_2)\zeta = w(\Omega_1)\Omega_2 - w(\Omega_2)\Omega_1 \quad (60)$$

and from (15) and (26) (or from (19), (47) and (52)), we have

$$R^S(\zeta,\Omega_1)\Omega_2 = w(\Omega_2)\Omega_1 - h(\Omega_1,\Omega_2)\zeta + \kappa(\zeta,\kappa(\Omega_1,\Omega_2)) - \kappa(\Omega_1,\kappa(\zeta,\Omega_2)). \quad (61)$$

From (60) (or from (61) and (45)), we get

$$R^S(\zeta,\Omega_1)\zeta = \Omega_1 - w(\Omega_1)\zeta. \quad (62)$$

From Theorem 3.2-(iv) and (60) (or from (26) and (56)), we obtain that

$$w(R^S(\Omega_1,\Omega_2)\Omega_3) = -w(\Omega_1)h(\Omega_2,\Omega_3) + w(\Omega_2)h(\Omega_1,\Omega_3). \quad (63)$$

At the end of this section, let us deal with the Ricci tensor of these curvatures on a PS-statistical manifold.

Let $\{\Lambda_i\}$, $i = 1,2,...,n$, be an orthonormal basis of the tangent space at any point p of the PS-statistical manifold. From (21), we have

$$\begin{aligned} Ric(\Omega_1,\Omega_2) &= \sum_{i=1}^{n} h(R(\Omega_1,\Lambda_i)\Lambda_i,\Omega_2) \\ &= \widehat{Ric}(\Omega_1,\Omega_2) + \sum_{i=1}^{n} h((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\Lambda_i) - (\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\Lambda_i) + \kappa(\Omega_1,\kappa(\Lambda_i,\Lambda_i)) - \kappa(\Lambda_i,\kappa(\Omega_1,\Lambda_i)),\Omega_2). \end{aligned} \quad (64)$$

From (12), (56) and (64), we get

$$Ric(\Omega_1,\zeta) = (1-n)w(\Omega_1) + \sum_{i=1}^{n} w((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\Lambda_i) - (\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\Lambda_i)). \tag{65}$$

Also, using the definition of Ricci tensor, from Lemma 3.1-(iv) and (46) we have

$$\begin{aligned} Ric(\Omega_1,\zeta) &= \sum_{i=1}^{n} h(R(\Omega_1,\Lambda_i)\Lambda_i,\zeta) \\ &= (1-n)w(\Omega_1) + \sum_{i=1}^{n} h((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\zeta) - (\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\zeta),\Lambda_i). \end{aligned} \tag{66}$$

From (65) and (66), we get

$$\sum_{i=1}^{n} w((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\Lambda_i) - (\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\Lambda_i)) = \sum_{i=1}^{n} h((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\zeta) - (\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\zeta),\Lambda_i)$$

and this equation is equivalent with the equation (59).

From (12) and (64) (or from the definition of Ricci tensor and (47)), we have

$$Ric(\zeta,\Omega_1) = (1-n)w(\Omega_1) + \sum_{i=1}^{n} h((\hat{D}_{\zeta}\kappa)(\Lambda_i,\Lambda_i) - (\hat{D}_{\Lambda_i}\kappa)(\zeta,\Lambda_i) + \kappa(\zeta,\kappa(\Lambda_i,\Lambda_i)) - \kappa(\Lambda_i,\kappa(\zeta,\Lambda_i)),\Omega_1). \tag{67}$$

Similarly, from (24)

$$\begin{aligned} Ric^*(\Omega_1,\Omega_2) &= \sum_{i=1}^{n} h(R^*(\Omega_1,\Lambda_i)\Lambda_i,\Omega_2) \\ &= \widehat{Ric}(\Omega_1,\Omega_2) - \sum_{i=1}^{n} h((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\Lambda_i) - (\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\Lambda_i) - \kappa(\Omega_1,\kappa(\Lambda_i,\Lambda_i)) + \kappa(\Lambda_i,\kappa(\Omega_1,\Lambda_i)),\Omega_2). \end{aligned} \tag{68}$$

From (12), (56) and (68), we get

$$Ric^*(\Omega_1,\zeta) = (1-n)w(\Omega_1) - \sum_{i=1}^{n} w((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\Lambda_i) - (\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\Lambda_i)). \tag{69}$$

Also, using the definition of Ricci tensor, from Lemma 3.1-(iv) and (51) we have

$$\begin{aligned} Ric^*(\Omega_1,\zeta) &= \sum_{i=1}^{n} h(R^*(\Omega_1,\Lambda_i)\Lambda_i,\zeta) \\ &= (1-n)w(\Omega_1) - \sum_{i=1}^{n} h((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\zeta) - (\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\zeta),\Lambda_i). \end{aligned} \tag{70}$$

From (69) and (70), we get

$$\sum_{i=1}^{n} w((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\Lambda_i) - (\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\Lambda_i)) = \sum_{i=1}^{n} h((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\Lambda_i) - (\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\Lambda_i),\Lambda_i)$$

and this equation is equivalent with the equation (59).

From (12) and (68) (or from the definition of Ricci tensor and (52)), we have

$$Ric^{*}(\zeta,\Omega_1) = (1-n)w(\Omega_1) - \sum_{i=1}^{n} h((\hat{D}_{\zeta}\kappa)(\Lambda_i,\Lambda_i) - (\hat{D}_{\Lambda_i}\kappa)(\zeta,\Lambda_i) - \kappa(\zeta,\kappa(\Lambda_i,\Lambda_i)) + \kappa(\Lambda_i,\kappa(\zeta,\Lambda_i)),\Omega_1). \quad (71)$$

Finally, let us give similar results for Ricci tensor of the curvature R^S on a PS-statistical manifold.

From (26), we have

$$Ric^{S}(\Omega_1,\Omega_2) = \widehat{Ric}(\Omega_1,\Omega_2) + \sum_{i=1}^{n} h(\kappa(\Omega_1,\kappa(\Lambda_i,\Lambda_i)) - \kappa(\Lambda_i,\kappa(\Omega_1,\Lambda_i)),\Omega_2). \quad (72)$$

From Theorem 3.2-(iv) and (60), we get

$$Ric^{S}(\Omega_1,\zeta) = \sum_{i=1}^{n} h(R^S(\Omega_1,\Lambda_i)\Lambda_i,\zeta) = -\sum_{i=1}^{n} h(R^S(\Omega_1,\Lambda_i)\zeta,\Lambda_i) = (1-n)w(\Omega_1). \quad (73)$$

From (12) and (72), we have

$$Ric^{S}(\zeta,\Omega_1) = (1-n)w(\Omega_1) + \sum_{i=1}^{n} h(\kappa(\zeta,\kappa(\Lambda_i,\Lambda_i)) - \kappa(\Lambda_i,\kappa(\zeta,\Lambda_i)),\Omega_1). \quad (74)$$

Since the Ricci tensor of R^S is symmetric, from (73) and (74) we obtain

$$\sum_{i=1}^{n} h(\kappa(\zeta,\kappa(\Lambda_i,\Lambda_i)) - \kappa(\Lambda_i,\kappa(\zeta,\Lambda_i)),\Omega_1) = 0. \quad (75)$$

Example 4.6. *Let we deal with the manifold $M = \{(x,y,z) \in \mathbb{R}^3,\ z \neq 0\}$ of dimension 3, where (x,y,z) are the standart coordinates in $\mathbb{R}^3$.*

We choose the vector fields $\{\Lambda_1,\Lambda_2,\Lambda_3\}$ as

$$\Lambda_1 = e^x\frac{\partial}{\partial y},\ \Lambda_2 = e^x\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right),\ \Lambda_3 = -\frac{\partial}{\partial x}, \quad (76)$$

which are linearly independent at each point of M.

Let h be the Riemannian metric defined by $h(\Lambda_i,\Lambda_j) = 0,\ i \neq j,\ i,j = 1,2,3$ and $h(\Lambda_\kappa,\Lambda_\kappa) = 1,\ \kappa = 1,2,3$.

Let w be the 1-form defined by $w(\Omega_3) = h(\Omega_3,\Lambda_3)$, for any $\Omega_3 \in \Gamma(TM^{(1,0)})$.

Let Ψ be the $(1,1)$-tensor field defined by

$$\Psi\Lambda_1 = \Lambda_1,\ \Psi\Lambda_2 = \Lambda_2,\ \Psi\Lambda_3 = 0. \quad (77)$$

Using the linearity of Ψ and h, we have $w(\Lambda_3) = 1$, $\Psi^2\Omega_3 = \Omega_3 - w(\Omega_3)\Lambda_3$ and $h(\Psi\Omega_3,\Psi\Omega_5) = h(\Omega_3,\Omega_5) - w(\Omega_3)w(\Omega_5)$, for any $\Omega_3,\Omega_5 \in \Gamma(TM^{(1,0)})$. Thus, for $\Lambda_3 = \zeta$, (Ψ,ζ,w,h) defines an almost paracontact metric structure on M.

Now, we have

$$[\Lambda_1,\Lambda_2] = 0,\ [\Lambda_1,\Lambda_3] = \Lambda_1,\ [\Lambda_2,\Lambda_3] = \Lambda_2. \quad (78)$$

The L-C connection $\hat{D}$ of h is given by Koszul's formula which is defined as

$$\begin{aligned} 2g(\hat{D}_{\Omega_1}\Omega_2,\Omega_3) &= \Omega_1 h(\Omega_2,\Omega_3) + \Omega_2 h(\Omega_1,\Omega_3) - \Omega_3 h(\Omega_1,\Omega_2) \\ &\quad - h(\Omega_1,[\Omega_2,\Omega_3]) - h(\Omega_2,[\Omega_1,\Omega_3]) + h(\Omega_3,[\Omega_1,\Omega_2]). \end{aligned} \quad (79)$$

Taking $\Lambda_3 = \zeta$ and using (79), we have

$$\begin{aligned} &\hat{D}_{\Lambda_1}\Lambda_1 = -\Lambda_3,\ \hat{D}_{\Lambda_1}\Lambda_2 = 0,\ \hat{D}_{\Lambda_1}\Lambda_3 = \Lambda_1, \\ &\hat{D}_{\Lambda_2}\Lambda_1 = 0,\ \hat{D}_{\Lambda_2}\Lambda_2 = -\Lambda_3,\ \hat{D}_{\Lambda_2}\Lambda_3 = \Lambda_2, \\ &\hat{D}_{\Lambda_3}\Lambda_1 = 0,\ \hat{D}_{\Lambda_3}\Lambda_2 = 0,\ \hat{D}_{\Lambda_3}\Lambda_3 = 0. \end{aligned} \tag{80}$$

From above, one can be easily see that (ϕ, ζ, w, h) is a PS-structure on M. Consequently, (M, ϕ, ζ, w, h) is a 3-dimensional PS-manifold (for detail, see [18]).

Now, let us suppose the PS-statistical structure (36) which is defined as $\tilde{D}_{\Omega_1}\Omega_2 = \hat{D}_{\Omega_1}\Omega_2 + w(\Omega_1)w(\Omega_2)\zeta$ ($\tilde{D}^*_{\Omega_1}\Omega_2 = \hat{D}_{\Omega_1}\Omega_2 - w(\Omega_1)w(\Omega_2)\zeta$ and $\kappa(\Omega_1, \Omega_2) = w(\Omega_1)w(\Omega_2)\zeta$) for this para-Sasakian manifold. Thus, from (80) we have

$$\begin{aligned} &\tilde{D}_{\Lambda_1}\Lambda_1 = \tilde{D}^*_{\Lambda_1}\Lambda_1 = -\Lambda_3,\ \tilde{D}_{\Lambda_1}\Lambda_2 = \tilde{D}^*_{\Lambda_1}\Lambda_2 = 0,\ \tilde{D}_{\Lambda_1}\Lambda_3 = \tilde{D}^*_{\Lambda_1}\Lambda_3 = \Lambda_1, \\ &\tilde{D}_{\Lambda_2}\Lambda_1 = \tilde{D}^*_{\Lambda_2}\Lambda_1 = 0,\ \tilde{D}_{\Lambda_2}\Lambda_2 = \tilde{D}^*_{\Lambda_2}\Lambda_2 = -\Lambda_3,\ \tilde{D}_{\Lambda_2}\Lambda_3 = \tilde{D}^*_{\Lambda_2}\Lambda_3 = \Lambda_2, \\ &\tilde{D}_{\Lambda_3}\Lambda_1 = \tilde{D}^*_{\Lambda_3}\Lambda_1 = 0,\ \tilde{D}_{\Lambda_3}\Lambda_2 = \tilde{D}^*_{\Lambda_3}\Lambda_2 = 0,\ \tilde{D}_{\Lambda_3}\Lambda_3 = -\tilde{D}^*_{\Lambda_3}\Lambda_3 = \Lambda_3. \end{aligned} \tag{81}$$

Actually, one can easily see from (81) that, $\tilde{T}(\Lambda_i, \Lambda_j) = 0$ and $(\tilde{D}_{\Lambda_i}h)(\Lambda_j, e_\kappa) = 0$ hold for all $i, j, \kappa = 1, 2, 3$. So, (D, h) is a statistical structure and since $\kappa(\Lambda_i, \Psi e_j) + \Psi\kappa(\Lambda_i, \Lambda_j) = 0$ holds for all $i, j = 1, 2, 3$, $(\tilde{D}, h, \Psi, w, \zeta)$ is a PS-statistical structure on M.

From the above results, we can obtain the components of the curvature tensors with respect to the connections D and D^*, respectively, as follows:

$$\begin{aligned} &\tilde{R}(\Lambda_1, \Lambda_2)\Lambda_1 = \Lambda_2,\ \tilde{R}(\Lambda_1, \Lambda_2)\Lambda_2 = -\Lambda_1,\ \tilde{R}(\Lambda_1, \Lambda_2)\Lambda_3 = 0, \\ &\tilde{R}(\Lambda_1, \Lambda_3)\Lambda_1 = 2\Lambda_3,\ \tilde{R}(\Lambda_1, \Lambda_3)\Lambda_2 = 0,\ \tilde{R}(\Lambda_1, \Lambda_3)\Lambda_3 = 0, \\ &\tilde{R}(\Lambda_2, \Lambda_3)\Lambda_1 = 0,\ \tilde{R}(\Lambda_2, \Lambda_3)\Lambda_2 = 2\Lambda_3,\ \tilde{R}(\Lambda_2, \Lambda_3)\Lambda_3 = 0. \end{aligned} \tag{82}$$

and

$$\begin{aligned} &\tilde{R}^*(\Lambda_1, \Lambda_2)\Lambda_1 = \Lambda_2,\ \tilde{R}^*(\Lambda_1, \Lambda_2)\Lambda_2 = -\Lambda_1,\ \tilde{R}^*(\Lambda_1, \Lambda_2)\Lambda_3 = 0, \\ &\tilde{R}^*(\Lambda_1, \Lambda_3)\Lambda_1 = 0,\ \tilde{R}^*(\Lambda_1, \Lambda_3)\Lambda_2 = 0,\ \tilde{R}^*(\Lambda_1, \Lambda_3)\Lambda_3 = -2\Lambda_1, \\ &\tilde{R}^*(\Lambda_2, \Lambda_3)\Lambda_1 = 0,\ \tilde{R}^*(\Lambda_2, \Lambda_3)\Lambda_2 = 0,\ \tilde{R}^*(\Lambda_2, \Lambda_3)\Lambda_3 = -2\Lambda_2. \end{aligned} \tag{83}$$

With the help of the equations (82) and (83), we get the Ricci tensors of the curvature tensors $\tilde{R}$ and $\tilde{R}^*$, respectively, as follows:

$$\begin{aligned} &\widetilde{Ric}(\Lambda_1, \Lambda_1) = -1,\ \widetilde{Ric}(\Lambda_1, \Lambda_2) = 0,\ \widetilde{Ric}(\Lambda_1, \Lambda_3) = 0, \\ &\widetilde{Ric}(\Lambda_2, \Lambda_1) = 0,\ \widetilde{Ric}(\Lambda_2, \Lambda_2) = -1,\ \widetilde{Ric}(\Lambda_2, \Lambda_3) = 0, \\ &\widetilde{Ric}(\Lambda_3, \Lambda_1) = 0,\ \widetilde{Ric}(\Lambda_3, \Lambda_2) = 0,\ \widetilde{Ric}(\Lambda_3, \Lambda_3) = -4 \end{aligned} \tag{84}$$

and

$$\begin{aligned} &\widetilde{Ric}^*(\Lambda_1, \Lambda_1) = -3,\ \widetilde{Ric}^*(\Lambda_1, \Lambda_2) = 0,\ \widetilde{Ric}^*(\Lambda_1, \Lambda_3) = 0, \\ &\widetilde{Ric}^*(\Lambda_2, \Lambda_1) = 0,\ \widetilde{Ric}^*(\Lambda_2, \Lambda_2) = -3,\ \widetilde{Ric}^*(\Lambda_2, \Lambda_3) = 0, \\ &\widetilde{Ric}^*(\Lambda_3, \Lambda_1) = 0,\ \widetilde{Ric}^*(\Lambda_3, \Lambda_2) = 0,\ \widetilde{Ric}^*(\Lambda_3, \Lambda_3) = 0. \end{aligned} \tag{85}$$

Furthermore, from the definition of the statistical curvature tensor, (82) and (83), we can obtain the components of the statistical curvature tensor as

$$\begin{aligned} &\tilde{R}^S(\Lambda_1, \Lambda_2)\Lambda_1 = \Lambda_2,\ \tilde{R}^S(\Lambda_1, \Lambda_2)\Lambda_2 = -\Lambda_1,\ \tilde{R}^S(\Lambda_1, \Lambda_2)\Lambda_3 = 0, \\ &\tilde{R}^S(\Lambda_1, \Lambda_3)\Lambda_1 = \Lambda_3\ ,\ \tilde{R}^S(\Lambda_1, \Lambda_3)\Lambda_2 = 0,\ \tilde{R}^S(\Lambda_1, \Lambda_3)\Lambda_3 = -\Lambda_1, \\ &\tilde{R}^S(\Lambda_2, \Lambda_3)\Lambda_1 = 0,\ \tilde{R}^S(\Lambda_2, \Lambda_3)\Lambda_2 = \Lambda_3,\ \tilde{R}^S(\Lambda_2, \Lambda_3)\Lambda_3 = -\Lambda_2 \end{aligned} \tag{86}$$

and from (86), we get the Ricci tensors of the statistical curvature tensor as

$$\begin{aligned}
&\widetilde{Ric}^S(\Lambda_1,\Lambda_1)=-2,\ \widetilde{Ric}^S(\Lambda_1,\Lambda_2)=0,\ \widetilde{Ric}^S(\Lambda_1,\Lambda_3)=0,\\
&\widetilde{Ric}^S(\Lambda_2,\Lambda_1)=0,\ \widetilde{Ric}^S(\Lambda_2,\Lambda_2)=-2,\ \widetilde{Ric}^S(\Lambda_2,\Lambda_3)=0,\\
&\widetilde{Ric}^S(\Lambda_3,\Lambda_1)=0,\ \widetilde{Ric}^S(\Lambda_3,\Lambda_2)=0,\ \widetilde{Ric}^S(\Lambda_3,\Lambda_3)=-2.
\end{aligned} \tag{87}$$

5. SOME CHARACTERIZATIONS FOR THESE MANIFOLDS

In this section, we investigate some special curvature conditions for a PS-statistical manifold. For this, firstly we study on Ricci semi-symmetric and Ricci pseudo-symmetric PS-statistical manifolds and after we give some results for ζ-projectively flat, projectively flat and Ψ-projectively semi-symmetric PS-statistical manifolds.

5.1. Ricci Semi-Symmetric and Ricci Pseudo-Symmetric PS-Statistical Manifolds

We know that, if (M,h) is a connected n-dimensional, $n\geq 3$, semi-Riemannian manifold of class C^∞, then for a $(0,k)$-tensor field T on M, $k\geq 1$, the $(0,k+2)$-tensors $\mathcal{R}\cdot T$ and $Q(h,T)$ are defined by

$$\begin{aligned}
(\mathcal{R}\cdot T)(X_1,...,X_k;\Omega_1,\Omega_2)&=(R(\Omega_1,\Omega_2)\cdot T)(X_1,...,X_k)\\
&=-T(R(\Omega_1,\Omega_2)X_1,X_2,...,X_k)\\
&\quad-...-T(X_1,...,X_{k-1},R(\Omega_1,\Omega_2)X_k)
\end{aligned} \tag{88}$$

and

$$\begin{aligned}
Q(h,T)(X_1,...,X_k;\Omega_1,\Omega_2)&=((\Omega_1\wedge_h\Omega_2)\cdot T)(X_1,...,X_k)\\
&=-T((\Omega_1\wedge_h\Omega_2)X_1,X_2,...,X_k)\\
&\quad-...-T(X_1,...,X_{k-1},(\Omega_1\wedge_h\Omega_2)X_k)
\end{aligned} \tag{89}$$

respectively, for all $X_1,...,X_k,\Omega_1,\Omega_2\in\Gamma(TM^{(1,0)})$. Here R is the Riemannian curvature tensor field of M and $\mathcal{R}$ is the Riemannian Christoffel tensor field given by $\mathcal{R}(\Omega_1,\Omega_2,\Omega_3,\Omega_4)=h(R(\Omega_1,\Omega_2)\Omega_3,\Omega_4)$. Also, the endomorphisms are defined by

$$R(\Omega_1,\Omega_2)\Omega_3=[D_{\Omega_1},D_{\Omega_2}]\Omega_3-D_{[\Omega_1,\Omega_2]}\Omega_3 \tag{90}$$

and

$$(\Omega_1\wedge_h\Omega_2)\Omega_3=h(\Omega_2,\Omega_3)\Omega_1-h(\Omega_1,\Omega_3)\Omega_2. \tag{91}$$

So, we can give the following definition for PS-statistical manifolds:

Definition 5.1. *Let M be an n-dimensional PS-statistical manifold. Then, M is called Ricci pseudo-symmetric with respect to R^S if at every point of M the tensor $\mathcal{R}^S\cdot Ric^S$ and $Q(h,Ric^S)$ are linearly dependent. This is equivalent to the fact that the equality*

$$\mathcal{R}^S\cdot Ric^S=L_{Ric^S}Q(h,Ric^S), \tag{92}$$

hold the set $U_{Ric^S}=\{x\in M: Q(h,Ric^S)\neq 0\}$, for some function L_{Ric^S} on U_{Ric^S}, where Ric^S is the Ricci tensor of R^S. Also, if $L_{Ric^S}=0$ holds in (92), i.e.,

$$\mathcal{R}^S\cdot Ric^S=0 \tag{93}$$

holds, then M is called Ricci semi-symmetric with respect to R^S.

Firstly let us assume that M is Ricci semi-symmetric with respect to R^S. Then, we can write

$$\begin{aligned}(\mathcal{R}^S \cdot Ric^S)(U,V;\Omega_1,\Omega_2) &= (R^S(\Omega_1,\Omega_2)\cdot Ric^S)(U,V)\\ &= -Ric^S(R^S(\Omega_1,\Omega_2)U,V) - Ric^S(U,R^S(\Omega_1,\Omega_2)V). \end{aligned} \tag{94}$$

Using (93) in (94), we have

$$Ric^S(R^S(\Omega_1,\Omega_2)U,V) + Ric^S(U,R^S(\Omega_1,\Omega_2)V) = 0. \tag{95}$$

Putting $\Omega_2 = V = \zeta$ in (95) and using (61), we get

$$\begin{aligned}&- w(U)Ric^S(\Omega_1,\zeta) + h(\Omega_1,U)Ric^S(\zeta,\zeta) - Ric^S(\kappa(\zeta,\kappa(\Omega_1,U)),\zeta) + Ric^S(\kappa(\Omega_1,\kappa(\zeta,U)),\zeta)\\ &+ w(\Omega_1)Ric^S(U,\zeta) - Ric^S(U,\Omega_1) = 0. \end{aligned} \tag{96}$$

Using (56) and (73) in (96), we have

$$Ric^S(U,\Omega_1) = (1-n)h(U,\Omega_1).$$

Hence, we can state the following Theorem:

Theorem 5.2. *Let M be a PS-statistical manifold with PS-statistical structure (D,h,Ψ,w,ζ). If M is Ricci semi-symmetric with respect to statistical curvature R^S, then M is Einstein with respect to Ricci tensor of R^S.*

Now, let us assume that M is Ricci pseudo-symmetric with respect to R^S. Then, from (92) we can write

$$(R^S(\Omega_1,\Omega_2)\cdot Ric^S)(U,V) = -L_{Ric^S}\left\{Ric^S((\Omega_1\wedge_h\Omega_2)U,V) + Ric^S(U,(\Omega_1\wedge_h\Omega_2)V)\right\}, \tag{97}$$

for all $\Omega_1,\Omega_2,U,V \in \Gamma(TM^{(1,0)})$. Using (91) in (97), we get

$$-Ric^S(R^S(\Omega_1,\Omega_2)U,V) - Ric^S(U,R^S(\Omega_1,\Omega_2)V) = -L_{Ric^S}\left\{\begin{array}{l} Ric^S(\Omega_1,V)h(\Omega_2,U) - Ric^S(\Omega_2,V)h(\Omega_1,U)\\ +Ric^S(U,\Omega_1)h(\Omega_2,V) - Ric^S(U,\Omega_2)h(\Omega_1,V)\end{array}\right\}. \tag{98}$$

Putting $\Omega_2 = V = \zeta$ in (98) and using (56), (61) and (73), we get

$$Ric^S(U,\Omega_1) = (1-n)h(U,\Omega_1).$$

So, we can give the following Theorem:

Theorem 5.3. *Let M be a PS-statistical manifold with PS-statistical structure (D,h,Ψ,w,ζ). If M is Ricci pseudo-symmetric with respect to statistical curvature R^S, then M is Einstein with respect to Ricci tensor of R^S.*

5.2. Projectively Flat and Ψ-Projectively Semi-Symmetric PS-Statistical Manifolds

Let M be an n-dimensional PS-statistical manifold. Then, the projective curvature tensor P^S of M with respect to the statistical curvature R^S is defined by

$$P^S(\Omega_1,\Omega_2)\Omega_3 = R^S(\Omega_1,\Omega_2)\Omega_3 - \frac{1}{n-1}\left\{Ric^S(\Omega_2,\Omega_3)\Omega_1 - Ric^S(\Omega_1,\Omega_3)\Omega_2\right\} \tag{99}$$

for all $\Omega_1,\Omega_2,\Omega_3 \in \Gamma(TM^{(1,0)})$.

Definition 5.4. *A PS-statistical manifold is called projectively flat with respect to the statistical curvature R^S, if the projective curvature tensor P^S vanishes at each point of the manifold. Also, a PS-statistical manifold is called ζ-projectively flat with respect to the statistical curvature R^S, if $P^S(\Omega_1,\Omega_2)\zeta = 0$ holds for all $\Omega_1,\Omega_2 \in \Gamma(TM^{(1,0)})$.*

Theorem 5.5. *Let M be a PS-statistical manifold with PS-statistical structure (D, h, Ψ, w, ζ). Then, M is ζ-projectively flat with respect to the statistical curvature R^S.*

Proof. It is obvious from (60), (73) and (99). □

Now, let us suppose that M is projectively flat with respect to the statistical curvature R^S. Then, since $P^S = 0$, from (99) we can write

$$R^S(\Omega_1, \Omega_2)\Omega_3 = \frac{1}{n-1}\left\{Ric^S(\Omega_2, \Omega_3)\Omega_1 - Ric^S(\Omega_1, \Omega_3)\Omega_2\right\}. \tag{100}$$

Taking $\Omega_1 = \zeta$ in (100) and using (61), we have

$$w(\Omega_3)\Omega_2 - h(\Omega_2, \Omega_3)\zeta + \kappa(\zeta, \kappa(\Omega_2, \Omega_3)) - \kappa(\Omega_2, \kappa(\zeta, \Omega_3)) = \frac{1}{n-1}\left\{Ric^S(\Omega_2, \Omega_3)\zeta - (1-n)w(\Omega_3)\Omega_2\right\}. \tag{101}$$

Applying w to (101), from (56) we get

$$Ric^S(\Omega_2, \Omega_3) = (1-n)h(\Omega_2, \Omega_3).$$

Thus, we have

Theorem 5.6. *Let M be a PS-statistical manifold with PS-statistical structure (D, h, Ψ, w, ζ). If M is projectively flat with respect to statistical curvature R^S, then M is Einstein with respect to Ricci tensor of R^S.*

Definition 5.7. *A PS-statistical manifold is called Ψ-projectively semi-symmetric with respect to the statistical curvature R^S, if it satisfies $(P^S(\Omega_1, \Omega_2)\Psi)\Omega_3 = 0$ holds for all $\Omega_1, \Omega_2, \Omega_3 \in \Gamma(TM^{(1,0)})$.*

Finally, let us assume that M is Ψ-projectively semi-symmetric with respect to the statistical curvature R^S. So, from $(P^S(\Omega_1, \Omega_2)\Psi)\Omega_3 = 0$ we can write

$$P^S(\Omega_1, \Omega_2)\Psi\Omega_3 - \Psi P^S(\Omega_1, \Omega_2)\Omega_3 = 0. \tag{102}$$

Using (99) in (102) and taking $\Omega_1 = \zeta$, from (61) and (74) we have

$$-h(\Omega_2, \Psi\Omega_3)\zeta + \kappa(\zeta, \kappa(\Omega_2, \Psi\Omega_3)) - \kappa(\Omega_2, \kappa(\zeta, \Psi\Omega_3)) - \frac{1}{n-1}Ric^S(\Omega_2, \Psi\Omega_3)\zeta - \Psi\kappa(\zeta, \kappa(\Omega_2, \Omega_3)) + \Psi\kappa(\Omega_2, \kappa(\zeta, \Omega_3)) = 0. \tag{103}$$

Applying w to (103), from (56) we get

$$Ric^S(\Omega_2, \Psi\Omega_3) = (1-n)h(\Omega_2, \Psi\Omega_3). \tag{104}$$

Taking $\Psi\Omega_2$ instead of Ω_2 and using (9), (29), (56) and (72), we obtain

$$Ric^S(\Omega_2, \Omega_3) + 2\sum_{i=1}^{n}\{-h(\kappa(\Omega_2, \kappa(\Lambda_i, \Lambda_i)), \Omega_3) + w(\kappa(\Omega_2, \kappa(\Lambda_i, \Lambda_i)))w(\Omega_3)\} = (1-n)h(\Omega_2, \Omega_3). \tag{105}$$

Hence, we can give the following Theorem:

Theorem 5.8. *Let M be a PS-statistical manifold with PS-statistical structure (D, h, Ψ, w, ζ). If M is Ψ-projectively semi-symmetric with respect to statistical curvature R^S and*

$$\sum_{i=1}^{n}\{h(\kappa(\Omega_2, \kappa(\Lambda_i, \Lambda_i)), \Omega_3) - w(\kappa(\Omega_2, \kappa(\Lambda_i, \Lambda_i)))w(\Omega_3)\} = 0$$

holds for all $\Omega_2, \Omega_3 \in \Gamma(TM^{(1,0)})$, then M is Einstein with respect to Ricci tensor of R^S.

6. CONCLUSION

One of the fundamental concept in information theory is that of the Fisher-Rao Information Matrix, which provides us with another measure of the distance between two different probability distributions. Such a measure endows the statistical manifold with a Riemannian structure. In fact, while the relative entropy does not define a real distance between distributions (for example, it is not symmetric), it can be shown that the Fisher-Rao Information Matrix arises as the Hessian of the relative entropy over a stationary point. The entries of such a matrix are in correspondence with the components of the metric tensor over the manifold of probability distributions [4].

On the other hand, the role played by differential geometry in statistics was not fully acknowledged until 1975 when Efron first introduced the concept of statistical curvature for one-parameter models and emphasized its importance in the theory of statistical estimation. Efron pointed out how any regular parametric family could be approximated locally by a curved exponential family and that the curvature of these models measures their departure from exponentiality. It turned out that this concept was intimately related to Fisher's theory of information loss. Efron's formal theory did not use all the bells and whistles of differential geometry. The first step to an elegant geometric theory was done by Dawid, who introduced a connection on the space of all positive probability distributions and showed that Efron's statistical curvature is induced by this connection. The use of differential geometry in its elegant splendor for the elaboration of previous ideas was systematically achieved by Amari, who studied the informational geometric properties of a manifold with a Fisher metric on it. This is the reason why sometimes this is also called the Fisher–Efron–Amari theory [5]. In the light of these studies, we focused on the curvature tensors of para-Sasakian (PS) statistical manifolds in terms of differential geometry. We started building this with Theorem 1, which we have used connections while doing it. In the context of PS geometry there is another connection of geometric significance which is parallel with respect to the metric and the other tensors defining the contact-metric strucuture. We have given our results using the connections ∇ and ∇^* on statistical manifolds. We have also studied the Ricci tensor of the statistical curvature and studied the cases of the manifold being Einstein under certain conditions in Theorem 3, Theorem 4 and Theorem 6. We have proved the projective flatness of the PS-statistical manifold. We believe that the concepts investigated in this work can be also studied in some new settings. The submanifolds of this subject can be examined as well as the inequality situation.

References

[1] T. Adati and K. Matsumoto; On Conformally Recurrent and Conformally Symmetric P-Sasakian Manifold, TRU Math., 13, 25-32, 1977.

[2] S. Amari; Differential-Geometrical Methods in Statistics, Lecture Notes in Statistics, 28, Springer, New York, 1985.

[3] A.M. Blaga and M. Crasmareanu; Golden Statistical Structures, Comptes rendus de l'Acad emie bulgare des Sciences, 69(9), 1113-1120, 2016.

[4] A. Bravetti and C. S. Lopez-Monsalvo; Para-Sasakian geometry in thermodynamic fluctuation theory, Journal of Physics A Mathematical and General 48:125206, DOI: 10.1088/1751-8113/48/12/125206, 2015.

[5] O. Calin and C. Udrişte; Geometric Modeling in Probability and Statistics, Springer, 2014.

[6] H. Furuhata; Hypersurfaces in Statistical Manifolds, Differential Geometry and its Applications, 27, 420-429, 2009.

[7] H. Furuhata, I. Hasegawa, Y. Okuyama, K. Sato and M.H. Shahid; Sasakian Statistical Manifolds, Journal of Geometry and Physics, 117, 179-186, 2017.

[8] H. Furuhata, I. Hasegawa, Y. Okuyama and K. Sato; Kenmotsu Statistical Manifolds and Warped Product, J. Geom., DOI 10.1007/s00022-017-0403-1, 2017.

[9] A. Kazan; Conformally-Projectively Trans-Sasakian Statistical Manifolds, Physica A: Statistical Mechanics and its Applications, 535, 122441, DOI: https://doi.org/10.1016/j.physa.2019.122441, 2019.

[10] S. Kazan; Anti-invariant $\xi^{\perp}$-cosymplectic-like statistical submersions, Thermal Science, 26(4A), 2991-3001, 2022.

[11] S. Kazan and K. Takano; Anti-Invariant Holomorphic Statistical Submersions, arXiv preprint arXiv:2209.03627, 2022.

[12] T. Kurose; Dual Connections and Affine Geometry, Math. Z., 203, 115-121, 1990.

[13] H. Matsuzoe, J-I. Takeuchi and S-I. Amari; Equiaffine Structures on Statistical Manifolds and Bayesian Statistics, Differential Geometry and its Applications, 24, 567–578, 2006.

[14] M. Noguchi; Geometry of Statistical Sanifolds, Differential Geometry and its Applications, 2, 197-222, 1992.

[15] C.R. Rao; Information and Accuracy Attainable in the Estimation of Statistical Parameters, Bulletin of the Calcutta Mathematical Society, 37, 81–91, 1945.

[16] S. Sasaki; On Differentiable Manifolds with Certain Structures which are Closely Related to Almost Contact Structure. I, The Tohoku Mathematical Journal. Second Series, vol. 12, 459–476, 1960.
[17] I. Sato; On a Structure Similar to the Almost Contact Structure, Tensor New Series, vol. 30, no. 3, pp. 219–224, 1976.
[18] S.S. Shukla and M.K. Shukla; On Ψ-Symmetric Para-Sasakian Manifolds, Int. Journal of Math. Analysis, 4(16), 761-769, 2010.
[19] S. Kazan and A. Kazan; Sasakian Statistical Manifolds with Semi-Symmetric Metric Connection, Universal Journal of Mathematics and Applications, 1(4), 226-232, 2018.
[20] K. Takano; Statistical manifolds with Almost Complex Structures and its Statistical Submersions, Tensor N.S., 65, 128–142, 2004.
[21] K. Takano; Statistical Manifolds with Almost Contact Structures and its Statistical Submersions, Journal of Geometry, 85, 171-187, 2006.
[22] A-D. Vilcu and G-E. Vilcu; Statistical Manifolds with almost Quaternionic Structures and Quaternionic Kähler-like Statistical Submersions, Entropy, 17, 6213-6228, 2015.
[23] J. Zhang; A Note on Curvature of α-Connections of a Statistical Manifold, Annals of the Institute of Statistical Mathematics, 59, 161-170, 2007.

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On the challenge of identifying space dependent coefficient in space-time fractional diffusion equations by fractional scaling transformations method

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Abstract. In this study, we get over the challenge of recovering unknown space dependent coefficient in space-time fractional diffusion equations by means of fractional scaling transformations method. Fractional differential equation is given in the sense of the conformable fractional derivative having substantial properties. By these properties and fractional scaling transformations method the fractional problem is reduced into integer order problem which allows us to tackle the problem better. Then we establish the solution and unknown coefficient of the reduced problem. Later, by employing inverse transformation, the solution and unknown coefficient of the fractional problem are obtained. Finally, some examples are presented to illustrate the implementation and effectiveness of the method.

1. Introduction

Last couple of decades fractional differential equations play a significant role in modelling of various processes. As a result, they attract growing attention of many scientists in diverse branches of sciences such as engineering, mathematics, chemistry and physics [1–8]. Consequently, numerous analytical and numerical methods have been utilized to construct solutions of mathematical problems including fractional differential equations [9–15].

Therefore, identification of unknown coefficients in fractional differential equations with or without additional measured data becomes one of the trend challenges in inverse problems [16–18]. Hence, many researchers in various research areas have been developing new methods to tackle with this kind of inverse problems including fractional derivatives [16–20].

In this research, our focus is on establishing space dependent diffusivity coefficient and the solution of the mathematical problem including space-time fractional diffusion equation by means of fractional scaling transformation methods. The main advantage of this method is that it turns fractional order differential equations into integer order differential equations which makes the problem easier to tackle with. We remark that this method works out for the fractional differential equations in the sense of conformable fractional derivative. The main goal in this article is to reveal the unknown coefficient of the following

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governing space-time fractional diffusion equation:

$$D_t^{\alpha}u(x,t) = D_x^{\beta}(k(x)D_x^{\beta}u(x,t)) + f(x,t), 0 < x < x_1, 0 < t < t_1, 0 < \alpha, \beta \leqslant 1, \tag{1}$$

where $u(x,t)$ and $k(x) > 0$ represent the temperature and thermal diffusivity, respectively. Associated to (1) the prescribed initial condition is

$$u(x,0) = \varphi(x), 0 \leqslant x \leqslant x_1, \tag{2}$$

and the prescribed Dirichlet boundary conditions are

$$D_x^{\beta}u(0,t) = 0, 0 < t \leqslant t_1, \tag{3}$$

$$u(x_1,t) = g(t), 0 < t \leqslant t_1, \tag{4}$$

with additional condition

$$u(x,t_1) = E(x), 0 < x \leqslant x_1. \tag{5}$$

Having the condition $k(x) > 0$ makes the problem (1)-(5) well-posed.

2. Preliminaries

Definition 1. [21] Given a function $f : [0,\infty) \to R$. Then the conformable fractional derivative of f of order α is defined by

$$T_t^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{6}$$

for all $t > 0$, $\alpha \epsilon (0,1)$.
Theorem 2. [21] Let $\alpha \epsilon (0,1]$ and f be α-differentiable at a point $t > 0$. Then $T_t^{\alpha} f(t) = t^{1-\alpha} f'(t)$.
Theorem 3. [21] Let $\alpha \epsilon (0,1]$ and f, g be α-differentiable at a point $t > 0$. Then,
$i) T^{\alpha}(af + bg) = aT^{\alpha}(f) + bT^{\alpha}(g)$ for all real constant a, b,
$ii) T^{\alpha}(t^p) = pt^{p-\alpha}$ for all $p \epsilon R$,
$iii) T^{\alpha}(fg) = fT^{\alpha}(g) + gT^{\alpha}(f)$,
$iv) T^{\alpha}(\frac{f}{g}) = \frac{gT^{\alpha}(f) - fT^{\alpha}(g)}{g^2}$,
$v) T^{\alpha}(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
Definition 4. [21] Let $\alpha \epsilon (n, n+1], n \epsilon N$ and f be an α-differentiable at t where $t > 0$, then the conformable fractional derivative of f of order α is defined as

$$T_t^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f^{(n)}(t+\varepsilon t^{n+1-\alpha}) - f^{(n)}(t)}{\varepsilon}, \tag{7}$$

where f is n-differentiable at $t > 0$.
Definition 5. [21] The conformable α-fractional integral of a function f is defined by

$$I_t^{\alpha}(f)(t) = \int_0^t \frac{f(x)}{x^{1-\alpha}} dx, \alpha \epsilon (0,1). \tag{8}$$

Theorem 6. [21, 22] Let $a \geqslant 0$ and $\alpha \in (0,1)$. Also, let $f : (a,b) \to R$ be a continuous function such that I_{α} exist, then for all $t > a$, we have

$$T_t^{\alpha}(I_t^{\alpha}(f)(t)) = f(t), t \geq 0, \tag{9}$$

$$I_t^{\alpha}(T_t^{\alpha}(f)(t)) = f(t) - f(a). \tag{10}$$

3. Analysis of the new fractional derivative

By means of the following fractional scaling transformations

$$X = \frac{x^{\beta}}{\beta}, T = \frac{t^{\alpha}}{\alpha}, u(x,t) = V(X,T), \tag{11}$$

the problem (1)-(5) is converted to into the following integer order problem

$$V_T = (k(X)V_X)_X + \overline{f}(X,T), 0 < X < \frac{x_1^{\beta}}{\beta}, 0 < T < \frac{t_1^{\alpha}}{\alpha}, \tag{12}$$

with initial conditions

$$V(X,0) = \overline{\varphi}(X), 0 < X \leqslant \frac{x_1^{\beta}}{\beta}, \tag{13}$$

and the prescribed Dirichlet boundary conditions are

$$V_X(0,T) = 0, 0 < T \leqslant \frac{t_1^{\alpha}}{\alpha}, \tag{14}$$

$$V(\frac{x_1^{\beta}}{\beta}, T) = \overline{g}(T), 0 < T \leqslant \frac{t_1^{\alpha}}{\alpha}, \tag{15}$$

and additional condition

$$V(X, \frac{t_1^{\alpha}}{\alpha}) = \overline{E}(X), 0 < X \leqslant \frac{x_1^{\beta}}{\beta}. \tag{16}$$

After establishing the solution and unknown coefficient of problem (12)-(16), by employing inverse transformation we obtain the solution $u(x,t)$ and an unknown diffusivity coefficient $k(x)$.

4. Illustrative Examples

In this section, we illustrate three examples of inverse problems about determination of unknown space dependent coefficient.

Example 1. Consider the inverse coefficient problem involving space-time fractional differential equations [23, 24]:

$$D_t^{\alpha}u(x,t) = D_x^{\beta}(k(x)D_x^{\beta}u(x,t)), 0 < x < \beta^{\frac{1}{\beta}}, 0 < t < \alpha^{\frac{1}{\alpha}}, \tag{17}$$

$$u(x,0) = \frac{x^{3\beta}}{\beta^3}, 0 \leqslant x \leqslant \beta^{\frac{1}{\beta}}, \tag{18}$$

$$D_x^{\beta}u(0,t) = 0, 0 < t \leqslant \alpha^{\frac{1}{\alpha}}, \tag{19}$$

$$u(\beta^{\frac{1}{\beta}}, t) = exp(t), 0 < t \leqslant \alpha^{\frac{1}{\alpha}}, \tag{20}$$

$$u(x, \alpha^{\frac{1}{\alpha}}) = \frac{x^{3\beta}}{\beta^3}exp(1), 0 \leqslant x \leqslant \beta^{\frac{1}{\beta}}. \tag{21}$$

By taking fractional scaling transformation methods into account the problem (17)-(21) turns into following integer order problem:

$$V_T = (k(X)V_X)_X, 0 < X < 1, 0 < T < 1, \tag{22}$$

with initial conditions

$$V(X,0) = X^3, 0 < X \leqslant 1, \tag{23}$$

and the prescribed Dirichlet boundary conditions are

$$V_X(0,T) = 0, 0 < T \leqslant 1, \tag{24}$$

$$V(1,T) = exp(T), 0 < T \leqslant 1, \tag{25}$$

and additional condition

$$V(X,1) = X^3 exp(1), 0 < X \leqslant 1. \tag{26}$$

This inverse problem have the solution $V(X,T) = X^3 exp(T)$ and unknown diffusivity coefficient becomes $k(X) = \frac{1}{12}X^2$. As seen from Figs.1-4, by means of inverse transformation the solution of problem (17)-(21) and unknown diffusivity coefficient are obtained in the following form respectively.

$$u(x,t) = \frac{x^{3\beta}}{\beta^3} exp(\frac{t^\alpha}{\alpha}) \tag{27}$$

and

$$k(x) = \frac{1}{12}\frac{x^{2\beta}}{\beta^2}. \tag{28}$$

Moreover, the values of exact and approximate solutions of problem (17)-(21) at $t = 0.8$ for different values of orders of α and β are presented in Table 1.

Table 1: The table of exact and approximate solutions of Ex. 1 at $t = 0.8$.

x	Exact	$\alpha = 1$ $\beta = 1$	$\alpha = 0.9$ $\beta = 1$	$\alpha = 0.8$ $\beta = 1$	$\alpha = 1$ $\beta = 0.9$	$\alpha = 0.9$ $\beta = 0.9$	$\alpha = 0.8$ $\beta = 0.9$
0.2	0.01780	0.01780	0.01780	0.02276	0.03958	0.04414	0.05060
0.4	0.14243	0.14243	0.15883	0.18209	0.25720	0.28680	0.32881
0.6	0.48072	0.48072	0.53605	0.61457	0.76863	0.85710	0.98264
0.8	1.13948	1.13948	1.27063	1.45675	1.67129	1.86366	2.13664
1	2.22554	2.22554	2.48171	2.84522	3.05287	3.40426	3.90290

Figure 1: The graphics of exact and approximate solution for $k(x)$ in Ex. 1 .

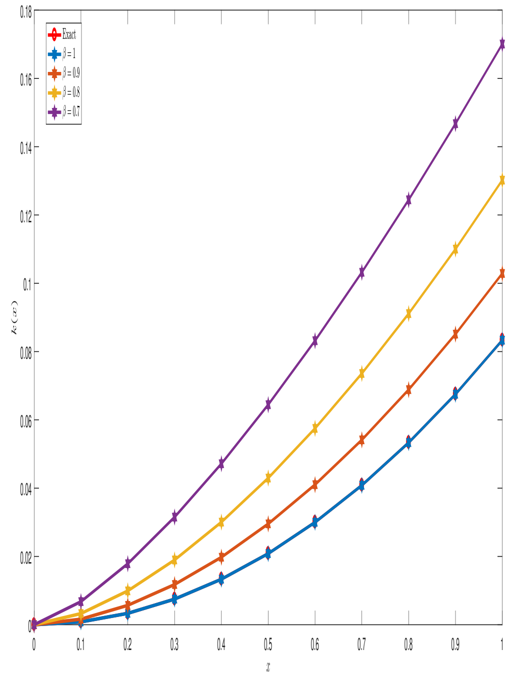


Figure 2: The graphics of exact solution for $u(x, t)$ in Ex. 1 .

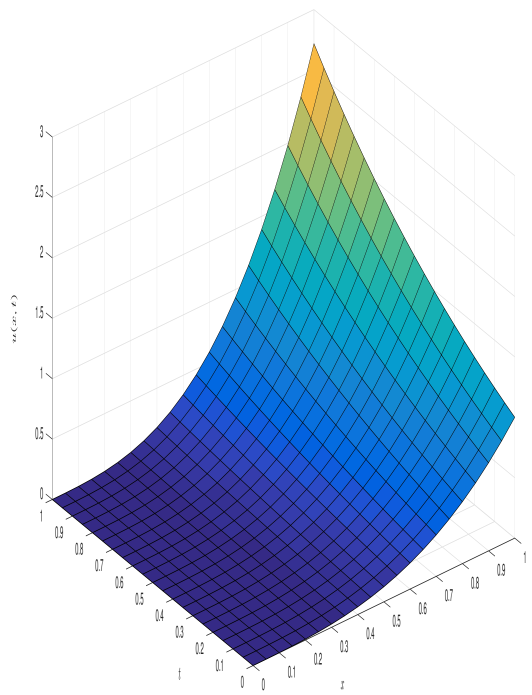


Figure 3: The graphics of approximate solution for $u(x,t)$ with $\alpha = 1$ and $\beta = 1$ in Ex. 1 .

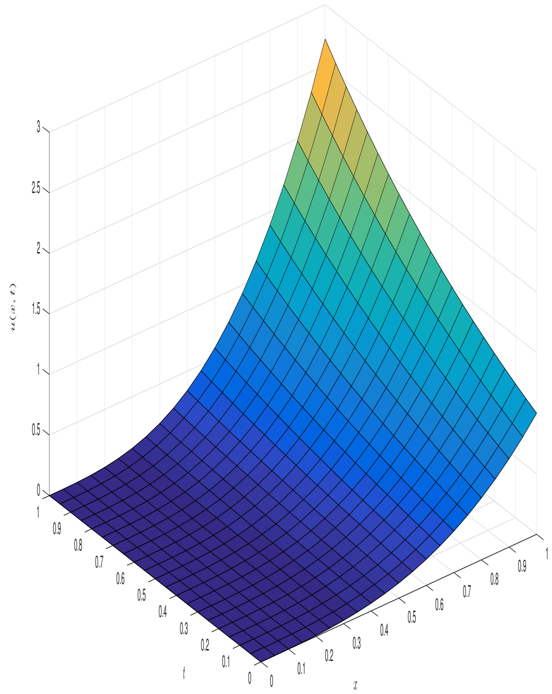
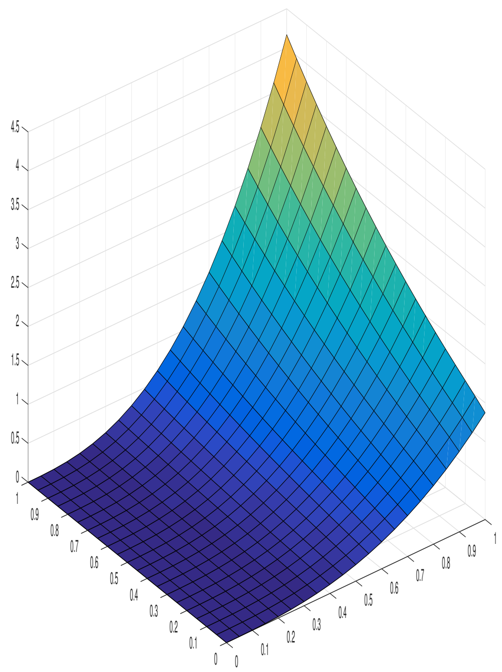


Figure 4: The graphics of approximate solution for $u(x,t)$ with $\alpha = 0.9$ and $\beta = 0.9$ in Ex. 1 .



Example 2. Consider the inverse coefficient problem involving space-time fractional differential equations [23, 24]:

$$D_t^{\alpha}u(x,t) = D_x^{\beta}(k(x)D_x^{\beta}u(x,t)), 0 < x < \beta^{\frac{1}{\beta}}, 0 < t < \alpha^{\frac{1}{\alpha}}, \tag{29}$$

$$u(x,0) = \frac{x^{2\beta}}{\beta^2}exp\Big(\frac{x^{\beta}}{\beta}\Big), 0 \leqslant x \leqslant \beta^{\frac{1}{\beta}}, \tag{30}$$

$$D_x^{\beta}u(0,t) = 0, 0 < t \leqslant \alpha^{\frac{1}{\alpha}}, \tag{31}$$

$$u(\beta^{\frac{1}{\beta}},t) = exp(1+\frac{t^{\alpha}}{\alpha}), 0 < t \leqslant \alpha^{\frac{1}{\alpha}}, \tag{32}$$

$$u(x,\alpha^{\frac{1}{\alpha}}) = \frac{x^{2\beta}}{\beta^2}exp\Big(\frac{x^{\beta}}{\beta}+1\Big), 0 \leqslant x \leqslant \beta^{\frac{1}{\beta}}. \tag{33}$$

By taking fractional scaling transformation methods into account the problem (29)-(33) turns into following integer order problem:

$$V_T = (k(X)V_X)_X, 0 < X < 1, 0 < T < 1, \tag{34}$$

with initial conditions

$$V(X,0) = X^2exp(X), 0 < X \leqslant 1, \tag{35}$$

and the prescribed Dirichlet boundary conditions are

$$V_X(0,T) = 0, 0 < T \leqslant 1, \tag{36}$$

$$V(1,T) = exp(1+T), 0 < T \leqslant 1, \tag{37}$$

and additional condition

$$V(X,1) = X^2exp(X+1), 0 < X \leqslant 1. \tag{38}$$

This inverse problem have the solution $V(X,T) = X^2exp(X+T)$ and unknown diffusivity coefficient becomes $k(X) = \frac{X^2-2X+2}{X^2+2X}$. As seen from Figs.5-8, by means of inverse transformation the solution of problem (34)-(38) and unknown diffusivity coefficient are obtained in the following form respectively.

$$u(x,t) = \frac{x^{2\beta}}{\beta^2}exp\Big(\frac{x^{\beta}}{\beta}+\frac{t^{\alpha}}{\alpha}\Big) \tag{39}$$

and

$$k(x) = \frac{x^{2\beta}-2\beta x^{\beta}+2\beta^2}{x^{2\beta}+2\beta x^{\beta}} \tag{40}$$

Moreover, the values of exact and approximate solutions of problem (28)-(32) at $t = 0.8$ for different values of orders of α and β are presented in Table 2.

Table 2: The table of exact and approximate solutions of Ex. 2 at $t = 0.8$.

x	Exact	$\alpha = 1$ $\beta = 1$	$\alpha = 0.9$ $\beta = 1$	$\alpha = 0.8$ $\beta = 1$	$\alpha = 1$ $\beta = 0.9$	$\alpha = 0.9$ $\beta = 0.9$	$\alpha = 0.8$ $\beta = 0.9$
0.2	0.10873	0.10873	0.12125	0.13901	0.19686	0.21952	0.25168
0.4	0.53122	0.53122	0.59236	0.67913	0.85941	0.95833	1.09870
0.6	1.45987	1.45987	1.62791	1.86636	2.20967	2.46401	2.82492
0.8	3.16994	3.16994	3.53481	4.05257	4.56313	5.08836	5.83369
1	6.04965	6.04965	6.74598	7.73410	8.34642	9.30711	10.67038

Figure 5: The graphics of exact and approximate solution for $k(x)$ in Ex. 2 .

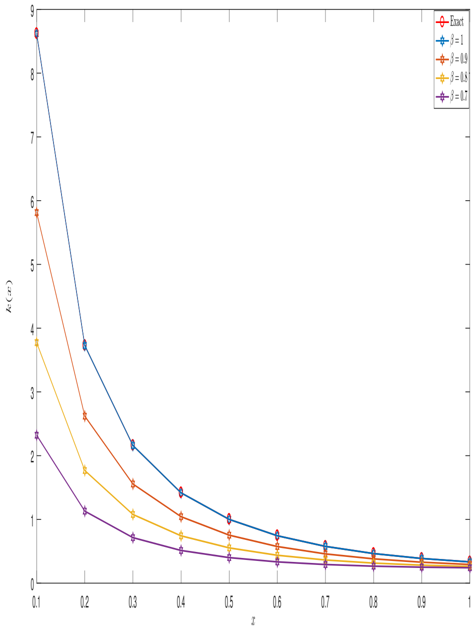


Figure 6: The graphics of exact solution for $u(x,t)$ in Ex. 2 .

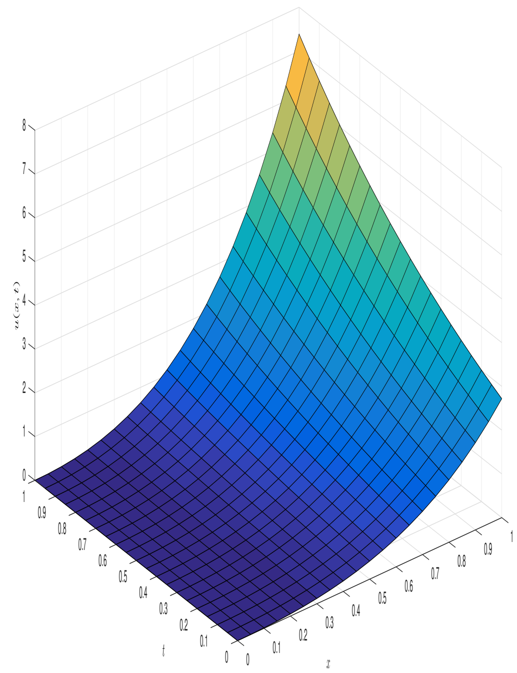


Figure 7: The graphics of approximate solution for $u(x,t)$ with $\alpha = 1$ and $\beta = 1$ in Ex. 2.

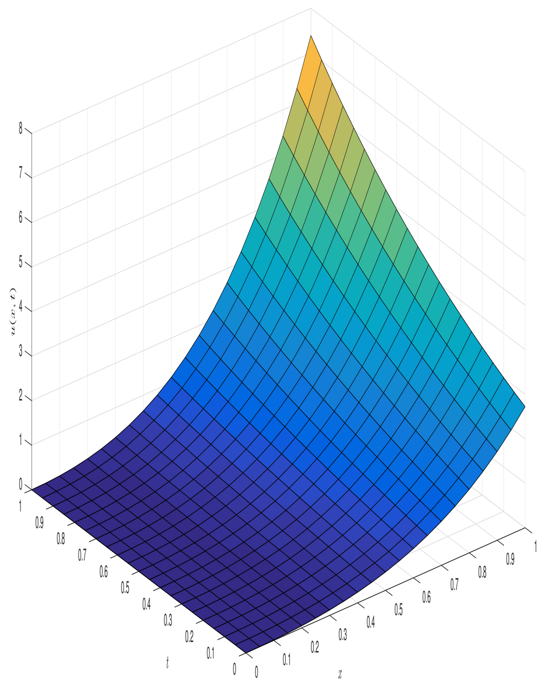
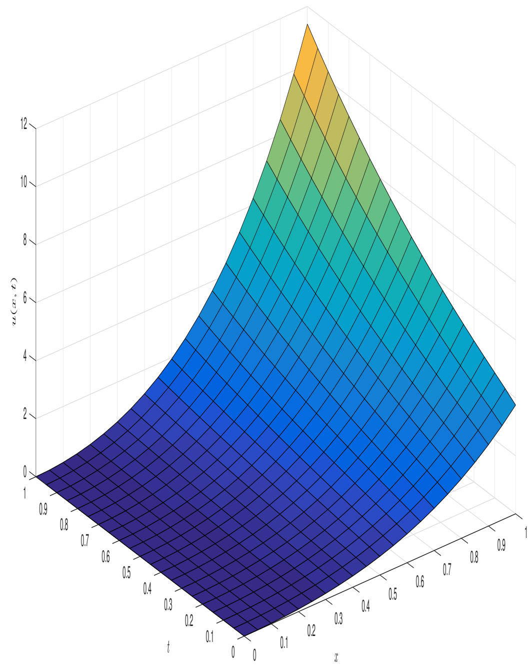


Figure 8: The graphics of approximate solution for $u(x,t)$ with $\alpha = 0.9$ and $\beta = 0.9$ in Ex. 2 .



Example 3. Consider the inverse coefficient problem involving space-time fractional differential equations [23, 24]:

$$D_t^{\alpha} u(x,t) = D_x^{\beta}(k(x) D_x^{\beta} u(x,t)) + (x^2 - 4x) exp(t),$$
$$0 < x < \beta^{\frac{1}{\beta}}, 0 < t < \alpha^{\frac{1}{\alpha}}, \quad (41)$$

$$u(x,0) = \frac{x^{2\beta}}{\beta^2}, 0 \leqslant x \leqslant \beta^{\frac{1}{\beta}}, \quad (42)$$

$$D_x^{\beta} u(0,t) = 0, 0 < t \leqslant \alpha^{\frac{1}{\alpha}}, \quad (43)$$

$$u(\beta^{\frac{1}{\beta}}, t) = exp(t), 0 < t \leqslant \alpha^{\frac{1}{\alpha}}, \quad (44)$$

$$u(x, \alpha^{\frac{1}{\alpha}}) = \frac{x^{2\beta}}{\beta^2} exp(1), 0 < x \leqslant \beta^{\frac{1}{\beta}}. \quad (45)$$

By taking fractional scaling transformation methods into account the problem (41)-(45) turns into following integer order problem:

$$V_T = (k(X) V_X)_X, 0 < X < 1, 0 < T < 1, \quad (46)$$

with initial conditions

$$V(X,0) = X^2, 0 < X \leqslant 1, \quad (47)$$

and the prescribed Dirichlet boundary conditions are

$$V_X(0,T) = 0, 0 < T \leqslant 1, \tag{48}$$

$$V(1,T) = exp(T), 0 < T \leqslant 1, \tag{49}$$

$$V(X,1) = X^2 exp(1), 0 < X \leqslant 1, \tag{50}$$

This inverse problem have the solution $V(X,T) = X^2 exp(T)$ and unknown diffusivity coefficient becomes $k(X) = X$. As seen from Figs.9-12, by means of inverse transformation the solution of problem (34)-(38) and unknown diffusivity coefficient are obtained in the following form respectively

$$u(x,t) = \frac{x^{2\beta}}{\beta^2} exp(\frac{t^\alpha}{\alpha}) \tag{51}$$

and

$$k(x) = \frac{x^\beta}{\beta}. \tag{52}$$

Moreover, the values of exact and approximate solutions of problem (41)-(45) at $t = 0.8$ for different values of orders of α and β are presented in Table 3.

Table 3: The table of exact and approximate solutions of Ex. 3 at $t = 0.8$.

x	Exact	$\alpha = 1$ $\beta = 1$	$\alpha = 0.9$ $\beta = 1$	$\alpha = 0.8$ $\beta = 1$	$\alpha = 1$ $\beta = 0.9$	$\alpha = 0.9$ $\beta = 0.9$	$\alpha = 0.8$ $\beta = 0.9$
0.2	0.08902	0.08902	0.09927	0.11381	0.15164	0.16909	0.19386
0.4	0.35609	0.35609	0.39707	0.45523	0.52803	0.58881	0.67505
0.6	0.80119	0.80119	0.89341	1.02428	1.09553	1.22162	1.40056
0.8	1.42435	1.42435	1.58829	1.82094	1.83871	2.05035	2.35067
1	2.22554	2.22554	2.48171	2.84522	2.74758	2.05035	3.51261

Figure 9: The graphics of exact and approximate solution for $k(x)$ in Ex. 3 .

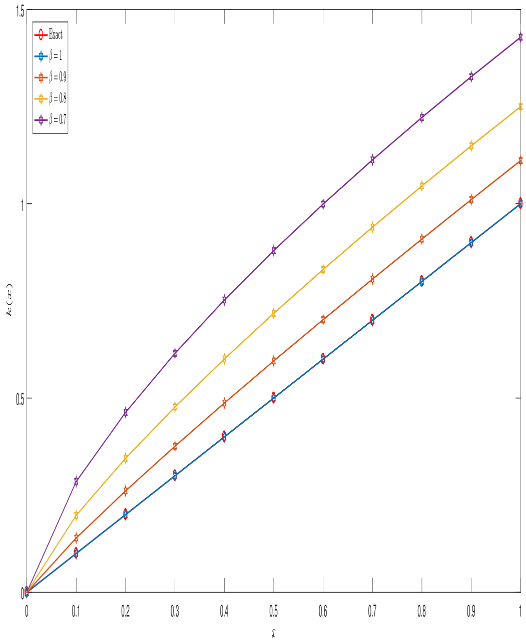


Figure 10: The graphics of exact solution for $u(x,t)$ in Ex. 3 .

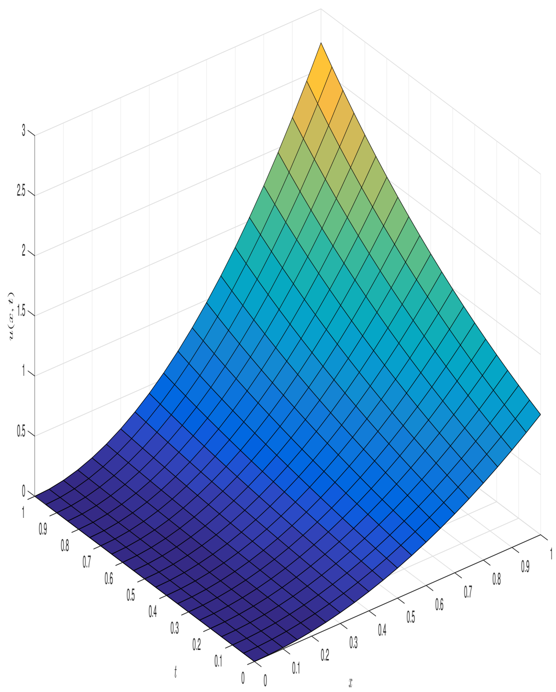


Figure 11: The graphics of approximate solution for $u(x,t)$ with $\alpha = 1$ and $\beta = 1$ in Ex. 3 .

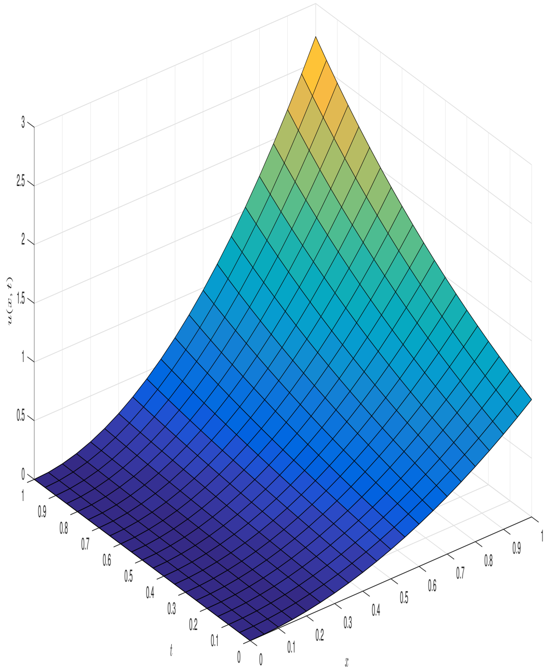
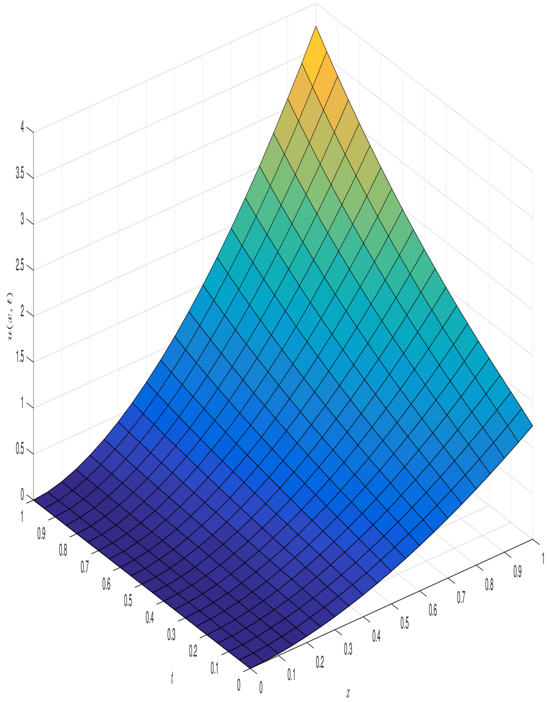


Figure 12: The graphics of approximate solution for $u(x,t)$ with $\beta = 0.9$ in Ex. 3 .



5. Conclusion

In this study, we tackle with the challenge of constructing the solution and unknown space dependent coefficient of space-time fractional diffusion equations by utilizing fractional scaling transformation method. This method enable us to reduce the problem into integer order inverse problem which gives us the opportunity to cope with easier problem. Then, taking the inverse transformation into account, the solution and the unknown coefficient are recovered. The outcomes illustrate that this method works better for the fractional problems in the sense of conformable fractional derivative. Future work will be concerned with the construction of the unknown parameter in space-time fractional differential equations with various boundary conditions.

References

[1] Oldham, KB, Spanier J. The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order. Academic Press. 1974.
[2] Miller, KS, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations. John Wiley and Sons. 1993.
[3] Debnath LA. Recent applications of fractional calculus to science and engineering. Int. J. Math. Math. Sci. 54, 2003, 3413 — 3442.
[4] Kilbas AA., Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. Elsevier. 2006.
[5] Podlubny I. Fractional differential equation. San Diego. CA: Academic Press. 1999.
[6] Ergun A. A half inverse problem for the singular diffusion operator with jump condition. Miskolch Mathematical Notes. 21(2), 2020, 805 – 821.
[7] Ergun A, Amirov R. A half-inverse problem for singular diffusion operator with certain boundary conditions. Numer. Methods Partial Diff. Eq. 38, 2022, 916 — 927.
[8] Ergun A, Amirov R. Half inverse problem for diffusion operators with jump conditions dependent on the spectral parameter. Numer Methods Partial Diff. Eq. 38, 2022, 577 -– 590.
[9] Sabatier J, Agarwal OP, Machado JAT(eds). Advances in fractional calculus: theoretical developments and applications in physics and engineering. Dordrecht: Springer. 2007.
[10] Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives theory and applications. Amsterdam: Gordon and Breach. 1993.
[11] Odibat Z. Approximations of fractional integrals and Caputo fractional derivatives. Appl. Math. Comput.178, 2006, 527 – 533.
[12] Momani S, Odibat Z. Analytical approach to linear fractional partial differential equations arising in fluid mechanics. Phys. Lett. A 355, 2006, 271 – 279.
[13] Atangana A. On the new fractional derivative and application to nonlinear Fisher-reaction- diffusion equation. Appl. Math. Comput. 273, 2016, 948 – 956..
[14] El-Ajou A, Abu Arqub O, Momani S, Baleanu D, Alsaedi A. A novel expansion iterative method for solving linear partial differential equations of fractional order. Appl. Math. Comput. 257, 2015, 119 – 133.
[15] Bayrak MA, Demir A. A new approach for space-time fractional partial differential equations by residual power series method. Appl. Math. Comput. 336, 2018, 215 – 230.
[16] Lesnic D, Yousefi SA, Ivanchov M. Determination of a time-dependent diffusivity from nonlocal conditions. J. Appl. Math. Comput. 41, 2013, 301 -– 320.
[17] Xiangtuan X, Hongbo G, Xiaohong L. An inverse problem for a fractional diffusion equation. J. of Comput. Appl. Math. 236, 2012, 4474 -– 4484.
[18] Mansur II, Muhammed C. Inverse source problem for a time-fractional diffusion equation with nonlocal boundary conditions. Appl. Math. Model. 40(4), 2016, 891 – 899.
[19] Songshu L, Lixin F. An Inverse Problem for a Two-Dimensional Time-Fractional Sideways Heat Equation. Math. Prob. Eng. 2020, Article ID 5865971, 13 pages.
[20] Zhiyuan L, Xing C, Gongsheng L. An inverse problem in time-fractional diffusion equations with nonlinear boundary condition. J. of Math. Phys. 60, 091502 , 2019.
[21] Khalil R, Al Horani M, Yousef A, Sababheh M. A new definition of fractional derivative. Journal of Computational and Applied Mathematics, 2014, 65 -– 70.
[22] Baleanu D, Diethelm K, Scalas E, Trujillo JJ. Fractional calculus models and numerical methods. Series on Complexity. Nonlinearity and Chaos. World Scientific. Boston. 2012.
[23] Jumarie G. Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results. Comput. Math. Appl, 51, 2006, 1367 -– 1376.
[24] Zhou G, Wu B, Ji W, Rho S. Time- or Space-Dependent Coefficient Recovery in Parabolic Partial Differential Equation for Sensor Array in the Biological Computing. Mathematical Problems in Engineering. 2015, Article ID 573932, 9 pages.