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Research Article

Empirical Voronoi wavelets

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ABSTRACT. Recently, the construction of 2D empirical wavelets based on partitioning the Fourier domain with the watershed transform has been proposed. If such approach can build partitions of completely arbitrary shapes, for some applications, it is desirable to keep a certain level of regularity in the geometry of the obtained partitions. In this paper, we propose to build such partition using Voronoi diagrams. This solution allows us to keep a high level of adaptability while guaranteeing a minimum level of geometric regularity in the detected partition.

Keywords: Empirical wavelet, Voronoi diagram, adaptive partitioning, harmonic mode decomposition.

2020 Mathematics Subject Classification: 42C40, 65T60.

1. INTRODUCTION

Empirical wavelets have been proposed in [8] in the 1D case, and then extended to 2D in [7] as an alternative to the empirical mode decomposition [9]. Its purpose is to build data-driven wavelets, i.e. a family of wavelets which is designed based on the content of the original signal/image to analyze. The corresponding wavelet filter bank aims at extracting the harmonic modes (i.e. amplitude modulated - frequency modulated components) plus some residue. This is achieved by considering that the expected modes should have a compact support (or at least are rapidly decreasing outside a compact support) in the Fourier domain. Therefore, the adaptability is obtained by detecting the supports of each mode instead of following some prescribed rule like classic wavelets. A wavelet filter is built for each support providing us the sought wavelet filter bank. A theoretical analysis of such construction in the 1D case is available in [2], considering arbitrary partitioning of the Fourier domain. If in 1D, partitions are made of intervals, partitions in 2D can have more variability in terms of their geometry. For instance, in [7], several types of geometries have been considered like rectangular boxes (analogous to a tensor approach), concentric rings centered at the origin (to represent Littlewood-Paley type operators), and polar wedges (to mimic the behavior of curvelets). A higher degree of flexibility have been achieved in [1], where partitions of arbitrary shapes are detected thanks to a watershed transform. Such level of adaptability is desirable for many applications, however it can lead to non-smooth geometries, affecting the degree of regularity of the wavelets themselves, which is frequently a desirable property for particular analyses. In this paper, we propose an alternative type of partitions based on Voronoi diagrams. This solution provides a trade-off between a high level of adaptability while keeping some simple geometric constraint on the partition to keep good properties of the obtained wavelets.

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The remaining of the paper is organized as follows. Section 2 gives a brief reminder about empirical wavelets, in particular the 2D case. Section 3 describes the Voronoi based empirical wavelets. Some experiments will be presented in Section 4 while conclusions will be given in Section 5.

2. Empirical wavelets

Empirical wavelets have proven to be very efficient in different problems from science and engineering, see for instance [3, 4, 5, 10, 11] to cite only a few. Their construction is originally inspired by the Empirical Mode Decomposition [9]. It aims at writing a signal f as the sum of harmonic modes f_k (i.e. amplitude modulated-frequency modulated components) and some residue r:

$$f(x) = r(x) + \sum_{k=1}^{N} f_k(x).$$

The key assumption is that the Fourier transform, $\hat{f}_k$, of each mode has a compact support, or is at least rapidly decaying outside of a compact support. Each wavelet filter, ψ_k, is then built on top of each support. The expected modes are obtained by filtering the signal f by ψ_k, i.e. $\hat{f}_k(\xi) = \hat{f}(\xi)\hat{\psi}_k(\xi)$, where ξ is the frequency, and then the inverse Fourier transform is applied to get f_k. Note that we will denote ψ_0 the scaling function that extracts the residue r (i.e. $f_0 = r$). Empirical wavelets follow the same principle as classic wavelet except that the supports of each wavelet filter in the Fourier domain are not given by a given rule (like the dyadic decomposition) but are detected from the spectrum of f itself. In practice, given a function f, we compute its magnitude spectrum, $|\hat{f}|$, then partition the Fourier domain to obtain the supports of the expected harmonic modes. Equipped with this partition, we build the wavelet filters and finally decompose f. This process describes the empirical wavelet transform. Note that this transform is not linear since the support detection step is, in general, not linear.

If in 1D, partitions of the Fourier domain are collections of intervals, in 2D, the partition cells can have very different geometries. For instance, in [7], the authors have re-visited some existing constructions of classic wavelets, and have shown that building empirical versions of them is equivalent to partition the 2D Fourier domain with 1) rectangular boxes those edges are parallel to the frequency axis, 2) concentric rings centered around the origin, 3) polar wedges. Each of these types of partitions correspond respectively to tensor wavelets (Figure 1.a), Littlewood-Paley wavelets (Figure 1.b), and curvelets (Figure 1.c). These partitions have strong geometric constraints since they are based on boxes, rings and angular sectors. A higher level of adaptability has been reached in [1], where the authors proposed to use a watershed transform [13, 14, 15] to find the lowest level lines in the Fourier domain that separate the expected supports, see Figure 1.d. Such approach removes all geometric constraints on the shape of the partition cells. However, if such flexibility is desirable for some applications, it also has some drawbacks. More specifically, the curves corresponding to the cell edges may lack some smoothness which will directly impact the level of regularity of the built wavelets, which can be an issue in particular circumstances. To mitigate such issue, we propose in the next section to use Voronoi partitions. This solution allows us to keep a comparable level of flexibility, since we use the same seeds than in the watershed case to find the Voronoi cells; and the partition geometry is smoother since the cells edges are made of linear segments.

3. Empirical Voronoid wavelets

In this section, we give the details on the construction of Empirical Voronoi Wavelets (EVW). In a nutshell, the different steps are: 1) detect the position of meaningful harmonic modes

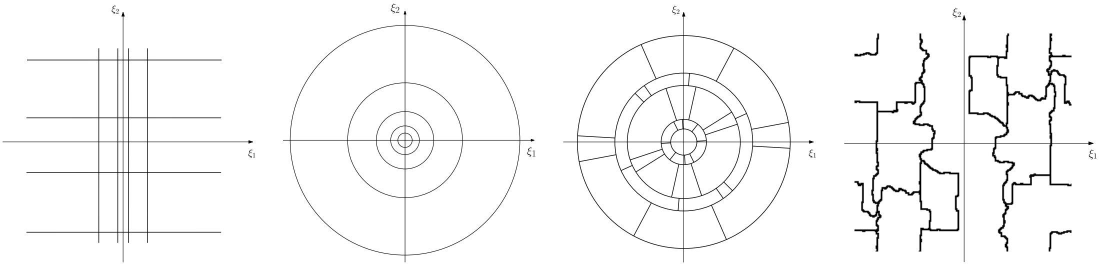


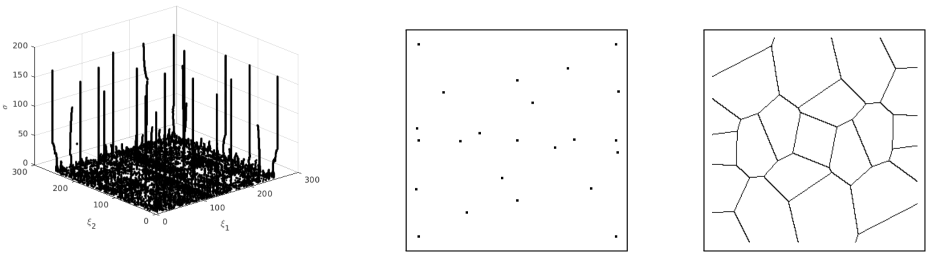
FIGURE 1. Existing 2D partitions of the Fourier domain. These different types of partitions correspond to a) tensor wavelets, b) Littlewood-Paley wavelets, c) curvelet type, d) watershed wavelets.

within the magnitude spectrum, 2) create the Voronoi partition, 3) build the wavelet filters accordingly to each Voronoi cell. Finally, the transform is obtained by performing each individual filtering. Hereafter, we provide details about these different steps.

3.1. **Detection of harmonic mode positions.** To detect the position of the meaningful harmonic modes within the magnitude spectrum, $|\hat{f}|$, we use the same method as in [1]. It consists in building a scale-space representation [6] of $|\hat{f}|$: $\mathcal{S}(\xi,\sigma) = (|\hat{f}| * g_\sigma)(\xi)$, where g_σ is a Gaussian kernel with variance σ. For each value of σ (we take the convention that $g_0 = \delta$, the Dirac function), we detect the set of local maxima, denoted $\{\xi_n^\sigma\}_{n=1}^{N_\sigma}$, in $\mathcal{S}$, where N_σ is the number of such local maxima for σ. By increasing σ, the spectrum becomes smoother removing the small variations within it, hence N_σ is decreasing (this property comes from one axiom of the scale-space theory that states that no extrema can appear while σ increases). We can then obtain a binary scale-space representation that is zero everywhere except where local maxima were detected, see Figure 2.a. The main idea is to notice that the maxima that do correspond to the expected meaningful modes are the ones corresponding to the "longest" curves in that representation. Therefore, if we denote $l_n = \arg\max_\sigma\{\xi_n^\sigma \text{ exist}\}$ the length of the curve associated to ξ_n^0, we can compute the histogram of $\{l_n\}_{n=1}^{N_0}$. This histogram will be bimodal: one mode will mostly count for the shortest curves while the second one for the longest ones. Then, we use Otsu's algorithm [12] to automatically find a threshold, T, that separates these two modes. The indices of the sought longest curves are then given by $\Lambda = \{n \,|\, l_n > T\}$, the position of the meaningful modes $\{\xi_n^0\}_{n\in\Lambda}$ extracted from Figure 2.a are depicted in Figure 2.b.

3.2. **Voronoi partitioning.** The next step consists in using the set $\{\xi_n^0\}_{n\in\Lambda}$, found previously, as the seeds of a Voronoi partitioning algorithm [16]. Each position in the domain is tagged with the label of the closest maxima position (we used the Euclidean distance). Note that, in the numerical implementation, if a given position is at equal distance from two maxima, some rule must be implemented to preserve the central symmetry that is expected when real images are processed. The Voronoi partition corresponding to the set of meaningful maxima depicted in Figure 2.b is given in Figure 2.c. To enforce a real transform, we pair together the Voronoi cells that are symmetric with respect to the origin.

3.3. **Empirical Voronoi Wavelet transform.** The construction of the wavelet filters follow the same procedure as in [1]. Given a Voronoi cell Ω, if we denote $\partial\Omega$ its edge, we define a distance



a) b) c)

FIGURE 2. a) existence of local maxima in the scale-space representation. Each curve correspond to one originally detected maxima. The vertical axis corresponds to the scale parameter σ. b) positions of maxima $\{\xi_n^0\}_{n\in\Lambda}$ corresponding to the meaningful modes. c) the Voronoi partition associated with $\{\xi_n^0\}_{n\in\Lambda}$.

Input : image f
Output: set of EVW filters $\{\widehat{\psi}_{\Omega_k}\}$, set of wavelet coefficients $\{f_k\}$

$\hat{f} \leftarrow \mathcal{F}(f)$
Detect position of harmonic modes $\{\xi_n^0\}_{n\in\Lambda}$ *from* $|\hat{f}|$
Create the Voronoi partition $\{\Omega_k\}_{k=1}^N$ *from the seeds* $\{\xi_n^0\}_{n\in\Lambda}$
for $k = 1$ *to* N **do**
    Build $\widehat{\psi}_{\Omega_k}$ *using Eq.*(3.3)
    Extract wavelet coefficients $f_k = \mathcal{F}^{-1}(\hat{f}\,\widehat{\psi}_{\Omega_k})$
end

Algorithm 1: Empirical Voronoi Wavelet Transform

transform by

$$D_\Omega(k,l) = \begin{cases} \frac{2\pi}{N}\min_{(p,q)\in\partial\Omega}\left(d(k,l,p,q)\right) & \text{if } (k,l)\in\Omega \\ -\frac{2\pi}{N}\min_{(p,q)\in\partial\Omega}\left(d(k,l,p,q)\right) & \text{if } (k,l)\notin\Omega \end{cases}, \tag{3.1}$$

where $d(k,l,p,q)$ is the quasi-Euclidean distance:

$$d(k,l,p,q) = \begin{cases} (\sqrt{2}-1)|q-l| + |p-k| & \text{if } \ |p-k| \geq |q-l| \\ (\sqrt{2}-1)|p-k| + |q-l| & \text{if } \ |p-k| < |q-l|. \end{cases} \tag{3.2}$$

The corresponding empirical Voronoi wavelet filter, $\widehat{\psi}_\Omega$, is then defined in the Fourier domain by

$$\widehat{\psi}_\Omega(k,l) = \begin{cases} 1 & \text{if } D_\Omega(k,l) > \tau \\ \cos\left(\frac{\pi}{2}\beta\left(\frac{\tau - D_\Omega(k,l)}{2\tau}\right)\right) & \text{if } D_\Omega(k,l) \leq |\tau| \\ 0 & \text{if } D_\Omega(k,l) < -\tau, \end{cases} \tag{3.3}$$

where τ defines the width of a transition area along $\partial\Omega$ and $\beta(x) = x^4(35 - 84x + 70x^2 - 20x^3)$. The Empirical Voronoi Wavelet transform (EVWT) is summarized in Algorithm 1 (we denote $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transform and its inverse, respectively).

It is straightforward to see that Proposition 1 in [1] remains valid in the present work since a Voronoi partition can be seen as a particular case of the more general partition considered in [1]. Therefore, the set $\{\psi_{\Omega_k}\}$ forms a frame. The direct consequence is the guaranty of the existence of the inverse transform by constructing the dual frame $\{\tilde{\psi}_{\Omega_k}\}$ via

$$\widehat{\tilde{\psi}}_{\Omega_k} = \frac{\widehat{\psi}_{\Omega_k}}{\sum_{k=0}^{N} |\widehat{\psi}_{\Omega_k}|^2}.$$

The inverse transform is thus given by

$$f = \mathcal{F}^{-1}\left(\sum_{k=0}^{N} \hat{f}_k \; \widehat{\tilde{\psi}}_{\Omega_k}\right).$$

4. Experiments

The example of an empirical Voronoi wavelet transform is given in Figure 3. The input image (on the top left of the figure) is a toy example made of piecewise objects (the oval and rectangle) on which four harmonic modes are superimposed. Two pairs of harmonic modes have similar frequencies but different orientations, i.e their corresponding positions in the Fourier domain lie in specific rings with different angular positions. The top right image shows the Voronoi partition plotted on top of the logarithm of the image magnitude spectrum. We can observe that the method indeed associates some specific cells to the particular harmonic modes. Finally, the wavelet coefficients, f_k, are given in the remaining images. We emphasize that each image has been re-normalized for visualization purposes most of them contain only information of very small magnitude compared to the main modes (given by the boxed images). On the other hand, the boxed images clearly show that some filters are indeed capable of extracting the different harmonic modes as well as the objects.

5. Conclusion

In this paper, we have proposed an alternative on how to create partitions in the Fourier domain for the purpose of building 2D empirical wavelets. Our solution, using Voronoi diagrams, provides a trade-off between having sub-domain with regular edges, and a high level of adaptability like the one previously proposed in the construction of empirical watershed wavelets. The corresponding Matlab code is publicly available at `https://www.mathworks.com/matlabcentral/fileexchange/42141-empirical-wavelet-transforms`.

6. Acknowledgement

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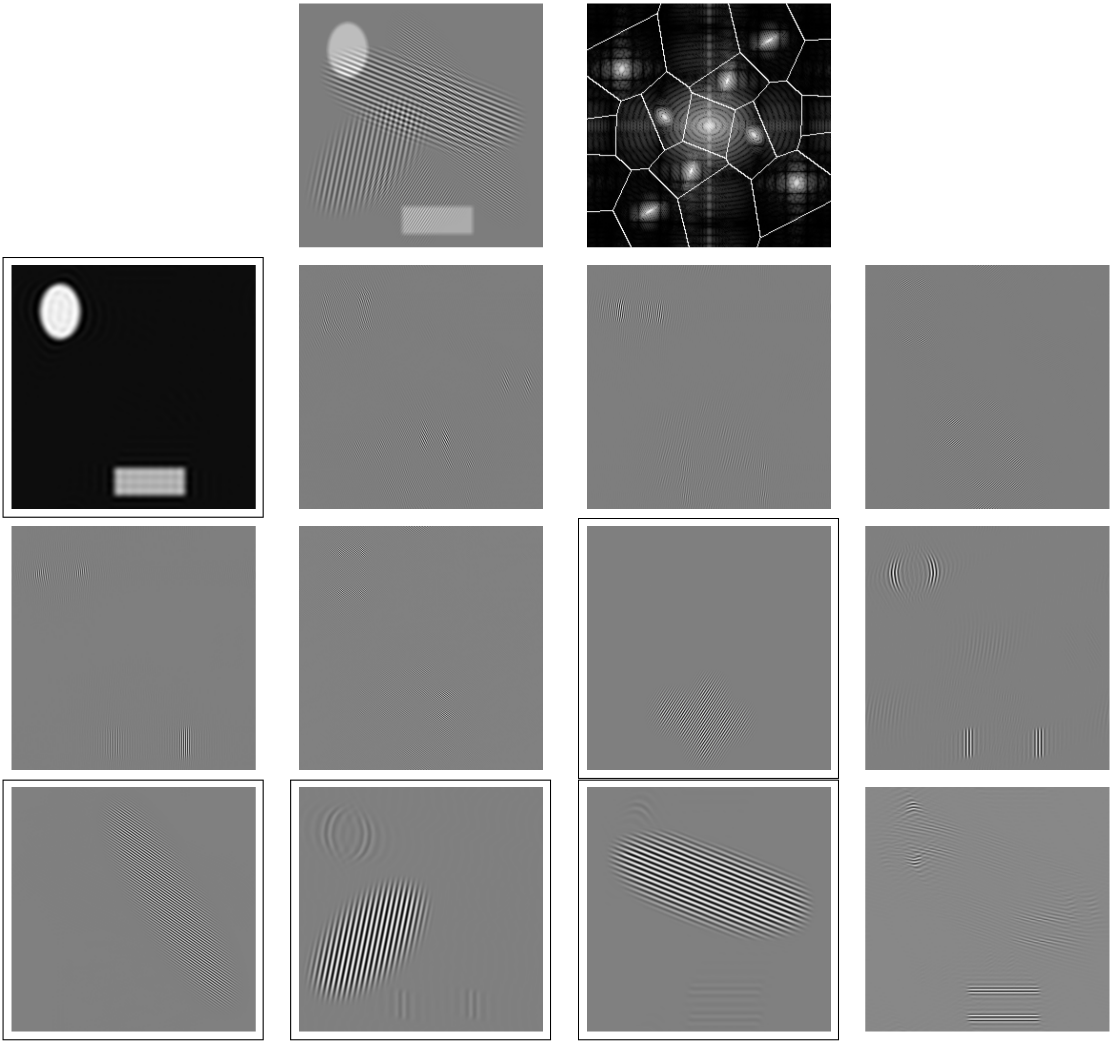

FIGURE 3. Example of an empirical Voronoi wavelet transform. The top left image is an synthetically generated input image, in particular it contains four harmonic modes. The top right image depicted the detected Voronoi partition superimposed on the logarithm of the magnitude spectrum of the input image. The remaining images correspond to the outputs of the different empirical Voronoi wavelet filters. Note that the images that look black do actually contain some information of very small energy compared to the main harmonic modes.

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Research Article

A fast converging sampling operator

BORISLAV R. DRAGANOV*

ABSTRACT. We construct a sampling operator with the property that the smoother a function is, the faster its approximation is. We establish a direct estimate and a weak converse estimate of its rate of approximation in the uniform norm by means of a modulus of smoothness and a K-functional. The case of weighted approximation is also considered. The weights are positive and power-type with non-positive exponents at infinity. This sampling operator preserves every algebraic polynomial.

Keywords: Sampling operator, sampling series, weighted approximation, direct estimate, weak converse estimate, modulus of smoothness, K-functional.

2020 Mathematics Subject Classification: 41A17, 41A25, 41A27, 41A35, 41A81, 94A20.

1. INTRODUCTION

The general form of the sampling series or operator of the function $f : \mathbb{R} \to \mathbb{R}$ is given for $w > 0$ by

$$(G_w^\chi f)(x) := \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\chi(wx-k), \quad x \in \mathbb{R}. \tag{1.1}$$

Here $\chi : \mathbb{R} \to \mathbb{R}$ is referred to as the kernel of the operator. Under certain assumptions on χ, we have that $G_w^\chi f$ is well-defined for any f in a given function class and $G_w^\chi f$ converges to f either point-wise, or in norm, as w tends to infinity. For example, if χ is continuous on $\mathbb{R}$, has compact support and

$$\sum_{k\in\mathbb{Z}} \chi(u-k) = 1, \quad u \in \mathbb{R}, \tag{1.2}$$

then (see [10, Theorem 1])

$$\lim_{w\to\infty} G_w^\chi f(x) = f(x)$$

at any point $x \in \mathbb{R}$ at which f is continuous, as, moreover, the convergence is uniform on $\mathbb{R}$ provided that f is bounded and uniformly continuous on $\mathbb{R}$. More general conditions on the kernel, which provide such approximation, can be found e.g. in [5, 6, 9, 14, 15, 17].

Clearly, (1.2) implies that G_w^χ reproduces the constant functions. Given any positive integer r, Butzer and Stens [10, pp. 165–168] constructed a kernel of compact support such that the corresponding sampling operator reproduces the algebraic polynomials up to degree $r-1$. Another approach to achieve the same goal is given in [6, Section 3.2]. The purpose of the present paper is to introduce a sampling operator which reproduces *all* algebraic polynomials.

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As a consequence, this sampling operator has the property that the smoother the function is, the faster its approximation is. We estimate the rate of approximation in unweighted and weighted uniform norm on the real line. The weights are of power-type with non-positive exponents at infinity.

The contents of the paper are organized as follows. In the next section, we construct the kernel of the sampling operator. We will consider and show that this sampling operator is well-defined for a certain broad class of continuous functions. Then, in Section 3, we state our main results about estimating its rate of approximation by a modulus of smoothness. Section 4 contains basic properties of the sampling operator, from which the main results are derived. In the last section, we provide proofs of the main results.

2. The definition of the sampling operator

Let $C(\mathbb{R})$ denote the space of the continuous (not necessarily bounded) functions on $\mathbb{R}$ and $CB(\mathbb{R})$ the space of the continuous bounded functions on $\mathbb{R}$. Further, let $\|\circ\|$ stand for the uniform norm in $CB(\mathbb{R})$. Let $C^r(\mathbb{R})$ and $C^\infty(\mathbb{R})$ be the spaces of the functions that are r-times and infinitely many times, respectively, continuously differentiable on $\mathbb{R}$. Also, as usual, let $L(\mathbb{R})$ denote the space of the Lebesgue summable functions on $\mathbb{R}$.

Let the function $\eta : \mathbb{R} \to \mathbb{R}$ be defined by

$$\eta(v) := \begin{cases} 1, & v = 0, \\ e^{-\dfrac{1}{e^{1/v^2} - e}}, & |v| < 1,\ v \neq 0, \\ 0, & |v| \geq 1. \end{cases}$$

Lemma 2.1. *We have that $\eta \in C^\infty(\mathbb{R})$.*

Proof. The assertion of the lemma is established by elementary calculus. For the sake of completeness, we include it.

Clearly, $\eta(v)$ is continuous on $\mathbb{R}$ and, for any $j \in \mathbb{N}_+$, $\eta^{(j)}(v)$ exists and is continuous on $\mathbb{R}\backslash\{0, \pm 1\}$. It remains to demonstrate that $\eta^{(j)}(v)$ exists and is continuous at $v = 0, \pm 1$. We set

$$\xi(v) := \frac{1}{e^{1/v^2} - e}, \quad v \in (-1, 1)\backslash\{0\}.$$

First, by means of Faà di Bruno's formula we get

$$\eta^{(j)}(v) = \eta(v) \sum_{m_1, m_2, \ldots, m_j} \frac{j!\,(-1)^{m_1+m_2\cdots+m_j}}{(m_1!\,1!^{m_1})(m_2!\,2!^{m_2})\cdots(m_j!\,j!^{m_j})} \times \prod_{n=1}^{j} \left(\xi^{(n)}(v)\right)^{m_n}, \quad v \in (-1, 1)\backslash\{0\}, \tag{2.1}$$

where the sum is over all non-negative integers $m_1, m_2, \ldots, m_j$ such that

$$1m_1 + 2m_2 + \cdots + jm_j = j.$$

Next, we verify by induction that

$$\xi^{(n)}(v) = \frac{\xi^{n+1}(v)}{v^{3n}} \sum_{\ell=1}^{n} e^{\ell/v^2} p_{n,\ell}(v^2), \quad v \in (-1, 1)\backslash\{0\}, \tag{2.2}$$

where $p_{n,\ell}(x)$ are algebraic polynomials of degree $n - 1$.

Now, using (2.1)-(2.2), we see by induction on $j \in \mathbb{N}_+$ that $\eta^{(j)}(v)$ exists at $v = 0, \pm 1$ and is equal to 0. Here, we use that $\eta(v)$ is continuous at these points and $\lim_{v \to 0, \pm 1 \mp 0} \eta^{(j)}(v) = 0$ for all $j \in \mathbb{N}_+$. $\square$

We need the Fourier transform of functions in $L(\mathbb{R})$. We use it in the form

$$\hat{f}(v) := \int_{\mathbb{R}} f(u) e^{-ivu} du, \quad u \in \mathbb{R}.$$

Lemma 2.2. *There exists $\theta \in L(\mathbb{R})$ such that $\hat{\theta}(v) = \eta(v)$, $v \in \mathbb{R}$. Moreover,*

$$\theta(u) = \frac{1}{\pi} \int_0^1 \eta(v) \cos uv \, dv, \quad u \in \mathbb{R}, \tag{2.3}$$

$\theta \in C^\infty(\mathbb{R})$ and $\theta^{(j)}(u) = O(|u|^{-n})$ as $u \to \pm\infty$ for all $j, n \in \mathbb{N}_0$.

Proof. The existence of θ as well as its representation (2.3) can be established by means of e.g. [8, Proposition 6.3.10] (see also its proof; let us note that the Fourier transform is normalized differently in [8]). The last two assertions of the lemma are established directly from the theorem for differentiation under the integral sign and by integration by parts, as it is more convenient to write $\theta(u)$ in the form

$$\theta(u) = \frac{1}{2\pi} \int_{-1}^1 \eta(v) \cos uv \, dv$$

and use that $\eta^{(n)}(\pm 1) = 0$ for all $n \in \mathbb{N}_0$. $\square$

We will consider the sampling operator

$$G_w := G_w^\theta,$$

where G_w^θ is defined in (1.1) with $\chi := \theta$ given in (2.3). As we will establish now, $G_w f(x)$, $x \in \mathbb{R}$, is well defined for any $f \in C(\mathbb{R})$ of at most polynomial growth at infinity. Actually, more is valid.

Proposition 2.1. *If $f \in C(\mathbb{R})$ is such that $f(x) = O(|x|^\nu)$ as $x \to \pm\infty$ with some $\nu \in \mathbb{N}_0$, then $G_w f \in C^\infty(\mathbb{R})$, $w > 0$, as, moreover,*

$$(G_w f)^{(j)}(x) = w^j \sum_{k \in \mathbb{Z}} f\left(\frac{k}{w}\right) \theta^{(j)}(wx - k), \quad x \in \mathbb{R}, \ j \in \mathbb{N}_0, \tag{2.4}$$

the series being uniformly convergent on the compact intervals of $\mathbb{R}$.

Proof. Below, we will denote by c positive constants, not necessarily the same at each occurrence, which are independent of $x \in \mathbb{R}$ and k. We have that

$$|f(x)| \le c(1 + |x|)^\nu, \quad x \in \mathbb{R}. \tag{2.5}$$

Let $j \in \mathbb{N}_0$. By Lemma 2.2, we have

$$|\theta^{(j)}(x)| \le c(1 + |x|)^{-\nu - 2}, \quad x \in \mathbb{R}. \tag{2.6}$$

Let $[a, b]$ be an arbitrary compact subinterval of $\mathbb{R}$. Let $\gamma := w \max\{|a|, |b|\}$. Then for all $x \in [a, b]$ and $k \in \mathbb{Z}$ such that $|k| \ge \gamma$, we have

$$1 + |wx - k| \ge 1 + |k| - \gamma \ge \frac{|k| + 1}{\gamma + 1};$$

hence, using (2.5) and (2.6), we arrive at the estimate

$$\left| f\left(\frac{k}{w}\right)\theta^{(j)}(wx-k)\right| \le c\left(1+\frac{|k|}{w}\right)^{\nu}(1+|wx-k|)^{-\nu-2}$$
$$\le \frac{c}{(|k|+1)^2}, \qquad x\in[a,b],\ |k|\ge\gamma.$$

Now, the Weierstrass M-test implies that the series

$$\sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\theta^{(j)}(wx-k)$$

is uniformly convergent on $[a,b]$ for each $j\in\mathbb{N}_0$. Consequently, $G_w f\in C^\infty(\mathbb{R})$ for every $w>0$ and (2.4) holds. □

Remark 2.1. *As it follows from Propositions 4.2 and 4.4 below, $(G_w f)^{(j)}(x)$, $j\in\mathbb{N}_0$, is at most of the same polynomial growth at infinity as $f(x)$.*

3. Estimates of the rate of approximation of G_w

We will consider approximation by G_w in the weighted uniform norm with the weight

$$\rho_{\alpha,\beta}(x) := \begin{cases} |x|^{-\alpha}, & x<-1,\\ 1, & -1\le x\le 1,\\ x^{-\beta}, & x>1,\end{cases}$$

where $\alpha,\beta\ge 0$. Let us explicitly note that the results obtained include the unweighted case $\rho_{0,0}(x)\equiv 1$. In the case $\alpha=\beta$, we can instead write $\rho_{\alpha,\alpha}$, equivalently, in the concise form

$$\rho_{\alpha,\alpha}(x) := \frac{1}{1+|x|^\alpha}, \qquad x\in\mathbb{R}.$$

As we observed earlier (Proposition 2.1), $G_w f$ is a well-defined infinitely continuously differentiable function on $\mathbb{R}$ for any $f\in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f\in CB(\mathbb{R})$ with some $\alpha,\beta\ge 0$.

Let $f\in C(\mathbb{R})$ be such that $\rho_{\alpha,\beta} f\in CB(\mathbb{R})$. We will use the modulus of smoothness of order $r\in\mathbb{N}_+$ of f, defined for $t>0$ by

$$\omega_r(f,t)_{\alpha,\beta} := \sup_{0<h\le t}\|\rho_{\alpha,\beta}\Delta_h^r f\|,$$

where $\Delta_h f(x) := f(x+h/2)-f(x-h/2)$, $x\in\mathbb{R}$, $h>0$, and $\Delta_h^r := \Delta_h(\Delta_h^{r-1})$. Clearly, $\rho_{\alpha,\beta}\Delta_h^r f\in CB(\mathbb{R})$ for every $f\in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f\in CB(\mathbb{R})$, and every $h>0$.

We will establish the following direct estimate of the rate of approximation of G_w.

Theorem 3.1. *Let $\alpha,\beta\ge 0$ and $r\in\mathbb{N}_+$. Then for all $f\in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f\in CB(\mathbb{R})$, and all $w\ge 1$ there holds*

$$\|\rho_{\alpha,\beta}(G_w f-f)\| \le c\,\omega_r(f,1/w)_{\alpha,\beta}.$$

Above c is a positive constant whose value is independent of f and w.

This theorem and basic properties of the modulus of smoothness, or more directly Proposition 4.3 below imply that if $f\in C^\infty(\mathbb{R})$ and $\rho_{\alpha,\beta} f^{(r)}\in CB(\mathbb{R})$ for all $r\in\mathbb{N}_0$, then

$$\|\rho_{\alpha,\beta}(G_w f-f)\| = O(w^{-r}) \text{ as } w\to\infty \quad \forall r\in\mathbb{N}_+;$$

in particular, if $f\in C^\infty(\mathbb{R})$ and $f^{(r)}\in CB(\mathbb{R})$ for all $r\in\mathbb{N}_0$, then

$$\|G_w f-f\| = O(w^{-r}) \text{ as } w\to\infty \quad \forall r\in\mathbb{N}_+.$$

Also, let us note that Theorem 3.1 yields that G_w preserves any algebraic polynomial (see also Corollary 4.1 and Remark 4.2 below).

Estimates of the rate of approximation of general sampling operators (1.1) in spaces of continuous functions associated with the weight $\rho_{2,2}$ have been recently obtained in [1]. Similar results for integral modifications of the general sampling operator were established in [2, 3]. Also, such results were proved for an integral form of general exponential sampling operators in function spaces equipped with a logarithmic weight in [4, Section 5].

The assertion of Theorem 3.1 in the unweighted case follows from the one-dimensional form of the general assertion in [15, Theorem 6, (8)]. A direct estimate of a different type than the one in Theorem 3.1 was established for a very general class of sampling operators, in particular, in the essential supremum norm with the weight $\rho_{\alpha,\alpha}$ under certain additional assumptions on f in [13, Theorem 31 and Remark 34]. There the rate of approximation of a general class of multivariate quasi-projection operators in weighted L_p-spaces was considered.

The direct estimate in Theorem 3.1 is essentially best possible–the following equivalence result holds.

Theorem 3.2. *Let $\alpha, \beta \geq 0$, $r \in \mathbb{N}_+$, $0 < \lambda < r$ and $f \in C(\mathbb{R})$ be such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$. Then*

$$\|\rho_{\alpha,\beta}(G_w f - f)\| = O(w^{-\lambda}) \qquad \iff \qquad \omega_r(f,t)_{\alpha,\beta} = O(t^\lambda).$$

We will prove these theorems in the last section.

4. Basic relations and estimates

We will often apply the following auxiliary result (cf. [1, Proposition 1]).

Lemma 4.3. *Let $\alpha, \beta \geq 0$ and $j, \ell \in \mathbb{N}_0$. Then for all $x \in \mathbb{R}$ and $w \geq 1$, there holds*

$$\sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} |wx-k|^\ell |\theta^{(j)}(wx-k)| \leq c\rho_{\alpha,\beta}(x)^{-1}. \tag{4.1}$$

Above c is a positive constant whose value is independent of $x \in \mathbb{R}$ and $w \geq 1$.

Proof. First, let us note that it is sufficient to establish (4.1) for $x \geq 0$. This readily follows from the relations

$$\rho_{\alpha,\beta}(x) = \rho_{\beta,\alpha}(-x), \quad x \in \mathbb{R},$$

and

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} |wx-k|^\ell |\theta^{(j)}(wx-k)|\\ &=\sum_{k\in\mathbb{Z}} \rho_{\beta,\alpha}\left(\frac{k}{w}\right)^{-1} |w(-x)-k|^\ell |\theta^{(j)}(w(-x)-k)|, \quad x \in \mathbb{R}.\end{aligned}$$

In the latter formula, we have taken into account that $\theta(u)$ is even; hence $\theta^{(2\ell)}(u)$ are even too, and $\theta^{(2\ell+1)}(u)$ are odd. Thus, let $x \geq 0$. We will estimate the sum on the negative k. We have

$$\rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} \leq 1 + \left|\frac{k}{w}\right|^\alpha \leq 2|k|^\alpha \leq 2(wx-k)^\alpha, \quad k \leq -1,\ w \geq 1. \tag{4.2}$$

Let $n \in \mathbb{N}_+$ be such that $\alpha + \ell - n < -1$. By virtue of Lemma 2.2, for any fixed $j \in \mathbb{N}_0$ there exists a positive constant c such that for all $u \geq 1$ there holds

$$|\theta^{(j)}(u)| \leq cu^{-n}. \tag{4.3}$$

Since $wx - k \geq 1$ for $k \leq -1$, using (4.2) and (4.3), we get

$$\begin{aligned}\sum_{k\leq -1} \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} |wx-k|^{\ell}|\theta^{(j)}(wx-k)| &\leq c \sum_{k\leq -1}(wx-k)^{\alpha+\ell-n} \\ &\leq c\sum_{k\geq 1} k^{\alpha+\ell-n} \leq c.\end{aligned} \tag{4.4}$$

To estimate the sum on the non-negative k, we take into account that

$$\begin{aligned}\rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} &\leq 1 + \left(\frac{k}{w}\right)^{\beta} \leq 1 + c\left(\left|\frac{k}{w}-x\right|^{\beta} + x^{\beta}\right) \\ &\leq c(1+|wx-k|^{\beta}+x^{\beta}), \quad k\geq 0,\ w\geq 1.\end{aligned}$$

Let $n \in \mathbb{N}_+$ be such that $\beta+\ell-n < -1$. By virtue of Lemma 2.2, there exists a positive constant c such that

$$|\theta^{(j)}(u)| \leq c(1+|u|)^{-n}, \quad u\in\mathbb{R}.$$

Consequently, similarly as in the previous case, we arrive at

$$\begin{aligned}&\sum_{k\geq 0}\rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} |wx-k|^{\ell}|\theta^{(j)}(wx-k)| \\ &\leq c\sum_{k\geq 0}(1+|wx-k|^{\beta}+x^{\beta})|wx-k|^{\ell}(1+|wx-k|)^{-n} \\ &\leq c\sum_{k\geq 0}(1+|wx-k|)^{\beta+\ell-n} + cx^{\beta}\sum_{k\geq 0}(1+|wx-k|)^{\ell-n} \\ &\leq c(1+x^{\beta})\sum_{k\geq 0}(1+|wx-k|)^{\beta+\ell-n} \\ &\leq c(1+x^{\beta}),\end{aligned}$$

as at the last estimate we have taken into consideration that the series $\sum_{k\geq 0}(1+|u-k|)^{\beta+\ell-n}$ is convergent for every $u \geq 0$ and its sum is bounded on $[0,\infty)$. The latter can be easily verified if we consider instead the series $\sum_{k\in\mathbb{Z}}(1+|u-k|)^{\beta+\ell-n}$ for $u\in\mathbb{R}$. Clearly, it is uniformly convergent on each compact interval; hence its sum is a continuous function on $\mathbb{R}$. In addition, the sum is 1-periodic; consequently, it is bounded on $\mathbb{R}$. Combining the estimates we established above on the sums on $k<0$ and $k\geq 0$, we arrive at

$$\sum_{k\in\mathbb{Z}}\rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} |wx-k|^{\ell}|\theta^{(j)}(wx-k)| \leq c(1+x^{\beta}), \quad x\geq 0;$$

hence (4.1) follows for $x \geq 0$. In view of the observation, we made in the beginning about the symmetry of the cases $x\leq 0$ and $x\geq 0$, the proof of the lemma is complete. □

We proceed to the basic properties of the operator G_w, which we will later use to establish estimates of its rate of approximation. We begin with showing that the family of operators $\{G_w\}_{w\geq 1}$ is uniformly bounded in the weighted spaces of continuous functions associated with the uniform norm with the weight $\rho_{\alpha,\beta}$.

Henceforward, c denotes positive constants, not necessarily the same at each occurrence, which are independent of the function and the operator order w.

Proposition 4.2. *Let $\alpha, \beta \ge 0$. Then for all $f \in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$, and all $w \ge 1$ there holds*

$$\|\rho_{\alpha,\beta} G_w f\| \le c\, \|\rho_{\alpha,\beta} f\|.$$

Proof. We have

$$|\rho_{\alpha,\beta}(x) G_w f(x)| \le \rho_{\alpha,\beta}(x) \sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} |\theta(wx-k)|\, \|\rho_{\alpha,\beta} f\|, \quad x \in \mathbb{R}.$$

Then the assertion follows immediately from Lemma 4.3 with $j = \ell = 0$. □

The discrete moment of θ of order $j \in \mathbb{N}_0$ is defined by

$$m_j(u) := \sum_{k\in\mathbb{Z}} (k-u)^j\, \theta(u-k), \quad u \in \mathbb{R}.$$

The following assertions for the discrete moments of θ holds true.

Lemma 4.4. *We have $m_0(u) = 1$ and $m_j(u) = 0$, $j \in \mathbb{N}_+$, for all $u \in \mathbb{R}$.*

Proof. The proof is standard–based on the Poisson summation formula (see e.g. [16, Theorem 4.2.8] or [8, Propositions 4.1.5, 5.1.28 and 5.1.29, and (3.1.22)]) and connected with certain Strang-Fix type conditions on θ (see e.g. [10, Lemma 3]). For the readers' convenience we include it. We apply the Poisson summation formula to the function $\theta_j(u) := u^j\theta(u)$, $j \in \mathbb{N}_0$. We have $\theta_j \in L(\mathbb{R})$ by virtue of Lemma 2.2. The Fourier transform of θ_j is

$$\widehat{\theta_j}(v) = i^j \hat{\theta}^{(j)}(v) = i^j \eta^{(j)}(v), \quad v \in \mathbb{R}, \tag{4.5}$$

where we have taken into account Lemma 2.2. Trivially, the series $\sum_{k\in\mathbb{Z}} |\widehat{\theta_j}(2\pi k)|$ is convergent.

Now, the Poisson summation formula, (4.5), $\eta^{(j)}(0) = 0$ for $j \ge 1$ (see Lemma 2.1) and $\eta^{(j)}(v) = 0$ for $|v| > 1$ and $j \ge 0$ yield

$$\begin{aligned} m_j(u) &= (-1)^j \sum_{k\in\mathbb{Z}} \theta_j(u-k) = (-1)^j \sum_{k\in\mathbb{Z}} \widehat{\theta_j}(2\pi k) e^{i2\pi k u} \\ &= (-i)^j \sum_{k\in\mathbb{Z}} \eta^{(j)}(2\pi k) e^{i2\pi k u} \\ &= \begin{cases} 1, & u \in \mathbb{R},\ j = 0, \\ 0, & u \in \mathbb{R},\ j \in \mathbb{N}_+. \end{cases} \end{aligned}$$

□

The following Jackson-type inequality holds true for G_w.

Proposition 4.3. *Let $\alpha, \beta \ge 0$ and $r \in \mathbb{N}_+$. Then for all $g \in C^r(\mathbb{R})$ such that $\rho_{\alpha,\beta} g, \rho_{\alpha,\beta} g^{(r)} \in CB(\mathbb{R})$, and all $w \ge 1$ there holds*

$$\|\rho_{\alpha,\beta}(G_w g - g)\| \le \frac{c}{w^r} \|\rho_{\alpha,\beta} g^{(r)}\|.$$

Proof. We expand $g(k/w)$ by Taylor's formula at the point $x \in \mathbb{R}$ to get

$$g\left(\frac{k}{w}\right) = \sum_{j=0}^{r-1} \frac{g^{(j)}(x)}{j!}\left(\frac{k}{w} - x\right)^j + \frac{1}{(r-1)!}\int_x^{k/w} \left(\frac{k}{w} - u\right)^{r-1} g^{(r)}(u)\, du. \tag{4.6}$$

Then, taking into account Lemma 4.4, we get the relation

$$(4.7)\qquad (G_w g)(x) - g(x) = \frac{1}{(r-1)!}\sum_{k\in\mathbb{Z}} \int_x^{k/w} \left(\frac{k}{w} - u\right)^{r-1} g^{(r)}(u)\, du\, \theta(wx-k), \quad x \in \mathbb{R}.$$

Consequently,

$$(4.8)\ |(G_w g)(x) - g(x)| \le \frac{\|\rho_{\alpha,\beta} g^{(r)}\|}{(r-1)!}\sum_{k\in\mathbb{Z}} \left| \int_x^{k/w} \left|\frac{k}{w} - u\right|^{r-1} \rho_{\alpha,\beta}(u)^{-1} du \right| |\theta(wx-k)|, \quad x \in \mathbb{R}.$$

The function $\rho_{\alpha,\beta}(u)^{-1}$ is positive, decreasing on $(-\infty, 0]$, and increasing on $[0, +\infty)$; hence

$$\rho_{\alpha,\beta}(u)^{-1} \le \rho_{\alpha,\beta}(x)^{-1} + \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} \quad \text{for } u \text{ between } x \text{ and } \frac{k}{w}.$$

Therefore, we deduce from (4.8) that

$$\begin{aligned} &|(G_w g)(x) - g(x)| \\ \le &\frac{\|\rho_{\alpha,\beta} g^{(r)}\|}{(r-1)!}\sum_{k\in\mathbb{Z}} \left| \int_x^{k/w} \left(\frac{k}{w} - u\right)^{r-1} du \right| \left(\rho_{\alpha,\beta}(x)^{-1} + \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1}\right) |\theta(wx-k)| \\ = &\frac{\|\rho_{\alpha,\beta} g^{(r)}\|}{r!\, w^r}\sum_{k\in\mathbb{Z}} |wx-k|^r \left(\rho_{\alpha,\beta}(x)^{-1} + \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1}\right) |\theta(wx-k)| \\ = &\frac{\|\rho_{\alpha,\beta} g^{(r)}\|}{r!\, w^r}\rho_{\alpha,\beta}(x)^{-1}\sum_{k\in\mathbb{Z}} |wx-k|^r |\theta(wx-k)| \\ + &\frac{\|\rho_{\alpha,\beta} g^{(r)}\|}{r!\, w^r}\sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} |wx-k|^r |\theta(wx-k)|, \quad x \in \mathbb{R}. \end{aligned}$$

Now, the assertion of the proposition follows from Lemma 4.3 with $j = 0$ and $\ell = r$, as to estimate the sum $\sum_{k\in\mathbb{Z}} |wx-k|^r |\theta(wx-k)|$, we apply it with $\alpha = \beta = 0$. □

If p is an algebraic polynomial of degree at most n, then $\rho_{n,n} p \in CB(\mathbb{R})$ and Proposition 4.3 with $r = n+1$ implies that G_w preserves p for all $w \ge 1$.

Corollary 4.1. *We have $G_w p = p$ for any algebraic polynomial p and all $w \ge 1$.*

Remark 4.2. *Actually, as it is quite easy to see, the assertion of the corollary holds for all $w > 0$.*

We will need a Bernstein-type inequality for G_w.

Proposition 4.4. *Let $\alpha, \beta \ge 0$ and $r \in \mathbb{N}_+$. Then for all $f \in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$, and all $w \ge 1$ there holds*

$$\|\rho_{\alpha,\beta}(G_w f)^{(r)}\| \le c w^r \|\rho_{\alpha,\beta} f\|.$$

Proof. Let us first recall that $G_w f \in C^\infty(\mathbb{R})$ (see Proposition 2.1). Then, by virtue of (2.4), we have

$$|\rho_{\alpha,\beta}(x)(G_w f)^{(r)} f(x)| \le w^r \rho_{\alpha,\beta}(x) \sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} |\theta^{(r)}(wx-k)|\, \|\rho_{\alpha,\beta} f\|, \quad x \in \mathbb{R}.$$

Now, the estimate in the proposition follows from Lemma 4.3 with $j = r$ and $\ell = 0$. □

The last auxiliary result for G_w, we will need, is an estimate of the weighted uniform norm of the derivatives of $G_w g$ for smooth g. In order to establish it, we will make use of a property of the discrete moments of the derivatives of θ, which is similar to Lemma 4.4. We set

$$m_{r,j}(u) := \sum_{k\in\mathbb{Z}} (k-u)^j\, \theta^{(r)}(u-k), \quad u \in \mathbb{R}.$$

The following assertions for the discrete moments of $\theta^{(r)}$ holds true.

Lemma 4.5. *Let $r \in \mathbb{N}_+$. We have $m_{r,j}(u) = 0$ for all $u \in \mathbb{R}$, where $j = 0, \ldots, r-1$.*

Proof. Just similarly as in the proof of Lemma 4.4, we apply the Poisson summation formula but to the function $\theta_{r,j}(u) := u^j \theta^{(r)}(u)$, $j \in \mathbb{N}_0$. Since $\widehat{\theta^{(r)}}(v) = (iv)^r \hat{\theta}(v) = (iv)^r \eta(v)$, $v \in \mathbb{R}$ (recall Lemma 2.2), we get

$$\widehat{\theta_{r,j}}(v) = i^j \widehat{\theta^{(r)}}^{(j)}(v) = i^{r+j} \big(v^r \eta(v)\big)^{(j)}, \quad v \in \mathbb{R}.$$

We have for $j = 0, \ldots, r-1$

$$\big(v^r \eta(v)\big)^{(j)} = \sum_{\ell=0}^{j} \binom{j}{\ell} r(r-1)\cdots(r-\ell+1) v^{r-\ell} \eta^{(j-\ell)}(v);$$

hence, we get $\widehat{\theta_{r,j}}(2\pi k) = 0$ for all $k \in \mathbb{Z}$.

Now, the Poisson summation formula yields

$$m_{r,j}(u) = (-1)^j \sum_{k\in\mathbb{Z}} \theta_{r,j}(u-k) = (-1)^j \sum_{k\in\mathbb{Z}} \widehat{\theta_{r,j}}(2\pi k) e^{i2\pi k u} \equiv 0.$$

□

Proposition 4.5. *Let $\alpha, \beta \geq 0$ and $r \in \mathbb{N}_+$. Then for all $g \in C^r(\mathbb{R})$ such that $\rho_{\alpha,\beta} g^{(r)} \in CB(\mathbb{R})$, and all $w \geq 1$ there holds*

$$\|\rho_{\alpha,\beta} (G_w g)^{(r)}\| \leq c\, \|\rho_{\alpha,\beta} g^{(r)}\|.$$

Proof. Since $\rho_{\alpha,\beta} g^{(r)} \in CB(\mathbb{R})$, then $g(x) = O(|x|^\nu)$ as $x \to \pm\infty$ with some $\nu \in \mathbb{N}_+$. Then, by (2.4), we have

$$(G_w g)^{(r)}(x) = w^r \sum_{k\in\mathbb{Z}} g\left(\frac{k}{w}\right) \theta^{(r)}(wx-k), \quad x \in \mathbb{R}. \tag{4.9}$$

We substitute $g(k/w)$ with its Taylor's expansion (4.6) and apply Lemma 4.5 to arrive at

$$(G_w g)^{(r)}(x) = \frac{w^r}{(r-1)!} \sum_{k\in\mathbb{Z}} \int_x^{k/w} \left(\frac{k}{w} - u\right)^{r-1} g^{(r)}(u)\, du\, \theta^{(r)}(wx-k), \quad x \in \mathbb{R}.$$

We complete the proof with the same argument, used to establish Proposition 4.3, but with $\theta^{(r)}$ in place of θ and we apply Lemma 4.3 with $j = \ell = r$. □

5. Estimates of the rate of approximation of G_w by a K-functional

In this section, we will establish a direct inequality and a matching weak converse inequality for the rate of approximation of G_w in the uniform norm on $\mathbb{R}$ with the weight $\rho_{\alpha,\beta}$ by means of a K-functional. These estimates follow from the basic properties of the operator given in the preceding section by means of standard techniques (see e.g. [11, Chapter 7, §§ 3 and 5] or [12, Chapters 9 and 10]).

The K-functional we will use is defined for $f \in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$, and $t > 0$ by

$$K_r(f,t)_{\alpha,\beta} := \inf\{\|\rho_{\alpha,\beta}(f-g)\| + t\,\|\rho_{\alpha,\beta} g^{(r)}\| : g \in C^r(\mathbb{R}),\ \rho_{\alpha,\beta} g, \rho_{\alpha,\beta} g^{(r)} \in CB(\mathbb{R})\}.$$

We proceed to the direct estimate.

Theorem 5.3. *Let $\alpha, \beta \geq 0$ and $r \in \mathbb{N}_+$. Then for all $f \in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$, and all $w \geq 1$ there holds*

$$\|\rho_{\alpha,\beta}(G_w f - f)\| \leq c\, K_r(f, w^{-r})_{\alpha,\beta}.$$

Proof. Let $g \in C^r(\mathbb{R})$ be such that $\rho_{\alpha,\beta} g, \rho_{\alpha,\beta} g^{(r)} \in CB(\mathbb{R})$. Then, by virtue of Propositions 4.2 and 4.3, we get

$$\begin{aligned}\|\rho_{\alpha,\beta}(G_w f - f)\| &\leq \|\rho_{\alpha,\beta} G_w (f-g)\| + \|\rho_{\alpha,\beta}(G_w g - g)\| + \|\rho_{\alpha,\beta}(g-f)\| \\ &\leq c\left(\|\rho_{\alpha,\beta}(f-g)\| + \frac{1}{w^r}\,\|\rho_{\alpha,\beta} g^{(r)}\|\right).\end{aligned}$$

Now, we take the infimum on g to arrive at the assertion of the theorem. □

The following weak converse inequality holds.

Theorem 5.4. *Let $\alpha, \beta \geq 0$ and $r \in \mathbb{N}_+$. Then for all $f \in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$, and all $w, v \geq 1$ there holds*

$$K_r(f, w^{-r})_{\alpha,\beta} \leq \|\rho_{\alpha,\beta}(G_v f - f)\| + c\left(\frac{v}{w}\right)^r K_r(f, v^{-r})_{\alpha,\beta}.$$

Proof. By virtue of Propositions 2.1, 4.2 and 4.4, we have $G_v f \in C^r(\mathbb{R})$ and $\rho_{\alpha,\beta} G_v f, \rho_{\alpha,\beta}(G_v f)^{(r)} \in CB(\mathbb{R})$. Then

$$K_r(f, w^{-r})_{\alpha,\beta} \leq \|\rho_{\alpha,\beta}(f - G_v f)\| + \frac{1}{w^r}\,\|\rho_{\alpha,\beta}(G_v f)^{(r)}\|. \tag{5.1}$$

Let $g \in C^r(\mathbb{R})$ be such that $\rho_{\alpha,\beta} g, \rho_{\alpha,\beta} g^{(r)} \in CB(\mathbb{R})$. Then, we use Propositions 4.4 and 4.5 to estimate the second term on the right above as follows:

$$\begin{aligned}\|\rho_{\alpha,\beta}(G_v f)^{(r)}\| &\leq \|\rho_{\alpha,\beta}(G_v (f-g))^{(r)}\| + \|\rho_{\alpha,\beta}(G_v g)^{(r)}\| \\ &\leq c v^r \left(\|\rho_{\alpha,\beta}(f-g)\| + \frac{1}{v^r}\|\rho_{\alpha,\beta} g^{(r)}\|\right).\end{aligned} \tag{5.2}$$

Combining (5.1) and (5.2), we arrive at

$$K_r(f, w^{-r})_{\alpha,\beta} \leq \|\rho_{\alpha,\beta}(G_v f - f)\| + c\left(\frac{v}{w}\right)^r \left(\|\rho_{\alpha,\beta}(f-g)\| + \frac{1}{v^r}\|\rho_{\alpha,\beta} g^{(r)}\|\right).$$

Finally, we take the infimum on g to derive the assertion of the theorem. □

Theorems 5.3 and 5.4 yield the following characterization of the rate of the approximation of G_w.

Corollary 5.2. *Let $\alpha, \beta \geq 0$, $r \in \mathbb{N}_+$, $0 < \lambda < r$ and $f \in C(\mathbb{R})$ be such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$. Then*

$$\|\rho_{\alpha,\beta}(G_w f - f)\| = O(w^{-\lambda}) \quad \Longleftrightarrow \quad K_r(f,t)_{\alpha,\beta} = O(t^{\lambda/r}).$$

Proof. If $K_r(f,t)_{\alpha,\beta} = O(t^{r/\lambda})$, then Theorem 5.3 implies $\|\rho_{\alpha,\beta}(G_w f - f)\| = O(w^{-\lambda})$.

To establish the inverse implication, we will use the Berens-Lorentz Lemma [7]. We will apply it in the form given in [11, Chapter 10, Lemma 5.2]. We set $\phi(x) := K_r(f, x^2)_{\alpha,\beta}$, $0 < x \le 1$, and $\mu := 2\lambda/r \in (0,2)$. Then Theorem 5.4 implies

$$\phi(x) \le c_f \left(y^\mu + \frac{x^2}{y^2}\, \phi(y) \right), \quad 0 < x \le y \le 1,$$

where c_f is a positive constant whose value may depend on f, but not on x and y. Now, the Berens-Lorentz Lemma yields

$$\phi(x) \le c' c_f x^\mu, \quad 0 < x \le 1,$$

with some positive constant c'; hence $K_r(f,t)_{\alpha,\beta} = O(t^{\lambda/r})$. □

The K-functional above and the modulus of smoothness given in Section 3 are equivalent, that is, there exist constants $c, t_0 > 0$ such that for all $f \in C(\mathbb{R})$ with $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$, and all $t \in (0, t_0]$ there hold (see [12, Theorem 6.1.1 with $\varphi \equiv 1$ and $p = \infty$])

$$c^{-1}\omega_r(f,t)_{\alpha,\beta} \le K_r(f,t^r)_{\alpha,\beta} \le c\,\omega_r(f,t)_{\alpha,\beta}. \tag{5.3}$$

Actually, it can be shown by means of the standard method to prove the above equivalence in the unweighted case (see e.g. [11, p. 177]) that it holds for any fixed positive t_0 (with c depending on t_0).

Combining Theorem 5.3 and Corollary 5.2 with relations (5.3) with $t_0 = 1$, we immediately get Theorems 3.1 and 3.2.

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Research Article

Directs estimates and a Voronovskaja-type formula for Mihesan operators

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ABSTRACT. We present an estimate for the rate of convergence of Mihesan operators in polynomial weighted spaces. A Voronovskaja-type theorem is included.

Keywords: Mihesan operators, rate of convergence, polynomial weighted spaces, Voronovskaja-type theorems.

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1. INTRODUCTION

For $|t| < 1$, let us consider the expansion

$$\frac{e^{at}}{(1-t)^y} = \sum_{k=0}^{\infty} p_{k,a}(y)\frac{t^k}{k!}, \qquad p_{k,a}(y) = \sum_{i=0}^{k} \binom{k}{i}(y)_i a^{k-i}. \tag{1.1}$$

Recall that, for $i \in \mathbb{N}$, $(x)_i = x(x+1)\cdots(x+i-1)$, while $(x)_0 = 1$. If we take $y = n$ and $t = x/(x+1)$, then

$$e^{ax/(x+1)}(1+x)^n = \sum_{k=0}^{\infty} \frac{p_{k,a}(n)}{k!}\left(\frac{x}{1+x}\right)^k.$$

For $a \geq 0$, Mihesan [9] defined

$$B_n^a(f,x) = \sum_{k=0}^{\infty} W_{n,k}^a(x) f\left(\frac{k}{n}\right), \quad \text{where} \quad W_{n,k}^a(x) = e^{-ax/(x+1)}\frac{p_{k,a}(n)x^k}{k!(1+x)^{n+k}}.$$

We also write $B_n^a(f(t),x)$ instead of $B_n^a(f,x)$. Notice that, for $a = 0$, $B_n^a(f)$ is just a Baskakov operator.

In this paper, we present a Voronovsakaja type result for the operators B_n^a in a weighted space $C_\varrho[0,\infty)$ defined as follows: for the weight $\varrho(x) = 1/(1+x)^q$ ($q \geq 0$ a fixed real),

$$C_\varrho[0,\infty) = \Big\{ f \in C[0,\infty) \,:\, \|f\|_\varrho < \infty \Big\},$$

where $\|f\|_\varrho = \sup_{x\geq 0} |\varrho(x) f(x)|$. In order to present a simple proof, here we only consider the case $q \geq 3/2$.

It is known (see [9]) that, if $f \in C[0,\infty)$ and there exist positive constants A and B such that $|f(x)| \leq Be^{Ax}$, then $B_n^a(f)$ is well defined. Hence, $B_n^a(f)$ is defined for all $f \in C_\varrho[0,\infty)$.

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In [11] and [13] some pointwise asymptotic expansions were given for Mihesan operators. But we remark that the results are not correct. For instance, in Theorem 4.1 of [11] and in Theorem 2.2 in [13] a term $g'(x)B_n^a(t-x,x)$ should be added.

The paper is organized as follows. In Section 2 we recall some known results. Section 3 is very technical. We need estimates for the moments of the operators $M_{n,k}^a(x)$ for $1 \le k \le 6$. Finally, in Section 5 we present the Voronovskaja type theorem. We remark that some different quantitative Voronovskaja theorems were given in [6] and [7]. An inverse result will appear in another paper.

In what follows C and C_i will denote absolute constants. They may be different on each occurrence.

2. Known results

It is known that (see [9])

$$B_n^a(1,x) = 1, \qquad B_n^a(t,x) = x + \frac{ax}{n(1+x)}, \tag{2.2}$$

and

$$B_n^a((t-x)^2,x) = \frac{\varphi^2(x)}{n} + \frac{1}{n^2}\frac{ax}{(1+x)}\frac{(a+1)x+1}{(1+x)} = \frac{\varphi^2(x)}{n}\Big(1+\frac{a}{n(1+x)^2}+\frac{a^2x}{n(1+x)^3}\Big). \tag{2.3}$$

It was verified in [10] that

$$\varphi^2(x)\Big(\frac{d}{dx}W_{n,k}^a(x)\Big) = \Big(k - nx - \frac{ax}{x+1}\Big)W_{n,k}^a(x). \tag{2.4}$$

Theorem 2.1. (see [4]) *If $a \ge 0$ and $q \ge 0$ are real numbers, there exists a constant $M_q(a)$ such that*

$$M_a(q) := \sup_{n>1}\sup_{x\ge 0}\frac{B_n^a((1+t)^q,x)}{(1+x)^q} < \infty.$$

Proposition 2.1. (see [4]) *If $a > 0$, $r \in [0,1]$, there exists a constant C such that for each integer $n > 1$ and each $x \ge 0$,*

$$B_n^a\Big(\frac{1}{(1+t)^r},x\Big) \le \Big(\frac{n}{n-1}\Big)^r\frac{1}{(1+x)^r}.$$

Remark 2.1. It is known a similar result for $a = 0$. We can use the arguments in [4] to verify that, if $a \ge 0$ and $r \in [0,2]$, there exists a constant $C = C(a,r)$ such that, for $n > 2$,

$$B_n^a\Big(\frac{1}{(1+t)^r},x\Big) \le \frac{C}{(1+x)^r} \tag{2.5}$$

(see also [5]).

Lemma 2.1. (see [8, Prop. 3.3]) *Assume $r \ge 0$, $m,p \in \mathbb{R}$, and $m-r+1>0$. Then for $x>0$ and $t \ge 0$, one has*

$$\Big|\int_x^t \frac{(t-s)^m}{s^r}(1+u)^p ds\Big| \le \frac{|t-x|^{m+1}}{(m-r+1)x^r}\Big((1+x)^p + (1+t)^p\Big).$$

3. Estimates for the moments

The moment of order $j \in \mathbb{N}_0$ of the operator B_n^a is defined by

$$M_{n,j}^a(x) = B_n^a((t-x)^j, x).$$

In this work, we need estimates of $M_{n,j}^a(x)$ for $0 \leq j \leq 6$. We remark that in Lemma 2.3 of [13] some computations were given for $M_{n,3}^a(x)$ and $M_{n,4}^a(x)$, but they are complicated. Here, we follow a different approach. In Lemma 1 of [2], it is asserted that $M_{n,j}^a(x) = O(n^{-[(j+1)/2]})$, where $[\alpha]$ denotes the integer part of α, but the estimate is not correct (see (2.3)). A similar assertion was given in [1], where the authors defined the Mihesan operators in the form

$$B_n^a(f, x) = \sum_{k=0}^{\infty} W_{n,k}^a(x) f\Big(\frac{k}{n+1}\Big).$$

Notice that

$$M_{n,0}^a(x) = 1 \qquad \text{and} \qquad M_{n,1}^a(x) = \frac{ax}{n(1+x)}. \tag{3.6}$$

We can use an iterative process to obtain representation for other moments of the operators.

Lemma 3.2. *If $a \geq 0$, $j, n \in \mathbb{N}$, $n > 1$, and $x \geq 0$, then*

$$M_{n,j+1}^a(x) = \frac{\varphi^2(x)}{n}\Big(jM_{n,j-1}^a(x) + \frac{a}{(1+x)^2} M_{n,j}^a(x) + \frac{d}{dx} M_{n,j}^a(x)\Big).$$

Proof. Taking into account (2.4), one has

$$\begin{aligned}
M_{n,j+1}^a(x) &= B_n^a((t-x)(t-x)^j, x) \\
&= \sum_{k=0}^{\infty} \Big(\frac{k}{n} - x\Big)\Big(\frac{k}{n} - x\Big)^j W_{n,k}^a(x) \\
&= \frac{1}{n}\frac{ax}{x+1} M_{n,j}^a(x) + \frac{1}{n}\sum_{k=0}^{\infty}\Big(k - nx - \frac{ax}{x+1}\Big)\Big(\frac{k}{n} - x\Big)^j W_{n,k}^a(x) \\
&= \frac{1}{n}\frac{ax}{x+1} M_{n,j}^a(x) + \frac{\varphi^2(x)}{n}\sum_{k=0}^{\infty}\Big(\frac{k}{n} - x\Big)^j \frac{d}{dx} W_{n,k}^a(x) \\
&= \frac{1}{n}\frac{ax}{x+1} M_{n,j}^a(x) + \frac{j\varphi^2(x)}{n} M_{n,j-1}^a(x) + \frac{\varphi^2(x)}{n}\frac{d}{dx} M_{n,j}^a(x) \\
&= \frac{\varphi^2(x)}{n}\Big(\frac{a}{(1+x)^2} M_{n,j}^a(x) + jM_{n,j-1}^a(x) + \frac{d}{dx} M_{n,j}^a(x)\Big).
\end{aligned}$$

□

Since in Lemma 3.2 the derivative appears, in order to estimate $M_{n,6}^a(x)$ we should study other derivatives of the previous moments.

Lemma 3.3. *Assume $a \geq 0$. There exists a constant C such that, if $n > 1$ and $x \geq 0$, then*

$$M_{n,2}^a(x) \leq C\frac{\varphi^2(x)}{n}, \qquad \Big|\frac{d}{dx} M_{n,2}^a(x)\Big| \leq C\frac{1+x}{n}, \qquad \Big|\frac{d^2}{dx^2} M_{n,2}^a(x)\Big| \leq \frac{C}{n},$$

$$\Big|\frac{d^3}{dx^3} M_{n,2}^a(x)\Big| \leq \frac{C}{n^2(1+x)^4} \qquad \text{and} \qquad \Big|\frac{d^4}{dx^4} M_{n,2}^a(x)\Big| \leq \frac{C}{n^2(1+x)^5}.$$

Proof. It follows from (2.3) that

$$\frac{d}{dx}M^a_{n,2}(x) = \frac{1}{n}\Big(1 + 2x + \frac{a}{n(1+x)^2} + \frac{2a^2x}{n(1+x)^3}\Big).$$

Hence

$$0 \le \frac{d}{dx}M^a_{n,2}(x) \le \frac{1 + 2x + a(1+2ax)}{n} \le \frac{C(1+x)}{n}.$$

The estimates for the other derivatives follow from the identities

$$\frac{d^2}{dx^2}M^a_{n,2}(x) = \frac{1}{n}\Big(2 - \frac{2a}{n(1+x)^3} + \frac{2a^2(1-2x)}{n(1+x)^4}\Big),$$
$$\frac{d^3}{dx^3}M^a_{n,2}(x) = \frac{1}{n^2}\Big(\frac{6a}{(1+x)^4} - \frac{4a^2(1-x)}{(1+x)^5}\Big),$$
$$\frac{d^4}{dx^4}M^a_{n,2}(x) = \frac{1}{n^2}\Big(\frac{4!\,a}{(1+x)^5} + \frac{4a^2(6-4x)}{(1+x)^6}\Big).$$

□

Assume $nx \ge 1$. Notice that $1 + x \le nx + x = (n+1)x$. Hence $(1+x)^2 \le (n+1)\varphi^2(x)$ and

$$\frac{1+x}{n} \le \frac{\sqrt{n+1}\varphi(x)}{n} \le \sqrt{2}\frac{\varphi(x)}{\sqrt{n}}.$$

Lemma 3.4. *Suppose that $a \ge 0$. There exists a constant C such that, for $n > 1$ and $x \ge 0$, one has*

$$|M^a_{n,3}(x)| \le C\frac{(1+x)\varphi^2(x)}{n^2}, \qquad \Big|\frac{d}{dx}M^a_{n,3}(x)\Big| \le \frac{C\varphi^2(x)}{n^2},$$

$$\Big|\frac{d^2}{dx^2}M^a_{n,3}(x)\Big| \le \frac{C(1+x)}{n^2} \quad \text{and} \quad \Big|\frac{d^3}{dx^3}M^a_{n,3}(x)\Big| \le \frac{C}{n^2}.$$

Proof. From Lemma 3.2, one has

$$\begin{aligned}M^a_{n,3}(x) &= \frac{\varphi^2(x)}{n}\Big\{2M^a_{n,1}(x) + \frac{a}{(1+x)^2}M^a_{n,2}(x) + \frac{d}{dx}M^a_{n,2}(x)\Big\}\\ &= \frac{\varphi^2(x)}{n}\Big\{\frac{2ax}{n(1+x)} + \frac{a}{(1+x)^2}M^a_{n,2}(x) + \frac{d}{dx}M^a_{n,2}(x)\Big\}.\end{aligned}$$

Taking into account (2.3) and Lemma 3.3, we obtain

$$|M^a_{n,3}(x)| \le C_1\frac{\varphi^2(x)}{n}\Big\{\frac{x}{n(1+x)} + \frac{x}{n(1+x)} + \frac{1+x}{n}\Big\} \le C_2\frac{(1+x)\varphi^2(x)}{n^2}.$$

Moreover, for $x \ge 0$,

$$\begin{aligned}n\frac{d}{dx}M^a_{n,3}(x) &= \frac{d}{dx}\Big\{\frac{2ax^2}{n} + \frac{ax}{(1+x)}M^a_{n,2}(x) + \varphi^2(x)\frac{d}{dx}M^a_{n,2}(x)\Big\}\\ &= \frac{4ax}{n} + \frac{a}{(1+x)^2}M^a_{n,2}(x) + \frac{ax}{(1+x)}\frac{d}{dx}M^a_{n,2}(x) + \varphi^2(x)\frac{d^2}{dx^2}M^a_{n,2}(x).\end{aligned}$$

Hence,

$$\Big|\frac{d}{dx}M^a_{n,3}(x)\Big| \le \frac{C_1\varphi^2(x)}{n}\Big(\frac{4}{n^2} + \frac{1}{n(1+x)^2} + \frac{a}{(1+x)}\frac{1}{n} + \frac{1}{n}\Big) \le \frac{C_2}{n^2}.$$

On the other hand

$$n\frac{d^2}{dx^2}M^a_{n,3}(x)=\frac{4a}{n}-\frac{2a}{(1+x)^3}M^a_{n,2}(x)+\frac{2a}{(1+x)^2}\frac{d}{dx}M^a_{n,2}(x)$$
$$+\Big(\frac{ax}{(1+x)}+(1+2x)\Big)\frac{d^2}{dx^2}M^a_{n,2}(x)+\varphi^2(x)\frac{d^3}{dx^3}M^a_{n,2}(x).$$

Therefore,

$$\Big|\frac{d^2}{dx^2}M^a_{n,3}(x)\Big|\le\frac{C_1}{n}\Big(\frac{1}{n}+\frac{\varphi^2(x)}{n(1+x)^3}+\frac{1+x}{n(1+x)^2}+\frac{1+x}{n}+\frac{\varphi^2(x)}{n^2(1+x)^4}\Big)$$
$$\le\frac{C_2(1+x)}{n^2}.$$

Finally

$$n\frac{d^3}{dx^3}M^a_{n,3}(x)=\frac{6aM^a_{n,2}(x)}{(1+x)^4}-\frac{6a}{(1+x)^3}\frac{d}{dx}M^a_{n,2}(x))+\varphi^2(x)\frac{d^4}{dx^4}M^a_{n,2}(x)$$
$$+\Big(\frac{3a}{(1+x)^2}+2\Big)\frac{d^2}{dx^2}M^a_{n,2}(x)+\Big(\frac{ax}{(1+x)}+2(1+2x)\Big)\frac{d^3}{dx^3}M^a_{n,2}(x)$$

and we obtain

$$\Big|\frac{d^3}{dx^3}M^a_{n,3}(x)\Big|\le\frac{C_1}{n^2}\Big(\frac{\varphi^2(x)}{(1+x)^4}+\frac{(1+x)}{(1+x)^3}+1+\frac{1+x}{n(1+x)^4}+\frac{\varphi^2(x)}{n(1+x)^5}\Big)\le\frac{C_2}{n^2}.$$

□

Lemma 3.5. *Suppose that $a\ge 0$. There exists a constant C such that, for $n>1$ and $x\ge 0$, one has*

$$0\le M^a_{n,4}(x)\le C\frac{\varphi^4(x)}{n^2},\quad\Big|\frac{d}{dx}M^a_{n,4}(x)\Big|\le\frac{C(1+x)\varphi^2(x)}{n^2}\quad\text{and}\quad\Big|\frac{d^2}{dx^2}M^a_{n,4}(x)\Big|\le\frac{C(1+x)^2}{n^2}.$$

Proof. First, we consider the representation

$$M^a_{n,4}(x)=\frac{\varphi^2(x)}{n}\Big(3M^a_{n,2}(x)+\frac{a}{(1+x)^2}M^a_{n,3}(x)+\frac{d}{dx}M^a_{n,3}(x)\Big).$$

It follows from Lemma 3.4 that

$$0\le M^a_{n,4}(x)\le C_1\frac{\varphi^2(x)}{n}\Big(\frac{\varphi^2(x)}{n}+\frac{(1+x)\varphi^2(x)}{n^2(1+x)^2}+\frac{\varphi^2(x)}{n^2}\Big)\le C_2\frac{\varphi^4(x)}{n^2}.$$

For the first derivative, we obtain

$$n\frac{d}{dx}M^a_{n,4}(x)=(1+2x)\Big(3M^a_{n,2}(x)+\frac{a}{(1+x)^2}M^a_{n,3}(x)+\frac{d}{dx}M^a_{n,3}(x)\Big)$$
$$+\varphi^2(x)\Big(3\frac{d}{dx}M^a_{n,2}(x)-\frac{2aM^a_{n,3}(x)}{(1+x)^3}+\frac{a}{(1+x)^2}\frac{d}{dx}M^a_{n,3}(x)+\frac{d^2}{dx^2}M^a_{n,3}(x)\Big)$$
$$=3(1+2x)M^a_{n,2}(x)+3\varphi^2(x)\frac{d}{dx}M^a_{n,2}(x)+\frac{a}{(1+x)^2}M^a_{n,3}(x)$$
$$+\Big(1+2x+\frac{ax}{(1+x)}\Big)\frac{d}{dx}M^a_{n,3}(x)+\varphi^2(x)\frac{d^2}{dx^2}M^a_{n,3}(x).$$

Hence

$$\begin{aligned}\Big|\frac{d}{dx}M^a_{n,4}(x)\Big| &\le \frac{C_1}{n}\Big(\frac{(1+x)\varphi^2(x)}{n}+\frac{(1+x)\varphi^2(x)}{n^2}+\frac{(1+x)\varphi^2(x)}{n^2}\Big)\\ &\le \frac{C_1(1+x)\varphi^2(x)}{n^2}.\end{aligned}$$

For the second derivative, we consider the identity

$$\begin{aligned}n\frac{d^2}{dx^2}M^a_{n,4}(x) &= 6M^a_{n,2}(x)+6(1+2x)\frac{d}{dx}M^a_{n,2}(x)+3\varphi^2(x)\frac{d^2}{dx^2}M^a_{n,2}(x)\\ &\quad-\frac{2a}{(1+x)^3}M^a_{n,3}(x)+\Big(\frac{a}{(1+x)^2}+\Big(2-\frac{a}{(1+x)^2}\Big)\frac{d}{dx}M^a_{n,3}(x)\\ &\quad+\Big((1+2x)-\frac{ax}{(1+x)}+(1+2x)\Big)\frac{d^2}{dx^2}M^a_{n,3}(x)+\varphi^2(x)\frac{d^3}{dx^3}M^a_{n,3}(x)\\ &= 6M^a_{n,2}(x)+6(1+2x)\frac{d}{dx}M^a_{n,2}(x)+3\varphi^2(x)\frac{d^2}{dx^2}M^a_{n,2}(x)-\frac{2a}{(1+x)^3}M^a_{n,3}(x)\\ &\quad+2\frac{d}{dx}M^a_{n,3}(x)+\Big(2(1+2x)-\frac{ax}{(1+x)}\Big)\frac{d^2}{dx^2}M^a_{n,3}(x)+\varphi^2(x)\frac{d^3}{dx^3}M^a_{n,3}(x).\end{aligned}$$

Therefore

$$\begin{aligned}\Big|\frac{d^2}{dx^2}M^a_{n,4}(x)\Big| &\le \frac{C_1}{n}\Big(\frac{\varphi^2(x)}{n}+\frac{(1+x)^2}{n}+\frac{\varphi^2(x)}{n}\\ &\quad+\frac{(1+x)\varphi^2(x)}{n^2(1+x)^3}+\frac{\varphi^2(x)}{n^2}+\frac{(1+x)^2}{n^2}+\frac{\varphi^2(x)}{n^2}\Big)\\ &= \frac{C_1(1+x)}{n^2}\Big(2x+(1+x)+\frac{1}{n}+\frac{2x}{n}+\frac{(1+x)}{n}\Big)\le\frac{C_2(1+x)^2}{n^2}.\end{aligned}$$

□

Corollary 3.1. *Suppose that $a \ge 0$. There exists a constant C such that, if $n > 1$ and $x \ge 0$, then*

$$B^a_{n,3}(|t-x|^3,x)\le C\frac{\varphi^3(x)}{n^{3/2}}.$$

Proof. It follows from the inequalities

$$B^a_{n,3}(|t-x|^3,x)\le\Big(M^a_{n,2}(x)M^a_{n,4}(x)\Big)^{1/2}\le C_1\frac{\varphi(x)}{\sqrt{n}}\frac{\varphi^2(x)}{n}.$$

□

Lemma 3.6. *Suppose that $a \ge 0$. There exists a constant C such that, if $n > 1$ and $x \ge 0$, then*

$$|M^a_{n,5}(x)|\le C\frac{(1+x)\varphi^4(x)}{n^3}\qquad\text{and}\qquad\Big|\frac{d}{dx}M^a_{n,5}(x)\Big|\le C\frac{(1+x)^2\varphi^2(x)}{n^3}.$$

Proof. For the fifth moment, one has

$$M^a_{n,5}(x)=\frac{\varphi^2(x)}{n}\Big(4M^a_{n,3}(x)+\frac{a}{(1+x)^2}M^a_{n,4}(x)+\frac{d}{dx}M^a_{n,4}(x)\Big).$$

It is clear that

$$\begin{aligned}|M^a_{n,5}(x)| &\leq C_1\frac{\varphi^2(x)}{n}\Big(\frac{(1+x)\varphi^2(x)}{n^2}+\frac{\varphi^4(x)}{n^2(1+x)^2}+\frac{(1+x)\varphi^2(x)}{n^2}\Big)\\ &\leq C_2\frac{(1+x)\varphi^4(x)}{n^3}.\end{aligned}$$

For the first derivative one has

$$\begin{aligned}\frac{d}{dx}M^a_{n,5}(x) &= \frac{(1+2x)}{n}\Big(4M^a_{n,3}(x)+\frac{a}{(1+x)^2}M^a_{n,4}(x)+\frac{d}{dx}M^a_{n,4}(x)\Big)\\ &+\frac{\varphi^2(x)}{n}\Big(4\frac{d}{dx}M^a_{n,3}(x)-\frac{2aM^a_{n,4}(x)}{(1+x)^3}+\frac{a}{(1+x)^2}\frac{d}{dx}M^a_{n,4}(x)+\frac{d^2}{dx^2}M^a_{n,4}(x)\Big).\end{aligned}$$

Therefore

$$\begin{aligned}\Big|\frac{d}{dx}M^a_{n,5}(x)\Big| &\leq \frac{C_1}{n}\Big(\frac{(1+x)^2\varphi^2(x)}{n^2}+\frac{\varphi^4(x)}{n^2(1+x)}+\frac{(1+x)^2\varphi^2(x)}{n^2}\\ &+\frac{\varphi^4(x)}{n^2}+\frac{\varphi^6(x)}{n^2(1+x)^3}+\frac{C1+x)\varphi^4(x)}{n^2(1+x)^2}+\frac{(1+x)^2\varphi^2(x)}{n^2}\Big)\\ &\leq C_2\frac{(1+x)^2\varphi^2(x)}{n^3}.\end{aligned}$$

□

Lemma 3.7. *Suppose that $a\geq 0$. There exists a constant C such that, if $n>1$ and $x\geq 0$, then*

$$0\leq M^a_{n,6}(x)\leq C\frac{(1+x)\varphi^4(x)}{n^3}\Big(x+\frac{1}{n}\Big).$$

Proof. As in Lemma 3.5, we obtain

$$M^a_{n,6}(x)=\frac{\varphi^2(x)}{n}\Big(5M^a_{n,4}(x)+\frac{a}{(1+x)^2}M^a_{n,5}(x)+\frac{d}{dx}M^a_{n,5}(x)\Big).$$

Taking into account Lemmas 3.5 and 3.6, we obtain

$$\begin{aligned}0\leq M^a_{n,6}(x) &\leq C_1\frac{\varphi^2(x)}{n}\Big(\frac{\varphi^4(x)}{n^2}+\frac{(1+x)\varphi^4(x)}{n^3(1+x)^2}+\frac{(1+x)^2\varphi^2(x)}{n^3}\Big)\\ &\leq C_1\frac{(1+x)\varphi^4(x)}{n^3}\Big(x+\frac{\varphi^2(x)}{n^3(1+x)^2}+\frac{(1+x)}{n}\Big)\leq C_2\frac{(1+x)\varphi^4(x)}{n^3}\Big(x+\frac{1}{n}\Big).\end{aligned}$$

□

4. The rate of convergence

Set

$$K(f,t)_\varrho=\inf_{g\in D(\varrho)}\Big(\|f-g\|_\varrho+t\left(\|\varphi^2g''\|_\varrho+a\|\varphi g'\|_\varrho\right)\Big). \tag{4.7}$$

Theorem 4.2. *Assume $a,q\geq 0$ are real numbers, and $\varrho(x)=1/(1+x)^q$. There exists a constant $C=C(a,q)$, such that for $n>2$ and $f\in C_\varrho[0,\infty)$, one has*

$$\|B^a_n(f)-f\|_\varrho\leq CK\Big(f,\frac{1}{n}\Big)_\varrho,$$

here $K(f,t)_\varrho$ is defined in (4.7).

Proof. If $x > 0$ and $g \in D(\varrho)$, we will use the representation

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t g''(s)(t-s)ds.$$

It follows from Proposition 2.1 that

$$\begin{aligned}\Big|\int_x^t g''(s)(t-s)ds\Big| &\le \|\varphi^2 g''\|_\varrho \Big|\int_x^t \frac{(t-s)}{\varphi^2(s)\varrho(s)}ds\Big| \\ &= \|\varphi^2 g''\|_\varrho \Big|\int_x^t \frac{(t-s)}{s}(1+s)^{q-1}ds\Big| \\ &\le \|\varphi^2 g''\|_\varrho \frac{(t-x)^2}{x}((1+x)^{q-1} + (1+t)^{q-1}).\end{aligned}$$

Taking into account (2.3), Hölder inequality, Theorem 2.1 (see also (2.5)), and Lemma 3.5, we obtain

$$\begin{aligned}&\Big|B_n^a\Big(\int_x^t g''(s)(t-s)ds, x\Big)\Big| \\ \le&\frac{\|\varphi^2 g''\|_\varrho}{x}\Big((1+x)^{q-1}B_n^a((t-x)^2, x) + B_n^a((t-x)^2(1+t)^{q-1}, x)\Big) \\ \le&\frac{\|\varphi^2 g''\|_\varrho}{x}\Big(C_1(1+x)^{q-1}\frac{\varphi^2(x)}{n} + (B_n^a((1+t)^{2q-2}, x))^{1/2}\sqrt{M_{n,4}^a(x)}\Big) \\ \le&C_2\frac{\|\varphi^2 g''\|_\varrho}{x}\Big(\frac{x(1+x)^q}{n} + (1+x)^{q-1}\frac{\varphi^2(x)}{n}\Big) \le C_3\frac{\|\varphi^2 g''\|_\varrho}{n\varrho(x)}.\end{aligned}$$

Since

$$B_n^a(g,x) - g(x) = g'(x)B_n^a(t-x,x) + B_n^a\Big(\int_x^t g''(s)(t-s)ds, x\Big),$$

from the previous estimate and (3.6), one has

$$\begin{aligned}\varrho(x)\Big|B_n^a(g,x) - g(x)\Big| &\le \Big|\varrho(x)g'(x)B_n^a(t-x,x)\Big| + \Big|B_n^a\Big(\int_x^t g''(s)(t-s)ds, x\Big)\Big| \\ &\le \varrho(x)\Big|g'(x)B_n^a(t-x,x)\Big| + B_n^a\Big(\Big|\int_x^t g''(s)(t-s)ds\Big|, x\Big) \\ &\le \varrho(x)|g'(x)|\frac{ax}{n(1+x)} + C_3\frac{\|\varphi^2 g''\|_\varrho}{n} \\ &\le \frac{a\sqrt{x}}{n(1+x)^{3/2}}\|\varphi g'\|_\varrho + C_3\frac{\|\varphi^2 g''\|_\varrho}{n} \le \frac{C_4}{n}\Big(\|\varphi^2 g''\|_\varrho + a\|\varphi g'\|_\varrho\Big).\end{aligned}$$

Therefore, for any $g \in D(\varrho)$,

$$\begin{aligned}\|B_n^a(f) - f\|_\varrho &\le \|f-g\|_\varrho + \|B_n^a(f-g)\|_\varrho + \|B_n^a(g) - g\|_\varrho \\ &\le C\Big\{\|f-g\|_\varrho + \frac{1}{n}\Big(\|\varphi^2 g''\|_\varrho + a\|\varphi g'\|_\varrho\Big)\Big\}.\end{aligned}$$

□

Remark 4.2. *For $g \in D(\varrho)$, the previous proof also yields the inequality*

$$\|B_n^a(f) - f\|_\varrho \le C\Big\{\|f-g\|_\varrho + \frac{1}{n}\Big(\|\varphi^2 g''\|_\varrho + a\|g'\|_\varrho\Big)\Big\}.$$

But, the estimate with $\|\varphi g'\|_\varrho$ is more convenient to study the inverse result.

5. A Voronovskaja-type theorem

Theorem 5.3. *Assume $a, q \geq 0$ and $\varrho(x) = 1/(1+x)^q$. There exists a constant C such that, if $n > 1$, $g \in C^3[0,\infty)$ and $g, \varphi^2 g'', \varphi^3 g''' \in C_\varrho[0,\infty)$, then*

$$\Big\| B_n^a(g) - g - \frac{ae_1}{n(1+e_1)} g' - \frac{\varphi^2 g''}{2n} \Big\|_\varrho \leq C\Big(\frac{\|\varphi^2 g''\|_\varrho + \|\varphi^2 g'''\|_\varrho}{n^2} + \frac{\|\varphi^3 g'''\|_\varrho}{n^{3/2}} \Big),$$

where $e_1(x) = x$.

Proof. For $g \in C^3[0,\infty)$, we use the Taylor expansion

$$g(t) = g(x) + g'(x)(t-x) + \frac{1}{2} g''(x)(t-x)^2 + \frac{1}{2}\int_x^t g'''(s)(t-s)^2 ds,$$

to obtain the representation

$$B_n^a(g,x) - Q_n^a(g,x) = \frac{1}{2} B_n^a\Big(\int_x^t g'''(s)(t-s)^2 ds, x \Big),$$

with

$$\begin{aligned} Q_n^a(g,x) &= g(x) + g'(x) B_n^a(t-x,x) + \frac{1}{2} g''(x) B_n^a((t-x)^2, x) \\ &= g(x) + g'(x) M_{n,1}^a(x) + \frac{1}{2} g''(x) M_{n,2}^a(x). \end{aligned}$$

We should estimate

$$B_n^a\Big(\Big| \int_x^t g'''(s)(t-s)^2 ds \Big|, x \Big) = \sum_{k=0}^\infty \Big| \int_x^{k/n} g'''(s)\Big(\frac{k}{n} - s\Big)^2 ds \Big| W_{n,k}^a(x).$$

(A) Suppose $0 \leq x < 1/n$. In this case, we consider the inequality

$$\begin{aligned} &\sum_{k=0}^\infty \Big| \int_x^{k/n} g'''(s)\Big(\frac{k}{n} - s\Big)^2 ds \Big| W_{n,k}^a(x) \\ \leq &\|\varphi^2 g'''\|_\varrho \sum_{k=0}^\infty \Big| \int_x^{k/n} \Big(\frac{k}{n} - s\Big)^2 \frac{ds}{\varphi^2(s)\varrho(s)} \Big| W_{n,k}^a(x). \end{aligned}$$

For $k = 0$, one has

$$\begin{aligned} W_{n,0}^a(x) \int_0^x \frac{s^2}{\varphi^2(s)\varrho(s)} ds = W_{n,0}^a(x) \int_0^x s(1+s)^{q-1} ds &\leq \frac{x^2(1+x)^{q-1}}{(1+x)^n} \\ &\leq \frac{(1+x)^q}{n^2}. \end{aligned}$$

On the other hand, if $q \geq 1$, then

$$\begin{aligned}
\sum_{k=1}^{\infty} W_{n,k}^a(x) \int_x^{k/n} \frac{(k/n-s)^2}{\varphi^2(s)\varrho(s)} ds &\leq \frac{1}{\sqrt{x}} \sum_{k=1}^{\infty} W_{n,k}^a(x) \int_x^{k/n} \frac{(k/n-s)^2}{\sqrt{s}(1+s)^{1-q}} ds \\
&\leq \frac{1}{\sqrt{x}} \sum_{k=1}^{\infty} W_{n,k}^a(x)(1+k/n)^{q-1}(k/n-x)^2 \int_x^{k/n} \frac{ds}{\sqrt{s}} \\
&= \frac{2}{\sqrt{x}} \sum_{k=1}^{\infty} W_{n,k}^a(x)(1+k/n)^{q-1}(k/n-x)^2(\sqrt{k/n}-\sqrt{x}) \\
&\leq \frac{2}{\sqrt{x}} \sum_{k=1}^{\infty} W_{n,k}^a(x)(1+k/n)^{q-1}(k/n-x)^{5/2} \\
&\leq \frac{2}{\sqrt{x}} \sqrt{M_{n,5}(x)} \Big(\sum_{k=0}^{\infty} (1+k/n)^{2q-2} W_{n,k}^a(x) \Big)^{1/2} \\
&\leq \frac{C_1\sqrt{1+x}}{\sqrt{x}} \frac{\varphi^2(x)}{n^{3/2}} (1+x)^{q-1} \leq C_2 \frac{\sqrt{x}}{n^{3/2}} (1+x)^q \leq \frac{C_1(1+x)^q}{n^2},
\end{aligned}$$

where we use Lemma 3.6 and Theorem 2.1.

If $q < 1$, the proof is simpler because, for $x < 1/n$,

$$\frac{1}{(1+k/n)^{1-q}} \leq \frac{1}{(1+x)^{1-q}}.$$

Therefore

$$\varrho(x)|B_n^a(g,x) - Q_n^a(g,x)| \leq \frac{C}{n^2} \|\varphi^2 g'''\|_{\varrho}.$$

(B) Assume that $nx \geq 1$. In this case, we consider the inequality

$$\begin{aligned}
&\sum_{k=0}^{\infty} \Big| \int_x^{k/n} g'''(s) \Big(\frac{k}{n} - s\Big)^2 ds \Big| W_{n,k}^a(x) \\
\leq &\|\varphi^3 g'''\|_{\varrho} \sum_{k=0}^{\infty} \Big| \int_x^{k/n} \Big(\frac{k}{n} - s\Big)^2 \frac{ds}{\varphi^3(s)\varrho(s)} \Big| W_{n,k}^a(x).
\end{aligned}$$

Since $1/n \leq x$, it follows from Lemma 3.7

$$M_{n,6}^a(x) \leq C \frac{(1+x)\varphi^4(x)}{n^3} \Big(x + \frac{1}{n} \Big) \leq 2C \frac{\varphi^6(x)}{n^3}.$$

Moreover

$$B_n^a(|t-x|^3, x) \leq \sqrt{M_{n,6}^a(x)} \leq 2C \frac{\varphi^3(x)}{n^{3/2}}.$$

Thus, we can use Lemma 2.1, Lemma 3.6 and Theorem 2.1 to obtain

$$\begin{aligned}
\varrho(x)\left|B_n^a(g,x)-Q_n^a(g,x)\right| &\le \frac{1}{2}\varrho(x)\|\varphi^3 g'''\|_\varrho B_n^a\Big(\Big|\int_x^t \frac{(t-s)^2}{\varphi^3(s)\varrho(s)}ds\Big|,x\Big)\\
&= \frac{1}{2}\varrho(x)\|\varphi^3 g'''\|_\varrho B_n^a\Big(\Big|\int_x^t \frac{(t-s)^2(1+s)^{q-3/2}}{s^{3/2}}ds\Big|,x\Big)\\
&\le \frac{1}{3}\varrho(x)\|\varphi^3 g'''\|_\varrho B_n^a\Big(|t-x|^3\Big(\frac{(1+x)^{q-3/2}}{x^{3/2}}+\frac{(1+t)^{q-3/2}}{x^{3/2}}\Big),x\Big)\\
&= \frac{1}{3}\|\varphi^3 g'''\|_\varrho\Big(\frac{B_n^a(|t-x|^3,x)}{\varphi^3(x)}+\frac{\varrho(x)}{x^{3/2}}\sqrt{M_{n,6}^a(x)}\sqrt{B_n^a((1+t)^{2q-3},x)}\\
&\le C_3\|\varphi^3 g'''\|_\varrho\Big(\frac{1}{n^{3/2}}+\frac{\varrho(x)}{x^{3/2}}\frac{\varphi^3(x)}{n^{3/2}}(1+x)^{q-3/2}\Big)\le C_4\frac{\|\varphi^3 g'''\|_\varrho}{n^{3/2}}.
\end{aligned}$$

Taking into account (2.2) and (2.3), we obtain

$$\begin{aligned}
&\Big\|B_n^a(g)-g-\frac{ae_1}{n(1+e_1)}g'-\frac{\varphi^2 g''}{2n}\Big\|_\varrho\\
\le&\Big\|B_n^a(g)-g-g'M_{n,1}^a-\frac{1}{2}g''M_{n,2}^a\Big\|_\varrho\\
&+\frac{1}{2}\Big\|\frac{\varphi^2 g''}{n}-\frac{\varphi^2(x)}{n}\Big(1+\frac{a}{n(1+x)^2}+\frac{a^2x}{n(1+x)^3}\Big)g''\Big\|_\varrho\\
=&\Big\|B_n^a(g)-g-g'M_{n,1}^a-\frac{1}{2}g''M_{n,2}^a\Big\|_\varrho+\frac{1}{n^2}\Big\|\Big(\frac{a}{(1+x)^2}+\frac{a^2x}{(1+x)^3}\Big)\varphi^2 g''\Big\|_\varrho\\
\le& C\Big(\frac{\|\varphi^2 g''\|_\varrho+\|\varphi^2 g'''\|_\varrho}{n^2}+\frac{\|\varphi^3 g'''\|_\varrho}{n^{3/2}}\Big).
\end{aligned}$$

□

Corollary 5.2. *Under the assumptions of Theorem 5.3, one has*

$$\lim_{n\to\infty}\Big\|n\Big(B_n^a(g)-g\Big)-\frac{ae_1}{(1+e_1)}g'-\frac{\varphi^2 g''}{2}\Big\|_\varrho=0,$$

where $e_1(x)=x$.

Remark 5.3. *In Theorem 3.2 of [13], a Voronovskaja-type theorem was given for functions* $f\in C^3[0,\infty)$ *such that* $f, f', f'', f'''\in C_\varrho[0,\infty)$*, but the authors only considered the case* $q\in\mathbb{N}_0$*. Moreover, they only obtained pointwise convergence.*

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Research Article

Exponential approximation in variable exponent Lebesgue spaces on the real line

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ABSTRACT. Present work contains a method to obtain Jackson and Stechkin type inequalities of approximation by integral functions of finite degree (IFFD) in some variable exponent Lebesgue space of real functions defined on $\boldsymbol{R} := (-\infty, +\infty)$. To do this, we employ a transference theorem which produce norm inequalities starting from norm inequalities in $\mathcal{C}(\boldsymbol{R})$, the class of bounded uniformly continuous functions defined on $\boldsymbol{R}$. Let $B \subseteq \boldsymbol{R}$ be a measurable set, $p(x) : B \to [1, \infty)$ be a measurable function. For the class of functions f belonging to variable exponent Lebesgue spaces $L_{p(x)}(B)$, we consider difference operator $(I - T_\delta)^r f(\cdot)$ under the condition that $p(x)$ satisfies the log-Hölder continuity condition and $1 \leq \operatorname{ess\,inf}_{x \in B} p(x)$, $\operatorname{ess\,sup}_{x \in B} p(x) < \infty$, where I is the identity operator, $r \in \mathrm{N} := \{1, 2, 3, \cdots\}$, $\delta \geq 0$ and

$$(*) \qquad T_\delta f(x) = \frac{1}{\delta} \int_0^\delta f(x+t)\, dt, \quad x \in \boldsymbol{R}, \quad T_0 \equiv I,$$

is the forward Steklov operator. It is proved that

$$(**) \qquad \|(I - T_\delta)^r f\|_{p(\cdot)}$$

is a suitable measure of smoothness for functions in $L_{p(x)}(B)$, where $\|\cdot\|_{p(\cdot)}$ is Luxemburg norm in $L_{p(x)}(B)$. We obtain main properties of difference operator $\|(I - T_\delta)^r f\|_{p(\cdot)}$ in $L_{p(x)}(B)$. We give proof of direct and inverse theorems of approximation by IFFD in $L_{p(x)}(\boldsymbol{R})$.

Keywords: Variable exponent Lebesgue space, one sided Steklov operator, integral functions of finite degree, best approximation, direct theorem, inverse theorem, modulus of smoothness, Marchaud inequality, K-functional.

2020 Mathematics Subject Classification: 41A10, 41A25, 41A27, 41A65.

1. INTRODUCTION

Some inequalities of Approximation Theory in a Homogenous Banach Spaces (HBS) can be obtained their uniform-norm counterparts. This information is known for a long time, (see e.g., [20] for definition of HBS). This elegant method was generalized to some variable exponent Lebesgue spaces functions defined on $\boldsymbol{R}$ (see Theorem 1 of [9]). Generally, these scale of function classes are non-translation invariant with respect to the ordinary translation $x \to f(x+a)$. Here, we give several uniform-norm inequalities on $C(\boldsymbol{R})$ and apply them to obtain several inequalities of approximation by IFFD in some variable exponent Lebesgue spaces $L_{p(x)}(\boldsymbol{R})$. Under some condition on $p(x)$ of $L_{p(x)}(\boldsymbol{R})$, we obtain main inequalities of exponential approximation by IFFD such as Jackson-Stechkin-Timan type estimates and equivalence of K-functional with suitable modulus of smoothness (**) given in abstract for functions of $L_{p(x)}(\boldsymbol{R})$. Note that many results of approximation by IFFD can be obtained easily their uniform-norm counterparts in $C(\boldsymbol{R})$.

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Consider an entire function $f(z)$ and put $M(r) = \max_{|z|=r} |f(z)|$ for $z = x + iy$. We say that an entire function f is of exponential type σ if $\limsup_{r\to\infty} r^{-1} \ln M(r) \le \sigma, \quad \sigma < \infty$.

The approximation by entire function of finite degree in the real line was originated in the beginning of twentieth century by Serge Bernstein [15] and became a separate branch of analysis due to the efforts of many mathematicians such as N. Wiener and R. Paley [45], N.I. Achiezer [4], S.M. Nikolskii [42], I.I. Ibragimov [29], A.F. Timan [52], M.F. Timan [53], R. Taberski [54, 55], F.G. Nasibov [41], V. Yu. Popov [46], A.A. Ligun [43], and others.

Studying function spaces with variable exponent is now an extensively developed field after their applications in elasticity theory [58], fluid mechanics [47, 48], differential operators [19, 48], nonlinear Dirichlet boundary value problems [40], nonstandard growth [58], and variational calculus. See the books [16, 18, 51] for more references. Nowadays, many mathematician solved many problems for the approximation of function in these type spaces defined on $[0, 2\pi] \subset \boldsymbol{R}$ (see e.g., [7, 8, 26, 30, 31, 34], [1, 2, 3, 11, 12], [5, 6, 9, 13, 14],[22, 24, 25, 28, 32, 33, 36],[37, 38, 44, 49, 50, 56]). In this paper, we propose generalized our last results in [10] which we obtained a direct and inverse theorems for approximation by entire functions of finite degree in variable exponent Lebesgue spaces on the whole real axis $\boldsymbol{R}$ with

(1.1) $$\sup_{0<h\le\delta} \|(I - T_h)f\|_{p(\cdot)}$$

as modulus of continuity $\Omega_1(f, \delta)_{p(\cdot)}$. Instead of (1.1), here we will use

(1.2) $$\|(I - T_\delta)^r f\|_{p(\cdot)}$$

as modulus smoothness $\Omega_r(f, \delta)_{p(\cdot)}$ and we obtain stronger Jackson inequality than obtained in [10].

Let $B \subseteq \boldsymbol{R}$ be a measurable set and $p(x) : B \to [1, \infty)$ be a measurable function. We define $\tilde{P}(B)$ as the class of measurable functions $p(x)$ satisfying the conditions

(1.3) $$1 \le p_B^- := \operatorname{ess\,inf}_{x\in B} p(x), \quad p_B^+ := \operatorname{ess\,sup}_{x\in B} p(x) < \infty.$$

We also set $p^- := p_{\boldsymbol{R}}^-$ and $p^+ := p_{\boldsymbol{R}}^+$. We define the $L_{p(\cdot)}(B)$ as the set of all functions $f : B \to \boldsymbol{R}$ such that

(1.4) $$I_{p(\cdot),B}\left(\frac{f}{\lambda}\right) := \int_B \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy < \infty$$

for some $\lambda > 0$. We set $I_{p(\cdot)}(f) := I_{p(\cdot),\boldsymbol{R}}(f)$. The set of functions $L_{p(\cdot)}(B)$, with norm

$$\|f\|_{p(\cdot),B} := \inf\left\{\eta > 0 : I_{p(\cdot),B}\left(\frac{f}{\eta}\right) < 1\right\}$$

is Banach space. We set $L_{p(\cdot)} := L_{p(\cdot)}(\boldsymbol{R})$.

For $i \in \mathrm{N}$, all constants $c_i(x, y, \cdots)$ will be some positive number such that they depend on the parameters $x, y, \cdots$ given in the brackets. Also, constants $c_i(x, y, \cdots)$ can be change only when the parameters $x, y, \cdots$ change. Absolute constants $\mathbf{c}_1, \mathbf{c}_2, \ldots$ will not change in each occurrence.

Definition 1.1. *For a measurable set $B \subseteq \boldsymbol{R}$, a measurable function $p(\cdot) : B \to \boldsymbol{R}$ is said to locally log-Hölder continuous on B if there is a positive constant $c_1(p)$ such that*

(1.5) $$|p(x) - p(y)| \log(e + 1/|x - y|) \le c_1(p) < \infty$$

for any $x, y \in B$. We say that p satisfies log-Hölder decay condition if there is a constant $c_2(p) > 0$ and $p_\infty > 1$ such that

(1.6) $$|p(x) - p_\infty| \log(e + |x|) \le c_2(p) < \infty$$

for any $x \in B$.

Define the class $P^{Log}(B) := \left\{p \in \tilde{P}(B) : \frac{1}{p} \text{ is satisfy (1.5)-(1.6)}\right\}$. *We set* $c_3(p) := \max\{c_1(p), c_2(p)\}$.

Definition 1.2. *([27, p.96]) Let* $\mathrm{N} := \{1, 2, 3, \cdots\}$ *be natural numbers and* $\mathrm{N}_0 := \mathrm{N} \cup \{0\}$.

(a) *A family* Q *of measurable sets* $E \subset \boldsymbol{R}$ *is called locally* N*-finite* ($N \in \mathrm{N}$) *if*

$$\sum_{E \in Q} \chi_E(x) \leq N$$

almost everywhere in $\boldsymbol{R}$, *where* χ_U *is the characteristic function of the set* U.

(b) *A family* Q *of open bounded sets* $U \subset \boldsymbol{R}$ *is locally* 1*-finite if and only if the sets* $U \in Q$ *are pairwise disjoint.*

(c) *Let* $U \subset \boldsymbol{R}$ *be a measurable set and*

$$A_U f := \frac{1}{|U|} \int_U |f(t)| \, dt.$$

(d) *For a family* Q *of open sets* $U \subset \boldsymbol{R}$, *we define averaging operator by*

$$T_Q : L^1_{loc} \to L^0,$$

$$T_Q f(x) := \sum_{U \in Q} \chi_U(x) A_U f = \sum_{U \in Q} \frac{\chi_U(x)}{|U|} \int_U |f(y)| \, dy, \quad x \in \boldsymbol{R},$$

where L^0 *is the set of measurable functions on* $\boldsymbol{R}$.

For a measurable set $A \subset \boldsymbol{R}$, symbol $|A|$ will represent the Lebesgue measure of A. We consider Transference result.

Definition 1.3. *For* $0 < \delta < \infty$, $\tau \in \boldsymbol{R}$, *we define family of Steklov operators*

$$\mathsf{S}_\delta f(x) := \frac{1}{\delta} \int_{x-\delta/2}^{x+\delta/2} f(t) \, dt = \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} f(x+t) \, dt, \quad x \in \boldsymbol{R}, \tag{1.7}$$

where f *is a locally integrable function defined on* $\boldsymbol{R}$.

The following result was obtained by Drihem for every cubes or balls in $\boldsymbol{R}^d$. We write below its restricted version with constants. The proof of this is the same with Theorem 2 of [23].

Proposition 1.1. *([23]) Suppose that* $p \in P^{Log}(\boldsymbol{R})$ *and* Q *is a bounded interval of* $\boldsymbol{R}$ *having Lebesgue measure* ≥ 1. *For every* $m > 0$, *there is* $c_4(m, c_3(p)) := \exp(-4mc_3(p)) \in (0,1)$ *such that*

$$\left(\frac{c_4(m, c_3(p))}{|Q|} \int_Q |f(y+\tau)| \, dy\right)^{p(x)} \leq \frac{3^{p^+}}{|Q|} \int_Q |f(y+\tau)|^{p(y+\tau)} \, dy + \frac{3^{p^+-1}}{(e+|x|)^m}$$
$$+ 3^{p^+-1} \int_Q \frac{dy}{(e+|y+\tau|)^m}$$

holds for all $x \in Q$, $\tau \in \boldsymbol{R}$ *and all* $f \in L_{p(\cdot)} + L_\infty(\boldsymbol{R})$ *with* $\|f\|_{p(\cdot)} + \|f\|_\infty \leq 1$.

Theorem 1.1. *Suppose that* $p \in P^{Log}(\boldsymbol{R})$. *Then, the family of operators* $\{\mathcal{U}_\tau f\}_{\tau \in \boldsymbol{R}}$ *defined by*

$$\mathcal{U}_\tau f(x) := \mathsf{S}_1 f(x+\tau) = \int_{-1/2}^{+1/2} f(x+\tau+t) \, dt, \quad x \in \boldsymbol{R}, \quad \tau \in \boldsymbol{R}$$

is uniformly bounded (in τ) in $L_{p(\cdot)}$, namely,

$$\|\mathcal{U}_\tau f\|_{p(\cdot)} \le c_5\left(p^+, c_3(p)\right)\|f\|_{p(\cdot)}$$

holds with $c_5\left(p^+, c_3(p)\right) := 2^{p^+ +1}3^{p^+}\left(1+2\cdot 3^{p^+}\left[\sum_{k=2}^{\infty} 2^{-k}+2\right]\right)\exp\left(8c_3(p)\right)$.

Proof of Theorem 1.1. Let us consider $f \in L_{p(\cdot)}$ with $\|f\|_{p(\cdot)} \le 1/2$. Suppose that

$$Q := \{U \subset \boldsymbol{R} : U \text{ open interval and } |U| = 1\}$$

be a locally 1-finite family of partition of $\boldsymbol{R}$. Choose $m = 2 > 1$ (constant $c_6\left(p^+\right)$ below becomes a finite number)

$$c_6\left(p^+\right) = 2^{p^+}3^{p^+}\left(1+2\cdot 3^{p^+}\left[\sum_{k=2}^{\infty} 2^{-k}+2\right]\right) < \infty.$$

We can select $c_4\left(2, c_3(p)\right) = \exp\left(-8c_3(p)\right) \in (0,1)$ as in Proposition 1.1. Then, using Corollary 2.2.2 of [27, p.20] we obtain

$$\begin{aligned}
\rho_{p(\cdot)}\left(\frac{c_4\left(2, c_3(p)\right)}{c_6\left(p^+\right)}\mathcal{U}_\tau f\right) &= \frac{1}{c_6\left(p^+\right)}\int_{\boldsymbol{R}}\left|c_4\left(2, c_3(p)\right)\int_{-1/2}^{+1/2} f(x+\tau+t)\,dt\right|^{p(x)} dx \\
&\le \frac{1}{c_6\left(p^+\right)}\sum_{U\in Q}\int_U\left|c_4\left(2, c_3(p)\right)\int_{-1/2}^{+1/2} f(x+\tau+t)\,dt\right|^{p(x)} dx \\
&\le \frac{2^{p^+}}{c_6\left(p^+\right)}\sum_{U\in Q}\int_U\left|\frac{c_4\left(2, c_3(p)\right)}{|2U|}\int_{2U}\chi_{2U}(y) f(y+\tau)\,dy\right|^{p(x)} dx \\
&\le \frac{2^{p^+}}{c_6\left(p^+\right)}\sum_{U\in Q}\int_U\left[\frac{3^{p^+}\chi_{2U}(y)}{|2U|}\int_{2U}|f(y+\tau)|^{p(y+\tau)}\,dy+\right. \\
&\left.+\frac{3^{p^+-1}}{(e+|x|)^2}+\frac{\chi_{2U}(y)}{|2U|}\int_{2U}\frac{3^{p^+-1}dy}{(e+|y+\tau|)^2}\right] dx \\
&\le \frac{2^{p^+-1}3^{p^+}}{c_6\left(p^+\right)}\sum_{U\in Q}\int_U\left[\chi_{2U}(y)\int_{2U+\tau}|f(s)|^{p(s)}\,ds\right. \\
&\left.+\frac{3^{p^+-1}2}{(e+|x|)^2}+\int_{2U+\tau}\frac{3^{p^+-1}ds}{(e+|s|)^2}\right] dx \\
&\le \frac{2^{p^+-1}3^{p^+}}{c_6\left(p^+\right)}\left(\sum_{U\in Q}\chi_{2U}\right)\left(1+3^{p^+}\int_{\boldsymbol{R}}\frac{ds}{(e+|s|)^2}\right) \\
&= \frac{2^{p^+}3^{p^+}}{c_6\left(p^+\right)}\left(1+3^{p^+}\int_{\boldsymbol{R}}\frac{ds}{(e+|s|)^2}\right) \\
&\le \frac{2^{p^+}3^{p^+}}{c_6\left(p^+\right)}\left(1+2\cdot 3^{p^+}\left[\sum_{k=2}^{\infty}\frac{1}{2^k}+2\right]\right) = 1
\end{aligned}$$

and hence

$$\|\mathcal{U}_{\tau} f\|_{p(\cdot)} \leq 2^{-1} c_5 \left(p^{+}, c_3 (p)\right).$$

General case $f \in L_{p(\cdot)}$ can be obtained easily by re-scaling:

$$\|\mathcal{U}_{\tau} f\|_{p(\cdot)} \leq c_5 \left(p^{+}, c_3 (p)\right) \|f\|_{p(\cdot)}.$$

□

Theorem 1.2. ([18, Theorem 4.4.8]) *Suppose that $p \in P^{Log}(\boldsymbol{R})$ and $f \in L_{p(\cdot)}$. If Q is locally 1-finite family of open bounded subintervals of $\boldsymbol{R}$ having Lebesgue measure 1, then the averaging operator T_Q is uniformly bounded in $L_{p(\cdot)}$, namely,*

$$\|T_Q f\|_{p(\cdot)} \leq c_7 (c_3 (p)) \|f\|_{p(\cdot)}$$

holds with $c_7 (c_3 (p)) := 2 \exp (8 c_3 (p))$.

Let $C(\boldsymbol{R})$ be the class of continuous functions defined on $\boldsymbol{R}$. For $r \in \mathrm{N}$, we define C^r consisting of every member $f \in C(\boldsymbol{R})$ such that the derivative $f^{(k)}$ exists and is continuous on $\boldsymbol{R}$ for $k = 1, ..., r$. We set $C^{\infty} := \{f \in C^r$ for any $r \in \mathrm{N}\}$. We denote by $C_c(\boldsymbol{R})$, the collection of real valued continuous functions on $\boldsymbol{R}$ and support of f is compact set in $\boldsymbol{R}$. We define $C_c^r := C^r \cap C_c(\boldsymbol{R})$ for $r \in \mathrm{N}$ and $C_c^{\infty} := C^{\infty} \cap C_c(\boldsymbol{R})$. Let $L_p(\boldsymbol{R})$, $1 \leq p \leq \infty$ be the classical Lebesgue space of functions on $\boldsymbol{R}$.

Theorem 1.3. [18, Corollary 4.6.6] *Let $p \in P^{Log}(\boldsymbol{R})$ and $f \in L_{p(\cdot)}$. Then*

$$\frac{\|f\|_{p(\cdot)}}{12 c_7 (c_3 (p))} \leq \sup_{g \in L_{p'(\cdot)} \cap C_c^{\infty} : \|g\|_{p'(\cdot)} \leq 1} \int_{\boldsymbol{R}} |f(x) g(x)| \, dx \leq 2 \|f\|_{p(\cdot)}. \tag{1.8}$$

Definition 1.4. *Let $p \in P^{Log}(\boldsymbol{R})$. For an $f \in L_{p(\cdot)}$, we define*

$$F_f(u) := \int_{\boldsymbol{R}} (\mathsf{S}_1 f)(x+u) |G(x)| \, dx, \quad u \in \boldsymbol{R}, \tag{1.9}$$

where $G \in L_{p'(\cdot)} \cap C_c^{\infty}$ and $\|G\|_{p'(\cdot)} \leq 1$.

Let $W_{p(\cdot)}^r$, $r \in \mathrm{N}$, be the class of functions $f \in L_{p(\cdot)}$ such that derivatives $f^{(k)}$ exist for $k = 1, ..., r-1$, $f^{(r-1)}$ absolutely continuous and $f^{(r)} \in L_{p(\cdot)}$.

Some properties of the function $F_f(\cdot)$ is given in the following theorem.

Theorem 1.4. *Let $p \in P^{Log}(\boldsymbol{R})$, $0 < \delta < \infty$, and $f \in L_{p(\cdot)}$. Then,*
(a) the function $F_f(\cdot)$ defined in (1.9) is a bounded, uniformly continuous on $\boldsymbol{R}$,
(b) $(\mathsf{S}_\delta f)' = \mathsf{S}_\delta (f')$ on $\boldsymbol{R}$ for $f \in W_{p(\cdot)}^1$.

Main theorem of this section is as follows.

Theorem 1.5. *Let $p \in P^{Log}(\boldsymbol{R})$. If $f, g \in L_{p(\cdot)}$ and*

$$\|F_f\|_{C(\boldsymbol{R})} \leq \mathbf{c}_1 \|F_g\|_{C(\boldsymbol{R})}$$

holds with an absolute constant $\mathbf{c}_1 > 0$, then norm inequality

$$\|f\|_{p(\cdot)} \leq c_8 \left(\mathbf{c}_1, p^{+}, c_3 (p)\right) \|g\|_{p(\cdot)} \tag{1.10}$$

also holds with $c_8 (\mathbf{c}_1, p^{+}, c_3 (p)) := 48 c_7 (c_3 (p)) \mathbf{c}_1 c_5 (p^{+}, c_3 (p))$.

Remark 1.1. *Theorem 1.5 is a powerful tool to obtain norm inequalities in $L_{p(\cdot)}$ (and other non-translation invariant Banach spaces of functions) for $p \in P^{Log}(\boldsymbol{R})$. In this work, we will use it frequently. See for example the following result.*

As a corollaries of Theorem 1.5, we get the following two results:

Theorem 1.6. *Suppose that $p \in P^{Log}(\boldsymbol{R})$, $0 < \delta < \infty$ and $\tau \in \boldsymbol{R}$. Then, the family of operators $\{\mathcal{S}_{\delta,\tau} f\}$ defined by*

$$\mathcal{S}_{\delta,\tau} f(x) := \mathsf{S}_\delta f(\cdot + \tau) = \frac{1}{\delta} \int_{x+\tau-\delta/2}^{x+\tau+\delta/2} f(s)\, ds, \quad x \in \boldsymbol{R}$$

is uniformly bounded (in δ and τ) in $L_{p(\cdot)}$, namely,

$$\|\mathcal{S}_{\delta,\tau} f\|_{p(\cdot)} \leq 48 c_7 (c_3(p))\, c_5 \left(p^+, c_3(p)\right) \|f\|_{p(\cdot)}$$

holds.

Corollary 1.1. *Let $p \in P^{Log}(\boldsymbol{R})$, $0 < \delta < \infty$, and $f \in L_{p(\cdot)}$. If $\tau = \delta/2$, then*

$$\begin{gathered} \mathcal{S}_{\delta,\delta/2} f(x) = \frac{1}{\delta} \int_0^\delta f(x+t)\, dt = T_\delta f(x), \\ \|T_\delta f\|_{p(\cdot)} \leq 48 c_7 (c_3(p))\, c_5 \left(p^+, c_3(p)\right) \|f\|_{p(\cdot)}, \\ \|(I - T_\delta)^r f\|_{p(\cdot)} \leq \left(1 + 48 c_7 (c_3(p))\, c_5 \left(p^+, c_3(p)\right)\right)^r \|f\|_{p(\cdot)}. \end{gathered} \tag{1.11}$$

For the proof of these results, we will need the following Propositions.

Proposition 1.2. *(a) $C_c(\boldsymbol{R})$ and C_c^∞ are dense subsets of $L_p(\boldsymbol{R})$, $1 \leq p < \infty$ (Theorems 17.10 and 23.59 of* [57, p. 415 and p. 575]*).*

(b) $C_c(\boldsymbol{R})$ contained $L_\infty(\boldsymbol{R})$, but not dense (Remark 17.11 of [57, p.416]*) in $L_\infty(\boldsymbol{R})$.*

(c) If $r \in \mathrm{N}$ and $f \in C_c^r$, then $\mathsf{S}_\delta(f) \in C_c^r$.

Proof of Proposition 1.2. (a) and (b) are known. (c) is follows from definitions. □

Proposition 1.3. ([18, Theorem 2.26]*) Let $B \subseteq \boldsymbol{R}$ be a measurable set. If $1 \leq p(x) < p_B^+ < \infty$, $p'(x) = p(x)/(p(x)-1)$, $f \in L_{p(\cdot)}(B)$, and $g \in L_{p'(\cdot)}(B)$, then Hölder's inequality*

$$\int_B f(x) g(x) dx \leq 2 \|f\|_{p(\cdot),B} \|g\|_{p'(\cdot),B} \tag{1.12}$$

holds.

Proof of Theorem 1.4. (a) Since $C_c(\boldsymbol{R})$ is a dense subset ([39, Theorem 4.1 (I)]) of $L_{p(\cdot)}$, we consider functions $H \in C_c(\boldsymbol{R})$ and prove that $F_H(\cdot) = \int_{\boldsymbol{R}} (S_1 H)(x+u_1) |G(x)|\, dx$ is bounded and uniformly continuous on $\boldsymbol{R}$, where $G \in L_{p'(\cdot)} \cap C_c^\infty$ and $\|G\|_{p'(\cdot)} \leq 1$. Boundedness of $F_H(\cdot)$ is easy consequence of the Hölder's inequality (1.12) and Theorem 1.1. On the other hand, note that H is uniformly continuous on $\boldsymbol{R}$, see e.g. Lemma 23.42 of [57, pp.557-558]. Take $\varepsilon > 0$ and $u_1, u_2, x \in \boldsymbol{R}$. Then, there exists a $\delta := \delta(\varepsilon) > 0$ such that

$$|H(x+u_1) - H(x+u_2)| \leq \frac{\varepsilon}{2(1 + |\mathrm{supp}(G)|)}$$

for $|u_1 - u_2| < \delta$. Then, for $|u_1 - u_2| < \delta$, $u_1, u_2 \in \boldsymbol{R}$ we have

$$\begin{aligned} |F_H(u_1) - F_H(u_2)| &= \left| \int_{\boldsymbol{R}} S_1 (H(x+u_1) - H(x+u_2)) |G(x)|\, dx \right| \\ &\leq \frac{1}{2(1 + |\mathrm{supp}(G)|)} \int_{\boldsymbol{R}} |S_1(\varepsilon)| |G(x)|\, dx = \frac{\varepsilon}{2(1 + |\mathrm{supp}(G)|)} \int_{\boldsymbol{R}} |G(x)|\, dx \\ &\leq \frac{\varepsilon}{(1 + |\mathrm{supp}(G)|)} (1 + |\mathrm{supp}(G)|) \|G\|_{p'(\cdot)} \leq \varepsilon. \end{aligned}$$

Now, the conclusion of Theorem 1.4 follows for the class $C_c(\boldsymbol{R})$. For the general case $f \in L_{p(\cdot)}$, there exists an $H \in C_c(\boldsymbol{R})$ so that

$$\|f-H\|_{p(\cdot)} < \xi/(8c_5(p^+, c_3(p)))$$

for any $\xi > 0$. Then, for this ξ,

$$\begin{aligned}
|F_f(u_1) - F_f(u_2)| &\leq \left|\int_{\boldsymbol{R}} \mathsf{S}_1(f-H)(x+u_1)|G(x)|\,dx\right| \\
&+ \left|\int_{\boldsymbol{R}} \mathsf{S}_1(H(x+u_1) - H(x+u_2))|G(x)|\,dx\right| \\
&+ \left|\int_{\boldsymbol{R}} \mathsf{S}_1(H-f)(x+u_2)|G(x)|\,dx\right| \\
&\leq 2\|\mathsf{S}_1(f-H)(\cdot+u_1)\|_{p(\cdot)} + \left|\int_{\boldsymbol{R}} \mathsf{S}_1(H(x+u_1) - H(x+u_2))|G(x)|\,dx\right| \\
&+ 2\|\mathsf{S}_1(f-H)(\cdot+u_2)\|_{p(\cdot)} \\
&\leq 4c_5(p^+, c_3(p))\|f-H\|_{p(\cdot)} + \xi/2 \leq \xi/2 + \xi/2 = \xi.
\end{aligned}$$

As a result F_f is bounded, uniformly continuous function defined on $\boldsymbol{R}$.

(b) can be obtained easily from definition. □

Proof of Theorem 1.5. Let $f \in L_{p(\cdot)}$ be non-negative. If $\|f\|_{p(\cdot)} = 0$, then the result (1.10) is obvious. So we assume that $\infty > \|f\|_{p(\cdot)} > 0$. In this case

$$\begin{aligned}
\|F_f\|_{C(\boldsymbol{R})} \leq \mathbf{c}_1 \|F_g\|_{C(\boldsymbol{R})} &= \mathbf{c}_1 \left\|\int_{\boldsymbol{R}} \mathsf{S}_1(g)(u+x)|G(x)|\,dx\right\|_{C(\boldsymbol{R})} \\
&= \mathbf{c}_1 \max_{u\in\boldsymbol{R}} \left|\int_{\boldsymbol{R}} \mathsf{S}_1(g)(u+x)|G(x)|\,dx\right| \\
&\leq 2\mathbf{c}_1 \max_{u\in\boldsymbol{R}} \|\mathsf{S}_1(g)(u+\cdot)\|_{p(\cdot)} \leq 2c_5(p^+, c_3(p))\,\mathbf{c}_1\|g\|_{p(\cdot)},
\end{aligned}$$

where we used hypothesis, Hölder's inequality and Theorem 1.1, respectively. On the other hand, for any $\varepsilon \in \left(0, \frac{\|f\|_{p(\cdot)}}{12c_7(c_3(p))}\right)$ and appropriately chosen $\tilde{G}_\varepsilon \in L_{p'(\cdot)}$ with $\left\|\tilde{G}_\varepsilon\right\|_{X'} \leq 1$ (see e.g. Theorem 1.3)

$$\int_{\boldsymbol{R}} |g(x)|\left|\tilde{G}_\varepsilon(x)\right| dx \geq \frac{1}{12c_7(c_3(p))}\|g\|_{p(\cdot)} - \varepsilon,$$

one can find

$$\begin{aligned}
\|F_f\|_{C(\boldsymbol{R})} \geq |F_f(0)| &\geq \int_{\boldsymbol{R}} \mathsf{S}_1(f)(x)|G(x)|\,dx \\
&= \mathsf{S}_1\left(\int_{\boldsymbol{R}} f(x)|G(x)|\,dx\right) \geq \mathsf{S}_1\left(\frac{1}{12c_7(c_3(p))}\|f\|_{p(\cdot)} - \varepsilon\right) \\
&= \frac{1}{12c_7(c_3(p))}\|f\|_{p(\cdot)} - \varepsilon.
\end{aligned}$$

In the last inequality, we take as $\varepsilon \to 0+$ and obtain

$$\|F_f\|_{C(\boldsymbol{R})} \geq \frac{1}{12c_7(c_3(p))}\|f\|_{p(\cdot)}.$$

Then for $f \in L_{p(\cdot)}$, we get

$$\|f\|_{p(\cdot)} \leq 24c_7 (c_3 (p)) \|F_f\|_{C(\boldsymbol{R})} \leq 24c_7 (c_3 (p)) \mathbf{c}_1 \|F_g\|_{C(\boldsymbol{R})}$$
$$\leq 48c_7 (c_3 (p)) \mathbf{c}_1 c_5 \left(p^+, c_3 (p)\right) \|g\|_{p(\cdot)} .$$

□

Definition 1.5. *For $p \in P^{Log}(\boldsymbol{R})$, $f \in L_{p(\cdot)}$, $0 < \delta < \infty$, $r \in \mathrm{N}_0$, we can define modulus of smoothness as*

$$\Omega_r(f,\delta)_{p(\cdot)} = \|(I - T_\delta)^r f\|_{p(\cdot)},$$
$$\Omega_0(f,\delta)_{p(\cdot)} := \|f\|_{p(\cdot)} =: \Omega_r(f,0)_{p(\cdot)}.$$

2. Uniform norm estimates

In this section, let $\Omega \subseteq \boldsymbol{R}$ be a measurable set and $C(\Omega)$ be the collection of functions continuous on Ω. If $\Omega \neq \boldsymbol{R}$ and $f \in C(\Omega)$, we will extend f to whole $\boldsymbol{R}$ by "$f(s) \equiv 0$ whenever $s \notin \Omega$." when necessary. For $f \in C(\Omega)$ and $\delta \geq 0$, we define the modulus of smoothness as

$$\Omega_r(f,\delta)_{C(\Omega)} := \|(I - T_\delta)^r f\|_{C(\Omega)}, \quad r \in \mathrm{N}, \tag{2.13}$$
$$\Omega_0(f,\cdot)_{C(\Omega)} := \|f\|_{C(\Omega)}$$

with $T_\delta f$ of ($*$).

Lemma 2.1. *Let $0 \leq \delta < \infty$, $r \in \mathrm{N}$ and $f \in C^r(\Omega)$. Then*

$$\frac{d^r}{dx^r} T_\delta f(x) = T_\delta \frac{d^r}{dx^r} f(x) \ \text{ on } \Omega. \tag{2.14}$$

The following theorem states the main properties of (2.13).

Theorem 2.7. *For $f \in C(\Omega)$, $0 \leq \delta < \infty$, and $r \in \mathrm{N}$, the following properties hold.*

(1) $\Omega_r(f,\delta)_{C(\Omega)}$ is non-negative, non-decreasing function of δ,
(2) $\Omega_r(f,\delta)_{C(\Omega)}$ is sub-additive with respect to f,
(3) $\|T_\delta f\|_{C(\Omega)} \leq \|f\|_{C(\Omega)}$,
(4) $\Omega_r(f,\delta)_{C(\Omega)} \leq 2\Omega_{r-1}(f,\delta)_{C(\Omega)} \leq \cdots \leq 2^{r-1}\Omega_1(f,\delta)_{C(\Omega)} \leq 2^r \|f\|_{C(\Omega)}$, (***)
(5) $\Omega_r(f,\delta)_{C(\Omega)} \leq 2^{-1}\delta\Omega_{r-1}(f',\delta)_{C(\Omega)} \leq \cdots \leq 2^{-r}\delta^r \left\|f^{(r)}\right\|_{C(\Omega)}$, if $f \in C^r(\Omega)$.

Let X be a Banach space with a norm $\|\cdot\|_X$ and $r \in \mathrm{N}$. We define Peetre's K-functional for the pair X and W_X^r as follows :

$$K_r(f,\delta,X)_X := \inf_{g \in W_X^r} \left\{ \|f - g\|_X + \delta^r \left\|g^{(r)}\right\|_X \right\}, \quad \delta > 0.$$

We set $T_\delta^r f := (T_\delta f)^r$.

Lemma 2.2. *Let $0 \leq \delta < \infty$, $r - 1 \in \mathrm{N}$, and $f \in C^r(\Omega)$ be given. Then*

$$\frac{d^r}{dx^r} T_\delta^r f(x) = \frac{d}{dx} T_\delta \frac{d^{r-1}}{dx^{r-1}} T_\delta^{r-1} f(x) \quad \text{on } \Omega. \tag{2.15}$$

Lemma 2.3. *(see e.g.[17, p.177]) Let $\Omega \subseteq \boldsymbol{R}$ be a measurable set, $\delta > 0$, $f \in C(\Omega)$ and $\tilde{T}_\delta f(\cdot) = f(\cdot + \delta)$. Then, for any $r \in \mathrm{N}$, there holds*

$$\frac{1}{r^r+2^r} \leq \frac{\sup\limits_{|h|\leq\delta}\left\|\left(I-\tilde{T}_h\right)^r f\right\|_{C(\Omega)}}{K_r\left(f,\delta,C\left(\Omega\right)\right)_{C(\Omega)}} \leq 2^r.$$

Main result of this section is the following theorem.

Theorem 2.8. *Let $\Omega \subseteq \boldsymbol{R}$ be a measurable set, $0<\delta<\infty$, $f \in C\left(\Omega\right)$, $r \in \mathrm{N}$ and $g \in C^2\left(\Omega\right)$. Then, the following inequalities*

$$\left\|\frac{d}{dx}T_\delta f\left(x\right)\right\|_{C(\Omega)} \leq \frac{2}{\delta}\left\|f\right\|_{C(\Omega)},$$

$$\left\|\frac{d^2}{dx^2}T_\delta f\left(x\right)\right\|_{C(\Omega)} \leq \frac{2}{\delta}\left\|\frac{d}{dx}T_\delta f\right\|_{C(\Omega)},$$

$$\left\|g\left(x\right)-T_\delta g\left(x\right)+\frac{\delta}{2}\frac{d}{dx}g\left(x\right)\right\|_{C(\Omega)} \leq \frac{\delta^2}{6}\left\|\frac{d^2}{dx^2}g\right\|_{C(\Omega)},$$

$$\left(c_8\left(r\right)\right)^{-1}K_r\left(f,\delta,C\left(\Omega\right)\right)_{C(\Omega)} \leq \left\|\left(I-T_\delta\right)^r f\right\|_{C(\Omega)} \leq 2^r K_r\left(f,\delta,C\left(\Omega\right)\right)_{C(\Omega)} \tag{2.16}$$

are hold with $c_8\left(1\right)=36$, $c_8\left(r\right)=2^r\left(r^r+\left(34\right)^r\right)$ for $r>1$.

As a corollary of Theorem 2.8, we can state the following result.

Proposition 2.4. *If $0<h\leq\delta<\infty$ and $f \in C\left(\Omega\right)$, then*

$$\left\|\left(I-T_h\right)f\right\|_{C(\Omega)} \leq 72\left\|\left(I-T_\delta\right)f\right\|_{C(\Omega)}. \tag{2.17}$$

As a corollary of (2.16) and Lemma 2.3, we can write

Corollary 2.2. *Let $\Omega \subseteq \boldsymbol{R}$ be a measurable set, $\delta>0$, $f \in C\left(\Omega\right)$ and $r \in \mathrm{N}$. Then,*

(i) there holds

$$1+2^{-r}r^r \leq \frac{\sup\limits_{|h|\leq\delta}\left\|\left(I-\tilde{T}_h\right)^r f\right\|_{C(\Omega)}}{\left\|\left(I-T_\delta\right)^r f\right\|_{C(\Omega)}} \leq 2^r c_8\left(r\right),$$

(ii) for $0<\delta_1\leq\delta_2$, there holds

$$\left(1+2^{-r}r^r\right)\Omega_r\left(f,\delta_1\right)_{C(\Omega)} \leq c_8\left(r\right)2^r\Omega_r\left(f,\delta_2\right)_{C(\Omega)}.$$

Remark 2.2. *From Theorem 23.62 of* [57, p.579], *we have*

$$\lim_{\delta\searrow 0}\Omega_1(f,\delta)_{C(\boldsymbol{R})} = \lim_{\delta\searrow 0}\left\|\left(I-T_\delta\right)f\right\|_{C(\boldsymbol{R})}=0. \tag{2.18}$$

Corollary 2.3. *If $f \in C\left(\boldsymbol{R}\right)$, $0<\delta<\infty$, and $r \in \mathrm{N}$, then, by (2.18) and (***),*

$$\lim_{\delta\searrow 0}\Omega_r(f,\delta)_{C(\boldsymbol{R})} = \lim_{\delta\searrow 0}\left\|\left(I-T_\delta\right)^r f\right\|_{C(\boldsymbol{R})}=0$$

holds.

Let $\mathcal{G}_\sigma\left(X\right)$ be the subspace of entire function of exponential type σ that belonging to a Banach space X. The quantity

$$A_\sigma(f)_X := \inf_g\{\|f-g\|_X : g \in \mathcal{G}_\sigma\left(X\right)\} \tag{2.19}$$

is called the deviation of the function $f \in X$ from $\mathcal{G}_\sigma\left(X\right)$.

Let $\mathcal{G}_{\sigma,p(\cdot)} := \mathcal{G}_{\sigma}\left(L_{p(\cdot)}\right)$ be the subspace of integral function f of exponential type σ that belonging to $L_{p(\cdot)}$. The quantity

$$A_{\sigma}(f)_{p(\cdot)} := \inf_{g}\{\|f-g\|_{p(\cdot)} : g \in \mathcal{G}_{\sigma,p(\cdot)}\}$$

is the deviation of the function $f \in L_{p(\cdot)}$ from $\mathcal{G}_{\sigma}$.

Remark 2.3. *Let $\sigma > 0$, $1 \le p \le \infty$, $f \in L_p(\boldsymbol{R})$,*

$$\vartheta(x) := \frac{2}{\pi}\frac{\sin(x/2)\sin(3x/2)}{x^2}$$

and

$$J(f,\sigma) = \sigma\int_{\boldsymbol{R}} f(x-u)\,\vartheta(\sigma u)\,du$$

be the de la Valèe Poussin operator ([13, definition given in (5.3)]*). It is known (see (5.4)-(5.5) of* [13]*) that, if $f \in L_p(\boldsymbol{R})$, $1 \le p \le \infty$, then*

(i) $J(f,\sigma) \in \mathcal{G}_{2\sigma}(L_p(\boldsymbol{R}))$,
(ii) $J(g_\sigma,\sigma) = g_\sigma$ *for any* $g_\sigma \in \mathcal{G}_\sigma(L_p(\boldsymbol{R}))$,
(iii) $\|J(f,\sigma)\|_{L_p(\boldsymbol{R})} \le \frac{3}{2}\|f\|_{L_p(\boldsymbol{R})}$,
(iv) $(J(f,\sigma))^{(r)} = J\left(f^{(r)},\sigma\right)$ *for any* $r \in \mathrm{N}$ *and* $f \in (L_p(\boldsymbol{R}))^r$,
(v) $\|J\left(f,\frac{\sigma}{2}\right) - f\|_{L_p(\boldsymbol{R})} \to 0$ *(as $\sigma \to \infty$) and hence*

$$\|\left(J\left(f,\frac{\sigma}{2}\right)\right)^{(k)} - f^{(k)}\|_{L_p(\boldsymbol{R})} \to 0 \text{ as } \sigma \to \infty$$

for $f \in W^r_{L_p(\boldsymbol{R})}$ *and* $1 \le k \le r$.

Corollary 2.4. *Let $0 < \sigma < \infty$.*

(i) If $1 \le p < \infty$, $f \in L_p(\boldsymbol{R})$. Then, using (v) of the last remark, we conclude

$$\lim_{\sigma\to\infty} A_\sigma(f)_{L_p(\boldsymbol{R})} = 0.$$

(ii) Let $g : \boldsymbol{R} \to \mathbb{C}$ be bounded on the real axis $\boldsymbol{R}$. Then (see [14]*)*

$$\lim_{\sigma\to\infty} A_\sigma(g)_{C(\boldsymbol{R})} = 0$$

if and only if g is uniformly continuous on $\boldsymbol{R}$.

Theorem 2.9. *Let $r \in \mathrm{N}$, $\sigma > 0$, $\delta \in (0,1)$ and $f \in \mathcal{C}(\boldsymbol{R})$. Then, the following Jackson type inequality*

$$A_\sigma(f)_{\mathcal{C}(\boldsymbol{R})} \le 5\pi 4^{r-1} c_8(r)\,\Omega_r(f,1/\sigma)_{\mathcal{C}(\boldsymbol{R})}, \tag{2.20}$$

and its weak inverse

$$\Omega_r(f,\delta)_{\mathcal{C}(\boldsymbol{R})} \le \left(1+2^{2r-1}\right)2^{r-1}\delta^r\left(A_0(f)_{\mathcal{C}(\boldsymbol{R})} + \int_{1/2}^{1/\delta} u^{r-1}A_u(f)_{\mathcal{C}(\boldsymbol{R})}\,du\right) \tag{2.21}$$

are hold.

We set $\lfloor\sigma\rfloor := \max\{n \in \mathbb{Z} : n \le \sigma\}$.

Theorem 2.10. *Let $r \in \mathrm{N}$, $f \in X^r_{\mathcal{C}(\boldsymbol{R})}$ and $\sigma > 0$. Then*

(a) (i) there exists (see [13, Proposition 25]*) a $g_\sigma \in \mathcal{G}_\sigma(\mathcal{C}(\boldsymbol{R}))$ such that*

$$A_\sigma(f)_{\mathcal{C}(\boldsymbol{R})} \le \|f-g_\sigma\|_{\mathcal{C}(\boldsymbol{R})} \le \frac{5\pi}{4}\frac{4^r}{\sigma^r}\|f^{(r)}\|_{\mathcal{C}(\boldsymbol{R})},$$

(ii) *and its weak inverse*

$$\|f^{(k)}\|_{\mathcal{C}(\boldsymbol{R})} \leq \left(1+2^{2k-1}\right) 2^{k+2}\pi^{k} c_8\left(k\right) \sum_{\nu=0}^{\infty} \frac{(\nu+1)^r}{\nu+1} A_\nu\left(f\right)_{\mathcal{C}(\boldsymbol{R})}$$

holds whenever $k=1,2,\cdots,r$ *and* $\sum_{\nu=0}^{\infty}(\nu+1)^{r-1}A_\nu\left(f\right)_{C(\boldsymbol{R})}<\infty$.

(b) (i) *the following inequality (see* [29, p.397]*)*

$$A_\sigma\left(f\right)_{\mathcal{C}(\boldsymbol{R})} \leq \frac{(5\pi)^r}{\sigma^r} A_\sigma\left(f^{(r)}\right)_{\mathcal{C}(\boldsymbol{R})},$$

(ii) *and its weak inverse*

$$A_\sigma\left(f^{(r)}\right)_{\mathcal{C}(\boldsymbol{R})} \leq \left\| f^{(r)} - \left(J\left(f^{(r)},\frac{\sigma}{2}\right)\right)\right\|_{\mathcal{C}(\boldsymbol{R})}$$

$$\leq \left(1+2^{2r-1}\right) 2^{r+2}\pi^r c_8\left(r\right)\left(A_\sigma\left(f\right)_{\mathcal{C}(\boldsymbol{R})}\sum_{k=0}^{\lfloor\sigma\rfloor}\frac{k^r}{k} + \sum_{\nu=\lfloor\sigma\rfloor+1}^{\infty}\frac{(\nu+1)^r}{\nu+1}A_\nu\left(f\right)_{\mathcal{C}(\boldsymbol{R})}\right)$$

hold when $\sum_{\nu=0}^{\infty}(\nu+1)^{r-1}A_\nu\left(f\right)_{\mathcal{C}(\boldsymbol{R})}<\infty$.

Theorem 2.11. *Let* $r,k\in\mathrm{N}$, $0<t\leq 1/2$, $0\leq\delta<\infty$ *and* $f\in\mathcal{C}(\boldsymbol{R})$. *Then*

(i) *there holds*

$$\Omega_{r+k}\left(f,\delta\right)_{\mathcal{C}(\boldsymbol{R})} \leq 2^k \Omega_r\left(f,\delta\right)_{\mathcal{C}(\boldsymbol{R})},$$

(ii) *and its weak inverse (Marchaud inequality)*

$$\Omega_r\left(f,t\right)_{\mathcal{C}(\boldsymbol{R})} \leq C_9\left(r,k\right)t^r\int_t^1 \frac{\Omega_{r+k}\left(f,u\right)_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du$$

with $C_9\left(r,k\right) = 10\pi\left(1+2^{2r-1}\right)2^{2r+3k}c_8\left(r+k\right)$.

Theorem 2.12. *Let* $\sigma>0$ *and* $f\in\mathcal{C}(\boldsymbol{R})$. *If* $\sum_{\nu=0}^{\infty}(\nu+1)^{k-1}A_\nu\left(f\right)_{\mathcal{C}(\boldsymbol{R})}<\infty$, *holds for some* $k\in\mathrm{N}$, *then*

(i) *the following Jackson type inequality for derivatives*

$$A_\sigma\left(f\right)_{\mathcal{C}(\boldsymbol{R})} \leq (5\pi)^{k+1} c_8\left(r\right)\sigma^{-k}\Omega_r\left(f^{(k)},\sigma^{-1}\right)_{\mathcal{C}(\boldsymbol{R})},$$

(ii) *and its weak inverse (see Theorem 6.3.4 of* [29, p.343]*)*

$$\Omega_r\left(f^{(k)},\frac{1}{\sigma}\right)_{\mathcal{C}(\boldsymbol{R})} \leq 2^{2k+r+1}\left(\frac{1}{\sigma^r}\sum_{\nu=0}^{\lfloor\sigma\rfloor}\frac{(\nu+1)^{r+k}}{\nu+1}A_\nu\left(f\right)_{\mathcal{C}(\boldsymbol{R})} + \sum_{\nu=\lfloor\sigma\rfloor+1}^{\infty}\frac{\nu^k}{\nu}A_\nu\left(f\right)_{\mathcal{C}(\boldsymbol{R})}\right)$$

are hold.

2.1. **Proofs of the results of section 2.**

Proof of Lemma 2.1. For $\delta=0$ (2.14) is obvious. For $0<\delta<\infty$, and $r=1$, one can find

$$\frac{d}{dx}T_\delta f(x) = \frac{d}{dx}\left(\frac{1}{\delta}\int_0^\delta f\left(x+t\right)dt\right) = \frac{1}{\delta}\int_0^\delta \frac{d}{dx}f\left(x+\tau\right)d\tau \tag{2.22}$$

$$= \frac{1}{\delta}\int_0^\delta\left(\frac{d}{dx}f\right)\left(x+\tau\right)d\tau = T_\delta\frac{d}{dx}f(x).$$

For $r>1$, (2.14) follows from (2.22). ∎

Proof of Theorem 2.7. (1)-(3) is known. (4) is seen from binomial expansion. To prove (5), it is sufficient to note inequality (see [10])

$$\|(I-T_\delta) f\|_{C(\Omega)} \leq 2^{-1}\delta \|f'\|_{C(\Omega)}, \quad \delta>0$$

for $f \in C^1(\Omega)$. Then

$$\|(I-T_\delta)^r f\|_{C(\Omega)} \leq 2^{-1}\delta \left\|(I-T_\delta)^{r-1} f'\right\|_{C(\Omega)} \leq \cdots \leq 2^{-r}\delta^r \left\|f^{(r)}\right\|_{C(\Omega)}$$

for $f \in C^r(\Omega)$, because

$$[(I-T_\delta)^r f]' = (I-T_\delta)^r f'.$$

□

Proof of Lemma 2.2. For $r=2$, by Lemma 2.1,

$$\begin{aligned}\frac{d^2}{dx^2}T_\delta^2 f &= \frac{d}{dx}\frac{d}{dx}T_\delta T_\delta f = \frac{d}{dx}\frac{d}{dx}T_\delta \Psi, \qquad [\Psi := T_\delta f]\\ &= \frac{d}{dx}T_\delta\frac{d}{dx}\Psi = \frac{d}{dx}T_\delta\frac{d}{dx}T_\delta f\end{aligned}$$

and the result (2.15) follows. For $r=3$, by Lemma 2.1,

$$\begin{aligned}\frac{d^3}{dx^3}T_\delta^3 f &= \frac{d}{dx}\frac{d^2}{dx^2}T_\delta^2 T_\delta f = \frac{d}{dx}\frac{d^2}{dx^2}T_\delta^2\Psi = \frac{d}{dx}\frac{d}{dx}T_\delta\frac{d}{dx}T_\delta\Psi\\ &= \frac{d}{dx}\frac{d}{dx}T_\delta\frac{d}{dx}T_\delta{}^2 f = \frac{d}{dx}T_\delta\frac{d}{dx}\frac{d}{dx}T_\delta{}^2 f = \frac{d}{dx}T_\delta\frac{d^2}{dx^2}T_\delta^2 f\end{aligned}$$

and (2.15) holds. Let (2.15) holds for $k \in \mathrm{N}$:

$$\frac{d^k}{dx^k}T_\delta^k f = \frac{d}{dx}T_\delta\frac{d^{k-1}}{dx^{k-1}}T_\delta^{k-1} f. \tag{2.23}$$

Then, for $k+1$, (2.23) and Lemma 2.1 implies that

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}T_\delta^{k+1} f &= \frac{d}{dx}\frac{d^k}{dx^k}T_\delta^k T_\delta f = \frac{d}{dx}\frac{d^k}{dx^k}T_\delta^k\Psi = \frac{d}{dx}\frac{d}{dx}T_\delta\frac{d^{k-1}}{dx^{k-1}}T_\delta^{k-1}\Psi\\ &= \frac{d}{dx}\frac{d}{dx}T_\delta\frac{d^{k-1}}{dx^{k-1}}T_\delta^k f = \frac{d}{dx}T_\delta\frac{d}{dx}\frac{d^{k-1}}{dx^{k-1}}T_\delta^k f = \frac{d}{dx}T_\delta\frac{d^k}{dx^k}T_\delta^k f.\end{aligned}$$

□

Proof of Theorem 2.8. For $f \in C(\Omega)$, we have

$$\begin{aligned}\left\|\frac{d}{dx}T_\delta f(x)\right\|_{C(\Omega)} &= \left\|\frac{d}{dx}\frac{1}{\delta}\int_0^\delta f(x+t)\,dt\right\|_{C(\Omega)}\\ &= \left\|\frac{1}{\delta}\frac{d}{dx}\int_x^{x+\delta} f(\tau)\,d\tau\right\|_{C(\Omega)} = \left\|\frac{1}{\delta}(f(x+\delta)-f(x))\right\|_{C(\Omega)} \leq \frac{2}{\delta}\|f\|_{C(\Omega)}.\end{aligned} \tag{2.24}$$

Inequality (2.24) also implies

$$\left\|\left(\frac{d}{dx}\right)^2 T_\delta f(x)\right\|_{C(\Omega)} \leq \frac{2}{\delta}\left\|\frac{d}{dx}T_\delta f\right\|_{C(\Omega)}$$

for $f \in C(\Omega)$. If $f \in C^2(\Omega)$, one can get

$$\left\|f(x)-T_\delta f(x)+\frac{\delta}{2}\frac{d}{dx}f(x)\right\|_{C(\Omega)} \leq \frac{\delta^2}{6}\left\|\frac{d^2}{dx^2}f\right\|_{C(\Omega)}. \tag{2.25}$$

To obtain (2.25), we will use the Taylor formula

$$f(x+t)=f(x)+t\frac{d}{dx}f(x)+\frac{t^2}{2}\frac{d^2}{dx^2}f(\xi)$$

for some $\xi\leq[x,x+t]$. Then, integrating the last equation with respect to t

$$\frac{1}{\delta}\int_0^\delta f(x+t)\,dt=f(x)+\frac{1}{\delta}\int_0^\delta tdt\frac{d}{dx}f(x)+\frac{1}{2}\frac{1}{\delta}\int_0^\delta t^2dt\frac{d^2}{dx^2}f(\xi),$$
$$T_\delta f(x)=f(x)+\frac{\delta}{2}\frac{d}{dx}f(x)+\frac{\delta^2}{6}\frac{d^2}{dx^2}f(\xi)$$

and (2.25) holds.

Now, (2.24) and (2.25) imply that

$$(1/36)\,K_1(f,\delta,C(\Omega))_{C(\Omega)}\leq\|(I-T_\delta)f\|_{C(\Omega)}\leq 2K_1(f,\delta,C(\Omega))_{C(\Omega)}\,. \tag{2.26}$$

Firstly, let us prove the right hand side of (2.26). For any $g\in C^1(\Omega)$

$$\|f-T_\delta f\|_{C(\Omega)}\leq\|f-g\|_{C(\Omega)}+\|g-T_\delta g\|_{C(\Omega)}+\|T_\delta(g-f)\|_{C(\Omega)}$$
$$\leq 2\|f-g\|_{C(\Omega)}+\frac{\delta}{2}\|g'\|_{C(\Omega)}\leq 2K_1(f,\delta,C(\Omega))_{C(\Omega)}\,.$$

For the left hand side of inequality (2.26), we need inequalities

$$\left\|f-T_\delta^2 f\right\|_{C(\Omega)}\leq 2\|f-T_\delta f\|_{C(\Omega)}\,, \tag{2.27}$$

$$\delta\left\|\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)}\leq 34\|f-T_\delta f\|_{C(\Omega)}\,. \tag{2.28}$$

First we prove (2.27). Then

$$\left\|f-T_\delta^2 f\right\|_{C(\Omega)}\leq\|f-T_\delta f\|_{C(\Omega)}+\|T_\delta f-T_\delta T_\delta f\|_{C(\Omega)}\leq 2\|f-T_\delta f\|_{C(\Omega)}\,.$$

Now, we consider inequality (2.28). In (2.25), we replace f by $T_\delta^2 f$ and obtain

$$\left\|T_\delta^2 f(x)-T_\delta T_\delta^2 f(x)+\frac{\delta}{2}\frac{d}{dx}T_\delta^2 f(x)\right\|_{C(\Omega)}\leq\frac{\delta^2}{6}\left\|\frac{d^2}{dx^2}T_\delta^2 f\right\|_{C(\Omega)}.$$

On the other hand, by (2.24),

$$\left\|\frac{d^2}{dx^2}T_\delta^2 f\right\|_{C(\Omega)}\leq\frac{2}{\delta}\left\|\frac{d}{dx}T_\delta f\right\|_{C(\Omega)}$$
$$\leq\frac{2}{\delta}\left\{\left\|\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)}+\left\|\frac{d}{dx}T_\delta(T_\delta f-f)\right\|_{C(\Omega)}\right\}$$
$$\leq\frac{2}{\delta}\left\|\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)}+\frac{4}{\delta^2}\|T_\delta f-f\|_{C(\Omega)}\,.$$

Hence,

$$
\begin{aligned}
\frac{\delta}{2}\left\|\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)} &\le \left\|T_\delta^2 f - T_\delta T_\delta^2 f - \frac{\delta}{2}\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)} + \left\|T_\delta^2 f - T_\delta T_\delta^2 f\right\|_{C(\Omega)} \\
&\le \frac{\delta^2}{6}\left\|\frac{d^2}{dx^2}T_\delta^2 f\right\|_{C(\Omega)} + \left\|T_\delta^2 f - T_\delta T_\delta^2 f\right\|_{C(\Omega)} \\
&\le \frac{\delta^2}{6}\frac{2}{\delta}\left\{\left\|\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)} + \frac{2}{\delta}\left\|T_\delta f - f\right\|_{C(\Omega)}\right\} + \left\|T_\delta^2 f - f\right\|_{C(\Omega)} \\
&+ \left\|T_\delta\left(T_\delta^2 f - f\right)\right\|_{C(\Omega)} + \left\|T_\delta f - f\right\|_{C(\Omega)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\delta}{6}\left\|\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)} &\le \frac{17}{3}\left\|T_\delta f - f\right\|_{C(\Omega)}, \\
\delta\left\|\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)} &\le 34\left\|T_\delta f - f\right\|_{C(\Omega)}.
\end{aligned}
$$

To finish proof of the left hand side of inequality (2.16) with $r = 1$, we proceed as

$$
K_1\left(f,\delta,C\left(\Omega\right)\right)_{C(\Omega)} \le \left\|f - T_\delta^2 f\right\|_{C(\Omega)} + \delta\left\|\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)} \le 36\left\|T_\delta f - f\right\|_{C(\Omega)}.
$$

The proof of (2.16) with $r = 1$ now completed.

Let $r > 1$ be a natural number and we define

$$
g\left(\cdot\right) = \sum_{l=1}^{r}\left(-1\right)^{l-1}\binom{r}{l}T_\delta^{2rl} f\left(\cdot\right).
$$

Then,

$$
\left\|f - g\right\|_{C(\Omega)} = \left\|\left(I - T_\delta^{2r}\right)^r f\right\|_{C(\Omega)} \le \left(2r\right)^r\left\|\left(I - T_\delta\right)^r f\right\|_{C(\Omega)}.
$$

On the other hand,

$$
\begin{aligned}
\delta^r\left\|\frac{d^r}{dx^r}T_\delta^{2r} f\right\|_{C(\Omega)} &= \delta^{r-1}\delta\left\|\frac{d}{dx}T_\delta^2\left(\frac{d^{r-1}}{dx^{r-1}}\right)T_\delta^{2r-2} f\right\|_{C(\Omega)} \\
&\le 34\delta^{r-1}\left\|\left(I - T_\delta\right)\frac{d^{r-1}}{dx^{r-1}}T_\delta^{2r-2} f\right\|_{C(\Omega)} \\
&\le \left(34\right)^2\delta^{r-2}\left\|\left(I - T_\delta\right)^2\frac{d^{r-2}}{dx^{r-2}}T_\delta^{2r-4} f\right\|_{C(\Omega)} \\
&\le \cdots \le \left(34\right)^r\left\|\left(I - T_\delta\right)^r f\right\|_{C(\Omega)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\delta^r\left\|\frac{d^r}{dx^r}T_\delta^{2rl} f\right\|_{C(\Omega)} &\le \left(34\right)^r\left\|\left(I - T_\delta\right)^r T_\delta^{2r(l-1)} f\right\|_{C(\Omega)} \\
&= \left(34\right)^r\left\|T_\delta^{2r(l-1)}\left(I - T_\delta\right)^r f\right\|_{C(\Omega)} \le \left(34\right)^r\left\|\left(I - T_\delta\right)^r f\right\|_{C(\Omega)}.
\end{aligned}
$$

Using the last inequality, we find

$$\begin{aligned}\delta^{r}\left\|\frac{d^{r}}{dx^{r}}g\right\|_{C(\Omega)}&=\delta^{r}\left\|\frac{d^{r}}{dx^{r}}\sum_{l=1}^{r}(-1)^{l-1}\binom{r}{l}T_{\delta}^{2rl}f\right\|_{C(\Omega)}\\&=\delta^{r}\left\|\sum_{l=1}^{r}(-1)^{l-1}\binom{r}{l}\frac{d^{r}}{dx^{r}}T_{\delta}^{2rl}f\right\|_{C(\Omega)}\\&\leq\sum_{l=1}^{r}\left|\binom{r}{l}\right|\delta^{r}\left\|\frac{d^{r}}{dx^{r}}T_{\delta}^{2rl}f\right\|_{C(\Omega)}\\&\leq 2^{r}(34)^{r}\left\|(I-T_{\delta})^{r}f\right\|_{C(\Omega)}\end{aligned}$$

and

$$\begin{aligned}K_{r}\left(f,\delta,C\left(\Omega\right)\right)_{C(\Omega)}&\leq\|f-g\|_{C(\Omega)}+\delta^{r}\left\|\frac{d^{r}}{dx^{r}}g\right\|_{C(\Omega)}\\&\leq 2^{r}\left(r^{r}+(34)^{r}\right)\left\|(I-T_{\delta})^{r}f\right\|_{C(\Omega)}.\end{aligned}$$

For the opposite direction of the last inequality, when $g\in W_{p(\cdot)}^{r}$,

$$\begin{aligned}\Omega_{r}\left(f,\delta\right)_{C(\Omega)}&\leq 2^{r}\|f-g\|_{C(\Omega)}+\Omega_{r}\left(g,\delta\right)_{C(\Omega)}\\&\leq 2^{r}\|f-g\|_{C(\Omega)}+2^{-r}\delta^{r}\left\|g^{(r)}\right\|_{C(\Omega)},\end{aligned}\tag{2.29}$$

and taking infimum on $g\in W_{p(\cdot)}^{r}$ in (2.29), we get

$$\Omega_{r}\left(f,\delta\right)_{C(\Omega)}\leq 2^{r}K_{r}\left(f,\delta,C\left(\Omega\right)\right)_{C(\Omega)}.$$

□

Proof of Proposition 2.4. Let $f\in C\left(\Omega\right)$. Then

$$\begin{aligned}\left\|(I-T_{h})f\right\|_{C(\Omega)}&\leq 2K_{1}\left(f,h,C\left(\Omega\right)\right)_{C(\Omega)}\\&\leq 2K_{1}\left(f,\delta,C\left(\Omega\right)\right)_{C(\Omega)}\leq 72\left\|(I-T_{\delta})f\right\|_{C(\Omega)}.\end{aligned}$$

□

Proof of Theorem 2.9. (i) We consider Jackson type inequality (2.20). For any $g\in X_{\mathcal{C}(\boldsymbol{R})}^{r}$, we have

$$\begin{aligned}A_{\sigma}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}&\leq A_{\sigma}\left(f-g\right)_{\mathcal{C}(\boldsymbol{R})}+A_{\sigma}\left(g\right)_{\mathcal{C}(\boldsymbol{R})}\\&\leq\|f-g\|_{\mathcal{C}(\boldsymbol{R})}+\frac{5\pi}{4}\frac{4^{r}}{\sigma^{r}}\left\|\frac{d^{r}}{dx^{r}}g\right\|_{\mathcal{C}(\boldsymbol{R})}.\end{aligned}$$

Taking infimum on $g\in X_{\mathcal{C}(\boldsymbol{R})}^{r}$ in the last inequality, we have

$$A_{\sigma}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}\leq\frac{5\pi 4^{r}}{4}K_{r}\left(f,\frac{1}{\sigma},\mathcal{C}(\boldsymbol{R})\right)_{\mathcal{C}(\boldsymbol{R})}\leq\frac{5\pi}{4}c_{8}\left(r\right)4^{r}\left\|\left(I-T_{\frac{1}{\sigma}}\right)^{r}f\right\|_{\mathcal{C}(\boldsymbol{R})}.$$

(ii) We give the proof of inverse estimate (2.21). Let $\sigma>0$ and $g_{\sigma}\in\mathcal{G}_{\sigma}\left(\mathcal{C}(\boldsymbol{R})\right)$ be the best approximating IFFD of $f\in\mathcal{C}(\boldsymbol{R})$. Suppose that $r\in\mathrm{N}$, $0<\delta<1$. Then, there exists a $m\in\mathrm{N}$ such that $\lfloor 1/\delta\rfloor=2^{m-1}$. Hence, $2^{m-1}\leq 1/\delta<2^{m}$. Now, we have

$$\begin{aligned}\Omega_{r}\left(f,\delta\right)_{\mathcal{C}(\boldsymbol{R})}&\leq\Omega_{r}\left(f-g_{2^{m}},\delta\right)_{\mathcal{C}(\boldsymbol{R})}+\Omega_{r}\left(g_{2^{m}},\delta\right)_{\mathcal{C}(\boldsymbol{R})}\\&\leq 2^{r}A_{2^{m}}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}+2^{-r}\delta^{r}\left\|\frac{d^{r}}{dx^{r}}g_{2^{m}}\right\|_{\mathcal{C}(\boldsymbol{R})}.\end{aligned}$$

On the other hand

$$
\begin{aligned}
\left\|\frac{d^r}{dx^r}g_{2^m}\right\|_{\mathcal{C}(\boldsymbol{R})} &= \left\|\sum_{\gamma=1}^{m}\left(\frac{d^r}{dx^r}g_{2^\gamma}-\frac{d^r}{dx^r}g_{2^{\gamma-1}}\right)+\left(\frac{d^r}{dx^r}g_1-\frac{d^r}{dx^r}g_0\right)\right\|_{\mathcal{C}(\boldsymbol{R})}\\
&\le \sum_{\gamma=1}^{m}2^{\gamma r}\left\|g_{2^\gamma}-g_{2^{\gamma-1}}\right\|_{\mathcal{C}(\boldsymbol{R})}+\left\|g_1-g_0\right\|_{\mathcal{C}(\boldsymbol{R})}\\
&\le A_0(f)_{\mathcal{C}(\boldsymbol{R})}+A_1(f)_{\mathcal{C}(\boldsymbol{R})}+\sum_{\gamma=1}^{m}2^{\gamma r}\left(A_{2^\gamma}(f)_{\mathcal{C}(\boldsymbol{R})}+A_{2^{\gamma-1}}(f)_{\mathcal{C}(\boldsymbol{R})}\right)\\
&\le A_0(f)_{\mathcal{C}(\boldsymbol{R})}+2^rA_1(f)_{\mathcal{C}(\boldsymbol{R})}+2\sum_{\gamma=1}^{m}2^{\gamma r}A_{2^{\gamma-1}}(f)_{\mathcal{C}(\boldsymbol{R})}\\
&\le 2\left(A_0(f)_{\mathcal{C}(\boldsymbol{R})}+\sum_{\gamma=1}^{m}2^{\gamma r}A_{2^{\gamma-1}}(f)_{\mathcal{C}(\boldsymbol{R})}\right).
\end{aligned}
$$

Then,

$$
\frac{\delta^r}{2^r}\left\|\frac{d^r}{dx^r}g_{2^m}\right\|_{\mathcal{C}(\boldsymbol{R})}\le\frac{2}{2^r}\delta^r\left(A_0(f)_{\mathcal{C}(\boldsymbol{R})}+\sum_{\gamma=1}^{m}2^{\gamma r}A_{q^{\gamma-1}}(f)_{\mathcal{C}(\boldsymbol{R})}\right).
$$

Hence,

$$
\begin{aligned}
\Omega_r(f,\delta)_{C(\boldsymbol{R})} &\le \frac{2^{(m+1)r}}{2^{mr}}A_{2^m}(f)_{\mathcal{C}(\boldsymbol{R})}+\frac{2}{2^r}\delta^r\left(A_0(f)_{\mathcal{C}(\boldsymbol{R})}+\sum_{\gamma=1}^{m}2^{\gamma r}A_{q^{\gamma-1}}(f)_{\mathcal{C}(\boldsymbol{R})}\right)\\
&\le \left(1+2^{2r-1}\right)2^{1-r}2^{2r}\delta^r\left(A_0(f)_{\mathcal{C}(\boldsymbol{R})}+\sum_{\gamma=1}^{m}\int_{2^{\gamma-2}}^{2^{\gamma-1}}u^{r-1}A_u(f)_{\mathcal{C}(\boldsymbol{R})}\,du\right)\\
&\le \left(1+2^{2r-1}\right)2^{r-1}\delta^r\left(A_0(f)_{\mathcal{C}(\boldsymbol{R})}+\int_{1/2}^{2^{m-1}}u^{r-1}A_u(f)_{\mathcal{C}(\boldsymbol{R})}\,du\right)\\
&\le \left(1+2^{2r-1}\right)2^{r-1}\delta^r\left(A_0(f)_{\mathcal{C}(\boldsymbol{R})}+\int_{1/2}^{1/\delta}u^{r-1}A_u(f)_{\mathcal{C}(\boldsymbol{R})}\,du\right).
\end{aligned}
$$

□

Proof of Theorem 2.10. Results a) (i) and b) (i) are known. Let us consider a) (ii). Suppose that $\sum\limits_{\nu=0}^{\infty}\frac{(\nu+1)^r}{\nu+1}A_\nu(f)_{\mathcal{C}(\boldsymbol{R})}<\infty$ and $k\in\{1,2,\cdots,r\}$. Then, using Nikolskii inequality, one gets

$$
\begin{aligned}
\|f^{(k)}\|_{\mathcal{C}(\boldsymbol{R})} &= \lim_{\sigma\to\infty}\left\|J\left(f^{(k)},\frac{\sigma}{2}\right)\right\|_{\mathcal{C}(\boldsymbol{R})}=\lim_{\sigma\to\infty}\left\|\left(J\left(f,\frac{\sigma}{2}\right)\right)^{(k)}\right\|_{\mathcal{C}(\boldsymbol{R})}\\
&\le \frac{\pi^k}{2^k}\frac{\sup\limits_{|h|\le\delta}\left\|\left(I-\tilde{T}_h\right)^k\left(J\left(f,\frac{\sigma}{2}\right)\right)\right\|_{\mathcal{C}(\boldsymbol{R})}}{\delta^k}\le\frac{\pi^k}{2^k}\frac{2^kc_8(k)\,\Omega_k\left(J\left(f,\frac{\sigma}{2}\right),\delta\right)_{\mathcal{C}(\boldsymbol{R})}}{\delta^k}
\end{aligned}
$$

$$\leq \left(1+2^{2k-1}\right)2^{k+2}\pi^{k}c_{8}\left(k\right)\sum_{\nu=0}^{\lfloor 1/\delta\rfloor}\frac{(\nu+1)^{k}}{\nu+1}A_{\nu}\left(J\left(f,\frac{\sigma}{2}\right)\right)_{\mathcal{C}(\boldsymbol{R})}$$
$$\leq \left(1+2^{2k-1}\right)2^{k+2}\pi^{k}c_{8}\left(k\right)\sum_{\nu=0}^{\infty}\frac{(\nu+1)^{r}}{\nu+1}A_{\nu}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}.$$

Note that (ii) b) is follow from (i) b). □

Proof of Theorem 2.11. (i) follows from properties of modulus of smoothness. We consider Marchaud type inequality (ii). Let $0<t<1/2$. Assume that $2^{m-1}\leq\frac{1}{t}<2^{m}$ for some $m\in\mathrm{N}$. Then,

$$\begin{aligned}
\Omega_{r}(f,t)_{\mathcal{C}(\boldsymbol{R})} &\leq \left(1+2^{2r-1}\right)2^{1-r}t^{r}\left(\sum_{\nu=1}^{m}2^{\nu r}A_{2^{\nu-1}}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}+A_{0}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}\right)\\
&\leq \frac{5\pi}{2}\left(1+2^{2r-1}\right)2^{r+2k}c_{8}\left(r+k\right)t^{r}\left(A_{0}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}+\sum_{\nu=1}^{m}2^{\nu r}\Omega_{k+r}(f,\frac{1}{2^{\nu}})_{\mathcal{C}(\boldsymbol{R})}\right)\\
&\leq \frac{5\pi}{2}\left(1+2^{2r-1}\right)2^{2r+3k}c_{8}\left(r+k\right)t^{r}\left(\Omega_{k+r}(f,\frac{1}{2})_{\mathcal{C}(\boldsymbol{R})}+\sum_{\nu=1}^{m}\int_{2^{-v}}^{2^{-v+1}}\frac{\Omega_{k+r}(f,u)_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du\right)\\
&\leq \frac{5\pi}{2}\left(1+2^{2r-1}\right)2^{2r+3k}c_{8}\left(r+k\right)t^{r}\left(\Omega_{k+r}(f,\frac{1}{2})_{\mathcal{C}(\boldsymbol{R})}+\int_{2^{-1}}^{2^{-m+1}}\frac{\Omega_{k+r}(f,u)_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du\right)\\
&\leq 5\pi\left(1+2^{2r-1}\right)2^{2r+3k}c_{8}\left(r+k\right)t^{r}\left(\int_{1/2}^{1}\frac{\Omega_{k+r}(f,u)_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du+\int_{t}^{1}\frac{\Omega_{k+r}(f,u)_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du\right)\\
&\leq 10\pi\left(1+2^{2r-1}\right)2^{2r+3k}c_{8}\left(r+k\right)t^{k}\int_{t}^{1}\frac{\Omega_{k+r}(f,u)_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du.
\end{aligned}$$

□

Using this section's estimates and Transference result Theorem 1.5, in the next section we will give several results on difference operator $\|(I-T_{\delta})^{r}f\|_{p(\cdot)}$ and approximation by IFFD in $L_{p(\cdot)}$.

3. Applications on Difference operator and Approximation

Notation . *Since the* $48c_{7}\left(c_{3}\left(p\right)\right)c_{5}\left(p^{+},c_{3}\left(p\right)\right)$ *of (1.11) will be used very frequently in the next parts, we will set* $c_{10}{:=}c_{10}\left(p^{+},c_{3}\left(p\right)\right){:=}48c_{7}\left(c_{3}\left(p\right)\right)c_{5}\left(p^{+},c_{3}\left(p\right)\right)$.

Lemma 3.4. *Let* $p\in P^{Log}\left(\boldsymbol{R}\right)$, $r\in\mathrm{N}$, *and* $0<\delta<\infty$. *Then*

$$\|(I-T_{\delta})^{r}f\|_{p(\cdot)}\leq c_{10}^{r}2^{-r}\delta^{r}\left\|f^{(r)}\right\|_{p(\cdot)},\quad f\in W_{L_{p(\cdot)}}^{r}$$

hold.

We will use notation $K_{r}\left(f,\delta,p\left(\cdot\right)\right):=K_{r}\left(f,\delta,L_{p(\cdot)}\right)_{L_{p(\cdot)}}$ for $r\in\mathrm{N}$, $p\in P^{Log}\left(B\right)$, $\delta>0$ and $f\in L_{p(\cdot)}\left(B\right)$.

As a corollary of Transference result, we can obtain the following Lemma.

Lemma 3.5. *Let $0 < h \leq \delta < \infty$, $p \in P^{Log}(\boldsymbol{R})$ and $f \in L_{p(\cdot)}$. Then*

$$\left\| (I - T_h) f \right\|_{p(\cdot)} \leq c_8 \left(72, p^+, c_3(p)\right) \left\| (I - T_\delta) f \right\|_{p(\cdot)} \tag{3.30}$$

holds.

In the following theorem, we show that K-functional $K_r(f, \delta, p(\cdot))$ and $\Omega_r(f, \delta)_{p(\cdot)}$ are equivalent.

Theorem 3.13. *Let $p(\cdot) \in P^{Log}(\boldsymbol{R})$. If $L_{p(\cdot)}$, then the K-functional $K_r(f, \delta, p(\cdot))$ and the modulus $\Omega_r(f, \delta)_{p(\cdot)}$ are equivalent, namely,*

$$\begin{aligned} \frac{1}{48 c_7 (c_3(p))\, 2^r c_5 (p^+, c_3(p))} &\leq \frac{K_r(f, \delta, p(\cdot))}{\Omega_r(f, \delta)_{p(\cdot)}} \\ &\leq 48 c_7 (c_3(p)) \left\{ (2r)^r + 2^r (34)^r \right\} c_5 \left(p^+, c_3(p)\right). \end{aligned}$$

Theorem 3.14. *For $p(\cdot) \in P^{Log}(\boldsymbol{R})$, $f, g \in L_{p(\cdot)}$ and $\delta > 0$, the modulus of smoothness $\Omega_r(f, \delta)_{p(\cdot)}$ has the following properties:*

(1) $\Omega_r(f, \delta)_{p(\cdot)}$ is non-negative; non-decreasing function of δ.
(2) For $f, g \in L_{p(\cdot)}$ and $\delta > 0$,

$$\Omega_r(f + g, \delta)_{p(\cdot)} \leq \Omega_r(f, \delta)_{p(\cdot)} + \Omega_r(g, \delta)_{p(\cdot)}. \tag{3.31}$$

(3) For $f \in L_{p(\cdot)}$,

$$\lim_{\delta \to 0} \Omega_r(f, \delta)_{p(\cdot)} = 0. \tag{3.32}$$

As a corollary of Theorem 3.13,

Corollary 3.5. *Let $p(\cdot) \in P^{Log}(\boldsymbol{R})$. If $\delta, \lambda \in (0, 1)$, $f \in L_{p(\cdot)}$, then*

$$\frac{\Omega_r(f, \lambda\delta)_{p(\cdot)}}{(1 + \lfloor \lambda \rfloor)^r \, \Omega_r(f, \delta)_{p(\cdot)}} \leq (48)^2 c_7^2 (c_3(p))\, 2^r c_5^2 \left(p^+, c_3(p)\right) \left((2r)^r + 2^r (34)^r\right)$$

holds.

Theorem 3.15. *Let $p(\cdot) \in P^{Log}(\boldsymbol{R})$, $r \in \mathrm{N}$, $\sigma > 0$ and $f \in L_{p(\cdot)}$. Then,*

$$A_\sigma(f)_{p(\cdot)} \leq c_{11} \left\| \left(I - T_{1/\sigma}\right)^r f \right\|_{p(\cdot)} \tag{3.33}$$

with $c_{11} := c_{11}(r, p^+, c_3(p)) := 30\pi 8^r c_5 (p^+, c_3(p))\, c_7 (c_3(p))\, c_8(r)$.

Now, we present the inverse theorem.

Theorem 3.16. *Let $p(\cdot) \in P^{Log}(\boldsymbol{R})$, $r \in \mathrm{N}$, $\delta \in (0, 1)$ and $f \in L_{p(\cdot)}$. Then,*

$$\Omega_r(f, \delta)_{p(\cdot)} \leq c_{12} \delta^r \left(A_0(f)_{p(\cdot)} + \int_{1/2}^{1/\delta} u^{r-1} A_{u/2}(f)_{p(\cdot)}\, du \right)$$

holds with $c_{12} := c_{12}(r, p^+, c_3(p)) := c_{13} 12 c_7 (c_3(p)) \left(1 + 2^{2r-1}\right) 2^r$, where $c_{13} := c_{13}(p^+, c_3(p)) := 2 c_5 (p^+, c_3(p)) \left(1 + 72 c_7 (c_3(p))\, c_5 (p^+, c_3(p))\right).$

In this section, we obtain Marchaud inequality.

Theorem 3.17. *Let $r,k\in\mathrm{N}$, $p\in P^{Log}\left(\boldsymbol{R}\right)$, $f\in L_{p(\cdot)}$ and $t\in(0,1/2)$. Then,*

$$\Omega_r\left(f,t\right)_{p(\cdot)}\leq c_{14}t^r\int_t^1\frac{\Omega_{r+k}\left(f,u\right)_{p(\cdot)}}{u^{r+1}}du$$

holds with $c_{14}:=c_{14}(r,k,p^+,c_3\left(p\right)):=48c_7\left(c_3\left(p\right)\right)C_9\left(r,k\right)c_5\left(p^+,c_3\left(p\right)\right).$

Theorem 3.18. *Let $p\in P^{Log}\left(\boldsymbol{R}\right)$, $r\in\mathrm{N}$ and $f\in L_{p(\cdot)}$. If*

$$\sum_{\nu=0}^{\infty}\nu^{k-1}A_{\nu/2}\left(f\right)_{p(\cdot)}<\infty$$

holds for some $k\in\mathrm{N}$, then $f^{(k)}\in L_{p(\cdot)}$ and

$$\Omega_r\left(f^{(k)},\frac{1}{\sigma}\right)_{p(\cdot)}\leq c_{14}\left(\frac{1}{\sigma^r}\sum_{\nu=0}^{\lfloor\sigma\rfloor}\left(\nu+1\right)^{r+k-1}A_{\nu/2}\left(f\right)_{p(\cdot)}+\sum_{\nu=\lfloor\sigma\rfloor+1}^{\infty}\nu^{k-1}A_{\nu/2}\left(f\right)_{p(\cdot)}\right)\tag{3.34}$$

with $c_{14}:=c_{14}(r,k,p^+,c_3\left(p\right)):=48c_7\left(c_3\left(p\right)\right)c_5\left(p^+,c_3\left(p\right)\right)2^{2k+r+2}$.

3.1. **Proofs of the results of section 3.**

Proof of Lemma 3.4. We note that (see [10]) the following inequality

$$\left\|\left(I-T_\delta\right)f\right\|_{p(\cdot)}\leq 2^{-1}c_{10}\delta\left\|f'\right\|_{p(\cdot)},\quad\delta>0\tag{3.35}$$

holds for $f\in L_{p(\cdot)}$. Then

$$\Omega_r\left(f,\delta\right)_{p(\cdot)}=\left\|\left(I-T_\delta\right)^rf\right\|_{p(\cdot)}\leq...\leq 2^{-r}c_{10}^r\delta^r\left\|f^{(r)}\right\|_{p(\cdot)},\delta>0$$

for $f\in W^r_{L_{p(\cdot)}}$. □

Proof of Theorem 3.13. For any $g\in W^r_{L_{p(\cdot)}}\left(\Omega\right)$, we have $F_g\in C^r\left(\Omega\right)$. Since F_f is linear in f,

$$\left(I-T_\delta\right)^rF_f=F_{(I-T_\delta)^rf}\quad\text{and}\quad\left(F_g\right)^{(r)}=F_{g^{(r)}},$$

using Theorem 1.5 we obtain

$$\begin{aligned}\left\|\left(I-T_\delta\right)^rf\right\|_{p(\cdot)}&\leq 24c_7\left(c_3\left(p\right)\right)\left\|F_{(I-T_\delta)^rf}\right\|_{C(\Omega)}=24c_7\left(c_3\left(p\right)\right)\left\|\left(I-T_\delta\right)^rF_f\right\|_{C(\Omega)}\\&\leq 24c_7\left(c_3\left(p\right)\right)2^rK_r\left(F_f,\delta,C\left(\Omega\right)\right)_{C(\Omega)}\\&\leq 24c_7\left(c_3\left(p\right)\right)2^r\left\{\left\|F_f-F_g\right\|_{C(\Omega)}+\delta^r\left\|\left(F_g\right)^{(r)}\right\|_{C(\Omega)}\right\}\\&=24c_7\left(c_3\left(p\right)\right)2^r\left\{\left\|F_{(f-g)}\right\|_{C(\Omega)}+\delta^r\left\|F_{g^{(r)}}\right\|_{C(\Omega)}\right\}\\&\leq 48c_7\left(c_3\left(p\right)\right)2^rc_5\left(p^+,c_3\left(p\right)\right)\left\{\left\|f-g\right\|_{p(\cdot)}+\delta^r\left\|g^{(r)}\right\|_{p(\cdot)}\right\}.\end{aligned}$$

Taking infimum and considering definition of K-functional one gets,

$$\left\|\left(I-T_\delta\right)^rf\right\|_{p(\cdot)}\leq 48c_7\left(c_3\left(p\right)\right)2^rc_5\left(p^+,c_3\left(p\right)\right)K_r\left(f,\delta,p\left(\cdot\right)\right).$$

Now, we consider the opposite direction of the last inequality. For

$$g\left(\cdot\right)=\sum_{l=1}^{r}\left(-1\right)^{l-1}\binom{r}{l}T_\delta^{2rl}f\left(\cdot\right),$$

we have

$$\begin{aligned}
K_r(f,\delta,p(\cdot)) &\leq \|f-g\|_{p(\cdot)} + \delta^r \left\| \frac{d^r}{dx^r} g \right\|_{p(\cdot)} \\
&\leq 24c_7(c_3(p)) \left\{ \left\| F_{(f-g)} \right\|_{C(\Omega)} + \delta^r \left\| F_{g^{(r)}} \right\|_{C(\Omega)} \right\} \\
&= 24c_7(c_3(p)) \left\{ \|F_f - F_g\|_{C(\Omega)} + \delta^r \left\| (F_g)^{(r)} \right\|_{C(\Omega)} \right\} \\
&\leq 24c_7(c_3(p)) \left\{ \left\| \left(I - T_\delta^{2r}\right)^r F_f \right\|_{C(\Omega)} + \delta^r \left\| \left(\sum_{l=1}^{r} (-1)^{l-1} \binom{r}{l} T_\delta^{2rl} F_f \right)^{(r)} \right\|_{C(\Omega)} \right\} \\
&= 24c_7(c_3(p)) \left\{ \left\| \left(I - T_\delta^{2r}\right)^r F_f \right\|_{C(\Omega)} + \sum_{l=1}^{r} \left| \binom{r}{l} \right| \delta^r \left\| \left(T_\delta^{2rl} F_f\right)^{(r)} \right\|_{C(\Omega)} \right\} \\
&\leq 24c_7(c_3(p)) \left\{ (2r)^r \|(I - T_\delta)^r F_f\|_{C(\Omega)} + 2^r(34)^r \|(I - T_\delta)^r F_f\|_{C(\Omega)} \right\} \\
&= 24c_7(c_3(p)) \{(2r)^r + 2^r(34)^r\} \left\| F_{(I-T_\delta)^r f} \right\|_{C(\Omega)} \\
&\leq 48c_7(c_3(p)) \{(2r)^r + 2^r(34)^r\} c_5\left(p^+, c_3(p)\right) \|(I - T_\delta)^r f\|_{p(\cdot)}.
\end{aligned}$$

□

Proof of Theorem 3.14. Properties (1) and (2), by definition of $\Omega_r(f,\delta)_{p(\cdot)}$ and the triangle inequality of $L_{p(\cdot)}$ are clearly valid. By using [21, Theorem 10.1] and [35, Lemma 2], the relation (3.32) is satisfied. □

Proof of Corollary 3.5. We have

$$\begin{aligned}
\frac{\Omega_r(f,\lambda\delta)_{p(\cdot)}}{(1+\lfloor\lambda\rfloor)^r \,\Omega_r(f,\delta)_{p(\cdot)}} &\leq \frac{48c_7(c_3(p))\,2^r c_5(p^+,c_3(p))}{(1+\lfloor\lambda\rfloor)^r} \frac{K_r(f,\lambda\delta,p(\cdot))}{\Omega_r(f,\delta)_{p(\cdot)}} \\
&\leq \frac{(48)^2 c_7^2(c_3(p))\,2^r c_5^2(p^+,c_3(p))}{(1+\lfloor\lambda\rfloor)^r} \frac{(1+\lfloor\lambda\rfloor)^r}{1} \{(2r)^r + 2^r(34)^r\} \\
&= (48)^2 c_7^2(c_3(p))\,2^r c_5^2\left(p^+,c_3(p)\right) \{(2r)^r + 2^r(34)^r\}.
\end{aligned}$$

□

Proof of Theorem 3.15. First we obtain

$$A_{2\sigma}(f)_{p(\cdot)} \leq 30\pi 8^r c_5\left(p^+,c_3(p)\right) c_7(c_3(p))\, c_8(r) \left\| \left(I - T_{1/(2\sigma)}\right)^r f \right\|_{p(\cdot)} \tag{3.36}$$

and (3.33) follows from (3.36). Let g_σ be an exponential type entire function of degree $\leq \sigma$, belonging to $\mathcal{C}(\boldsymbol{R})$, as best approximation of $F_f \in \mathcal{C}(\boldsymbol{R})$. Since $F_{V_\sigma f} = V_\sigma F_f$ and $V_\sigma g_\sigma = g_\sigma$, there holds

$$\begin{aligned}
A_{2\sigma}(f)_{p(\cdot)} \leq \|f - V_\sigma f\|_{p(\cdot)} &\leq 24c_7\left(c_3(p)\right)\|F_{f-V_\sigma f}\|_{\mathcal{C}(\boldsymbol{R})} \\
&= 24c_7\left(c_3(p)\right)\|F_f - V_\sigma F_f\|_{\mathcal{C}(\boldsymbol{R})} \\
&= 24c_7\left(c_3(p)\right)\|F_f - g_\sigma + g_\sigma - V_\sigma F_f\|_{\mathcal{C}(\boldsymbol{R})} \\
&= 24c_7\left(c_3(p)\right)\|F_f - g_\sigma + V_\sigma g_\sigma - V_\sigma F_f\|_{\mathcal{C}(\boldsymbol{R})} \\
&\leq 24c_7\left(c_3(p)\right)\left(A_\sigma(F_f)_{\mathcal{C}(\boldsymbol{R})} + \frac{3}{2}A_\sigma(F_f)_{\mathcal{C}(\boldsymbol{R})}\right) \\
&= 12c_7\left(c_3(p)\right)A_\sigma(F_f)_{\mathcal{C}(\boldsymbol{R})}.
\end{aligned}$$

For any $g \in W^r_{\mathcal{C}(\boldsymbol{R})}$

$$\begin{aligned}
A_\sigma(u)_{\mathcal{C}(\boldsymbol{R})} &\leq A_\sigma(u-g)_{\mathcal{C}(\boldsymbol{R})} + A_\sigma(g)_{\mathcal{C}(\boldsymbol{R})} \\
&\leq \|u-g\|_{\mathcal{C}(\boldsymbol{R})} + \frac{5\pi}{4}\frac{4^r}{\sigma^r}\left\|\frac{d^r}{dx^r}g\right\|_{\mathcal{C}(\boldsymbol{R})} \\
&\leq \frac{5\pi 4^r}{4}K_r\left(u, \frac{1}{\sigma}, \mathcal{C}(\boldsymbol{R})\right)_{\mathcal{C}(\boldsymbol{R})} \leq \frac{5\pi 8^r}{4}K_r\left(u, \frac{1}{2\sigma}, \mathcal{C}(\boldsymbol{R})\right)_{\mathcal{C}(\boldsymbol{R})} \\
&\leq \frac{5\pi 8^r}{4}c_8(r)\left\|\left(I - T_{\frac{1}{2\sigma}}\right)^r u\right\|_{\mathcal{C}(\boldsymbol{R})}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_{2\sigma}(f)_{p(\cdot)} &\leq 12c_7\left(c_3(p)\right)A_\sigma(F_f)_{\mathcal{C}(\boldsymbol{R})} \\
&\leq 15\pi 8^r c_7\left(c_3(p)\right)c_8(r)\left\|\left(I - T_{\frac{1}{2\sigma}}\right)^r F_f\right\|_{\mathcal{C}(\boldsymbol{R})} \\
&= 15\pi 8^r c_7\left(c_3(p)\right)c_8(r)\left\|F_{\left(I-T_{1/(2\sigma)}\right)^r f}\right\|_{\mathcal{C}(\boldsymbol{R})} \\
&\leq 30\pi 8^r c_5\left(p^+, c_3(p)\right)c_7\left(c_3(p)\right)c_8(r)\left\|\left(I - T_{1/(2\sigma)}\right)^r f\right\|_{p(\cdot)}.
\end{aligned}$$

□

Proof of Theorem 3.16. Let g_σ be an exponential type entire function of degree $\leq \sigma$, belonging to $L^{p(\cdot)}$, as best approximation of $f \in L^{p(\cdot)}$. Then

$$\begin{aligned}
\Omega_r(f,\delta)_{p(\cdot)} &= \|(I - T_\delta)^r f\|_{p(\cdot)} \\
&\leq 24c_7\left(c_3(p)\right)\left\|F_{(I-T_\delta)^r f}\right\|_{\mathcal{C}(\boldsymbol{R})} \\
&= 24c_7\left(c_3(p)\right)\|(I - T_\delta)^r F_f\|_{\mathcal{C}(\boldsymbol{R})} \\
&\leq 12c_7\left(c_3(p)\right)\left(1 + 2^{2r-1}\right)2^r\delta^r\left(A_0(F_f)_{\mathcal{C}(\boldsymbol{R})} + \int_{1/2}^{1/\delta} u^{r-1}A_u(F_f)_{\mathcal{C}(\boldsymbol{R})}\,du\right) \\
&\leq c_{13}12c_7\left(c_3(p)\right)\left(1 + 2^{2r-1}\right)2^r\delta^r\left(A_0(f)_{p(\cdot)} + \int_{1/2}^{1/\delta} u^{r-1}A_{u/2}(f)_{p(\cdot)}\,du\right),
\end{aligned}$$

because

$$A_{2\sigma}(F_f)_{\mathcal{C}(\boldsymbol{R})} \leq \|F_f - V_\sigma F_f\|_{\mathcal{C}(\boldsymbol{R})} = \|F_{f-V_\sigma f}\|_{\mathcal{C}(\boldsymbol{R})} \leq 2c_5\left(p^+, c_3(p)\right)\|f - V_\sigma f\|_{p(\cdot)}$$

$$
\begin{aligned}
&= 2c_5\left(p^{+},c_3(p)\right)\|f-g_\sigma+g_\sigma-V_\sigma f\|_{p(\cdot)}\\
&\le 2c_5\left(p^{+},c_3(p)\right)\left(\|f-g_\sigma\|_{p(\cdot)}+\|V_\sigma g_\sigma-V_\sigma f\|_{p(\cdot)}\right)\\
&\le 2c_5\left(p^{+},c_3(p)\right)\left(\|f-g_\sigma\|_{p(\cdot)}+72c_7(c_3(p))\,c_5\left(p^{+},c_3(p)\right)\|g_\sigma-f\|_{p(\cdot)}\right)\\
&= 2c_5\left(p^{+},c_3(p)\right)\left(1+72c_7(c_3(p))\,c_5\left(p^{+},c_3(p)\right)\right)A_\sigma(f)_{p(\cdot)}.
\end{aligned}
$$

□

Proof of Theorem 3.17. Let g_σ be an exponential type entire function of degree $\le\sigma$, belonging to $L^{p(\cdot)}$, as best approximation of $f\in L_{p(\cdot)}$. Then

$$
\begin{aligned}
\Omega_r(f,t)_{p(\cdot)} &= \|(I-T_t)^r f\|_{p(\cdot)} \le 24c_7(c_3(p))\left\|F_{(I-T_t)^r f}\right\|_{\mathcal{C}(\boldsymbol{R})}\\
&= 24c_7(c_3(p))\,\|(I-T_t)^r F_f\|_{\mathcal{C}(\boldsymbol{R})}\\
&\le 24c_7(c_3(p))\,C_9(r,k)\,t^r\int_t^1\frac{\left\|(I-T_u)^{r+k}F_f\right\|_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du\\
&= 24c_7(c_3(p))\,C_9(r,k)\,t^r\int_t^1\frac{\left\|F_{(I-T_u)^{r+k}f}\right\|_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du\\
&\le 48c_7(c_3(p))\,C_9(r,k)\,c_5\left(p^{+},c_3(p)\right)t^r\int_t^1\frac{\left\|(I-T_u)^{r+k}f\right\|_{p(\cdot)}}{u^{r+1}}du\\
&= 48c_7(c_3(p))\,C_9(r,k)\,c_5\left(p^{+},c_3(p)\right)t^r\int_t^1\frac{\Omega_{r+k}(f,u)_{p(\cdot)}}{u^{r+1}}du.
\end{aligned}
$$

□

Proof of Theorem 3.18. Proof of (3.34) is similar to that of proof of Theorem 3.17. □

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Research Article

Localization of the spectra of dual frames multipliers

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ABSTRACT. This paper concerns dual frames multipliers, i.e. operators in Hilbert spaces consisting of analysis, multiplication and synthesis processes, where the analysis and the synthesis are made by two dual frames, respectively. The goal of the paper is to give some results about the localization of the spectra of dual frames multipliers, i.e. to identify regions of the complex plane containing the spectra using some information about the frames and the symbols.

Keywords: Multipliers, dual frames, spectrum.

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1. INTRODUCTION

Frame multipliers have been objects of several studies [6, 9, 17, 18, 19, 20, 21] and applications in physics [13], signal processing (in particular, Gabor multipliers [12, 16] attracted interest as time-variant filters) and acoustics [4, 5]. Details about applications are discussed also in the survey [22].

Frame multipliers are part of the Bessel multipliers which were introduced in [2] and we are going to recall. A *Bessel sequence* of a separable Hilbert space $\mathcal{H}$ is a sequence $\varphi = \{\varphi_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{H}$ such that there exists $B_\varphi > 0$ and

$$\sum_{n\in\mathbb{N}} |\langle f, \varphi_n\rangle|^2 \leq B_\varphi \|f\|^2, \qquad \forall f \in \mathcal{H}.$$

The constant B_φ is called a *Bessel bound* of φ. A sequence $\varphi = \{\varphi_n\}_{n\in\mathbb{N}}$ is a *frame* of $\mathcal{H}$ if there exist $A, B > 0$ (*lower bound* and *upper bound* of φ, respectively) such that

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, \varphi_n\rangle|^2 \leq B\|f\|^2, \qquad \forall f \in \mathcal{H}. \tag{1.1}$$

Now, let $\varphi = \{\varphi_n\}_{n\in\mathbb{N}}, \psi = \{\psi_n\}_{n\in\mathbb{N}}$ be Bessel sequences of $\mathcal{H}$ and $m = \{m_n\}_{n\in\mathbb{N}} \in \ell^\infty$, i.e. a bounded complex sequence. The operator $M_{m,\varphi,\psi}$ given by

$$M_{m,\varphi,\psi} f = \sum_{n\in\mathbb{N}} m_n \langle f, \psi_n\rangle \varphi_n, \qquad f \in \mathcal{H},$$

is called the *Bessel multiplier* of φ, ψ with *symbol* m. Correspondent versions of Bessel multipliers have been studied also in continuous and distributional contexts (see [3, 10, 23]). A Bessel multiplier $M_{m,\varphi,\psi}$ is called a *frame multiplier* if φ and ψ are frames.

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This paper deals with the spectra of *dual frames multipliers*, i.e. multipliers $M_{m,\varphi,\psi}$, where φ and ψ are dual frames and $m \in \ell^\infty$. Two frames φ and ψ of $\mathcal{H}$ are called *dual* if $f = \sum_{n\in\mathbb{N}}\langle f, \varphi_n\rangle\psi_n$ for every $f \in \mathcal{H}$ (or, equivalently, $f = \sum_{n\in\mathbb{N}}\langle f, \psi_n\rangle\varphi_n$ for every $f \in \mathcal{H}$). In particular, the study of this paper is inspired by the result for Bessel multipliers shown in Proposition 1 below, which is an immediate consequence of [2, Theorem 6.1], i.e. of the fact that

$$\|M_{m,\varphi,\psi}\| \leq \sup_{n\in\mathbb{N}}|m_n| B_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}}, \tag{1.2}$$

where $M_{m,\varphi,\psi}$ is any Bessel multiplier with B_φ and B_ψ some Bessel bounds of φ and ψ, respectively.

Proposition 1. *The spectrum of any Bessel multiplier $M_{m,\varphi,\psi}$ is contained in the closed disk centered the origin with radius $\sup_{n\in\mathbb{N}} |m_n| B_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}}$, where B_φ and B_ψ are Bessel bounds of φ and ψ, respectively.*

Proposition 1 provides information about the location of the spectra of Bessel multipliers in the complex plane. However, the given estimate may be too large for the spectra of dual frames multipliers[1]. The main results of the paper, Theorems 4.1 and 5.2, provide more accurate localization results for the spectra of dual frames multipliers $M_{m,\varphi,\psi}$ under some conditions on φ and ψ. We also stress that these conditions are satisfied by many frames used in applications (see Remark 4.2).

A localization of the spectrum of $M_{m,\varphi,\psi}$ may show that $M_{m,\varphi,\psi}$ is invertible. The invertibility of multipliers was a subject faced in [6, 9, 17, 18, 19, 20] and Theorems 4.1 and 5.2 bring new results in this direction (see Remark 5.4).

Moreover, the knowledge of a region containing the spectrum of $M_{m,\varphi,\psi}$ gives, in particular, information about the distribution of the possible eigenvalues of $M_{m,\varphi,\psi}$. In connection with this subject, recently in [9] some types of dual frames multipliers with at most countable spectra have been studied.

The paper is organized as follows. In Section 2, we recall some basic notions of frame theory, while we give some preliminary localization results about the spectra of dual frames multipliers in Section 3. Finally, Sections 4 and 5 contain the main results mentioned above together with examples.

2. Preliminaries

Throughout the paper, $\mathcal{H}$ indicates a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Given an operator T acting between two Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $R(T)$ and $N(T)$ the *range* and *kernel* of T, respectively, and by T^* its *adjoint* when T is bounded.

If $T : \mathcal{H} \to \mathcal{H}$ is a bounded operator, then we write $\rho(T)$ and $\sigma(T)$ for the *resolvent set* and *spectrum* of T, respectively. We recall that $\rho(T)$ is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ has bounded inverse $(T - \lambda I)^{-1}$ everywhere defined on $\mathcal{H}$ and $\sigma(T)$ is the complement set of $\rho(T)$. We recall that two bounded operators $T, T' : \mathcal{H} \to \mathcal{H}$ are said to be *similar* if there exists a bounded and bijective operator $S : \mathcal{H} \to \mathcal{H}$ such that $T = S^{-1}TS$. Throughout the paper, we will apply the following standard perturbation result (for a reference see, for instance, Theorem IV.1.16 of [15]).

[1] Indeed, if $m \in \ell^\infty$, φ and ψ are dual Riesz bases (for the definition see the end part of Section 2), then the spectrum of $M_{m,\varphi,\psi}$ is the closure of the set $\{m_n\}_{n\in\mathbb{N}}$ (see [1, Proposition 2.1] or [8, Section 5.1]), which is in general smaller than the closed disk centered the origin with radius $\sup_{n\in\mathbb{N}}|m_n| B_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}}$.

Lemma 2.1. *Let $T, B : \mathcal{H} \to \mathcal{H}$ be bounded operators. If T is bijective and $\|B\| < \|T^{-1}\|^{-1}$, then $T + B$ is bijective.*

We denote by ℓ^2 (respectively, ℓ^∞) the usual spaces of square summable (respectively, bounded) complex sequences indexed by $\mathbb{N}$. A *limit point* for $m \in \ell^\infty$ is the limit of a converging subsequence of m.

In the introduction, we gave the definitions of Bessel sequences and frames. Here, we recall some other notions and elementary results about frame theory [7]. A sequence $\varphi = \{\varphi_n\}_{n\in\mathbb{N}}$ is *complete* in $\mathcal{H}$ if its linear span is dense in $\mathcal{H}$ if and only if $\langle \varphi_n, f\rangle = 0$ for every $n \in \mathbb{N}$ implies $f = 0$. A frame for $\mathcal{H}$ is, in particular, complete in $\mathcal{H}$.

Let φ be a frame for $\mathcal{H}$. We say that φ is a *Parseval frame* if (1.1) holds with $A = B = 1$. The operator $S : \mathcal{H} \to \mathcal{H}$, defined by

$$Sf = \sum_{n\in\mathbb{N}} \langle f, \varphi_n\rangle \varphi_n, \qquad f \in \mathcal{H},$$

is well-defined, bounded, bijective and it called the *frame operator* of φ. The sequence $\{S^{-1}\varphi_n\}_{n\in\mathbb{N}}$ is a dual frame of φ, called the *canonical dual*, and $\{S^{-\frac{1}{2}}\varphi_n\}_{n\in\mathbb{N}}$ is a Parseval frame for $\mathcal{H}$.

A *Riesz basis* φ for $\mathcal{H}$ is a complete sequence in $\mathcal{H}$ satisfying for some $A, B > 0$

$$A \sum_{n\in\mathbb{N}} |c_n|^2 \le \left\| \sum_{n\in\mathbb{N}} c_n \varphi_n \right\|^2 \le B \sum_{n\in\mathbb{N}} |c_n|^2, \qquad \forall \{c_n\} \in \ell^2. \tag{2.3}$$

A Riesz basis φ for $\mathcal{H}$ is a frame for $\mathcal{H}$, the constants in (1.1) can be chosen as in (2.3) and the canonical dual of φ is a Riesz basis too (called *dual Riesz basis* of φ).

3. Basic localization results

In this section, we give two preliminary localization results of the spectra of dual frames multipliers (Propositions 2 and 3) without requiring specific properties of the two frames. For the first one, we need the notion of numerical range. Given a bounded operator $T : \mathcal{H} \to \mathcal{H}$ the *numerical range* of T is the set $\mathfrak{n}_T = \{\langle Tf, f\rangle : f \in \mathcal{H}, \|f\| = 1\}$. We recall also that the spectrum of T is contained in the closure of $\mathfrak{n}_T$ (see [15, Corollary V.3.3]).

In addition, we are going to use the following lemma, which states that to examine the spectrum of a dual frame multiplier $M_{m,\varphi,\psi}$ where ψ is, in particular, the canonical dual frame of φ, we can just consider a multiplier determined by a Parseval frame.

Lemma 3.2. *Let φ be a frame for $\mathcal{H}$, ψ its canonical dual frame and $m \in \ell^\infty$. Then $M_{m,\varphi,\psi}$ is similar to $M_{m,\rho,\rho}$, where ρ is the Parseval frame associated to φ, and so $\sigma(M_{m,\varphi,\psi}) = \sigma(M_{m,\rho,\rho})$. In particular, if m is a real (resp., non-negative) sequence, then $M_{m,\varphi,\psi}$ is similar to a self-adjoint (resp., non-negative) operator and $\sigma(M_{m,\varphi,\psi})$ is real (resp., non-negative).*

Proof. Let S be the frame operator of φ, which is a bijective operator. It is immediate to see that $S^{-\frac{1}{2}} M_{m,\varphi,\psi} S^{\frac{1}{2}} = M_{m,\rho,\rho}$, where $\rho = S^{-\frac{1}{2}}\varphi$ is the Parseval frame associated to φ. The rest of the statement follows easily. □

Proposition 2. *Let φ be a frame for $\mathcal{H}$, ψ its canonical dual and $m \in \ell^\infty$. Then $\sigma(M_{m,\varphi,\psi})$ is contained in the closed convex hull of m, i.e. the closure of the set $\{\sum_{n\in\mathbb{N}} a_n m_n : \sum_{n\in\mathbb{N}} |a_n|^2 = 1\}$.*

Proof. By Lemma 3.2, we can confine to the case where $\varphi = \psi$ is a Parseval frame. We note that

$$\langle M_{m,\varphi,\varphi} f, f\rangle = \sum_{n\in\mathbb{N}} m_n |\langle f, \varphi_n\rangle|^2,$$

therefore, because $\|f\|^2 = \sum_{n\in\mathbb{N}}|\langle f, \varphi_n\rangle|^2$, the numerical range (and then also the spectrum) of $M_{m,\varphi,\varphi}$ is contained in the closed convex hull of m. $\square$

Remark 3.1.

(i) Under the hypothesis of Proposition 2, if in addition m is a real sequence, we have that $\sigma(M_{m,\varphi,\psi}) \subseteq [\inf_{n\in\mathbb{N}} m_n, \sup_{n\in\mathbb{N}} m_n]$.

(ii) If φ is not a Parseval frame and ψ is its canonical dual, then the numerical range of $M_{m,\varphi,\psi}$ is not necessarily contained in the closed convex hull of m (even though by Proposition 2 $\sigma(M_{m,\varphi,\psi})$ is). For example, let

$$\varphi = \{e_1, e_1 + e_2, e_3, \dots, e_n, \dots\} \text{ and } \psi = \{e_1 - e_2, e_2, e_3, \dots, e_n, \dots\},$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$ and $m = \{2, 1, 1, \dots\}$. The sequences are frames and ψ is the canonical dual of φ. Moreover, $\frac{3+i}{2}$ belongs to the numerical range of $M_{m,\varphi,\psi}$, because $\langle M_{m,\varphi,\psi} f, f\rangle = \frac{3+i}{2}$, where $f = \frac{e_1+ie_2}{\sqrt{2}}$. Nevertheless, $\frac{3+i}{2}$ is not in the convex hull of m.

The statement of Proposition 2 may not hold if ψ is just a dual frame of φ. For example, take

$$\varphi = \{e_1, e_1, e_2, \dots, e_n, \dots\} \text{ and } \psi = \{ie_1, (1-i)e_1, e_2, \dots, e_n, \dots\},$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$ and $m = \{2, 1, 1, \dots\}$. Then, a straightforward calculation shows that $M_{m,\varphi,\psi} f = f + (1-i)\langle f, e_1\rangle e_1$ for every $f \in \mathcal{H}$, so $M_{m,\varphi,\psi}$ is not self-adjoint and, in particular, its spectrum is not contained in the closed convex hull of m, which is a subset of the real line. For generic dual frames, we can actually state the following.

Proposition 3. *Let φ, ψ be dual frames for $\mathcal{H}$ with upper bounds B_φ, B_ψ, respectively. Let $m \in \ell^\infty$. If $\lambda, \mu \in \mathbb{C}$ and*

$$\sup_{n\in\mathbb{N}}|m_n - \mu| B_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}} < |\mu - \lambda|, \tag{3.4}$$

then $\lambda \in \rho(M_{m,\varphi,\psi})$. In particular,

(1) if m is contained in the disk of center μ with radius r, then $\sigma(M_{m,\varphi,\psi})$ is contained in the disk of center μ with radius $r B_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}}$;

(2) if m is real, then $\sigma(M_{m,\varphi,\psi})$ is contained in the disk of center $\frac{1}{2}(\sup_{n\in\mathbb{N}} m_n + \inf_{n\in\mathbb{N}} m_m)$ with radius $\frac{1}{2}(\sup_{n\in\mathbb{N}} m_n - \inf_{n\in\mathbb{N}} m_m) B_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}}$.

Proof. For simplicity, we write $m - \lambda$ and $m - \mu$ for the complex sequences $\{m_n - \lambda\}$ and $\{m_n - \mu\}$, respectively. If (3.4) holds, then by (1.2) we have $\|M_{m-\mu,\varphi,\psi}\| < |\mu - \lambda|$ and by Lemma 2.1 we have $\lambda - \mu \in \rho(M_{m-\mu,\varphi,\psi})$, i.e. $\lambda \in \rho(M_{m,\varphi,\psi})$. Now, the rest of the statement is immediate. $\square$

When ψ is the canonical dual of φ, Proposition 2 gives a more accurate result than Proposition 3.

4. Main result 1

For the localization result in this section, we make the assumption that a frame contains a Riesz basis. Note that this is not a very strong requirement. Indeed, it is satisfied by many frames used in applications (see Remark 4.2 below for a consideration about Gabor and wavelet frames). For simplicity, we write the statement of the result in terms of the resolvent set.

Theorem 4.1. *Let φ, ψ be frames for $\mathcal{H}$ such that for some $I \subseteq \mathbb{N}$, $\{\varphi_n : n \in I\}$, $\{\psi_n : n \in I\}$ be Riesz bases for $\mathcal{H}$ with lower frame bounds $A_{\varphi,1}$ and $A_{\psi,1}$, respectively. Moreover, let $B_{\varphi,2}$ and $B_{\psi,2}$ be Bessel bounds of $\{\varphi_n : n \in \mathbb{N}\backslash I\}$ and $\{\psi_n : n \in \mathbb{N}\backslash I\}$, respectively. Let $m \in \ell^\infty$. If*

$$\sup_{n\in\mathbb{N}\backslash I} |m_n| {B_{\varphi,2}}^{\frac{1}{2}} {B_{\psi,2}}^{\frac{1}{2}} < \inf_{n\in I} |m_n| {A_{\varphi,1}}^{\frac{1}{2}} {A_{\psi,1}}^{\frac{1}{2}}, \tag{4.5}$$

then $M_{m,\varphi,\psi}$ is bijective.
If, in addition, φ and ψ are dual frames and

$$\sup_{n\in\mathbb{N}\backslash I} |m_n - \lambda| {B_{\varphi,2}}^{\frac{1}{2}} {B_{\psi,2}}^{\frac{1}{2}} < \inf_{n\in I} |m_n - \lambda| {A_{\varphi,1}}^{\frac{1}{2}} {A_{\psi,1}}^{\frac{1}{2}}, \tag{4.6}$$

then $\lambda \in \rho(M_{m,\varphi,\psi})$.

Proof. We can write $M_{m,\varphi,\psi} = M_1 + M_2$, where $M_1 = M_{m^{(1)},\varphi^{(1)},\psi^{(1)}}$ and $M_2 = M_{m^{(2)},\varphi^{(2)},\psi^{(2)}}$, $m^{(1)} = \{m_n : n \in I\}$, $m^{(2)} = \{m_n : n \in \mathbb{N}\backslash I\}$, $\varphi^{(1)} = \{\varphi_n : n \in I\}$, $\varphi^{(2)} = \{\varphi_n : n \in \mathbb{N}\backslash I\}$, $\psi^{(1)} = \{\psi_n : n \in I\}$, $\psi^{(2)} = \{\psi_n : n \in \mathbb{N}\backslash I\}$. First of all, $\inf_{n\in I} |m_n| > 0$ holds by (4.5), so M_1 is bijective by [17, Theorem 5.1]. Moreover, (4.5) allows to apply Lemma 2.1 because

$$\|M_2\| \le \sup_{n\in\mathbb{N}\backslash I} |m_n| {B_{\varphi,2}}^{\frac{1}{2}} {B_{\psi,2}}^{\frac{1}{2}}$$

by (1.2), and

$$\inf_{n\in I} |m_n| {A_{\varphi,1}}^{\frac{1}{2}} {A_{\psi,1}}^{\frac{1}{2}} \le \|M_1^{-1}\|^{-1}$$

by Propositions 7.7 and 7.2 of [2] and the fact that a Bessel bound of the canonical dual of φ (resp., ψ) is ${A_{\varphi,1}}^{-1}$ (resp., ${A_{\psi,1}}^{-1}$). The second part of the statement now follows from the fact that $M_{m,\varphi,\psi} - \lambda I = M_{m-\lambda,\varphi,\psi}$ when φ and ψ are dual frames (here, we write $m - \lambda$ for the sequence $\{m_n - \lambda\}$). □

We show an application of Theorem 4.1 with an example of multiplier with $0-1$ symbol (i.e. a sequence made only of 0 and 1)[2].

Example 4.1. *Let φ be a Parseval frame for $\mathcal{H}$ such that $\{\varphi_{2n}\}_{n\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$ with lower bound A. Clearly, we have $0 < A < 1$. Consequently, $\{\varphi_{2n-1}\}$ is a Bessel sequence with bound $1 - A$. Let, moreover, m be a sequence of 0 and 1. With these choices, we apply Theorem 4.1 to $M_{m,\varphi,\varphi}$. Condition (4.6) is*

$$\sup_{n\in\mathbb{N}} |m_{2n-1} - \lambda|(1 - A) < \inf_{n\in\mathbb{N}} |m_{2n} - \lambda| A. \tag{4.7}$$

We have

$$\inf_{n\in\mathbb{N}} \{|-\lambda|, |1-\lambda|\} = \begin{cases} -\lambda, & \lambda < 0, \\ \lambda, & 0 \le \lambda \le \frac{1}{2}, \\ 1-\lambda, & \frac{1}{2} < \lambda \le 1, \\ \lambda - 1, & 1 < \lambda, \end{cases}$$

and

$$\sup_{n\in\mathbb{N}} \{|-\lambda|, |1-\lambda|\} = \begin{cases} 1-\lambda, & \lambda < 0, \\ 1-\lambda, & 0 \le \lambda \le \frac{1}{2}, \\ \lambda, & \frac{1}{2} < \lambda \le 1, \\ \lambda, & 1 < \lambda. \end{cases}$$

[2]Such a multiplier often occurs in applications, see e.g. [22].

Since, by Proposition 2, we already know that $\sigma(M_{m,\varphi,\varphi}) \subseteq [0,1]$, we need to check the validity of (4.7) only for $0 \le \lambda \le 1$. We note that if $0 \le \lambda \le \frac{1}{2}$, then (4.7) is true if and only if $\lambda > 1 - A$, which makes sense only if $A > \frac{1}{2}$. On the other hand, if $\frac{1}{2} < \lambda \le 1$, then (4.7) is true if and only if $\lambda < A$, which makes sense again only if $A > \frac{1}{2}$. Thus, by Theorem 4.1, we can write that if $A > \frac{1}{2}$

$$\sigma(M_{m,\varphi,\varphi}) \subseteq [0, 1-A] \cup [A, 1].$$

As particular case of Theorem 4.1, we get the following.

Corollary 1. *Let φ be a frame for $\mathcal{H}$ with bounds A and B such that for some $I \subseteq \mathbb{N}$ $\{\varphi_n : n \in I\}$ is a Riesz basis for $\mathcal{H}$ with lower frame bound A'. Let ψ be the canonical dual of φ and $m \in \ell^\infty$. If*

$$\sup_{n \in \mathbb{N}\setminus I} |m_n - \lambda| \frac{B - A'}{A} < \inf_{n \in I} |m_n - \lambda| \frac{A'}{B},$$

then $\lambda \in \rho(M_{m,\varphi,\psi})$.

Proof. The statement follows by Theorem 4.1 once noticed that $\{\psi_n\}_{n\in I} = \{S^{-1}\varphi_n\}_{n\in I}$ is a Riesz basis with lower bound $\frac{A'}{B^2}$, $\{\psi_n\}_{n\in\mathbb{N}\setminus I}$ has Bessel bound $B - A'$ and $\{\psi_n\}_{n\in\mathbb{N}\setminus I} = \{S^{-1}\varphi_n\}_{n\in\mathbb{N}\setminus I}$ has Bessel bound $\frac{B-A'}{A^2}$. □

Remark 4.2. *Gabor and wavelet frames are classical frames which occur in applications (see [7, 11, 14]). A (regular) Gabor frame for $L^2(\mathbb{R})$ is a frame of the form*

$$\mathcal{G}(g, a, b) = \{E_b^m T_a^n g\}_{m,n\in\mathbb{Z}},$$

where $g \in L^2(\mathbb{R})$, $a, b > 0$, $(T_a f)(x) = f(x-a)$ and $(E_b f)(x) = e^{2\pi i b x} f(x)$ for $x \in \mathbb{R}$. A Gabor frame which is a finite union of Riesz bases can be easily constructed in this way. Let $N \in \mathbb{N}$ and $\mathcal{G}(g, a, b)$ a Riesz basis for $L^2(\mathbb{R})$. A simple calculation shows that $\mathcal{G}(g, \frac{a}{N}, b)$ (as well as $\mathcal{G}(g, a, \frac{b}{N})$) is a frame for $L^2(\mathbb{R})$ which is a union of N Riesz bases.

Frames which are unions of Riesz bases can be found also in the context of wavelet frames. In particular, the frame multiresolution analysis technique (see [7, Ch. 17]) gives a way to construct wavelet frames which are unions of Riesz bases.

5. Main result 2

In this section, we consider Parseval frames φ for $\mathcal{H}$ which are unions of multiples of orthonormal bases. In other words, we can think that there exists $k \in \mathbb{N}$ such that

$$\{\varphi_{(i-1)k+j} : i \in \mathbb{N}\} = \{\alpha_j e_i^j : i \in \mathbb{N}\}, \tag{5.8}$$

where $\alpha_j \in \mathbb{C}\setminus\{0\}$ and $\{e_i^j : i \in \mathbb{N}\}$ is orthonormal basis for $\mathcal{H}$ for $j = 1, \dots, k$. Also here, we remark that this condition occurs for frames used in application. For instance, following Remark 4.2, if $\mathcal{G}(g, a, b)$ is an orthonormal basis for $L^2(\mathbb{R})$, then $\frac{1}{N}\mathcal{G}(g, \frac{a}{N}, b)$ and $\frac{1}{N}\mathcal{G}(g, a, \frac{b}{N})$ are Parseval frames and unions of N multiples of orthonormal bases.

Theorem 5.2. *Let φ be as in (5.8), $m \in \ell^\infty$ and $l_1, \dots, l_k \in \mathbb{C}$. If $\lambda \in \mathbb{C}$ and*

$$\sum_{j=1}^{k} |\alpha_j|^2 \sup_{i\in\mathbb{N}} |m_{(i-1)k+j} - l_j| < \left| \sum_{j=1}^{k} |\alpha_j|^2 l_j - \lambda \right|, \tag{5.9}$$

then $\lambda \in \rho(M_{m,\varphi,\varphi})$. As a consequence, if m is a real sequence, then

$$\sigma(M_{m,\varphi,\varphi}) \subseteq \left[\sum_{j=1}^{k} |\alpha_j|^2 \inf_{i\in\mathbb{N}} m_{(i-1)k+j}, \sum_{j=1}^{k} |\alpha_j|^2 \sup_{i\in\mathbb{N}} m_{(i-1)k+j} \right]. \tag{5.10}$$

Proof. First of all, we note that $\lambda \neq \sum_{j=1}^{k}|\alpha_j|^2 l_j$ by (5.9). We have $M_{m,\varphi,\varphi}f = \sum_{j=1}^{k}|\alpha_j|^2 l_j f + M_{m',\varphi,\varphi}$, where $m' = \{m'_n\}$ and $m'_{(i-1)k+j} = m_{(i-1)k+j} - l_j$ for $i \in \mathbb{N}$ and $j = 1, \dots, k$. Thus, the first statement follows by Lemma 2.1 noting that

$$\|M_{m',\varphi,\varphi}\| \leq \sum_{j=1}^{k}|\alpha_j|^2 \sup_{i\in\mathbb{N}}|m_{(i-1)k+j} - l_j|.$$

Now assume that m is real, i.e. $M_{m,\varphi,\varphi}$ is self-adjoint. Therefore, (5.9) implies that $\sigma(M_{m,\varphi,\varphi})$ is contained in the interval

$$\left[\sum_{j=1}^{k}|\alpha_j|^2(l_j - \sup_{i\in\mathbb{N}}|m_{(i-1)k+j} - l_j|), \sum_{j=1}^{k}|\alpha_j|^2(l_j + \sup_{i\in\mathbb{N}}|m_{(i-1)k+j} - l_j|)\right]. \tag{5.11}$$

Choosing in (5.11), first $l_j < \inf_{i\in\mathbb{N}} m_{(i-1)k+j}$ and then $l_j > \sup_{i\in\mathbb{N}} m_{(i-1)k+j}$ for every $j = 1, \dots, k$, we find (5.10). □

Example 5.2. *Let $\varphi = \{\frac{1}{\sqrt{2}}e_1, \frac{1}{\sqrt{2}}f_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}f_2, \dots\}$, where $\{e_n\}$ and $\{f_n\}$ are orthonormal bases for $\mathcal{H}$. Furthermore, let $m = \{m_n\}$ be such that $m_{4n-3} = 0$, $m_{4n-2} = \frac{1}{3}$, $m_{4n-1} = \frac{2}{3}$ and $m_{4n} = 1$, $n \in \mathbb{N}$. Taking into account Proposition 2, the spectrum of $M_{m,\varphi,\varphi}$ is contained in $[0,1]$. This estimate can be improved by Theorem 5.2: in particular, we obtain that $\sigma(M_{m,\varphi,\varphi}) \subseteq [\frac{1}{6}, \frac{5}{6}]$.*

Remark 5.3. *Theorem 5.2 is not a special case of Theorem 4.1 (and vice-versa). In particular, Theorem 4.1 gives no improvement on the localization of the spectrum in Example 5.2. On the other hand, Theorem 5.2 does not add any further information about the spectrum of the multiplier in Example 4.1, even in the case where $\{\varphi_{2n}\}_{n\in\mathbb{N}}$ and $\{\varphi_{2n+1}\}_{n\in\mathbb{N}}$ are multiples of orthonormal bases.*

Remark 5.4. *Theorems 4.1 and 5.2 give, in particular, new criteria of invertibility in comparison to the results in [17]. For instance, let*

$$\varphi = \left\{\frac{1}{\sqrt{2}}e_1, \frac{1}{\sqrt{2}}f_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}f_2, \dots\right\},$$

where $\{e_n\}$ and $\{f_n\}$ are orthonormal bases for $\mathcal{H}$ and $m = \{m_n\}$ is such that $m_{2n-1} = \frac{1}{n+1}$ and $m_{2n} = 2 - \frac{1}{n+1}$, $n \geq 1$. Both Theorems 4.1 and 5.2 show that $\sigma(M_{m,\varphi,\varphi}) \subseteq [\frac{3}{4}, \frac{5}{4}]$. In particular, $M_{m,\varphi,\varphi}$ is invertible. However, Propositions 4.1, 4.2 and 4.4 of [17] do not apply to this multiplier $M_{m,\varphi,\varphi}$.

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