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Research Article

The algebra of thin measurable operators is directly finite

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ABSTRACT. Let $\mathcal{M}$ be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ equipped with a faithful normal semifinite trace τ, $S(\mathcal{M},\tau)$ be the *-algebra of all τ-measurable operators. Let $S_0(\mathcal{M},\tau)$ be the *-algebra of all τ-compact operators and $T(\mathcal{M},\tau) = S_0(\mathcal{M},\tau) + \mathbb{C}I$ be the *-algebra of all operators $X = A + \lambda I$ with $A \in S_0(\mathcal{M},\tau)$ and $\lambda \in \mathbb{C}$. It is proved that every operator of $T(\mathcal{M},\tau)$ that is left-invertible in $T(\mathcal{M},\tau)$ is in fact invertible in $T(\mathcal{M},\tau)$. It is a generalization of Sterling Berberian theorem (1982) on the subalgebra of thin operators in $\mathcal{B}(\mathcal{H})$. For the singular value function $\mu(t;Q)$ of $Q = Q^2 \in S(\mathcal{M},\tau)$, the inclusion $\mu(t;Q) \in \{0\} \bigcup [1,+\infty)$ holds for all $t > 0$. It gives the positive answer to the question posed by Daniyar Mushtari in 2010.

Keywords: Hilbert space, von Neumann algebra, semifinite trace, τ-measurable operator, τ-compact operator, singular value function, idempotent.

2020 Mathematics Subject Classification: 16E50, 46L51.

1. INTRODUCTION

In this paper, we extend the Sterling Berberian's result [2] (see also [12]) on direct finiteness of the algebra of thin operators on an infinite-dimensional Hilbert space to the Irving Segal's non-commutative integration setting [16]. Let $\mathcal{M}$ be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ equipped with a faithful normal semifinite trace τ, $S(\mathcal{M},\tau)$ be the *-algebra of all τ-measurable operators. Let $S_0(\mathcal{M},\tau)$ be the *-algebra of all τ-compact operators and $T(\mathcal{M},\tau) = S_0(\mathcal{M},\tau) + \mathbb{C}I$ be the *-algebra of all operators $X = A + \lambda I$ with $A \in S_0(\mathcal{M},\tau)$ and a complex number λ. We prove that every operator of $T(\mathcal{M},\tau)$ left-invertible in $T(\mathcal{M},\tau)$ is actually invertible in $T(\mathcal{M},\tau)$ (Theorem 3.1). Assume that $A \in S(\mathcal{M},\tau)$ and $B \in T(\mathcal{M},\tau)$. We have $AB \in T(\mathcal{M},\tau)$ if and only if $BA \in T(\mathcal{M},\tau)$ (Theorem 3.2). For the singular value function $\mu(t;Q)$ of $Q = Q^2 \in S(\mathcal{M},\tau)$, we have $\mu(t;Q) \in \{0\} \bigcup [1,+\infty)$ for all $t > 0$ (Theorem 3.3). It is the positive answer to the question by Daniyar Mushtari of year 2010.

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2. PRELIMINARIES

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, let $\mathcal{P}(\mathcal{M})$ be the lattice of projections in $\mathcal{M}$, I be the unit of $\mathcal{M}$. Also $\mathcal{M}^+$ denotes the cone of positive elements in $\mathcal{M}$. A mapping $\varphi : \mathcal{M}^+ \to [0,+\infty]$ is called *a trace*, if $\varphi(X+Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$); $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is called *faithful*, if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$; *normal*, if $X_i \uparrow X$ $(X_i, X \in \mathcal{M}^+) \Rightarrow$

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$\varphi(X) = \sup \varphi(X_i)$; *semifinite*, if $\varphi(X) = \sup\{\varphi(Y) : \ Y \in \mathcal{M}^+, \ Y \leq X, \ \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated to the von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. Let τ be a faithful normal semifinite trace on $\mathcal{M}$. A closed operator X, affiliated to $\mathcal{M}$ and possesing a domain $\mathfrak{D}(X)$ everywhere dense in $\mathcal{H}$ is said to be *τ-measurable* if, for any $\varepsilon > 0$, there exists a $P \in \mathcal{P}(\mathcal{M})$ such that $P\mathcal{H} \subset \mathfrak{D}(X)$ and $\tau(I-P) < \varepsilon$. The set $S(\mathcal{M},\tau)$ of all τ-measurable operators is a *-algebra under passage to the adjoint operator, multiplication by a scalar, and operations of strong addition and multiplication resulting from the closure of the ordinary operations [16], [14]. Let $\mathcal{L}^+$ and $\mathcal{L}^{\mathrm{h}}$ denote the positive and Hermitian parts of a family $\mathcal{L} \subset S(\mathcal{M},\tau)$, respectively. We denote by $\leq$ the partial order in $S(\mathcal{M},\tau)^{\mathrm{h}}$ generated by its proper cone $S(\mathcal{M},\tau)^+$. If $X \in S(\mathcal{M},\tau)$, then $|X| = \sqrt{X^*X} \in S(\mathcal{M},\tau)^+$. The generalized singular value function $\mu(X) : t \to \mu(t;X)$ of the operator X is defined by setting

$$\mu(s;X) = \inf\{\|XP\| : \ P \in \mathcal{P}(\mathcal{M}) \text{ and } \tau(I-P) \leq s\}.$$

Lemma 2.1. (see [10]) *We have* $\mu(s+t;XY) \leq \mu(s;X)\mu(t;Y)$ *for all* $X, Y \in S(\mathcal{M},\tau)$ *and* $s, t > 0$.

The sets $U(\varepsilon,\delta) = \{X \in S(\mathcal{M},\tau) : \ (\|XP\| \leq \varepsilon \text{ and } \tau(I-P) \leq \delta \text{ for some } P \in \mathcal{P}(\mathcal{M}))\}$, where $\varepsilon > 0$, $\delta > 0$, form a base at 0 for a metrizable vector topology t_τ on $S(\mathcal{M},\tau)$, called *the measure topology* [14]. Equipped with this topology, $S(\mathcal{M},\tau)$ is a complete metrizable topological *-algebra in which $\mathcal{M}$ is dense. We will write $X_n \xrightarrow{\tau} X$ if a sequence $\{X_n\}_{n=1}^\infty$ converges to $X \in S(\mathcal{M},\tau)$ in the measure topology on $S(\mathcal{M},\tau)$.

The set of τ-compact operators $S_0(\mathcal{M},\tau) = \{X \in S(\mathcal{M},\tau) : \lim\limits_{t\to\infty} \mu(t;X) = 0\}$ is an ideal in $S(\mathcal{M},\tau)$. For any closed and densely defined linear operator $X : \mathfrak{D}(X) \to \mathcal{H}$, the *null projection* $\mathrm{n}(X) = \mathrm{n}(|X|)$ is the projection onto its kernel $\mathrm{Ker}(X)$, the *range projection* $\mathrm{r}(X)$ is the projection onto the closure of its range $\mathrm{Ran}(X)$ and the *support projection* $\mathrm{supp}(X)$ of X is defined by $\mathrm{supp}(X) = I - \mathrm{n}(X)$.

The two-sided ideal $\mathcal{F}(\mathcal{M},\tau)$ in $\mathcal{M}$ consisting of all elements of τ-finite range is defined by

$$\mathcal{F}(\mathcal{M},\tau) = \{X \in \mathcal{M} : \ \tau(\mathrm{r}(X)) < \infty\} = \{X \in \mathcal{M} : \ \tau(\mathrm{supp}(X)) < \infty\}.$$

Equivalently, $\mathcal{F}(\mathcal{M},\tau) = \{X \in \mathcal{M} : \ \mu(t;X) = 0 \text{ for some } t > 0\}$. Clearly, $S_0(\mathcal{M},\tau)$ is the closure of $\mathcal{F}(\mathcal{M},\tau)$ with respect to the measure topology [9].

3. Main results

Throughout the sequel, let $\mathcal{M}$ be an arbitrary semifinite von Neumann algebra, with some distinguished faithful normal semifinite trace τ.

Lemma 3.2. *We have* $|X| \in T(\mathcal{M},\tau)$ *for every* $X \in T(\mathcal{M},\tau)$.

Proof. The ideal $\mathcal{F}(\mathcal{M},\tau)$ is a C^*-subalgebra in $\mathcal{M}$. Hence $F(\mathcal{M},\tau) = \mathcal{F}(\mathcal{M},\tau) + \mathbb{C}I$ is an unital C^*-subalgebra in $\mathcal{M}$ and if $X \in F(\mathcal{M},\tau)$, then $|X| \in F(\mathcal{M},\tau)$. Assume that $X \in T(\mathcal{M},\tau)$, i.e., $X = A + \lambda I$ with $A \in S_0(\mathcal{M},\tau)$ and $\lambda \in \mathbb{C}$. Since $\mathcal{F}(\mathcal{M},\tau)$ is t_τ-dense in $S_0(\mathcal{M},\tau)$, there exists a sequence $\{A_n\}_{n=1}^\infty \subset \mathcal{F}(\mathcal{M},\tau)$ such that $A_n \xrightarrow{\tau} A$ as $n \to \infty$. Then the sequence $X_n = A_n + \lambda I$, $n \in \mathbb{N}$, lies in $F(\mathcal{M},\tau)$ and t_τ-converges to the operator X as $n \to \infty$. According to the results given above, $|X_n| = B_n + |\lambda| I$ with some $B_n \in F(\mathcal{M},\tau)^{\mathrm{h}}$, $n \in \mathbb{N}$. Since $X_n \xrightarrow{\tau} X$ as $n \to \infty$, we have $X_n^* \xrightarrow{\tau} X^*$ as $n \to \infty$ by t_τ-continuity of the involution in $S(\mathcal{M},\tau)$. Then via joint t_τ-continuity of the multiplication in $S(\mathcal{M},\tau)$, we have $X_n^*X_n \xrightarrow{\tau} X^*X$ as $n \to \infty$. Therefore, we obtain $|X_n| \xrightarrow{\tau} |X|$ as $n \to \infty$ by t_τ-continuity of the real function $f(t) = \sqrt{t}$, $t \geq 0$ [18]. Thus the sequence $\{B_n\}_{n=1}^\infty$ t_τ-converges to a some operator $B \in S_0(\mathcal{M},\tau)^{\mathrm{h}}$ and $|X| = B + |\lambda| I$. $\square$

Lemma 3.3. (see [4, Corollary 2.4]) *If $X \in T(\mathcal{M},\tau)$ and $XX^* \le X^*X$, then $XX^* = X^*X$.*

Lemma 3.4. *The idempotents of $T(\mathcal{M},\tau)$ are the operators $P, I-P$, where P runs over the idempotent operators of $S_0(\mathcal{M},\tau)$.*

Proof. Assume that $X = A + \lambda I \in T(\mathcal{M},\tau)$ and $X^2 = X$. Then $A^2 + 2\lambda A + \lambda^2 I = A + \lambda I$, i.e., $\lambda \in \{0,1\}$. If $\lambda = 0$, then $A^2 = A$ and $A \in S_0(\mathcal{M},\tau)$ is an idempotent operator. Then $I - A \in T(\mathcal{M},\tau)$ and is also an idempotent. If $\lambda = 1$, then $A^2 = -A = (-A)^2$ and $-A \in S_0(\mathcal{M},\tau)$ is an idempotent operator. Then $I - (-A) \in T(\mathcal{M},\tau)$ and is also an idempotent. $\square$

Consider $F_0(\mathcal{M},\tau) = \{A \in S_0(\mathcal{M},\tau) : \ \tau(\mathrm{r}(A)) < +\infty\}$ and $\mathcal{A}(\mathcal{M},\tau) = F_0(\mathcal{M},\tau) + \mathbb{C}I$. Then $\mathcal{A}(\mathcal{M},\tau)$ is a *-subalgebra of $T(\mathcal{M},\tau)$.

Lemma 3.5. *$\mathcal{A}(\mathcal{M},\tau)$ contains every idempotent of $T(\mathcal{M},\tau)$.*

Proof. Let Q be an idempotent operator of $S(\mathcal{M},\tau)$. Then

$$(Q + Q^* - I)^2 = I + (Q - Q^*)(Q - Q^*)^*$$

and by [6, Theorem 2.21] there exists a unique "range" projection $Q^\sharp \in \mathcal{P}(\mathcal{M})$, defined by the formula $Q^\sharp = Q(Q + Q^* - I)^{-1}$ with $(Q + Q^* - I)^{-1} \in \mathcal{M}$ and subject to the condition $Q^\sharp \cdot S(\mathcal{M},\tau) = Q \cdot S(\mathcal{M},\tau)$. By [6, Theorem 2.23], there exists a unique decomposition $Q = P + Z$, where $P = Q^\sharp \in \mathcal{P}(\mathcal{M})$ and $Z \in S(\mathcal{M},\tau)$ is a nilpotent so that $Z^2 = 0$ and $ZP = 0$, $PZ = Z$. Thus $QP = P$ and $PQ = Q$. Assume that $Q \in S_0(\mathcal{M},\tau)$. Since $QP = P$, we have $P \in S_0(\mathcal{M},\tau)$. Since the singular function $\mu(t;P) = \chi_{(0,\tau(P)]}(t)$ for all $t > 0$, we conclude that $P \in \mathcal{F}(\mathcal{M},\tau)$. Then by equality $PQ = Q$, we have $Q \in F_0(\mathcal{M},\tau)$ and apply Lemma 3.4. $\square$

Lemma 3.6. *$F_0(\mathcal{M},\tau)$ is a regular ring.*

Proof. We show that for every operator $A \in F_0(\mathcal{M},\tau)$ the equation $AXA = A$ possesses a solution in $F_0(\mathcal{M},\tau)$. For $A \in F_0(\mathcal{M},\tau)$, the range projection $\mathrm{r}(A)$ and the support projection $\mathrm{supp}(A)$ lie in $\mathcal{F}(\mathcal{M},\tau)$. Consider the projection $P = \mathrm{r}(A) \bigvee \mathrm{supp}(A)$ in $\mathcal{F}(\mathcal{M},\tau)$ and the reduced von Neumann algebra $\mathcal{M}_P = P\mathcal{M}P$, the reduced faithful normal finite trace τ_P with $\tau_P(X) = \tau(PXP)$, $X \in \mathcal{M}_P^+$. The algebra $\mathcal{M}_P$ is finite, therefore $S(\mathcal{M}_P,\tau_P)$ is a regular ring by [15, Theorem 4.3]. Since $A \in S(\mathcal{M}_P,\tau_P)$, the equation $AXA = A$ admits a solution in $S(\mathcal{M}_P,\tau_P) \subset F_0(\mathcal{M},\tau)$. $\square$

Idempotents P, Q of a ring $\mathcal{R}$ are said to be *equivalent* (in $\mathcal{R}$), written $P \sim Q$, if there exist elements $X, Y \in \mathcal{R}$ such that $XY = P$ and $YX = Q$ (replacing X, Y by PXQ, QYP, one can suppose that $X \in P\mathcal{R}Q$, $Y \in Q\mathcal{R}P$ [13, p. 22]). Projections (=self-adjoint idempotents) P, Q of a ring with involutions are said to be **-equivalent* if there exists an element X such that $XX^* = P$ and $X^*X = Q$.

Theorem 3.1. *If $X, Y \in T(\mathcal{M},\tau)$ such that $XY = I$, then $YX = I$.*

Proof. In the terms of ring theory, we assert that the ring $T(\mathcal{M},\tau)$ is "directly finite" [11, p. 49]. Since $F_0(\mathcal{M},\tau)$ (by Lemma 3.6) and $\mathcal{A}(\mathcal{M},\tau)/F_0(\mathcal{M},\tau) \cong \mathbb{C}$ are both regular rings, $\mathcal{A}(\mathcal{M},\tau)$ is a regular ring [11, p. 2, Lemma 1.3]; since, moreover, the involution of $\mathcal{A}(\mathcal{M},\tau)$ is proper ($AA^* = 0$ implies $A = 0$), the algebra $\mathcal{A}(\mathcal{M},\tau)$ is *-regular in the sense of von Neumann [1, p. 229].

If X, Y are elements of $T(\mathcal{M},\tau)$ such that $XY = I$, then $P = YX$ is an idempotent of $T(\mathcal{M},\tau)$ such that $P \sim I$ in $T(\mathcal{M},\tau)$. By Lemma 3.5, we have $P \in \mathcal{A}(\mathcal{M},\tau)$; since $\mathcal{A}(\mathcal{M},\tau)$ is *-regular, there exists a projection $Q \in \mathcal{A}(\mathcal{M},\tau)$ such that $Q \cdot \mathcal{A}(\mathcal{M},\tau) = P \cdot \mathcal{A}(\mathcal{M},\tau)$ [1, p. 229, Proposition 3]. Then $P \sim Q$ in $\mathcal{A}(\mathcal{M},\tau)$ [13, p. 21, Theorem 14], a fortiori $P \sim Q$ in $T(\mathcal{M},\tau)$; already $P \sim I$ in $T(\mathcal{M},\tau)$, so $Q \sim I$ in $T(\mathcal{M},\tau)$ by transitivity. Since $T(\mathcal{M},\tau)$

satisfies the "square root" axiom (SR) and contains square roots of its positive elements (see Lemma 3.2 and [13, p. 90]), it follows that the projections P, I are *-equivalent in $T(\mathcal{M},\tau)$ [13, p. 35, Theorem 27], say $X \in T(\mathcal{M},\tau)$ with $XX^* = P$, $X^*X = I$. By Lemma 3.3, $P = I$; then $Q \cdot \mathcal{A}(\mathcal{M},\tau) = P \cdot \mathcal{A}(\mathcal{M},\tau) = \mathcal{A}(\mathcal{M},\tau)$ shows that $P = I$, that is, $YX = I$. □

Theorem 3.1 can obviously be reformulated as follows: if $A, B \in S_0(\mathcal{M},\tau)$ and $A+B+AB = 0$, then $AB = BA$. On invertibility in $S(\mathcal{M},\tau)$, see [17], [7] and [8].

Theorem 3.2. *Assume that $A \in S(\mathcal{M},\tau)$ and $B \in T(\mathcal{M},\tau)$. Then $AB \in T(\mathcal{M},\tau)$ if and only if $BA \in T(\mathcal{M},\tau)$.*

Proof. "$\Rightarrow$". If $B \in S_0(\mathcal{M},\tau)$, then $BA \in S_0(\mathcal{M},\tau) \subset T(\mathcal{M},\tau)$. Assume that $B \notin S_0(\mathcal{M},\tau)$. Then $B = \lambda I + K$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $K \in S_0(\mathcal{M},\tau)$. Hence,

$$AB = \lambda A + AK = \mu I + K_1 \tag{3.1}$$

for some $\mu \in \mathbb{C}$ and $K_1 \in S_0(\mathcal{M},\tau)$.

Case 1: $\mu = 0$. Then we have $A \in S_0(\mathcal{M},\tau)$ by (3.1); hence $BA \in S_0(\mathcal{M},\tau) \subset T(\mathcal{M},\tau)$.

Case 2: $\mu \neq 0$. Then by (3.1), we have $\lambda A = \mu I + K_2$ with $K_2 = K_1 - AK \in S_0(\mathcal{M},\tau)$. Therefore, $A = \frac{\mu}{\lambda} I + \frac{1}{\lambda} K_2$ and

$$BA = (\lambda I + K)\Big(\frac{\mu}{\lambda} I + \frac{1}{\lambda} K_2\Big) = I + K_3$$

with $K_3 = K_1 - AK + \frac{\mu}{\lambda} K + \frac{1}{\lambda} KK_1 - \frac{1}{\lambda} KAK \in S_0(\mathcal{M},\tau)$. Thus $BA \in T(\mathcal{M},\tau)$.

"$\Leftarrow$". We know that $X \in T(\mathcal{M},\tau)$ if and only if $X^* \in T(\mathcal{M},\tau)$, and apply the proof given above to the pair $\{A^*, B^*\}$. □

Corollary 3.1. *If $A \in S(\mathcal{M},\tau)$ and $B \in T(\mathcal{M},\tau) \setminus S_0(\mathcal{M},\tau)$ then the following conditions are equivalent:*

(i) $AB \in T(\mathcal{M},\tau)$;

(ii) $BA \in T(\mathcal{M},\tau)$;

(iii) $A \in T(\mathcal{M},\tau)$.

Proof. "(i)$\Rightarrow$(iii)". Let $B = \lambda I + K$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $K \in S_0(\mathcal{M},\tau)$. Then $AB = \lambda A + AK = \mu I + K_1$ for some $\mu \in \mathbb{C}$ and $K_1 \in S_0(\mathcal{M},\tau)$. Thus $\lambda A = \mu I + K_1 - AK$ and $A = \frac{\mu}{\lambda} I + \frac{1}{\lambda} K_1 - \frac{1}{\lambda} AK \in T(\mathcal{M},\tau)$. □

Theorem 3.3. *If $Q \in S(\mathcal{M},\tau)$ is such that $Q^2 = Q$, then $\mu(t;Q) \in \{0\} \bigcup [1,+\infty)$ for all $t > 0$. For the symmetry $U = 2Q - I$, we have $\mu(t;U) \geq 1$ for all $t > 0$.*

Proof. For $Q = Q^2 \notin S_0(\mathcal{M},\tau)$, we have $\mu(t;Q) \geq 1$ for all $t > 0$, see [5, Lemma 3.8]. Let $Q = Q^2 \in S_0(\mathcal{M},\tau)$ and P be "the range" projection of the idempotent Q, see the proof of Lemma 3.5. Since $QP = P$ and $P \in \mathcal{P}(\mathcal{M}) \bigcap \mathcal{F}(\mathcal{M},\tau)$, by Lemma 2.1 we have

$$1 = \mu(s+t;P) = \chi_{(0,\tau(P)]}(s+t) = \mu(s+t;QP) \leq \mu(s;P)\mu(t;Q) = \mu(t;Q)$$

for all $s,t > 0$ with $s + t \leq \tau(P)$. By tending s to $0+$, we obtain $\mu(t;Q) \geq 1$ for all $0 < t < \tau(P)$. By the right continuity of the function $\mu(t;\cdot)$, we have $\mu(\tau(P);Q) \geq 1$. If $t > \tau(P)$ then $\mu(t;P) = 0$; by the equality $PQ = Q$ and by Lemma 2.1, we obtain

$$0 \leq \mu(t;Q) = \mu(t;PQ) \leq \mu(t-\varepsilon;P)\mu(\varepsilon;Q) = 0$$

for all $\varepsilon > 0$ with $t - \varepsilon > \tau(P)$.

Let $Q \in S(\mathcal{M},\tau)$ be such that $Q^2 = Q$. For the symmetry $U = 2Q - I$, we have $U^2 = I$ and by Lemma 2.1 obtain

$$1 = \mu(2t;I) = \mu(2t;U^2) \leq \mu(t;U)\mu(t;U) = \mu(t;U)^2$$

for all $t > 0$. $\square$

Note that for $Q \in \mathcal{M}$ such that $Q^2 = Q$ the relation $\mu(t; Q) \in \{0\} \bigcup [1, \|Q\|]$ for all $t > 0$ was obtained by another way in [3, item 1) of Lemma 3.8]. Theorem 3.3 gives the positive answer to the question by Daniyar Mushtari of year 2010.

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References

[1] S. K. Berberian: *Baer *-rings. Die Grundlehren der mathematischen Wissenschaften, Band 195*, Springer-Verlag, New York-Berlin (1972).

[2] S. K. Berberian: *The algebra of thin operators is directly finite*, Publ. Sec. Mat. Univ. Autònoma Barcelona, **26** (2) (1982), 5-7.

[3] A. M. Bikchentaev: *Local convergence in measure on semifinite von Neumann algebras*, Proc. Steklov Inst. Math., **255** (2006), 35-48.

[4] A. M. Bikchentaev: *On normal τ-measurable operators affiliated with semifinite von Neumann algebras*, Math. Notes, **96** (3-4) (2014), 332-341.

[5] A. M. Bikchentaev: *Concerning the theory of τ-measurable operators affiliated to a semifinite von Neumann algebra*, Math. Notes, **98** (3-4) (2015), 382-391.

[6] A. M. Bikchentaev: *On idempotent τ-measurable operators affiliated to a von Neumann algebra*, Math. Notes, **100** (3-4) (2016), 515-525.

[7] A. M. Bikchentaev: *On τ-essentially invertibility of τ-measurable operators*, Internat. J. Theoret. Phys., **60** (2) (2021), 567-575.

[8] A. M. Bikchentaev: *Essentially invertible measurable operators affiliated to a semifinite von Neumann algebra and commutators*, Sib. Math. J., **63** (2) (2022), 224-232.

[9] P. Dodds, B. de Pagter: *Normed Köthe spaces: A non-commutative viewpoint*, Indag. Math. (N.S.), **25** (2) (2014), 206-249.

[10] T. Fack, H. Kosaki: *Generalized s-numbers of τ-measurable operators*, Pacific J. Math., **123** (2) (1986), 269-300.

[11] K. R. Goodearl: *von Neumann regular rings, Monographs and Studies in Mathematics, vol. 4. Pitman (Advanced Publishing Program)*, Boston, Mass.-London (1979).

[12] I. Halperin: *On a theorem of Sterling Berberian*, C. R. Math. Rep. Acad. Sci. Canada, **3** (1) (1981), 33-35.

[13] I. Kaplansky: *Rings of operators*, W.A. Benjamin, Inc., New York-Amsterdam (1968).

[14] E. Nelson: *Notes on non-commutative integration*, J. Funct. Anal., **15** (2) (1974), 103-116.

[15] K. Saitô: *On the algebra of measurable operators for a general AW^*-algebra. II*, Tohoku Math. J. (2), **23** (3) (1971), 525-534.

[16] I. E. Segal: *A non-commutative extension of abstract integration*, Ann. Math., **57** (3) (1953), 401-457.

[17] I. D. Tembo: *Invertibility in the algebra of τ-measurable operators, in: Operator algebras, operator theory and applications, Oper. Theory Adv. Appl., vol. 195*, Birkhäuser Verlag, Basel (2010), 245-256.

[18] O. E. Tikhonov: *Continuity of operator functions in topologies connected with a trace on a von Neumann algebra*, Soviet Math. (Iz. VUZ), **31** (1) (1987), 110-114.

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Research Article

Banach-valued Bloch-type functions on the unit ball of a Hilbert space and weak spaces of Bloch-type

THAI THUAN QUANG*

ABSTRACT. In this article, we study the space $\mathcal{B}_\mu(B_X, Y)$ of Y-valued Bloch-type functions on the unit ball B_X of an infinite dimensional Hilbert space X with μ is a normal weight on B_X and Y is a Banach space. We also investigate the characterizations of the space $\mathcal{WB}_\mu(B_X)$ of Y-valued, locally bounded, weakly holomorphic functions associated with the Bloch-type space $\mathcal{B}_\mu(B_X)$ of scalar-valued functions in the sense that $f \in \mathcal{WB}_\mu(B_X)$ if $w \circ f \in \mathcal{B}_\mu(B_X)$ for every $w \in \mathcal{W}$, a separating subspace of the dual Y' of Y.

Keywords: Operators on Hilbert spaces, Bloch spaces, weak holomorphic.

2020 Mathematics Subject Classification: 47B38, 30H30, 47B02, 47B33, 47B91.

1. INTRODUCTION

The space of classical Bloch functions on the unit disk $\mathbb{B}_1$ of the complex plane $\mathbb{C}$ was extended to the higher dimension cases. In 1975, using terminology from differential geometry [5], K. T. Hahn introduced the notion of Bloch functions on bounded homogeneous domains in $\mathbb{C}^n$. Further, Bloch functions on bounded homogeneous domains in terms of the Bergman metric was studied by R. M. Timoney in [12, 13]. In [7], S. G. Krantz and D. Ma considered function theoretic and functional analytic properties of Bloch functions on strongly pseudoconvex domain.

Recently, O. Blasco and his colleagues extended the notion to the infinite dimensional setting by considering Bloch functions on the unit ball of an infinite dimensional Hilbert space (see [1, 2, 3]) and, after that, Z. Xu continued the study this topic (see [14]). C. Chu, H. Hamada, T. Honda, G. Kohr generalized the Bloch space to a bounded symmetric domain in a complex Banach space realized as the open unit ball of a JB^*-triple (see [4]). H. Hamada [6] introduced Bloch-type spaces on the unit ball of a complex Banach space.

Motivated by the above results, in this article, the space of Banach-valued Bloch-type functions on the unit ball B_X of an infinite dimensional Hilbert space X with a normal weight (say Bloch-type space) is introduced. We will consider two possible extensions of the classical Bloch space. The first one extends the classical Bloch space by considering the Bloch-type spaces $\mathcal{B}_\mu(B_X, Y)$ of holomorphic functions f on B_X with values in a Banach space Y such that $\sup_{z\in B_X} \mu(z)\|\lozenge f(z)\| < \infty$ where μ is a normal weight on B_X and $\lozenge f$ denotes either the holomorphic gradient ∇f or the radial derivative Rf of f. Basing on the idea in [1] with minor modifications, we give the connection between functions in $\mathcal{B}_\mu(B_X, Y)$ and their restrictions to finite dimensional ones, which leads to the fact that if for a given $m \geq 2$, the restrictions of

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the function to the m-dimensional subspaces have their Bloch-type norms uniformly bounded, then the function is a Bloch-type one and conversely. The second one gives the characterizations of the space $\mathcal{WB}_\mu(B_X)$ of Banach-valued, locally bounded, weakly holomorphic functions associated with the Bloch-type space $\mathcal{B}_\mu(B_X)$ of scalar-valued functions in the sense that $f \in \mathcal{WB}_\mu(B_X)$ if $w \circ f \in \mathcal{B}_\mu(B_X)$ for every $w \in \mathcal{W}$, a separating subspace of the dual Y' of Banach space Y.

Finally, some open problems are proposed at the end of the paper.

2. The Bloch-type spaces on the unit ball of a Hilbert space

Throughout the forthcoming, unless otherwise specified, we shall denote by X a complex Hilbert space with the open unit ball B_X and Y a Banach space. By $\mathscr{H}(B_X, Y)$, we denote the vector space of Y-valued holomorphic functions on B_X. We write $\mathscr{H}(B_X)$ instead of $\mathscr{H}(B_X, \mathbb{C})$. Denote

$$\mathscr{H}^\infty(B_X, Y) = \Big\{ f \in \mathscr{H}(B_X, Y) : \sup_{z \in B_X} \|f(z)\| < \infty \Big\}.$$

It is easy to check that $\mathscr{H}^\infty(B_X, Y)$ is Banach under the sup-norm

$$\|f\|_\infty := \sup_{z \in B_X} \|f(z)\|.$$

Let $(e_k)_{k\in\Gamma}$ be an orthonormal basis of X that we fix at once. Then every $z \in X$ can be written as

$$z = \sum_{k\in\Gamma} z_k e_k, \quad \overline{z} = \sum_{k\in\Gamma} \overline{z_k} e_k.$$

Given $f \in \mathscr{H}(B_X, Y)$ and $z \in B_X$. We will denote, as usual, by $\nabla f(z)$ the gradient of f at z; that is, the unique element representing the linear operator $f'(z) \in L(X, Y)$. We can write

$$\nabla f(z) = \Big(\frac{\partial f}{\partial z_k}(z) \Big)_{k\in\Gamma}$$

and hence

$$f'(z)(x) = \sum_{k\in\Gamma} \frac{\partial f}{\partial z_k}(z)(x_k e_k) \quad \forall x \in X.$$

We define the radial derivative of f at $z \in B_X$ as follows:

$$Rf(z) := \sum_{k\in\Gamma} \frac{\partial f}{\partial z_k}(z)(z_k e_k) = f'(z)(z).$$

It is obvious that

$$\|Rf(z)\| \le \|\nabla f(z)\| \|z\| \quad \forall z \in B_X$$

and

$$\|\nabla f(z)\| := \sup_{u\in Y', \|u\|=1} \|\nabla(u \circ f)(z)\|, \quad \|Rf(z)\| := \sup_{u\in Y', \|u\|=1} |R(u \circ f)(z)|.$$

Definition 2.1. *A positive, continuous function μ on the interval $[0,1)$ is called normal if there are three constants $0 \le \delta < 1$ and $0 < a < b < \infty$ such that*

$$(W_1) \qquad \frac{\mu(t)}{(1-t)^a} \text{ is decreasing on } [\delta, 1), \quad \lim_{t\to 1} \frac{\mu(t)}{(1-t)^a} = 0,$$

$$(W_2) \qquad \frac{\mu(t)}{(1-t)^b} \text{ is increasing on } [\delta, 1), \quad \lim_{t\to 1} \frac{\mu(t)}{(1-t)^b} = \infty.$$

If we say that a function $\mu : B_X \to [0,\infty)$ is normal, we also assume that it is radial, that is, $\mu(z) = \mu(\|z\|)$ for every $z \in B_X$.

Then, it follows from (W_1) that a normal function μ is strictly decreasing on $[\delta, 1)$ and $\mu(t) \to 0$ as $t \to 1$. Note that, for every non-increasing, normal weight μ,

$$S_\mu := \sup_{t\in[0,1)} \frac{(1-t)^b}{\mu(t)} < \infty. \tag{2.1}$$

Throughout this paper, a weight always is assumed to be normal. For a normal weight μ on B_X, we denote

$$I_\mu(z) := \int_0^{\|z\|} \frac{dt}{\mu(t)} \quad \forall z \in B_X.$$

In the sequel, when no confusion can arise, we will use the symbol $\Diamond$ to denote either ∇ or R. We define Bloch-type spaces on the unit ball B_X as follows:

$$\mathcal{B}^{\Diamond}_\mu(B_X, Y) := \Big\{ f \in \mathscr{H}(B_X, Y) : \ \|f\|_{s\mathcal{B}^{\Diamond}_\mu(B_X,Y)} := \sup_{z\in B_X} \mu(z)\|\Diamond f(z)\| < \infty \Big\}.$$

It is easy to check $\|\cdot\|_{s\mathcal{B}^{\Diamond}_\mu(B_X,Y)}$ is a semi-norm on $\mathcal{B}^{\Diamond}_\mu(B_X, Y)$ and this space is Banach under the sup-norm

$$\|f\|_{\mathcal{B}^{\Diamond}_\mu(B_X,Y)} := \|f(0)\| + \|f\|_{s\mathcal{B}^{\Diamond}_\mu(B_X,Y)}.$$

We also define little Bloch-type spaces on the unit ball B_X as follows:

$$\mathcal{B}^{\Diamond}_{\mu,0}(B_X, Y) := \Big\{ f \in \mathcal{B}^{\Diamond}_\mu(B_X, Y) : \ \lim_{\|z\|\to 1} \mu(z)\|\Diamond f(z)\| = 0 \Big\}$$

endowed with the norm induced by $\mathcal{B}^{\Diamond}_\mu(B_X, Y)$. In the case $Y = \mathbb{C}$, we write $\mathcal{B}^{\Diamond}_\mu(B_X)$, $\mathcal{B}^{\Diamond}_{\mu,0}(B_X)$ instead of the respective notations. For $\mu(z) = 1 - \|z\|^2$, we write $\mathcal{B}^{\Diamond}(B_X, Y)$ instead of $\mathcal{B}^{\Diamond}_\mu(B_X, Y)$ and when $\dim X = m$, $Y = \mathbb{C}$ we obtain correspondingly the classical Bloch space $\mathcal{B}^{\Diamond}(\mathbb{B}_m)$. We will show below that the study of Bloch-type spaces on the unit ball can be reduced to studying functions defined on finite dimensional subspaces.

Now, for each finite subset $F \subset \Gamma$, in symbol $|F| = m < \infty$, we denote by $\mathbb{B}_{[F]}$ the unit ball of $\operatorname{span}\{e_k, \ k \in F\}$. Without loss of generality we may assume that $F = \{1, \dots, m\}$, and hence $\mathbb{B}_{[F]} = \mathbb{B}_m$. For each $m \in \mathbb{N}$, we denote

$$z_{[m]} := (z_1, \dots, z_m) \in \mathbb{B}_m.$$

For $m \geq 2$ by

$$OS_m := \{x = (x_1, \dots, x_m), \ x_k \in X, \langle x_k, x_j\rangle = \delta_{kj}\},$$

we denote the family of orthonormal systems of order m. It is clear that OS_1 is the unit sphere of X. For every $x \in OS_m$, $f \in \mathscr{H}(B_X, Y)$, we define

$$f_x(z_{[m]}) = f\Big(\sum_{k=1}^m z_k x_k\Big).$$

Then

$$\Big\|\nabla f_x(z_{[m]})\Big\| = \Big\|\nabla f\Big(\sum_{k=1}^m z_k x_k\Big)\Big\|. \tag{2.2}$$

Definition 2.2. *Let $\mathbb{B}_1$ be the open unit ball in $\mathbb{C}$ and $f \in \mathscr{H}(B_X, Y)$. We define an affine semi-norm as follows*

$$\|f\|_{s\mathcal{B}_\mu^{\mathrm{aff}}(B_X,Y)} := \sup_{\|x\|=1} \|f(\cdot\, x)\|_{s\mathcal{B}_\mu(\mathbb{B}_1,Y)},$$

where $f(\cdot\, x) : \mathbb{B}_1 \to Y$ given by $f(\cdot\, x)(\lambda) = f(\lambda x)$ for every $\lambda \in \mathbb{B}_1$, and

$$\|f(\cdot\, x)\|_{s\mathcal{B}_\mu^R(\mathbb{B}_1,Y)} = \sup_{\lambda\in\mathbb{B}_1} \mu(\lambda x)\|f'(\cdot\, x)(\lambda)\|.$$

It is easy to see that $\|\cdot\|_{s\mathcal{B}_\mu^{\mathrm{aff}}(B_X,Y)}$ is a semi-norm on $\mathcal{B}_\mu(B_X, Y)$. We denote

$$\mathcal{B}_\mu^{\mathrm{aff}}(B_X, Y) := \{f \in \mathcal{B}_\mu(B_X, Y) :\ \|f\|_{s\mathcal{B}_\mu^{\mathrm{aff}}(B_X,Y)} < \infty\}.$$

It is also easy to check that $\mathcal{B}_\mu^{\mathrm{aff}}(B_X, Y)$ is Banach under the norm

$$\|f\|_{\mathcal{B}_\mu^{\mathrm{aff}}(B_X,Y)} := \|f(0)\| + \|f\|_{s\mathcal{B}_\mu^{\mathrm{aff}}(B_X,Y)}.$$

We also define little affine Bloch-type spaces on the unit ball B_X as follows:

$$\mathcal{B}_{\mu,0}^{\mathrm{aff}}(B_X, Y) := \Big\{f \in \mathcal{B}_\mu^{\mathrm{aff}}(B_X, Y) : \lim_{|\lambda|\to 1} \sup_{\|x\|=1} \mu(\lambda x)\|f'(\cdot\, x)(\lambda)\| = 0\Big\}.$$

As the above, for $\mu(z) = 1 - \|z\|^2$ we use notation $\mathcal{B}$ and $\mathcal{B}_0$ instead of $\mathcal{B}_\mu$ and $\mathcal{B}_{\mu,0}$, respectively.

Proposition 2.1. *Let $f \in \mathscr{H}(B_X, Y)$. The following are equivalent:*

(1) $f \in \mathcal{B}_\mu^\nabla(B_X, Y)$;
(2) $\sup_{x\in OS_m} \|f_x\|_{\mathcal{B}_\mu^\nabla(\mathbb{B}_m,Y)} < \infty$ *for every* $m \geq 2$;
(3) *There exists* $m \geq 2$ *such that* $\sup_{x\in OS_m} \|f_x\|_{\mathcal{B}_\mu^\nabla(\mathbb{B}_m,Y)} < \infty$.

Moreover, for each $m \geq 2$

$$\|f\|_{s\mathcal{B}_\mu^\nabla(B_X,Y)} = \sup_{x\in OS_m} \|f_x\|_{s\mathcal{B}_\mu^\nabla(\mathbb{B}_m,Y)}. \tag{2.3}$$

Proof. (1) $\Rightarrow$ (2): Let $m \geq 2$ and $z_{[m]} \in \mathbb{B}_m$. According to (2.2)

$$\big\|\nabla f_x\big(z_{[m]}\big)\big\| = \bigg\|\nabla f\bigg(\sum_{j=1}^m z_j e_j\bigg)\bigg\|.$$

Denote $\mu^{[m]} = \mu\big|_{\mathbb{B}_m}$. Since $\big\|\sum_{j=1}^m z_j e_j\big\| = \big\|z_{[m]}\big\|$ we get

$$\begin{aligned} \|f_x\|_{s\mathcal{B}_{\mu^{[m]}}^\nabla(\mathbb{B}_m,Y)} &= \sup_{z_{[m]}\in\mathbb{B}_m} \mu^{[m]}(z_{[m]})\|\nabla f_x(z_{[m]})\| \\ &\leq \sup_{z\in B_X} \mu^{[m]}(z_{[m]})\bigg\|\nabla f\bigg(\sum_{j=1}^m z_j e_j\bigg)\bigg\| \\ &\leq \|f\|_{s\mathcal{B}_\mu^\nabla(B_X,Y)}. \end{aligned} \tag{2.4}$$

In particular, we obtain (2).

(2) $\Rightarrow$ (1): Let $z = \sum_{k\in\Gamma} z_k e_k$. We denote the partial sums of this series by s_n. Because f is holomorphic, $\frac{\partial f}{\partial z_j}$ are continuous. Then with $e_{[m]} := (e_1, \ldots, e_m)$ we have

$$\begin{aligned}\|\nabla f(z)\| &= \sup_{u\in Y', \|u\|=1} \|\nabla(u\circ f)(z)\| \\ &= \sup_{u\in Y', \|u\|=1} \lim_{n\to\infty} \|\nabla(u\circ f)(s_n)\| \\ &\le \sup_{u\in Y', \|u\|=1} \sup_{m\ge 2} \|\nabla(u\circ f_{e_{[m]}})(z_{[m]})\| \\ &= \sup_{x\in OS_m, m\ge 2} \|\nabla f_x(z_{[m]})\|.\end{aligned}$$

Then, it follows from the assumption (2) and $\|z_{[m]}\| \le \|z\|$, that

$$\begin{aligned}\mu^{[m]}(z_{[m]})\|\nabla f(z)\| &\le \mu^{[m]}(z_{[m]})\|\nabla f(z)\| \\ &\le \sup_{x\in OS_m, m\ge 2} \mu^{[m]}(z_{[m]})\|\nabla f_x(z_{[m]})\| < \infty.\end{aligned} \tag{2.5}$$

Thus $f \in \mathcal{B}^{\nabla}_{\mu}(B_X, Y)$.

(2) $\Rightarrow$ (3): It is obvious.

(3) $\Rightarrow$ (1): Assume that there exists $m \ge 2$ such that $\sup_{x\in OS_m} \|f_x\|_{\mathcal{B}_\mu(B_X,Y)} < \infty$. We fix $z \in B_X$, $z \ne 0$. Consider $x = (\frac{z}{\|z\|}, x_2, \ldots, x_m) \in OS_m$ and put $z_{[m]} := (\|z\|, 0, \ldots, 0) \in \mathbb{B}_m$. Then $\|z_{[m]}\| = \|z\|$ and

$$\|\nabla f_x(z_{[m]})\| = \left\|\nabla f\Big(\sum_{k=1}^{m} z_k x_k\Big)\right\| = \|\nabla f(z)\|. \tag{2.6}$$

This implies that

$$\begin{aligned}\|f\|_{s\mathcal{B}^{\nabla}_{\mu}(B_X,Y)} &= \sup_{z\in\mathcal{B}_X} \mu(z)\|\nabla f(z)\| \\ &\le \sup_{z\in\mathcal{B}_X} \mu(z_{[m]})\|\nabla f_x(z_{[m]})\| \\ &\le \sup_{x\in OS_m} \|f_x\|_{\mathcal{B}_\mu(\mathbb{B}_m,Y)} < \infty.\end{aligned} \tag{2.7}$$

Thus $f \in \mathcal{B}^{\nabla}_{\mu}(B_X, Y)$. On the other hand, it is obvious that

$$\sup_{x\in OS_m} \|f_x\|_{\mathcal{B}^{\nabla}_{\mu}(\mathbb{B}_m,Y)} \le \|f\|_{s\mathcal{B}^{\nabla}_{\mu}(B_X,Y)} \quad \forall m \ge 2. \tag{2.8}$$

Hence, we obtain (2.3) from (2.4), (2.5), (2.7) and (2.8). $\square$

Remark 2.1. *The proposition is not true for the case $m = 1$. Indeed, let X be a Hilbert space with the orthonormal basis $\{e_n\}_{n\ge 1}$. Consider $f : B_X \to \mathbb{C}$ given by*

$$f(z) := \sum_{n=1}^{\infty} \frac{\langle z, e_n\rangle}{\sqrt{n}} \quad \forall z \in B_X.$$

Then $f \in \mathscr{H}(B_X)$ because

$$\sum_{n=1}^{\infty} \frac{|\langle z, e_n\rangle|^2}{n} \le \sum_{n=1}^{\infty} |\langle z, e_n\rangle|^2 = \|z\|^2 < 1.$$

For each $x = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n \in OS_1$ *and for every* $z_{[1]} := z_1 \in \mathbb{B}_1$*, we have*

$$\nabla f_x(z_{[1]}) = \nabla f(z_1 x_1) = \nabla f\bigg(\sum_{n=1}^{\infty} \frac{\langle z_1 x_1, e_n\rangle}{\sqrt{n}}\bigg),$$

and thus, since $\|\nabla f_x(z_{[1]})\|^2 = |x_1|^2 \leq 1$ *we get*

$$\sup_{x\in OS_1} \|f_x(z_{[1]})\|_{\mathcal{B}_\nabla(\mathbb{B}_1)} = \sup_{x\in OS_1} (1-\|z_{[1]}\|^2)\|\nabla f_x(z_{[1]})\| \leq 1.$$

However, $f \notin \mathcal{B}^\nabla(B_X)$ *because for every* $z \in B_X$*, we have*

$$\|\nabla f(z)\|^2 = \sum_{n=1}^{\infty}\Big|\frac{\partial f}{\partial z_n}(z)\Big|^2 = \sum_{n=1}^{\infty}\frac{1}{n}.$$

Proposition 2.2. *Let* $f \in \mathscr{H}(B_X, Y)$*. The following are equivalent:*

(1) $f \in \mathcal{B}^\nabla_{\mu,0}(B_X, Y)$;
(2) $\forall\varepsilon > 0\ \exists\varrho > 0\ \forall z \in B_X$ *with* $\|z_{[m]}\| > \varrho$ *for every* $m \geq 2$

$$\sup_{m\geq 2}\sup_{x\in OS_m} \mu(z_{[m]})\|\nabla f_x(z_{[m]})\| < \varepsilon;$$

(3) $\exists m \geq 2\ \forall\varepsilon > 0\ \exists\varrho > 0\ \forall z \in B_X$ *with* $\|z_{[m]}\| > \varrho$

$$\sup_{x\in OS_m} \mu(z_{[m]})\|\nabla f_x(z_{[m]})\| < \varepsilon.$$

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1): The proof is straight-forward by putting $x \in OS_m$ and $z_{[m]} \in \mathbb{B}_m$ as in the proof of (3) $\Rightarrow$ (1) in Proposition 2.1 for each $z \in B_X$ with $\|z\| > \varrho$. □

In the next proofs below we need the following lemma.

Lemma 2.1. *For every* $f \in \mathcal{B}^\nabla_\mu(B_X, Y)$ *and* $x \in X$ *with* $\|x\| = 1$*, we have*

$$Rf(\lambda x) = \lambda f'(\cdot\, x)(\lambda) \quad \forall\lambda \in \mathbb{B}_1 \tag{2.9}$$

and

$$f'(\cdot\, x)(\lambda)(\mu) = f'(\lambda x)(\mu x) \quad \forall\lambda, \mu \in \mathbb{B}_1. \tag{2.10}$$

Proof. First, it follows from the Bessel inequality that every $x \in X$ has only a countable number of non-zero Fourier coefficients $\langle x, e_j\rangle$. Indeed, for every $\varepsilon > 0$ the set $\{j \in \Gamma : |\langle x, e_j\rangle| > \varepsilon\}$ is finite. Then we still have $x = \sum_{j\in\Gamma}\langle x, e_j\rangle e_j = \sum_{j\in\Gamma} x_j e_j$ where the sum is in fact a countable one, and it is independent of the particular enumeration of the countable number of non-zero summands. Hence, we can write $x = \sum_{j=1}^{\infty} x_j e_j$. Then, by the definitions of $f(\cdot\, x)$

and $f'(\cdot\, x)(\lambda)$, we have

$$\begin{aligned}
&\Big\|\frac{1}{t}\sum_{k=1}^{\infty}\Big(f\Big(\sum_{j=1}^{k}\lambda x_je_j+t\lambda x_ke_k+\sum_{j=k+1}^{\infty}(\lambda+t\lambda)x_je_j\Big)\\
&\qquad -f\Big(\sum_{j=1}^{k}\lambda x_je_j+\sum_{j=k+1}^{\infty}(\lambda+t\lambda)x_je_j\Big)\Big)-\lambda f'(\cdot\, x)(\lambda)\Big\|\\
&=\Big\|\frac{f((\lambda+t\lambda)x)-f(\lambda x)}{h}-\lambda f'(\cdot\, x)(\lambda)\Big\|\\
&=\Big\|\frac{f(\cdot\, x)(\lambda+t\lambda)-f(\cdot\, x)(\lambda)}{t}-\lambda f'(\cdot\, x)(\lambda)\Big\|\to 0 \quad \text{as } t\to 0.
\end{aligned}$$

Hence (2.9) is proved.

For $\lambda,\eta\in\mathbb{B}_1$ we have

$$\begin{aligned}
&\|\eta f'(\cdot\, x)(\lambda)-f'(\lambda x)(\eta x)\|\\
&=\Big\|\frac{f(\cdot\, x)(\lambda+t\eta)-f(\cdot\, x)(\lambda)}{t}-\eta f'(\cdot\, x)(\lambda)-\frac{f(\lambda x+t\eta x)-f(\lambda x)}{t}+f'(\lambda x)(\eta x)\Big\|\\
&\le\Big\|\frac{f(\cdot\, x)(\lambda+t\eta)-f(\cdot\, x)(\lambda)}{t}-\eta f'(\cdot\, x)(\lambda)\Big\|+\Big\|\frac{f(\lambda x+t\eta x)-f(\lambda x)}{t}+f'(\lambda x)(\eta x)\Big\|\\
&\to 0 \quad \text{as} \quad t\to 0.
\end{aligned}$$

Then $f'(\cdot\, x)(\lambda)(\eta)=\eta f'(\cdot\, x)(\lambda)=f'(\lambda x)(\eta x)$, and (2.10) is proved. □

Proposition 2.3. (1) *The spaces $\mathcal{B}^R_\mu(B_X,Y)$ and $\mathcal{B}^{\mathrm{aff}}_\mu(B_X,Y)$ coincide. Moreover,*

$$\|f\|_{s\mathcal{B}^R_\mu(B_X,Y)}\le\|f\|_{s\mathcal{B}^{\mathrm{aff}}_\mu(B_X,Y)}\lesssim\|f\|_{s\mathcal{B}^R_\mu(B_X,Y)}\quad\forall f\in\mathcal{B}^R_\mu(B_X,Y).$$

(2) *The spaces $\mathcal{B}^R_{\mu,0}(B_X,Y)$ and $\mathcal{B}^{\mathrm{aff}}_{\mu,0}(B_X,Y)$ coincide.*

Proof. (1) Let $f\in\mathcal{B}^{\mathrm{aff}}_\mu(B_X,Y)$. In order to prove $f\in\mathcal{B}_R(B_X,Y)$ it suffices to show that

$$Rf(z)=\|z\|f'\Big(\cdot\,\frac{z}{\|z\|}\Big)(\|z\|)\quad\forall z\in B_X\setminus\{0\}. \tag{2.11}$$

It is easy to see that (2.11) follows immediately from (2.9) for $y=\frac{z}{\|z\|}$ and $\lambda=\|z\|$ for every $z\in B_X\setminus\{0\}$. Moreover, it follows from (2.11) that

$$\|f\|_{s\mathcal{B}^R_\mu(B_X,Y)}\le\|f\|_{s\mathcal{B}^{\mathrm{aff}}_\mu(B_X,Y)}.$$

Thus, the first inequality in (1) is proved. Now, let $f\in\mathcal{B}^R_\mu(B_X,Y)$ and $x\in X$ be such that $\|x\|=1$. Since f is holomorphic at $0\in B_X$, its derivative $f':B_X\to L(X,Y)$ is also holomorphic, and thus there are $r\in(0,1)$ and $M>0$ such that

$$\|f'(z)\|_{L(X,Y)}\le M\quad\forall z\in\overline{B}(0,r):=\{u\in X:\ \|u\|\le r\}.$$

Then, by (2.10) we have

$$\begin{aligned}\sup_{|\lambda|\le r}\mu(\lambda x)\|f'(\cdot x)(\lambda)\| &= \sup_{|\lambda|\le r}\mu(\lambda x)\sup_{|\eta|\le 1}\|f'(\cdot x)(\lambda)(\eta)\|\\ &= \sup_{|\lambda|\le r}\mu(\lambda x)\sup_{|\eta|\le 1}\|f'(\lambda x)(\eta x)\|\\ &\le \sup_{|\lambda|\le r}\mu(\lambda x)\|f'(\lambda x)\|\le M.\end{aligned}$$

For the case where $\|z\| > r$, by (2.9), (2.10) and the increasing monotony of the function $\frac{1-t}{t}$, similar calculation to [1, Proposition 2.4], we have

$$\mu(\lambda x)|f'(\cdot x)(\lambda)|\le\Big(\mu(\lambda x)\frac{1-r}{r}+\mu(\lambda x)\Big)\|Rf(\lambda x)\|. \tag{2.12}$$

This implies that

$$\sup_{|\lambda|>r}\mu(\lambda x)|f'(\cdot x)(\lambda)|\le\frac{1}{r}\sup_{z\in B_X}\mu(z)\|Rf(z)\|.$$

Therefore, $f\in\mathcal{B}^{\mathrm{aff}}_\mu(B_X,Y)$, and we also obtain $\|f\|_{s\mathcal{B}_\mu{}^{\mathrm{aff}}(B_X,Y)}\le\frac{1}{r}\|f\|_{s\mathcal{B}^R_\mu(B_X,Y)}$. Hence, the second inequality in (1) is proved

(2) Let $f\in\mathcal{B}^{\mathrm{aff}}_{\mu,0}(B_X,Y)$. Then, using (2.11) it is easy to see that $f\in\mathcal{B}^R_{\mu,0}(B_X,Y)$. In the converse direction, it follows from (2.12) that $f\in\mathcal{B}^{\mathrm{aff}}_{\mu,0}(B_X,Y)$ if $f\in\mathcal{B}^R_{\mu,0}(B_X,Y)$. □

Next, we will compare the spaces $\mathcal{B}^\nabla_\mu(B_X,Y)$ and $\mathcal{B}^R_\mu(B_X,Y)$. We need a vector-valued version of Lemma 4.11 in [12]. First we note that

$$f\in\mathcal{B}_\mu(\mathbb{B}_1,Y)\quad\text{if and only if}\quad u\circ f\in\mathcal{B}_\mu(\mathbb{B}_1)\text{ for all }u\in Y' \tag{2.13}$$

and, interchanging the suprema, we have

$$\|f\|_{\mathcal{B}^\nabla_\mu(\mathbb{B}_1,Y)}\asymp\sup_{\|u\|=1}\|u\circ f\|_{\mathcal{B}^\nabla_\mu(\mathbb{B}_1)}. \tag{2.14}$$

Lemma 2.2. *Let $f\in\mathcal{B}^{\mathrm{aff}}(\mathbb{B}_2,Y)$. If there exists $M>0$ such that $\|f(\cdot x)\|_{s\mathcal{B}^{\mathrm{aff}}(\mathbb{B}_1,Y)}\le M$ for any $x=(x_1,x_2)\in\mathbb{B}_2$, then*

$$\mu((x_1,0))\|\nabla f(x_1,0)\|\le 2\sqrt{2}MR_\mu\quad\forall x_1\in\mathbb{C},|x_1|<1, \tag{2.15}$$

where $R_\mu:=1+\max_{t\in[0,\delta]}\mu(t)I_\mu(\delta)$.

Proof. We modify the proof of Lemma 4.11 in [12]. Fix $u\in Y'$ with $\|u\|=1$. By the hypothesis, $f(\cdot x)\in\mathcal{B}(\mathbb{B}_1,Y)$. Then it follows from (2.13) that $u\circ f(\cdot x)\in\mathcal{B}_\mu(\mathbb{B}_1)$.

$$\|u\circ f(\cdot x)\|_{s\mathcal{B}^\nabla_\mu}\le\|u\|\|f(\cdot x)\|_{s\mathcal{B}^{\mathrm{aff}}_\mu}\le M.$$

First of all, the hypotheses imply that

$$\mu((x_1,0))\Big|\frac{\partial(u\circ f)}{\partial x_1}(x_1,0)\Big|\le M,$$

and so it is sufficient to show that

$$\mu((x_1,0))\Big|\frac{\partial(u\circ f)}{\partial x_2}(x_1,0)\Big|\le 2\sqrt{2}M.$$

Indeed, from the hypotheses, we have

$$|f(z)-f(0)|=\Big|\int_0^1\langle\nabla f(tz),\overline{z}\rangle dt\Big|\le M\int_0^{\|z\|}\frac{dt}{\mu(t)}=MI_\mu(z).$$

Then, using the Cauchy integral formula and a simple estimate, we obtain

$$
\begin{aligned}
&\mu((x_1,0))\Big|\frac{\partial(u\circ f)}{\partial x_2}(x_1,0)\Big| \\
\le&\mu((x_1,0))\frac{1}{2\pi}\int_{|w|=1/\sqrt[4]{2}}\frac{\|u\||f(x_1,w)-f(0)+f(0)-f(x_1,0)|}{|w|^2}dw \\
\le&\mu((x_1,0))\frac{2MI_\mu(x_1)}{2\pi}\int_{|w|=1/\sqrt[4]{2}}\frac{dw}{w^2}\le 2\sqrt{2}MR_\mu
\end{aligned}
$$

as required. □

Theorem 2.1. (1) *The spaces* $\mathcal{B}^\nabla_\mu(B_X,Y)$ *and* $\mathcal{B}^R_\mu(B_X,Y)$ *coincide. Moreover,*

$$\|f\|_{\mathcal{B}^R_\mu(B_X,Y)}\asymp\|f\|_{\mathcal{B}^\nabla_\mu(B_X,Y)}.$$

(2) *The spaces* $\mathcal{B}^\nabla_{\mu,0}(B_X,Y)$ *and* $\mathcal{B}^R_{\mu,0}(B_X,Y)$ *coincide.*

Proof. We prove this theorem by modifying the method of Timoney which was used in [12].

(1) Let us show that $\|f\|_{s\mathcal{B}^\nabla_\mu(B_X,Y)}\le 2\sqrt{2}R_\mu\|f\|_{s\mathcal{B}^{\mathrm{aff}}_\mu(B_X,Y)}$ and the result follows using Pr-position 2.3. Fix $u\in Y'$ with $\|u\|=1$. Let $z\in B_X$ and $v\in X$ with $\|v\|=1$ be fixed. We may assume that $\dim X\ge 2$. Then there exist orthonormal unit vectors $e_1,e_2\in X$ and $s,t_1,t_2\in\mathbb{C}$ with $|s|<1$ and $|t_1|^2+|t_2|^2=1$ such that $z=se_1$, $v=t_1e_1+t_2e_2$. For $f\in\mathcal{B}^R_\mu(B_X,Y)$ put

$$F(z_1,z_2)=(u\circ f)(z_1e_1+z_2e_2),\quad (z_2,z_2)\in\mathbb{B}_2.$$

Then $F\in H(B_X)$ and it is easy to check that F satisfies the assumptions of Lemma 2.2. Then

$$\mu(z)|\nabla(u\circ f)(z)|=\mu(s)|\nabla(u\circ f)(se_1)|=\mu(s,0)|\nabla F(s,0)|\le 2\sqrt{2}MR_\mu,$$

hence, $\|f\|_{s\mathcal{B}^\nabla_\mu(B_X,Y)}\le 2\sqrt{2}R_\mu\|f\|_{s\mathcal{B}^{\mathrm{aff}}_\mu(B_X,Y)}$ as required.

(2) Because $\|Rf(z)\|<\|\nabla f(z)\|$ for every $z\in B_X$, it suffices to show that $\mathcal{B}^R_{\mu,0}(B_X,Y)\subset\mathcal{B}^\nabla_{\mu,0}(B_X,Y)$. Let $f\in\mathcal{B}^R_{\mu,0}(B_X,Y)$ and consider the function $F(z_1,z_2)$ defined in the proof of the part (1). In exactly the same estimates in [6, Theorem 2.8(i)] we obtain that

$$\Big|\frac{\partial F}{\partial z_2}(z_1,0)\Big|\le\frac{\pi|z_1|}{2\mu(|z_1|)\delta}\sup_{r_0\le\|z\|<1}\mu(z)|Rf(z)|\quad\text{for } |z_1|\ge r_0 \tag{2.16}$$

and

$$\Big|\frac{\partial F}{\partial z_1}(z_1,0)\Big|=\Big|\frac{Rf(z_1e_1)}{z_1}\Big|\le\frac{1}{\mu(|z_1|)\delta}\sup_{r_0\le\|z\|<1}\mu(z)|Rf(z)|\quad\text{for } |z_1|\ge r_0. \tag{2.17}$$

From (2.16) and (2.17), we obtain

$$
\begin{aligned}
\mu(z)|\langle\nabla f(z),v\rangle|&=\mu(s)|\langle\nabla f(se_1),t_1e_1+t_2e_2\rangle| \\
&=\mu(s)\Big|t_1\frac{\partial F}{\partial z_1}(s,0)+t_2\frac{\partial F}{\partial z_2}(s,0)\Big| \\
&\le\mu(s)\Big(\Big|\frac{\partial F}{\partial z_1}(s,0)\Big|^2+\Big|\frac{\partial F}{\partial z_2}(s,0)\Big|^2\Big)^{1/2} \\
&\le\frac{\pi}{\sqrt{2}\delta}\sup_{r_0\le\|z\|<1}\mu(z)|Rf(z)|,\quad \|z\|\ge r_0,\|v\|=1.
\end{aligned} \tag{2.18}
$$

Now, by the hypothesis, for every $\varepsilon>0$ we can find $r_0\in(\delta,1)$ such that $\mu(z)\|Rf(z)\|<\varepsilon$ for $\|z\|>r_0$. Therefore, it follows from (2.18) that $\lim_{\|z\|\to 1}\mu(z)\|\nabla f(z)\|=0$, that means $f\in\mathcal{B}^\nabla_{\mu,0}(B_X,Y)$. □

We can now combine the results of Proposition 2.3 and Lemma 2.2 with an argument analogous to the Theorem 2.6 in [1] and obtain the following theorem:

Theorem 2.2. *The spaces $\mathcal{B}_\mu^\nabla(B_X,Y)$, $\mathcal{B}_\mu^R(B_X,Y)$ and $\mathcal{B}_\mu^{\mathrm{aff}}(B_X,Y)$ coincide. The spaces $\mathcal{B}_{\mu,0}^\nabla(B_X,Y)$, $\mathcal{B}_{\mu,0}^R(B_X,Y)$ and $\mathcal{B}_{\mu,0}^{\mathrm{aff}}(B_X,Y)$ coincide. Moreover,*

$$\|f\|_{\mathcal{B}_\mu^R(B_X,Y)} \le \|f\|_{\mathcal{B}_\mu^\nabla(B_X,Y)} \le 2\sqrt{2}R_\mu \|f\|_{\mathcal{B}_\mu^{\mathrm{aff}}(B_X,Y)}.$$

Next, we present a Möbius invariant norm for the Bloch-type space $\mathcal{B}(B_X,Y)$. Möbius transformations on a Hilbert space X are the mappings φ_a, $a \in B_X$, defined as follows:

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \quad z \in \mathcal{B}_X, \tag{2.19}$$

where $s_a = \sqrt{1-\|a\|^2}$, P_a is the orthogonal projection from X onto the one dimensional subspace $[a]$ generated by a, and Q_a is the orthogonal projection from X onto $X \ominus [a]$. It is clear that

$$P_a(z) = \frac{\langle z,a\rangle}{\|a\|^2}a,\ (z \in X) \quad \text{and} \quad Q_a(z) = z - \frac{\langle z,a\rangle}{\|a\|^2}a,\ (z \in B_X).$$

When $a = 0$, we simply define $\varphi_a(z) = -z$. It is obvious that each φ_a is a holomorphic mapping from B_X into X. We will also need the following facts about the pseudohyperbolic distance in B_X. It is given by

$$\varrho_X(x,y) := \|\varphi_{-y}(x)\| \quad \text{for any } x, y \in B_X.$$

For details concerning Möbius transformations and the pseudohyperbolic distance, we refer to the book of K. Zhu [15]. It is well known that, in the case $n \ge 2$, the equality $\|f\circ\varphi\|_{\mathcal{B}^\nabla(\mathbb{B}_n,Y)} = \|f\|_{\mathcal{B}^\nabla(\mathbb{B}_n,Y)}$ is false. Our goal is to find a semi-norm on $\mathcal{B}(B_X,Y)$ which is invariant under the automorphisms of the ball B_X.

Definition 2.3. *Let X be a complex Hilbert space, Y be a Banach space and $f \in H(B_X,Y)$. Consider the invariant gradient norm*

$$\|\widetilde{\nabla} f(z)\| := \|\nabla(f\circ\varphi_z)(0)\| \quad \textit{for any } z \in B_X.$$

We recall the following result of Blasco and his colleagues in [1]:

Lemma 2.3 (Lemma 3.5, [1])**.** *Let $f \in H(B_X)$. Then*

$$\|\widetilde{\nabla} f(z)\| = \sup_{w\neq 0} \frac{|\langle\nabla f(z), w\rangle|(1-\|z\|^2)}{\sqrt{(1-\|z\|^2)\|w\|^2 + |\langle w,z\rangle|^2}}.$$

We define invariant semi-norm as follows

$$\|f\|_{s\mathcal{B}^{\mathrm{inv}}(B_X,Y)} := \sup_{z\in B_X} \|\widetilde{\nabla} f(z)\| = \sup_{z\in B_X} \sup_{u\in Y',\|u\|\le 1} \|\widetilde{\nabla}(u\circ f)(z)\|.$$

We denote

$$\mathcal{B}^{\mathrm{inv}}(B_X,Y) := \{f \in \mathcal{B}(B_X,Y) :\ \|f\|_{s\mathcal{B}^{\mathrm{inv}}(B_X,Y)} < \infty\}.$$

It is also easy to check that $\mathcal{B}^{\mathrm{inv}}(B_X,Y)$ is Banach under the norm

$$\|f\|_{\mathcal{B}^{\mathrm{inv}}(B_X,Y)} := \|f(0)\| + \|f\|_{s\mathcal{B}^{\mathrm{inv}}(B_X,Y)}.$$

Now, applying Theorem 3.8 in [1] to the functions $u\circ f$ for every $u \in Y'$, we obtain the following:

Theorem 2.3. *The spaces $\mathcal{B}^\nabla(B_X,Y)$, and $\mathcal{B}^{\mathrm{inv}}(B_X,Y)$ coincide. Moreover,*

$$\|f\|_{\mathcal{B}^\nabla(B_X,Y)} \le \|f\|_{\mathcal{B}^{\mathrm{inv}}(B_X,Y)} \lesssim \|f\|_{\mathcal{B}^\nabla(B_X,Y)}.$$

3. Weak Holomorphic Spaces Associated with Bloch-Type Spaces

Let X, Y be complex Banach spaces and $\mathcal{W} \subset Y'$ be a separating subspace of the dual Y' of Y. Let $\mathcal{E} \subset \mathscr{H}(B_X)$ be a Banach space. We say that the space

$$\mathcal{W}\mathcal{E} := \{f : B_X \to Y : \ f \text{ is locally bounded and } w \circ f \in \mathcal{E}, \ \forall w \in \mathcal{W}\}$$

equipped with the norm

$$\|f\|_{\mathcal{W}\mathcal{E}} := \sup_{w \in \mathcal{W}, \|w\| \le 1} \|w \circ f\|_{\mathcal{E}} \tag{3.1}$$

is the Banach space $\mathcal{W}$-associated with $\mathcal{E}$ of Y-valued functions.

Remark 3.2. *In the case the norm $\|\cdot\|_{\mathcal{E}}$ of Banach space $\mathcal{E}$ is written in the form*

$$\|f\| = |f(0)| + \|f\|_{s\mathcal{E}} \quad \forall f \in \mathcal{E}$$

the space $\mathcal{W}\mathcal{E}$ can be equipped with the norm

$$\|f\|_{\mathcal{W}\mathcal{E}^+} := \sup_{w \in \mathcal{W}, \|w\| \le 1} |w \circ f(0)| + \sup_{w \in \mathcal{W}, \|w\| \le 1} \|w \circ f\|_{s\mathcal{E}} \quad \forall f \in \mathcal{E}. \tag{3.2}$$

However, it is easy to check that $\mathcal{W}\mathcal{E} = \mathcal{W}\mathcal{E}^+$ and

$$\|\cdot\|_{\mathcal{W}\mathcal{E}} \asymp \|\cdot\|_{\mathcal{W}\mathcal{E}^+}$$

on $\mathcal{W}\mathcal{E}$ where

$$\mathcal{W}\mathcal{E}^+ := \Big\{ f : B_X \to Y : \ f \text{ is locally bounded and } \sup_{w \in \mathcal{W}, \|w\| \le 1} \|w \circ f\|_{s\mathcal{E}} < \infty \Big\}.$$

Therefore, by $\mathcal{W}\mathcal{E}$ we always mean that is $(\mathcal{W}\mathcal{E}, \|\cdot\|_{\mathcal{W}\mathcal{E}})$.

Suppose now that $\mathcal{E} \subset \mathscr{H}(B_X)$ is a Banach space such that

(e1) $\mathcal{E}$ contains the constant functions,
(e2) the closed unit ball $B_{\mathcal{E}}$ is compact in the compact open topology τ_{co} of B_X.

It is easy to check that the properties (e1), (e2) are satisfied by a large number of well-known function spaces, such as classical Hardy, Bergman, BMOA, and Bloch spaces.

Proposition 3.4. *Let X, Y be complex Banach spaces and $\mathcal{W} \subset Y'$ be a separating subspace. Let $\mathcal{E} \subset \mathscr{H}(B_X)$ a Banach space satisfying (e1)-(e2) and $\mathcal{W}\mathcal{E}$ be the Banach space $\mathcal{W}$-associated with $\mathcal{E}$. Then, the following assertions hold:*

(we1) *$f \mapsto f \otimes y$ defines a bounded linear operator $P_y : \mathcal{E} \to \mathcal{W}\mathcal{E}$ for any $y \in Y$, where $(f \otimes y)(z) = f(z)y$ for $z \in B_X$,*
(we2) *$g \mapsto w \circ g$ defines a bounded linear operator $Q_w : \mathcal{W}\mathcal{E} \to \mathcal{E}$ for any $w \in \mathcal{W}$,*
(we3) *For all $z \in B_X$ the point evaluations $\widetilde{\delta}_z : \mathcal{W}\mathcal{E} \to (Y, \sigma(Y, W))$, where $\widetilde{\delta}_z(g) = g(z)$, are continuous.*

In the case the hypothesis "separating" of $\mathcal{W}$ is replaced by a stronger one that $\mathcal{W}$ is "almost norming", we obtain the assertion (we3′) below instead of (we3):

(we3′) *For all $z \in B_X$ the point evaluations $\widetilde{\delta}_z : \mathcal{W}\mathcal{E} \to Y$ are bounded.*

Here, the subspace $\mathcal{W}$ of Y' is called *almost norming* if

$$q_{\mathcal{W}}(x) := \sup_{w \in \mathcal{W}, \|w\| \le 1} |w(x)|$$

defines an equivalent norm on Y.

Proof. (i) Fix $y \in Y$. In fact, for every $f \in \mathcal{E}$ we have $w \circ (f \otimes y) = w(y)f$. Then

$$\begin{aligned}\|P_y(f)\|_{\mathcal{WE}} = \sup_{\|w\|\le 1} \|w \circ (f \otimes y)\|_{\mathcal{E}} &= \sup_{\|w\|\le 1} \|w(y)f\|_{\mathcal{E}} \\ &\le \|w\| \cdot \|y\| \cdot \|f\|_{\mathcal{E}} \\ &\le \|y\| \cdot \|f\|_{\mathcal{E}}.\end{aligned}$$

Thus (we1) holds.

(ii) Fix $w \in \mathcal{W}$, for every $g \in \mathcal{WE}$ we have

$$\begin{aligned}\|Q_w(g)\|_{\mathcal{E}} = \|w \circ g\|_{\mathcal{E}} &= \|w\| \Big\| \frac{w}{\|w\|} \circ g \Big\|_{\mathcal{E}} \\ &\le \|w\| \sup_{\|u\|\le 1} \|u \circ g\|_{\mathcal{E}} \\ &= \|w\| \cdot \|g\|_{\mathcal{WE}}.\end{aligned}$$

Thus (we2) is true.

(iii) Fix $z \in B_X$. Note first that since $\mathcal{E}$ satisfies (e1) and (e2), then the evaluation maps $\delta_z \in \mathcal{E}'$ for $z \in B_X$ where $\delta_z(f) = f(z)$ for $f \in \mathcal{E}$. It is obvious that $w(\widetilde{\delta}_z(g)) = \delta_z(w \circ g)$ for every $g \in \mathcal{WE}$ and for every $w \in \mathcal{W}$. Let V be a $\sigma(Y, \mathcal{W})$-neighbourhood of 0 in Y. Without loss of genarality we may assume $V = \{y \in Y : |w(y)| < 1\}$ for some $w \in \mathcal{W}$. Then $\widetilde{\delta}_z(\|\delta_z\|^{-1}\|w\|^{-1}B_{\mathcal{WE}}) \subset V$, where $B_{\mathcal{WE}}$ is the unit ball of $\mathcal{WE}$. Indeed, for every $g \in B_{\mathcal{WE}}$ we have

$$\begin{aligned}|w(\widetilde{\delta}_z(\|\delta_z\|^{-1}\|w\|^{-1}g))| &= \|\delta_z\|^{-1}|\delta_z(\|w\|^{-1}w \circ g)| \\ &\le \|\delta_z\|^{-1}\|\delta_z\| \cdot \big\|\|w\|^{-1}w \circ g\big\|_{\mathcal{E}} \\ &\le \sup_{u\in\mathcal{W}, \|u\|\le 1} \|u \circ g\|_{\mathcal{E}} \\ &= \|g\|_{\mathcal{WE}} < 1.\end{aligned}$$

Thus, (we3) holds.

In the case where $\mathcal{W}$ is almost norming, since $q_{\mathcal{W}}$ defines an equivalent norm, there exists $C > 0$ such that

$$\begin{aligned}\|\widetilde{\delta}_z(g)\| = \|g(z)\| &\le C q_{\mathcal{W}}(g(z)) \\ &= C \sup_{w\in\mathcal{W}, \|w\|\le 1} |w(g(z))| \\ &\le C \sup_{w\in\mathcal{W}, \|w\|\le 1} \|w \circ g\| \\ &= C\|g\|_{\mathcal{WE}} \quad \forall g \in \mathcal{WE}.\end{aligned}$$

The assertion (we3′) is proved. □

Now let $\mathcal{W} \subset Y'$ be a separating subspace of the dual Y'. Applying Proposition 2.3, Theorems 2.2 and 2.3 to functions $w \circ f$ for each $f \in H(B_X, Y)$ and $w \in \mathcal{W}$, we obtain the equivalence of the norms in $\mathcal{W}$-associated Bloch-type spaces:

$$\|\cdot\|_{\mathcal{WB}^R_\mu(B_X)} \cong \|\cdot\|_{\mathcal{WB}^\nabla_\mu(B_X)} \cong \|\cdot\|_{\mathcal{WB}^{\mathrm{aff}}_\mu(B_X)},$$

$$\|\cdot\|_{\mathcal{WB}^R(B_X)} \cong \|\cdot\|_{\mathcal{WB}^\nabla(B_X)} \cong \|\cdot\|_{\mathcal{WB}^{\mathrm{aff}}(B_X)} \cong \|\cdot\|_{\mathcal{WB}^{\mathrm{inv}}(B_X)}.$$

Hence, for the sake of simplicity, from now on we write $\mathcal{B}_\mu$ instead of $\mathcal{B}^R_\mu$. Recall that, the space $\mathcal{WB}_\mu(B_X)$ equipped with the norm in the form either (3.1) or (3.2). It is clear that for every separating subspace $\mathcal{W}$ of Y' we have

$$\mathcal{B}^\diamondsuit_\mu(B_X, Y) \subset \mathcal{WB}^\diamondsuit_\mu(B_X), \quad \mathcal{B}^\diamondsuit_{\mu,0}(B_X, Y) \subset \mathcal{WB}^\diamondsuit_{\mu,0}(B_X).$$

The main result of this section is the following:

Theorem 3.4. *Let $\mathcal{W} \subset Y'$ be a separating subspace. Let μ be a normal weight on B_X. Then $\mathcal{W}\mathcal{B}_\mu(B_X)$ and $\mathcal{W}\mathcal{B}_{\mu,0}(B_X)$ satisfy (we1)-(we3).*

We need the following lemma whose proof parallels that of Lemma 13 in [11] and will be omitted.

Lemma 3.4. *Let μ be a normal weight on B_X. Then there exists $C_\mu > 0$ such that*

$$C_\mu \leq \frac{\mu(r)}{\mu(r^2)} \leq 1 \quad \forall r \in [0,1).$$

Proof of Theorem 3.4. By Proposition 3.4, it suffices to show that $\mathcal{B}_\mu(B_X)$, $\mathcal{B}_{\mu,0}(B_X)$ satisfy (e1) and (e2). It is obvious that $\mathcal{B}_\mu(B_X)$, $\mathcal{B}_{\mu,0}(B_X)$ satisfy (e1). Because $\mathcal{B}_{\mu,0}(B_X)$ is the subspace of $\mathcal{B}_\mu(B_X)$, it suffices to check (e2) for the space $\mathcal{B}_\mu(B_X)$. In order to prove (e2) holds for $\mathcal{B}_\mu(B_X)$, we will show that the closed unit ball U of $\mathcal{B}_\mu(B_X)$ is pointwise bounded and equicontinuous.

(i) First, we prove that U is pointwise bounded. It suffices to prove that

$$|f(z)| \leq \max\Big\{1, I_\mu(z)\Big\}\|f\|_{\mathcal{B}_\mu(B_X)} \quad \forall f \in \mathcal{B}_\mu(B_X), \forall z \in B_X. \tag{3.3}$$

Fix $f \in \mathcal{B}_\mu(B_X)$ and put $g(z) = f(z) - f(0)$ for every $z \in B_X$. Note that $g(0) = 0$ and $\|g\|_{\mathcal{B}_\mu(B_X)} = \|f\|_{s\mathcal{B}_\mu(B_X)}$. As in Lemma 2.2 by Cauchy-Schwarz inequality, we have

$$|g(z)| \leq \int_0^1 \frac{\|f\|_{s\mathcal{B}_\mu(\mathcal{B}_X)}\|z\|}{\mu(tz)}dt = \|f\|_{s\mathcal{B}_\mu(B_X)} I_\mu(z) = \|g\|_{\mathcal{B}_\mu(B_X)} I_\mu(z).$$

Consequently,

$$\begin{aligned} |f(z)| &\leq |f(0)| + |g(z)| \leq |f(0)| + \|g\|_{\mathcal{B}_\mu(B_X)} I_\mu(z) \\ &\leq \max\Big\{1, I_\mu(z)\Big\}\big(|f(0)| + \|f\|_{s\mathcal{B}_\mu(B_X)}\big) \\ &= \max\Big\{1, I_\mu(z)\Big\}\|f\|_{\mathcal{B}_\mu(B_X)}. \end{aligned}$$

(ii) Next, we show that U is equicontinuous. For each $f \in U$, by Proposition 2.1 we can find $m \geq 2$ such that

$$\|f\|_{s\mathcal{B}_\mu(B_X)} = \sup_{y \in OS_m} \|f_y\|_{s\mathcal{B}_\mu(\mathbb{B}_m)}.$$

Fix $e_{[m]} = (e_1, \ldots, e_m) \in OS_m$. Then, for every $z = (z_k)_{k\in\Gamma}, w = (w_k)_{k\in\Gamma} \in B_X$, we consider $z_{[m]} := (z_1, \ldots, z_m)$, $w_{[m]} := (w_1, \ldots, w_m)$. By Theorem 3.6 in [15] and Lemma 3.4, we have

$$\begin{aligned} &|f_{e_{[m]}}(z_{[m]}) - f_{e_{[m]}}(w_{[m]})| \\ \leq &\beta(z_{[m]}, w_{[m]}) \sup_{x_{[m]} \in \mathbb{B}_m} \|\widetilde{\nabla} f_{e_{[m]}}(x_{[m]})\| \\ \leq &\beta(z_{[m]}, w_{[m]}) \sup_{x_{[m]} \in \mathbb{B}_m} \sup_{y \in \mathbb{B}_m \setminus \{0\}} \frac{|\langle \nabla f_{e_{[m]}}(x_{[m]}), y\rangle|(1 - \|x_{[m]}\|^2)}{\sqrt{(1 - \|x_{[m]}\|^2)\|y\|^2 + |\langle y, x_{[m]}\rangle|^2}} \\ \leq &\beta(z_{[m]}, w_{[m]}) C_\mu^{-1} \sup_{x_{[m]} \in \mathbb{B}_m} \frac{\mu^{[m]}(\|x_{[m]}\|)|\nabla f_{e_{[m]}}(x_{[m]})|\sqrt{1 - \|x_{[m]}\|^2}}{\mu^{[m]}(\|x_{[m]}\|^2)} \\ \leq &\beta(z_{[m]}, w_{[m]}) C_\mu^{-1} \|f_{e_{[m]}}\|_{\mathcal{B}_\mu(\mathbb{B}_m)} \frac{\sqrt{1 - \|x_{[m]}\|^2}}{\mu^{[m]}(\|x_{[m]}\|^2)}, \end{aligned}$$

where β is the Bergman metric on $\mathbb{B}_m$ given by

$$\beta(s,t) = \frac{1}{2}\log\frac{1+|(\varphi_m)_s(t)|}{1-|(\varphi_m)_s(t)|}$$

with $(\varphi_m)_s$ is the involutive automorphism of $\mathbb{B}_m$ that interchanges 0 and s. If $\|x_{[m]}\|^2 \le \delta$ it is clear that

$$\frac{\sqrt{1-\|x_{[m]}\|^2}}{\mu^{[m]}(\|x_{[m]}\|^2)} \le \frac{1}{m_{\mu,\delta}} < \infty,$$

where $m_{\mu,\delta} = \min_{t\in[0,\delta]}\mu(t) > 0$; if $\|x_{[m]}\|^2 > \delta$ and $b \ge 1/2$, by (2.1) we have

$$\frac{\sqrt{1-\|x_{[m]}\|^2}}{\mu^{[m]}(\|x_{[m]}\|^2)} \le \frac{(1-\|x_{[m]}\|^2)^b}{\mu^{[m]}(\|x_{[m]}\|^2)} < S_\mu < \infty;$$

if $\|x_{[m]}\|^2 > \delta$ and $b < 1/2$, we get

$$\frac{\sqrt{1-\|x_{[m]}\|^2}}{\mu^{[m]}(\|x_{[m]}\|^2)} = \frac{(1-\|x_{[m]}\|^2)^b}{\mu^{[m]}(\|x_{[m]}\|^2)}(1-\|x_{[m]}\|^2)^{1/2-b} \le S_\mu(1-\delta)^{1/2-b} < \infty.$$

Consequently,

$$|f_{e_{[m]}}(z_{[m]}) - f_{e_{[m]}}(w_{[m]})| \le \beta(z_{[m]}, w_{[m]})\widehat{S}_\mu \|f_{e_{[m]}}\|_{\mathcal{B}_\mu(\mathbb{B}_m)},$$

where

$$\widehat{S}_\mu := C_\mu^{-1}\max\left\{m_{\mu,\delta}^{-1}, S_\mu(1-\delta)^{1/2-b}\right\}.$$

Since $\beta(s,t)$ is the infimum of the set consisting of all $\ell(\gamma)$ where γ is a piecewise smooth curve in $\mathbb{B}_m$ from s to t (see [15, p. 25]) we have

$$|f_{e_{[m]}}(z_{[m]}) - f_{e_{[m]}}(w_{[m]})| \le \|z_{[m]} - w_{[m]}\|\widehat{S}_\mu\|f_{e_{[m]}}\|_{\mathcal{B}_\mu(\mathbb{B}_m)} \le \widehat{S}_\mu\|z-w\|.$$

Consequently,

$$|f(z) - f(w)| = \lim_{m\to\infty}|f_{e_{[m]}}(z_{[m]}) - f_{e_{[m]}}(w_{[m]})| \le \widehat{S}_\mu\|z-w\|.$$

This yields that U is equicontinuous. □

Remark 3.3. *In fact, the estimate (3.3) can be written as follows*

$$|f(z)| \le |f(0)| + I_\mu(z)\|f\|_{s\mathcal{B}_\mu}.$$

Finally, we discuss the linearization theorem for spaces $\mathcal{W}\mathcal{B}_\mu(B_X)$ which will be usefull in investigation some related problems, especially the theory of operators between these spaces. In fact, this theorem holds for spaces $\mathcal{W}\mathcal{E}$ where $\mathcal{E} \subset \mathscr{H}(B_X)$ is a Banach space satisfying (e1)-(e2). In this paper, we will state and prove this theorem for the general case.

Theorem 3.5 (Linearization). *Let X, Y be complex Banach spaces and $\mathcal{W} \subset Y'$ be a separating subspace. Let $\mathcal{E} \subset \mathscr{H}(B_X)$ be a Banach space satisfying (e1)-(e2). Then there exist a Banach space ${}^*\mathcal{E}$ and a mapping $\delta_X \in \mathscr{H}(B_X, {}^*\mathcal{E})$ with the following universal property: A function $f \in \mathcal{W}\mathcal{E}$ if and only if there is a unique mapping $T_f \in L({}^*\mathcal{E}, Y)$ such that $T_f \circ \delta_X = f$. This property characterize ${}^*\mathcal{E}$ uniquely up to an isometric isomorphism.*

Moreover, the mapping

$$\Phi : f \in \mathcal{W}\mathcal{E} \mapsto T_f \in L({}^*\mathcal{E}, Y)$$

is a topological isomorphism.

We will prove this theorem by Mujica's method [8, Theorem 2.1], which is based on the Dixmier–Ng theorem, with a little improvement.

Proof. Let us denote by ${}^*\mathcal{E}$ the closed subspace of all linear functionals $u \in \mathcal{E}'$ such that $u\big|_{B_\mathcal{E}}$ is τ_{co}-continuous. As in the proof of Mujica [8, Theorem 2.1], we get the evaluation mapping $\delta_X : B_X \to {}^*\mathcal{E}$ given by $\delta_X(x) = \delta_x$ with $\delta_x(g) := g(x)$ for all $g \in \mathcal{E}$, is holomorphic and

$$\text{span}\{\delta_x : x \in B_X\} \text{ is a dense subspace of } {}^*\mathcal{E}. \tag{3.4}$$

Now, we show that ${}^*\mathcal{E}$ and δ_X have required universal property. First, given a locally bounded function $f : B_X \to Y$. Assume that there exists $T_f \in L({}^*\mathcal{E}, Y)$ such that $T_f \circ \delta_X = f$. Since T_f is continuous and δ_X is holomorphic, it follows that $u \circ f \in \mathscr{H}(B_X)$ for every $u \in \mathcal{W}$. Since $\mathcal{W}$ is separating, according to [10, Lemma 4.2] we have $f \in \mathscr{H}_{LB}(B_X, Y)$. Next, it follows from $({}^*\mathcal{E})' = \mathcal{E}$ (see [9]) that $u \circ f \in \mathcal{E}$ for each $u \in \mathcal{W}$, and then $f \in \mathcal{W}\mathcal{E}$. Now, we will prove the converse of the statement. Fix $f \in \mathcal{W}\mathcal{E}$.

(i) *The case of* $Y = \mathbb{C}$: We define $T_f := Jf$, where $J : \mathcal{E} \to ({}^*\mathcal{E})'$ is the evaluation mapping given by $(Jf)(u) = u(f)$ for all $u \in {}^*\mathcal{E}$, which is a topological isomorphism by the Ng Theorem [9, Theorem 1]. Since $(Jg) \circ \delta_X(x) = \delta_x(g) = g(x)$ for all $g \in \mathcal{E}$, $x \in B_X$, it implies that $T_f \circ \delta_X = f$. From (3.4) we obtain the uniqueness of T_f.

(ii) *The case of* Y *is Banach:* We define $T_f : {}^*\mathcal{E} \to \mathcal{W}'$ by

$$(T_f u)(\varphi) = T_{\varphi\circ f}(u) = u(\varphi \circ f) \quad \forall u \in {}^*\mathcal{E}, \forall \varphi \in \mathcal{W}, \tag{3.5}$$

i.e. $T_{\varphi\circ f}$ is defined as in the case (i).

It is easy to check that $T_f \in L({}^*\mathcal{E}, \mathcal{W}')$ and $\|T_f\| = \|f\|_{\mathcal{W}\mathcal{E}}$, hence, Φ is a isometric isomorphism. Furthermore,

$$(T_f\delta_x)(\varphi) = (\varphi \circ f)(x)$$

for every $x \in B_X$ and $\varphi \in \mathcal{W}$ and, therefore, since $\mathcal{W}$ is separating we get $T_f\delta_x = f(x) \in Y$ for every $x \in B_X$. Then, by (3.4) $T_f \in L({}^*\mathcal{E}, Y)$. The uniqueness of T_f follows also from the fact (3.4) that $\delta_X(B_X)$ generates a dense subspace of ${}^*\mathcal{E}$.

Finally, the uniqueness of ${}^*\mathcal{E}$ up to an isometric isomorphism follows from the universal property, together with the isometry $\|T_f\| = \|f\|_{\mathcal{W}\mathcal{E}}$. This completes the proof. □

Our results suggest the following questions.

Problems.

(1) Let $\mathcal{E}_i$, $i = 1, 2$, be spaces of holomorphic functions on the unit ball B_X of a Banach space X and $\mathcal{W}\mathcal{E}_i$ be the Banach spaces $\mathcal{W}$-associated with $\mathcal{E}_i$ of Y-valued functions. Let ψ be a holomorphic on B_X and φ a holomorphic self-map of B_X. Consider the extended Cesàro composition operators $C_{\psi,\varphi} : \mathcal{E}_1 \to \mathcal{E}_2$, $\widetilde{C}_{\psi,\varphi} : \mathcal{W}\mathcal{E}_1 \to \mathcal{W}\mathcal{E}_2$, and the weighted composition operators $W_{\psi,\varphi} : \mathcal{E}_1 \to \mathcal{E}_2$, $\widetilde{W}_{\psi,\varphi} : \mathcal{W}\mathcal{E}_1 \to \mathcal{W}\mathcal{E}_2$.

Is there any relationship between the boundedness as well as the (weak) compactness of $C_{\psi,\varphi}$, $W_{\psi,\varphi}$ and of $\widetilde{C}_{\psi,\varphi}$, $\widetilde{W}_{\psi,\varphi}$? How does separating subspace $\mathcal{W} \subset Y'$ affect that relationship?

(2) In the case where $\mathcal{E}_1 = \mathcal{B}_\nu(B_X)$, $\mathcal{E}_2 = \mathcal{B}_\mu(B_X)$ with ν and μ are normal weights on the unit ball B_X of a infinite dimensional Hilbert space X, is it possible to characterize the boundedness as well as the (weak) compactness of $\widetilde{C}_{\psi,\varphi}$, $\widetilde{W}_{\psi,\varphi}$ via the estimates for the restrictions of ψ and φ to a m-dimensional subspace of X for some $m \geq 2$?

References

[1] O. Blasco, P. Galindo and A. Miralles: *Bloch functions on the unit ball of an infinite dimensional Hilbert space*, J. Func. Anal., **267** (2014), 1188–1204.

[2] O. Blasco, P. Galindo, M. Lindström and A. Miralles: *Composition operators on the Bloch space of the unit ball of a Hilbert space*, Banach J. Math. Anal., **11**(2) (2017), 311–334.

[3] O. Blasco, P. Galindo, M. Lindström and A. Miralles: *Interpolating sequences for weighted spaces of analytic functions on the unit ball of a Hilbert space,* Revista Mate., Complutense, **32** (2019), 115–139.
[4] C. Chu, H. Hamada, T. Honda and G. Kohr: *Bloch functions on bounded symmetric domains*, J. Funct. Anal., **272**(6) (2017), 2412–2441.
[5] K. T. Hahn: *Holomorphic mappings of the hyperbolic space into the complex Euclidean space and the Bloch theorem*, Canad. J. Math., **27** (1975), 446–458.
[6] H. Hamada: *Bloch-type spaces and extended Cesàro operators in the unit ball of a complex Banach space*, Sci. China Math., **62**(4) (2019), 617–628.
[7] S. G. Krantz: *Lipschitz spaces, smoothness of functions, and approximation theory*, Exposition. Math., **1**(3) (1983), 193–260.
[8] J. Mujica: *A completeness criterion for indutive limits of Banach spaces,* In: Functional Analysis, Holomorphy and Approximation Theory II, pp. 319–329, North-Holland Math. Studies, 86, North-Holland Amsterdam, (1984).
[9] K. F. Ng: *On a theorem of Dixmier,* Math. Scand., **29** (1971), 279–280.
[10] T. T. Quang, L. V. Lam and N. V. Dai: *On $\sigma(\cdot, W)$-holomorphic functions and theorems of Vitali-type*, Complex Anal. and Oper. Theory, **7**(1) (2013), 237–259.
[11] A. L. Shields and D. L. Williams: *Bounded projections, duality, and multipliers in spaces of analytic functions*, Trans. Amer. Math. Soc., **162** (1971), 287–302.
[12] R. M. Timoney: *Bloch functions in several complex variables I*, Bull. Lond. Math. Soc., **12** (1980), 241–267.
[13] R. M. Timoney: *Bloch functions in several complex variables II*, J. Reine Angew. Math., **319** (1980), 1–22.
[14] Z. Xu: *Bloch Type Spaces on the Unit Ball of a Hilbert Space*, Czechoslovak Math. J., **69** (2019), 695–711.
[15] K. Zhu: *Spaces of Holomorphic Functions in the Unit Ball,* Graduate Texts in Mathematics, vol. 226, Springer-Verlag, New York, (2005).

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Research Article

Branched continued fraction representations of ratios of Horn's confluent function H_6

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ABSTRACT. In this paper, we derive some branched continued fraction representations for the ratios of the Horn's confluent function H_6. The method employed is a two-dimensional generalization of the classical method of constructing of Gaussian continued fraction. We establish the estimates of the rate of convergence for the branched continued fraction expansions in some region Ω (here, region is a domain (open connected set) together with all, part or none of its boundary). It is also proved that the corresponding branched continued fractions uniformly converge to holomorphic functions on every compact subset of some domain Θ, and that these functions are analytic continuations of the ratios of double confluent hypergeometric series in Θ. At the end, several numerical experiments are represented to indicate the power and efficiency of branched continued fractions as an approximation tool compared to double confluent hypergeometric series.

Keywords: Hypergeometric function, branched continued fraction, convergence.

2020 Mathematics Subject Classification: 33C65, 32A17, 40A99.

1. INTRODUCTION

This paper deals with branched continued fraction representations for the ratios of the Horn's confluent function H_6, which occurs in [27] (see also [24, Subsection 5.7.1]) of second-order hypergeometric series of two variables. The branched continued fraction representations under consideration will be two-dimensional generalization of the classical Gaussian continued fraction, or rather its confluent case. Necessarily, due to the convergence of branched continued fractions, this requires restrictions on the allowed values of the parameters of the Horn's confluent function H_6.

J. Horn [27] listed all convergent hypergeometric series of the second order: 14 complete series, including Appell's hypergeometric series F_1, F_2, F_3, and F_4, dating back to 1880 [6], and 20 of their confluent cases. In [24, Section 5.9], for each function in Horn's list a system of two partial differential equations is given, which has this function as a solution. For the basics of hypergeometric functions of two variables, see, for instance, [7, Chapter 9], [24, Section 5.9–2.12], and [25, Chapter 1].

In order for a branched continued fraction to be a representation of a function, it is required to solve such problems: to construct the branched continued fraction expansion, to prove the convergence of the constructed expansion, and last, more important, to prove the convergence of the branched continued fraction to the function of which it is an expansion.

For Appell's hypergeometric functions, branched continued fraction representations were derived in [8, pp. 244–252] for F_1, in [15] for F_3, and in [16, 26] for F_4. A branched continued

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fraction expansion for function F_2 was constructed in [13], but the problem of its convergence remains open. In [1], it is represented a branched continued fraction representations for the Horn's function H_3. At last, in [18], it is indicated which three- and four-term recurrent relations give similar expansions for the Horn's function H_4. Some interesting and different branched continued fraction representations of other hypergeometric series can be found in [2, 3, 14, 28, 29, 31], and some special analytic functions of one or several variables in [19, 20, 21, 22, 23, 30, 32].

The contents of this paper are as follows. In Section 2, we derive three different formal branched continued fraction expansions for three different ratios of the Horn's confluent function H_6. In Section 3, we establish the estimates of the rate of convergence for the branched continued fractions mentioned above. We also prove that the branched continued fraction expansions converge to the functions, which are analytic continuations of Horn's confluent function H_6 ratios in some domain (here, domain is an open connected set), i.e., our main result is formulated in the Theorem 3.3. Finally, in Section 4, we present some numerical experiments to indicate the power and efficiency of branched continued fractions as an approximation tool compared to double confluent hypergeometric series.

2. Expansions

The Horn's confluent function H_6 [27] is defined as double power series by

$$\mathrm{H}_6(a,c;\mathbf{z})=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}}{(c)_{m+n}}\frac{z_1^m z_2^n}{m!\,n!},\quad |z_1|<1/4, \tag{2.1}$$

where a,c are complex numbers, $c\notin\{0,-1,-2,\ldots\}$, $(\cdot)_k$ is the Pochhammer symbol, $\mathbf{z}=(z_1,z_2)\in\mathbb{C}^2$.

Throughout the paper, let $[\cdot]$ be an integer part of a number. We set $\mathcal{I}_0=\{1,2,3\}$ and for $k\in\mathbb{N}$ we introduce the following sets of multiindices

$$\mathcal{I}_k=\{i(k)=(i_0,i_1,i_2,\ldots,i_k):\ i_0\in\mathcal{I}_0,\ 2-[(i_{r-1}-1)/2]\le i_r\le 3-[(i_{r-1}-1)/2],\ 1\le r\le k\}.$$

Using the idea of combining several branched continued fraction expansions into one form using the Kronecker delta symbol, proposed in [1], we will prove the following theorem.

Theorem 2.1. *Let for all $i_0\in\mathcal{I}_0$*

$$R_{i_0}(a,c;\mathbf{z})=\frac{\mathrm{H}_6(a,c;\mathbf{z})}{\mathrm{H}_6(a+\delta_{i_0}^1+\delta_{i_0}^2,c+\delta_{i_0}^2+\delta_{i_0}^3;\mathbf{z})}, \tag{2.2}$$

where δ_i^j is the Kronecker delta. Then for each $i_0\in\mathcal{I}_0$, the ratio $R_{i_0}(a,c;\mathbf{z})$ has a formal branched continued fraction expansion of the form

$$1-\frac{a}{2c}\delta_{i_0}^3+\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{P_{i(1)}(\mathbf{z})}{Q_{i(1)}}+\sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]}\frac{P_{i(2)}(\mathbf{z})}{Q_{i(2)}}+\cdots, \tag{2.3}$$

where for $i(1) \in \mathcal{I}_1$

$$P_{i(1)}(\mathbf{z}) = p_{i_0,i_1}(a,c;\mathbf{z})$$

$$= \begin{cases} -2\dfrac{a+1}{c}z_1, & \text{if } i_0 = 1,\ i_1 = 2, \\ -\dfrac{z_2}{c}, & \text{if } i_0 = 1,\ i_1 = 3, \\ -\dfrac{(2c-a)(a+1)}{c(c+1)}z_1, & \text{if } i_0 = 2,\ i_1 = 2, \\ -\dfrac{c-a}{c(c+1)}z_2, & \text{if } i_0 = 2,\ i_1 = 3, \\ \dfrac{a}{2c}, & \text{if } i_0 = 3,\ i_1 = 1, \\ \dfrac{a}{2c(c+1)}z_2, & \text{if } i_0 = 3,\ i_1 = 2, \end{cases} \tag{2.4}$$

for $i(k+1) \in \mathcal{I}_{k+1}$, $k \geq 1$,

$$P_{i(k+1)}(\mathbf{z}) = p_{i_k,i_{k+1}}\left(a+k-\sum_{r=0}^{k-1}\delta^3_{i_r}, c+k-\sum_{r=0}^{k-1}\delta^1_{i_r};\mathbf{z}\right)$$

$$= \begin{cases} -\dfrac{2(a+k-\sum_{r=0}^{k-1}\delta^3_{i_r}+1)}{c+k-\sum_{r=0}^{k-1}\delta^1_{i_r}}z_1, & \text{if } i_k = 1,\ i_{k+1} = 2, \\ -\dfrac{z_2}{c+k-\sum_{r=0}^{k-1}\delta^1_{i_r}}, & \text{if } i_k = 1,\ i_{k+1} = 3, \\ -\dfrac{(2c-a+k+\sum_{r=0}^{k-1}(\delta^3_{i_r}-2\delta^1_{i_r}))(a+k-\sum_{r=0}^{k-1}\delta^3_{i_r}+1)}{(c+k-\sum_{r=0}^{k-1}\delta^1_{i_r})(c+k-\sum_{r=0}^{k-1}\delta^1_{i_r}+1)}z_1, & \text{if } i_k = 2,\ i_{k+1} = 2, \\ -\dfrac{c-a+\sum_{r=0}^{k-1}(\delta^3_{i_r}-\delta^1_{i_r})}{(c+k-\sum_{r=0}^{k-1}\delta^1_{i_r})(c+k-\sum_{r=0}^{k-1}\delta^1_{i_r}+1)}z_2, & \text{if } i_k = 2,\ i_{k+1} = 3, \\ \dfrac{a+k-\sum_{r=0}^{k-1}\delta^3_{i_r}}{2(c+k-\sum_{r=0}^{k-1}\delta^1_{i_r})}, & \text{if } i_k = 3,\ i_{k+1} = 1, \\ \dfrac{a+k-\sum_{r=0}^{k-1}\delta^3_{i_r}}{2(c+k-\sum_{r=0}^{k-1}\delta^1_{i_r})(c+k+1-\sum_{r=0}^{k-1}\delta^1_{i_r})}z_2, & \text{if } i_k = 3,\ i_{k+1} = 2, \end{cases} \tag{2.5}$$

and for $i(k) \in \mathcal{I}_k$, $k \geq 1$,

$$Q_{i(k)} = q_{i_k}\left(a+k-\sum_{r=0}^{k-1}\delta^3_{i_r}, c+k-\sum_{r=0}^{k-1}\delta^1_{i_r}\right)$$

$$= 1 - \frac{a+k-\sum_{r=0}^{k-1}\delta^3_{i_r}}{2(c+k-\sum_{r=0}^{k-1}\delta^1_{i_r})}\delta^3_{i_k}. \tag{2.6}$$

Proof. The formal identities

$$\mathrm{H}_6(a,c;\mathbf{z}) = \mathrm{H}_6(a+1,c;\mathbf{z}) - \frac{2(a+1)}{c}z_1\mathrm{H}_6(a+2,c+1;\mathbf{z}) - \frac{1}{c}z_2\mathrm{H}_6(a+1,c+1;\mathbf{z}), \tag{2.7}$$

$$\mathrm{H}_6(a,c;\mathbf{z}) = \mathrm{H}_6(a+1,c+1;\mathbf{z}) - \frac{(a+1)(2c-a)}{c(c+1)}z_1\mathrm{H}_6(a+2,c+2;\mathbf{z})$$
$$- \frac{c-a}{c(c+1)}z_2\mathrm{H}_6(a+1,c+2;\mathbf{z}), \tag{2.8}$$

and

$$\mathrm{H}_6(a,c;\mathbf{z}) = \frac{a}{2c}\mathrm{H}_6(a+1,c+1;\mathbf{z}) + \frac{2c-a}{2c}\mathrm{H}_6(a,c+1;\mathbf{z}) + \frac{a}{2c(c+1)}z_2\mathrm{H}_6(a+1,c+2;\mathbf{z}) \tag{2.9}$$

are easily verified from (2.1). Dividing (2.7) by $\mathrm{H}_6(a+1,c;\mathbf{z})$, (2.8) by $\mathrm{H}_6(a+1,c+1;\mathbf{z})$, and (2.9) by $\mathrm{H}_6(a,c+1;\mathbf{z})$, we get

$$R_1(a,c;\mathbf{z}) = 1 - \frac{2(a+1)}{c}z_1\frac{1}{R_2(a+1,c;\mathbf{z})} - \frac{1}{c}z_2\frac{1}{R_3(a+1,c;\mathbf{z})}, \tag{2.10}$$

$$R_2(a,c;\mathbf{z}) = 1 - \frac{(a+1)(2c-a)}{c(c+1)}z_1\frac{1}{R_2(a+1,c+1;\mathbf{z})} - \frac{c-a}{c(c+1)}z_2\frac{1}{R_3(a+1,c+1;\mathbf{z})} \tag{2.11}$$

and

$$R_3(a,c;\mathbf{z}) = \frac{2c-a}{2c} + \frac{a}{2c}\frac{1}{R_1(a,c+1;\mathbf{z})} + \frac{a}{2c(c+1)}z_2\frac{1}{R_2(a,c+1;\mathbf{z})}, \tag{2.12}$$

respectively. It is obvious that for $i \in \mathcal{I}_0$ the identities (2.10)–(2.12) can be written as

$$R_i(a,c;\mathbf{z}) = 1 - \frac{a}{2c}\delta_i^3 + \sum_{j=2-[(i-1)/2]}^{3-[(i-1)/2]} \frac{p_{i,j}(a,c;\mathbf{z})}{R_j(a+1-\delta_i^3, c+1-\delta_i^1;\mathbf{z})}, \tag{2.13}$$

where $p_{i,j}(a,c;\mathbf{z})$, $(i,j) \in \mathcal{I}_1$ are defined as (2.4).

Now, we can construct branched continued fractions for ratios $R_{i_0}(a,c;\mathbf{z})$ for all $i_0 \in \mathcal{I}_0$. Setting $i = i_0$, on the first step, from (2.13) for $i_0 \in \mathcal{I}_0$, we obtain

$$\begin{aligned} R_{i_0}(a,c;\mathbf{z}) &= 1 - \frac{a}{2c}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{p_{i_0,i_1}(a,c;\mathbf{z})}{R_{i_1}(a+1-\delta_{i_0}^3, c+1-\delta_{i_0}^1;\mathbf{z})} \\ &= 1 - \frac{a}{2c}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{P_{i(1)}(\mathbf{z})}{R_{i_1}(a+1-\delta_{i_0}^3, c+1-\delta_{i_0}^1;\mathbf{z})}. \end{aligned}$$

It follows from (2.13) that for $i_1 \in \mathcal{I}_0$

$$\begin{aligned} &R_{i_1}(a+1-\delta_{i_0}^3, c+1-\delta_{i_0}^1;\mathbf{z}) \\ =&q_{i_1}(a+1-\delta_{i_0}^3, c+1-\delta_{i_0}^1) + \sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} \frac{p_{i_1,i_2}(a+1-\delta_{i_0}^3, c+1-\delta_{i_0}^1;\mathbf{z})}{R_{i_2}(a+2-\sum_{r=0}^1\delta_{i_r}^3, c+2-\sum_{r=0}^1\delta_{i_r}^1;\mathbf{z})} \\ =&Q_{i(1)} + \sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} \frac{P_{i(2)}(\mathbf{z})}{R_{i_2}(a+2-\sum_{r=0}^1\delta_{i_r}^3, c+2-\sum_{r=0}^1\delta_{i_r}^1;\mathbf{z})}, \end{aligned}$$

where $P_{i(2)}(\mathbf{z})$, $i(2) \in \mathcal{I}_2$, and $Q_{i(1)}$, $i(1) \in \mathcal{I}_1$, are defined by (2.5) and (2.6), respectively. Then, on the second step for $i_0 \in \mathcal{I}_0$, we have

$$\begin{aligned} R_{i_0}(a,c;\mathbf{z}) = 1 - \frac{a}{2c}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{P_{i(1)}(\mathbf{z})}{Q_{i(1)}} \\ + \sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} \frac{P_{i(2)}(\mathbf{z})}{R_{i_2}(a+2-\sum_{r=0}^1\delta_{i_r}^3, c+2-\sum_{r=0}^1\delta_{i_r}^1;\mathbf{z})}. \end{aligned}$$

Next, applying (2.13) after nth steps, for $i_0 \in \mathcal{I}_0$ we get

$$R_{i_0}(a,c;\mathbf{z}) = 1 - \frac{a}{2c}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{P_{i(1)}(\mathbf{z})}{Q_{i(1)}} + \sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} \frac{P_{i(2)}(\mathbf{z})}{Q_{i(2)}}$$
$$+ \cdots + \sum_{i_{n-1}=2-[(i_{n-2}-1)/2]}^{3-[(i_{n-2}-1)/2]} \frac{P_{i(n-1)}(\mathbf{z})}{Q_{i(n-1)}}$$
$$+ \sum_{i_n=2-[(i_{n-1}-1)/2]}^{3-[(i_{n-1}-1)/2]} \frac{P_{i(n)}(\mathbf{z})}{R_{i_n}(a+n-\sum_{r=0}^{n-1}\delta_{i_r}^3, c+2-\sum_{r=0}^{n-1}\delta_{i_r}^1;\mathbf{z})},$$

where $P_{i(1)}(\mathbf{z})$, $i(1) \in \mathcal{I}_1$, $P_{i(k)}(\mathbf{z})$, $i(k) \in \mathcal{I}_k$, $2 \le k \le n$, and $Q_{i(k)}$, $i(k) \in \mathcal{I}_k$, $1 \le k \le n-1$, are defined by (2.4), (2.5), and (2.6), respectively. Finally, by (2.13), we obtain the formal branched continued fraction expansions (2.3) for ratios (2.2) for all $i_0 \in \mathcal{I}_0$. □

3. Convergence

In this section, we consider some question of convergence of the branched continued fractions (2.3). We refer the readers to [1, 5, 12] for the notations and definitions used below.

Let i_0 be an arbitrary index from the set $\mathcal{I}_0$. For the 'tails' of the approximants of the branched continued fraction (2.3), we set

$$G_{i(r)}^{(r)}(\mathbf{z}) = Q_{i(r)}, \quad i(r) \in \mathcal{I}_r,\ r \ge 1, \tag{3.14}$$

and

$$G_{i(k)}^{(r)}(\mathbf{z}) = Q_{i(k)} + \sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} \frac{P_{i(k+1)}(\mathbf{z})}{Q_{i(k+1)}} + \sum_{i_{k+2}=2-[(i_{k+1}-1)/2]}^{3-[(i_{k+1}-1)/2]} \frac{P_{i(k+2)}(\mathbf{z})}{Q_{i(k+2)}}$$
$$+ \cdots + \sum_{i_r=2-[(i_{r-1}-1)/2]}^{3-[(i_{r-1}-1)/2]} \frac{P_{i(r)}(\mathbf{z})}{Q_{i(r)}},$$

where $i(k) \in \mathcal{I}_k,\ 1 \le k \le r-1,\ r \ge 2$. Then, it is easily seen that relations

$$G_{i(k)}^{(r)}(\mathbf{z}) = Q_{i(k)} + \sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} \frac{P_{i(k+1)}(\mathbf{z})}{G_{i(k+1)}^{(r)}(\mathbf{z})}, \quad i(k) \in \mathcal{I}_k,\ 1 \le k \le r-1,\ r \ge 2 \tag{3.15}$$

hold. It follows that for each $n \ge 1$ the nth approximant

$$f_n^{(i_0)}(\mathbf{z}) = 1 - \frac{a}{2c}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{P_{i(1)}(\mathbf{z})}{Q_{i(1)}} + \sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} \frac{P_{i(2)}(\mathbf{z})}{Q_{i(2)}}$$
$$+ \cdots + \sum_{i_n=2-[(i_{n-1}-1)/2]}^{3-[(i_{n-1}-1)/2]} \frac{P_{i(n)}(\mathbf{z})}{Q_{i(n)}}$$

can be written as

$$f_n^{(i_0)}(\mathbf{z}) = 1 - \frac{a}{2c}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{P_{i(1)}(\mathbf{z})}{G_{i(1)}^{(n)}(\mathbf{z})}.$$

In addition, it can be shown (see [12, p. 28]) that for $m > n$ and $n \geq 1$

$$f_m^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z}) = (-1)^n \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} \dots \sum_{i_{n+1}=2-[(i_n-1)/2]}^{3-[(i_n-1)/2]} \frac{\prod_{k=1}^{n+1} P_{i(k)}(\mathbf{z})}{\prod_{k=1}^{n+1} G_{i(k)}^{(m)}(\mathbf{z}) \prod_{k=1}^{n} G_{i(k)}^{(n)}(\mathbf{z})},$$

provided $G_{i(k)}^{(r)}(\mathbf{z}) \neq 0$ for all $i(k) \in \mathcal{I}_k$, $1 \leq k \leq r$, $r \in \{m, n\}$, which for convenience we will write as

$$f_m^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z}) = (-1)^n \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \dots \sum_{i_{n+1}=2-[(i_n-1)/2]}^{3-[(i_n-1)/2]} \frac{P_{i(1)}(\mathbf{z})}{G_{i(1)}^{(q)}(\mathbf{z})}$$
$$\times \prod_{k=1}^{[(n+1)/2]} \frac{P_{i(2k)}(\mathbf{z})}{G_{i(2k-1)}^{(r)}(\mathbf{z}) G_{i(2k)}^{(r)}(\mathbf{z})} \prod_{k=1}^{[n/2]} \frac{P_{i(2k+1)}(\mathbf{z})}{G_{i(2k)}^{(q)}(\mathbf{z}) G_{i(2k+1)}^{(q)}(\mathbf{z})}, \tag{3.16}$$

where $q = m$, $r = n$, if n is even, and $q = n$, $r = m$, if n is odd.

To prove the main result, we will state the following theorem.

Theorem 3.2. *Let a and c be real constants such that*

$$a \geq 0, \quad c \geq a + 1 + \delta_{i_0}^1 \quad \textit{for all} \quad i_0 \in \mathcal{I}_0. \tag{3.17}$$

Then for each $i_0 \in \mathcal{I}_0$:

(**A**) *The branched continued fraction (2.3) converges to a finite value $f^{(i_0)}(\mathbf{z})$ for each $\mathbf{z} \in \Omega$, where*

$$\Omega = \{\mathbf{z} \in \mathbb{R}^2 : \ -L_1 \leq z_1 \leq 0, \ -L_2 \leq z_2 \leq 0\}, \tag{3.18}$$

L_1 and L_2 are positive constants such that $2L_2 < c + 1$, and it converges uniformly on every compact subset of an interior of Ω.

(**B**) *If $f_n^{(i_0)}(\mathbf{z})$ denotes the nth approximant of the branched continued fraction (2.3), then for each $\mathbf{z} \in \Omega$ and $n \geq 1$*

$$|f^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z})| \leq M_{i_0} \left(\frac{\eta}{\eta + 1}\right)^n, \tag{3.19}$$

where

$$M_{i_0} = \begin{cases} \dfrac{2(a+1)L_1}{c} + \dfrac{2(c+1)L_2}{c(c+1-L_2)}, & \textit{if } i_0 = 1, \\ \dfrac{(2c-a)(a+1)L_1}{c(c+1)} + \dfrac{2(c-a)L_2}{c(c+1-L_2)}, & \textit{if } i_0 = 2, \\ \dfrac{a}{2c} + \dfrac{aL_2}{2c(c+1)}, & \textit{if } i_0 = 3, \end{cases} \tag{3.20}$$

and

$$\eta = \max\left\{2L_1 + \frac{2L_2(c+1)}{c(c+1-L_2)}, \ \frac{c+1+L_2}{c+1-2L_2}\right\}. \tag{3.21}$$

Proof. The proof is similar to that of Theorem 1 in [1]. In this case, it follows directly from (2.6) that for all $i(k) \in \mathcal{I}_k$, $k \geq 1$, the elements $Q_{i(k)} = 1$ if $i_k \neq 3$. When $i_k = 3$ from (2.6), we have

$$Q_{i(k)} = 1 - \frac{a + k - \sum_{r=0}^{k-1} \delta_{i_r}^3}{2c + 2k - 2\sum_{r=0}^{k-1} \delta_{i_r}^1} \geq \frac{1}{2} \quad \text{for all} \quad i(k) \in \mathcal{I}_k, \ k \geq 1. \tag{3.22}$$

Since

$$-\sum_{r=0}^{k-1}\delta_{i_r}^3=\sum_{r=0}^{k-1}(\delta_{i_r}^1-\delta_{i_r}^3)-\sum_{r=0}^{k-1}\delta_{i_r}^1\quad\text{for all}\quad i(k)\in\mathcal{I}_k,\ k\ge 1,$$

then to prove the validity of (3.22), provided (3.17), it suffices to show that

$$\sum_{r=0}^{k-1}(\delta_{i_r}^1-\delta_{i_r}^3)\le\delta_{i_0}^1\quad\text{for all}\quad i(k)\in\mathcal{I}_k,\ k\ge 1. \tag{3.23}$$

Indeed, if $k=1$, then for any $i_0\in\mathcal{I}_0$ inequalities (3.23) are obvious. If $i(k)$ is a fixed arbitrary multiindex in $\mathcal{I}_k$, $k\ge 2$, then for any r, $1\le r\le k-1$, there is a possible pair of indices (i_{r-1},i_r), such as $(1,2)$, $(1,3)$, $(2,2)$, $(2,3)$, $(3,1)$, or $(3,2)$. It clearly follows that (3.23) is valid in these cases.

Let $\mathbf{z}$ be an arbitrary fixed point in (3.18) and n be an arbitrary natural number. It is easy to see from (2.4)–(2.6), (3.14), (3.15), and (3.18) that inequalities

$$G_{i(k)}^{(n)}(\mathbf{z})\ge 1\quad\text{for all}\quad i(k)\in\mathcal{I}_k,\ 1\le k\le n, \tag{3.24}$$

hold for all $i_k\ne 3$. By induction on k, we show that the following inequalities

$$G_{i(k)}^{(n)}(\mathbf{z})\ge\frac{c+1-L_2}{2(c+1)}\quad\text{for all}\quad i(k)\in\mathcal{I}_k,\ 1\le k\le n, \tag{3.25}$$

valid for $i_k=3$.

For $k=n$ and for each $i(n)\in\mathcal{I}_n$, inequalities (3.25) are obvious. By induction hypothesis that (3.25) hold for $k=r+1$, where $r+1\le n$, and for each $i(r+1)\in\mathcal{I}_{r+1}$, using (2.4), (2.5), (3.14), (3.18), and (3.22) for any $i(r)\in\mathcal{I}_r$ we get

$$\begin{aligned}
G_{i(r)}^{(n)}(\mathbf{z})&=Q_{i(r)}+\frac{P_{i(r),1}(\mathbf{z})}{G_{i(r),1}^{(n)}(\mathbf{z})}+\frac{P_{i(r),2}(\mathbf{z})}{G_{i(r),2}^{(n)}(\mathbf{z})}\\
&\ge Q_{i(r)}-\frac{|P_{i(r),2}(\mathbf{z})|}{G_{i(r),2}^{(n)}(\mathbf{z})}\\
&\ge\frac{1}{2}-\frac{a+r-\sum_{p=0}^{r-1}\delta_{i_p}^3}{2(c+r-\sum_{p=0}^{r-1}\delta_{i_p}^1)(c+r+1-\sum_{p=0}^{r-1}\delta_{i_p}^1)}|z_2|\\
&\ge\frac{c+1-L_2}{2(c+1)}.
\end{aligned}$$

Next, we prove that

$$\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}\frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})G_{i(k)}^{(n)}(\mathbf{z})|}\le\frac{\eta}{\eta+1}\quad\text{for all}\quad i(k)\in\mathcal{I}_k,\ k\ge 1, \tag{3.26}$$

where η is defined by (3.21), which are equivalent to

$$\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}\frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})|}\le\eta\left(|G_{i(k)}^{(n)}(\mathbf{z})|-\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}\frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})|}\right)$$

for all $i(k)\in\mathcal{I}_k$, $k\ge 1$. Again, let n be an arbitrary natural number. Since it follows from (2.4)–(2.6), (3.14), (3.15), (3.18), (3.22), (3.24), and (3.25) that, for any k, $1\le k\le n$, and $i(k)\in\mathcal{I}_k$, and for any $\mathbf{z} \in \Omega$

$$|G_{i(k)}^{(n)}(\mathbf{z})| - \sum_{i_{k+1}=2}^{3} \frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})|} = Q_{i(k)} = 1,$$

if $i_k \neq 3$, and

$$\begin{aligned}|G_{i(k)}^{(n)}(\mathbf{z})| - \sum_{i_{k+1}=1}^{2} \frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})|} &\geq Q_{i(k)} - 2\frac{|P_{i(k),2}(\mathbf{z})|}{|G_{i(k),2}^{(n)}(\mathbf{z})|} \\ &\geq \frac{1}{2} - \frac{a+k-1-\sum_{r=0}^{k-2}\delta_{i_r}^3}{(c+k-1-\sum_{r=0}^{k-2}\delta_{i_r}^1)(c+k-\sum_{r=0}^{k-2}\delta_{i_r}^1)}|z_2| \\ &\geq \frac{1}{2} - \frac{|z_2|}{c+1} \\ &\geq \frac{c+1-2L_2}{2(c+1)},\end{aligned}$$

if $i_k = 3$, then we obtain

$$\begin{aligned}\sum_{i_{k+1}=2}^{3} \frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})|} &\leq \frac{2(a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3+1)}{c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1}|z_1| + \frac{2(c+1)}{(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+1-L_2)}|z_2| \\ &\leq 2L_1 + \frac{2L_2(c+1)}{c(c+1-L_2)} \\ &\leq \eta,\end{aligned}$$

if $i_k = 1$,

$$\begin{aligned}\sum_{i_{k+1}=2}^{3} \frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})|} &\leq \frac{(2c-a+k+\sum_{r=0}^{k-1}(\delta_{i_r}^3-2\delta_{i_r}^1))(a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3+1)}{(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1+1)}|z_1| \\ &\quad + \frac{(c-a+\sum_{r=0}^{k-1}(\delta_{i_r}^3-\delta_{i_r}^1))(2(c+1))}{(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1+1)(c+1-L_2)}|z_2| \\ &\leq 2L_1 + \frac{2L_2}{c+1-L_2} \\ &\leq \eta,\end{aligned}$$

if $i_k = 2$, and

$$\begin{aligned}\sum_{i_{k+1}=1}^{2} \frac{|P_{i(k+1)}(\mathbf{z})|}{|G_{i(k+1)}^{(n)}(\mathbf{z})|} &\leq \frac{a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)} + \frac{a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1+1)}|z_2| \\ &\leq \frac{1}{2} + \frac{L_2}{2(c+1)} \\ &= \frac{c+1+L_2}{c+1-2L_2}\frac{c+1-2L_2}{2(c+1)} \\ &\leq \frac{c+1-2L_2}{2(c+1)}\eta,\end{aligned}$$

if $i_k=3$. Now, it is easy see from (2.4), (3.18), (3.20), (3.24) and (3.25) that

$$\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{|P_{i(1)}(\mathbf{z})|}{|G^{(q)}_{i(1)}(\mathbf{z})|}\le M_{i_0}\quad\text{for all}\quad i_0\in\mathcal{I}_0\quad\text{and}\quad q\ge 1. \tag{3.27}$$

From (3.18), (3.24), and (3.25) it follows that $G^{(q)}_{i(k)}(\mathbf{z})\ne 0$ for all $i(k)\in\mathcal{I}_k$, $1\le k\le q$, $q\ge 1$, and for all $\mathbf{z}\in\Omega$. Hence, applying (3.26) and (3.27) to (3.16) for any $m>n\ge 1$ and for any $\mathbf{z}\in\Omega$, we obtain

$$\begin{aligned}|f^{(i_0)}_m(\mathbf{z})-f^{(i_0)}_n(\mathbf{z})|&\le\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{|P_{i(1)}(\mathbf{z})|}{|G^{(q)}_{i(1)}(\mathbf{z})|}\left(\frac{\eta}{\eta+1}\right)^n\\&\le M_{i_0}\left(\frac{\eta}{\eta+1}\right)^n,\end{aligned}$$

where $q=m$, if n is even, and $q=n$, if n is odd. From this (A) follows if $n\to\infty$. At last, passing to the limit as $m\to\infty$, we get (B). □

Now, we prove our main result.

Theorem 3.3. *Let a and c be real constants satisfying the inequalities (3.17), and $\nu_1,\nu_2,\nu_3,\mu_1,\mu_2,\mu_3$ be positive numbers such that*

$$\frac{2\nu_1}{\mu_2}\le\min\left\{1-\mu_1-\frac{\nu_2}{c\mu_3},\ 1-\mu_2-\frac{\nu_2}{(c+1)\mu_3}\right\},\quad \frac{\nu_3}{(c+1)\mu_2}\le\frac{1}{2}-\mu_3. \tag{3.28}$$

Then for each $i_0\in\mathcal{I}_0$:

(**A**) *The branched continued fraction (2.3) converges uniformly on every compact subset of*

$$\Theta=\{\mathbf{z}\in\mathbb{C}^2:\ |z_1|+\mathrm{Re}(z_1)<2\nu_1,\ |z_2|+\mathrm{Re}(z_2)<2\nu_2,\ |z_2|-\mathrm{Re}(z_2)<2\nu_3\} \tag{3.29}$$

to the function $f^{(i_0)}(\mathbf{z})$ holomorphic in Θ.

(**B**) *The function $f^{(i_0)}(\mathbf{z})$ is an analytic continuation of (2.2) in the domain (3.29).*

Proof. The proof of (A) is similar to the proof of Theorem 2 [1]. Let $\mathbf{z}$ be an arbitrary fixed point in (3.29). Since a and c satisfy (3.17), it follows from the proof of Theorem 2 that inequalities (3.22) hold for $i_k=3$, and that for all $i(k)\in\mathcal{I}_k$, $k\ge 1$, the elements $Q_{i(k)}=1$ if $i_k\ne 3$. Now, for any $i(k)\in\mathcal{I}$, $k\ge 1$, from (2.4)–(2.6) and (3.29) with $i_k=1$, we have

$$\begin{aligned}|P_{i(k),2}(\mathbf{z})|-\mathrm{Re}(P_{i(k),2}(\mathbf{z}))&=\frac{2(a+k-\sum_{r=0}^{k-1}\delta^3_{i_r}+1)}{c+k-\sum_{r=0}^{k-1}\delta^1_{i_r}}(|z_1|+\mathrm{Re}(z_1))\\&<4\nu_1,\\|P_{i(k),3}(\mathbf{z})|-\mathrm{Re}(P_{i(k),3}(\mathbf{z}))&=\frac{|z_2|+\mathrm{Re}(z_2)}{c+k-\sum_{r=0}^{k-1}\delta^1_{i_r}}\\&<\frac{2\nu_2}{c},\end{aligned}$$

and, thus,

$$\sum_{i_{k+1}=2}^{3}\frac{|P_{i(k+1)}(\mathbf{z})|-\mathrm{Re}(P_{i(k+1)}(\mathbf{z}))}{\mu_{i_{k+1}}}<\frac{4\nu_1}{\mu_2}+\frac{2\nu_2}{c\mu_3}$$
$$\le 2(1-\mu_1)$$
$$=2(\mathrm{Re}(Q_{i(k)})-\mu_1).$$

If $i_k=2$, we obtain

$$|P_{i(k),2}(\mathbf{z})|-\mathrm{Re}(P_{i(k),2}(\mathbf{z}))$$
$$=\frac{(2c-a+k+\sum_{r=0}^{k-1}(\delta_{i_r}^3-2\delta_{i_r}^1))(a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3+1)}{(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1+1)}(|z_1|+\mathrm{Re}(z_1))$$
$$<4\nu_1,$$
$$|P_{i(k),3}(\mathbf{z})|-\mathrm{Re}(P_{i(k),3}(\mathbf{z}))$$
$$=\frac{c-a+\sum_{r=0}^{k-1}(\delta_{i_r}^3-\delta_{i_r}^1)}{(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1+1)}(|z_2|+\mathrm{Re}(z_2))$$
$$<\frac{2\nu_2}{c+1},$$

and, thus,

$$\sum_{i_{k+1}=2}^{3}\frac{|P_{i(k+1)}(\mathbf{z})|-\mathrm{Re}(P_{i(k+1)}(\mathbf{z}))}{\mu_{i_{k+1}}}<\frac{4\nu_1}{\mu_2}+\frac{2\nu_2}{(c+1)\mu_3}$$
$$\le 2(\mathrm{Re}(Q_{i(k)})-\mu_2).$$

At last, if $i_k=3$ we get

$$|P_{i(k),1}(\mathbf{z})|-\mathrm{Re}(P_{i(k),1}(\mathbf{z}))=\frac{a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)}-\frac{a+k-\sum_{p=r}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)}$$
$$=0,$$
$$|P_{i(k),2}(\mathbf{z})|-\mathrm{Re}(P_{i(k),2}(\mathbf{z}))=\frac{a+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(c+k+1-\sum_{r=0}^{k-1}\delta_{i_r}^1)}(|z_2|-\mathrm{Re}(z_2))$$
$$<\frac{2\nu_3}{c+1},$$

and, thus,

$$\sum_{i_{k+1}=1}^{2}\frac{|P_{i(k+1)}(\mathbf{z})|-\mathrm{Re}(P_{i(k+1)}(\mathbf{z}))}{\mu_{i_{k+1}}}<2\left(\frac{1}{2}-\mu_3\right)$$
$$\le 2(\mathrm{Re}(Q_{i(k)})-\mu_3).$$

Thus, by Lemma 1 [4], for all $i(k)\in\mathcal{I}_k$, $1\le k\le n$, $n\ge 1$, and for all $\mathbf{z}\in\Theta$ the following inequalities hold

$$\mathrm{Re}(G_{i(k)}^{(n)}(\mathbf{z}))\ge\mu_k,$$

where $G_{i(k)}^{(n)}(\mathbf{z})$, $i(k)\in\mathcal{I}_k$, $1\le k\le n$, $n\ge 1$, are defined by (3.14) and (3.15). The approximants $f_n^{(i_0)}(\mathbf{z})$, $n\ge 1$, of (2.3) form a sequence of functions holomorphic in (3.29).

At last, it remains to show that the branched continued fraction (2.3) converges uniformly on compact subsets of Θ. Let $\mathcal{K}$ is an arbitrary compact subset of (3.29). Then there exists an open ball around the origin with radius R, containing $\mathcal{K}$. By (2.4), for the any $\mathbf{z} \in \mathcal{K}$ and for any $n \geq 1$, we get

$$\begin{aligned} |f_n^{(i_0)}(\mathbf{z})| &\leq 1 + \frac{a}{2c}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{|P_{i(1)}(\mathbf{z})|}{\mu_{i(1)}} \\ &= C_{i_0}(\mathcal{K}), \end{aligned}$$

where

$$C_{i_0}(\mathcal{K}) = \begin{cases} \dfrac{2(a+1)R}{c\mu_2} + \dfrac{R}{c\mu_3}, & \text{if } i_0 = 1, \\ \dfrac{(2c-a)(a+1)R}{c(c+1)\mu_2} + \dfrac{(c-a)R}{c(c+1)\mu_3}, & \text{if } i_0 = 2, \\ \dfrac{a}{2c\mu_1} + \dfrac{aR}{2c(c+1)\mu_2}, & \text{if } i_0 = 3. \end{cases}$$

It follows that for each $i_0 \in \mathcal{I}_0$ the sequence $\{f_n^{(i_0)}(\mathbf{z})\}$ is uniformly bounded on $\mathcal{K}$, and hence it is uniformly bounded on every compact subset of the domain (3.29). We set $\delta = \min\{c/4, \nu_1, \nu_3\}$. Then, by Theorem 2, the sequence $\{f_n^{(i_0)}(\mathbf{z})\}$ converges in

$$\Delta = \{\mathbf{z} \in \mathbb{C}^2 : \ -\delta < \mathrm{Re}(z_k) < 0, \ \mathrm{Im}(z_k) = 0, \ k = 1, 2\},$$

which is the real neighborhood of the point $\mathbf{z}^{(0)} = (-\delta/2, -\delta/2)$ in Θ. Furthermore, it is clear that $\Delta \subset \Theta$. Thus, by Theorem 3 [1] (see also Theorem 2.17 [12]), for each $i_0 \in \mathcal{I}_0$ the branched continued fraction (2.3) converges uniformly on compact subsets of Θ to the function $f^{(i_0)}(\mathbf{z})$ holomorphic in Θ. This proves (A).

Finally, the proof of (B) is analogous to the proof of Theorem 2 [1]; hence it is omitted. □

Setting $a = 0$ and $i_0 = 1$ (or $i_0 = 2$ and replacing c by $c - 1$) in Theorem 3.3, we get a corollary.

Corollary 3.1. *Let c be real constant such that $c \geq 2$, and $\nu_1, \nu_2, \nu_3, \mu_1, \mu_2, \mu_3$ be positive numbers satisfying the inequalities (3.28). Then for $i_0 = 1$ (or $i_0 = 2$):*

(**A**) *The branched continued fraction*

$$\frac{1}{1} + \sum_{i_1=2}^{3} \frac{P_{i(1)}(\mathbf{z})}{Q_{i(1)}} + \sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} \frac{P_{i(2)}(\mathbf{z})}{Q_{i(2)}} + \cdots + \sum_{i_k=2-[(i_{k-1}-1)/2]}^{3-[(i_{k-1}-1)/2]} \frac{P_{i(k)}(\mathbf{z})}{Q_{i(k)}} + \cdots, \tag{3.30}$$

where for $i(1) \in \mathcal{I}_1$

$$P_{i(1)}(\mathbf{z}) = \begin{cases} -\dfrac{2}{c} z_1, & \text{if } i_1 = 2, \\ -\dfrac{z_2}{c}, & \text{if } i_1 = 3, \end{cases} \tag{3.31}$$

for $i(k+1) \in \mathcal{I}_{k+1}$, $k \geq 1$,

$$P_{i(k+1)}(\mathbf{z})$$

$$(3.32)\qquad = \begin{cases} -\dfrac{2(k-\sum_{r=1}^{k-1}\delta_{i_r}^3+1)}{c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1-1}z_1, & \textit{if } i_k=1,\ i_{k+1}=2,\\ -\dfrac{z_2}{c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1-1}, & \textit{if } i_k=1,\ i_{k+1}=3,\\ -\dfrac{(2(c-1)+k+\sum_{r=1}^{k-1}(\delta_{i_r}^3-2\delta_{i_r}^1))(k-\sum_{r=1}^{k-1}\delta_{i_r}^3+1)}{(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1-1)(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1)}z_1, & \textit{if } i_k=2,\ i_{k+1}=2,\\ -\dfrac{c+\sum_{r=1}^{k-1}(\delta_{i_r}^3-\delta_{i_r}^1)-1}{(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1-1)(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1)}z_2, & \textit{if } i_k=2,\ i_{k+1}=3,\\ \dfrac{k-\sum_{r=1}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1-1)}, & \textit{if } i_k=3,\ i_{k+1}=1,\\ \dfrac{k-\sum_{r=1}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1-1)(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1)}z_2, & \textit{if } i_k=3,\ i_{k+1}=2, \end{cases}$$

and for $i(k) \in \mathcal{I}_k$, $k \geq 1$,

$$(3.33)\qquad Q_{i(k)} = 1 - \frac{k-\sum_{r=1}^{k-1}\delta_{i_r}^3}{2(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1-1)}\delta_{i_k}^3,$$

converges uniformly on every compact subset of (3.29) to the function $f(\mathbf{z})$ holomorphic in Θ.

($\mathbf{B}$) *The function $f(\mathbf{z})$ is an analytic continuation of $\mathrm{H}_6(1,c;\mathbf{z})$ in the domain (3.29).*

4. Numerical Experiments

From [24, Formula (37), p. 236], it follows that Horn's confluent function $\mathrm{H}_6(1,2;\mathbf{z})$ satisfies the system of two partial differential equations

$$(4.34)\qquad \begin{cases} z_1(1-4z_1)\dfrac{\partial^2 u}{\partial z_1^2}+z_2(1-4z_1)\dfrac{\partial^2 u}{\partial z_1\partial z_2}-z_2^2\dfrac{\partial^2 u}{\partial z_2^2}+(2-10z_1)\dfrac{\partial u}{\partial z_1}-4z_2\dfrac{\partial u}{\partial z_2}-2u=0,\\ z_1\dfrac{\partial^2 u}{\partial z_1\partial z_2}+z_2\dfrac{\partial^2 u}{\partial z_2^2}-2z_1\dfrac{\partial u}{\partial z_1}+(2-z_2)\dfrac{\partial u}{\partial z_2}-u=0, \end{cases}$$

where $u=u(\mathbf{z})$ is an unknown function of independent variables z_1 and z_2. If the conditions of Corollary 3.1 are satisfied, the branched continued fraction (3.30) satisfies (4.34).

Setting $c=2$, $\nu_1=\nu_2=\nu_3=1/20$, and $\mu_1=\mu_2=\mu_3=1/5$ it is easy to see that the conditions (3.28) are satisfied. Thus, by Corollary 3.1, the approximations of (3.30) with $c=2$ can be used to approximate the solution of (4.34) in the domain (3.29). From (3.31)–(3.32), we have such the approximations as

$$f_1(\mathbf{z})=1,\quad f_2(\mathbf{z})=\frac{3}{3-3z_1-2z_2},\quad \text{etc.}\quad .$$

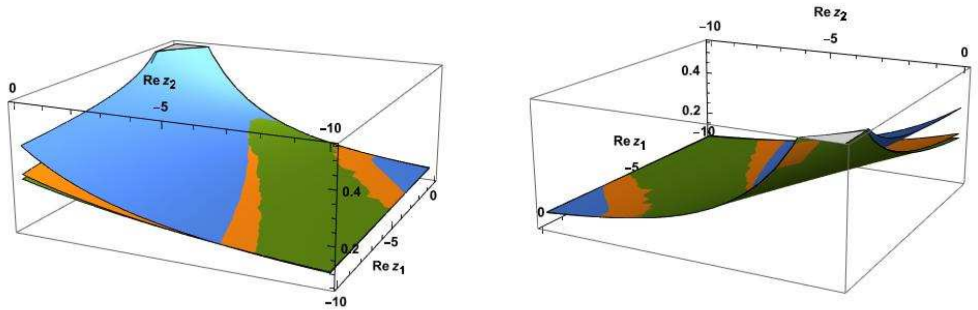
The values of these approximations $f_n(\mathbf{z})$ are given in Table 1 together with the values of the partial sums $S_n(\mathbf{z})$ of $\mathrm{H}_6(1,2,\mathbf{z})$ for $1\leq n\leq 10$ and for the various values of $\mathbf{z}$. This table shows the rate of convergence of $f_n(\mathbf{z})$ and $S_n(\mathbf{z})$ to $u(\mathbf{z})$ as n increases. We also see that the branched continued fraction gives better approximations of the solution of (4.34) than double confluent hypergeometric series.

TABLE 1. Approximation of the solution of (4.34) by branched continued fraction (3.30) with $c = 2$ and confluent hypergeometric series $\mathrm{H}_6(1, 2, \mathbf{z})$

n	$f_n(-0.2, -0.04)$	$S_n(-0.2, -0.04)$	$f_n(0.04, 0.04)$	$S_n(0.04, 0.04)$
1	1	0.78	1	1.06
2	0.8152173913043479	0.8682666666666666	1.0714285714285714	1.0650666666666666
3	0.8436283082662936	0.824104	1.066142202005891	1.0655813333333333
4	0.8390655552756958	0.8488421546666667	1.0656278844624396	1.0656393813333334
5	0.8397705605715909	0.8339969065244445	1.0656448968469723	1.0656463745422222
6	0.8396627464248548	0.8433291477625905	1.065647430637354	1.0656472558998347
7	0.8396790254380795	0.8372627668078485	1.0656473978749492	1.0656473706761305
8	0.8396765860675122	0.8413072281608152	1.0656473883827595	1.0656473859992222
9	0.8396769494594616	0.8385568865940254	1.0656473883916724	1.0656473880852033
10	0.839676895549416	0.8404571814141544	1.0656473884206237	1.0656473883736706

From [17, §3.4], it follows that

$$\begin{aligned}\mathrm{H}_6(1,2,\mathbf{z}) = \int_0^1 \Bigg(&\frac{(1-4tz_1)^{-1/2}}{B(1,1)}\,{}_1F_2\left(\frac{1}{2};\frac{1}{2},1;\frac{t(1-t)z_2^2}{1-4tz_1}\right) \\ &+\frac{2(t-t^2)^{1/2}z_2}{(1-4tz_1)B(1/2,3/2)}\,{}_1F_2\left(1;\frac{3}{2},\frac{3}{2};\frac{t(1-t)z_2^2}{1-4tz_1}\right)\Bigg)\,dt. \end{aligned} \tag{4.35}$$



(A) ■ – 5th, ■ – (4.35), ■ – 10th (B) ■ – 5th, ■ – (4.35), ■ – 10th

FIGURE 1. The plots of values of the nth approximants of (3.30)

In Figure 1 (A)–(B), we can see the plots of the values of 5th and 10th approximations of (3.30) approaches to the plot of the function (4.35). Figure 2 (A)–(D) shows the plots where the 10th approximants of (3.30) guarantees certain truncation error bounds for function (4.35). Finally, in Table 2, we can see that the 5th approximant of (3.30) is eventually a better approximation to (4.35) than the corresponding 5th partial sum of (2.2).

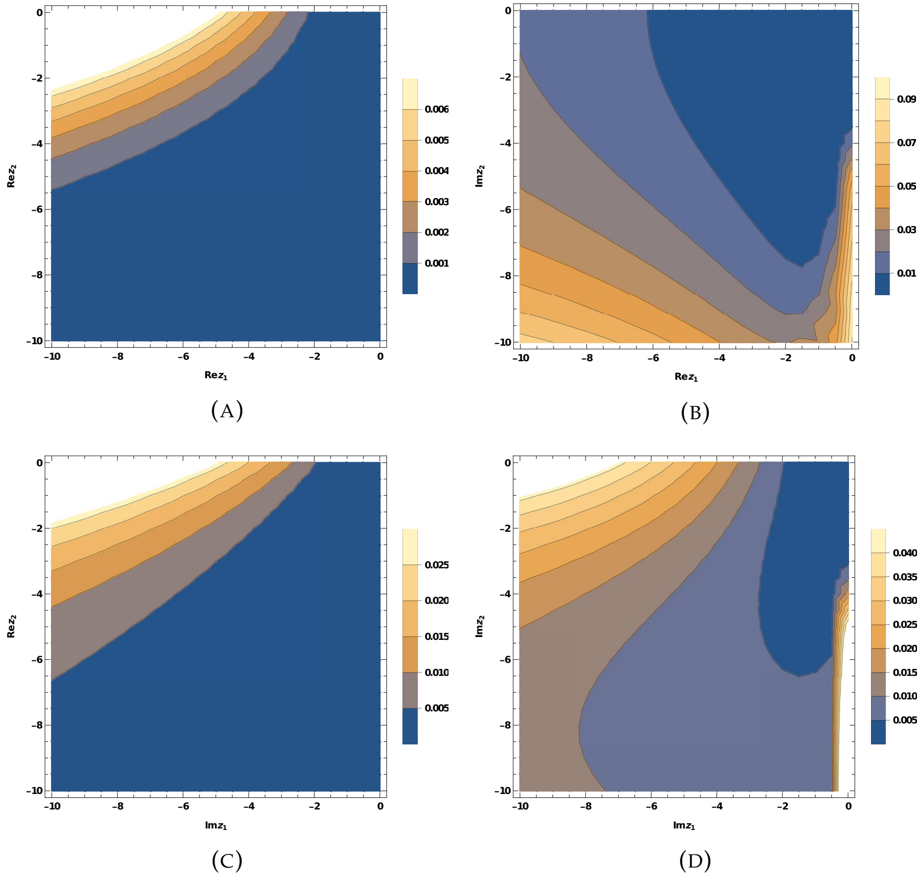


FIGURE 2. The plots where the 10th approximants of (3.30) guarantees certain truncation error bounds for (4.35)

TABLE 2. Relative errors of 5th partial sum and 5th approximant for the Horn's confluent function $H_6(1, 2, \mathbf{z})$

$\mathbf{z}$	**(4.35)**	**(2.2)**	**(3.30)**
$(-0.01, 0.01)$	0.9951138277	3.8606×10^{-08}	8.8026×10^{-09}
$(-0.1, 0.1)$	0.9593510752	6.2346×10^{-05}	9.4458×10^{-06}
$(-0.1, -0.01)$	0.9118965224	1.1498×10^{-04}	6.5181×10^{-06}
$(0.09, 0.05)$	1.1425549298	1.1470×10^{-04}	5.0158×10^{-06}
$(-0.15, -0.2)$	0.8094560924	2.3880×10^{-03}	2.0638×10^{-04}
$(0.2, 0.2)$	1.5918307333	2.6823×10^{-02}	2.7319×10^{-03}
$(0.2, -5.0)$	0.1998004145	$2.0382 \times 10^{+00}$	2.5676×10^{-03}
$(-5.0, 0.3)$	0.3782185176	$3.1579 \times 10^{+05}$	2.0912×10^{-01}
$(-10.0, -10.0)$	0.0932899388	$7.0858 \times 10^{+07}$	3.8248×10^{-02}
$(-25.0, -25.0)$	0.0395665845	$1.6635 \times 10^{+10}$	6.6127×10^{-01}

5. Conclusions

The paper considers the problem of representing the ratios of the confluent hypergeometric Horn's function H_6 by branched continued fractions. It is proved that the branched continued fractions converge to the ratios of the confluent hypergeometric series of which they are expansions, but the conditions of their convergence impose additional restrictions on the parameters of the function. The expediency and effectiveness of using branched continued fractions as an approximation tool are confirmed by numerical experiments. Nevertheless, the problems of improving and developing new methods of researching the convergence of such and similar branched continued fractions are open. Along the way, let us note the recent interesting and promising ideas regarding the study of the convergence of branched continued fractions proposed in papers [9, 10, 11].

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References

[1] T. Antonova, R. Dmytryshyn and V. Kravtsiv: *Branched continued fraction expansions of Horn's hypergeometric function H_3 ratios*, Mathematics, **9** (2) (2021), 148.

[2] T. Antonova, R. Dmytryshyn and R. Kurka: *Approximation for the ratios of the confluent hypergeometric function $\Phi_D^{(N)}$ by the branched continued fractions*, Axioms, **11** (9) (2022), 426.

[3] T. Antonova, R. Dmytryshyn and S. Sharyn: *Generalized hypergeometric function ${}_3F_2$ ratios and branched continued fraction expansions*, Axioms, **10** (4) (2021), 310.

[4] T. M. Antonova, N. P. Hoyenko: *Approximation of Lauricella's functions F_D ratio by Nörlund's branched continued fraction in the complex domain*, Mat. Metody Fiz. Mekh. Polya, **47** (2) (2004) 7–15. (In Ukrainian)

[5] T. M. Antonova: *On convergence of branched continued fraction expansions of Horn's hypergeometric function H_3 ratios*, Carpathian Math. Publ., **13** (3) (2021), 642–650.

[6] P. Appell: *Sur les séries hypergéométriques de deux variables et sur des équations différentielles linéaires aux dérivées partielles*, C. R. Acad. Sci. Paris, **90** (1880), 296–298.

[7] W. N. Bailey: *Generalised Hypergeometric Series*, Cambridge University Press, Cambridge (1935).

[8] P. I. Bodnarchuk, V. Y. Skorobogatko: *Branched Continued Fractions and Their Applications*, Naukova Dumka, Kyiv (1974). (In Ukrainian)

[9] D. I. Bodnar, I. B. Bilanyk, *Estimation of the rates of pointwise and uniform convergence of branched continued fractions with inequivalent variables*, J. Math. Sci., **265** (3) (2022), 423–437.

[10] D. I. Bodnar, I. B. Bilanyk: *On the convergence of branched continued fractions of a special form in angular domains*, J. Math. Sci., **246** (2) (2020), 188–200.

[11] D. I. Bodnar, I. B. Bilanyk: *Parabolic convergence regions of branched continued fractions of the special form*, Carpathian Math. Publ., **13** (3) (2021), 619–630.

[12] D. I. Bodnar: *Branched Continued Fractions*, Naukova Dumka, Kyiv (1986). (In Russian)

[13] D. I. Bodnar: *Expansion of a ratio of hypergeometric functions of two variables in branching continued fractions*, J. Math. Sci., **64** (32) (1993), 1155–1158.

[14] D. I. Bodnar, N. P. Hoyenko *Approximation of the ratio of Lauricella functions by a branched continued fraction*, Mat. Studii, **20** (2) (2003), 210–214.

[15] D. I. Bodnar, O. S. Manzii: *Expansion of the ratio of Appel hypergeometric functions F_3 into a branching continued fraction and its limit behavior*, J. Math. Sci., **107** (1) (2001), 3550–3554.

[16] D. I. Bodnar: *Multidimensional* C*-fractions*, J. Math. Sci., **90** (5) (1998), 2352–2359.

[17] Yu. A. Brychkov, N. V. Savischenko: *On some formulas for the Horn functions $H_3(a,b;c;w,z)$, $H_6^{(c)}(a;c;w,z)$ and Humbert function $\Phi_3(b;c;w,z)$*, Integral Transforms Spec. Funct., **32** (9) (2020), 661–676.

[18] R. I. Dmytryshyn, I.-A. V. Lutsiv: *Three- and four-term recurrence relations for Horn's hypergeometric function H_4*, Res. Math., **30** (1) (2022), 21–29.

[19] R. I. Dmytryshyn: *Multidimensional regular* C*-fraction with independent variables corresponding to formal multiple power series*, Proc. R. Soc. Edinb. Sect. A, **150** (5) (2020), 1853–1870.

[20] R. I. Dmytryshyn: *On the expansion of some functions in a two-dimensional* g*-fraction independent variables*, J. Math. Sci., **181** (3) (2012), 320–327.

[21] R. I. Dmytryshyn, S. V. Sharyn: *Approximation of functions of several variables by multidimensional* S*-fractions with independent variables*, Carpathian Math. Publ., **13** (3) (2021), 592–607.

[22] R. I. Dmytryshyn: *The multidimensional generalization of* g*-fractions and their application*, J. Comput. Appl. Math., **164–165** (2004), 265–284.

[23] R. I. Dmytryshyn: *Two-dimensional generalization of the Rutishauser qd-algorithm*, J. Math. Sci., **208** (3) (2015), 301–309.

[24] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi: *Higher Transcendental Functions*, Vol. 1, McGraw-Hill Book Co., New York (1953).

[25] H. Exton: *Multiple Hypergeometric Functions and Applications*, Halsted Press, Chichester (1976).

[26] V. R. Hladun, N. P. Hoyenko, O. S. Manzij and L. Ventyk: *On convergence of function $F_4(1,2;2,2;z_1,z_2)$ expansion into a branched continued fraction*, Math. Model. Comput., **9** (3) (2022), 767–778.

[27] J. Horn: *Hypergeometrische Funktionen zweier Veränderlichen*, Math. Ann., **105** (1931), 381–407.

[28] N. Hoyenko, T. Antonova and S. Rakintsev: *Approximation for ratios of Lauricella–Saran fuctions F_S with real parameters by a branched continued fractions*, Math. Bul. Shevchenko Sci. Soc., **8** (2011), 28–42. (In Ukrainian)

[29] N. Hoyenko, V. Hladun and O. Manzij: *On the infinite remains of the Nórlund branched continued fraction for Appell hypergeometric functions*, Carpathian Math. Publ., **6** (1) (2014), 11–25. (In Ukrainian)

[30] J. A. Murphy, M. R. O'Donohoe: *A two-variable generalization of the Stieltjes-type continued fraction*, J. Comput. Appl. Math., **4** (3) (1978), 181–190.

[31] M. Pétréolle, A. D. Sokal and B. X. Zhu: *Lattice paths and branched continued fractions: An infinite sequence of generalizations of the Stieltjes-Rogers and Thron-Rogers polynomials, with coefficientwise Hankel-total positivity*, arXiv, (2020), arXiv:1807.03271v2.

[32] W. Siemaszko: *Thile-type branched continued fractions for two-variable functions*, J. Comput. Appl. Math., **6** (2) (1983), 121–125.

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Research Article

Moving least squares approximation using variably scaled discontinuous weight function

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ABSTRACT. Functions with discontinuities appear in many applications such as image reconstruction, signal processing, optimal control problems, interface problems, engineering applications and so on. Accurate approximation and interpolation of these functions are therefore of great importance. In this paper, we design a moving least-squares approach for scattered data approximation that incorporates the discontinuities in the weight functions. The idea is to control the influence of the data sites on the approximant, not only with regards to their distance from the evaluation point, but also with respect to the discontinuities of the underlying function. We also provide an error estimate on a suitable *piecewise* Sobolev Space. The numerical experiments are in compliance with the convergence rate derived theoretically.

Keywords: MLS approximation, Meshfree methods, variably scaled discontinuous kernels, discontinuous function approximation.

2020 Mathematics Subject Classification: 65D05, 41A05.

1. INTRODUCTION

In practical applications, over a wide range of studies such as surface reconstruction, numerical solution of differential equations and kernel learning [6, 10, 15], one has to solve the problem of reconstructing an unknown function $f : \Omega \longrightarrow \mathbb{R}$ sampled at some finite set of data sites $X = \{\mathbf{x}_i\}_{1\leq i\leq N} \subset \Omega \subset \mathbb{R}^d$ with corresponding data values $f_i = f(\mathbf{x}_i),\ 1 \leq i \leq N$. Since in practice the function values f_i are sampled at scattered points, and not at a uniform grid, Meshless (or meshfree) Methods (MMs) are used as an alternative of numerical mesh-based approaches, such as Finite Elements Method (FEM) and Finite Differences (FD). The idea of MMs could be traced back to [18]. Afterwards, multivariate MMs existed under many names and were used in different contexts; interested readers are referred to [23] for an overview over MMs. In a general setting, MMs are designed, at least partly, to avoid the use of an underlying mesh or triangulation. The approximant of f at X can be expressed in the form

$$s_{f,X}(\mathbf{x}) = \sum_{i=1}^{N} \alpha_i(\mathbf{x}) f_i. \tag{1.1}$$

One might seek a function $s_{f,X}$ that interpolates the data, i.e. $s_{f,X}(\mathbf{x}_i) = f_i,\ 1 \leq i \leq N$, and in this case $\alpha_i(\mathbf{x})$ will be the *cardinal functions*. However, one might consider a more generalized framework known as *quasi-interpolation* in which $s_{f,X}$ only approximates the data, i.e.,

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$s_{f,X}(\mathbf{x}_i) \approx f_i$. The latter case means that we prefer to let the approximant $s_{f,X}$ only nearly fits the function values. This is useful, for instance, when the given data contain some noise, or the number of data is too large. The standard approach to deal with such a problem is to compute the Least-Squares (LS) solution, i.e., one minimizes the error (or cost) function

$$\sum_{i=1}^{N}[s_{f,X}(\mathbf{x}_i) - f_i]^2. \tag{1.2}$$

A more generalized setting of LS is known as the weighted LS , in which (1.2) turns to

$$\sum_{i=1}^{N}[s_{f,X}(\mathbf{x}_i) - f_i]^2 w(\mathbf{x}_i), \tag{1.3}$$

which is ruled by the *weighted* discrete ℓ_2 inner product. In practice, the role of $w(\mathbf{x}_i)$ is to add more flexibility to the LS formulation for data f_i that influence the approximation process, which are supposed, for example, to be affected by some noise. However, these methods are global in the sense that all data sites have influence on the solution at any evaluation point $\mathbf{x} \in \Omega$. Alternatively, for a fixed evaluation point $\mathbf{x}$, one can consider only n-th closest data sites $\mathbf{x}_i$, $i = 1, \ldots, n$ of $\mathbf{x}$ such that $n \ll N$. The *Moving Least-Squares* (MLS) method, which is a *local* variation of the classical weighted least-squares technique, has been developed following this idea. To be more precise, in the MLS scheme, for each evaluation point $\mathbf{x}$ one needs to solve a *weighted least-squares* problem, minimizing

$$\sum_{i=1}^{N}[s_{f,X}(\mathbf{x}_i) - f_i]^2 w(\mathbf{x}, \mathbf{x}_i) \tag{1.4}$$

by choosing the weight functions $w(\mathbf{x}, \mathbf{x}_i) : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ to be localized around $\mathbf{x}$, so that *few* data sites are taken into account. The key difference with respect to (1.3) is that the weight function is indeed *moving* with the evaluation point, meaning that it depends on both the $\mathbf{x}_i$ and $\mathbf{x}$. Consequently, for each evaluation point $\mathbf{x}$, a small linear system needs to be solved. Also, one can let $w(\cdot, \mathbf{x}_i)$ be a radial function i.e., $w(\mathbf{x}, \mathbf{x}_i) = \varphi(\|\mathbf{x} - \mathbf{x}_i\|_2)$ for some non-negative univariate function $\varphi : [0, \infty) \longrightarrow \mathbb{R}$. Doing in this way, $w(\cdot, \mathbf{x}_i)$ inherits the translation invariance property of radial basis functions. We mention that (1.4) could be generalized as well by letting $w_i(\cdot) = w(\cdot, \mathbf{x}_i)$ moves with respect to a *reference* point $\mathbf{y}$ such that $\mathbf{y} \neq \mathbf{x}$, (See e.g [13, Chap 22]).

The earliest idea of MLS approximation technique can be traced back to Shepard's seminal paper [25], in which the author considered the approximation by constants. Later on, the general framework of MLS was introduced by Lancaster and Salkauskas in [16], where they presented the analysis of MLS methods for smoothing and interpolation of scattered data. Afterwards, in [8] the author analyzed the connection between MLS and the Backus-Gilbert approach [4], and showed that the method is effective for derivatives approximations as well. Since then, MLS method showed its effectiveness in different applications [20, 21]. The error analysis of MLS approximation has been provided by some authors, mainly based on the work of Levin [17]. In [27, Chap. 3 & 4] and [26], the author suggested error bounds that take into account the so-called *fill-distance*, whose definition is recalled in Subsection 2.1. Other works focusing on the theoretical aspects of MLS method include [3], in which the authors provided error estimates in L_∞ for the function and its first derivatives in the one dimensional case, then [2], where they generalized this approach to the multi-dimensional case. In both these works, the error analysis is based on the *support* of the weight functions and not on the fill distance.

More recently, in [19] the author obtained an error estimate for MLS approximation of functions that belong to integer or fractional-order Sobolev spaces, which shows similarities to the bound previously studied in [22] for kernel-based interpolation.

The MLS method has rarely been used for approximating piecewise-continuous functions, i.e, functions that possess some discontinuities or jumps. In this case, it would be essential that the approximant takes into account the location of the discontinuities. To this end, in this paper, we let the weight function be a *Variably Scaled Discontinuous Kernel* (VSDK) [12]. VSDK interpolant have been employed to mitigate the Gibbs phenomenon, outperforming classical kernel-based interpolation in [11]. Similarly in MLS approximation framework, the usage of VSDK weights allows the construction of data-dependent approximants (as discussed in [17, §4]) that are able to overcome the performances of classical MLS approximants, as indicated by a careful theoretical analysis and then assessed by various numerical experiments.

The paper is organized as follows. In Section 2, we recall necessary notions of the MLS, VSDKs and Sobolev spaces. Section 3, presents the original contribution of this work, consisting in the use of variably scaled discontinuous weights for reconstructing discontinuous functions in the framework of MLS approximation. The error analysis shows that the MLS-VSDKs approximation can outperform classical MLS schemes as the discontinuities of the underlying function are assimilated into the weight function. In Section 4, we discuss some numerical experiments that support our theoretical findings, and in Section 5, we draw some conclusions.

2. Preliminaries on MLS and VSKs

2.1. **Moving Least Squares (MLS) approximation.** In this introduction to MLS, we resume and deepen what outlined in the previous section. The interested readers are also referred to [13, Chap. 22].

Let Ω be a non-empty and bounded domain in $\mathbb{R}^d$ and X be the set of N distinct data sites (or centers). We consider the target function f, and the corresponding function values f_i as defined above. Moreover, $\mathbb{P}_\ell^d$ indicates the space of d-variate polynomials of degree at most $\ell \in \mathbb{N}$, with basis $\{p_1, ..., p_Q\}$ and dimension $Q = \binom{\ell+d}{d}$.

Several equivalent formulations exist for the MLS approximation scheme. As the standard formulation, the MLS approximant looks for the best weighted approximation to f at the evaluation point $\mathbf{x}$ in $\mathbb{P}_\ell^d$ (or any other linear space of functions $\mathcal{U}$), with respect to the discrete ℓ_2 norm induced by the weighted inner product $\langle f, g\rangle_{w_\mathbf{x}} = \sum_{i=1}^N w(\mathbf{x}_i, \mathbf{x}) f(\mathbf{x}_i) g(\mathbf{x}_i)$. Mathematically speaking, the MLS approximant will be the linear combination of the polynomial basis i.e.,

$$s_{f,X}(\mathbf{x}) = \sum_{j=1}^{Q} c_j(\mathbf{x}) p_j(\mathbf{x}), \tag{2.5}$$

where the coefficients are obtained by locally minimizing the weighted least square error in (1.4), which is equivalent to minimizing $\|f - s_f\|_{w_\mathbf{x}}$. We highlight that the local nature of the approximant is evident from the fact that the coefficient $c_j(\mathbf{x})$ must be computed for each evaluation point $\mathbf{x}$.

In another formulation of MLS approximation known as the *Backus-Gilbert* approach, one considers the approximant $s_{f,X}(\mathbf{x})$ to be a *quasi interpolant* of the form (1.1). In this case, one seeks the values of the basis functions $\alpha_i(\mathbf{x})$ (also known as generating or shape functions) as the minimizers of

$$\frac{1}{2}\sum_{i=1}^{N} \alpha_i^2(\mathbf{x}) \frac{1}{w(\mathbf{x}_i, \mathbf{x})}$$

subject to the polynomial reproduction constraints

$$\sum_{i=1}^{N} p(\mathbf{x}_i)\alpha_i(\mathbf{x}) = p(\mathbf{x}) \quad \text{for all } p \in \mathbb{P}_\ell^d.$$

Such a constrained quadratic minimization problem can be converted to a system of linear equations by introducing Lagrange multipliers $\boldsymbol{\lambda}(\mathbf{x}) = [\lambda_1(\mathbf{x}), ..., \lambda_Q(\mathbf{x})]^T$. Consequently (e.g see [27, Corollary 4.4]), the MLS basis function α_i evaluated at $\mathbf{x}$ is given by

$$\alpha_i(\mathbf{x}) = w(\mathbf{x}, \mathbf{x}_i) \sum_{k=1}^{Q} \lambda_k(\mathbf{x}) p_k(\mathbf{x}_i), \quad 1 \leq i \leq N, \tag{2.6}$$

such that $\lambda_k(\mathbf{x})$ are the unique solution of

$$\sum_{k=1}^{Q} \lambda_k(\mathbf{x}) \sum_{i=1}^{N} w(\mathbf{x}, \mathbf{x}_i) p_k(\mathbf{x}_i) p_s(\mathbf{x}_i) = p_s(\mathbf{x}), \quad 1 \leq s \leq Q. \tag{2.7}$$

We observe that the weight function $w_i(\mathbf{x}) = w(\mathbf{x}, \mathbf{x}_i)$ controls the influence of the center $\mathbf{x}_i$ over the approximant, so it should be *small* when evaluated at a point that is far from $\mathbf{x}$, that is it should decay to zero fast enough. To this end, we may let $w_i(\mathbf{x})$ be positive on a ball centered at $\mathbf{x}$ with radius r, $B(\mathbf{x}, r)$, and zero outside. For example, a compactly supported radial kernel satisfies such a behaviour. Thus, let $I(\mathbf{x}) = \{i \in \{1, \ldots, N\}, \|\mathbf{x} - \mathbf{x}_i\|_2 \leq r\}$ be the family of indices of the centers X, for which $w_i(\mathbf{x}) > 0$, with $|I| = n \ll N$. Only the centers $\mathbf{x}_i \in I$ influence the approximant $s_{f,X}(\mathbf{x})$. Consequently, the matrix representation of (2.6) and (2.7) is

$$\begin{aligned} \boldsymbol{\alpha}(\mathbf{x}) &= W(\mathbf{x}) P^T \boldsymbol{\lambda}(\mathbf{x}), \\ \boldsymbol{\lambda}(\mathbf{x}) &= (P W(\mathbf{x}) P^T)^{-1} \mathbf{p}(\mathbf{x}), \end{aligned}$$

where $\boldsymbol{\alpha}(\mathbf{x}) = [\alpha_1(\mathbf{x}), ..., \alpha_n(\mathbf{x})]^T$, $W(\mathbf{x}) \in \mathbb{R}^{n \times n}$ is the diagonal matrix carrying the weights $w_i(\mathbf{x})$ on its diagonal, $P \in \mathbb{R}^{Q \times n}$ such that its k-th row contains p_k evaluated at data sites in $I(\mathbf{x})$, and $\mathbf{p}(\mathbf{x}) = [p_1(\mathbf{x}), ..., p_Q(\mathbf{x})]^T$. More explicitly, the basis functions are given by

$$\boldsymbol{\alpha}(\mathbf{x}) = W(\mathbf{x}) P^T (P W(\mathbf{x}) P^T)^{-1} \mathbf{p}(\mathbf{x}). \tag{2.8}$$

Moreover, it turns out that the solution of (2.5) is identical to the solution offered by the Backus-Gilbert approach (see e.g. [27, Chap. 3 & 4]).

In the MLS literature, it is known that a local polynomial basis shifted to the evaluation point $\mathbf{x} \in \Omega$ leads to a more stable method (see e.g. [27, Chap. 4]). Accordingly, we let the polynomial basis to be $\{1, (\cdot - \mathbf{x}), \ldots, (\cdot - \mathbf{x})^\ell\}$, meaning that different bases for each evaluation point are employed. In this case, since with standard monomials basis we have $p_1 \equiv 1$ and $p_k(0) = 0$ for $2 \leq k \leq Q$, then $\mathbf{p}(\mathbf{x}) = [1, 0, ..., 0]^T$.

To ensure the invertibility of $PW(\mathbf{x})P^T$ in (2.8), the centers in $I(\mathbf{x})$ needs to be $\mathbb{P}_\ell^d$-unisolvent. Then as long as $w_i(\mathbf{x})$ is positive, $PW(\mathbf{x})P^T$ will be a positive definite matrix, and so invertible; more details are available in [13, Chap. 22].

Furthermore, thanks to equation (2.6), it is observable that the behaviour of $\alpha_i(\mathbf{x})$ is heavily influenced by the behaviour of the weight functions $w_i(\mathbf{x})$, in particular it includes continuity and the support of the basis functions $\alpha_i(\mathbf{x})$. Another significant feature is that the weight functions $w_i(\mathbf{x})$ which are singular at the data sites lead to cardinal basis functions i.e., $\alpha_i(\mathbf{x}_j) = \delta_{i,j},\ i, j = 1, ..., n$, meaning that MLS scheme interpolates the data (for more details see [17, Theorem 3]).

We also recall the following definitions that we will use for the error analysis.

(1) A set $\Omega \subset \mathbb{R}^d$ is said to satisfy an **interior cone condition** if there exists an angle $\Theta \in (0, \pi/2)$ and a radius $r > 0$ so that for every $\mathbf{x} \in \Omega$ a unit vector $\xi(\mathbf{x})$ exists such that the cone

$$C(\mathbf{x}, \xi, \Theta, r) = \{\mathbf{x} + t\mathbf{y} : \mathbf{y} \in \mathbb{R}^d, \|\mathbf{y}\|_2 = 1, \cos(\Theta) \leq \mathbf{y}^T\xi, t \in [0, r]\}$$

is contained in Ω.

(2) A set $X = \{\mathbf{x}_1, ..., \mathbf{x}_N\}$ with $Q \leq N$ is called $\mathbb{P}^d_\ell$-unisolvent if the zero polynomial is the only polynomial from $\mathbb{P}^d_\ell$ that vanishes on X.

(3) The **fill distance** is defined as

$$h_{X,\Omega} = \sup_{\mathbf{x}\in\Omega} \min_{1\leq j\leq N} \|\mathbf{x} - \mathbf{x}_j\|_2.$$

(4) The **separation distance**

$$q_X = \frac{1}{2}\min_{i\neq j}\|\mathbf{x}_i - \mathbf{x}_j\|.$$

(5) The set of data sites X is said to be **quasi-uniform** with respect to a constant $c_{qu} > 0$ if

$$q_X \leq h_{X,\Omega} \leq c_{qu} q_X.$$

2.2. **Sobolev spaces and error estimates for MLS.** Assume $k \in \mathbb{N}_0$ and $p \in [1, \infty)$, then the *integer-order* Sobolev space $W^k_p(\Omega)$ consists of all u with distributional (weak) derivatives $D^{\boldsymbol{\delta}} u \in L^p$, $|\boldsymbol{\delta}| \leq k$. The semi-norm and the norm associated with these spaces are

$$|u|_{W^k_p(\Omega)} := \Big(\sum_{|\boldsymbol{\delta}|=k} \|D^{\boldsymbol{\delta}} u\|^p_{L^p(\Omega)}\Big)^{1/p}, \qquad \|u\|_{W^k_p(\Omega)} := \Big(\sum_{|\boldsymbol{\delta}|\leq k} \|D^{\boldsymbol{\delta}} u\|^p_{L^p(\Omega)}\Big)^{1/p}.$$

Moreover, letting $0 < s < 1$, the *fractional-order* Sobolev space $W^{k+s}_p(\Omega)$ is the space of the functions u for which semi-norm and norm are defined as

$$|u|_{W^{k+s}_p(\Omega)} := \Big(\sum_{|\boldsymbol{\delta}|=k} \int_\Omega \int_\Omega \frac{|D^{\boldsymbol{\delta}} u(\mathbf{x}) - D^{\boldsymbol{\delta}} u(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^{d+ps}}\Big)^{1/p}$$

$$\|u\|_{W^{k+s}_p(\Omega)} := \Big(\|u\|_{W^k_p(\Omega)} + |u|_{W^{k+s}_p(\Omega)}\Big)^{1/p}.$$

Consider certain Sobolev spaces $W^k_p(\Omega)$ with the condition that $1 < p < \infty$ and $k > m + d/p$ (for $p = 1$ the equality is also possible), then according to [22, Theorem 2.12] the sampling inequality

$$\|u\|_{W^m_p(\Omega)} \leq C h^{k-m-d(1/p-1/p)_+}_{X,\Omega} \|u\|_{W^k_p}$$

holds for a function u that satisfies $u(X) = 0$, with $h_{X,\Omega}$ being the *fill distance* associated with X and $(\mathbf{y})_+ = \max\{0, \mathbf{y}\}$. For more information regarding Sobolev Spaces and sampling inequalities, we refer the reader to [1, 7] and [24], respectively.

Getting back to the MLS scheme, let $D^{\boldsymbol{\delta}}$ be a derivative operator such that $|\boldsymbol{\delta}| \leq \ell$ (we recall that ℓ is the maximum degree of the polynomials). Assuming $w \in C^\ell(\Omega)$, [19, Theorem 3.11] shows that $\{D^{\boldsymbol{\delta}}\alpha_i(\mathbf{x})\}_{1\leq i\leq n}$ forms a *local polynomial reproduction* in a sense that there exist constants h_0, $C_{1,\boldsymbol{\delta}}$, C_2 such that for every evaluation point $\mathbf{x}$

- $\sum_{i=1}^N D^{\boldsymbol{\delta}}\alpha_i(\mathbf{x})p(\mathbf{x}_i) = p(\mathbf{x})$ for all $p \in \mathbb{P}^d_\ell$
- $\sum_{i=1}^N |D^{\boldsymbol{\delta}}\alpha_i(\mathbf{x})| \leq C_{1,\boldsymbol{\delta}} h^{-|\boldsymbol{\delta}|}_{X,\Omega}$
- $D^{\boldsymbol{\delta}}\alpha_i(\mathbf{x}) = 0$ provided that $\|\mathbf{x} - \mathbf{x}_i\|_2 \geqslant C_2 h_{X,\Omega}$

for all X with $h_{X,\Omega} \leq h_0$.

The particular case of $|\boldsymbol{\delta}| = 0$ was previously discussed in [27, Theorem 4.7] in which it is shown that $\{\alpha_i(\mathbf{x})\}_{1\leq i\leq n}$ forms a local polynomial reproduction. However, in this case the basis functions $\{\alpha_i(\cdot)\}_{1\leq i\leq n}$ could be even discontinuous but it is necessary that $w_i(\mathbf{x})$ are bounded (for more details see [27, Chap 3,4]). Consequently, we restate the the MLS error bound in Sobolev Spaces developed in [19].

Theorem 2.1. [19, Theorem 3.12] *Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded set with a Lipschitz boundary. Let ℓ be a positive integer, $0 \leq s < 1$, $p \in [1, \infty)$, $q \in [1, \infty]$ and let $\boldsymbol{\delta}$ be a multi-index satisfying $\ell > |\boldsymbol{\delta}| + d/p$ for $p > 1$ and $\ell \geqslant |\boldsymbol{\delta}| + d$ for $p = 1$. If $f \in W_p^{\ell+s}(\Omega)$ and $w \in C^\ell(\Omega)$, there exist constants $C > 0$ and $h_0 > 0$ such that for all $X = \{\mathbf{x}_1, ..., \mathbf{x}_N\} \subset \Omega$ which are quasi-uniform with $h_{X,\Omega} \leq \min\{h_0, 1\}$, the error estimate holds*

$$\|f - s_{f,X}\|_{W_q^{|\boldsymbol{\delta}|}(\Omega)} \leq C h_{X,\Omega}^{\ell+s-|\boldsymbol{\delta}|-d(1/p-1/q)_+} \|f\|_{W_p^{\ell+s}(\Omega)}. \tag{2.9}$$

The employed polynomial basis are shifted to the evaluation point $\mathbf{x}$ and scaled with respect to the fill distance $h_{X,\Omega}$, and $w_i(\cdot)$ is positive on $[0, 1/2]$, supported in $[0, 1]$ such that its even extension is non-negative and continuous on $\mathbb{R}$.

Remark 2.1. *The above error bounds holds also when $s = 1$. However, recalling the definition of (semi-)norms in fractional-order Sobolev space, we see that in this case we reach to an integer-order Sobolev space of $\ell + 1$. Therefore, it requires that $\ell + 1 > |\boldsymbol{\delta}| + d/p$ for $p > 1$ or $\ell + 1 \geqslant |\boldsymbol{\delta}|$ for $p = 1$ in order that (2.9) holds true. The key point is that in this case, the polynomial space is still $\mathbb{P}_\ell^d$ and not $\mathbb{P}_{\ell+1}^d$.*

2.3. **Variably Scaled Discontinuous Kernels (VSDKs).** Variably Scaled Kernels (VSKs) were firstly introduced in [9]. The basic idea behind them is to map the data sites from $\mathbb{R}^d$ to $\mathbb{R}^{d+1}$ via a scaling function $\psi : \Omega \longrightarrow \mathbb{R}$ and to construct an augmented approximation space in which the data sites are $\{(\mathbf{x}_i, \psi(\mathbf{x}_i))\ i = 1, ..., N\}$ (see [9, Def. 2.1]). Though the first goal of doing so was getting a *better* nodes distribution in the augmented dimension, later on in [12] the authors came up with the idea of also encoding the behavior of the underlying function f inside the scale function ψ. Precisely, for the target function f that possesses some jumps, the key idea is the following.

Definition 2.1. *Let $\mathcal{P} = \{\Omega_1, ..., \Omega_n\}$ be a partition of Ω and let $\boldsymbol{\beta} = (\beta_1, ..., \beta_n)$ be a vector of real distinct values. Moreover, assume that all the jump discontinuities of the underlying function f lie on $\bigcup_{j=1}^n \partial\Omega_j$. The piecewise constant scaling function $\psi_{\mathcal{P},\boldsymbol{\beta}}$ with respect to the partition $\mathcal{P}$ and the vector $\boldsymbol{\beta}$ is defined as*

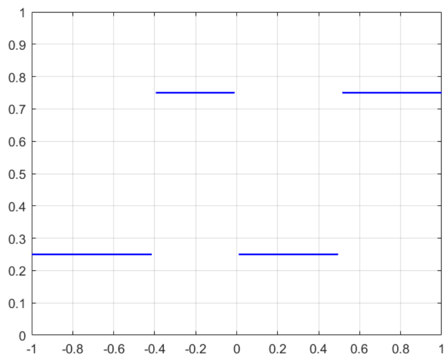
$$\psi_{\mathcal{P},\boldsymbol{\beta}}(\mathbf{x})|_{\Omega_j} = \beta_j, \ \mathbf{x} \in \Omega.$$

Successively, let Φ^ε be a positive definite radial kernel on $\Omega \times \Omega$ that depends on the shape parameter $\varepsilon > 0$. A variably scaled discontinuous kernel on $(\Omega \times \mathbb{R}) \times (\Omega \times \mathbb{R})$ is defined as

$$\Phi_\psi^\varepsilon(\mathbf{x}, \mathbf{y}) = \Phi^\varepsilon\big(\Psi(\mathbf{x}), \Psi(\mathbf{y})\big), \quad \mathbf{x}, \mathbf{y} \in \Omega \tag{2.10}$$

such that $\Psi(\mathbf{x}) = (\mathbf{x}, \psi(\mathbf{x}))$.

Moreover, we point out that if Φ^ε is (strictly) positive definite then so is Φ_ψ^ε, and if Φ^ε and ψ are continuous then so is Φ_ψ^ε [9, Theorem 2.2]. Figure 1 shows two different choices for the discontinuous scale function for the univariate case. In any case, it matters that the discontinuities of the target function f are assimilated into the kernel Φ_Ψ^ε.



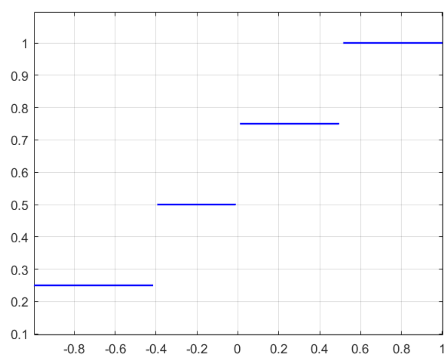


FIGURE 1. Discontinuous scale functions

3. MLS-VSDKs

Let f be a function with some jump discontinuities defined on Ω, $\mathcal{P}$ and $\psi_{\mathcal{P},\beta}$ as in Definition 2.1. We look for the MLS approximant with *variably scaled discontinuous weight function* such that

$$w_{\psi}(\mathbf{x}, \mathbf{x}_i) = w(\Psi(\mathbf{x}), \Psi(\mathbf{x}_i)). \tag{3.11}$$

We assume that the node points that remain in the support of the weight functions after the scaling retain the unisolvency with respect to $\mathbb{P}^d_{\ell}$. In this case $PW(\mathbf{x})P^T$ is positive definite, meaning that (2.8) is solvable and so, the basis functions $\alpha(\mathbf{x})$ uniquely exist. However, with new weight functions, from (3.11) also $\alpha(\mathbf{x})$ might be continuous or discontinuous regarding to the given data values f_i. Therefore, our basis functions are indeed data-dependent thanks to (3.11). From now on, we call this scheme MLS-VSDK, and we will denote the corresponding approximant as $s^{\psi}_{f,X}$.

Since the basis functions are data dependent, one might expect that the space in which we express the error bound should be data dependent as well. Towards this idea, for $k \in \mathbb{Z},\ k \geq 0$, and $1 \leq p \leq \infty$, we define the *piecewise* Sobolev Spaces

$$\mathcal{W}^k_p(\Omega) = \{f : \Omega \longrightarrow \mathbb{R} \text{ s.t. } f_{|\Omega_j} \in W^k_p(\Omega_j), \quad j \in \{1, ..., n\}\},$$

where $f_{|\Omega_j}$ denotes the restriction of f to Ω_j, and $W^k_p(\Omega_j)$ denote the Sobolev space on Ω_i. We endow $\mathcal{W}^k_p(\Omega)$ with the norm

$$\|f\|_{\mathcal{W}^k_p(\Omega)} = \sum_{j=1}^{n} \|f\|_{W^k_p(\Omega_j)}.$$

When $k = 0$ we simply denote $\mathcal{W}^0_p(\Omega)$ by $\mathcal{L}^p(\Omega)$, which is the space that contains functions that are piecewise L^p on Ω. Moreover, it could be shown that for any partition of Ω the standard Sobolev space $W^k_p(\Omega)$ is contained in $\mathcal{W}^k_p(\Omega)$ (see [11] and reference therein). We assume that every set $\Omega_j \in \mathcal{P}$ satisfies Lipschitz boundary conditions which will be essential for our error analysis.

Lemma 3.1. *Let $\mathcal{P}$ be as in Definition 2.1 and set the derivative order $\boldsymbol{\delta} = 0$. Then, by assuming $\ell > d/p$ (equality also holds for $p = 1$) and using Theorem 2.1, the error satisfies the inequality*

$$\|f - s^{\psi}_{f,X}\|_{L^2(\Omega_j)} \leq C_j h^{\ell+1-d(1/p-1/2)_+}_{X,\Omega_j} \|f\|_{W^{\ell+1}_p(\Omega_j)} \qquad \textit{for all } \ \Omega_j \in \mathcal{P} \tag{3.12}$$

with $h_{X,\Omega_j} < \min\{h_0, 1\}$ *the fill distance with respect to* Ω_j.

Proof. Recalling Definition 2.1, we know that the discontinuities of f and subsequently $w_i(\cdot)$ are located only at the boundary and not on the domain Ω_j, meaning that $w_i(\cdot)$ is C^ℓ Ω_j. Furthermore, the basis $\{\alpha_i(\mathbf{x})\}_{1\leq i\leq n}$ forms a local polynomial reproduction i.e., there exists a constant C such that $\sum_{i=1}^{N}|\alpha_i|\leq C$. Letting $s=1$ and $q=2$, by noticing that $W_q^0(\Omega_j)=L^q(\Omega_j)$, then the error bound (3.12) is an immediate consequence of Theorem 2.1 and the Remark 2.1. □

From the above proposition, it could be understood that $s_{f,X}^{\psi}$ behaves similarly to $s_{f,X}$ in the domain Ω_j, where there is no discontinuity. This is in agreement with Definition 2.1. Consequently, it is required to extend the error bound (3.12) to the whole domain Ω.

Theorem 3.2. *Let* f, $\mathcal{P}$, $\psi_{\mathcal{P},\beta}$ *be as before, and the weight functions as in* (3.11)*. Then, for* $f \in \mathcal{W}_p^{\ell+1}(\Omega)$*, as long as* $\ell > |\boldsymbol{\delta}|+d/p$ *(equality also holds for* $p=1$*), the MLS-VSDK approximant* $s_{f,X}^{\psi}$ *error can be bounded as follows:*

$$\|f-s_{f,X}^{\psi}\|_{\mathcal{L}^2(\Omega)}\leq Ch^{\ell+1-d(1/p-1/2)_+}\|f\|_{\mathcal{W}_p^{\ell+1}(\Omega)}, \tag{3.13}$$

where $h=\max\{h_{X,\Omega_1},\dots,h_{X,\Omega_n}\}$ *such that* h_{X,Ω_j} *is the fill distance associated to the subdomain* Ω_j.

Proof. By Lemma 3.1, we know that (3.12) holds for each Ω_j. Let h_{X,Ω_j} and C_j be the fill distance and a constant associated with each Ω_j, respectively. Then, we have

$$\sum_{j=1}^{n}\|f-s_{f,X}^{\psi}\|_{L^2(\Omega_j)}\leq\sum_{j=1}^{n}C_j h_{X,\Omega_j}^{\ell+1-d(1/p-1/2)_+}\|f\|_{W_p^{\ell+1}(\Omega_j)}.$$

By definition, we get $\sum_{j=1}^{n}\|f-s_{f,X}^{\psi}\|_{L^2(\Omega_j)}=\|f-s_{f,X}^{\psi}\|_{\mathcal{L}^2(\Omega)}$. Moreover, letting $C=\max\{C_1,\dots,C_n\}$ and $h=\max\{h_{X,\Omega_1},\dots,h_{X,\Omega_n}\}$, then the right hand side can be bounded by

$$Ch^{\ell+1-d(1/p-1/2)_+}\|f\|_{\mathcal{W}_p^{\ell+1}(\Omega)}.$$

Putting these together we conclude. □

Some remarks are in order.

(1) One might notice that the error bound in (2.9) is indeed local (the basis functions are local by assumption), meaning that if f is less smooth in a subregion of Ω, say it possesses only $\ell' \leq \ell$ continuous derivatives there, then the approximant (interpolant) has order $\ell'+1$ in that region and this is the best we can get. On the other hand according to (3.13), thanks to the definition of piecewise Sobolev space, the regularity of the underlying function in the interior of the subdomain Ω_j matters. In other words, as long as f possesses regularity of order ℓ in subregions, say Ω_j and Ω_{j+1}, the approximant order of $\ell+1$ is achievable, regardless of the discontinuities on the boundary of Ω_j and Ω_{j+1}.

(2) Another interesting property of the MLS-VSDK scheme is that it is indeed data dependent. To clarify, for the evaluation point $\mathbf{x}\in\Omega_j$ take two data sites $\mathbf{x}_i,\ \mathbf{x}_{i+1}\in B(\mathbf{x},r)$ with the same distance from $\mathbf{x}$ such that $\mathbf{x}_i\in\Omega_j$ and $\mathbf{x}_{i+1}\in\Omega_{j+1}$. Due to the Definition (2.10), $w_\psi(\mathbf{x},\mathbf{x}_{i+1})$ decays to zero faster than $w_\psi(\mathbf{x},\mathbf{x}_i)$ i.e., the data sites from the same subregion Ω_j pay more contribution to the approximant (interpolant) $s_{f,X}^{\psi}$, rather than the one from another subregion Ω_{j+1} beyond a discontinuity line. On the other hand in the classical MLS scheme, this does not happen as the weight function gives the same value to both $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$.

(3) We highlight that in MLS-VSDK scheme we do not scale polynomials and so the polynomial space $\mathbb{P}_\ell^d$ is not changed. We scale only the weight functions and thus, in case the given function values bear discontinuities, the basis functions $\{\alpha_i(\cdot)\}_{1\leq i\leq n}$ are modified.

We end this section by recalling that the MLS approximation convergence order is achievable only in the *stationary setting*, i.e., the shape parameter ε must be scaled with respect to the fill distance. It leads to *peaked basis functions* for densely spaced data and *flat basis function* for coarsely spaced data. In other words, the local support of the weight functions $B(\mathbf{x}, r)$, and subsequently basis functions must be tuned with regards to the $h_{X,\Omega}$ using the shape parameter ε. Consequently, this holds also in MLS-VSDK scheme, meaning that after scaling w_i we still need to take care of ε. This is different with respect to VS(D)Ks interpolation, where $\varepsilon = 1$ was kept fixed [9, 12].

4. Numerical experiments

In this section, we compare the performance of the MLS-VSDK with respect to the classical MLS method. In all numerical, tests we fix the polynomials space up to degree 1. Considering the evaluation points as $Z = \{z_1, ..., z_s\}$, we compute root mean square error and maximum error by

$$RMSE = \sqrt{\frac{1}{s}\sum_{i=1}^{s}(f(z_i) - s_{f,X}(z_i))^2}, \quad MAE = \max_{z_i \in Z}|f(z_i) - s_{f,X}(z_i)|.$$

We consider four different weight functions to verify the convergence order of $s_{f,x}^{\psi}$ to a given f, as presented in Theorem 3.2.

(1) $w^1(\mathbf{x}, \mathbf{x}_i) = (1 - \varepsilon\|\mathbf{x} - \mathbf{x}_i\|)_+^4 \cdot (4\varepsilon\|\mathbf{x} - \mathbf{x}_i\| + 1)$, which is the well-known C^2 *Wendland* function. Since each w_i^1 is locally supported on the open ball $B(0, 1)$, then it verifies the conditions required by Theorem 3.2.

(2) $w^2(\mathbf{x}, \mathbf{x}_i) = \exp(-\varepsilon\|\mathbf{x} - \mathbf{x}_i\|^2)$, i.e. the Gaussian RBF. We underline that when Gaussian weight functions are employed, with decreasing separation distance of the approximation centers, the calculation of the basis functions in (2.8) can be badly conditioned. Therefore, in order to make the computations stable, in this case we regularize the system by adding a small multiple, say $\lambda = 10^{-8}$, of the identity to the diagonal matrix W.

(3) $w^3(\mathbf{x}, \mathbf{x}_i) = \exp(-\varepsilon\|\mathbf{x} - \mathbf{x}_i\|)(15 + 15\|\mathbf{x} - \mathbf{x}_i\| + 6\|\mathbf{x} - \mathbf{x}_i\|^2 + \|\mathbf{x} - \mathbf{x}_i\|^3)$, that is a C^6 *Matérn* function.

(4) $w^4(\mathbf{x}, \mathbf{x}_i) = (\exp(\varepsilon\|\mathbf{x} - \mathbf{x}_i\|)^2 - 1)^{-1}$, suggested in [17], which enjoys an additional feature which leads to interpolatory MLS, since it possesses singularities at the centers.

One might notice that w^2, w^3 and w^4 are not locally supported. However, the key point is that they are all decreasing with the distance from the centers and so, in practice, one can overlook the data sites that are so far from the center $\mathbf{x}$. As a result, one generally considers a *local stencil* containing n nearest data sites of the set Z of evaluation points. While there is no clear theoretical background concerning the stencil size, in MLS literature, one generally lets $n = 2 \times Q$ (see e.g [5]). However, it might be possible that in some special cases one could reach a better accuracy using different stencil sizes. This aspect is covered by our numerical tests, which are outlined in the following.

(1) In Section 4.1, we present an example in the one-dimensional framework, where the stencil size is fixed to be $n = 2 \times Q$. Moreover, we consider w^1, w^2 and w^3.

(2) In Section 4.2, we move to the two-dimensional framework and we keep the same stencil size. Here, we restrict the test to the weight function w^1 and verify Theorem 3.2.
(3) In Section 4.3, we remain in the two-dimensional setting but the best accuracy is achieved with $n = 20$. Moreover, in addition to w^2 and w^3, we test the interpolatory case by considering w^4 as weight function.
(4) In Section 4.4, we present a two-dimensional, experiments where the data sites have been perturbed via some white noise. We fix $n = 25$ and w^2, w^3 are involved.

4.1. **Example 1.** On $\Omega = (-1, 1)$, we assess MLS approximant for

$$f_1(x) = \begin{cases} e^{-x}, & -1 < x < -0.5 \\ x^3, & -0.5 \le x < 0.5 \\ 1, & 0.5 \le x < 1 \end{cases}$$

with discontinuous scale function

$$\psi(x) = \begin{cases} 1, & x \in (-1, 0.5) \text{ and } [0.5, 1) \\ 2, & x \in [-0.5, 0.5)\,. \end{cases}$$

We note that the function ψ is defined only by two cases. The important fact is that has a jump as f_1.

To evaluate the approximant consider the evaluation grid of equispaced points with step size $5.0e-4$. Tables 1 and 2 include RMSE of f_1 approximation using w^1 as the weight function. Again, in order to investigate the convergence rate, consider two sets of uniform

number of centers	ε value	RMSE MLS-VSDK	RMSE classic MLS
9	0.25	3.58e-1	3.95e-1
17	0.5	1.99e-1	3.02e-1
33	1	3.10e-3	2.17e-1
65	2	8.42e-4	1.54e-1
257	4	5.67e-5	7.68e-2
513	8	1.43e-5	5.35e-2

TABLE 1. Comparison of the RMSE for f_1 approximation at *uniform* data sites

number of centers	ε value	RMSE MLS-VSDK	RMSE classic MLS
9	0.25	3.53e-1	3.77e-1
17	0.5	1.99e-1	3.01e-1
33	1	3.08e-3	2.17e-1
65	2	8.39e-4	1.54e-1
257	4	5.67e-5	7.73e-2
513	8	1.43e-5	5.41e-2

TABLE 2. Comparison of the RMSE for f_1 approximation at *Halton* data sites

and Halton nodes with the size from Table 1. In order to generalize our results to globally supported weight functions, we take into account w^2 and w^3, Gaussian and Matérn C^6 radial functions, respectively. For the uniform data sites let the shape parameter values to be

$\varepsilon^U_{GA} = [5, 20, 40, 80, 160, 320]$ and $\varepsilon^U_{Mat} = [5, 10, 20, 40, 80, 160]$ for w^2 and w^3. Our computation shows convergence rates of 2.54 and 2.26 for MLS-VSDK scheme, shown in Figure 2. Accordingly, for *Halton* points let $\varepsilon^H_{Mat} = [5, 10, 20, 50, 200, 400]$, $\varepsilon^H_{GA} = [10, 20, 30, 50, 100, 200]$. The corresponding convergence rates are 2.38 and 2.33. On the other hand, using non-scaled

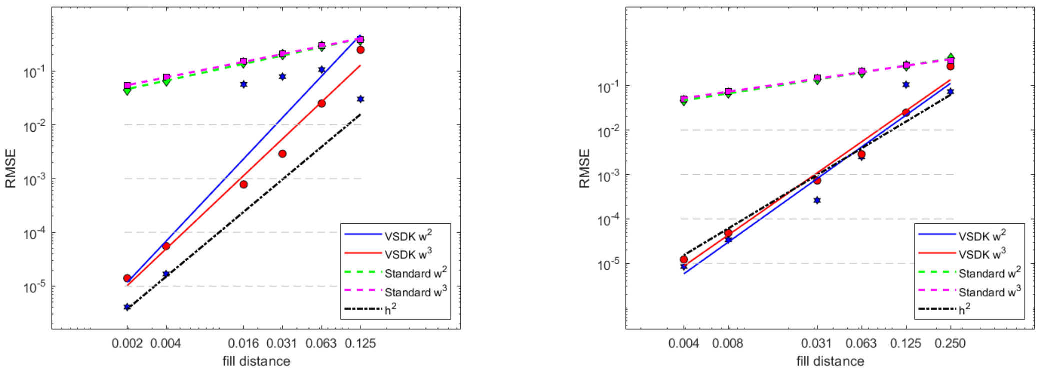


FIGURE 2. Convergence rates for approximating f_1 with MLS-VSDK and MLS-Standard schemes using *uniform* data sites (left) and *Halton* data sites (right)

weight functions, the standard MLS scheme can hardly reach an approximation order of 1, in both cases.

4.2. **Example 2.** Consider on $\Omega = (-1, 1)^2$ the discontinuous function

$$f_2(x, y) = \begin{cases} \exp(-(x^2 + y^2)), & x^2 + y^2 \le 0.6 \\ x + y, & x^2 + y^2 > 0.6 \end{cases}$$

and the discontinuous scale function

$$\psi(x, y) = \begin{cases} 1, & x^2 + y^2 \le 0.6 \\ 2, & x^2 + y^2 > 0.6 \end{cases}.$$

As evaluation points, we take the grid of equispaced points with mesh size $1.00e - 2$. Figure 3 shows both the *RMSE* and *absolute error* for the classical *MLS* and *MLS-VSDK* approximation of f_2 sampled from $1089 = 33^2$ uniform data sites taking w^1 as the weight function. Figure 3 shows that using classical MLS, the approximation error significantly increases near the discontinuities, while using MLS-VSDK the approximant can overcome this issue. In order to investigate the convergence rate, we consider increasing sets of $\{25, 81, 289, 1089, 4225, 16641\}$ Halton and uniform points as the data sites. To find an appropriate value for the shape parameter, we fix an initial value and we multiply it by a factor of 2 at each step. Thus, let $\varepsilon = [0.25, 0.5, 1, 2, 4, 8]$ be the vector of shape parameter which is modified with respect to the number of the centers in both cases of uniform and Halton data sites. The left plot of Figure 4 shows a convergence rate of 2.58 for the MLS-VSDK and only 0.66 for classical MLS methods, while these values are 2.04 and 0.70 in the right plot.

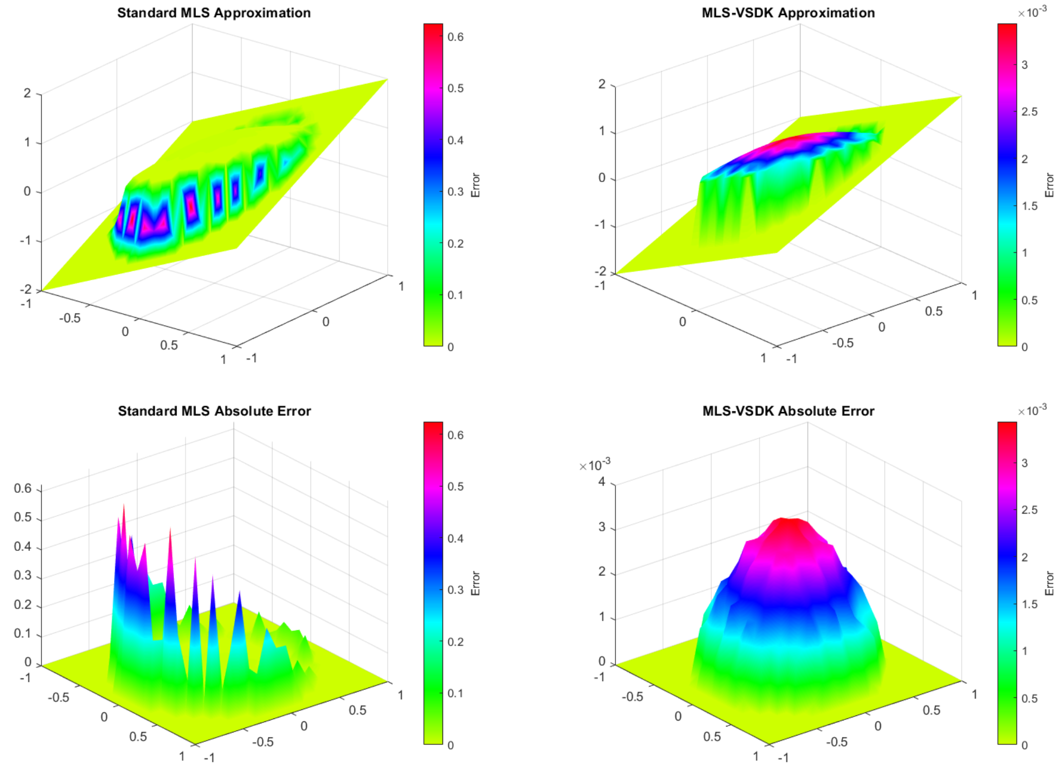


FIGURE 3. RMSE and abs-error of f_2 MLS (left) and MLS-VSDK (right) aproximation schemes using w^1 weight function

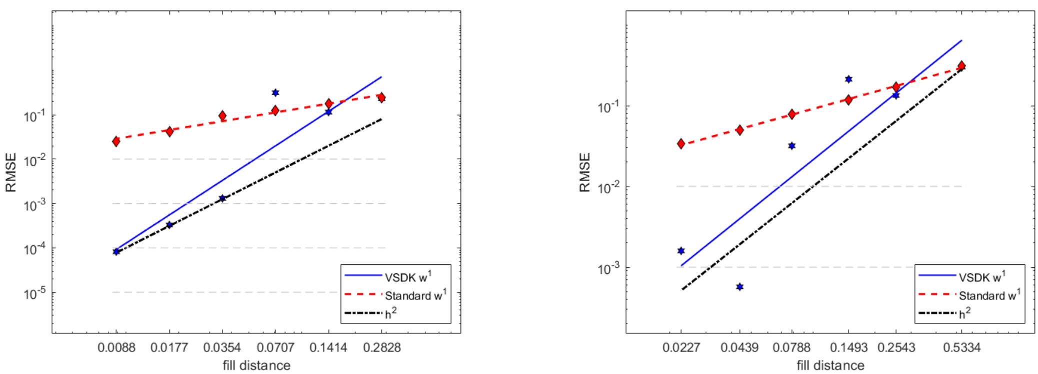


FIGURE 4. Convergence rates for approximation of function f_2 with MLS-VSDK and MLS standard schemes using *Uniform* data sites (left) and *Halton* data sites (right)

4.3. **Example 3.** Consider the following function

$$f_3(x,y) = \begin{cases} 2\big(1-\exp(-(y+0.5)^2)\big), & |x|\leq 0.5,\ |y|\leq 0.5 \\ 4(x+0.8), & -0.8\leq x\leq -0.65, |y|\leq 0.8 \\ 0.5, & 0.65\leq x\leq 0.8, |y|\leq 0.2 \\ 0, & \text{otherwise} \end{cases}$$

defined on $\Omega = (-1,1)^2$. Regarding the discontinuities of f_3, the scale function is considered to be

$$\psi(x,y) = \begin{cases} 1, & |x|\leq 0.5,\ |y|\leq 0.5 \\ 2, & -0.8\leq x\leq -0.65, |y|\leq 0.8 \\ 3, & 0.65\leq x\leq 0.8, |y|\leq 0.2 \\ 0, & \text{otherwise} \end{cases}$$

Moreover, let the centers and evaluation points be the same as the Example 4.1. Table 3 and 4 shows RMSE of MLS-VSDK and conventional MLS approximation of f_3 using w^4 which interpolates the data. We underline that our experiments show that the stencil of size $n = 20$ leads to the best accuracy. Figure 5 shows **RMSE** and **Absolute Error** for *standard MLS*

number of centers	ε value	RMSE MLS-VSDK	RMSE classic MLS
25	1	3.67e-1	1.47e+0
81	2	3.68e-1	8.86e-1
289	4	1.49e-2	7.44e-1
1089	8	4.23e-3	7.72e-1
4225	16	1.06e-3	6.64e-1
16641	32	2.65e-4	5.25e-1

TABLE 3. RMSE of f_3 interpolation with *uniform* data sites

number of centers	ε value	RMSE MLS-VSDK	RMSE classic MLS
25	1	8.84e-1	1.53e+0
81	2	8.95e-2	1.05e+0
289	4	1.42e-2	8.74e-1
1089	8	4.18e-3	6.48e-1
4225	16	1.09e-3	6.68e-1
16641	32	3.02e-4	7.07e-1

TABLE 4. RMSE of f_3 interpolation with *Halton* data sites

and *MLS-VSDK approximation* of f_3 sampled from 1089 uniform points using w_4 as weight function. Once again, Figure 5 shows how MLS-VSDK scheme can improve the accuracy by reducing the error near the jumps. Eventually, letting $\varepsilon^U_{GA} = [2,4,8,16,32,64]$ and $\varepsilon^U_{Mat} = [10,20,40,80,160,320]$, Figure 6 shows that h^2 convergence is achievable. To be more precise, the rate of convergence in the left plot is 2.54 and 2.69 for w_2 and w_3, respectively. On the other hand, letting $\varepsilon^H_{GA} = [1,2,4,8,16,32]$ and ε^H_{Mat} as the uniform case, convergence rates of 2.50 and 2.73 is achievable when *Halton* data sites are employed.

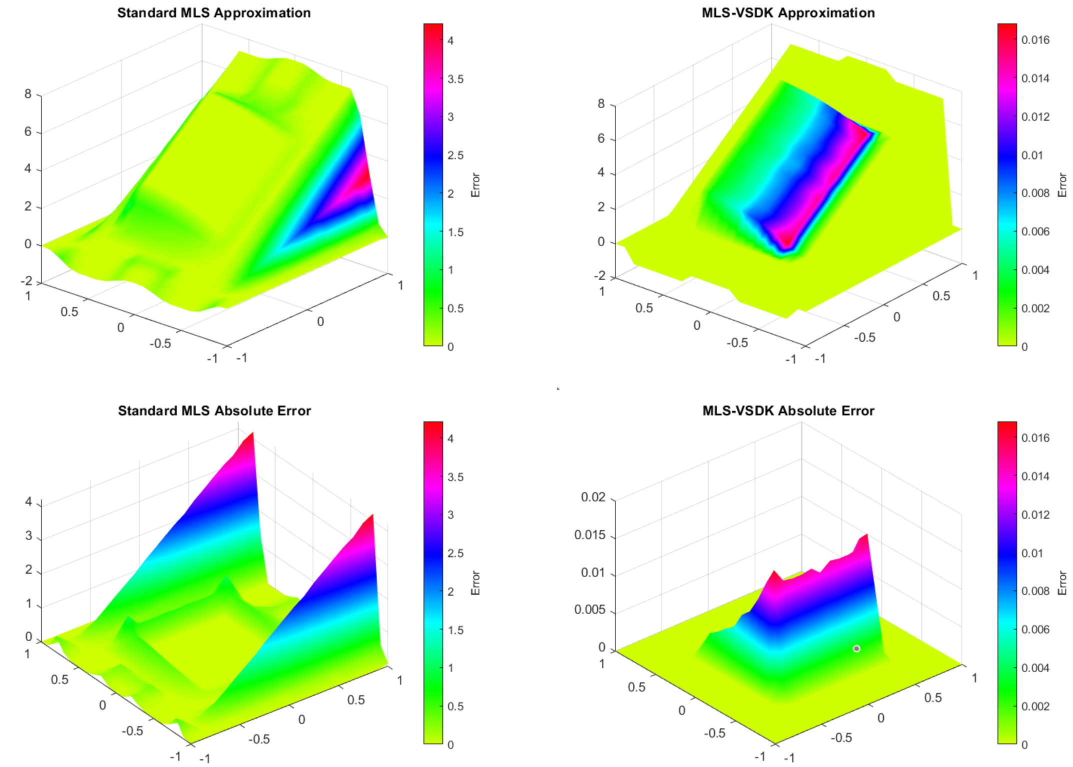


FIGURE 5. RMSE and abs-error of f_3 MLS(left) and MLS-VSDK(right) aproximation(interpolation) schemes using w^4 weight function

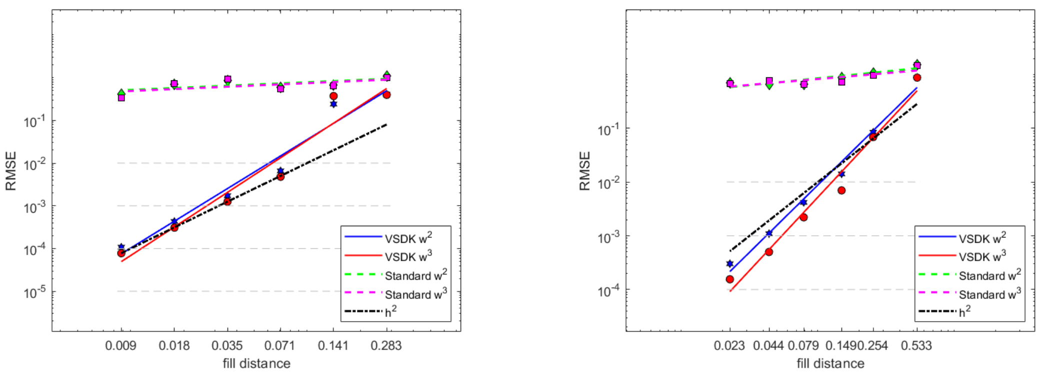


FIGURE 6. Convergence rates for approximation of function f_3 with MLS-VSDK and MLS standard schemes using *Uniform* data sites (left) and *Halton* data sites (right)

4.4. **Example 4.** In applications, the discontinuities are likely to be unknown. To overcome this problem, one can consider edge detector method to extract the discontinuities. However, in this way the approximation depends also on the performance of the edge detector method as well [11]. In this direction, in this final experiment the location of the discontinuities are not exact. This is modeled by adding some noise drawn from the standard normal distribution multiplied by 0.01 to the edges of $\Omega_i \in \mathcal{P}$. We take the test function f_2 and the data sites in Section 4.2. We fix $n = 25$, and $\varepsilon_{GA} = [0.25, 0.5, 1, 2, 4, 8]$, $\varepsilon_{Mat} = [1, 2, 4, 816, 32]$ for both *Halton* and *uniform* centers. Figure 7 shows that the suggested MLS-VSDK is still able to obtain a good convergence rate compared to classical MLS even when the discontinuities are not known exactly.

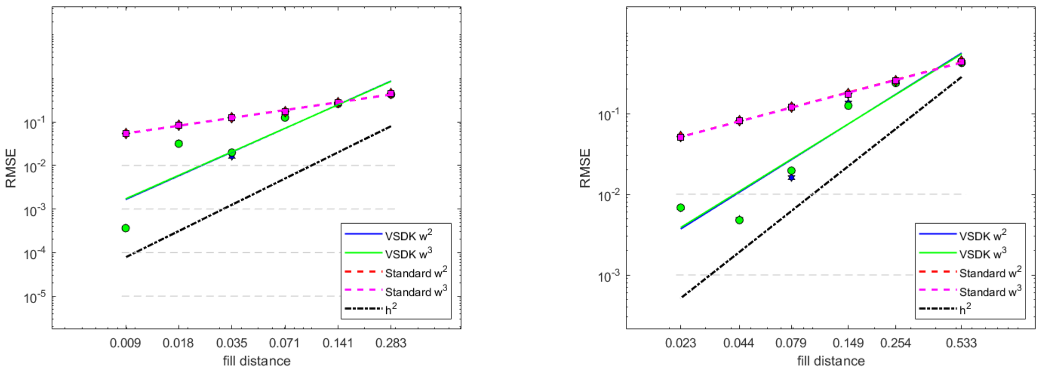


FIGURE 7. Convergence rates for approximation of function f_2, based on noisy given data values, with MLS-VSDK and MLS standard schemes using *Uniform* data sites (left) and *Halton* data sites (right)

5. CONCLUSIONS

To approximate a discontinuous function using scattered data values, we studied a new technique based on the use of discontinuously scaled weight functions, that we called the MLS-VSDK scheme, that is the application of discontinuous scaled weight functions to the MLS. It enabled us to move toward a data-dependent scheme, meaning that MLS-VSDK is able to encode the behavior of the underlying function. We obtained a theoretical Sobolev-type error estimate which justifies why MLS-VSDK can outperform conventional MLS. The numerical experiments confirmed the theoretical convergence rates. Besides, our numerical tests showed that the suggested scheme can reach high accuracy even if the position of the data values are slightly perturbed.

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REFERENCES

[1] R. A. Adams, J. Fournier: *Sobolev Spaces*, Elsevier, London, (2003).
[2] M. G. Armentano: *Error estimates in Sobolev spaces for moving least square approximations*, SIAM J. Numer. Anal., **39** (2001), 38–51.
[3] M. G. Armentano, R. G. Duran: *Error estimates for moving least square approximations*, Appl. Numer. Math., **37** (2001), 397–416.

[4] G. E. Backus, J. F. Gilbert: *Numerical applications of a formalism for geophysical inverse problems*, Geophys. J. Int., **13** (1967), 247–276.

[5] V. Bayona: *Comparison of moving least squares and RBF+poly for interpolation and derivative approximation*, J. Sci. Comput., **81** (2019), 486–512.

[6] V. Bayona, N. Flyer, B. Fornberg and G. A. Barnett: *On the role of polynomials in RBF-FD approximations: II. Numerical solution of elliptic PDEs*, J. Comput. Phys., **332** (2017), 257–273.

[7] S. C. Bernard, L. R. Scott: *The mathematical theory of finite element methods*, Springer, New York, (2003).

[8] L. Bos, K. Salkauskas: *Moving least-squares are Backus-Gilbert optimal*, J. Approx. Theory, **59** (1989), 267–275.

[9] M. Bozzini, L. Lenarduzzi, M. Rossini and R. Schaback: *Interpolation with variably scaled kernels*, SIAM J. Numer. Anal., **35** (2015), 199–219.

[10] S. Cuomo, A. Galletti, G. Giunta and A. Starace: *Surface reconstruction from scattered point via RBF interpolation on GPU*, Federated Conference on Computer Science and Information Systems, Krakow (Poland) (2013), 433–440.

[11] S. De Marchi, W. Erb, F. Marchetti, E. Perracchione and M. Rossini: *Shape-Driven interpolation with discontinuous kernels: Error analysis, edges extraction and application in magnetic particle imaging*, J. Sci. Comput., **42** (2020), 472–491.

[12] S. De Marchi, F. Marchetti and E. Perracchione: *Jumping with variably scaled discontinuous kernels*, BIT Numer. Math., **60** (2019), 441–463.

[13] G. E. Fasshauer: *Meshfree Approximation Methods*, World Scientific Publishing: Singapore, (2007).

[14] G. E. Fasshauer, M. J. McCourt: *Kernel based approximation methods using MATLAB*, World Scientific Publishing: Singapore, (2015).

[15] S. Guastavino, F. Benvenuto: *Convergence rates of spectral regularization methods: A comparison between ill-posed inverse problems and statistical kernel learning*, SIAM J. Numer. Anal., **58** (6) (2020), 3504–3529.

[16] P. Lancaster, K. Salkuaskas: *Surfaces generated by moving least squares methods*, Math. Comput., **37** (1981), 141–158.

[17] D. Levin: *The approximation power of moving least-squares*. Math. Comp., **67** (1998), 1517–1531.

[18] L. B. Lucy: *A numerical approach to the testing of the fission hypothesis*, AJ, **82** (1982), 1013–1024.

[19] D. Mirzaei: *Analysis of moving least square approximation revisited*, J. Comput. Appl. Math., **282** (2015), 237–250.

[20] D. Mirzaei, R. Schaback: *Direct meshless local Petrov–Galerkin (DMLPG) method: A generalized MLS approximation*, Appl. Numer. Math., **68** (2013), 73–82.

[21] D. Mirzaei, R. Schaback and M. Dehghan: *On generalized moving least squares and diffuse derivatives*, SIAM J. Numer. Anal., **32** (2012), 983–1000.

[22] F. J. Narcowich, J. D. Ward and H. Wendland: *Sobolev bounds on functions with scattered zeros, with applications to radial basis function surface fitting*, Math. Comput., **78** (2015), 743–763.

[23] V. P. Nguyen, T. Rabczuk, S. Bordas and M. Duflot: *Meshless methods: A review and computer implementation aspects*, Math. Comput. Simul., **79** (3) (2008), 763–813.

[24] C. Rieger, B. Zwicknagl: *Sampling inequalities for infinitely smooth functions, with applications to interpolation and machine learning*, Adv. Comput. Math., **32** (1) (2010), 103–129.

[25] D. Shepard: *A two-dimensional interpolation function for irregularly-spaced data*, Proceedings of the 1968 23rd ACM national conference, New York, (U.S.A) (1968), 27–29.

[26] H. Wendland: *Local polynomial reproduction and moving least squares approximation*, SIAM J. Numer. Anal., **21** (2001), 285–300.

[27] H. Wendland: *Scattered Data Approximation*, Cambridge University Press: Cambridge (2005).

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Research Article

Construction of rational interpolations using Mamquist-Takenaka systems

ZSUZSANNA NAGY-CSIHA, MARGIT PAP, AND FERENC WEISZ*

ABSTRACT. Rational functions have deep system-theoretic significance. They represent the natural way of modeling linear dynamical systems in the frequency (Laplace) domain. Using rational functions, the goal of this paper to compute models that match (interpolate) given data sets of measurements. In this paper, the authors show that using special rational orthonormal systems, the Malmquist-Takenaka systems, it is possible to write the rational interpolant $r_{(n,m)}$, for $n = N-1, m = N$ using only N sampling nodes (instead of $2N$ nodes) if the interpolating nodes are in the complex unit circle or on the upper half-plane. Moreover, the authors prove convergence results related to the rational interpolant. They give an efficient algorithm for the determination of the rational interpolant.

Keywords: Rational interpolation, Hardy spaces, Malmquist-Takenaka systems, discrete biorthogonality.

2020 Mathematics Subject Classification: 42C05, 30H10, 30J10, 31C20, 65D05, 65D25.

1. INTRODUCTION

Rational functions have deep system-theoretic significance. They represent the natural way of modeling linear dynamical systems in the frequency (Laplace) domain, because the Laplace transform of a sum of complex exponentials is a rational function; more precisely, the transfer functions (or frequency responses) of such systems are rational functions. Using rational functions, our goal is to compute models that match (interpolate) given data sets of measurements.

We give first a short summary related to the general solution of the rational interpolation problem. Let us consider a function $f : H \to \mathbb{C}$, $H \subset \mathbb{C}$, and a general rational function of the form:

$$r_{(n,m)}(x) = \frac{\sum_{i=0}^{n} \alpha_i x^i}{\sum_{j=0}^{m} \beta_j x^j},$$

where $\alpha_i, \beta_j, x \in \mathbb{C}$, and m and n are not necessarily equal natural numbers. To find a rational interpolant $r_{(n,m)}$ of type (n, m) requires $n + m + 1$ sample points (or in other word nodes), because we have to determine the α_i and β_j coefficients (one coefficient can be set to 1). Knowing $(x_k, f(x_k))$, $k = 1, \ldots, n + m + 1$, we search the solution of the interpolation problem satisfying the following conditions

$$r_{(n,m)}(x_k) = f(x_k),\ \ k = 1, \ldots, n + m + 1.$$

In this paper, we show that using special rational orthonormal systems, the Malmquist-Takenaka systems, it is possible to write the rational interpolant $r_{(n,m)}$, for $m = N, n = N-1$ using only N sampling nodes (instead of $2N$ nodes) if the interpolating nodes are in the unit circle or on

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the upper half-plane, moreover we can prove convergence results related to the rational interpolant. We give an efficient algorithm for the determination of the rational interpolant. We will introduce new rational interpolation operators of type $(N-1,N)$ using N special nodes in the closed unit disc. These nodes are solutions of certain equation related to the Malmquist-Takenaka systems and its dual systems, and we will study the properties of the new interpolation operators. We will study the analogue of the problem also for the closed upper half-plane. Before we present our results, let us revise the classical method to find $r_{(n,m)}$ (see for example Berrut, Trefethen or Ionita [1, 11] and the reference list therein). We write our interpolation conditions in the following form:

$$\left(\sum_{i=0}^{n}\alpha_i x_k^i\right) - f(x_k)\left(\sum_{j=0}^{m}\beta_j x_k^j\right) = 0.$$

In matrix form, this is equivalent to

$$\mathbf{Ab} = \mathbf{0},$$

where

$$\mathbf{A} := \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n & -f(x_0) & -f(x_0)x_0 & -f(x_0)x_0^2 & \dots & -f(x_0)x^m \\ 1 & x_1 & x_1^2 & \dots & x_1^n & -f(x_1) & -f(x_1)x_1 & -f(x_1)x_1^2 & \dots & -f(x_1)x^m \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_M & x_M^2 & \dots & x_M^n & -f(x_M) & -f(x_M)x_M & -f(x_M)x_M^2 & \dots & -f(x_M)x_M^m \end{bmatrix}$$

and

$$\mathbf{b} := [\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n, \beta_0, \beta_1, \beta_2, \dots \beta_m]^T.$$

However, there is no any guarantee that the solution exists, and it is unique. It is possible that there are more $\mathbf{b}$ vectors satisfying the equation -if it exists at all. When $\beta_0 = 1, \beta_1 = \beta_2 = \dots = \beta_m = 0$, then the problem reduces to the construction of a polynomial interpolant. In this case, if the nodes x_k are different from each other and we have $M = n+1$ samples, the problem has unique solution. If we want to express the interpolation polynomial $r_{(n,0)}(x) = P_n(x) = \sum_{i=0}^{n} c_i x^i$ in the basis $\Phi_k(x) = x^k$ satisfying the condition $P_n(x) = f(x_k)$, $k = 1,\dots,M = n+1$, then the solution $c = (c_0, c_1, \dots, c_n)$ of the system is $\mathbf{c} = \Phi^{-1}\mathbf{f}$, where $\mathbf{f} = (f(x_1),\dots,f(x_{n+1}))^T$ and

$$\Phi = V(x_0, x_1, \dots, x_n) = \begin{bmatrix} 1 & x_0^1 & \dots & x_0^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n^1 & \dots & x_n^n \end{bmatrix}.$$

We don't have to solve the linear equation system if we write the interpolation polynomial in Lagrange form. In this way, we reduce the number of operations. Let us consider the Lagrange interpolation polynomials corresponding to the $n+1$ sample points defined by

$$l_i(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)} = \frac{\prod_{j=0,j\neq i}^{n}(x-x_j)}{\prod_{j=0,j\neq i}^{n}(x_i-x_j)}.$$

Because

$$l_i(x_k) = \delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k, \end{cases} \tag{1.1}$$

the solution of the interpolation problem has the following form:

$$P_n(x) = L_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i).$$

The set $\{l_i(x),\, i = 0, \ldots, n\}$ is the so-called Lagrange basis, thus the resulted interpolation polynomial is the linear combination of the Lagrange basis. There is only one unique Lagrange polynomial basis perfectly fitting to the set of different sample points $\{(x_i, f(x_i)),\, i = 0, \ldots, n\}$. Unfortunately, using the Lagrange method, the basis have to be recalculated when we add a new sample point, requiring $\mathcal{O}(n^2)$ operations. A solution for this problem, to diminish the number (cost) of the operations, is the Barycentric Lagrange polynomial interpolation. Using the divided differences method, we get a much faster algorithm than the Lagrange interpolation, mainly when we have a new, additional sample point. First let us consider the Lagrange polynomial of constant function 1:

$$e_n(x) = \sum_{i=0}^{n} \left\{ \frac{\prod_{j=0, j\neq i}^{n} (x - x_j)}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)} \right\} = 1.$$

Using this, we can write for any function f the interpolant $L_n(x)$ in the following form:

$$L_n(x) = \frac{L_n(x)}{e_n(x)} = \frac{\sum_{i=0}^{n} \left\{ \frac{\prod_{j=0, j\neq i}^{n} (x - x_j)}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)} f(x_i) \right\}}{\sum_{i=0}^{n} \left\{ \frac{\prod_{j=0, j\neq i}^{n} (x - x_j)}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)} \right\}}.$$

Simplifying by $\prod_{j=0}^{n} (x - x_j)$, if we consider that $\prod_{j=0, j\neq i}^{n} (x - x_j) = \prod_{j=0}^{n} (x - x_j) \frac{1}{x - x_i}$, we arrive to the following:

$$L_n(x) = \frac{\sum_{i=0}^{n} \left\{ \frac{1}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)} \frac{f(x_i)}{x - x_i} \right\}}{\sum_{i=0}^{n} \left\{ \frac{1}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)} \frac{1}{x - x_i} \right\}}.$$

Let be $\frac{1}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)} = \lambda_i$, then

$$L_n(x) = \frac{\sum_{i=0}^{n} \left\{ \lambda_i \frac{f(x_i)}{x - x_i} \right\}}{\sum_{i=0}^{n} \left\{ \lambda_i \frac{1}{x - x_i} \right\}}$$

is called polynomial Barycentic formula. After the determination of each λ_i, it is relatively fast to calculate the polynomial in this form, it requires $\mathcal{O}(n)$ operations. Another advantage of the Barycentric formula is that it is numerically stable. In case if we choose λ_i freely, we get a rational interpolant $r_{(n,n)}$ fitting to the sample points (when $\lambda_i = \frac{1}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)}$, we get the Lagrange polynomial $L_n(x)$). These rational functions satisfy the interpolation condition $r_{(n,n)}(x_k) = f(x_k), k = 0, \ldots, n$. These rational interpolants are called Lagrange rational interpolants. Freely choosing the λ_i-s, there are more rational functions fitting to the sample points. In order to determine $r_{(n,n)}$ uniquely, we need to fix $n + 1$ more λ_i-s. We can get the Lagrange rational interpolation which satisfies

$$r_{(n,n)}(x_k) = f(x_k), \quad k = 1, \ldots, 2n + 1$$

without solving the system of equation in the following way:

1. We have to divide the $2n+1$ sample points $\{x_j\}$ into two subgroups, $n+1$ Lagrange nodes denoted by η_i and the remaining n sample points denoting by (μ_j). Similarly, also the set of the corresponding function values $(\{f(x_j)\}\ )$ has to be partitioned.

2. We express $r_{(n,n)}$ with the Lagrange basis $\prod_{j=0,j\neq i}^{n}(x-\eta_j)$ associated to the Lagrange nodes η_i in the following form:

$$r_{(n,n)}(x) = \frac{\sum_{i=0}^{n}\left\{a_i \prod_{j=0,j\neq i}^{n}(x-\eta_j)\right\}}{\sum_{i=0}^{n}\left\{b_i \prod_{j=0,j\neq i}^{n}(x-\eta_j)\right\}}.$$

This rational function can be written in barycentric form:

$$r_{(n,n)}(x) = \frac{\sum_{i=0}^{n}\left\{a_i \frac{1}{x-\eta_i}\right\}}{\sum_{i=0}^{n}\left\{b_i \frac{1}{x-\eta_i}\right\}}.$$

Similarly to the polynomial barycentric formula, also this rational barycentric formula requires only $\mathcal{O}(n)$ operations. When $x=\eta_i$, then $r_{(n,n)}(\eta_i)=\frac{a_i}{b_i}=f(\eta_i)$, so if we set $a_i=b_i f(\eta_i)$, then $r_{(n,n)}$ will exactly interpolate f at the η_i nodes and we get back our earlier formula when the coefficients were the same in the numerator and denominator.

3. Now, what we have to do is only to determine the unknown coefficients b_i using the remaining n sample points (μ_j) and the corresponding $f(\mu_j)$ values (which were still not used). Using that $a_i=b_i f(\eta_i)$, for the μ_j points $j=1,\ldots,n$ the rational function satisfies the following:

$$r_{(n,n)}(\mu_j) = f(\mu_j) = \frac{\sum_{i=0}^{n}\left\{\frac{b_i f(\eta_i)}{\mu_j-\eta_i}\right\}}{\sum_{i=0}^{n}\left\{\frac{b_i}{\mu_j-\eta_i}\right\}}.$$

Rearranging these equations, we get the equivalent forms for $j=1,\ldots,n$:

$$\sum_{i=0}^{n}\left\{\frac{b_i f(\mu_j)}{\mu_j-\eta_i}\right\} = \sum_{i=0}^{n}\left\{\frac{b_i f(\eta_i)}{\mu_j-\eta_i}\right\},$$

$$\sum_{i=0}^{n}\left\{\frac{b_i(f(\mu_j)-f(\eta_i))}{\mu_j-\eta_i}\right\} = 0.$$

These conditions can be written in a matrix form:

$$\begin{bmatrix} \frac{(f(\mu_0)-f(\eta_0))}{\mu_0-\eta_0} & \cdots & \frac{(f(\mu_0)-f(\eta_n))}{\mu_0-\eta_n} \\ \vdots & \ddots & \vdots \\ \frac{(f(\mu_{n-1})-f(\eta_0))}{\mu_{n-1}-\eta_0} & \cdots & \frac{(f(\mu_{n-1})-f(\eta_n))}{\mu_{n-1}-\eta_n} \end{bmatrix} \begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix} = 0.$$

The matrix is called Loewner matrix ($\mathbb{L}$) and it is an $n\cdot(n+1)$ matrix.

4. The Loewner matrix and its null space has to be computed using the partitioned nodes and the corresponding sample values solving the $\mathbb{L}\bar{b}=0$ equation. In this way, we get the $\bar{b}$ vector of the coefficients.

5. After the above-mentioned steps, we can form the Lagrange rational polynomial using the barycentric formula:

$$r_{(n,n)}(x) = \frac{\sum_{i=0}^{n} \left\{ \frac{b_i f(\eta_i)}{x-\eta_i} \right\}}{\sum_{i=0}^{n} \left\{ \frac{b_i}{x-\eta_i} \right\}}.$$

If the number of sample points is large, then the number of operations to determine the rational interpolant fitting the data is still high. Our goal is to find new methods to write the rational interpolant using less initial data and to reduce the number of operations if it is possible.

The paper is organized as follows. In Section 2, we present rational interpolation using Malmquist-Takenaka systems for the unit disc and also for the upper half plane. In both cases, we give the algorithms how the rational interpolant can be described. We study also the convergence properties of the interpolants. In Section 3, we introduce new rational interpolation operators with special nodes related to discrete biorthogonality of Malmquist-Takenaka systems and we study their properties.

2. Rational interpolation using Malmquist-Takenaka systems

In what follows, we focus on the determination of a rational interpolant of type $(N-1, N)$. According to the algorithms presented in the previous section, to write a rational interpolant of type $(N-1, N)$ in general, we would need $2N$ nodes and the values of the function in these nodes. In this section, we show that choosing a good basis of rational functions, the Mamquist-Takenaka system, we can reduce the number of the data and we can avoid to solve the system of equations associated to the interpolation problem. We will work with some assumptions regarding the nodes and the function f. We assume that the nodes are in the unit disc or in the upper half-plane and the function f belongs to the Hardy space of the unit disc or the Hardy space of the upper half-plane, respectively. Using the corresponding Malmquist-Takenaka systems, we show that it is possible to write a rational interpolant of type $(N-1, N)$ using only N nodes and the values of the function in these nodes. Moreover, we give an algorithm for the determination of the rational interpolant, and we study the convergence properties of the rational interpolant.

2.1. **Rational interpolation with nodes in the unit disc related to Malmquist-Takenaka system of the unit disc.** Let $\mathbb{D}$ denote the open and $\overline{\mathbb{D}}$ denote the closed unit disc, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, and let us denote the unit circle with $\mathbb{T}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let us denote the set of analytic functions over $\mathbb{D}$ with $A(\mathbb{D})$, the Hardy space of the unit disc with

$$H^2(\mathbb{D}) = \left\{ f \in A(\mathbb{D}) : \|f\|_{H^2(\mathbb{D})} = \sup_{r<1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^2 dr \right)^{1/2} < \infty \right\}.$$

For every function $f \in H^2(\mathbb{D})$ and for a.e. $t \in [-\pi, \pi)$, there exists the finite limit $f(e^{it}) := \lim_{r\to 1} f(re^{it})$. Moreover for the limit function holds that $f \in L^2(\mathbb{T})$, and $\|f\|_{H^2(\mathbb{D})} = \|f\|_{L^2(\mathbb{T})}$. The set of the limit functions of $H^2(\mathbb{D})$ is the Hardy space of the unit circle denoted by $H^2(\mathbb{T})$. The Malmquist–Takenaka system ([13, 20]) is an orthonormal system of rational functions, products of Blaschke factors, in the Hardy space of unit disc, which contains as special case the classical ”trigonometric” system. In system identification, it is frequently applied in order to approximate the transfer functions of the systems. Let us consider a sequence $a = (a_1, a_2, \ldots)$

of complex numbers, $a_n \in \mathbb{D}$ of the unit disc $\mathbb{D}$, and denote the Blaschke functions by

$$b_a(z) := \frac{z-a}{1-\overline{a}z} \qquad (a \in \mathbb{D},\ z \in \mathbb{C},\ 1-\overline{b}z \neq 0).$$

The Malmquist-Takenaka (MT) system $\Phi_n = \Phi_n^a$ $(n \in \mathbb{N}^*)$ is defined by

$$\Phi_1(z) = \frac{\sqrt{1-|a_1|^2}}{1-\overline{a_1}z}, \quad \Phi_n(z) = \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}z}\prod_{k=1}^{n-1} b_{a_k}(z),\ n \geq 2. \tag{2.2}$$

When all parameters are equal, i.e., $a_n = a$, $n \in \mathbb{N}^*$, we obtain the so called discrete Laguerre system and particularly, when $a_n = 0$, $n \in \mathbb{N}^*$, we obtain the trigonometric system. Consequently, these systems can be viewed as extensions of the trigonometric system on the unit circle. These functions form an orthonormal system on the unit circle, i.e.,

$$\langle \Phi_n, \Phi_m \rangle = \frac{1}{2\pi}\int_0^{2\pi} \Phi_n(e^{it})\overline{\Phi_m(e^{it})}dt = \delta_{mn} \quad (m, n \in \mathbb{N}^*).$$

If the sequence $a = (a_1, a_2, \ldots)$ satisfies the non-Blaschke condition

$$\sum_{n\geq 1}(1-|a_n|) = +\infty, \tag{2.3}$$

then the corresponding MT system is complete in the Hardy space of the unit disc. Let us consider the orthogonal projection operator of order N of an arbitrary function $f \in H^2(\mathbb{T})$ with respect to the MT system:

$$P_N f(z) = \sum_{k=1}^{N} \langle f, \Phi_k \rangle \Phi_k(z). \tag{2.4}$$

For a special sequence $a = (a_1, a_2, \ldots)$, Pap proved in [15] that the analytic continuation in the unit disc of the projection $P_N f$ is at the same time a rational interpolation operator in the unit disc for the analytic continuation of f in the unit disc. In this paper, we show that this interpolation property is true in general for any sequence $a = (a_1, a_2, \ldots)$, with elements from $\mathbb{D}$, different from each other.

Theorem 2.1. *Let us consider a sequence $a = (a_1, a_2, \ldots)$, with elements from $\mathbb{D}$, different from each other ($a_k \neq a_j$, $k \neq j$). For every $f \in H^2(\mathbb{T})$, the projection operator $P_N f$ is a rational interpolation operator of type $(N-1, N)$ at the points $a_1, a_2, \ldots, a_N$ for the analytic continuation of f in the unit disc.*

Proof. In order to prove the interpolation property of $P_N f$, let us consider the kernel function of this projection operator:

$$K_N(z, \xi) = \sum_{k=1}^{N} \overline{\Phi_k(\xi)}\Phi_k(z). \tag{2.5}$$

According to the Christoffel-Darboux formula (see [12, 16, 2]), the kernel function can be written in closed form

$$K_N(z, \xi) = (1-z\overline{\xi})^{-1}\left(1 - \overline{\prod_{k=1}^{N}\frac{\xi-a_k}{1-\overline{a_k}\xi}}\prod_{k=1}^{N}\frac{z-a_k}{1-\overline{a_k}z}\right). \tag{2.6}$$

From this relation, it follows that the values of the kernel-function at the points $(a_m,\ m = 1,\dots,N)$ are equal to localized Cauchy kernels

$$K(a_m,\xi) = \frac{1}{1 - a_m\overline{\xi}}.$$

From this property and the Cauchy integral formula, we get that the interpolation property holds, i.e.,

$$P_N f(a_m) = \langle f, K_N(.,a_m)\rangle = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{it})}{1 - a_m e^{-it}}dt = f(a_m) \qquad (m = 1,\dots,N).$$

□

For special choice of $a = (a_1, a_2, \dots)$, $a_i \in \mathbb{D}, i \in \{1,\dots,N\}$ (in Pap [15]), it has been shown that the coefficients of the projection operator $P_N f$ can be computed exactly if we know f at $a_1, a_2, \dots, a_N$. We show that this algorithm can be extended in general, when we can measure f at $a_1, a_2, \dots, a_N \in \mathbb{D}$ with $a_i \neq a_j,\ i \neq j,\ i,j, \in \{1,\dots,N\}$. Consequently, $P_N f$ can be written exactly if we know the values of $f(a_i)$. We present here the steps of the algorithm.

1. Step: For $k = 1,\dots,N$, we write the partial fraction decomposition of Φ_k:

$$\Phi_k(\xi) = \sum_{k'=1}^{k} c_{kk'} \frac{1}{1 - \overline{a_{k'}}\xi}.$$

Using the orthonormality of the functions $\{\Phi_{k'},\ k' = 1,\dots,k\}$ and the Cauchy formula, we get that

$$\delta_{kn} = \langle \Phi_n, \Phi_k\rangle = \sum_{k'=1}^{k} \overline{c_{kk'}}\Phi_n(a_{k'}), \quad (n = 1,\dots,k).$$

If we order these equality's so that we write first the relations for $n = k$ then for $n = k-1$ etc., this is equivalent to

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = \begin{pmatrix} \Phi_k(a_k) & 0 & 0 & 0 & \dots & 0 \\ \Phi_{k-1}(a_k) & \Phi_{k-1}(a_{k-1}) & 0 & 0 & \dots & 0 \\ \Phi_{k-2}(a_k) & \Phi_{k-2}(a_{k-1}) & \Phi_{k-2}(a_{k-2}) & 0 & \dots & 0 \\ \vdots & & & \vdots & & \\ \Phi_1(a_k) & \Phi_1(a_{k-1}) & \Phi_1(a_{k-2}) & \dots & & \Phi_1(a_1) \end{pmatrix} \begin{pmatrix} \overline{c_{kk}} \\ \overline{c_{kk-1}} \\ \overline{c_{kk-2}} \\ \vdots \\ \overline{c_{k1}} \end{pmatrix}.$$

2. Step: We solve the previous system of equations. Because of the elements from the main diagonal are different from zero, this system has a unique solution

$$(\overline{c_{kk}}, \overline{c_{kk-1}}, \overline{c_{kk-2}}, \dots, \overline{c_{k1}})^T.$$

3. Step: For $k = 1,\dots,N$, we determine the vectors $(\overline{c_{kk}}, \overline{c_{kk-1}}, \overline{c_{kk-2}}, \dots, \overline{c_{k1}})^T$, then based on Cauchy formula, we can compute the exact value of $\langle f, \Phi_k\rangle$ knowing the values of f on the set $a_1,\dots,a_N$. Indeed, using again the partial fraction decomposition of ψ_k and the Cauchy integral formula, we get that

$$\begin{aligned} \langle f, \Phi_k\rangle &= \sum_{k'=1}^{k} \overline{c_{kk'}} \left\langle f(\xi), \frac{1}{1 - \overline{a_{k'}}\xi} \right\rangle \\ &= \sum_{k'=1}^{k} \overline{c_{kk'}} f(a_{k'}). \end{aligned}$$

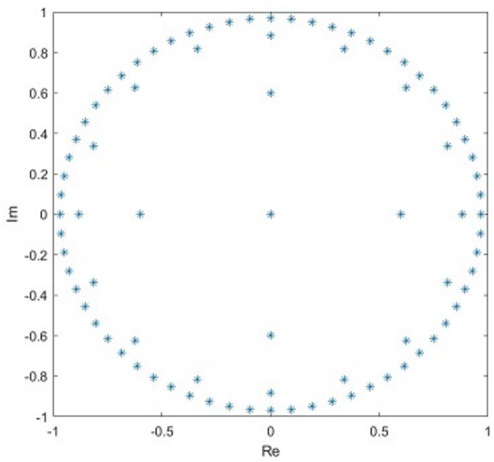

FIGURE 1. The first 85 elements of the sequence a

4. Step: We write

$$P_N f(z) = \sum_{k=1}^{N} \langle f, \Phi_k \rangle \Phi_k(z),$$

which is in the same time a projection operator and a rational interpolation operator of type $(N-1, N)$ at the points $a_1, a_2, \ldots, a_N$. A Matlab code was developed for the interpolation process on the unit disc (see **code**). In the code, we defined the sequence $a = (a_1, a_2, \ldots)$ as it is given in [15], in equations (2.4), (2.6) and (2.7), where the points of the sequence form concentric circles. For $k = 3$, we get the first 85 elements of the sequence (see on Figure 1). We apply the Steps 1–4 mentioned above to create $P_n f$ for the function

$$f(z) = \frac{1}{2 - z^2}.$$

We plot the function f and the projection operator $P_N f$ at the points $z_i = a_i$. As one can see on Figure 2, the values of the function and the projection operator are equal at these points, as it was stated in Theorem 2.1. In general, it is a hard task to study the convergence properties of an interpolation operator. In this case using that $P_N f$ is at the same time projection operator, we can derive more easily convergence results. The properties of orthogonal projection $P_N f$ on the unit circle were studied by Malmquist and Takenaka [13, 20]. If the sequence a is non-Blaschke sequence, i.e., $\sum_{n=0}^{\infty}(1 - |a_n|) = \infty$, then the Malmquist-Takenaka system is complete in the $H^p(\mathbb{T})$ for $0 < p < \infty$ (it follows from K. Hoffman, (1962, pp. 64) [10], J. B. Garnett, (1981, pp. 53) [9] and Z. Szabó [19, 18]), and $P_N f$ converge to f in norm on the circle and the convergence is compactly uniform on the disc for every $f \in H^2(\mathbb{D})$.

2.2. **Rational interpolation with nodes on the upper half-plane related to the Malmquist-Takenaka systems on the upper half-plane.** Let us denote the upper half-plane with $\mathbb{C}_+$, $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$. Let us denote the set of analytic functions over $\mathbb{C}_+$ with $A(\mathbb{C}_+)$, respectively,

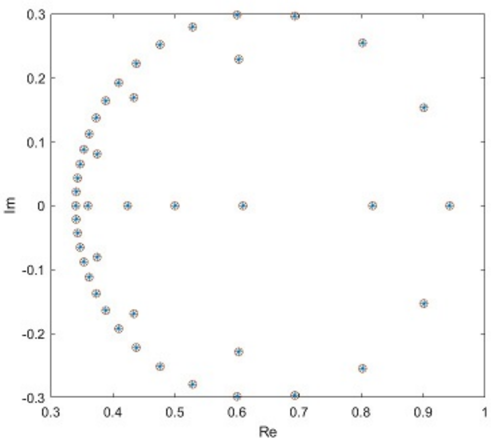

FIGURE 2. The interpolated function f (star) and the interpolation operator $P_N f$ (circle) at $z_i = a_i$

and consider the Hardy space of the upper half-plane

$$H^2(\mathbb{C}_+) = \left\{ f \in A(\mathbb{C}_+) : \|f\|_{H^2(\mathbb{C}_+)} = \sup_{0<y} \left(\int |f(x+iy)|^2 dx \right)^{1/2} < \infty \right\}.$$

If $f \in H^2(\mathbb{C}_+)$, for a.e. $x \in \mathbb{R}$ there exist the finite limit $f(x) := \lim_{y \to 0_+} f(x+iy)$, the limit function of f satisfies the following conditions $f \in L^2(\mathbb{R})$ and $\|f\|_{L^2(\mathbb{R})} = \|f\|_{H^2(\mathbb{C}_+)}$. The set of limit functions is the Hardy space of the real line denoted by $H^2(\mathbb{R})$. The Hardy space of the upper half-plane and the Hardy space of the unit disc $\mathrm{H}^2(\mathbb{D})$ may be connected through the Cayley transform. The conformal mapping from $\mathbb{C}_+$ to $\mathbb{D}$ defined by

$$C(\omega) = \frac{i-\omega}{i+\omega} \qquad (\omega \in \mathbb{C}_+) \tag{2.7}$$

is called Cayley transform and it extends continuously as a bijective mapping from the extended real line to $\mathbb{T}$. With the Cayley transform, the linear transformation from $\mathrm{H}^2(\mathbb{D})$ to $\mathrm{H}^2(\mathbb{C}_+)$ defined for $f \in \mathrm{H}^2(\mathbb{D})$ by

$$Tf(z) := \frac{1}{\sqrt{\pi}} \frac{1}{i+z} (f \circ C)(z) \tag{2.8}$$

is an isomorphism. Consequently, the theory of the real line is a close analogy with what we have for the circle. Using the Caley transform given by (2.7) and (2.8), we can make the transition of MT system to the upper half-plane. The system

$$\Psi_n(z) := c_n (T\Phi_n)(z) = (Tf)(z) := c_n \frac{1}{\sqrt{\pi}} \frac{1}{i+z} \Phi_n(C(z)) \qquad (\Im z \geq 0,\ n \in \mathbb{N}^*)$$

is the analogue of the Malmquist-Takeneka system for the upper half-plane. It is easy to check that for $a \in \mathbb{D}$ with $a^* := 1/\overline{a}$,

$$\lambda_a := C^{-1}(a) = i\frac{1-a}{1+a} \in \mathbb{C}_+, \quad \lambda_{a^*} = \overline{\lambda}_a, \quad \frac{\sqrt{1-|a|^2}}{|1+\overline{a}|} = \sqrt{\Im \lambda_a}, \tag{2.9}$$

and

$$\tilde{b}_a(z) = b_a(-1)\frac{z-\lambda_a}{z-\overline{\lambda}_a}, \quad \tilde{r}_a(z) = r_a(-1)\frac{z+i}{z-\overline{\lambda}_a} \ \ (z \in \overline{\mathbb{C}}_+). \tag{2.10}$$

This implies that the functions $\Psi_n = c_n T\Phi_n \ (n \in \mathbb{N}^*)$, $c_n = \frac{\sqrt{\Im\lambda_{a_n}}}{\Phi_n(-1)}$ are of the form

$$\Psi_1(z) = \frac{1}{\sqrt{\pi}}\frac{\sqrt{\Im\lambda_{a_1}}}{z-\overline{\lambda}_{a_1}}, \quad \Psi_n(z) = \frac{1}{\sqrt{\pi}}\frac{\sqrt{\Im\lambda_{a_n}}}{z-\overline{\lambda}_{a_n}}\prod_{k=1}^{n-1}\frac{z-\lambda_{a_k}}{z-\overline{\lambda}_{a_k}}. \tag{2.11}$$

The system of functions $\{\Psi_n\}_{n=1}^{\infty}$ is orthonormal on the entire axis in the following sense

$$\int_{-\infty}^{+\infty}\Psi_n(t)\overline{\Psi_m(t)}dt = \delta_{mn}. \tag{2.12}$$

Moreover, if the following non-Blaschke condition for the upper half-plane is satisfied

$$\sum_{k=1}^{\infty}\frac{\Im\lambda_{a_k}}{1+|\lambda_{a_k}|^2} = \infty,$$

then $(\Psi_n, \ n \in \mathbb{N}^*)$ is a complete orthonormal system for $H^2(\mathbb{C}_+)$. Let us consider the orthogonal projection operator of order N of an arbitrary function $f \in \mathrm{H}^2(\mathbb{C}_+)$ with respect to $\mathbf{\Psi}_N = \{\Psi_n, n = 1, 2, \cdots, N\}$ given by

$$Q_N f(z) = \sum_{k=1}^{N}\langle f, \Psi_k\rangle\Psi_k(z). \tag{2.13}$$

Let us consider the kernel function of this projection operator

$$\widetilde{K}_N(\omega, w) = \sum_{k=1}^{N}\overline{\Psi_k(w)}\Psi_k(\omega).$$

Then the projection operator can be expressed as a scalar product:

$$Q_N f(z) = \int_{-\infty}^{\infty} f(t)\widetilde{K}_N(z,t)dt = \langle f(.), \widetilde{K}_N(., z)\rangle. \tag{2.14}$$

According to [3], the kernel function can be written in the following form:

$$\widetilde{K}(\omega, w)_N = \sum_{k=1}^{N}\overline{\Psi_k(w)}\Psi_k(\omega) = \frac{1-\overline{\widetilde{B}_N(w)}\widetilde{B}_N(\omega)}{2i\pi(\overline{w}-\omega)}, \quad \omega \neq \overline{w},$$

where

$$\widetilde{B}_N(\omega) = \prod_{k=1}^{N}\frac{\omega-\lambda_{a_k}}{\omega-\overline{\lambda_{a_k}}}\tau_k, \quad \tau_k = \frac{|1+\lambda_{a_k}^2|}{1+\lambda_{a_k}^2}$$

is the Blaschke product on the upper half-plane. Eisner and Pap [4] proved the following interpolation property of the projection operator:

Theorem 2.2 (Eisner, Pap [4])**.** *For any $f \in \mathrm{H}^2(\mathbb{C}_+)$, the projection operator $Q_N f$ is an interpolation operator of type $(N-1, N)$ on the set $\{\lambda_{a_k}, \ j, k = 1, \ldots, N\}$, $\lambda_{a_k} \neq \lambda_{a_j}, k \neq j$, i.e.*

$$Q_N f(\lambda_{a_k}) = f(\lambda_{a_k}) \qquad (k = 1, \ldots, N).$$

If condition (2.2) is satisfied, $\{\Psi_k,\ k=1,\dots\infty\}$ is a complete orthonormal set in the Hilbert space $\mathrm{H}^2(\mathbb{C}_+)$ and we have $\|f-Q_Nf\|_{\mathrm{H}^2(\mathbb{C}_+)}\to 0$ as $N\to\infty$. Since convergence in $\mathrm{H}^2(\mathbb{C}_+)$ implies uniform convergence to the analytic continuation of f on the upper half-plane on every compact subset, we conclude that $Q_Nf\to f$ uniformly on every compact subset of the upper half-plane. For $\lambda_a=(\lambda_{a_1},\lambda_{a_2},\dots)$, $\lambda_{a_n}\in\mathbb{C}_+$ and $\lambda_{a_n}\neq\lambda_{a_k}$, $n\neq k$, we show that the coefficients of the projection operator Q_Nf can be computed exactly if we know f in $\lambda_{a_1},\lambda_{a_2},\dots$. Consequently, Q_Nf can be written exactly if we know the values of $f(\lambda_{a_i})$. We present here the steps of the algorithm.

1. Step: For $k=1,\dots,N$, we write the partial fraction decomposition of Φ_k:

$$\Psi_k(\xi)=\sum_{k'=1}^{k} b_{kk'}\frac{1}{\xi-\overline{\lambda_{a_{k'}}}}.$$

Using the orthonormality of the functions $\{\Psi_{k'},\ k'=1,\dots,k\}$ and the Cauchy formula, we get that

$$\delta_{kn}=\langle\Psi_n,\Psi_k\rangle=\sum_{k'=1}^{k}\overline{b_{kk'}}\Psi_n(\lambda_{a_{k'}})\qquad(n=1,\dots,k).$$

If we order these equality's so that we write first the relations for $n=k$ then for $n=k-1$ etc., this is equivalent to

$$\begin{pmatrix}1\\0\\0\\.\\.\\.\\0\end{pmatrix}=\begin{pmatrix}\Psi_k(\lambda_{a_k}) & 0 & 0 & 0 & \dots & 0\\ \Psi_{k-1}(\lambda_{a_k}) & \Psi_{k-1}(\lambda_{a_{k-1}}) & 0 & 0 & \dots & 0\\ \Psi_{k-2}(\lambda_{a_k}) & \Psi_{k-2}(\lambda_{a_{k-1}}) & \Psi_{k-2}(\lambda_{a_{k-2}}) & 0 & \dots & 0\\ \vdots & & & \vdots & & \\ \Psi_1(\lambda_{a_k}) & \Psi_1(\lambda_{a_{k-1}}) & \Psi_1(\lambda_{a_{k-2}}) & \dots & & \Psi_1(\lambda_{a_1})\end{pmatrix}\begin{pmatrix}\overline{b_{kk}}\\ \overline{b_{kk-1}}\\ \overline{b_{kk-2}}\\ \vdots\\ \overline{b_{k1}}\end{pmatrix}.$$

2. Step: We solve the previous system of equations. Because of the elements from the main diagonal are different from zero, this system has a unique solution

$$(\overline{b_{kk}},\overline{b_{kk-1}},\overline{b_{kk-2}},\dots,\overline{b_{k1}})^T.$$

3. Step: If we determine the vector $(\overline{b_{kk}},\overline{b_{kk-1}},\overline{b_{kk-2}},\dots,\overline{b_{k1}})^T$, then based on Cauchy formula, we can compute the exact value of $\langle f,\Psi_k\rangle$ knowing the values of f on the set $\lambda_{a_1},\dots,\lambda_{a_n}$. Indeed, using again the partial fraction decomposition of Ψ_k and the Cauchy integral formula for upper half-plane, we get that

$$\begin{aligned}\langle f,\Psi_k\rangle&=\sum_{k'=1}^{k}\overline{b_{kk'}}\left\langle f(\omega),\frac{1}{\omega-\overline{\lambda_{a_{k'}}}}\right\rangle\\&=\sum_{k'=1}^{k}\overline{b_{kk'}}f(\lambda_{a_{k'}}).\end{aligned}$$

4. Step: We write

$$Q_Nf(z)=\sum_{k=1}^{N}\langle f,\Psi_k\rangle\Psi_k(z),$$

which is in the same time a projection operator and a rational interpolation operator of type $(N-1,N)$ at the points $\lambda_{a_1},\lambda_{a_2},\dots,\lambda_{a_n}$. We also developed a Matlab code for the interpolation process on the upper half-plane (see **code**). In the code, we use the sequence $a=(a_1,a_2,\dots)$ as it is given in [15], in equations (2.4), (2.6) and (2.7), and we defined the $\lambda_a=(\lambda_{a_1},\lambda_{a_2},\dots)$

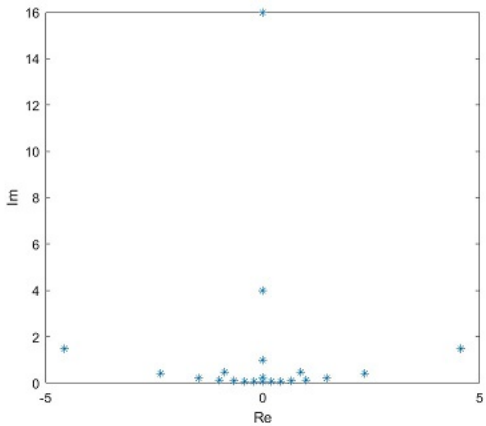

FIGURE 3. The first 21 elements of the sequence λ_a

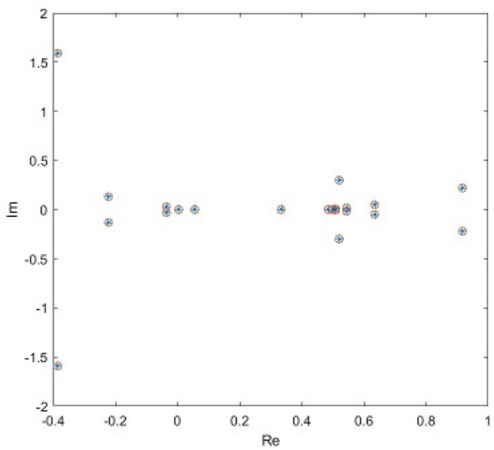

FIGURE 4. The interpolated function f (star) and the interpolation operator $Q_N f$ (circle) at $z_i = \lambda_{a_i}$

sequence with Cayley transformation, see in (2.9). On Figure 3, the first 21 elements of the λ_a sequence can be seen. Following Steps 1–4 mentioned above, we create $Q_N f$ for the function

$$f(z) = \frac{1}{2 - z^2}.$$

Representing the function f and the projection operator $Q_N f$ at the points $z_i = \lambda_{a_i}$, we can see that the values are equal at these points as it was stated in Theorem 2.2 (see Figure 4).

3. RATIONAL INTERPOLATION WITH SPECIAL NODES RELATED TO DISCRETE BIORTHOGONALITY OF MAMQUIST-TAKENAKA SYSTEMS

Discretization results connected to MT systems for unit disc and the upper half-plane were published in [16, 17, 4, 8]. Based on these results, an analogue of discrete Fourier transform

(DFT) was developed and the discrete versions was applied successfully for compression and representation of human ECG signals [5, 6]. In paper [7], Fridli and Schipp introduced the dual of the Malmquist-Takenaka system on the unit disc and proved discrete biorthogonal property on a set of points of the unit disc. Nagy-Csiha and Pap recently introduced the dual system of the Malmquist-Takenaka system on the upper half-plane and proved discrete biorthogonality result on a set of discretization points on upper half-plane [14].

In this section, using the discretization points as nodes on closed disc and on closed upper half-plane respectively, we introduce new rational interpolation operators and we study their properties.

3.1. **Rational interpolation based on the dual of the Mamquist-Takenaka system in the unit disc and discrete biorthogonality.** Let us denote by $z^* = 1/\overline{z}$. Let $\mathcal{Q}$ denote the set of rational functions. For any $f \in \mathcal{Q}$, the domain will be extended to $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by $f(a) = \infty$ if a is a pole of f and $f(\infty) := \lim_{z\to\infty} f(z)$. Let us consider the following two types of inversions:

$$f^*(z) := (f(z))^*, \quad f^\star(z) := f(z^*) \quad (z \in \overline{\mathbb{C}}, f \in \mathcal{Q}).$$

It is obvious that for any $z \in \mathbb{T}$, we have

$$z = z^*. \quad f^*(z) = f^\star(z) = f(z) \quad (f \in \mathcal{Q}).$$

Moreover, in case of Blaschke-products $B_N(z) = \prod_{k=1}^N b_{a_k}(z)$, the operations coincide:

$$B_N^*(z) = B_N^\star(z) = B_N(z^*) \quad (z \in \overline{\mathbb{C}}).$$

Let us consider the following functions:

$$\begin{aligned} \Phi_1^\star &= \overline{z}\frac{\sqrt{1-|a_1|^2}}{\overline{z}-\overline{a}_1} = r_{a_1}^\star(z), \\ \Phi_n^\star &= \Phi_n(z^*) = \overline{z}\frac{\sqrt{1-|a_n|^2}}{\overline{z}-\overline{a}_n}\prod_{k=1}^{n-1}\frac{1-a_k\overline{z}}{\overline{z}-\overline{a}_k} = r_{a_n}^\star(z)\prod_{k=1}^{n-1} b_{a_k}^\star(z) \quad (n \in \mathbb{N}^*), z \in \mathbb{C}\setminus\mathbb{D}. \end{aligned} \tag{3.15}$$

The system $\mathbf{\Phi}^\star := ((\Phi_n)^\star, n \in \mathbb{N}^*)$ is called the dual of the MT system $\mathbf{\Phi} = (\Phi_n, n \in \mathbb{N}^*)$. If $z \in \mathbb{T}$, then $\Phi_n^\star = \Phi_n, n \in \mathbb{N}^*$. If $|u| \le 1$, it is easy to see that the equation $B_N(z) = u$ has exactly N solutions in the closed unit disc counting with multiplicities. In particular, if $u \in \mathbb{T}$, then all of the roots are of multiplicity one and they are on the unit circle. If $|u| \ge 1$, then $|u^*| \le 1$. In that case $B_N(z) = u$ if and only if $B_N^*(z) = u^*$. But $B_N^*(z) = B_N(z^*)$, which implies that the equation $B_N(z) = u$ has N solutions outside of the open unit disc. In the following, we will consider an $u \in \overline{\mathbb{D}}$ for which the equation has N distinct roots. Let us introduce the set:

$$\mathcal{Z}_{N,u}^{\mathbf{a}} := \{z \in \mathbb{C} : B_N(z) = u, \ (B_N)'(z) \ne 0\} \quad (0 < |u| \le 1).$$

If it has N different elements, denote the elements by z_k and $\mathcal{Z}_{N,u}^{\mathbf{a}} = \{z_k, \ k = 1, \dots, N\}$. We recall Theorem 2.1. of Fridli and Schipp in [7]. It is easy to verify (see the proof in [7]) that the following theorem holds not just for $0 < |u| \le 1$ as it is mentioned in [7], but for $u \in \mathbb{C}\setminus\{0\}$.

Theorem 3.3 (Fridli, Schipp [7]). *Let $0 < |u| \le 1$ be a parameter for which the set $\mathcal{Z}_{N,u}^{\mathbf{a}}$ has N different elements. Then the $\Phi_n, \Phi_n^\star$ $(1 \le n \le N)$ systems are biorthogonal with respect to the following discrete scalar product*

$$[\Phi_n, \Phi_m^\star]_{\mathbf{a},u} := \sum_{z\in\mathcal{Z}_{N,u}^{\mathbf{a}}} \Phi_n(z)\overline{\Phi_m^\star(z)}/K_N(z, z^*) = \delta_{mn} \quad (1 \le m, n \le N),$$

where $K_N(z, z^*)$ is the Dirichlet kernel,

$$K_N(z,z^*) = \sum_{k=1}^{N} \Phi_k(z)\overline{\Phi_k(z^*)} = \sum_{k=1}^{N} \frac{(1-|a_k|^2)z}{(1-\overline{a_k}z)(z-a_k)}.$$

On the unit circle, the MT system is orthonormal with respect to the continuous measure, i.e.

$$\int_{\mathbb{T}} \Phi_n(z)\overline{\Phi_m(z)}dz = \frac{1}{2\pi}\int_0^{2\pi} \Phi_n(e^{it})\overline{\Phi_m(e^{it})}dt = \delta_{mn}.$$

From the definition of the dual system, it follows that the original system and the dual system are equal on the unite circle $\mathbb{T}$, i.e., if $z \in \mathbb{T}$, then $\Phi_n^\star = \Phi_n, n \in \mathbb{N}$. As a consequence, on the unit circle the original system and the dual system are biorthogonal with respect to the scalar product generated by the continuous measure:

$$\int_{\mathbb{T}} \Phi_n(z)\overline{\Phi_m^\star(z)}dz = \frac{1}{2\pi}\int_0^{2\pi} \Phi_n(e^{it})\overline{\Phi_m^\star(e^{it})}dt = \delta_{mn}.$$

The continuous projection operators connected to the MT and dual MT system for $f \in H^2(\mathbb{D})$ are the following:

$$P_N f(z) = \sum_{k=1}^{N} \langle f, \Phi_k^\star\rangle \Phi_k(z),\ z \in \overline{\mathbb{D}},\ f \in H^2(\mathbb{D}),$$

$$P_N^\bullet f(z) = \sum_{k=1}^{N} \langle f, \Phi_k\rangle \Phi_k^\star(z),\ z \in \mathbb{C}\setminus\mathbb{D},\ f \in H^2(\mathbb{C}\setminus\mathbb{D}),$$

where

$$\langle f, \Phi_k\rangle = \int_{\mathbb{T}} f(z)\overline{\Phi_k(z)}dz = \int_{\mathbb{T}} f(z)\overline{\Phi_k^\star(z)}dz = \langle f, \Phi_k^\star\rangle.$$

Taking into account, that on the circle $\Phi_k = \Phi_k^\star$, the projection $P_N f$ is the same projection which was studied in the previous section and the projections are related to each other in the following way:

$$P_N^\bullet f(z) = \sum_{k=1}^{N} \langle f, \Phi_k^\star\rangle \Phi_k^\star(z) = \sum_{k=1}^{N} \langle f, \Phi_k^\star\rangle \Phi_k(z^*) = P_N f(z^*),\ z \in \mathbb{C}\setminus\mathbb{D}.$$

For $z \in \mathbb{T}$, the two projection operator are the same: $P_N^\bullet f(z) = P_N f(z)$. Consequently, it is enough to study the properties of $P_N f(z), z \in \overline{\mathbb{D}}$. In the previous section, we proved that $P_N f(z)$ is a rational interpolant of type $(N-1, N)$ of f in a_k, $k = 1, \ldots, N$. Then $P_N^\bullet f(z)$ will interpolate the analytic continuation of f (if this exists) outside of the disc in a_k^*, $k = 1, \ldots, N$. In analog way, we can consider the discrete projection operator associated to the discrete scalar product denoted by $P_N^\circ f$, expressed as follows:

$$P_N^\circ f(z) = \sum_{k=1}^{N} [f, \Phi_k^\star]_{\mathbf{a},u} \Phi_k(z),$$

where the coefficients are expressed by the discrete scalar product as follows

$$[f, \Phi_k^\star]_{\mathbf{a},u} = \sum_{z_j \in \mathcal{Z}_{N,u}^{\mathbf{a}}} \frac{f(z_j)\overline{\Phi_k^\star(z_j)}}{K_N(z_j, z_j^*)}.$$

The question naturally arises, weather $P_N^\circ f$ is an interpolation operator or not. In what follows, we will study the properties of this discrete projection operator. In [19], Szabó studied the properties of $P_N^\circ f$ for $u = 1$ and $z \in \mathbb{T}$. For this special case, he proved that this projection operator is also an interpolation operator on the set of discretazation points

$$\mathcal{Z}_{N,1}^{\mathbf{a}} := \{z \in \mathbb{C} : B_N(z) = 1\},$$

which are on the the unit circle. In this paper, we will extend the results obtained by Szabó for $P_N^\circ f(z)$ when $u \in \overline{\mathbb{D}} \setminus \{0\},\ z \in \overline{\mathbb{D}}$ and the discretization point are in the closed unit disc.

Theorem 3.4. *Let $0 < |u| \leq 1$ be a parameter for which the set $\mathcal{Z}_{N,u}^{\mathbf{a}}$ has N different elements and $P_N^\circ f$ defined as before. For every $f \in C(\overline{\mathbb{D}})$, the projection operator $P_N^\circ f$ is a Lagrange type rational interpolation operator of type $(N-1, N)$ at $z_k \in \mathcal{Z}_{N,u}^{\mathbf{a}}$, i.e.,*

$$P_N^\circ f(z_k) = f(z_k), \ \ z_k \in \mathcal{Z}_{N,u}^{\mathbf{a}}.$$

Proof. Let us consider the kernel function of the discrete projection operator:

$$K_N^\circ(z,\xi) := \overline{\sum_{k=1}^{N} \overline{\Phi_k^\star(\xi)}\Phi_k(z)} = \overline{\sum_{k=1}^{N} \overline{\Phi_k(\xi^*)}\Phi_k(z)} = \overline{K_N(z,\xi^*)}. \tag{3.16}$$

The discrete projection can be expressed using $K_N^\circ(z,\xi)$ and the discrete scalar product more explicitly:

$$P_N^\circ f(z) = [f(.), K_N^\circ(z,\cdot)]_{\mathbf{a},u} = \sum_{z_j \in \mathcal{Z}_{N,u}^{\mathbf{a}}} \frac{K_N(z,z_j^*)}{K_N(z_j,z_j^*)} f(z_j).$$

Let us consider

$$\ell_{N,\xi}(z) = \frac{K_N(z,\xi^*)}{K_N(\xi,\xi^*)}.$$

From the definition, it follows that $\ell_{N,\xi}(\xi) = 1$. According to the Christoffel-Darboux formula for $z \neq \xi$, the kernel function can be written in closed form

$$K_N(z,\xi^*) = (1 - z\overline{\xi^*})^{-1}\left(1 - \overline{B_N(\xi^*)}B_N(z)\right). \tag{3.17}$$

$\ell_{N,\xi}(z)$ is a rational function in z of type $(N-1, N)$. Because of $B_N(z^*) = B_N^*(z) = 1/\overline{B_N(z)}$,

$$\ell_{N,\xi}(z) = \frac{1 - \frac{B_N^a(z)}{B_N^a(\xi)}}{K_N(\xi,\xi^*)(1 - z\overline{\xi^*})}.$$

From here and the definition of $\ell_{N,\xi}(z)$, we get that for $z_j, z_k \in \mathcal{Z}_{N,u}^{\mathbf{a}},\ z_j \neq z_k$ we have $\ell_{N,z_j}(z_k) = \delta_{jk}$, so these functions, behave like the Lagrange interpolation polynomials. Consequently, $P_N^\circ f$ has the following interpolation property

$$P_N^\circ f(z_k) = \sum_{z_j \in \mathcal{Z}_{N,u}^{\mathbf{a}}} \frac{K_N(z_k,z_j^*)}{K_N(z_j,z_j^*)} f(z_j) = f(z_k), \ \ z_k \in \mathcal{Z}_{N,u}^{\mathbf{a}}.$$

□

We consider ℓ_{N,z^*}, the dual of $\ell_{N,z}$. For these functions, we can prove the following orthogonality properties.

Theorem 3.5. *Let $0 < |u| \leq 1$ be a parameter for which the set $\mathcal{Z}_{N,u}^{\mathbf{a}}$ has N different elements. For $z_j, z_m \in \mathcal{Z}_{N,u}^{\mathbf{a}}$, the functions ℓ_{N,z^*}, and $\ell_{N,z}$ satisfy the following biorthogonality relation*

$$\frac{1}{2\pi}\int_0^{2\pi} \ell_{N,z_j}(e^{it})\overline{\ell_{N,z_m^*}(e^{it})}dt = \frac{1}{K_N(z_m,z_m^*)}\delta_{mj}.$$

In addition, $\ell_{N,z}$ satisfies the following discrete orthogonality relation:

$$[\ell_{N,z_n}, \ell_{N,z_m}]_{\mathbf{a},u} = \delta_{nm} \frac{1}{K_N^{\mathbf{a}}(z_n, z_n^*)}.$$

Proof.

$$\begin{aligned}
\langle \ell_{N,z_j}, \ell_{N,z_m^*} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} \ell_{N,z_j}(e^{it}) \overline{\ell_{N,z_m^*}(e^{it})} dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{K_N(e^{it}, z_j^*)}{K_N(z_j, z_j^*)} \overline{\left(\frac{K_N(e^{it}, z_m)}{K_N(z_m^*, z_m)} \right)} dt \\
&= \frac{1}{2\pi} \frac{1}{K_N(z_j, z_j^*) K_N(z_m, z_m^*)} \int_0^{2\pi} \sum_{k=1}^N \Phi_k(e^{it}) \overline{\Phi_k(z_j^*)} \overline{\sum_{k'=1}^N \Phi_{k'}(e^{it}) \overline{\Phi_{k'}(z_m)}} dt \\
&= \frac{1}{K_N(z_j, z_j^*) K_N(z_m, z_m^*)} \sum_{k=1}^N \sum_{k'=1}^N \frac{1}{2\pi} \int_0^{2\pi} \Phi_k(e^{it}) \overline{\Phi_k(z_j^*)} \overline{\Phi_{k'}(e^{it}) \overline{\Phi_{k'}(z_m)}} dt \\
&= \frac{1}{K_N(z_j, z_j^*) K_N(z_m, z_m^*)} \sum_{k=1}^N \sum_{k'=1}^N \overline{\Phi_k(z_j^*)} \Phi_{k'}(z_m) \frac{1}{2\pi} \int_0^{2\pi} \Phi_k(e^{it}) \overline{\Phi_{k'}(e^{it})} dt \\
&= \frac{1}{K_N(z_j, z_j^*) K_N(z_m, z_m^*)} \sum_{k=1}^N \sum_{k'=1}^N \overline{\Phi_k(z_j^*)} \Phi_{k'}(z_m) \delta_{kk'} \\
&= \frac{1}{K_N(z_j, z_j^*) K_N(z_m, z_m^*)} \sum_{k=1}^N \overline{\Phi_k(z_j^*)} \Phi_k(z_m) \\
&= \frac{1}{K_N(z_j, z_j^*) K_N(z_m, z_m^*)} K_N(z_m, z_j^*).
\end{aligned}$$

From here if $z_m = z_j$, then $\langle \ell_{N,z_m}, \ell_{N,z_m^*} \rangle = \frac{1}{K_N(z_m, z_m^*)}$. If $z_m \neq z_j$, then

$$K_N(z_m, z_j^*) = \frac{1 - B_N(z_m)\overline{B_N(z_j^*)}}{1 - z_m \overline{z_j^*}} = \frac{1 - B_N(z_m)\overline{B_N^*(z_j)}}{1 - z_m \overline{z_j^*}} = \frac{1 - u\overline{u^*}}{1 - z_m \overline{z_j^*}} = 0.$$

We get that for every $u \in \overline{\mathbb{D}} \setminus \{0\}$

$$\frac{1}{2\pi} \int_0^{2\pi} \ell_{N,z_j}(e^{it}) \overline{\ell_{N,z_m^*}(e^{it})} dt = \frac{1}{K_N(z_m, z_m^*)} \delta_{mj}.$$

If $u \in \mathbb{T}$, then $z_j \in \mathbb{T}$ for every $j = 1, \ldots, N$, consequently $\ell_{N,z_j} = \ell_{N,z_j^*}$. In this case, we have that the system $\left\{\ell_{N,z_j},\, j=1,\ldots,N\right\}$ is orthogonal. If specially $u = 1$, then we get the result of Szabó [19]. The discrete orhtogonality of the system $\left\{\ell_{N,z_j},\ j = 1, \ldots, N\right\}$ is true for every $u \in \overline{\mathbb{D}} \setminus \{0\}$. Indeed

$$\begin{aligned}
[\ell_{N,z_n}, \ell_{N,z_m}]_{\mathbf{a},u} &= \sum_{z_j \in \mathcal{Z}_{N,u}^{\mathbf{a}}} \ell_{N,z_n}(z_j) \overline{\ell_{N,z_m}(z_j)} \frac{1}{K_N^{\mathbf{a}}(z_j, z_j^*)} \\
&= \sum_{z_j \in \mathcal{Z}_{N,u}^{\mathbf{a}}} \delta_{nj} \delta_{mj} \frac{1}{K_N^{\mathbf{a}}(z_j, z_j^*)} \\
&= \delta_{nm} \frac{1}{K_N^{\mathbf{a}}(z_n, z_n^*)}.
\end{aligned}$$

□

Beside the interpolation property, $P_N f$ and $P_N^{\circ} f$ reconstruct f exactly in some cases if we measure the function in the N interpolation points. Let us denote by $\mathcal{P}_k$ the space of polynomials of degree at most k. Let us consider the polynomials of the following form: $\eta(z) = \prod_{n=1}^{N}(1 - z\overline{a}_n)$ and the set

$$\mathcal{R}_N := \left\{ \frac{p}{\eta} : \, p \in \mathcal{P}_{N-1} \right\}.$$

The system

$$\Phi_N^a = \{\Phi_n, n = 1, \ldots, N\}$$

forms an orthonormal basis in $\mathcal{R}_N$:

$$\mathcal{R}_N = \text{span}\{\Phi_l, l = 1, \ldots, N\}.$$

For every $f \in \mathcal{R}_N$, we have $f = P_N f = P_N^{\circ} f$. Indeed, if $f(z) = \sum\limits_{k=1}^{N} c_k \Phi_k(z)$, the continuous biorthogonality implies that

$$\langle f, \Phi_j^{\star} \rangle = \sum_{k=1}^{N} c_k \langle \Phi_k, \Phi_j^{\star} \rangle = \sum_{k=1}^{N} c_k \delta_{kj} = c_j,$$

from which we get that $f = P_N f$. Similarly, from discrete biorthogonality, we get

$$[f, \Phi_j^{\star}]_{\mathbf{a},u} = \sum_{k=1}^{N} c_k [\Phi_k, \Phi_j^{\star}]_{\mathbf{a},u} = \sum_{k=1}^{N} c_k \delta_{kj} = c_j,$$

which implies that $f = P_N^{\circ} f$.

3.2. **Rational interpolation based on the dual of the Mamquist-Takenaka system on the upper half-plane and discrete biorthogonality.** Recently Nagy-Csiha and Pap introduced the dual system for the Malmquist-Takenaka system on the upper half-plane. It was proved that these systems are also discrete biorthogonal with respect to the discrete inner product over a set of discratization points in closed upper half-plane (see [14]). In this subsection, we prove that on the discretisation nodes belonging to the closed upper half-plane, we can construct an interpolation operator of type $(N-1, N)$.

First, we introduce the notations and we present a short summary of the discrete biorthogonality of Malmquist-Takenaka and it's dual on the upper half-plane. We consider the isometric transform of the Malmquist-Takenaka and it's dual to the upper half-plane. With straightforward computation, it is easy to see that for

$$a_k = K(\lambda_k) = \frac{i - \lambda_k}{i + \lambda_k}, \quad \lambda_k \in \mathbb{C}_+, \quad k = 1, \ldots, \infty$$

the dual system of (2.11) is equal to

$$\widetilde{\Psi}_1^{\lambda}(z) := \frac{i + \overline{z}}{i + z} \frac{\frac{\sqrt{\Im \lambda_1}}{\sqrt{\pi}}}{\overline{z} - \overline{\lambda_1}} = \frac{i + \overline{z}}{i + z} \Psi_1^{\lambda}(\overline{z}),$$

$$\widetilde{\Psi}_n^{\lambda}(z) = \frac{i + \overline{z}}{i + z} \frac{\frac{\sqrt{\Im \lambda_n}}{\sqrt{\pi}}}{\overline{z} - \overline{\lambda_n}} \prod_{k=1}^{n-1} \frac{\overline{z} - \lambda_k}{\overline{z} - \overline{\lambda_k}} = \frac{i + \overline{z}}{i + z} \Psi_n^{\lambda}(\overline{z}).$$

For arbitrary values of the variables ω and w from $\mathbb{C}_+$ and for any N, $1 \le N < \infty$, the kernel function corresponding to the system (2.11) and its dual can be written also in closed form as follows [3]:

$$\widetilde{K}_N(\omega, w) = \sum_{k=1}^{N} \Psi_k^\lambda(\omega)\overline{\widetilde{\Psi}_k^\lambda(w)} = \overline{\left(\frac{i+\overline{w}}{i+w}\right)} \sum_{k=1}^{N} \Psi_k^\lambda(\omega)\overline{\Psi_k^\lambda(\overline{w})} = \frac{w-i}{\overline{w}-i}\frac{1-\overline{\widetilde{B}_N(\overline{w})}\widetilde{B}_N(\omega)}{2i\pi(w-\omega)},\ \omega \neq w,$$

$$\widetilde{K}_N(\omega, \omega) = \sum_{k=1}^{N} \Psi_k^\lambda(\omega)\overline{\widetilde{\Psi}_k^\lambda(\omega)} =: \frac{1}{\widetilde{\rho}_N(\omega)} = \frac{w-i}{\overline{w}-i}\sum_{k=1}^{N}\frac{\Im\lambda_k}{\pi(\omega-\lambda_k)(\omega-\overline{\lambda}_k)}.$$

For $a_k = K(\lambda_k) = \frac{i-\lambda_k}{i+\lambda_k}$, we assume that the following equation has N different solutions denoted by z_k:

$$\frac{z-a_1}{1-\overline{a}_1 z}\frac{z-a_2}{1-\overline{a}_2 z}\cdots\frac{z-a_N}{1-\overline{a}_N z} = u,\ u \in \mathbb{D}\setminus\{0\}. \tag{3.18}$$

We present the analogue of Theorem 3.3 for the upper half-plane. Let us consider t_k, where $z_k = K(t_k) = \frac{i-t_k}{i+t_k}$ is the solution of the equation (3.18), and the following set of nodes on the closed upper half-plane

$$\mathbb{C}_N = \{t_k : k = 1, \dots, N\}. \tag{3.19}$$

Let us denote by $\omega = K^{-1}(z) = i\frac{1-z}{1+z}$, $w = K^{-1}(\xi) = i\frac{1-\xi}{1+\xi}$, $a_k = K(\lambda_k) = \frac{i-\lambda_k}{i+\lambda_k}$, $z_k = K(t_k) = \frac{i-t_k}{i+t_k}$. Then

$$\overline{\left(\frac{i\frac{1-\xi}{1+\xi}-\lambda_k}{i\frac{1-\xi}{1+\xi}-\overline{\lambda_k}}\frac{|1+\lambda_k^2|}{1+\lambda_k^2}\right)}\frac{i\frac{1-z}{1+z}-\lambda_k}{i\frac{1-z}{1+z}-\overline{\lambda_k}}\frac{|1+\lambda_k^2|}{1+\lambda_k^2} = \overline{\left(\frac{\xi-a_k}{1-\overline{a_k}\xi}\right)}\frac{z-a_k}{1-\overline{a_k}z}. \tag{3.20}$$

According to (3.20) and the property $\overline{w} = K^{-1}(\xi^*)$, we get

$$\overline{\widetilde{B}_N(\overline{w})}\widetilde{B}_N(\omega) = \overline{B_N(\xi^*)}B_N(z).$$

From this and the definition of z_k, it follows that

$$\overline{\widetilde{B}_N(\overline{t}_j)}\widetilde{B}_N(t_i) = \overline{B_N(z_j^*)}B_N(z_i) = \frac{u}{u} = 1. \tag{3.21}$$

Consider the following discrete scalar product:

$$\langle F, G\rangle_N = \sum_{t\in\mathbb{C}_N} F(t)\overline{G(t)}\widetilde{\rho}_N(t).$$

Theorem 3.6 (Nagy-Csiha, Pap, [14]). *The finite collection of $\Psi_n^\lambda, (1 \le n \le N)$ and $\widetilde{\Psi}_n^\lambda, (0 \le n \le N)$ are discrete biorthogonal systems with respect to the scalar product*

$$\langle F, G\rangle_N = \sum_{t\in\mathbb{C}_N} F(t)\overline{G(t)}\widetilde{\rho}_N(t),$$

namely

$$\langle \Psi_m^\lambda, \widetilde{\Psi}_n^\lambda\rangle_N = \delta_{mn} \quad (1 \le m, n \le N).$$

For $\omega \in \mathbb{R}$, $\Psi_n^\lambda(\omega) = \widetilde{\Psi}_n^\lambda(\omega)$. If we choose in the proof of the theorem $u \in \mathbb{T}$, then the discretisation points are all real numbers, i.e., $t_k \in \mathbb{R}$, $k = 1, \dots, N$, and from Theorem 3.6, we reobtain Theorem 2.2 of Eisner and Pap [4]. For the Hardy space of the upper half-plane, it is possible to introduce similar projection operators by using the biorthogonal systems $(\Psi_n, \widetilde{\Psi}_n, n \in \mathbb{N}^*)$.

They are also biorthogonal with respect to the continuous measure on the real line. Indeed, for $t \in \mathbb{R}$, we have that $\widetilde{\Psi}_k^\lambda(t) = \Psi_k^\lambda(t)$, consequently

$$\int_{-\infty}^{\infty} \Psi_n^\lambda(t)\overline{\widetilde{\Psi}_m^\lambda(t)}dt = \int_{-\infty}^{\infty} \Psi_n^\lambda(t)\overline{\Psi_m^\lambda(t)}dt = \delta_{mn}.$$

Similarly as in the case of unit disc, we can consider the following projection operators:

$$Q_N f(t) = \sum_{k=1}^{N} \langle f, \widetilde{\Psi}_k^\lambda \rangle \Psi_k^\lambda(t), \ t \in \overline{\mathbb{C}}_+, \ f \in H_2(\mathbb{C}_+),$$

$$Q_N^\bullet f(t) = \sum_{k=1}^{N} \langle f, \Psi_k^\lambda \rangle \widetilde{\Psi}_k^\lambda(t), \ t \in \mathbb{C} \setminus \mathbb{C}_+, \ f \in H_2(\mathbb{C} \setminus \mathbb{C}_+),$$

where

$$\langle f, \widetilde{\Psi}_k^\lambda \rangle = \langle f, \Psi_k^\lambda \rangle = \int_{-\infty}^{\infty} f(t)\overline{\Psi_k^\lambda(t)}dt. \tag{3.22}$$

If $t \in \mathbb{R}$, then the two projection operators are the same, $Q_N f(t) = Q_N^\bullet f(t)$, and for $t \in \mathbb{C}_+$ we have $\frac{i+t}{i+\bar{t}} Q_N f(t) = Q_N^\bullet f(\bar{t})$. In the previous section, we saw that $Q_N f(z)$ is a rational interpolant of type $(N-1, N)$ of f in λ_k, $k = 1, \ldots, N$. With $Q_N^\bullet f(z)$, we can construct interpolation for the analytic continuation of f (if this exists) outside of the disc with nodes $\overline{\lambda_k}$, $k = 1, \ldots, N$. In the case of the upper half-plane, the discrete projection operator is the following:

$$Q_N^\circ f(t) = \sum_{k=1}^{N} \langle f, \widetilde{\Psi}_k^\lambda \rangle_N \Psi_k^\lambda(t),$$

where

$$\langle f, \widetilde{\Psi}_k^\lambda \rangle_N = \sum_{t_j \in \mathbb{C}_N}^{N} \frac{f(t_j)\overline{\widetilde{\Psi}_k^\lambda(t_j)}}{\widetilde{K}_N(t_j, t_j)}.$$

The question, whether $Q_N^\circ f$ is an interpolation operator or not, naturally arises. In what follows, we will study the properties of this discrete projection operator.

Theorem 3.7. *Assume that $\mathbb{C}_N$ defined by (3.19) has N different elements. For every $f \in C(\overline{\mathbb{C}}_+)$, the projection operator $Q_N^\circ f$ is a Lagrange type rational interpolation operator of type $(N-1, N)$ at $t_k \in \mathbb{C}_N$, i.e.,*

$$Q_N^\circ f(t_k) = f(t_k), \ \ t_k \in \mathbb{C}_N.$$

Proof. Let us consider the kernel function of the discrete projection operator:

$$\widetilde{K}_N^\circ(z, \xi) := \overline{\sum_{k=1}^{N} \overline{\widetilde{\Psi}_k^\lambda(\xi)} \Psi_k^\lambda(z)} = \overline{\widetilde{K}_N(z, \xi)}. \tag{3.23}$$

The discrete projection can be expressed also by the kernel function, i.e.,

$$Q_N^\circ f(t) = \langle f(.), \widetilde{K}_N^\circ(t, .) \rangle_N = \sum_{t_j \in \mathbb{C}_N} \frac{\widetilde{K}_N(t, t_j)}{\widetilde{K}_N(t_j, t_j)} f(t_j).$$

Let us consider

$$q_{N,\xi}(t) = \frac{\widetilde{K}_N(t, \xi)}{\widetilde{K}_N(\xi, \xi)}.$$

If $t \neq \xi$, then

$$q_{N,\xi}(t) = \frac{\xi - i}{\bar{\xi} - i}\frac{1 - \overline{\widetilde{B}_N(\bar{\xi})}\widetilde{B}_N(t)}{\widetilde{K}_N(\xi,\xi)2i\pi(\xi - t)}.$$

If $u \neq 0$, then for $t_j, t_k \in \mathbb{C}_N$, according to (3.21), we have $q_{N,t_j}(t_k) = \delta_{ik}$, so these rational functions behave like the Lagrange interpolation polynomials. Consequently, the discrete projection operator $Q_N^\circ f$ has the following interpolation property

$$Q_N^\circ f(t_k) = \sum_{t_j \in \mathbb{C}_N} \frac{\widetilde{K}_N(t_k,t_j)}{\widetilde{K}_N(t_j,t_j)} f(t_j) = f(t_k), \quad t_k \in \mathbb{C}_N.$$

$\square$

We introduce $q_{N,\bar{t}}$, the dual of $q_{N,t}$. Similarly to the disc, a biorthogonal property and a discrete orthogonality of these functions can be proved for these functions.

Theorem 3.8. *Assume that $\mathbb{C}_N$ defined by (3.19) has N different elements. For $t_j, t_m \in \mathbb{C}_N$, the functions $q_{N,\bar{t}_m}$, and ℓ_{N,t_j} satisfy the following biorthogonality relation*

$$\langle q_{N,t_j}, q_{N,\bar{t}_m}\rangle = \int_{-\infty}^{\infty} q_{N,t_j}(t)\overline{q_{N,\bar{t}_m}(t)}dt = \frac{t_m - i}{\bar{t}_m - i}\frac{1}{\widetilde{K}(t_m,t_m)}\delta_{jm}.$$

In addition, q_{N,t_n} satisfies the following discrete orthogonality relation:

$$\langle q_{N,t_n}, q_{N,t_m}\rangle_N = \delta_{nm}\frac{1}{\widetilde{K}_N(t_n,t_n)}.$$

Proof. We have

$$\begin{aligned}
\langle q_{N,t_j}, q_{N,\bar{t}_m}\rangle &= \int_{-\infty}^{\infty} q_{N,t_j}(t)\overline{q_{N,\bar{t}_m}(t)}dt = \int_{-\infty}^{\infty} \frac{\widetilde{K}_N(t,t_j)}{\widetilde{K}_N(t_j,t_j)}\frac{\overline{\widetilde{K}_N(t,\bar{t}_m)}}{\overline{\widetilde{K}_N(\bar{t}_m,\bar{t}_m)}}dt \\
&= \frac{1}{\widetilde{K}_N(t_j,t_j)\overline{\widetilde{K}_N(\bar{t}_m,\bar{t}_m)}}\int_{-\infty}^{\infty} \widetilde{K}_N(t,t_j)\overline{\widetilde{K}_N(t,\bar{t}_m)}dt \\
&= \frac{1}{\widetilde{K}_N(t_j,t_j)\overline{\widetilde{K}_N(\bar{t}_m,\bar{t}_m)}}\int_{-\infty}^{\infty}\sum_{k=0}^{N-1}\Psi_k^\lambda(t)\overline{\widetilde{\Psi}_k^\lambda(t_j)}\overline{\sum_{k'=0}^{N-1}\Psi_{k'}^\lambda(t)\overline{\widetilde{\Psi}_{k'}^\lambda(\bar{t}_m)}}dt \\
&= \frac{1}{\widetilde{K}_N(t_j,t_j)\overline{\widetilde{K}_N(\bar{t}_m,\bar{t}_m)}}\sum_{k=0}^{N-1}\sum_{k'=0}^{N-1}\overline{\widetilde{\Psi}_k^\lambda(t_j)}\widetilde{\Psi}_{k'}^\lambda(\bar{t}_m)\int_{-\infty}^{\infty}\Psi_k^\lambda(t)\overline{\Psi_{k'}^\lambda(t)}dt \\
&= \frac{1}{\widetilde{K}_N(t_j,t_j)\overline{\widetilde{K}_N(\bar{t}_m,\bar{t}_m)}}\sum_{k=0}^{N-1}\sum_{k'=0}^{N-1}\overline{\widetilde{\Psi}_k^\lambda(t_j)}\widetilde{\Psi}_{k'}^\lambda(\bar{t}_m)\delta_{kk'} \\
&= \frac{1}{\widetilde{K}_N(t_j,t_j)\overline{\widetilde{K}_N(\bar{t}_m,\bar{t}_m)}}\sum_{k=0}^{N-1}\overline{\widetilde{\Psi}_k^\lambda(t_j)}\widetilde{\Psi}_k^\lambda(\bar{t}_m) \\
&= \frac{1}{\widetilde{K}_N(t_j,t_j)\overline{\widetilde{K}_N(\bar{t}_m,\bar{t}_m)}}\frac{i+t_m}{i+\bar{t}_m}\sum_{k=0}^{N-1}\overline{\widetilde{\Psi}_k^\lambda(t_j)}\Psi_k^\lambda(t_m) \\
&= \frac{1}{\widetilde{K}_N(t_j,t_j)\overline{\widetilde{K}_N(\bar{t}_m,\bar{t}_m)}}\frac{i+t_m}{i+\bar{t}_m}\widetilde{K}_N(t_m,t_j).
\end{aligned}$$

If $t_m \neq t_j$, then if $u \neq 0$, according to (3.21),

$$\widetilde{K}_N(t_m, t_j) = \frac{t_j - i}{\bar{t}_j - i} \frac{1 - \overline{\widetilde{B}_N(\bar{t}_j)} \widetilde{B}_N(t_m)}{2i\pi(t_j - t_m)} = 0.$$

Since

$$\begin{aligned}
\overline{\widetilde{K}_N(\bar{t}_m, \bar{t}_m)} &= \overline{\sum_{k=0}^{N-1} \Psi_k^\lambda(\bar{t}_m) \overline{\widetilde{\Psi}_k^\lambda \bar{t}_m)}} \\
&= \overline{\sum_{k=0}^{N-1} \frac{i + t_m}{i + \bar{t}_m} \widetilde{\Psi}_k^\lambda(t_m) \overline{\frac{i + t_m}{i + \bar{t}_m} \Psi_j^\lambda(t_m)}} \\
&= \frac{i + t_m}{i + \bar{t}_m} \frac{\bar{t}_m - i}{t_m - i} \sum_{k=0}^{N-1} \Psi_k^\lambda(t_m) \overline{\widetilde{\Psi}_k^\lambda(t_m)} \\
&= \frac{i + t_m}{i + \bar{t}_m} \frac{\bar{t}_m - i}{t_m - i} \widetilde{K}(t_m, t_m),
\end{aligned}$$

then for every $u \in \overline{\mathbb{D}} \setminus \{0\}$, we get

$$\int_{-\infty}^{\infty} q_{N,t_j}(t) \overline{q_{N,\bar{t}_m}(t)} dt = \frac{t_m - i}{\bar{t}_m - i} \frac{1}{\widetilde{K}(t_m, t_m)} \delta_{jm}.$$

When the nodes t_m are all on the real line, then $\bar{t}_m = t_m$, we obtain that the system $\{q_{N,t_j}(t), t_j \in \mathbb{C}_N\}$ is orthogonal, and we reobtain the result proved by Eisner, Pap in [4]

$$\langle q_{N,t_n}, q_{N,t_m} \rangle_N = \sum_{t_j \in \mathbb{C}_N} \frac{\widetilde{K}_N(t_j, t_n)}{\widetilde{K}_N(t_n, t_n)} \overline{\left(\frac{\widetilde{K}_N(t_j, t_m)}{\widetilde{K}_N(t_m, t_m)} \right)} \frac{1}{\widetilde{K}_N(t_j, t_j)} = \delta_{nm} \frac{1}{\widetilde{K}_N(t_n, t_n)}.$$

□

Beside the interpolation property $Q_N f$ and $Q_N^\circ f$ reconstruct exactly f in some cases if we measure the function in the N interpolation points. If f has the form $f(t) = \sum_{k=1}^{N} c_k \Psi_k^\lambda(t)$, then the continuous biorthogonality implies that

$$\langle f, \widetilde{\Psi}_j^\lambda \rangle = \sum_{k=0}^{N-1} c_k \langle \Psi_k^\lambda, \widetilde{\Psi}_j^\lambda \rangle = \sum_{k=0}^{N-1} c_k \delta_{kj} = c_j,$$

therefore we get that $f = Q_N f$. From discrete orthogonality, we get that

$$[f, \widetilde{\Psi}_j^\lambda]_{\lambda,u} = \sum_{k=0}^{N-1} c_k [\Psi_k^\lambda, \widetilde{\Psi}_j^\lambda]_{\lambda,u} = \sum_{k=0}^{N-1} c_k \delta_{kj} = c_j,$$

which implies, that

$$f = Q_N^\circ f.$$

References

[1] J. P. Berrut, L. N. Trefethen: *Barycentric lagrange interpolation*, SIAM REVIEW, **46** (3) (2004), 501–517.

[2] M. M. Džrbašjan: *Expansions in systems of rational functions with fixed poles*, Izv. Akad. Nauk Arm. SSR, Ser. Mat., **2** (1) (1967), 3–51.

[3] M. M. Džrbašjan: *Biorthogonal Systems of Rational Functions and best Approximation of the Cauchy Kernel on the Real Axis*, Sbornik: Mathematics, **24** (3) (1974), 409–433.

[4] T. Eisner, M. Pap: *Discrete orthogonality of the Malmquist-Takenaka system of the upper half plane and rational interpolation*, J. Fourier Anal. Appl., **20** (4) (2014), 1–16.

[5] S. Fridli, P. Kovács, L. Lócsi and F. Schipp: *Rational modeling of multi-lead QRS complexes in ECG signals*, Annales Univ. Sci, Budapest., Sect. Comp., **36** (2012), 145–155.
[6] S. Fridli, L. Lócsi and F. Schipp: *Rational function systems in ECG processing*, in R. Moreno-Diaz et al. (Eds.) EUROCAST 2011 (2011), 88–95.
[7] S. Fridli, F. Schipp: *Discrete rational biorthogonal systems on the disc*, Annales Univ. Sci. Budapest., Sect. Comp., **50** (2020), 127–134.
[8] M. Gaál, B Nagy, Zs. Nagy-Csiha and Sz. Gy. Révész: *Minimal energy point system on the unit circle and the real line*, SIAM J. Math. Anal., **52** (6) (2020), 6281–6296.
[9] J. B. Garnett: *Bounded Analytic Functions*, Academic Press (1981).
[10] K. Hoffman: *Banach spaces of analytic functions*, Prentice Hall Inc. (1962).
[11] A. C. Ionita: *Lagrange rational interpolation and its applications to approximation of large-scale dynamical systems*, Rice University's digital scholarship archive (2013).
[12] G. G. Lorentz, M. Golitschek and Y. Makovoz: *Constructive Approximation: Advanced Problems*, Springer Berlin Heidelberg (1996).
[13] F. Malmquist: *Sur la détermination d'une classe functions analytiques par leurs dans un esemble donné de points*, Compute Rendus Six. Cong. math. scand. Kopenhagen, Denmark (1925), 253–259.
[14] Zs. Nagy-Csiha, M. Pap: *Discrete biorthogonal systems and equilibrium condition in the Hardy space of unit disc and upper half-plane*, Mathematical Methods for Engineering Applications. ICMASE 2021. Springer Proceedings in Mathematics and Statistics, vol 384. Springer (2022), 291–301.
[15] M. Pap: *Hyperbolic wavelets and multiresolution in $H^2(T)$*, Journal of Fourier Analysis and Applications, **17** (2011), 755–776.
[16] M. Pap, F. Schipp: *Malmquist-Takenaka systems and equilibrium conditions*, Math. Pannon., **12** (2) (2001), 185–194.
[17] M. Pap, F. Schipp: *Equilibrium conditions for the Malmquist-Takenaka systems*, Acta Sci. Math. (Szeged), **81** (3-4) (2015), 469–482.
[18] Z. Szabó: *Rational orthonormal functions and applications. PhD Thesis*, Eötvös Loránd University, Budapest (2001).
[19] Z. Szabó: *Interpolation and quadrature formulae for rational systems on the unit circle*, Annales Univ. Sci. Budapest., Sect. Comp., **21** (2004), 41–56.
[20] S. Takenaka: *On the orthogonal functions and a new formula of interpolation*, Japanese J. Math. II. (1925), 129–145.

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