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New Traveling Wave Solutions for the Sixth-order Boussinesq Equation

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Abstract

In this paper, we investigate the new traveling wave solutions for the sixth-order Boussinesq equation using the tanh-coth method. Such a method is a type of expansion method that represents the solutions of partial differential equations as polynomials of tanh and coth functions. We discover several new traveling wave solutions for the sixth-order Boussinesq equation with different parameters, which are of fundamental importance for various applications.

1. Introduction

In this paper, we consider the following sixth-order Boussinesq equation (1.1)

$$u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} + (u^2)_{xx} = 0, \tag{1.1}$$

where $\beta = 1$ or -1. The Boussinesq approximation for water waves was originally derived by Joseph Boussinesq in 1871 [1]. The fourth-order Boussinesq equations were then introduced in the following year [2]. Since then, a great number of mathematical models have been referred as Boussinesq equations, which are usually called Boussinesq-type equations. Among the wide range of Boussinesq-type equations, the sixth-order Boussinesq equations have attracted great attentions from the researchers all over the world. In particular, the Boussinesq-type equations with linear strong damping and nonlinear source [3], fourth-order dispersion term and nonlinear source [4], cubic nonlinearity [5], and the linear Boussinesq-type equation [6] have been considered. In addition to the aforementioned work, Christov, Maugin and Velarde [7] reexamined the Boussinesq-type equations for the shallow fluid layers and derived equation (1.1). The exact controllability and stability of the equation has been studied in [8]. However, the traveling wave solutions for (1.1) has not been considered. In this paper, we will fill in the gap by discussing the traveling wave solutions in the closed form.
The methodology that we use for the derivation of the traveling wave solutions is called the tanh-coth method, which belongs to the broader category of expansion methods. The expansion methods are analytical methods that look for a summation of finite terms in specific forms, including the tanh function and extended tanh expansion method, Jacobi elliptic functions method, extended direct algebraic method, sine-cosine method, and modified (G'/G)-expansion method. In particular, Amirov and Anutgan [9] applied the tanh function and polynomial function methods to derive the analytical solitary wave solutions for the sixth-order modified Boussinesq equation. A similar method named tanh-coth method has also been used to find the exact solutions for various partial differential equations. In [10], the author used tanh-coth method to derive the solitons and kink solutions for nonlinear parabolic equations, including the Fisher equation, Newell-Whithead equation, Allen-Cahn equation, FitzHugh-Nagumo equation and the Burgers-Fisher equation. The tanh-coth method for some nonlinear pseudo-parabolic equations, including the Benjamin-Bona-Mahony-Peregrine-Burgers equation, the Oskolkov-Benjamin-Bona-Mahony-Burgers equation, the Oskolkov equation and the generalized hyperelastic-rod wave equation, were discussed

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in [11]. Recently, the method has also been successfully applied to stochastic differential equations [12, 13] and fractional differential equations [14, 15]. Some extended methods including the extended tanh method [16, 17] and the modified tanh-coth method [18], were developed for the Zakharov-like equation, fourth-order Boussinesq equation, the Klein-Gordon equations, the Khokhlov-Zabolotskaya-Kuznetsov, the Newell-Whitehead-Segel and the Rabinovich wave equations.
Other than the tanh related methods, the Jacobi elliptic function method has also been applied to find the traveling wave and soliton solutions for partial differential equations and fractional differential equations. In [19], the authors used the F-expansion technique to solve the sine-Gordon equation in terms of the Jacobi elliptic functions. Also in [20], Fang and Dai discussed three different approaches for obtaining the bright and dark soliton solutions for a time-fractional higher-order nonlinear Schrodinger equation. More specifically, the Jacobi elliptic function method, Riccati equation method and the double function method have been used to study the time-fractional Schrodinger equation with Kerr law, power law and log law of nonlinearity. Similar to the aforementioned methods, the extended direct algebraic method also assumes that the solution to a given differential equation can be expressed as a finite sum of certain functions. But it requires that each of the function satisfies a specific first order differential equation with parameters. The extended direct algebraic method has been used to find the traveling wave solutions for the coupled systems of KdV equations, the variant Boussinesq equations and the coupled Burgers equations [21]. An alternative method named sine-cosine method was also employed to construct the traveling wave solutions for nonlinear Schrodinger equations [22]. Instead of looking for an analytical solution in the form of a summation of some particular functions, the sine-cosine method simply looks for an ansatz in the form of a power of a truncated sine or cosine function with some unknown parameters. Such a method has been successfully utilized to obtain some traveling wave solutions for several nonlinear Schrodinger equations. Another popular method called modified (G'/G)-expansion method has also been developed for finding exact wave solutions of various PDEs. The main idea of the method is to assume that the exact solution can be expressed as a polynomial in (G'/G) and that G satisfies a specific second-order ODE with parameters to be determined by balancing the derivatives and nonlinear terms in the given PDE. Interest readers can check the work by Bansal and Gupta in [23] where they used such a method to solve the Klein-Gordon-Schrodinger equation.
In this paper, we investigate the traveling wave solutions for the sixth-order Boussinesq equation (1.1) by utilizing the tanh-coth method due to its powerfulness and simplicity. The rest of the paper is organized as follows: in section 2, we describe the framework of the tanh-coth method for general PDEs. In section 3-5, we establish the procedure of finding the traveling wave solutions for the sixth-order Boussinesq equation and discuss different cases for the values of parameters β in (1.1). In particular, we discuss the Boussinesq equation with $\beta = 1$ and $\beta = -1$ in section 4 and section 5, respectively. Some concluding remarks are given in section 6.

2. Description of the tanh-coth method

Consider a PDE in the following form

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}, u_{xxx}, \ldots) = 0, \tag{2.1}$$

where P is a polynomial in terms of the unknown function $u(x,t)$ and its various derivatives. We look for a traveling wave solution $u(\xi)$ with $\xi = x - vt$, where v is the wave speed. Then equation (2.1) can be written as

$$P(u, u', u'', u''', \ldots) = 0, \tag{2.2}$$

which is an ODE with respect to $u(\xi)$, the traveling wave solution.
Next, we let $Y = \tanh(\mu\xi)$ and assume that $u(\xi)$ can be expressed as a finite expansion given in the following equation

$$u(\xi) = a_0 + \sum_{i=1}^{M} a_i Y^i(\xi) + \sum_{i=1}^{M} b_i Y^{-i}(\xi). \tag{2.3}$$

Here a_i for $0 \le i \le M$ and b_j for $1 \le j \le M$ are unknown constants to be determined, and we assume that $a_M \neq 0$. We then substitute (2.3) into (2.2) and balance the coefficients of the various powers of Y. One key component in such a process is to apply the following equality for Y:

$$Y' = \mu - \mu Y^2, \tag{2.4}$$

so that the various derivatives of Y can be converted to powers of Y. Also note that we need to consider the change of variables before we apply (2.4). That is, when we calculate $u'(\xi)$, the following change of derivative is needed:

$$u'(\xi) = \mu(1-Y^2)\frac{du}{dY} = \mu(1-Y^2)\left(\sum_{i=1}^{M} i a_i Y^{i-1} - \sum_{i=1}^{M} i b_i Y^{-i-1}\right).$$

Note that the highest power of Y in $u'(\xi)$ is $(M+1)$ which is one more than the highest power of Y in $u(\xi)$. In addition, we can further calculate the second derivative of $u(\xi)$ to get

$$u''(\xi) = \mu(1-Y^2)\frac{du'(\xi)}{dY} = \mu^2(1-Y^2)\left(-2Y\frac{du}{dY} + (1-Y^2)\frac{d^2u}{dY^2}\right).$$

Since the leading terms in $Y\frac{du}{dY}$ and $(1-Y^2)\frac{d^2u}{dY^2}$ are both Y^M, the highest power of Y in $u''(\xi)$ is $(M+2)$, which is two more than the highest power of Y in $u(\xi)$. Similarly, one can show that if the highest order of derivatives for all the linear terms in (2.2) is K, then the leading term for all the linear terms in the equation is a constant times Y^{M+k}. Usually, one can calculate the value of M by balancing the linear terms of the highest order and the leading nonlinear terms. For example, if P in (2.1) is defined to be $P(u,u_t,u_{xx}) = u_t - u_{xx} + u - u^3$, then the linear term of the highest order in (2.2) is u'' which leads to Y^{M+2} terms, and the leading nonlinear term is u^3 which leads to Y^{3M} terms. By matching the highest power of these two terms, we can get $M = 1$.

Once the value of M is determined, we can rewrite (2.2) as a finite expansion in terms of Y using (2.4) and (2.3). We then further collect all the coefficients of Y^i for all i and derive a system of equations by setting theses coefficients to be equal to zero. By solving the algebraic system, we can obtain the values of a_i (for $0 \le i \le M$), b_j (for $1 \le j \le M$), μ and v, which leads to an analytical solution in the form of (2.3). Note that if we assume that $b_j = 0$ for $1 \le j \le M$ in (2.3), then the method recovers the standard tanh method. The tanh-coth method works very well for PDEs in the form of (2.1). Even for PDEs that are not in the given form as in (2.1), we may still apply the tanh-coth method if the PDEs can be transformed to (2.1). Interested readers can refer to [24] for a thorough discussion about finding the exact solutions of the sine-Gordon and the sinh-Gordon equations using the tanh method.

3. The tanh-coth Method for the sixth-order Boussinesq equation

We now discuss how to solve the sixth-order Boussinesq equation (1.1) using the tanh-coth method. Let $u(x,t) = u(\xi)$ be the traveling wave solution to (1.1) where $\xi = x - vt$ with v being the constant speed of the traveling wave. Then, equation (1.1) becomes

$$v^2u'' - u'' + \beta u^{(4)} - u^{(6)} + (u^2)'' = 0. \tag{3.1}$$

Here u'', $u^{(4)}$ and $u^{(6)}$ represent $\frac{d^2u}{d\xi^2}$, $\frac{d^4u}{d\xi^4}$ and $\frac{d^6u}{d\xi^6}$, respectively. We then integrate (3.1) with respect to ξ twice, and set the integration constants to zero, to obtain the following equation

$$(v^2-1)u + \beta u'' - u^{(4)} + u^2 = 0. \tag{3.2}$$

We now use the tanh-coth method by letting

$$u(\xi) = \sum_{i=0}^{M} a_iY^i(\xi) + \sum_{i=1}^{M} b_iY^{-i}(\xi), \tag{3.3}$$

where $Y = \tanh(\mu\xi)$ satisfies

$$Y' = \mu - \mu Y^2, \tag{3.4}$$

and a_i for $i = 0,1,\ldots,M$ and b_j for $j = 1,2,\ldots,M$ are constants to be determined. Based on the ansarz of $u(\xi)$ given in (3.3) and the derivative of Y in (3.4), as well as the description of the tanh-coth method in section 2, we can balance the highest power of Y in the leading nonlinear term u^2 with the power of Y in the linear term of the highest order, i.e.,$u^{(4)}$, in (3.2). Thus, we can get

$$2M = M + 4,$$

which leads to $M = 4$. Therefore, equation (3.3) becomes

$$u(\xi) = \sum_{i=0}^{4} a_iY^i(\xi) + \sum_{i=1}^{4} b_iY^{-i}(\xi). \tag{3.5}$$

Detailed calculations show that

$$\begin{aligned} u'' \;=\;& 20\mu^2a_4Y^6 + 12\mu^2a_3Y^5 + (6\mu^2a_2 - 32\mu^2a_4)Y^4 + (2\mu^2a_1 - 18\mu^2a_3)Y^3 + (12\mu^2a_4 - 8\mu^2a_2)Y^2 \\ & + (6\mu^2a_3 - 2\mu^2a_1)Y + (2\mu^2a_2 + 2\mu^2b_2) + (6\mu^2b_3 - 2\mu^2b_1)Y^{-1} + (12\mu^2b_4 - 8\mu^2b_2)Y^{-2} \\ & + (2\mu^2b_1 - 18\mu^2b_3)Y^{-3} + (6\mu^2b_2 - 32\mu^2b_4)Y^{-4} + 12\mu^2b_3Y^{-5} + 20\mu^2b_4Y^{-6}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} u^{(4)} \;=\;& 840\mu^4a_4Y^8 + 360\mu^4a_3Y^7 + (120\mu^4a_2 - 2080\mu^4a_4)Y^6 + (24\mu^4a_1 - 816\mu^4a_3)Y^5 \\ & + (1696\mu^4a_4 - 240\mu^4a_2)Y^4 + (576\mu^4a_3 - 40\mu^4a_1)Y^3 + (136\mu^4a_2 - 480\mu^4a_4)Y^2 \\ & + (16\mu^4a_1 - 120\mu^4a_3)Y + (24\mu^4a_4 - 16\mu^4a_2 - 16\mu^4b_2 + 24\mu^4b_4) \\ & + (16\mu^4b_1 - 120\mu^4b_3)Y^{-1} + (136\mu^4b_2 - 480\mu^4b_4)Y^{-2} + (576\mu^4b_3 - 40\mu^4b_1)Y^{-3} \\ & + (1696\mu^4b_4 - 240\mu^4b_2)Y^{-4} + (24\mu^4b_1 - 816\mu^4b_3)Y^{-5} + (120\mu^4b_2 - 2080\mu^4b_4)Y^{-6} \\ & + 360\mu^4b_3Y^{-7} + 840\mu^4b_4Y^{-8}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} u^2 &= a_4^2Y^8 + 2a_3a_4Y^7 + (a_3^2 + 2a_2a_4)Y^6 + (2a_1a_4 + 2a_2a_3)Y^5 + (a_2^2 + 2a_0a_4 + 2a_1a_3)Y^4 \\ &\quad + (2a_0a_3 + 2a_1a_2 + 2a_4b_1)Y^3 + (a_1^2 + 2a_0a_2 + 2a_3b_1 + 2a_4b_2)Y^2 \\ &\quad + (2a_0a_1 + 2a_2b_1 + 2a_3b_2 + 2a_4b_3)Y + (a_0^2 + 2a_1b_1 + 2a_2b_2 + 2a_3b_3 + 2a_4b_4) \\ &\quad + (2a_0b_1 + 2a_1b_2 + 2a_2b_3 + 2a_3b_4)Y^{-1} + (b_1^2 + 2a_0b_2 + 2a_1b_3 + 2a_2b_4)Y^{-2} \\ &\quad + (2a_0b_3 + 2a_1b_4 + 2b_1b_2)Y^{-3} + (b_2^2 + 2a_0b_4 + 2b_1b_3)Y^{-4} + (2b_1b_4 + 2b_2b_3)Y^{-5} \\ &\quad + (b_3^2 + 2b_2b_4)Y^{-6} + 2b_3b_4Y^{-7} + b_4^2Y^{-8}. \end{aligned} \tag{3.8}$$

We then substitute (3.5), (3.6), (3.7) and (3.8) into (3.2), collect all the coefficients of Y^i for $i = -8, -7, \ldots, 8$, and set them equal to zero so that we can obtain a system of equations. Next, we discuss the results for $\beta = 1$ and -1.

4. The Boussinesq equation with $\beta = 1$

For the case of $\beta = 1$, we get the following system

$$\begin{aligned}
O(Y^8):&\quad a_4^2 - 840\mu^4 a_4 = 0,\\
O(Y^7):&\quad -360a_3\mu^4 + 2a_3a_4 = 0,\\
O(Y^6):&\quad 2a_2a_4 + 20\mu^2 a_4 - 120\mu^4 a_2 + 2080\mu^4 a_4 + a_3^2 = 0,\\
O(Y^5):&\quad 2a_1a_4 + 2a_2a_3 + 12\mu^2 a_3 - 24\mu^4 a_1 + 816\mu^4 a_3 = 0,\\
O(Y^4):&\quad 240\mu^4 a_2 - 1696a_4\mu^4 + 6\mu^2 a_2 - 32a_4\mu^2 + a_2^2 + a_4v^2 - a_4 + 2a_0a_4 + 2a_1a_3 = 0,\\
O(Y^3):&\quad 2a_0a_3 - a_3 + 2a_1a_2 + 2a_4b_1 + 2\mu^2 a_1 - 18\mu^2 a_3 + 40\mu^4 a_1 - 576\mu^4 a_3 + a_3v^2 = 0,\\
O(Y^2):&\quad 2a_0a_2 - a_2 + 2a_3b_1 + 2a_4b_2 - 8\mu^2 a_2 + 12\mu^2 a_4 - 136\mu^4 a_2 + 480\mu^4 a_4 + a_2v^2 + a_1^2 = 0,\\
O(Y):&\quad 2a_0a_1 - a_1 + 2a_2b_1 + 2a_3b_2 + 2a_4b_3 - 2\mu^2 a_1 + 6\mu^2 a_3 - 16\mu^4 a_1 + 120\mu^4 a_3 + a_1v^2 = 0,\\
O(Y^0):&\quad 2a_1b_1 - a_0 + 2a_2b_2 + 2a_3b_3 + 2a_4b_4 + 2\mu^2 a_2 + 16\mu^4 a_2 - 24\mu^4 a_4 + 2\mu^2 b_2 + 16\mu^4 b_2\\
&\qquad -24\mu^4 b_4 + a_0v^2 + a_0^2 = 0,\\
O(Y^{-1}):&\quad 2a_0b_1 - b_1 + 2a_1b_2 + 2a_2b_3 + 2a_3b_4 - 2\mu^2 b_1 + 6\mu^2 b_3 - 16\mu^4 b_1 + 120\mu^4 b_3 + b_1v^2 = 0,\\
O(Y^{-2}):&\quad 2a_0b_2 - b_2 + 2a_1b_3 + 2a_2b_4 - 8\mu^2 b_2 + 12\mu^2 b_4 - 136\mu^4 b_2 + 480\mu^4 b_4 + b_2v^2 + b_1^2 = 0,\\
O(Y^{-3}):&\quad 2a_0b_3 - b_3 + 2a_1b_4 + 2b_1b_2 + 2\mu^2 b_1 - 18\mu^2 b_3 + 40\mu^4 b_1 - 576\mu^4 b_3 + b_3v^2 = 0,\\
O(Y^{-4}):&\quad 240\mu^4 b_2 - 1696b_4\mu^4 + 6\mu^2 b_2 - 32b_4\mu^2 + b_2^2 + b_4v^2 - b_4 + 2a_0b_4 + 2b_1b_3 = 0,\\
O(Y^{-5}):&\quad 2b_1b_4 + 2b_2b_3 + 12\mu^2 b_3 - 24\mu^4 b_1 + 816\mu^4 b_3 = 0,\\
O(Y^{-6}):&\quad 2b_2b_4 + 20\mu^2 b_4 - 120\mu^4 b_2 + 2080\mu^4 b_4 + b_3^2 = 0,\\
O(Y^{-7}):&\quad -360b_3\mu^4 + 2b_3b_4 = 0,\\
O(Y^{-8}):&\quad -840\mu^4 b_4 + b_4^2 = 0.
\end{aligned}$$

4.1. When $a_4 = 0$

We can show that if $a_4 = 0$, then $a_3 = a_2 = a_1 = 0$ based on the coefficients of $O(Y^i)$ with $i = 1, 2, \ldots, 8$. Then the coefficient of $O(Y^0)$ leads to

$$-a_0 + 2\mu^2 b_2 + 16\mu^4 b_2 - 24\mu^4 b_4 + a_0v^2 + a_0^2 = 0.$$

If we further assume $b_4 = 0$, then $b_3 = b_2 = b_1 = 0$, and the equation above leads to the trivial solutions to (3.2), namely, $u = 0$ or $u = 1 - v^2$. Therefore, for the case of $a_4 = 0$, we assume $b_4 \neq 0$ so that the coefficient of Y^{-8} gives

$$b_4 = 840\mu^4.$$

Thus we can solve for b_3, b_2, b_1 using the coefficients of Y^{-i} for $i = 7, 6$ and 5 to get

$$b_3 = 0, \quad b_2 = -\frac{140}{13}\mu^2 - 1120\mu^4, \quad b_1 = 0.$$

We further substitute the value of b_1, b_2, b_3 and a_i with $1 \le i \le 4$ into the coefficients of Y^{-4}, Y^{-2} and Y^0, respectively, to obtain

$$1568\mu^4 + \frac{560}{13}\mu^2 - \left(3v^2 + 6a_0 - \frac{476}{169}\right) = 0, \tag{4.1}$$

$$3968\mu^6 + \frac{1904}{13}\mu^4 + \left(-8v^2 - 16a_0 + \frac{112}{13}\right)\mu^2 + \left(-\frac{1}{13}v^2 - \frac{2}{13}a_0 + \frac{1}{13}\right) = 0, \tag{4.2}$$

$$38080\mu^8 + \frac{31360}{13}\mu^6 + \frac{280}{13}\mu^4 - a_0^2 - a_0v^2 + a_0 = 0. \tag{4.3}$$

Equation (4.1) and (4.2) lead to

$$v^2+2a_0=\frac{3968\mu^6+\frac{1904}{13}\mu^4+\frac{112}{13}\mu^2+\frac{1}{13}}{8\mu^2+\frac{1}{13}}=\frac{1568\mu^4+\frac{560}{13}\mu^2+\frac{476}{169}}{3}. \tag{4.4}$$

Thus we get the following equation about μ:

$$640\mu^6+\frac{336}{13}\mu^4-\frac{31}{2197}=\left(\mu^2-\frac{13}{676}\right)\left(640\mu^4+\frac{496}{13}\mu^2+\frac{124}{169}\right)=0.$$

The roots of the equation above are $\mu_1=-\frac{\sqrt{13}}{26}$, $\mu_2=\frac{\sqrt{13}}{26}$, $\mu_3=\sqrt{\frac{-31+3\sqrt{31}i}{1040}}$, $\mu_4=\sqrt{\frac{-31-3\sqrt{31}i}{1040}}$, $\mu_5=-\sqrt{\frac{-31+3\sqrt{31}i}{1040}}$ and $\mu_6=-\sqrt{\frac{-31-3\sqrt{31}i}{1040}}$.

4.1.1. For $\mu=\mu_1=-\frac{\sqrt{13}}{26}$

We substitute the value of μ into equation (4.4) to get

$$v^2+2a_0=\frac{238}{169}. \tag{4.5}$$

We then substitute the value of μ into equation (4.3), and obtain

$$a_0^2+a_0v^2-a_0=\frac{3465}{114244}. \tag{4.6}$$

Solving (4.5) and (4.6) leads to

$$(1)\,a_0=\frac{105}{338}, v=\frac{\sqrt{133}}{13};\quad (2)\,a_0=\frac{105}{338}, v=-\frac{\sqrt{133}}{13};$$
$$(3)\,a_0=\frac{33}{338}, v=\frac{\sqrt{205}}{13};\quad (4)\,a_0=\frac{33}{338}, v=-\frac{\sqrt{205}}{13}.$$

In addition, we can calculate that

$$b_4=\frac{105}{338},\quad b_3=0,\quad b_2=-\frac{105}{169},\quad b_1=0.$$

Based on the discussion above, we can obtain four traveling wave solutions:

$$\begin{aligned}
u_1(x,t) &= \frac{105}{338}-\frac{105}{169}\coth^2\left(-\frac{\sqrt{13}}{26}\left(x-\frac{\sqrt{133}}{13}t\right)\right)+\frac{105}{338}\coth^4\left(-\frac{\sqrt{13}}{26}\left(x-\frac{\sqrt{133}}{13}t\right)\right).\\
u_2(x,t) &= \frac{105}{338}-\frac{105}{169}\coth^2\left(-\frac{\sqrt{13}}{26}\left(x+\frac{\sqrt{133}}{13}t\right)\right)+\frac{105}{338}\coth^4\left(-\frac{\sqrt{13}}{26}\left(x+\frac{\sqrt{133}}{13}t\right)\right).\\
u_3(x,t) &= \frac{33}{338}-\frac{105}{169}\coth^2\left(-\frac{\sqrt{13}}{26}\left(x-\frac{\sqrt{205}}{13}t\right)\right)+\frac{105}{338}\coth^4\left(-\frac{\sqrt{13}}{26}\left(x-\frac{\sqrt{205}}{13}t\right)\right).\\
u_4(x,t) &= \frac{33}{338}-\frac{105}{169}\coth^2\left(-\frac{\sqrt{13}}{26}\left(x+\frac{\sqrt{205}}{13}t\right)\right)+\frac{105}{338}\coth^4\left(-\frac{\sqrt{13}}{26}\left(x+\frac{\sqrt{205}}{13}t\right)\right).
\end{aligned}$$

The traveling wave solution $u_1(x,t)$ at $T=1$ and $T=3$ is given in Figure 4.1. The figure is generated using MATLAB 2019a. Note that $u_1(x,t)$ is defined for $x\neq\frac{\sqrt{133}}{13}t$, thus we only plot part of the spatial domain such that $x-\frac{\sqrt{133}}{13}t$ is large enough. The formulation of $u_1(x,t)$ indicates that the wave travels from left to right, and it is consistent with the observation from Figure 4.1. The behavior of u_2, u_3 and u_4 are very similar to that of u_1. Therefore, we skip the plots of these solutions.

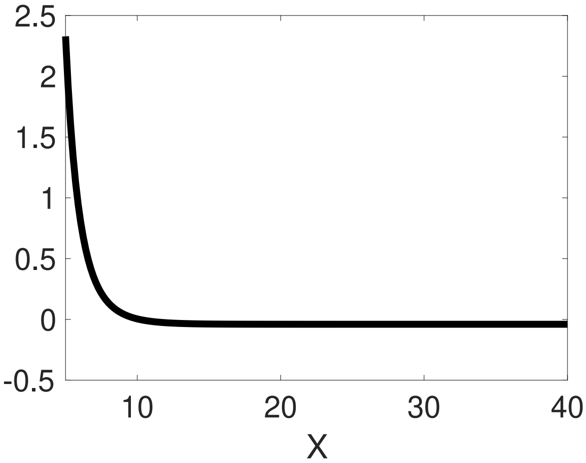
4.1.2. For $\mu=\mu_2=\frac{\sqrt{13}}{26}$

It is easy to show that the values of b_i $(1\le i\le 4)$, a_j $(0\le j\le 4)$ and v_0 are the same as their values in the case when $\mu=\mu_1=-\frac{\sqrt{13}}{26}$. Also note that $\coth^2(-\xi)=\coth^2(\xi)$ and $\coth^4(-\xi)=\coth^4(\xi)$. Therefore, the traveling wave solutions for this case are exactly the same as $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$ and $u_4(x,t)$ in the previous section.

4.1.3. For $\mu=\mu_3=\sqrt{\frac{-31+3\sqrt{31}i}{1040}}$

We substitute the value of μ into equation (4.4) to get

$$v^2+2a_0=\frac{14203-819\sqrt{31}i}{16900}. \tag{4.7}$$



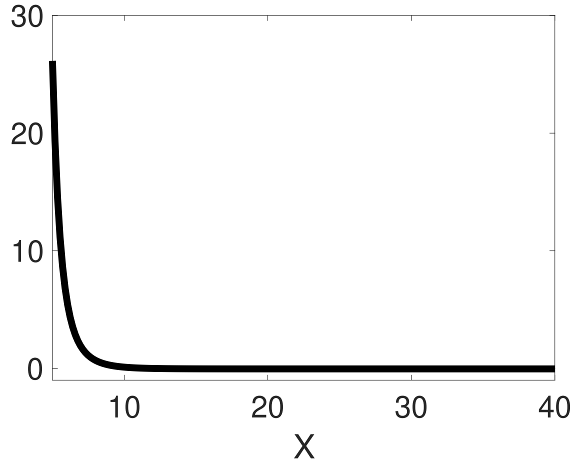


Figure 4.1: The traveling wave solution $u_1(x,t)$ at $T = 1$ (the left figure) and $T = 3$ (the right figure).

We then substitute the value of μ into equation (4.3) to get

$$a_0^2 + a_0 v^2 - a_0 = \frac{-6595281 + 2189313\sqrt{31}i}{456976000}. \tag{4.8}$$

(4.7) and (4.8) lead to

$$a_0^2 + \frac{2697 + 819\sqrt{31}i}{16900} a_0 + \frac{-6595281 + 2189313\sqrt{31}i}{456976000} = 0.$$

Thus $a_0 = \frac{-5394-1638\sqrt{31}i \pm \sqrt{11873682-4222386\sqrt{31}i}}{67600}$ and v can be solved using (4.7).

4.2. When $a_4 \neq 0$ and $b_4 = 0$

Note that the coefficients of $O(Y^i)$ and $O(Y^{-i})$ for $i = 1, \ldots, 8$ are symmetric in the sense that if we interchange a_i, b_i in the formulations of $O(Y^i)$, we can obtain the formulations of $O(Y^{-i})$. Thus, we can show that

$$b_4 = b_3 = b_2 = b_1 = 0,$$

and

$$a_4 = 840\mu^4, \quad a_3 = 0, \quad a_2 = -\frac{140}{13}\mu^2 - 1120\mu^4, \quad a_1 = 0.$$

In addition, it is also easy to verify that equations (4.1), (4.2) and (4.3) are also satisfied. Therefore, the solution of μ is the same as that in the case when $a_4 = 0$, i.e., $\mu_1 = -\frac{\sqrt{13}}{26}$, $\mu_2 = \frac{\sqrt{13}}{26}$, $\mu_3 = \sqrt{\frac{-31+3\sqrt{31}i}{1040}}$, $\mu_4 = \sqrt{\frac{-31-3\sqrt{31}i}{1040}}$, $\mu_5 = -\sqrt{\frac{-31+3\sqrt{31}i}{1040}}$ and $\mu_6 = -\sqrt{\frac{-31-3\sqrt{31}i}{1040}}$. So we can obtain another four traveling wave solutions for $\mu = \mu_1$ and $\mu = \mu_2$:

$$\begin{aligned}
u_5(x,t) &= \frac{105}{338} - \frac{105}{169}\tanh^2\left(\frac{\sqrt{13}}{26}\left(x - \frac{\sqrt{133}}{13}t\right)\right) + \frac{105}{338}\tanh^4\left(\frac{\sqrt{13}}{26}\left(x - \frac{\sqrt{133}}{13}t\right)\right). \\
u_6(x,t) &= \frac{105}{338} - \frac{105}{169}\tanh^2\left(\frac{\sqrt{13}}{26}\left(x + \frac{\sqrt{133}}{13}t\right)\right) + \frac{105}{338}\tanh^4\left(\frac{\sqrt{13}}{26}\left(x + \frac{\sqrt{133}}{13}t\right)\right). \\
u_7(x,t) &= \frac{33}{338} - \frac{105}{169}\tanh^2\left(\frac{\sqrt{13}}{26}\left(x - \frac{\sqrt{205}}{13}t\right)\right) + \frac{105}{338}\tanh^4\left(\frac{\sqrt{13}}{26}\left(x - \frac{\sqrt{205}}{13}t\right)\right). \\
u_8(x,t) &= \frac{33}{338} - \frac{105}{169}\tanh^2\left(\frac{\sqrt{13}}{26}\left(x + \frac{\sqrt{205}}{13}t\right)\right) + \frac{105}{338}\tanh^4\left(\frac{\sqrt{13}}{26}\left(x + \frac{\sqrt{205}}{13}t\right)\right).
\end{aligned}$$

We further use MATLAB 2019a to visualize the traveling wave solutions u_5, u_6, u_7 and u_8 for $t \in [0,30]$. Note that these functions are defined for all real numbers. Figure 4.2 shows that u_5 travels in the positive x-direction and u_6 travels in the negative x-direction. As one can observe in Figure 4.3, the solutions u_7 and u_8 have quite similar behavior as u_5 and y_6, though they have slightly different magnitudes and propagating speeds.

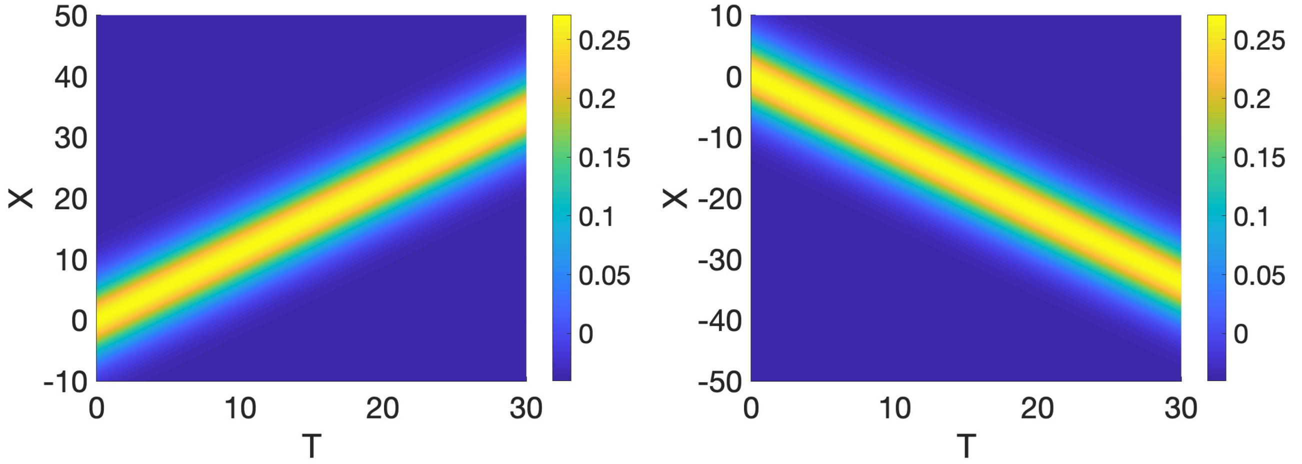


Figure 4.2: The traveling wave solution $u_5(x,t)$ (the left figure) and $u_6(x,t)$ (the right figure).

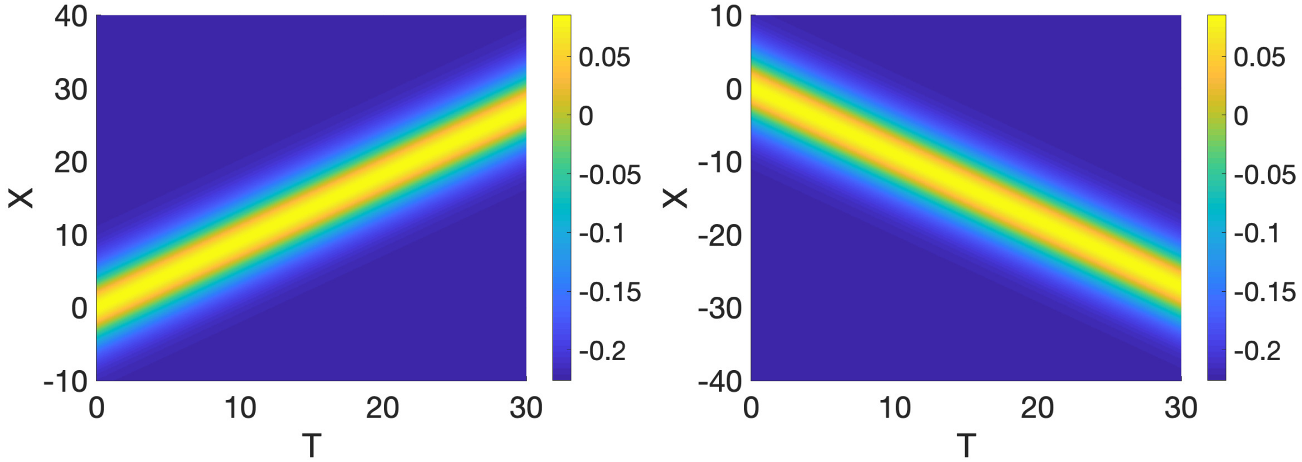


Figure 4.3: The traveling wave solution $u_7(x,t)$ (the left figure) and $u_8(x,t)$ (the right figure).

4.3. When $a_4 \neq 0$ and $b_4 \neq 0$

Similar to the procedure discussed in the previous sections, we can solve that

$$a_4 = b_4 = 840\mu^4, \quad a_3 = b_3 = a_1 = b_1 = 0, \quad a_2 = b_2 = -\frac{140}{13}\mu^2 - 1120\mu^4.$$

We can also show that equations (4.1), (4.2) and (4.4) are also satisfied, and the solutions of μ are $\mu_1 = -\frac{\sqrt{13}}{26}$, $\mu_2 = \frac{\sqrt{13}}{26}$, $\mu_3 = \sqrt{\frac{-31+3\sqrt{31}i}{1040}}$, $\mu_4 = \sqrt{\frac{-31-3\sqrt{31}i}{1040}}$, $\mu_5 = -\sqrt{\frac{-31+3\sqrt{31}i}{1040}}$ and $\mu_6 = -\sqrt{\frac{-31-3\sqrt{31}i}{1040}}$. The coefficient of $O(Y^0)$ leads to

$$-2(2\mu^2 a_2 + 16\mu^4 a_2 - 24\mu^4 a_4) - 2a_2^2 - 2a_4^2 - a_0^2 - a_0 v^2 + a_0 = 0.$$

For $\mu = \mu_1$ or $\mu = \mu_2$, the equation above leads to

$$a_0^2 + a_0 v^2 - a_0 = -\frac{23380}{28561}.$$

Since $v^2 + 2a_0 = \frac{238}{169}$, we have

$$a_0^2 - \frac{69}{169}a_0 - \frac{23380}{28561} = 0.$$

Its solution is $a_0 = \frac{69 \pm \sqrt{98281}}{338}$. Thus, we have

$$v = \sqrt{\frac{238}{169} - 2a_0} = \sqrt{1 \pm \frac{\sqrt{98281}}{169}}.$$

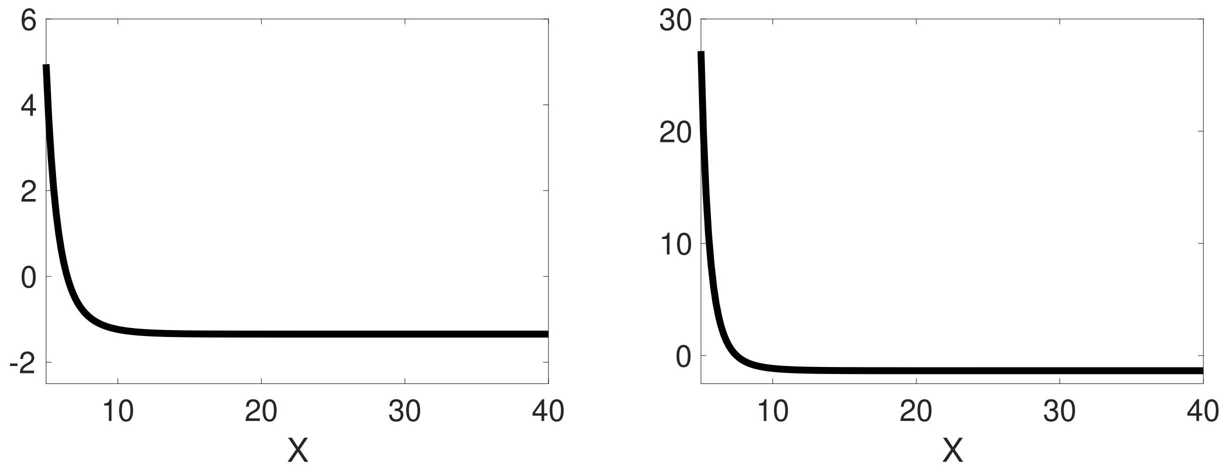


Figure 4.4: The traveling wave solution $u_9(x,t)$ at $T=1$ (the left figure) and $T=1.6$ (the right figure).

Since we assume the constant velocity v of the traveling wave solution is a real number, here we only take $v=\sqrt{1+\frac{\sqrt{98281}}{169}}$, and the corresponding value of a_0 is $a_0=\frac{69-\sqrt{98281}}{338}$. We then use the value of μ to find $a_4=b_4=\frac{105}{338}$, and $a_2=b_2=-\frac{105}{169}$. Note that $\tanh^2(-\xi)=\tanh^2(\xi)$, $\tanh^4(-\xi)=\tanh^4(\xi)$, $\coth^2(-\xi)=\coth^2(\xi)$ and $\coth^4(-\xi)=\coth^4(\xi)$. Therefore, we can obtain the following traveling wave solutions for $\mu=\mu_1$ and μ_2:

$$\begin{aligned} u_9(x,t) \quad = \quad & \frac{69-\sqrt{98281}}{338}-\frac{105}{169}\tanh^2\left(\frac{\sqrt{13}}{26}\left(x-\sqrt{1+\frac{\sqrt{98281}}{169}}t\right)\right) \\ & -\frac{105}{169}\coth^2\left(\frac{\sqrt{13}}{26}\left(x-\sqrt{1+\frac{\sqrt{98281}}{169}}t\right)\right)+\frac{105}{338}\tanh^4\left(\frac{\sqrt{13}}{26}\left(x-\sqrt{1+\frac{\sqrt{98281}}{169}}t\right)\right) \\ & +\frac{105}{338}\coth^4\left(\frac{\sqrt{13}}{26}\left(x-\sqrt{1+\frac{\sqrt{98281}}{169}}t\right)\right). \end{aligned}$$

Due to the coth function in the formulation of $u_9(x,t)$, the domain of u_9 is not the entire real axis. We plot the traveling wave solution $u_9(x,t)$ at $T=1$ and $T=1.6$ in Figure 4.4. The figure shows that the solution is traveling in the positive x-direction. When $\mu=\mu_3$, μ_4, μ_5 or μ_6, there is no real solution for v, thus we do not consider the other cases of μ.

5. The Boussinesq equation with $\beta=-1$

For the case of $\beta=-1$, we get the following system

$$\begin{aligned} O(Y^8): &\quad a_4^2-840\mu^4 a_4=0, \\ O(Y^7): &\quad -360a_3\mu^4+2a_3a_4=0, \\ O(Y^6): &\quad 2a_2a_4-20\mu^2a_4-120\mu^4a_2+2080\mu^4a_4+a_3^2=0, \\ O(Y^5): &\quad 2a_1a_4+2a_2a_3-12\mu^2a_3-24\mu^4a_1+816\mu^4a_3=0, \\ O(Y^4): &\quad 240\mu^4a_2-1696a_4\mu^4-6\mu^2a_2+32a_4\mu^2+a_2^2+a_4v^2-a_4+2a_0a_4+2a_1a_3=0, \\ O(Y^3): &\quad 2a_0a_3-a_3+2a_1a_2+2a_4b_1-2\mu^2a_1+18\mu^2a_3+40\mu^4a_1-576\mu^4a_3+a_3v^2=0, \\ O(Y^2): &\quad 2a_0a_2-a_2+2a_3b_1+2a_4b_2+8\mu^2a_2-12\mu^2a_4-136\mu^4a_2+480\mu^4a_4+a_2v^2+a_1^2=0, \\ O(Y): &\quad 2a_0a_1-a_1+2a_2b_1+2a_3b_2+2a_4b_3+2\mu^2a_1-6\mu^2a_3-16\mu^4a_1+120\mu^4a_3+a_1v^2=0, \end{aligned}$$

$$
\begin{aligned}
O(Y^0): &\quad 2a_1b_1 - a_0 + 2a_2b_2 + 2a_3b_3 + 2a_4b_4 - 2\mu^2 a_2 + 16\mu^4 a_2 - 24\mu^4 a_4 - 2\mu^2 b_2 + 16\mu^4 b_2 \\
&\qquad -24\mu^4 b_4 + a_0 v^2 + a_0^2 = 0, \\
O(Y^{-1}): &\quad 2a_0b_1 - b_1 + 2a_1b_2 + 2a_2b_3 + 2a_3b_4 + 2\mu^2 b_1 - 6\mu^2 b_3 - 16\mu^4 b_1 + 120\mu^4 b_3 + b_1 v^2 = 0, \\
O(Y^{-2}): &\quad 2a_0b_2 - b_2 + 2a_1b_3 + 2a_2b_4 + 8\mu^2 b_2 - 12\mu^2 b_4 - 136\mu^4 b_2 + 480\mu^4 b_4 + b_2 v^2 + b_1^2 = 0, \\
O(Y^{-3}): &\quad 2a_0b_3 - b_3 + 2a_1b_4 + 2b_1b_2 - 2\mu^2 b_1 + 18\mu^2 b_3 + 40\mu^4 b_1 - 576\mu^4 b_3 + b_3 v^2 = 0, \\
O(Y^{-4}): &\quad 240\mu^4 b_2 - 1696 b_4 \mu^4 - 6\mu^2 b_2 + 32 b_4 \mu^2 + b_2^2 + b_4 v^2 - b_4 + 2a_0b_4 + 2b_1b_3 = 0, \\
O(Y^{-5}): &\quad 2b_1b_4 + 2b_2b_3 - 12\mu^2 b_3 - 24\mu^4 b_1 + 816\mu^4 b_3 = 0, \\
O(Y^{-6}): &\quad 2b_2b_4 - 20\mu^2 b_4 - 120\mu^4 b_2 + 2080\mu^4 b_4 + b_3^2 = 0, \\
O(Y^{-7}): &\quad -360 b_3 \mu^4 + 2b_3b_4 = 0, \\
O(Y^{-8}): &\quad -840\mu^4 b_4 + b_4^2 = 0.
\end{aligned}
$$

5.1. When $a_4 = 0$ and $b_4 \neq 0$

By considering the coefficients for Y^7, Y^6 and Y^5, we can show that $a_4 = a_3 = a_2 = a_1 = 0$. Then we use the coefficient for Y^0 terms to get

$$-a_0 - 2\mu^2 b_2 + 16\mu^4 b_2 - 24\mu^4 b_4 + a_0 v^2 + a_0^2 = 0.$$

Since $b_4 \neq 0$, we have $b_4 = 840\mu^4$ using the coefficient for Y^{-8}. Similarly, we can calculate the values of b_1, b_2 and b_3, i.e.,

$$b_1 = b_3 = 0, \quad b_2 = \frac{140}{13}\mu^2 - 1120\mu^4.$$

Similar to equations (4.1)-(4.3), we can use the coefficients of Y^{-4}, Y^{-2} and Y^0 to derive the following equalities:

$$1568\mu^4 - \frac{560}{13}\mu^2 - \left(3v^2 + 6a_0 - \frac{476}{169}\right) = 0, \tag{5.1}$$

$$3968\mu^6 - \frac{1904}{13}\mu^4 + \left(-8v^2 - 16a_0 + \frac{112}{13}\right)\mu^2 + \left(-\frac{1}{13}v^2 - \frac{2}{13}a_0 + \frac{1}{13}\right) = 0, \tag{5.2}$$

$$38080\mu^8 - \frac{31360}{13}\mu^6 + \frac{280}{13}\mu^4 - a_0^2 - a_0 v^2 + a_0 = 0. \tag{5.3}$$

Equation (5.1) and (5.2) lead to

$$v^2 + 2a_0 = \frac{3968\mu^6 - \frac{1904}{13}\mu^4 + \frac{112}{13}\mu^2 + \frac{1}{13}}{8\mu^2 + \frac{1}{13}} = \frac{1568\mu^4 - \frac{560}{13}\mu^2 + \frac{476}{169}}{3}. \tag{5.4}$$

Eventually, we can obtain the equation about μ. That is,

$$640\mu^6 + \frac{2800}{13}\mu^4 - \frac{1120\mu^2}{169} - \frac{31}{2197} = 0.$$

There are two real roots and four pure imaginary roots to the equation above, but here we only consider the two real roots, i.e.,

$$\mu = \pm\frac{6263491387804093}{36028797018963968} \approx \pm 0.1738468. \tag{5.5}$$

For either two values of μ, we can substitute it into equation (5.4) to get

$$v^2 + 2a_0 = \frac{8847763345396973}{9007199254740992} \approx 0.9822991. \tag{5.6}$$

We then use the value of μ in equation (5.3) to get

$$a_0^2 + a_0 v^2 - a_0 = -\frac{4366459107829337}{288230376151711744} \approx -0.0151492. \tag{5.7}$$

(5.6) and (5.7) lead to

$$a_0^2 + \frac{159435909344019}{9007199254740992}a_0 - \frac{4366459107829337}{288230376151711744} = 0.$$

Therefore, we have

$$a_0 = \frac{\pm\sqrt{4941615711925531876692800332649} - 159435909344019}{18014398509481984} \approx -0.1322503 \text{ or } 0.1145494.$$

Next, we use the values of a_0 to calculate the value of v so that we can eventually get

$$v = \frac{\sqrt{2}\sqrt{11230173773696513}}{134217728} \approx 1.1166090 \text{ or } \frac{\sqrt{2}\sqrt{27136898943141883}}{268435456} \approx 0.8678711.$$

We then further calculate

$$b_4 = 840\mu^4 \approx 0.7672664 \text{ and } b_2 = \frac{140}{13}\mu^2 - 1120\mu^4 \approx -0.6975465.$$

Therefore, the four traveling wave solutions for this case are of the following form $u(x,t) = a_0 + b_2\coth^2(\mu(x-vt)) + b_4\coth^4(\mu(x-vt))$, where a_0, b_2, b_4, μ and v are given in the previous calculations. Since there exist two distinct values for μ and v, there are four traveling wave solutions in such a form.

5.2. When $a_4 \neq 0$ and $b_4 = 0$

Since the coefficients for Y^i and Y^{-i} terms (for $i = 1,2,\ldots,8$) are symmetric, the calculations from the previous sections can be directly applied here. Therefore, the four traveling wave solutions for this case are of the form $u(x,t) = a_0 + a_2\tanh^2(\mu(x-vt)) + a_4\tanh^4(\mu(x-vt))$. Here, the values of a_0, μ and v are the same as that in the previous section, and the values of a_2 and a_4 are equal to the values of b_2 and b_4 in the previous section, respectively.

5.3. When $a_4 \neq 0$ and $b_4 \neq 0$

Using the algebraic equations for the coefficients of Y^i and Y^{-i} for $i = 1,2,\ldots,8$, we can find the values of a_i and b_j for $i, j = 1,2,3,4$:

$$a_4 = b_4 = 840\mu^4, \quad a_1 = a_3 = b_1 = b_3 = 0, \quad a_2 = b_2 = \frac{140}{13}\mu^2 - 1120\mu^4. \tag{5.8}$$

One can also show that the value of μ is the same as in (5.5). That is, $\mu = \pm\frac{6263491387804093}{36028797018963968}$. In addition, we can show that equation (5.6) also holds. So we have $v^2 + 2a_0 = \frac{8847763345396973}{9007199254740992}$. We further use the equation about $O(Y^0)$ to get

$$-2(2\mu^2 a_2 + 16\mu^4 a_2 - 24\mu^4 a_4) + 2a_2^2 + 2a_4^2 + a_0^2 + a_0 v^2 - a_0 = 0.$$

We then substitute the values of a_4 and a_2 from (5.8) into the equation above to get

$$a_0^2 + a_0 v^2 - a_0 + 3996160\mu^8 - \frac{573440}{13}\mu^6 + \frac{31920}{160}\mu^4 = 0,$$

which can be further reduced to

$$a_0^2 + a_0 v^2 - a_0 + \frac{5154129393924667}{2251799813685248} = 0$$

using the value of μ. Eventually, we can use the equation about $v^2 + 2a_0$ to derive a quadratic equation with respect to a_0:

$$a_0^2 + \frac{159435909344019}{9007199254740992}a_0 - \frac{5154129393924667}{2251799813685248} = 0.$$

The two roots to the equation above are

$$a_0 = \frac{\pm\sqrt{742813746781938776364458008666985} - 15943590934019}{18014398509481984} \approx -1.5217852 \text{ or } 1.5040843.$$

Since $v^2 + 2a_0 \approx 0.9822991$ and we look for a real number v, here we only choose the negative number for a_0, i.e.,

$$a_0 = -\frac{\sqrt{742813746781938776364458008666985} + 15943590934019}{18014398509481984} \approx -1.5217852.$$

Finally, we can calculate the two values of v, i.e., $v \approx \pm 2.0064570$, and we have $a_2 \approx -0.6975465$ and $a_4 \approx 0.7672664$. Therefore, the two traveling wave solutions for this case are in the following form:

$$u(x,t) = a_0 + a_2\tanh^2(\mu(x-vt)) + a_4\tanh^4(\mu(x-vt)) + a_2\coth^2(\mu(x-vt)) + a_4\coth^4(\mu(x-vt)),$$

where the values of μ, v, a_0, a_2 and a_4 can be found in the discussion above.

6. Conclusion

In this paper, we apply the tanh-coth method to obtain several new traveling wave solutions for the sixth-order Boussinesq equation with $\beta = 1$ or $\beta = -1$. By balancing the nonlinear quadratic term and the sixth-order derivative term in the equation, we are able to determine the number of terms in the expansion solution. By further solving the algebraic system about the unknown parameters, we obtain new solutions for the equation. These new exact solutions can also be used to assess the performance of various numerical methods for the sixth-order Boussinesq equation.

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On Complete Group Classification of Time Fractional Systems Evolution Differential Equation with a Constant Delay

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Abstract

A fractional order system of evolution partial differential equations with a constant delay is considered. By exploiting the Lie symmetry method, we give a complete group classification of the system. Furthermore, we establish the corresponding symmetry reductions and construct some analytical solutions to the system.

1. Introduction

Fractional differential equations arise in cases where the extension of differential models to non-integer orders is imperative for more generalized analysis. The theory of fractional differential equations has been considerably developed over the years with some applications in engineering, natural sciences, economic models among others [1]-[5].

Since the extension of Lie symmetry analysis to the theory of differential equations by Ovsiannikov [6], it has remained one of the most powerful technique of studying and constructing analytical solutions to both deterministic and stochastic differential equations, moreover, it has advanced substantially leading to new generalizations and vast applications. For some of the recent work about the classical Lie symmetry theory of differential equations, its applications and extensions, we refer the reader to [7]-[18] and references therein.

Time lags occur naturally in most physical processes because they involve transfer of material or information. Although, time delay effect may improve the system performance [19], oftentimes, it is diagnosed as source of instability [20, 21]. It is therefore important that time delays are included in differential models in order to comprehensively investigate their effect on the systems' performance. Among the most recent applications of Lie symmetry theory is to the functional fractional differential equations [22]-[27].

Analytical solutions to evolution equations play a vital role in mechanics and dynamical systems, because they can naturally represent numerous physical phenomena, for instance, finite speed propagation, perturbations, heat transfer, solitons, among others. Different approaches have been introduced by mathematicians and engineers to construct exact solutions to fractional evolution equations [28]-[31].

In this article, we extend the Lie symmetry theory to the class of time fractional order system of differential equation with a constant delay, and carry out a complete group classification of time fractional system evolution delay differential equation. i.e.,

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$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = w_x g(w,\overline{w}), \quad g_{\overline{w}} \neq 0, \\ \\ \frac{\partial^\alpha w}{\partial t^\alpha} = f(u,\overline{u}) u_x, \quad f_{\overline{u}} \neq 0, \end{cases} \tag{1.1}$$

where $w(t-s,x) = \overline{w}$, $u(t-s,x) = \overline{u}$.
Let us recall that, for a differential equation involving some arbitrary function(s), the group classification problem consists of firstly, finding the Lie symmetries of differential equation with arbitrary function(s) and then determining all possible function(s) for which larger symmetry groups exist.
The motivation for the study in this paper is twofold, symmetry analysis for systems of fractional differential equations carried out in [26, 27]. Secondly, the group classification question for delay differential equations in [32, 33].
We proceed by introducing one of the definition of fractional derivative which will be used throughout this paper, that is, the Riemann-Liouville derivative defined by

$$D_t^\alpha u(t,x) = \frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \frac{\partial^n u}{\partial t^n}, & \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(\mu,x)}{(t-\mu)^{\alpha+1-n}} d\mu, & n-1 < \alpha < n, n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where Γ is a gamma function.
The rest of the article is organized as follow, a complete group classification of the time evolution fractional system of equations with a constant delay is presented in Section 2 and then followed by symmetry reductions and invariant solutions in Section 3.

2. Admitted Lie group transformation

In this section, we apply the method used in [6, 26, 32], to obtain the admitted Lie groups transformation of equation (1.1).
The vector field associated with the one-parameter group of transformation is

$$H = \xi \partial_x + \tau \partial_t + \phi \partial_u + \zeta \partial_w,$$

where the infinitismal with infinitismals ξ, τ, ϕ, ζ depend on the variables t, x, u and v.
The Lie-B*ä*cklund generator up to the fractional order corresponding to (1.1) is

$$\overline{H}^\alpha = \phi^u \partial_u + \phi^{u_x} \partial u_x + \zeta^{w_x} \partial w_x + \phi^{u_\alpha} \partial u_\alpha + \zeta^{w_\alpha} \partial w_\alpha + \overline{\phi^u} \partial_{\overline{u}} + \overline{\zeta^u} \partial_{\overline{w}} \tag{2.1}$$

where the coefficient in (2.1) are define as follows;

$$\phi^u = \phi - \xi u_x - \tau u_t, \quad \zeta^w = \zeta - \xi w_x - \tau w_t, \quad \overline{\phi^u} = \overline{\phi} - \overline{\xi} \overline{u_x} - \overline{\tau} \overline{u_t}, \quad \overline{\zeta^w} = \overline{\zeta} - \overline{\xi} \overline{w_x} - \overline{\tau} \overline{w_t} \tag{2.2}$$

$$\phi^{u_x} = D_x(\phi^u), \quad \zeta^{w_x} = D_x(\zeta^w) \quad \phi^{u_\alpha} = D_t^\alpha(\phi^u), \quad \zeta^{w_\alpha} = D_t^\alpha(\zeta^w) \tag{2.3}$$

$$\phi^{u_x} = \phi_x - \xi_x u_x - \tau_x u_t + \phi_u u_x - \xi_u u_x^2 - \tau_u u_x u_t - u_{xt} \tau - u_{xx} \xi + \phi_w w_x - w_t u_x \xi_w - w_x u_t \tau_w \tag{2.4}$$

and

$$\zeta^{w_x} = \zeta_x - \xi_x w_x - \tau_x w_t + \zeta_w w_x - \xi_w w_x^2 - \tau_w w_x w_t - w_{xt} \tau - w_{xx} \xi + \zeta_u u_x - u_t w_x \xi_u - u_x w_t \tau_u. \tag{2.5}$$

The prolongations of the fractional terms above can be expanded as follows;

$$\begin{aligned} \phi^{u_\alpha} &= \frac{\partial^\alpha \phi}{\partial t^\alpha} + \phi_w \frac{\partial^\alpha w}{\partial t^\alpha} + (\phi_u - \alpha D_t(\tau)) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \phi_u}{\partial t^\alpha} - w \frac{\partial^\alpha \phi_w}{\partial t^\alpha} - \tau D_t^{\alpha+1}(u) - \xi D_t^\alpha(u_x) \\ &- \sum_{n=1}^{+\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x) + \sum_{n=1}^{+\infty} \left[\binom{\alpha}{n} \frac{\partial^n \phi_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}(u) \\ &+ \sum_{n=1}^{+\infty} \left[\binom{\alpha}{n} \frac{\partial^n \phi_w}{\partial t^n} \right] D_t^{\alpha-n}(w) + \eta_{\alpha_1} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \zeta^{w_\alpha} &= \frac{\partial^\alpha \zeta}{\partial t^\alpha} + \zeta_u \frac{\partial^\alpha u}{\partial t^\alpha} + (\zeta_w - \alpha D_t(\tau)) \frac{\partial^\alpha w}{\partial t^\alpha} - w \frac{\partial^\alpha \zeta_w}{\partial t^\alpha} - u \frac{\partial^\alpha \zeta_u}{\partial t^\alpha} - \tau D_t^{\alpha+1}(w) - \xi D_t^\alpha(w_x) \\ &- \sum_{n=1}^{+\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(w_x) + \sum_{n=1}^{+\infty} \left[\binom{\alpha}{n} \frac{\partial^n \zeta_w}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}(w) \\ &+ \sum_{n=1}^{+\infty} \left[\binom{\alpha}{n} \frac{\partial^n \zeta_u}{\partial t^n} \right] D_t^{\alpha-n}(u) + \eta_{\alpha_2} \end{aligned} \tag{2.7}$$

with

$$\begin{aligned}\eta_{\alpha_1} &= \sum_{n=2}^{+\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\times\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\phi}{\partial t^{n-m}\partial u^k}\\ &+\sum_{n=2}^{+\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\times\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-w]^r\frac{\partial^m}{\partial t^m}(w^{k-r})\frac{\partial^{n-m+k}\phi}{\partial t^{n-m}\partial w^k},\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\eta_{\alpha_2} &= \sum_{n=2}^{+\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\times\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-w]^r\frac{\partial^m}{\partial t^m}(w^{k-r})\frac{\partial^{n-m+k}\zeta}{\partial t^{n-m}\partial w^k}\\ &+\sum_{n=2}^{+\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\times\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\zeta}{\partial t^{n-m}\partial u^k}.\end{aligned} \tag{2.9}$$

Applying the infinitesimal generator and equation(1.1), we obtain the invariance criterion of the system as;

$$\begin{cases}(\phi^{u_\alpha}-g\zeta^{w_x}-w_xg_w\zeta^{w}-w_xg_{\overline{w}}\zeta^{\overline{w}})\Big|_{(1.1)}=0\\[2ex] (\zeta^{w_\alpha}-f\phi^{u_x}-u_xf_u\phi^{u}-u_xf_{\overline{u}}\phi^{\overline{u}})\Big|_{(1.1)}=0.\end{cases} \tag{2.10}$$

Substituting (2.2)–(2.7) into the invariance criterion of the system (2.10) , and then equating the coefficients of various derivatives of $u,\overline{u}$ and $w,\overline{w}$ to zero, we have the simplified system of determining equations as follows;

$$\tau_u=\xi_w=\tau_w=\xi_t=\tau_x=\xi_u=0, \tag{2.11}$$

$$\zeta_{uu}=\zeta_{ww}=\zeta_{ut}=\zeta_{wt}=0, \tag{2.12}$$

$$\phi_{uu}=\phi_{ww}=\phi_{ut}=\phi_{wt}=0, \tag{2.13}$$

$$\overline{\tau}=\tau,\quad \overline{\xi}=\xi, \tag{2.14}$$

$$g\zeta_u-f\phi_w=0, \tag{2.15}$$

$$f\zeta_w+f\xi_x-f_{\overline{u}}\overline{\phi}-f_u\phi-f\phi_u=0, \tag{2.16}$$

$$g\phi_u+g\xi_x-g\zeta_w-g_w\zeta-g_{\overline{w}}\overline{\zeta}=0, \tag{2.17}$$

$$\frac{\partial^\alpha\zeta}{\partial t^\alpha}-f\phi_x=0, \tag{2.18}$$

$$\frac{\partial^\alpha\phi}{\partial t^\alpha}-g\zeta_x=0. \tag{2.19}$$

The lower limit in the definition of fractional derivative (1.2) is fixed, the presence of 0 requires that the manifold $t=0$ is invariant i.e.,

$$\tau(t,x,u,\overline{u})\Big|_{t=0}=0. \tag{2.20}$$

Differentiating (2.18) and (2.19) with respect to w and u respectively we have

$$\phi_{xw}=0,\quad \textit{and}\quad \zeta_{xw}=0. \tag{2.21}$$

Solving the system of equation (2.11)–(2.14) using (2.21) and (2.20), we obtain the following infinitesimals

$$\tau=0,\quad \xi=\psi_1(x)\quad \phi=uwc_2+u\psi_2(x)+c_3w+\psi_3(t,x) \tag{2.22}$$

and

$$\zeta = c_4uw + c_5u + w\psi_4(x) + \psi_5(t,x). \tag{2.23}$$

Differentiating (2.18) with respect to u and $u\overline{u}$, we get the following system

$$\begin{cases} f_u\phi_x + f\phi_{ux} = 0 \\ f_{u\overline{u}}\phi_x + f_{\overline{u}}\phi_{ux} = 0. \end{cases} \tag{2.24}$$

Similarly, differentiating (2.19) with respect to w and $w\overline{w}$, we obtain the following system

$$\begin{cases} g_u\zeta_x + g\zeta_{ux} = 0 \\ g_{w\overline{w}}\zeta_x + g_{\overline{w}}\zeta_{ux} = 0. \end{cases} \tag{2.25}$$

Using system (2.24) and (2.25) to eliminate ϕ_{ux} and ζ_{ux} respectively we have

$$\begin{cases} (f_uf_{\overline{u}} - ff_{u\overline{u}})\phi_x = 0 \\ (g_wg_{\overline{w}} - gg_{w\overline{w}})\zeta_x = 0. \end{cases} \tag{2.26}$$

From the system (2.26) and equation (2.18), (2.19) it implies that, ϕ and ζ has no dependency in x i.e.,

$$\phi_x = \zeta_x = 0 \tag{2.27}$$

similarly, using (2.15) and (2.22)

$$\zeta_u = \phi_w = 0. \tag{2.28}$$

This reduces the infinitesimals (2.22) and (2.23) to

$$\tau = 0, \quad \xi = \psi_1(x) \quad \phi = c_6u + \psi_6(t), \quad \zeta = c_7w + \psi_7(t). \tag{2.29}$$

Using (2.27), (2.18), (2.29) and (2.19) we have

$$\frac{\partial^\alpha \psi_7(t)}{\partial t^\alpha} = 0, \quad \frac{\partial^\alpha \psi_6(t)}{\partial t^\alpha} = 0. \tag{2.30}$$

Finally, equations (2.16) and (2.17) are classification equations and are assumed to be satisfied without any restriction on $f(u,\overline{u})$ and $g(w,\overline{w})$. This implies that, to get a minimal symmetry algebra for any choice of functions $f(u,\overline{u}), g(w,\overline{w})$, we have to assume

$$\xi_x = \phi = \zeta = 0.$$

Therefore, for any arbitrary functions $f(u,\overline{u}), g(w,\overline{w})$, the system (1.1) admits a one dimensional symmetry algebra spanned by the infinitesimal generators

$$H_1 = \partial_x.$$

2.1. Classification

To search for a functions $f(u,\overline{u}), g(w,\overline{w})$, that may admit a larger symmetry algebra we have to consider the case when,

$$\xi_x = \phi = \zeta \neq 0. \tag{2.31}$$

Substituting the infinitesimals in (2.29) into equation (2.16) and (2.17), we respectively get

$$c_7f + f\psi_{1_x} - c_6\overline{u}f_{\overline{u}} - f_{\overline{u}}\overline{\psi_6} - c_6f - c_6uf_u - \psi_6f_u = 0 \tag{2.32}$$

and

$$c_6g + g\psi_{1_x} - c_7g - c_7wg_w - g_w\psi_7 - c_7\overline{w}g_{\overline{w}} - g_{\overline{w}}\overline{\psi_7} = 0. \tag{2.33}$$

Differentiating (2.32) with respect to u and $\overline{u}$ we get

$$\begin{cases} -c_6(\overline{u}f_{\overline{u}u} + 2f_u + uf_{uu}) + c_7f_u + f_u\psi_{1_x} = \psi_6f_{uu} + f_{\overline{u}u}\overline{\psi_6} \\ -c_6(\overline{u}f_{\overline{u}\overline{u}} + 2f_{\overline{u}} + uf_{\overline{u}u}) + c_7f_{\overline{u}} + f_{\overline{u}}\psi_{1_x} = \psi_6f_{\overline{u}u} + f_{\overline{u}\overline{u}}\overline{\psi_6}. \end{cases} \tag{2.34}$$

The system (2.34) is algebraic with respect to $\overline{\psi_6}$ and ψ_6, with the determinant of the matrix as;

$$\Delta_1 = f_{u\overline{u}}^2 - f_{uu} f_{\overline{u}\overline{u}}.$$

Similarly, differentiating (2.33) with respect to w and $\overline{w}$, we obtain the system

$$\begin{cases} -c_7(\overline{w} g_{\overline{w}w} + 2g_w + w g_{ww}) + c_6 g_w + g_w \psi_{1_x} = \psi_7 g_{ww} + g_{\overline{w}w} \overline{\psi_7} \\ -c_7(\overline{w} g_{\overline{w}\overline{w}} + 2g_{\overline{w}} + w g_{\overline{w}w}) + c_6 g_{\overline{w}} + g_{\overline{w}} \psi_{1_x} = \psi_7 g_{\overline{w}w} + g_{\overline{w}\overline{w}} \overline{\psi_7}. \end{cases} \tag{2.35}$$

The system (2.35) is algebraic with respect to $\overline{\psi_7}$ and ψ_7, with the determinant of the matrix as

$$\Delta_2 = g_{w\overline{w}}^2 - g_{ww} g_{\overline{w}\overline{w}}.$$

In the subsequent subsections, we discus the cases when the determinants of both matrices are equal to zero and otherwise;

2.2. $\Delta_1 \neq 0$, $\Delta_2 \neq 0$.

Solving the system (2.34) and (2.35) for $\psi_6, \overline{\psi_6}$ and $\psi_7, \overline{\psi_7}$ respectively to get

$$\begin{cases} \psi_6 = \dfrac{c_6(u f_{u\overline{u}}^2 + 2 f_{\overline{u}} f_{u\overline{u}} - 2 f_u f_{\overline{u}\overline{u}} - u f_{uu} f_{\overline{u}\overline{u}}) + c_7(f_u f_{\overline{u}\overline{u}} - f_{\overline{u}} f_{\overline{u}u}) + (f_u f_{\overline{u}\overline{u}} - f_{\overline{u}} f_{u\overline{u}}) \psi_{1_x}}{\Delta_1}, \\[2ex] \overline{\psi_6} = \dfrac{(\overline{u} f_{u\overline{u}}^2 + 2 f_u f_{u\overline{u}} - 2 f_{uu} f_{\overline{u}} - \overline{u} f_{uu} f_{\overline{u}\overline{u}}) c_6 + (f_{uu} f_{\overline{u}} - f_u f_{u\overline{u}}) c_7 + (f_{uu} f_{\overline{u}} - f_u f_{u\overline{u}}) \psi_{1_x}}{\Delta_1} \end{cases} \tag{2.36}$$

and

$$\begin{cases} \psi_7 = \dfrac{c_6(w g_{w\overline{w}}^2 + 2 g_{\overline{w}} g_{w\overline{w}} - 2 g_w g_{\overline{w}\overline{w}} - w g_{ww} g_{\overline{w}\overline{w}}) + c_7(g_w g_{\overline{w}\overline{w}} - g_{\overline{w}} g_{\overline{w}w}) + (g_w g_{\overline{w}\overline{w}} - g_{\overline{w}} g_{w\overline{w}}) \psi_{1_x}}{\Delta_1}, \\[2ex] \overline{\psi_7} = \dfrac{(\overline{w} g_{w\overline{w}}^2 + 2 g_w g_{w\overline{w}} - 2 g_{ww} g_{\overline{w}} - \overline{w} g_{ww} g_{\overline{w}\overline{w}}) c_6 + (g_{ww} g_{\overline{w}} - g_w g_{w\overline{w}}) c_7 + (g_{ww} g_{\overline{w}} - g_w g_{w\overline{w}}) \psi_{1_x}}{\Delta_1}. \end{cases} \tag{2.37}$$

Since $\psi_6, \overline{\psi_6}, \psi_7, \overline{\psi_7}$ and ψ_1 are independent of $u, \overline{u}, w, \overline{w}$, we can consider the case when $\psi_6, \overline{\psi_6}, \psi_7, \overline{\psi_7}$ are all constant and ψ_1 is linear in x. However, by the virtue of equation (2.30), i.e., the fractional derivative of a non-zero constant is not zero, consequently it follows that;

$$\psi_6(t) = \psi_7(t) = constant = 0, \quad and \quad \psi_1 = c_9 x + c_8. \tag{2.38}$$

Therefore, equation (2.32) and (2.33) using (2.38) become;

$$c_7 f + f c_9 - c_6 \overline{u} f_{\overline{u}} - c_6 f - c_6 u f_u = 0 \tag{2.39}$$

and

$$c_6 g + g c_9 - c_7 g - c_7 w g_w - c_7 \overline{w} g_{\overline{w}} = 0. \tag{2.40}$$

Any function $f(u, \overline{u})$ and $g(w, \overline{w})$ satisfying (2.39), (2.40) will also satisfy the system (2.36) and (2.37) respectively. The general solutions of (2.39) and (2.40) are;

$$f(u, \overline{u}) = h_1\left(\frac{\overline{u}}{u}\right) u^{\frac{c_7 + c_9 - c_6}{c_6}}, \quad g(w, \overline{w}) = h_2\left(\frac{\overline{w}}{w}\right) w^{\frac{c_6 + c_9 - c_7}{c_7}}.$$

Here, h_1, h_2 are an arbitrary functions.
Thus, the Lie algebra extended by three dimension and is spanned by the following generators

$$H_1, \quad H_2 = x\partial_x, \quad H_3 = u\partial_u, \quad H_4 = w\partial_w.$$

2.3. $\Delta_1 = 0$, $\Delta_2 = 0$.

In this subsection, we investigate all the possible functions that are solutions to the determinants of the matrices of the systems (2.34) and (2.35) i.e.

$$\Delta_1 = f_{u\overline{u}}^2 - f_{uu} f_{\overline{u}\overline{u}} = 0, \quad \Delta_2 = g_{w\overline{w}}^2 - g_{ww} g_{\overline{w}\overline{w}} = 0 \tag{2.41}$$

as well as deducing the extra symmetry algebras. The following cases are considered.

2.3.1. $f_{\overline{u}\overline{u}} \neq 0, \quad g_{\overline{w}\overline{w}} \neq 0.$

The general solutions of equation (2.41) are;

$$f_u = \delta_1(f_{\overline{u}}), \quad g_w = \delta_1(g_{\overline{w}}). \tag{2.42}$$

where $\delta_1(u)$ and $\delta_2(w)$ are an arbitrary functions of integrations.
Substituting equation (2.42) into the systems of equations (2.34) and (2.35)

$$\begin{cases} (2c_6 - c_7 - \psi_{1_x})(\delta_1' f_{\overline{u}} - f_u) = 0 \\ (2c_7 - c_6 - \psi_{1_x})(\delta_2' g_{\overline{w}} - g_w) = 0 \end{cases} \tag{2.43}$$

This leads to two cases;
Case I: $\delta_1' f_{\overline{u}} = f_u, \ \delta_2' g_{\overline{w}} = g_w$ This implies

$$\delta_1' f_{\overline{u}} = f_u, \quad \delta_2' g_{\overline{w}} = g_w. \tag{2.44}$$

Equation (2.44) has a general solutions

$$f(u,\overline{u}) = f_1(c_{16}u + \overline{u}), \quad g(w,\overline{w}) = g_1(c_{17}w + \overline{w}). \tag{2.45}$$

Substituting equation (2.45) into (2.32) and (2.33), we have

$$\frac{c_7 - c_6 + \psi_{1x}}{c_6\overline{u} + c_6c_{16}u + c_{16}\psi_6 + \overline{\psi}_6} = \frac{f_1{}'}{f_1} \tag{2.46}$$

and

$$\frac{c_6 - c_7 + \psi_{1x}}{c_7\overline{w} + c_7c_{17}w + c_{17}\psi_7 + \overline{\psi}_7} = \frac{g_1{}'}{g_1}. \tag{2.47}$$

Now considering the fact that f_1, g_1 are not functions of x, t, it is clear that, ψ_1, ψ_6, ψ_7 must be constants and from equation (2.30), it follows that ψ_6, ψ_7 are not constant except zero, since fractional derivatives of non-zero constant are not zero. Therefore, equation (2.46) and (2.47) reduces to

$$\frac{f_1'}{f_1} = \frac{c_7 - c_6 + c_{18}}{c_6(c_{16}u + \overline{u})} \tag{2.48}$$

and

$$\frac{g_1'}{g_1} = \frac{c_6 - c_7 + c_{18}}{c_7(c_{17}w + \overline{w})} \tag{2.49}$$

with

$$\psi_1 = c_{18}x + c_{19}. \tag{2.50}$$

Solving equation (2.48) and (2.49) we obtain another set of functions

$$f(u,\overline{u}) = (c_{16}u + \overline{u})^{\frac{c_7 - c_6 + c_{18}}{c_6}}, \quad g(w,\overline{w}) = (c_{17}w + \overline{w})^{\frac{c_6 - c_7 + c_{18}}{c_7}}, \tag{2.51}$$

that give an extra algebra. The Lie algebra is extended by three dimension and is spanned by the following generators;

$$H_1, \quad H_2, \quad H_3, \quad H_4.$$

Case II: $\delta_1' f_{\overline{u}} \neq f_u, \ \delta_2' g_{\overline{w}} \neq g_w$ Under this case, it is clear from equation (2.43) that;

$$\psi_1 = c_6x + c_{16}, \qquad c_6 = c_7. \tag{2.52}$$

Substituting (2.52) into the systems (2.34) and (2.35), we have

$$\begin{cases} (c_6u + \psi_6)\delta_1' = -(\overline{\psi_6} + c_6\overline{u}) \\ (c_7w + \psi_7)\delta_2' = -(\overline{\psi_7} + c_7\overline{w}). \end{cases} \tag{2.53}$$

If $\delta_1' = \delta_2' = 0$, no extra symmetry algebra is possible, so we proceed by considering the case when $\delta_1' \neq 0$ and $\delta_2' \neq 0$.

Differentiating the system (2.53) with respect to $\overline{u}, \overline{w}$ we get

$$\begin{cases} (c_6 u+\psi_6)\delta_1'' f_{\overline{u}\overline{u}} = -c_6 \\ (c_7 w+\psi_7)\delta_2'' g_{\overline{w}\overline{w}} = -c_7 \end{cases} \tag{2.54}$$

and differentiating (2.54) with respect to temporal variable we obtain

$$\begin{cases} \psi_{6_t}\delta_1'' = 0 \\ \psi_{7_t}\delta_2'' = 0. \end{cases}$$

1. If $\delta_1'' \neq 0, \delta_2'' \neq 0$
   It implies ψ_6, ψ_7 are constant, and by using equation (2.30) we obtain specifically;
   $$\psi_6 = \psi_7 = 0.$$
   From (2.53), we can assumes the following functions
   $$\begin{cases} f(u,\overline{u}) = uF\left(\frac{\overline{u}}{u}\right) + c_{21}, \quad F_{u\overline{u}} \neq 0 \\ g(w,\overline{w}) = wG\left(\frac{\overline{w}}{w}\right) + c_{22}, \quad G_{u\overline{u}} \neq 0. \end{cases} \tag{2.55}$$
   Substituting equation (2.55) into (2.32) and (2.33) using (2.52) we note that an extra symmetry algebra can obtained if $c_{21} = c_{22} = 0$. Thus, the symmetry algebra is spanned by the generators;
   $$H_1, \quad H_5 = x\partial_x + u\partial_u + w\partial_w.$$
2. If $\delta_1'' = 0, \delta_2'' = 0$ then,
   $$f_u = c_{23} f_{\overline{u}} + c_{24}, \quad g_w = c_{25} g_{\overline{w}} + c_{26}.$$
   This can be solved to obtain
   $$f(u,\overline{u}) = c_{24}u + F(\overline{u} + c_{23}u), \quad g(w,\overline{w}) = c_{26}w + G(\overline{w} + c_{25}w),$$
   with $F_{u\overline{u}} \neq 0$ and $F_{w\overline{w}} \neq 0$.
   From the system (2.54) and (2.53), we have $c_6 = 0$ and
   $$c_{23}\psi_6 = -\overline{\psi_6}, \quad c_{25}\psi_7 = -\overline{\psi_7}. \tag{2.56}$$
   However, ψ_6 and ψ_7 have to satisfy equation (2.30), which in turn pushes them to zero, so in this case there is no any extra symmetry algebra possible.

2.3.2. $f_{\overline{u}\overline{u}} = 0, \quad g_{\overline{w}\overline{w}} = 0.$

Since $f_{\overline{u}} \neq 0, \quad g_{\overline{w}} \neq 0$, equations (2.41) have the following solutions;

$$f(u,\overline{u}) = c_9\overline{u} + h_1(u), \quad g(w,\overline{w}) = c_{10}\overline{w} + h_2(w) \tag{2.57}$$

where $h_1(u)$, $h_2(w)$ are an arbitrary functions of integrations.
Substituting equation (2.57) into the systems of equations (2.34) and (2.35) gives;

$$\begin{cases} (c_6 u+\psi_6)h_1'' = 0 \\ (c_7 u+\psi_7)h_2'' = 0 \end{cases}$$

as well as;

$$\psi_1 = c_6 x + c_{11}, \qquad c_6 = c_7. \tag{2.58}$$

For $h_i'' \neq 0$, we have no extra symmetry i.e., we get the minimal algebra, therefore we consider the case $h_i'' = 0$, from which equation (2.57) becomes;

$$f(u,\overline{u}) = c_9\overline{u} + c_{12}u + c_{13}, \quad g(w,\overline{w}) = c_{10}\overline{w} + c_{14}w + c_{15}. \tag{2.59}$$

Substituting equation (2.58) and (2.59) into (2.31) and (2.32), it follows that extra symmetry algebra are possible if

$$\begin{cases} c_{13} = c_{15} = 0 \\ c_9\overline{\psi_6(t)} = -c_{12}\psi_6(t), \quad c_{10}\overline{\psi_7(t)} = -c_{13}\psi_7(t), \end{cases}$$

but ψ_6 and ψ_7 have to satisfy equation (2.30), which implies that $\psi_6 = \psi_7 = 0$. Therefore, the equation admits one additional symmetry algebra which is the linear combination of H_2, H_3, H_4 and it is spanned by;

$$H_1, \quad H_5 = x\partial_x + u\partial_u + w\partial_w.$$

2.4. Summary of the classification

In the previous sections, we have carried out a complete group classification of the fractional evolution systems of partial differential equations with a constant delay i.e.,

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = w_x g(w,\overline{w}), \quad g_{\overline{w}} \neq 0 \\ \\ \frac{\partial^\alpha w}{\partial t^\alpha} = f(u,\overline{u})u_x, \quad f_{\overline{u}} \neq 0, \end{cases}$$

where $w(t-s,x) = \overline{w}$, $u(t-s,x) = \overline{u}$. We have proved that for any arbitrary functions f,g, the system admits one dimensional symmetry Lie algebra, which is a shift in the temporal and spatial variables i.e.,

$$H_1 = \partial_x.$$

We have also shown that, the Lie symmetry algebra can be extended up to dimension four in the following cases;

1. For functions;

$$f(u,\overline{u}) = h_1\left(\frac{\overline{u}}{u}\right)u^{\frac{c_7+c_9-c_6}{c_6}}, \quad g(w,\overline{w}) = h_2\left(\frac{\overline{w}}{w}\right)w^{\frac{c_6+c_9-c_7}{c_7}},$$

and

$$f(u,\overline{u}) = (c_{16}u+\overline{u})^{\frac{c_7-c_6+c_{18}}{c_6}}, \quad g(w,\overline{w}) = (c_{17}w+\overline{w})^{\frac{c_6-c_7+c_{18}}{c_7}}$$

The Lie algebra is extended by three dimension and is spanned by the following infinitesimal generators.

$$H_1, \quad H_2 = x\partial_x, \quad H_3 = u\partial_u, \quad H_4 = w\partial_w.$$

2. For functions

$$\begin{cases} f(u,\overline{u}) = uF\left(\frac{\overline{u}}{u}\right), \quad F_{u\overline{u}} \neq 0 \\ g(w,\overline{w}) = wG\left(\frac{\overline{w}}{w}\right), \quad G_{u\overline{u}} \neq 0. \end{cases}$$

and

$$f(u,\overline{u}) = c_9\overline{u}+c_{12}u, \quad g(w,\overline{w}) = c_{10}\overline{w}+c_{14}w.$$

The Lie symmetry algebra was extended by one dimension and it is spanned by;

$$H_1, \quad H_5 = x\partial_x + u\partial_u + w\partial_w.$$

It is easy to check that all the generators obtained form a Lie algebra and this takes us to the next section, were we implement one of the applications of group classification to determine symmetry reductions and invariant solutions of the system.

3. Applications

3.1. $\Delta_1 \neq 0, \Delta_2 \neq 0$

In the case above, the system (1.1) becomes

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = w_x h_2\left(\frac{\overline{w}}{w}\right)w^{k_1}, \\ \frac{\partial^\alpha w}{\partial t^\alpha} = u_x h_1\left(\frac{\overline{u}}{u}\right)u^{k_1}, \end{cases} \tag{3.1}$$

We carry out the symmetry reduction using the sub-algebras X_1 and X_2 as presented below.

3.1.1. Sub-algebra $X_1 = H_2 + H_3 + H_4$,

$$\frac{dx}{x} = \frac{du}{u} = \frac{dw}{w}.$$

The similarity transformation obtained are

$$z = t, \quad u(t,x) = V_1(z)x, \quad w(t,x) = V_2(z)x.$$

which reduces the system (3.1) to

$$\begin{cases} x\frac{\partial^\alpha V_1}{\partial z^\alpha} = h_2\left(\frac{\overline{V_2}}{V_2}\right)V_2^{(1+k_1)}, \\ x\frac{\partial^\alpha V_2}{\partial z^\alpha} = h_1\left(\frac{\overline{V_1}}{V_1}\right)V_1^{(1+k_1)}. \end{cases}$$

3.2. $\Delta_1 = 0, \Delta_2 = 0.$

In this section, we discuss case by case resulting from the different functions obtained in Section 2.3 above. Symmetry reductions are obtained and some invariant solutions of the system are constructed.

3.2.1. $f_{\overline{u}\overline{u}} \neq 0, \quad g_{\overline{w}\overline{w}} \neq 0$ **and** $\delta_1^{'} f_{\overline{u}} = f_u,\ \delta_2^{'} g_{\overline{w}} = g_w.$

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = w_x (c_{17} w + \overline{w})^{k_1} \\ \frac{\partial^\alpha w}{\partial t^\alpha} = u_x (c_{16} u + \overline{u})^{k_1}, \end{cases}$$

Sub-algebra $X_1 = H_2 + H_3 + H_4,$

$$\frac{dx}{x} = \frac{du}{u} = \frac{dw}{w}.$$

The similarity variables obtained are

$$z = t, \quad u(t,x) = V_1(z)x, \quad w(t,x) = V_2(z)x,$$

which are used to transform the system as below;

$$\begin{cases} \frac{\partial^\alpha V_1}{\partial z^\alpha} = V_2 \left(c_{17} V_2 + \overline{V_2}\right)^{k_1} x^{(k_1 - 1)}, \\ \frac{\partial^\alpha V_2}{\partial z^\alpha} = V_1 \left(c_{16} V_1 + \overline{V_1}\right)^{k_1} x^{(k_1 - 1)}. \end{cases}$$

3.2.2. $f_{\overline{u}\overline{u}} \neq 0, \quad g_{\overline{w}\overline{w}} \neq 0$ **and** $\delta_1^{'} f_{\overline{u}} \neq f_u,\ \delta_2^{'} g_{\overline{w}} \neq g_w.$

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = w_x w G\left(\frac{\overline{w}}{w}\right), \quad G_{u\overline{u}} \neq 0, \\ \frac{\partial^\alpha w}{\partial t^\alpha} = u_x u F\left(\frac{\overline{u}}{u}\right), \quad F_{u\overline{u}} \neq 0, \end{cases}$$

Sub-algebra $X_2 = H_6$

$$\frac{dx}{x} = \frac{du}{u} = \frac{dw}{w}.$$

The similarity transformations obtained are

$$z = t, \quad u(t,x) = V_1(z)x, \quad w(t,x) = V_2(z)x,$$

which are utilized to reduce the system to single variable fractional delay differential equations;

$$\begin{cases} \frac{\partial^\alpha V_1}{\partial t^\alpha} = V_2^2 G\left(\frac{\overline{V_2}}{V_2}\right), \\ \frac{\partial^\alpha V_2}{\partial t^\alpha} = V_1^1 F\left(\frac{\overline{V_1}}{V_1}\right). \end{cases} \tag{3.2}$$

3.2.3. $f_{\overline{u}\overline{u}} = 0, \quad g_{\overline{w}\overline{w}} = 0$

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = (c_{10}\overline{w} + c_{14} w) w_x, \\ \frac{\partial^\alpha w}{\partial t^\alpha} = (c_9 \overline{u} + c_{12} u) u_x, \end{cases} \tag{3.3}$$

Sub-algebra $X_2 = H_6$

$$\frac{dx}{x} = \frac{du}{u} = \frac{dw}{w}.$$

The similarity variables obtained are

$$z = t, \quad u(t,x) = V_1(z)x, \quad w(t,x) = V_2(z)x.$$

which leads to

$$\begin{cases} \frac{\partial^\alpha V_1}{\partial z^\alpha} = (c_{10} + c_{14}) V_2^2, \\ \frac{\partial^\alpha V_2}{\partial z^\alpha} = (c_9 + c_{12}) V_1^2. \end{cases} \tag{3.4}$$

The system (3.4) has a solution of the form $V_1(z) = k_2 z^{\lambda_1}$, and $V_2(z) = k_3 z^{\lambda_2}$. Substituting this back to the system, we have

$$\begin{cases} k_2 \frac{\Gamma(\lambda_1+1)}{\Gamma(\lambda_1+1-\alpha)} z^{\lambda_1-\alpha} = (c_{10}+c_{14}) k_3^2 z^{2\lambda_2}, \\ k_3 \frac{\Gamma(\lambda_2+1)}{\Gamma(\lambda_2+1-\alpha)} z^{\lambda_2-\alpha} = (c_9+c_{12}) k_2^2 z^{2\lambda_1}. \end{cases}$$

To obtain the values of the constants k_2 and k_3, we assume the powers of z in the system to be the same leading to, $\lambda_1 = \lambda_2 = -\alpha$. As a result, we have

$$\begin{cases} k_2 = \left(\frac{a_2^2}{a_1^5}\right)^{\frac{1}{3}} \frac{\Gamma(-\alpha+1)}{\Gamma(1-2\alpha)}, \\ k_3 = \left(\frac{a_2}{a_1^4}\right)^{\frac{1}{3}} \frac{\Gamma(-\alpha+1)}{\Gamma(1-2\alpha)}. \end{cases}$$

Therefore, the exact solution of the system (3.3) is

$$\begin{cases} u(t,x) = \left(\frac{a_2^2}{a_1^5}\right)^{\frac{1}{3}} \frac{\Gamma(-\alpha+1)}{\Gamma(1-2\alpha)} x t^{-\alpha}, \\ w(t,x) = \left(\frac{a_2}{a_1^4}\right)^{\frac{1}{3}} \frac{\Gamma(-\alpha+1)}{\Gamma(1-2\alpha)} x t^{-\alpha}, \end{cases} \tag{3.5}$$

where $a_1 = (c_{10}+c_{14})$ and $a_2 = (c_9+c_{12})$.

Figures 3.1 and 3.2 illustrate the solution of the system (3.5).

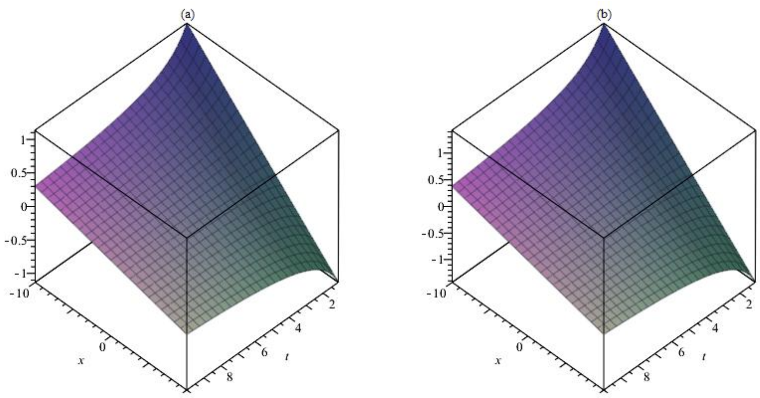

Figure 3.1: u, w when $a_1 = 2,\ a_2 = 1,\ \alpha = 0.6$ with $x = -10 \ldots 10,\ t = 1.1 \ldots 10$.

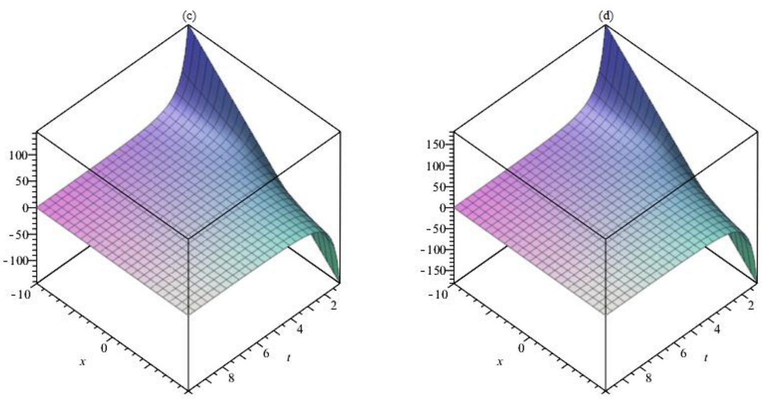

Figure 3.2: u, w when $a_1 = 2,\ a_2 = 1,\ \alpha = 1.6$ with $x = -10 \ldots 10,\ t = 1.1 \ldots 10$.

4. Conclusion

Lie symmetry analysis of fractional order evolution system equations with a constant delay was investigated. A one dimensional minimal symmetry algebra corresponding to an arbitrary function was obtained

$$H_1 = \partial_x.$$

Functions that lead to larger symmetry algebra were also found as well as their extended symmetry algebras. Further more, invariant solutions were obtained in addition to one new exact solution.

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The Form of Solutions and Periodic Nature for Some System of Difference Equations

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Abstract

In this paper, we study the form of the solution of the following systems of difference equations of order two

$$w_{n+1}=\frac{w_n s_{n-1}}{w_n+s_{n-1}}, \quad s_{n+1}=\frac{s_n w_{n-1}}{\pm s_n \pm w_{n-1}},$$

with nonzero real numbers initial conditions.

1. Introduction

Difference equations naturally occur as discrete analogs and numerical solutions to differential and delay differential equations that have applications in biology, ecology, economy, physics, and other fields. Thus, there has recently been an increase in interest in the study of qualitative analysis of systems of difference equations and rational difference equations. Although the form of difference equations is quite straightforward, it is extremely challenging to fully comprehend the behaviors of their solutions, see [1]-[7].

The periodicity of the solutions of the system of difference equations

$$w_{n+1}=\frac{m}{s_n}, \quad s_{n+1}=\frac{p s_n}{w_{n-1}s_{n-1}},$$

was studied by Cinar in [8].
El-Dessoky and Elsayed [9] have analyzed the form of the solutions and the periodicity character of the following systems :

$$w_{n+1}=\frac{w_n s_{n-1}}{s_{n-1}\pm s_n}, \quad s_{n+1}=\frac{s_n w_{n-1}}{w_{n-1}\pm w_n}.$$

Kurbanli et al. [10] discussed the periodicity of solutions of the system of difference equations

$$w_{n+1}=\frac{w_{n-1}+s_n}{w_{n-1}s_n-1}, \quad s_{n+1}=\frac{s_{n-1}+w_n}{s_{n-1}w_n-1}.$$

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El-Dessoky [11] investigated the form of the solutions and the periodicity character of the following systems :

$$w_{n+1}=\frac{s_{n-1}s_{n-2}}{w_n(\pm 1\pm s_{n-1}s_{n-2})},\quad s_{n+1}=\frac{w_{n-1}w_{n-2}}{s_n(\pm 1\pm w_{n-1}w_{n-2})}.$$

Touafek and Elsayed [12] have investigated the periodicity and determined the shape of the solutions of the systems of difference equations of order two:

$$w_{n+1}=\frac{s_n w_{n-1}}{\pm w_{n-1}\pm s_n},\quad s_{n+1}=\frac{w_n s_{n-1}}{\pm w_n\pm w_{n-1}}.$$

Yalcınkaya [13] has found the sufficient conditions for the global asymptotic stability of the system of difference equations

$$w_{n+1}=\frac{w_n+s_{n-1}}{w_n s_{n-1}-1},\quad s_{n+1}=\frac{s_n+w_{n-1}}{s_n w_{n-1}-1}.$$

Elsayed et al. [14] foud the form of the solutions of the systems of difference equations

$$w_{n+1}=\frac{s_n(w_{n-3}+s_{n-4})}{s_{n-4}+w_{n-3}-s_n},\quad s_{n+1}=\frac{w_{n-2}(w_{n-2}+s_{n-3})}{2w_{n-2}+s_{n-3}}.$$

$$w_{n+1}=\frac{(s_{n-4}-w_{n-3})s_n}{s_{n-4}-w_{n-3}+s_n},\quad s_{n+1}=\frac{(s_{n-3}-w_{n-2})w_{n-2}}{s_{n-3}}.$$

Yang et al. [15] has studied the behavior of the solutions of the systems

$$w_n=\frac{a}{s_{n-p}},\quad s_n=\frac{bs_{n-p}}{w_{n-q}s_{n-q}}.$$

Touafek et al. [16] examined periodicity and provided the form of the solutions of the systems of nonlinear difference equations

$$w_{n+1}=\frac{w_{n-3}}{\pm 1\pm w_{n-3}s_{n-1}},\quad s_{n+1}=\frac{s_{n-3}}{\pm 1\pm s_{n-3}w_{n-1}}.$$

Turki et al. [17] studied the dynamics of the twelfth-order difference equations

$$w_{n+1}=aw_{n-5}-\frac{bw_{n-5}}{cw_{n-5}-dw_{n-11}}.$$

Similarly, difference equations and nonlinear systems of the rational difference equations were investigated see [18]-[25]. This paper's main goal is to consider the systems of difference equations below

$$w_{n+1}=\frac{w_n s_{n-1}}{w_n+s_{n-1}},\quad s_{n+1}=\frac{s_n w_{n-1}}{\pm s_n\pm w_{n-1}},$$

where the initial conditions are w_0, w_{-1}, s_0 and s_{-1} arbitrary positive real numbers.

2. The system $w_{n+1}=\frac{w_n s_{n-1}}{w_n+s_{n-1}}$, $s_{n+1}=\frac{s_n w_{n-1}}{s_n+w_{n-1}}$

In this section, we examine the solutions of the system of the difference equations in the form :

$$w_{n+1}=\frac{w_n s_{n-1}}{w_n+s_{n-1}},\quad s_{n+1}=\frac{s_n w_{n-1}}{s_n+w_{n-1}}. \tag{2.1}$$

Theorem 2.1. *Assume that $\{w_n,s_n\}$ is a solution of system (2.1). Then for $n=1,2,\ldots$, we have*

$$w_{2n}=\frac{abcd}{cd((\xi_n-\eta_{n-1})a+\eta_n b)+ab((\eta_n-\xi_{n-1})c+\xi_n d)},$$

$$w_{2n+1}=\frac{abcd}{cd(\xi_n a+(\eta_{n+1}-\xi_n)b)+ab(\eta_n c+(\xi_{n+1}-\eta_n)d)},$$

$$s_{2n}=\frac{abcd}{cd((\eta_n-\xi_{n-1})a+\xi_n b)+ab((\xi_n-\eta_{n-1})c+\eta_n d)},$$

$$s_{2n+1}=\frac{abcd}{cd(\eta_n a+(\xi_{n+1}-\eta_n)b)+ab(\xi_n c+(\eta_{n+1}-\xi_n)d)},$$

where $\{\eta_n\}_{n=1}^{\infty}=\{1,2,7,17,44,...\}$, $\eta_0=1$, $\eta_{-1}=0,\eta_{-2}=-1$, $\{\xi_n\}_{n=1}^{\infty}=\{1,3,6,17,...\}$, $\xi_0=0$, $\xi_n=\eta_{n+1}-\eta_n-\eta_{n-1}-\eta_{n-2}-\eta_{n-3}$, $w_0=a$, $w_{-1}=b$, $s_0=c$ *and* $s_{-1}=d$.

Proof. For n=0, the result holds. Now, suppose that $n>0$ and that our assumption holds for $n-1$. That is

$$w_{2n-2}=\frac{abcd}{cd((\xi_{n-1}-\eta_{n-2})a+\eta_{n-1}b)+ab((\eta_{n-1}-\xi_{n-2})c+\xi_{n-1}d)},$$

$$w_{2n-1}=\frac{abcd}{cd(\xi_{n-1}a+(\eta_n-\xi_{n-1})b)+ab(\eta_{n-1}c+(\xi_n-\eta_{n-1})d)},$$

$$s_{2n-2}=\frac{abcd}{cd((\eta_{n-1}-\xi_{n-2})a+\xi_{n-1}b)+ab((\xi_{n-1}-\eta_{n-2})c+\eta_{n-1}d)},$$

$$s_{2n-1}=\frac{abcd}{cd(\eta_{n-1}a+(\xi_n-\eta_{n-1})b)+ab(\xi_{n-1}c+(\eta_n-\xi_{n-1})d)}.$$

Now, it follows from Eq.(2.1) that

$$\begin{aligned}
w_{2n}&=\frac{w_{2n-1}s_{2n-2}}{w_{2n-1}+s_{2n-2}}\\
&=\frac{\left(\frac{abcd}{cd(\xi_{n-1}a+(\eta_n-\xi_{n-1})b)+ab(\eta_{n-1}c+(\xi_n-\eta_{n-1})d)}\right)\left(\frac{abcd}{cd((\eta_{n-1}-\xi_{n-2})a+\xi_{n-1}b)+ab((\xi_{n-1}-\eta_{n-2})c+\eta_{n-1}d)}\right)}{\frac{abcd}{cd(\xi_{n-1}a+(\eta_n-\xi_{n-1})b)+ab(\eta_{n-1}c+(\xi_n-\eta_{n-1})d)}+\frac{abcd}{cd((\eta_{n-1}-\xi_{n-2})a+\xi_{n-1}b)+ab((\xi_{n-1}-\eta_{n-2})c+\eta_{n-1}d)}}\\
&=\frac{abcd}{cd((\xi_{n-1}+\eta_{n-1}-\xi_{n-2})a+(\eta_n-\xi_{n-1}+\xi_{n-1})b)+ab((\eta_{n-1}+\xi_{n-1}-\eta_{n-2})c+(\xi_n-\eta_{n-1}+\eta_{n-1})d)}\\
&=\frac{abcd}{cd((\xi_n-\eta_{n-1})a+\eta_n b)+ab((\eta_n-\xi_{n-1})c+\xi_n d)}.
\end{aligned}$$

And

$$\begin{aligned}
s_{2n}&=\frac{s_{2n-1}w_{2n-2}}{s_{2n-1}+w_{2n-2}}\\
&=\frac{\left(\frac{abcd}{cd(\eta_{n-1}a+(\xi_n-\eta_{n-1})b)+ab(\xi_{n-1}c+(\eta_n-\xi_{n-1})d)}\right)\left(\frac{abcd}{cd((\xi_{n-1}-\eta_{n-2})a+\eta_{n-1}b)+ab((\eta_{n-1}-\xi_{n-2})c+\xi_{n-1}d)}\right)}{\frac{abcd}{cd(\eta_{n-1}a+(\xi_n-\eta_{n-1})b)+ab(\xi_{n-1}c+(\eta_n-\xi_{n-1})d)}+\frac{abcd}{cd((\xi_{n-1}-\eta_{n-2})a+\eta_{n-1}b)+ab((\eta_{n-1}-\xi_{n-2})c+\xi_{n-1}d)}}\\
&=\frac{abcd}{cd((\eta_{n-1}+\xi_{n-1}-\eta_{n-2})a+(\xi_n-\eta_{n-1}+\eta_{n-1})b)+ab((\xi_{n-1}+\eta_{n-1}-\xi_{n-2})c+(\eta_n-\xi_{n-1}+\xi_{n-1})d)}\\
&=\frac{abcd}{cd((\eta_n-\xi_{n-1})a+\xi_n b)+ab((\xi_n-\eta_{n-1})c+\eta_n d)}.
\end{aligned}$$

Also,

$$w_{2n+1}=\frac{w_{2n}s_{2n-1}}{w_{2n}+s_{2n-1}}$$

$$=\frac{(\frac{abcd}{cd((\xi_n-\eta_{n-1})a+\eta_n b)+ab((\eta_n-\xi_{n-1})c+\xi_n)d)})(\frac{abcd}{cd(\eta_{n-1}a+(\xi_n-\eta_{n-1})b)+ab(\xi_{n-1}c+(\eta_n-\xi_{n-1})d)})}{\frac{abcd}{cd((\xi_n-\eta_{n-1})a+\eta_n b)+ab((\eta_n-\xi_{n-1})c+\xi_n d)}+\frac{abcd}{cd(\eta_{n-1}a+(\xi_n-\eta_{n-1})b)+ab(\xi_{n-1}c+(\eta_n-\xi_{n-1})d)}}$$

$$=\frac{abcd}{cd((\xi_n-\eta_{n-1}+\eta_{n-1})a+(\eta_n+\xi_n-\eta_{n-1})b+ab((\eta_n-\xi_{n-1}+\xi_{n-1})c+(\xi_n+\eta_n-\xi_{n-1})d)}$$

$$=\frac{abcd}{cd(\xi_n a+(\eta_{n+1}-\xi_n)b)+ab(\eta_n c+(\xi_{n+1}-\eta_n)d)}.$$

And

$$s_{2n+1}=\frac{s_{2n}w_{2n-1}}{s_{2n}+w_{2n-1}}$$

$$=\frac{(\frac{abcd}{cd((\eta_n-\xi_{n-1})a+\xi_n b)+ab((\xi_n-\eta_{n-1})c+\eta_n d)})(\frac{abcd}{cd(\xi_{n-1}a+(\eta_n-\xi_{n-1})b)+ab(\eta_{n-1}c+(\xi_n-\eta_{n-1})d)})}{\frac{abcd}{cd((\eta_n-\xi_{n-1})a+\xi_n b)+ab((\xi_n-\eta_{n-1})c+\eta_n d)}+\frac{abcd}{cd(\xi_{n-1}a+(\eta_n-\xi_{n-1})b)+ab(\eta_{n-1}c+(\xi_n-\eta_{n-1})d)}}$$

$$=\frac{abcd}{cd((\eta_n-\xi_{n-1}+\xi_{n-1})a+(\xi_n+\eta_n-\xi_{n-1})b+ab((\xi_n-\eta_{n-1}+\eta_{n-1})c+(\eta_n+\xi_n-\eta_{n-1})d)}$$

$$=\frac{abcd}{cd(\eta_n a+(\xi_{n+1}-\eta_n)b)+ab(\xi_n c+(\eta_{n+1}-\xi_n)d)}.$$

□

Example 2.2. *Figure 2.1 demonstrates the behavior of the solutions of the system of difference equations (2.1) with* $w_{-1}=4,\ w_0=1,\ s_{-1}=-2$ *and* $s_0=2$.

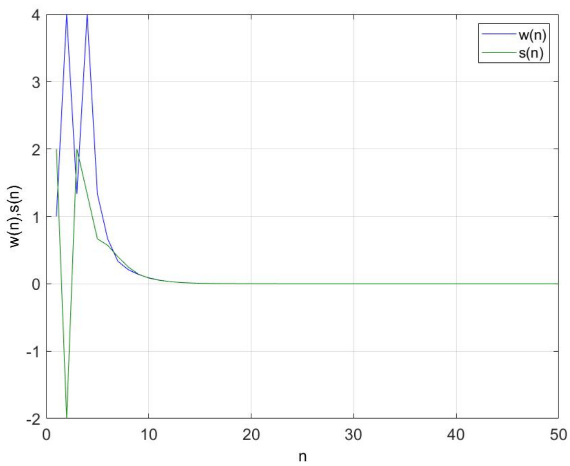

Figure 2.1

3. The system $w_{n+1}=\frac{w_n s_{n-1}}{w_n+s_{n-1}}$, $s_{n+1}=\frac{s_n w_{n-1}}{s_n-w_{n-1}}$

In this section, we investigate the solutions of the following system of the difference equations :

$$w_{n+1}=\frac{w_n s_{n-1}}{w_n+s_{n-1}}, \quad s_{n+1}=\frac{s_n w_{n-1}}{s_n-w_{n-1}}. \tag{3.1}$$

Theorem 3.1. *Assume that $\{w_n,s_n\}$ is a solution of system (3.1). Then for $n=1,2,...,$ we have*

$$w_{2n-1}=\frac{abd}{a(f_{n+2}b+f_{n+1}d)+f_{n+2}bd}, \quad w_{2n}=\frac{acd}{a(f_{n+2}d+f_{n+2}c)+f_{n+3}cd},$$

$$s_{2n-1}=\frac{bcd}{c(f_{n+2}d+f_{n+1}b)-f_{n+2}bd}, \quad s_{2n}=\frac{abc}{a(f_{n+3}b-f_{n+2}c)+f_{n+2}bc},$$

where $\{f_m\}_{m=0}^{\infty}=\{0,1,0,1,1,2,3,...\}$.

Proof. For n=0, the result holds. Now, suppose that $n>0$ and that our assumption holds for $n-1$ and $n-2$. That is

$$w_{2n-3}=\frac{abd}{a(f_{n+1}b+f_n d)+f_{n+1}bd}, \quad w_{2n-2}=\frac{acd}{a(f_{n+1}d+f_{n+1}c)+f_{n+2}cd},$$

$$w_{2n-5}=\frac{abd}{a(f_n b+f_{n-1}d)+f_n bd}, \quad w_{2n-4}=\frac{acd}{a(f_n d+f_n c)+f_{n+1}cd},$$

$$s_{2n-3}=\frac{bcd}{c(f_{n+1}d+f_n b)-f_{n+1}bd}, \quad s_{2n-2}=\frac{abc}{a(f_{n+2}b-f_{n+1}c)+f_{n+1}bc},$$

$$s_{2n-5}=\frac{bcd}{c(f_n d+f_{n-1}b)-f_n bd}, \quad s_{2n-4}=\frac{abc}{a(f_{n+1}b-f_n c)+f_n bc}.$$

Now, from Eq.(3.1) we get :

$$\begin{aligned}
w_{2n} &= \frac{w_{2n-1}s_{2n-2}}{w_{2n-1}+s_{2n-2}} \\
&= \frac{\left(\frac{abd}{a(f_{n+2}b+f_{n+1}d)+f_{n+2}bd}\right)\left(\frac{abc}{a(f_{n+2}b-f_{n+1}c)+f_{n+1}bc}\right)}{\frac{abd}{a(f_{n+2}b+f_{n+1}d)+f_{n+2}bd}+\frac{abc}{a(f_{n+2}b-f_{n+1}c)+f_{n+1}bc}} \\
&= \frac{a^2b^2cd}{ab((f_{n+2}b-f_{n+1}c)ad+f_{n+1}bcd+(f_{n+2}b+f_{n+1}d)ac+f_{n+2}bcd)} \\
&= \frac{abcd}{f_{n+2}bad-f_{n+1}cad+f_{n+1}bcd+f_{n+2}bac+f_{n+1}dac+f_{n+2}bcd} \\
&= \frac{abcd}{b(f_{n+2}ad+(f_{n+1}+f_{n+2})cd+f_{n+2}ac)} \\
&= \frac{acd}{(f_{n+2}d+f_{n+2}c)a+f_{n+3}cd}.
\end{aligned}$$

And

$$s_{2n}=\frac{s_{2n-1}w_{2n-2}}{s_{2n-1}-w_{2n-2}}$$

$$= \frac{\left(\frac{bcd}{c(f_{n+2}d+f_{n+1}b)-f_{n+2}bd}\right)\left(\frac{acd}{a(f_{n+1}d+f_{n+1}c)+f_{n+2}cd}\right)}{\frac{bcd}{c(f_{n+2}d+f_{n+1}b)-f_{n+2}bd}-\frac{acd}{a(f_{n+1}d+f_{n+1}c)+f_{n+2}cd}}$$

$$= \frac{abc^2d^2}{cd((f_{n+1}d+f_{n+1}c)ad+f_{n+2}bcd-(f_{n+2}d+f_{n+1}b)ac+f_{n+2}abd)}$$

$$= \frac{abcd}{f_{n+1}bad+f_{n+1}cab+f_{n+2}bcd-f_{n+2}dac-f_{n+1}bac+f_{n+2}abd}$$

$$= \frac{abcd}{d(f_{n+2}cb+(f_{n+1}+f_{n+2})ba-f_{n+2}ac)}$$

$$= \frac{acb}{(f_{n+3}b-f_{n+2}c)a+f_{n+2}cb}.$$

Also,

$$w_{2n-1} = \frac{w_{2n-2}s_{2n-3}}{w_{2n-2}+s_{2n-3}}$$

$$= \frac{\left(\frac{acd}{a(f_{n+1}d+f_{n+1}c)+f_{n+2}cd}\right)\left(\frac{bcd}{c(f_{n+1}d+f_nb)-f_{n+1}bd}\right)}{\frac{acd}{a(f_{n+1}d+f_{n+1}c)+f_{n+2}cd}+\frac{bcd}{c(f_{n+1}d+f_nb)-f_{n+1}bd}}$$

$$= \frac{abc^2d^2}{cd((f_{n+1}d+f_nb)ac-f_{n+1}bad+(f_{n+1}d+f_{n+1}c)ab+f_{n+2}bcd)}$$

$$= \frac{abcd}{f_{n+1}adc+f_ncab-f_{n+1}bad+f_{n+1}bad+f_{n+1}bac+f_{n+2}bcd}$$

$$= \frac{abcd}{c(f_{n+1}ad+(f_n+f_{n+1})ba+f_{n+2}bd)}$$

$$= \frac{abd}{(f_{n+2}b+f_{n+1}d)a+f_{n+2}bd}.$$

And

$$s_{2n-1} = \frac{s_{2n-2}w_{2n-3}}{s_{2n-2}-w_{2n-3}}$$

$$= \frac{\left(\frac{abc}{a(f_{n+2}b-f_{n+1}c)+f_{n+1}bc}\right)\left(\frac{abd}{a(f_{n+1}b+f_nd)+f_{n+1}bd}\right)}{\frac{abc}{a(f_{n+2}b-f_{n+1}c)+f_{n+1}bc}-\frac{abd}{a(f_{n+1}b+f_nd)+f_{n+1}bd}}$$

$$= \frac{a^2b^2cd}{ab((f_{n+1}b+f_nd)ac+f_{n+1}bcd-(f_{n+2}b-f_{n+1}c)ad-f_{n+1}cbd)}$$

$$= \frac{abcd}{f_{n+1}bac+f_ncad+f_{n+1}bcd-f_{n+2}dab+f_{n+1}dac-f_{n+1}cbd}$$

$$= \frac{abcd}{a(f_{n+1}cb+(f_n+f_{n+1})dc-f_{n+2}bd)}$$

$$= \frac{bcb}{(f_{n+2}d+f_{n+1}b)c-f_{n+2}bd}.$$

□

Example 3.2. *We assume a numerical example for Eq.(3.1) where $w_{-1}=3,\ w_0=0.5,\ s_{-1}=-4$ and $s_0=1$. See Fig. 3.1.*

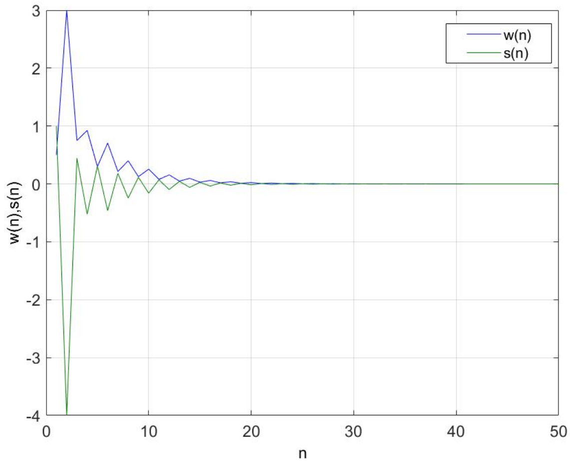

Figure 3.1

4. The system $w_{n+1} = \frac{w_n s_{n-1}}{w_n + s_{n-1}}$, $s_{n+1} = \frac{s_n w_{n-1}}{-s_n + w_{n-1}}$

In this section, we give a specific form the solutions of the system of the difference equation in the form:

$$w_{n+1} = \frac{w_n s_{n-1}}{w_n + s_{n-1}}, \quad s_{n+1} = \frac{s_n w_{n-1}}{-s_n + w_{n-1}}. \tag{4.1}$$

Theorem 4.1. *Assume that $\{w_n, s_n\}$ is a solution of system (4.1). Then for $n = 1, 2, ...,$ we have*

$$w_{2n} = \frac{abcd}{cd((\eta_{n+1} - \xi_n)a + \xi_n b) + ab(\eta_n c + (\eta_{n+1} - \xi_{n+1})d)},$$

$$w_{2n+1} = \frac{abcd}{cd((\xi_{n+1} - \eta_{n+1})a + \eta_{n+1} b) + ab(\xi_n c + (\xi_{n+1} - \eta_{n+2})d)},$$

$$s_{2n} = \frac{abcd}{cd(-\eta_n a + (\xi_{n+1} - \eta_{n+1})b) + ab((\eta_{n+1} - \xi_n)c + \xi_n d)},$$

$$s_{2n+1} = \frac{abcd}{cd(-\xi_n a + (\eta_{n+2} - \xi_{n+1})b) + ab((\xi_{n+1} - \eta_{n+1})c + \eta_{n+1} d)},$$

where $\{\xi_n\}_{n=1}^{\infty} = \{1, 0, -5, -13, -12, ...\}$, $\xi_0 = 1$, $\xi_{-1} = 0$, $\eta_n = ((\xi_{n-2} + \xi_{n-1} + \xi_n) \div 2)$ and $\{\eta_n\}_{n=1}^{\infty} = \{1, 1, -2, -9, -15, ...\}$.

Proof. For n=0, the result holds. Now, suppose that $n > 0$ and that our assumption holds for $n-1$. That is

$$w_{2n-2} = \frac{abcd}{cd((\eta_n - \xi_{n-1})a + \xi_{n-1} b) + ab(\eta_{n-1} c + (\eta_n - \xi_n)d)},$$

$$w_{2n-1} = \frac{abcd}{cd((\xi_n - \eta_n)a + \eta_n b) + ab(\xi_{n-1} c + (\xi_n - \eta_{n+1})d)},$$

$$s_{2n-2} = \frac{abcd}{cd(-\eta_{n-1} a + (\xi_n - \eta_n)b) + ab((\eta_n - \xi_{n-1})c + \xi_{n-1} d)},$$

$$s_{2n-1} = \frac{abcd}{cd(-\xi_{n-1} a + (\eta_{n+1} - \xi_n)b) + ab((\xi_n - \eta_n)c + \eta_n d)}.$$

Now, it follows from Eq.(4.1) that

$$
\begin{aligned}
w_{2n} &= \frac{w_{2n-1}s_{2n-2}}{w_{2n-1}+s_{2n-2}} \\
&= \frac{\left(\frac{abcd}{cd((\xi_n-\eta_n)a+\eta_n b)+ab(\xi_{n-1}c+(\xi_n-\eta_{n+1})d)}\right)\left(\frac{abcd}{cd(-\eta_{n-1}a+(\xi_n-\eta_n)b)+ab((\eta_n-\xi_{n-1})c+\xi_{n-1}d)}\right)}{\frac{abcd}{cd((\xi_n-\eta_n)a+\eta_n b)+ab(\xi_{n-1}c+(\xi_n-\eta_{n+1})d)}+\frac{abcd}{cd(-\eta_{n-1}a+(\xi_n-\eta_n)b)+ab((\eta_n-\xi_{n-1})c+\xi_{n-1}d)}} \\
&= \frac{abcd}{cd((\xi_n-\eta_n-\eta_{n-1})a+(\eta_n+\xi_n-\eta_n)b)+ab((\xi_{n-1}+\eta_n-\xi_{n-1})c+(\xi_n-\eta_{n+1}+\xi_{n-1})d)} \\
&= \frac{abcd}{cd((\eta_{n+1}-\xi_n)a+\xi_n b)+ab(\eta_n c+(\eta_{n+1}-\xi_{n+1})d)}.
\end{aligned}
$$

And

$$
\begin{aligned}
s_{2n} &= \frac{s_{2n-1}w_{2n-2}}{-s_{2n-1}+w_{2n-2}} \\
&= \frac{\left(\frac{abcd}{cd(-\xi_{n-1}a+(\eta_{n+1}-\xi_n)b)+ab((\xi_n-\eta_n)c+\eta_n d)}\right)\left(\frac{abcd}{cd((\eta_n-\xi_{n-1})a+\xi_{n-1}b)+ab(\eta_{n-1}c+(\eta_n-\xi_n)d)}\right)}{\frac{-abcd}{cd(-\xi_{n-1}a+(\eta_{n+1}-\xi_n)b)+ab((\xi_n-\eta_n)c+\eta_n d)}+\frac{abcd}{cd((\eta_n-\xi_{n-1})a+\xi_{n-1}b)+ab(\eta_{n-1}c+(\eta_n-\xi_n)d)}} \\
&= \frac{abcd}{cd((-\xi_{n-1}-\eta_n+\xi_{n-1})a+(\eta_{n+1}-\xi_n-\xi_{n-1})b)+ab((\xi_n-\eta_n-\eta_{n-1})c+(\eta_n-\eta_n+\xi_n)d} \\
&= \frac{abcd}{cd(-\eta_n a+(\xi_{n+1}-\eta_{n+1})b)+ab((\eta_{n+1}-\xi_n)c+\xi_n d)}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
w_{2n+1} &= \frac{w_{2n}s_{2n-1}}{w_{2n}+s_{2n-1}} \\
&= \frac{\left(\frac{abcd}{cd((\eta_{n+1}-\xi_n)a+\xi_n b)+ab(\eta_n c+(\eta_{n+1}-\xi_{n+1})d)}\right)\left(\frac{abcd}{cd(-\xi_{n-1}a+(\eta_{n+1}-\xi_n)b)+ab((\xi_n-\eta_n)c+\eta_n d)}\right)}{\frac{abcd}{cd((\eta_{n+1}-\xi_n)a+\xi_n b)+ab(\eta_n c+(\eta_{n+1}-\xi_{n+1})d)}+\frac{abcd}{cd(-\xi_{n-1}a+(\eta_{n+1}-\xi_n)b)+ab((\xi_n-\eta_n)c+\eta_n d)}} \\
&= \frac{abcd}{cd((\eta_{n+1}-\xi_n-\xi_{n-1})a+(\xi_n+\eta_{n+1}-\xi_n)b+ab((\eta_n+\xi_n-\eta_n)c+(\eta_{n+1}-\xi_{n+1}+\eta_n)d)} \\
&= \frac{abcd}{cd((\xi_{n+1}-\eta_{n+1})a+\eta_{n+1}b)+ab(\xi_n c+(\xi_{n+1}-\eta_{n+2})d)}.
\end{aligned}
$$

And

$$
\begin{aligned}
s_{2n+1} &= \frac{s_{2n}w_{2n-1}}{-s_{2n}+w_{2n-1}} \\
&= \frac{\left(\frac{abcd}{cd(-\eta_n a+(\xi_{n+1}-\eta_{n+1})b)+ab((\eta_{n+1}-\xi_n)c+\xi_n d)}\right)\left(\frac{abcd}{cd((\xi_n-\eta_n)a+\eta_n b)+ab(\xi_{n-1}c+(\xi_n-\eta_{n+1})d)}\right)}{\frac{-abcd}{cd(-\eta_n a+(\xi_{n+1}-\eta_{n+1})b)+ab((\eta_{n+1}-\xi_n)c+\xi_n d)}+\frac{abcd}{cd((\xi_n-\eta_n)a+\eta_n b)+ab(\xi_{n-1}c+(\xi_n-\eta_{n+1})d)}} \\
&= \frac{abcd}{cd((-\eta_n-\xi_n+\eta_n)a+(\xi_{n+1}-\eta_{n+1}-\eta_n)b)+ab((\eta_{n+1}-\xi_n-\xi_{n-1})c+(\xi_n-\xi_n+\eta_{n+1})d} \\
&= \frac{abcd}{cd(-\xi_n a+(\eta_{n+2}-\xi_{n+1})b)+ab((\xi_{n+1}-\eta_{n+1})c+\eta_{n+1}d)}.
\end{aligned}
$$

□

Example 4.2. *Consider the solutions of Eq.(4.1) when $w_{-1}=3,\ w_0=-1,\ s_{-1}=4$ and $s_0=-1$. See Fig. 4.1.*

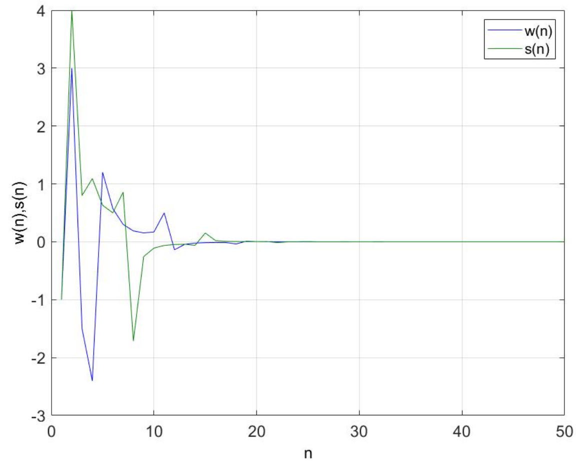

Figure 4.1

5. The system $w_{n+1}=\frac{w_n s_{n-1}}{w_n+s_{n-1}}$, $s_{n+1}=\frac{s_n w_{n-1}}{-s_n-w_{n-1}}$

In this section, we obtain the solutions form for the system of two difference equations :

$$w_{n+1}=\frac{w_n s_{n-1}}{w_n+s_{n-1}}, \quad s_{n+1}=\frac{s_n w_{n-1}}{-s_n-w_{n-1}}. \tag{5.1}$$

Theorem 5.1. *Suppose that $\{w_n,s_n\}$ are solutions of system (5.1). Then every solutions of system (5.1) are periodic with period twelve and given by the following formulas for n=0,1,2,... ,*

$$w_{12n-1}=b, \qquad w_{12n}=a,$$

$$w_{12n+1}=\frac{ad}{a+d}, \qquad w_{12n+2}=\frac{adc}{(a+d)c+ad},$$

$$w_{12n+3}=\frac{adb}{(a+d)b-ad}, \qquad w_{12n+4}=\frac{dc}{d+c},$$

$$w_{12n+5}=-b, \qquad w_{12n+6}=-a,$$

$$w_{12n+7}=\frac{-ad}{a+d}, \qquad w_{12n+8}=\frac{-adc}{(a+d)c+ad},$$

$$w_{12n+9}=\frac{-adb}{(a+d)b-ad}, \qquad w_{12n+10}=\frac{-dc}{d+c},$$

$$s_{12n-1}=d, \qquad s_{12n}=c,$$

$$s_{12n+1}=\frac{-cb}{c+b}, \qquad s_{12n+2}=\frac{-abc}{cb-(c+b)a},$$

$$s_{12n+3}=\frac{-cdb}{(b+d)c+bd}, \qquad s_{12n+4}=\frac{-ba}{b-a},$$

$$s_{12n+5}=-d, \qquad s_{12n+6}=-c,$$

$$s_{12n+7}=\frac{cb}{c+b},\qquad s_{12n+8}=\frac{abc}{cb-(c+b)a},$$

$$s_{12n+9}=\frac{cdb}{(b+d)c+bd},\qquad s_{12n+10}=\frac{ba}{b-a}.$$

Proof. For n=0, the result holds. Now, suppose that $n>0$ and that our assumption holds for $n-1$. That is

$$w_{12n-13}=b,\qquad w_{12n-12}=a,$$

$$w_{12n-11}=\frac{ad}{a+d},\qquad w_{12n-10}=\frac{adc}{(a+d)c+ad},$$

$$w_{12n-9}=\frac{adb}{(a+d)b-ad},\qquad w_{12n-8}=\frac{dc}{d+c},$$

$$w_{12n-7}=-b,\qquad w_{12n-6}=-a,$$

$$w_{12n-5}=\frac{-ad}{a+d},\qquad w_{12n-4}=\frac{-adc}{(a+d)c+ad},$$

$$w_{12n-3}=\frac{-adb}{(a+d)b-ad},\qquad w_{12n-2}=\frac{-dc}{d+c},$$

$$s_{12n-13}=d,\qquad s_{12n-12}=c,$$

$$s_{12n-11}=\frac{-cb}{c+b},\qquad s_{12n-10}=\frac{-abc}{cb-(c+b)a},$$

$$s_{12n-9}=\frac{-cdb}{(b+d)c+bd},\qquad s_{12n-8}=\frac{-ba}{b-a},$$

$$s_{12n-7}=-d,\qquad s_{12n-6}=-c,$$

$$s_{12n-5}=\frac{cb}{c+b},\qquad s_{12n-4}=\frac{abc}{cb-(c+b)a},$$

$$s_{12n-3}=\frac{cdb}{(b+d)c+bd},\qquad s_{12n-2}=\frac{ba}{b-a}.$$

Now, from Eq.(5.1) that

$$w_{12n}=\frac{w_{12n-1}s_{12n-2}}{w_{12n-1}+s_{12n-2}}=\frac{\frac{b^2a}{b-a}}{b+\frac{ba}{b-a}}=\frac{b^2a}{b^2-ab+ab}=a.$$

$$s_{12n}=\frac{s_{12n-1}w_{12n-2}}{-s_{12n-1}-w_{12n-2}}=\frac{\frac{-d^2c}{c+d}}{-d+\frac{cd}{c+d}}=\frac{-d^2c}{-d^2-cd+cd}=c.$$

Also,

$$w_{12n+5}=\frac{w_{12n+4}s_{12n+3}}{w_{12n+4}+s_{12n+3}}=\frac{\left(\frac{dc}{d+c}\right)\left(\frac{-cbd}{(b+d)c+bd}\right)}{\frac{dc}{d+c}-\frac{cbd}{(b+d)c+bd}}=\frac{-c^2bd^2}{cd(bc+dc+bd-bd-cb)}=-b.$$

$$s_{12n+5}=\frac{s_{12n+4}w_{12n+3}}{-s_{12n+4}-w_{12n+3}}=\frac{\left(\frac{-ba}{b-a}\right)\left(\frac{abd}{(a+d)b-ad}\right)}{-\frac{-bd}{b-a}-\frac{abd}{(a+d)b-ad}}=\frac{-a^2b^2d}{ba(ab+db-ad-bd+ad)}=-d.$$

Similarly, obtaining the other relations is very simple. Thus, the proof is completed. □

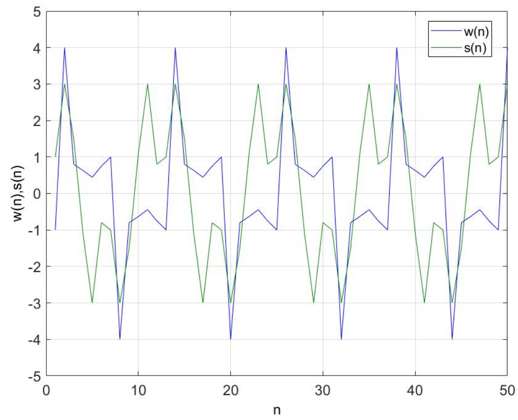

Figure 5.1

Example 5.2. *Figure 5.1 considers the solution of Eq.(5.1) with* $w_{-1} = 4,\ w_0 = -1,\ s_{-1} = 3$ *and* $s_0 = 1$.

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Holomorphically Planar Conformal Vector Field On Almost $\alpha-$Cosymplectic $(\kappa,\mu)-$ Spaces

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Abstract

The aim of the present paper is to study holomorphically planar conformal vector (HPCV) fields on almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces. This is done assuming various conditions such as i) U is pointwise collinear with ξ (in this case, the integral manifold of the distribution D is totally geodesic, or totally umbilical), ii) M has a constant $\xi-$sectional curvature (under this condition the integral manifold of the distribution D is totally geodesic (or totally umbilical) or the manifold is isometric to sphere $S^{2n+1}(\sqrt{c})$ of radius $\frac{1}{\sqrt{c}}$), iii) M an almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces (in this case the manifold has constant curvature, or the integral manifold of the distribution D is totally geodesic(or totally umbilical) or U is an eigenvector of h).

1. Introduction

Killing vector fields are of great importance in terms of having an impact on the the geometry in addition to the topology of Riemannian manifolds and being incompressible fields has a significant role in physics. The importance of Killing vector fields in Riemannian geometry is associated with the fact that the flows preserve the given metric and determine the symmetry degree of the manifold. Also, in terms of physics, Killing vectors allow the energy and momentum of a freely moving particle to be conserved in flat space- times. In a general manner, special vector fields such as Killing vector fields are conformal vector fields of which flow maintains a conformal class of metrics. A vector field V satisfying $£_V g = 2fg$ on a Riemannian manifold (M,g) is said to be a conformal vector field or conformal transformation on M, where $£$ denotes the Lie derivative on M and f is a smooth function. If f is constant, then V is called homothetic. Also, it has been stated that if the metrically associated $1-$form of V is closed, it is described as closed. V is named as the gradient conformal vector field in case of the fact that the conformal vector field V is the gradient of any differentiable function. The conformal vector fields have been carried out in numerous studies ([1]- [3]).
As a result of these studies, Sharma [4] introduced a holomorphically planar conformal vector (HPCV) field U as a generalization of a closed conformal vector field on an almost Hermitian manifolds. Then, Ghosh–Sharma [5] extensively studied this concept in various conditions. Later, in [6], HPCV fields studied on contact metric manifolds and Einstein contact metric manifolds under some curvature conditions. An HPCV field on a contact metric manifold refers to a vector field U on (M,φ,ζ,η) which satisfies

$$\nabla_X U = aX + b\varphi X \tag{1.1}$$

for arbitrary $X \in \chi(M)$, where a and b are smooth functions on M.

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It must be considered that another class of almost contact manifold, named almost cosymplectic manifold, has gained much attention in various studies [7]. The notion was introduced in [8] as an almost contact metric manifold of which the fundamental $2-$form Φ and $1-$form η are closed. It has been reported that if the almost contact structure is normal then the manifold is called cosymplectic (in the present case the term "cosymplectic" has been adopted to refer to the term "coKähler" in [7]). Also, Endo [9] defined almost cosymplectic $(\kappa,\mu)-$spaces that the curvature tensor of the manifold satisfies

$$R(X,Y)\zeta = \kappa(\eta(Y)X-\eta(X)Y)+\mu(\eta(Y)hX-\eta(X)hY) \tag{1.2}$$

for any $X,Y\in\chi(M)$, where κ,μ are constant and $h=\frac{1}{2}\pounds_{\zeta}\varphi$. On the other hand, Kenmotsu [10] defined the almost Kenmotsu manifold, an almost contact manifold satisfying $d\eta=0$ and $d\Phi=2\eta\wedge\Phi$. According to this definition, Kim [11] introduced the notion of almost $\alpha-$cosymplectic manifold, referring to an almost contact manifold satisfying $d\eta=0$ and $d\Phi=2\alpha\eta\wedge\Phi$ (α is real constant). Aktan et al. [12] carried out an extensive study on almost $\alpha-$cosymplectic $(\kappa,\mu,\upsilon)-$spaces and revealed some outcomes of substantial importance. In recent studies numerous studies were carried out on this subject (cf. [13]-[17]). In the light of these studies, the aim of the paper is to study the HPCV fields on almost $\alpha-$cosymplectic manifolds and almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces. Firstly, we give some basic definitions and properties of such structures. In main section, we consider an almost $\alpha-$cosymplectic and almost cosymplectic $(\kappa,\mu)-$ spaces admits a non zero HPCV field U. This is done assuming various conditions such as i) U is pointwise collinear with ζ (in this case the integral manifold of the distribution D is totally geodesic or totally umbilical), ii) M has a constant $\zeta-$sectional curvature (under this condition the integral manifold of the distribution D is totally geodesic (or totally umbilical) or the manifold is isometric to sphere $S^{2n+1}(\sqrt{c})$ of radius $\frac{1}{\sqrt{c}}$), iii) M an almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces (i this case the manifold is constant negative curvature or the integral manifold of the distribution D is totally geodesic(or totally umbilical) or U is an eigenvector of h).

2. Preliminaries

Let M be a $(2n+1)-$dimensional smooth manifold. An almost contact structure on M is a triple (φ,ζ,η) which carries a field φ of endomorphisms of the tangent spaces, a vector field ζ, called characteristic vector field, and a $1-$form η satisfying

$$\varphi^2=-I+\eta\otimes\zeta,\ \eta(\varphi)=0,\ \varphi(\zeta)=0. \tag{2.1}$$

A smooth manifold with such a structure is called an almost contact manifold. It is known that any almost contact manifold M admits a Riemannian metric g satisfying

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y),\ g(X,\zeta)=\eta(X),$$

then g is called compatible metric with the structure. Then the manifold (M,φ,ζ,η,g) is called an almost contact metric manifold. An almost contact structure (φ,ζ,η) is said to be normal if the Nijenheus tensor of φ vanishes identically. The fundamental $2-$form Φ on M is defined by $\Phi(X,Y)=g(\varphi X,Y)$ for any vector fields $X,\ Y\in\chi(M)$.
An almost $\alpha-$cosymplectic manifold is an almost contact metric manifold defined by $d\eta=0$ and $d\Phi=2\alpha\eta\wedge\Phi$, for any real number α . If $\alpha=0$ then the manifold reduce to almost cosymplectic manifold. Furthermore, normal almost $\alpha-$cosymplectic manifold is called $\alpha-$cosymplectic manifold (For more detail [11], [12]).
Let M be an almost $\alpha-$cosymplectic manifold and $D=\{X:\eta(X)=0\}$ which denote the distribution orthogonal to ζ. Since the $1-$form is closed, then we have $\pounds_{\zeta}\eta=0$ and $[X,\zeta]\in D$ for any $X\in D$. The Levi-Civita connection satisfies $\nabla_{\zeta}\zeta=0$ and $\nabla_{\zeta}\varphi\in D$, which implies that $\nabla_{\zeta}X\in D$ for any $X\in D$.
In addition, an almost $\alpha-$cosymplectic manifold satisfies the following equations [11]:

$$h\zeta=0,\ g(hX,Y)=g(X,hY),\ trace(h)=0,\ \varphi h+h\varphi=0,$$

$$\nabla_X\zeta=-\alpha\varphi^2X-\varphi hX=-A,$$

$$(\nabla_X\varphi)Y+(\nabla_{\varphi X}\varphi)\varphi Y=-\alpha[\eta(Y)\varphi X+2g(X,\varphi Y)\zeta]-\eta(Y)hX, \tag{2.2}$$

$$(\nabla_X\eta)Y=\alpha[g(X,Y)-\eta(X)\eta(Y)]+g(\varphi Y,hX), \tag{2.3}$$

$$tr(A\varphi)=tr(\varphi A)=0,\ tr(h\varphi)=tr(\varphi h)=0,$$

$$tr(A)=-2\alpha n,\ tr(h)=0,$$

for any vector fields $X,Y\in\chi(M)$.

3. Holomorphically planar conformal vector fields on almost $\alpha-$cosymplectic manifolds

In this part, we study almost $\alpha-$cosymplectic manifolds with respect to a HPCV field U. We first state and prove the following lemma for the our main theorem.

Lemma 3.1. *Let M be an almost $\alpha-$cosymplectic manifold with respect to a HPCV field U.Then*

$$\varphi U(a) = -U(b) + (\zeta(b) + 2\alpha nb)\eta(U) \tag{3.1}$$

holds on M.

Proof. Using equation (1.1) in the formula Riemannian curvature tensor $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$, we have

$$R(X,Y)U = X(a)Y - Y(a)X + X(b)\varphi Y - Y(b)\varphi X + b[(\nabla_X \varphi)Y - (\nabla_Y \varphi)X]. \tag{3.2}$$

Replacement of X with φX and Y with φY in the previous equation yields

$$\begin{aligned} R(\varphi X, \varphi Y)U &= \varphi X(a)\varphi Y - \varphi Y(a)\varphi X - \varphi X(b)Y + \varphi X(b)\eta(Y)\zeta + \varphi Y(b)X \\ &\quad -\varphi Y(b)\eta(X)\zeta + b\left[(\nabla_{\varphi X}\varphi)\varphi Y - (\nabla \varphi_Y \varphi)\varphi X\right]. \end{aligned} \tag{3.3}$$

By adding (3.2) and (3.3) and using (2.2), we obtain

$$\begin{aligned} R(X,Y)U + R(\varphi X, \varphi Y)U &= \varphi X(a)\varphi Y - \varphi Y(a)\varphi X - \varphi X(b)Y + \varphi X(b)\eta(Y)\zeta \\ &\quad +\varphi Y(b)X - \varphi Y(b)\eta(X)\zeta + X(a)Y - Y(a)X \\ &\quad +X(b)\varphi Y - Y(b)\varphi X + b\alpha[\eta(X)\varphi Y + 2g(\varphi X, Y)\zeta \\ &\quad +\eta(X)hY - \eta(Y)\varphi X - 2g(X, \varphi Y)\zeta + \eta(Y)hX]. \end{aligned} \tag{3.4}$$

Using (2.1) and considering the inner product of (3.4) with U and then replacing X with φX and Y with φY, we have

$$\begin{aligned} &[X - \eta(X)\zeta](a)g(\varphi Y, \varphi U) - [Y - \eta(Y)\zeta](a)g(\varphi X, \varphi U) \\ &-[-X + \eta(X)\zeta](a)g(\varphi Y, U) + [-Y + \eta(Y)\zeta](a)g(\varphi X, U) \\ &+\varphi X(a)g(\varphi Y, U) - \varphi Y(a)g(\varphi X, U) - \varphi X(b)g(\varphi Y, \varphi U) \\ &+\varphi Y(b)g(\varphi X, \varphi U) - 4\alpha bg(X, \varphi Y)\eta(U) \\ = \;&0. \end{aligned} \tag{3.5}$$

Putting φY for Y in (3.5), we have

$$\begin{aligned} &g(Da, \varphi X)[-g(Y,U) + \eta(Y)\eta(U)] - [-g(Da, Y) + \eta(Y)\zeta(a)]g(\varphi X, U) \\ &-g(Db, \varphi X)g(\varphi Y, U) - [-g(Db, Y) + \eta(Y)\zeta(b)][-g(X,U) + \eta(X)\eta(U)] \\ &+[g(Da, X) - \eta(X)\zeta(a)]g(\varphi Y, U) + g(Da, \varphi Y)[-g(X,U) + \eta(X)\eta(U)] \\ &+[g(Db, X) - \eta(X)\zeta(b)][-g(Y,U) + \eta(Y)\eta(U)] - g(Db, \varphi Y)g(\varphi X, U) \\ &-4\alpha b[g(X,Y) - \eta(X)\eta(Y)] \\ = \;&0. \end{aligned} \tag{3.6}$$

Contracting X and Y in (3.6), we obtain

$$\varphi U(a) = -U(b) + \zeta(b)\eta(U) + 2\alpha bn\eta(U).$$

□

Theorem 3.2. *([11]) Let M be an almost $\alpha-$cosymplectic $f-$manifold and $\widetilde{M}$ be integral manifold of D. Then*

i) when $\alpha = 0$, $\widetilde{M}$ is totally geodesic if and only if all the operators h_i vanishes.
ii) when $\alpha \neq 0$ $\widetilde{M}$ is totally umbilical if and only if all the operators h_i vanishes.

Theorem 3.3. *Let M be an almost $\alpha-$cosymplectic manifold. If M admits a non zero HPCV field U such that U is pointwise collinear with ζ, then the integral manifold of the distribution D is totally geodesic or totally umbilical.*

Proof. Let U be a nonzero HPCV field on M and U is pointwise collinear with ζ such that

$$U = \rho\zeta \tag{3.7}$$

where $\rho \neq 0$ a smooth function on M. Substituting (3.7) in (3.1), we obtain $2n\alpha b\rho = 0$, which implies $b = 0$ since $\rho \neq 0$.Using these obtained result together with equations (3.7) and (1.1), we get $\rho = \eta(U)$. From

$$X(\rho) = \nabla_X \rho = \nabla_X(\eta(U)) = (\nabla_X \eta)U + \eta(\nabla_X U)$$

and using (1.1), (2.3), (3.7) , we have

$$X(\rho) = a\eta(X).$$

Taking the covariant derivative of equation (3.7) along X, we get

$$aX = -\alpha\rho\varphi^2 X - \rho\varphi hX + \alpha a\eta(X)\zeta. \tag{3.8}$$

Considering by the inner product of (3.8) with Y, then contracting X and Y last equation $\rho\alpha = a$. Then equation (3.8) reduces to

$$a\varphi hX = 0, \tag{3.9}$$

since $\rho \neq 0$ a smooth function, so $a \neq 0$. Hence from (3.9) we obtain the result $h = 0$. The proof is completed. □

Theorem 3.4. *Let M be a complete almost $\alpha-$cosymplectic manifold that admits an HPCV field U. If M has a constant $\zeta-$sectional curvature, then*

i) the integral manifold of the distribution D is totally geodesic or totally umbilical,
ii) the manifold M is isometric to sphere $S^{2n+1}(\sqrt{c})$ of radius $\frac{1}{\sqrt{c}}$.

Proof. Let $K(\zeta, X) = c$ is the positive constant sectional curvature of an almost $\alpha-$cosymplectic manifold. Then by a simple calculation, we obtain:

$$R(\zeta, X)\zeta = -c\left[X - \eta(X)\zeta\right]. \tag{3.10}$$

for any vector field $X \in \chi(M)$.
Using $X = \zeta$ in (3.2) yield

$$R(\zeta, Y)U = \zeta(a)Y - Y(a)\zeta + \zeta(b)\varphi Y + b\alpha\varphi Y + bhY. \tag{3.11}$$

By considering the inner product of (3.11) with ζ, we obtain

$$g(R(\zeta, Y)U, \zeta) = \zeta(a)\eta(Y) - Y(a). \tag{3.12}$$

Then, using (3.10), we obtain

$$g(R(\zeta, Y)U, \zeta) = -g(R(\zeta, Y)\zeta, U) = c\left[g(Y, U) - \eta(Y)\eta(U)\right]. \tag{3.13}$$

From the equations (3.12) and (3.13), we obtain

$$Da - \zeta(a)\zeta + cU - c\eta(U)\zeta = 0. \tag{3.14}$$

By considering the inner product of (3.11) with U implies

$$\zeta(a)U - (Da)\eta(U) - \zeta(b)\varphi U - b\alpha\varphi U + bhU = 0. \tag{3.15}$$

If we eleminate Da from the last two equations, we obtain

$$-\zeta(a)\varphi^2 U - c\eta(U)\varphi^2 U - \zeta(b)\varphi(U) - b\alpha\varphi U + bhU = 0. \tag{3.16}$$

Then differentiating (3.14) covariantly along any vector field X and further the inner product with Y, we obtain

$$\begin{aligned} g(\nabla_X Da, Y) \;=\; & \zeta(a)\left[\alpha g(X,Y)\zeta - \alpha\eta(X)\eta(Y) - g(\varphi hX, Y)\right] \\ & + X(\zeta(a))\eta(Y) - c\left[ag(X,Y) + bg(\varphi X, Y)\right] \\ & + c\eta(Y)\left[a\eta(X) + \alpha g(X,U) - \alpha\eta(X)\eta(U)\right] \\ & + c\eta(U)\left[\alpha g(X,Y) - \alpha\eta(X)\eta(Y) - g(\varphi hX, Y)\right] \\ & - cg(U, \varphi hX)\eta(Y). \end{aligned} \tag{3.17}$$

If we recall the Hessian operator, that is, $Hess_a(X,Y) = g(\nabla_X Da, Y) = g(\nabla_Y Da, X)$ and using the antisymmetrizing of the preceding equation

$$\begin{aligned} & X(\zeta(a))\eta(Y) + \alpha c\eta(Y)g(X,U) - cg(\varphi hX,U)\eta(Y) - Y(\zeta(a))\eta(X) \\ & -\alpha c\eta(X)g(Y,U) + cg(\varphi hY,U)\eta(X) + 2bcg(\varphi X,Y) \\ = \; & 0. \end{aligned} \tag{3.18}$$

Replacing X by φX and Y by φY in (3.18), we obtain $2bcg(\varphi X,Y) = 0$, then from $b = 0$ ($c \neq 0$). Thus using (3.15), we obtain

$$\zeta(a)U = Da\eta(U). \tag{3.19}$$

On the other hand, since $b = 0$ equation (3.16) reduce to

$$[\zeta(a) + c\eta(U)]\varphi^2 U = 0,$$

which implies either $\varphi^2 U = 0$ or $\zeta(a) = -c\eta(U)$.

Case1. If $\varphi^2 U = 0$, then $U = \eta(U)\zeta$ which implies U is pointwise collinear with ζ. Thus, from Theorem 3.3. the integral submanifold of the distribution D is totally geodesic or totally umbilical.

Case2. If $\zeta(a) = -c\eta(U)$, then from (3.19) we have $(Da + cU)\eta(U) = 0$.Thus, in both cases, $Da = -cU$ obtained. By considering (1.1) and using covariant differentiation, we obtain $\nabla_X Da = -caX$, any $X \in \chi(M)$. By view of ([18]-Theorem 3), condition ii) is proved.

□

In this part, we suppose that (M,φ,ζ,η) is an almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces, namely the Riemannian curvature tensor satisfies (1.2). Furthermore, the following relations are provided.

4. Holomorphically planar conformal vector fields on almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces

Proposition 4.1. *[12] Let M be an almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces. Then the following relations are hold.*

$$h^2 = \left(\kappa + \alpha^2\right)\varphi^2, \text{ for } \kappa \leq -\alpha^2, \tag{4.1}$$

$$\nabla_\zeta h = -\mu\varphi h,$$

$$R(\zeta,X)Y = \kappa\left(g(Y,X)\zeta - \eta(Y)X\right) + \mu\left(g(hY,X)\zeta - \eta(Y)hX\right), \tag{4.2}$$

$$(\nabla_X\varphi)Y = g(\alpha\varphi X + hX, Y)\zeta - \eta(Y)(\alpha\varphi X + hX),$$

for any $X,Y \in \chi(M)$,where h is symmetric operator $h = \frac{1}{2}\pounds_\zeta\varphi$.
From (4.1), we find easily that $\kappa \leq 0$ and $\kappa = 0$ if and only if M is a cosymplectic manifold, thus in the following we always suppose $\kappa < 0$.

Theorem 4.2. *Let (M,φ,ζ,η) be an almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces that admits an HPCV field U, then*

i) the manifold has constant curvature,
ii) the integral manifold of the distribution D is totally geodesic or totally umbilical,
iii) U is an eigenvector of h.

Proof. Using $X = \zeta$ in (3.2), we obtain

$$R(\zeta,Y)U = \zeta(a)Y - Y(a)\zeta + \zeta(b)\varphi Y + b\alpha\varphi Y + bhY. \tag{4.3}$$

By considering the inner product of (4.3) with ζ, we have

$$g(R(\zeta,Y)U,\zeta) = \zeta(a)\eta(Y) - Y(a). \tag{4.4}$$

Using the (4.2) in the preceding equation, we obtain

$$g(R(\zeta,Y)U,\zeta) = \kappa g(Y,U) - \kappa\eta(Y)\eta(U) + \mu g(hY,U). \tag{4.5}$$

Eqs. (4.4)-(4.5) yield to

$$\zeta(a)\eta(Y) - Y(a) = \kappa g(Y,U) - \kappa\eta(Y)\eta(U) + \mu g(hY,U) \tag{4.6}$$

which implies

$$-\kappa\eta(U)\zeta + \mu hY + \kappa U = \zeta(a)\zeta - Da. \tag{4.7}$$

On the other hand,taking the inner product both sides of equation (4.3) with U,

$$\zeta(a)g(Y,U) - Y(a)\eta(U) + \zeta(b)g(\varphi Y,U) + b\alpha g(\varphi Y,U) + bg(hY,U) = 0.$$

Remove Y in preceding equation

$$\zeta(a)U - Da\eta(U) - \zeta(b)\varphi(U) - b\alpha\varphi U + bhU = 0. \tag{4.8}$$

Eliminating Da from (4.7) and (4.8), we have

$$\mu\eta(U)hY + bhU - \zeta(b)\varphi(U) - b\alpha\varphi U - \zeta(a)\varphi^2 U - \kappa\eta(U)\varphi^2 U = 0. \tag{4.9}$$

On the other hand, substituting $Y = \zeta$ and then taking the covariant derivative of equation (4.7) along X, we have

$$-\kappa[\eta(\nabla_X U)\zeta + g(U,\nabla_X\zeta)\zeta + \eta(U)\nabla_X\zeta - \nabla_X U] = X(\zeta(a))\zeta - \zeta(a)\nabla_X\zeta - \nabla_X Da.$$

Then using (1.1) and $AX = -\nabla_X\zeta$ in preceding equation, also taking the inner product with Y, we obtain

$$\begin{aligned} & -\kappa[a\eta(X)\eta(Y) - g(AX,U)\eta(Y) - g(AX,Y)\eta(U) - ag(X,X_2) - bg(\varphi X,Y)] \\ = \; & X(\zeta(a))\eta(Y) + \zeta(a)g(AX,Y) - g(\nabla_X Da,Y) \end{aligned} \tag{4.10}$$

Using the symmetry of the Hessian operator, we have

$$\kappa[2bg(\varphi X,Y) - g(U,AX)\eta(Y) - g(U,AY)\eta(X)] - X(\zeta(a))\eta(Y) - Y(\zeta(a))\eta(X) = 0$$

Replacing X with φX and Y with φY in the previous equation, we obtain that

$$2\kappa bg(\varphi X,Y) = 0$$

which implies $b = 0$ as $\kappa < 0$. Therefore, from equation (4.8), we get

$$\zeta(a)U = (Da)\eta(U). \tag{4.11}$$

Considering that $Y \in [\lambda]'$ in (4.6), we obtain that

$$(\kappa + \mu\lambda)g(Y,U) = -Y(a). \tag{4.12}$$

Substituting $Y = \zeta$ in the last equality, then (4.11) and (4.12) implies that

$$\zeta(a) = -(\kappa + \mu\lambda)\eta(U) \text{ and } Da = -(\kappa + \mu\lambda)U. \tag{4.13}$$

By using equality of $\zeta(a)$ and $b = 0$ in (4.9), we obtain

$$-\zeta(a)\varphi^2 U - \kappa\eta(U)\varphi^2 U + \mu\eta(U)hY = 0$$

which implies

$$\mu\left[\lambda\eta(U)\varphi^2 U + hU\right]\eta(U) = 0.$$

Case1. If $\mu = 0$, then from (4.2) the manifold has constant curvature.

Case2. If $\lambda\eta(U)\varphi^2 U + hU = 0$, then from (2.1), we obtain $hU = \lambda U - \lambda\eta(U)\zeta$. If we apply h to both sides of the equation, we infer that $h^2 U = \lambda hU$. From that, $tr(h^2) = 0$, so we obtain $h = 0$. Under the same conditions of Theorem 3.3 the integral manifold of the distribution D is totally geodesic or totally umbilical.

Case3. If $\eta(U) = 0$, then from (4.13), $\zeta(a) = 0$. Using that value in (4.7), we obtain

$$Da = \kappa\eta(U)\zeta - \mu hY - \kappa U.$$

When this result is considered together with the value of Da in (4.13), we infer that

$$hU = \lambda U.$$

which implies that V is an eingenvector of h.

□

5. Conclusion and discussion

The notion of conformality is an important object that appears in various fields of mathematics (e.g., real and complex analysis, Riemannian geometry, classical geometry) as well as in physics (e.g., general relativity, conformal field theory) and also,has many applications in military aircraft, electronics, cartography, and so on. The notion, which started with conformal functions between Euclidean spaces, conformal maps between Riemannian or semi-Riemannian manifolds, was later extended to conformal vector fields. Recently, it is an important tool used in many mathematical and physical subjects with many special types.
Considering the importance and wide application of this notion, we characterize and classify almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces admitting holomorphically conformal vector fields which a generalization of the conformal vector field. In this direction, many results have been given in the third section and an important characterization of the given structure has been obtained. This study will shed light on our future investigations. Our further studies will be denote applications of some types of conformal vector fields like $\varphi-$holomorphic planar conformal vector fields and Ricci biconformal vector fields.

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Approximate Fuzzy Inverse Matrix Calculation Method using Scenario-based Inverses and Bisection

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Abstract

In this paper, we introduce a numerical method to construct the inverse of a square matrix whose elements are trapezoidal or triangular fuzzy numbers (FNs). A set of fuzzy linear equations is required to be solved in order to determine the fuzzy inverse matrix. The proposed technique first iteratively searches the possible solution intervals and then narrows those too-wide estimated intervals via bisection. Using interval arithmetic in left and right matrix multiplication, we aim to approximate the identity matrix as a result of product operations. The dissimilarity of the endpoints of intervals belonging to multiplication matrices with the identity matrix is considered to be an error function to be minimized. In this way, even if the entries of a matrix are uncertain, the fuzzy inverse matrix containing all inverse matrices can be found quickly with the use of computer technology. The method is explained, and comparisons are drawn with inverse stable examples from the literature.

1. Introduction

Real-world models are too complicated to be presented with accurate values of parameters since precise knowledge is often unavailable. To overcome imprecision, we may substitute crisp numbers with FNs provided by experts. For the purpose of solving fully fuzzy linear systems of equations and fully fuzzy matrix equations, an analysis of the fuzzy inverse matrix is beneficial. The narrowest family of matrices that contain all the inverses is known as the inverse of a given fuzzy matrix, whose entries are FNs. Notice that interval and fuzzy matrices are connected since each $\alpha-$cut of a FN is represented by an interval.

Chanas and Nowakowski [1] developed the fuzzy numerical simulation (FNS) technique that allows one to assign an exact numerical value to a fuzzy variable by generating two values from a uniform distribution. Rohn [2] examined the invertibility of interval matrices by providing necessary and sufficient conditions. For the solution of simultaneous linear equations with fuzzy parameters, Rao and Chen [3] provided a computational methodology based on a search algorithm and bisection method. The idea of finding the inverse matrix using scenario-based inverse matrices was first proposed by Dehghan et al. [4]. Basaran [5] presented a method to calculate the inverse of a fuzzy matrix that consists of $LR-$type FNs. Refer to [7] and [8] for suggested alterations and revisions for [5]. Mosleh [6] used fuzzy neural networks with fuzzy weights for fuzzy inputs and fuzzy targets to find approximate solutions to fully fuzzy matrix equations, which can be utilized to determine fuzzy inverse matrices. Guo et al. [9] designed a numerical procedure to derive the fuzzy inverse of the matrix with $LR-$type FNs and supplied a sufficient condition for its existence. Chen and Huang [10] proposed a mathematical programming model to acquire the fuzzy weights of the fuzzy analytical network process by utilizing the fuzzy inverse matrix on the basis of the criterion of the minimum spread of FNs. Babakordi and Taghi-Nezhad [11] also employed linear programming for the calculation of fuzzy inverse matrix. Farahani and Ebadi [12] discussed the sufficient and necessary conditions for the invertibility of a fuzzy matrix by analyzing a system of fuzzy polynomial equations. Akdemir and Kocken [13] proposed an FNS-based bisection procedure

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for a fuzzy linear regression model with crisp inputs and fuzzy outputs.

Fuzzy matrices whose entries are symmetrical triangular FNs have been considered mostly in the literature. As far as we know, studies on inverse calculations of fuzzy square matrices with asymmetrical and/or trapezoidal fuzzy entries are scarce. Some attempts using approximate operations had high errors. In some cases, inverse stable matrices were not considered. The motivation for this study is to contribute to the literature by proposing a computational technique making use of crisp scenario-based inverses and the bisection method. In this paper, we suggest a new method that first solves a large number of possible equation scenarios and then attains a very large range of solutions. After that, it attempts to reduce the error by narrowing these initial ranges via bisection. For a given square matrix with fuzzy entries, parameters are repeatedly generated via the single-value FNS method, thus the inverses of crisp scenario-based matrices are calculated too many times, different from the approach of Dehghan et al. [4]. This way, we can estimate the intervals in which the inverses are located. But the fuzzinesses of the solutions, namely the lengths of the corresponding ranges of solutions, are too large at the beginning since an excessive number of scenarios is affecting the estimation. After the left and right matrix multiplication, we aim to approximate fuzzy units or fuzzy zeros by increasing the similarity of both sides' endpoints. The fuzzy identity matrix is considered to be a nearly crisp identity matrix.

The rest of this paper is constructed as follows: In the following section, we give fundamental definitions of fuzzy set theory along with the notation we use. In Section 3, we develop a numerical method for the determination of the approximate fuzzy inverse matrix. In Section 4, we provide numerical examples to illustrate our methodology. We conclude in Section 5.

2. Preliminaries

In this section, we review some basic concepts of fuzzy set theory that will be used in the subsequent sections.

Definition 2.1. *A standard fuzzy set $\widetilde{a}$ on the universe of discourse X is defined as*

$$\widetilde{a} = \{\, \langle x, \mu_{\widetilde{a}}(x) \rangle \,|\, x \in X \}$$

where the membership function $\mu_{\widetilde{a}} : X \to [0,1]$ denotes the grade of belonging of the element x to the set $\widetilde{a}$.

A normal fuzzy subset of the real line, i.e., there exists $\exists x \in \mathbb{R}$ such that $\mu(x) = 1$, with a convex membership function is called a FN.

Definition 2.2 ($LR-$type FN)**.** *Let $L,\ R : \mathbb{R}^+ \to [0,1]$ be decreasing functions with $L(0)=1,\ R(0)=1,\ L(1)=0,\ R(1)=0,\ L(x),\ R(x)<1$ for $x>0$. The FN $\widetilde{a}$ is called $LR-$type if there exist $m_L, m_R\ (m_L < m_R), \alpha > 0$ and $\beta > 0$, with the membership function:*

$$\mu_{\widetilde{a}}(x) = \begin{cases} L\left(\frac{m_L - x}{\alpha}\right), & \text{if } x < m_L \\ 1, & \text{if } m_L \le x < m_R \\ R\left(\frac{x - m_R}{\beta}\right), & \text{if } m_R \le x \end{cases}$$

where α, β are called left spread and right spread, respectively. The lengths of the lower and upper ranges α and β are also known as left fuzziness and right fuzziness. Then, we denote $\widetilde{a} = (m_L, m_R, \alpha, \beta)_{LR}$. Also, the endpoints $\widetilde{a}_L = m_L - \alpha$ and $\widetilde{a}_R = m_R + \beta$ are termed as lower and upper bounds, respectively. The interval $[m_L, m_R]$ is additionally referred to as mode interval.

If $L(x) = R(x) = \max\{0, 1-x\}$, then $\widetilde{a}$ is called a Trapezoidal FN. Similarly, if $L(x) = R(x) = \max\{0, 1-x\}$ and $m_L = m_R$, then we get a Triangular FN. Also, sets consisting of these two types of numbers are closed under the operations given in the definition below.

Definition 2.3. *Let $\widetilde{a_1} = (m_L, m_R, \alpha_1, \beta_1)_{LR}$ and $\widetilde{a_2} = (n_L, n_R, \alpha_2, \beta_2)_{LR}$ be two trapezoidal (or triangular) FNs, and $\lambda \in \mathbb{R}$, then we have:*

- *Addition*

$$\widetilde{a_1} + \widetilde{a_2} = (m_L + n_L, m_R + n_R, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR},$$

- *Scalar Multiplication*

$$\lambda \widetilde{a_1} = \begin{cases} (\lambda m_L, \lambda m_R, \lambda \alpha_1, \lambda \beta_1)_{LR}, & \text{if } \lambda \ge 0 \\ (\lambda m_R, \lambda m_L, -\lambda \beta_1, -\lambda \alpha_1)_{LR}, & \text{if } \lambda < 0 \end{cases}$$

- *Multiplication*

$$\widetilde{a_1} \times \widetilde{a_2} = (l, r, \alpha, \beta)_{LR}, \tag{2.1}$$

where the endpoints of the mode interval and bounds are

$$\begin{aligned} l &= \min\{m_L n_L, m_L n_R, m_R n_L, m_R n_R\}, \\ r &= \max\{m_L n_L, m_L n_R, m_R n_L, m_R n_R\}, \\ l-\alpha &= \min\{(m_L-\alpha_1)(n_L-\alpha_2), (m_L-\alpha_1)(n_R+\beta_2), (m_R+\beta_1)(n_L-\alpha_2), (m_R+\beta_1)(n_R+\beta_2)\}, \\ r+\beta &= \max\{(m_L-\alpha_1)(n_L-\alpha_2), (m_L-\alpha_1)(n_R+\beta_2), (m_R+\beta_1)(n_L-\alpha_2), (m_R+\beta_1)(n_R+\beta_2)\}, \end{aligned}$$

respectively.

Definition 2.4. *[5] A FN $\widetilde{1} = (1, 1, \varepsilon, \delta)_{LR}$ is called fuzzy unit if the spreads $\varepsilon, \delta > 0$ are sufficiently small.*

Definition 2.5. *[5] A FN $\widetilde{0} = (0, 0, \rho, \sigma)_{LR}$ is called fuzzy zero if the spreads $\rho, \sigma > 0$ are sufficiently small.*

Definition 2.6. *[5] A fuzzy matrix is a fuzzy identity matrix if its main diagonal and remaining elements are respectively fuzzy units and zeros. It is denoted by $\widetilde{I}$ as follows:*

$$\widetilde{I} = \begin{bmatrix} \widetilde{1} & \widetilde{0} & \dots & \widetilde{0} \\ \widetilde{0} & \widetilde{1} & \dots & \widetilde{0} \\ \vdots & \ddots & \ddots & \vdots \\ \widetilde{0} & \widetilde{0} & \dots & \widetilde{1} \end{bmatrix}.$$

3. Proposed method

We attempt to determine the approximate fuzzy (multiplicative) inverse of a square matrix whose parameters are all arbitrary trapezoidal (or triangular) FNs, i.e., not necessarily non-negative. Consider a square fuzzy matrix $\widetilde{A}$, represented by mutually independent $LR-$type fuzzy entries. If there exists an $LR-$type fuzzy matrix $\widetilde{X}$ whose entries are unknown trapezoidal (or triangular) FNs, such that:

$$\widetilde{A} \times \widetilde{X} = \widetilde{X} \times \widetilde{A} = \widetilde{I},$$

we refer the fuzzy matrix $\widetilde{X}$ is the inverse of the fuzzy matrix $\widetilde{A}$ and denote it by $\widetilde{A}^{-1} = \widetilde{X}$. We apply the interval arithmetic-based equation, Equation (2.1) for multiplication.

First, let us consider the right inverse matrix via $\widetilde{A} \times \widetilde{X} = \widetilde{I}$. For the $LR-$type fuzzy entries $\widetilde{a_{ij}} = \left(m_{ij}^L, m_{ij}^R, \alpha_{ij}, \beta_{ij}\right)_{LR}$ and $LR-$type fuzzy unknowns $\widetilde{x_{jk}} = \left(n_{jk}^L, n_{jk}^R, \tau_{jk}, \phi_{jk}\right)_{LR}$, the following fully fuzzy system of linear equations occurred:

$$\begin{aligned} \sum_{j=1}^{n} \widetilde{a_{ij}} \times \widetilde{x_{jk}} &= \sum_{j=1}^{n} \left(m_{ij}^L, m_{ij}^R, \alpha_{ij}, \beta_{ij}\right)_{LR} \times \left(n_{jk}^L, n_{jk}^R, \tau_{jk}, \phi_{jk}\right)_{LR} \\ &= \begin{cases} (1, 1, \varepsilon_{ik}, \delta_{ik})_{LR}, & \text{if } i = k \\ (0, 0, \rho_{ik}, \sigma_{ik})_{LR}, & \text{otherwise} \end{cases} \end{aligned}$$

where $i, k = 1, 2, ..., n$. Similarly, for the left inverse matrix, the following conditions:

$$\sum_{j=1}^{n} \widetilde{x_{ij}} \times \widetilde{a_{jk}} = \begin{cases} \widetilde{1}, & \text{if } i = k \\ \widetilde{0}, & \text{otherwise} \end{cases}$$

must hold for $i, k = 1, 2, ..., n$. Each system is $n \times n$ by $n \times n$ since there are $n \times n$ unknowns and $n \times n$ equations in total.

The error function we have considered in order to produce close matrices to the fuzzy identity matrix as a result of both left and right multiplication is given as follows:

$$error = \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{r \in \{R, L\}} \left(e_{ikr}^R + e_{ikr}^L\right) \tag{3.1}$$

where

$$e_{ikr}^{R}=\begin{cases}\left|\left(\sum_{j=1}^{n}\widetilde{a_{ij}}\times\widetilde{x_{jk}}\right)_r-1\right|, & \text{if } i=k\\ \left|\left(\sum_{j=1}^{n}\widetilde{a_{ij}}\times\widetilde{x_{jk}}\right)_r-0\right|, & \text{otherwise}\end{cases}\tag{3.2}$$

and

$$e_{ikr}^{L}=\begin{cases}\left|\left(\sum_{j=1}^{n}\widetilde{x_{ij}}\times\widetilde{a_{jk}}\right)_r-1\right|, & \text{if } i=k\\ \left|\left(\sum_{j=1}^{n}\widetilde{x_{ij}}\times\widetilde{a_{jk}}\right)_r-0\right|, & \text{otherwise.}\end{cases}\tag{3.3}$$

Four separate sorts of errors e_{ikR}^{R}, e_{ikL}^{R}, e_{ikR}^{L}, and e_{ikL}^{L} arise for each entry (i,k) of multiplication matrices. The first two of them are a result of the difference between the upper and lower endpoints of the multiplication entry from the right and the corresponding element of the identity matrix. Similarly, the last two of them correspond to left-hand multiplication. In addition, if the spreads ε,δ,ρ, and σ of fuzzy units and zeros are specified as in [5], the error functions (3.2) and (3.3) can be revised accordingly. See also the definition of $\varepsilon-$inverse in [4].
Our approach consists of two phases. At the first one, the lower and upper bounds of potential inverse matrices are searched, and then the absolute error values are calculated. The bisection algorithm is then applied, starting with the spread with the largest total error $\left(e_{ikr}^{R}+e_{ikr}^{L}\right)$ which belongs to the entry (i,k) having the highest r fuzziness, such that $r=R$ and $r=L$ correspond to right and left, respectively. With a specified number of bisection iterations, the original spread is narrowed to a specific extent depending on whether or not the overall error value decreases. The opted narrowing gives a minimum overall error value, amongst others. The bisection process is then performed for the spread with the second-largest inaccuracy. When all spreads' errors are decreased, the process is finished. Let us now explain those two phases of our method separately in Subsections 3.1 and 3.2.

3.1. Search algorithm based on FNS

The foundation of our search algorithm is based on the generation of single values of FNs. We can estimate the intervals in which the inverse entries are located by computing the inverses of crisp matrices, which are obtained by substituting the generated values of the fuzzy parameters. It is obvious that this process needs to be repeated too many times in order to yield meaningful results. The literature contains a wide variety of efficient crisp techniques for calculating inverses or pseudo-inverses. We therefore try to employ these non-fuzzy methods to offer more accurate estimates. To generate given $LR-$type fuzzy entry $\widetilde{a_{ij}}=\left(m_{ij}^{L},m_{ij}^{R},\alpha_{ij},\beta_{ij}\right)_{LR}$, we first simulate two independent uniformly distributed random numbers, u_{ij} and v_{ij}. The weighted arithmetic mean of the endpoints of $u_{ij}-$cut interval is a sample of $\widetilde{a_{ij}}$ as follows:

$$\overline{a_{ij}}=v_{ij}\left[m_{ij}^{L}-\alpha_{ij}L^{-1}\left(u_{ij}\right)\right]+\left(1-v_{ij}\right)\left[\beta_{ij}R^{-1}\left(u_{ij}\right)+m_{ij}^{R}\right]\tag{3.4}$$

where v_{ij} and $(1-v_{ij})$ are considered as weights for $i,j=1,2,\ldots,n$.
The algorithm initially calculates 1-cut inverse matrices by Rohn's method [2], if any. If these inverses are inconsistent, the process is terminated. To find the lower and upper bounds of the inverse matrix's entries, the process first generates single values of the fuzzy parameters and repeatedly computes the inverses of the resulting crisp matrices. For any entry, it is regarded as a potential value for the lower (upper) bound if the value is less (greater) than the left (right) endpoint of the mode interval of the 1-cut inverse. After a large number of endpoints are determined and then averaged, the infimum and supremum of these mean values are used to derive the right and left endpoints, respectively. These bounds, as well as errors, which are too big initially, are required for the execution of the second phase, namely the bisection algorithm. Refer to Algorithm 1 for this phase of our method. Algorithm 1 is a similar process to the search algorithm in [3]. The sequence of the simulation repetitions is represented by the iteration counters s and t, which are integers from 1 to S and 1 to T, respectively.

3.2. Bisection algorithm

The bisection algorithm is applied to the spreads in accordance with the order in which the total errors $\left(e_{ikr}^{R}+e_{ikr}^{L}\right)$, which are obtained at the end of Algorithm 1, are listed in descending order. A one-to-one mapping between the total errors and unknowns is considered. For example, an error $\left(e_{ikr}^{R}+e_{ikr}^{L}\right)$ leads to an update in variable $\left(\widetilde{x_{jk}}\right)_r$ where $j=i$. The fuzzy matrix

Algorithm 1 FNS-based Search Algorithm

INPUT: $LR-$type fuzzy entries $\widetilde{a_{ij}} = \left(m_{ij}^L, m_{ij}^R, \alpha_{ij}, \beta_{ij}\right)_{LR}$ where $i, j = 1, 2, \ldots, n$, and large positive integers T, S

OUTPUT: $LR-$type fuzzy unknowns $\widetilde{x_{jk}} = \left(n_{jk}^L, n_{jk}^R, \tau_{jk}, \phi_{jk}\right)_{LR}$ and errors e_{ikR}^R, e_{ikL}^R, e_{ikR}^L, and e_{ikL}^L where $i, j, k = 1, 2, \ldots, n$

1: **procedure** FNSBASEDSEARCH
2: Compute the inverse matrix of the interval matrix that accepts mode intervals $\left[m_{ij}^L, m_{ij}^R\right]$ as entries by using Rohn's method. Continue if the solutions are consistent, namely $n_{jk}^L \leq n_{jk}^R$ for all $j, k = 1, 2, \ldots, n$. Otherwise, report that the inverse matrix does not exist and break.
3: **for** $t \leftarrow 1, T$ **do**
4: **for** $s \leftarrow 1, S$ **do**
5: Simulate $\overline{A}$ using the Formula (3.4)
6: Determine $\left(\overline{A}\right)^{-1}$ using matrix inversion and store the entries as $\widetilde{x_{jk}}^s$ for all $j, k = 1, 2, \ldots, n$
7: **end for**
8: For each j and k, store the mean value of the $\widetilde{x_{jk}}^s$ values less (or similarly greater) than n_{jk}^L (or respectively n_{jk}^R) in $\left(\widetilde{x_{jk}}\right)_L^t$ (or $\left(\widetilde{x_{jk}}\right)_R^t$).
9: **end for**
10: Set $\left(\widetilde{x_{jk}}\right)_L := \max\limits_t \left(\widetilde{x_{jk}}\right)_L^t$ and $\left(\widetilde{x_{jk}}\right)_R := \min\limits_t \left(\widetilde{x_{jk}}\right)_R^t$ for all $j, k = 1, 2, \ldots, n$
11: Set the fuzzinesses as $\tau_{jk} := n_{jk}^L - \left(\widetilde{x_{jk}}\right)_L$ and $\phi_{jk} := \left(\widetilde{x_{jk}}\right)_R - n_{jk}^R$ for all $j, k = 1, 2, \ldots, n$
12: Using Eqs. (3.2) and (3.3), calculate the errors e_{ikR}^R, e_{ikL}^R, e_{ikR}^L, and e_{ikL}^L for the attained inverse
13: **end procedure**

under consideration must therefore be square. Our method consists of applying the bisection algorithm $2n^2$ times after the search algorithm, Algorithm 1. For the obtained inverse after running Algorithm 1, denote the left endpoint matrix as $(\widetilde{x})_L$ and the right endpoint matrix as $(\widetilde{x})_R$. Similarly, denote the matrix consisting of left endpoints of the mode intervals as n^L and the matrix consisting of right endpoints of the mode intervals as n^R, if existed. Refer to Algorithm 2 for this phase of our method. The sequence of the bisection repetitions is expressed by the loop counter m, which ranges from 1 to M. ω, on the other hand, is the order of the bisection iteration with the lowest error value.

4. Illustrative examples

On a computer running MS Windows 10 Pro and equipped with an Intel Core i5-7400 CPU (3.00 GHz) and 4 GB of RAM, all computational tests are carried out using MATLAB R2019a. We use the assumptions that $T = S = 1000$ and $M = 20$ in our computations. MATLAB's "rand" function is used as a pseudo-random generator. We display the values rounded to four significant digits. We take into consideration the following instances to illustrate the method suggested in this paper:

Example 4.1. *Our first example is from [4]. Consider the following fuzzy matrix with triangular entries:*

$$\widetilde{A} = \begin{bmatrix} (80,80,4,8)_{LR} & (25,25,5,5)_{LR} \\ (50,50,10,6)_{LR} & (120,120,6,10)_{LR} \end{bmatrix}.$$

We do not need to use Rohn's method since the entries are triangular FNs and mode intervals consist of only one number. Therefore, inconsistency is never an issue at the line-2 of Algorithm 1, so we simply calculate the inverse of a crisp matrix if it exists. The inverse matrix of the mode matrix is as follows:

$$n^L = n^R = \begin{bmatrix} 80 & 25 \\ 50 & 120 \end{bmatrix}^{-1} = \begin{bmatrix} 0.0144 & -0.0030 \\ -0.0060 & 0.0096 \end{bmatrix}.$$

Dehghan et al. [4] reported the left and right spread matrices for the inverse matrix as follows:

$$\left(\tau_{jk}\right)_1 = \begin{bmatrix} 0.0030 & 0.0013 \\ 0.0021 & 0.0019 \end{bmatrix} \text{ and } \left(\phi_{jk}\right)_1 = \begin{bmatrix} 0.0020 & 0.0016 \\ 0.0032 & 0.0013 \end{bmatrix},$$

respectively. Additionally, the following corresponding spreads are obtained with Rohn's method [2]:

$$\left(\tau_{jk}\right)_2 = \begin{bmatrix} 0.0022 & 0.0013 \\ 0.0020 & 0.0013 \end{bmatrix} \text{ and } \left(\phi_{jk}\right)_2 = \begin{bmatrix} 0.0019 & 0.0011 \\ 0.0022 & 0.0013 \end{bmatrix}.$$

Without applying the bisection algorithm, we find those spreads as follows:

$$\left(\tau_{jk}\right)_3 = \begin{bmatrix} 0.0004 & 0.0002 \\ 0.0002 & 0.0002 \end{bmatrix} \text{ and } \left(\phi_{jk}\right)_3 = \begin{bmatrix} 0.0002 & 0.0002 \\ 0.0004 & 0.0001 \end{bmatrix}.$$

Algorithm 2 Bisection Algorithm

INPUT: Left and right endpoint matrices $(\widetilde{x})_L$, $(\widetilde{x})_R$, boundaries of modes n^L, n^R, errors computed in Algorithm 1, and number of bisection repetitions M
OUTPUT: Updated *error* (3.1), and endpoint $\left(\widetilde{x_{jk}}\right)_r$ where $j = i$

```
procedure BISECTION(i, k, r)
    Initialize h_L = 0 and h_R = 1 (If the total error (e^R_{ikr} + e^L_{ikr}) is relatively much higher than the other ones, then initialize
    h_L = 0.5 and h_R = 1 to speed up the process)
    Set m := 1, (x̃_jk)^{new}_{r,m} := (x̃_jk)_r and error_m := error computed via Eq. (3.1)
    for m ← 2, M do
        h := (h_L + h_R)/2
        (x̃_jk)^{new}_{r,m} := h n^r_{jk} + (1 − h) (x̃_jk)_r
        Update the error using the new value (x̃_jk)^{new}_{r,m} instead of (x̃_jk)_r, and store it in error_m
        if error_m < error_(m−1) then
            h_L := h
        else
            h_R := h
        end if
    end for
    Return (x̃_jk)_r = (x̃_jk)^{new}_{r,ω} such that error_ω = min_{m∈{1,2,...,M}} error_m
    Update error
end procedure
```

The authors [4] used $\alpha-$cuts to view the fuzzy matrix as an interval matrix and investigated five scenario-based matrices for $\alpha \in \{0, 0.25, 0.50, 0.75, 1\}$. In the first phase, on the other hand, we perform $T \times S = 10^6$ inverse calculations.
After the bisection algorithm, we obtain the following spreads:

$$\left(\tau_{jk}\right)_4 = \begin{bmatrix} 7.8317e-10 & 3.0507e-10 \\ 3.8743e-10 & 4.6294e-10 \end{bmatrix} \text{ and } \left(\phi_{jk}\right)_4 = \begin{bmatrix} 3.8099e-10 & 3.8117e-10 \\ 7.8932e-10 & 2.6150e-10 \end{bmatrix}.$$

The right and left multiplications of the inverse matrices obtained by the four methods mentioned above with the matrix $\widetilde{A}$ are calculated as follows:

$$\begin{aligned}
\left(\widetilde{A} \times \widetilde{X}\right)_1 &= \begin{bmatrix} [0.6234, 1.3872] & [-0.2244, 0.2206] \\ [-0.5970, 0.5992] & [0.6370, 1.3610] \end{bmatrix}, \\
\left(\widetilde{X} \times \widetilde{A}\right)_1 &= \begin{bmatrix} [0.6256, 1.3872] & [-0.3310, 0.3324] \\ [-0.4048, 0.3976] & [0.6348, 1.3610] \end{bmatrix}, \\
\left(\widetilde{A} \times \widetilde{X}\right)_2 &= \begin{bmatrix} [0.6872, 1.3584] & [-0.2124, 0.1826] \\ [-0.5520, 0.4796] & [0.7054, 1.3410] \end{bmatrix}, \\
\left(\widetilde{X} \times \widetilde{A}\right)_2 &= \begin{bmatrix} [0.6864, 1.3584] & [-0.3150, 0.2724] \\ [-0.3720, 0.3216] & [0.7062, 1.3410] \end{bmatrix}, \\
\left(\widetilde{A} \times \widetilde{X}\right)_3 &= \begin{bmatrix} [0.8753, 1.1708] & [-0.0908, 0.0792] \\ [-0.2464, 0.1805] & [0.8879, 1.1516] \end{bmatrix}, \\
\left(\widetilde{X} \times \widetilde{A}\right)_3 &= \begin{bmatrix} [0.8844, 1.1705] & [-0.1308, 0.1186] \\ [-0.1713, 0.1206] & [0.8788, 1.1518] \end{bmatrix}, \\
\left(\widetilde{A} \times \widetilde{X}\right)_4 &= \begin{bmatrix} [0.9126, 1.1449] & [-0.0719, 0.0599] \\ [-0.2036, 0.1222] & [0.9246, 1.1257] \end{bmatrix}, \\
\left(\widetilde{X} \times \widetilde{A}\right)_4 &= \begin{bmatrix} [0.9246, 1.1449] & [-0.1018, 0.0898] \\ [-0.1437, 0.0814] & [0.9126, 1.1257] \end{bmatrix}.
\end{aligned}$$

The sum of errors are 6.0826, 5.3212, 2.2564, and 1.7413, respectively. If we make an interpretation based on these findings, our method allows us to get better results that are more similar to the crisp identity matrix than the other two methods.
Now let us discuss some details of the implementation of the method. In the first phase, we reach the initial errors

$$\begin{aligned}
e^R_{11L} + e^L_{11L} = 0.2403, &\quad e^R_{11R} + e^L_{11R} = 0.3412, \\
e^R_{12L} + e^L_{12L} = 0.2216, &\quad e^R_{12R} + e^L_{12R} = 0.1978, \\
e^R_{21L} + e^L_{21L} = 0.4177, &\quad e^R_{21R} + e^L_{21R} = 0.3011, \\
e^R_{22L} + e^L_{22L} = 0.2333, &\quad e^R_{22R} + e^L_{22R} = 0.3034,
\end{aligned}$$

which measure lower and upper bound dissimilarities with the identity matrix. Thus, the bisection order to apply to spreads is as follows: $n^L_{21}-(\widetilde{x_{21}})_L$, $(\widetilde{x_{11}})_R-n^R_{11}$, $(\widetilde{x_{22}})_R-n^R_{22}$, $(\widetilde{x_{21}})_R-n^R_{21}$, $n^L_{11}-(\widetilde{x_{11}})_L$, $n^L_{22}-(\widetilde{x_{22}})_L$, $n^L_{12}-(\widetilde{x_{12}})_L$, $(\widetilde{x_{12}})_R-n^R_{12}$.

Example 4.2. *Our second example is from [5]. Consider the following fuzzy matrix with triangular entries:*

$$\widetilde{A}=\begin{bmatrix} (10,10,4,4)_{LR} & (8,8,3,3)_{LR} \\ (6,6,2,2)_{LR} & (4,4,3,3)_{LR} \end{bmatrix}.$$

Basaran [5] specified the fuzzy identity matrix as follows:

$$\widetilde{I}=\begin{bmatrix} (1,1,0.5,0.5)_{LR} & (0,0,0.5,0.5)_{LR} \\ (0,0,0.5,0.5)_{LR} & (1,1,0.5,0.5)_{LR} \end{bmatrix}.$$

The inverse matrices found in [5], [7], [8], and with the proposed method are respectively given as follows:

$$\begin{aligned}
\left(\widetilde{A}^{-1}\right)_1 &= \begin{bmatrix} (-0.5,-0.5,0.88,0.88)_{LR} & (1,1,2.13,2.13)_{LR} \\ (0.75,0.75,1.63,1.63)_{LR} & (-1.25,-1.25,2.57,2.57)_{LR} \end{bmatrix},\\
\left(\widetilde{A}^{-1}\right)_2 &= \begin{bmatrix} (-0.5,-0.5,0.875,0.875)_{LR} & (1,1,1.625,1.625)_{LR} \\ (0.75,0.75,0.625,0.625)_{LR} & (-1.25,-1.25,1.125,1.125)_{LR} \end{bmatrix},\\
\left(\widetilde{A}^{-1}\right)_3 &= \begin{bmatrix} (-0.5,-0.5,0.5,0.11)_{LR} & (1,1,1,0.65)_{LR} \\ (0.75,0.75,0.75,0.35)_{LR} & (-1.25,-1.25,1.25,0.52)_{LR} \end{bmatrix},\\
\left(\widetilde{A}^{-1}\right)_4 &= \begin{bmatrix} (-0.5,-0.5,2.0191e-06,1.4934e-06)_{LR} & (1,1,2.5932e-06,3.6078e-06)_{LR} \\ (0.75,0.75,1.9759e-06,2.8142e-06)_{LR} & (-1.25,-1.25,4.9302e-06,3.4408e-06)_{LR} \end{bmatrix}.
\end{aligned}$$

The right and left multiplications of the inverse matrices obtained by the four methods given above with the matrix $\widetilde{A}$ *are calculated as follows:*

$$\begin{aligned}
\left(\widetilde{A}\times\widetilde{X}\right)_1 &= \begin{bmatrix} [-29.0000,31.5000] & [-57.8400,58.3400] \\ [-17.2000,19.7000] & [-35.7800,34.2800] \end{bmatrix},\\
\left(\widetilde{X}\times\widetilde{A}\right)_1 &= \begin{bmatrix} [-28.3600,30.3600] & [-23.0900,26.0900] \\ [-42.8800,43.8800] & [-36.4200,35.4200] \end{bmatrix},\\
\left(\widetilde{A}\times\widetilde{X}\right)_2 &= \begin{bmatrix} [-18.6250,20.3750] & [-34.8750,36.1250] \\ [-10.8750,12.6250] & [-21.6250,20.8750] \end{bmatrix},\\
\left(\widetilde{X}\times\widetilde{A}\right)_2 &= \begin{bmatrix} [-24.2500,26.2500] & [-19.5000,22.5000] \\ [-18.2500,18.7500] & [-16.0000,15.0000] \end{bmatrix},\\
\left(\widetilde{A}\times\widetilde{X}\right)_3 &= \begin{bmatrix} [-14.0000,9.7600] & [-27.5000,19.4500] \\ [-8.0000,6.1400] & [-17.5000,12.4700] \end{bmatrix},\\
\left(\widetilde{X}\times\widetilde{A}\right)_3 &= \begin{bmatrix} [-14.0000,10.8600] & [-11.0000,9.6000] \\ [-20.0000,12.4800] & [-17.5000,11.3700] \end{bmatrix},\\
\left(\widetilde{A}\times\widetilde{X}\right)_4 &= \begin{bmatrix} [-3.2500,5.2500] & [-7.7501,7.7501] \\ [-3.2500,3.2500] & [-4.7500,6.7500] \end{bmatrix},\\
\left(\widetilde{X}\times\widetilde{A}\right)_4 &= \begin{bmatrix} [-3.0000,5.0000] & [-4.5000,4.5000] \\ [-5.5001,5.5001] & [-5.0000,7.0000] \end{bmatrix}.
\end{aligned}$$

The sum of errors are 542.1400, 328.5000, 213.6300, and 74.0007, respectively.

Example 4.3. *Our third example is from [5]. Consider the following fuzzy matrix with triangular entries:*

$$\widetilde{A}=\begin{bmatrix} (-6,-6,3,3)_{LR} & (-4,-4,2,2)_{LR} \\ (-4,-4,1,1)_{LR} & (-3,-3,1,1)_{LR} \end{bmatrix}.$$

with negative modes. With the same specified fuzzy identity matrix, the inverse matrices found in [5], [7], and with the proposed method are respectively given as follows:

$$\begin{aligned}
\left(\widetilde{A}^{-1}\right)_1 &= \begin{bmatrix} (-1.5,-1.5,1.5,1.5)_{LR} & (2,2,2,2)_{LR} \\ (2,2,2.25,2.25)_{LR} & (-3,-3,3.5,3.5)_{LR} \end{bmatrix},\\
\left(\widetilde{A}^{-1}\right)_2 &= \begin{bmatrix} (-1.5,-1.5,6,6)_{LR} & (2,2,8.25,8.25)_{LR} \\ (2,2,7,7)_{LR} & (-3,-3,9.5,9.5)_{LR} \end{bmatrix},\\
\left(\widetilde{A}^{-1}\right)_3 &= \begin{bmatrix} (-1.5,-1.5,4.6516e-06,3.9282e-06)_{LR} & (2,2,5.5567e-06,6.5507e-06)_{LR} \\ (2,2,5.3942e-06,6.9193e-06)_{LR} & (-3,-3,9.7823e-06,7.6952e-06)_{LR} \end{bmatrix}.
\end{aligned}$$

The sum of errors are 416.5000, 1.4305e+03, and 105.0009, respectively.

Example 4.4. *Our fourth example is from [11]. Consider the following fuzzy matrix with triangular entries:*

$$\widetilde{A}=\begin{bmatrix}(5,5,1,1)_{LR} & (6,6,2,2)_{LR}\\ (4,4,2,2)_{LR} & (7,7,1,1)_{LR}\end{bmatrix}.$$

Babakordi and Taghi-Nezhad [11] reported the following inverse as a non-fuzzy matrix:

$$\widetilde{A}^{-1}=\begin{bmatrix}(0.6364,0.6364,0,0)_{LR} & (-0.5455,-0.5455,0,0)_{LR}\\ (-0.3636,-0.3636,0,0)_{LR} & (0.4545,0.4545,0,0)_{LR}\end{bmatrix}.$$

Their total error is 24.0000. Our inverse with a total error of 24.0001 is as follows:

$$\widetilde{A}^{-1}=\begin{bmatrix}(0.6364,0.6364,0.6306e-06,1.0550e-06)_{LR} & (-0.5455,-0.5455,1.0876e-06,0.6242e-06)_{LR}\\ (-0.3636,-0.3636,0.7781e-06,0.4623e-06)_{LR} & (0.4545,0.4545,0.4354e-06,0.7651e-06)_{LR}\end{bmatrix}.$$

The fuzzy matrix considered here has a non-fuzzy inverse, which is the case where we observe the minimum possible error value. Our method approximates this error.

Example 4.5. *Our next example is from [6]. Consider the following fuzzy matrix with triangular entries:*

$$\widetilde{A}=\begin{bmatrix}(10,10,0.1,0.1)_{LR} & (8,8,0.1,0.01)_{LR}\\ (6,6,0.15,0.1)_{LR} & (4,4,0.09,0.19)_{LR}\end{bmatrix}$$

with relatively small spreads. The author [6] specified the fuzzy identity matrix as follows:

$$\widetilde{I}=\begin{bmatrix}(1,1,0.78,0.4157)_{LR} & (0,0,0.5,0.642)_{LR}\\ (0,0,0.5,0.4183)_{LR} & (1,1,0.6038,0.43)_{LR}\end{bmatrix}.$$

The following fuzzy inverse matrix with an overall error of 0.7035 was given as a result:

$$\widetilde{A}^{-1}=\begin{bmatrix}(-0.5,-0.5,0.05,0.02)_{LR} & (1,1,0.02,0.01)_{LR}\\ (0.75,0.75,0.02,0.02)_{LR} & (-1.25,-1.25,0.023,0.04)_{LR}\end{bmatrix}.$$

With our approximation having a total error of 1.0160, the following inverse matrix is shown below:

$$\widetilde{A}^{-1}=\begin{bmatrix}(-0.5,-0.5,0.0526,0.0163)_{LR} & (1,1,0.0000,0.0115)_{LR}\\ (0.75,0.75,0.0152,0.0169)_{LR} & (-1.25,-1.25,0.0298,0.0369)_{LR}\end{bmatrix}.$$

The 0-cuts of the inverses are respectively given as:

$$\begin{bmatrix}[-0.5500,-0.4800] & [0.9800,1.0100]\\ [0.7300,0.7700] & [-1.2730,-1.2100]\end{bmatrix} \text{ and } \begin{bmatrix}[-0.5526,-0.4837] & [1.0000,1.0115]\\ [0.7348,0.7669] & [-1.2798,-1.2131]\end{bmatrix},$$

which end up being quite similar despite not being able to achieve a better error value.

Example 4.6. *Our last example is adapted from [14]. Consider the following fuzzy matrix with trapezoidal entries:*

$$\widetilde{A}=\begin{bmatrix}(2,4,3,1)_{LR} & (-2,1,0,5)_{LR}\\ (-1,2,4,4)_{LR} & (2,4,2,0)_{LR}\end{bmatrix}.$$

According to [2] and [15], we only need to consider the inverses of the following permutations:

$$A_{11}=\begin{bmatrix}2 & -2\\ -1 & 2\end{bmatrix}, A_{12}=\begin{bmatrix}4 & 1\\ 2 & 4\end{bmatrix}, A_{13}=\begin{bmatrix}4 & -2\\ 2 & 2\end{bmatrix}, A_{14}=\begin{bmatrix}2 & 1\\ -1 & 4\end{bmatrix},$$
$$A_{31}=\begin{bmatrix}4 & 1\\ -1 & 2\end{bmatrix}, A_{32}=\begin{bmatrix}2 & -2\\ 2 & 4\end{bmatrix}, A_{33}=\begin{bmatrix}2 & 1\\ 2 & 2\end{bmatrix}, A_{34}=\begin{bmatrix}4 & -2\\ -1 & 4\end{bmatrix}$$

where

$$\begin{aligned} A_I &= \begin{bmatrix}[2,4] & [-2,1]\\ [-1,2] & [2,4]\end{bmatrix}, \underline{A}=\begin{bmatrix}2 & -2\\ -1 & 2\end{bmatrix}, \overline{A}=\begin{bmatrix}4 & 1\\ 2 & 4\end{bmatrix}\\ A_c &= \frac{\overline{A}+\underline{A}}{2}=\begin{bmatrix}3 & -0.5\\ 0.5 & 3\end{bmatrix}, \Delta=\frac{\overline{A}-\underline{A}}{2}=\begin{bmatrix}1 & 1.5\\ 1.5 & 1\end{bmatrix},\\ A_{yz}=A_c-T_y\Delta T_z\,,\ T_1 &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, T_2=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}, T_3=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, T_4=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.\end{aligned}$$

Thus, the inverse matrix of the interval matrix is as follows:

$$A_I^{-1}=\left[\min_{(y,z)}A_{yz}^{-1},\max_{(y,z)}A_{yz}^{-1}\right]=\begin{bmatrix}[1/6,1] & [-1/2,1]\\ [-1,1/2] & [1/6,1]\end{bmatrix}.$$

By implementing our algorithm, the following inverse matrix with a total error of 125.0003 is computed as follows:

$$\widetilde{A}^{-1}=\begin{bmatrix}(1/6,1,0.2846,0.2425e-05)_{LR} & (-1/2,1,0.0000,0.1855e-05)_{LR}\\ (-1,1/2,0.0000,0.1881e-05)_{LR} & (1/6,1,0.0936,0.2233e-05)_{LR}\end{bmatrix}.$$

The right and left multiplications of the inverse matrix obtained by our method with the matrix $\widetilde{A}$ are calculated as follows:

$$\begin{aligned}\widetilde{A}\times\widetilde{X} &= \begin{bmatrix}[-7.0000,8.0000] & [-4.5000,11.0000]\\ [-9.0000,8.0000] & [-5.0000,10.0000]\end{bmatrix},\\ \widetilde{X}\times\widetilde{A} &= \begin{bmatrix}[-6.0000,11.0000] & [-4.0000,10.0000]\\ [-10.0000,8.5000] & [-6.0000,7.0000]\end{bmatrix}.\end{aligned}$$

Notice that the error of the Rohn method, 125, is a lower bound for our overall error. The fact that the error term is very close to the lower bound shows the efficiency of the method.

5. Conclusion

We provide a computational approach to find the multiplicative inverse of a square fuzzy matrix where the components are triangular or trapezoidal FNs. This method first obtains a large number of potential inverses to get intervals for the entries. After that, it tries to reduce the error by making these ranges smaller. The algorithm is implemented and successfully tested by solving several examples. At first glance, it seems that our algorithm reduces the total error of the approximate inverses obtained with the existing techniques. It is an effort to improve efficacy by making use of well-known, reliable, efficient, and non-fuzzy techniques. In the literature, fuzzy matrices whose entries are symmetrical triangular FNs have been considered mostly. But, inverse calculations of square fuzzy matrices with asymmetrical and/or trapezoidal fuzzy entries can be achieved with our methodology. Moreover, we are able to reduce the error because we employ interval arithmetic rather than approximate multiplication operations. Future developments of this study include the investigation of solving linear and non-linear equation systems with standard, intuitionistic, or picture fuzzy coefficients and extensional work into the applications of the proposed approach to other areas, such as mathematical programming.

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The data used to support the findings of this study are included within the article. All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Some Important Properties of Almost Kenmotsu $(\kappa,\mu,\nu)-$Space on the Concircular Curvature Tensor

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Abstract

In this article, pseudoparallel submanifolds for almost Kenmotsu $(\kappa,\mu,\nu)-$space are investigated. The almost Kenmotsu $(\kappa,\mu,\nu)-$space is considered on the concircular curvature tensor. Submanifolds of these manifolds with properties such as concircular pseudoparallel, concircular $2-$pseudoparallel, concircular Ricci generalized pseudoparallel, and concircular $2-$Ricci generalized pseudoparallel has been characterized. Necessary and sufficient conditions are given for the invariant submanifolds of almost Kenmotsu $(\kappa,\mu,\nu)-$space to be total geodesic according to the behavior of the κ,μ,ν functions.

1. Introduction

Let $\left(M^{2n+1},\phi,\xi,\eta,g\right)$ be a $(2n+1)-$dimensional contact metric manifold. We know that here R is the curvature tensor, ξ is the characteristic vector field and the condition $R(\rho_1,\rho_2)\xi=0$ is satisfied, for any vector field $\rho_1,\rho_2\in M^{2n+1}$. The contact metric manifold that satisfies this condition also satisfies the condition

$$R(\rho_1,\rho_2)\xi=\eta(\rho_2)(\kappa I+\mu h)\rho_1-\eta(\rho_1)(\kappa I+\mu h)\rho_2, \tag{1.1}$$

and this condition is called (κ,μ) nullity condition, where κ,μ are constants and h is the self adjoint $(1,1)-$tensor field. E. Boeckx in [1] and D. E. Blair et al. in [2], (κ,μ) nullity conditions on contact metric manifolds are considered when κ and μ are constant. E. Boeckx proved that non-Sasakian contact metric manifold is completely determined locally by its dimension for the constant values of κ and μ. If vector field ξ relate to the $(\kappa,\mu)-$nullity distribution, then (1.1) is provided and the manifold $\left(M^{2n+1},\phi,\xi,\eta,g\right)$ is described $(\kappa,\mu)-$contact metric manifold.
In particular, if κ and μ are not constant smooth functions on M^{2n+1}, then the manifold $\left(M^{2n+1},\phi,\xi,\eta,g\right)$ is described generalized $(\kappa,\mu)-$contact metric manifold [2].
T. Koufogiorgos et al. introduced $(\kappa,\mu,\nu)-$contact metric manifold in [3]. Riemann curvature tensor of $(\kappa,\mu,\nu)-$contact metric manifoldis in the form

$$\tilde{R}(\rho_1,\rho_2)\xi=\kappa[\eta(\rho_2)\rho_1-\eta(\rho_1)\rho_2]+\mu[\eta(\rho_2)h\rho_1-\eta(\rho_1)h\rho_2]+\nu[\eta(\rho_2)\phi h\rho_1-\eta(\rho_1)\phi h\rho_2], \tag{1.2}$$

for all $\rho_1,\rho_2\in\Gamma(TM)$, where κ,μ,ν are smooth functions on M^{2n+1}.
If $d\eta=0$ and $d\Phi=2\eta\wedge\Phi$, then this manifold is an almost Kenmotsu manifold, where $\Phi(\rho_1,\rho_2)=g(\rho_1,\phi\rho_2)$ is the fundamental $2-$form of M^{2n+1}. If an almost Kenmotsu manifold provide a $(\kappa,\mu,\nu)-$nullity distribution, it is described an almost Kenmotsu $(\kappa,\mu,\nu)-$space [4].

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Later on, manifolds that do not have a contact metric structure but satisfy condition (1.2) have been studied. The almost cosymplectic $(\kappa,\mu,\nu)-$space is defined by P. Dacko and Z. Olszak in [5]. M. Atçeken obtained very important properties of almost Kenmotsu $(\kappa,\mu,\nu)-$space in [6]. Pseudoparallel submanifolds of many different structures have been investigated in [7–18].
The concept of submanifold for a manifold is quite interesting. For example, it plays a very important role in fields such as applied mathematics, analysis and physics, contributing to the illumination of these fields.
In this article, pseudoparallel submanifolds for almost Kenmotsu $(\kappa,\mu,\nu)-$space are investigated. The almost Kenmotsu $(\kappa,\mu,\nu)-$space is considered on the concircular curvature tensor. Submanifolds of these manifolds with properties such as concircular pseudoparallel, concircular $2-$pseudoparallel, concircular Ricci generalized pseudoparallel, and concircular $2-$Ricci generalized pseudoparallel has been characterized. Necessary and sufficient conditions are given for the invariant submanifolds of almost Kenmotsu $(\kappa,\mu,\nu)-$space to be total geodesic according to the behavior of the κ,μ,ν functions.

2. Preliminary

Let $\tilde{N}$ be $(2n+1)-$dimensional contact metric manifold. This manifold admits an almost contact metric structure (ϕ,ξ,η,g) such that

$$\phi^2\rho_1=-\rho_1+\eta(\rho_1)\xi,\quad \eta(\rho_1)=g(\rho_1,\xi),\quad \eta(\xi)=1,\eta\circ\phi=0, \tag{2.1}$$

$$g(\phi\rho_1,\phi\rho_2)=g(\rho_1,\rho_2)-\eta(\rho_1)\eta(\rho_2), \tag{2.2}$$

for all vector fields $\rho_1,\rho_2\in\Gamma\left(T\tilde{N}\right)$, where $\Gamma\left(T\tilde{N}\right)$ denotes the set of differentiable vector fields on $\tilde{N}$ [3]. $\tilde{N}$ together with the (ϕ,ξ,η,g) is called a contact metric manifold.
The Riemannian curvature tensor $\tilde{R}$ of $\tilde{N}$ is given

$$\tilde{R}(\rho_1,\rho_2)=\tilde{\nabla}_{\rho_1}\tilde{\nabla}_{\rho_2}-\tilde{\nabla}_{\rho_2}\tilde{\nabla}_{\rho_1}-\tilde{\nabla}_{[\rho_1,\rho_2]},$$

for all $\rho_1,\rho_2\in\Gamma\left(T\tilde{N}\right)$, where $\tilde{\nabla}$ is the Levi-Civita connection of g.
Let h be tensor field $(1,1)-$type and l_ξ be the Lie-derivative in the direction of ξ. Thus, we can write

$$2h\rho_1=\left(l_\xi\phi\right)\rho_1,$$

for all $\rho_1\in\Gamma\left(T\tilde{N}\right)$. On the other hand h is self-adjoint and satisfies

$$\phi h+h\phi=0, trh=tr\phi h=0, h\xi=0. \tag{2.3}$$

In addition, contact metric manifolds provide the formula given by

$$\tilde{\nabla}_{\rho_1}\xi=\phi\rho_1-\phi h\rho_1, \tilde{\nabla}_\xi\phi=0. \tag{2.4}$$

The $(\kappa,\mu)-$nullity distribution of a contact metric manifold $\tilde{N}$ for the pair $(\kappa,\mu)\in\mathbb{R}^2$ is distribution

$$\tilde{R}(\rho_1,\rho_2)\rho_3=\kappa[g(\rho_2,\rho_3)\rho_1-g(\rho_1,\rho_3)\rho_2]+\mu[g(\rho_2,\rho_3)h\rho_1-g(\rho_1,\rho_3)h\rho_2],$$

for all $\rho_1,\rho_2\in\Gamma\left(T\tilde{N}\right)$.
Now let's give some equations below which are important for almost Kenmotsu $(\kappa,\mu,\nu)-$space. Let $\tilde{N}^{2n+1}(\phi,\eta,\xi,g)$ be $(2n+1)-$dimensional almost Kenmotsu $(\kappa,\mu,\nu)-$space. Then the following relations are provided.

$$h^2=(\kappa+1)\phi^2, \kappa\leq-1, \tag{2.5}$$

$$\xi(\kappa)=2(\kappa+1)(\nu-2), \tag{2.6}$$

$$\left(\tilde{\nabla}_{\rho_1}\phi\right)\rho_2=g(\phi\rho_1+h\rho_1,\rho_2)\xi-\eta(\rho_2)(\phi\rho_1+h\rho_1), \tag{2.7}$$

$$\tilde{\nabla}_{\rho_1}\xi=-\phi^2\rho_1-\phi h\rho_1, \tag{2.8}$$

$$S(\rho_1,\xi)=2n\kappa\eta(\rho_1), \tag{2.9}$$

$$\tilde{R}(\xi,\rho_1)\rho_2=\kappa[g(\rho_1,\rho_2)\xi-\eta(\rho_2)\rho_1]+\mu[g(h\rho_1,\rho_2)\xi-\eta(\rho_2)h\rho_1]+\nu[g(\phi h\rho_1,\rho_2)\xi-\eta(\rho_2)\phi h\rho_1]. \tag{2.10}$$

Let N be the immersed submanifold of an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. Let the tangent and normal subspaces of N in $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$ be $\Gamma(TN)$ and $\Gamma\left(T^{\perp}N\right)$, respectively. Gauss and Weingarten formulas for $\Gamma(TM)$ and $\Gamma\left(T^{\perp}M\right)$ are

$$\tilde{\bigtriangledown}_{\rho_1}\rho_2=\bigtriangledown_{\rho_1}\rho_2+\sigma(\rho_1,\rho_2), \tag{2.11}$$

$$\tilde{\bigtriangledown}_{\rho_1}\rho_5=-A_{\rho_5}\rho_1+\bigtriangledown^{\perp}_{\rho_1}\rho_5, \tag{2.12}$$

respectively, for all $\rho_1,\rho_2\in\Gamma\left(T\tilde{M}\right)$ and $\rho_5\in\Gamma\left(T^{\perp}\tilde{M}\right)$, where $\bigtriangledown$ and $\bigtriangledown^{\perp}$ are the connections on N and $\Gamma\left(T^{\perp}N\right)$, respectively, σ and A are the second fundamental form and the shape operator of N. There is a relation

$$g\left(A_{\rho_5}\rho_1,\rho_2\right)=g\left(\sigma(\rho_1,\rho_2),\rho_5\right) \tag{2.13}$$

between the second basic form and shape operator defined as above. The covariant derivative of the second fundamental form σ is defined as

$$\left(\tilde{\bigtriangledown}_{\rho_1}\sigma\right)(\rho_2,\rho_3)=\bigtriangledown^{\perp}_{\rho_1}\sigma(\rho_2,\rho_3)-\sigma\left(\bigtriangledown_{\rho_1}\rho_2,\rho_3\right)-\sigma\left(\rho_2,\bigtriangledown_{\rho_1}\rho_3\right), \tag{2.14}$$

for all $\rho_1,\rho_2,\rho_3\in\Gamma(TN)$. Specifically, if $\tilde{\bigtriangledown}\sigma=0$, N is said to be its second fundamental form is parallel.
Let R be the Riemann curvature tensor of N. In this case, the Gauss equation can be expressed as

$$\tilde{R}(\rho_1,\rho_2)\rho_3=R(\rho_1,\rho_2)\rho_3+A_{\sigma(\rho_1,\rho_3)}\rho_2-A_{\sigma(\rho_2,\rho_3)}\rho_1+\left(\tilde{\bigtriangledown}_{\rho_1}\sigma\right)(\rho_2,\rho_3)-\left(\tilde{\bigtriangledown}_{\rho_2}\sigma\right)(\rho_1,\rho_3), \tag{2.15}$$

for all $\rho_1,\rho_2,\rho_3\in\Gamma(TN)$.
$\tilde{R}\cdot\sigma$ is given by

$$\left(\tilde{R}(\rho_1,\rho_2)\cdot\sigma\right)(\rho_4,\rho_5)=R^{\perp}(\rho_1,\rho_2)\sigma(\rho_4,\rho_5)-\sigma\left(R(\rho_1,\rho_2)\rho_4,\rho_5\right)-\sigma\left(\rho_4,R(\rho_1,\rho_2)\rho_5\right), \tag{2.16}$$

where the Riemannian curvature tensor of normal bundle $\Gamma\left(T^{\perp}N\right)$ is given

$$R^{\perp}(\rho_1,\rho_2)=\left[\nabla^{\perp}_{\rho_1},\nabla^{\perp}_{\rho_2}\right]-\nabla^{\perp}_{[\rho_1,\rho_2]}.$$

On the other hand, the concircular curvature tensor for Riemannian manifold $\left(N^{2n+1},g\right)$ is given by

$$C(\rho_1,\rho_2)\rho_3=\tilde{R}(\rho_1,\rho_2)\rho_3-\frac{r}{2n(2n+1)}\left[g(\rho_2,\rho_3)\rho_1-g(\rho_1,\rho_3)\rho_2\right], \tag{2.17}$$

where r denotes the scalar curvature of N.
Similarly, the tensor $C\cdot\sigma$ is defined by

$$\left(C(\rho_1,\rho_2)\cdot\sigma\right)(\rho_4,\rho_5)=R^{\perp}(\rho_1,\rho_2)\sigma(\rho_4,\rho_5)-\sigma\left(C(\rho_1,\rho_2)\rho_4,\rho_5\right)-\sigma\left(\rho_4,C(\rho_1,\rho_2)\rho_5\right), \tag{2.18}$$

for all $\rho_1,\rho_2,\rho_4,\rho_5\in\Gamma(TN)$.
Let N be a Riemannian manifold, T is $(0,k)-$type tensor field and A is $(0,2)-$type tensor field. In this case, Tachibana tensor field $Q(A,T)$ is defined as

$$Q(A,T)(X_1,...,X_k;\rho_1,\rho_2)=-T\left((\rho_1\wedge_A\rho_2)X_1,...,X_k\right)-...-T\left(X_1,...,X_{k-1},(\rho_1\wedge_A\rho_2)X_k\right), \tag{2.19}$$

where

$$(\rho_1\wedge_A\rho_2)\rho_3=A(\rho_2,\rho_3)\rho_1-A(\rho_1,\rho_3)\rho_2, \tag{2.20}$$

$k\geq 1,X_1,X_2,...,X_k,\rho_1,\rho_2\in\Gamma(TN)$.

Definition 2.1 ([8]). *A submanifold N of a Riemannian manifold $\left(\tilde{N},g\right)$ is said to be concircular pseudoparallel, concircular 2−pseudoparallel, concircular Ricci-generalized pseudoparallel and concircular 2−Ricci generalized pseudoparallel if*

$C\cdot\sigma$ *and* $Q(g,\sigma)$

$C\cdot\tilde{\nabla}\sigma$ *and* $Q\left(g,\tilde{\nabla}\sigma\right)$

$C\cdot\sigma$ *and* $Q(S,\sigma)$

$C\cdot\tilde{\nabla}\sigma$ *and* $Q\left(S,\tilde{\nabla}\sigma\right)$

are linearly dependent, respectively.

3. Invariant Pseudoparalel Submanifolds of an Almost Kenmotsu $(\kappa,\mu,\nu)-$Space

Let N be the immersed submanifold of an $(2n+1)-$dimensional an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. If $\phi\left(T_{\rho_1}N\right)\subset T_{\rho_1}N$ in every ρ_1 point, the manifold N is called invariant submanifold. We note that all of properties of an invariant submanifold inherit the ambient manifold. From this section of the article, we will assume that the manifold N is the invariant submanifold of the an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. So, it is clear that the following proposition.

Proposition 3.1. *Let N be an invariant submanifold of an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$ such that ξ is tangent to N. Then the following equalities hold on N.*

$$R(\rho_1,\rho_2)\xi=\kappa[\eta(\rho_2)\rho_1-\eta(\rho_1)\rho_2]+\mu[\eta(\rho_2)h\rho_1-\eta(\rho_1)h\rho_2]+\nu[\eta(\rho_2)\phi h\rho_1-\eta(\rho_1)\phi h\rho_2], \tag{3.1}$$

$$R(\xi,\rho_1)\rho_2=\kappa[g(\rho_1,\rho_2)\xi-\eta(\rho_2)\rho_1]+\mu[g(h\rho_1,\rho_2)\xi-\eta(\rho_2)h\rho_1]+\nu[g(\phi h\rho_1,\rho_2)\xi-\eta(\rho_2)\phi h\rho_1], \tag{3.2}$$

$$\left(\nabla_{\rho_1}\phi\right)\rho_2=g(\phi\rho_1+h\rho_1,\rho_2)\xi-\eta(\rho_4)(\phi\rho_1+h\rho_1), \tag{3.3}$$

$$\nabla_{\rho_1}\xi=-\phi^2\rho_1-\phi h\rho_1, \tag{3.4}$$

$$C(\rho_1,\rho_2)\xi=\left[\kappa-\frac{r}{2n(2n+1)}\right][\eta(\rho_2)\rho_1-\eta(\rho_1)\rho_2]+\mu[\eta(\rho_2)h\rho_1-\eta(\rho_1)h\rho_2]+\nu[\eta(\rho_2)\phi h\rho_1-\eta(\rho_1)\phi h\rho_2], \tag{3.5}$$

$$\begin{aligned}C(\xi,\rho_1)\rho_2=&\left[\kappa-\frac{r}{2n(2n+1)}\right][g(\rho_1,\rho_2)\xi-\eta(\rho_2)\rho_1]+\mu[g(h\rho_1,\rho_2)\xi-\eta(\rho_2)h\rho_1]\\&+\nu[g(\phi h\rho_1,\rho_2)\xi-\eta(\rho_2)\phi h\rho_1],\end{aligned} \tag{3.6}$$

$$C(\xi,\rho_1)\xi=\left[\kappa-\frac{r}{2n(2n+1)}\right][\eta(\rho_1)\xi-\rho_1] \tag{3.7}$$

$$\sigma(\rho_1,\xi)=0,\quad \sigma(\phi\rho_1,\rho_2)=\sigma(\rho_1,\phi\rho_2)=\phi\sigma(\rho_1,\rho_2), \tag{3.8}$$

for all $\rho_1,\rho_2\in\Gamma(TN)$, where ∇,σ and R denote the induced Levi-Civita connection on N, the shape operator and Riemannian curvature tensor of N, respectively.

Lemma 3.2 ([6]). *Let N be the invariant submanifold of an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. Then the second fundamental form σ of N is parallel if and only if N is the total geodesic submanifold provided $\kappa\neq 0$.*

Let us now consider the invariant submanifolds of the almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$ on the concircular curvature tensor.
Equivalent to the definition of concircular pseudoparallel given above, it can be said that there is a function $\digamma_1$ on the set $M_1=\{x\in N|\,\sigma(x)\neq g(x)\}$ such that

$$C\cdot\sigma=\digamma_1 Q(g,\sigma).$$

If $\digamma_1=0$ specifically, N is called a concircular semiparallel submanifold.

Theorem 3.3. *Let N be the invariant submanifold of the $(2n+1)-$dimensional an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. If N is concircular pseudoparallel submanifold, then N is either a total geodesic submanifold or*

$$\digamma_1=\left(\kappa-\frac{r}{2n(2n+1)}\right)\mp\sqrt{(\kappa+1)(\nu^2-\mu^2)},\mu\cdot\nu(\kappa+1)=0.$$

Proof. Let's assume that N is a concircular pseudoparallel submanifold. So, we can write

$$(C(\rho_1,\rho_2)\cdot\sigma)(\rho_4,\rho_5)=\digamma_1 Q(g,\sigma)(\rho_4,\rho_5;\rho_1,\rho_2), \tag{3.9}$$

for all $\rho_1,\rho_2,\rho_4,\rho_5\in\Gamma(TN)$. From (2.18), it is clear that

$$R^{\perp}(\rho_1,\rho_2)\sigma(\rho_4,\rho_5)-\sigma(C(\rho_1,\rho_2)\rho_4,\rho_5)-\sigma(\rho_4,C(\rho_1,\rho_2)\rho_5)=-\digamma_1\left\{\sigma((\rho_1\wedge_g\rho_2)\rho_4,\rho_5)+\sigma(\rho_4,(\rho_1\wedge_g\rho_2)\rho_5)\right\}.$$

Easily from here, we can write

$$R^{\perp}(\rho_1,\rho_2)\sigma(\rho_4,\rho_5)-\sigma(C(\rho_1,\rho_2)\rho_4,\rho_5)-\sigma(\rho_4,C(\rho_1,\rho_2)\rho_5)=-F_1\{g(\rho_2,\rho_4)\sigma(\rho_1,\rho_5)-g(\rho_1,\rho_4)\sigma(\rho_2,\rho_5) +g(\rho_2,\rho_5)\sigma(\rho_4,\rho_1)-g(\rho_1,\rho_5)\sigma(\rho_4,\rho_2)\}. \tag{3.10}$$

If we choose $\rho_1=\rho_4=\xi$ in (3.10) and make use of (3.7), we get

$$\sigma(C(\xi,\rho_2)\xi,\rho_5)=-F_1\sigma(\rho_2,\rho_5). \tag{3.11}$$

If we use (3.7) out of (3.11), we obtain

$$\left[F_1-\left(\kappa-\frac{r}{2n(2n+1)}\right)\right]\sigma(\rho_2,\rho_5)=\mu\sigma(h\rho_2,\rho_5)+\nu\phi\sigma(h\rho_2,\rho_5). \tag{3.12}$$

Substituting $h\rho_2$ for ρ_2 in (3.12) by view of (2.5) and (3.8), we have

$$\left[F_1-\left(\kappa-\frac{r}{2n(2n+1)}\right)\right]\sigma(h\rho_2,\rho_5)=-(\kappa+1)[\mu\sigma(\rho_2,\rho_5)+\nu\phi\sigma(\rho_2,\rho_5)]. \tag{3.13}$$

From (3.12) and (3.13), one can easily see that

$$\left\{(\kappa+1)(\mu^2-\nu^2)+\left[F_1-\left(\kappa-\frac{r}{2n(2n+1)}\right)\right]^2\right\}\sigma(\rho_2,\rho_5)+2(\kappa+1)\mu\nu\phi\sigma(\rho_2,\rho_5)=0. \tag{3.14}$$

This tell us that N is either totally geoesic submanifold or

$$(\kappa+1)(\mu^2-\nu^2)+\left[F_1-\left(\kappa-\frac{r}{2n(2n+1)}\right)\right]^2=(\kappa+1)\mu\nu=0.$$

This completes the proof. □

Corollary 3.4. *Let N be an invariant pseudoparallel submanifold of the $(2n+1)-$dimensional an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. Then N is concircular semiparallel if and only if N is totally geodesic provided*

$$(\kappa+1)(\mu^2-\nu^2)+\left(\kappa-\frac{r}{2n(2n+1)}\right)^2\neq 0 \text{ or } (\kappa+1)\mu\nu\neq 0.$$

Equivalent to the definition of concircular Ricci generalized pseudoparallel given above, it can be said that there is a function F_2 on the set
$M_2=\{x\in N|S(x)\neq\sigma(x)\}$ such that

$$C\cdot\sigma=F_2Q(S,\sigma).$$

If $F_2=0$ specifically, N is called a concircular Ricci generalized semiparallel submanifold.

Theorem 3.5. *Let N be the invariant submanifold of the $(2n+1)-$dimensional an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. If N is concircular Ricci generalized pseudoparallel submanifold, then N is either a total geodesic submanifold or*

$$F_2=\frac{2n(2n+1)\kappa-r}{4n^2\kappa(2n+1)}\mp\frac{1}{2n\kappa}\sqrt{(\kappa+1)(\nu^2-\mu^2)},\mu\cdot\nu(\kappa+1)=0.$$

Proof. Let's assume that N is a concircular Ricci generalized pseudoparallel submanifold. So, we can write

$$(C(\rho_1,\rho_2)\cdot\sigma)(\rho_4,\rho_5)=F_2Q(S,\sigma)(\rho_4,\rho_5;\rho_1,\rho_2), \tag{3.15}$$

for all $\rho_1,\rho_2,\rho_4,\rho_5\in\Gamma(TN)$. From (2.18), it is clear that

$$R^{\perp}(\rho_1,\rho_2)\sigma(\rho_4,\rho_5)-\sigma(C(\rho_1,\rho_2)\rho_4,\rho_5)-\sigma(\rho_4,C(\rho_1,\rho_2)\rho_5)=-F_2\{\sigma((\rho_1\wedge_S\rho_2)\rho_4,\rho_5)+\sigma(\rho_4,(\rho_1\wedge_S\rho_2)\rho_5)\}.$$

Easily from here, we can write

$$R^{\perp}(\rho_1,\rho_2)\sigma(\rho_4,\rho_5)-\sigma(C(\rho_1,\rho_2)\rho_4,\rho_5)-\sigma(\rho_4,C(\rho_1,\rho_2)\rho_5)=-F_2\{S(\rho_2,\rho_4)\sigma(\rho_1,\rho_5)-S(\rho_1,\rho_4)\sigma(\rho_2,\rho_5) +S(\rho_2,\rho_5)\sigma(\rho_4,\rho_1)-S(\rho_1,\rho_5)\sigma(\rho_4,\rho_2)\}. \tag{3.16}$$

If we choose $\rho_1=\rho_5=\xi$ in (3.16) and make use of (3.7), we get

$$\sigma\left(\rho_4, C\left(\xi, \rho_2\right) \xi\right)=-\digamma_2 S(\xi, \xi) \sigma\left(\rho_4, \rho_2\right). \tag{3.17}$$

If we use (2.9) and (3.7) in (3.17), we obtain

$$\left[\left(\kappa-\frac{r}{2n(2n+1)}\right)-2n\kappa\digamma_2\right] \sigma\left(\rho_4, \rho_2\right)=\mu \sigma\left(\rho_4, h\rho_2\right)+v \phi \sigma\left(\rho_4, h\rho_2\right). \tag{3.18}$$

Substituting $h\rho_2$ for ρ_2 in (3.18) by view of (2.5) and (3.8), we have

$$\left[\left(\kappa-\frac{r}{2n(2n+1)}\right)-2n\kappa\digamma_2\right] \sigma\left(h\rho_2, \rho_4\right)=-(\kappa+1)\left[\mu \sigma\left(\rho_2, \rho_4\right)+v \phi \sigma\left(\rho_2, \rho_4\right)\right]. \tag{3.19}$$

From (3.18) and (3.19), one can easily see that

$$\left\{\left[\left(\kappa-\frac{r}{2n(2n+1)}\right)-2n\kappa\digamma_2\right]^2+(\kappa+1)\left(\mu^2-v^2\right)\right\} \sigma\left(\rho_4, \rho_2\right)+2(\kappa+1) \mu v \phi \sigma\left(\rho_4, \rho_2\right)=0. \tag{3.20}$$

This tell us that N is either totally geoesic submanifold or

$$\left[\left(\kappa-\frac{r}{2n(2n+1)}\right)-2n\kappa\digamma_2\right]^2+(\kappa+1)\left(\mu^2-v^2\right)=(\kappa+1) \mu v=0.$$

This completes the proof. □

Corollary 3.6. *Let N be an invariant pseudoparallel submanifold of the $(2n+1)-$dimensional an almost Kenmotsu $(\kappa,\mu,v)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. Then N is concircular Ricci generalized semiparallel if and only if N is totally geodesic provided*

$$(\kappa+1)\left(\mu^2-v^2\right)+\left(\kappa-\frac{r}{2n(2n+1)}\right)^2 \neq 0 \text{ or } (\kappa+1) \mu v \neq 0.$$

Equivalent to the definition of concircular $2-$pseudoparallel given above, it can be said that there is a function $\digamma_3$ on the set $M_3=\left\{x \in N \mid g(x) \neq \tilde{\nabla} \sigma(x)\right\}$ such that

$$C \cdot \tilde{\nabla} \sigma=\digamma_3 Q(g, \tilde{\nabla} \sigma).$$

If $\digamma_3=0$ specifically, N is called a concircular $2-$semiparallel submanifold.

Theorem 3.7. *Let N be the invariant submanifold of the $(2n+1)-$dimensional an almost Kenmotsu $(\kappa,\mu,v)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. If N is concircular $2-$pseudoparallel submanifold, then N is either a total geodesic submanifold or*

$$\digamma_3=\left[\kappa-\frac{r}{2n(2n+1)}\right] \mp \sqrt{(\kappa+1)\left(v^2-\mu^2\right)}, \mu \cdot v(\kappa+1)=0.$$

Proof. Let's assume that $\tilde{M}$ is a concircular $2-$pseudoparallel submanifold. So, we can write

$$\left(C\left(\rho_1, \rho_2\right) \cdot \tilde{\triangledown} \sigma\right)\left(\rho_4, \rho_5, \rho_3\right)=\digamma_3 Q(S, \tilde{\triangledown} \sigma)\left(\rho_4, \rho_5, \rho_3 ; \rho_1, \rho_2\right), \tag{3.21}$$

for all $\rho_1, \rho_2, \rho_4, \rho_5, \rho_3 \in \Gamma(TM)$. If we choose $\rho_1=\rho_5=\xi$ in (3.21), we can write

$$\begin{aligned} & R^{\perp}\left(\xi, \rho_2\right)\left(\tilde{\triangledown}_{\rho_4} \sigma\right)\left(\xi, \rho_3\right)-\left(\tilde{\triangledown}_{C\left(\xi, \rho_2\right) \rho_4} \sigma\right)\left(\xi, \rho_3\right)-\left(\tilde{\triangledown}_{\rho_4} \sigma\right)\left(C\left(\xi, \rho_2\right) \xi, \rho_3\right)-\left(\tilde{\triangledown}_{\rho_4} \sigma\right)\left(\xi, C\left(\xi, \rho_2\right) \rho_3\right) \\ & =-\digamma_3\left\{\left(\tilde{\triangledown}_{\left(\xi \wedge_g \rho_2\right) \rho_4} \sigma\right)\left(\xi, \rho_3\right)+\left(\tilde{\triangledown}_{\rho_4} \sigma\right)\left(\left(\xi \wedge_g \rho_2\right) \xi, \rho_3\right)+\left(\tilde{\triangledown}_{\rho_4} \sigma\right)\left(\xi,\left(\xi \wedge_g \rho_2\right) \rho_3\right)\right\}. \end{aligned} \tag{3.22}$$

Let's calculate all the expressions in (3.22). In view of (2.14), (2.19), (3.4), and (3.8), we can derive

$$\begin{aligned} R^{\perp}\left(\xi, \rho_2\right)\left(\tilde{\triangledown}_{\rho_4} \sigma\right)\left(\xi, \rho_3\right)= & R^{\perp}\left(\xi, \rho_2\right)\left\{\triangledown_{\rho_4}^{\perp} \sigma\left(\xi, \rho_3\right)-\sigma\left(\triangledown_{\rho_4} \xi, \rho_3\right)-\sigma\left(\xi, \triangledown_{\rho_4} \rho_3\right)\right\} \\ = & -R^{\perp}\left(\xi, \rho_2\right) \sigma\left(\triangledown_{\rho_4} \xi, \rho_3\right) \\ = & R^{\perp}\left(\xi, \rho_2\right)\left\{\sigma\left(\phi h \rho_4, \rho_3\right)-\sigma\left(\rho_4, \rho_3\right)\right\}, \end{aligned} \tag{3.23}$$

$$\begin{aligned}
\left(\tilde{\bigtriangledown}_{C(\xi,\rho_2)\rho_4}\sigma\right)(\xi,\rho_3) &= \bigtriangledown^{\perp}_{C(\xi,\rho_2)\rho_4}\sigma(\xi,\rho_3)-\sigma\left(\bigtriangledown_{C(\xi,\rho_2)\rho_4}\xi,\rho_3\right)-\sigma\left(\xi,\bigtriangledown_{C(\xi,\rho_2)\rho_4}\rho_3\right)\\
&= \sigma\left(\phi^2C(\xi,\rho_2)\rho_4+\phi hC(\xi,\rho_2)\rho_4,\rho_3\right)\\
&= \eta(\rho_4)\left\{\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\rho_2,\rho_3)+\mu\sigma(h\rho_2,\rho_3)\right.\\
&\quad +\nu\sigma(\phi h\rho_2,\rho_3)-\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\phi\sigma(h\rho_2,\rho_3)\\
&\quad \left.+(\kappa+1)\mu\phi\sigma(\rho_2,\rho_3)-(\kappa+1)\nu\sigma(\rho_2,\rho_3)\right\}
\end{aligned}\tag{3.24}$$

$$\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(C(\xi,\rho_2)\xi,\rho_3)=\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)\left(\left[\kappa-\frac{r}{2n(2n+1)}\right][\eta(\rho_2)\xi-\rho_2]-\mu h\rho_2-\nu\phi h\rho_2,\rho_3\right)\tag{3.25}$$

$$\begin{aligned}
\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\xi,C(\xi,\rho_2)\rho_3) &= \nabla^{\perp}_{\rho_4}\sigma(\xi,C(\xi,\rho_2)\rho_3)-\sigma\left(\nabla_{\rho_4}\xi,C(\xi,\rho_2)\rho_3\right)\\
&\quad -\sigma\left(\xi,\nabla_{\rho_4}C(\xi,\rho_2)\rho_3\right)\\
&= -\sigma\left(\nabla_{\rho_4}\xi,C(\xi,\rho_2)\rho_3\right)\\
&= \sigma\left(\phi^2\rho_4+\phi h\rho_4,C(\xi,\rho_2)\rho_3\right)\\
&= \eta(\rho_3)\left\{\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\rho_4,\rho_2)+\mu\sigma(\rho_4,h\rho_2)\right.\\
&\quad +\nu\phi\sigma(\rho_4,h\rho_2)-\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\phi h\rho_4,\rho_2)\\
&\quad \left.+\mu(\kappa+1)\phi\sigma(\rho_4,\rho_2)-\nu(\kappa+1)\sigma(\rho_4,\rho_2)\right\},
\end{aligned}\tag{3.26}$$

$$\begin{aligned}
\left(\tilde{\bigtriangledown}_{(\xi\wedge_g\rho_2)\rho_4}\sigma\right)(\xi,\rho_3) &= \bigtriangledown^{\perp}_{(\xi\wedge_g\rho_2)\rho_4}\sigma(\xi,\rho_3)-\sigma\left(\bigtriangledown_{(\xi\wedge_g\rho_2)\rho_4}\xi,\rho_3\right)-\sigma\left(\xi,\bigtriangledown_{(\xi\wedge_g\rho_2)\rho_4}\rho_3\right)\\
&= \sigma\left(\phi^2(\xi\wedge_g\rho_2)\rho_4+\phi h(\xi\wedge_g\rho_2)\rho_4,\rho_3\right)\\
&= \eta(\rho_4)\{\sigma(\rho_2,\rho_3)-\phi\sigma(h\rho_2,\rho_3)\},
\end{aligned}\tag{3.27}$$

$$\begin{aligned}
\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)((\xi\wedge_g\rho_2)\xi,\rho_3) &= \left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\eta(\rho_2)\xi-\rho_2,\rho_3)\\
&= \left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\eta(\rho_2)\xi,\rho_3)-\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\rho_2,\rho_3)\\
&= -\sigma\left(\nabla_{\rho_4}\eta(\rho_2)\xi,\rho_3\right)-\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\rho_2,\rho_3)\\
&= -\sigma\left(\rho_4[\eta(\rho_2)]\xi+\eta(\rho_2)\nabla_{\rho_4}\xi,\rho_3\right)-\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\rho_2,\rho_3)\\
&= \eta(\rho_2)\{\sigma(\phi h\rho_4,\rho_3)-\sigma(\rho_4,\rho_3)\}-\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\rho_2,\rho_3)
\end{aligned}\tag{3.28}$$

$$\begin{aligned}
\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\xi,(\xi\wedge_g\rho_2)\rho_3) &= -\sigma\left(\nabla_{\rho_4}\xi,(\xi\wedge_g\rho_2)\rho_3\right)\\
&= \sigma\left(\phi^2\rho_4+\phi h\rho_4,g(\rho_2,\rho_3)\xi-g(\xi,\rho_3)\rho_2\right)\\
&= \eta(\rho_3)\{\sigma(\rho_4,\rho_2)-\sigma(\phi h\rho_4,\rho_2)\}
\end{aligned}\tag{3.29}$$

If we substitute (3.22), (3.23), (3.24), (3.25), (3.26), (3.27), (3.28) in (3.21), we obtain

$$\begin{aligned}
&R^{\perp}(\xi,\rho_2)\{\sigma(\phi h\rho_4,\rho_3)-\sigma(\rho_4,\rho_3)\}-\eta(\rho_4)\left\{\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\rho_2,\rho_3)+\mu\sigma(h\rho_2,\rho_3)\right.\\
&\left.+\nu\sigma(\phi h\rho_2,\rho_3)-\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\phi\sigma(h\rho_2,\rho_3)+(\kappa+1)\mu\phi\sigma(\rho_2,\rho_3)-(\kappa+1)\nu\sigma(\rho_2,\rho_3)\right\}\\
&-\left(\tilde{\nabla}_{\rho_4}\sigma\right)\left(\left[\kappa-\tfrac{r}{2n(2n+1)}\right][\eta(\rho_2)\xi-\rho_2]-\mu h\rho_2-\nu\phi h\rho_2,\rho_3\right)\\
&-\eta(\rho_3)\left\{\begin{array}{l}\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\rho_4,\rho_2)+\mu\sigma(\rho_4,h\rho_2)+\nu\phi\sigma(\rho_4,h\rho_2)-\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\phi h\rho_4,\rho_2)\\
+\mu(\kappa+1)\phi\sigma(\rho_4,\rho_2)-\nu(\kappa+1)\sigma(\rho_4,\rho_2)\end{array}\right\}\\
&=-F_3\{\eta(\rho_4)\{\sigma(\rho_2,\rho_3)-\phi\sigma(h\rho_2,\rho_3)\}+\eta(\rho_2)\{\sigma(\phi h\rho_4,\rho_3)-\sigma(\rho_4,\rho_3)\}-\left(\tilde{\nabla}_{\rho_4}\sigma\right)(\rho_2,\rho_3)\\
&\quad+\eta(\rho_3)\{\sigma(\rho_4,\rho_2)-\sigma(\phi h\rho_4,\rho_2)\}\}
\end{aligned}\tag{3.30}$$

If we choose $\rho_3=\xi$ in (3.30), we get

$$\begin{aligned}
&-\left(\tilde{\nabla}_{\rho_4}\sigma\right)\left(\left[\kappa-\tfrac{r}{2n(2n+1)}\right][\eta(\rho_2)\xi-\rho_2]-\mu h\rho_2-\nu\phi h\rho_2,\xi\right)-\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\rho_4,\rho_2)-\mu\sigma(\rho_4,h\rho_2)\\
&-\nu\phi\sigma(\rho_4,h\rho_2)+\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\phi h\rho_4,\rho_2)-\mu(\kappa+1)\phi\sigma(\rho_4,\rho_2)+\nu(\kappa+1)\sigma(\rho_4,\rho_2)\\
&=-F_3\left\{-\left(\tilde{\nabla}_{\rho_4}\sigma\right)(\rho_2,\xi)+\sigma(\rho_4,\rho_2)-\sigma(\phi h\rho_4,\rho_2)\right\}.
\end{aligned}\tag{3.31}$$

By direct calculations, one can easily see that

$$\begin{aligned}
&\left(\tilde{\nabla}_{\rho_4}\sigma\right)\left(\left[\kappa-\tfrac{r}{2n(2n+1)}\right][\eta(\rho_2)\xi-\rho_2]-\mu h\rho_2-\nu\phi h\rho_2,\xi\right)\\
&=\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\rho_2,\rho_4)+\mu\sigma(h\rho_2,\rho_4)+\nu\phi\sigma(\rho_4,h\rho_2)-\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\phi\sigma(\rho_2,h\rho_4)\\
&\quad+\mu(\kappa+1)\phi\sigma(\rho_4,\rho_2)-\nu(\kappa+1)\sigma(\rho_4,\rho_2),
\end{aligned}\tag{3.32}$$

and

$$\left(\tilde{\nabla}_{\rho_4}\sigma\right)(\rho_2,\xi)=\phi\sigma(h\rho_4,\rho_2)-\sigma(\rho_4,\rho_2).\tag{3.33}$$

If (3.32) and (3.33) are out in (3.31), we obtain

$$\left[F_3-\left(\kappa-\tfrac{r}{2n(2n+1)}\right)+(\nu-\mu\phi)(\kappa+1)\right]\sigma(\rho_4,\rho_2)-\left[\left(F_3-\left(\kappa-\tfrac{r}{2n(2n+1)}\right)\right)\phi+(\mu+\phi\nu)\right]\sigma(\rho_4,h\rho_2)=0.\tag{3.34}$$

Substituting $h\rho_2$ instead of ρ_2 in (3.34), we can easily see that

$$\begin{aligned}
&\left[F_3-\left(\kappa-\tfrac{r}{2n(2n+1)}\right)+(\nu-\mu\phi)(\kappa+1)\right]\sigma(\rho_4,h\rho_2)\\
&-\left[\left(F_3-\left(\kappa-\tfrac{r}{2n(2n+1)}\right)\right)\phi+(\mu+\phi\nu)\right](\kappa+1)\sigma(\rho_4,\rho_2)=0.
\end{aligned}\tag{3.35}$$

From common solutions of (3.34) and (3.35), we can infer

$$\begin{aligned}
&\left\{\left[F_3-\left(\kappa-\tfrac{r}{2n(2n+1)}\right)+(\nu-\mu\phi)(\kappa+1)\right]^2\right.\\
&\left.+\left[\left(F_3-\left(\kappa-\tfrac{r}{2n(2n+1)}\right)\right)\phi+(\mu+\phi\nu)\right]^2(\kappa+1)\right\}\sigma(\rho_4,\rho_2)=0
\end{aligned}\tag{3.36}$$

This implies that N is either totally geodesic or

$$F_3=\left[\kappa-\frac{r}{2n(2n+1)}\right]\mp\sqrt{(\kappa+1)(\nu^2-\mu^2)},\mu\cdot\nu(\kappa+1)=0$$

This completes of the proof. □

Corollary 3.8. *Let N be an invariant pseudoparallel submanifold of the* $(2n+1)-$*dimensional an almost Kenmotsu* $(\kappa,\mu,\nu)-$*space* $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. *Then N is concircular* $2-$*semiparallel if and only if N is totally geodesic provided*

$$\left[\kappa-\frac{r}{2n(2n+1)}\right]^2-(\kappa+1)\left(\nu^2-\mu^2\right)\neq 0 \; or \; (\kappa+1)\mu\nu\neq 0.$$

Equivalent to the definition of concircular $2-$Ricci generalized pseudoparallel given above, it can be said that there is a function $\digamma_4$ on the set
$M_4=\left\{x\in N|\, S(x)\neq\tilde{\nabla}\sigma(x)\right\}$ such that

$$C\cdot\tilde{\nabla}\sigma=\digamma_4 Q\left(S,\tilde{\nabla}\sigma\right).$$

If $\digamma_4=0$ specifically, N is called a concircular $2-$Ricci generalized semiparallel submanifold.

Theorem 3.9. *Let N be the invariant submanifold of the* $(2n+1)-$*dimensional an almost Kenmotsu* $(\kappa,\mu,\nu)-$*space* $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. *If N is concircular* $2-$*Ricci generalized pseudoparallel submanifold, then N is either a total geodesic submanifold or*

$$\digamma_4=\frac{1}{2n}\left(1\mp\frac{2n(2n+1)}{2n(2n+1)\kappa-r}\sqrt{(\kappa+1)(\nu^2-\mu^2)}\right),\mu\cdot\nu(\kappa+1)=0.$$

Proof. The proof of the theorem can be easily done similar to the proof of the previous theorem. □

4. Conclusion

In this article, pseudoparallel submanifolds for almost Kenmotsu $(\kappa,\mu,\nu)-$space are investigated. The almost Kenmotsu $(\kappa,\mu,\nu)-$space is considered on the concircular curvature tensor. Submanifolds of these manifolds with properties such as concircular pseudoparallel, concircular $2-$pseudoparallel, concircular Ricci generalized pseudoparallel, and concircular $2-$Ricci generalized pseudoparallel has been characterized. Necessary and sufficient conditions are given for the invariant submanifolds of almost Kenmotsu $(\kappa,\mu,\nu)-$space to be total geodesic according to the behavior of the κ,μ,ν functions.

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On Some Spectral Properties of Discrete Sturm-Liouville Problem

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Abstract

Time scale theory helps us to combine differential equations with difference equations. Especially in models such as biology, medicine, and economics, since the independent variable is handled discrete, it requires us to analyze in discrete clusters. In these cases, the difference equations defined in $\mathbb{Z}$ are considered. Boundary value problems (BVP's) are used to solve and model problems in many physical areas. In this study, we examined spectral features of the discrete Sturm-Liouville problem. We have given some examples to make the subject understandable. The discrete Sturm-Liouville problem is solved by using the discrete Laplace transform. In the classical case, the discrete Laplace transform is preferred because it is a very useful method in differential equations and it is thought that the discrete Laplace transform will show similar properties. The other method obtained for the solution of this problem is the solutions obtained according to the states of the characteristic equation and λ parameter. In this solution, discrete Wronskian and Cramer methods are used.

1. Introduction

A time scale $\mathbb{T}$ is a non-empty, arbitrary, closed subset of $\mathbb{R}$. This theory was first studied by Hilger in his doctoral thesis [1]. Later, Bohner and Peterson expressed Δ-derivative, Δ-integral and some properties in [2]. Bohner and Georgiev studied the concepts for multivariate functions on time scale [3] . There are many studies in different years on this theory [4, 5]. For instance, time scale population model is used in many important areas such as wound healing, maximization and minimization problems in economy, epidemic problems.
The special case of $\mathbb{T}=\mathbb{Z}$ has many applications in literature. Due to the difficulties posed by derivative and integral in general case, the special cases of time scale are frequently used in many applications. Difference equation is a type of equation that have applications in many fields such as biology, medicine and population. Examples of these applications can be given such as population growth model, logistics surplus model, competition model and infectious disease model. First studies on BVP's for linear Δ-difference equations on time scale are in the relevant references [6, 7]. In addition, various studies have been carried out on the general theory of difference equations [8], non-regular cases of linear ordinary difference equations, asymptotic behavior of difference equation systems and difference equations [9], finite difference calculus. Other important studies that deal with the properties of difference equations are [10] and [11]. In addition, there are many studies that examine discrete versions of Sturm-Liouville, Bessel, Dirac on time scales (see [12–29]).
To give basic results, we should recall substantial concepts of time scale theory. For $t \in \mathbb{T}$, forward and backward jump operators [2] are expressed by

$$\sigma(t)=inf\{s\in\mathbb{T}; s>t\}$$

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and

$$\rho(t) = sup\{s \in \mathbb{T}; s < t\}.$$

Let's state a set $\mathbb{T}^{\kappa}$ which is derived from $\mathbb{T}$, and necessary for the definition of delta derivatives. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - m$. In other cases, $\mathbb{T}^{\kappa} = \mathbb{T}$. Moreover, let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$. Then, one can define $f^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\varepsilon < 0$, there is a neighborhood $U = (t-\delta, t+\delta) \cap \mathbb{T}$ of t for some $\delta > 0$ such that

$$\left|[f(\sigma(t) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \varepsilon |\sigma(t) - s|,$$

for all $s \in U$. $f^{\Delta}(t)$ is known as Δ-derivative of f at $t \in \mathbb{T}^{\kappa}$. Now, let's express another important concept that is necessary when defining an integral on $\mathbb{T}$. $f : \mathbb{T} \to \mathbb{R}$ is regulated if its right-sided limit exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. f is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. $C_{rd}(\mathbb{T})$ indicates the set of all rd-continuous functions. By the special selections of $\mathbb{T}$, we can have below derivatives [7].

1. $f^{\Delta}(t) = f^{'}(t)$ for all $t \in \mathbb{R}$ if $\mathbb{T} = \mathbb{R}$.
2. $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$ for all $t \in \mathbb{Z}$ if $\mathbb{T} = \mathbb{Z}$.

Let's start to express the concept of integral, which is very important for the subject we are working on, gradually. There exists a function F which is pre-differentiable with region of differentiation D where $F^{\Delta}(t) = f(t)$ holds for all $t \in D$ where f is regulated. For functions f and F satisfying these conditions and an arbitrary constant C, we express indefinite delta integral of f by

$$\int f(t)\Delta t = F(t) + C.$$

In same logic, Cauchy integral of f on $[r,s]$ is defined by

$$\int_r^s f(t)\Delta t = F(s) - F(r),$$

for all $r,s \in \mathbb{T}$. The definitions of delta derivative and delta integral, which are generally given in arbitrary time scales, have different representations in different time scales.
We will now express some spectral results for a special case of $\mathbb{T}$. In this study, we take into account below discrete Sturm-Liouville problem

$$L_{\Delta}y(t) = -\Delta^2 y(t) + q(t)y(t) = \lambda y(t), \quad 0 < t < N, \tag{1.1}$$

with separated discrete boundary conditions

$$\Delta y(0) - hy(0) = 0, \tag{1.2}$$

$$\Delta y(N) + Hy(N) = 0, \tag{1.3}$$

where λ is spectral parameter, $q \in L_2^{\mathbb{Z}}[0,N]$, $N \in \mathbb{Z}^+$ and $H, h \in \mathbb{R}$. Considering this problem, which is very important in terms of mathematical physics in classical analysis, in a discrete situation will give very important results. By setting $\mathbb{T} = \mathbb{R}$ in (1.1), it reduces to Sturm-Liouville equation on $\mathbb{R}$ as

$$Ly(t) = -y^{''}(t) + q(t)y(t) = \lambda y(t).$$

Discrete version of $L_2[0,N]$ will play a key role in the study, while the main results are obtained in spectral point of view. So let's define this space.The discrete $L_2^{\mathbb{Z}}[0,N]$ space is defined by [according to Theorem 1.79 (iv) in [2]]

$$L_2^{\mathbb{Z}}[0,N] = \left\{ x(t) : \sum_{t=0}^{N-1} |x(t)|^2 < \infty \right\}.$$

Inner product on $L_2^{\mathbb{Z}}[0,N]$ is defined by

$$\langle x, y \rangle = \int_0^N x(t)y(t)\Delta t = \sum_{t=0}^{N-1} x(t)y(t),$$

where $x,y \in L_2^{\mathbb{Z}}[0,N]$. Another concept that we will use in the study is discrete Laplace transform (or L-transform) [2, 11]. Assume that $f : \mathbb{N}_a \to \mathbb{R}$ is regulated where $N_a = \{a, a+1, a+2, \ldots\}$ and $a \in \mathbb{R}$. Then, discrete L-transform of f based at a is defined by

$$L_a\{f\}(s) = F_a(k) = \int_0^\infty \frac{f(a+k)}{(s+1)^{k+1}} \Delta k = \sum_{k=0}^{\infty} \frac{f(a+k)}{(s+1)^{k+1}},$$

for all complex numbers $s \neq -1$ when the improper integral converges. Here, $L_a\{f\}(s) = F_a(k)$, $L_a^{-1}\{F_a(k)\} = f(s)$ where L_a^{-1} is inverse discrete L-transform [11].

Let's continue with another concept that is extremely important for L-transform. f is of exponential order $r > 0$ if there exists a constant $A > 0$ such that $|f(t)| \leq Ar^t$ for $t \in \mathbb{N}_a$. Let f be exponential order $r > 0$. Then, for any $N \in \mathbb{Z}^+$ [11],

$$L_a\left\{\Delta^N f\right\}(s) = s^N F_a(s) - \sum_{j=0}^{N-1} s^j \Delta^{N-1-j} f(a),$$

for $|s+1| > r$. Let $f, g : \mathbb{N}_a \to \mathbb{R}$ and discrete L-transforms of f and g converge for $|s+1| > r$ where $r > 0$. Then, discrete L-transform of $c_1 f + c_2 g$ converges for $|s+1| > r$ and

$$L_a\{c_1 f + c_2 g\}(s) = c_1 L_a\{f\}(s) + c_2 L_a\{g\}(s),$$

for $|s+1| > r$, $c_1, c_2 \in \mathbb{R}$. Addition, assume that $p \neq \pm i$. Then,

$$L_a\{\cos_p(t,a)\}(s) = \frac{s}{s^2+p^2},$$

and

$$L_a\{\sin_p(t,a)\}(s) = \frac{p}{s^2+p^2}.$$

This study is planned as follows: In Section 2, we give proofs of some basic theorems for spectral properties of discrete Strum-Liouville equation. Using some methods, we get eigenfunctions of (1.1)-(1.3) discrete Sturm-Liouville problem in Section 3.

2. Some spectral properties of discrete Sturm-Liouville equation

The eigenvalues and eigenfunctions of differential operators need to be found in solving problems encountered in many fields such as analysis, applied mathematics and mathematical physics. For this reason, spectral properties of Sturm-Liouville problem, which has applications in many fields, have been an important subject of study.

Orthogonality of eigenfunctions, simplicity and reality of eigenvalues, formally self-adjointness property of operator are well-known properties in usual spectral analysis. The following results are generalized to discrete case. All the features that will be given below will allow to better understand and explain the physical phenomenon expressed by the problem expressed in the discrete situation.

Theorem 2.1. *The eigenfunctions corresponding to the distinct eigenvalues of problem (1.1)-(1.3) are orthogonal.*

Proof. We have to show that there are $\lambda_1 \neq \lambda_2$ for $y_1(t,\lambda_1)$ and $y_2(t,\lambda_2)$ such that $\langle y_1, y_2 \rangle = 0$.

$$-\Delta^2 y_1 + q(t) y_1 = \lambda_1 y_1,$$

$$-\Delta^2 y_2 + q(t) y_2 = \lambda_2 y_2.$$

If necessary adjustments are made here, we get

$$-y_2(\Delta^2 y_1) + y_1(\Delta^2 y_2) = (\lambda_1 - \lambda_2) y_1 y_2.$$

Let's take discrete integral of both sides on $[0,N]$ to get

$$-\int_0^N y_2(\Delta^2 y_1)\Delta t + \int_0^N y_1(\Delta^2 y_2)\Delta t = (\lambda_1 - \lambda_2)\int_0^N y_1 y_2 \Delta t,$$

$$\int_0^N \Delta[(\Delta y_2)y_1 - (\Delta y_1)y_2]\Delta t = (\lambda_1 - \lambda_2)\int_0^N y_1 y_2 \Delta t,$$

$$[(\Delta y_2)y_1 - (\Delta y_1)y_2]_0^N = (\lambda_1 - \lambda_2)\int_0^N y_1 y_2 \Delta t,$$

$$\Delta y_2(N)y_1(N) - \Delta y_1(N)y_2(N) - \Delta y_2(0)y_1(0) + \Delta y_1(0)y_2(0) = (\lambda_1 - \lambda_2)\int_0^N y_1 y_2 \Delta t.$$

If we substitute the boundary conditions of the (1.1)-(1.3) problem, we get

$$-Hy_2(N)y_1(N) + Hy_1(N)y_2(N) - hy_2(0)y_1(0) + hy_1(0)y_2(0) = (\lambda_1 - \lambda_2)\int_0^N y_1 y_2 \Delta t$$

$$0 = (\lambda_1 - \lambda_2)\int_0^N y_1 y_2 \Delta t.$$

Since $\lambda_1 \neq \lambda_2$, we get

$$\int_0^N y_1 y_2 \Delta t = 0$$

or

$$\sum_{t=0}^{N-1} y_1(t)y_2(t) = 0.$$

Thus,

$$\langle y_1, y_2 \rangle = 0.$$

□

Theorem 2.2. *Eigenvalues corresponding to the discrete Sturm-Liouville problem (1.1)-(1.3) are all real.*

Proof. Let λ be an eigenvalue and u be eigenfunction corresponding to λ. Since L_Δ is symmetric where $L_\Delta u = \lambda u$, we get

$$\langle L_\Delta u, u \rangle = \langle \lambda u, u \rangle,$$

$$\langle L_\Delta u, u \rangle = \langle u, \bar{L_\Delta} u \rangle = \langle u, \bar{\lambda} u \rangle.$$

Then,

$$\langle \lambda u, u \rangle = \langle u, \bar{\lambda} u \rangle,$$

$$\langle \lambda u, u \rangle - \langle u, \bar{\lambda} u \rangle = 0,$$

$$(\lambda - \bar{\lambda})\langle u, u \rangle = 0.$$

Since u is the eigenvalue, $\langle u, u \rangle \neq 0$, we get

$$(\lambda - \bar{\lambda}) = 0$$

and

$$\lambda = \bar{\lambda}.$$

It yields that, the eigenvalues are all real. This completes the proof. □

Theorem 2.3. *Eigenvalues corresponding to discrete Sturm-Liouville problem (1.1)-(1.3) are all simple.*

Proof. To prove this, we will consider two eigenfunctions corresponding to the same eigenvalue. We will show that these eigenfunctions are linearly dependent.

$$L_\Delta u = -\Delta^2 u + q(t)u = \lambda u,$$

$$L_\Delta v = -\Delta^2 v + q(t)v = \lambda v.$$

If necessary adjustments are made here, we get

$$v(-\Delta^2 u + q(t)u) - u(-\Delta^2 v + q(t)v) = 0,$$

$$\Delta[u(\Delta v) - v(\Delta u)] = 0,$$

$$u(\Delta v) - v(\Delta u) = c, \quad c \in \mathbb{R}. \tag{2.1}$$

Now let us set $t = 0$. Then, we get

$$u(0)\Delta v(0) - v(0)\Delta u(0) = c,$$

$$u(0)hv(0) - v(0)hu(0) = c,$$

and

$$c = 0.$$

If this expression is substituted in (2.1), we get

$$u\Delta v - v\Delta u = 0.$$

If the fact $\Delta\left(\frac{u}{v}\right) = \frac{(\Delta u)v-(\Delta v)u}{v^2}$ is used in the above equation, we get

$$\Delta\left(\frac{u}{v}\right) = 0,$$

$$\frac{u}{v} = c_1,$$

$$u = c_1 v, \quad c_1 \in \mathbb{R}.$$

This means u and v are linearly dependent. Proof is completed. □

Theorem 2.4. *Discrete Sturm-Liouville operator L_Δ is formally self-adjoint on $L_2^{\mathbb{Z}}[0,N]$.*

Proof. Let u and v be two eigenfunctions. We have to show that $\langle v, L_\Delta u\rangle = \langle L_\Delta v, u\rangle$. Let's consider the following equations.

$$L_\Delta u = -\Delta^2 u + q(t)u$$

$$L_\Delta v = -\Delta^2 v + q(t)v$$

If necessary arrangements are made, it yields

$$vL_\Delta u - uL_\Delta v = \Delta[-(\Delta u)v + (\Delta v)u].$$

Let's take discrete integral for both sides on $[0,N]$ to obtain

$$\begin{aligned}
\int_0^N vL_\Delta u\Delta t - \int_0^N uL_\Delta v\Delta t &= \int_0^N \Delta[-(\Delta u)v + (\Delta v)u]\Delta t \\
&= [(\Delta u)v - (\Delta v)u]_0^N \\
&= \Delta u(N)v(N) + \Delta v(N)u(N) + \Delta u(0)v(0) - \Delta v(0)u(0) \\
&= Hu(N)v(N) - Hu(N)v(N) + hu(0)v(0) - hv(0)u(0) \\
&= 0.
\end{aligned}$$

From here, we get $\langle v, L_\Delta u\rangle - \langle L_\Delta v, u\rangle = 0$ and $\langle v, L_\Delta u\rangle = \langle L_\Delta v, u\rangle$. □

The feature of being formally self-adjointness is important in terms of making sense of the problem that is handled physically.

3. Some examples on discrete Sturm-Liouville equation

In this section, the discrete eigenfunctions of the Sturm-Liouville problem (1.1)-(1.3) with various types of conditions will be obtained. In these examples, it is seen that obtaining the eigenfunctions in the discrete case is more troublesome and difficult than in the classical case.

Example 3.1. *Let us consider the below discrete BVP*

$$-\Delta^2 y = \lambda y,$$

$$y(0) = 0, \Delta y(2) = 0.$$

In this example, the discrete boundary value problem will be solved in this particular case. By taking the necessary derivatives in $-\Delta^2 y = \lambda y$, the following equation is obtained.

$$y(t+2) - 2y(t+1) + (\lambda+1)y(t) = 0.$$

Characteristic equation of this equation is $k^2 - 2k + (1+\lambda) = 0$ and its characteristic roots are $k_{1,2} = 1 \pm i\sqrt{-\lambda}$. There are three situations for these roots as $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$.

1. Let $\lambda = 0$. Since k is double-decker root as $k_{1,2} = 1$, we get

$$y(t) = c_1 + c_2 t.$$

   Since $c_1 = 0$, $c_2 = 0$ by the given conditions, λ is not an eigenvalue.
2. Let $\lambda < 0$. We get $k_1 = 1 - \sqrt{-\lambda}$, $k_2 = 1 + \sqrt{-\lambda}$ and

$$y(t) = c_1(1 - \sqrt{-\lambda})^t + c_2(1 + \sqrt{-\lambda})^t.$$

   By the conditions, we get $c_1 = 0$ and $c_2 = 0$. So, λ is not an eigenvalue.
3. Let $\lambda > 0$. We get $k_1 = 1 - i\sqrt{\lambda}$, $k_2 = 1 + i\sqrt{\lambda}$. Since $r = \sqrt{1+\lambda}$, and $\theta = \tan^{-1}(\sqrt{\lambda})$, y has the following form.

$$y(t) = (c_1 \cos\theta t + c_2 \sin\theta t)(1+\lambda)^{\frac{t}{2}}.$$

   Since $y(0) = 0$ and $c_1 = 0$, it should be $\sin 3\theta = 0$ for $c_2 \neq 0$. Then, it yields $\theta = \frac{\pi z}{3}, z = 1, 2, 3, \ldots$ Therefore,

$$y(t) = c_2(1+\lambda)^{\frac{t}{2}} \sin\frac{\pi z}{3} t.$$

Example 3.2. *Consider discrete L-transform to solve below discrete IVP*

$$-\Delta^2 y(t) = \lambda y(t),$$

$$y(0) = 2, \quad \Delta y(0) = 4.$$

Let's apply discrete L-transform to both sides of equation as

$$L_\alpha\left(-\Delta^2 y(t)\right) = L_\alpha\left(\lambda y(t)\right),$$

$$-s^2 Y(s) + sy(0) + \Delta y(0) = \lambda Y(s),$$

$$Y(s) = \frac{2s+4}{\lambda + s^2}.$$

By using inverse discrete L-transform, we get

$$y(t) = 2\cos_{\sqrt{\lambda}}(t,0) + \frac{4}{\sqrt{\lambda}} \sin_{\sqrt{\lambda}}(t,0).$$

Now let's get the eigenfunctions of discrete Sturm-Liouville problem in general case, which includes $q(t) \in L_2^{\mathbb{Z}}[0,N]$.

Example 3.3. *Consider following discrete equation*

$$-\Delta^2 y(t) + q(t)y(t) = \lambda y(t).$$

First, let's find the homogeneous solution of this equation. Since $y(t+2)-2y(t+1)+(\lambda+1)y(t)=0$, characteristic equation is $k^2-2k+(1+\lambda)=0$. There are three situations for the roots as $\lambda>0$, $\lambda=0$ and $\lambda<0$.

1. Let $\lambda>0$. Then, homogeneous solution is

$$y_h(t)=c_1(1+\lambda)^{\frac{t}{2}}\cos\theta t+c_2(1+\lambda)^{\frac{t}{2}}\sin\theta t.$$

Here, $y_p(t)=u_1(t)y_1(t)+u_2(t)y_2(t)$ can be written as a particular solution. Since $y_1=(1+\lambda)^{\frac{t}{2}}\cos\theta t$ and $y_2=(1+\lambda)^{\frac{t}{2}}\sin\theta t$, we get

$$y_p(t)=u_1(t)(1+\lambda)^{\frac{t}{2}}\cos\theta t+u_2(t)(1+\lambda)^{\frac{t}{2}}\sin\theta t.$$

The following operations can be performed.

$$\Delta u_1(t)(1+\lambda)^{\frac{t+1}{2}}\cos\theta(t+1)+\Delta u_2(t)(t+1)(1+\lambda)^{\frac{t+1}{2}}\sin\theta(t+1)=0,$$

$$\Delta u_1(t)(1+\lambda)^{\frac{t+2}{2}}\cos\theta(t+2)+\Delta u_2(t)(t+2)(1+\lambda)^{\frac{t+2}{2}}\sin\theta(t+2)=-q(t)y(t).$$

When necessary solutions are made, we get

$$u_1(t)=\sum_{i=0}^{t-1}\frac{-q(i)y_2(i+1)}{W_{\mathbb{Z}}(i+1)}$$

and

$$u_2(t)=\sum_{i=0}^{t-1}\frac{q(i)y_1(i+1)}{W_{\mathbb{Z}}(i+1)}$$

where

$$W_{\mathbb{Z}}(y_1(i+1),y_2(i+2))=\begin{vmatrix}y_1(i+1) & y_2(i+1)\\ y_1(i+2) & y_2(i+2)\end{vmatrix}.$$

$$\begin{aligned}W_{\mathbb{Z}}(y_1(i+1),y_2(i+2)) &= \begin{vmatrix}(1+\lambda)^{\frac{i+1}{2}}\cos\theta(i+1) & (1+\lambda)^{\frac{i+1}{2}}\sin\theta(i+1)\\ (1+\lambda)^{\frac{i+2}{2}}\cos\theta(i+2) & (1+\lambda)^{\frac{i+2}{2}}\sin\theta(i+2)\end{vmatrix}\\ &= (1+\lambda)^{\frac{2i+3}{2}}\sin\theta\end{aligned}$$

Finally, general solution of given discrete equation is

$$\begin{aligned}y(t,\lambda) &= c_1(1+\lambda)^{\frac{t}{2}}\cos\theta t+c_2(1+\lambda)^{\frac{t}{2}}\sin\theta t+(1+\lambda)^{\frac{t}{2}}\cos\theta t\sum_{i=0}^{t-1}\frac{-q(i)y(i)(1+\lambda)^{\frac{i+1}{2}}\sin\theta(i+1)}{(1+\lambda)^{\frac{2i+5}{2}}\sin\theta}\\ &\quad+(1+\lambda)^{\frac{t}{2}}\sin\theta t\sum_{i=0}^{t-1}\frac{q(i)y(i)(1+\lambda)^{\frac{i+1}{2}}\cos\theta(i+1)}{(1+\lambda)^{\frac{2i+5}{2}}\sin\theta}.\end{aligned}$$

2. Let $\lambda=0$. Homogeneous solution is

$$y_h(t)=c_1+c_2t.$$

Similarly, $y_p(t)=u_1(t)y_1(t)+u_2(t)y_2(t)$ can be written as a particular solution.

$$\Delta u_1(t)+\Delta u_2(t)(t+1)=0$$

$$\Delta u_1(t)+\Delta u_2(t)(t+2)=-q(t)y(t).$$

Wronskian of y_1 and y_2 is as follows.

$$W_{\mathbb{Z}}(y_1(i+1),y_2(i+2))=\begin{vmatrix}1 & i+1\\ 1 & i+2\end{vmatrix}=1.$$

As a result, general solution is obtained as

$$y(t)=c_1+c_2t-\sum_{i=0}^{t-1}q(i)y(i)+t\sum_{i=0}^{t-1}q(i)y(i)(i+1).$$

3. Let $\lambda < 0$. In this case, homogeneous solution is

$$y_h(t) = c_1(1-\sqrt{-\lambda})^t + c_2(1+\sqrt{-\lambda})^t.$$

When the necessary calculations are made, we get

$$W_{\mathbb{Z}}(y_1(i+1), y_2(i+2)) = (1+i)^i(-2\sqrt{-\lambda} - 2\lambda\sqrt{-\lambda}).$$

Considering the definitions of u_1 and u_2, general solution is obtained as follows.

$$\begin{aligned} y(t) &= c_1(1-\sqrt{-\lambda})^t + c_2(1+\sqrt{-\lambda})^t - (1+\sqrt{-\lambda})^t \sum_{i=0}^{t-1} \frac{q(i)y(i)(1-\sqrt{-\lambda})^t}{(1+i)^i(-2\sqrt{-\lambda} - 2\lambda\sqrt{-\lambda})} \\ &\quad + (1-\sqrt{-\lambda})^t \sum_{i=0}^{t-1} \frac{q(i)y(i)(1+\sqrt{-\lambda})^t}{(1+i)^i(-2\sqrt{-\lambda} - 2\lambda\sqrt{-\lambda})} \end{aligned}$$

Example 3.4. *Consider discrete L-transform to solve below discrete IVP*

$$-\Delta^2 y(t) + q(t)y(t) = \lambda y(t)$$

$$y(0) = c_1, \Delta y(0) = c_2.$$

Let $q(t)y(t) = f(t)$. Applying discrete L-transform to both sides of equation, it yields

$$L\{-\Delta^2 y\}(s) + L\{f\}(s) = \lambda L\{y\}(s)$$

$$-s^2 Y(s) + sy(0) + \Delta y(0) + L\{f\}(s) = \lambda Y(s)$$

$$Y(s) = c_1 \frac{s}{s^2 + (\sqrt{\lambda})^2} + \frac{c_2}{\sqrt{\lambda}} \frac{\sqrt{\lambda}}{s^2 + (\sqrt{\lambda})^2} + \frac{1}{\lambda + s^2} L\{f\}(s).$$

By applying discrete inverse L-transform in last equation, we get

$$y(t) = c_1 \cos_{\sqrt{\lambda}}(t,0) + \frac{c_2}{\sqrt{\lambda}} \sin_{\sqrt{\lambda}}(t,0) + L^{-1}\left\{\frac{1}{\lambda + s^2} L\{f\}(s)\right\}.$$

Let's apply discrete convolution to last expression on right-hand side of equation.

$$\begin{aligned} L^{-1}\left\{\frac{1}{\lambda + s^2} L\{f\}(s)\right\} &= \frac{\sin_{\sqrt{\lambda}}(t,0)}{\sqrt{\lambda}} * q(t)y(t) \\ &= \sum_{r=0}^{t-1} \frac{\sin_{\sqrt{\lambda}}(r,0)}{\sqrt{\lambda}} q(t-\sigma(r))y(t-\sigma(r)) \end{aligned}$$

If we consider expression that we found in $y(t)$ solution, we get

$$y(t) = c_1 \cos_{\sqrt{\lambda}}(t,0) + \frac{c_2}{\sqrt{\lambda}} \sin_{\sqrt{\lambda}}(t,0) + \sum_{r=0}^{t-1} \frac{\sin_{\sqrt{\lambda}}(r,0)}{\sqrt{\lambda}} q(t-\sigma(r))y(t-\sigma(r)).$$

Finally, let us express two important concepts in the solution of the examples given in this section as a reminder.

Remark 3.5 ([2]). *Let y_1 and y_2 be delta differentiable functions. Discrete Wronskian of these functions is defined by*

$$W_{\mathbb{Z}} = \begin{pmatrix} y_1(t) & y_2(t) \\ \Delta y_1(t) & \Delta y_2(t) \end{pmatrix}.$$

Remark 3.6 ([11]). *Let $f, g : \mathbb{N}_a \to \mathbb{R}$. Delta convolution product of f and g is defined by*

$$(f * g)(t) = \sum_{r=a}^{t-1} f(r) g(t-\sigma(r)+a)$$

for $t \in \mathbb{N}_a$.

4. Conclusion

Difference equations are used in mathematical models and numerical solutions of differential equations in various fields. Sturm-Liouville problems are used to solve problems in many physical fields. In this study, we examined discrete Sturm-Liouville operator and its spectral properties. We have obtained a solution for the discrete Sturm-Liouville problem we are considering using some methods. We did one of these solutions by considering the existing discrete L-transform. We proved the basic spectral properties of the operator for the discrete Sturm-Liouville difference equation, such as self-adjointness, orthogonality of eigenfunctions, and realness of eigenvalues. We hope that the study will guide researchers for discrete case of Sturm-Liouville problem.

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An Examination for the Intersection of Two Ruled Surfaces

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Abstract

In this study, firstly, each natural lift curve of the main curve is corresponded to the ruled surface by exploiting E. Study mapping and the relation among the subset of the tangent bundle of unit 2-sphere, $T\bar{M}$ and ruled surfaces in $\mathbb{R}^3$. Secondly, the intersection of two ruled surfaces, which are obtained by using the relation given above, is examined for the condition of the zero-set of $\lambda(u,v) = 0$. Then, all redundant and non-redundant solutions of the zero-set are investigated. Furthermore, the degenerate situations $(u,v) = 0$, where the whole plane is degenerated by the zero-set, are denoted. Finally, some examples are given to verify the results.

1. Introduction

In mathematics, the ruled surface, whose parametric expression is given as below, is defined as the set of points drawn by a moving straight line. That is, the parametric representation of the ruled surface ϕ, acquired by the set $\{\vec{k}(u), \vec{q}(u)\}$, is

$$\vec{h}(u,v) = \vec{k}(u) + v\vec{q}(u), \quad u \in I, \quad v \in \mathbb{R}$$

for the set $\{\vec{k}(u), \vec{q}(u)\}$. Here, $k = \vec{k}(u)$ is a point and $q = \vec{q}(u)$ is a non-null vector in $\mathbb{R}^3$. Moreover, $\vec{k}(u)$ and $\vec{q}(u)$ are called the base curve and various of the generating lines, respectively, see [1]. From several significant applications of this surface, many mathematicians dealt with this surface in literature. Some of them are as follows: in [2], considering geometric invariants of space curves, a categorization of special developable surfaces under special condition was investigated. In [3], the theory of Mannheim curves was extended to the ruled surfaces. In [4], sectional curvature of ruled surfaces was computed in Minkowski space. In [5], some important properties of special ruled surfaces were investigated according to modified orthogonal frame. In [6], the evolution of several associated type ruled surfaces was defined. Furthermore, the Mannheim offset of developable ruled surface was defined. In [7], a correspondence among unit dual sphere, DS^2, the tangent bundle of the unit 2-sphere, TS^2, and non-null ruled surfaces was mentioned in detail. In [8], taking this correspondence into consideration, the ruled surface was described by using E. Study mapping and the relation between the tangent bundle of unit 2-sphere and non-null ruled surfaces. In [9], the non-null ruled surfaces were introduced by exploiting E. Study mapping and the isomorphism between pseudo-spheres and the tangent bundles of pseudo-spheres in E^3_1. In [10], Frenet vector fields and invariants of timelike ruled surfaces were explored. In [11], the ruled surface according to the Darboux frame was introduced. In [12], in 3-dimensional contact metric manifold, the properties of the ruled surface were defined. In [13], some geometric interpretations for timelike ruled surfaces were examined.
The surface intersection problem has important research fields in differential geometry, geometric modeling, architecture, computer aided design, etc. Several algorithms were considered for the intersection of two surfaces in literature. In [14], the intersection of two ruled surfaces was investigated under some special conditions. In [15], in higher dimension, the problems of curve and surface intersections were formulated. Additionally, the algebraic set to a lower dimensional space was constructed. In [16], an adaptive algorithm was developed for finding the intersection curves. In [17], a boundary method for surface

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intersection was studied for smooth parametric surfaces defined over rectangular and triangular domains. In [18], a hybrid algorithm for the calculation of the intersection of an algebraic surface and a rational polynomial parametric surface patch was computed.
The theory of curves has substantial field in geometry, engineering, computer modeling mentioned above, etc. The relation among Frenet operators for given two smooth curves opened new research areas for many mathematicians in literature. One of the curves, which is compared with Frenet operators of the main curve, is the natural lift curve. Namely, the natural lift curve, which was firstly encountered in J. A. Thorpe's book in [19], is defined as a smooth curve obtained by the unit tangent vectors of any given smooth curve:
for the curve Γ, $\bar{\Gamma}$ is called the natural lift of Γ on $T\bar{M}$, which provides the following equation:

$$\bar{\Gamma}(u) = (\bar{q}(u), \bar{\vartheta}(u)) = (q^{'}(u)|_{\gamma(u)}, \vartheta^{'}(u)|_{\vartheta(u)}).$$

There are some studies about the geometric interpretations about the natural lift curve. Some of them are as follows: in [20], some properties of the natural lift curve were investigated in $\mathbb{R}^3$. In [21], the correspondence between the natural lift curve and its involute curve was given. In [22], the Frenet frames of the natural lift curve and its Bertrand mate were examined. In [23], dual spherical curves of the natural lift curve were denoted in terms of Frenet vector fields. In [24], the condition being the natural lifts of the spherical indicatrix of the curve is an integral curve of the geodesic spray was introduced. In [25], the authors proved that if the natural lifts geodesic spray of spherical indicator curvatures of Mannheim partner curve was an integral curve, Mannheim Curve was obtained. In [26], the condition being the natural lifts of the spherical indicatrix of the evolute curve are an integral curve of geodesic spray was expressed.
There is no research about the intersection of two ruled surfaces generated by the natural lift curves in literature. Therefore, in this paper, using the mentioned isomorphism and using some properties about dual numbers given in [27], we obtain two ruled surfaces acquired by the natural lift curves. Furthermore, we analyze the cases for the intersection of two ruled surfaces by examining the zero-set of $\lambda(u,v) = 0$. Then, we categorize all redundant and non-redundant solutions of the zero-set. Moreover, in the subsections, we consider the degenerate situations $(u,v) = 0$, where the whole plane is degenerated by the zero-set.
This paper is organized as follows: in Section 2, some basic definitions and theorems about the dual numbers and the ruled surfaces acquired by the natural lift curve are mentioned. In Section 3, the intersection of two ruled surfaces acquired by the natural lift curves is examined by calculating the zero-set $\lambda(u,v) = 0$. Additionally, all redundant and non-redundant solutions are denoted. Then, some examples are given to verify the results. In Section 4, obtained results are discussed in detail.

2. Preliminaries

The set of dual numbers is

$$\mathbb{D} = \{X = x + \varepsilon x^*; (x, x^*) \in \mathbb{R} \times \mathbb{R}, \ \ \varepsilon^2 = 0\},$$

where $\vec{x}$ and $\vec{x}^*$ are real and dual parts of $\vec{X}$, respectively. If $\vec{x}$ and $\vec{x}^*$ are vectors in $\mathbb{R}^3$, $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$ is called a dual vector. For $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$ and $\vec{Y} = \vec{y} + \varepsilon\vec{y}^*$, the basic operations are given as follows:
the addition is

$$\vec{X} + \vec{Y} = (\vec{x} + \vec{y}) + \varepsilon(\vec{x}^* + \vec{y}^*),$$

and the inner product is

$$\langle\vec{X}, \vec{Y}\rangle = \langle\vec{x}, \vec{y}\rangle + \varepsilon(\langle\vec{x}^*, \vec{y}\rangle + \langle\vec{x}, \vec{y}^*\rangle).$$

Moreover, the vector product is

$$\vec{X} \times \vec{Y} = \vec{x} \times \vec{y} + \varepsilon(\vec{x} \times \vec{y}^* + \vec{x}^* \times \vec{y}).$$

The norm of $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$ is

$$|\vec{X}| = \sqrt{\langle\vec{x}, \vec{x}\rangle} + \varepsilon\frac{\langle\vec{x}, \vec{x}^*\rangle}{\sqrt{\langle\vec{x}, \vec{x}\rangle}}, \ \ \vec{x} \neq 0.$$

The dual vector is called unit dual vector, if $|\vec{X}| = 1$. The unit dual sphere, which consists of all unit dual vectors, is given as the following set:

$$DS^2 = \{\vec{X} = \vec{x} + \varepsilon\vec{x}^* \in \mathbb{D}^3 : |\vec{X}| = 1\}. \tag{2.1}$$

For more information about dual vectors, see [27].
Let S^2 be a unit 2-sphere in $\mathbb{R}^3$. The tangent bundle of S^2 is given by

$$TS^2 = \{(q, \vartheta) \in \mathbb{R}^3 \times \mathbb{R}^3 : |q| = 1, \ \ \langle q, \vartheta\rangle = 0\},$$

where "$\langle , \rangle$" is the inner product and "$|,|$" is the norm in $\mathbb{R}^3$, respectively, see [8]. Let $T\bar{M}$ also be a subset of TS^2, defined by

$$T\bar{M} = \{(\bar{q},\bar{\vartheta}) \in \mathbb{R}^3 \times \mathbb{R}^3 : |\bar{q}| = 1, \ \langle \bar{q},\bar{\vartheta}\rangle = 0\}. \tag{2.2}$$

Here, $\bar{q}$ and $\bar{\vartheta}$ are the derivatives of q and ϑ, respectively, see [23]. From Eqs. (2.1) and (2.2), the correspondence between the unit dual sphere and the subset of the tangent bundle of unit 2-sphere is given by

$$\begin{array}{rcl} T\bar{M} & \longrightarrow & DS^2, \\ \bar{\Gamma} = (\bar{q},\bar{\vartheta}) & \longmapsto & \vec{\bar{\Gamma}} = \vec{\bar{q}} + \varepsilon \vec{\bar{\vartheta}}. \end{array}$$

Theorem 2.1 (E. Study mapping)**.** *There is one-to-one relation between the oriented lines in $\mathbb{R}^3$ and the points of DS^2.*

Theorem 2.2. *Let $\bar{\Gamma}(u) = (\vec{\bar{q}}(u), \bar{\vartheta}(u)) \in T\bar{M}$. In $\mathbb{R}^3$, the ruled surface acquired by the natural lift curve $\bar{\Gamma}(u)$ can be expressed by*

$$\bar{\phi}(u,v) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u) + v\vec{\bar{q}}(u),$$

where

$$\beta(u) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u)$$

is the base curve of $\bar{\phi}$.

As a result, the isomorphism among $T\bar{M}$, DS^2 and $\mathbb{R}^3$ can be written by

$$\begin{array}{rcl} T\bar{M} & \longrightarrow & DS^2 \longrightarrow \mathbb{R}^3, \\ \bar{\Gamma}(u) = (\bar{q}(u),\bar{\vartheta}(u)) & \longmapsto & \vec{\bar{\Gamma}}(u) = \vec{\bar{q}}(u) + \varepsilon \vec{\bar{\vartheta}}(u) \longmapsto \bar{\phi}(u,v) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u) + v\vec{\bar{q}}(u). \end{array}$$

Here $\bar{\phi}(u,v)$ is the ruled surface in $\mathbb{R}^3$ related to the dual curve $\bar{\Gamma}(u) = \vec{\bar{q}}(u) + \varepsilon\vec{\bar{\vartheta}}(u) \in DS^2$ (or to the natural lift curve $\bar{\Gamma}(u) \in T\bar{M}$), see [23].

3. Some characterizations for the intersection of two ruled surfaces

Let $\bar{\phi}_1(u,s)$ and $\bar{\phi}_2(v,t)$ be the ruled surfaces acquired by the natural lift curves $\bar{\alpha}(u) = (\alpha_1(u),\alpha_1^*(u))$ and $\tilde{\alpha}(v) = (\alpha_2(v),\alpha_2^*(v))$, where $\bar{\alpha}(u)$ and $\tilde{\alpha}(v)$ on $T\bar{M}$, respectively. Considering the isomorphism mentioned above, the ruled surface generated by $\bar{\alpha}(u) = (\alpha_1(u),\alpha_1^*(u))$ is

$$\bar{\phi}_1(u,s) = \alpha_1(u) \times \alpha_1^*(u) + s\alpha_1^*(u),$$

where the base curve is

$$C(u) = \alpha_1(u) \times \alpha_1^*(u).$$

Moreover, the ruled surface generated by $\tilde{\alpha}(v) = (\alpha_2(v),\alpha_2^*(v))$ is

$$\bar{\phi}_2(v,t) = \alpha_2(v) \times \alpha_2^*(v) + t\alpha_2^*(v),$$

where the base curve is

$$D(v) = \alpha_2(v) \times \alpha_2^*(v).$$

Let L_1 and L_2 denote the rullings of $\bar{\phi}_1$ and $\bar{\phi}_2$, respectively. As $\bar{\phi}_1$ and $\bar{\phi}_2$ intersect, we write

$$\bar{\phi}_1(u,s) = \bar{\phi}_2(v,t).$$

That is,

$$C(u) - D(v) = -s\alpha_1^*(u) + t\alpha_2^*(v). \tag{3.1}$$

It is obvious that $C(u)-D(v)$ is the linear combination of $\alpha_1^*(u)$ and $\alpha_2^*(v)$. In this equation, $C(u)-D(v)$, $\alpha_1^*(u)$ and $\alpha_2^*(v)$ are linearly dependent. Hence, the following condition is satisfied:

$$\lambda(u,v) = det(\alpha_1^*(u),\alpha_2^*(v),C(u)-D(v)) = 0.$$

Now, we will investigate the solutions of the determinant given as above in detail.

3.1. Redundant and non-redundant solutions

The condition of being $\lambda(u,v)=0$ is a desired condition. However, it is not sufficient condition for examining the intersection of two rulling lines L_1 and L_2. Some redundant points, which do not related to real intersection points of these ruled surfaces, could be contained by the solution set of $\lambda(u,v)=0$. Hence, we categorize all probabilities for redundant solutions as below: the solutions of $\lambda(u,v)=0$ denote the linear dependency of $C(u)-D(v), \alpha_1^*(u)$ and $\alpha_2^*(v)$:

$$c_1\alpha_1^*(u)+c_2\alpha_2^*(v)+c_3(C(u)-D(v))=0 \tag{3.2}$$

for some non-zero constants c_1,c_2,c_3. There are two conditions for the solution of Eq. (3.2):
(i) if $c_3\neq 0$, then Eq. (3.1) is obtained. We say that two rulling lines intersect. Furthermore, there has not been any redundant solution of $\lambda(u,v)=0$ under this condition.
(ii) If $c_3=0$, we get

$$\alpha_1^*(u)=-\frac{c_1}{c_2}\alpha_2^*(v)$$

for $c_1\neq 0$ and $c_2\neq 0$. It is deduced that $\alpha_1^*(u)$ and $\alpha_2^*(v)$ are parallel or opposite.
Additionally, (u,v) provides $\lambda(u,v)=0$ without checking the intersection of L_1 and L_2. In this way, (u,v) is called redundant if $\alpha_1^*(u)$ and $\alpha_2^*(v)$ are parallel or opposite. However, the related rullings do not coincide. The condition of being parallel or opposite for $\alpha_1^*(u)$ and $\alpha_2^*(v)$ can be expressed by employing the zero-set of another function as below:

$$\Delta(u,v)=\|\alpha_1^*(u)\times\alpha_2^*(v)\|^2=\|\alpha_1^*(u)\|^2\|\alpha_2^*(v)\|^2-\langle\alpha_1^*(u),\alpha_2^*(v)\rangle^2=0.$$

The zero set of $\Delta(u,v)=0$ is a subset of the zero set of $\lambda(u,v)=0$. Furthermore, L_1 and L_2 overlap each other iff $\Delta(u,v)=0$ and there exist the following equations:

$$\begin{aligned}\delta_1(u,v) &= \|\alpha_1^*(u)\times(C(u)-D(v))\|^2\\ &= \|\alpha_1^*(u)\|^2\|C(u)-D(v)\|^2-\langle\alpha_1^*(u),C(u)-D(v)\rangle^2=0,\\ \delta_2(u,v) &= \|\alpha_2^*(v)\times(C(u)-D(v))\|^2\\ &= \|\alpha_2^*(v)\|^2\|C(u)-D(v)\|^2-\langle\alpha_2^*(v),C(u)-D(v)\rangle^2=0.\end{aligned}$$

$\Delta(u,v)=\delta_1(u,v)=\delta_2(u,v)=0$ iff two rulling lines L_1 and L_2 overlap each other.
As a result, $\Delta(u,v)+\delta_1(u,v)+\delta_2(u,v)=0$, since $\Delta(u,v),\delta_1(u,v),\delta_2(u,v)\geq 0$. Consequently, the solution of $\lambda(u,v)=0$ is redundant iff $\Delta(u,v)=0$ and $\Delta(u,v)+\delta_1(u,v)+\delta_2(u,v)\neq 0$.

3.2. Birational correspondence

As L_1 and L_2 intersect, s and t are expressed as rational bivariate functions u and v. Calculating the inner product of Eq. (3.1) with $\alpha_1^*(u)$ and $\alpha_2^*(v)$, we have

$$\begin{pmatrix}\|\alpha_1^*(u)\|^2 & -\langle\alpha_1^*(u),\alpha_2^*(v)\rangle\\ -\langle\alpha_1^*(u),\alpha_2^*(v)\rangle & \|\alpha_2^*(v)\|^2\end{pmatrix}\begin{pmatrix}s\\ t\end{pmatrix}=\begin{pmatrix}\langle\alpha_1^*(u),D(v)-C(u)\rangle\\ \langle\alpha_2^*(v),C(u)-D(v)\rangle\end{pmatrix}.$$

As $\Delta(u,v)$ is not equal to 0, this matrix equation becomes non-singular. Furthermore, there exists unique rational solutions of $s(u,v)$ and $t(u,v)$. So, we find

$$\begin{aligned}s(u,v) &= \frac{\|\alpha_2^*(v)\|^2\langle\alpha_1^*(u),D(v)-C(u)\rangle+\langle\alpha_1^*(u),\alpha_2^*(v)\rangle\langle\alpha_2^*(v),C(u)-D(v)\rangle}{\|\alpha_1^*(u)\|^2\|\alpha_2^*(v)\|^2-\langle\alpha_1^*(u),\alpha_2^*(v)\rangle^2},\\ t(u,v) &= \frac{\|\alpha_1^*(u)\|^2\langle\alpha_2^*(v),C(u)-D(v)\rangle+\langle\alpha_1^*(u),\alpha_2^*(v)\rangle\langle\alpha_1^*(u),D(v)-C(u)\rangle}{\|\alpha_1^*(u)\|^2\|\alpha_2^*(v)\|^2-\langle\alpha_1^*(u),\alpha_2^*(v)\rangle^2}.\end{aligned}$$

In this situation, as $\Delta(u,v)\approx 0$, the calculation of $s(u,v)$ and $t(u,v)$ is unstable numerically. In this situation, the squared distance $\delta(u,v)$ is measured by using two parallel ruling lines and distinguish the lines if their squared distance is bigger than a particular contribution: $\delta(u,v)\geq\varepsilon^2$, where $\delta(u,v)$ is expressed as $\frac{\delta_1(u,v)}{\|\alpha_1^*(u)\|^2}$, the squared distance between $D(v)$ and L_1 or $\frac{\delta_2(u,v)}{\|\alpha_2^*(v)\|^2}$, the squared distance between $C(u)$ and L_2.
Assume that $\tilde{C}$, which has the projection C onto the uv-plane, is a section of the intersection curve of $\bar{\phi}_1$ and $\bar{\phi}_2$, respectively. In this situation, these ruled surfaces do not coincide. If $\tilde{C}$ is a connected curve section, C is accepted as a connected section of $\lambda(u,v)=0$. Generally, the opposite condition is not true. There is not any unique solution for $s(u,v)$ and $t(u,v)$ as a connected curve section C of $\lambda(u,v)=0$ consists of a point (u,v) of $\Delta(u,v)=0$. Moreover, in some degenerate cases, the intersection curve could be empty or a single point, while the zero-set of $\lambda(u,v)\equiv 0$ is the all plane. In these conditions, there is not any relation between an intersection curve $\tilde{C}$ and C of $\lambda(u,v)=0$. Genarally, with the exception of the following conditions, there exists birational correspondence between $\tilde{C}$ and C on $\lambda(u,v)=0$:

(i) parallel ruling lines,
(ii) degenerate cases
(iii) peaks and self-intersections.

Assume that $\bar{\phi}_1$ has an peak P such that $\bar{\phi}_1(u,s)=P$, for $u_0 \leq u \leq u_1$, and P lies on the other ruled surface $\bar{\phi}_2(v_0,t_0)$. Moreover, the zero-set of $\lambda(u,v)=0$ includes a line section: $\{(u,v_0): u_0 \leq u \leq u_1\}$. The whole line section is related to a single point P in the intersection of $\bar{\phi}_1$ and $\bar{\phi}_2$. Q lies on the intersection curve: $Q=\bar{\phi}_1(u_1,s_1)=\bar{\phi}_1(u_2,s_2)=\bar{\phi}_2(v_1,t_1)$. The same intersection point Q relates to the two different solutions (u_1,v_1) and (u_2,v_1). Therefore, there is not birational relation between C and $\tilde{C}$ in these situations.
All singular points of $\bar{\phi}_1(u,s)$ are on the striction curve:

$$\bar{C}(u)=(\alpha_1(u)\times\alpha_1^*(u))-\frac{\langle(\alpha_1(u)\times\alpha_1^*(u))',(\alpha_1^*)'(u)\rangle}{\langle(\alpha_1^*)'(u),(\alpha_1^*)'(u)\rangle}.$$

If the curve $\bar{C}$ degenerates into a point, this point is the peak of a conical surface $\bar{\phi}_1(u,s)$. Let $\bar{\phi}_1(u,s)$ be noncylindrical ruled surface. Then, all singular points of $\bar{\phi}_1(u,s)$ could be distinguished along the striction curve $\bar{C}$ by controlling the condition as follows:

$$\langle\bar{C}'(u)\times\alpha_1(u),(\alpha_1)'(u)\rangle=0.$$

Self-intersection points of $\bar{\phi}_1$ could be distinguished by the intersection of $\bar{\phi}_1(u,s)$ with $\bar{\phi}_1(v,t)$. $u-v=0$, which is described as the diogonal line, is included in the zero-sets of all variate functions. By eliminating the diagonal line from these zero-sets, the self-intersection of $\bar{\phi}_1(u,s)$ could be characterized.

Example 3.1. *Let us consider $\alpha_1(u)=(0,0,1)$ and the vector $\alpha_1^*(u)=(\frac{1-u^2}{1+u^2},\frac{2u}{1+u^2},0)$ in $\mathbb{R}^3$. Since $\|\alpha_1(u)\|=1$ and $\langle\alpha_1(u),\alpha_1^*(u)\rangle=0$, the natural lift curve $\alpha(u)=(\alpha_1(u),\alpha_1^*(u))\in T\bar{M}$. Then, the ruled surface corresponding to the natural lift curve $\alpha(u)=(\alpha_1(u),\alpha_1^*(u))$ is given as*

$$\bar{\phi}_1(u,s)=\left(\frac{-2u}{1+u^2},\frac{u^2-1}{1+u^2},0)+s(\frac{1-u^2}{1+u^2},\frac{2u}{1+u^2},0\right)$$

where the base curve is

$$C(u)=\left(\frac{-2u}{1+u^2},\frac{u^2-1}{1+u^2},0\right).$$

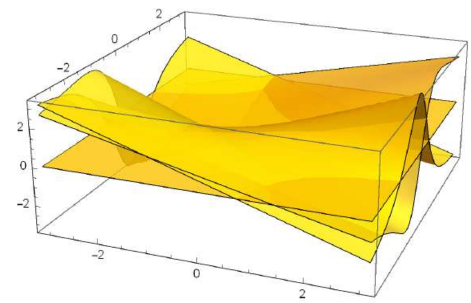

Figure 3.1: The ruled surface $\bar{\phi}_1(u,s)$ acquired by $\bar{\alpha}(u)$

Let us consider another vector couple as $\alpha_2(v)=(\frac{-2v}{1+v^2},\frac{1-v^2}{1+v^2},0)$ and the vector $\alpha_2^*(v)=(\frac{1-v^2}{1+v^2},\frac{2v}{1+v^2},0)$ in $\mathbb{R}^3$. Since $\|\alpha_2(v)\|=1$ and $\langle\alpha_2(v),\alpha_2^*(v)\rangle=0$, the natural lift curve $\tilde{\alpha}(v)=(\alpha_2(v),\alpha_2^*(v))\in T\bar{M}$. Then, the ruled surface corresponding to the natural lift curve $\tilde{\alpha}(v)=(\alpha_2(v),\alpha_2^*(v))$ is given as

$$\bar{\phi}_2(v,t)=\left(0,0,\frac{-v^4-2v^2-1}{v^4+2v^2+1})+t(\frac{1-v^2}{1+v^2},\frac{2v}{1+v^2},0\right)$$

where the base curve is

$$D(v)=\left(0,0,\frac{-v^4-2v^2-1}{v^4+2v^2+1}\right).$$

If these ruled surfaces intersect, we get

$$\bar{\phi}_1(u,s)=\bar{\phi}_2(v,t).$$

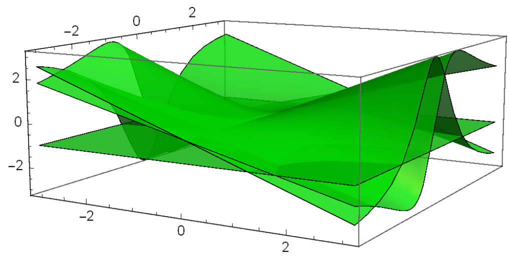

Figure 3.2: The ruled surface $\bar{\phi}_2(v,t)$ generated by $\bar{\alpha}(v)$

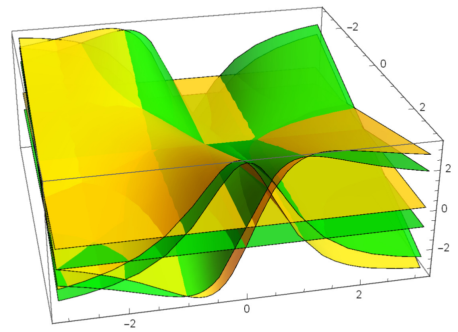

Figure 3.3: The intersection of $\bar{\phi}_1(u,s)$ and $\bar{\phi}_2(v,t)$

Considering these ruled surfaces, we calculate

$$\begin{aligned}
\lambda(u,v) &= \frac{2(v-u)(1-uv)}{(1+u^2)(1+v^2)},\\
\Delta(u,v) &= \left\|\left(0,0,\frac{2(v-u)(1-uv)}{(1+u^2)(1+v^2)}\right)\right\|^2,\\
\delta_1(u,v) &= \left\|\left(\frac{2u}{1+u^2},\frac{1-u^2}{u^2+1},\frac{u^4+4u^2-1}{u^4+2u^2+1}\right)\right\|^2,\\
\delta_2(u,v) &= \left\|\left(\frac{2v}{1+v^2},\frac{v^2-1}{v^2+1},\frac{(1-v^2)(u^2-1)+4uv}{(u^2+1)(v^2+1)}\right)\right\|^2.
\end{aligned}$$

The real solutions of $\lambda(u,v)=0$ and $\Delta(u,v)=0$ represents a planar curve $(v-u)(1-uv)=0$. Hence, the zero-set of $\Delta(u,v)=0$ is the subset of $\lambda(u,v)=0$. It is simply to control that $\Delta(u,v)+\delta_1(u,v)+\delta_2(u,v)>0$ for all (u,v). Thus, all solutions of $\Delta(u,v)=0$ are redundant solutions of $\lambda(u,v)=0$.
The non-redundant solution of $C=\{(u,v):\lambda(u,v)=0,\Delta(u,v)\neq 0\}$ is comprised of four components in the $uv-$ plane, given as follows:

$$\begin{aligned}
C_1 &= \{(u,v): uv=1, u<-1\},\\
C_2 &= \{(u,v): uv=1, -1<u<0\},\\
C_3 &= \{(u,v): uv=1, 0<u<1\},\\
C_4 &= \{(u,v): uv=1, u>1\}.
\end{aligned}$$

Hence, the intersection curve contains four connected components. Moreover, the limit points of C are $(1,1)$ and $(-1,-1)$. However, they are not included in the solution set C. In a small neighborhood of these limit points, the parameter values of $s(u,v)$ and $t(u,v)$ diverge to $\mp\infty$.

3.3. Degenerate cases

In some situations, the whole plane is degenerated by the solution set. These situations cover all probabilities for degenerate cases of $\lambda(u,v)=0$. The exceptions occurs when two ruled surfaces coincide. We denote that the two ruled surfaces, which overlap each other, are planes or rational bilinear surfaces as below. Therefore, the determination of all degenerate cases may be decreased for categorizing the special types of input surfaces: whether the surface is a plane, cylinder, cone, quadric, etc. As $\alpha_1^*(u)$ and $\alpha_2^*(v)$ are parallel or opposite for all pairs of (u,v), $\bar{\phi}_1(u,s)$ and $\bar{\phi}_2(v,t)$ are cylindrical surfaces that are parallel to each other. In the contrary case, $\alpha_1^*(u)$ and $\alpha_2^*(v)$ are parallel or opposite for the couple of (u,v) providing the condition of $\Delta(u,v)=0$. Generally, $\Delta(u,v)=0$ could not be space-filling curve. Thus, there exists an area $[u_a,u_b]\times[v_a,v_b]$, where $\Delta(u,v)\neq 0$. For $\lambda(u,v)=0, u_a\leq u\leq u_b, v_a\leq v\leq v_b$, we write

$$C(u)-D(v)=-s(u,v)\alpha_1^*(u)+t(u,v)\alpha_2^*(v).$$

From this equation, we conclude that each ruling line L_1 of $\bar{\phi}_1$ intersects with all other rulling lines L_2 of $\bar{\phi}_2$ and the converse is true. There are three different cases:
first of all is that there is a couple of lines L_1 and L_2 which intersect at P. Each ruling line L_2 of $\bar{\phi}_2$ coincides with both L_1 and L_2. There are two subcases to examine:
if there are infinitely many L_2 running through the point P, $\bar{\phi}_2$ must be a conical surface at P.
Otherwise, infinitely many lines L_2 must be contained in the plane identified by L_1 and L_2. Then, a plane is degenerated by the whole surface $\bar{\phi}_2$.
Similarly, the surface type of $\bar{\phi}_1$ has also been identified. If $\bar{\phi}_2$ is a non-planar conical surface, all ruling lines L_1 of $\bar{\phi}_1$ run through the apex P. Hence, $\bar{\phi}_1$ becomes a conical surface.
Otherwise, $\bar{\phi}_2$ is a plane. All ruling lines L_1 of $\bar{\phi}_1$ are the subset of the plane for $\bar{\phi}_2$. Hence, $\bar{\phi}_1$ and $\bar{\phi}_2$ degenerate into the same plane.
Second of all is that there is a pair of parallel lines L_1 and L_2.
Then, there has been a unique plane identified by these two parallel lines. All ruling lines L_2 of $\bar{\phi}_2$ are subsets of the plane. Therefore, the whole surface $\bar{\phi}_2$ degenerates into the plane. Likewise, $\bar{\phi}_1$ has also been contained in the same plane.
Third of all is that any two different lines L_1 and L_2 are skew. Furthermore, any two different lines L_1 and L_2 are also skew. (Otherwise, we will result in first or second cases given above examined before.) Assume that T is the intersection point of L_1 and L_2. Let us consider as $\bar{\phi}_1=T$ for all $(u,v)\in[u_a,u_b]\times[v_a,v_b]$. Then $\bar{\phi}_1$ and $\bar{\phi}_2$ coincide. Therefore, $\bar{\phi}_1$ and $\bar{\phi}_2$ indicate the same surface. $\bar{\phi}_1$ generates a rational bilinear surface with special conditions of u and v. Additionally, this surface are considered as a quadric surface.

4. Conclusion

In this paper, different from literature, the ruled surfaces acquired by the natural lift curves are defined by using E. Study mapping and the isomorphism between the subset of the unit tangent bundle of unit 2-sphere, $T\bar{M}$, and unit dual sphere, DS^2. Taking the the intersection of ruled surfaces obtained in this way into consideration, the cases for the intersection are investigated by exploiting $\lambda(u,v)=0$. Therefore, redundant and non-redundant solutions are scrunutized under some conditions. Moreover, being to be degenerate conditions are denoted in detail. Then, obtained results are illustrated by some significiant examples.

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