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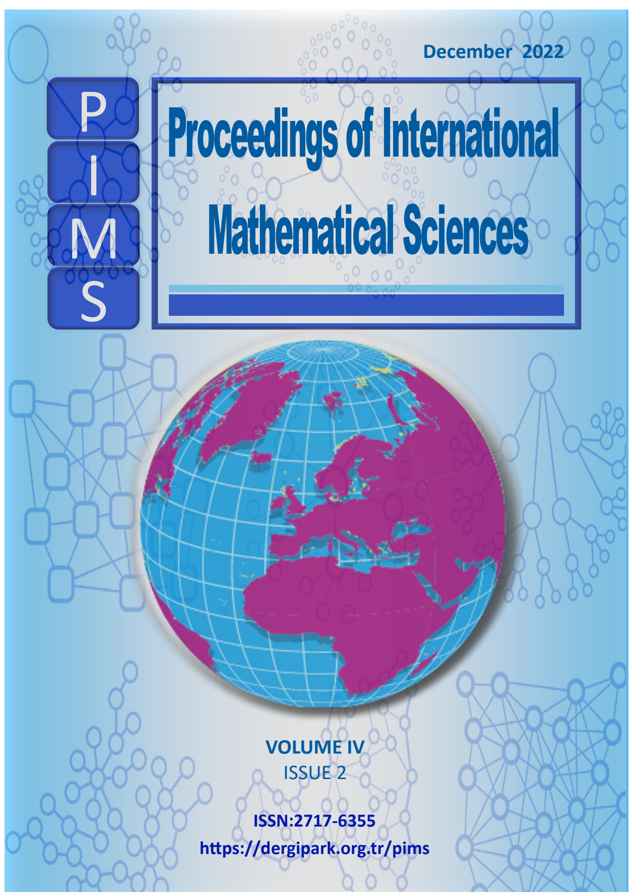

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APPROXIMATE ANALYTICAL SOLUTIONS OF THE SCHRÖDİNGER EQUATION IN CENTRAL POTENTIAL FIELD

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Abstract. We investigate the approximate l-state solutions of the Schrödinger equation for Hulthn plus a class of Yukawa potential. In this context, we construct the bound-state energy equation and the wave function expressed by the Gauss hypergeometric function by means of asymptotic iteration approach in detail.

1. Introduction

Bound state solutions of the Schrödinger equation for a quantum system interacting with spherical symmetric potential models are among the most important in various fields of physics. The l-state solutions of the non-relativistic wave equation for exponential potentials especially are of great interest in literature[1, 2, 3, 4]. Under consideration of this problem, it cannot be possible to obtain analytical solutions without approximations. For this reason, approximations and their applications are essential in quantum mechanical models. In the present work, we choose the proper approximate expression for investigating of analytical solutions. Asymptotic iteration method proposed by Ciftci et al. [5, 6, 7] is a powerful tool for solving second-order homogeneous linear differential equation. This method gives a precise way to probe the bound state solutions of the Schrödinger wave equation for any l-state. In this context, the purpose of this work is to apply the asymptotic iteration approach to investigate the non-relativistic treatment of Hulthn plus a class of Yukawa potential.
A combination of two potentials has aroused extensive research interest in literature. Many researchers have adopted this type of potential to carry out some works[8, 9, 10, 11, 12, 13, 14, 15]. Motivated by these works, we consider the following form of potential model which is the superposition of Hulthn[16] and a class of Yukawa[17] potentials

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$$V\left(r\right)=-\frac{2\alpha Ze^{2}e^{-2\alpha r}}{1-e^{-2\alpha r}}-\frac{Ae^{-\alpha r}}{r}-\frac{Be^{-2\alpha r}}{r^{2}} \tag{1.1}$$

where the parameter Z denotes the atomic number. α is the screening parameter which determines the range of potential, A and B are the coupling strengths of the potential. Hulthn plus a class of Yukawa potential has been newly proposed by Ahmadov et al.[15]. Bound state solutions of the Dirac equation under the spin and pseudospin symmetries for this potential including Coulomb-like tensor interaction have been presented in [15]. As far as we know, no report has been made so far in literature employing this combined potential within the framework of non-relativistic theory. For this reason, we focus on studying the model of a quantum system with Hulthn plus a class of Yukawa potential by means of asymptotic iteration method.

2. Overview of the asymptotic iteration method

The purpose of this section is to briefly outline the asymptotic iteration approach used to solve the second-order differential equations. The details of this approach have been reported in [5, 6, 7]. We start with the approach by writing a general form of the second-order differential equation

$$y^{''}\left(r\right)=\lambda_{0}\left(r\right)y^{'}\left(r\right)+s_{0}\left(r\right)y(r) \tag{2.1}$$

where $\lambda_0(r)$ and $s_0(r)$ functions in $C_\infty(a,b)$ are sufficiently differentiable. The general solution of Equation (2.1) can be obtained in the following form

$$y\left(r\right)=\exp(-\smallint^{r}\alpha(r')dr')\left[C_{2}+C_{1}\smallint^{r}\exp\left(\smallint^{r'}\left[\lambda_{0}(\tau)+2\alpha(\tau)\right]d\tau\right)dr'\right] \tag{2.2}$$

For sufficiently large k,

$$\frac{s_{k}(r)}{\lambda_{k}(r)}=\frac{s_{k-1}(r)}{\lambda_{k-1}(r)}=\alpha(r) \tag{2.3}$$

in which

$$\begin{aligned}\lambda_{k}\left(r\right)&=\lambda_{k-1}^{'}\left(r\right)+s_{k-1}\left(r\right)+\lambda_{0}(r)\lambda_{k-1}\left(r\right)\\ s_{k}\left(r\right)&=s_{k-1}^{'}\left(r\right)+s_{0}\left(r\right)\lambda_{k-1}\left(r\right)\end{aligned} \tag{2.4}$$

If the eigenvalue problem has exact analytical solutions, the termination condition Equation (2.3), or equivalently

$$\delta_{k}\left(r\right)=\lambda_{k}\left(r\right)s_{k-1}\left(r\right)-\lambda_{k-1}\left(r\right)s_{k}\left(r\right)=0 \tag{2.5}$$

produces, at each iteration, an expression that is independent of r. It is noted that k displays the iteration number. Physically meaningful solution of Equation (2.1) is provided by the first term of Equation (2.2) not the second term, so we can use the first term as the wave function generator

$$y\left(r\right)=C_{2}exp\left(-\int^{r}\frac{s_{k}\left(r'\right)}{\lambda_{k}\left(r'\right)}dr'\right) \tag{2.6}$$

in which C_2 denotes the integrant constant which can be determined by normalization.

There is also an alternative way to determine the wave function within the framework of AIM. The following second-order homogeneous linear differential equation allows us to find the wave function

$$y^{''}=2\left(\frac{ax^{N+1}}{1-bx^{N+2}}-\frac{(m+1)}{x}\right)y'-\frac{wx^{N}}{1-bx^{N+2}}y \tag{2.7}$$

in which N=-1,0,1... and a , w , m are the real numbers which are to be determined. The general solution of Eq.(2.7) is found in the following form

$$y_n(x) = (-1)^n C_2 (N+2)^n (\mu)_n {}_2F_1\left(-n,\ t+n;\ \mu; bx^{N+2}\right) \tag{2.8}$$

where $(\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(\mu)}$, $\mu = \frac{2m+N+3}{N+2}$, $t = \frac{(2m+1)b+2a}{(N+2)b}$ and ${}_2F_1$ denotes to the Gauss hypergeometric function being defined as

$${}_2F_1(-n,\ b, c, x) = \sum_{k=0}^{n} \frac{(-n)_k (b)_k x^k}{(c)_k k!}$$

the Pochhammer symbol $(\alpha)_k$ is defined by $(\alpha)_0 = 1$ and $(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\ldots L(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ for $k = 1, 2, 3\ldots$ It should be mentioned that the details concerning AIM can be found in [5, 6, 7].

3. Bound state solutions of Hulthn plus a class of Yukawa potential in approximate analytic form

Firstly, we focus on the separation of variables for the Schrödinger equation. The motion of a particle in central potential field is described in non-relativistic theory as follows

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)\right]\psi(\overrightarrow{r}) = E\psi(\overrightarrow{r}) \tag{3.1}$$

where E and μ define non-relativistic energy and reduced mass, $V(r)$ is the central potential, $\hbar$ is the Planck constant.

The following expression is the Laplace operator in three dimensions

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2(\sin\theta)^2}\frac{\partial^2}{\partial\varphi^2} \tag{3.2}$$

We employ a form of the total spherical wave function as

$$\psi(r,\theta,\varphi) = \frac{R(r)}{r} Y_l^m(\theta,\varphi) \tag{3.3}$$

where $R(r)$ and $Y_l^m(\theta,\phi)$ are the radial wave function and the spherical harmonics. The way of separating variables has been applied to Schrödinger equation. Based on this way, we obtain the non-relativistic wave equation with respect to $R(r)$

$$\frac{d^2R(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2}(E - V(r)) - \frac{l(l+1)}{r^2}\right]R(r) = 0 \tag{3.4}$$

When inserted Equation (1.1) into Equation (3.4), the radial Schrödinger wave equation becomes

$$\frac{d^2R(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2}\left(E - \left(-\frac{2Ze^2\alpha e^{-2\alpha r}}{(1-e^{-2\alpha r})} - \frac{Ae^{-\alpha r}}{r} - \frac{Be^{-2\alpha r}}{r^2}\right)\right) - \frac{l(l+1)}{r^2}\right]R(r) = 0 \tag{3.5}$$

Equation (3.5) cannot be solved analytically for any l-state because of the centrifugal term. Therefore, to solve this equation, we need to use an approximation of the following form

$$\frac{1}{r} \approx \frac{2\alpha e^{-\alpha r}}{(1-e^{-2\alpha r})},\ \frac{1}{r^2} \approx \frac{4\alpha^2 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \tag{3.6}$$

This scheme is called as the Greene-Aldrich approximation[18] which is only suitable for a short range (small α) potential. If we apply this approximation and transformation $x = e^{-2\alpha r}$ to Equation (3.5), then we can rewrite the radial wave equation in non-relativistic theory

$$\frac{d^2R(x)}{dx^2} + \frac{1}{x}\frac{dR(x)}{dx} + \left[-\frac{\gamma^2}{x^2} + \frac{\beta^2}{x\,(1-x)} - \frac{\sigma^2}{(1-x)^2} - \frac{l(l+1)}{x(1-x)^2}\right] R\,(x) = 0 \quad (3.7)$$

In Equation (3.7), we take the abbreviations as

$$\beta^2 = \frac{\mu\,(V_0 + V_0')}{2\alpha^2\hbar^2}, \gamma^2 = -\frac{\mu E}{2\alpha^2\hbar^2}, \sigma^2 = -\frac{\mu B'}{2\alpha^2\hbar^2} \quad (3.8)$$

For simplicity, we take as $V_0 = 2\alpha Z e^2$, $V_0' = 2A\alpha$ and $B' = 4B\alpha^2$ in above expressions. By analyzing the asymptotic behaviour of Equation (3.7) at the origin and infinity, we can propose the wave function in terms of $R(x)$

$$R\,(x) = x^{\gamma}(1-x)^{\delta+1} f(x) \quad (3.9)$$

with

$$\delta = -\frac{1}{2} + \frac{\sqrt{1 + 4\sigma^2 + 4l(l+1)}}{2} \quad (3.10)$$

After taking the proposed wave function given in Equation (3.9) and inserting this into Equation (3.7), we obtain the second-order homogeneous linear differential equation as

$$\frac{d^2f(x)}{dx^2} = \left[\frac{(2\gamma + 2\delta + 3)\,x - (2\gamma + 1)}{x(1-x)}\right]\frac{df(x)}{dx} + \left[\frac{(2\gamma + 1)\,(\delta + 1) - \beta^2 + l(l+1)}{x(1-x)}\right] f(x) \quad (3.11)$$

This equation is convenient to apply the asymptotic iteration approach. Comparison of Equation (3.11) and Equation (2.1) gives the values of λ_0 and s_0. With Equation (2.4), it is then easy to obtain the values of $\lambda_n(x)$ and $s_n(x)$ in the following forms

$$\begin{aligned}
\lambda_0 &= \frac{(2\gamma + 2\delta + 3)\,x - (2\gamma + 1)}{x(1-x)}\\
s_0 &= \frac{(2\gamma + 1)\,(\delta + 1) - \beta^2 + l(l+1)}{x(1-x)}\\
\lambda_1 &= \frac{(2\gamma + 2\delta + 3)\,x}{x(1-x)} - \frac{(2\gamma + 2\delta + 3)\,x - (2\gamma + 1)}{x^2\,(1-x)} + \frac{(2\gamma + 2\delta + 3)\,x - (2\gamma + 1)}{x(1-x)^2}\\
&\quad + \frac{(1+\delta)\,(2\gamma + 1) + l\,(l+1) - \beta^2}{x\,(1-x)} + \frac{((2\gamma + 2\delta + 3)\,x - (2\gamma + 1))^2}{x^2(1-x)^2}\\
s_1 &= -\frac{(1+\delta)\,(2\gamma + 1) + l\,(l+1) - \beta^2}{x^2\,(1-x)} + \frac{(1+\delta)\,(2\gamma + 1) + l\,(l+1) - \beta^2}{x(1-x)^2}\\
&\quad + \frac{\left((1+\delta)\,(2\gamma + 1) + l\,(l+1) - \beta^2\right)((2\gamma + 2\delta + 3)\,x - (2\gamma + 1))}{x^2(1-x)^2}\\
&\vdots
\end{aligned} \quad (3.12)$$

To calculate the radial energy eigenvalues, we employ the termination condition given by Equation (2.3). Thus, these energy eigenvalues are obtained as

$$\frac{s_0}{\lambda_0} = \frac{s_1}{\lambda_1} \Rightarrow \gamma_0 = -\frac{l(l+1) - \beta^2 + \delta + 1}{2\,(\delta + 1)}$$

$$
\begin{aligned}
&\frac{s_1}{\lambda_1}=\frac{s_2}{\lambda_2} \Rightarrow \gamma_1=-\frac{l(l+1)-\beta^2+3\delta+4}{2\,(\delta+2)}\\
&\frac{s_2}{\lambda_2}=\frac{s_3}{\lambda_3} \Rightarrow \gamma_2=-\frac{l(l+1)-\beta^2+5\delta+9}{2\,(\delta+3)}\\
&\qquad\vdots
\end{aligned}
\tag{3.13}
$$

Based on the preceding expressions, we can generalize in the following form

$$
\gamma_n=-\frac{l(l+1)-\beta^2+(2n+1)\delta+(n+1)^2}{2\,(\delta+n+1)}, n=0,1,2,\ldots \tag{3.14}
$$

Substituting the values of γ, β and σ given in Equation (3.8) and the value of δ given in Equation (3.10) into Equation (3.14), it can be built as

$$
E=-\frac{2\alpha^2\hbar^2}{\mu}\left(\frac{\frac{\mu}{\hbar^2\alpha}\left(Ze^2+A\right)-l\,(l+1)-\left(n^2+n+\frac{1}{2}\right)-(2n+1)\sqrt{\frac{1}{4}-\frac{2\mu B}{\hbar^2}+l\,(l+1)}}{2n+1+2\sqrt{\frac{1}{4}-\frac{2\mu B}{\hbar^2}+l\,(l+1)}}\right)^2 \tag{3.15}
$$

Thus, we find the energy spectrum for Hulthn plus a class of Yukawa potential in spherical coordinates. By comparing with Equation (3.11) and Equation (2.7) and following expressions below Equation (2.8), we can easily find as

$$
\begin{gathered}
b=1, N=-1, a=\delta+1, m=2\gamma-1\\
\mu=2\gamma+1, \mathrm{t}=2\gamma+2\delta+2
\end{gathered}
\tag{3.16}
$$

Directly the function of $f(x)$ can be obtained from Equation (2.8) with the substitution of Equation (3.16) in the following form

$$
f\,(x)=(-1)^n C_2 (2\gamma+1)_n {}_2F_1\,(-n,\;\; 2\gamma+2\delta+2+n; 2\gamma+1\;; x\;) \tag{3.17}
$$

If we put Equation (3.17) into Equation (3.9), we can obtain the unnormalized radial wave function for Hulthn plus a class of Yukawa potential

$$
\begin{aligned}
R\,(x)=&(-1)^n C_2 (2\gamma+1)_n (1-x)^{\delta+1} x^{\gamma}\\
&\times\; {}_2F_1\quad(-n,\;\; 2\gamma+2\delta+2+n; 2\gamma+1\;; x\;)
\end{aligned}
\tag{3.18}
$$

Then, substituting $x=e^{-2\alpha r}$ into Equation (3.18), we write the unnormalized radial wave function for considered potential with respect to r

$$
\begin{aligned}
R\,(r)=&(-1)^n C_2 (2\gamma+1)_n \left(1-e^{-2\alpha r}\right)^{\delta+1} e^{-2\alpha\gamma r}\\
&\times {}_2F_1\left(-n,\;\; 2\gamma+2\delta+2+n; 2\gamma+1\;; e^{-2\alpha r}\;\right)
\end{aligned}
\tag{3.19}
$$

in which C_2 is the integration constant.

4. Conclusion

We consider Hulthn plus a class of Yukawa potential because of the importance of the combined potentials. In this connection, bound state solutions of the Schrödinger equation have been established for any l-state within the framework of asymptotic iteration method. To achieve this, we apply a proper approximation scheme which is called as Greene-Aldrich approximation. Therefore, we construct the energy eigenvalues and unnormalized wave function in approximate analytic form. We note that the theoretical results obtained for the considered potential may shed light on the applications in different fields.

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SOME RESULTS ON THE STUDY OF Ξ-HILFER TYPE FUZZY FRACTIONAL DIFFERENTIAL EQUATIONS WITH TIME DELAY

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Abstract. This paper is concerned with the finite-time stability of Ξ-Hilfer type fuzzy fractional differential equations (FFDEs) with time delay. By applying standard theorems and a hypothetical condition, we explore the existence of solution and stabilty results.

1. Introduction

In this manuscript, we will explore the existence and stabilty of the following Ξ-Hilfer type FFDE with time delay

$$\begin{cases} \mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi} w(t) = g(t, w_t), \quad t \in (0,b], \\ \mathscr{I}_{0^+}^{1-\gamma,\Xi} w(0^+) = w_0, \quad \gamma = \zeta_1 + \zeta_2 - \zeta_1\zeta_2, \\ w(t) = \chi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where $w \in \mathbb{R}^c$, $g : [0,b] \times C([-\tau,b], E_c) \to E_c$ is fuzzy function, where $\chi \in C([-\tau,0], E_c)$ and E_c is the space of fuzzy sets. Moreover $\mathscr{I}_{0^+}^{1-\gamma,\Xi}$, $\mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi}$ denotes the Ξ-Hilfer fractional integral and derivative of order $\zeta_1 \in (0,1)$ and type $\zeta_2 \in [0,1]$. Compared to the literature [1] to [35], the main contributions and novality of this paper are reflected in the following aspects:

(i) The system (1.1) has delay terms, which can be truly reflected the object process of change.

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(ii) In view of different systems, although the method used to study the existence and stability, but there are many differences in the processes of proof.
(iii) We are able to prove time-stability by using new assumptions.

2. Elementary results

This section of research paper is devoted to basic results and definitions that we need for investigation of the main results.

Let us take $\mathfrak{J} = [0, b]$. Let $C([-\tau, b], E_c)$ be the family of all continuous fuzzy functions from $[-\tau, b]$ into E_c, which endowed with the supremum metric: $D_{[-\tau,b]}[w, \widehat{0}] = \sup_{t\in[-\tau,b]} D_0[w(t), \widehat{0}]$ and $AC(\mathfrak{J}, E_c)$ be the family of all absolutely continuous fuzzy functions on the interval $\mathfrak{J}$ with the values in E_c. Let $\gamma \in (0, 1)$, by $C_{\gamma,\Xi}(\mathfrak{J}, E_c)$. We denote the family of continuous functions defined by $C_{\gamma,\Xi}(\mathfrak{J}, E_c) = \{w : (0, b] \to E_c|(\Xi(t) - \Xi(0))^{1-\gamma}w(t) \in C(\mathfrak{J}, E_c)\}$.

Let E_c denote the space of all fuzzy numbers on $\mathbb{R}^c$, if $w : \mathbb{R}^c \to [0, 1]$ satisfies normal, convex, upper semicontinuous and compactly supported.
The q-level set of w is defined by

$$\begin{aligned}
&[w]^q = \{t \in \mathbb{R}^c : w(t) \geq q\}, \quad q \in [0, 1] \quad \text{and}\\
&[w]^0 = \{t \in \mathbb{R}^c | w(t) > 0\}.
\end{aligned}$$

It follows that the q-level set of $w \in E_c$, $[w]^q$ is a nonempty compact interval, for any $q \in [0, 1]$. We denote by $[\underline{w}(q), \overline{w}(q)]$ the q-level of a fuzzy number w.

Definition 2.1. [12] *Ler w_1 and w_2 be two fuzzy sets defined on E_c and $\mu \in \mathbb{R}^c$. Due to Zadeh's extension principle, $w_1 + w_2$ and μw_1 are in E_c and defined as*

$$\begin{aligned}
[w_1 + w_2]^q =& [w_1]^q + [w_2]^q,\\
[\mu w]^q =& \mu[w]^q, \quad \textit{for all} \quad q \in [0, 1],
\end{aligned}$$

where $[w_1]^q + [w_2]^q$ represents the usual addition of two intervals of $\mathbb{R}^c$ and $\mu[w_1]^q$ represents the usual scalar product between μ and an real interval.

Definition 2.2. [12] *The distance $D_0[w_1, w_2]$ between two fuzzy numbers is defined by*

$$D_0[w_1, w_2] = \sup_{0\leq q\leq 1} H([w_1]^q, [w_2]^q) \quad \textit{for all} \quad w_1, w_2 \in E_c, \tag{2.1}$$

where $H([w_1]^q, [w_2]^q) = max\{|\underline{w_1}(q) - \underline{w_2}(q)|, |\overline{w_1}(q) - \overline{w_2}(q)|\}$ is the Hausdroff distance between $[w_1]^q$ and $[w_2]^q$.

Definition 2.3. [12] *Let $w_1, w_2 \in E_c$. There exists $w_3 \in E_c$ such that $w_1 = w_2+w_3$, that is., $w_3 = w_1 \ominus w_2$, where w_3 is Hukuhara difference of w_1 and w_2.*
The generalized Hukuhara difference of two fuzzy numbers $w_1, w_2 \in E_c$ [gH-difference] is defined as

$$w_1 \ominus_{gH} w_2 = w_3 \Leftrightarrow \begin{cases} (i) w_1 = w_2 + w_3, \quad or\\ (ii) w_2 = w_1 + (-1)w_3, \end{cases} \tag{2.2}$$

where $w_1 \ominus_{gH} w_2$ is called as gH-difference of w_1 and w_2 in E_c.

In the q-levels, we have that for all $q \in [0, 1]$,

$$[w_1 \ominus_{gH} w_2]^q = [min\{\underline{w_1}(s) - \underline{w_2}(s), \overline{w_1}(s) - \overline{w_2}(s)\}, \tag{2.3}$$

$$max\{\underline{w_1}(s) - \underline{w_2}(s), \overline{w_1}(s) - \overline{w_2}(s)\}]. \tag{2.4}$$

Also, the condition for the existence of $w_1 \ominus_{gH} w_2$ *in the case(i) is* $d([w_1]^q) \geq d([w_2]^q)$*, and the condition for the existence of* $w_1 \ominus_{gH} w_2$ *in the case(ii) is* $d([w_2]^q) \geq d([w_1]^q)$*.*

Definition 2.4. [12] *A function* $w : [0, b] \to E_c$ *is said to be d-increasing (d-decreasing) on* $[0, b]$ *if for every* $q \in [0, 1]$ *the function* $t \mapsto d[w(t)]^q$ *is nondecreasing (nonincreasing) on* $[0, b]$*. Let* w *be a d-increasing or d-decreasing on* $[0, b]$*, then we say that* w *is d-monotone on* $[0, b]$*.*

Definition 2.5. [12] *The generalized Hukuhara derivative of a fuzzy-valued function* $w : (0, b) \to E_c$ *at* t *is defined as*

$$w'_{gH}(t) = \lim_{h \to 0} \frac{w(t+h) \ominus_{gH} w(t)}{h},$$

if $w'_{gH}(t) \in E_c$*, we say that* w *is generalized Hukuhara differentiable (gH-differentiable) at* t*.*

Moreover, we say that w *is* $[(i) - gH]$*-differentiable at* t *if*

$$\begin{aligned}[w'_{gh}(t)]^q &= \left[\left[\lim_{h \to 0} \frac{\underline{w}(t+h) \ominus_{gH} \underline{w}(t)}{h}\right]^q, \left[\lim_{h \to 0} \frac{\overline{w}(t+h) \ominus_{gH} \overline{w}(t)}{h}\right]^q\right], \\ &= [(\underline{w})'(q, t), (\overline{w})'(q, t)], \end{aligned} \tag{2.5}$$

and that w *is* $[(ii) - gH]$*-differentiable at* t *if*

$$[w'_{gH}(t)]^q = [(\overline{w})'(q, t), (\underline{w})'(q, t)]. \tag{2.6}$$

Definition 2.6. [12] *Let us consider* $w \in \mathscr{L}(\mathfrak{J}, E_c)$ *as a fuzzy function and* $\zeta_1 \in (0, 1)$*, then the fuzzy* Ξ*-type Riemann-Liouville integral of fuzzy-valued function* w *is defined as follows:*

$$(\mathscr{I}_{0^+}^{\zeta_1, \Xi} w)(t) = \frac{1}{\Gamma(\zeta_1)} \int_0^t \Xi'(s)(\Xi(t) - \Xi(s))^{\zeta_1 - 1} w(s) ds, \quad \text{for all} \quad t \in \mathfrak{J}, \tag{2.7}$$

where $\Gamma(\zeta_1)$ *is the Gamma function.*

Definition 2.7. [12] *Let* $w : \mathfrak{J} \to E_c$ *be a continuous fuzzy mapping. The fuzzy* Ξ*-type Riemann-Liouville fractional derivative of order* $n-1 < \alpha < n$ *for fuzzy-valued function* w *is defined by*

$$(\mathscr{D}_{0^+}^{\zeta_1, \Xi} w)(t) = \frac{1}{\Gamma(n - \zeta_1)} \left(\frac{1}{\Xi'(t)} \frac{d}{dt}\right)^n \int_0^t \Xi'(s)(\Xi(t) - \Xi(s))^{n - \zeta_1 - 1} w(s) ds, \forall \quad t \in \mathfrak{J}. \tag{2.8}$$

If $w \in C(\mathfrak{J}, E_c)$*, then the* Ξ*-Hilfer fractional integral of order* ζ_1 *of the fuzzy-valued function* w *is defined as follows:*

$$w_{\zeta_1, \Xi}(t) = (\mathscr{I}_{0^+}^{\zeta_1, \Xi} w)(t) = \frac{1}{\Gamma(\zeta_1)} \int_0^t \Xi'(t)(\Xi(t) - \Xi(s))^{\zeta_1 - 1} w(s) ds.$$

Since $[w(t)]^q = [\underline{w}(q, t), \overline{w}(q, t)]$ *and* $0 < \zeta_1 < 1$*, let us consider the fuzzy* Ξ*-fractional integral of the fuzzy-valued function* w *based on lower and upper functions, that is,*

$$[(\mathscr{I}_{0^+}^{\zeta_1, \Xi} w)(t)]^q = [(\mathscr{I}_{0^+}^{\zeta_1, \Xi} \underline{w})(q, t), (\mathscr{I}_{0^+}^{\zeta_1, \Xi} \overline{w})(q, t)],$$

where

$\big(\mathscr{I}_{0^+}^{\zeta_1,\Xi}\underline{w}\big)(q,t) = \frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}\underline{w}(q,s)(s)ds,$

and

$\big(\mathscr{I}_{0^+}^{\zeta_1,\Xi}\overline{w}\big)(q,t) = \frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}\overline{w}(q,s)(s)ds.$

In addition, it follows that the opeartor $w_{\zeta_1,\Xi}(t)$ is linear and bounded from $C([\mathfrak{J},E_c)$ to $C(\mathfrak{J},E_c)$. Indeed, we have

$$c \leq \|w\|_0\frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}ds = \frac{\|w\|_0}{\Gamma(\zeta_1+1)}\big(\Xi(t)-\Xi(0)\big)^{\zeta_1},$$

where $\|w\|_0 = \sup_{t\in\mathfrak{J}} D_0[w(t),\widehat{0}]$.

Definition 2.8. [12] *Let order ζ_1 and type ζ_2 satisfy $n-1<\zeta_1\leq n$ and $0\leq\zeta_2\leq 1$, with $n\in N$. The fuzzy Ξ-Hilfer generalized Hukuhara fractional derivative(or Ξ-Hilfer gH-fractional derivative) (left-sided/right-sided), with respect to t, with a function $w\in C_{1-\gamma,\Xi}(\mathfrak{J},E_c)$, is defined as follows:*

$$\begin{aligned}\big(\mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi}w\big)(t) =&\big(\mathscr{I}_{0^+}^{\zeta_2(1-\zeta_1),\Xi}\big)\left(\frac{1}{\Xi'(t)}\frac{d}{dt}\right)\big(\mathscr{I}_{0^+}^{(1-\zeta_2)(1-\zeta_1),\Xi}w\big)(t)\\ =&\big(\mathscr{I}_{0^+}^{\zeta_2(1-\zeta_1),\Xi}f^{\Xi}\mathscr{I}_{0^+}^{(1-\zeta_2)(1-\zeta_1),\Xi}w\big)(t),\end{aligned}$$

if the gH-derivative $w^{'}_{(1-\zeta_1),\Xi}(t)$ exists for $t\in\mathfrak{J}$, where

$$w_{(1-\zeta_1),\Xi}(t) := \big(\mathscr{I}_{0^+}^{(1-\zeta_1),\Xi}w\big)(t) = \frac{1}{\Gamma(1-\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{-\zeta_1}w(s)ds.$$

Definition 2.9. [11] *Let $\zeta_1>0$, $\zeta_2>0$. Then the two parameters Mittag-Leffler function is defined as*

$$\mathbb{E}_{\zeta_1,\zeta_2}(w) = \sum_{k=0}^{\infty}\frac{w^k}{\Gamma(\zeta_1 k+\zeta_2)},\quad w\in E_c. \tag{2.9}$$

If $\zeta_2=1$, the one-parameter Mittag-Leffler function defined by

$$\mathbb{E}_{\zeta_1}(w) = \sum_{k=0}^{\infty}\frac{w^k}{\Gamma(\zeta_1 k+1)},\quad w\in E_c,\zeta_1>0. \tag{2.10}$$

Definition 2.10. [11] *The fuzzy problem (1.1) is said to be finite time stable with respect to $\{0,J,\tau,\sigma,\epsilon\}$, $0<\delta_1<\epsilon$, $\epsilon\in E_c$, such that for any solution w of fuzzy problem (1.1), if and if $D_0[w_0,\widehat{0}]<\sigma$ and $D_0[\chi,\widehat{0}]<\sigma$, implies a solution w of fuzzy problem (1.1) satisfying $D_{\mathfrak{J}}^{\gamma}[w,\widehat{0}]<\epsilon$.*
For our convenience, we define $\mathscr{N}(w)=\{w\in C_\gamma(\mathfrak{J},E_c).\ w$ satisfies (3.1)$\}$.

Lemma 2.1. [12] *Let $\zeta_1,\zeta_2,\vartheta_1>0$. Then*

(i) $\mathscr{I}_{0^+}^{\zeta_1\Xi}\mathscr{I}_{0^+}^{\zeta_2,\Xi}w(t) = \mathscr{I}_{0^+}^{\zeta_1,\zeta_2}w(t).$

(ii) $\mathscr{I}_{0^+}^{\zeta_1,\Xi}(\Xi(t)-\Xi(0))^{\vartheta_1-1} = \dfrac{\Gamma(\vartheta_1)}{\Gamma(\zeta_1+\vartheta_1)}(\Xi(t)-\Xi(0))^{\zeta_1+\vartheta_1-1.}$

Lemma 2.2. [12] *Let* $\zeta_1 > 0$, $0 \leq \gamma < 1$. *If* $w \in C_{\gamma,\Xi}[0,b]$ *and* $\mathscr{I}_{0^+}^{1-\zeta_1,\Xi}w \in C^1_{\gamma,\Xi}[0,b]$, *then*

$$\mathscr{I}_{0^+}^{\zeta_1,\Xi}\mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi}w(t) = w(t) - \frac{\mathscr{I}_{0^+}^{1-\zeta_1,\Xi}w(t)}{\Gamma(\zeta_1)}(\Xi(t)-\Xi(0))^{\zeta_1-1}.$$

Lemma 2.3. [12] *Let* $w \in L^1(0,b)$. *If* $\mathscr{D}_{0^+}^{\zeta_2(1-\zeta_1),\Xi}w$ *exists on* $L^1(0,b)$, *then*

$$\mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi}\mathscr{I}_{0^+}^{\zeta_1,\Xi}w = \mathscr{I}_{0^+}^{\zeta_2(1-\zeta_2),\Xi}\mathscr{D}_{0^+}^{\zeta_2(1-\zeta_1),\Xi}w, \quad \text{for all} \quad t \in (0,b].$$

Theorem 2.4. [25]**(Schauder fixed point theorem)** *Let* $H \neq 0$ *be a bounded, closed, convex subset of a fuzzy Banach space in* X. *If* $T : H \to H$ *be a continuous compact operator. Then,* T *has at least one fixed point in* H.

Lemma 2.5. [11]**(Generalized Gronwall's Inequality)** *Let* $\zeta_1 > 0$ *and* $x_1(t), x_2(t)$ *be two nonnegative function locally integrable on* $[0,T]$. *Assume that* g *is nonnegative and nondecreasing, and let* $\Xi \in C^1([0,T], E_c)$ *an increasing function such that* $\Xi^{'}(t) \neq 0$ *for all* $t \in [0,T]$. *If*

$$x_1(t) \leq x_2(t) + g(t)\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1}x_1(s)ds, \quad t \in [0,T].$$

Then

$$x_1(t) \leq x_2(t) + \int_0^t \sum_{n=1}^{\infty}\frac{[g(t)\Gamma(\zeta_1)]}{\Gamma(n\zeta_1)}\Xi^{'}(s)(\Xi(t)-\Xi(s))^{n\zeta_1}x_2(s)ds, \quad t \in [0,T].$$

If x_2 *be a nondecreasing function on* $[0,T]$. *Then*

$$x_1(t) \leq x_2(t)\mathbb{E}_{\zeta_1}\{g(t)\Gamma(\zeta_1)[\Xi(t)-\Xi(0)]^{\zeta_1}\}, \quad t \in [0,T].$$

Lemma 2.6. [12] *Let* $g : (0,b] \times E_c \to E_c$ *be a continuous fuzzy function. Then the following problem*

$$\begin{cases} \mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi}w(t) = g(t,w_t), \quad t \in (0,b], \\ \mathscr{I}_{0^+}^{1-\gamma,\Xi}w(0^+) = w_0, \quad \gamma = \zeta_1+\zeta_2-\zeta_1\zeta_2, \end{cases}$$

is equivalent to integral equation

$$w(t) = \frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0 + \frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1}g(s,w_s)ds.$$

3. Existence and stability theory

In this section, we estabilish and demonstrate the existence and stabilty of (1.1). We assume the following assumptions before begining and examining the key outcomes. $(A1)$ There exists a positive constants $\mathscr{L}$ such that

$$D_0[g(t,w),\widehat{0}] \leq \mathscr{L}(\Xi(t)-\Xi(0))^{1-\gamma}D_{[-\tau,0]}[w,\widehat{0}],$$
$$\text{for all} \quad w \in C([-\tau,0],E_c), w \in E_c, t \in \mathfrak{J}.$$

with $\mathscr{L} \in \big[0, \Gamma(\zeta_1+1)\big(\frac{1}{\Xi(b)-\Xi(0)}\big)^{1+\zeta_1-\gamma}\big]$.
$(A2)$ There exists a positive constants $\mathscr{L}^*$ such that

$$\begin{aligned} D_0[g(t,w_t),g(t,w_t^*)] &\leq \mathscr{L}^*D_{[-\tau,0]}[w_t,w_t^*] \\ &= \mathscr{L}^*D_{[t-\tau,t]}[w,w^*], \quad \text{for all} \quad w,w^* \in E_c. \end{aligned}$$

Lemma 3.1. *Let $g : [0, b] \times C([-\tau, 0], E_c) \to E_c$ be a continuous fuzzy function, $\chi \in C([-\tau, 0], E_c)$. Then a d- monotone fuzzy function $w \in C(\mathfrak{J}, E_c)$ is a solution of initial value problem (1.1) if and only if w satisfies the integral equation*

$$w(t) \ominus_{gH} \frac{(\Xi(t) - \Xi(0))^{\gamma-1}}{\Gamma(\gamma)} w_0 = \frac{1}{\Gamma(\zeta_1)} \int_0^t \Xi^{'}(s)(\Xi(t) - \Xi(s))^{\zeta_1 - 1} g(s, w_s) ds, \quad t \in \mathfrak{J} \tag{3.1}$$

and $w(t) = \chi(t)$, $t \in [-\tau, 0]$, and the fuzzy function $t \mapsto \mathscr{I}_{0^+}^{1-\gamma,\Xi} g(t, w_t)$ is d-increasing on $\mathfrak{J}$.

Proof. Let us assume $w \in C(\mathfrak{J}, E_c)$ be a d-monotone solution of (1.1) and let $y(t) = w(t) \ominus_{gH} (\mathscr{I}_{0^+}^{1-\gamma,\Xi})$, $t \in \mathfrak{J}$. Since w is d-monotone on $\mathfrak{J}$, it follows that $t \mapsto y(t)$ is d-increasing on $\mathfrak{J}$. It follow from (1.1) and Lemma 2.12 we have

$$\mathscr{I}_{0^+}^{\zeta_1,\Xi} \mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi} w(t) = w(t) \ominus_{gH} \frac{w_0}{\Gamma(\gamma)} (\Xi(t) - \Xi(0))^{1-\gamma}, \quad t \in \mathfrak{J}.$$

Since $g(t, w) \in C_{\gamma,\Xi}(\mathfrak{J}, E_c)$ for any $w \in E_c$ and by using the Eqn.(1.1), it follows that

$$\begin{aligned} \mathscr{I}_{0^+}^{\zeta_1,\Xi} \mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi} w(t) &= \mathscr{I}_{0^+}^{\zeta_1,\Xi} g(t, w_t) \\ &= \frac{1}{\Gamma(\zeta_1)} \int_0^t \Xi^{'}(s)(\Xi(t) - \Xi(s))^{\zeta_1 - 1} g(s, w_s) ds, \quad t \in \mathfrak{J}. \end{aligned}$$

In addition, since $y(t)$ is d-increasing on $(0, b]$, due to $t \mapsto g_{\zeta_1,\Xi}(t, w)$ is also d-increasing on $(0, b]$. We obtain that

$$w(t) \ominus_{gH} \frac{(\Xi(t) - \Xi(0))^{\gamma-1}}{\Gamma(\gamma)} w_0 = \frac{1}{\Gamma(\zeta_1)} \int_0^t \Xi^{'}(s)(\Xi(t) - \Xi(s))^{\zeta_1 - 1} g(s, w_s) ds, \quad t \in \mathfrak{J}$$

For every $t \in [-\tau, 0]$, we have $w(t) = \chi(t)$. This implies that (3.1) is satisfied.
Conversely, assume that $w \in C(\mathfrak{J}, E_c)$ satisfies (1.1). If $t \in [0, b]$, then $w(0^+) = w_0$, and applying $\mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi}$ on both sides, we obtain
$\mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi} w(t) = g(t, w_t), \quad t \in (0, b]$.
And we can easily prove that $w(t) = \chi(t)$ for $t \in [-\tau, 0]$. □

Lemma 3.2. *Let $g : [0, b] \times C([-\tau, 0], E_c) \to E_c$ be a continuous fuzzy function, $\chi \in C([-\tau, 0], E_c)$. Assume that (A1) is satisfied. Then for any $w \in C([-\tau, b], E_c)$ of Eqn.(1.1), there exists a constant $\eta > 0$ such that $D_{[-\tau,b]}[w, \widehat{0}] \leq \eta$.*

Proof. Let us assume $w \in C([-\tau, 0], E_c)$. If $t \in [-\tau, 0]$, then we have that $w(t) = \chi(t)$. In according to the boundedness of χ, which gives $w(t)$ is bounded.
Suppose $t \in \mathfrak{J}$, which is $w \in \mathscr{N}(y)$. Then, for $\xi \in [0, t]$, $t \in (0, b]$, it follows that , we have

$$\begin{aligned} D_{[-\tau,0]}[w_\xi, \widehat{0}] &= \sup_{\theta \in [-\tau,0]} D_0[y_\xi(\theta), \widehat{0}] \\ &= \sup_{\theta \in [-\tau,0]} D_0[y(\xi + \theta), \widehat{0}] \\ &\leq \sup_{r \in [-\tau,0]} D_0[y_r, \widehat{0}] + \sup_{r \in [0,\xi]} D_0[y(r), \widehat{0}] \\ &\leq D_{[-\tau,0]}[\chi, \widehat{0}] + \sup_{r \in [0,\xi]} D_0[y_r, \widehat{0}]. \end{aligned} \tag{3.2}$$

Hence, for $t \in (0, b]$, by using (3.1), (3.2), $(A1)$, Definition 2 and the Beta Function $B(\cdot,\cdot)$, we have

$$
\begin{aligned}
&D_0[(\Xi(t)-\Xi(0))^{1-\gamma}w(t),\widehat{0}]\\
&\quad\le D_0\bigg[(\Xi(t)-\Xi(0))^{1-\gamma}\bigg(\frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0\bigg),\widehat{0}\bigg]\\
&\quad+D_0\bigg[(\Xi(t)-\Xi(0))^{1-\gamma}\frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}g(s,w_s)ds,\widehat{0}\bigg]\\
&\quad\le \frac{1}{\Gamma(\gamma)}D_0[w_0,\widehat{0}]\\
&\quad+\frac{(\Xi(b)-\Xi(0))^{1-\gamma}}{\Gamma(\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}\{\mathscr{L}(\Xi(s)-\Xi(0))^{1-\gamma}D_{[-\tau,0]}[w_s,\widehat{0}]\}ds\\
&\quad\le \frac{1}{\Gamma(\gamma)}D_0[w_0,\widehat{0}]\\
&\quad+\mathscr{L}(\Xi(b)-\Xi(0))^{1-\gamma}\bigg\{D_{[-\tau,0]}[\chi,\widehat{0}]\frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}(\Xi(s)-\Xi(0))^{1-\gamma}ds\\
&\quad+\frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}\sup_{r\in[0,\xi]}D_0[(\Xi(s)-\Xi(0))^{1-\gamma}y_s,\widehat{0}]\bigg\}\\
&\quad\le \frac{1}{\Gamma(\gamma)}D_0[w_0,\widehat{0}]\\
&\quad+\frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma,\zeta_1)D_{[-\tau,0]}[\chi,\widehat{0}]\\
&\quad+\frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{1-\gamma}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}\sup_{r\in[0,\xi]}D_0[(\Xi(s)-\Xi(0))^{1-\gamma}y_s,\widehat{0}]ds.
\end{aligned}
$$

It follows from the generalized Gronwall inequality gives that,

$$N(t) \le M^*\mathbb{E}_{\zeta_1}(\mathscr{L}(\Xi(b)-\Xi(0))^{1-\gamma}(\Xi(t)-\Xi(0))^{\zeta_1}) = \eta,$$

where

$$
\begin{aligned}
N(t) &= \sup_{r\in[0,\xi]}D_0[(\Xi(s)-\Xi(0))^{1-\gamma}y_s,\widehat{0}],\\
M^* &= \frac{1}{\Gamma(\gamma)}D_0[w_0,\widehat{0}]+\frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma,\zeta_1)D_{[-\tau,0]}[\chi,\widehat{0}].
\end{aligned}
$$

This implies that, there exists a constant $\eta > 0$ such that $D_{[-\tau,b]}[w,\widehat{0}] \le \eta$. □

Theorem 3.3. *Let $g : [0,b]\times C([-\tau,0],E_c) \to E_c$ be a continuous fuzzy function, $\chi \in C([-\tau,0],E_c)$. Assume that $(A1)$ is satisfied. Then the fuzzy problem (1.1) has at least one solution $w \in C([-\tau,b],E_c)\cap C_\gamma(\mathfrak{J},E_c)$.*

Proof. Let us define the operator $\Theta : C([-\tau,b],E_c) \to C([-\tau,b],E_c)\cap C_\gamma(\mathfrak{J},E_c)$ is given by

$$
(\Theta w)(t) = \begin{cases}
(Tw)(t) \ominus_{gH} \frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0 = \frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}g(s,w_s)ds,\\
t\in\mathfrak{J},\\
w(t)=\chi(t), \quad t\in[-\tau,0],
\end{cases}
$$

where $T : C_\gamma(\mathfrak{J}, E_c) \to C_\gamma(\mathfrak{J}, E_c)$, let us assume $w \in C([-\tau, 0], E_c)$. Because $w(t) = \chi(t)$, $t \in [-\tau, 0]$.

Step 1. $T(B_{\eta_1}) \subseteq B_{\eta_1}$.

Let us define a bounded, closed and convex set $B_{\eta_1} \in C_{1-\gamma}[0, b]$ as follows

$$B_{\eta_1} = \{y \in C_\gamma(\mathfrak{J}, E_c) | D^\gamma_{[0,b]}[y, \widehat{0}] \leq \eta_1\}.$$

with

$$\eta_1 \geq max\Bigg\{\left(\frac{1}{\Gamma(\gamma)}D_0[w_0, \widehat{0}] + \frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b) - \Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma, \zeta_1)D_{[-\tau,0]}[\chi, \widehat{0}]\right)$$
$$\times \frac{\Gamma(\zeta_1+1)}{\Gamma(\zeta_1+1) - \mathscr{L}(\Xi(b) - \Xi(0))^{1+\zeta_1-\gamma}}, \eta\Bigg\}.$$

If $w \in B_{\eta_1}$. Then, for $r \in [0, t]$, $t \in (0, b]$, we get

$$\begin{aligned} D_{[-\tau,0]}[w_t, \widehat{0}] &\leq \sup_{r\in[-\tau,0]} D_0[w_t(s), \widehat{0}] \\ &= \sup_{\xi\in[t-\tau,t]} D_0[w(\xi), \widehat{0}] \\ &\leq D_{[-\tau,0]}[\chi, \widehat{0}] + D_{[0,b]}[w, \widehat{0}]. \end{aligned} \tag{3.3}$$

Therefore, for each $t \in (0, b]$, we get

$$\begin{aligned} &D_0[(\Xi(t) - \Xi(0))^{1-\gamma}(Tw)(t), \widehat{0}] \\ &\leq \frac{1}{\Gamma(\gamma)}D_0[w_0, \widehat{0}] + \frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b) - \Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma, \zeta_1)D_{[-\tau,0]}[\chi, \widehat{0}] \\ &+ \frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b) - \Xi(0))^{1-\gamma}\int_0^t \Xi'(s)(\Xi(t) - \Xi(s))^{\zeta_1-1} \sup_{r\in[0,\xi]} D_0[(\Xi(s) - \Xi(0))^{1-\gamma}y_s, \widehat{0}]]ds \\ &\leq \frac{1}{\Gamma(\gamma)}D_0[w_0, \widehat{0}] + \frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b) - \Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma, \zeta_1) + D_{[-\tau,0]}[\chi, \widehat{0}] \\ &+ \frac{\kappa}{\Gamma(\zeta_1+1)}(\Xi(b) - \Xi(0))^{1+\zeta_1-\gamma}\sup_{t\in\mathfrak{J}} D_0[\Xi(t) - \Xi(0))^{1-\gamma}w_t, \widehat{0}]. \end{aligned}$$

This proves that $T(B_{\eta_1}) \subseteq B_{\eta_1}$.

Step 2. T is continuous on B_{η_1}.

Let $\{w_n\}^\infty_{n\geq1}$ $(n = 1, 2, ...)$ be a sequence in B_{η_1} such that $w_n \to w$ in $C([-\tau, 0], E_c)$. Then, for each $t \in \mathfrak{J}$, we have

$$\begin{aligned} &D_0[(\Xi(t) - \Xi(0))^{1-\gamma}(Tw_n)(t), (\Xi(t) - \Xi(0))^{1-\gamma}(Tw)(t)] \\ &\leq D_0\Bigg[(\Xi(t) - \Xi(0))^{1-\gamma}\left((Tw_n)(t) \ominus_{gH} \frac{(\Xi(t) - \Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0\right), \\ &(\Xi(t) - \Xi(0))^{1-\gamma}\left((Tw)(t) \ominus_{gH} \frac{(\Xi(t) - \Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0\right)\Bigg] \\ &\leq (\Xi(b) - \Xi(0))^{1-\gamma}\frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi'(s)(\Xi(t) - \Xi(s))^{\zeta_1-1}D_0[g(s, w_{ns}), g(s, w_s)]ds \\ &\leq (\Xi(b) - \Xi(0))^{1-\gamma}\frac{\mathscr{L}^*}{\Gamma(\zeta_1)}\int_0^t \Xi'(s)(\Xi(t) - \Xi(s))^{\zeta_1-1}D_{[-\tau,0]}[w_{ns}, w_s]ds \\ &\to 0 \quad as \quad n \to \infty. \end{aligned}$$

Now, $\lim_{n\to\infty} w_n = w \in B_{\eta_1}$. Then, for each $t \in (0, b]$, we get $\lim_{n\to\infty}(\Xi(t)-\Xi(0))^{1-\gamma}w_n = (\Xi(t)-\Xi(0))^{1-\gamma}w$. Furthermore, for each $t \in [-\tau, 0]$, due to $\chi \in C([-\tau,0],E_c)$ that is, $\lim_{n\to\infty} w_n(t) = \chi(t) = g_1(w)(t) = w(t)$. Moreover, one has $\|w_n\|_{C_{1-\zeta_1}} \leq \eta_1$ and $\|w\|_{C_{1-\zeta_1}} \leq \eta_1$. Hence, for $r \in [0,t]$, $t \in (0,b]$, we get $\lim_{n\to\infty} w_{ns} = w_s, \quad D_{[-\tau,0]}[w_{ns},\widehat{0}] \leq \eta_1 + D_{[-\tau,0]}[\chi,\widehat{0}]$, $D_{[-\tau,0]}[w_s,\widehat{0}] \leq \eta_1 + D_{[-\tau,0]}[\chi,\widehat{0}]$. This implies that, it follows from $(A1)$ $(A2)$, for $r \in (0,t)$, $t \in (0,b]$, one's get

$$\begin{aligned} D_0[g(s,w_{ns}),g(s,w_s)] &\leq D_0[g(s,w_{ns}),\widehat{0}] + D_0[g(s,w_s),\widehat{0}] \\ &\leq 2\mathscr{L}((\Xi(b)-\Xi(0))D_{[-\tau,0]}[\chi,\widehat{0}] + \kappa\eta_1). \end{aligned}$$

Taking into account the fact that T is continuous, that is, $D_0[g(s,w_{ns}),g(s,w_s)] \to 0 \quad \text{as} \quad w_n \to w$, which gives $\|w_{ns}-w_s\|_0 \to 0 \quad \text{as} \quad w_n \to w$,
where $\sup_{t\in\mathfrak{J}} D_0[(Tw_n)(t),(Tw)(t)] \leq \|Tw_n - Tw\|_0$. Thus T is continuous
Step 3. T is compact in B_{η_1}
First, we have to prove T maps bounded sets into equicontinuous sets in B_{η_1}.
For any $t_1,t_2 \in (0,b]$, $t_1 < t_2$ and $w \in B_{\eta_1}$, we get
$D_0[(\Xi(t_2)-\Xi(0))^{1-\gamma}(Tw)(t_2),(\Xi(t_1)-\Xi(0))^{1-\gamma}(Tw)(t_1)]$

$$\begin{aligned} &\leq D_0\Bigg[(\Xi(t_2)-\Xi(0))^{1-\gamma}\Bigg((Tw)(t_2) \ominus_{gH} \frac{(\Xi(t_2)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0\Bigg), \\ &(\Xi(t_1)-\Xi(0))^{1-\gamma}\Bigg((Tw)(t_1) \ominus_{gH} \frac{(\Xi(t_1)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0\Bigg)\Bigg] \\ &\leq \frac{1}{\Gamma(\zeta_1)}(\Xi(t_2)-\Xi(0))^{1-\gamma}\int_{t_1}^{t_2} \Xi^{'}(s)(\Xi(t_2)-\Xi(s))^{\zeta_1-1}D_0[g(s,w_s),\widehat{0}]ds \\ &+ \frac{1}{\Gamma(\zeta_1)}\int_0^{t_1} \Xi^{'}(s)[(\Xi(t_2)-\Xi(0))^{1-\gamma}(\Xi(t_2)-\Xi(s))^{\zeta_1-1} - (\Xi(t_1) \\ &-\Xi(0))^{1-\gamma}(\Xi(t_1)-\Xi(s))^{\zeta_1-1}]D_0[g(s,w_s),\widehat{0}]ds \\ &\to 0 \quad as \quad t_2 \to t_2. \end{aligned}$$

The right hand sides of the above equation tends to zero independently of $w \in B_{\eta_1}$ as $t_2 \to t_1$, which means that $D_0[(Tw)(t_2),(Tw)(t_1)] \to 0$. Thus, it follows from the Arzela-Ascoli theorem gives that the operator T is completely continuos. Consequently, by using the Schauder's fixed point theorem gives that the operator T has at least one fixed point. Hence Eqn.(1.1) has at least one solution on $\mathfrak{J}$. This completes the proof. □

Theorem 3.4. *Assume that $g : [0,b] \times C([-\tau,0],Ec) \to E_c$ be a continuous fuzzy function, $\chi \in C([-\tau,0],E_c)$. Assume that $(A1)$-$(A2)$ is satisfied, then the Eqn.(1.1) is finite-time stable with respect to $\{0,[-\tau,b],\tau,\sigma,\epsilon\}$, $0<\sigma<\epsilon$, $\sigma,\epsilon \in \mathbb{R}^c$.*
If $M_1^*\mathbb{E}_{\zeta_1}(\mathscr{L}(\Xi(b)-\Xi(0))^{1-\gamma}(\Xi(b)-\Xi(0))^{\zeta_1}) < 1$, $t \in \mathfrak{J}$
where

$$M_1^* = \frac{1}{\Gamma(\gamma)} + \frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma,\zeta_1).$$

Proof. According to the similar proof (**??**) and by Definition 2.10, we have $D_0[(\Xi(t)-\Xi(0))^{1-\gamma}w(t),\widehat{0}]$

$$
\begin{aligned}
&\leq D_0\left[(\Xi(t)-\Xi(0))^{1-\gamma}\left(\frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0\right),\widehat{0}\right]\\
&+ D_0\left[(\Xi(t)-\Xi(0))^{1-\gamma}\frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}g(s,w_s)ds,\widehat{0}\right]\\
&\leq \frac{1}{\Gamma(\gamma)}D_0[w_0,\widehat{0}]+\frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma,\zeta_1)D_{[-\tau,0]}[\chi,\widehat{0}]\\
&+\frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{1-\gamma}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}\sup_{r\in[0,\xi]}D_0[(\Xi(s)-\Xi(0))^{1-\gamma}y(s),\widehat{0}]ds,\\
&\leq \frac{1}{\Gamma(\gamma)}\sigma+\frac{\mathscr{L}}{\Gamma(\gamma)}(\Xi(b)-\Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma,\zeta_1)\sigma+\frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{1-\gamma}\\
&\times\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}\sup_{r\in[0,\xi]}D_0[(\Xi(s)-\Xi(0))^{1-\gamma}y(s),\widehat{0}]ds
\end{aligned}
$$

Now, we put

$$
\begin{aligned}
N(t)&=\sup_{r\in[0,t]}D_0[(\Xi(s)-\Xi(0))^{1-\gamma}y(s),\widehat{0}],\\
M_1^*&=\frac{1}{\Gamma(\gamma)}+\frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{\zeta_1+2-2\zeta_1}B(2-\gamma,\zeta_1).
\end{aligned}
$$

It follows from the generalized Gronwall inequality gives that, we have

$$
N(t)=D^{\gamma}_{[0,b]}[w,\widehat{0}]\leq \sigma M_1^*\mathbb{E}_{\zeta_1}(\mathscr{L}(\Xi(b)-\Xi(0))^{1-\gamma}(\Xi(t)-\Xi(0))^{\zeta_1})<\sigma<\epsilon.
$$

Therefore, Eqn.(1.1) is finite-time stable. This completes the proof. □

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COMPARATIVE ANALYSIS OF FIRST AND SECOND ORDER METHODS FOR OPTIMIZATION IN NEURAL NETWORKS

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Abstract. Artificial Neural Networks are fine tuned to yield the best performance through an iterative process where the values of their parameters are altered. Optimization is the preferred method to determine the parameters that yield the minima of the loss function, an evaluation metric for ANN's. However, the process of finding an optimal model which has minimum loss faces several obstacles, the most notable being the efficiency and rate of convergence to the minima of the loss function. Such optimization efficiency is imperative to reduce the use of computational resources and time when training Neural Network models. This paper reviews and compares the intuition and effectiveness of existing optimization algorithms such as Gradient Descent, Gradient Descent with Momentum, RMSProp and Adam that implement first order derivatives, and Newton's Method that utilizes second order derivatives for convergence. It also explores the possibility to combine and leverage first and second order optimization techniques for improved performance when training Artificial Neural Networks.

1. Introduction

In Mathematics, **optimization** is the process of maximizing or minimizing a real function by finding the best set of input values under certain conditions or constraints. It can be defined as:

$$\arg\min_{\theta} f(\theta) \;\; or \;\; \arg\max_{\theta} f(\theta) \tag{1.1}$$

Here, θ represents the arguments for the function. The concept of optimization is used in abundance in real life: in GPS Systems, financial companies and airline reservations. Similarly, optimization methods are equally important in the field of Artificial Intelligence and Machine Learning, especially for their application to Artificial Neural Network Models.

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1.1. **Artificial Neural Networks.** Artificial Neural Networks are Machine Learning Models that mimic the functionality of biological neurons. They implement learning algorithms which are fine-tuned by training on data to improve their accuracy. ANN's comprise of an Input layer, a single or multiple hidden layers, and an output layer. Each layer consists of a particular number of artificial neurons or nodes and each node receives an input from all the nodes in the previous layer and outputs values to all the nodes in the next layer. Each of the individual connections between the nodes are assigned parameter values called weights and biases. These parameters are altered and tuned during the learning process using the input training data for optimal performance and accuracy.

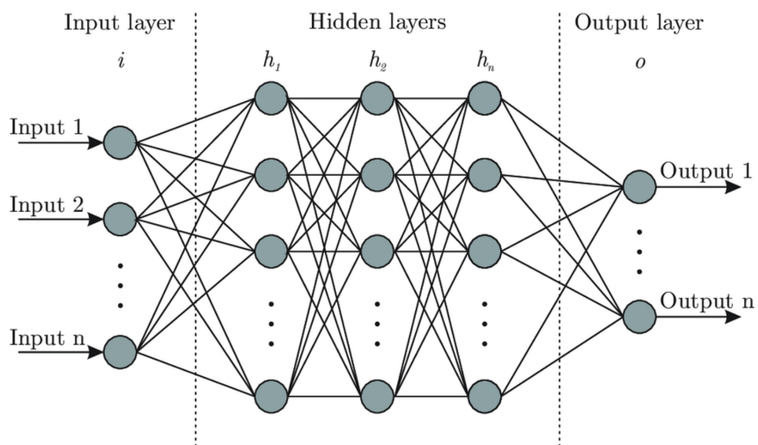

FIGURE 1. Artificial Neural Network with 3 hidden layers [1]

During the initial training or learning process of a Neural Network, the features or attributes of each individual data record are passed into the first hidden layer as an input. Each of these input features have some weight value attached to them and all the inputs are connected to each neuron in the hidden layer. Using the inputs, the output z of neuron j in the hidden layer is,

$$z_j = f\left(b + \sum_{i=1}^{n} x_i w_i\right), \tag{1.2}$$

where n is the total number of input features, b denotes the bias value, w denotes the weights for particular features and f is the activation or transfer function of the layer. The activation function is a linear or non linear function that determines the output of the neuron. Examples of activation functions are Linear function, Sigmoid function, Hyperbolic Tangent function, Rectified Linear Unit function, etc. The output of each neuron in a layer is passed as an input to all the neurons in the next layer with their own weights and biases values. Similarly, each layer receives its own input and calculates an output and passes it to the next layer. This process is also known as forward propagation. The last layer, which is the output layer receives the inputs from the neurons of the last hidden layer and provides the output of the neural network.

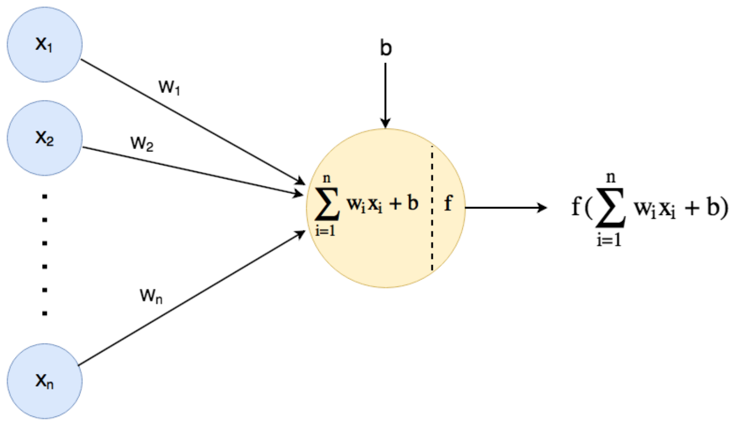

FIGURE 2. An artificial neuron with n inputs [2]

1.2. **Optimization in Neural Networks.** The output of a neural network represents the prediction for a particular input. The initialized values of the weights and biases do not usually produce an accurate prediction. Hence, ANN's use an iterative process where these parameters are adjusted in each iteration to increase the prediction accuracy. This step is also known as Backpropagation. Backpropagation is the process of updating and finding the optimal values of weights or coefficients which helps the model to minimize the error i.e difference between the actual and predicted values.[3] The difference between the predicted and the actual value for an individual data sample is calculated using a loss function. The most common loss function for regression problems is given by,

$$L\left(\widehat{y}\right) = \frac{1}{2}\left(\widehat{y} - y\right)^2, \tag{1.3}$$

where $\widehat{y}$ represents the neural network prediction and y denotes the actual value. The difference between the predicted and the actual values for all the records in the data set is given by a Cost Function J,

$$J = \frac{1}{2m}\sum_{k=1}^{m} L_k\left(\widehat{y}\right), \tag{1.4}$$

where m denotes the total number of records or data samples.

Neural Networks strive to adjust the parameters of the model to minimize the loss function. Hence, optimization is implemented to find the values of the weights and biases that will engender the minimum value of the loss function (closest to zero).

$$\arg\min_{(W,b)} \sum_{k=1}^{m} \frac{1}{2m}\left(f\left(W^T X_k + b\right) - y_k\right)^2, \tag{1.5}$$

where W denotes an $n \times j$ weight matrix of the output layer(n is the number of input features coming from the previous layer, and j is the number of outputs of the network), b denotes the bias of the output layer, and f denotes the activation function of the output layer. Equation 1.5 can be understood as a combination of equations 1.2,1.3 and 1.4.

There are several different approaches of optimization to find the minimum of a function. These methods usually utilize the first and second derivatives of the function with respect to the parameters. The efficiency of such methods are evaluated through the computational cost (memory) and time cost for optimization.

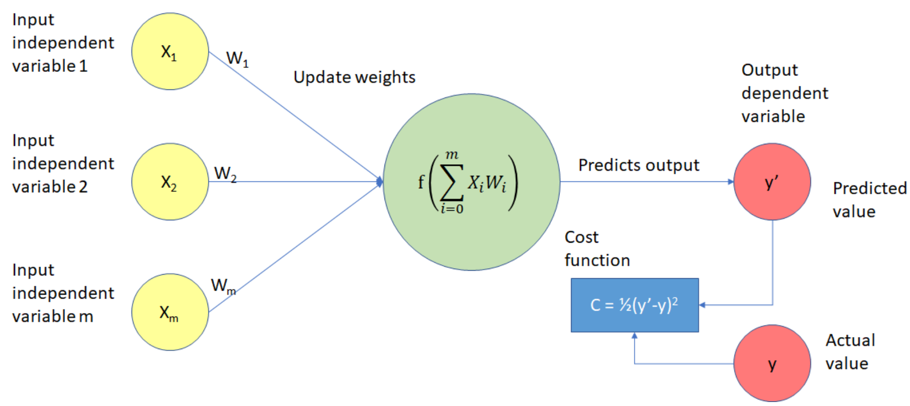

FIGURE 3. Forward and Backpropagation in a single neuron network.[4]

2. First Order Optimization Methods

First Order Optimization refers to methods that utilize the first derivative of the target function with respect to the parameters. It can only be applied to functions that are differentiable and continuous. One of the most commonly used first order methods is Gradient Descent where the gradient is used to descend down the curve of the function.

2.1. **Gradient Descent.** Gradient Descent is a first order iterative method for optimization where the idea is to take repeated steps to update the parameters in the opposite direction of the gradient. For a vector $\theta = [\theta_1, \theta_2, \ldots \theta_n]$ where θ_n represents the parameters for the cost function J, the updated values after a particular iteration is given by,

$$\theta = \theta - \eta \nabla_\theta J(\theta), \tag{2.1}$$

where η denotes the learning rate or the size of the steps that are taken to reach the minimum and

$$\nabla_\theta J(\theta) = \left[\frac{\partial J}{\partial \theta_1}, \frac{\partial J}{\partial \theta_2}, \ldots \frac{\partial J}{\partial \theta_n}\right]$$

With respect to Neural Networks, there are three variants of gradient descent that are used for convergence: Batch Gradient Descent, Mini Batch Gradient Descent and Stochastic Gradient Descent. These variants differ in terms of the number of samples used to calculate the loss function gradient for each parameter update step.

2.1.1. *Batch Gradient Descent.* Batch or Vanilla Gradient Descent computes the gradient of the cost function with respect to the parameters θ for the entire training dataset.[5] Here, $\nabla_\theta J(\theta) = \sum_{i=1}^{m} \nabla_\theta J(\theta)$, where m denotes the total number of records in the dataset. Hence, the summation of the gradients of the entire dataset needs to be calculated to perform just one parameter update. In other words, the parameters are updated once in each epoch (one epoch refers to one iteration through the entire training data).[6] Therefore, for a large dataset, Batch Gradient Descent is really slow since a single update is performed after going through all the records. However, because the information of the entire dataset is being evaluated each time the parameters are updated, the convergence path taken using Batch Gradient Descent is smooth and free of noise which accounts for a more direct path towards the minimum.

2.1.2. *Stochastic Gradient Descent.* In contrast, Stochastic Gradient Descent performs a parameter update after each record in the dataset. In terms of SGD, $\nabla_\theta J(\theta) = \nabla_\theta^{(i)} J(\theta)$, where (i) denotes a random record. Hence, the parameters are updated m times in one epoch. As the gradient of the cost function with respect to the parameters for each record will vary largely, the convergence path using SGD is full of noise and oscillations in different directions compared to Batch Gradient Descent. Hence, SGD requires higher number of iterations to reach the minima. Furthermore, SGD performs frequent updates with a high variance that causes the objective function to fluctuate heavily. The random and drastic changes in parameter values due to the nature of SGD enables it to jump out of local minimas into new and potentially better minima. Due to the high variance of the gradients for each record, SGD never actually converges completely to the minima but rather oscillates around the region. However, SGD provides advantages of updating the parameters almost instantly and the escaping local minimas. Furthermore, it is less computationally expensive and converges faster than Batch Gradient Descent when the dataset is very large.

2.1.3. *Mini-Batch Gradient Descent.* Batch Gradient Descent updates parameters after going through the entire dataset while Stochastic Gradient Descent performs updates after each record. Mini-Batch Gradient Descent leverages the efficiency of both these methods. It divides the total dataset into small batches and updates the parameters after going through each batch. Hence, $\nabla_\theta J(\theta) = \sum_{i=1}^{k} \nabla_\theta J(\theta)$, where $k << m$. By taking a small sample of the total data, Mini-Batch Gradient Descent eliminates the heavy computational cost for large datasets

using Batch Gradient Descent while reducing the noise and variance of Stochastic Gradient Descent leading to a more stable convergence.

2.2. **Limitations of Gradient Descent.** Although it addresses the limitations of the previous two Gradient Descent methods, Mini-Batch Gradient Descent has several limitations.

(1) **Choice of Learning Rate**

Learning Rates denotes the size of the steps taken during convergence. If a learning rate is too small, the parameter updates will be insignificant and the number of iterations required to converge will be huge. If a learning rate is too big, the update might overshoot and jump over the minima to

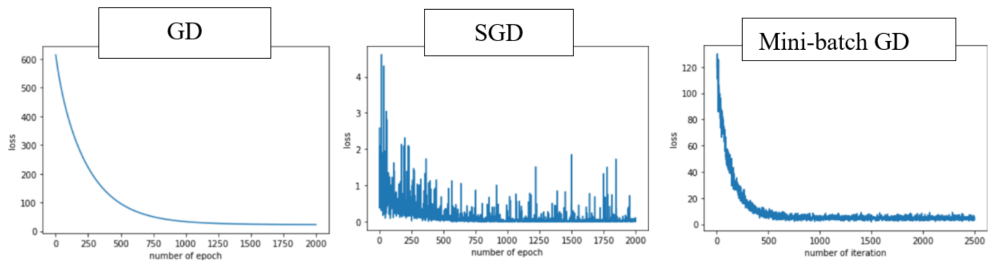

FIGURE 4. Loss Function fluctuations after parameter updates.[7]

the opposite side causing the loss function to fluctuate around the minima or in worst cases, even diverge.

(2) **Saddle Points and Local Minima**

Saddle points are flat regions of a function where the partial derivatives of the cost function with respect to its parameters are opposite in nature i.e areas where one dimension has a positive partial and another has a negative partial. Such points are usually surrounded by a plateau which makes it very difficult for Gradient Descent to escape as the gradient is close to zero in all dimensions.[5]

(3) **Noisy Convergence** The performance of Artificial Neural Networks increases with the size of the data. Hence, Batch Gradient Descent is rarely used due to its high computational cost. However, as SGD and Mini-Batch Gradient Descent update parameters using only a small portion of the total data, the convergence path using these methods have high variance. This increases the number of oscillations to reach the minimum as the path taken is not direct. The extent of noise depends on the size of the batch used (the larger the batch the smoother the convergence path).

Hence, due to the impracticality in application of Mini-Batch Gradient Descent, other first order methods were developed to address and eliminate such limitations.

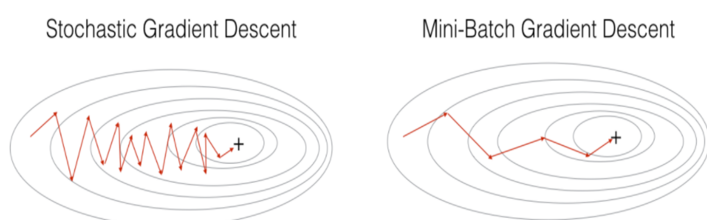

FIGURE 5. Convergence path using a contour map.[8]

2.3. **Gradient Descent with Momentum.** One of the major challenges with Gradient Descent is that the updated value of a parameter depends only on the gradient of the cost function at the previous parameter value. Therefore, it gets stuck in areas where the gradient is very close to zero in all dimensions. Furthermore, the number of iterations to converge is higher due to the large variance in the gradient using SGD and Mini-Batch Gradient Descent. Gradient Descent with Momentum addresses both these challenges by using Exponentially Weighted Averages of the gradients to update the parameters. Exponentially Weighted Averages is used in sequential noisy data to reduce the noise and smoothen the data.[7] Referring to Figure 5 above, the vertical oscillations slows gradient descent and prevents the use of a high learning rate. By using the exponentially weighted averages, the partial derivative with respect to the vertical direction has an average closer to zero as it is in both (positive and negative) directions.[9] In contrast, the partial derivative with respect to the horizontal direction is always positive, hence the average in that direction will be large. This allows for a more direct convergence path to the minima. The exponentially weighted average of the gradient for a given iteration t is,

$$M_{d\theta_t} = \beta M_{d\theta_{t-1}} + (1-\beta)\,\nabla_{\theta_t} J(\theta)\,,$$

where $0 < \beta \leq 1$. Similarly,

$$\begin{aligned} M_{d\theta_{t-1}} &= \beta M_{d\theta_{t-2}} + (1-\beta)\,\nabla_{\theta_{t-1}} J(\theta) \\ M_{d\theta_{t-2}} &= \beta M_{d\theta_{t-3}} + (1-\beta)\,\nabla_{\theta_{t-2}} J(\theta)\,, \end{aligned}$$

and $M_{d\theta_1} = 0$. $M_{d\theta_1}$ is also called the momentum term. Expanding $M_{d\theta_t}$,

$$M_{d\theta_t} = \beta\left(\beta(\beta M_{d\theta_{t-3}} + (1-\beta)\,\nabla_{\theta_{t-2}} J(\theta)\right) + (1-\beta)\,\nabla_{\theta_{t-1}} J(\theta)) + (1-\beta)\,\nabla_{\theta_t} J(\theta) \tag{2.2}$$

Hence, using Gradient Descent with Momentum, for an iteration t, the new parameter value is dependent on the gradients of all previous iterations. Here, β is the hyper-parameter that determines the degree of smoothness of the convergence path and is usually 0.9. Consequently, the weight assigned to the averages of the previous iterations is larger compared to the weight assigned to the gradient at the current point. The parameters are then updated using the formula,

$$\theta = \theta - \eta M_{d\theta_t} \tag{2.3}$$

For partial derivatives with respect to dimensions whose values oscillate between positive and negative, as the averages are closer to zero, the oscillations in those dimensions are reduced. Inversely, the momentum term increases for dimensions whose gradients point in the same directions, resulting in a larger update step after each iteration and gaining faster convergence.

2.4. **Root Mean Square Propagation.** Although Gradient Descent with Momentum reduces oscillations in dimensions where the partials point in opposite directions in each iteration, it is more effective in accelerating the convergence in dimensions whose derivatives point in the same direction. However, this poses some challenges. For a large t (located closer to minima), the momentum term for these dimensions and consequently, the parameter updates will be very large. Hence, for parameter updates near the minima, the risk of overshooting and missing the minima due to the large momentum term is very high. Similar to a ball rolling down a large cliff and going back and forth across the bottom several times, GD with Momentum increases the number of iterations to settle down into the minima. Root Mean Square Propagation or RMSProp eliminates this risk. While momentum accelerates our search in the direction of the minima, RMSProp impedes our search in the direction of the oscillations.[10] RMSProp implements this by using an adaptive learning rate, or a learning rate that changes in each iteration. In RMSProp, after each iteration the learning rate decreases independently for each dimension based on the partial derivative of that dimension at that particular iteration . If a dimension/ parameter has a higher partial derivative, than its learning rate is lower compared to a parameter with a lower partial derivative value. Using RMSProp, the new parameter values θ,

$$\theta = \theta - \frac{\eta}{\sqrt{V_{d\theta_t}} + \epsilon} \cdot \nabla_{\theta_t} J\left(\theta\right) \tag{2.4}$$

where

$$V_{d\theta_t} = \alpha V_{d\theta_{t-1}} + \left(1 - \alpha\right) \nabla_{\theta_t} J\left(\theta\right)^2 ,$$

Here, ϵ is a very small value, usually 10^{-8} and $0 < \alpha \leq 1$.

Similar to GD with Momentum, RMSProp also uses an exponential weighted average. However, instead of taking the average of the gradient, it uses the exponentially weighted average of the squared gradient. This is to ensure the sole dependency of the adaptive learning rate on the magnitude of the partials of the different parameters and not the signs. Hence, in contrast to the Momentum term $M_{d\theta_t}$, if the partials of a parameter at different iterations has opposite signs, their exponentially weighted average won't cancel out but instead be added, increasing $V_{d\theta_t}$. The square root is added in the denominator of 2.4 to ensure that the learning rate isn't too small. Therefore for situations like the one shown in Figure 5, the step taken in both directions will decrease for increasing iterations. However, the direction that is steeper will have a significantly smaller step compared to shallower directions.

2.5. **Adaptive Moment Estimation.** In Gradient Descent with Momentum, the step size for parameters which have the same signed partial derivatives increases after each iteration. In contrast, in RMSProp the step size for parameters decreases after each iteration but to a greater extent for ones which have a higher magnitude partial derivative. Adaptive Moment Estimation or Adam is an optimization algorithm that is the combination of the features of both of these first order methods. Adam utilizes the acceleration that is provided by GD with Momentum, but to ensure that the step size doesn't infinitely increase towards latter iterations, it uses the RMSProp term to decrease the learning rate to limit the updates as the iterations increase. Hence, it leverages the increase in the Momentum term by decreasing the learning rate. The Momentum and RMSProp terms are given by,

$$M_{d\theta_t} = \beta M_{d\theta_{t-1}} + \left(1 - \beta\right) \nabla_{\theta_t} J\left(\theta\right) ,$$

and

$$V_{d\theta_t} = \alpha V_{d\theta_{t-1}} + \left(1 - \alpha\right) \nabla_{\theta_t} J\left(\theta\right)^2 ,$$

For a parameter θ, the updated value using Adam is given by,

$$\theta = \theta - \frac{\eta}{\sqrt{V_{d\theta_t}} + \epsilon} \cdot M_{d\theta_t} \tag{2.5}$$

Adam is the most widely used optimization method because it performs really well in optimization test functions compared to other algorithms.

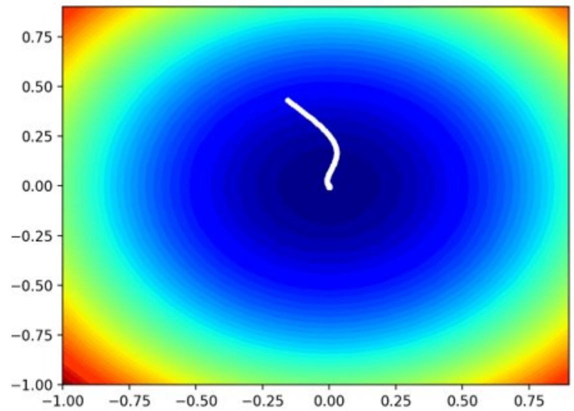

FIGURE 6. Contour Plot of the Test Objective Function With Adam [11]

3. SECOND ORDER OPTIMIZATION METHODS

Second Order Optimization methods are a separate set of methods for optimization that differ from the traditional gradient descent ideology. Instead of using the gradient of the objective function at a particular point to update the parameters, second order methods utilize the Hessian of the function. The gradient of a function is a vector where each element represents the partial derivative of the function with respect to the individual parameters. For a function with a total of p parameters, the Hessian is a $p \times p$ matrix where each element represents the second partial derivative of the functions with respect to the parameters. The Hessian of function $f(x_1, x_2, \ldots, x_n)$ is given by,

$$H_f(x_1, x_2, \ldots, x_n) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

3.1. **Newton's Method.** Newton's method is an optimization technique that utilizes the approximation of a function using Taylor's Expansion to the second order. For a function $f(x)$, the Taylor expansion to the second order about a certain point x_0 in the domain is given by,

$$f_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$

The approximation function f_2 is a quadratic function about point x_0. Newton's method finds the value of x, where f_2 is a minimum, and assigns that value to x_0, i.e $x_0 = \arg\min_x f_2(x)$. Taylor's expansion is used to approximate a new f_2 about the updated x_0. The minima of the new approximation is found and x_0 is updated again. Hence, Newton's method iteratively updates the parameter x_0 to values where newly approximated Taylor Functions are a minimum until the minimum of the actual function f is reached.

For an approximation f_2, a point x is a maximum or minimum only if the gradient at that point is 0.

$$\frac{df_2}{d\left(x-x_0\right)} = 0$$

$$\frac{d\left[f\left(x_0\right)+f'\left(x_0\right)\left(x-x_0\right)+\frac{1}{2}f''\left(x_0\right)\left(x-x_0\right)^2\right]}{d\left(x-x_0\right)} = 0$$

$$f'\left(x_0\right)+f''\left(x_0\right)\left(x-x_0\right) = 0$$

$$x = x_0 - \frac{f'\left(x_0\right)}{f''\left(x_0\right)} \tag{3.1}$$

For an approximation of a multi-variable function, $f_2(\theta)$ where θ is a vector of the parameters,

$$\theta_{t+1} = \theta_t - \left[H\left(\theta_t\right)\right]^{-1}\nabla_\theta f\left(\theta_t\right) \tag{3.2}$$

Compared to gradient descent, Newton's method is extremely fast. For a suitably chosen learning , gradient descent takes 229 steps to converge to the minimum whereas Newton's method converges to the minimum in only 6 steps.[12]

Although it is very efficient, Newton's method has numerous limitations in application to Neural Networks. ANN's usually have thousands of parameters and are non-convex by nature. However, Newton's method isn't applicable to non-convex functions. The initialization of the parameters to areas closer to the maximum of the function or at points where the Hessian is negative-definite can lead to a quadratic approximation that is concave. For such an approximation, the parameter update will lead towards the maximum point of the approximated concave function instead of the minimum that is required. Hence, Newton's method can lead to an increase in the value of the loss which is undesired. Furthermore, Newtonian methods are very computationally expensive. The calculation of the Hessian is itself an $0(N^2)$ and inverting the Hessian is $0(N^3)$ compared to Gradient Descent methods which scale at $O(N)$.[12] Additionally, saddle points where the Gradient and Hessian are almost zero might lead to computationally inaccurate values and slow updates of the parameters. Such extreme limitations render the use of Newtonian methods in Neural Networks useless.

However, there are methods not covered in this paper called Quasi-Newton methods that eliminate the large computational cost of traditional Newton's method while preserving optimization efficiency. Quasi-Newton methods utilize an approximation of the Hessian using a generalized secant method, eliminating the need to invert the Hessian. Hence, they have a computational complexity of $O(N^2)$ compared to $O(N^3)$ for Newton's method, while retaining most of the efficiency when converging using Newton's method.

4. Conclusion and Future Work

Although Quasi-Newton methods such as the Broyden—Fletcher—Goldfarb—Shanno algorithm represent significant progress in the field of second order optimization, the lack of precision in the calculation of the Hessian can sometimes lead to slower convergence.[13] Hence, instead of approximating the Hessian, leveraging Gradient Descent and Newton's Method can possibly lead to better performance. An optimization algorithm can be introduced which consists of two stages, beginning with Gradient Descent and ending with Newton's method. The transition into Newton's method can be implemented at a point where the Hessian is a positive-definite matrix. However, since the calculation of the Hessian is computationally expensive, this prompts further research on finding a general method to determine the point of transition into Newton's Method.

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FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS OF FENG-LIU TYPE Θ-CONTRACTIONS ON M-METRIC SPACES

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Abstract. In this paper, we present a new fixed point result for multivalued θ-contractions on M-complete M-metric spaces using Feng-Liu's technique. Our results extend and generalize some related fixed point theorems in the literature.

1. Introduction and preliminaries

Matthews [9] introduced the notion of the partial metric space, which is more general than the metric space, and presented a fundamental fixed point theorem on partial metric spaces. Then, Asadi, Karapınar and Salimi [5] extended the concept of partial metric spaces to M-metric spaces and presented some fixed point theorems for single valued mappings on M-metric spaces.

Definition 1.1 ([5]). *Let X be a nonempty set. A function $m : X \times X \to [0, \infty)$ is called an M-metric if the following conditions are satisfied: for all $x, y, z \in X$*

(m1) $m(x, x) = m(y, y) = m(x, y) \Leftrightarrow x = y$,
(m2) $m_{xy} = \min\{m(x, x), m(y, y)\} \leq m(x, y)$,
(m3) $m(x, y) = m(y, x)$,
(m4) $m(x, x) - m_{xy} \leq m(x, z) - m_{xz} + m(z, y) - m_{zy}$.

Then, the pair (X, m) is called an M-metric space.

Next, Altun et al. [4] studied on the topological structures of M-metric space, and then presented some fixed point theorems for multivalued mappings of Feng-Liu type on M-metric space (see [4, 14, 15] and references therein). Let (X, m) be an M-metric space, $x \in X$ and $\varepsilon > 0$. The open ball with centered $x \in X$ and

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radius ε is defined by

$$B_m(x,\varepsilon) = \{y \in X : m(x,y) < m_{xy} + \varepsilon\}.$$

Then, the family

$$\{B_m(x,\varepsilon) : x \in X, \varepsilon > 0\}$$

is a base of a topology on X. This topology is defined by τ_m and the closure of a subset A of X with respect to τ_m by $\overline{A^m}$.

Example 1.1. *Let $X = \left\{\frac{1}{n^2} : n \in \{1,2,3,\cdots\}\right\} \cup \{0\}$ and $m : X \times X \to [0,\infty)$ be defined by $m(x,y) = \min\{x,y\}$. Then, (X,m) is a M-metric space. In this case, we have $\tau_m = \{\emptyset, X\}$.*

Definition 1.2. *Let (X,m) be an M-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then,*

(1) *$\{x_n\}$ is said to be M-converges to x if and only if*
$$\lim_{n\to\infty} [m(x_n,x) - m_{x_n x}] = 0.$$
(2) *$\{x_n\}$ is said to be M-Cauchy sequence if $\lim_{n,m\to\infty}[m(x_n,x_m) - m_{x_n x_m}]$ exists and is finite.*
(3) *(X,m) is said to be M-complete if every M-Cauchy sequence M-converges to a point $x \in X$.*

Note that the M-convergence of a sequence on an M-metric space coincides with the convergence with respect to τ_m.

Altun et al [4] proved the following fixed point theorem, which is M-metric version of Feng-Liu's fixed point theorem [12].

Theorem 1.1. *Let (X,m) be a M-complete M-metric space and $T : X \to C_m(X)$ (the family of all nonempty closed subsets of X) be a multivalued map. If there exist two constants $b,c \in (0,1)$ such that for all $x \in X$ with $m(x,Tx) > 0$ there is $y \in T_b^x(m)$ satisfying*

$$m(y,Ty) \le cm(x,y),$$

where

$$T_b^x(m) = \{y \in Tx : bm(x,y) \le m(x,Tx)\},$$

and

$$m(x,Tx) = \inf\{m(x,y) : y \in Tx\}.$$

Then, T has a fixed point in X provided that $c < b$ and the function $f(x) = m(x,Tx)$ is lower semicontinuous with respect to τ_m.

On the other hand, Jleli and Samet [12] introduced the concept of θ-contraction and then gave a fixed point theorem. So that, they generalize Banach contraction principle which is a quite different from many results in literature.

Let Θ be the family of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(Θ1) θ is non-decreasing;
(Θ2) for each sequence $\{t_n\} \subset (0,\infty)$, $\lim\limits_{n\to\infty} t_n = 0$ if and only if $\lim\limits_{n\to\infty} \theta(t_n) = 1$;
(Θ3) there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim\limits_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$.

Example 1.2. *Let us consider the functions* $\theta_1(t) = e^{\sqrt{t}}$, $\theta_2(t) = e^{\sqrt{te^t}}$, $\theta_3(t) = 2 - \frac{2}{\pi}\arctan\left(\frac{1}{t^\alpha}\right)$ *for* $0 < \alpha < 1$ *and* $\theta_4(t) = e^{\sqrt{t^2+t}}$. *Then it can be seen that* $\theta_i \in \Theta$ *for* $i \in \{1, 2, 3, 4\}$.

Jleli and Samet [12] proved the following theorem.

Theorem 1.2. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping. Suppose that there exist* $\theta \in \Theta$ *and* $k \in (0, 1)$ *such that*

$$x, y \in X,\ d(Tx, Ty) > 0 \Rightarrow \theta\left(d(Tx, Ty)\right) \leq \left[\theta\left(d(x, y)\right)\right]^k.$$

Then, T *has a unique fixed point.*

Then, taking into account the family Θ, many authors have presented some fixed point results for both single valued and multivalued mappings on metric space. For example, in [2] the authors obtained a fixed point theorem for compact set valued mappings on metric space. Also, a similar result for closed set valued mappings on metric spaces have been provided by taking the following condition ($\Theta 4$) into consideration (see [1, 2, 3, 6, 7, 8, 10, 11, 13] and references therein):

($\Theta 4$) $\theta(\inf A) = \inf \theta(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

We denote by Ξ the set of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying ($\Theta 1$)-($\Theta 4$).

In this paper, we present Feng-Liu type fixed point theorems for multivalued mappings considering the both families Θ and Ξ in M-metric spaces.

2. Main Result

Let (X, m) be an M-metric space. $P_m(X)$ and $C_m(X)$ denotes the family of all nonempty subsets and the family of all nonempty closed (w.r.t. τ_m) subsets of X, respectively. Also, we indicate the family of all subsets A of X satisfying the following property by $A_m(X)$: for all $x \in X$

$$\left.\begin{array}{l} m(x, A) = 0 \Rightarrow x \in A \\ \text{and} \\ m(x, A) > 0 \Rightarrow \exists a_x \in A,\, m(x, A) = m(x, a_x) \end{array}\right\}.$$

If (X, m) is a metric space, then it is clear that

$$A_m(X) = \{A \subseteq X : \forall x \in X, \exists a_x \in A, m(x, A) = m(x, a_x)\}$$

and also $A_m(X) \subseteq C_m(X)$. Let $T : X \to P_m(X)$ be a mapping, $\theta \in \Theta$ and $b \in (0, 1]$. For $x \in X$ with $m(x, Tx) > 0$, consider the set

$$\Theta_b^x(m) = \left\{y \in Tx : \left[\theta\left(m(x, y)\right)\right]^b \leq \theta\left(m(x, Tx)\right)\right\}.$$

It is clear that if $b_1 \leq b_2$, then $\Theta_{b_1}^x(m) \subseteq \Theta_{b_2}^x(m)$ for fixed $x \in X$.

Theorem 2.1. *Let* (X, m) *be an* M*-complete* M*-metric space and* $T : X \to A_m(X)$ *be a multivalued map* $\theta \in \Theta$. *If there exists a constant* $k \in (0, 1)$ *such that for any* $x \in X$ *with* $m(x, Tx) > 0$, *there is* $y \in \Theta_b^x(m)$ *for* $b \in (0, 1]$ *satisfying*

$$\theta\left(m(y, Ty)\right) \leq \left[\theta\left(m(x, y)\right)\right]^k, \tag{2.1}$$

then T *has a fixed point in* X *provided that* $k < b$ *and the function* $f(x) = m(x, Tx)$ *is lower semi-continuous with respect to* τ_m.

Proof. Suppose that T has no fixed point. Then, for all $x \in X$ we have $m(x, Tx) > 0$. Since $Tx \in A_m(X)$ for every $x \in X$, the set $\Theta_b^x(m)$ is nonempty for any $b \in (0,1]$. Let $x_0 \in X$ be any initial point, then there exists $x_1 \in \Theta_b^{x_0}(m)$ such that

$$\Theta\left(m\left(x_1, Tx_1\right)\right) \leq \left[\Theta\left(m(x_0, x_1)\right)\right]^k$$

and for $x_1 \in X$, there exists $x_2 \in \Theta_b^{x_1}(m)$ satisfying

$$\Theta\left(m\left(x_2, Tx_2\right)\right) \leq \left[\Theta\left(m(x_1, x_2)\right)\right]^k.$$

Continuing this process, we get an iterative sequence $\{x_n\}$, where $x_{n+1} \in \Theta_b^{x_n}(m)$ and

$$\theta\left(m\left(x_{n+1}, Tx_{n+1}\right)\right)) \leq \left[\theta\left(m(x_n, x_{n+1})\right)\right]^k. \tag{2.2}$$

We will show that $\{x_n\}$ is a Cauchy sequence. Since $x_{n+1} \in \Theta_b^{x_n}(m)$, we have

$$\left[\theta\left(m\left(x_n, x_{n+1}\right)\right)\right]^b \leq \theta\left(m(x_n, Tx_n)\right). \tag{2.3}$$

From (2.2) and (2.3), we have

$$\theta\left(m\left(x_{n+1}, Tx_{n+1}\right)\right) \leq \left[\theta\left(m\left(x_n, Tx_n\right)\right)\right]^{\frac{k}{b}}$$

and

$$\theta\left(m\left(x_{n+1}, x_{n+2}\right)\right) \leq \left[\theta\left(m\left(x_n, x_{n+1}\right)\right)\right]^{\frac{k}{b}}.$$

By the way, we can obtain

$$1 < \theta\left(m\left(x_n, x_{n+1}\right)\right) \leq \left[\theta\left(m\left(x_0, x_1\right)\right)\right]^{\left(\frac{k}{b}\right)^n} \tag{2.4}$$

and

$$1 < \theta\left(m\left(x_n, Tx_n\right)\right) \leq \left[\theta\left(m\left(x_0, Tx_0\right)\right)\right]^{\left(\frac{k}{b}\right)^n}. \tag{2.5}$$

Letting $n \to \infty$ in (2.4),

$$\lim_{n\to\infty} \theta\left(m\left(x_n, x_{n+1}\right)\right) = 1.$$

From (Θ_2),

$$\lim_{n\to\infty} m\left(x_n, x_{n+1}\right) = 0^+.$$

Similarly, we can obtain

$$\lim_{n\to\infty} m\left(x_n, Tx_n\right) = 0.$$

So from (Θ_3), there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta\left(m\left(x_n, x_{n+1}\right)\right) - 1}{\left(m\left(x_n, x_{n+1}\right)\right)^r} = \ell.$$

Suppose that $\ell < \infty$. In this case, let $\varepsilon = \ell/2 > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$

$$\left|\frac{\theta\left(m\left(x_n, x_{n+1}\right)\right) - 1}{\left(m\left(x_n, x_{n+1}\right)\right)^r} - \ell\right| \leq \varepsilon.$$

This implies that, for all $n \geq n_0$,

$$\frac{\theta\left(m\left(x_n, x_{n+1}\right)\right) - 1}{\left(m\left(x_n, x_{n+1}\right)\right)^r} \geq \ell - \varepsilon = \varepsilon.$$

Then, for all $n \geq n_0$,

$$n\left[m\left(x_n, x_{n+1}\right)\right]^r \leq An\left[\theta\left(m\left(x_n, x_{n+1}\right)\right) - 1\right],$$

where $A = \frac{1}{\varepsilon}$.

Suppose now that $\ell = \infty$. Let $\varepsilon > 0$ be arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$

$$\frac{\theta\left(m\left(x_n, x_{n+1}\right)\right) - 1}{\left(m\left(x_n, x_{n+1}\right)\right)^r} \geq \varepsilon.$$

This implies that, for all $n \geq n_0$,

$$n\left[m\left(x_n, x_{n+1}\right)\right]^r \leq An\left[\theta\left(m\left(x_n, x_{n+1}\right)\right) - 1\right],$$

where $A = \frac{1}{\varepsilon}$. Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$

$$n\left[m\left(x_n, x_{n+1}\right)\right]^r \leq An\left[\theta\left(m\left(x_n, x_{n+1}\right)\right) - 1\right].$$

Using (2.4), we obtain for all $n \geq n_0$

$$n\left[m\left(x_n, x_{n+1}\right)\right]^r \leq An\left[\left[\theta\left(m\left(x_0, x_1\right)\right)\right]^{\left(\frac{k}{b}\right)^n} - 1\right].$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n\left[m\left(x_n, x_{n+1}\right)\right]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$

$$m\left(x_n, x_{n+1}\right) \leq \frac{1}{n^{1/r}}. \tag{2.6}$$

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using (m4) and from (2.6), we have

$$\begin{aligned}
m\left(x_n, x_m\right) - m_{x_n x_m} &\leq \left[m\left(x_n, x_{n+1}\right) - m_{x_n x_{n+1}}\right] + \left[m\left(x_{n+1}, x_m\right) - m_{x_{n+1} x_m}\right] \\
&\leq \left[m\left(x_n, x_{n+1}\right) - m_{x_n x_{n+1}}\right] + \cdots + \left[m\left(x_{m-1}, x_m\right) - m_{x_{m-1} x_m}\right] \\
&\leq m\left(x_n, x_{n+1}\right) + m\left(x_{n+1}, x_{n+2}\right) + \cdots + m\left(x_{m-1}, x_m\right) \\
&\leq \sum_{i=n}^{m-1} m(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} m(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.
\end{aligned}$$

By the convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/k}}$, letting to limit $n \to \infty$, we get

$$\lim_{n,m\to\infty} \left[m\left(x_n, x_m\right) - m_{x_n x_m}\right] = 0.$$

Hence, we find that $\{x_n\}$ is an M-Cauchy sequence. Because X is M-complete, one sees that there exists $z \in X$ such that

$$\lim_{n\to\infty} \left[m\left(x_n, z\right) - m_{x_n z}\right] = 0$$

that is, $\{x_n\}$ converges to z with respect to τ_m. Now, we show that z is fixed point of T. On the other hand, from (2.5) and (Θ_2), we have $\lim_{n\to\infty} m\left(x_n, Tx_n\right) = 0$. Since $f(x) = m(x, Tx)$ is lower semi-continuous with respect to τ_m, then

$$0 < m\left(z, Tz\right) = f(z) \leq \liminf_{n\to\infty} f\left(x_{n'}\right) = \liminf_{n\to\infty} m\left(x_n, Tx_n\right) = 0.$$

This is a contradiction. Hence, T has a fixed point. $\square$

To give a fixed point result for $C_m(X)$ valued multivalued mappings, we will consider the family Ξ.

Theorem 2.2. *Let (X, m) be an M-complete M-metric space and $T : X \to C_m(X)$ be a multivalued map $\theta \in \Xi$. If there exists a constant $k \in (0, 1)$ such that for all any $x \in X$ with $m(x, Tx) > 0$, there is $y \in \Theta_b^x(m)$ for $b \in (0, 1)$ satisfying*

$$\theta\left(m(y, Ty)\right) \leq \left[\theta\left(m(x, y)\right)\right]^k.$$

Then, T has a fixed point in X provided that $k < b$ and the function $f(x) = m(x, Tx)$ is lower semi-continuous with respect to τ_m.

Proof. Suppose that T has no fixed point. Then, for all $x \in X$ we have $m(x, Tx) > 0$. Indeed, if $m(x, Tx) = 0$, then $x \in \overline{Tx}^m = Tx$. Since $\theta \in \Xi$, for any $x \in X$ with $m(x, Tx) > 0$, the set $\Theta_b^x(m)$ is nonempty for any $b \in (0, 1)$. Indeed, using the property (Θ_4), we obtain

$$\begin{aligned} \Theta_b^x(m) &= \left\{ y \in Tx : \left[\theta\left(m(x, y)\right)\right]^b \leq \theta\left(m(x, Tx)\right) \right\} \\ &= \left\{ y \in Tx : \left[\theta\left(m(x, y)\right)\right]^b \leq \theta\left(\inf\left\{m(x, y) : y \in Tx\right\}\right) \right\} \\ &= \left\{ y \in Tx : \left[\theta\left(m(x, y)\right)\right]^b \leq \inf\left\{\theta\left(m(x, y) : y \in Tx\right)\right\} \right\} \\ &\neq \emptyset. \end{aligned}$$

The rest of the proof can be completed as in the proof of Theorem 2.1 by considering the $Tz \in C_m(X)$. □

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