Communications in Advanced Mathematical Sciences

VOLUME V
ISSUE III

ISSN 2651-4001

VOLUME 5 ISSUE 3
ISSN 2651-4001

September 2022
www.dergipark.org.tr/tr/pub/cams

COMMUNICATIONS IN ADVANCED MATHEMATICAL SCIENCES

Editors in Chief

Managing Editor

Editorial Board

Contents

1 **Existence Results for BVP of a Class of Generalized Fractional-Order Implicit Differential Equations**
Kadda MAAZOUZ, Dvivek VİVEK, Elsayed ELSAYED **114 - 123**

2 **Shift Filter of Quasi-ordered Residuated Systems**
Daniel A. ROMANO **124 - 130**

3 **Stability of Solutions for a Krichhoff-Type Plate Equation with Degenerate Damping**
Fatma EKİNCİ, Erhan PİŞKİN **131 - 136**

4 **Global Weak Solution, Uniqueness and Exponential Decay for a Class of Degenerate Hyperbolic Equation**
Ducival PEREİRA, Carlos RAPOSO **137 - 149**

5 **$\mathcal{I}_2$-Uniform Convergence of Double Sequences of Functions In 2-Normed Spaces**
Sevim YEGÜL GÜZEY, Erdinç DÜNDAR, Mukaddes ARSLAN **150 - 160**

Communications in Advanced Mathematical Sciences
Vol. 5, No. 3, 114-123, 2022
Research Article
e-ISSN: 2651-4001
DOI: 10.33434/cams.1069182

Existence Results for BVP of a Class of Generalized Fractional-Order Implicit Differential Equations

Kadda Maazouz[1], Dvivek Vivek[2]*, Elsayed M. Elsayed[3]

Abstract
In this paper, we study and deal with the existence of solutions to boundary value problem for implicit differential equations involving generalized fractional derivative, this study is based on the approach of Nonlinear alternative and Krasnoselskii fixed points.

Keywords: Boundary value problem, Fixed point, Generalized fractional derivative, Integral equation
2010 AMS: 26A33, 34A08, 34B15

[1] Department of Mathematics, University of Tiaret, Algeria
[2] Department of Mathematics, PSG College of Arts & Science, Coimbatore, India-641 014, ORCID: 0000-0003-0951-8060
[3] Department of Mathematics, Faculty of Science, Mansura 35516, Egypt, ORCID: 0000-0003-0894-8472
*Corresponding author: peppyvivek@gmail.com
Received: 7 February 2022, **Accepted:** 12 September 2022, **Available online:** 30 September 2022

1. Introduction

Because of its wide applicability in biology, medicine and in more and more fields, the theory of fractional differential equations has recently been attracting increasing interest. Especially, many research papers had devoted to generalized fractional differential operator, this concept of generalized integral and derivative was given through Katugampola [11, 12]. The use of Katugampola fractional derivative (KFD) is to generalize the Hadamard and Riemann-Liouville integrals and derivatives which widely discussed by many researchers, one can refer to [8, 11, 12, 22]. Anderson et al. [1] studied some properties of KFD with potential application in quantum mechanics. In [8], Janaki et al. established existence and uniqueness of solutions to the impulsive differential equations with inclusions, and the authors also established some conditions for the uniqueness and existence of solutions for a class of fractional implicit differential equations with KFD [9]. Recently, Vivek et al. [22] investigated existence and stability of solutions for impulsive type integro-differential equations. Followed by the work, the existence and Ulam stability of solutions for impulsive type pantograph equations was considered in [23].

As a result of unifying different techniques for initial or boundary conditions, nonlinear boundary conditions received more and more attention, see [5, 6, 10], [13]-[18].

In this paper, we consider the following boundary value problem for implicit differential equations with KFD of the form

$$\begin{cases} {}^{\rho}D^{\alpha}u(t)=\Psi(t,u(t),{}^{\rho}D^{\alpha}u(t)), \quad t\in J:=[a,b], \quad 1<\alpha<2, \quad \rho>0,\\ c_1u(a)-d_1u'(a)=u_1,\\ c_2u(b)-d_2u'(b)=u_2, \end{cases} \tag{1.1}$$

where ${}^{\rho}D^{\alpha}$ is the generalized fractional derivative of order α, $\Psi: J\times R\times R\longrightarrow R$, is given function, $c_1,c_2,d_1,d_2,u_1,u_2\in R$ and $0\leqslant a<b<\infty$.

The paper is organized as follows: In Section 2, we present definitions, lemmas, and some results. Section 3 is devoted to establish our main results. Finally, two explanatory examples are given to illustrate the theoretical results.

2. Fundamental Results

We now introduce some definitions, preliminary facts about the fractional calculus, notations, and some auxiliary results, which will be used later.

Definition 2.1. [12] *The generalized left-sided fractional integral of order $\alpha \in \mathbb{C}$, $(Re(\alpha) > 0)$ is defined for $t > a$ by*

$$ {}^{\rho}I^{\alpha}h(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \left(t^{\rho} - s^{\rho}\right)^{\alpha-1} s^{\rho-1}h(s)ds, \tag{2.1} $$

if the integral exists, where $\Gamma(.)$ is the Gamma function.

Definition 2.2. [12] *The generalized left-sided fractional derivative, corresponding to the generalized fractional integral (2.1) is defined for $t > a$ by*

$$ {}^{\rho}D^{\alpha}h(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right) \int_a^t \left(t^{\rho} - s^{\rho}\right)^{n-\alpha-1} s^{\rho-1}h(s)ds, $$

where $n = [\alpha] + 1$, if the integral exists.

Lemma 2.3. *Let $\alpha > 0$ and $\rho > 0$, then the differential equation*

$$ {}^{\rho}D^{\alpha}f(t) = 0, $$

has solutions

$$ f(t) = a_0 + \sum_{k=1}^{n-2} a_k \left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\alpha-k}, \quad a_k \in R, k = 0,1,2,\ldots,n-2; \quad n = [\alpha] + 1. $$

Lemma 2.4. *Let $\alpha > 0$ and $\rho > 0$, then*

$$ {}^{\rho}I^{\alpha}\left({}^{\rho}D^{\alpha}f(t)\right) = f(t) + a_0 + \sum_{k=1}^{n-2} a_k \left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\alpha-k}, $$

for some

$$ a_k \in R, k = 0,1,2,\ldots,n-2; \quad n = [\alpha] + 1. $$

Theorem 2.5. *[7](Nonlinear alternative)*
Let X be a Banach space with $C \subset X$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $T : \overline{U} \longrightarrow C$ is a compact. Then either,

1. *T has a fixed point in $\overline{U}$, or*
2. *there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda Tu$.*

Theorem 2.6. *[19] (Krasnoselskii's fixed point theorem)*
Let E be a bounded closed convex and nonempty subset of a Banach space X. Let A, B two operators such that $Ax + By \in E$ for every pair $x, y \in E$. If A is a contraction and B is completely continuous then there exists $z \in E$ such that $Az + Bz = z$.

3. Main Results

The following lemma is essential to state and prove our main result

Lemma 3.1. *Let $1 < \alpha < 2$, $\rho > 0$ and $\psi \in C(J,R)$ be a continuous function. Then the following boundary value problem*

$$ \begin{cases} {}^{\rho}D^{\alpha}u(t) = \psi(t), \quad t \in J, \\ c_1u(a) - d_1u'(a) = u_1, \\ c_2u(b) - d_2u'(b) = u_2, \end{cases} \tag{3.1} $$

has a unique solution given by

$$u(t)=\frac{c_2u_1}{c_1}-u_2+\phi_{a,b}\sigma_t+\int_a^t K_t^{\alpha}(s)\psi(s)ds,$$

where

$$K_t^{\alpha}(s)=\frac{\rho^{\alpha-1}}{\Gamma(\alpha)}(t^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1},\qquad \sigma_t=\Big(\frac{t^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-1},$$

$$\phi_{a,b}=\frac{1}{\delta}\Big(\frac{c_2u_1}{c_1}-u_2+\int_a^b K(s)\psi(s)ds\Big),\quad K(s)=c_2K_b^{\alpha}(s)-d_2\rho b^{\rho-1}K_b^{\alpha-1}(s),$$

and

$$\delta=d_2(\alpha-1)b^{\rho-1}\Big(\frac{b^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-2}-c_2\Big(\frac{b^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-1}.$$

Proof. Let u satisfies the problem (3.1) then, by Lemmas 2.3 and 2.3 we have

$$\begin{aligned}u(t)&=a_0+a_1\Big(\frac{t^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-1}+\frac{\rho^{\alpha-1}}{\Gamma(\alpha)}\int_a^t(t^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}\psi(s)ds\\&=a_0+a_1\Big(\frac{t^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-1}+\int_a^t K_t^{\alpha}(s)\psi(s)ds.\end{aligned}$$

Then

$$u'(t)=a_1(\alpha-1)t^{\rho-1}\Big(\frac{t^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-2}+\rho t^{\rho-1}\int_a^t K_t^{\alpha-1}(s)\psi(s)ds.$$

Therefore

$$u(a)=a_0\quad\text{and}\quad u'(a)=0,$$

so we have

$$c_1u(a)-d_1u'(a)=c_1a_0=u_1$$

it follows that

$$a_0=\frac{u_1}{c_1}\cdot$$

On the other hand, we have

$$c_2u(b)=c_2a_0+c_2a_1\Big(\frac{b^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-1}+c_2\int_a^b K_b^{\alpha}(s)\psi(s)ds,$$

and

$$d_2u'(b)=d_2a_1(\alpha-1)b^{\rho-1}\Big(\frac{b^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-2}+d_2\rho b^{\rho-1}\int_a^b K_b^{\alpha-1}(s)\psi(s)ds.$$

Then we obtain

$$\begin{aligned}c_2u(b)-d_2u'(b)&=c_2a_0+c_2a_1\Big(\frac{b^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-1}-d_2a_1(\alpha-1)b^{\rho-1}\Big(\frac{b^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-2}\\&\quad+\int_a^b\Big[c_2K_b^{\alpha}(s)-d_2\rho b^{\rho-1}K_b^{\alpha-1}(s)\Big]\psi(s)ds=u_2\\&=\frac{c_2u_1}{c_1}+c_2a_1\Big(\frac{b^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-1}-d_2a_1(\alpha-1)b^{\rho-1}\Big(\frac{b^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-2}\\&\quad+\int_a^b\Big[c_2K_b^{\alpha}(s)-d_2\rho b^{\rho-1}K_b^{\alpha-1}(s)\Big]\psi(s)ds=u_2\\&=\frac{c_2u_1}{c_1}-a_1\left(d_2(\alpha-1)b^{\rho-1}\Big(\frac{b^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-2}-c_2\Big(\frac{b^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-1}\right)\\&\quad+\int_a^b\Big[c_2K_b^{\alpha}(s)-d_2\rho b^{\rho-1}K_b^{\alpha-1}(s)\Big]\psi(s)ds.\end{aligned}$$

From $c_2u(b)-d_2u'(b)=u_2$ we deduce that

$$\begin{aligned}
a_1 &= \frac{1}{\delta}\Big(\frac{c_2u_1}{c_1}-u_2+\int_a^b\Big[c_2K_b^{\alpha}(s)-d_2\rho b^{\rho-1}K_b^{\alpha-1}(s)\Big]\psi(s)ds\Big)\\
&= \frac{1}{\delta}\Big[\frac{c_2u_1}{c_1}-u_2+\int_a^b\Big(c_2K_b^{\alpha}(s)-d_2\rho b^{\rho-1}K_b^{\alpha-1}(s)\Big)\psi(s)ds\Big]\\
&= \frac{1}{\delta}\Big(\frac{c_2u_1}{c_1}-u_2+\int_a^b K(s)\psi(s)ds\Big)=\phi_{a,b}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
u(t) &= \frac{c_2u_1}{c_1}-u_2+\phi_{a,b}\Big(\frac{t^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-1}+\int_a^t K_t^{\alpha}(s)\psi(s)ds\\
&= \frac{c_2u_1}{c_1}-u_2+\phi_{a,b}\sigma_t+\int_a^t K_t^{\alpha}(s)\psi(s)ds.
\end{aligned}$$

Then we can accomplish the purpose desired, which complete the proof.

For sake of brevity, we need the following proposition which is very useful in what follows.

Proposition 3.2. *For* $1<\alpha<2,\ \rho>0$, *and* $t,s\in J$ *we have*
(*i*) $\int_a^t K_t^{\alpha}(s)ds\leqslant\int_a^b K_b^{\alpha}(s)ds=\frac{\rho^{\alpha-2}}{\Gamma(\alpha+1)}(b^{\rho}-a^{\rho})^{\alpha}$
(*ii*) $\int_a^b K_b^{\alpha-1}(s)ds=\frac{\rho^{\alpha-2}}{(\alpha-1)\Gamma(\alpha)}(b^{\rho}-a^{\rho})^{\alpha-1}$
(*iii*) $\int_a^b|K(s)|ds\leqslant\frac{(b^{\rho}-a^{\rho})^{\alpha-1}\rho^{\alpha-2}}{(\alpha-1)\Gamma(\alpha)}\Big(|c_2|(b^{\rho}-a^{\rho})+|d_2|\rho b^{\rho-1}\Big):=K^*.$

Proof. The proof of (i) and (ii) is immediate, it remains to prove (iii). Indeed, we have

$$\begin{aligned}
\int_a^b|K(s)|ds &= \int_a^b\Big|c_2K_b^{\alpha}(s)-d_2\rho b^{\rho-1}K_b^{\alpha-1}(s)\Big|ds\\
&\leqslant \frac{|c_2|\rho^{\alpha-2}}{\Gamma(\alpha+1)}(b^{\rho}-a^{\rho})^{\alpha}+\frac{|d_2|\rho b^{\rho-1}\rho^{\alpha-2}}{(\alpha-1)\Gamma(\alpha)}(b^{\rho}-a^{\rho})^{\alpha-1}\\
&\leqslant \frac{(b^{\rho}-a^{\rho})^{\alpha-1}\rho^{\alpha-2}}{\Gamma(\alpha)}\Big(|c_2|\frac{b^{\rho}-a^{\rho}}{\alpha}+|d_2|\frac{\rho b^{\rho-1}}{\alpha-1}\Big)\\
&\leqslant \frac{(b^{\rho}-a^{\rho})^{\alpha-1}\rho^{\alpha-2}}{(\alpha-1)\Gamma(\alpha)}\Big(|c_2|(b^{\rho}-a^{\rho})+|d_2|\rho b^{\rho-1}\Big)=K^*.
\end{aligned}$$

3.1 Existence results

Now, we are in position to first result which is based on Theorem 2.5.

Theorem 3.3. *Assume that*
$(\mathscr{A}_1)$ Ψ *is continuous.*
$(\mathscr{A}_2)$ *There exist constants* $k>0$ *and* $0<l<1$ *such that*

$$|\Psi(t,u_2,v_2)-\Psi(t,u_1,v_1)|\leqslant k|u_2-u_1|+l|v_2-v_1|$$

for any $u_1,v_1,\ u_2,v_2\in R$, *and* $t\in J$.
Then the problem (1.1) *has at least one solution.*

Proof. Let us consider the operator $\chi:\mathscr{C}(J,R)\longrightarrow\mathscr{C}(J,R)$ defined by

$$(\chi u)(t)=\frac{c_2u_1}{c_1}-u_2+\phi_{a,b}\sigma_t+\int_a^t K_t^{\alpha}(s)\psi(s)ds$$

where

$$\psi(s)=\Psi(s,u(s),\psi(s)).$$

Step 1: χ is continuous.
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $\mathscr{C}(J,R)$. Then for each $t \in J$, we have

$$\begin{aligned}
\big|(\chi u_n)(t) - (\chi u)(t)\big| &= \Big|\frac{\sigma_t}{\delta}\int_a^b K(s)\big(\psi_n(s) - \psi(s)\big)ds \\
&+ \int_a^t K_t^\alpha(s)\big(\psi_n(s) - \psi(s)\big)ds\Big| \\
&\leqslant \frac{\sigma_b}{|\delta|}\int_a^b |K(s)|\big|\psi_n(s) - \psi(s)\big|ds \\
&+ \int_a^t |K_t^\alpha(s)|\big|\psi_n(s) - \psi(s)\big|ds
\end{aligned}$$

where

$$\psi_n(s) = \Psi(s, u_n(s), \psi_n(s)).$$

In virtue of $(\mathscr{A}_2)$, we have

$$|\psi_n(s) - \psi(s)| \leq \frac{k}{1-l}|u_n(s) - u(s)|.$$

It follows that

$$\begin{aligned}
|\chi u_n(t) - \chi u(t)| &\leqslant \frac{k}{1-l}\Big(\frac{\sigma_b K^*}{|\delta|} + \int_a^b K_b^\alpha(s)ds\Big)|u_n(s) - u(s)| \\
&\leqslant \frac{k}{1-l}\Big(\frac{\sigma_b K^*}{|\delta|} + \frac{\rho^{\alpha-2}}{\Gamma(\alpha+1)}(b^\rho - a^\rho)^\alpha\Big)\|u_n - u)\|_\infty.
\end{aligned}$$

Since $u_n \to u$, we get that $\|\chi u_n - \chi u\|_\infty \to 0$ as $n \to \infty$. Hence χ is continuous.
Step 2: χ maps bounded sets into bounded sets in $\mathscr{C}(J,R)$.
It is enough to show that there exists a positive constant m for $r > 0$ such that for each $u \in \mathscr{D}_r = \{u \in \mathscr{C}(J,R) : \|u\|_\infty \leqslant r\}$ we have $\|\chi u\|_\infty \leqslant m$. Indeed for each $t \in J$, and $u \in \mathscr{D}_r$ we have

$$\begin{aligned}
|(\chi u)(t)| &= \Big|\frac{c_2 u_1}{c_1} - u_2 + \phi_{a,b}\sigma_t + \int_a^t K_t^\alpha(s)\psi(s)ds\Big| \\
&\leqslant \Big|\frac{c_2 u_1}{c_1}\Big| + |u_2| + |\phi_{a,b}|\sigma_b + \int_a^t K_t^\alpha(s)|\psi(s)|ds.
\end{aligned}$$

According to $(\mathscr{A}_2)$ we have

$$\begin{aligned}
|\psi(s)| &= |\Psi(s,u(s),\psi(s)) - \Psi(s,0,0) + \Psi(s,0,0)| \\
&\leqslant \frac{k\|u\|_\infty + \sup_{s\in J}|\Psi(s,0,0)|}{1-l} \\
&\leqslant \frac{kr + \Psi^*}{1-l}, \qquad \text{where} \quad \Psi^* = \sup_{s\in J}|\Psi(s,0,0)|.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|\phi_{a,b}| &= \left|\frac{1}{\delta}\Big(\frac{c_2 u_1}{c_1} - u_2 + \int_a^b K(s)\psi(s)ds\Big)\right| \\
&\leqslant \frac{1}{|\delta|}\left(\Big|\frac{c_2 u_1}{c_1}\Big| + |u_2| + \int_a^b |K(s)||\psi(s)|ds\right) \\
&\leqslant \frac{1}{|\delta|}\left(\Big|\frac{c_2 u_1}{c_1}\Big| + |u_2| + \frac{kr + \Psi^*}{1-l}\int_a^b |K(s)||ds\right) \\
&\leqslant \frac{1}{|\delta|}\left(\Big|\frac{c_2 u_1}{c_1}\Big| + |u_2| + \frac{(kr + \Psi^*)K^*}{1-l}\right) := \phi_{a,b}^*.
\end{aligned}$$

Then,

$$\begin{aligned} |(\chi u)(t)| &\leqslant \left|\frac{c_2u_1}{c_1}\right| + |u_2| + \phi^*_{a,b}\sigma_b + \frac{kr+\Psi^*}{1-l}\int_a^t K_t^\alpha(s)ds \\ &\leqslant \left|\frac{c_2u_1}{c_1}\right| + |u_2| + \phi^*_{a,b}\sigma_b + \frac{kr+\Psi^*}{1-l}\int_a^b K_b^\alpha(s)ds := m. \end{aligned}$$

It follows that

$$\|\chi u\|_\infty \leqslant m$$

which implies that χ maps bounded sets into bounded sets of $\mathscr{C}(J,R)$.
Step 3: χ maps bounded sets into a equicontinuous set of $\mathscr{C}(J,R)$.
Let $u \in \mathscr{D}_r$ (as defined in **Step 2**), and t_1, $t_2 \in J$ with $t_1 < t_2$, then

$$\begin{aligned} &|\chi u(t_2) - \chi u(t_1)| \\ &\leq |\phi_{a,b}||\sigma_{t_2} - \sigma_{t_1}| + \left|\int_a^{t_2} K_{t_2}^\alpha(s)\psi(s)ds - \int_a^{t_1} K_{t_1}^\alpha(s)\psi(s)ds\right| \\ &\leq \phi^*_{a,b}|\sigma_{t_2} - \sigma_{t_1}| + \left|\int_a^{t_1}(K_{t_2}^\alpha - K_{t_1}^\alpha)(s)\psi(s)ds + \int_{t_1}^{t_2} K_{t_2}^\alpha(s)\psi(s)ds\right| \\ &\leq \phi^*_{a,b}|\sigma_{t_2} - \sigma_{t_1}| + \frac{(k\|u\|_\infty + \Psi^*)\rho^{\alpha-2}}{(1-l)\Gamma(\alpha+1)}\left|\int_a^{t_1}(K_{t_2}^\alpha - K_{t_1}^\alpha)(s)ds + \int_{t_1}^{t_2} K_{t_2}^\alpha(s)ds\right| \\ &\leq \phi^*_{a,b}|\sigma_{t_2} - \sigma_{t_1}| + \frac{(kr+\Psi^*)\rho^{\alpha-2}}{(1-l)\Gamma(\alpha+1)}\left[2\left(t_2^\rho - t_1^\rho\right)^\alpha + t_1^{\rho\alpha} - t_2^{\rho\alpha}\right]. \end{aligned}$$

As $t_2 \longrightarrow t_1$ the right-hand side of above inequality tends to zero. As a sequence of Steps 1 to 3 together with Arzelà-Ascoli theorem, we conclude that χ is completely continuous.
Step 4: A priori bounds.
We show there exists an open set $\mathscr{O} \subset \mathscr{C}(J,R)$ with $u \neq \lambda\chi(u)$ where $\lambda \in (0,1)$ and $u \in \partial\mathscr{O}$. Let $u \in \mathscr{C}(J,R)$ and $u = \lambda\chi(u)$, with $\lambda \in (0,1)$, then for each $t \in J$ we have

$$\begin{aligned} |u(t)| &= \lambda\left|\frac{c_2u_1}{c_1} + u_2 + \phi_{a,b}\sigma_t + \int_a^t K_t^\alpha(s)\psi(s)ds\right| \\ &\leqslant \left|\frac{c_2u_1}{c_1}\right| + |u_2| + |\phi_{a,b}|\sigma_b + \int_a^b K_b^\alpha(s)|\psi(s)|ds \\ &\leqslant \left|\frac{c_2u_1}{c_1}\right| + |u_2| + \phi^*_{a,b}\sigma_b + \frac{kr+\Psi^*}{1-l}\int_a^b K_b^\alpha(s)ds. \end{aligned}$$

Thus

$$\|u\|_\infty \leqslant m.$$

Let

$$\mathscr{O} = \{u \in \mathscr{C}(J,R) : \|u\|_\infty < m+1\}.$$

By our choosing of $\mathscr{O}$, there is no $u \in \partial\mathscr{O}$, such that $u = \lambda\chi(u)$, for $\lambda \in (0,1)$. As a consequence of Theorem 3.3 and the nonlinear alternative of Leray-Schauder's fixed point theorem, χ has a fixed point $u \in O$ which is a solution of our problem (1.1).

The second result is based on Theorem 2.6.

Theorem 3.4. *Assume that* $(\mathscr{A}_1)$, $(\mathscr{A}_2)$, *and*

$$\theta = \frac{k\sigma_b K^*}{|\delta|(1-l)} < 1. \tag{3.2}$$

Then the problem (1.1) *has at least one solution.*

Proof. Let

$$\mathscr{M} = \{u \in \mathscr{C}(J,R) : \|u\|_\infty \leqslant r_1 + r_2 \leqslant r\},$$

where

$$r_1 = \Big|\frac{c_2 u_1}{c_1}\Big| + |u_2| + \phi^*_{a,b}\sigma_b, \quad r_2 = \frac{(kr+\Psi^*)(b^\rho - a^\rho)^\alpha \rho^{\alpha-2}}{(1-l)\Gamma(\alpha+1)}.$$

We define two operators S_1 and S_2 by

$$S_1 u(t) = \frac{c_2 u_1}{c_1} - u_2 + \phi_{a,b}\sigma_t$$

$$S_2 u(t) = \int_a^t K_t^\alpha(s)\psi(s)ds$$

where

$$\psi(s) = \Psi(s, u(s), \psi(s)).$$

Step 1: We will show that $S_1 u + S_2 v \in \mathscr{M}$.
Let $u, v \in \mathscr{M}$, and $t \in J$ so we have

$$\begin{aligned}
|S_1 u(t)| &\leqslant \Big|\frac{c_2 u_1}{c_1}\Big| + |u_2| + |\phi_{a,b}|\sigma_t \\
&\leqslant \Big|\frac{c_2 u_1}{c_1}\Big| + |u_2| + |\phi_{a,b}|\sigma_b \\
&\leqslant \Big|\frac{c_2 u_1}{c_1}\Big| + |u_2| + \phi^*_{a,b}\sigma_b \\
&\leqslant r_1,
\end{aligned}$$

and

$$\begin{aligned}
|S_2 v(t)| &\leqslant \int_a^t K_t^\alpha(s)|\psi(s)|ds \\
&\leqslant \frac{(kr+\Psi^*)}{1-l}\int_a^b K_b^\alpha(s)ds \\
&\leqslant \frac{(kr+\Psi^*)(b^\rho - a^\rho)^\alpha \rho^{\alpha-2}}{(1-l)\Gamma(\alpha+1)} \\
&\leqslant r_2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|S_1 u + S_2 v\|_\infty &\leqslant \|S_1 u\|_\infty + \|S_2 v\|_\infty \\
&\leqslant r_1 + r_2 \\
&\leqslant r.
\end{aligned}$$

We deduce that $S_1 u + S_2 v \in \mathscr{M}$.
Step 2: S_1 is a contraction on $\mathscr{M}$.
For each $t \in J$, $u, v \in \mathscr{M}$, $\psi(s) = \Psi(s, u(s), \psi(s))$, and $\phi(s) = \Psi(s, v(s), \phi(s))$, we have

$$\begin{aligned}
\big|S_1 u(t) - S_1 v(t)\big| &= \Big|\frac{\sigma_t}{\delta}\int_a^b K(s)\big(\psi(s) - \phi(s)\big)ds\Big| \\
&\leqslant \frac{\sigma_b}{|\delta|}\int_a^b |K(s)|\big|\psi(s) - \phi(s)\big|ds \\
&\leqslant \frac{k\sigma_b}{|\delta|(1-l)}\int_a^b |K(s)||u(s) - v(s)|ds \\
&\leqslant \frac{k\sigma_b K^*}{|\delta|(1-l)}|u(s) - v(s)|.
\end{aligned}$$

Therefore

$$\|S_1u - S_1v\|_\infty \leqslant \frac{k\sigma_b K^*}{|\delta|(1-l)}\|u-v\|_\infty.$$

By (3.2) we deduce that S_1 is a contraction.
Step 3: S_2 is compact.
It is clear that S_2 is continuous and uniformly bounded on $\mathscr{M}$ ($\|S_2u\|_\infty \leqslant r_2$).
It remains to show that S_2 maps bounded sets into a equicontinuous set of $\mathscr{C}(J,R)$.
Let $u \in \mathscr{M}$, and $t_1, t_2 \in J$ with $t_1 < t_2$, then

$$\begin{aligned} |S_2u(t_2) - S_2u(t_1)| &= \Big|\int_a^{t_2} K_{t_2}^\alpha(s)\psi(s)ds - \int_a^{t_1} K_{t_1}^\alpha(s)\psi(s)ds\Big| \\ &= \Big|\int_a^{t_1} (K_{t_2}^\alpha - K_{t_1}^\alpha)(s)\psi(s)ds + \int_{t_1}^{t_2} K_{t_2}^\alpha(s)\psi(s)ds\Big| \\ &\leqslant \frac{(k\|u\|_\infty + \Psi^*)\rho^{\alpha-2}}{(1-l)\Gamma(\alpha+1)}\left|\int_a^{t_1} (K_{t_2}^\alpha - K_{t_1}^\alpha)(s)ds + \int_{t_1}^{t_2} K_{t_2}^\alpha(s)ds\right| \\ &\leqslant \frac{(kr+\Psi^*)\rho^{\alpha-2}}{(1-l)\Gamma(\alpha+1)}\Big[2\big(t_2^\rho - t_1^\rho\big)^\alpha + t_1^{\rho\alpha} - t_2^{\rho\alpha}\Big]. \end{aligned}$$

It is obvious that since $t_2 \longrightarrow t_1$ we get $|S_2u(t_2) - S_2u(t_1)| \longrightarrow 0$. It means that S_2 is compact. By Theorem 3.4 we conclude that our problem (1.1) has a solution in $\mathscr{C}(J,R)$.

4. Examples

Example 4.1. *Let us consider the following boundary problem*

$$\begin{cases} {}^{\frac{1}{3}}D^{\frac{3}{2}}u(t) = \frac{|u(t)|}{5+|u(t)|} + \frac{1}{2}\tan\left|{}^{\frac{1}{3}}D^{\frac{3}{2}}u(t)\right|, & t \in [0, \frac{\pi}{3}], \\ u(0) - u'(0) = \frac{3}{2}, \\ u(\frac{\pi}{3}) + u'(\frac{\pi}{3}) = \frac{\pi}{6}. \end{cases} \tag{4.1}$$

Let the function Ψ defined by

$$\Psi(t,u,v) = \frac{u}{5+u} + \frac{1}{2}\tan v, \qquad u,v \in R^+, \quad t \in [0, \frac{\pi}{3}].$$

Obviously the function Ψ is continuous. Now we check assumption $(\mathscr{A}_2)$. Indeed for each $t \in [0, \frac{\pi}{3}]$ and $u, v \in R^+$, we have

$$\begin{aligned} |\Psi(t,u_2,v_2) - \Psi(t,u_1,v_1)| &= \Big|\frac{u_2}{5+u_2} - \frac{u_1}{5+u_1} + \frac{1}{2}\big(\tan v_2 - \tan v_1\big)\Big| \\ &\leqslant \Big|\frac{5(u_2-u_1)}{(5+u_2)(5+u_1)}\Big| + \frac{1}{2}\big|\tan v_2 - \tan v_1\big| \\ &\leqslant \frac{1}{5}\big|u_2 - u_1\big| + \frac{2}{3}\big|v_2 - v_1\big|. \end{aligned}$$

Therefore $(\mathscr{A}_2)$ holds for $k = \frac{1}{5}$, and $l = \frac{2}{3}$. Then according to Theorem 3.3 the problem (4.1) has at least one solution.

Example 4.2. *Let us consider the following boundary problem*

$$\begin{cases} {}^{\frac{1}{4}}D^{\frac{5}{2}}u(t) = \frac{|u(t)|}{3+\left|{}^{\frac{1}{4}}D^{\frac{5}{2}}u(t)\right|} + \frac{\left|{}^{\frac{1}{4}}D^{\frac{5}{2}}u(t)\right|}{3+|u(t)|}, & t \in [0,1], \\ u(0) - u'(0) = 1, \\ u(1) + u'(1) = \frac{1}{2}. \end{cases} \tag{4.2}$$

Set the function Ψ as

$$\Psi(t,u,v) = \frac{u}{3+v} + \frac{v}{3+u}, \qquad u,v \in R^+, \quad t \in [0,1].$$

It is easy to see that the function Ψ is continuous. On the other hand for each $t \in [0,1]$ and $u, v \in R^+$, we have

$$\begin{aligned} |\Psi(t,u_2,v_2) - \Psi(t,u_1,u_1)| &= \left| \frac{u_2}{3+v_2} + \frac{v_2}{3+u_2} - \frac{u_1}{3+v_1} - \frac{v_1}{3+u_1} \right| \\ &\leqslant \left| \frac{3u_2 + u_2 v_1 - 3u_1 - u_1 v_2}{(3+u_2)(3+v_2)} \right| + \left| \frac{3v_2 + v_2 u_1 - 3v_1 - u_2 v_1}{(3+v_1)(3+u_1)} \right| \\ &\leqslant \frac{1}{9}\Big(|3u_2 - 3u_1| + |3v_2 - 3v_1| \Big) \\ &\leqslant \frac{1}{3}\Big(|u_2 - u_1| + |v_2 - v_1| \Big). \end{aligned}$$

Therefore the assumption $(\mathscr{A}_2)$ holds for $k = l = \frac{1}{3}$. On the other hand we have

$$\theta = \frac{\frac{1}{3} \times 8 \times \frac{16}{15\Gamma(\frac{5}{2})}}{11 \times \frac{2}{3}} = \frac{128}{495\sqrt{\pi}} < 1$$

By Theorem 3.4 we conclude that the problem (4.2) has at least one solution.

5. Conclusion

In this paper, we studied some existence results of certain type of differential fractional problem involving the concept of the generalized fractional derivative, in this study we focused on Nonlinear alternative and Krasnoselskii fixed points.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] D. R. Anderson, D. J. Ulness, *Properties of Katugampola fractional derivative with potential application in quantum mechanics*, J. Math. Phys., **56** (2015).

[2] M. S. Abdo, *Further results on the existence of solutions for generalized fractional quadratic functional integral equations*, J. Math. Anal & Model, **1**(1) (2020), 33-46.

[3] A. Bashir, S. Sivasundaram, *Some existence results for fractional integro-differential equations with nonlinear conditions*, Comm. Appl. Analysis, **12** (2) (2008), 107-112.

[4] B. N. Abood, S. S. Redhwan, O. Bazighifan, K. Nonlaopon, *Investigating a generalized fractional quadratic integral equation*, Fractal Fract., 6 (2022), 251.

[5] M. Benchohra, K. Maazouz, *Existence and uniqueness results for implicit fractional differential equations with integral boundary conditions*, Comm. App. Analysis, **20** (2016), 355-366.

[6] A. Cabada, K. Maazouz, *Results for Fractional Differential Equations with Integral Boundary Conditions Involving the Hadamard Derivative*, In: Area I. et al. (eds) Nonlinear Analysis and Boundary Value Problems, NABVP 2018, Springer Proceedings in Mathematics & Statistics, vol 292. Springer, Cham, 2019.

[7] G. J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[8] M. Janaki, K. Kanagarajan, E. M. Elsayed, *Existence criteria for Katugampola fractional type impulsive differential equations with inclusions*, J. Math. Sci. Model., **2**(1) (2019), 51-63.

[9] M. Janaki, K. Kanagarajan, D. Vivek, *Analytic study on fractional implicit differential equations with impulses via Katugampola fractional Derivative*, Int. J. Math. Appl., **6**(2-A) (2018), 53-62.

[10] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[11] U. N. Katugampola, *New approach to generalized fractional integral*, Appl. Math. Comput., **218**(3) (2011), 860-865.

[12] U. N Katugampola, *New approach to generalized fractional derivative*, Bull. Math. Anal. Appl., **6** (4) (2014), 1-15.

[13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differenatial Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B. V. Amsterdam, 2006.

[14] V. Lakshmikantham, S. Leela, J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[15] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity. An Introduction to Mathematical Models*, Imperial College Press, London, 2010.

[16] K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[17] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[18] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[19] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge 1974, Bull. Math. Anal. Appl., **6** (4) (2014), 1-15.

[20] S. S. Redhwan, S. L. Shaikh, M. S. Abdo, *Caputo-Katugamola type implicit fractional differential equation with two-point anti-periodic boundary conditions*, Results in Nonlinear Analysis, **5**(1) (2022), 12-28.

[21] S. S. Redhwan, S. L. Shaikh, M. S. Abdo, W. Shatanawi, K. Abodayeh, M. A. Almalahi, T. Aljaaidi, *Investigating a generalized Hilfer-type fractional differential equation with two-point and integral boundary conditions*, AIMS Math., **7**(2) (2021), 1856-187.

[22] D. Vivek, E. M. Elsayed, K. Kanagarajan, *Dynamics and stability results for impulsive type integro-differential equations with generalized fractional derivative*, Math. Nat. Sci., **4** (2019), 1-12.

[23] D. Vivek, K. Kanagarajan, S. Harikrishnan, *Theory and analysis of impulsive type pantograph equations with Katugampola fractioanl derivative*, J. Vabration Testing and System Dynamic, **2** (1) 2018, 9-20.

[24] D. Vivek, E. M. Elsayed, K. Kanagarajan, *Theory of fractional implicit differential equations with complex order*, J. Uni. Math., **2**(2) (2019), 154-165.

[25] H. Wang, *Existence results for fractional functional differential equations with impulses*, J. Appl. Math. Comput., **38** (2012), 85-101.

[26] E. Zeidler, *Nonlinear functional Analysis and its Applications-I Fixed Point Theorem*, Springer, New-York, 1993.

Communications in Advanced Mathematical Sciences
Vol. 5, No. 3, 124-130, 2022
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1089222

Shift Filter of Quasi-ordered Residuated Systems

Daniel A. Romano[1]*

Abstract
The concept of residuated relational systems ordered under a quasi-order relation was introduced in 2018 by S. Bonzio and I. Chajda as a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq \rangle$, where $(A, \cdot)$ is a commutative semigroup with the identity 1 as the top element in this ordered monoid under a quasi-order $\preccurlyeq$. In 2020, the author introduced and analyzed the concepts of filters in this type of algebraic structures. In addition to the previous, the author continued to investigate some of the types of filters in quasi-ordered residuated systems such as, for example, implicative and comparative filters. In this article, as a continuation of previous author's research, the author introduced and analyzed the concepts of shift filters of quasi-ordered residuated systems and then compared it with other types of filters.

Keywords: Comparative filter, Filter, Implicative filter, Shift filter, Quasi-ordered residuated system
2010 AMS: Primary 08A02, Secondary 06A11

[1] *International Mathematical Virtual Institute, 78000 Banja Luka, Bosnia and Herzegovina, ORCID: 0000-0003-1148-3258*
***Corresponding author**: daniel.a.romano@hotmail.com
Received: 17 March 2022; **Accepted:** 12 September 2022; **Available online:** 30 September 2022

1. Introduction

Let $(A, \cdot, 1)$ be a commutative semigroup with the identity 1. Suppose that on the carrier A there exists another operation $\rightarrow$ and one relation R that with multiplication in A have a link $(x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R$ for each $x, y, z \in A$. A relational system designed in this way, when R is a quasi-ordered relation on A, is in the focus of this paper.

The concept of residuated relational systems ordered under a quasi-order relation was introduced in 2018 by S. Bonzio and I. Chajda in [2]. Previously, this concept was discussed in [1]. This paper continues the investigations of quasi-ordered residuated systems and of their filters which were started in the author article [3]. In particular, the concept of shift filters of a quasi-ordered residuated system is introduced and analyzed. This type of filter is compared to the concept of filter and the concept of implicative (introduced in [4]) and comparative filters ([6]) in this algebraic system. It is shown (Theorem 3.2) that every comparative filter is a shift filter and vice versa does not have to be. In addition, it is shown (Theorem 3.3) that if the implicative filter F satisfies the added condition

$$(\forall u, v \in A)((u \rightarrow v) \rightarrow v \in F \Longrightarrow (v \rightarrow u) \rightarrow u \in F))$$

then F is a shift filter. The reverse, of course, does not have to be.

It should be said here that a quasi-ordered residuated system, in the general case, it does not have to be a commutative residauted lattice (see Example 2.8).

2. Preliminaries

2.1 Concept of quasi-ordered residuated systems

In article [2], S. Bonzio and I. Chajda introduced and analyzed the concept of residual relational systems.

Definition 2.1 ([2], Definition 2.1). *A* residuated relational system *is a structure* $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, *where* $\langle A, \cdot, \rightarrow, 1\rangle$ *is an algebra of type* $\langle 2, 2, 0\rangle$ *and R is a binary relation on A and satisfying the following properties:*

(1) $(A, \cdot, 1)$ *is a commutative monoid;*
(2) $(\forall x \in A)((x, 1) \in R)$*;*
(3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$.

We will refer to the operation $\cdot$ *as multiplication, to* $\rightarrow$ *as its residuum and to condition (3) as residuation.*

The basic properties for residuated relational systems are subsumed in the following:

Theorem 2.2 ([2], Proposition 2.1). *Let* $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$ *be a residuated relational system. Then*

(4) $(\forall x, y \in A)(x \rightarrow y = 1 \Longrightarrow (x, y) \in R)$*;*
(5) $(\forall x \in A)((x, 1 \rightarrow 1) \in R)$*;*
(6) $(\forall x \in A)((1, x \rightarrow 1) \in R)$*;*
(7) $(\forall x, y, z \in A)(x \rightarrow y = 1 \Longrightarrow (z \cdot x, y) \in R)$*;*
(8) $(\forall x, y \in A)((x, y \rightarrow 1) \in R)$.

Recall that a *quasi-order relation* $'\preccurlyeq'$ on a set *A* is a binary relation which is reflexive and transitive (Some authors use the term pre-order relation).

Definition 2.3 ([2], Definition 3.1). *A* quasi-ordered residuated system *is a residuated relational system* $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$, *where* $\preccurlyeq$ *is a quasi-order relation in the monoid* $(A, \cdot)$

Example 2.4. *Let* $A = \{1, a, b, c, d\}$ *and operations '*$\cdot$*' and '*$\rightarrow$*' defined on A as follows:*

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
s	a	a	d	c	d
b	b	d	b	d	d
c	c	c	d	c	d
d	d	d	d	d	d

and

$\rightarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	a	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ *is a quasi-ordered residuated systems where the relation '*$\preccurlyeq$*' is defined as follows*

$$\preccurlyeq := \{(1,1), (a,1), (b,1), (c,1), (d,1), (b,b), (a,a), (c,a), (d,a), (d,b), (d,c)\}.$$

Example 2.5. *For a commutative monoid A, let* $\mathfrak{P}(A)$ *denote the powerset of A ordered by set inclusion and '*$\cdot$*' the usual multiplication of subsets of A. Then* $\langle \mathfrak{P}(A), \cdot, \rightarrow, A, \subseteq\rangle$ *is a quasi-ordered residuated system in which the residuum are given by*

$$(\forall X, Y \in \mathfrak{P}(A))(Y \rightarrow X := \{z \in A : Yz \subseteq X\}).$$

Example 2.6. *Let* $\mathbb{R}$ *be a field of real numbers. Define two binary operations '*$\cdot$*' and '*$\rightarrow$*' on* $A = [0, 1] \subset \mathbb{R}$ *by*

$$(\forall x, y \in [0, 1])(x \cdot y := max\{0, x + y - 1\}) \text{ and } x \rightarrow y := min\{1, 1 - x + y\}).$$

Then, A is a commutative monoid with the identity 1 *and* $\langle A, \cdot, \rightarrow, <, 1\rangle$ *is a quasi-ordered residuated system.*

Example 2.7. *Any commutative residuated lattice* $\langle A, \cdot, \rightarrow, 0, 1, \wedge, \vee, R\rangle$, *where R is a lattice quasi-order, is a quasi-ordered residuated system.*

The following example shows that a quasi-ordered residuated system *A* does not have to be a lattice because:
- in the general case, *A* does not have to have a common lower bound,
- *A* doesn't have to be a lattice.

Example 2.8. *Let* $A = \langle -\infty, 1]\subset \mathbb{R}$ *(the real numbers field). If we define '*$\cdot$*' and '*$\rightarrow$*' as follows,* $(\forall y, v \in A)(u \cdot v := min\{u, v\})$ *and* $u \rightarrow v := 1$ *if* $u \leqslant v$ *and* $u \rightarrow v := v$ *if* $v < u$ *for all* $u, v \in A$, *then* $\mathfrak{A} := \langle A, \cdot, \rightarrow, 1, <\rangle$ *is a quasi-ordered residuated system.*

The following proposition shows the basic properties of quasi-ordered residuated systems.

Proposition 2.9 ([2], Proposition 3.1). *Let A be a quasi-ordered residuated system. Then*

(9) $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y))$*;*
(10) $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow (y \rightarrow z \preccurlyeq x \rightarrow z \wedge z \rightarrow x \preccurlyeq z \rightarrow y))$*;*
(11) $(\forall x, y \in A)(x \cdot y \preccurlyeq x \wedge x \cdot y \preccurlyeq y)$.

Estimating that this topic is interesting ([1]-[3]), it is certain that there is interest in the development of the concept of some substructures such as some types of filters [4]-[7] in these systems.

2.2 Concepts of filters

In the article [3], in order to determine the concept of filters of quasi-ordered residuated systems, the relationships between the following conditions are analyzed:

(F0) $1 \in F$;

(F1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$;

(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$;

(F3) $(\forall u, v \in A)((u \in F \wedge u \rightarrow v \in F) \implies v \in F)$.

It is shown ([3], Proposition 3.2) that $(F2) \implies (F1)$. In addition, it is shown ([3], Proposition 3.4) that for every nonempty subset F of system $\mathfrak{A}$ is valid $(F2) \implies (F0)$.

Based on our previous analysis of the interrelationship between conditions (F1), (F2) and (F3) in a quasi-ordered residual system, we introduced the concept of filters in the following definition.

Definition 2.10 ([3], Definition 3.1)**.** *For a subset F of a quasi-ordered residuated system $\mathfrak{A}$, we say that it is a* filter *of $\mathfrak{A}$ if it satisfies conditions (F2) and (F3).*

Example 2.11. *Let $\mathfrak{A}$ be as in Example 2.8. All filters in this quasi-ordered residuated system are of the form $\langle -\infty, 1]$, where $x < 1$.*

Lemma 2.12 ([4], Lemma 3.1)**.** *Let a subset F of a quasi-ordered residuated system $\mathfrak{A}$ satisfy the condition* (F2). *Then the following holds*

(12) $(\forall u \in A)(u \in F \iff 1 \rightarrow u \in F)$.

Lemma 2.13 ([4], Lemma 3.4)**.** *Let a subset F of a quasi-ordered residuated system $\mathfrak{A}$ satisfy the condition* (F2). *Then the following holds*

(13) $(\forall u, v, z \in A)(u \rightarrow (v \rightarrow z) \in F \iff v \rightarrow (u \rightarrow z) \in F)$.

Lemma 2.14 ([6])**.** *Let $\mathfrak{A}$ be a quasi-ordered residuated system. Then*

(14) $(\forall u, v, z \in A)(u \rightarrow v \preccurlyeq (v \rightarrow z) \rightarrow (u \rightarrow z))$ *and*

(15) $(\forall u, v, z \in A)(v \rightarrow z \preccurlyeq (u \rightarrow v) \rightarrow (u \rightarrow z))$.

Terms covering some of the requirements used herein to identify various types of filters in the observed algebraic structure are mostly taken from papers on UP-algebras.

Definition 2.15 ([4], Definition 3.1)**.** *For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$, we say that the* implicative filter *of $\mathfrak{A}$ if (F2) and the following condition*

(IF) $(\forall u, v, z \in A)((u \rightarrow (v \rightarrow z) \in F \wedge u \rightarrow v \in F) \implies u \rightarrow z \in F)$

are valid.

It is known that every implicative filter of a quasi-ordered residuated system $\mathfrak{A}$ is a filter of $\mathfrak{A}$ ([4], Theorem 3.1) but that the reverse does not have to be.

Definition 2.16 ([6], Definition 5)**.** *For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that a* comparative filter *of $\mathfrak{A}$ if (F2) and the following condition*

(FC) $(\forall u, v, z \in A)((u \rightarrow ((v \rightarrow z) \rightarrow v) \in F \wedge u \in F) \implies v \in F)$

are valid.

Example 2.17. *Let $\mathfrak{A}$ be a quasi-ordered residuated system as in Example 2.4. Then the set $F := \{1, a, b\}$ is a comparative filter of $\mathfrak{A}$.*

Since any comparative filter F of $\mathfrak{A}$ satisfies the condition (F2), F also satisfies the condition (F0): $1 \in F$.

Proposition 2.18 ([6], Theorem 3.2)**.** *Let F be a filter of a quasi-ordered residuated system $\mathfrak{A}$. Then F is a comparative filter of $\mathfrak{A}$ if and only if the condition*

(16) $(\forall v, z \in A)((v \rightarrow z) \rightarrow v \in F \implies v \in F)$

is valid.

3. The Concept of Shift Filters

In this section, which is the main part of this article, we introduce and analyze the concept of shift filters of quasi-ordered residuated system.

Definition 3.1. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. A non empty subset F of A is a* shift filter *of $\mathfrak{A}$ if it satisfies the conditions (F2) and the following condition*
(SF) $(\forall u,v,z \in A)((u \rightarrow (v \rightarrow z) \in F \wedge u \in F) \Longrightarrow ((z \rightarrow v) \rightarrow v) \rightarrow z \in F)$.

Remark 3.2. *In some other algebraic systems, request (SF) is recognized as a fantastic filter.*

Example 3.3. *Let $A = \{1,a,b,c\}$ and operations '$\cdot$' and '$\rightarrow$' defined on A as follows:*

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	a	c
b	b	a	b	c
c	c	a	c	c

and

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	a	1	c
c	1	a	b	1

.

Then $\mathfrak{A} = \langle A,\cdot,\rightarrow,1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq = \{(1,1),(a,a),(b,b),(c,c),(a,1),(b,1),(c,1),(a,b)\}.$$

Then the subsets $\{1,b\}$ is a shift filter of $\mathfrak{A}$.

Example 3.4. *Let $A = \{1,a,b,c\}$ and operations '$\cdot$' and '$\rightarrow$' be defined on A as follows:*

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	c	c
b	b	c	b	c
c	c	c	c	c

and

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	c
c	1	a	b	1

.

Then $\mathfrak{A} = \langle A,\cdot,\rightarrow,1\rangle$ is a quasi-ordered residuated systems, where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq = \{(1,1),(a,a),(b,b),(c,c),(a,1),(b,1),(c,1),(a,b),(a,c)\}.$$

Then the subsets $\{1,b\}$ is a filter of $\mathfrak{A}$ but it is not a shift filter of $\mathfrak{A}$. For example, for $u = b$, $v = a$ and $z = c$, we have $b \rightarrow (a \rightarrow c) = b \rightarrow 1 = 1 \in \{1,b\}$ and $b \in \{1,b\}$, but $((c \rightarrow a) \rightarrow a) \rightarrow c = (a \rightarrow a) \rightarrow c = 1 \rightarrow c = c \notin \{1,b\}$.

It can be verified that a shift filter of a quasi-ordered residuated system $\mathfrak{A}$ has the following property:

Proposition 3.5. *Let F be a shift filter of a quasi-ordered residuated system $\mathfrak{A}$. Then*
(17) $(\forall u,v \in A)(u \rightarrow v \in F \Longrightarrow ((v \rightarrow u) \rightarrow u) \rightarrow v \in F)$.

Proof. Let F be a shift filter of $\mathfrak{A}$. If we put $u = 1$, $v = u$ and $z = v$ in (SF), we get

$$(1 \rightarrow (u \rightarrow v) \in F \wedge 1 \in F) \Longrightarrow ((v \rightarrow u) \rightarrow u) \rightarrow v \in F$$

whence it follows

$$u \rightarrow v \in F \Longrightarrow ((v \rightarrow u) \rightarrow u) \rightarrow v \in F$$

by (F0) and Lemma 2.12. □

Let us show now that the condition (17) is sufficient for a filter F of a quasi-ordered system $\mathfrak{A}$ satisfying the condition (17) to be a shift filter of $\mathfrak{A}$.

Theorem 3.6. *Let F be a filter of a quasi-ordered residuated system $\mathfrak{A}$ and suppose that F satisfies the condition (17). Then F is a shift filter of $\mathfrak{A}$.*

Proof. Suppose that F is a filter of $\mathfrak{A}$ that satisfies the condition (17). Let $u,v,z \in A$ be such that $u \to (v \to z) \in F$ and $u \in F$. Then $v \to z$ by (F3). Thus $((z \to v) \to v) \to z \in F$ by (17). So, F is a shift filter of $\mathfrak{A}$. □

Our second theorem on this class of filters of quasi-ordered residuated systems is the following:

Theorem 3.7. *Every comparative filter of a quasi-ordered residuared system $\mathfrak{A}$ is a shift filter of $\mathfrak{A}$.*

Proof. Suppose that F is a comparative filter of $\mathfrak{A}$. To prove that F is a shift filter of $\mathfrak{A}$, we will show that F satisfies the condition (17). For this purpose, let us take elements $u,v \in A$ such that $u \to v \in F$.

From $u \to v \in F$ and from the valid formula (14) in the form $u \to v \preccurlyeq ((v \to u) \to u) \to ((v \to u) \to v)$, it follows $((v \to u) \to u) \to ((v \to u) \to v) \in F$ according (F2), which it is equivalent to

$$(v \to u) \to (((v \to u) \to u) \to v \in F$$

according to (13).

On the other hand, from the valid formula (11), in the form $(v \preccurlyeq (v \to u) \to u) \to v$, with respect to (14), we obtain

$$(((v \to u) \to u) \to v) \to u \preccurlyeq (v \to u).$$

From here, by acting with $((v \to u) \to u) \to v$ on the last inequality by the right, taking into account the valid formula (14), we obtain

$$(v \to u) \to (((v \to u) \to u) \to v) \preccurlyeq$$
$$((((v \to u) \to u) \to v) \to u) \to (((v \to u) \to u) \to v).$$

From here it follows

$$((((v \to u) \to u) \to v) \to u) \to (((v \to u) \to u) \to v) \in F.$$

Since F is a comparative filter in $\mathfrak{A}$, we get $((v \to u) \to u) \to v \in F$ in accordance with (16). Therefore, F is a shift filter. □

The following example shows that any shift filter of a quasi-ordered residuated system $\mathfrak{A}$ does not have to be a comparative filter of $\mathfrak{A}$.

Example 3.8. *Let $A = \{1,a,b,c\}$ and operations '$\cdot$' and '$\to$' be defined on A as follows:*

$\cdot$	1	a	b	c
1	a	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

and

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	a	b
b	1	a	1	b
c	1	1	1	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems, where the relation '$\preccurlyeq$' is defined as follows:

$$\preccurlyeq = \{(1,1),(a,a),(b,b),(c,c),(a,1),(b,1),(c,1),(b,a),(c,a),(c,b)\}.$$

Then the subsets $\{1\}$ is a shift filter of $\mathfrak{A}$ but it is not a comparative filter of $\mathfrak{A}$. For example, for $u = 1$, $v = a$ and $z = b$, we have $1 \to ((a \to b) \to a) = 1 \to (a \to a) = 1 \to 1 = 1 \in \{1\}$ and $1 \in \{1\}$, but $a \notin \{1\}$.

Theorem 3.9. *Let F be an implicative filter of a quasi-ordered residuated system $\mathfrak{A}$ satisfying*
(18) $(\forall u,v \in A)((u \to v) \to v \in F \Longrightarrow (v \to u) \to u \in F)$.
Then F is a shift of $\mathfrak{A}$.

Proof. The proof of this theorem is obtained by combining Theorem 4 in [6] and Theorem 3.7. □

The following example shows that any shift filter of a quasi-ordered residuated system $\mathfrak{A}$ does not have to be an implicative filter of $\mathfrak{A}$.

Example 3.10. *Let $A = \{1,a,b,c\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:*

$\cdot$	*1*	*a*	*b*	*c*
1	*a*	*a*	*b*	*c*
a	*a*	*a*	*b*	*c*
b	*b*	*b*	*b*	*c*
c	*c*	*c*	*c*	*c*

and

$\rightarrow$	*1*	*a*	*b*	*c*
1	*1*	*a*	*b*	*c*
a	*1*	*1*	*a*	*b*
b	*1*	*1*	*1*	*b*
c	*1*	*1*	*1*	*1*

.

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ is a quasi-ordered residuated systems, where the relation '$\preccurlyeq$' is defined as follows:

$$\preccurlyeq = \{(1,1),(a,a),(b,b),(c,c),(a,1),(b,1),(c,1),(b,a),(c,a),(c,b)\}.$$

Then the subsets $\{1\}$ is a shift filter of $\mathfrak{A}$ but it is not an implicative filter of $\mathfrak{A}$. For example, for $u = b$, $v = b$ and $z = c$, we have $b \rightarrow (b \rightarrow c) = 1 \in \{1\}$ and $b \rightarrow b = 1 \in \{1\}$, but $b \rightarrow c = b \notin \{1\}$.

We end this section with the following theorem.

Theorem 3.11. *The family $\mathfrak{F}_s(A)$ of all shift filters of a quasi-ordered residuated system $\mathfrak{A}$ forms a complete lattice.*

Proof. Let $\{F_k\}_{k\in\Lambda}$ be a family of shift filters of $\mathfrak{A}$ where Λ is index set. It is clear that $1 \in \bigcap_{k\in\Lambda} F_k$. Let $u, v \in \bigcap_{k\in\Lambda} F_k$ and $u \preccurlyeq v$. Then $u \in F_k$ and $u \preccurlyeq v$ for any $k \in \Lambda$. Thus $v \in F_k$ by (F2) since F_k is a shift filter in $\mathfrak{A}$. Hence $v \in \bigcap_{k\in\Lambda} F_k$.

Let $u, v, z \in A$ be such that $u \rightarrow (v \rightarrow z) \in \bigcap_{k\in\Lambda} F_k$ and $u \in \bigcap_{k\in\Lambda} F_k$. Then $u \rightarrow (v \rightarrow z) \in F_k$ and $u \in F_k$ for any $k \in \Lambda$. Thus $((z \rightarrow v) \rightarrow v) \rightarrow z \in F_k$ for all $k \in \Lambda$. Hence $((z \rightarrow v) \rightarrow v) \rightarrow z \in \bigcap_{k\in\Lambda} F_k$. So, the intersection $\bigcap_{k\in\Lambda} F_k$ satisfies the condition (SF). Therefore $\bigcap_{k\in\Lambda} F_k$ is a shift filter of $\mathfrak{A}$.

Let $\mathfrak{X}$ be the family of all shift filters containing the union $\bigcup_{k\in\Lambda} F_k$. Then $\cap\mathfrak{X}$ is a shift filter of $\mathfrak{A}$ according to the first part of this proof.

If we put $\sqcap_{k\in\Lambda} F_k = \bigcap_{k\in\Lambda} F_k$ and $\sqcup_{k\in\Lambda} F_k = \cap\mathfrak{X}$, then $(\mathfrak{F}_s(A), \sqcap, \sqcup)$ is a complete lattice. $\square$

Let $\mathfrak{A}$ be a quasi-ordered residuated system. Before embarking on further conclusions, let us recall the terms 'minimum filter' and 'maximum filter' in a quasi-ordered residuated system: We shall say that a filter A is a minimal filter of $\mathfrak{A}$ if there is no a filter B of $\mathfrak{A}$ such that $B \subset A$. Also, dually, we shall say that a filter A is a maximal filter of $\mathfrak{A}$ if there is no a filter B of $\mathfrak{A}$ such that $A \subset B$. It is easy to conclude that if A and B are two minimum interiors filters of a quasi-ordered residuated system $\mathfrak{A}$, then $A \cap B = \emptyset$, because, otherwise, according to the previous theorem, $A \cap B$ would be a filter of $\mathfrak{A}$ contained in A and contained in B, which is impossible.

Corollary 3.12. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. For any subset T of A, there is the unique minimum shift filter of $\mathfrak{A}$ that contains T.*

Proof. The proof of this Corollary follows directly from the second part of the proof of the previous theorem. $\square$

Corollary 3.13. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. For any element x of A, there is the unique minimum shift filter of $\mathfrak{A}$ that contains x.*

Proof. The proof of this Corollary follows from the previous Corollary if we take $T = \{x\}$. $\square$

4. Conclusion

The concept of quasi-ordered residuated systems was introduced in 2018 by S. Bonzio and I. Chajda. as a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where $(A, \cdot)$ is a commutative semigroup with the identity 1 as the top element in this ordered monoid under a quasi-order R. In such algebraic systems, the author introduced the concept of filters, and then several types of filters such as implicative [4], associated [5] and comparative filters [6]. It is shown that a comparative filter is an implicative filter and vice versa does not have to be.

The concept of shift filters of such algebraic systems was introduced and analyzed in this paper. Also, this class of filters was compared with previously introduced filters. It is shown (Theorem 3.2) that every comparative filter is a shift filter and vice versa does not have to be. In addition, it is shown (Theorem 3.3) that if the implicative filter F satisfies the added condition

$$(\forall u, v \in A)((u \rightarrow v) \rightarrow v \in F \implies (v \rightarrow u) \rightarrow u \in F))$$

then F is a shift filter. The reverse, of course, does not have to be.

In our paper [8], we analyze a quasi-ordered residuated system (which we call the 'strong quasi-ordered residuated system') in which implicative and comparative filters are coincide. It is a quasi-ordered residuated system in which the formula

$$(\forall u, v \in A)((u \rightarrow v) \rightarrow v \preccurlyeq (v \rightarrow u) \rightarrow u \wedge (v \rightarrow u) \rightarrow u \preccurlyeq (u \rightarrow v) \rightarrow v)$$

is a valid formula. We also analyze the possibility of the existence of some new types of filters in such systems as prime and irreducible filters and their interrelationships ([9, 10]).

Acknowledgements

The author thanks to Akbar Reazei (Kerman, Iran) for patiently reading the draft version of this article. The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Competing interests

The authors declare that they have no competing interests.

References

[1] S. Bonzio, *Algebraic structures from quantum and fuzzy logics,* Ph.D Thesis, Cagliari: Universit'a degli studi di Cagliari, 2015.

[2] S. Bonzio, I. Chajda, *Residuated relational systems*, Asian-European J. Math., **11**(2) (2018), 1850024

[3] D. A. Romano, *Filters in residuated relational system ordered under quasi-order*, Bull. Int. Math. Virtual Inst., **10**(3) (2020), 529-534.

[4] D. A. Romano, *Implicative filters in quasi-ordered residuated dystem*, Proyecciones J. Math., **40**(2) (2021), 417-424.

[5] D. A. Romano, *Associated filters in quasi-ordered residuated systems*, Contributions to Mathematics, **1** (2020), 22-26.

[6] D. A. Romano, *Comparative filter in quasi-ordered residuated sysyem*, Bull. Int. Math. Virtual Inst., **11**(1) (2021), 177-184.

[7] D. A. Romano, *Ideals in quasi-ordered residuated system*, Contrib. Math., **3** (2021), 68-76.

[8] D. A. Romano, *Strong quasi-ordered residuated system*, Open J. Math. Sci., **5** (2021), 73-79.

[9] D. A. Romano, *Prime and irreducible filters in strong quasi-ordered residuated systems*, Open J. Math. Sci., **5** (2021), 172-181.

[10] D. A. Romano, *Three types of prime filters in strong quasi-ordered residuated systems*, An. Univ. Craiova, Ser. Mat. Inf., **49**(1) (2022), 135-145.

Communications in Advanced Mathematical Sciences
Vol. 5, No. 3, 131-136, 2022
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1118409

Stability of Solutions for a Krichhoff-Type Plate Equation with Degenerate Damping

Fatma Ekinci[1]*, Erhan Pişkin[2]

Abstract
We investigate a Kirchhoff type plate equation with degenerate damping term. By potential well theory, we show the asymptotic stability of energy in the presence of a degenerate damping.

Keywords: Degenerate damping, Kirchhoff-type equation, Stability
2010 AMS: 35B40, 35B44, 35G61, 35L75

[1] *Department of Mathematics, Dicle University, 21280 Diyarbakır, Turkey, ORCID: 0000-0002-9409-3054*
[2] *Department of Mathematics, Dicle University, 21280 Diyarbakır, Turkey, ORCID: 0000-0001-6587-4479*
***Corresponding author**: ekincifatma2017@gmail.com
Received: 18 May 2022, **Accepted:** 2 August 2022, **Available online:** 30 September 2022

1. Introduction and Preliminaries

In this paper, we focus on the stability of solutions under the sufficent conditions for the following problem

$$\begin{cases} u_{tt}+\Delta^2 u-\Delta u-\left(\int\limits_{\Omega}|\nabla u|^2\,dx\right)^{\gamma}\Delta u+|u|^{\rho}\,j'(u_t)=|u|^{q-1}u \ \text{ in } \ \Omega\times(0,+\infty), \\ u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x) \qquad \text{on } \ x\in\Omega, \\ u(x,t)=\frac{\partial}{\partial n}u(x,t)=0 \qquad \text{on } \ x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\gamma>0$, j' denotes the derivative of $j(\alpha)$ [1], n is the outer normal and Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$. Also, here

$$\Delta u-\left(\int\limits_{\Omega}|\nabla u|^2\,dx\right)^{\gamma}\Delta u \ \text{ and } \ |u|^{\rho}\,j'(u_t)$$

represent Kirchhoff-type term and degenerate damping term, respectively.

1.1 Kirchhoff-type plate problems

To motivation for this problem comes from the following equation so called Beam equation model

$$u_{tt}+\Delta^2 u-\left(\alpha+\beta\int\limits_{\Omega}|\nabla u|^2\,dx\right)\Delta u=|u|^{q-2}u, \tag{1.2}$$

without source term $\left(|u|^{q-2}u\right)$ was firstly introduced by Woinowsky-Krieger [2] to describe the dynamic bucking of a hinged extensible beam under an axial force. It was extensively studied by several researcers in different contexs. In [3, 4], the authors showed the global attractor, convergence and unboundedness of solutions with $|u_t|^{p-2}u_t$ nonlinear damping term. Then, the model also was investigated in [5, 6] and the authors obtained the existence, decay estimates of solutions and blow up of solutions with both negative and positive initial energy with $|u_t|^{p-2}u_t$ nonlinear damping term.

Recently, Pereira et al. [7] and Pişkin and Yüksekkaya [8] studied the model (1.2) with u_t. Pereira et al. studied existence of the global solutions through the Faedo-Galerkin approximations and obtained the asymptotic behavior by using the Nakao method. Pişkin and Yüksekkaya proved the blow up of solutions with positive and negative initial energy.

1.2 Problems with degenerate damping

This kind of degenerate damping effects was firstly investigated by Levine and Serrin [9] and considered the following equation

$$\left(|u_t|^{l-2}u_t\right)_t - a\nabla.\left(|\nabla u|^{q-2}\nabla u\right) + b|u|^{\rho}|u_t|^{m-2}u_t = c|u|^{p-2}u.$$

The authors considered the blow up of solutions with negative initial energy for the case $\rho+m<p$ under several other restrictions imposed on the paremeters m,ρ,p,q. But Levine and Serrin obtain only blow up solution with negative initial energy without any guarantees that the solution has a local solution. Then, Pitts and Rammaha [10] proved global and local existence for $\rho+m\geq p$ and for the case $\rho<1$ established uniqueness. Also, the authors obtained blow up solutions for negative initial energy and $\rho+m<p$.

On the other hand, the hyperbolic models with degenerate damping are of much interest in material science and physics. It particularly appears in physics when the friction is modulated by the strains. There is a wide literature has degenerate damping terms, namely $\delta(u)h(u_t)$ where $\delta(u)$ is a positive function and h is nonlinear, (see [11]-[27]).

The remaining part of this paper is organized as follows: In the next section, we study the stability result.

Now, we present some preliminary material which will be helpful for the proof of our result. Throughout this paper, we denote the standart $L^2(\Omega)$ norm by $\|.\| = \|.\|_{L^2(\Omega)}$ and $L^q(\Omega)$ norm $\|.\|_q = \|.\|_{L^q(\Omega)}$.

(**A1**) $\rho,p,q\geq 0$; $\rho\leq\frac{n}{n-2}$, $q+1\leq\frac{2n}{n-2}$ if $n\geq 3$. There exist positive constants c, c_0, c_1 such that for all $\alpha,\beta\in R$ $j(\alpha):R\to R$ be a C^1 convex real function satisfies

- $j(\alpha)\geq c|\alpha|^{p+1}$,
- $j'(\alpha)$ is single valued and $|j'(\alpha)|\leq c_0|\alpha|^p$,
- $(j'(\alpha)-j'(\beta))(\alpha-\beta)\geq c_1|\alpha-\beta|^{p+1}$.

(**A2**) $u_0(x)\in H_0^2(\Omega)$, $u_1(x)\in L^2(\Omega)$, $|u(\tau)|^{\rho}j(u_t)\in L^2(\Omega\times(0,T))$.

The said solution of (1.1) satisfies the energy identity

$$E'(t) = -\int_{\Omega}|u(\tau)|^{\rho}j(u_t)(\tau)\,dxd\tau\leq 0 \tag{1.3}$$

where

$$E(t) = \frac{1}{2}\left[\|u_t\|^2+\|\Delta u\|^2+\|\nabla u\|^2+\frac{1}{\gamma+1}\|\nabla u\|^{2(\gamma+1)}\right]-\frac{1}{q+1}\|u\|_{q+1}^{q+1} \tag{1.4}$$

and

$$E(0) = \frac{1}{2}\left[\|u_1\|^2+\|\Delta u_0\|^2+\|\nabla u_0\|^2+\frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)}\right]-\frac{1}{q+1}\|u_0\|_{q+1}^{q+1}. \tag{1.5}$$

Moreover, by computation, we get $E(t)$ is a non-increasing function, then

$$E(t)\leq E(0). \tag{1.6}$$

Now, we define

$$\alpha_1 = \lambda_1^{-\frac{2}{q-1}},\quad E_1 = \left(\frac{1}{2(\gamma+1)}-\frac{1}{q+1}\right)\alpha_1^{q+1},$$

$$\alpha_2 = \left(\frac{1}{(q+1)\lambda_1^2}\right)^{\frac{1}{q-1}},\quad E_2 = \frac{q+1}{2}\left(\frac{1}{2}-\frac{1}{q+1}\right)\alpha_2^{q+1},$$

$$W_0 = \left\{(\alpha,E)\in R^2, 0\leq\alpha<\alpha_2, 0<E<E_2\right\},$$

$$V = \left\{(\alpha,E)\in R^2, \alpha>\alpha_1, 0<E<E_1\right\}$$

where λ_1 is the embedding constant (where $H_0^2(\Omega)$ is embedded into $L^{q+1}(\Omega)$).

2. Stability

This section is devoted to prove the stability of solutions for problem (1.1).

Lemma 2.1. *Assume that* $(A1)$ *and* $(A2)$ *hold and* $\left(\|u_0\|_{q+1}, E(0)\right) \in W_0$, *then*

$$\left(\|u(t)\|_{q+1}, E(t)\right) \in W_0, \ \ t \geq 0, \tag{2.1}$$

and

$$E(t) \geq \frac{1}{2}\left[\|u_t(t)\|^2 + \|\Delta u(t)\|^2 + \frac{1}{\gamma+1}\|\nabla u(t)\|^{2(\gamma+1)}\right] + \frac{1}{4}\|\nabla u(t)\|^2, \ \ t \geq 0. \tag{2.2}$$

Proof. By using the embedding theorem and (1.6), we get

$$\begin{aligned} E_2 &> E(0) \geq E(t) \\ &\geq \frac{1}{2}\left[\|u_t(t)\|^2 + \|\Delta u(t)\|^2 + \frac{1}{\gamma+1}\|\nabla u(t)\|^{2(\gamma+1)}\right] \\ &\quad + \frac{1}{4}\|\nabla u(t)\|^2 + \frac{1}{4}\lambda_1^{-2}\|u(t)\|_{q+1}^2 - \frac{1}{2}\|u(t)\|_{q+1}^{q+1} \\ &\geq \frac{1}{2}\left[\|u_t(t)\|^2 + \|\Delta u(t)\|^2 + \frac{1}{\gamma+1}\|\nabla u(t)\|^{2(\gamma+1)}\right] \\ &\quad + \frac{1}{4}\|\nabla u(t)\|^2 + h\left(\|u(t)\|_{q+1}\right), \end{aligned} \tag{2.3}$$

where $h(\alpha) = \frac{1}{4}\lambda_1^{-2}\alpha^2 - \frac{1}{2}\alpha^{q+1}$, for $\alpha \geq 0$. It is not difficult to verify that $h(\alpha)$ reachs its maximum E_2 for $\alpha = \alpha_2$, $h(\alpha)$ is strictly decreasing for $\alpha \geq \alpha_2$ and $h(\alpha) \to -\infty$ as $\alpha \to \infty$. By the continuity of $\|u(t)\|_{q+1}$ and $\alpha(0) = \|u_0\|_{q+1} < \alpha_2$, $\alpha(t) < \alpha_2$ for all $t \geq 0$. Further, $E(t) < E_2$ by (2.3). Then, (2.1) holds.

To obtain (2.2), it remains to the note that $h(\alpha) \geq 0$ whenever $0 \leq \alpha < \alpha_2$. Then (2.2) comes after at once. □

Lemma 2.2. *Assume that* $(A1)$ *and* $(A2)$ *hold, then*

$$\|\nabla u(t)\|^2 \geq 2\|u(t)\|_{q+1}^{q+1} \ \ or \ \ \|\nabla u(t)\|^2 - \|u(t)\|_{q+1}^{q+1} \geq \frac{1}{2}\|\nabla u(t)\|^2. \tag{2.4}$$

Furthermore, we have for constant C

$$\begin{cases} \|u_t(t)\| \in L^2(\Omega), \\ \|\nabla u(t)\| \leq C, \ \ \|u(t)\|_{q+1} \leq C, \ \ \|u_t(t)\| \leq C, \ \ \|\Delta u(t)\| \leq C. \end{cases} \tag{2.5}$$

Proof. By using the embedding theorem, we get

$$\begin{aligned} \frac{1}{2}\|\nabla u(t)\|^2 - \frac{1}{2}\|u(t)\|_{q+1}^{q+1} &\geq \frac{1}{4}\|\nabla u(t)\|^2 + \frac{1}{4}\lambda_1^{-2}\|u(t)\|_{q+1}^2 - \frac{1}{2}\|u(t)\|_{q+1}^{q+1} \\ &= \frac{1}{4}\|\nabla u(t)\|^2 + h\left(\|u(t)\|_{q+1}\right). \end{aligned}$$

Since $h(\alpha) \geq 0$, if $0 \leq \alpha < \alpha_2$ and $0 \leq \|u(t)\|_{q+1} < \alpha_2$ by Lemma 1, (2.4) is true.

The initial result in (2.5) comes from the assumption $(A2)$. The remainder of results in (2.5) follows (1.6), (2.2) and (2.4). □

Lemma 2.3. *Let* $\left(\|u_0\|_{q+1}, E(0)\right) \in W_0$ *and* $E(t) \geq \eta$, *where* $\eta > 0$ *is a constant, then there exists* $\delta = \delta(\eta) > 0$ *such that*

$$\|u_t(t)\|^2 + \|\Delta u(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla u(t)\|^{2(\gamma+1)} - \|u(t)\|_{q+1}^{q+1} \geq \delta, \ \ t \geq 0. \tag{2.6}$$

Proof. From the definition of $E(t)$ and $E(t) \geq \eta$, we get

$$\|u_t(t)\|^2 + \|\Delta u(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla u(t)\|^{2(\gamma+1)} \geq 2\eta, \ \ t \geq 0. \tag{2.7}$$

Now, we suppose by contradiction that (2.6) does not hold. By (2.4), there is a sequences $t_n \subset R^+$ as follows

$$\begin{aligned} & \|u_t(t_n)\|^2 + \|\Delta u(t_n)\|^2 + \|\nabla u(t_n)\|^2 + \|\nabla u(t_n)\|^{2(\gamma+1)} - \|u(t_n)\|_{q+1}^{q+1} \\ \geq \ & \|u_t(t_n)\|^2 + \|\Delta u(t_n)\|^2 + \|\nabla u(t_n)\|^{2(\gamma+1)} + \frac{1}{2}\|\nabla u(t_n)\|^2 \to 0, \ \ (n \to \infty). \end{aligned}$$

Then, we get

$$\|u_t(t_n)\|^2 \to 0, \|\Delta u(t_n)\|^2 \to 0, \|\nabla u(t_n)\|^{2(\gamma+1)} \to 0, \ \ \|\nabla u(t_n)\|^2 \to 0, \ \ n \to \infty.$$

This is imposible since (2.7) and yield the desired result. This completes the proof of lemma. □

Theorem 2.4. *Assume that* $(A1)$ *and* $(A2)$ *hold, we get*

$$\lim_{t\to\infty} E(t_n) = 0, \ \lim_{t\to\infty}\|\Delta u(t_n)\|^2 = 0. \tag{2.8}$$

Proof. Assume that (2.8) fails, then there exists $\eta > 0$ such that $E(t) \geq \eta$ for all $t \geq 0$ since (1.6) and $E(t) \geq 0$. Multiplying both sides of (1.1) by u, integrating them over $[T,t] \times \Omega$ $(0 < T \leq t \leq \infty)$ and integrating by parts, we have

$$\begin{aligned} & (u_t(s), u(s))|_{s=T}^{t} \\ = \ & \int_T^t \Big[2\|u_t(s)\|^2 - \Big(\|u_t(s)\|^2 + \|\Delta u(s)\|^2 + \|\nabla u(s)\|^2 + \|\nabla u(s)\|^{2(\gamma+1)} - \|u(s)\|_{q+1}^{q+1}\Big) \\ & - \int_\Omega |u(s)|^\rho u(s) j'(u_t)(s)\,dx\Big]\,ds \\ = \ & \int_T^t (K_1 + K_2 + K_3)\,ds. \end{aligned} \tag{2.9}$$

By (1.6), (2.2) and (2.5), we have

$$\int_T^t K_1 ds = \int_T^t 2\|u_t(s)\|^2 ds \leq 4E^{\frac{1}{2}}(0)\left(\int_T^t \|u_t(s)\|^2 ds\right)^{\frac{1}{2}}\left(\int_T^t ds\right)^{\frac{1}{2}} \leq C_1\left(\int_T^t ds\right)^{\frac{1}{2}}. \tag{2.10}$$

Here and in the next positive constant C_i not depend on t and T. From Lemma 3, we have

$$\begin{aligned} \int_T^t K_2 ds \ &= \ -\int_T^t \Big(\|u_t(s)\|^2 + \|\Delta u(s)\|^2 + \|\nabla u(s)\|^2 + \|\nabla u(s)\|^{2(\gamma+1)} - \|u(s)\|_{q+1}^{q+1}\Big)\,ds \\ &\leq \ -\delta\int_T^t ds. \end{aligned} \tag{2.11}$$

Set

$$H(t) = E_1 - E(t).$$

From (1.3), we have

$$H'(t) = -E'(t) = \int_\Omega |u(t)|^\rho j(u_t)(t)\,dx \geq 0. \tag{2.12}$$

Form (2.12) and since $E(t) \geq 0$ for $t \geq 0$ and $H(t) \in C(0,\infty)$ we reach at $\int_\Omega |u(t)|^\rho j'(u_t)(t)\,dx \in L^1(0,\infty)$, using Holder inequality, (2.4) and embedding theorem $H_0^2(\Omega) \hookrightarrow L^{\rho+p}(\Omega)$, we have

$$\begin{aligned}
\int_T^t K_3 ds &= -\int_T^t \int_\Omega |u(s)|^\rho u(s)\, j'(u_t)(s)\,dxds \\
&\leq \int_T^t \int_\Omega |u(s)|^{\rho+1-\frac{\rho+p+1}{p+1}} |u(s)|^{\frac{\rho+p+1}{p+1}} |u_t(s)|^p\,dxds \\
&\leq \left(\int_T^t \int_\Omega |u|^\rho j(u_t)(s)\,dxds\right)^{\frac{p}{p+1}} \left(\int_T^t \int_\Omega |u(s)|^{\rho+p+1} dxds\right)^{\frac{1}{p+1}} \\
&\leq C_2 \left(\int_T^t H'(s)\,ds\right)^{\frac{p}{p+1}} \left(\int_T^t \|u(s)\|_{\rho+p+1}^{\rho+p+1} ds\right)^{\frac{1}{p+1}} \\
&\leq C_3 \left(\int_T^t \|\nabla u(s)\|^{\rho+p+1} ds\right)^{\frac{1}{p+1}} \leq C_4 \left(\int_T^t ds\right)^{\frac{1}{p+1}}. \qquad (2.13)
\end{aligned}$$

Then from (2.9)-(2.13), as $p+1 \leq 2$, we know

$$(u_t(s), u(s))|_{s=T}^t \leq C_1 \left(\int_T^t ds\right)^{\frac{1}{2}} + C_4 \left(\int_T^t ds\right)^{\frac{1}{p+1}} - \delta \int_T^t ds \leq C_5 \left(\int_T^t ds\right)^{\frac{1}{p+1}} - \delta \int_T^t ds. \qquad (2.14)$$

Moreover, by applying Holder inequality and (2.5),

$$|(u_t(s), u(s))| \leq C_6 \left(\|u_t(s)\|^2 + \|\Delta u(s)\|^2 + \|\nabla u(s)\|^2 + \|\nabla u(s)\|^{2(\gamma+1)}\right) < \infty.$$

In turn, we arrive a result that is in contradiction with (2.14) for fixing T when $t \to \infty$. Therefore, we derive $\lim_{t\to\infty} E(t) = 0$ and $\lim_{t\to\infty} \|\Delta u(t)\|^2 = 0$ by (2.2). This completes the proof. □

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] V. Barbu, I. Lasiecka, M. A. Rammaha, *Existence and uniqueness of solutions to wave equations with nonlinear degenerate damping and source terms*, Control Cybernet., **34**(3) (2005), 665-687.

[2] S. Woinowsky-Krieger, *The effect of axial force on the vibration of hinged bars*, J Appl Mech., **17** (1950), 35-36.

[3] J. A. Esquivel-Avila, *Dynamic analysis of a nonlinear Timoshenko equation*, Abstr Appl Anal., **2010** (2011), 1-36.

[4] J. A. Esquivel-Avila, *Global attractor for a nonlinear Timoshenko equation with source terms*, Math Sci., **7**(32) (2013), 1-8.

[5] E. Pişkin, *Existence, decay and blow up of solutions for the extensible beam equation with nonlinear damping and source terms*, Open Math., **13** (2005), 408-420.

[6] E. Pişkin, N. Irkıl, *Blow up positive initial-energy solutions for the extensible beam equation with nonlinear damping and source terms*, Ser. Math. Inform., **31**(3) (2016), 645-654.

[7] D. C. Pereira, H. Nguyen, C. A. Raposo, C. H. M. Maranhao, *On the solutions for an extensible beam equation with internal damping and source terms*, Differential Equations & Applications, **11**(3) (2019), 367-377.

[8] E. Pişkin, H. Yüksekkaya, *Non-existence of solutions for a Timoshenko equations with weak dissipation*, Math Morav., **22**(2) (2018), 1-9.

[9] H. A. Levine, J. Serrin, *Global nonexistence theorems for quasilinear evolution with dissipation*, Arch. Rational Mech. Anal, **137** (1997), 341-361.

[10] D. R. Pitts, M. A. Rammaha, *Global existence and nonexistence theorems for nonlinear wave equations*, Indiana Uni. Math. J., **51**(6) (2002), 1479-1509.

[11] V. Barbu, I. Lasiecka, M. A. Rammaha, *Blow-up of generalized solutions to wave equations with nonlinear degenerate damping and source terms*, Indiana Uni. Math. J., **56**(3) (2007), 995-1022.

[12] V. Barbu, I. Lasiecka, M.A. Rammaha, *On nonlinear wave equations with degenerate damping and source terms*, Trans. Amer. Math. Soc., **357**(7) (2005), 2571-2611.

[13] Q. Hu, H. Zhang, *Blow up and asymptotic stability of weak solutions to wave equations with nonlinear degenerate damping and source terms*, Electron J. Differ. Eq., **2007** (76) (2007), 1-10.

[14] S. Xiao, W. Shubin, *A blow-up result with arbitrary positive initial energy for nonlinear wave equations with degenerate damping terms*, J. Part. Diff. Eq., **32** (2019), 181-190.

[15] F. Ekinci, E. Pişkin, *Nonexistence of global solutions for the Timoshenko equation with degenerate damping*, Menemui Mat., **43**(1) (2021), 1-8.

[16] E. Pişkin, F. Ekinci, *General decay and blowup of solutions for coupled viscoelastic equation of Kirchhoff type with degenerate damping terms*, Math. Meth. App. Sci., **42**(16) (2019), 1-21.

[17] E. Pişkin, F. Ekinci, *Local existence and blow up of solutions for a coupled viscoelastic Kirchhoff-type equations with degenerate damping*, Miskolc Math. Notes, **22**(2) (2021), 861-874.

[18] E. Pişkin, F. Ekinci, *Blow up of solutions for a coupled Kirchhoff-type equations with degenerate damping terms*, Applications & Applied Mathematics, **14**(2) (2019), 942-956.

[19] E. Pişkin, F. Ekinci, K. Zennir, *Local existence and blow-up of solutions for coupled viscoelastic wave equations with degenerate damping terms*, Theor. Appl. Mech., **47**(1) (2020), 123-154.

[20] E. Pişkin, F. Ekinci, *Global existence of solutions for a coupled viscoelastic plate equation with degenerate damping terms*, Tbilisi Math. J., **14**(2021), 195-206.

[21] E. Pişkin, F. Ekinci, H. Zhang, *Blow up, lower bounds and exponential growth to a coupled quasilinear wave equations with degenerate damping terms*, Dynamics of Continuous, Discrete and Impulsive Systems, In press.

[22] F. Ekinci, E. Pişkin, S. M. Boulaaras, I. Mekawy, *Global existence and general decay of solutions for a quasilinear system with degenerate damping terms*, J. Funct. Spaces, **2021** (2021), 4316238.

[23] F. Ekinci, E. Pişkin, *Blow up and exponential growth to a Petrovsky equation with degenerate damping*, Univers. J. Math. Appl., **4**(2) (2021), 82-87.

[24] F. Ekinci, E. Pişkin, *Global existence and growth of solutions to coupled degeneratly damped Klein-Gordon equations*, Al-Qadisiyah Journal of Pure Science, **27**(1) (2022), 29-40.

[25] F. Ekinci, E. Pişkin, *Growth of solutions for fourth order viscoelastic system*, Sigma Journal of Engineering and Natural Sciences, (2021), 1-7.

[26] F. Ekinci, E. Pişkin, K. Zennir, *Existence, blow up and growth of solutions for a coupled quasi-linear viscoelastic Petrovsky equations with degenerate damping terms*, Journal of Information and Optimization Sciences, (2021), 1-29.

[27] E. Pişkin, F. Ekinci, *Blow up, exponential growth of solution for a reaction-diffusion equation with multiple nonlinearities*, Tbilisi Math. J., **12**(4) (2019), 61-70.

Communications in Advanced Mathematical Sciences
Vol. 5, No. 3, 137-149, 2022
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1012330

Global Weak Solution, Uniqueness and Exponential Decay for a Class of Degenerate Hyperbolic Equation

Ducival C. Pereira[1], Carlos A. Raposo[2]*

Abstract
This paper deals with the existence, uniqueness, and energy decay of solutions for a degenerate hyperbolic equation given by

$$K(x,t)u'' - M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u - \Delta u' = 0,$$

with operator coefficient $K(x,t)$ satisfying suitable properties and $M(\cdot) \in C^1([0,\infty))$ is a function such that the greatest lower bound is zero. For global weak solutions and uniqueness, we apply the Faedo-Galerkin method. For the exponential decay, we use a theorem due to M. Nakao.

Keywords: Degenerate hyperbolic equations, Exponential decay, Global weak solution
2010 AMS: Primary 35K22

[1] *Department of Mathematics, State University of Pará, Belém, PA, Brazil, ORCID: 0000-0003-4511-0185*
[2] *Department of Mathematics, Federal Univesity of Bahia, Salvador, BA, Brazil, ORCID: 0000-0001-8014-7499*
***Corresponding author**: carlos.raposo@ufba.br
Received: 20 October 2021, **Accepted:** 2 August 2022, **Available online:** 30 September 2022

Dedicated in memory to Raúl Moisés Izaguirre Maguiña,
Academic Vice Rector from Universidad Nacional Mayor de San Marcos, Lima, Perú.

1. Introduction

In this work, we will be focused on the existence, uniqueness, and exponential decay of global weak solution to the problem associated with the degenerate hyperbolic equation

$$K(x,t)u'' - M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u - \Delta u' = 0, \text{ in } Q = \Omega \times (0,T), \tag{1.1}$$

$$u(x,t) = 0, \text{ on } \Sigma = \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x),\ u'(x,0) = u_1(x),\ x \in \Omega, \tag{1.3}$$

where Ω is a bounded open set of $\mathbb{R}^n$ $(n \geq 1)$, with smooth boundary $\partial\Omega$ and $T > 0$ is a fixed but arbitrary real number. $u(x,t)$ represents the transversal displacement of a spacial variable $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ at time $t > 0$, u' denotes the derivative of u with respect to time. $M(\cdot)$ is a $C^1([0,\infty))$ function such that $M(\lambda)\geq 0$, for all $\lambda\in[0,\infty)$ and the operator coefficient $K(x,t)\in C^1([0,T],L^\infty(\Omega))$ satisfying suitable properties. By standard notation,

$$|\nabla u(x,t)|^2=\sum_{i=1}^n\left|\frac{\partial u(x,t)}{\partial x_i}\right|^2 \text{ and } \Delta u(x,t)=\sum_{i=1}^n\frac{\partial^2 u(x,t)}{\partial x_i^2} \text{ is the Laplace operator.}$$

Equation (1.1) with $K(x,t)=1$ has its origin in the nonlinear vibration of an a stretched string and was considered in [1]. Existence of global solution was proved for $K(x,t)\geq 0$ and $M=1$ in [2], see also [3]. For a background and physical properties of this model we refer the reader to [4]-[7].

In fact,

$$u''-M\left(\int_\Omega|\nabla u|^2\,dx\right)\Delta u+\alpha u'=0 \text{ in } Q=\Omega\times(0,T), \tag{1.4}$$

when $M(\lambda)\geq m_0>0$ is known as non-degenerate, and for $\alpha=0$, global solutions have been obtained by several authors under various assumption, see [8]-[13].

The operator coefficient $K(x,t)$ plays an important role in the asymptotic behaviour for equation (1.1). The energy of the equation (1.1) is given by

$$E(t)=\frac{1}{2}\left[|K^{1/2}u'(t)|^2+\widehat{M}(a(u(t)))\right]$$

being

$$\widehat{M}(t)=\int_0^t M(s)\,ds \tag{1.5}$$

and

$$a(u,v)=\int_\Omega\nabla u\nabla v\,dx \text{ the Dirichlet's form, for which we write } a(u) \text{ instead of } a(u,u).$$

When $K(x,t)=1$, for non-degenerate case, with $\alpha>0$, exponential decay properties was studied in [23]-[26]. However, the decay rate of the solutions is not so fast in the degenerate case. In fact, in [1], for example was showed that the problem (1.4) was a polynomial rate of decay given by $E(t)\leq Ct^{-\left(\frac{\alpha+1}{\alpha}\right)}$.

Another example presented by J. G. Dix [27], fully transcribed here, shows that decay of solutions is not necessarily exponential. Consider for $\Omega=(0,2\pi)\in\mathbb{R}$,

$$\begin{aligned}
u''-M\left(\|u_x\|^2\right)u_{xx}+u'=0,\ x\in\Omega,\ t\geq 1+\sqrt{2},\\
u\left(x,1+\sqrt{2}\right)=\frac{1}{\sqrt{\pi}}e^{1/(1+\sqrt{2})}\sin(x),\\
u'\left(x,1+\sqrt{2}\right)=\frac{1}{9\sqrt{\pi}}e^{1/(1+\sqrt{2})}\sin(x),\\
u(0,t)=0,\ \ u(2\pi,t)=0,\ \text{for } t\geq 1+\sqrt{2},
\end{aligned}$$

where M is the non-negative and continuous function defined as

$$M(r)=\begin{cases}\dfrac{1}{16}\ln^2(r)(4-4\ln(r)-\ln^2(r)), & \text{if } 1\leq r\leq e^{2/(1+\sqrt{2})},\\ 0, & \text{otherwise.}\end{cases}$$

Then $u(x,t)=\dfrac{1}{\sqrt{\pi}}e^{1/t}\sin(x)$ is a solution. Since

$$u'=-\frac{1}{t^2}u,\ \ u''=\left(\frac{1}{t^4}+\frac{2}{t^3}\right)u,\ \ u_x=\frac{1}{\sqrt{\pi}}e^{1/t}\cos(x),\ \ u_{xx}=-u,$$

$\|u_x\|^2 = e^{2/t}$, and $M(e^{2/t}) = \dfrac{1}{t^2} - \dfrac{2}{t^3} - \dfrac{1}{t^4}$ for $t \geq 1+\sqrt{2}$, it follows that u satisfies the initial-value problem. Notice that $\|u'\|$ decays polynomially rather than exponentially as $t \to \infty$. In fact, $\|u'\|^2 = \dfrac{1}{t^4}e^{2/t}$.

Moreover, when is considered the nonhomogeneous equation $u'' - M\left(\|u_x\|^2\right)u_{xx} + u' = f(x,t)$, and a general non-constant function M, in spite of the convergence of $\|u'\|$ to zero remains illusive, that is, was not verified it and was not presented a counter-example, was proved in [27] that if $||f(x,t)||$ is square integrable on $[0,\infty)$ then $\|u'\|$ is square integrable on $[0,\infty)$.

On the other hand, when the greatest lower bound for $M(\lambda)$ is zero, the equation (1.4) is known as degenerate, see [14]-[16]. The degenerate equation (1.1) studied in this manuscript has been considered in just a few publications, see for instance [17, 18] and references therein.

It is well known that the Cauchy problem is well-posed for strictly hyperbolic differential equations. However, in dimension one, the Cauchy problem associated with degenerate hyperbolic equations is not well-posed. See [19]. Despite this, nonlinear degenerate hyperbolic equations are one of the most important classes of partial differential equations. We present some results in the literature in several contexts. For linear and semilinear equations of Tricomi type, existence, uniqueness, and qualitative properties of weak solutions to the degenerate hyperbolic Goursat problem, which play a very important part in applied and engineering sciences, was established in [20]. In [21] was considered the generalized Riemann problem for the Suliciu relaxation system in Lagrangian coordinates. The Suliciu relaxation system can be considered as a simplified viscoelastic shallow fluid model. Recently, the mixed Cauchy problem with lateral boundary condition for noncharacteristic degenerate hyperbolic equations was analyzed in [22], where, unlike other works on mixed Cauchy that the problems under consideration are obtained in weighted spaces, authors obtained all solutions in classical Sobolev spaces. Then, in the context above, the degenerate equation gives us a feature yield several striking phenomena that require new mathematical ideas, approaches, and theories.

The outline of this manuscript is the following. In Section 2 we introduce the notation, necessary assumptions and the main results. The proof of the existence theorem is performed in section 3, in three steps: approximate problem, a priori estimates and passage to the limit in the approximated equation. The uniqueness of the solution is given in section 4. Finally in section 5 the asymptotic behaviour is studied where we prove the exponential decay by using the Nakao method.

2. Preliminaries and Main Results

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with sufficiently smooth boundary $\partial\Omega$. By $H^m(\Omega)$, m a non-negative integer, we denote the Sobolev space of order m. For $m=0$, $H^0(\Omega) = L^2(\Omega)$. Further, we set $H_0^m(\Omega) =$ the closure of $\mathscr{D}(\Omega)$ in $H^m(\Omega)$, where $\mathscr{D}(\Omega)$ is the space of infinitely continuously differentiable functions with compact support contained in Ω. The inner product and norm in $L^2(\Omega)$ and $H_0^1(\Omega)$ are represented by $(\cdot,\cdot)$, $|\cdot|$ and $((\cdot,\cdot))$, $||\cdot||$ respectively. The space $H_0^1(\Omega)\cap H^2(\Omega)$ is equipped with the norm $|\Delta u|$.

As in [29] for $T>0$ a real number and B a Banach space, we denote

$$L^p(0,T,B) = \left(\text{u mensurable from [0,T] into B} \;\middle|\; \begin{array}{l} \left(\displaystyle\int_0^T ||u(t)||_B^p dt\right)^{\frac{1}{p}} < \infty, \text{ if } 1 \leq p < \infty, \\ \displaystyle\sup_{0<t<T} \text{ess}\, ||u(t)||_B < \infty, \text{ if } p = \infty. \end{array} \right).$$

From now and on, let us assume that the volume density function $K(x,t)$ satisfies:

(H.1) $K(x,t) \in C^1([0,T], L^\infty(\Omega))$, $K(x,t) \geq 0$ and $K(x,0) \geq C_0 > 0$ for some $C_0 \in \mathbb{R}$.

(H.2) $\left|\dfrac{\partial K(x,t)}{\partial t}\right| \leq \gamma + C(\gamma)K(x,t)$, for all $\gamma > 0$.

In this manuscript, we deal with a degenerate case, then we consider that $M(\lambda)$, $\lambda > 0$, a real function satisfying

(H.3) $M(\lambda) \in C^1([0,\infty))$ with $M(\lambda) \geq 0$, for all, $\lambda > 0$.

The well-posedness of problem (1.1) is ensured by

Theorem 2.1. *For $u_0, u_1 \in H_0^1(\Omega)\cap H^2(\Omega)$ there exists a unique function $u : [0,T] \to L^2(\Omega)$ with the following regularity*

$$u \in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{2.1}$$
$$u' \in L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{2.2}$$
$$u'' \in L^2(0,T;H_0^1(\Omega)), \tag{2.3}$$

such that

$$K(x,t)u'' - M(a(u(t))\Delta u - \Delta u' = 0 \text{ in } L^2(Q), \tag{2.4}$$
$$u(x,t) = 0 \text{ on } \Sigma = \partial\Omega\times(0,T), \tag{2.5}$$
$$u(x,0) = u_0(x),\ u'(x,0) = u_1(x),\ x\in\Omega. \tag{2.6}$$

Remark 2.2. *From (2.1), (2.2), (2.3) we have that $u \in C^0([0,T],H_0^1(\Omega)\cap H^2(\Omega))$ and $u' \in C^0([0,T],H_0^1(\Omega))$ so the initial conditions (2.6) are well set.*

For asymptotic behaviour the exponential stability is given by

Theorem 2.3. *Under the hypothesis of Theorem 2.1, the energy $E(t)$ associated to equation (1.1) satisfies*

$$E(t) \le C_0 e^{-\alpha t}, \text{ for all } t \ge 0, \text{ where } C_0 \text{ and } \alpha \text{ are positive constants.}$$

3. Existence of Solution

The aim of this section is to prove the theorem (2.1). For this goal, we use the Faedo-Galerkin method, a standard technique well described in the book by Temam [30].

3.1 Step 1. Perturbed approximate problem

Let $(w_\nu)_{\nu\in\mathbb{N}}$ be a basis of $H_0^1(\Omega)\cap H^2(\Omega)$ consisting of eigenvectors of the operator $-\Delta$, that is,

$$-\Delta w_j = \lambda_j w_j,\ j = 1,2,\cdots,n,\cdots$$

where $0<\lambda_1<\lambda_2\le\cdots\le\lambda_n\le\cdots,\ \lambda_n\to\infty$ as $n\to\infty$, $w_j\big|_{\partial\Omega}=0,\ j=1,2,\cdots,$ and $V_m=[w_1,\cdots,w_m]$ is the $H_0^1(\Omega)\cap H^2(\Omega)$ subspace generated by the first m eigenfunctions.

For all $w\in V_m$, let

$$u_{\varepsilon m}(t) = \sum_{j=1}^m g_{j\varepsilon m}(t)w_j,\ 0<\varepsilon<1,$$

be a local solution of the approximated problem

$$((K+\varepsilon)u''_{\varepsilon m},v) + M(a(u_{\varepsilon m}))a(u_{\varepsilon m},v) + a(u'_{\varepsilon m},v) = 0,\ \forall v\in V_m \tag{3.1}$$
$$u_{\varepsilon m}(0) = u_{0m} \longrightarrow u_0 \text{ strongly in } H_0^1(\Omega)\cap H^2(\Omega), \tag{3.2}$$
$$u'_{\varepsilon m}(0) = u_{1m} \longrightarrow u_1 \text{ strongly in } H_0^1(\Omega)\cap H^2(\Omega), \tag{3.3}$$

which exists in a interval $[0,T_{\varepsilon m})$, $0<T_{\varepsilon m}\le T$, by virtue of Carathéodory's theorem, see [28]. The extension of the solution to the whole interval $[0,T]$ is a consequence of the following priori estimates.

3.2 Step 2. Priori estimates

(I) Replacing $w = u'_{\varepsilon m}(t)$ in perturbed approximate equation (3.1), we get

$$\frac{1}{2}\frac{d}{dt}(K,u'^2_{\varepsilon m}) + \frac{\varepsilon}{2}\frac{d}{dt}|u'_{\varepsilon m}|^2 + \frac{1}{2}M(a(u_{\varepsilon m}))\frac{d}{dt}a(u_{\varepsilon m}) + ||u'_{\varepsilon m}||^2 = \frac{1}{2}(\frac{\partial K}{\partial t},u'^2_{\varepsilon m}). \tag{3.4}$$

From (1.5) we get

$$\frac{d}{dt}\widehat{M}(a(u_{\varepsilon m})) = M(a(u_{\varepsilon m}))\frac{d}{dt}a(u_{\varepsilon m}),$$

then, (H.2), (3.4) leads to

$$\frac{d}{dt}\left[(K,u_{\varepsilon m}'^2)+\varepsilon|u_{\varepsilon m}'|^2+\widehat{M}(a(u_{\varepsilon m}))\right]+2||u_{\varepsilon m}'||^2\le\gamma\mu||u_{\varepsilon m}'||^2+C(\gamma)(K,u_{\varepsilon m}'^2),$$

where $\mu^{1/2}$ is the Poincaré constant. Performing integration from 0 to t, $0<t\le T_{\varepsilon m}$ we obtain

$$(K,u_{\varepsilon m}'^2)+\varepsilon|u_{\varepsilon m}'|^2+\widehat{M}(a(u_{\varepsilon m}))+(2-\gamma\mu)\int_0^t||u_{\varepsilon m}'||^2\,ds\le(K(0),u_{1m}^2)+\varepsilon|u_{1m}|^2+\widehat{M}(a(u_{0m}))+C(\gamma)\int_0^t(K,u_{\varepsilon m}'^2)\,ds. \tag{3.5}$$

Since $K(0)\in L^\infty(\Omega)$, by using (3.2), (3.3) and choosing $\gamma<2/C$ we obtain

$$(K,u_{\varepsilon m}'^2)+\varepsilon|u_{\varepsilon m}'|^2+\widehat{M}(a(u_{\varepsilon m}))+(2-\gamma\mu)\int_0^t||u_{\varepsilon m}'||^2\,ds\le C_1+C(\gamma)\int_0^t(K,u_{\varepsilon m}'^2)\,ds, \tag{3.6}$$

being $C_1>0$ a real constant independent of ε,m and t. Now, applying Gronwall's inequality in (3.6), we come to

$$(K,u_{\varepsilon m}'^2)+\varepsilon|u_{\varepsilon m}'|^2+\widehat{M}(a(u_{\varepsilon m}))+(2-\gamma\mu)\int_0^t||u_{\varepsilon m}'||^2\,ds\le C_2,$$

with $C_2>0$ a real constant independent of ε,m and t. Therefore,

$$\begin{aligned}&(K^{1/2}u_{\varepsilon m}')\text{ is bounded in }L^\infty(0,T;L^2(\Omega)),\\&(\sqrt{\varepsilon}\,u_{\varepsilon m}')\text{ is bounded in }L^\infty(0,T;L^2(\Omega)),\\&(u_{\varepsilon m}')\text{ is bounded in }L^2(0,T;H_0^1(\Omega)).\end{aligned} \tag{3.7}$$

From (3.7) and of Fundamental Theorem of Calculus, that is, $u_{\varepsilon m}(t)=u_{\varepsilon m}(0)+\int_0^t u_{\varepsilon m}'(s)\,ds$, we have

$$(u_{\varepsilon m})\text{ is bounded in }L^\infty(0,T;H_0^1(\Omega)). \tag{3.8}$$

(II) Replacing $v=u_{\varepsilon m}''(t)$ in equation (3.1), we get

$$(K,u_{\varepsilon m}''^2)+\varepsilon|u_{\varepsilon m}''|^2+M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m},u_{\varepsilon m}'')+\frac{1}{2}\frac{d}{dt}||u_{\varepsilon m}'||^2=0. \tag{3.9}$$

Note that

$$\begin{aligned}M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m},u_{\varepsilon m}'')&=M(a(u_{\varepsilon m}))\left[\frac{d}{dt}a(u_{\varepsilon m},u_{\varepsilon m}')-a(u_{\varepsilon m}')\right]\\&=\frac{d}{dt}\left[M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m},u_{\varepsilon m}')\right]-2M'(a(u_{\varepsilon m}))a(u_{\varepsilon m},u_{\varepsilon m}')a(u_{\varepsilon m},u_{\varepsilon m}')-M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m}').\end{aligned}$$

Thereby

$$\begin{aligned}\left|\int_0^t M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m},u_{\varepsilon m}'')\,ds\right|&\le\left|M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m},u_{\varepsilon m}')\right|+\left|M(a(u_{0m}))\,a(u_{0m},u_{1m}')\right|\\&\quad+2\int_0^t\left|M'(a(u_{\varepsilon m}))a(u_{\varepsilon m},u_{\varepsilon m}')^2\right|ds+\int_0^t\left|M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m}')\right|ds.\end{aligned}$$

Since, $M(\lambda)\in C^1([0,\infty))$, then

$$M(a(u_{\varepsilon m}))\le\sup_{m\ge1}\{M(\lambda)\,:\,0\le\lambda\le\sup||u_{\varepsilon m}||_{L^\infty(0,T;H_0^1(\Omega))}\}\le c$$

and

$$M'(a(u_{\varepsilon m}))\le\sup_{m\ge1}\{M(\lambda)\,:\,0\le\lambda\le\sup||u_{\varepsilon m}||_{L^\infty(0,T;H_0^1(\Omega))}\}\le\overline{c},$$

with $c,\overline{c}$ positive constants independent of ε,m and t.

Then,

$$\left|\int_0^t M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m},u''_{\varepsilon m})\,ds\right| \le c\,\|u_{\varepsilon m}\|\,\left\|u'_{\varepsilon m}\right\| + C_3 + 2\overline{c}\int_0^t \|u_{\varepsilon m}\|^2\left\|u'_{\varepsilon m}\right\|^2 ds + c\int_0^t \left\|u'_{\varepsilon m}\right\|^2 ds.$$

From (3.7) and (3.8) we have

$$\left|\int_0^t M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m},u''_{\varepsilon m})\,ds\right| \le C_4 + \alpha\left\|u'_{\varepsilon m}\right\|^2, \text{ with } C_4,\alpha \text{ positive constants independent of } \varepsilon, m \text{ and } t. \tag{3.10}$$

Integrating (3.9) from 0 to t, $0<t\le T$, and using the estimate (3.10) we obtain

$$\int_0^t (K,u''^2_{\varepsilon m})\,ds + \varepsilon\int_0^t |u''_{\varepsilon m}|^2 ds + (\frac{1}{2}-\alpha)||u'_{\varepsilon m}||^2 \le C_4. \tag{3.11}$$

Choosing properly $0<\alpha<1/2$ we obtain directly from estimate (3.11)

$$\begin{aligned} (K^{1/2}u''_{\varepsilon m}) &\text{ is bounded in } L^2(Q),\\ (\sqrt{\varepsilon}\,u''_{\varepsilon m}) &\text{ is bounded in } L^2(Q),\\ (u'_{\varepsilon m}) &\text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)). \end{aligned} \tag{3.12}$$

(III) Now we will get an estimate for $u''_{\varepsilon m}(t)$. At this point we have an additional degree of difficulty. We first obtain an estimate for $u''_{\varepsilon m}(0)$. In this direction, taking $t=0$ and $v=u''_{\varepsilon m}(0)$ in equation (3.1) we obtain

$$((K(0),u''^2_{\varepsilon m}(0)) + \varepsilon|u''_{\varepsilon m}(0)|^2 + M(a(u_{0m}))\,a(u_{0m},u''_{\varepsilon m}(0)).+a(u'_{1m},u''_{\varepsilon m}(0)) = 0.$$

Since $K(0)\ge C_0>0$ we have

$$(C_0+\varepsilon)|u''_{\varepsilon m}(0)|^2 \le |M(a(u_{0m}))\Delta u_{0m} + \Delta u_{1m}|\,|u''_{\varepsilon m}(0)|,$$

therefore

$$|u''_{\varepsilon m}(0)| \le \tilde{c}, \text{ where } \tilde{c} \text{ is a positive constant independent of } \varepsilon, m \text{ and } t. \tag{3.13}$$

Deriving the approximate equation (3.1) with respect to t and making $v=u''_{\varepsilon m}(t)$ we obtain

$$(Ku'''_{\varepsilon m},u''_{\varepsilon m}) + (\frac{\partial K}{\partial t}u''_{\varepsilon m},u''_{\varepsilon m}) + \varepsilon(u'''_{\varepsilon m},u''_{\varepsilon m}) + \frac{d}{dt}\left[M(a(u_{\varepsilon m}))\right]a(u_{\varepsilon m},u''_{\varepsilon m}) + M(a(u_{\varepsilon m}))a(u'_{\varepsilon m},u''_{\varepsilon m}) + a(u''_{\varepsilon m}) = 0,$$

that is,

$$\frac{1}{2}\frac{d}{dt}(K,u''^2_{\varepsilon m}) + \frac{1}{2}(\frac{\partial K}{\partial t},u''^2_{\varepsilon m}) + \frac{\varepsilon}{2}\frac{d}{dt}|u''_{\varepsilon m}|^2 + \|u''_{\varepsilon m}\|^2 = -2M'(a(u_{\varepsilon m}))a(u_{\varepsilon m},u'_{\varepsilon m})a(u_{\varepsilon m},u''_{\varepsilon m}) - M(a(u_{\varepsilon m}))a(u'_{\varepsilon m},u''_{\varepsilon m}),$$

and then,

$$\frac{1}{2}\frac{d}{dt}\left[(K,u''^2_{\varepsilon m}) + \varepsilon|u''_{\varepsilon m}|^2\right] + \|u''_{\varepsilon m}\|^2 \le C_5 + \mu\frac{\gamma}{2}\|u''_{\varepsilon m}\|^2 + \frac{C(\gamma)}{2}(K,u''^2_{\varepsilon m}), \text{ with } C_5 \text{ independent of } \varepsilon, m \text{ and } t. \tag{3.14}$$

Integrating (3.14) from 0 to t, we obtain

$$\frac{1}{2}\left[(K,u''^2_{\varepsilon m}) + \varepsilon|u''_{\varepsilon m}|^2\right] + (1-\mu\frac{\gamma}{2})\int_0^t \|u''_{\varepsilon m}\|^2 ds \le C_5 + C(\gamma)\int_0^t (K,u''^2_{\varepsilon m})\,ds + \frac{1}{2}\left[K(0),u''^2_{\varepsilon m}(0)) + \varepsilon|u''_{\varepsilon m}(0)|^2\right]. \tag{3.15}$$

By using (3.13) and Gronwall's inequality, (3.15) leads to

$$\frac{1}{2}\left[(K,u''^2_{\varepsilon m}) + \varepsilon|u''_{\varepsilon m}|^2\right] + (1-\mu\frac{\gamma}{2})\int_0^t \|u''_{\varepsilon m}\|^2 ds \le C_6, \text{ with } C_6 \text{ a positive constant independent of } \varepsilon, m \text{ and } t.$$

Therefore,

$$(K^{1/2}u''_{\varepsilon m}) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)), \tag{3.16}$$

$$(\sqrt{\varepsilon}\,u''_{\varepsilon m}) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)), \tag{3.17}$$

$$(u''_{\varepsilon m}) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)). \tag{3.18}$$

(IV) Replacing $v = -\Delta u_{\varepsilon m}$ in the approximate equation (3.1), we obtain

$$((K+\varepsilon)u''_{\varepsilon m}, -\Delta u_{\varepsilon m}) + M(a(u_{\varepsilon m}))\, a(u_{\varepsilon m}, -\Delta u_{\varepsilon m}) + a(u'_{\varepsilon m}, -\Delta u_{\varepsilon m}) = 0,$$

that leads us to

$$\frac{1}{2}\frac{d}{dt}|-\Delta u_{\varepsilon m}|^2 \leq K_0|-\Delta u_{\varepsilon m}||u''_{\varepsilon m}| + \varepsilon|-\Delta u_{\varepsilon m}||u''_{\varepsilon m}| + |M(a(u_{\varepsilon m}))||-\Delta u_{\varepsilon m}|^2, \text{ where } K_0 = \max_{0\leq s\leq T}\left(\operatorname{supess}_{x\in\Omega} K(x,s)\right).$$

Performing integration from 0 to t, using Young's inequality and (3.18), we obtain

$$|-\Delta u_{\varepsilon m}|^2 \leq C_7 + C_8\int_0^t |-\Delta u_{\varepsilon m}(s)|^2 ds.$$

Applying Gronwall's inequality we get

$$|-\Delta u_{\varepsilon m}|^2 \leq C_9. \tag{3.19}$$

Then we obtain,

$$\|u_{\varepsilon m}\|^2_{H^2(\Omega)} \leq C_9, \text{ where the constants } C_7, C_8, C_9 \text{ are positives and independent of } \varepsilon, m \text{ and } t.$$

In fact we have the following regularity

$$(u_{\varepsilon m}) \text{ is bounded in } L^\infty(0,T;H^2(\Omega)). \tag{3.20}$$

(V) Replacing $v = -\Delta u'_{\varepsilon m}$ in approximated equation (3.1), we get

$$((K+\varepsilon)u''_{\varepsilon m}, -\Delta u'_{\varepsilon m}) + M(a(u_{\varepsilon m}))\, a(u_{\varepsilon m}, -\Delta u'_{\varepsilon m}) + a(u'_{\varepsilon m}, -\Delta u'_{\varepsilon m}) = 0,$$

then,

$$|-\Delta u'_{\varepsilon m}|^2 \leq K_0|-\Delta u'_{\varepsilon m}||u''_{\varepsilon m}| + |M(a(u_{\varepsilon m}))||-\Delta u_{\varepsilon m}||-\Delta u'_{\varepsilon m}| + \varepsilon|u''_{\varepsilon m}||-\Delta u'_{\varepsilon m}|.$$

Performing integration from 0 to t, using Young's inequality, (3.18) and (3.19) we obtain

$$\int_0^t |-\Delta u'_{\varepsilon m}(s)|^2 ds \leq C_{10} + \alpha\int_0^t |-\Delta u'_{\varepsilon m}(s)|^2 ds, \text{ thus } (1-\alpha)\int_0^t |-\Delta u'_{\varepsilon m}(s)|^2 ds \leq C_{10}.$$

Then

$$\|u'_{\varepsilon m}\|^2_{H^2(\Omega)} \leq C_{10},\ C_{10} \text{ independent of } \varepsilon, m \text{ and } t.$$

Therefore

$$(u'_{\varepsilon m}) \text{ is bounded in } L^2(0,T;H^2(\Omega)). \tag{3.21}$$

3.3 Step 3. Passage to the limit

From estimates (3.9), (3.12), (3.16), (3.17), (3.18), (3.20), and (3.21), there exists a subsequence of $(u_{\varepsilon m})$, denoted by same way, such that,

$$u_{\varepsilon m} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{3.22}$$

$$u'_{\varepsilon m} \rightharpoonup u' \text{ in } L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{3.23}$$

$$u''_{\varepsilon m} \rightharpoonup u'' \text{ in } L^2(0,T;H_0^1(\Omega)),$$

$$\sqrt{\varepsilon}\, u''_{\varepsilon m} \rightharpoonup 0 \text{ in } L^2(0,T;L^2(\Omega)).$$

$$Ku''_{\varepsilon m} \rightharpoonup Ku'' \text{ in } L^2(Q).$$

From compact immersion $H_0^1(\Omega)\cap H^2(\Omega)\hookrightarrow H_0^1(\Omega)$, by Aubin-Lions's lemma [29] follows that $u_{\varepsilon m}\to u$ in $L^2(0,T;H_0^1(\Omega))$, and so $a(u_{\varepsilon m})\to a(u)$ in $L^2(0,T)$, and, as $M\in C^1([0,\infty))$ we obtain

$$M(a(u_{\varepsilon m}))\to M(a(u)) \text{ in } L^2(0,T).$$

From (3.22) and (3.23) we wave that $\Delta u_{\varepsilon m}\rightharpoonup \Delta u$ in $L^2(Q)$, and $\Delta u'_{\varepsilon m}\rightharpoonup \Delta u'$ in $L^2(Q)$. Thereby,

$$M(a(u_{\varepsilon m}))\Delta u_{\varepsilon m}\rightharpoonup M(a(u))\Delta u \text{ in } L^2(Q).$$

Now consider the approximated equation

$$(K+\varepsilon)u''_{\varepsilon m}-M(a(u_{\varepsilon m}))\Delta u_{\varepsilon m}-\Delta u'_{\varepsilon m}=0.$$

Making the inner product in $L^2(\Omega)$ by $\varphi\in L^2(\Omega)$ we obtain

$$((K+\varepsilon)u''_{\varepsilon m},\varphi)-(M(a(u_{\varepsilon m}))\Delta u_{\varepsilon m},\varphi)-(\Delta u'_{\varepsilon m},\varphi)=0.$$

Taking the limit with $m\to\infty$ and $\varepsilon\to 0$, we get

$$((Ku'',\varphi)-(M(a(u))\Delta u,\varphi)-(\Delta u',\varphi)=0, \text{ for all } \varphi\in L^2(Q), \text{ and then (2.4) is proven.}$$

The verification of the initial data (2.6) is obtained in a standard way.

4. Uniqueness of Solution

Consider u and $\widehat{u}$ with the hypotheses of regularity (2.1), (2.2) of Theorem 2.1. Then, $w=u-\widehat{u}$ is solution of the equation

$$Kw''-(M(a(u))\Delta w-[M(a(u))-M(a(\widehat{u}))]\Delta\widehat{u}-\Delta w'=0, \tag{4.1}$$

with initial conditions

$$w(0)=0 \text{ and } w'(0)=0. \tag{4.2}$$

Taking the inner product in $L^2(\Omega)$ on both sides of (4.1) with w,w' and w'' respectively, we get

$$\begin{aligned}
(Kw'',w)+(M(a(u))a(w)+[M(a(u))-M(a(\widehat{u}))]\,a(\widehat{u},w)+a(w',w)&=0,\\
(Kw'',w')+(M(a(u))a(w,w')+[M(a(u))-M(a(\widehat{u}))]\,a(\widehat{u},w')+a(w')&=0,\\
(K,w''^2)+(M(a(u))a(w,w'')+[M(a(u))-M(a(\widehat{u}))]\,a(\widehat{u},w'')+a(w',w'')&=0,
\end{aligned}$$

that is

$$\begin{aligned}
(Kw'',w)+(M(a(u))\|w\|^2+[M(a(u))-M(a(\widehat{u}))]\,a(\widehat{u},w)+\frac{1}{2}\frac{d}{dt}\|w\|^2&=0,\\
\frac{1}{2}\frac{d}{dt}(K,w'^2)-\frac{1}{2}(\frac{\partial K}{\partial t},w'^2)+\frac{1}{2}(M(a(u))\frac{d}{dt}\|w\|^2+\|w'\|^2+[M(a(u))-M(a(\widehat{u}))]\,a(\widehat{u},w')&=0,\\
(K,w''^2)+(M(a(u))a(w,w'')+[M(a(u))-M(a(\widehat{u}))]\,a(\widehat{u},w'')+\frac{1}{2}\frac{d}{dt}\|w'\|^2&=0.
\end{aligned}$$

Adding the last three equations above and integrating from 0 to t, we obtain

$$\begin{aligned}
&\int_0^t(K,w''^2)\,ds+\frac{1}{2}(K,w'^2)+\frac{1}{2}M(a(u))\|w\|^2+\frac{1}{2}\|w\|^2+\frac{1}{2}\|w'\|^2+\int_0^t\|w'\|^2\,dx\\
&\quad=\int_0^t\left\{\frac{1}{2}(\frac{\partial K}{\partial t},w'^2)-(Kw'',w)-M(a(u))\|w\|^2-M(a(u))a(w,w'')\right\}ds\\
&\qquad+\int_0^t\left\{[M(a(\widehat{u}))-M(a(u))]\,[a(\widehat{u},w)+a(\widehat{u},w')+a(\widehat{u},w'')]+M'(a(u))a(u,u')\|w\|^2\right\}ds.
\end{aligned}$$

Note that

$$\frac{1}{2}(\frac{\partial K}{\partial t},w'^2)\leq\delta C\|w'\|^2+C(\delta)(K,w'^2),$$

and

$$\begin{aligned}\int_0^t (Kw'',w)\,ds &= (Kw',w) - \int_0^t (\frac{\partial K}{\partial t}, w'w)\,ds - \int_0^t (K,w'^2)\,ds \\ &\leq C_1\|w'\|\|w\| + C_2\int_0^t \|w'\|\|w\|\,ds + C(\delta)C_1\int_0^t \|w'\|\|w\|\,ds + \int_0^t (K,w'^2)\,ds.\end{aligned}$$

Then we have,

$$\begin{aligned}\int_0^t (Kw'',w)\,ds &\leq \alpha\|w'\|^2 + \frac{C_3}{\alpha}\|w\|^2 + \int_0^t (K,w'^2)\,ds \\ &\leq \alpha\|w'\|^2 + C_4\int_0^t \|w\|^2\,ds + C_5\int_0^t \|w'\|^2\,ds + \int_0^t (K,w'^2)\,ds.\end{aligned}$$

Besides that,

$$\begin{aligned}[M(a(\widehat{u})) - M(a(u))]\left[a(\widehat{u},w) + a(\widehat{u},w') + a(\widehat{u},w'')\right] &\leq |M'(\xi)||a(\widehat{u}) - a(u)|\|\widehat{u}\|\|w\| + \|\widehat{u}\|\|w'\| + \|\widehat{u}\|w''\| \\ &= |M'(\xi)||(\|\widehat{u}\| - \|u\|)(\|\widehat{u}\| + \|u\|)|\|\widehat{u}\|(\|w\| + \|w'\| + \|w''\|) \\ &\leq |M'(\xi)||\widehat{u} - u\|)(\|\widehat{u}\| + \|u\|)|\|\widehat{u}\|(\|w\| + \|\|w'\| + \|w''\|) \\ &= |M'(\xi)|\|w\|(\|\widehat{u}\| + \|u\|)\|\widehat{u}\|(\|w\| + \|\|w'\| + \|w''\|) \\ &\leq C_6\|w\|^2 + C_7\|w'\|^2 + C_8\|w\|\|w''\|\end{aligned}$$

and

$$\begin{aligned}M(a(u))a(w,w'') &= M(a(u))\left[\frac{d}{dt}a(w,w') - a(w')\right] \\ &= \frac{d}{dt}\left[M(a(u))a(w,w')\right] - 2M(a(u))a(u,u')a(w,w') - M(a(u))a(w'),\end{aligned}$$

then,

$$\begin{aligned}\int_0^t M(a(u))a(w,w')\,ds &\leq C_9\|w\|\|w'\| + C_{10}\int_0^t \|w\|\|w'\|\,ds + C_{11}\int_0^t \|w'\|^2\,ds \\ &\leq \alpha\|w'\|^2 + C_{12}\int_0^t \|w\|^2\,ds + C_{13}\int_0^t \|w'\|^2\,ds.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{2}(K,w'^2) + \frac{1}{2}M(a(u))\|w\|^2 + \frac{1}{2}\|w\|^2 + \left(\frac{1}{2} - 2\alpha\right)\|w'\|^2 \\ &\quad\leq \int_0^t \left[(1 + C(\gamma))(K,w'^2) + M(a(u))\|w\|^2 + C_{14}\|w\|^2 + C_5\|w'\|^2\right]\,ds + C_8\int_0^t \|w\|\|w''\|\,ds.\end{aligned}$$

Then,

$$\begin{aligned}&(K,w'^2) + M(a(u))\|w\|^2 + \|w\|^2 + (1-4\alpha)\|w'\|^2 \\ &\quad\leq c\int_0^t \left[(K,w'^2) + M(a(u))\|w\|^2 + \|w\|^2 + (1-4\alpha)\|w'\|^2\right]\,ds + \overline{c}\int_0^t \|w\|\|w''\|\,ds.\end{aligned}$$

Now, we denote

$$\varphi(t) = (K,w'^2) + M(a(u))\|w\|^2 + \|w\|^2 + (1-4\alpha)\|w'\|^2$$

and we obtain

$$\varphi(t) \leq c\int_0^t \varphi(s)\,ds + \overline{c}\int_0^t g(s)\varphi^{1/2}(s)\,ds,\ \text{where } g(s) = \|w''\| \in L^1(0,T).$$

Then, we have $\varphi(t) = 0,$ for all $t \in [0,T]$ and finally $w = 0$, that is, $u = \widehat{u}$ which proves the uniqueness of solution.

5. Asymptotic Behaviour

In this section we prove the exponential decay of solution to the problem (1.1)-(1.3). Let start by present the following result:

Lemma 5.1 (Nakao's Lemma, [31]). *Suppose that $E(t)$ is a bounded nonnegative function on $\mathbb{R}^+$, satisfying*

$$\operatorname*{supess}_{t\leq s\leq t+1} E(s) \leq C[E(t)-E(t+1)], \ \ \text{for } t\geq 0, \text{ where } C \text{ is a positive constant.}$$

Then, we have

$$E(t)\leq Ce^{-\alpha t}, \text{ with } \alpha = \frac{1}{C+1}, \text{ for all } t\geq 0.$$

The main result of this section is given by the following theorem:

Theorem 5.2. *Under the hypotheses of Theorem 2.1, the energy associated with the system (1.1)-(1.3) satisfies*

$$E(t)\leq Ce^{-\alpha t}, \ \ \text{for all } t\geq 0, \text{ where } C \text{ and } \alpha \text{ are positive constants.}$$

Proof. Multiplying (1.1) by u_t and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[|K^{1/2}u'(t)|^2+\widehat{M}(a(u(t))\right]+\|u'(t)\|^2=\frac{1}{2}(\frac{\partial K}{\partial t},u'(t)), \text{ where, } \widehat{M}(t)=\int_0^t M(s)\,ds.$$

By (H.2) we have

$$|(\frac{\partial K}{\partial t},u'^2(t))|\leq \gamma|u'^2(t))|^2+C(\gamma)|(K,u'^2(t))|\leq \mu(\delta+C(\gamma)K_0)|u'(t)|^2,$$

with

$$K_0=\max_{t\leq s\leq T}\left(\operatorname*{supess}_{x\in\Omega}K(x,s)\right), \text{ and } \mu>0 \text{ is a constant such that } |\varphi|^2\leq\mu\|\varphi\|^2,\ \varphi\in H_0^1(\Omega).$$

Whence follows that

$$\frac{1}{2}\frac{d}{dt}\left[|K^{1/2}u'(t)|^2+\widehat{M}(a(u(t))\right]+[1-\mu(\gamma+C(\gamma)K_0)]\,\|u'(t)\|^2\leq 0, \tag{5.1}$$

where $\gamma>0$ is sufficiently small such that $1-\mu(\gamma+C(\gamma)K_0)>0$.

Now, its important to remember that $E(t)=\frac{1}{2}\left[|K^{1/2}u'(t)|^2+\widehat{M}(a(u(t)))\right]$.

Integrating (5.1) from t to $t+1$, we obtain

$$\int_t^{t+1}|u'(s)|^2\,ds\leq\mu\int_t^{t+1}\|u'(s)\|^2\,ds\leq C_{15}\,[E(t)-E(t+1)]\overset{\text{def}}{=}F^2(t), \text{ with } C_{15}=\frac{\mu}{1-\mu(\gamma+C(\gamma)K_0)}>0. \tag{5.2}$$

Therefore, from (5.2), there exist $t_1\in\left[t,t+\frac{1}{4}\right]$ and $t_2\in\left[t+\frac{3}{4},t+1\right]$ such that $|u'(t_i)|\leq 2F(t),\ \ i=1,2.$

The inner product in $L^2(\Omega)$ of (1.1) with $u(t)$ implies

$$\frac{d}{dt}(Ku'(t),u(t))-|K^{1/2}u'(t)|^2+M(a(u))a(u)+((u'(t),u(t)))=(\frac{\partial K}{\partial t}u'(t),u(t)).$$

Integrating from t_1 to t_2 and by using (H.2) we have

$$\begin{aligned}\int_{t_1}^{t_2}M(a(u))a(u)\,dt\leq{}&K_0|u'(t_1)||u(t_1)|+K_0|u'(t_2)||u(t_2)|\\&+\mu K_0\int_{t_1}^{t_2}\|u'(s)\|^2\,ds+\int_{t_1}^{t_2}\|u'(s)\|\|u(s)\|\,ds\\&+\gamma\sqrt{\mu}\int_{t_1}^{t_2}|u'(s)|\|u(s)\|\,ds+C(\gamma)K_0\sqrt{\mu}\int_{t_1}^{t_2}|u'(s)|\|u(s)\|\,ds.\end{aligned} \tag{5.3}$$

Now,

$$M(a(u))a(u) \geq m_0 a(u) = m_0\|u\|^2, \text{ where } m_0 = \min_{0\leq s\leq a(u)} M(s) > 0. \tag{5.4}$$

Then, by (5.2), (5.3) and (5.4), we obtain

$$m_0 \int_{t_1}^{t_2} \|u(s)\|^2\, ds \leq 4\mu K_0 F(t) \operatorname*{supess}_{t\leq s\leq t+1} \|u(s)\| + C_{16}F^2(t) + \frac{3}{4}m_0 \int_{t_1}^{t_2} \|u(s)\|^2\, ds,$$

where $C_{16} = \mu K_0 + \dfrac{1}{m_0} + \dfrac{\mu\gamma^2}{m_0} + \dfrac{\mu C^2(\gamma)K_0^2}{m_0} > 0$.

Then we have,

$$\int_{t_1}^{t_2} \|u(s)\|^2 ds \leq C_{17}F(t) \operatorname*{supess}_{t\leq s\leq t+1} \|u(s)\| + C_{18}F^2(t) \overset{\text{def}}{=} G^2(t), \text{ being } C_{17} = \frac{4\mu K_0}{m_0} \text{ and } C_{18} = \frac{4C}{m_0}. \tag{5.5}$$

From (5.2) and (5.5) we obtain

$$\int_{t_1}^{t_2} \left[|u'(s)|^2 + \|u(s)\|^2\right] ds \leq F^2(t) + G^2(t). \tag{5.6}$$

Thus, by (5.6) there exists $t^* \in [t_1,t_2]$ such that $|u'(t^*)|^2 + \|u(t^*)\|^2 \leq 2[F^2(t)+G^2(t)]$. (5.7)

Now, not that,

$$\widehat{M}(a(u((t^*)))) \leq m_1\|u(t^*)\|^2 \leq 2m_1[F^2(t)+G^2(t)], \text{ with } m_1 = \max_{0\leq s\leq a(u(t^*))} M(s). \tag{5.8}$$

From (5.7) and (5.8), we have $E(t^*) \leq C_{16}[F^2(t)+G^2(t)]$. (5.9)

Since that $E(t)$ is increasing, we obtain $\operatorname*{supess}_{t\leq s\leq t+1} E(s) \leq E(t^*) + [1-\mu(\gamma+C(\gamma))K_0 \int_t^{t+1} \|u'(s)\|^2 ds$. (5.10)

Now, by (5.2), (5.9) and (5.10), we get $\operatorname*{supess}_{t\leq s\leq t+1} E(s) \leq C_{17}[F^2(t) + F(t) \operatorname*{supess}_{t\leq s\leq t+1} \|u'(s)\| \leq C_{18}F^2(t) + \dfrac{1}{2} \operatorname*{supess}_{t\leq s\leq t+1} E(s)$.

Then, by (5.2) $\operatorname*{supess}_{t\leq s\leq t+1} E(s) \leq C[E(t)-E(t+1)]$, where $C_i,\ i = 15,16,17,18$ and C are positive constants.

Therefore, by Nakao's lemma, we obtain $E(t) \leq Ce^{-\alpha t}$, with $\alpha = \dfrac{1}{C+1}$, for all $t \geq 0$.

The exponential decay of the solution was been proven. □

6. Conclusion

We prove the existence, uniqueness, and exponential stability of the solution to a degenerate hyperbolic equation where the greatest lower bound for Kirchhoff function $M(\cdot)$ is zero. We consider strong damping as a stabilization mechanism. We have improved previous results in the literature, mainly because the exponential decay for this type of problem, as far as we know, has not been previously considered.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous referees for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] K. Nishihara, Y. Yamada, *On global solutions of some degenerate quasilinear hyperbolic equations with dissipative terms*, Funkcial. Ekvac., **33** (1990), 151-159.

[2] J. Ferreira, D. C. Pereira, *On a nonlinear degenerate evolution equation with strong damping*, Internat. J. Math. & Math. Sci., **15**(3) (1992), 543-552.

[3] D. C. Pereira, *Mixed problem for a nonlinear vibrations equation*, Bol. Soc. Parana. Mat., **9**(2) (1988), 31-42.

[4] G. F. Carrier, *On the nonlinear vibration problem of the elastic string*, Q. Appl. Math, **3** (1945), 157-165.

[5] R. Dickey, *Free vibration in dynamic buckling of extensible beam*, J. Math. Analysis Applic., **29** (1970), 439-451.

[6] R. Narasihma, *Nonlinear vibrations of an elastic string*, J. Sound Vibrations, **8** (1968), 134–146.

[7] S. I. Pohozaev, *On a class of quasilinear hyperbolic equations*, Math. USSR Sbornik, **25** (1975), 145-158.

[8] R. W. Dickey, *Infinite systems of nonlinear oscillation equations related the string*, Amer. Math. Soc., **23** (1969), 459-468.

[9] J. L. Lions, *On Some Questions in Boundary Value Problems of Mathematical Physics*, Contemporary Developments in Continuum Mechanics and Partial Differential Equations, edited by G. M. De La Penha and L. A. Medeiros, North-Holland, Amsterdam, 1978.

[10] R. F. C. Lobato, D. C. Pereira, M. L. Santos, *Exponential decay to the degenerate nonlinear coupled beams system with weak damping*, Math. Phys., (2012), 1-14, Article ID 659289.

[11] S. D. B. Menezes, E. A. Oliveira, D. C. Pereira, J. Ferreira, *Existence, uniqueness and uniform decay for the nonlinear beam degenerate equation with weak damping*, Appl. Math. Comput. **154** (2004), 555-565.

[12] K. Nishihara, *On a global solution of some quasilinear hyperbolic equation*, Tokyo J. Math., **7** (1984), 437-459.

[13] P. H. Rivera Rodriguez, *On local strong solutions of a nonlinear partial differential equation*, Appl. Anal., **10** (1984), 93-104.

[14] A. Arosio, S. Spagnolo, *Global Solutions to the Cauchy Problem for a Nonlinear Hyperbolic equation*, Nonlinear Partial Differential Equations and Their Applications, College de France Seminar Vol. 6, edited by Bresis, H. and Lions, J. L., Pitman, London, 1984.

[15] Y. Yamada, *Some nonlinear degenerate wave equations*, Nonlinear Anal., **11** (1987), 1155-1168.

[16] T. Yamazaki, *On local solutions of some quasilinear degenerate hyperbolic equations*, Funkcial. Ekvac., **31** (1988), 439-457.

[17] N. A. Lar'kin, *A class of degenerate hyperbolic equations*, Dinamika Sploshn. Sredy, **36** (1978), 71–77.

[18] N. A. Lar'kin, *On a class of quasi-linear hyperbolic equations having global solutions*, Sov. Math., Dokl. **20** (1979), 28-31.

[19] F. Colombini, S. Spagnolo, *An example of a weakly hyperbolic Cauchy problem not well posed in C^∞*, Acta Math., **148** (1982), 243-253.

[20] D. Lupo, K. R. Payne, N. I. Popivanov, *On the degenerate hyperbolic Goursat problem for linear and nonlinear equations of Tricomi type*, Nonlinear Anal. Theory Methods Appl., **108** (2014), 29-56.

[21] R. L. Cruz, J. C. Juajibioy, L. Rendon, *Generalized Riemann problem for a totally degenerate hyperbolic system*, Electron. J. Differ. Equ., **196** (2018), 1-14.

[22] N. Kakharman, T. Kal'menov, *Mixed Cauchy problem with lateral boundary condition for noncharacteristic degenerate hyperbolic equations,* Bound. Value. Probl., **35** (2022), 1-11.

[23] E. H. Brito, *The damped elastic stretched string equation generalized: existence, uniqueness, regularity and stability*, Appl. Anal., **13** (1982), 219-233.

[24] K. Nishihara, *Exponential decay of solutions of some quasilinear hyperbolic equations with linear damping*, Nonlinear Anal., **8** (1984), 623-636.

[25] K. Nishihara, *Global existence and asymptotic behaviour of the solution of some quasilinear hyperbolic equation with linear damping*, Funkcial. Ekvac., **32** (1989), 343-355.

[26] Y. Yamada, *On some quasilinear wave equations with dissipative terms*, Nagoya Math. J., **87** (1982), 17-39.

[27] J. G. Dix, *Decay of solutions of a degenerate hyperbolic equation*, Electron. J. Differ. Equ., **21** (1998), 1–10.

[28] E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill Inc., New York, 1955.

[29] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes Aux Limites Non Liné aires*, Dunod, Paris, 1969.

[30] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and physics*, Springer Verlag, Applied Math. Sci., 68, 1988.

[31] M. Nakao, *Decay of solutions for some nonlinear evolution equations*, J. Math. Analysis Appl., **60** (1977), 542-549.

Communications in Advanced Mathematical Sciences
Vol. 5, No. 3, 150-160, 2022
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1177174

$\mathscr{I}_2$-Uniform Convergence of Double Sequences of Functions In 2-Normed Spaces

Sevim Yegül Güzey[1], Erdinç Dündar[2]*, Mukaddes Arslan[3]

Abstract
In this work, we discuss various types of $\mathscr{I}_2$-uniform convergence and equi-continuous for double sequences of functions. Also, we introduce the concepts of $\mathscr{I}_2$-uniform convergence, $\mathscr{I}_2^*$-uniform convergence, $\mathscr{I}_2$-uniformly Cauchy sequences and $\mathscr{I}_2^*$-uniformly Cauchy sequences for double sequences of functions in 2-normed spaces. Then, we show the relationships between these new concepts.

Keywords: Double sequence of functions, $\mathscr{I}$-Convergence, Uniformly convergence, 2-normed spaces.
2010 AMS: Primary 40A05, Secondary 40A30, 40A35, 46A70

[1] *Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey, ORCID: 0000-0001-8301-0252*
[2] *Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey, ORCID: 0000-0002-0545-7486*
[3] *Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey, ORCID: 0000-0002-5798-670X*
***Corresponding author**: carlos.raposo@ufba.br
Received: 19 September 2022, **Accepted:** 30 September 2022, **Available online:** 30 September 2022

1. Introduction

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [15] and Schoenberg [34].

The idea of $\mathscr{I}$-convergence was introduced by Kostyrko et al. [27] as a generalization of statistical convergence which is based on the structure of the ideal $\mathscr{I}$ of subset of $\mathbb{N}$ [15, 16]. Das et al. [8] introduced the concept of $\mathscr{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Gökhan et al. [20] introduced the notions of pointwise and uniform statistical convergence of double sequences of real-valued functions. Gezer and Karakuş [19] investigated $\mathscr{I}$-pointwise and $\mathscr{I}$-uniform convergence and $\mathscr{I}^*$-pointwise and $\mathscr{I}^*$-uniform convergence of function sequences. Also, they examined the relationships between them. Baláz et al. [5] investigated $\mathscr{I}$-convergence and $\mathscr{I}$-continuity of real functions. Balcerzak et al. [6] studied statistical convergence and ideal convergence for sequences of functions. Dündar and Altay [10, 11] studied the concepts of $\mathscr{I}_2$-pointwise and $\mathscr{I}_2$-uniform convergence and $\mathscr{I}_2^*$- pointwise and $\mathscr{I}_2^*$-uniform convergence of double sequences of functions and investigated some properties about them. Furthermore, Dündar [12] investigated some results of $\mathscr{I}_2$-convergence of double sequences of functions.

The concept of 2-normed spaces was initially introduced by Gähler [17, 18] in the 1960's. Since then, this concept has been studied by many authors. Gürdal and Pehlivan [24] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Şahiner et al. [36] and Gürdal [26] studied $\mathscr{I}$-convergence in 2-normed spaces. Gürdal and Açık [25] investigated $\mathscr{I}$-Cauchy and $\mathscr{I}^*$-Cauchy sequences in 2-normed spaces. Sarabadan and Talebi [32] presented various kinds of statistical convergence and $\mathscr{I}$-convergence for sequences of functions with values in 2-normed spaces and also defined the notion of $\mathscr{I}$-equistatistically convergence and study $\mathscr{I}$-equistatistically convergence of sequences of functions. Recently, Savaş and Gürdal [33] concerned with $\mathscr{I}$-convergence of sequences of functions in random 2-normed spaces and introduce the concepts of ideal uniform convergence and ideal pointwise convergence in the topology induced by random 2-normed spaces, and gave some basic properties of these concepts. Arslan and Dündar [1, 2] investigated the concepts of $\mathscr{I}$-convergence, $\mathscr{I}^*$-convergence, $\mathscr{I}$-Cauchy and $\mathscr{I}^*$-Cauchy sequences of functions in 2-normed spaces and showed relationships between them. Yegül and Dündar [39] studied statistical convergence of sequence of functions in 2-normed spaces. Also, Dündar et al. [13] investigated $\mathscr{I}$-uniform convergence of sequences of functions in 2-normed spaces. Futhermore, a lot of development have been made in this area (see [7, 28, 29, 30, 35, 37]).

2. Definitions and Notations

Now, we recall the concept of 2-normed space, ideal convergence and some fundamental definitions and notations (See [1, 2, 3, 4, 5, 6, 13, 14, 15, 16, 19, 21, 22, 23, 24, 25, 26, 27, 31, 32, 36, 38]).

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x,y\| = 0$ if and only if x and y are linearly dependent.

(ii) $\|x,y\| = \|y,x\|$.

(iii) $\|\alpha x,y\| = |\alpha|\|x,y\|$, $\alpha \in \mathbb{R}$.

(iv) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$.

The pair $(X,\|\cdot,\cdot\|)$ is then called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x,y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x,y\| = |x_1y_2 - x_2y_1|; \;\; x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2.$$

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$.

A sequence (x_n) in 2-normed space $(X,\|\cdot,\cdot\|)$ is said to be convergent to L in X if $\lim_{n\to\infty} \|x_n - L, y\| = 0$, for every $y \in X$. In such a case, we write $\lim_{n\to\infty} x_n = L$ and call L the limit of (x_n).

Let $X \neq \emptyset$. A class $\mathscr{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathscr{I}$, (ii) $A,B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$, (iii) $A \in \mathscr{I}$, $B \subset A$ implies $B \in \mathscr{I}$.

$\mathscr{I}$ is called a nontrivial ideal if $X \notin \mathscr{I}$. A nontrivial ideal $\mathscr{I}$ in X is called admissible if $\{x\} \in \mathscr{I}$, for each $x \in X$.

Throughout the paper, we let $\mathscr{I} \subset 2^{\mathbb{N}}$ be an admissible ideal.

Let $\mathscr{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then, $\mathscr{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathscr{I}_f$ convergence is the usual convergence.

Throughout the paper we take $\mathscr{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $\mathscr{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathscr{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

$\mathscr{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i,j \geq m(A) \Rightarrow (i,j) \notin A)\}$. Then $\mathscr{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathscr{I}_2$ is strongly admissible if and only if $\mathscr{I}_2^0 \subset \mathscr{I}_2$.

We say that an admissible ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathscr{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathscr{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathscr{I}_2$ (hence $B_j \in \mathscr{I}_2$ for each $j \in \mathbb{N}$).

Let $X \neq \emptyset$. A non empty class $\mathscr{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathscr{F}$, (ii) $A,B \in \mathscr{F}$ implies $A \cap B \in \mathscr{F}$, (iii) $A \in \mathscr{F}$, $A \subset B$ implies $B \in \mathscr{F}$.

If $\mathscr{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class

$$\mathscr{F}(\mathscr{I}) = \{M \subset X : (\exists A \in \mathscr{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathscr{I}$.

A sequence (x_n) in 2-normed space $(X,\|\cdot,\cdot\|)$ is said to be $\mathscr{I}$-convergent to $L \in X$, if for each $\varepsilon > 0$ and each nonzero $z \in X$, $A(\varepsilon,z) = \{n \in \mathbb{N} : \|x_n - L, z\| \geq \varepsilon\} \in \mathscr{I}$. In this case, we write $\mathscr{I} - \lim_{n\to\infty} \|x_n - L, z\| = 0$ or $\mathscr{I} - \lim_{n\to\infty} \|x_n, z\| = \|L, z\|$.

A sequence (x_n) in 2-normed space $(X,\|\cdot,\cdot\|)$ is said to be $\mathscr{I}^*$-convergent to $L \in X$ if and only if there exists a set $M \in \mathscr{F}$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ such that $\lim_{k\to\infty} \|x_{m_k} - L, z\| = 0$, for each nonzero $z \in X$.

Throughout the paper, we let X and Y be two 2-normed spaces, $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of functions and f,g be two functions from X to Y.

The sequence $\{f_n\}_{n\in\mathbb{N}}$ is equi-continuous on X if

$$(\forall z\in X)\,(\forall \varepsilon>0)\,(\exists\delta>0)\,(\forall x,x_0\in X)\,\|x-x_0,z\|_X<\delta\Rightarrow\|f_n(x)-f_n(x_0)\|_\infty<\varepsilon.$$

The sequence $\{f_n\}$ is said to be $\mathscr{I}$-uniformly convergent to f (on X) if and only if

$$(\forall z\in Y)\,(\forall\varepsilon>0)\,(\exists M\in\mathscr{I})\,(\forall n\in\mathbb{N}\backslash M)\,(\forall x\in X)\,\|f_n(x)-f(x),z\|_Y\leq\varepsilon.$$

We write $f_n \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}} f$.

The sequence of functions $\{f_n\}$ is said to be $\mathscr{I}^*$-uniformly convergent to f on X, if for every $\varepsilon>0$ there exists a set $K\in\mathscr{F}(\mathscr{I})$ $(\mathbb{N}\backslash K\in\mathscr{I})$ and $\exists n_0=n_{0(\varepsilon)}\in K$ such that for all $n\geq n_0$, $n\in K$ and for each nonzero $z\in Y$, $\|f_n(x)-f(x),z\|<\varepsilon$, for each $x\in X$ and in this case, we write $f_n \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}^*} f$.

$\{f_n\}$ is said to be $\mathscr{I}$-uniformly Cauchy if for every $\varepsilon>0$ there exists $s=s(\varepsilon)\in\mathbb{N}$ such that for each nonzero $z\in Y$,

$$\{n\in\mathbb{N}:\|f_n(x)-f_s(x),z\|\geq\varepsilon\}\in\mathscr{I},\ for\ each\ \ x\in X. \tag{2.1}$$

The sequence of functions $\{f_n\}$ is said to be $\mathscr{I}^*$-uniformly Cauchy sequence, if there exist a set $M\in\mathscr{F}(\mathscr{I}), M=\{m_1<m_2<...<m_k<...\}\subset\mathbb{N}$ such that for every $\varepsilon>0$ there is an $k_0=k_0(\varepsilon)$ such that for each nonzero $z\in Y$, $\|f_{m_k}(x)-f_{m_p}(x),z\|<\varepsilon$, for each $x\in X$ and for all $k,p>k_0$.

Throughout the paper, we let $\mathscr{I}_2\subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, X and Y be two 2-normed spaces, $\{f_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$, $\{g_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ and $\{h_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ be three double sequences of functions, f, g and k be three functions from X to Y.

The double sequence of functions $\{f_{mn}\}$ in 2-normed space $(X,\|.,.\|)$ is said to be convergent (pointwise) to f if, for each point $x\in X$ and every $\varepsilon>0$, there exists a positive integer $k_0=k_0(x,\varepsilon)$ such that for all $m,n\geq k_0$ implies $\|f_{mn}(x)-f(x),z\|<\varepsilon$, for every $z\in Y$. In this case we write $f_{mn}\overset{\|.,.\|_Y}{\rightarrow} f$.

The double sequence of functions $\{f_{mn}\}$ in 2-normed space $(X,\|.,.\|)$ is said to be uniformly convergent to f if, for every $\varepsilon>0$ there exists a positive integer $k_0=k_0(\varepsilon)$ such that for all $m,n\geq k_0$ implies $\|f_{mn}(x)-f(x),z\|<\varepsilon$, for all $x\in X$ and every $z\in Y$. In this case we write $f_{mn}\overset{\|.,.\|_Y}{\rightrightarrows} f$.

The double sequence of functions $\{f_{mn}\}$ in 2-normed space $(X,\|.,.\|)$ is said to be $\mathscr{I}_2$-convergent (pointwise sense) to f if, for each $x\in X$ and every $\varepsilon>0$, $A(\varepsilon,z)=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|\geq\varepsilon\}\in\mathscr{I}_2$, for each nonzero $z\in Y$.

This can be expressed by the formula

$$(\forall z\in Y)\ (\forall x\in X)\ (\forall\varepsilon>0)\ (\exists H\in\mathscr{I}_2)\ (\forall(m,n)\notin H)\ \|f_{mn}(x)-f(x),z\|<\varepsilon.$$

In this case, we write $\mathscr{I}_2-\lim\limits_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|\ or\ f_{mn}\overset{\|.,.\|_Y}{\longrightarrow}_{\mathscr{I}_2} f$.

The double sequence of functions $\{f_{mn}\}$ in 2-normed space $(X,\|.,.\|)$ is said to be $\mathscr{I}_2^*$-convergent (pointwise sense) to f, if there exists a set $M\in\mathscr{F}(\mathscr{I}_2)$ $(i.e., H=\mathbb{N}\times\mathbb{N}\setminus M\in\mathscr{I}_2)$ such that for each $x\in X$, each nonzero $z\in Y$ and all $(m,n)\in M$ $\lim\limits_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|$ and we write $\mathscr{I}_2^*-\lim\limits_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|$ or $f_{mn}\overset{\|.,.\|_Y}{\longrightarrow}_{\mathscr{I}_2^*} f$.

A double sequence of functions $\{f_{mn}\}$ is said to be $\mathscr{I}_2$-Cauchy sequence, if for every $\forall\varepsilon>0$ and each $x\in X$ there exist $s=s(\varepsilon,x)$, $t=t(\varepsilon,x)\in\mathbb{N}$ such that

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f_{st}(x),z\|\geq\varepsilon\}\in\mathscr{I}_2,$$

for each nonzero $z\in Y$.

A double sequence of functions $\{f_{mn}\}$ is said to be $\mathscr{I}_2^*$- Cauchy sequence, if there exists a set $M\in\mathscr{F}(\mathscr{I}_2)$ $(i.e., H=\mathbb{N}\times\mathbb{N}\setminus M\in\mathscr{I}_2)$ and for every $\varepsilon>0$ and each $x\in X$, $k_0=k_0(\varepsilon,x)\in\mathbb{N}$ such that for all $(m,n),(s,t)\in M$ and each $z\in Y$ $\|f_{mn}(x)-f_{st}(x),z\|<\varepsilon$, whenever $m,n,s,t>k_0$. In this case, we write $\lim\limits_{m,n,s,t\to\infty}\|f_{mn}(x)-f_{st}(x),z\|=0$.

Now we begin with quoting the lemmas due to Yegül and Dündar [40, 41, 42] which are needed throughout the paper.

Lemma 2.1 ([41]). *For each $x\in X$ and each nonzero $z\in Y$,*

$$\mathscr{I}_2^*-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|\ implies\ \mathscr{I}_2-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|.$$

Lemma 2.2 ([41]). *Let $\mathscr{I} \subset 2^{\mathbb{N}\times\mathbb{N}}$ be an admissible ideal having the property $(AP2)$. For each $x \in X$ and each nonzero $z \in Y$,*

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ implies } \mathscr{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

Lemma 2.3 ([42]). *If $\{f_{mn}\}$ is $\mathscr{I}_2$-convergent if and only if it is $\{f_{mn}\}$ is $\mathscr{I}_2$-Cauchy double sequence in 2-normed spaces.*

Lemma 2.4 ([40]). *Let D be a compact subset of X and f and f_{mn}, $(m,n = 1,2,...)$, be continuous functions on D. Then, $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows} f$ on D if and only if $\lim_{m,n\to\infty} c_{mn} = 0$, where $c_{mn} = \max_{x\in D} \|f_{mn}(x) - f(x), z\|$.*

3. Main Results

In this paper, we define concepts of $\mathscr{I}_2$-uniform convergence, $\mathscr{I}_2^*$-uniform convergence, $\mathscr{I}_2$-uniformly Cauchy and $\mathscr{I}_2^*$-uniformly Cauchy sequence of functions and investigate relationships between them and some properties such as continuity in 2-normed spaces.

Definition 3.1. *The double sequence $\{f_{mn}\}$ is said to be $\mathscr{I}_2$-uniformly convergent to f (on X) if for every $\varepsilon > 0$ and each nonzero $z \in Y$,*

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\} \in \mathscr{I}_2, \text{ for each } x \in X.$$

This can be written by the formula

$$(\forall z \in Y)\,(\forall \varepsilon > 0)\,(\exists M \in \mathscr{I}_2)\,(\forall m,n \in \mathbb{N}\backslash M)\,(\forall x \in X)\, \|f_{mn}(x) - f(x), z\|_Y \leq \varepsilon.$$

We write $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$.

Theorem 3.2. *For each $x \in X$ and each nonzero $z \in Y$,*

$$f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows} f \text{ implies } f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f.$$

Proof. Let $\varepsilon > 0$ be given. Since

$$\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$$

for each $x \in X$ and each nonzero $z \in Y$, therefore, there exists a positive integer $k_0 = k_0(\varepsilon)$ such that $\|f_{mn}(x) - f(x), z\| < \varepsilon$, whenever $m,n \geq k_0$. This implies that for each nonzero $z \in Y$,

$$\begin{aligned} A(\varepsilon, z) &= \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| < \varepsilon\} \\ &\subset ((\mathbb{N}\times\{1,2,..,(k_0-1)\}) \cup (\{1,2,..,(k_0-1)\}\times\mathbb{N})). \end{aligned}$$

Since $\mathscr{I}_2$ be a strongly admissible ideal, therefore

$$((\mathbb{N}\times\{1,2,..,(k_0-1)\}) \cup (\{1,2,..,(k_0-1)\}\times\mathbb{N})) \in \mathscr{I}_2.$$

Hence, it is clear that $A(\varepsilon, z) \in \mathscr{I}_2$ and consequently we have

$$f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f.$$

□

Theorem 3.3. *Let D be a compact subset of X and f, $\{f_{mn}\}$, $m,n = 1,2,...$ be continuous functions on D. Then,*

$$f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$$

on D if and only if for each nonzero $z \in Y$,

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|c_{mn}(x), z\| = 0,$$

where

$$c_{mn} = \max_{x\in D} \|f_{mn}(x) - f(x), z\|.$$

Proof. Assume that $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$ on D. Since f and $\{f_{mn}\}$ be continuous functions on D, so $(f_{mn}(x) - f(x))$ is continuous on D, for each $m,n \in \mathbb{N}$. By $\mathscr{I}_2-$uniform convergence, for every $\varepsilon > 0$ and each nonzero $z \in Y$

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{2}\right\} \in \mathscr{I}_2,$$

for each $x \in D$. Hence, for every $\varepsilon > 0$ and each nonzero $z \in Y$, it is clear that

$$c_{mn} = \max_{x\in D} \|f_{mn}(x) - f(x), z\| \geq \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{2},$$

for each $x \in D$. Thus, we have

$$\mathscr{I}_2 - \lim_{m,n\to\infty} c_{mn} = 0.$$

Now, conversely, suppose that $\mathscr{I}_2 - \lim_{m,n\to\infty} c_{mn} = 0$. For every $\varepsilon > 0$ and each nonzero $z \in Y$, we let following sets

$$A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \max_{x\in D}\|f_{mn}(x) - f(x), z\| \geq \varepsilon\}$$

and

$$B(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\},$$

for each $x \in D$. Since $\mathscr{I}_2 - \lim_{m,n\to\infty} c_{mn} = 0$, then $A(\varepsilon) \in \mathscr{I}_2$. Now, we let $(m,n) \in A^c(\varepsilon)$. Since for every $\varepsilon > 0$ and each nonzero $z \in Y$,

$$\|f_{mn}(x) - f(x), z\| \leq \max_{x\in D} \|f_{mn}(x) - f(x), z\| < \varepsilon,$$

for each $x \in D$, then $(m,n) \in B^c(\varepsilon)$ and so, $A^c(\varepsilon) \subset B^c(\varepsilon)$. Hence, we have $B(\varepsilon) \subset A(\varepsilon)$ and so, $B(\varepsilon) \in \mathscr{I}_2$. This proves the theorem. □

Definition 3.4. *The sequence of functions $\{f_{mn}\}$ is said to be $\mathscr{I}_2^*$-uniformly convergent to f on X, if for every $\varepsilon > 0$ there exists a set $K \in \mathscr{F}(\mathscr{I}_2)$ (i.e., $\mathbb{N}\times\mathbb{N}\backslash K \in \mathscr{I}_2$) and $\exists n_0 = n_0(\varepsilon) \in K$ such that for all $m,n \geq n_0$, $(m,n) \in K$ and for each nonzero $z \in Y$,*

$$\|f_{mn}(x) - f(x), z\| < \varepsilon,$$

for each $x \in X$ and in this case, we write $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2^*} f$.

Theorem 3.5. *Let $\{f_{mn}\}$ be a sequence of continuous functions and f be function from X to Y. If $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2^*} f$, then f is continuous on X.*

Proof. Assume $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2^*} f$ on X. Then, for every $\varepsilon > 0$, there exists a set $K \in \mathscr{F}(\mathscr{I}_2)$ $(i.e.,\ H = \mathbb{N}\times\mathbb{N}\setminus K \in \mathscr{I}_2)$ and $k_0 = k_0(\varepsilon), l_0 = l_0(\varepsilon) \in \mathbb{N}$ such that

$$\|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{3}, \ \ (m,n \in K)$$

for each nonzero $z \in Y$, each $x \in X$ and all $m > k_0, n > l_0$. Now, we let $x_0 \in X$ is arbitrary. Since $\{f_{k_0 l_0}\}$ is continuous at $x_0 \in X$, there is a $\delta > 0$ such that for each nonzero $z \in Y$,

$$\|x - x_0, z\| < \delta$$

implies

$$\|f_{k_0 l_0}(x) - f_{k_0 l_0}(x_0), z\| < \frac{\varepsilon}{3}.$$

Then, for all $x \in X$ for which $\|x - x_0, z\| < \delta$, we have

$$\begin{aligned} \|f(x) - f(x_0), z\| &\leq \|f(x) - f_{k_0 l_0}(x_0), z\| + \|f_{k_0 l_0}(x) - f_{k_0 l_0}(x_0), z\| \\ &+ \|f_{k_0 l_0}(x) - f(x_0), z\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

for each nonzero $z \in Y$. Since $x_0 \in X$ is arbitrary, f is continuous on X. □

Theorem 3.6. *Let $\mathscr{I} \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with the property $(AP2)$, D be a compact subset of X and $\{f_{mn}\}$ be a sequence of continuous function on D. Assume that $\{f_{mn}\}$ be monotonic decreasing on D, i.e.,*

$$f_{(m+1),(n+1)}(x) \leq f_{mn}(x), (m,n = 1,2,...)$$

for every $x \in D$, f is continuous and for each nonzero $z \in Y$,

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$$

on D. Then,

$$f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$$

on D.

Proof. Let

$$g_{mn} = f_{mn} - f \tag{3.1}$$

be a sequence of functions on D. Since $\{f_{mn}\}$ is continuous and monotonic decreasing and f is continuous on D, then $\{g_{mn}\}$ is continuous and monotonic decreasing on D. Since

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|,$$

for each $x \in D$ and nonzero $z \in Y$, then by (3.1),

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|g_{mn}(x), z\| = 0$$

on D and since $\mathscr{I}_2$ satisfies the condition $(AP2)$ then, by Lemma 2.2, for each nonzero $z \in Y$, we have

$$\mathscr{I}_2^* - \lim_{m,n\to\infty} \|g_{mn}(x), z\| = 0,$$

for each $x \in D$. Hence, for every $\varepsilon > 0$ and each $x \in D$ there exists $K_x \in \mathscr{F}(\mathscr{I}_2)$ such that

$$0 \leq g_n(x) < \frac{\varepsilon}{2}, \; ((m,n), (m(x) = m(x,\varepsilon), n(x) = n(x,\varepsilon)) \in K_x)$$

for $m \geq m(x)$ and $n \geq n(x)$, $(m,n) \in K_x$. Since $\{g_{mn}\}$ is continuous at $x \in D$, for every $\varepsilon > 0$ there exists an open set $A(x)$ which contains x such that for each nonzero $z \in Y$,

$$\|g_{m(x)n(x)}(t) - g_{m(x)n(x)}(x), z\| \leq \frac{\varepsilon}{2},$$

for all $t \in A(x)$. Then, for every $\varepsilon > 0$, by monotonicity for each nonzero $z \in Y$, we have

$$\begin{aligned} 0 \leq g_{mn}(x) \leq g_{mn}(t) \leq g_{m(x)n(x)}(t) &= g_{m(x)n(x)}(t) - g_{m(x)n(x)}(x) + g_{m(x)n(x)}(x) \\ &\leq \|g_{m(x)n(x)}(t) - g_{m(x)n(x)}(x), z\| + g_{m(x)n(x)}(x) \end{aligned}$$

for every $t \in A(x)$ and for all $m \geq m(x)$, $n \geq n(x)$ and for each $x \in D$. Since $D \subset \bigcup_{x\in D} A(x)$ and D is a compact set, by the Heine-Borel theorem D has a finite open covering such that

$$D \subset A(x_1) \cup A(x_2) \cup A(x_3) ... \cup A(x_i).$$

Now, let

$$K = K_{x_1} \cap K_{x_2} \cap K_{x_3} \cap ... \cap K_{x_i}$$

and define

$$M = max\{m(x_1), m(x_2), m(x_3), ..., m(x_i)\},$$

$$N = max\{n(x_1), n(x_2), n(x_3), ..., n(x_i)\}.$$

Since for every K_{x_i} belong to $\mathscr{F}(\mathscr{I}_2)$, we have $K \in \mathscr{F}(\mathscr{I}_2)$. Then, when all $(m,n) \geq (M,N)$

$$0 \leq g_{mn}(t) < \varepsilon, \, (m,n) \in K,$$

for every $t \in A(x)$. So

$$g_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2^*} 0,$$

on D. Since $\mathscr{I}$ is an admissible ideal

$$g_n \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} 0$$

on D and by (3.1) we have

$$f_n \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$$

on D. □

Definition 3.7. *The sequence $\{f_{mn}\}_{n\in\mathbb{N}}$ is equi-continuous on X if*

$$(\forall z \in X)\,(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall x, x_0 \in X)\,\|x - x_0, z\|_X < \delta \Rightarrow \|f_{mn}(x) - f_{mn}(x_0), z\|_\infty < \varepsilon.$$

Theorem 3.8. *Let $\mathscr{I} \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, X and Y be two 2-normed spaces with $dimY < \infty$. Assume that $f_{mn} \overset{\|.,.\|_Y}{\longrightarrow}_{\mathscr{I}_2} f$ on X, where $f_{mn} : X \to Y$, $m,n \in \mathbb{N}$ are equi-continuous on X and $f : X \to Y$, then f is continuous on X. If X is compact then, we have $f_n \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$ on X.*

Proof. First we will prove that f is continuous on X. Let $x_0 \in X$ and $\varepsilon > 0$. By the equi-continuity of f_{mn}'s there exists $\delta > 0$ and for each nonzero $z \in Y$ such that

$$\|f_{mn}(x) - f_{mn}(x_0), z\| < \frac{\varepsilon}{3}$$

for every $m,n \in \mathbb{N}$, $x \in B_\delta(x_0)$ ($B_\delta(x_0)$ stands for an open ball in X with center x_0 and radius δ.) Since $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$. The set

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x_0) - f(x_0), z\| \geq \frac{\varepsilon}{3}\right\} \cup \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{3}\right\}$$

is in $\mathscr{I}_2$ and is different from $\mathbb{N}\times\mathbb{N}$. Hence, for each nonzero $z \in Y$, there exists $(m,n) \in \mathbb{N}\times\mathbb{N}$ such that

$$\|f_{mn}(x_0) - f(x_0), z\| < \frac{\varepsilon}{3} \text{ and } \|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{3}.$$

Thus, for each nonzero $z \in Y$ we have

$$\begin{aligned} \|f(x_0) - f(x), z\| &\leq \|f(x_0) - f_{mn}(x_0), z\| + \|f_{mn}(x_0) - f_{mn}(x), z\| + \|f_{mn}(x) - f(x), z\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

so f is continuous on X. We assume that X is compact. Let $\varepsilon > 0$. Since X is compact, it follows that f is uniformly continuous and f_{mn}'s are equi-uniformly continuous on X. So, pick $\delta > 0$ such that for any $x, x' \in X$ with

$$\|x - x', z\| < \delta,$$

then, by equi-uniformly and uniformly continuous for each nonzero $z \in Y$, we have

$$\|f_{mn}(x) - f_{mn}(x'), z\| < \frac{\varepsilon}{3} \text{ and } \|f(x) - f(x'), z\| < \frac{\varepsilon}{3}.$$

By the compactness of X, we can choose a finite subcover

$$B_{x_1}(\delta), B_{x_2}(\delta), B_{x_3}(\delta), ..., B_{x_k}(\delta)$$

from the cover $\{B_x(\delta)\}_{x\in X}$ of X. Using $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$ pick a set $M \in \mathscr{I}_2$ such that for each nonzero $z \in Y$,

$$\|f_{mn}(x_i) - f(x_i), z\| < \frac{\varepsilon}{3}, \ i \in \{1,2,...,k\},$$

for all $m,n \notin M$. Let $m,n \notin M$ and $x \in X$. Thus, $x \in B_{x_i}(\delta)$ for since $i \in \{1,2,...,k\}$. Hence, for each nonzero $z \in Y$ we have

$$\begin{aligned} \|f_{mn}(x) - f(x), z\| &\le \|f_{mn}(x) - f_{mn}(x_i), z\| + \|f_{mn}(x_i) - f(x_i), z\| + \|f(x_i) - f(x), z\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

and so $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$ on X. □

Definition 3.9. *$\{f_{mn}\}$ is said to be $\mathscr{I}_2$-uniformly Cauchy if for every $\varepsilon > 0$ there exists $s = s(\varepsilon) \in \mathbb{N}$, $t = t(\varepsilon) \in \mathbb{N}$ such that for each nonzero $z \in Y$,*

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f_{st}(x), z\| \ge \varepsilon\} \in \mathscr{I}_2, \ for\ each\ \ x \in X. \tag{3.2}$$

Now, we give $\mathscr{I}_2$-Cauchy criteria for $\mathscr{I}_2$-uniformly convergence in 2-normed space.

Theorem 3.10. *Let $\mathscr{I}_2 \subset 2^{\mathbb{N}} \times \mathbb{N}$ be a strongly admissible ideal with the property $(AP2)$ and let $\{f_{mn}\}$ be a sequence of bounded function on X. Then, $\{f_{mn}\}$ is $\mathscr{I}_2$-uniformly convergent if and only if it is $\mathscr{I}_2$-uniformly Cauchy sequence on X.*

Proof. Assume that $\{f_{mn}\}$ $\mathscr{I}_2$-uniformly convergent to a function f defined on X. Let $\varepsilon > 0$. Then, for each nonzero $z \in Y$, we have

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2}\right\} \notin \mathscr{I}_2$$

for each $x \in X$. We can select an $m(\varepsilon), n(\varepsilon) \in \mathbb{N}$ such that for each nonzero $z \in Y$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{m(\varepsilon)n(\varepsilon)}(x) - f(x), z\| < \frac{\varepsilon}{2}\right\} \notin \mathscr{I}_2,$$

for each $x \in X$. The triangle inequality yields that for each nonzero $z \in Y$

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f_{m(\varepsilon)n(\varepsilon)}(x), z\| < \varepsilon\} \notin \mathscr{I}_2,$$

for each $x \in X$. Since ε is arbitrary, $\{f_{mn}\}$ is $\mathscr{I}_2$-uniformly Cauchy on X.

Conversely, assume that $\{f_{mn}\}$ is $\mathscr{I}_2$-uniformly Cauchy on X. Let $x \in X$ be fixed. By (3.2) for every $\varepsilon > 0$ there is an $s = s(\varepsilon)$ and $t = t(\varepsilon) \in \mathbb{N}$ such that for each nonzero $z \in Y$,

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f_{st}(x), z\| < \varepsilon\} \notin \mathscr{I}_2.$$

Hence, $\{f_{mn}\}$ is $\mathscr{I}_2$-Cauchy, so by Lemma 2.3 we have that $\{f_{mn}\}$ is $\mathscr{I}_2$-convergent to f.Then, $f_{mn} \xrightarrow{\|.,.\|_Y}_{\mathscr{I}_2} f$ on X.

Now we shall show that this convergence must be uniform. Note that since $\mathscr{I}_2$ satisfy the condition $(AP2)$, by (3.2) there is a $K \notin \mathscr{I}_2$ such that for each nonzero $z \in Y$,

$$\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon, \ ((m,n),(s,t) \in K) \tag{3.3}$$

for all $m,n,s,t \ge N$ and $N = N(\varepsilon) \in \mathbb{N}$ and for each $x \in X$. By (3.3) for $s,t \to \infty$ and each nonzero $z \in Y$,

$$\|f_{mn}(x) - f(x), z\| < \varepsilon, \ \ ((m,n) \in K)$$

for all $n,m > N$ and each $x \in X$. This shows that

$$f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2^*} f$$

on X. Since $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ is a strongly admissible ideal we have

$$f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$$

on X. □

Definition 3.11. *Let $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal and $\{f_{mn}\}$ be a double sequence of function on X. $\{f_{mn}\}$ is said to be $\mathscr{I}_2^*$-uniformly Cauchy sequence, if there exist a set $K \in \mathscr{F}(\mathscr{I}_2), (i.e., H = \mathbb{N}\times\mathbb{N}\setminus K \in \mathscr{I}_2)$, for every $\varepsilon > 0$ and each $x \in X$, $k_0 = k_0(\varepsilon, x)$ such that for all $((m,n),(s,t)) \in K$ and each nonzero $z \in Y$,*

$$\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon,$$

whenever $m,n,s,t, > k_0$. In this case, we write

$$\lim_{m,n,s,t\to\infty} \|f_{mn}(x) - f_{st}(x), z\| = 0.$$

Theorem 3.12. *If $\{f_{mn}\}$ is a $\mathscr{I}_2^*$-uniformly Cauchy sequence then it is $\mathscr{I}_2$-uniformly Cauchy sequence in 2-normed spaces.*

Proof. Let $\{f_{mn}\}$ be a $\mathscr{I}_2^*$-uniformly Cauchy sequence in 2-normed spaces then, by definition there exists the set $K \in \mathscr{F}(\mathscr{I}_2), (i.e., H = \mathbb{N}\times\mathbb{N}\setminus K \in \mathscr{I}_2)$ such that for every $\varepsilon > 0$ and for each nonzero $z \in Y$, $k_0 = k_0(\varepsilon)$ and $((m,n),(s,t)) \in K$

$$\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon,$$

for each $x \in X$ and $m,n,s,t > k_0$. Let $N = N(\varepsilon, z)$. Then for $\varepsilon > 0$ and for each nonzero $z \in Y$, we have

$$\|f_{mn}(x) - f_N(x), z\| < \varepsilon,$$

for each $x \in X$ and $m,n > k_0$. Now put $H = \mathbb{N}\times\mathbb{N}\setminus K$. It is clear that $H \in \mathscr{I}_2$ and

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|f_{mn}(x) - f_N(x)\| \geq \varepsilon\} \subset H \cup K.$$

Since $\mathscr{I}_2$ is an admissible ideal then $H \cup K \in \mathscr{I}_2$. Hence, for every $\varepsilon > 0$ we find $N = N(\varepsilon, z)$ such that $A(\varepsilon, z) \in \mathscr{I}_2$, i.e., $\{f_{mn}\}$ is $\mathscr{I}_2$-uniformly Cauchy sequence. □

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous referees for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] M. Arslan, E. Dündar, *$\mathscr{I}$-Convergence and $\mathscr{I}$-Cauchy Sequence of Functions In 2-Normed Spaces*, Konuralp Journal of Mathematics, **6**(1) (2018), 57–62.

[2] M. Arslan, E. Dündar, *On $\mathscr{I}$-Convergence of sequences of functions in 2-normed spaces*, Southeast Asian Bulletin of Mathematics, **42** (2018), 491–502.

[3] M. Arslan, E. Dündar, *Rough convergence in 2-normed spaces*, Bulletin of Mathematical Analysis and Applications, **10**(3) (2018), 1–9.

[4] M. Arslan, E. Dündar, On Rough Convergence in 2-Normed Spaces and Some Properties Filomat **33**(16) (2019), 5077–5086.

[5] V. Baláz, J. Červeňanský, P. Kostyrko, T. Šalát, *I-convergence and I-continuity of real functions*, Acta Mathematica, Faculty of Natural Sciences, Constantine the Philosopher University, Nitra, **5** (2004), 43–50.

[6] M. Balcerzak, K. Dems, A. Komisarski, *Statistical convergence and ideal convergence for sequences of functions*, J. Math. Anal. Appl. **328**(1) (2007), 715-729.

[7] H. Çakallı and S. Ersan, *New types of continuity in 2-normed spaces,* Filomat, **30**(3) (2016), 525–532.

[8] P. Das, P. Kostyrko, W. Wilczyński, P. Malik, *$\mathscr{I}$ and $\mathscr{I}^*$-convergence of double sequences*, Math. Slovaca, **58** (2008), No. 5, 605–620.

[9] P. Das, P. Malik, *On extremal $\mathscr{I}$-limit points of double sequences*, Tatra Mt. Math. Publ. **40** (2008), 91–102.

[10] E. Dündar, B. Altay, *$\mathscr{I}_2$-convergence of double sequences of functions*, Electronic Journal of Mathematical Analysis and Applications, **3**(1) (2015), 111–121.

[11] E. Dündar, B. Altay, *$\mathscr{I}_2$-uniform convergence of double sequences of functions*, Filomat, **30**(5) (2016), 1273–1281.

[12] E. Dündar, *On some results of $\mathscr{I}_2$-convergence of double sequences of functions*, Mathematical Analysis Sciences and Applications E-notes, **3**(1) (2015), 44–52.

[13] E. Dündar, M. Arslan, S. Yegül, *On $\mathscr{I}$-Uniform convergence of sequences of functions in 2-normed spaces*, Rocky Mountain Journal of Mathematics **50**(5) (2020), 1637–1646.

[14] E. Dündar, U. Ulusu, N. Pancaroğlu, *Strongly $\mathscr{I}_2$-lacunary convergence and $\mathscr{I}_2$-lacunary Cauchy double sequences of sets*, The Aligarh Bulletin of Mathematisc, **35**(1-2) (2016), 1-15.

[15] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[16] J.A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.

[17] S. Gähler, *2-metrische Räume und ihre topologische struktur*, Math. Nachr. **26** (1963), 115–148.

[18] S. Gähler, *2-normed spaces*, Math. Nachr. **28** (1964), 1–43.

[19] F. Gezer, S. Karakuş, *$\mathscr{I}$ and $\mathscr{I}^*$ convergent function sequences*, Math. Commun. **10** (2005), 71-80.

[20] A. Gökhan, M. Güngör and M. Et, *Statistical convergence of double sequences of real-valued functions*, Int. Math. Forum, **2(8)** (2007), 365-374.

[21] H. Gunawan, M. Mashadi, *On n-normed spaces*, Int. J. Math. Math. Sci. **27** (10) (2001), 631–639.

[22] H. Gunawan, M. Mashadi, *On finite dimensional 2-normed spaces*, Soochow J. Math. **27** (3) (2001), 321–329.

[23] M. Gürdal, S. Pehlivan, *The statistical convergence in 2-Banach spaces*, Thai J. Math. **2** (1) (2004), 107–113.

[24] M. Gürdal, S. Pehlivan, *Statistical convergence in 2-normed spaces*, Southeast Asian Bull. Math. **33** (2009), 257–264.

[25] M. Gürdal, I. Açık, *On $\mathscr{I}$-Cauchy sequences in 2-normed spaces*, Math. Inequal. Appl. **11**(2) (2008), 349–354.

[26] M. Gürdal, *On ideal convergent sequences in 2-normed spaces*, Thai J. Math. **4**(1) (2006), 85–91.

[27] P. Kostyrko, T. Šalát, W. Wilczyński, *$\mathscr{I}$-convergence*, Real Anal. Exchange **26** (2) (2000), 669–686.

[28] M. Mursaleen, S.A. Mohiuddine, *On ideal convergence in probabilistic normed spaces*, Math. Slovaca **62**(1) (2012), 49–62.

[29] M. Mursaleen, A. Alotaibi, *On $\mathscr{I}$-convergence in random 2-normed spaces*, Math. Slovaca **61**(6) (2011), 933–940.

[30] F. Nuray, W.H. Ruckle, *Generalized statistical convergence and convergence free spaces*, J. Math. Anal. Appl. **245** (2000), 513–527.

[31] F. Nuray, E. Dündar, U. Ulusu, *Wijsman $\mathscr{I}_2$-convergence of double sequences of cosed sets*, Pure and Applied Mathematics Letters, **2** (2014), 35-39.

[32] S. Sarabadan, S. Talebi, *Statistical convergence and ideal convergence of sequences of functions in 2-normed spaces*, Int. J. Math. Math. Sci. **2011** (2011), 10 pages. doi:10.1155/2011/517841.

[33] E. Savaş, M. Gürdal, *Ideal Convergent Function Sequences in Random 2-Normed Spaces,* Filomat, **30**(3) (2016), 557–567.

[34] I.J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361–375.

[35] A. Sharma, K. Kumar, *Statistical convergence in probabilistic 2-normed spaces,* Mathematical Sciences, **2**(4) (2008), 373–390.

[36] A. Şahiner, M. Gürdal, S. Saltan, H. Gunawan, *Ideal convergence in 2-normed spaces*, Taiwanese J. Math. **11**(5) (2007), 1477–1484.

[37] B.C. Tripathy, M. Sen, S. Nath, *$\mathscr{I}$-convergence in probabilistic n-normed space*, Soft Comput. **16**(6) (2012), 1021–1027.

[38] U. Ulusu, F. Nuray, E. Dündar, *$\mathscr{I}$-limit and $\mathscr{I}$-cluster points for functions defined on amenable semigroups*, Fundamental Journal of Mathematics and Applications, **4**(2) (2021), 45-48.

[39] S. Yegül, E. Dündar, *On Statistical convergence of sequences of functions in 2-normed spaces*, Journal of Classical Analysis, **10**(1) (2017), 49–57.

[40] S. Yegül, E. Dündar, *Statistical convergence of double sequences of functions and some properties in 2-normed spaces*, Facta Universitatis, Series Mathematics and Informatics, **33**(5) (2018), 705–719.

[41] S. Yegül, E. Dündar, *$\mathscr{I}_2$-convergence of double sequences of functions in 2-normed spaces*, Universal Journal of Mathematics and Applications **2**(3) (2019) 130–137.

[42] S. Yegül, E. Dündar, *On $\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy double sequences of functions in 2-normed spaces*, Facta Universitatis Series Mathematics and Informatics **35**(3) (2020) 801–814.