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email: gcuvalcioglu@mersin.edu.tr

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THEORY OF GENERALIZED CONNECTEDNESS ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-CONNECTEDNESS) IN GENERALIZED TOPOLOGICAL SPACES ($\mathscr{T}_{\mathfrak{g}}$-SPACES)

M. I. KHODABOCUS AND N. -UL. -H. SOOKIA

0000-0003-2252-4342 and 0000-0002-3155-0473

Abstract. In this paper, the definitions of novel classes of generalized connected sets (briefly, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*connected sets*) and generalized disconnected sets (briefly, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*disconnected sets*) in generalized topological spaces (briefly, $\mathscr{T}_{\mathfrak{g}}$-*spaces*) are defined in terms of generalized sets (briefly, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*sets*) and, their properties and characterizations with respect to set-theoretic relations are presented. The basic properties and characterizations of the notions of local, pathwise, local pathwise and simple $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness are also presented. The study shows that local pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness implies local $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness, pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness implies $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness, and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is a $\mathscr{T}_{\mathfrak{g}}$-property. Diagrams establish the various relationships amongst these types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness presented here and in the literature, and a nice application supports the overall theory.

1. Introduction

Among the most important topological properties (briefly, $\mathscr{T}$-properties relative to ordinary topology, and $\mathscr{T}_{\mathfrak{g}}$-properties relative to generalized topology), the $\mathscr{T}$-properties[1] called $\mathfrak{T}$-connectedness and $\mathfrak{g}$-$\mathfrak{T}$-connectedness in $\mathscr{T}$-spaces (ordinary and generalized connectedness in ordinary topological spaces) and the $\mathscr{T}_{\mathfrak{g}}$-properties

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Key words and phrases. Generalized topological space ($\mathscr{T}_{\mathfrak{g}}$-*spaces*), generalized local connectedness (*local* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*connectedness*), generalized pathwise connectedness (*pathwise* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*connectedness*), generalized local pathwise connectedness (*local pathwise* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*connectedness*), generalized simple connectedness (*simple* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*connectedness*), generalized components ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*components*).

[1]Notes to the reader: $\mathfrak{T} = (\Omega, \mathscr{T})$, $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ are topological spaces (briefly, $\mathscr{T}$-space and $\mathscr{T}_{\mathfrak{g}}$-space) with ordinary and generalized topologies $\mathscr{T}$ and $\mathscr{T}_{\mathfrak{g}}$ (briefly, topology and $\mathfrak{g}$-topology). Subsets of $\mathfrak{T}$, $\mathfrak{T}_{\mathfrak{g}}$, respectively, are called $\mathfrak{T}$, $\mathfrak{T}_{\mathfrak{g}}$-sets; subsets of $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$, respectively, are called $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-open sets, and their complements are called $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-closed sets. Generalizations of $\mathfrak{T}$-sets, $\mathscr{T}$-open and $\mathscr{T}$-closed sets, respectively, are called $\mathfrak{g}$-$\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathscr{T}$-open and $\mathfrak{g}$-$\mathscr{T}$-closed sets; generalizations of $\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets, respectively, are called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-closed sets. Connectedness in $\mathfrak{T}$ with $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-sets are called $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-connectedness, respectively; connectedness in $\mathfrak{T}_{\mathfrak{g}}$ with $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets are called $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness, respectively.

called $\mathfrak{T}_{\mathfrak{g}}$-connectedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in $\mathscr{T}_{\mathfrak{g}}$-spaces (ordinary and generalized connectedness in generalized topological spaces) are no doubt the most important invariant properties [1, 2, 3]. Indeed, $\mathfrak{T}$-connectedness is an absolute property of a $\mathfrak{T}$-set [1, 4, 5], and $\mathfrak{g}$-$\mathfrak{T}$-connectedness, $\mathfrak{T}_{\mathfrak{g}}$-connectedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness, respectively, are absolute properties of a $\mathfrak{g}$-$\mathfrak{T}$-set, a $\mathfrak{T}_{\mathfrak{g}}$-set, and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set [3, 6, 7, 8, 9, 10, 11]. Typical examples of $\mathfrak{g}$-$\mathfrak{T}$-connectedness in $\mathscr{T}$-spaces are α, β, γ-connectedness [12, 13, 14]; examples of $\mathfrak{T}_{\mathfrak{g}}$-connectedness in $\mathscr{T}_{\mathfrak{g}}$-spaces are semi$^{*}\alpha$, s, gb-connectedness [2, 15, 16], whereas examples of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in $\mathscr{T}_{\mathfrak{g}}$-spaces are bT$^{\mu}$, μ-rgb, π p-connectedness [17, 18, 19], among others.

In the literature of $\mathscr{T}_{\mathfrak{g}}$-spaces, the study of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets by various authors has produced some new classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in $\mathscr{T}_{\mathfrak{g}}$-spaces, similar in descriptions to $\mathfrak{g}$-$\mathfrak{T}$-connectedness in $\mathscr{T}$-spaces [17, 20, 21]. By using the θ-modification generalized topology and $\gamma\theta$-operator introduced by [22], [23] have extended the notion of θ-connectedness [24] to the setting of $\mathscr{T}_{\mathfrak{g}}$-spaces and studied its $\mathscr{T}_{\mathfrak{g}}$-properties accordingly. Based on the work of [12], [20] have introduced a new type of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in $\mathscr{T}_{\mathfrak{g}}$-spaces called hyperconnected and studied the $\mathscr{T}_{\mathfrak{g}}$-properties associated with it and its analogue in the generalized sense. In the same year, [25] have introduced, studied and exemplified the notion of extremally μ-disconnected $\mathscr{T}_{\mathfrak{g}}$-spaces, just to name a few.

In view of the above references, it would appear that, from every new type of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set introduced in a $\mathscr{T}_{\mathfrak{g}}$-space, there can be introduced a new type of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in the $\mathscr{T}_{\mathfrak{g}}$-space. Having introduced a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets and studied from it some $\mathscr{T}_{\mathfrak{g}}$-properties in a $\mathscr{T}_{\mathfrak{g}}$-space [6, 7, 8, 9, 10], it seems, therefore, reasonable to introduce a new type of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in the $\mathscr{T}_{\mathfrak{g}}$-space and discuss its $\mathscr{T}_{\mathfrak{g}}$-properties. In this paper, we attempt to make a contribution to such a development by introducing a new theory, called *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Connectedness*, in which it is presented a new generalized version of $\mathfrak{T}_{\mathfrak{g}}$-connectedness in terms of the notion of $\mathfrak{g}$-$\mathfrak{T}$-set, discussing the fundamental properties and giving its characterizations on this ground.

The paper is organised as follows: In Sect. 2, preliminary notions are described in Sect. 2.1 and the main results of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in a $\mathscr{T}_{\mathfrak{g}}$-space are reported in Sect. 3. In Sect. 4, the establishment of the relationships among various types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness are discussed in Sect. 4.1. To support the work, a nice application of the concept of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in a $\mathscr{T}_{\mathfrak{g}}$-space is presented in Sect. 4.2. Finally, Sect. 5 provides concluding remarks and future directions of the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in a $\mathscr{T}_{\mathfrak{g}}$-space.

2. Theory

2.1. **Preliminaries.** Notations and notions not presented below are found in the standard references [6, 7, 8, 9, 10]. Everywhere, $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-spaces are designated by the topological structures $\mathfrak{T} \overset{\text{def}}{=} (\Omega, \mathscr{T})$ and $\mathfrak{T}_{\mathfrak{g}} \overset{\text{def}}{=} (\Omega, \mathscr{T}_{\mathfrak{g}})$, respectively, on both of which no separation axioms are assumed unless otherwise mentioned [8, 10, 26, 27, 28]. The symbols I_n^0, $I_n^* \subset \mathbb{N}^0$ designate 0-included and 0-excluded finite index sets while I_∞^0, $I_\infty^* \subseteq \mathbb{N}^0$ the corresponding infinite index sets [10]. By $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}} \subseteq \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ are meant a pair of $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets [10]. The operators $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{cl}_{\mathfrak{g}} : \mathscr{P}(\Lambda) \longrightarrow \mathscr{P}(\Lambda)$ carrying any $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ into its interior $\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and closure $\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ are called $\mathfrak{g}$-interior and $\mathfrak{g}$-closure operators [9]. The totality of all possible compositions of these $\mathfrak{g}$-operators forms the class $\mathscr{L}_{\mathfrak{g}}[\Omega] \stackrel{\text{def}}{=} \{\mathbf{op}_{\mathfrak{g},\nu\mu}(\cdot) = (\mathrm{op}_{\mathfrak{g},\nu}(\cdot), \neg\,\mathrm{op}_{\mathfrak{g},\mu}(\cdot)) : (\nu,\mu) \in I_3^0 \times I_3^0\}$ [9]. Then, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set if and only if there exist $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$ and $\mathbf{op}_{\mathfrak{g}}(\cdot) \in \mathscr{L}_{\mathfrak{g}}[\Omega]$ such that the following statement holds:

$$(\exists\xi)\big[(\xi \in \mathscr{S}_{\mathfrak{g}}) \wedge \big((\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})) \vee (\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}}))\big)\big]. \tag{2.1}$$

The derived class $\mathfrak{g}$-ν-$\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}$-$\nu$-$\mathrm{E}[\mathfrak{T}_{\mathfrak{g}}]$ is called the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of category $\nu \in I_3^0$ (briefly, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-*sets*) [9, 10]. Accordingly, the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets [10] are

$$\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{(\nu,\mathrm{E})\in I_3^0\times\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\nu\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{g}}]. \tag{2.2}$$

Notations and notions utilized in the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in $\mathscr{T}_{\mathfrak{g}}$-spaces are now presented. By $\pi : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ is meant a $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-map between $\mathscr{T}_{\mathfrak{g}}$-spaces $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$. A map $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ is called a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-map if and only if, for every $(\mathscr{O}_{\mathfrak{g},\omega}, \mathscr{K}_{\mathfrak{g},\omega}) \in \mathscr{T}_{\mathfrak{g},\Omega} \times \neg\mathscr{T}_{\mathfrak{g},\Omega}$, there exists $(\mathscr{O}_{\mathfrak{g},\sigma}, \mathscr{K}_{\mathfrak{g},\sigma}) \in \mathscr{T}_{\mathfrak{g},\Sigma} \times \neg\mathscr{T}_{\mathfrak{g},\Sigma}$ such that:

$$\big[\pi_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\omega}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\sigma})\big] \vee \big[\pi_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\omega}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\sigma})\big]. \tag{2.3}$$

It is said to be of category ν if and only if $\pi_{\mathfrak{g}} \in \mathfrak{g}$-$\nu$-$\mathrm{M}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ where,

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{M}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}} :\ &(\forall \mathscr{O}_{\mathfrak{g},\omega}, \mathscr{K}_{\mathfrak{g},\omega})(\exists \mathscr{O}_{\mathfrak{g},\sigma}, \mathscr{K}_{\mathfrak{g},\sigma}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot))\\ &\big[(\pi_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\omega}) \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},\sigma})) \vee (\pi_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\omega}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g},\sigma}))\big]\big\}.\end{aligned} \tag{2.4}$$

Let $\mathfrak{g}$-ν-$\mathrm{M}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}$-$\nu$-$\mathrm{M}_{\mathrm{E}}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ where,

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{M}_{\mathrm{O}}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] &\stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}} : (\forall \mathscr{O}_{\mathfrak{g},\omega})(\exists \mathscr{O}_{\mathfrak{g},\sigma}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot))\\ &\qquad \big[\pi_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\omega}) \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},\sigma})\big]\big\},\\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{M}_{\mathrm{K}}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] &\stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}} : (\forall \mathscr{K}_{\omega})(\exists \mathscr{K}_{\mathfrak{g},\sigma}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot))\\ &\qquad \big[\pi_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\omega}) \supseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g},\sigma})\big]\big\}.\end{aligned} \tag{2.5}$$

Then, if $\pi_{\mathfrak{g}} \in \mathfrak{g}$-$\nu$-$\mathrm{M}_{\mathrm{O}}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$, it is called a $\mathfrak{g}$-ν-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-open map; if $\pi_{\mathfrak{g}} \in \mathfrak{g}$-$\nu$-$\mathrm{M}_{\mathrm{K}}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$, it is called a $\mathfrak{g}$-ν-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-closed map. Accordingly, the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-maps [10] are

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{M}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] &= \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{M}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]\\ &= \bigcup_{(\nu,\mathrm{E})\in I_3^0\times\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\nu\text{-}\mathrm{M}_{\mathrm{E}}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]\\ &= \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\mathrm{M}_{\mathrm{E}}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}].\end{aligned} \tag{2.6}$$

A map $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ is said to be $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous if and only if, for every $(\mathscr{O}_{\mathfrak{g},\sigma}, \mathscr{K}_{\mathfrak{g},\sigma}) \in \mathscr{T}_{\mathfrak{g},\Sigma} \times \neg\mathscr{T}_{\mathfrak{g},\Sigma}$, there exists $(\mathscr{O}_{\mathfrak{g},\omega}, \mathscr{K}_{\mathfrak{g},\omega}) \in \mathscr{T}_{\mathfrak{g},\Omega} \times \neg\mathscr{T}_{\mathfrak{g},\Omega}$ such that the following statement holds:

$$\big[\pi_{\mathfrak{g}}^{-1}(\mathscr{O}_{\mathfrak{g},\sigma}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\omega})\big] \vee \big[\pi_{\mathfrak{g}}^{-1}(\mathscr{K}_{\mathfrak{g},\sigma}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\omega})\big]. \tag{2.7}$$

It is said to be of category ν if and only if $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ where,

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] \stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}}:\ & \left(\forall\mathscr{O}_{\mathfrak{g},\sigma},\mathscr{K}_{\mathfrak{g},\sigma}\right)\left(\exists\mathscr{O}_{\mathfrak{g},\omega},\mathscr{K}_{\mathfrak{g},\omega},\mathbf{op}_{\mathfrak{g},\nu}(\cdot)\right) \\ & \left[\left(\pi_{\mathfrak{g}}^{-1}\left(\mathscr{O}_{\mathfrak{g},\sigma}\right) \subseteq \mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{O}_{\mathfrak{g},\omega}\right)\right) \vee \left(\pi_{\mathfrak{g}}^{-1}\left(\mathscr{K}_{\mathfrak{g},\sigma}\right) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{K}_{\mathfrak{g},\omega}\right)\right)\right]\big\}.\end{aligned} \tag{2.8}$$

Obviously, $\mathfrak{g}\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] = \bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$. A map $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ is said to be $\mathfrak{g}\text{-}\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-irresolute if and only if, for every $\left(\mathscr{O}_{\mathfrak{g},\sigma},\mathscr{K}_{\mathfrak{g},\sigma}\right) \in \mathscr{T}_{\mathfrak{g},\Sigma} \times \neg\mathscr{T}_{\mathfrak{g},\Sigma}$, there exists $\left(\mathscr{O}_{\mathfrak{g},\omega},\mathscr{K}_{\mathfrak{g},\omega}\right) \in \mathscr{T}_{\mathfrak{g},\Omega} \times \neg\mathscr{T}_{\mathfrak{g},\Omega}$ such that the following statement holds:

$$\left[\pi_{\mathfrak{g}}^{-1}\left(\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\sigma}\right)\right) \subseteq \mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\omega}\right)\right] \vee \left[\pi_{\mathfrak{g}}^{-1}\left(\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\sigma}\right)\right) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\omega}\right)\right]. \tag{2.9}$$

It is said to be a $\mathfrak{g}\text{-}\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-irresolute map of category ν if and only if $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ where,

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] \stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}}:\ & \left(\forall\mathscr{O}_{\mathfrak{g},\sigma},\mathscr{K}_{\mathfrak{g},\sigma}\right)\left(\exists\mathscr{O}_{\mathfrak{g},\omega},\mathscr{K}_{\mathfrak{g},\omega},\mathbf{op}_{\mathfrak{g},\nu}(\cdot)\right) \\ & \left[\left(\pi_{\mathfrak{g}}^{-1}\left(\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{O}_{\mathfrak{g},\sigma}\right)\right) \subseteq \mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{O}_{\mathfrak{g},\omega}\right)\right) \vee \left(\pi_{\mathfrak{g}}^{-1}\left(\neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{K}_{\mathfrak{g},\sigma}\right)\right) \supseteq \right.\right. \\ & \left.\left. \neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{K}_{\mathfrak{g},\omega}\right)\right)\right]\big\}.\end{aligned} \tag{2.10}$$

Evidently, $\mathfrak{g}\text{-}\mathrm{I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] = \bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$. The classes $\mathrm{M}_{\mathrm{O}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ and $\mathrm{M}_{\mathrm{K}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ denote the families of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed maps, respectively, from $\mathfrak{T}_{\mathfrak{g},\Omega}$ into $\mathfrak{T}_{\mathfrak{g},\Sigma}$, with $\mathrm{M}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}}\mathrm{M}_{\mathrm{E}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$.

Definition 2.1 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Separation, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Connected)**.** A $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation of category ν of two nonempty $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is realised if and only if there exists either a pair $\left(\mathscr{U}_{\mathfrak{g},\xi},\mathscr{U}_{\mathfrak{g},\zeta}\right) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets or a pair $\left(\mathscr{V}_{\mathfrak{g},\xi},\mathscr{V}_{\mathfrak{g},\zeta}\right) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets such that:

$$\left(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{R}_{\mathfrak{g}} \sqcup \mathscr{S}_{\mathfrak{g}}\right) \bigvee \left(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{R}_{\mathfrak{g}} \sqcup \mathscr{S}_{\mathfrak{g}}\right). \tag{2.11}$$

Two nonempty $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ which are not $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated of category ν are said to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected of category ν.

The definitions of classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated sets of category ν follow.

Definition 2.2. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

– I. The $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is said to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected of category ν if and only if it belongs to the following class of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-connected sets:

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right] \stackrel{\text{def}}{=} \Big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : & \left(\forall\left(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right) \\ & \left[\neg\left(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\right) \bigwedge \neg\left(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\right)\right]\Big\}.\end{aligned} \tag{2.12}$$

– II. The $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is said to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated of category ν if and only if it belongs to the following class of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-separated sets:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g}}\right] \stackrel{\text{def}}{=} \Big\{ & \mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \left(\exists\left(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right) \\ & \left[\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big)\right]\Big\}. \end{aligned} \tag{2.13}$$

The dependence of $\mathfrak{g}$-ν-$\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathfrak{g}$-ν-$\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ on both $\mathfrak{g}$-ν-$\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathfrak{g}$-ν-$\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ is immediate. Thus, to define the pairs $\left(\nu\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right], \nu\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right)$, $\left(\mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\left[\mathfrak{T}\right], \mathfrak{g}\text{-}\nu\text{-}\mathrm{D}\left[\mathfrak{T}\right]\right)$, and $\left(\nu\text{-}\mathrm{Q}\left[\mathfrak{T}\right], \nu\text{-}\mathrm{D}\left[\mathfrak{T}\right]\right)$, respectively, it suffices to let them be dependent on the pairs $\left(\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right], \nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right)$, $\left(\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}\right], \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}\right]\right)$, and $\left(\nu\text{-}\mathrm{O}\left[\mathfrak{T}\right], \nu\text{-}\mathrm{K}\left[\mathfrak{T}\right]\right)$; the characters of these classes are found in our previous works [9, 10]. The notations $\mathfrak{g}$-$\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathfrak{g}$-$\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ stand for $\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathfrak{g}\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g}}\right] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, respectively.

Remark 2.3. In defining the classes $\mathfrak{g}$-ν-$\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathfrak{g}$-ν-$\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, it is clear that by the statement $\left(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ is meant a pair of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets. Furthermore, by $\Omega \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ or $\Omega \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ is meant a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connection of category ν or a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation of category ν of the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is realised.

Definition 2.4. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then:

– I. $\mathfrak{T}_{\mathfrak{g}}$ is called a $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}$-space $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle} \stackrel{\text{def}}{=} \left(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}\right)$ if and only if it is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected of category ν.

– II. $\mathfrak{T}_{\mathfrak{g}}$ is called a $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}$-space $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle} \stackrel{\text{def}}{=} \left(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}\right)$ if and only if it is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated of category ν.

In the sequel, by $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{LC} \rangle} \stackrel{\text{def}}{=} \left(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{LC} \rangle}\right)$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{PC} \rangle} \stackrel{\text{def}}{=} \left(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{PC} \rangle}\right)$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{LPC} \rangle} \stackrel{\text{def}}{=} \left(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{LPC} \rangle}\right)$, and $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{SC} \rangle} \stackrel{\text{def}}{=} \left(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{SC} \rangle}\right)$, respectively, are meant *locally*, *pathwise*, *locally pathwise*, and *simply* $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}$-spaces. Finally, by a $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{A} \rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{A} \rangle} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{A} \rangle}\right)$ is meant $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{A} \rangle} = \bigvee_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{A} \rangle} = \left(\Omega, \bigvee_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{A} \rangle}\right) = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{A} \rangle}\right)$, where $\mathrm{A} \in \{\mathrm{C}, \mathrm{LC}, \mathrm{PC}, \mathrm{LPC}, \mathrm{SC}, \mathrm{D}\}$.

By omitting the subscript $\mathfrak{g}$ in almost all symbols of the above descriptions, very similar descriptions are obtained but in a $\mathscr{T}$-space [10]. In the following sections, the main results of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness are presented.

3. Main Results

If for all $\left(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ neither $\mathscr{U}_{\mathfrak{g},\xi} \sqcup \mathscr{V}_{\mathfrak{g},\zeta} = \Omega$ nor $\mathscr{U}_{\mathfrak{g},\zeta} \sqcup \mathscr{V}_{\mathfrak{g},\xi} = \Omega$ is satisfied, then a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is said to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated. Hence, the following theorem:

Theorem 3.1. *If* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}\right)$ *be a* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}$*-space, then there exists a pair* $\left(\mathscr{U}_{\mathfrak{g},\xi}, \mathscr{V}_{\mathfrak{g},\zeta}\right) \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ *or a pair* $\left(\mathscr{U}_{\mathfrak{g},\zeta}, \mathscr{V}_{\mathfrak{g},\xi}\right) \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ *such that*

$$\left[\mathscr{U}_{\mathfrak{g},\xi} \sqcup \mathscr{V}_{\mathfrak{g},\zeta} = \Omega\right] \vee \left[\mathscr{U}_{\mathfrak{g},\zeta} \sqcup \mathscr{V}_{\mathfrak{g},\xi} = \Omega\right]. \tag{3.1}$$

Proof. Let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$ be a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space. Then, there exists a pair $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega\Big).$$

If $\mathscr{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ then $\mathscr{U}_{\mathfrak{g},\zeta} = \complement\left(\mathscr{U}_{\mathfrak{g},\xi}\right) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, and if $\mathscr{V}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ then $\mathscr{V}_{\mathfrak{g},\zeta} = \complement\left(\mathscr{V}_{\mathfrak{g},\xi}\right) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Thus, if $\mathscr{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it suffices to set $\mathscr{V}_{\mathfrak{g},\zeta} = \complement\left(\mathscr{U}_{\mathfrak{g},\xi}\right)$, and if $\mathscr{V}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it suffices to set $\mathscr{U}_{\mathfrak{g},\zeta} = \complement\left(\mathscr{V}_{\mathfrak{g},\xi}\right)$. By substitutions, it follows, then, that

$$\big[\mathscr{U}_{\mathfrak{g},\xi} \sqcup \mathscr{V}_{\mathfrak{g},\zeta} = \Omega\big] \vee \big[\mathscr{U}_{\mathfrak{g},\zeta} \sqcup \mathscr{V}_{\mathfrak{g},\xi} = \Omega\big],$$

which was to be proved. □

Remark 3.2. Given $\left(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\right) \subseteq \mathfrak{T}_{\mathfrak{g}}$ and $\neg\,\mathrm{op}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, the statement $\big(\mathscr{R}_{\mathfrak{g}} \cap \neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{S}_{\mathfrak{g}}\right)\big) \cup \big(\neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{R}_{\mathfrak{g}}\right) \cap \mathscr{S}_{\mathfrak{g}}\big) = \emptyset$, when $\nu = 0$, may be called the *Hausdorff-Lennes Separation Condition* in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = \left(\Omega, \mathscr{T}_{\mathfrak{g}}\right)$.

If a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = \left(\Omega, \mathscr{T}_{\mathfrak{g}}\right)$ is $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-connected, then either $\complement\left(\mathscr{U}_{\mathfrak{g},\lambda}\right) = \mathscr{U}_{\mathfrak{g},\eta}$, so that $\mathscr{U}_{\mathfrak{g},\lambda} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ or, $\complement\left(\mathscr{V}_{\mathfrak{g},\lambda}\right) = \mathscr{V}_{\mathfrak{g},\eta}$, so that $\mathscr{V}_{\mathfrak{g},\lambda} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, where $(\lambda, \eta) \in \{(\xi,\zeta), (\zeta,\xi)\}$. Therefore, $\mathfrak{T}_{\mathfrak{g}}$ is $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-connected if it has no nonempty proper $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Hence, these theorems follow:

Theorem 3.3. *If a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = \left(\Omega, \mathscr{T}_{\mathfrak{g}}\right)$ has a nonempty proper $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, then it is a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$:*

$$\exists \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \implies \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big). \tag{3.2}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be a nonempty proper $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open-closed set in $\mathfrak{T}_{\mathfrak{g}}$. Then, there exists $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\big[\mathscr{U}_{\mathfrak{g},\xi} \supseteq \mathscr{S}_{\mathfrak{g}} \supseteq \complement\left(\mathscr{U}_{\mathfrak{g},\zeta}\right)\big] \vee \big[\complement\left(\mathscr{V}_{\mathfrak{g},\xi}\right) \supseteq \mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{V}_{\mathfrak{g},\zeta}\big].$$

Consequently, the following relation holds:

$$\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} \supseteq \mathscr{S}_{\mathfrak{g}} \cup \mathscr{U}_{\mathfrak{g},\zeta} \supseteq \complement\left(\mathscr{U}_{\mathfrak{g},\zeta}\right) \cup \mathscr{U}_{\mathfrak{g},\zeta}\Big)$$
$$\bigvee \Big(\complement\left(\mathscr{V}_{\mathfrak{g},\xi}\right) \cup \mathscr{V}_{\mathfrak{g},\xi} \supseteq \mathscr{S}_{\mathfrak{g}} \cup \mathscr{V}_{\mathfrak{g},\xi} \supseteq \bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda}\Big).$$

Since $\complement\left(\mathscr{U}_{\mathfrak{g},\zeta}\right) \cup \mathscr{U}_{\mathfrak{g},\zeta} = \Omega$, $\mathscr{S}_{\mathfrak{g}} \cup \mathscr{U}_{\mathfrak{g},\zeta} = \Omega$ and, consequently, $\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega$; observe that, $\mathscr{U}_{\mathfrak{g},\xi} = \mathscr{S}_{\mathfrak{g}} = \complement\left(\mathscr{U}_{\mathfrak{g},\zeta}\right)$ because $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ is a nonempty proper $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open-closed set in $\mathfrak{T}_{\mathfrak{g}}$. Since $\complement\left(\mathscr{V}_{\mathfrak{g},\xi}\right) \sqcup \mathscr{V}_{\mathfrak{g},\xi} = \Omega$ and $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, $\complement\left(\mathscr{V}_{\mathfrak{g},\xi}\right) = \mathscr{S}_{\mathfrak{g}} = \mathscr{V}_{\mathfrak{g},\zeta}$. Therefore, $\complement\left(\mathscr{V}_{\mathfrak{g},\xi}\right) \cup \mathscr{V}_{\mathfrak{g},\xi} = \mathscr{S}_{\mathfrak{g}} \cup \mathscr{V}_{\mathfrak{g},\xi} = \bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda}$. By substitutions, it consequently follows that

$$\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega\Big),$$

which was to be proved. □

The converse of the above theorem also holds as demonstrated below.

Theorem 3.4. *If $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$ be a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space, then it has a nonempty proper $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$:*

$$\exists \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \iff \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big). \tag{3.3}$$

Proof. Let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$ be a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space. Then, there exists a pair $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega\Big).$$

But $\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega$ implies either $\mathscr{U}_{\mathfrak{g},\xi} = \complement(\mathscr{U}_{\mathfrak{g},\zeta})$ or $\mathscr{U}_{\mathfrak{g},\zeta} = \complement(\mathscr{U}_{\mathfrak{g},\xi})$, and on the other hand, $\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega$ implies either $\mathscr{V}_{\mathfrak{g},\xi} = \complement(\mathscr{V}_{\mathfrak{g},\zeta})$ or $\mathscr{V}_{\mathfrak{g},\zeta} = \complement(\mathscr{V}_{\mathfrak{g},\xi})$. Consequently, there exists a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ such that

$$\big[\mathscr{U}_{\mathfrak{g},\xi} \supseteq \mathscr{S}_{\mathfrak{g}} \supseteq \complement(\mathscr{U}_{\mathfrak{g},\zeta})\big] \vee \big[\complement(\mathscr{V}_{\mathfrak{g},\xi}) \supseteq \mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{V}_{\mathfrak{g},\zeta}\big].$$

Hence, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$ has a nonempty proper $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$; this completes the proof of the theorem. □

Combined together, the above theorems establish the necessary and sufficient conditions for a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, and hence the following corollary.

Corollary 3.5. *Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then it is a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$ if and only if $\mathfrak{T}_{\mathfrak{g}}$ has a nonempty proper $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$:*

$$\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big) \iff \exists \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]. \tag{3.4}$$

A $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation is realised if the only $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathfrak{T}_{\mathfrak{g}}$ which are both $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed sets are the improper $\mathfrak{T}_{\mathfrak{g}}$-sets $\emptyset,\, \Omega \subseteq \mathfrak{T}_{\mathfrak{g}}$. The theorem follows.

Theorem 3.6. *A $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is said to be a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$ if the only $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathfrak{T}_{\mathfrak{g}}$ which are both $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed sets are the improper $\mathfrak{T}_{\mathfrak{g}}$-sets $\emptyset,\, \Omega \subseteq \mathfrak{T}_{\mathfrak{g}}$.*

Proof. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set in $\mathfrak{T}_{\mathfrak{g}}$. Then, there exists $\big(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that $\mathscr{U}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{V}_{\mathfrak{g}}$. Consequently, $\complement(\mathscr{U}_{\mathfrak{g}}) \subseteq \complement(\mathscr{S}_{\mathfrak{g}}) \subseteq \complement(\mathscr{V}_{\mathfrak{g}})$. Since $\big(\complement(\mathscr{V}_{\mathfrak{g}}), \complement(\mathscr{U}_{\mathfrak{g}})\big) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it follows that, $\complement(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Since $\mathscr{S}_{\mathfrak{g}} \cap \complement(\mathscr{S}_{\mathfrak{g}}) = \emptyset$, implying $\mathscr{S}_{\mathfrak{g}} \cup \complement(\mathscr{S}_{\mathfrak{g}}) = \Omega$, it results, obviously, that,

$$\big[\big(\mathscr{S}_{\mathfrak{g}}, \complement(\mathscr{S}_{\mathfrak{g}})\big) \in (\emptyset, \Omega)\big] \vee \big[\big(\mathscr{S}_{\mathfrak{g}}, \complement(\mathscr{S}_{\mathfrak{g}})\big) \in (\Omega, \emptyset)\big].$$

This completes the proof of the theorem. □

The logical relationship between $\mathfrak{T}_{\mathfrak{g}}$-connectedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is contained in the following theorem.

Theorem 3.7. *If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}\big)$, then it is also a $\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-space $\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle} = \big(\Omega, \mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}\big)$:*

$$\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}\big) \implies \mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle} = \big(\Omega, \mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}\big). \tag{3.5}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$. Then it has a nonempty proper $\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{S}_{\mathfrak{g}} \in \mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \cap \mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ in $\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$. Since $\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \cap \mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right] \subseteq \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \cap \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, it follows that, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \cap \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ in $\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$. This proves that $\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$ is also a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$. In other words, if $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}\big)$, then it is also a $\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-space $\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle} = \big(\Omega, \mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}\big)$, and the proof is complete. $\square$

By virtue of the above theorem, the following corollary follows.

Corollary 3.8. *If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$, then it is also a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$:*

$$\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big) \impliedby \mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big). \tag{3.6}$$

A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-connected if and only if it is $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-connected as a $\mathscr{T}_{\mathfrak{g}}$-subspace. The theorem follows.

Theorem 3.9. *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Gamma}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$, then $\mathscr{S}_{\mathfrak{g}}$ is $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Gamma}$-connected if and only if it is $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}$-connected:*

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big] \iff \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]. \tag{3.7}$$

Proof. – *Necessity.* Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Gamma}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and suppose that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$. Then, $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$ and, hence, for all $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$,

$$\neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big),$$

But, $\mathscr{T}_{\mathfrak{g},\Gamma} \times \neg\mathscr{T}_{\mathfrak{g},\Gamma} \subseteq \mathscr{T}_{\mathfrak{g},\Omega} \times \neg\mathscr{T}_{\mathfrak{g},\Omega}$, and on the other hand, $\mathbf{op}_{\mathfrak{g},\Gamma}(\cdot) \in \mathscr{L}_{\mathfrak{g}}\big[\Gamma\big]$ implies $\mathrm{op}_{\mathfrak{g},\Gamma}(\mathscr{O}_{\mathfrak{g},\lambda}) = \Gamma \cap \mathrm{op}_{\mathfrak{g},\Omega}(\mathscr{O}_{\mathfrak{g},\lambda}) = \mathrm{op}_{\mathfrak{g},\Omega}(\mathscr{O}_{\mathfrak{g},\lambda})$ and $\neg\,\mathrm{op}_{\mathfrak{g},\Gamma}(\mathscr{K}_{\mathfrak{g},\lambda}) = \Gamma \cap \neg\,\mathrm{op}_{\mathfrak{g},\Omega}(\mathscr{K}_{\mathfrak{g},\lambda}) = \neg\,\mathrm{op}_{\mathfrak{g},\Omega}(\mathscr{K}_{\mathfrak{g},\lambda})$ for any $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g},\Gamma} \times \neg\mathscr{T}_{\mathfrak{g},\Gamma}$, where $\mathbf{op}_{\mathfrak{g},\Omega}(\cdot) \in \mathscr{L}_{\mathfrak{g}}\big[\Omega\big]$. Thus, for all $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$,

$$\neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big).$$

Consequently, $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ and, hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$.

– *Sufficiency.* Conversely, suppose that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$. This implies that $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$. Therefore, for all $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$,

$$\neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big).$$

But the statement $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \subseteq \big(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\big) \subseteq \mathfrak{T}_{\mathfrak{g},\Gamma} \times \mathfrak{T}_{\mathfrak{g},\Gamma}$ implies, evidently, $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$, since $\mathbf{op}_{\mathfrak{g},\Gamma}(\cdot) \in \mathscr{L}_{\mathfrak{g}}\big[\Gamma\big]$ is equivalent to $\mathbf{op}_{\mathfrak{g},\Gamma}(\cdot) = \Gamma \cap \mathbf{op}_{\mathfrak{g},\Omega}(\cdot) \in \mathscr{L}_{\mathfrak{g}}\big[\Omega\big]$. Consequently, for all $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in$

$\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$,

$$\neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big),$$

Therefore, $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-D}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$ and, thus, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$. □

There are some very fundamental $\mathscr{T}_{\mathfrak{g}}$-properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets which follow from the next theorem.

Theorem 3.10. *If $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set of a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$, then there exists $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that*

$$\Big(\bigvee_{\lambda=\xi,\zeta} \big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{U}_{\mathfrak{g},\lambda}\big)\Big) \bigvee \Big(\bigvee_{\lambda=\xi,\zeta} \big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{V}_{\mathfrak{g},\lambda}\big)\Big). \tag{3.8}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set in a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$. Then, for all $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\sigma,\zeta_\sigma} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$,

$$\begin{aligned}
&\neg\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big) \\
\Rightarrow\ &\neg\Big(\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda} = \emptyset\Big) \bigwedge \neg\Big(\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda} = \emptyset\Big) \\
\Rightarrow\ &\Big(\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda} \neq \emptyset\Big) \bigwedge \Big(\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda} \neq \emptyset\Big).
\end{aligned}$$

Since $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$, there exists, therefore, pairs $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\begin{aligned}
&\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega\Big) \\
\Rightarrow\quad &\Big(\bigsqcup_{\lambda=\xi,\zeta} \big(\mathscr{S}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\lambda}\big) = \mathscr{S}_{\mathfrak{g}}\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \big(\mathscr{S}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g},\lambda}\big) = \mathscr{S}_{\mathfrak{g}}\Big) \\
\Rightarrow\quad &\Big(\bigvee_{\lambda=\xi,\zeta} \big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{U}_{\mathfrak{g},\lambda}\big)\Big) \bigvee \Big(\bigvee_{\lambda=\xi,\zeta} \big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{V}_{\mathfrak{g},\lambda}\big)\Big).
\end{aligned}$$

Since $\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda}, \bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda} \neq \emptyset$ hold, and, moreover, $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$, the proof at once follows. □

Equivalently stated, the following proposition states that, any $\mathfrak{T}_{\mathfrak{g}}$-set which is contained in a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set is also a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected.

Proposition 1. Let $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be $\mathfrak{T}_{\mathfrak{g}}$-sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ and $\mathscr{R}_{\mathfrak{g}}$ satisfies

$$\big[\mathscr{R}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big] \vee \big[\mathscr{R}_{\mathfrak{g}} \subseteq \neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big], \tag{3.9}$$

then $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$.

Proof. Let $\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be $\mathfrak{T}_{\mathfrak{g}}$-sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, where $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$, and, by hypothesis, $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$. Since $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$, there exists, then, $(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda})_{\lambda=\xi_\rho,\zeta_\rho} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ such that,

$$\Big(\bigsqcup_{\lambda=\xi_\rho,\zeta_\rho} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{R}_{\mathfrak{g}}\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi_\rho,\zeta_\rho} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{R}_{\mathfrak{g}}\Big).$$

Since $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$, it must be contained in either of $\mathscr{U}_{\mathfrak{g},\xi_\rho}$, $\mathscr{U}_{\mathfrak{g},\zeta_\rho}$, or in either of $\mathscr{V}_{\mathfrak{g},\xi_\rho}$, $\mathscr{V}_{\mathfrak{g},\zeta_\rho}$. Consequently,

$$\begin{aligned} &\Big(\bigvee_{\lambda=\xi_\rho,\zeta_\rho} (\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{U}_{\mathfrak{g},\lambda})\Big) \bigvee \Big(\bigvee_{\eta=\xi_\rho,\zeta_\rho} (\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{V}_{\mathfrak{g},\eta})\Big) \\ &\Rightarrow \Big(\bigvee_{\lambda=\xi_\rho,\zeta_\rho} \big(\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{U}_{\mathfrak{g},(\rho,\lambda)}\big)\Big) \bigvee \Big(\bigvee_{\eta=\xi_\rho,\zeta_\rho} \big(\neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{V}_{\mathfrak{g},(\rho,\eta)}\big)\Big), \end{aligned}$$

where $\mathscr{U}_{\mathfrak{g},(\rho,\lambda)} = \mathrm{op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g},\lambda})$ and $\mathscr{V}_{\mathfrak{g},(\rho,\eta)} = \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g},\eta})$ for every pair $(\lambda,\eta) \in \{(\xi_\rho,\zeta_\rho),(\zeta_\rho,\xi_\rho)\}$. With no loss of generality, let it be supposed that

$$\big[\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{U}_{\mathfrak{g},(\rho,\lambda)}\big] \vee \big[\neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{V}_{\mathfrak{g},(\rho,\eta)}\big]$$

holds for a $(\lambda,\eta) \in \{(\xi_\rho,\zeta_\rho),(\zeta_\rho,\xi_\rho)\}$. Then, since the relations $\mathscr{R}_{\mathfrak{g}} = \bigsqcup_{\sigma=\lambda,\eta} \mathscr{U}_{\mathfrak{g},\sigma} \subseteq \bigsqcup_{\sigma=\lambda,\eta} \mathscr{U}_{\mathfrak{g},(\rho,\sigma)}$ and $\emptyset = \bigcap_{\sigma=\lambda,\eta} \mathscr{V}_{\mathfrak{g},\sigma} \supseteq \bigcap_{\sigma=\lambda,\eta} \mathscr{V}_{\mathfrak{g},(\rho,\sigma)}$ hold, it follows that,

$$\begin{aligned} \mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{U}_{\mathfrak{g},\eta} &\subseteq \mathscr{U}_{\mathfrak{g},(\rho,\lambda)} \cap \mathscr{U}_{\mathfrak{g},\eta} \subseteq \bigcap_{\sigma=\lambda,\eta} \mathscr{U}_{\mathfrak{g},(\rho,\sigma)} = \emptyset; \\ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{V}_{\mathfrak{g},\lambda} &\subseteq \mathscr{V}_{\mathfrak{g},(\rho,\eta)} \cap \mathscr{V}_{\mathfrak{g},\lambda} \subseteq \bigcap_{\sigma=\lambda,\eta} \mathscr{V}_{\mathfrak{g},\sigma} = \emptyset. \end{aligned}$$

Therefore, $\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{U}_{\mathfrak{g},\eta}, \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{V}_{\mathfrak{g},\lambda} = \emptyset$. On the other hand, since $\mathscr{R}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ satisfies $\big[\mathscr{R}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big] \vee \big[\mathscr{R}_{\mathfrak{g}} \subseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big]$, it results that,

$$\begin{aligned} \mathscr{U}_{\mathfrak{g},\eta} &= \mathscr{R}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\eta} = \mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{U}_{\mathfrak{g},\eta}, \\ \mathscr{V}_{\mathfrak{g},\lambda} &= \mathscr{R}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g},\lambda} = \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{U}_{\mathfrak{g},\lambda}. \end{aligned}$$

From these and $\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{U}_{\mathfrak{g},\eta}$, $\neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{V}_{\mathfrak{g},\lambda} = \emptyset$, it follows that, $\mathscr{U}_{\mathfrak{g},\eta}$, $\mathscr{V}_{\mathfrak{g},\lambda} = \emptyset$, which contradict the hypothesis that $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$. ☐

The following proposition states that, if it be given a collection of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets with non-void intersection, then $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is preserved under the operation of union.

Proposition 2. Let $\{\mathscr{S}_{\mathfrak{g},\nu} : \nu \in I_n^*\} \subseteq \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ be a collection of $n \geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\bigcap_{\nu\in I_n^*} \mathscr{S}_{\mathfrak{g},\nu} \neq \emptyset$, then $\bigcup_{\nu\in I_n^*} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$:

$$\bigcap_{\nu\in I_n^*} \big(\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]\big) \neq \emptyset \implies \bigcup_{\nu\in I_n^*} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]. \tag{3.10}$$

Proof. Let $\{\mathscr{S}_{\mathfrak{g},\nu} : \nu \in I_n^*\} \subseteq \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ be a collection of $n \geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, and suppose, by hypothesis, that $\bigcup_{\nu\in I_n^*} \mathscr{S}_{\mathfrak{g},\nu} \in$

$\mathfrak{g}$-$\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, where $\bigcap_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\neq\emptyset$. Since $\bigcup_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\in\mathfrak{g}$-$\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, there exists, then, $\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta}\in\mathfrak{g}$-$\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}$-$\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\Big(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{U}_{\mathfrak{g},\lambda}=\textstyle\bigcup_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\Big)\bigvee\Big(\displaystyle\bigsqcup_{\lambda=\xi,\zeta}\mathscr{V}_{\mathfrak{g},\lambda}=\textstyle\bigcup_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\Big).$$

Since $\bigcap_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\neq\emptyset$, there exists a unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\eta\}\subset\mathfrak{T}_{\mathfrak{g}}$ satisfying $\{\eta\}\subseteq\bigcap_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\neq\emptyset$. But, by hypothesis, $\bigcup_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\in\mathfrak{g}$-$\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Consequently,

$$\begin{aligned}&\Big(\bigvee_{\lambda=\xi,\zeta}\Big(\{\eta\}\subseteq\mathscr{U}_{\mathfrak{g},\lambda}\cap\Big(\bigcap_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\Big)\Big)\Big)\\&\bigvee\Big(\bigvee_{\lambda=\xi,\zeta}\Big(\{\eta\}\subseteq\mathscr{V}_{\mathfrak{g},\lambda}\cap\Big(\bigcap_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\Big)\Big)\Big).\end{aligned}$$

Clearly, for every $\nu\in I_n^*$,

$$\begin{aligned}&\Big(\bigvee_{\lambda=\xi,\zeta}\big(\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{U}_{\mathfrak{g},\lambda}\neq\emptyset\big)\Big)\bigvee\Big(\bigvee_{\lambda=\xi,\zeta}\big(\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{V}_{\mathfrak{g},\lambda}\neq\emptyset\big)\Big)\\&\Rightarrow\Big(\bigvee_{\lambda=\xi,\zeta}\big(\mathscr{S}_{\mathfrak{g},\nu}\subseteq\mathscr{U}_{\mathfrak{g},\lambda}\big)\Big)\bigvee\Big(\bigvee_{\lambda=\xi,\zeta}\big(\mathscr{S}_{\mathfrak{g},\nu}\subseteq\mathscr{V}_{\mathfrak{g},\lambda}\big)\Big).\end{aligned}$$

Therefore,

$$\Big(\bigvee_{\lambda=\xi,\zeta}\Big(\bigcup_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\subseteq\mathscr{U}_{\mathfrak{g},\lambda}\Big)\Big)\bigvee\Big(\bigvee_{\lambda=\xi,\zeta}\Big(\bigcup_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\subseteq\mathscr{V}_{\mathfrak{g},\lambda}\Big)\Big),$$

which contradicts the hypothesis that $\bigcup_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\in\mathfrak{g}$-$\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. □

Stated differently, the following proposition states that, if every two-point $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathfrak{T}_{\mathfrak{g}}$-set is a $\mathfrak{T}_{\mathfrak{g}}$-subset of some $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected subset of the $\mathfrak{T}_{\mathfrak{g}}$-set, then the $\mathfrak{T}_{\mathfrak{g}}$-set is also a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set.

Proposition 3. Let $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. If every two-point $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{Q}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}$ satisfies the relation $\mathscr{Q}_{\mathfrak{g}}\subseteq\mathscr{R}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}$, where $\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}$-$\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, then $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}$-$\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$:

$$\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}\supseteq\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\implies\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big].\tag{3.11}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and suppose that every two-point $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{Q}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}$ satisfies the relation $\mathscr{Q}_{\mathfrak{g}}\subseteq\mathscr{R}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}$, where $\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}$-$\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, and by hypothesis, $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}$-$\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Then, there exists a pair $\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\sigma,\zeta_\sigma}\in\mathfrak{g}$-$\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}$-$\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{U}_{\mathfrak{g},\lambda}=\mathscr{S}_{\mathfrak{g}}\Big)\bigvee\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{V}_{\mathfrak{g},\lambda}=\mathscr{S}_{\mathfrak{g}}\Big).$$

Since $\mathscr{U}_{\mathfrak{g},\lambda}$, $\mathscr{V}_{\mathfrak{g},\lambda}\neq\emptyset$ for every $\lambda\in\{\xi_\sigma,\zeta_\sigma\}$, assume that

$$\begin{aligned}\{\xi\}&=\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{U}_{\mathfrak{g},\xi_\sigma}=\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{V}_{\mathfrak{g},\xi_\sigma},\\\{\zeta\}&=\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{U}_{\mathfrak{g},\zeta_\sigma}=\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{V}_{\mathfrak{g},\zeta_\sigma},\ \mathscr{Q}_{\mathfrak{g}}=\{\xi\}\cup\{\zeta\}.\end{aligned}$$

In other words, $\mathscr{Q}_{\mathfrak{g}} \subset \times_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda}$ or $\mathscr{Q}_{\mathfrak{g}} \subset \times_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda}$. Since $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, for all $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\rho,\zeta_\rho} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$,

$$\begin{aligned}
&\neg\Big(\bigsqcup_{\lambda=\xi_\rho,\zeta_\rho} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{R}_{\mathfrak{g}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi_\rho,\zeta_\rho} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{R}_{\mathfrak{g}}\Big) \\
&\Rightarrow \neg\Big(\bigsqcup_{\lambda=\xi_\rho,\zeta_\rho} \big(\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\lambda}\big) = \mathscr{Q}_{\mathfrak{g}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi_\rho,\zeta_\rho} \big(\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g},\lambda}\big) = \mathscr{Q}_{\mathfrak{g}}\Big) \\
&\Rightarrow \neg\big(\mathscr{Q}_{\mathfrak{g}} = \mathscr{Q}_{\mathfrak{g}}\big) \wedge \neg\big(\mathscr{Q}_{\mathfrak{g}} = \mathscr{Q}_{\mathfrak{g}}\big),
\end{aligned}$$

which contradicts the hypothesis that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. □

A $\mathscr{T}_{\mathfrak{g}}$-space is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected if any two-point $\mathfrak{T}_{\mathfrak{g}}$-set can be enclosed in some $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set, and hence the following proposition.

Proposition 4. If any two-point $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{Q}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ can be enclosed in some $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\,[\mathfrak{T}_{\mathfrak{g}}]$, then the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}\big)$:

$$\big(\forall \mathscr{Q}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}\big)\big(\exists \mathscr{S}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}} \cup \mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\,[\mathfrak{T}_{\mathfrak{g}}]\big) \implies \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}\big). \tag{3.12}$$

Proof. Let $\xi \in \mathfrak{T}_{\mathfrak{g}}$ be fixed and, for every $\zeta \in \mathfrak{T}_{\mathfrak{g}}$, let $\mathscr{Q}_{\mathfrak{g}(\xi,\zeta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a two-point $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ containing ξ, $\zeta \in \mathfrak{T}_{\mathfrak{g}}$. Then, $\bigcup_{\zeta \in \mathfrak{T}_{\mathfrak{g}}} \mathscr{Q}_{\mathfrak{g}(\xi,\zeta)} \in \mathfrak{g}\text{-}\mathrm{Q}\,[\mathfrak{T}_{\mathfrak{g}}]$ and, by hypothesis, it is the entire $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Hence $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}\big)$. □

The theorem given below states that, any pair of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets which is contained in some pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated sets is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected.

Theorem 3.11. *Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. If $(\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-}\mathrm{D}\,[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{D}\,[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated sets and $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-}\mathrm{S}\,[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{S}\,[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets satisfying the statement $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \subseteq (\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta})$, then $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-}\mathrm{D}\,[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{D}\,[\mathfrak{T}_{\mathfrak{g}}]$:*

$$(\mathscr{S}_{\mathfrak{g},\lambda} = \mathscr{R}_{\mathfrak{g},\lambda} \cup \mathscr{S}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-}\mathrm{D}\,[\mathfrak{T}_{\mathfrak{g}}] \implies (\mathscr{R}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-}\mathrm{D}\,[\mathfrak{T}_{\mathfrak{g}}]\,. \tag{3.13}$$

Proof. Let $(\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-}\mathrm{D}\,[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{D}\,[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated sets and let $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-}\mathrm{S}\,[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{S}\,[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets satisfying $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \subseteq (\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta})$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, there exists $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\sigma,\zeta_\sigma} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\begin{aligned}
&\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda} = \bigsqcup_{\eta=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\eta}\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda} = \bigsqcup_{\eta=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\eta}\Big) \\
&\Leftrightarrow \Big(\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda} = \emptyset\Big) \bigvee \Big(\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda} = \emptyset\Big).
\end{aligned}$$

Since $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \subseteq (\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta})$, $\bigcap_{\eta=\alpha,\beta} \mathscr{R}_{\mathfrak{g},\eta} \subseteq \bigcap_{\eta=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\eta}$. If the relation $\bigsqcup_{\eta=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\eta} = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda}$ is satisfied, then $\bigcap_{\eta=\alpha,\beta} \mathscr{R}_{\mathfrak{g},\eta} \subseteq \bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda} = \emptyset$; if

$\bigsqcup_{\eta=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\eta}=\bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}}\mathscr{V}_{\mathfrak{g},\lambda}$, then $\bigcap_{\eta=\alpha,\beta}\mathscr{R}_{\mathfrak{g},\eta}\subseteq\bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}}\mathscr{V}_{\mathfrak{g},\lambda}=\emptyset$ holds. Hence, there exists $\left(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\xi_{\rho},\zeta_{\rho}}\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\Big(\bigsqcup_{\lambda=\xi_{\rho},\zeta_{\rho}}\mathscr{U}_{\mathfrak{g},\lambda}=\bigsqcup_{\eta=\alpha,\beta}\mathscr{R}_{\mathfrak{g},\eta}\Big)\bigvee\Big(\bigsqcup_{\lambda=\xi_{\rho},\zeta_{\rho}}\mathscr{V}_{\mathfrak{g},\lambda}=\bigsqcup_{\eta=\alpha,\beta}\mathscr{R}_{\mathfrak{g},\eta}\Big).$$

This proves the theorem. □

The basic relation between $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separateness follows:

Theorem 3.12. *In order that a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected it is necessary and sufficient that there exists no $(\mathscr{R}_{\mathfrak{g},\alpha},\mathscr{R}_{\mathfrak{g},\beta})\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ or $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\in\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that it be expressible as*

$$\Big(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{R}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Big)\bigvee\Big(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Big). \tag{3.14}$$

Proof. – *Necessity.* Let $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and let there exists $(\mathscr{R}_{\mathfrak{g},\lambda},\mathscr{S}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\Big(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{R}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Big)\bigvee\Big(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Big).$$

Since $(\mathscr{R}_{\mathfrak{g},\lambda},\mathscr{S}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, there exists $\left(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that $\left(\mathscr{R}_{\mathfrak{g},\lambda},\mathscr{S}_{\mathfrak{g},\lambda}\right)_{\lambda=\alpha,\beta}=\left(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\alpha,\beta}$. Consequently,

$$\Big(\bigsqcup_{\lambda=\alpha,\beta}\mathscr{U}_{\mathfrak{g},\lambda}=\mathscr{S}_{\mathfrak{g}}\Big)\bigvee\Big(\bigsqcup_{\lambda=\alpha,\beta}\mathscr{V}_{\mathfrak{g},\lambda}=\mathscr{S}_{\mathfrak{g}}\Big).$$

This shows that $(\mathscr{R}_{\mathfrak{g},\lambda},\mathscr{S}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Hence, $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. The condition of the theorem is, therefore, necessary.

– *Sufficiency.* Conversely, suppose that $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, there exists, then, a pair $\left(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\Big(\bigsqcup_{\lambda=\alpha,\beta}\mathscr{U}_{\mathfrak{g},\lambda}=\mathscr{S}_{\mathfrak{g}}\Big)\bigvee\Big(\bigsqcup_{\lambda=\alpha,\beta}\mathscr{V}_{\mathfrak{g},\lambda}=\mathscr{S}_{\mathfrak{g}}\Big).$$

But, $\left(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Therefore, it follows that there exists $\left(\mathscr{R}_{\mathfrak{g},\lambda},\mathscr{S}_{\mathfrak{g},\lambda}\right)_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that the relations expressible by $\left(\mathscr{R}_{\mathfrak{g},\lambda},\mathscr{S}_{\mathfrak{g},\lambda}\right)_{\lambda=\alpha,\beta}=\left(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\alpha,\beta}$ hold. Hence, the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ is expressible as

$$\Big(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{R}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Big)\bigvee\Big(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Big).$$

The condition of the theorem is, therefore, sufficient. □

If $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, then $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets in $\mathscr{S}_{\mathfrak{g}}$ are clearly also in $\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, and conversely. Likewise, if $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, then $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets in $\mathscr{S}_{\mathfrak{g}}$ are clearly also in $\in\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, and conversely. Hence, an immediate consequence of the above theorem is the following corollary:

Corollary 3.13. *Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:*

– I. *If $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, then in order that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it is necessary and sufficient that there exists no $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that it be expressible as $\mathscr{S}_{\mathfrak{g}} = \bigsqcup_{\lambda=\alpha,\beta} \mathscr{R}_{\mathfrak{g},\lambda}$.*

– II. *If $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, then in order that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it is necessary and sufficient that there exists no $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that it be expressible as $\mathscr{S}_{\mathfrak{g}} = \bigsqcup_{\lambda=\alpha,\beta} \mathscr{R}_{\mathfrak{g},\lambda}$.*

The following remark contains classifications of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness with respect to openness and closedness.

Remark 3.14. Suppose $\bigsqcup_{\sigma=\alpha,\beta} \mathscr{R}_{\mathfrak{g},\sigma} = \mathscr{S}_{\mathfrak{g}}$ hold, then it is no error to call $\mathscr{S}_{\mathfrak{g}}$ a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*connected open set* if $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*connected closed set* if $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$.

From the above corollary, it would appear that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness depends on the existence of certain $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated sets or, equivalently, on the existence of certain disjoint $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open, closed sets. As another simple ways of characterizing $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness, the proposition follows.

Proposition 5. A $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}\big)$ if and only if any one of the following statements holds:

– I. $\exists\, (\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \in \big(\mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2 : \quad \bigsqcup_{\lambda=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\lambda} = \Omega;$

– II. $\exists\, (\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \in \big(\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2 : \quad \bigsqcup_{\lambda=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\lambda} = \Omega;$

– III. $\exists\, (\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \in \big(\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2 : \quad \bigsqcup_{\lambda=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\lambda} = \Omega.$

Proof. – *Necessity.* Let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}\big)$ be a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}$-space. Then, there exists $\mathscr{S}_{\mathfrak{g}} \in \big(\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big) \setminus \{\emptyset, \Omega\}$. Consequently, there exists $\big(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\mathscr{U}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{V}_{\mathfrak{g}} \;\Rightarrow\; \complement(\mathscr{U}_{\mathfrak{g}}) \subseteq \complement(\mathscr{S}_{\mathfrak{g}}) \subseteq \complement(\mathscr{V}_{\mathfrak{g}}).$$

Therefore, $\complement(\mathscr{S}_{\mathfrak{g}}) \in \big(\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big) \setminus \{\emptyset, \Omega\}$. Hence,

$$\big(\mathscr{S}_{\mathfrak{g}}, \complement(\mathscr{S}_{\mathfrak{g}})\big) \in \big(\mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2 \cup \big(\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2 \cup \big(\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2.$$

– *Sufficiency.* Conversely, suppose that

$$\exists\, (\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \in \big(\mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2 \cup \big(\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2 \cup \big(\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2,$$

such that $\Omega = \bigsqcup_{\lambda=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\lambda}$. Then if $(\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \in \big(\mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2$, there exists $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that,

$$\begin{aligned} &\Big(\bigsqcup_{\lambda=\alpha,\beta} \mathscr{U}_{\mathfrak{g},\lambda} = \bigsqcup_{\lambda=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\lambda}\Big) \bigvee \Big(\bigsqcup_{\lambda=\alpha,\beta} \mathscr{V}_{\mathfrak{g},\lambda} = \bigsqcup_{\lambda=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\lambda}\Big) \\ \Rightarrow \quad &\Big(\bigsqcup_{\lambda=\alpha,\beta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega\Big) \bigvee \Big(\bigsqcup_{\lambda=\alpha,\beta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega\Big). \end{aligned}$$

If $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\in\big(\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\setminus\{\emptyset\}\big)^2$, there exists $\big(\mathscr{U}_{\mathfrak{g},\alpha},\mathscr{U}_{\mathfrak{g},\beta}\big)\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that the statement $\big(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta}\big)\subseteq\big(\mathscr{U}_{\mathfrak{g},\alpha},\mathscr{U}_{\mathfrak{g},\beta}\big)$ holds. Consequently, it follows that $\Omega=\bigsqcup_{\lambda=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\lambda}\subseteq\bigsqcup_{\lambda=\alpha,\beta}\mathscr{U}_{\mathfrak{g},\lambda}$. Hence, $\bigsqcup_{\lambda=\alpha,\beta}\mathscr{U}_{\mathfrak{g},\lambda}=\Omega$.

If $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\in\big(\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\setminus\{\emptyset\}\big)^2$, then there exists $\big(\mathscr{V}_{\mathfrak{g},\alpha},\mathscr{V}_{\mathfrak{g},\beta}\big)\in\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that the statement $\big(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta}\big)\supseteq\big(\mathscr{V}_{\mathfrak{g},\alpha},\mathscr{V}_{\mathfrak{g},\beta}\big)$ holds. Thus, it results that $\emptyset=\bigcap_{\lambda=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\lambda}\supseteq\bigcap_{\lambda=\alpha,\beta}\mathscr{V}_{\mathfrak{g},\lambda}$. Hence, $\bigsqcup_{\lambda=\alpha,\beta}\mathscr{V}_{\mathfrak{g},\lambda}=\Omega$. These complete the proof of the proposition. □

The following lemma is a useful tool for the proof of the theorem following it.

Lemma 3.15. *Let $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space, and let $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets in $\mathfrak{T}_{\mathfrak{g}}$. If there exists a unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\xi\}\subset\mathfrak{T}_{\mathfrak{g}}$ such that $\bigcap_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}=\{\xi\}$, then $\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ in $\mathfrak{T}_{\mathfrak{g}}$:*

$$\exists\{\xi\}=\bigcap_{\sigma=\alpha,\beta}\big(\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big)\ \Rightarrow\ \bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big].\tag{3.15}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space, let $\mathscr{S}_{\mathfrak{g}}=\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}$, and suppose that there exists a unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\xi\}\subset\mathfrak{T}_{\mathfrak{g}}$ such that $\{\xi\}=\bigcap_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}$, where $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, and assume that

$$\Bigg(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{U}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Bigg)\bigvee\Bigg(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{V}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Bigg),$$

for some $\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Since $\{\xi\}\subseteq\mathscr{S}_{\mathfrak{g}}$,

$$\Bigg(\bigvee_{\sigma=\alpha,\beta}\big(\{\xi\}\subseteq\mathscr{U}_{\mathfrak{g},\sigma}\big)\Bigg)\bigvee\Bigg(\bigvee_{\sigma=\alpha,\beta}\big(\{\xi\}\subseteq\mathscr{V}_{\mathfrak{g},\sigma}\big)\Bigg),$$

meaning that, with respect to $\big(\mathscr{U}_{\mathfrak{g},\alpha},\mathscr{U}_{\mathfrak{g},\beta}\big)\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, either $\xi\in\mathscr{U}_{\mathfrak{g},\alpha}$ or $\xi\in\mathscr{U}_{\mathfrak{g},\beta}$; with respect to $\big(\mathscr{V}_{\mathfrak{g},\alpha},\mathscr{V}_{\mathfrak{g},\beta}\big)\in\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, either $\xi\in\mathscr{V}_{\mathfrak{g},\alpha}$ or $\xi\in\mathscr{V}_{\mathfrak{g},\beta}$. Therefore, set

$$\big(\{\xi\}\subseteq\mathscr{U}_{\mathfrak{g},\alpha}\big)\vee\big(\{\xi\}\subseteq\mathscr{V}_{\mathfrak{g},\alpha}\big).$$

Clearly, $\mathscr{U}_{\mathfrak{g},\beta},\ \mathscr{V}_{\mathfrak{g},\beta}\neq\emptyset$; $\mathscr{U}_{\mathfrak{g},\beta}\subseteq\bigsqcup_{\sigma=\alpha,\beta}\mathscr{U}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}$ and, $\mathscr{V}_{\mathfrak{g},\beta}\subseteq\bigsqcup_{\sigma=\alpha,\beta}\mathscr{V}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}$. Therefore, for at least a $\sigma\in\{\alpha,\beta\}$,

$$\big(\mathscr{U}_{\mathfrak{g},\beta}\cap\mathscr{S}_{\mathfrak{g},\sigma}\neq\emptyset\big)\vee\big(\mathscr{V}_{\mathfrak{g},\beta}\cap\mathscr{S}_{\mathfrak{g},\sigma}\neq\emptyset\big).$$

Choose a $\eta\in\{\alpha,\beta\}$. Then, for every $\sigma\in\{\alpha,\beta\}$,

$$\big(\mathscr{U}_{\mathfrak{g},\sigma}\cap\mathscr{S}_{\mathfrak{g},\eta}\subseteq\mathscr{U}_{\mathfrak{g},\sigma}\big)\vee\big(\mathscr{V}_{\mathfrak{g},\sigma}\cap\mathscr{S}_{\mathfrak{g},\eta}\subseteq\mathscr{V}_{\mathfrak{g},\sigma}\big).$$

Therefore, with respect to $\big(\mathscr{U}_{\mathfrak{g},\alpha},\mathscr{U}_{\mathfrak{g},\beta}\big)\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, $\mathscr{U}_{\mathfrak{g},\alpha}\cap\mathscr{S}_{\mathfrak{g},\eta}$ and $\mathscr{U}_{\mathfrak{g},\beta}\cap\mathscr{S}_{\mathfrak{g},\eta}$ are $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated sets; with respect to $\big(\mathscr{V}_{\mathfrak{g},\alpha},\mathscr{V}_{\mathfrak{g},\beta}\big)\in\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, $\mathscr{V}_{\mathfrak{g},\alpha}\cap\mathscr{S}_{\mathfrak{g},\eta}$ and $\mathscr{V}_{\mathfrak{g},\beta}\cap\mathscr{S}_{\mathfrak{g},\eta}$ are also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated sets. Consequently,

$$\Bigg(\mathscr{S}_{\mathfrak{g},\eta}\cap\Bigg(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{U}_{\mathfrak{g},\sigma}\Bigg)=\mathscr{S}_{\mathfrak{g},\eta}\Bigg)\bigvee\Bigg(\mathscr{S}_{\mathfrak{g},\eta}\cap\Bigg(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{V}_{\mathfrak{g},\sigma}\Bigg)=\mathscr{S}_{\mathfrak{g},\eta}\Bigg)$$

$$\Rightarrow\Bigg(\bigsqcup_{\sigma=\alpha,\beta}\big(\mathscr{U}_{\mathfrak{g},\sigma}\cap\mathscr{S}_{\mathfrak{g},\eta}\big)=\mathscr{S}_{\mathfrak{g},\eta}\Bigg)\bigvee\Bigg(\bigsqcup_{\sigma=\alpha,\beta}\big(\mathscr{V}_{\mathfrak{g},\sigma}\cap\mathscr{S}_{\mathfrak{g},\eta}\big)=\mathscr{S}_{\mathfrak{g},\eta}\Bigg).$$

Therefore, $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\notin\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, contrary to hypothesis. Hence, it follows that $\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}$ must be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g}}$, that is $\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. □

For the case of $n\geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets, the theorem follows.

Theorem 3.16. *Let $\mathscr{S}_{\mathfrak{g},1}$, $\mathscr{S}_{\mathfrak{g},2}$, $\ldots$, $\mathscr{S}_{\mathfrak{g},n}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be $n\geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets in $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. If, for every $(\alpha,\beta)\in I_n^*\times I_n^*$, there exists a unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\xi\}\subset\mathfrak{T}_{\mathfrak{g}}$ such that $\bigcap_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}=\{\xi\}$, then $\bigcup_{\sigma\in I_n^*}\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ in $\mathfrak{T}_{\mathfrak{g}}$:*

$$\exists\,\{\xi\}=\bigcap_{\sigma\in\{\alpha,\beta\}\subseteq I_n^*\times I_n^*}\big(\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big)\implies\bigcup_{\sigma\in I_n^*}\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big].\tag{3.16}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space, let $\mathscr{S}_{\mathfrak{g}}=\bigcup_{\sigma\in I_n^*}\mathscr{S}_{\mathfrak{g},\sigma}$, and suppose that, for every $(\alpha,\beta)\in I_n^*\times I_n^*$, there exists a unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\xi\}\subset\mathfrak{T}_{\mathfrak{g}}$ such that $\bigcap_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}=\{\xi\}$, where $\mathscr{S}_{\mathfrak{g},1}$, $\mathscr{S}_{\mathfrak{g},2}$, $\ldots$, $\mathscr{S}_{\mathfrak{g},n}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ are $n\geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets in $\mathfrak{T}_{\mathfrak{g}}$. If $(\xi_\alpha,\xi_\beta)\in\mathscr{S}_{\mathfrak{g}}\times\mathscr{S}_{\mathfrak{g}}$ be any pair of elements of $\mathscr{S}_{\mathfrak{g}}$, then there is a pair $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets such that $(\xi_\alpha,\xi_\beta)\in(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})$. Set $\mathscr{Q}_{\mathfrak{g},(\alpha,\beta)}=\{\xi_\alpha,\xi_\beta\}$ and $\mathscr{R}_{\mathfrak{g},(\alpha,\beta)}=\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}$; clearly, $\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\neq\emptyset$ by hypothesis. Then, for every $(\alpha,\beta)\in I_n^*\times I_n^*$, the relation $\mathscr{Q}_{\mathfrak{g},(\alpha,\beta)}\subseteq\mathscr{R}_{\mathfrak{g},(\alpha,\beta)}\subseteq\mathscr{S}_{\mathfrak{g}}$ holds. Since, for every $(\alpha,\beta)\in I_n^*\times I_n^*$, there exists a unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\xi\}\subset\mathfrak{T}_{\mathfrak{g}}$ such that $\bigcap_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}=\{\xi\}$, it follows that $\mathscr{R}_{\mathfrak{g},(\alpha,\beta)}=\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ in $\mathfrak{T}_{\mathfrak{g}}$. Since, for every $(\alpha,\beta)\in I_n^*\times I_n^*$, $\mathscr{Q}_{\mathfrak{g},(\alpha,\beta)}\subseteq\mathscr{S}_{\mathfrak{g}}$ is a two-point $\mathfrak{T}_{\mathfrak{g}}$-set satisfying the relation $\mathscr{Q}_{\mathfrak{g},(\alpha,\beta)}\subseteq\mathscr{R}_{\mathfrak{g},(\alpha,\beta)}\subseteq\mathscr{S}_{\mathfrak{g}}$, where $\mathscr{R}_{\mathfrak{g},(\alpha,\beta)}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ in $\mathfrak{T}_{\mathfrak{g}}$, it follows that $\mathscr{S}_{\mathfrak{g}}=\bigcup_{\sigma\in I_n^*}\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ in $\mathfrak{T}_{\mathfrak{g}}$. This proves the theorem. □

When a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated, it is natural that we should attempt to obtain some information about the various $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets into which it can be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated. The maximal $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets of the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ are particularly interesting.

Definition 3.17. If $\zeta\in\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ is a point of a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, then

$$\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\overset{\mathrm{def}}{=}\big\{\xi\in\mathscr{S}_{\mathfrak{g}}:\ \big(\exists\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big)\big[(\xi,\zeta)\in\mathscr{R}_{\mathfrak{g}}^2\subseteq\mathscr{S}_{\mathfrak{g}}^2\big]\big\}\tag{3.17}$$

is called the "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component of $\mathscr{S}_{\mathfrak{g}}$ corresponding to ζ."

According to this definition, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component is nonempty, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, and is not a proper $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set of any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set of a $\mathscr{T}_{\mathfrak{g}}$-space. The theorem follows.

Theorem 3.18. *For each point $\zeta\in\mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ of $\mathscr{S}_{\mathfrak{g}}$ corresponding to ζ is the largest $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set in $\mathfrak{T}_{\mathfrak{g}}$ which contains the point ζ:*

$$\big(\forall\zeta\in\mathfrak{T}_{\mathfrak{g}}\big)\big(\nexists\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big)\big[\mathscr{R}_{\mathfrak{g}}\supset\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\big].\tag{3.18}$$

Proof. Let $(\xi,\zeta)\in\big(\mathscr{R}_{\mathfrak{g},\zeta}\setminus\{\zeta\}\big)\times\mathscr{R}_{\mathfrak{g},\zeta}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, where $\mathscr{R}_{\mathfrak{g},\zeta}\in\mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ is any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set which contains $\zeta\in\mathfrak{T}_{\mathfrak{g}}$, and $\xi\in\mathscr{R}_{\mathfrak{g},\zeta}$. Clearly, $\mathscr{Q}_{\mathfrak{g},(\xi,\zeta)}=\{\xi,\zeta\}\subseteq\mathscr{R}_{\mathfrak{g},\zeta}$ and, therefore, $\xi\in\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$, implying $\mathscr{R}_{\mathfrak{g},\zeta}\subseteq\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$. To prove the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness of $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$, consider an arbitrary point $\eta\in$

$\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$. Since $\mathscr{Q}_{\mathfrak{g},(\eta,\zeta)} = \{\eta,\zeta\} \subseteq \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{R}_{\mathfrak{g},\eta} \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\mathscr{Q}_{\mathfrak{g},(\eta,\zeta)} = \{\eta,\zeta\} \subseteq \mathscr{R}_{\mathfrak{g},\eta}$. Therefore, $\mathscr{R}_{\mathfrak{g},\eta} \subseteq \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ and, consequently, $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] = \bigcup_{\eta\in\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]} \mathscr{R}_{\mathfrak{g},\eta}$. But, this is the union of the collection $\{\mathscr{R}_{\mathfrak{g},\eta} :\ \eta \in \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\} \subseteq \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets with a common point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$. Hence, $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$. □

In a $\mathscr{T}_{\mathfrak{g}}$-space, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-components are $\mathfrak{g}$-$\mathfrak{T}$-closed sets, as demonstrated in the following theorem.

Theorem 3.19. *For each point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \subset \mathfrak{T}_{\mathfrak{g}}$ of $\mathscr{S}_{\mathfrak{g}}$ corresponding to ζ is a $\mathfrak{g}$-$\mathfrak{T}$-closed set of $\mathfrak{T}_{\mathfrak{g}}$:*

$$(\forall\zeta \in \mathfrak{T}_{\mathfrak{g}})\big[\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]\big]. \tag{3.19}$$

Proof. Let $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \subset \mathfrak{T}_{\mathfrak{g}}$ be the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component of $\mathscr{S}_{\mathfrak{g}}$ corresponding to $\zeta \in \mathfrak{T}_{\mathfrak{g}}$. Then, $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ is the largest $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set in $\mathfrak{T}_{\mathfrak{g}}$ containing the point ζ. Suppose that $\xi \in \neg\,\mathrm{op}_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\big)$. Since $\neg\,\mathrm{op}_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\big) \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set, and $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ is the largest $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set in $\mathfrak{T}_{\mathfrak{g}}$ which contains the point ζ, $\mathscr{Q}_{\mathfrak{g},(\xi,\zeta)} = \{\xi,\zeta\} \subseteq \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$. Hence, $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\big)$, meaning that $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ must be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set in $\mathfrak{T}_{\mathfrak{g}}$. This proves the theorem. □

A central fact about the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-components of a $\mathscr{T}_{\mathfrak{g}}$-space is that, to each point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ there corresponds a unique $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ of $\mathscr{S}_{\mathfrak{g}}$. This fact is contained in the following theorem.

Theorem 3.20. *The class $\{\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] :\ \zeta \in \mathfrak{T}_{\mathfrak{g}}\}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-components of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ forms a partition of $\mathfrak{T}_{\mathfrak{g}}$:*

$$\{\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] :\ \zeta \in \mathfrak{T}_{\mathfrak{g}}\} \implies \bigsqcup_{\zeta\in\mathfrak{T}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] = \Omega. \tag{3.20}$$

Proof. Let $\{\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] :\ \zeta \in \mathfrak{T}_{\mathfrak{g}}\}$ be the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-components of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Clearly, $\Omega = \bigcup_{\zeta\in\mathfrak{T}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$. Let $\eta \in \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \cap \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\xi]$. Then, since $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta], \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\xi] \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ and contain the point $\eta \in \mathfrak{T}_{\mathfrak{g}}$, it follows that, $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\eta] \supseteq \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ and $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\eta] \supseteq \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\xi]$. But $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta], \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\xi]$ are $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-components and, hence, $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] = \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\eta] = \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\xi]$. This shows that distinct $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-components are disjoint or, equivalently, $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \cap \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\xi] \neq \emptyset$ implies $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] = \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\xi]$. □

By virtue of this theorem, it thus follows that, each $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ belongs to a unique $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ of $\mathscr{S}_{\mathfrak{g}}$. The corollary follows.

Corollary 3.21. *For each point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, there corresponds a unique $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ of $\mathscr{S}_{\mathfrak{g}}$ containing it:*

$$(\forall\zeta \in \mathfrak{T}_{\mathfrak{g}})\big(\exists!\,\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\zeta \in \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\big]. \tag{3.21}$$

A $\mathscr{T}_{\mathfrak{g}}$-space that is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected has at most one $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component, as demonstrated in the following proposition.

Proposition 6. If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}\big)$, then it has at most one $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}$-$\mathrm{C}_{\Omega}[\zeta] = \Omega$:

$$\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}\big) \iff \exists!\, \mathfrak{g}\text{-}\mathrm{C}_{\Omega}[\zeta] = \Omega. \tag{3.22}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}\big)$, and let it be supposed that it has $\alpha \in I_{\infty}^{*}$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-components $\mathfrak{g}$-$\mathrm{C}_{\Omega}[\zeta_1]$, $\mathfrak{g}$-$\mathrm{C}_{\Omega}[\zeta_2]$, $\ldots$, $\mathfrak{g}$-$\mathrm{C}_{\Omega}[\zeta_\alpha]$. Then, $\bigsqcup_{\mu \in I_{\alpha}^{*}} \mathfrak{g}\text{-}\mathrm{C}_{\Omega}[\zeta_\mu] = \Omega$ because $\bigcap_{\mu \in I_{\alpha}^{*}} \mathfrak{g}\text{-}\mathrm{C}_{\Omega}[\zeta_\mu] = \emptyset$. Hence, $\mathfrak{T}_{\mathfrak{g}}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated, which contradicts the fact that $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}\big)$. $\square$

The combination of an additional concept called *path* with the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness will bring forth a further refinement of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness called *pathwise* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-connectedness.*

Definition 3.22. A path from an initial point $\xi \in \mathfrak{T}_{\mathfrak{g}}$ to a terminal point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $([0,1], \mathfrak{T}_{\mathfrak{g}})$-continuous map $\varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ with $\big(\varphi_{\mathfrak{g},\zeta}(0), \varphi_{\mathfrak{g},\zeta}(1)\big) = (\xi, \zeta)$. A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is said to be pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected if and only if, for every $(\xi, \zeta) \in \mathscr{S}_{\mathfrak{g}} \times \mathscr{S}_{\mathfrak{g}}$,

$$\big(\exists \mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big)\big(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}\big)\big[\mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{Q}_{\mathfrak{g}} \supseteq \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big)\big]. \tag{3.23}$$

Evidently, $\mathrm{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big)$ signifies the *image* of the $([0,1], \mathfrak{T}_{\mathfrak{g}})$-continuous map $\varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ from the initial point $\xi = \varphi_{\mathfrak{g},\zeta}(0)$ to the terminal point $\zeta = \varphi_{\mathfrak{g},\zeta}(1)$. The following theorem is an immediate consequence of the above definition.

Theorem 3.23. *A subset $\Gamma \subseteq \Omega$ of Ω of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$, with the absolute $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g},\Omega} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, is said to be pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected if and only if, with the relative $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g},\Gamma} : \mathscr{P}(\Gamma) \longmapsto \mathscr{T}_{\mathfrak{g},\Gamma} = \big\{\mathscr{O}_{\mathfrak{g}} \cap \Gamma :\ \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\Omega}\big\}$, the $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected.*

Proof. – *Necessity.* Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ be a $\mathscr{T}_{\mathfrak{g}}$-space, and suppose that a subset $\Gamma \subseteq \Omega$ of Ω, with the absolute $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g},\Omega} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Omega}$. Then, for every $(\xi, \zeta) \in \Gamma \times \Gamma \subseteq \Omega \times \Omega$,

$$\big(\exists \mathscr{Q}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]\big)\big(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g},\Omega}\big)\big[\Gamma \supseteq \mathscr{Q}_{\mathfrak{g},\omega} \supseteq \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big)\big].$$

Since $\mathscr{Q}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ and $\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big] \supseteq \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$, it follows that $\mathscr{Q}_{\mathfrak{g},\gamma} = \mathscr{Q}_{\mathfrak{g},\omega} \cap \Gamma$ for every $\mathscr{Q}_{\mathfrak{g},\gamma} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$. Since $\big\{\varphi_{\mathfrak{g},\zeta}(0), \varphi_{\mathfrak{g},\zeta}(1)\big\} \subset \Gamma \times \Gamma$, it also follows that $\Gamma \supseteq \mathscr{Q}_{\mathfrak{g},\gamma} \supseteq \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big)$. Therefore, for every $(\xi, \zeta) \in \Gamma \times \Gamma$,

$$\big(\exists \mathscr{Q}_{\mathfrak{g},\gamma} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]\big)\big(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g},\Gamma}\big)\big[\Gamma \supseteq \mathscr{Q}_{\mathfrak{g},\gamma} \supseteq \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big)\big].$$

Hence, with the relative $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g},\Gamma} : \mathscr{P}(\Gamma) \longmapsto \mathscr{T}_{\mathfrak{g},\Gamma} = \big\{\mathscr{O}_{\mathfrak{g}} \cap \Gamma :\ \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\Omega}\big\}$, the $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected.

– *Sufficiency.* Conversely, suppose that, with the relative $\mathfrak{g}$-topology given by $\mathscr{T}_{\mathfrak{g},\Gamma} : \mathscr{P}(\Gamma) \longmapsto \mathscr{T}_{\mathfrak{g},\Gamma} = \big\{\mathscr{O}_{\mathfrak{g}} \cap \Gamma :\ \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\Omega}\big\}$, the $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. Then, for every $(\xi, \zeta) \in \Gamma \times \Gamma$,

$$\big(\exists \mathscr{Q}_{\mathfrak{g},\gamma} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]\big)\big(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g},\Gamma}\big)\big[\Gamma \supseteq \mathscr{Q}_{\mathfrak{g},\gamma} \supseteq \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big)\big].$$

Since $\mathscr{Q}_{\mathfrak{g},\gamma} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$ and $\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big] \subseteq \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$, it follows that a $\mathscr{Q}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ exists such that $\mathscr{Q}_{\mathfrak{g},\omega} \cap \Gamma = \mathscr{Q}_{\mathfrak{g},\gamma}$. Furthermore, since $\mathscr{Q}_{\mathfrak{g},\omega} \subseteq \Gamma$ and

$\mathscr{Q}_{\mathfrak{g},\omega} \supseteq \mathscr{Q}_{\mathfrak{g},\gamma}$, it follows that $\Gamma \supseteq \mathscr{Q}_{\mathfrak{g},\omega} \supseteq \operatorname{im}\bigl(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\bigr)$. Therefore, for every $(\xi,\zeta) \in \Gamma\times\Gamma \subseteq \Omega\times\Omega$,

$$\bigl(\exists \mathscr{Q}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-}\mathrm{Q}\bigl[\mathfrak{T}_{\mathfrak{g},\Omega}\bigr]\bigr)\bigl(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g},\Omega}\bigr)\bigl[\Gamma \supseteq \mathscr{Q}_{\mathfrak{g},\omega} \supseteq \operatorname{im}\bigl(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\bigr)\bigr].$$

Hence, the subset $\Gamma \subseteq \Omega$, with the absolute $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g},\Omega} : \mathscr{P}(\Omega) \to \mathscr{P}(\Omega)$, is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. ☐

The relationship between the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness and pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness follows.

Theorem 3.24. *Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\mathscr{S}_{\mathfrak{g}}$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, then $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\bigl[\mathfrak{T}_{\mathfrak{g}}\bigr]$.*

Proof. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be an arbitrary pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\mathscr{S}_{\mathfrak{g}} = \emptyset$, then $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-}\mathrm{D}\bigl[\mathfrak{T}_{\mathfrak{g}}\bigr]$ and, therefore, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\bigl[\mathfrak{T}_{\mathfrak{g}}\bigr]$. Suppose $\mathscr{S}_{\mathfrak{g}} \neq \emptyset$, consider any point $\xi \in \mathscr{S}_{\mathfrak{g}}$. Since $\mathscr{S}_{\mathfrak{g}}$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, for every $\zeta \in \mathscr{S}_{\mathfrak{g}}$, there is a path $\varphi_{\mathfrak{g},\zeta} : [0,1] \to \mathscr{S}_{\mathfrak{g}}$ from the initial point $\xi \in \mathscr{S}_{\mathfrak{g}}$ to the terminal point $\zeta \in \mathscr{S}_{\mathfrak{g}}$, and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set $\mathscr{Q}_{\mathfrak{g},(\xi,\zeta)} \in \mathfrak{g}\text{-}\mathrm{Q}\bigl[\mathfrak{T}_{\mathfrak{g}}\bigr]$, containing ξ, $\zeta \in \mathscr{S}_{\mathfrak{g}}$, such that $\mathscr{Q}_{\mathfrak{g},(\xi,\zeta)} \supseteq \operatorname{im}\bigl(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\bigr)$. Clearly, $\operatorname{im}\bigl(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\bigr) \in \mathrm{C}\bigl[[0,1]\,;\mathfrak{T}_{\mathfrak{g}}\bigr]$. Moreover, $\mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{Q}_{\mathfrak{g},(\xi,\zeta)} \supseteq \operatorname{im}\bigl(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\bigr)$ and, consequently, $\mathscr{S}_{\mathfrak{g}} \supseteq \bigcup_{\zeta\in\mathscr{S}_{\mathfrak{g}}} \mathscr{Q}_{\mathfrak{g},(\xi,\zeta)} \supseteq \bigcup_{\zeta\in\mathscr{S}_{\mathfrak{g}}} \operatorname{im}\bigl(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\bigr) = \mathscr{S}_{\mathfrak{g}}$, since $\xi \in \mathscr{S}_{\mathfrak{g}}$. But, $\xi \in \operatorname{im}\bigl(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\bigr)$ for every $\zeta \in \mathscr{S}_{\mathfrak{g}}$ and, hence, $\bigcap_{\zeta\in\mathscr{S}_{\mathfrak{g}}} \operatorname{im}\bigl(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\bigr) \neq \emptyset$. Furthermore, $\operatorname{im}\bigl(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\bigr) \in \mathfrak{g}\text{-}\mathrm{Q}\bigl[\mathfrak{T}_{\mathfrak{g}}\bigr]$ for every $\zeta \in \mathscr{S}_{\mathfrak{g}}$, and by the relation $\mathscr{S}_{\mathfrak{g}} = \bigcup_{\zeta\in\mathscr{S}_{\mathfrak{g}}} \operatorname{im}\bigl(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\bigr)$, it follows, then, that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\bigl[\mathfrak{T}_{\mathfrak{g}}\bigr]$. This proves the theorem. ☐

Thus, pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is a stronger form of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness. For this reason, we stated that pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is a further refinement of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. An immediate consequence of such a statement is the following proposition.

Proposition 7. If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space, then it is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected:

$$\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{PC}\rangle} = \bigl(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{PC}\rangle}\bigr) \implies \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle} = \bigl(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}\bigr). \tag{3.24}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space, and suppose it be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated. Then, $\mathfrak{T}_{\mathfrak{g}}$ has a nonempty proper $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right] \cap \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$. There exists, then, $(\xi,\zeta) \in \mathscr{S}_{\mathfrak{g}} \times \complement(\mathscr{S}_{\mathfrak{g}})$. Let $\varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ be a path from ξ to ζ. Clearly, $[0,1] \supset \operatorname{im}\bigl(\varphi^{-1}_{\mathfrak{g},\zeta_{|\mathscr{S}_{\mathfrak{g}}}}\bigr)$ for $0 \in \operatorname{im}\bigl(\varphi^{-1}_{\mathfrak{g},\zeta_{|\mathscr{S}_{\mathfrak{g}}}}\bigr)$ and $1 \notin \operatorname{im}\bigl(\varphi^{-1}_{\mathfrak{g},\zeta_{|\mathscr{S}_{\mathfrak{g}}}}\bigr)$, or for $0 \notin \operatorname{im}\bigl(\varphi^{-1}_{\mathfrak{g},\zeta_{|\mathscr{S}_{\mathfrak{g}}}}\bigr)$ and $1 \in \operatorname{im}\bigl(\varphi^{-1}_{\mathfrak{g},\zeta_{|\mathscr{S}_{\mathfrak{g}}}}\bigr)$. Since $\varphi_{\mathfrak{g},\zeta} \in \mathrm{C}\bigl[[0,1]\,;\mathfrak{T}_{\mathfrak{g}}\bigr]$, it follows that $\operatorname{im}\bigl(\varphi^{-1}_{\mathfrak{g},\zeta_{|\mathscr{S}_{\mathfrak{g}}}}\bigr)$ is both open and closed. But, this contradicts the fact that $[0,1]$ is connected. Hence, the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. ☐

Definition 3.25. Let $\phi_{\mathfrak{g},\zeta}$, $\varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ be two paths in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ satisfying $\bigl(\phi_{\mathfrak{g},\zeta}(0), \phi_{\mathfrak{g},\zeta}(1)\bigr) = \bigl(\varphi_{\mathfrak{g},\zeta}(0), \varphi_{\mathfrak{g},\zeta}(1)\bigr) = (\xi,\zeta)$. Then, $\phi_{\mathfrak{g},\zeta}$ is said to be "homotopic" to $\varphi_{\mathfrak{g},\zeta}$, written $\phi_{\mathfrak{g},\zeta} \simeq \varphi_{\mathfrak{g},\zeta}$, if there exists a $\bigl([0,1]^2, \mathfrak{T}_{\mathfrak{g}}\bigr)$-continuous map $\mathfrak{h}_{\mathfrak{g}} : [0,1]^2 \longrightarrow \mathfrak{T}_{\mathfrak{g}}$, called a "homotopy" from $\phi_{\mathfrak{g},\zeta}$ to $\varphi_{\mathfrak{g},\zeta}$, written

$\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\varphi_{\mathfrak{g},\zeta}$, satisfying,

$$\begin{aligned}\mathfrak{h}_{\mathfrak{g}}(\lambda,\mu) &= (1-\mu)\,\phi_{\mathfrak{g},\zeta}(\lambda)+\mu\varphi_{\mathfrak{g},\zeta}(\lambda) \quad \forall\mu\in\{0,1\},\\ \mathfrak{h}_{\mathfrak{g}}(\lambda,\mu) &= (1-\lambda)\,\xi+\lambda\zeta \quad \forall\lambda\in\{0,1\}.\end{aligned} \tag{3.25}$$

The homotopy $\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\varphi_{\mathfrak{g},\zeta}$ is said to establish a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopy from $\phi_{\mathfrak{g},\zeta}$ to $\varphi_{\mathfrak{g},\zeta}$ in a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set $\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ if it belongs to the class:

$$\mathfrak{g}\text{-}\mathrm{H}\big[[0,1]^2;\mathscr{R}_{\mathfrak{g}}\big]\stackrel{\mathrm{def}}{=}\big\{\mathfrak{h}_{\mathfrak{g}}:\ \big(\exists\phi_{\mathfrak{g},\zeta},\varphi_{\mathfrak{g},\zeta}:[0,1]\longrightarrow\mathscr{R}_{\mathfrak{g}}\big)\big[\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\varphi_{\mathfrak{g},\zeta}\big]\big\}. \tag{3.26}$$

For any $\phi_{\mathfrak{g},\zeta},\varphi_{\mathfrak{g},\zeta},\psi_{\mathfrak{g},\zeta}:[0,1]\longrightarrow\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$, the statements $\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\phi_{\mathfrak{g},\zeta}$, $\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\varphi_{\mathfrak{g},\zeta}$ implies $\mathfrak{h}_{\mathfrak{g}}:\varphi_{\mathfrak{g},\zeta}\simeq\phi_{\mathfrak{g},\zeta}$ and, $\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\varphi_{\mathfrak{g},\zeta}$ and $\mathfrak{h}_{\mathfrak{g}}:\varphi_{\mathfrak{g},\zeta}\simeq\psi_{\mathfrak{g},\zeta}$ implies $\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\psi_{\mathfrak{g},\zeta}$ hold, as shown in the following theorem.

Theorem 3.26. *The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopy relation is an equivalence relation in the collection of all paths in any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set $\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$.*

Proof. – *Reflexivity.* Let $\phi_{\mathfrak{g},\zeta}:[0,1]\longrightarrow\mathscr{R}_{\mathfrak{g}}$ be any path, where $\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ is any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then the $\big([0,1]^2,\mathscr{R}_{\mathfrak{g}}\big)$-continuous map $\mathfrak{h}_{\mathfrak{g}}:[0,1]^2\longrightarrow\mathscr{R}_{\mathfrak{g}}$ defined, for every $(\lambda,\mu)\in[0,1]^2$, by $\mathfrak{h}_{\mathfrak{g}}(\lambda,\mu)=\phi_{\mathfrak{g},\zeta}(\lambda)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopy from $\phi_{\mathfrak{g},\zeta}$ to $\phi_{\mathfrak{g},\zeta}$, and that defined, for every $(\lambda,\mu)\in[0,1]^2$, by $\mathfrak{h}_{\mathfrak{g}}(\lambda,\mu)=\varphi_{\mathfrak{g},\zeta}(\mu)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopy from $\varphi_{\mathfrak{g},\zeta}$ to $\varphi_{\mathfrak{g},\zeta}$. Hence, $\mathfrak{h}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{H}\big[[0,1]^2;\mathscr{R}_{\mathfrak{g}}\big]$, and $\simeq$ is reflexive.

– *Symmetry.* Let $\mathfrak{h}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{H}\big[[0,1]^2;\mathscr{R}_{\mathfrak{g}}\big]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopy $\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\varphi_{\mathfrak{g},\zeta}$. Then the $\big([0,1]^2,\mathscr{R}_{\mathfrak{g}}\big)$-continuous map $\hat{\mathfrak{h}}_{\mathfrak{g}}:[0,1]^2\longrightarrow\mathscr{R}_{\mathfrak{g}}$ defined, for every $(\lambda,\mu)\in[0,1]^2$, by $\hat{\mathfrak{h}}_{\mathfrak{g}}(\lambda,\mu)=\mathfrak{h}_{\mathfrak{g}}(\lambda,1-\mu)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopy $\hat{\mathfrak{h}}_{\mathfrak{g}}:\varphi_{\mathfrak{g},\zeta}\simeq\phi_{\mathfrak{g},\zeta}$. Hence, $\hat{\mathfrak{h}}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{H}\big[[0,1]^2;\mathscr{R}_{\mathfrak{g}}\big]$, and $\simeq$ is symmetric.

– *Transitivity.* Let $\mathfrak{h}_{\mathfrak{g},\alpha},\mathfrak{h}_{\mathfrak{g},\beta}\in\mathfrak{g}\text{-}\mathrm{H}\big[[0,1]^2;\mathscr{R}_{\mathfrak{g}}\big]$ be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopies $\mathfrak{h}_{\mathfrak{g},\alpha}:\phi_{\mathfrak{g},\zeta}\simeq\varphi_{\mathfrak{g},\zeta}$ and $\mathfrak{h}_{\mathfrak{g},\beta}:\varphi_{\mathfrak{g},\zeta}\simeq\psi_{\mathfrak{g},\zeta}$, respectively. Consider the $\big([0,1]^2,\mathscr{R}_{\mathfrak{g}}\big)$-continuous map $\mathfrak{h}_{\mathfrak{g}}:[0,1]^2\longrightarrow\mathscr{R}_{\mathfrak{g}}$ defined, for every $(\lambda,\mu)\in[0,1]^2$, by $\mathfrak{h}_{\mathfrak{g}}(\lambda,\mu)=\mathfrak{h}_{\mathfrak{g},\alpha}(\lambda,\eta\mu)$ if $\mu\in\left[0,\dfrac{1}{\eta}\right]$ and $\mathfrak{h}_{\mathfrak{g}}(\lambda,\mu)=\mathfrak{h}_{\mathfrak{g},\beta}(\lambda,\eta\mu-1)$ if $\mu\in\left[\dfrac{1}{\eta},1\right]$, where $\eta\in(1,\infty)$. Clearly, $\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\psi_{\mathfrak{g},\zeta}$. Hence, it follows that, $\mathfrak{h}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{H}\big[[0,1]^2;\mathscr{R}_{\mathfrak{g}}\big]$, and $\simeq$ is transitive. □

The concept of simply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space is defined below.

Definition 3.27. Let $\varphi_{\mathfrak{g},\zeta}:[0,1]\longrightarrow\mathfrak{T}_{\mathfrak{g}}$ be a path from $\xi\in\mathfrak{T}_{\mathfrak{g}}$ to $\zeta\in\mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ with $\big(\varphi_{\mathfrak{g},\zeta}(0),\varphi_{\mathfrak{g},\zeta}(1)\big)=(\xi,\zeta)$. Then:

– I. If $\varphi_{\mathfrak{g},\zeta}:[0,1]\longrightarrow\{\zeta\}$, then $\varphi_{\mathfrak{g},\zeta}$ is called a "constant path" at $\zeta\in\mathfrak{T}_{\mathfrak{g}}$: $\varphi_{\mathfrak{g},\zeta}(\lambda)\stackrel{\mathrm{def}}{=}\mathfrak{c}_{\mathfrak{g}}(\lambda)$ for all $\lambda\in[0,1]$.

– II. If $\varphi_{\mathfrak{g},\zeta}:\{0,1\}\longrightarrow\{\zeta\}$, then $\varphi_{\mathfrak{g},\zeta}$ is called a "closed path" at $\zeta\in\mathfrak{T}_{\mathfrak{g}}$: $\varphi_{\mathfrak{g},\zeta}(\lambda)\stackrel{\mathrm{def}}{=}\mathfrak{k}_{\mathfrak{g}}(\lambda)$ for all $\lambda\in[0,1]$

– III. If, for every $\lambda\in[0,1]$, $\varphi_{\mathfrak{g},\zeta}(\lambda)=\mathfrak{k}_{\mathfrak{g}}(\lambda)$ and $\varphi_{\mathfrak{g},\zeta}(\lambda)\simeq\mathfrak{c}_{\mathfrak{g}}(\lambda)$, then $\varphi_{\mathfrak{g},\zeta}$ is said to be "contractable to the point $\zeta\in\mathfrak{T}_{\mathfrak{g}}$."

A $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ is "simply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected" if and only if, at each point $\zeta\in\mathfrak{T}_{\mathfrak{g}}$, any closed path $\mathfrak{k}_{\mathfrak{g}}:[0,1]\longrightarrow\mathfrak{T}_{\mathfrak{g}}$ is contractable to ζ.

The necessary and sufficient conditions for a pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space to be simply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected are contained in the following theorem.

Theorem 3.28. *Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space. Then, $\mathfrak{T}_{\mathfrak{g}}$ is simply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected if and only if, at each $\zeta \in \mathfrak{T}_{\mathfrak{g}}$, any closed path $\mathfrak{k}_{\mathfrak{g}} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ at ζ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopic to the constant path $\mathfrak{c}_{\mathfrak{g}} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ at $\zeta \in \mathfrak{T}_{\mathfrak{g}}$: $\mathfrak{h}_{\mathfrak{g}} : \mathfrak{k}_{\mathfrak{g}} \simeq \mathfrak{c}_{\mathfrak{g}}$ for each $\zeta \in \mathfrak{T}_{\mathfrak{g}}$.*

Proof. – *Necessity.* Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space, and suppose it be simply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. Since $\mathfrak{T}_{\mathfrak{g}}$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, for every $(\xi, \zeta) \in \Omega \times \Omega$,

$$\big(\exists \mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big)\big(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}\big)\big[\Omega \supseteq \mathscr{Q}_{\mathfrak{g}} \supseteq \operatorname{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big)\big].$$

If $\varphi_{\mathfrak{g},\zeta} : \{0,1\} \longrightarrow \{\zeta\}$, then $\varphi_{\mathfrak{g},\zeta}$ is a closed path at $\zeta \in \mathfrak{T}_{\mathfrak{g}}$: $\varphi_{\mathfrak{g},\zeta}(\lambda) = \mathfrak{k}_{\mathfrak{g}}(\lambda)$ for all $\lambda \in [0,1]$. Since $\mathfrak{T}_{\mathfrak{g}}$ is simply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, it follows that, at each point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$, the closed path $\mathfrak{k}_{\mathfrak{g}} : [0,1] \to \mathfrak{T}_{\mathfrak{g}}$ is contractable to ζ. Thus, at each $\zeta \in \mathfrak{T}_{\mathfrak{g}}$, the closed path $\mathfrak{k}_{\mathfrak{g}} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ at ζ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopic to the constant path $\mathfrak{c}_{\mathfrak{g}} : [0,1] \to \mathfrak{T}_{\mathfrak{g}}$ at $\zeta \in \mathfrak{T}_{\mathfrak{g}}$: $\mathfrak{h}_{\mathfrak{g}} : \mathfrak{k}_{\mathfrak{g}} \simeq \mathfrak{c}_{\mathfrak{g}}$ for each $\zeta \in \mathfrak{T}_{\mathfrak{g}}$. The condition of the theorem is, therefore, necessary.

– *Sufficiency.* Conversely, suppose that, at every point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ in a pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, any closed path $\mathfrak{k}_{\mathfrak{g}} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ at ζ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopic to the constant path $\mathfrak{c}_{\mathfrak{g}} : [0,1] \to \mathfrak{T}_{\mathfrak{g}}$ at $\zeta \in \mathfrak{T}_{\mathfrak{g}}$: $\mathfrak{h}_{\mathfrak{g}} : \mathfrak{k}_{\mathfrak{g}} \simeq \mathfrak{c}_{\mathfrak{g}}$ for every $\zeta \in \mathfrak{T}_{\mathfrak{g}}$. Then, there exists a path $\varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ satisfying $\varphi_{\mathfrak{g},\zeta} : \{0,1\} \longrightarrow \{\zeta\}$ and, therefore, contractable to $\zeta \in \mathfrak{T}_{\mathfrak{g}}$. Thus, at each point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$, any closed path $\mathfrak{k}_{\mathfrak{g}} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ is contractable to ζ. The $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is, then, simply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. The condition of the theorem is, therefore, sufficient. □

The definition of local $\mathfrak{g}$-$\mathfrak{T}$-connectedness at a point $\xi \in \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is now given.

Definition 3.29. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then:

– I. $\mathfrak{T}_{\mathfrak{g}}$ is said to be "locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected at a point $\xi \in \mathfrak{T}_{\mathfrak{g}}$" if and only if,

$$\big(\forall \mathscr{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}\,[\mathfrak{T}_{\mathfrak{g}}]\big)\big(\exists \mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\,[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\xi \in \mathscr{Q}_{\mathfrak{g}} \subseteq \mathscr{U}_{\mathfrak{g}}\big]. \tag{3.27}$$

– II. $\mathfrak{T}_{\mathfrak{g}}$ is said to be "locally pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected" if and only if, given any $(\xi, \mathscr{U}_{\mathfrak{g},\xi}) \in \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{g}\text{-O}\,[\mathfrak{T}_{\mathfrak{g}}]$, there exists $(\xi, \mathscr{Q}_{\mathfrak{g},\xi}) \in \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{g}\text{-Q}\,[\mathfrak{T}_{\mathfrak{g}}]$ such that $(\zeta, \eta) \in \mathscr{Q}_{\mathfrak{g}} \times \mathscr{Q}_{\mathfrak{g}}$, with $\zeta \neq \eta$, implies that,

$$\big(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}\big)\big[\{\zeta, \eta\} \subseteq \operatorname{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big) \subseteq \mathscr{Q}_{\mathfrak{g},\xi} \subseteq \mathscr{U}_{\mathfrak{g},\xi}\big]. \tag{3.28}$$

The $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is said to be "locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected" if and only if it is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected at every point $\xi \in \mathfrak{T}_{\mathfrak{g}}$.

As an immediate consequence of the above definition, it is shown below that local pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness implies locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected.

Theorem 3.30. *If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a locally pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space, then it is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected:*

$$\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{LPC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{LPC}\rangle}\big) \implies \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{LC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{LC}\rangle}\big). \tag{3.29}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a locally pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space. Then, for any given $(\xi, \mathscr{U}_{\mathfrak{g},\xi}) \in \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, there exists $(\xi, \mathscr{Q}_{\mathfrak{g},\xi}) \in \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ such that $(\zeta, \xi) \in \mathscr{Q}_{\mathfrak{g}} \times \mathscr{Q}_{\mathfrak{g}}$, with $\zeta \neq \xi$, implies that,

$$\big(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \to \mathfrak{T}_{\mathfrak{g}}\big)\big[\{\zeta, \xi\} \subseteq \operatorname{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big) \subseteq \mathscr{Q}_{\mathfrak{g},\xi} \subseteq \mathscr{U}_{\mathfrak{g},\xi}\big].$$

Consequently, $\xi \in \operatorname{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big) \subseteq \mathscr{Q}_{\mathfrak{g},\xi} \subseteq \mathscr{U}_{\mathfrak{g},\xi}$ and, therefore, $\xi \in \mathscr{Q}_{\mathfrak{g},\xi} \subseteq \mathscr{U}_{\mathfrak{g},\xi}$. Hence, it follows that

$$\big(\forall \mathscr{U}_{\mathfrak{g}}, \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\big)\big(\exists \mathscr{Q}_{\mathfrak{g}}, \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\xi \in \mathscr{Q}_{\mathfrak{g},\xi} \subseteq \mathscr{U}_{\mathfrak{g},\xi}\big].$$

The $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is therefore locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. ☐

In a locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set, as demonstrated in the following theorem.

Theorem 3.31. *If $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ be the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component of $\mathscr{S}_{\mathfrak{g}}$ corresponding to ζ in a locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$:*

$$\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \subseteq \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{LC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{LC}\rangle}\big) \implies \mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-O}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{LC}\rangle}\big]. \tag{3.30}$$

Proof. Let $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ be the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component of $\mathscr{S}_{\mathfrak{g}}$ corresponding to ζ in a locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, local $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness at $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ implies

$$\big(\forall \mathscr{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\big)\big(\exists \mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\zeta \in \mathscr{Q}_{\mathfrak{g}} \subseteq \mathscr{U}_{\mathfrak{g}}\big].$$

Consequently, $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] = \bigcup_{\mathscr{Q}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}} \mathscr{Q}_{\mathfrak{g}} \subseteq \bigcup_{\mathscr{U}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g}}$. But, since every $\mathscr{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ satisfies $\mathscr{U}_{\mathfrak{g}} \subseteq \operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g}}\big)$ for some $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}}$, it follows that the relation $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}} \operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g}}\big) = \operatorname{op}_{\mathfrak{g}}\big(\bigcup_{\mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}} \mathscr{O}_{\mathfrak{g}}\big)$ holds. Thus, $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$. ☐

The necessary and sufficient conditions for a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ to be locally $\mathfrak{g}$-$\mathfrak{T}$-connected at a point $\xi \in \mathfrak{T}_{\mathfrak{g}}$ is contained in the following theorem.

Theorem 3.32. *A $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected at a point $\xi \in \mathfrak{T}_{\mathfrak{g}}$ if and only if,*

$$\big(\forall \mathscr{O}_{\mathfrak{g},\xi} \in \mathscr{T}_{\mathfrak{g}}\big)\big(\exists \mathscr{Q}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\xi \in \mathscr{Q}_{\mathfrak{g},\xi} \subseteq \operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\xi}\big)\big]. \tag{3.31}$$

Proof. – Necessity. Let it be assumed that the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected at $\xi \in \mathfrak{T}_{\mathfrak{g}}$, and let $\mathscr{O}_{\mathfrak{g},\xi} \in \mathscr{T}_{\mathfrak{g}}$ be an arbitrary $\mathscr{T}_{\mathfrak{g}}$-open neighbourhood of ξ. There exists, then, a $\mathscr{T}_{\mathfrak{g}}$-open neighbourhood $\hat{\mathscr{O}}_{\mathfrak{g},\xi} \in \mathscr{T}_{\mathfrak{g}}$ of ξ such that $\xi \in \operatorname{op}_{\mathfrak{g}}\big(\hat{\mathscr{O}}_{\mathfrak{g},\xi}\big) \subseteq \operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\xi}\big)$ and, for every $\{\zeta, \eta\} \subseteq \hat{\mathscr{O}}_{\mathfrak{g},\xi}$,

$$\big(\exists \mathscr{Q}_{\mathfrak{g},(\zeta,\eta)} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\{\zeta, \eta\} \subseteq \mathscr{Q}_{\mathfrak{g},(\zeta,\eta)} \subseteq \operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\xi}\big)\big].$$

Suppose $\eta \in \hat{\mathscr{O}}_{\mathfrak{g},\xi}$ be the arbitrary point. Then, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set $\mathscr{Q}_{\mathfrak{g},(\xi,\eta)} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ satisfying $\{\xi, \eta\} \subseteq \mathscr{Q}_{\mathfrak{g},(\xi,\eta)} \subseteq \operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\xi}\big)$. Let $\mathscr{Q}_{\mathfrak{g},\xi} = \bigcup_{\eta \in \hat{\mathscr{O}}_{\mathfrak{g},\xi}} \mathscr{Q}_{\mathfrak{g},(\xi,\eta)} \subseteq \operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\xi}\big)$. Since $\mathscr{Q}_{\mathfrak{g},\xi} \supseteq \hat{\mathscr{O}}_{\mathfrak{g},\xi}$ and $\bigcup_{\eta \in \hat{\mathscr{O}}_{\mathfrak{g},\xi}} \mathscr{Q}_{\mathfrak{g},(\xi,\eta)} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that $\mathscr{Q}_{\mathfrak{g},\xi}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected neighbourhood of ξ contained in $\operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\xi}\big)$. The condition of the theorem is, therefore, necessary.

– *Sufficiency.* Conversely, suppose the following condition holds:

$$\big(\forall \mathscr{O}_{\mathfrak{g},\xi} \in \mathscr{T}_{\mathfrak{g}}\big)\big(\exists \mathscr{Q}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\xi \in \mathscr{Q}_{\mathfrak{g},\xi} \subseteq \operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\xi}\big)\big].$$

Let $\mathscr{O}_{\mathfrak{g},\xi} \in \mathscr{T}_{\mathfrak{g}}$ be an arbitrary $\mathscr{T}_{\mathfrak{g}}$-open neighbourhood of ξ. Then, $\mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\xi})$ contains a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected neighbourhood $\mathscr{Q}_{\mathfrak{g},\xi}$ of ξ. Since $\mathscr{Q}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$, for any $\{\zeta,\eta\} \subseteq \mathscr{Q}_{\mathfrak{g},\xi}$, there exists $\mathscr{Q}_{\mathfrak{g},(\xi,\eta)} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\{\zeta,\eta\} \subseteq \mathscr{Q}_{\mathfrak{g},(\zeta,\eta)}$. But $\mathscr{Q}_{\mathfrak{g},\xi} \supseteq \mathscr{Q}_{\mathfrak{g},(\zeta,\eta)}$ and, consequently, $\{\zeta,\eta\} \subseteq \mathscr{Q}_{\mathfrak{g},(\zeta,\eta)} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\xi})$. Hence, the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected at $\xi \in \mathfrak{T}_{\mathfrak{g}}$. The condition of the theorem is, therefore, sufficient. $\square$

The notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness between any $\mathscr{T}_{\mathfrak{g}}$-spaces $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$ and the relevant basic theorems are now discussed.

Theorem 3.33. *Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces, let $\mathscr{S}_{\mathfrak{g},1}$, $\mathscr{S}_{\mathfrak{g},2}$, $\ldots$, $\mathscr{S}_{\mathfrak{g},n} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Lambda}]$ be $n \geq 1$ mutually disjoint $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathfrak{T}_{\mathfrak{g},\Lambda}$, where $\Lambda \in \{\Omega, \Sigma\}$, and let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-B}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-bijective map $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$. Then*

– I. $\pi_{\mathfrak{g}}\big(\bigsqcup_{\alpha\in I_n^*} \mathscr{S}_{\mathfrak{g},\alpha}\big) = \bigsqcup_{\alpha\in I_n^*} \pi_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$,
– II. $\pi_{\mathfrak{g}}^{-1}\big(\bigsqcup_{\alpha\in I_n^*} \mathscr{S}_{\mathfrak{g},\alpha}\big) = \bigsqcup_{\alpha\in I_n^*} \pi_{\mathfrak{g}}^{-1}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$.

Proof. – I. Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces, let $\mathscr{S}_{\mathfrak{g},1}$, $\mathscr{S}_{\mathfrak{g},2}$, $\ldots$, $\mathscr{S}_{\mathfrak{g},n} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Omega}]$, and let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-B}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$. If $\zeta \in \pi_{\mathfrak{g}}\big(\bigsqcup_{\alpha\in I_n^*} \mathscr{S}_{\mathfrak{g},\alpha}\big)$, then, since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-B}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$, there exists $\xi \in \bigsqcup_{\alpha\in I_n^*} \mathscr{S}_{\mathfrak{g},\alpha}$ such that, $\pi_{\mathfrak{g}}^{-1}(\zeta) = \xi \in \bigsqcup_{\alpha\in I_n^*} \mathscr{S}_{\mathfrak{g},\alpha}$. Consequently,

$$\begin{aligned} \pi_{\mathfrak{g}}^{-1}(\zeta) \in \bigsqcup_{\alpha\in I_n^*} \mathscr{S}_{\mathfrak{g},\alpha} \quad &\Rightarrow \quad \bigvee_{\alpha\in I_n^*} \big(\pi_{\mathfrak{g}}^{-1}(\zeta) \in \mathscr{S}_{\mathfrak{g},\alpha}\big) \\ &\Rightarrow \quad \bigvee_{\alpha\in I_n^*} \big(\zeta \in \pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\alpha})\big) \;\Rightarrow\; \zeta \in \bigsqcup_{\alpha\in I_n^*} \pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\alpha}). \end{aligned}$$

Hence, $\pi_{\mathfrak{g}}\big(\bigsqcup_{\alpha\in I_n^*} \mathscr{S}_{\mathfrak{g},\alpha}\big) \subseteq \bigsqcup_{\alpha\in I_n^*} \pi_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$. Conversely, if it be assumed that $\zeta \in \bigsqcup_{\alpha\in I_n^*} \pi_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$ then,

$$\begin{aligned} \bigvee_{\alpha\in I_n^*} \big(\zeta \in \pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\alpha})\big) \quad &\Rightarrow \quad \bigvee_{\alpha\in I_n^*} \big(\pi_{\mathfrak{g}}^{-1}(\zeta) \in \mathscr{S}_{\mathfrak{g},\alpha}\big) \\ &\Rightarrow \quad \pi_{\mathfrak{g}}^{-1}(\zeta) \in \bigsqcup_{\alpha\in I_n^*} \mathscr{S}_{\mathfrak{g},\alpha} \;\Rightarrow\; \zeta \in \pi_{\mathfrak{g}}\big(\textstyle\bigsqcup_{\alpha\in I_n^*} \mathscr{S}_{\mathfrak{g},\alpha}\big). \end{aligned}$$

Hence, $\pi_{\mathfrak{g}}\big(\bigsqcup_{\alpha\in I_n^*} \mathscr{S}_{\mathfrak{g},\alpha}\big) \supseteq \bigsqcup_{\alpha\in I_n^*} \pi_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$.

– II. If $\xi \in \pi_{\mathfrak{g}}^{-1}\big(\bigsqcup_{\alpha\in I_n^*} \mathscr{S}_{\mathfrak{g},\alpha}\big)$, where $\mathscr{S}_{\mathfrak{g},1}$, $\mathscr{S}_{\mathfrak{g},2}$, $\ldots$, $\mathscr{S}_{\mathfrak{g},n} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$, then,

$$\begin{aligned} \pi_{\mathfrak{g}}(\xi) \in \bigsqcup_{\alpha\in I_n^*} \mathscr{S}_{\mathfrak{g},\alpha} \quad &\Rightarrow \quad \bigvee_{\alpha\in I_n^*} \big(\pi_{\mathfrak{g}}(\xi) \in \mathscr{S}_{\mathfrak{g},\alpha}\big) \\ &\Rightarrow \quad \bigvee_{\alpha\in I_n^*} \big(\xi \in \pi_{\mathfrak{g}}^{-1}(\mathscr{S}_{\mathfrak{g},\alpha})\big) \;\Rightarrow\; \xi \in \bigsqcup_{\alpha\in I_n^*} \pi_{\mathfrak{g}}^{-1}(\mathscr{S}_{\mathfrak{g},\alpha}). \end{aligned}$$

Hence, $\pi_{\mathfrak{g}}^{-1}\big(\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}\big)\subseteq\bigsqcup_{\alpha\in I_n^*}\pi_{\mathfrak{g}}^{-1}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$. Conversely, if it be supposed that $\xi\in\bigsqcup_{\alpha\in I_n^*}\pi_{\mathfrak{g}}^{-1}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$ then,

$$\begin{aligned}\bigvee_{\alpha\in I_n^*}\left(\xi\in\pi_{\mathfrak{g}}^{-1}\left(\mathscr{S}_{\mathfrak{g},\alpha}\right)\right) \quad &\Rightarrow \quad \bigvee_{\alpha\in I_n^*}\left(\pi_{\mathfrak{g}}\left(\xi\right)\in\mathscr{S}_{\mathfrak{g},\alpha}\right)\\ &\Rightarrow \quad \pi_{\mathfrak{g}}\left(\xi\right)\in\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha} \;\Rightarrow\; \xi\in\pi_{\mathfrak{g}}^{-1}\big(\textstyle\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}\big).\end{aligned}$$

Hence, $\pi_{\mathfrak{g}}^{-1}\big(\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}\big)\supseteq\bigsqcup_{\alpha\in I_n^*}\pi_{\mathfrak{g}}^{-1}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$. □

The following theorem shows, among others, that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is a $\mathscr{T}_{\mathfrak{g}}$-property.

Theorem 3.34. *Let $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega}\longrightarrow\mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous map and let $\mathscr{S}_{\mathfrak{g},\omega}\subset\mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set. If $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Omega}$, then $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Sigma}$:*

$$\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]\implies\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]. \tag{3.32}$$

Proof. Let $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, $\mathscr{S}_{\mathfrak{g},\omega}=\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$, and suppose that $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, that is, $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\notin\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$. There exists, therefore, $\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\sigma,\zeta_\sigma}\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]$ such that,

$$\Bigg(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{U}_{\mathfrak{g},\lambda}=\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\Bigg)\bigvee\Bigg(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{V}_{\mathfrak{g},\lambda}=\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\Bigg).$$

Set $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)=\bigsqcup_{\lambda=\xi_\omega,\zeta_\omega}\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)$, where $\mathscr{S}_{\mathfrak{g},\omega}\supseteq\bigcup_{\lambda=\xi_\omega,\zeta_\omega}\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}$ and, for every $(\lambda,\mu)\in\left\{(\xi_\sigma,\xi_\omega),(\zeta_\sigma,\zeta_\omega)\right\}$, set

$$\big[\mathscr{U}_{\mathfrak{g},\lambda}=\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\mu)}}\big)\big]\vee\big[\mathscr{V}_{\mathfrak{g},\lambda}=\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\mu)}}\big)\big].$$

In other words, $\mathscr{S}_{\mathfrak{g},(\omega,\xi_\omega)}\subseteq\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ denotes the $\mathfrak{T}_{\mathfrak{g}}$-set of all $\xi\in\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ for which $\pi_{\mathfrak{g}}\left(\xi\right)\in\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\xi_\omega)}}\big)$, and $\mathscr{S}_{\mathfrak{g},(\omega,\zeta_\omega)}\subseteq\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ denotes the $\mathfrak{T}_{\mathfrak{g}}$-set of all $\zeta\in\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ for which $\pi_{\mathfrak{g}}\left(\zeta\right)\in\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\zeta_\omega)}}\big)$. Since the inequality $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)\neq\emptyset$ holds for every $\lambda\in\{\xi_\omega,\zeta_\omega\}$, and both the relations $\bigsqcup_{\lambda=\xi_\omega,\zeta_\omega}\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)=\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ and $\bigcap_{\lambda=\xi_\omega,\zeta_\omega}\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)=\emptyset$ hold, it follows that, $\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}\neq\emptyset$ for every $\lambda\in\{\xi_\omega,\zeta_\omega\}$, $\bigcup_{\lambda=\xi_\omega,\zeta_\omega}\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}=\mathscr{S}_{\mathfrak{g},\omega}$ and $\bigcap_{\lambda=\xi_\omega,\zeta_\omega}\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}=\emptyset$. Since $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, for any $\lambda\in\{\xi_\omega,\zeta_\omega\}$, there exists, for every $\big(\mathscr{O}_{\mathfrak{g},(\sigma,\lambda)},\mathscr{K}_{\mathfrak{g},(\sigma,\lambda)}\big)\in\mathscr{T}_{\mathfrak{g},\Sigma}\times\neg\mathscr{T}_{\mathfrak{g},\Sigma}$, $\big(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)},\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\big)\in\mathscr{T}_{\mathfrak{g},\Omega}\times\neg\mathscr{T}_{\mathfrak{g},\Omega}$, with $\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}$, $\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\subset\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}$ and $\mathscr{O}_{\mathfrak{g},(\sigma,\lambda)}$, $\mathscr{K}_{\mathfrak{g},(\sigma,\lambda)}\subset\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)$, such that,

$$\big[\pi_{\mathfrak{g}}^{-1}\left(\mathscr{O}_{\mathfrak{g},(\sigma,\lambda)}\right)\subseteq\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}\right)\big]\vee\big[\pi_{\mathfrak{g}}^{-1}\left(\mathscr{K}_{\mathfrak{g},(\sigma,\lambda)}\right)\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\right)\big].$$

Since $\bigcap_{\lambda=\xi_\omega,\zeta_\omega}\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)=\emptyset$ implies $\bigcap_{\lambda=\xi_\omega,\zeta_\omega}\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}=\emptyset$, it follows, evidently, that,

$$\Bigg(\bigcap_{\lambda=\xi_\omega,\zeta_\omega}\pi_{\mathfrak{g}}^{-1}\left(\mathscr{O}_{\mathfrak{g},(\sigma,\lambda)}\right)=\emptyset\Bigg)\bigwedge\Bigg(\bigcap_{\lambda=\xi_\omega,\zeta_\omega}\pi_{\mathfrak{g}}^{-1}\left(\mathscr{K}_{\mathfrak{g},(\sigma,\lambda)}\right)=\emptyset\Bigg).$$

Therefore, the setting $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) = \bigsqcup_{\lambda=\xi_\omega,\zeta_\omega} \mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)$ holds. It now remains to prove that it is the case and the supposition that $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ is a contradiction. Since $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ it follows that, for all pair $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\omega,\zeta_\omega} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$,

$$\begin{aligned}
&\neg\Big(\bigsqcup_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{U}_{\mathfrak{g},\lambda} = \mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{V}_{\mathfrak{g},\lambda} = \mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\Big)\\
\Leftrightarrow{}& \neg\Big(\bigcap_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{U}_{\mathfrak{g},\lambda} = \emptyset\Big) \bigwedge \neg\Big(\bigcap_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{V}_{\mathfrak{g},\lambda} = \emptyset\Big)\\
\Rightarrow{}& \Big(\bigcap_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{U}_{\mathfrak{g},\lambda} \neq \emptyset\Big) \bigwedge \Big(\bigcap_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{V}_{\mathfrak{g},\lambda} \neq \emptyset\Big).
\end{aligned}$$

There exists, then, a unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\eta_\omega\} \subset \mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ such that,

$$\Big(\bigcap_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{U}_{\mathfrak{g},\lambda} \supseteq \{\eta_\omega\}\Big) \bigwedge \Big(\bigcap_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{V}_{\mathfrak{g},\lambda} \supseteq \{\eta_\omega\}\Big).$$

Since $\{\eta_\omega\} \subset \bigcup_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{S}_{\mathfrak{g},(\omega,\lambda)} = \mathscr{S}_{\mathfrak{g},\omega}$ and $\bigcap_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{S}_{\mathfrak{g},(\omega,\lambda)} = \emptyset$, it results that,

$$\big[\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \supset \mathscr{S}_{\mathfrak{g},(\omega,\xi_\omega)} \supset \{\eta_\omega\}\big] \vee \big[\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \supset \mathscr{S}_{\mathfrak{g},(\omega,\zeta_\omega)} \supset \{\eta_\omega\}\big].$$

On the other hand, since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, it follows that, for every unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\eta_\sigma\} \subset \bigsqcup_{\lambda=\xi_\omega,\zeta_\omega} \mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big) = \mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$,

$$\bigvee_{\lambda=\xi_\omega,\zeta_\omega} \big[\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \supset \mathscr{S}_{\mathfrak{g},(\omega,\lambda)} \supset \pi_{\mathfrak{g}}^{-1}\left(\{\eta_\sigma\}\right)\big].$$

In particular, if $\pi_{\mathfrak{g}}^{-1}\left(\{\eta_\sigma\}\right) = \{\eta_\omega\}$, then $\{\eta_\sigma\} = \pi_{\mathfrak{g}}\left(\{\eta_\omega\}\right)$, leading to a contradiction. There exists, therefore, $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\omega,\zeta_\omega} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ such that,

$$\Big(\bigsqcup_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{U}_{\mathfrak{g},\lambda} = \mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{V}_{\mathfrak{g},\lambda} = \mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\Big).$$

This proves that the supposition $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ is a contradiction and, hence, $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$. □

The following corollary is another way of saying that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectivity of $\mathfrak{T}_{\mathfrak{g},\Omega}$ implies the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectivity of $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

Corollary 3.35. *Let $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-continuous map and let $\mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set. If $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated in $\mathfrak{T}_{\mathfrak{g},\Sigma}$, then $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated in $\mathfrak{T}_{\mathfrak{g},\Omega}$:*

$$\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right] \implies \mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]. \tag{3.33}$$

If the image of a $\mathfrak{g}$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-continuous map is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, then it is also $\mathfrak{T}_{\mathfrak{g}}$-connected, as proved in the following proposition.

Proposition 8. Let $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega}\longrightarrow\mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous map and let $\mathscr{S}_{\mathfrak{g},\omega}\subset\mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set. If $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Sigma}$, then $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)$ is $\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Sigma}$:

$$\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\in\mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]\implies\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\in\text{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right].\tag{3.34}$$

Proof. Let $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, $\mathscr{S}_{\mathfrak{g},\omega}=\operatorname{dom}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\in\mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ and, suppose that $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\in\text{D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, that is, $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\notin\text{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$. There exists, then, $\bigl(\mathscr{R}_{\mathfrak{g},\lambda},\mathscr{S}_{\mathfrak{g},\lambda}\bigr)_{\lambda=\xi_\sigma,\zeta_\sigma}\in\text{O}\bigl[\mathfrak{T}_{\mathfrak{g},\Sigma}\bigr]\times\text{K}\bigl[\mathfrak{T}_{\mathfrak{g},\Sigma}\bigr]$ such that,

$$\Bigl(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{R}_{\mathfrak{g},\lambda}=\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\Bigr)\bigvee\Bigl(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{S}_{\mathfrak{g},\lambda}=\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\Bigr).$$

Since $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\in\text{D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, set $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\bigr)$ and, for every $\lambda\in\{\xi_\sigma,\zeta_\sigma\}$, let

$$\bigl[\mathscr{R}_{\mathfrak{g},\lambda}=\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\bigr)\bigr]\vee\bigl[\mathscr{S}_{\mathfrak{g},\lambda}=\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\bigr)\bigr].$$

On the other hand, since $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, there exists, for any $\lambda\in\{\xi_\sigma,\zeta_\sigma\}$, $\bigl(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)},\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\bigr)\in\mathscr{T}_{\mathfrak{g},\Omega}\times\neg\mathscr{T}_{\mathfrak{g},\Omega}$, satisfying $\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}$, $\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\subset\operatorname{dom}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)$, such that,

$$\begin{aligned}&\bigl[\pi_{\mathfrak{g}}^{-1}\left(\mathscr{R}_{\mathfrak{g},\lambda}\right)\subseteq\operatorname{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}\right)\bigr]\vee\bigl[\pi_{\mathfrak{g}}^{-1}\left(\mathscr{S}_{\mathfrak{g},\lambda}\right)\supseteq\neg\operatorname{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\right)\bigr]\\ \Rightarrow\quad&\bigl[\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\bigr)\subseteq\pi_{\mathfrak{g}}\bigl(\operatorname{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}\right)\bigr)\bigr]\vee\bigl[\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\bigr)\\ &\qquad\supseteq\pi_{\mathfrak{g}}\bigl(\neg\operatorname{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\right)\bigr)\bigr].\end{aligned}$$

Since $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\bigr)$, it is plain that $\pi_{\mathfrak{g}}\bigl(\operatorname{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\omega}\right)\bigr)=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}\bigl(\operatorname{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}\right)\bigr)$, and also $\emptyset=\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}\bigl(\neg\operatorname{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\right)\bigr)$, implying $\pi_{\mathfrak{g}}\bigl(\neg\operatorname{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\omega}\right)\bigr)=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}\bigl(\neg\operatorname{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\right)\bigr)$, for some $\bigl(\mathscr{O}_{\mathfrak{g},\omega},\mathscr{K}_{\mathfrak{g},\omega}\bigr)\in\mathscr{T}_{\mathfrak{g},\Omega}\times\neg\mathscr{T}_{\mathfrak{g},\Omega}$. But, clearly the relation $\bigl(\pi_{\mathfrak{g}}\bigl(\operatorname{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\omega}\right)\bigr),\pi_{\mathfrak{g}}\bigl(\neg\operatorname{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\omega}\right)\bigr)\bigr)\in\mathfrak{g}\text{-O}\bigl[\mathfrak{T}_{\mathfrak{g},\Sigma}\bigr]\times\mathfrak{g}\text{-K}\bigl[\mathfrak{T}_{\mathfrak{g},\Sigma}\bigr]$ holds. Thus, $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\in\text{D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ implies $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\in\mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, or equivalently, $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\in\mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ implies $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\in\text{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$. This proves the proposition. $\square$

The following corollary is another way of saying that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectivity of $\mathfrak{T}_{\mathfrak{g},\Omega}$ implies the $\mathfrak{T}_{\mathfrak{g}}$-connectivity of $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

Corollary 3.36. *Let $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega}\longrightarrow\mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous map and let $\mathscr{S}_{\mathfrak{g},\omega}\subset\mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set. If $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)$ is $\mathfrak{T}_{\mathfrak{g}}$-separated in $\mathfrak{T}_{\mathfrak{g},\Sigma}$, then $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated in $\mathfrak{T}_{\mathfrak{g},\Sigma}$, then:*

$$\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\in\text{D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]\implies\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\in\mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right].\tag{3.35}$$

Theorem 3.37. *Let $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega}\longrightarrow\mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute map and let $\mathscr{S}_{\mathfrak{g},\omega}\subset\mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set. If $\operatorname{dom}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Omega}$, then $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Sigma}$:*

$$\operatorname{dom}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\in\mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]\implies\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)\in\mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right].\tag{3.36}$$

Proof. Let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ be a $\mathfrak{g}\text{-}\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-irresolute map $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$, let $\mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set, and suppose $\operatorname{im}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right)$ be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. Since $\operatorname{im}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) \notin \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, or equivalently $\operatorname{im}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, by hypothesis, there exists $\left(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\xi_\sigma,\zeta_\sigma} \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right] \times \mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ such that,

$$\Bigg(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{U}_{\mathfrak{g},\lambda} = \operatorname{im}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right)\Bigg) \bigvee \Bigg(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{V}_{\mathfrak{g},\lambda} = \operatorname{im}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right)\Bigg).$$

On the other hand, since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, there exists, for any $\lambda \in \{\xi_\sigma,\zeta_\sigma\}$, $\left(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)},\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\right) \in \mathscr{T}_{\mathfrak{g},\Omega} \times \neg\mathscr{T}_{\mathfrak{g},\Omega}$, satisfying $\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}$, $\mathscr{K}_{\mathfrak{g},(\omega,\lambda)} \subset \operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right)$, such that,

$$\left[\pi_{\mathfrak{g}}^{-1}\left(\mathscr{U}_{\mathfrak{g},\lambda}\right) \subseteq \operatorname{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}\right)\right] \vee \left[\pi_{\mathfrak{g}}^{-1}\left(\mathscr{V}_{\mathfrak{g},\lambda}\right) \supseteq \neg\operatorname{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\right)\right].$$

Since both the relation $\pi_{\mathfrak{g}}^{-1}\left(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{U}_{\mathfrak{g},\lambda}\right) = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}^{-1}\left(\mathscr{U}_{\mathfrak{g},\lambda}\right)$ and the relation $\pi_{\mathfrak{g}}^{-1}\left(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{V}_{\mathfrak{g},\lambda}\right) = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}^{-1}\left(\mathscr{V}_{\mathfrak{g},\lambda}\right)$. Evidently, $\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}^{-1}\left(\mathscr{U}_{\mathfrak{g},\lambda}\right) \subset \operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right)$, and also $\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}^{-1}\left(\mathscr{V}_{\mathfrak{g},\lambda}\right) \subset \operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right)$, and from which it follows that a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation $\operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\right)$ is realised in $\mathfrak{T}_{\mathfrak{g},\Omega}$. Consequently, $\operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$. Therefore, $\operatorname{im}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) \in \text{D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ implies $\operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$, or equivalently, $\operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) \in \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ implies $\operatorname{im}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) \in \text{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$. This proves proves the proposition. ☐

In actual fact, between any two such $\mathscr{T}_{\mathfrak{g}}$-spaces $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness, being a $\mathscr{T}_{\mathfrak{g}}$-property, is preserved by a $\mathfrak{g}\text{-}\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-homeomorphism $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \cong \mathfrak{T}_{\mathfrak{g},\Sigma}$.

Theorem 3.38. *Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces, and let $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \cong \mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}\text{-}\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-homeomorphism. If $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, then $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected:*

$$\mathfrak{T}_{\mathfrak{g},\Omega} \cong \mathfrak{T}_{\mathfrak{g},\Sigma} : \quad \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{\langle\mathrm{C}\rangle} = \left(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\Omega}^{\langle\mathrm{C}\rangle}\right) \implies \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{\langle\mathrm{C}\rangle} = \left(\Sigma,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\Sigma}^{\langle\mathrm{C}\rangle}\right). \tag{3.37}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces, let $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \cong \mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}\text{-}\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-homeomorphism, and suppose that $\mathfrak{T}_{\mathfrak{g},\Sigma}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated. There exists, then, $\left(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\xi_\sigma,\zeta_\sigma} \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right] \times \mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ such that,

$$\Bigg(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{U}_{\mathfrak{g},\lambda} = \operatorname{dom}\left(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma}\right)\Bigg) \bigvee \Bigg(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{V}_{\mathfrak{g},\lambda} = \operatorname{dom}\left(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma}\right)\Bigg).$$

Clearly, $\operatorname{dom}\left(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma}\right) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ and, with no loss of generality, consider the setting $\operatorname{dom}\left(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma}\right) = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\operatorname{dom}\left(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\right)$ so that, for every $\lambda \in \{\xi_\sigma,\zeta_\sigma\}$, either $\mathscr{U}_{\mathfrak{g},\lambda} = \operatorname{dom}\left(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\right)$ or $\mathscr{V}_{\mathfrak{g},\lambda} = \operatorname{dom}\left(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\right)$. Since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-Hom}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, $\pi_{\mathfrak{g}}^{-1} : \mathfrak{T}_{\mathfrak{g},\Sigma} \cong \mathfrak{T}_{\mathfrak{g},\Omega}$ and, for any $\left(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta}\right) \in \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g},\Lambda}\right] \times \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g},\Lambda}\right]$, $\pi_{\mathfrak{g}}^{-1}\left(\bigsqcup_{\lambda=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\lambda}\right) =$

$\bigsqcup_{\lambda=\alpha,\beta}\pi_{\mathfrak{g}}^{-1}\left(\mathscr{S}_{\mathfrak{g},\lambda}\right)$, where $\Lambda\in\{\Omega,\Sigma\}$, it results that,

$$\begin{aligned}&\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}^{-1}\big(\mathscr{U}_{\mathfrak{g},\lambda}\big)=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\operatorname{im}\big(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\big)\Big)\\ \bigvee&\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}^{-1}\big(\mathscr{V}_{\mathfrak{g},\lambda}\big)=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\operatorname{im}\big(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\big)\Big),\end{aligned}$$

where $\operatorname{im}\big(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma}\big)=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\operatorname{im}\big(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\big)$. On the other hand, since $\pi_{\mathfrak{g}}^{-1}\big(\mathscr{U}_{\mathfrak{g},\lambda}\big)\in\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ and $\pi_{\mathfrak{g}}^{-1}\big(\mathscr{V}_{\mathfrak{g},\lambda}\big)\in\mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ hold for every $\lambda\in\{\xi_\sigma,\zeta_\sigma\}$, there exist, therefore, $\big(\mathscr{U}_{\mathfrak{g},\eta},\mathscr{V}_{\mathfrak{g},\eta}\big)_{\eta=\xi_\omega,\zeta_\omega}\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ such that,

$$\begin{aligned}\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}^{-1}\big(\mathscr{U}_{\mathfrak{g},\lambda}\big)&=\bigsqcup_{\eta=\xi_\omega,\zeta_\omega}\mathscr{U}_{\mathfrak{g},\eta},\\ \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}^{-1}\big(\mathscr{V}_{\mathfrak{g},\lambda}\big)&=\bigsqcup_{\eta=\xi_\omega,\zeta_\omega}\mathscr{V}_{\mathfrak{g},\eta}.\end{aligned}$$

By substitution, then, it follows that,

$$\begin{aligned}&\Big(\bigsqcup_{\eta=\xi_\omega,\zeta_\omega}\mathscr{U}_{\mathfrak{g},\eta}=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\operatorname{im}\big(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\big)\Big)\\ \bigvee&\Big(\bigsqcup_{\eta=\xi_\omega,\zeta_\omega}\mathscr{V}_{\mathfrak{g},\eta}=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\operatorname{im}\big(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\big)\Big).\end{aligned}$$

Since $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Hom}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ and $\pi_{\mathfrak{g}}^{-1}\in\mathfrak{g}\text{-}\mathrm{Hom}\left[\mathfrak{T}_{\mathfrak{g},\Sigma};\mathfrak{T}_{\mathfrak{g},\Omega}\right]$, for each $\operatorname{im}\big(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\big)$, there exists a unique $\operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega_\eta}\big)$, with $\operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega}\big)=\bigsqcup_{\eta=\xi_\omega,\zeta_\omega}\operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega_\eta}\big)$. Thus, there exists $\big(\mathscr{U}_{\mathfrak{g},\eta},\mathscr{V}_{\mathfrak{g},\eta}\big)_{\eta=\xi_\omega,\zeta_\omega}\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ such that,

$$\Big(\bigsqcup_{\eta=\xi_\omega,\zeta_\omega}\mathscr{U}_{\mathfrak{g},\eta}=\operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega}\big)\Big)\bigvee\Big(\bigsqcup_{\eta=\xi_\omega,\zeta_\omega}\mathscr{V}_{\mathfrak{g},\eta}=\operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega}\big)\Big).$$

Hence, $\mathfrak{T}_{\mathfrak{g},\Omega}=(\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated. □

An immediate consequence of the above theorem is the following corollary.

Corollary 3.39. *Let $\mathfrak{T}_{\mathfrak{g},\Omega}=(\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma}=(\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces, and let $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega}\cong\mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-homeomorphism. If $\mathfrak{T}_{\mathfrak{g},\Sigma}=(\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated, then $\mathfrak{T}_{\mathfrak{g},\Omega}=(\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated:*

$$(3.38)\ \mathfrak{T}_{\mathfrak{g},\Omega}\cong\mathfrak{T}_{\mathfrak{g},\Sigma}:\quad \mathfrak{g}\text{-}\mathfrak{T}^{\langle\mathrm{D}\rangle}_{\mathfrak{g},\Omega}=\big(\Omega,\mathfrak{g}\text{-}\mathscr{T}^{\langle\mathrm{D}\rangle}_{\mathfrak{g},\Omega}\big)\ \Leftarrow\ \mathfrak{g}\text{-}\mathfrak{T}^{\langle\mathrm{D}\rangle}_{\mathfrak{g},\Sigma}=\big(\Sigma,\mathfrak{g}\text{-}\mathscr{T}^{\langle\mathrm{D}\rangle}_{\mathfrak{g},\Sigma}\big).$$

For every $\mu\in I_n^*$, let $\mathfrak{g}\text{-}\mathfrak{T}^{\langle\mathrm{C}\rangle}_{\mathfrak{g},\mu}=\big(\Omega_\mu,\mathfrak{g}\text{-}\mathscr{T}^{\langle\mathrm{C}\rangle}_{\mathfrak{g},\mu}\big)$ stand for the shortened form of $\mathfrak{g}\text{-}\mathfrak{T}^{\langle\mathrm{C}\rangle}_{\mathfrak{g}}(\Omega_\mu)=\big(\Omega_\mu,\mathfrak{g}\text{-}\mathscr{T}^{\langle\mathrm{C}\rangle}_{\mathfrak{g}}(\Omega_\mu)\big)$. In the following lemma, it is proved that the Cartesian product of two $\mathfrak{g}$-$\mathscr{T}^{\langle\mathrm{C}\rangle}_{\mathfrak{g}}$-spaces is also a $\mathfrak{g}$-$\mathscr{T}^{\langle\mathrm{C}\rangle}_{\mathfrak{g}}$-space.

Lemma 3.40. *If $\mathfrak{g}\text{-}\mathfrak{T}^{\langle\mathrm{C}\rangle}_{\mathfrak{g},\mu}=\big(\Omega_\mu,\mathfrak{g}\text{-}\mathscr{T}^{\langle\mathrm{C}\rangle}_{\mathfrak{g},\mu}\big)$, $\mu\in\{\alpha,\beta\}$, be two $\mathfrak{g}$-$\mathscr{T}^{\langle\mathrm{C}\rangle}_{\mathfrak{g}}$-spaces, then $\bigtimes_{\mu=\alpha,\beta}\mathfrak{g}\text{-}\mathfrak{T}^{\langle\mathrm{C}\rangle}_{\mathfrak{g},\mu}$ is also a $\mathfrak{g}$-$\mathscr{T}^{\langle\mathrm{C}\rangle}_{\mathfrak{g}}$-space.*

Proof. Let $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle} = \big(\Omega_\mu, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}\big)$, $\mu \in \{\alpha, \beta\}$, be two $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-spaces, and suppose $\xi = (\xi_\alpha, \xi_\beta) \in \times_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$ and $\zeta = (\zeta_\alpha, \zeta_\beta) \in \times_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$ be any two pairs of points in $\times_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$. Then,

$$\big[\{\xi_\alpha\} \times \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\beta}^{\langle \mathrm{C}\rangle} \cong \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\beta}^{\langle \mathrm{C}\rangle}\big] \wedge \big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\alpha}^{\langle \mathrm{C}\rangle} \times \{\zeta_\beta\} \cong \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\alpha}^{\langle \mathrm{C}\rangle}\big].$$

Consequently, $\{\xi_\alpha\} \times \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\beta}^{\langle \mathrm{C}\rangle}$ and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\alpha}^{\langle \mathrm{C}\rangle} \times \{\zeta_\beta\}$ are both $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. But,

$$\big(\{\xi_\alpha\} \times \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\beta}^{\langle \mathrm{C}\rangle}\big) \cap \big(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\alpha}^{\langle \mathrm{C}\rangle} \times \{\zeta_\beta\}\big) = \{(\xi_\alpha, \zeta_\beta)\} \neq \emptyset.$$

Hence, $\big(\{\xi_\alpha\} \times \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\beta}^{\langle \mathrm{C}\rangle}\big) \cup \big(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\alpha}^{\langle \mathrm{C}\rangle} \times \{\zeta_\beta\}\big)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. Accordingly, $\xi,\ \zeta \in \times_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$ belong to the same $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component. That is, $\xi,\ \zeta \in \mathfrak{g}\text{-}\mathrm{C}_{\Omega}[\eta] \subseteq \times_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component of $\Omega = \times_{\mu=\alpha,\beta} \Omega_\mu$ corresponding to the point $\eta \in \times_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$. But $\xi,\ \zeta \in \times_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$ were arbitrary. Hence the Cartesian product $\times_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$ has one $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}\text{-}\mathrm{C}_{\Omega}[\eta] = \times_{\mu=\alpha,\beta} \Omega_\mu$, and is therefore a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-space. □

More generally, the Cartesian product of $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-spaces is also a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-space; that is, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is a product invariant $\mathscr{T}_{\mathfrak{g}}$-property. The theorem follows.

Theorem 3.41. *If* $\big\{\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle} = \big(\Omega_\mu, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}\big) :\ \mu \in I_n^*\big\}$ *be a collection of* $n \geq 1$ $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$*-spaces, then* $\times_{\mu\in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$ *is also a* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$*-space:*

$$\big\{\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle} = \big(\Omega_\mu, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}\big) :\ \mu \in I_n^*\big\} \ \Rightarrow\ \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle} = \bigtimes_{\mu\in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}. \tag{3.39}$$

Proof. Let $\big\{\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle} = \big(\Omega_\mu, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}\big) :\ \mu \in I_n^*\big\}$ be a collection of $n \geq 1$ $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-spaces, and let $\times_{\mu\in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$ be the Cartesian product of these $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-spaces. Moreover, let $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \times_{\mu\in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$, and let $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \subseteq \times_{\mu\in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$ be the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component of $\mathscr{S}_{\mathfrak{g}} \subseteq \times_{\mu\in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$ corresponding to $\zeta \in \times_{\mu\in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$. By hypothesis, let it be claimed that, for every $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \times_{\mu\in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$, $\xi \in \neg\,\mathrm{op}_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\big)$ and, thus, $\xi \in \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ since $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\big)$, meaning that $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ must be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set in $\times_{\mu\in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$. For every $\big(\mu, \sigma(\mu), \mathscr{O}_{\mathfrak{g},\sigma(\mu)}\big) \in \big\{\mu\big\} \times I_\infty^* \times \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$, there exists $I_{\sigma(\mu)} \subseteq I_\infty^*$ such that $\mathscr{O}_{\mathfrak{g},\sigma(\mu)} = \bigcup_{\nu\in I_{\sigma(\mu)}^*} \mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)}$. Thus, the class $\mathscr{B}\big[\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}\big] \overset{\text{def}}{=} \big\{\mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)} :\ (\nu, \mu, \sigma(\nu,\mu)) \in I_\infty^* \times \big\{\mu\big\} \times I_\infty^*\big\}$ is a $\mathscr{T}_{\mathfrak{g}}$-basis for $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle} : \mathscr{P}(\Omega_\mu) \to \mathscr{P}(\Omega_\mu)$. Therefore, for any $\xi \in \mathscr{O}_{\mathfrak{g},\sigma(\mu)} \in \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$, there exists $\mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)} \in \mathscr{B}\big[\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}\big]$ with $\xi \in \mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)} \subseteq \mathscr{O}_{\mathfrak{g},\sigma(\mu)} \in \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}$. Now let

$$\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathscr{R}_{\mathfrak{g}} = \Bigg(\bigtimes_{\mu\in I_n^*\setminus J_n} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}\Bigg) \times \Bigg(\bigtimes_{\mu\in J_n\subset I_n^*} \mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)}\Bigg).$$

Now the following relation holds,

$$\mathscr{S}_{\mathfrak{g}} = \Bigg(\bigtimes_{\mu\in I_n^*\setminus J_n} \{\zeta_\mu\}\Bigg) \times \Bigg(\bigtimes_{\mu\in J_n\subset I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle}\Bigg) \cong \bigtimes_{\mu\in J_n\subset I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C}\rangle},$$

and, hence, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. Furthermore, $\zeta \in \mathscr{S}_{\mathfrak{g}}$ and, consequently, it follows that $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \supseteq \mathscr{S}_{\mathfrak{g}}$. But, by the property of the intersection of Cartesian products,

$$\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}} = \Bigg(\bigtimes_{\mu \in I_n^* \setminus J_n} \{\zeta_\mu\} \Bigg) \times \Bigg(\bigtimes_{\mu \in J_n \subset I_n^*} \mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)} \Bigg) \neq \emptyset.$$

Therefore, $\mathscr{R}_{\mathfrak{g}} \subset \bigtimes_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C} \rangle}$ contains a point of $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$. Accordingly, $\xi \in \neg \operatorname{op}_{\mathfrak{g}}(\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]) \subseteq \mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$. Hence the Cartesian product $\bigtimes_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C} \rangle}$ has one $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}\text{-C}_{\Omega}[\zeta] = \bigtimes_{\mu \in I_n^*} \Omega_\mu$, and is therefore a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}$-space. □

The concept of $(\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma})$-surjective map between any such $\mathscr{T}_{\mathfrak{g}}$-spaces $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$ is now defined.

Definition 3.42 ($(\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma})$-Surjective Map)**.** A $(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-map $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ is said to be surjective if and only if it belongs the following class:

$$\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}] \stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}} :\ (\forall \zeta \in \mathfrak{T}_{\mathfrak{g},\Sigma})(\exists \xi \in \mathfrak{T}_{\mathfrak{g},\Omega})\big[\pi_{\mathfrak{g}}(\xi) = \zeta\big]\big\}. \tag{3.40}$$

If the domain of a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute surjective map is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, then its codomain is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, as demonstrated in the theorem below.

Theorem 3.43. *Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces. If $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}] \cap \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute surjective map $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \to \mathfrak{T}_{\mathfrak{g},\Sigma}$ and $\mathfrak{T}_{\mathfrak{g},\Omega}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, then $\mathfrak{T}_{\mathfrak{g},\Sigma}$ is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected.*

Proof. Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces of which $\mathfrak{T}_{\mathfrak{g},\Sigma}$ is assumed to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated, and let $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \to \mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute surjective map. Since $\operatorname{dom}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$, there exists $(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda})_{\lambda=\xi_\sigma,\zeta_\sigma} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ such that,

$$\begin{aligned}&\Bigg(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda} = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \operatorname{dom}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}) \Bigg)\\ \bigvee &\Bigg(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda} = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \operatorname{dom}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}) \Bigg),\end{aligned}$$

where $\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \operatorname{dom}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}) = \operatorname{dom}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ so that, for every $\lambda \in \{\xi_\sigma, \zeta_\sigma\}$, either $\mathscr{U}_{\mathfrak{g},\lambda} = \operatorname{dom}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda})$ or $\mathscr{V}_{\mathfrak{g},\lambda} = \operatorname{dom}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda})$. Since the relation $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$ holds, there exists $(\mathscr{O}_{\mathfrak{g},\eta}, \mathscr{K}_{\mathfrak{g},\eta}) \in \mathscr{T}_{\mathfrak{g},\Omega} \times \neg\mathscr{T}_{\mathfrak{g},\Omega}$, $\eta \in \{\xi_\omega, \zeta_\omega\}$ with $(\lambda, \eta) \in \{(\xi_\sigma, \xi_\omega), (\zeta_\sigma, \zeta_\omega)\}$, such that,

$$\big[\pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\lambda}) \subseteq \operatorname{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\eta})\big] \vee \big[\pi_{\mathfrak{g}}^{-1}(\mathscr{V}_{\mathfrak{g},\lambda}) \supseteq \neg \operatorname{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\eta})\big].$$

Evidently, $\operatorname{dom}(\pi_{\mathfrak{g}|\Omega}) = \operatorname{im}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma})$ and $\operatorname{dom}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma}) = \operatorname{im}(\pi_{\mathfrak{g}|\Omega})$. Since $\operatorname{im}(\pi_{\mathfrak{g}|\Omega})$, $\operatorname{dom}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$, set $\operatorname{im}(\pi_{\mathfrak{g}|\Omega}) = \bigsqcup_{\eta=\xi_\omega,\zeta_\omega} \operatorname{im}(\pi_{\mathfrak{g}|\Omega_\eta})$ and $\operatorname{dom}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma}) = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \operatorname{dom}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda})$, and for any $(\lambda, \eta) \in \{(\xi_\sigma, \xi_\omega), (\zeta_\sigma, \zeta_\omega)\}$ set

$$\begin{aligned}&\big[\operatorname{im}(\pi_{\mathfrak{g}|\Omega_\eta}) = \operatorname{dom}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}) = \mathscr{U}_{\mathfrak{g},\lambda}\big]\\ \vee&\big[\operatorname{im}(\pi_{\mathfrak{g}|\Omega_\eta}) = \operatorname{dom}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}) = \mathscr{V}_{\mathfrak{g},\lambda}\big].\end{aligned}$$

Since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, for every $(\lambda,\eta) \in \{(\xi_\sigma,\xi_\omega),(\zeta_\sigma,\zeta_\omega)\}$,

$$\left(\bigcap_{\eta=\xi_\omega,\zeta_\omega} \operatorname{dom}\bigl(\pi_{\mathfrak{g}|\Omega_\eta}\bigr) = \bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \operatorname{im}\bigl(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\bigr) = \bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \pi_{\mathfrak{g}}^{-1}\bigl(\mathscr{U}_{\mathfrak{g},\lambda}\bigr) = \emptyset\right)$$
$$\bigvee\left(\bigcap_{\eta=\xi_\omega,\zeta_\omega} \operatorname{dom}\bigl(\pi_{\mathfrak{g}|\Omega_\eta}\bigr) = \bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \operatorname{im}\bigl(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\bigr) = \bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \pi_{\mathfrak{g}}^{-1}\bigl(\mathscr{V}_{\mathfrak{g},\lambda}\bigr) = \emptyset\right),$$

Thus, $\operatorname{dom}\bigl(\pi_{\mathfrak{g}|\Omega}\bigr) = \bigsqcup_{\eta=\xi_\omega,\zeta_\omega} \operatorname{dom}\bigl(\pi_{\mathfrak{g}|\Omega_\eta}\bigr)$. Since the relation $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] \cap \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ holds, it follows that

$$\bigsqcup_{\eta=\xi_\omega,\zeta_\omega} \operatorname{dom}\bigl(\pi_{\mathfrak{g}|\Omega_\eta}\bigr) = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \operatorname{im}\bigl(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\bigr) \subseteq \bigsqcup_{\eta=\xi_\omega,\zeta_\omega} \operatorname{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\eta}\right),$$
$$\bigcap_{\eta=\xi_\omega,\zeta_\omega} \operatorname{dom}\bigl(\pi_{\mathfrak{g}|\Omega_\eta}\bigr) = \bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \operatorname{im}\bigl(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\bigr) \supseteq \bigcap_{\eta=\xi_\omega,\zeta_\omega} \neg\operatorname{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\eta}\right).$$

Thus, $\bigsqcup_{\eta=\xi_\omega,\zeta_\omega} \operatorname{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\eta}\right) = \operatorname{dom}\bigl(\pi_{\mathfrak{g}|\Omega}\bigr)$ and $\bigcap_{\eta=\xi_\omega,\zeta_\omega} \neg\operatorname{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\eta}\right) = \emptyset$. There exists, then, $\bigl(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\bigr)_{\lambda=\xi_\omega,\zeta_\omega} \in \mathfrak{g}\text{-O}\bigl[\mathfrak{T}_{\mathfrak{g},\Omega}\bigr] \times \mathfrak{g}\text{-K}\bigl[\mathfrak{T}_{\mathfrak{g},\Omega}\bigr]$ such that,

$$\left(\bigsqcup_{\eta=\xi_\omega,\zeta_\omega} \mathscr{U}_{\mathfrak{g},\eta} = \operatorname{dom}\bigl(\pi_{\mathfrak{g}|\Omega}\bigr)\right)\bigvee\left(\bigsqcup_{\eta=\xi_\omega,\zeta_\omega} \mathscr{V}_{\mathfrak{g},\eta} = \operatorname{dom}\bigl(\pi_{\mathfrak{g}|\Omega}\bigr)\right).$$

Thus, $\operatorname{dom}\bigl(\pi_{\mathfrak{g}|\Omega}\bigr) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$, or equivalently $\operatorname{dom}\bigl(\pi_{\mathfrak{g}|\Omega}\bigr) \notin \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ which contradicts the assumption that $\operatorname{dom}\bigl(\pi_{\mathfrak{g}|\Omega}\bigr) \in \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$. Hence, $\mathfrak{T}_{\mathfrak{g},\Sigma}$ must be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. □

Pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is also preserved under a $\mathfrak{g}$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-continuous map, as proved below.

Theorem 3.44. *Let* $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ *be a* $\mathfrak{g}$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$*-continuous map and let* $\mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ *be a* $\mathfrak{T}_{\mathfrak{g}}$*-set. If* $\operatorname{dom}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)$ *is pathwise* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-connected in* $\mathfrak{T}_{\mathfrak{g},\Omega}$*, then* $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)$ *is also pathwise* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-connected in* $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

Proof. Let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, $\mathscr{S}_{\mathfrak{g},\omega} = \operatorname{dom}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr) \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ be pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Omega}$, and suppose ξ_σ, $\zeta_\sigma \in \operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)$. Then, there exist ξ_ω, $\zeta_\omega \in \mathfrak{T}_{\mathfrak{g},\Omega}$ such that $\bigl(\pi_{\mathfrak{g}}(\xi_\omega),\pi_{\mathfrak{g}}(\zeta_\omega)\bigr) = (\xi_\sigma,\zeta_\sigma)$. But $\operatorname{dom}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Omega}$ and, therefore, there exists a path $\varphi_{\mathfrak{g},\zeta} : [0,1] \to \mathfrak{T}_{\mathfrak{g},\Omega}$ such that $\varphi_{\mathfrak{g},\zeta}(0) = \xi_\omega$, $\varphi_{\mathfrak{g},\zeta}(1) = \zeta_\omega$, and $\operatorname{im}\bigl(\varphi_{\mathfrak{g},\zeta|[0,1]}\bigr) \subseteq \operatorname{dom}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)$. Since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ and $\varphi_{\mathfrak{g},\zeta} \in \text{C}\bigl[[0,1];\mathfrak{T}_{\mathfrak{g},\Omega}\bigr]$, it follows that $\pi_{\mathfrak{g}} \circ \varphi_{\mathfrak{g},\zeta} \in \mathfrak{g}\text{-C}\bigl[[0,1];\mathfrak{T}_{\mathfrak{g},\Sigma}\bigr]$. Moreover, $\pi_{\mathfrak{g}}\circ\varphi_{\mathfrak{g},\zeta}(0) = \pi_{\mathfrak{g}}(\xi_\omega) = \xi_\sigma$, $\pi_{\mathfrak{g}}\circ\varphi_{\mathfrak{g},\zeta}(1) = \pi_{\mathfrak{g}}(\zeta_\omega) = \zeta_\sigma$, and $\operatorname{im}\bigl(\pi_{\mathfrak{g}}\circ\varphi_{\mathfrak{g},\zeta}\bigr) \subseteq \operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)$. Hence, $\operatorname{im}\bigl(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\bigr)$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. □

In the discussion section, categorical classifications of the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-disconnectedness are presented. Thereafter, a nice application is given and, finally, the work is terminated with a concluding remarks section.

4. Discussion

4.1. **Categorical Classifications.** Having adopted a categorical approach in the classifications of the $\mathscr{T}_{\mathfrak{g}}$-properties in $\mathscr{T}_{\mathfrak{g}}$-spaces, called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-disconnectedness, the dual aims of the present section are, to establish the various relations amongst the elements of the sequence $\big\langle \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}\big)\big\rangle_{\nu\in\ I_3^0}$ of $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-spaces and the elements of the sequence $\big\langle \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}\big)\big\rangle_{\nu\in\ I_3^0}$ of $\mathfrak{g}$-$\mathscr{T}^{\langle \mathrm{C}\rangle}$-spaces, and to illustrate them through diagrams.

If a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-separated, then $\mathfrak{T}_{\mathfrak{g}}$ has a nonempty proper $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, where $\nu \in I_3^0$. But, for every $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, the relation $\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds; for every $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, the relation given by $\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds. Consequently,

$$\begin{aligned}
&\mathrm{op}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{op}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}};\\
&\neg\,\mathrm{op}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) \subseteq \neg\,\mathrm{op}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}}) \ \forall \mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}.
\end{aligned}$$

Therefore, if $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is a nonempty proper $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set then,

$$\begin{array}{ccc}
\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}0\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}0\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] & \Longrightarrow & \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}1\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}1\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\\
& & \Downarrow\\
\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}2\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}2\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] & \Longrightarrow & \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}3\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}3\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big].
\end{array}$$

In other words, $\mathfrak{g}$-3-$\mathfrak{T}_{\mathfrak{g}}$-separation implies $\mathfrak{g}$-1-$\mathfrak{T}_{\mathfrak{g}}$-separation and the latter in turn implies $\mathfrak{g}$-0-$\mathfrak{T}_{\mathfrak{g}}$-separation. On the other hand, $\mathfrak{g}$-2-$\mathfrak{T}_{\mathfrak{g}}$-separation is implied by $\mathfrak{g}$-3-$\mathfrak{T}_{\mathfrak{g}}$-separation. Similar implications also hold for $\mathfrak{g}$-$\mathfrak{T}$-separateness in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$. For, if $\mathscr{S} \subset \mathfrak{T}$ is a nonempty proper $\mathfrak{g}$-$\mathfrak{T}$-open-closed set then,

$$\begin{array}{ccc}
\mathscr{S} \in \mathfrak{g}\text{-}0\text{-}\mathrm{O}\big[\mathfrak{T}\big] \cap \mathfrak{g}\text{-}0\text{-}\mathrm{K}\big[\mathfrak{T}\big] & \Longrightarrow & \mathscr{S} \in \mathfrak{g}\text{-}1\text{-}\mathrm{O}\big[\mathfrak{T}\big] \cap \mathfrak{g}\text{-}1\text{-}\mathrm{K}\big[\mathfrak{T}\big]\\
& & \Downarrow\\
\mathscr{S} \in \mathfrak{g}\text{-}2\text{-}\mathrm{O}\big[\mathfrak{T}\big] \cap \mathfrak{g}\text{-}2\text{-}\mathrm{K}\big[\mathfrak{T}\big] & \Longrightarrow & \mathscr{S} \in \mathfrak{g}\text{-}3\text{-}\mathrm{O}\big[\mathfrak{T}\big] \cap \mathfrak{g}\text{-}3\text{-}\mathrm{K}\big[\mathfrak{T}\big].
\end{array}$$

As above, $\mathfrak{g}$-3-$\mathfrak{T}_{\mathfrak{g}}$-separation implies $\mathfrak{g}$-1-$\mathfrak{T}_{\mathfrak{g}}$-separation and the latter in turn implies $\mathfrak{g}$-0-$\mathfrak{T}_{\mathfrak{g}}$-separation. On the other hand, $\mathfrak{g}$-2-$\mathfrak{T}_{\mathfrak{g}}$-separation is implied by $\mathfrak{g}$-3-$\mathfrak{T}_{\mathfrak{g}}$-separation.

For visualization, a so-called *categorical connectedness diagram*, expressing the various relations amongst $\mathfrak{g}$-$\mathfrak{T}$-connected and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected spaces, is presented in Fig. **??** and that, expressing the various relationships amongst $\mathfrak{g}$-$\mathfrak{T}$-connected and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected spaces, so-called *categorical disconnectedness diagram*, is presented in Fig. **??**. The categorical classifications of $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{LC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{LC}\rangle}\big)$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{PC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{PC}\rangle}\big)$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{LPC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{LPC}\rangle}\big)$, and $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{SC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{SC}\rangle}\big)$ called, respectively, *locally*, *pathwise*, *locally pathwise*, and *simply* $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-spaces can be diagrammed in an analogous manner. The following implications concern the transformations of $\mathfrak{g}$-$\mathfrak{T}$-connected sets under some types of $\mathfrak{g}$-$\mathfrak{T}$-maps.

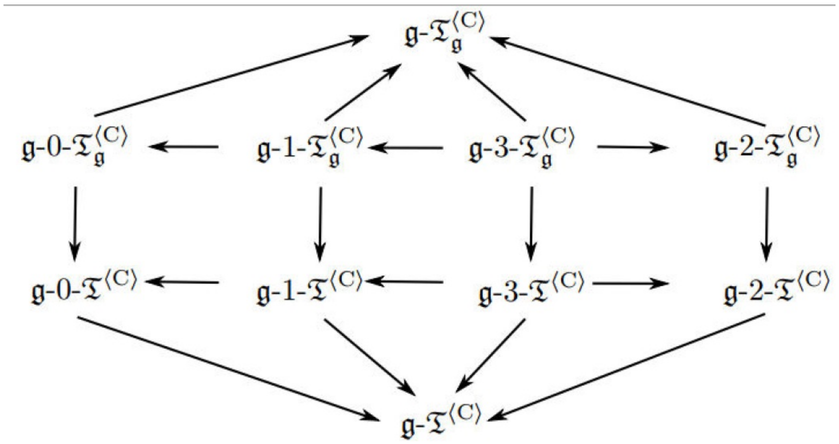

FIGURE 1. Relationships: $\mathfrak{g}$-$\mathfrak{T}$-connected and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected spaces.

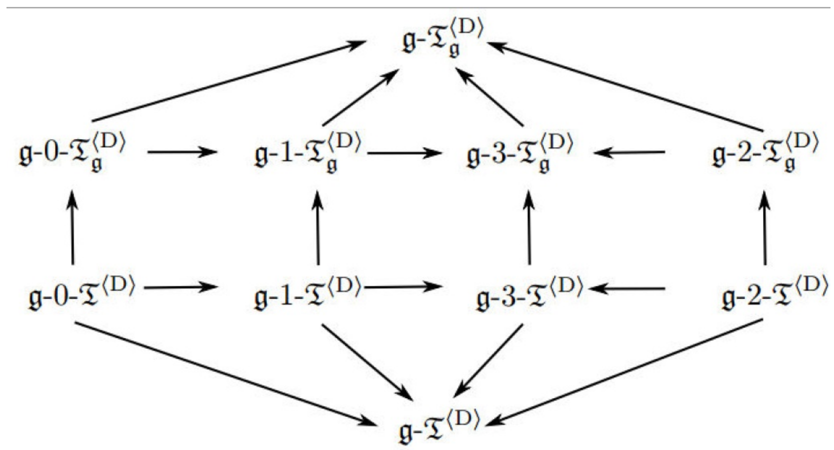

FIGURE 2. Relationships: $\mathfrak{g}$-$\mathfrak{T}$-separated and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated spaces.

For every $\nu \in I_3^0$, if $\pi_{\mathfrak{g}} \in \mathfrak{g}$-$\nu$-$\mathrm{I}\,[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}] \cap \mathfrak{g}$-$\nu$-$\mathrm{S}\,[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$ holds, then $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}\big) \in \mathfrak{g}$-$\nu$-$\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ implies $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}\big) \in \mathfrak{g}$-$\nu$-$\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]$, and hence the following implication:

$$\begin{array}{c} \big(\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}\big) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]\big) \wedge \big(\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{I}\,[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}] \cap \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\,[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]\big) \\ \Downarrow \\ \mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}\big) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]. \end{array} \tag{4.1}$$

For every $\nu \in I_3^0$, if $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ holds, then $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}\big) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ implies $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}\big) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]$, and hence the following implication:

$$\begin{gathered}\big(\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}\big) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]\big) \wedge \big(\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]\big)\\ \Downarrow\\ \mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}\big) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big].\end{gathered}\tag{4.2}$$

In the following section a nice application comprising of some interesting cases is discussed.

4.2. **A Nice Application.** Focusing on basic concepts from the point of view of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness, we shall now present a nice application comprising of some interesting cases. Let $\Omega_\sigma = \big\{\xi_\nu :\ \nu \in I_\sigma^*\big\}$ denote the underlying set, conditioned by the parameter $\sigma \in I_\infty^*$, and consider the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\sigma} = (\Omega_\sigma, \mathscr{T}_{\mathfrak{g},\sigma})$, where $\mathscr{T}_{\mathfrak{g},\sigma} : \mathscr{P}(\Omega_\sigma) \longrightarrow \mathscr{P}(\Omega_\sigma)$ will be defined in the following cases.

– Case i. Set $\sigma = 1$. Then, $\Omega_1 = \{\xi_1\}$, $\mathscr{T}_{\mathfrak{g},1} = \big\{\emptyset, \Omega_1\big\} = \big\{\mathscr{O}_{\mathfrak{g},1}, \mathscr{O}_{\mathfrak{g},2}\big\}$, $\neg\mathscr{T}_{\mathfrak{g},1} = \big\{\Omega_1, \emptyset\big\} = \big\{\mathscr{K}_{\mathfrak{g},1}, \mathscr{K}_{\mathfrak{g},2}\big\}$, and, for every $(\mu,\nu) \in I_2^* \times I_3^0$ it results that $\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},\mu})$, $\neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g},\mu}) \in \big\{\mathscr{O}_{\mathfrak{g},1}, \mathscr{K}_{\mathfrak{g},1}, \mathscr{O}_{\mathfrak{g},2}, \mathscr{K}_{\mathfrak{g},2}\big\} = \big\{\emptyset, \Omega_1\big\}$. Therefore, for every $\nu \in I_3^0$, $\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},1}\big] = \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},1}\big] = \big\{\emptyset, \Omega_1\big\}$. Thus, for every $\nu \in I_3^0$, there exists neither a pair $(\mathscr{U}_{\mathfrak{g},\xi}, \mathscr{U}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},1}\big] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},1}\big]$ of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets nor a pair $(\mathscr{V}_{\mathfrak{g},\xi}, \mathscr{V}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},1}\big] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},1}\big]$ of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets such that:

$$\Bigg(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega\Bigg) \bigvee \Bigg(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega\Bigg).$$

Evidently, the $\mathfrak{T}_{\mathfrak{g}}$-sets $\emptyset$, $\Omega_1 \subseteq \mathfrak{U}$ are the only $\mathfrak{T}_{\mathfrak{g}}$-open-closed sets, and $\mathfrak{g}\text{-}\mathrm{C}_{\Omega_1}[\xi_1] = \{\xi_1\} = \Omega_1$ is the unique $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component in $\mathfrak{T}_{\mathfrak{g},1}$. Thus, the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},1} = (\Omega_1, \mathscr{T}_{\mathfrak{g},1})$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},1}^{\langle\mathrm{C}\rangle} = \big(\Omega_1, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},1}^{\langle\mathrm{C}\rangle}\big)$, and the latter in turn implies that it is also a $\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}$-space $\mathfrak{T}_{\mathfrak{g},1}^{\langle\mathrm{C}\rangle} = \big(\Omega_1, \mathscr{T}_{\mathfrak{g},1}^{\langle\mathrm{C}\rangle}\big)$. Hence, *every indiscrete $\mathscr{T}_{\mathfrak{g}}$-space which is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected is also $\mathfrak{T}_{\mathfrak{g}}$-connected.* Furthermore, the underlying set $\Omega_1 = \{\xi_1\}$ being a 1-point set, it also follows that, *every discrete $\mathscr{T}_{\mathfrak{g}}$-space that has at most one point is both $\mathfrak{T}_{\mathfrak{g}}$-connected and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected.*

– Case ii. Set $\sigma = 2$. Then, $\Omega_2 = \big\{\xi_1, \xi_2\big\}$. Choose $\mathscr{T}_{\mathfrak{g},2} = \big\{\emptyset, \Omega_2\big\} = \big\{\mathscr{O}_{\mathfrak{g},1}, \mathscr{O}_{\mathfrak{g},2}\big\}$ so that, $\neg\mathscr{T}_{\mathfrak{g},2} = \big\{\Omega_2, \emptyset\big\} = \big\{\mathscr{K}_{\mathfrak{g},1}, \mathscr{K}_{\mathfrak{g},2}\big\}$. Then, the collection of $\mathfrak{T}_{\mathfrak{g}}$-open sets is $\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big] = \big\{\emptyset, \Omega_2\big\}$, and $\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},2}\big] = \mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big]$ stands for the collection of $\mathfrak{T}_{\mathfrak{g}}$-closed sets. On the other hand, for every $\nu \in I_3^0$, $\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big] = \mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big] \cup \big\{\{\xi_1\}, \{\xi_2\}\big\} = \mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},2}\big] \cup \big\{\{\xi_1\}, \{\xi_2\}\big\} = \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},2}\big]$. Clearly, there exists a pair $(\mathscr{U}_{\mathfrak{g},\xi}, \mathscr{U}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big]$ of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets or a pair $(\mathscr{V}_{\mathfrak{g},\xi}, \mathscr{V}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},2}\big] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},2}\big]$ of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets such that:

$$\Bigg(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega\Bigg) \bigvee \Bigg(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega\Bigg).$$

This description is realised when either $(\mathscr{U}_{\mathfrak{g},\xi}, \mathscr{U}_{\mathfrak{g},\zeta}) = \big(\{\xi_1\}, \{\xi_2\}\big)$ or $(\mathscr{V}_{\mathfrak{g},\xi}, \mathscr{V}_{\mathfrak{g},\zeta}) = \big(\{\xi_2\}, \{\xi_1\}\big)$. On the other hand, there exists neither a pair $(\mathscr{U}_{\mathfrak{g},\xi}, \mathscr{U}_{\mathfrak{g},\zeta}) \in \mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big] \times \mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big]$ of nonempty $\mathfrak{T}_{\mathfrak{g}}$-open sets nor a pair $(\mathscr{V}_{\mathfrak{g},\xi}, \mathscr{V}_{\mathfrak{g},\zeta}) \in \mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},2}\big] \times \mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},2}\big]$

of nonempty $\mathfrak{T}_{\mathfrak{g}}$-closed sets such that the above statement holds. Thus, the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},2}=(\Omega_2,\mathscr{T}_{\mathfrak{g},2})$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},2}^{\langle\mathrm{D}\rangle}=\left(\Omega_2,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},2}^{\langle\mathrm{D}\rangle}\right)$ but not a $\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle}$-space $\mathfrak{T}_{\mathfrak{g},2}^{\langle\mathrm{D}\rangle}=\left(\Omega_2,\mathscr{T}_{\mathfrak{g},2}^{\langle\mathrm{D}\rangle}\right)$. Alternatively said, *every* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}$*-space is a* $\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}$*-space but the converse need not be true in general.* Moreover, the underlying set $\Omega_2=\{\xi_1,\xi_2\}$ being a 2-point set, it follows that *every discrete* $\mathscr{T}_{\mathfrak{g}}$*-space that has at least two points is* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-separated.* It is plain that every $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\big\{\left(\{\xi_1\},\{\xi_1,\xi_2\}\right),\left(\{\xi_2\},\{\xi_1,\xi_2\}\right)\big\}$ is a pair of nonempty $\mathfrak{T}_{\mathfrak{g}}$-sets which are not $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated, for, $\{\xi_1\}=\{\xi_1\}\cap\{\xi_1,\xi_2\}=\complement(\{\xi_2\})=\complement(\{\xi_2\}\cap\{\xi_1,\xi_2\})\neq\emptyset$, and $\mathscr{S}_{\mathfrak{g}}=\{\xi_1,\xi_2\}$ is the only $\mathfrak{T}_{\mathfrak{g}}$-set satisfying $\mathscr{S}_{\mathfrak{g}}=\{\xi_1\}\sqcup\{\xi_2\}$. Accordingly, $\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},2}]=\big\{\{\xi_1\},\{\xi_2\}\big\}$ and $\mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},2}]=\big\{\{\xi_1,\xi_2\}\big\}$. Observe in passing that, $\Omega_2=\bigsqcup_{\zeta=\xi_1,\xi_2}\mathfrak{g}\text{-C}_{\Omega_2}[\zeta]$. Thus, *if a* $\mathscr{T}_{\mathfrak{g}}$*-space has more than one* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-component, then it is a* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle}$*-space.*

– Case iii. Set $\sigma>2$. Then, $\Omega_{>2}=\{\xi_\alpha:\ \alpha\in I_{\sigma>2}^{*}\}$. Let $\mathscr{T}_{\mathfrak{g},>2}:\mathscr{P}(\Omega_{>2})\longrightarrow\mathscr{P}(\Omega_{>2})$ generate the elements of $\mathscr{T}_{\mathfrak{g},>2}=\big\{\mathscr{O}_{\mathfrak{g},(\alpha,\beta)}:\ (\alpha,\beta)\in I_{\infty}^{0}\times I_{\infty}^{0}\big\}$ and $\neg\mathscr{T}_{\mathfrak{g},>2}=\big\{\mathscr{K}_{\mathfrak{g},(\alpha,\beta)}=\complement\big(\mathscr{O}_{\mathfrak{g},(\alpha,\beta)}\big):\ (\alpha,\beta)\in I_{\infty}^{0}\times I_{\infty}^{0}\big\}$ as thus:

$$\mathscr{O}_{\mathfrak{g},(\alpha,\beta)}=\begin{cases}\emptyset\quad\forall(\alpha,\beta)\in\{0\}\times\{0\};\\ \big\{\xi_{\alpha+\mu}:\ \mu\in I_{\beta}^{0}\big\}\quad\forall(\alpha,\beta)\in I_{\infty}^{*}\times I_{\infty}^{0};\\ \Omega_{>2}\quad\forall(\alpha,\beta)\in\{1\}\times\{\infty\}.\end{cases}\tag{4.3}$$

Clearly, $\Omega_{>2}\subseteq\mathfrak{U}$ is an ∞-point set. Furthermore, it is easily verified that, $\mathscr{T}_{\mathfrak{g},>2}(\emptyset)=\emptyset$, $\mathscr{T}_{\mathfrak{g},>2}\left(\mathscr{O}_{\mathfrak{g},(\alpha,\beta)}\right)\subseteq\mathscr{O}_{\mathfrak{g},(\alpha,\beta)}$ for every $(\alpha,\beta)\in I_{\infty}^{0}\times I_{\infty}^{0}$, and, finally, $\mathscr{T}_{\mathfrak{g},>2}\big(\bigcup_{(\alpha,\beta)\in I_{\infty}^{0}\times I_{\infty}^{0}}\mathscr{O}_{\mathfrak{g},(\alpha,\beta)}\big)=\bigcup_{(\alpha,\beta)\in I_{\infty}^{0}\times I_{\infty}^{0}}\mathscr{T}_{\mathfrak{g},>2}\left(\mathscr{O}_{\mathfrak{g},(\alpha,\beta)}\right)$. Hence, it follows that $\mathscr{T}_{\mathfrak{g},>2}:\mathscr{P}(\Omega_{>2})\longrightarrow\mathscr{P}(\Omega_{>2})$ is a $\mathfrak{g}$-topology on the ∞-point set $\Omega_{>2}$. On the other hand, it can be shown that, for every $(\alpha,\beta,\nu)\in I_{\infty}^{*}\times I_{\infty}^{0}\times I_{3}^{0}$,

$$\mathscr{O}_{\mathfrak{g},(\alpha,0)}\cap\mathscr{O}_{\mathfrak{g},(\alpha,\beta)}\subseteq\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{O}_{\mathfrak{g},(\alpha,0)}\right)\cap\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{O}_{\mathfrak{g},(\alpha,\beta)}\right)=\{\xi_\alpha\}\in\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},>2}].$$

This implies that the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},>2}=(\Omega_{>2},\mathscr{T}_{\mathfrak{g},>2})$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{LC}\rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},>2}^{\langle\mathrm{LC}\rangle}=\left(\Omega_{>2},\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},>2}^{\langle\mathrm{LC}\rangle}\right)$, and hence, it is also a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},>2}^{\langle\mathrm{C}\rangle}=\left(\Omega_{>2},\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},>2}^{\langle\mathrm{C}\rangle}\right)$.

Moreover, $\mathscr{T}_{\mathfrak{g}}$-properties relative to such $\mathscr{T}_{\mathfrak{g}}$-spaces $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{LC}\rangle}=\left(\Omega,\mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{LC}\rangle}\right)$, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{PC}\rangle}=\left(\Omega,\mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{PC}\rangle}\right)$, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{LPC}\rangle}=\left(\Omega,\mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{LPC}\rangle}\right)$, and, also, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{SC}\rangle}=\left(\Omega,\mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{SC}\rangle}\right)$ called, respectively, *locally*, *pathwise*, *locally pathwise*, and *simply* $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}$-spaces can be discussed in an analogous manner by slight modifications of some $\mathscr{T}_{\mathfrak{g}}$-properties found in those cases. The next section provides concluding remarks and future directions of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness discussed in the preceding sections.

5. Conclusion

In this paper, a new theory called *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Connectedness* has been developed, the foundation of which was based on the theories of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-maps. A careful perusal of the Mathematical developments of the earlier sections will show that the proposed theory has, in its own rights, several advantages. The very first advantage is that the theory holds equally well when $(\Lambda,\mathscr{T}_{\mathfrak{g},\Lambda})=(\Lambda,\mathscr{T}_{\Lambda})$, where $\Lambda\in\big\{\Omega,\Sigma\big\}$, and other characteristics adapted on this basis, in which case it might be called *Theory of* $\mathfrak{g}$-$\mathfrak{T}$*-Connectedness.*

Hence, in a $\mathscr{T}_{\mathfrak{g}}$-space the theoretical framework categorises such pairs of concepts as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected closed, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected semi-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected semi-closed, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected preopen and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected preclosed, and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected semi-preopen and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected semi-preclosed as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way; in a $\mathscr{T}$-space it categorises such pairs of concepts as $\mathfrak{g}$-$\mathfrak{T}$-connected open and $\mathfrak{g}$-$\mathfrak{T}$-connected closed, $\mathfrak{g}$-$\mathfrak{T}$-connected semi-open and $\mathfrak{g}$-$\mathfrak{T}$-connected semi-closed, $\mathfrak{g}$-$\mathfrak{T}$-connected preopen and $\mathfrak{g}$-$\mathfrak{T}$-connected preclosed, and $\mathfrak{g}$-$\mathfrak{T}$-connected semi-preopen and $\mathfrak{g}$-$\mathfrak{T}$-connected semi-preclosed as $\mathfrak{g}$-$\mathfrak{T}$-connected of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way.

It is an interesting topic for future research to develop the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness of mixed categories. More precisely, for some pair $(\nu,\mu)\in I_3^0\times I_3^0$ such that $\nu\neq\mu$, to develop the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness with respect to the elements of the classes $\big\{(\mathscr{U}_{\mathfrak{g},\nu},\mathscr{U}_{\mathfrak{g},\mu}):\ (\mathscr{U}_{\mathfrak{g},\nu},\mathscr{U}_{\mathfrak{g},\mu})\in\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big\}$ and $\big\{(\mathscr{V}_{\mathfrak{g},\nu},\mathscr{V}_{\mathfrak{g},\mu}):\ (\mathscr{V}_{\mathfrak{g},\nu},\mathscr{V}_{\mathfrak{g},\mu})\in\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big\}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, as [31] and [32] developed the theory of b-open and b-closed sets in a $\mathscr{T}$-space $\mathfrak{T}$. Such a theory is what we thought would certainly be worth considering, and the discussion of this paper terminates here.

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(M. I. Khodabocus) University of Mauritius, Faculty of Science, Department of Mathematics, Réduit 80837, MAURITIUS

Current address: Université des Mascareignes, Faculty of Sustainable Development and Engineering, Department of Emerging Technologies, Rose Hill Campus, MAURITIUS

Email address, M. I. Khodabocus: `ikhodabocus@udm.ac.mu`

(N. -Ul. -H. Sookia) University of Mauritius, Faculty of Science, Department of Mathematics, Réduit 80837, MAURITIUS

Email address, N. -Ul. -H. Sookia: `sookian@uom.ac.mu`

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PAIR DIFFERENCE CORDIAL LABELING OF CERTAIN BROKEN WHEEL GRAPHS

R. PONRAJ, A. GAYATHRI, AND S. SOMASUNDARAM

0000-0001-7593-7429, 0000-0002-1004-0175 and 0000-0001-6980-7468

Abstract. Let $G = (V, E)$ be a (p, q) graph.
Define

$$\rho = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm \rho\}$ called the set of labels.
Consider a mapping $f : V \longrightarrow L$ by assigning different labels in L to the different elements of V when p is even and different labels in L to p-1 elements of V and repeating a label for the remaining one vertex when p is odd.The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1. A graph G for which there exists a pair difference cordial labeling is called a pair difference cordial graph. In this paper we investigate the pair difference cordial labeling behavior of Certain broken wheel graphs.

1. Introduction

In this paper we consider only finite, undirected and simple graphs. The origin of graph labeling is graceful labeling [17]. In 1980, Cahit introduced the cordial labeling of graphs in [1]. In the sequal several cordial related labeling was studied in [13,14,15,16,18,19,20,21,22]. Ponraj et al introduced the pair difference cordial labeling in [4]. Also we have investigated various graphs [4,5,6,7,8,9,10] for pair difference cordial labeling. In this paper we investigate pair difference cordial labeling behavior of Certain broken wheel graphs.

2. Preliminaries

Definition 2.1. [2] Let $W_3 = C_3 + K_1$ where C_3 is abca and $V(K_1) = \{u\}$. The broken wheel $W(l, m, n)$ is obtained from the wheel W_3 with $V(W(l, m, n)) =$

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$V(W_3) \cup \{u_i : 1 \leq i \leq l-1\} \cup \{v_i : 1 \leq i \leq m-1\} \cup \{w_i : 1 \leq i \leq n-1\}$ and $E(W(l,m,n)) = E(W_3) \cup \{u_i u_{i+1} : 2 \leq i \leq l-2\} \cup \{v_i v_{i+1} : 2 \leq i \leq m-2\} \cup \{w_i w_{i+1} : 2 \leq i \leq n-1\} \cup \{au_1, u_{l-1}, bv_1, v_{m-1}c, aw_1, cw_{n-1}\}$.

3. Pair difference cordial labeling

Definition 3.1. [4] Let $G = (V, E)$ be a (p, q) graph.
Define

$$\rho = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm \rho\}$ called the set of labels.
Consider a mapping $f : V \longrightarrow L$ by assigning different labels in L to the different elements of V when p is even and different labels in L to p-1 elements of V and repeating a label for the remaining one vertex when p is odd.The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1.A graph G for which there exists a pair difference cordial labeling is called a pair difference cordial graph.

Theorem 3.2. [4] *The wheel W_n is pair difference cordial if and only if n is even.*

4. Main Results

Theorem 4.1. *The broken wheel $W(n, n, n)$ is pair difference cordial for all values of $n \geq 2$.*

Proof. Take the vertex and edges set from the definition 2.1. Now there are four cases arises.
Case(i). $n \equiv 0 \pmod 4$

Fix the label $1, -1$ to the vertex u_1, v_1 respectively and assign the labels $2, 4$ to the vertices u_2, u_3 respectively and assign the labels $3, 5$ respectively to the vertices u_4, u_5. Next assign the labels $6, 8$ to the vertices u_6, u_7 respectively and assign the labels $7, 9$ respectively to the vertices u_8, u_9. Proceeding like this until we reach the vertex u_{n-1}. Note that in this process the vertex u_{n-1} gets the label $n-1$.

Now we assign the labels $-2, -4$ to the vertices v_2, v_3 respectively and assign the labels $-3, -5$ respectively to the vertices v_4, v_5. Next assign the labels $-6, -8$ to the vertices v_6, v_7 respectively and assign the labels $-7, -9$ respectively to the vertices v_8, v_9. Proceeding like this until we reach the vertex v_{n-1}. Here we notice that in this process the vertex v_{n-1} gets the label $-n+1$.

Next we assign the labels $n, n+1$ to the vertices w_{n-1}, w_{n-2} respectively and assign the labels $-n, -(n+1)$ respectively to the vertices w_{n-3}, w_{n-4}. Now we assign the labels $n+2, n+3$ to the vertices w_{n-5}, w_{n-6} respectively and assign the labels $-(n+2), -(n+3)$ respectively to the vertices w_{n-7}, w_{n-8}. Proceeding this process until we reach the vertices w_1, a. Finally assign the labels $\frac{3n}{2}, \frac{3n-2}{2}, -\frac{3n}{2}$

respectively to the vertices u, b, c.

Case(ii). $n \equiv 1 \pmod 4$

First we assign the labels $1, 2, 4, 3$ to the vertices u_1, u_2, u_3, u_4 respectively and assign the labels $5, 6, 8, 7$ respectively to the vertices u_5, u_6, u_7, u_8. Next assign the labels $9, 10, 12, 11$ to the vertices $u_9, u_{10}, u_{11}, u_{12}$ respectively and assign the labels $13, 14, 16, 15$ respectively to the vertices $u_{13}, u_{14}, u_{15}, u_{16}$. Proceeding like this until we reach the vertex u_{n-1}. Note that in this process the vertices $u_{n-4}, u_{n-3}, u_{n-2}$, u_{n-1} gets the label $n-4, n-3, n-1, n-2$.

Now we assign the labels $-1, -2, -4, -3$ to the vertices v_1, v_2, v_3, v_4 respectively and assign the labels $-5, -6, -8, -7$ respectively to the vertices v_5, v_6, v_7, v_8. Next assign the labels $-9, -10, -12, -11$ to the vertices $v_9, v_{10}, v_{11}, v_{12}$ respectively and assign the labels $-13, -14, -16, -15$ respectively to the vertices $v_{13}, v_{14}, v_{15}, v_{16}$. Proceeding like this until we reach the vertex v_{n-4}. We notice that in this process the vertex v_{n-4} gets the label $-n+4$ and assign the labels $-(n-3), -(n-2), -(n-1)$ respectively to the vertices $v_{n-3}, v_{n-2}, v_{n-1}$.

Next we assign the labels $n, n+1$ to the vertices c, w_{n-1} respectively and assign the labels $-n, -(n+1)$ respectively to the vertices w_{n-2}, w_{n-3}. Now we assign the labels $n+2, n+3$ to the vertices w_{n-4}, w_{n-5} respectively and assign the labels $-(n+2), -(n+3)$ respectively to the vertices w_{n-6}, w_{n-7}. Proceeding this process until we reach the vertices w_2. Finally assign the labels $\frac{3n-1}{2}, \frac{3n+1}{2}, -\frac{3n-1}{2}, -\frac{3n+1}{2}$ respectively to the vertices w_1, a, u, c.

Case(iii). $n \equiv 2 \pmod 4$
There are subcases arises.

Subcase(i). $n = 2$

Assign the labels $1, 2, 3, -1, -2, -3, 1$ respectively to the vertices $a, u_1, b, v_1, c, w_1, u$.

Subcase(i). $n > 2$

Assign the labels to the vertices u_i, v_i , $(1 \leq i \leq n-1)$ as in technique of case (i). Next we assign the labels $-n, -(n+1)$ to the vertices c, w_{n-1} respectively and assign the labels $n, (n+1)$ respectively to the vertices w_{n-2}, w_{n-3}. Now we assign the labels $-(n+2), -(n+3)$ to the vertices w_{n-4}, w_{n-5} respectively and assign the labels $(n+2), (n+3)$ respectively to the vertices w_{n-6}, w_{n-7}. Proceeding this process until we reach the vertices w_1, a.Note that the vertices w_4, w_3 gets the labels $\frac{3n-6}{2}, \frac{3n-4}{2}$. Lastly assign the labels $-\frac{3n}{2}, -\frac{3n-2}{2}, \frac{3n-2}{2}, \frac{3n}{2}, \frac{3n-2}{2}$ respectively to the vertices w_2, w_1, a, u, b.

Case(iv). $n \equiv 3 \pmod 4$

There are subcases arises.

Subcase(i). $n = 3$

Assign the labels $1, 2, 3, 4, -1, -2, -3, -4, 5, -5$ respectively to the vertices $a, u_1, u_2, b, v_1, v_2, c, w_1, w_2, u$.

Subcase(i). $n > 2$

Assign the labels to the vertices u_i, v_i , $(1 \le i \le n-1)$ as in technique of case (i). Next we assign the labels $-n, -(n+1)$ to the vertices c, w_{n-1} respectively and assign the labels $n, (n+1)$ respectively to the vertices w_{n-2}, w_{n-3}. Now we assign the labels $-(n+2), -(n+3)$ to the vertices w_{n-4}, w_{n-5} respectively and assign the labels $(n+2), (n+3)$ respectively to the vertices w_{n-6}, w_{n-7}. Proceeding this process until we reach the vertices w_3. Note that the vertices w_1, a gets the labels $\frac{3n-3}{2}, \frac{3n-1}{2}$ Finally assign the labels $\frac{3n+1}{2}, -\frac{3n+1}{2}$ respectively to the vertices u, b.

The Table 1 given below establish that this vertex labeling f is a pair difference cordial of $W(n, n, n)$.

Nature of n	Δ_{f_1}	$\Delta_{f_1^c}$
$n \equiv 0 \pmod 4$	$\frac{3n+2}{2}$	$\frac{3n+4}{2}$
$n \equiv 1 \pmod 4$	$\frac{3n+3}{2}$	$\frac{3n+3}{2}$
$n \equiv 2 \pmod 4$	$\frac{3n+2}{2}$	$\frac{3n+4}{2}$
$n \equiv 3 \pmod 4$	$\frac{3n+3}{2}$	$\frac{3n+3}{2}$

TABLE 1

□

Theorem 4.2. *The broken wheel $W(l, m, m)$ is pair difference cordial for all values of $l, m \ge 2$.*

Proof. Take the vertex and edges set from the definition 2.1. Now there are four cases arises.
Case(1). $l \equiv 0 \pmod 4$
Here four subcases arises.
Sub Case(i). $m \equiv 0 \pmod 4$

First we assign the labels $1, 2, 4, 3$ to the vertices v_1, v_2, v_3, v_4 respectively and assign the labels $5, 6, 8, 7$ respectively to the vertices v_5, v_6, v_7, v_8. Next assign the labels $9, 10, 12, 11$ to the vertices $v_9, v_{10}, v_{11}, v_{12}$ respectively and assign the labels $13, 14, 16, 15$ respectively to the vertices $v_{13}, v_{14}, v_{15}, v_{16}$. Proceeding like this until we reach the vertex v_{m-1}. Note that in this process the vertices $v_{m-4}, v_{m-3}, v_{m-2}, v_{m-1}$ gets the label $m-4, m-3, m-1, m-2$.

Now we assign the labels $-1, -2, -4, -3$ to the vertices $w_{m-1}, w_{m-2}, w_{m-3}, w_{m-4}$ respectively and assign the labels $-5, -6, -8, -7$ respectively to the vertices $w_{m-5}, w_{m-6}, w_{m-7}, w_{m-8}$. Next assign the labels $-9, -10, -12, -11$ to the vertices $w_{m-9}, w_{m-10}, w_{m-11}, w_{m-12}$ respectively and assign the labels $-13, -14, -16, -15$ respectively to the vertices $w_{m-13}, w_{m-14}, w_{m-15}, w_{m-16}$. Proceeding like this until we reach the vertex w_1. We notice that in this process the vertex w_1 gets the label $-n+1$.

Next we assign the labels $m, m+1, -m, -(m+1)$ to the vertices a, u_1, u_2, u_3 respectively and assign the labels $m+2, m+3, -(m+2), -(m+3)$ respectively to the vertices u_4, u_5, u_6, u_7. Now we assign the labels $m+4, m+5, -(m+4), -(m+5)$ to the vertices u_8, u_9, u_{10}, u_{11} respectively and assign the labels $m+6, m+7, -(m+6), -(m+7)$ respectively to the vertices $u_{12}, u_{13}, u_{14}, u_{15}$. Proceeding this process until we reach the vertices u_{l-1}.Note that in this process the vertices $u_{l-4}, u_{l-3}, u_{l-2}, u_{l-1}$ gets the label $\frac{l+2m-4}{2}, \frac{l+2m-2}{2}, -\frac{l+2m-4}{2}, -\frac{l+2m-2}{2}$ respectively . Finally assign the labels $-\frac{l+2m}{2}, \frac{l+2m}{2}, \frac{l+2m}{2}$ respectively to the vertices b, u, c.

Sub Case(ii). $m \equiv 1 \pmod 4$
Assign the labels to the vertices v_i , $(1 \leq i \leq m-1)$, u_i , $(1 \leq i \leq l-1)$ and w_i , $(1 \leq i \leq m-5)$ as in technique of case (i). Next we assign the labels $-(m-4), -(m-3), -(m-2), -(m-1)$ to the vertices $w_{m-4}, w_{m-3}, w_{m-2}, w_{m-1}$ respectively and assign the labels $\frac{l+2m}{2}, -\frac{l+2m}{2}, -\frac{l+2m-2}{2}$ respectively to the vertices b, u, c.

Sub Case(iii). $m \equiv 2 \pmod 4$
Assign the labels to the vertices v_i, w_i , $(1 \leq i \leq m-1)$ as in technique of case (i). Next we assign the labels $-n, -(n+1), n, n+1$ to the vertices a, u_1, u_2, u_3 respectively and assign the labels $-(n+2), -(n+3), n+2, n+3$ respectively to the vertices u_4, u_5, u_6, u_7. Next assign the labels $-(n+4), -(n+5), n+4, n+5$ respectively to the vertices u_8, u_9, u_{10}, u_{11}. Proceeding this process until we reach the vertices u_{l-1}.Note that in this process the vertices $u_{l-4}, u_{l-3}, u_{l-2}, u_{l-1}$ gets the label $-\frac{l+2m-4}{2}, -\frac{l+2m-2}{2}, \frac{l+2m-4}{2}, \frac{l+2m-2}{2}$ respectively . Finally assign the labels $\frac{l+2m}{2}, -\frac{l+2m}{2}, -\frac{l+2m-2}{2}$ respectively to the vertices b, u, c.

Sub Case(iv). $m \equiv 3 \pmod 4$
Assign the labels to the vertices v_i, w_i , $(1 \leq i \leq m-1)$, u_i , $(1 \leq i \leq l-1)$ and a, b, c, u as in technique of case (i).

The Table 2 given below establish that this vertex labeling f is a pair difference cordial of $W(l, m, m)$.

Nature of n	Δ_{f_1}	$\Delta_{f_1^c}$
$n \equiv 0 \pmod 4$	$\frac{l+2m+2}{2}$	$\frac{l+2m+4}{2}$
$n \equiv 1 \pmod 4$	$\frac{l+2m+2}{2}$	$\frac{l+2m+4}{2}$
$n \equiv 2 \pmod 4$	$\frac{l+2m+2}{2}$	$\frac{l+2m+4}{2}$
$n \equiv 3 \pmod 4$	$\frac{l+2m+4}{2}$	$\frac{l+2m+2}{2}$

TABLE 2

Case(2). $l \equiv 1 \pmod 4$
Here four subcases arises.

Sub Case(i). $m \equiv 0 \pmod 4$
Assign the labels to the vertices v_i, w_i , $(1 \le i \le m-1)$, u_i , $(1 \le i \le l-2)$ and a as in technique of subcase (i) in case 1. Finally assign the labels $\frac{l+2m-1}{2}, \frac{l+2m+1}{2}, -\frac{l+2m-1}{2}, -\frac{l+2m+1}{2}$ respectively to the vertices u_{n-1}, b, u, c.

Sub Case(ii). $m \equiv 1 \pmod 4$
Assign the labels to the vertices v_i, w_i , $(1 \le i \le m-1)$ as in technique of subcase (i) in case 1. Next we assign the labels $-m, -(m+1), m, m+1$ to the vertices a, u_1, u_2, u_3 respectively and assign the labels $-(m+2), -(m+3), m+2, m+3$ respectively to the vertices u_4, u_5, u_6, u_7. Next assign the labels $-(m+4), -(m+5), m+4, m+5$ respectively to the vertices u_8, u_9, u_{10}, u_{11}. Proceeding this process until we reach the vertices u_{l-2}.Note that in this process the vertices $u_{l-5}, u_{l-4}, u_{l-3}, u_{l-2}$ gets the label $\frac{l+2m-5}{2}, \frac{l+2m-3}{2}, -\frac{l+2m-5}{2}, -\frac{l+2m-3}{2}$ respectively . Finally assign the labels $\frac{l+2m-1}{2}, \frac{l+2m+1}{2}, -\frac{l+2m-1}{2}, -\frac{l+2m+1}{2}$ respectively to the vertices u_{n-1}, b, u, c.

Sub Case(iii). $m \equiv 2 \pmod 4$
Assign the labels to the vertices v_i, w_i , $(1 \le i \le m-1)$, u_i , $(1 \le i \le l-1)$ and a, b, c, u as in technique of subcase (ii) in case 2.

Sub Case(iv). $m \equiv 3 \pmod 4$
Assign the labels to the vertices v_i, w_i , $(1 \le i \le m-1)$, u_i , $(1 \le i \le l-2)$ and a as in technique of subcase (ii) in case 2. Finally assign the labels $-\frac{l+2m-1}{2}, -\frac{l+2m+1}{2}, \frac{l+2m-1}{2}, \frac{l+2m+1}{2}$ respectively to the vertices u_{l-1}, b, u, c.

The Table 3 given below establish that this vertex labeling f is a pair difference cordial of $W(l, m, m)$.

Nature of n	Δ_{f_1}	$\Delta_{f_1^c}$
$n \equiv 0 \pmod 4$	$\frac{l+2m+3}{2}$	$\frac{l+2m+3}{2}$
$n \equiv 1 \pmod 4$	$\frac{l+2m+3}{2}$	$\frac{l+2m+3}{2}$
$n \equiv 2 \pmod 4$	$\frac{l+2m+3}{2}$	$\frac{l+2m+3}{2}$
$n \equiv 3 \pmod 4$	$\frac{l+2m+3}{2}$	$\frac{l+2m+3}{2}$

Table 3

Case(3). $l \equiv 2 \pmod 4$
Here four subcases arises.

Sub Case(i). $m \equiv 0 \pmod 4$
Assign the labels to the vertices v_i, w_i , $(1 \le i \le m-1)$, u_i , $(1 \le i \le l-1)$ and a as in technique of subcase (i) in case 1. Finally assign the labels $-\frac{l+2m-2}{2}, -\frac{l+2m}{2}, -\frac{l+2m-2}{2}$ respectively to the vertices b, u, c.

Sub Case(ii). $m \equiv 1 \pmod 4$
Assign the labels to the vertices v_i, w_i , $(1 \le i \le m-1)$, u_i , $(1 \le i \le l-1)$ and a, b, c, u as in technique of subcase (ii) in case 3.

Sub Case(iii). $m \equiv 2 \pmod 4$
Assign the labels to the vertices v_i, w_i , $(1 \leq i \leq m-1)$, u_i , $(1 \leq i \leq l-1)$ and a as in technique of subcase (i) in case 1. Finally assign the labels $\frac{l+2m-2}{2}, \frac{l+2m}{2}, \frac{l+2m-2}{2}$ respectively to the vertices b, u, c.

Sub Case(iv). $m \equiv 3 \pmod 4$
Assign the labels to the vertices v_i, w_i , $(1 \leq i \leq m-1)$, u_i , $(1 \leq i \leq l-1)$ and a, b, c, u as in technique of subcase (iii) in case 3.

The Table 4 given below establish that this vertex labeling f is a pair difference cordial of $W(l, m, m)$.

Nature of n	Δ_{f_1}	$\Delta_{f_1^c}$
$n \equiv 0 \pmod 4$	$\frac{l+2m+4}{2}$	$\frac{l+2m+2}{2}$
$n \equiv 1 \pmod 4$	$\frac{l+2m+2}{2}$	$\frac{l+2m+4}{2}$
$n \equiv 2 \pmod 4$	$\frac{l+2m+2}{2}$	$\frac{l+2m+4}{2}$
$n \equiv 3 \pmod 4$	$\frac{l+2m+4}{2}$	$\frac{l+2m+2}{2}$

Table 4

Case(4). $l \equiv 3 \pmod 4$
Here four subcases arises.

Sub Case(i). $m \equiv 0 \pmod 4$
Assign the labels to the vertices v_i, w_i , $(1 \leq i \leq m-1)$ as in technique of subcase (i) in case 1. Next we assign the labels $m, m+1, -m, -(m+1)$ to the vertices a, u_1, u_2, u_3 respectively and assign the labels $m+2, m+3, -(m+2), -(m+3)$ respectively to the vertices u_4, u_5, u_6, u_7. Next assign the labels $m+4, m+5, -(m+4), -(m+5)$ respectively to the vertices u_8, u_9, u_{10}, u_{11}. Proceeding this process until we reach the vertices u_{l-1}, b.Note that in this process the vertices $u_{l-3}, u_{l-2}, u_{l-1}, b$ gets the label $\frac{l+2m-3}{2}, \frac{l+2m-1}{2}, -\frac{l+2m-3}{2}, -\frac{l+2m-1}{2}$ respectively . Finally assign the labels $-\frac{l+2m+1}{2}, \frac{l+2m+1}{2}$ respectively to the vertices u, c.

Sub Case(ii). $m \equiv 1 \pmod 4$
Assign the labels to the vertices v_i, w_i , $(1 \leq i \leq m-1)$, u_i , $(1 \leq i \leq l-1)$ and a, b, c, u as in technique of subcase (i) in case 4.

Sub Case(iii). $m \equiv 2 \pmod 4$
Assign the labels to the vertices v_i, w_i , $(1 \leq i \leq m-1)$, u_i , $(1 \leq i \leq l-1)$ and a, b, c, u as in technique of subcase (i) in case 4.

Sub Case(iv). $m \equiv 3 \pmod 4$
Assign the labels to the vertices v_i, w_i , $(1 \leq i \leq m-1)$, u_i , $(1 \leq i \leq l-1)$ and a, b, c, u as in technique of subcase (i) in case 4.

The Table 5 given below establish that this vertex labeling f is a pair difference cordial of $W(l,m,m)$.

Nature of n	Δ_{f_1}	$\Delta_{f_1^c}$
$n \equiv 0 \pmod 4$	$\frac{l+2m+3}{2}$	$\frac{l+2m+3}{2}$
$n \equiv 1 \pmod 4$	$\frac{l+2m+3}{2}$	$\frac{l+2m+3}{2}$
$n \equiv 2 \pmod 4$	$\frac{l+2m+3}{2}$	$\frac{l+2m+3}{2}$
$n \equiv 3 \pmod 4$	$\frac{l+2m+3}{2}$	$\frac{l+2m+3}{2}$

Table 5

□

5. Discussion

The pair sum labeling was introduced by Ponraj and Parthipan in [11]. Recently the difference cordial labeling of graphs was introduced in [12]. Follows from these two concepts, we have defined a new concept of pair difference cordial labeling of graphs [4]. The pair difference cordial labeling behaviour of broken wheel have been investigated in this paper.

6. Conclusion

The pair difference cordial labeling behaviour of some broken wheel graphs have been investigated in this paper. Presently, it is difficult to investigate the pair difference cordial labeling behaviour of generalized web, broken web graphs. The pair difference cordial labeling behaviour of subdivison of broken whell graphs are the open problems.

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The Declaration of Research and Publication Ethics
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R. Ponraj, Department of Mathematics, Sri Paramakalyani College, Alwarkurichi–627 412, India.
Email address: ponrajmaths@gmail.com

A. Gayathri, Research Scholor,Reg.No:20124012092023, Department of Mathematics, Manonmaniam Sundaranar University, Abhishekapati,Tirunelveli–627 012, India.
Email address: gayugayathria555@gmail.com

S. Somasundaram, Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli– 627 012, India
Email address: somutvl@gmail.com

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ON AUTOMORPHISMS OF LIE ALGEBRA OF SYMMETRIC POLYNOMIALS

ŞEHMUS FINDIK AND NAZAR ŞAHİN ÖĞÜŞLÜ

0000-0001-5717-4413 and 0000-0001-7407-9178

Abstract. Let L_n be the free Lie algebra of rank n over a field K of characteristic zero, $L_{n,c} = L_n/(L_n''+\gamma_{c+1}(L_n))$ be the free metabelian nilpotent of class c Lie algebra, and $F_n = L_n/L_n''$ be the free metabelian Lie algebra generated by $x_1, \dots, x_n$ over a field K of characteristic zero. We call a polynomial $p(X_n)$ in these Lie algebras *symmetric* if $p(x_1, \dots, x_n) = p(x_{\pi(1)}, \dots, x_{\pi(n)})$ for each element of the symmetric group S_n. The sets $L_n^{S_n}$, $F_n^{S_n}$, and $L_{n,c}^{S_n}$ of symmetric polynomials coincides with the algebras of invariants of the group S_n in L_n, F_n, and $L_{n,c}$, respectively. We determine the groups $\mathrm{Inn}(L_{n,c}^{S_n}) \cap \mathrm{Inn}(L_{n,c})$ and $\mathrm{Inn}(F_n^{S_n}) \cap \mathrm{Inn}(F_n)$ of inner automorphisms of the algebras $L_{n,c}^{S_n}$ and $F_n^{S_n}$ in the groups $\mathrm{Inn}(L_{n,c})$ and $\mathrm{Inn}(F_n)$, respectively. In particular, we obtain the descriptions of the groups $\mathrm{Aut}(L_2^{S_2}) \cap \mathrm{Aut}(L_2)$ and $\mathrm{Aut}(F_2^{S_2}) \cap \mathrm{Aut}(F_2)$ of automorphisms of the algebras $L_2^{S_2}$ and $F_2^{S_2}$ in the groups $\mathrm{Aut}(L_2)$ and $\mathrm{Aut}(F_2)$, respectively.

1. Introduction

Let A_n be the free algebra of rank n over a field K of characteristic zero in a variety of algebras generated by $X_n = \{x_1, \dots, x_n\}$. A polynomial $p(X_n) \in A_n$ is said to be symmetric if $p(x_1, \dots, x_n) = p(x_{\pi(1)}, \dots, x_{\pi(n)})$ for all $\pi \in S_n$. The set of such polynomials is equal to the algebra $A_n^{S_n}$ of invariants of the symmetric group S_n. The algebra $A_n^{S_n}$ is well known, when $A_n = K[X_n]$ is the commutative associative unitary algebra by the fundamental theorem on symmetric polynomials: $K[X_n]^{S_n} = K[\sigma_1, \dots, \sigma_n]$, $\sigma_i = x_1^i + \cdots + x_n^i$. For the case $A_n = K\langle X_n \rangle$, the associative algebra of rank n, see e.g. [6].

Now let $A_n = F_n$ be the free metabelian Lie algebra of rank n over K. It is well known, see e.g. [3], that the algebra $F_n^{S_n}$ of symmetric polynomials is not finitely generated. Recently the authors [5] have provided an infinite set of generators for $F_2^{S_2}$, later the result was generalized in [4].

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One may consider the group $\mathrm{Aut}(F_n^{S_n})$ of automorphisms preserving the algebra $F_n^{S_n}$. The group $\mathrm{Aut}(F_n)$ is a semidirect product of the general linear group $\mathrm{GL}_n(K)$ and the group $\mathrm{IAut}(F_n)$ of automorphisms which are equivalent to the identity map modulo the commutator ideal F_n'. Hence it is natural to work in $\mathrm{IAut}(F_n)$ approaching the group $\mathrm{Aut}(F_n^{S_n}) \cap \mathrm{IAut}(F_n)$. Additionally, the cases $\mathrm{Aut}(F_n^{S_n}) \cap \mathrm{Aut}(F_n)$ and $\mathrm{Aut}(L_{n,c}^{S_n}) \cap \mathrm{Aut}(L_{n,c})$ are of the interest in the present study, where L_n and $L_{n,c}$ are the free Lie algebra and the free metabelian Lie algebra of nilpotency class c, respectively.

In this study, we determine the inner automorphisms of $F_n^{S_n}$ and of $L_{n,c}^{S_n}$, which are inner automorphisms of F_n and of $L_{n,c}$. Later we describe the groups $\mathrm{Aut}(L_2^{S_2}) \cap \mathrm{Aut}(L_2)$, $\mathrm{Aut}(F_2^{S_2}) \cap \mathrm{Aut}(F_2)$ and $\mathrm{Aut}(L_{2,c}^{S_2}) \cap \mathrm{Aut}(L_{2,c})$.

2. Preliminaries

Let L_n be the free Lie algebra of rank $n \geq 2$ generated by $X_n = \{x_1, \ldots, x_n\}$ over a field K of characteristic zero. We denote by $F_n = L_n/L_n''$ the free metabelian Lie algebra, and $L_{n,c} = L_n/(L_n'' + \gamma^{c+1}(L_n))$ the free metabelian nilpotent Lie algebra of nilpotency class c, where $\gamma^1(L_n) = [L_n, L_n] = L_n'$ is the commutator ideal of L_n, $\gamma^k(L_n) = [\gamma^{k-1}(L_n), L_n]$, $k \geq 2$, and $L_n'' = [L_n', L_n']$. We assume that the algebras F_n and $L_{n,c}$ of rank n are generated by the same set X_n.

The commutator ideal F_n' of the free metabelian Lie algebra F_n is of a natural $K[X_n]$-module structure as a consequence of the metabelian identity

$$[[z_1, z_2], [z_3, z_4]] = 0, \quad z_1, z_2, z_3, z_4 \in F_n,$$

with action:

$$f(X_n)g(x_1, \ldots, x_n) = f(X_n)g(\mathrm{ad}x_1, \ldots, \mathrm{ad}x_n), \quad f(X_n) \in F_n', \quad g(X_n) \in K[X_n],$$

where $K[X_n]$ is the (commutative, associative, unitary) polynomial algebra, and the adjoint action is defined as $z_1\mathrm{ad}z_2 = [z_1, z_2]$, for $z_1, z_2 \in F_n$. One may define a similar action on the free metabelian nilpotent Lie algebra $L_{n,c}$. It is well known by Bahturin [1] that the monomials $[x_{k_1}, x_{k_2}]x_{k_3} \cdots x_{k_l}$, $k_1 > k_2 \leq k_3 \leq k_l$, forms a basis for F_n', which is a basis for $L_{n,c}'$ when $l \leq c-2$.

An element $s(X_n)$ in L_n, F_n, or $L_{n,c}$ is called symmetric if

$$s(x_1, \ldots, x_n) = s(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \pi s(x_1, \ldots, x_n)$$

for each permutation π in the symmetric group S_n. The sets $L_n^{S_n}$, $F_n^{S_n}$, and $L_{n,c}^{S_n}$ of symmetric polynomials coincide with the algebras of invariants of the group S_n. See the work [5] for $F_2^{S_2}$, and its generalization [4] for the full description of the algebra $F_n^{S_n}$. The decription of the algebra $L_{n,c}^{S_n}$ is a direct consequence of the known results on the algebra $F_n^{S_n}$.

Let $A_n = L_n, F_n, L_{n,c}$. It is well known that the automorphism group $\mathrm{Aut}(A_n)$ is a semidirect product of the general linear group $\mathrm{GL}_n(K)$ and the group $\mathrm{IAut}(A_n)$ of automorphisms which are equivalent to the identity map modulo the commutator ideal A_n'. Hence it is natural to work in $\mathrm{IAut}(A_n)$ when determining the whole group $\mathrm{Aut}(A_n)$. Now consider the group $\mathrm{Aut}(A_n^{S_n})$ of automorphisms consisting of automorphisms of A_n preserving each symmetric polynomial in the algebra $A_n^{S_n}$.

In the next section, approaching the group $\mathrm{IAut}(A_n^{S_n})$, we describe the inner automorphism group $\mathrm{Inn}(A_n^{S_n}) \cap \mathrm{Inn}(A_n)$ in the group $\mathrm{Inn}(A_n)$ for $A_n = F_n$, and $A_n = L_{n,c}$. The case $\mathrm{Inn}(L_n^{S_n}) \cap \mathrm{Inn}(L_n)$ is skipped since the free Lie algebra

L_n does not have nonidentical inner automorphisms. Later, we obtain the groups $\mathrm{Aut}(L_2^{S_2})$, $\mathrm{Aut}(F_2^{S_2}) \cap \mathrm{Aut}(F_2)$, and $\mathrm{Aut}(L_{2,c}^{S_2}) \cap \mathrm{Aut}(L_{2,c})$ as a consequence of the results obtained.

3. Main Results

3.1. **Inner automorphisms of $F_n^{S_n}$.** Let $u \in F_n'$ be an element from the commutator ideal of the free metabelian Lie algebra F_n. Then the adjoint operator $\mathrm{ad}u : v \to [v, u]$ is a nilpotent derivation of F_n, and $\psi_u = \exp(\mathrm{ad}u) = 1 + \mathrm{ad}u$ is an automorphism of the Lie algebra F_n. The inner automorphism group $\mathrm{Inn}(F_n)$ of F_n consisting of such automorphisms is abelian: $\psi_{u_1}\psi_{u_2} = \psi_{u_1+u_2}$, $\psi_u^{-1} = \psi_{-u}$.

In the following theorem we determine the group $\mathrm{Inn}(F_n^{S_n}) \cap \mathrm{Inn}(F_n)$ of inner automorphisms preserving the algebra $F_n^{S_n}$.

Theorem 3.1. *The automorphism $\psi_u \in \mathrm{Inn}(F_n^{S_n}) \cap \mathrm{Inn}(F_n)$ if and only if $u \in (F_n')^{S_n}$.*

Proof. If $u \in (F_n')^{S_n}$, then clearly $\psi_u(v) = v + [v, u] \in F_n^{S_n}$ for every $v \in F_n^{S_n}$. Conversely let $v \in F_n^{S_n}$ be a symmetric polynomial, and $u \in F_n'$ be an arbitrary element. We may assume that the linear (symmetric) summand v_l of v is nonzero, since ψ_u acts identically on the commutator ideal F_n' of the free metabelian Lie algebra F_n. Then $\psi_u(v) \in F_n^{S_n}$ implies that $[v, u] = [v_l, u] \in F_n^{S_n}$. For each $\pi \in S_n$, we have that

$$[v_l, u] = \pi[v_l, u] = [\pi v_l, \pi u] = [v_l, \pi u],$$

and $[v_l, u - \pi u] = 0$, which gives that $u - \pi u = 0$ or $u = \pi u$. $\square$

3.2. **Inner automorphisms of $L_{n,c}^{S_n}$.** Let $u \in L_{n,c}$ be an element from the free metabelian Lie algebra $L_{n,c}$. Then the adjoint operator $\mathrm{ad}u(v) = [v, u]$ is a nilpotent derivation of $L_{n,c}$, since $\mathrm{ad}^c u = 0$ and

$$\varepsilon_u = \exp(\mathrm{ad}u) = 1 + \mathrm{ad}u + \frac{1}{2}\mathrm{ad}^2 u + \cdots + \frac{1}{(c-1)!}\mathrm{ad}^{c-1} u$$

is an inner automorphism of $L_{n,c}$. The set $\mathrm{Inn}(L_{n,c}) = \{\varepsilon_u \mid u \in L_{n,c}\}$ is the inner automorphism group of $L_{n,c}$. In this subsection, we investigate the group $\mathrm{Inn}(L_{n,c}^{S_n}) \cap \mathrm{Inn}(L_{n,c})$ of inner automorphisms of the algebra $L_{n,c}^{S_n}$.

Lemma 3.2. *Let $u = \sum_{i=1}^n \alpha_i x_i$ for some $\alpha_i \in K$, and $v = \sum_{i=1}^n x_i \in L_{n,c}^{S_n}$ such that $[u, v] \in L_{n,c}^{S_n}$. Then $u = \alpha v$ for some $\alpha \in K$.*

Proof. Let $\pi = (1k) \in S_n$ be a fixed transposition for $k = 2, \ldots, n$. Then

$$\pi u = \alpha_1 x_{\pi(1)} + \cdots + \alpha_n x_{\pi(n)} = \alpha_1 x_k + \alpha_k x_1 + \sum_{i \neq 1,k} \alpha_i x_i,$$

and $u - \pi u = \alpha_1(x_1 - x_k) + \alpha_k(x_k - x_1) = \alpha_{1k}(x_1 - x_k)$, where $\alpha_{1k} = \alpha_1 - \alpha_k$. Now $[u, v] \in L_{n,c}^{S_n}$ gives that $[u, v] = \pi[u, v] = [\pi u, \pi v] = [\pi u, v]$, and hence

$$\begin{aligned} 0 = [u - \pi u, v] &= [\alpha_{1k}(x_1 - x_k), x_1 + \cdots + x_n] \\ &= \alpha_{1k}\left(2[x_1, x_k] + \sum_{i \neq 1,k} [x_1, x_i] + \sum_{i \neq 1,k} [x_i, x_k]\right) \end{aligned}$$

where the elements in the paranthesis are basis elements of $L_{n,c}$, which implies that $\alpha_{1k} = 0$, $k \geq 2$. This completes the proof by the choice $\alpha = \alpha_1 = \cdots = \alpha_n$. $\square$

Theorem 3.3. *The automorphism* $1 \neq \psi_u \in \mathrm{Inn}(L_{n,c}^{S_n}) \cap \mathrm{Inn}(L_{n,c})$ *if and only if* $u \in L_{n,c}^{S_n}$.

Proof. If $u, v \in L_{n,c}^{S_n}$ then it is straighforward to see that

$$\varepsilon_u(v) = v + [v,u] + \cdots + (1/(c-1)!)[[\cdots [v,u], \ldots], u] \in L_{n,c}^{S_n}.$$

Conversely, let $u = u_l + u_0 \in L_{n,c}$ be an arbitrary element and $v = v_l + v_0 \in L_{n,c}^{S_n}$ be a symmetric polynomial such that $\varepsilon_u(v) \in L_{n,c}^{S_n}$, where u_l and v_l are the linear components of u and v, respectively. In the expression of $\varepsilon_u(v)$, the component of degree 2 is $[v_l, u_l]$ which is symmetric by the natural grading on the Lie algebra $L_{n,c}^{S_n}$. Hence $u_l = \alpha v_l$ for some $\alpha \in K$ by Lemma 3.2, and $[v_l, u_l] = 0$. Note that $[v_0, u_0] = 0$ by metabelian identity. The computations

$$\begin{aligned}\varepsilon_u(v) =& v + [v_l + v_0, u_l + u_0] \sum_{k=0}^{c-2} \frac{u_l^k}{(k+1)!} \\ =& v + [v_0, v_l] \sum_{k=0}^{c-3} \frac{\alpha^{k+1} v_l^k}{(k+1)!} + [v_l, u_0] \sum_{k=0}^{c-3} \frac{\alpha^k v_l^k}{(k+1)!}\end{aligned}$$

give that $[v_l, u_0] \sum_{k=0}^{c-3} \frac{\alpha^k v_l^k}{(k+1)!} \in L_{n,c}^{S_n}$, and that $u_0 \in L_{n,c}^{S_n}$ by Theorem 3.1. □

3.3. **On automorphisms of $L_2^{S_2}$, $F_2^{S_2}$, and $L_{2,c}^{S_2}$.** In the sequel, we fix the notation $x_1 = x$, $x_2 = y$, for the sake of simplicity. It is well known by [2] that each automorphism of L_2 is linear. The next theorem determines the automorphism group $\mathrm{Aut}(L_2^{S_2})$.

Theorem 3.4. *Let* $\xi \in \mathrm{Aut}(L_2^{S_2}) \cap \mathrm{Aut}(L_2)$. *Then* ξ *and its inverse* ξ^{-1} *are of the form*

$$\xi(x) = ax + by, \quad \xi(y) = bx + ay,$$
$$\xi^{-1}(x) = c^{-1}ax - c^{-1}by, \quad \xi^{-1}(y) = -c^{-1}bx + c^{-1}ay,$$

such that $c = a^2 - b^2 \neq 0$, $a, b \in K$.

Proof. Let ξ be of the form $\xi : x \to ax + by,\ y \to cx + dy$ such that $ad \neq bc$, where $a, b, c, d \in K$. Since $x + y \in L_2^{S_2}$, then $\xi(x + y) = (a + c)x + (b + d)y \in L_2^{S_2}$, which is contained in $K\{x + y\}$. Hence $a + c = b + d$. On the other hand $[[x, y], x] - [[x, y], y] \in L_2^{S_2}$, and

$$\xi([[x, y], x] - [[x, y], y]) = \beta\left((a - c)[[x, y], x] + (b - d)[[x, y], y]\right)$$

where $\beta = ad - bc \neq 0$. The fact that $\xi \in \mathrm{Aut}(L_2^{S_2}) \cap \mathrm{Aut}(L_2)$ gives $a - c = -b + d$. Consequently, $a = d$ and $b = c$. Conversely, it is straightforward to show that the automorphism stated in the theorem preserves symmetric polynomials. □

It is well known, see the work by Drensky [3], that each automorphism of F_2 is a product of a linear automorphism and an inner automorphism of F_2. Using this fact, we obtain the following result as a consequence of Theorem 3.4 and Theorem 3.1.

Theorem 3.5. *Let* $\varphi \in \mathrm{Aut}(F_2^{S_2}) \cap \mathrm{Aut}(F_2)$. *Then* φ *is a product of a linear automorphism* $\xi : x \to ax + by,\ y \to bx + ay$, *and an inner automorphism* ψ_u *where* $u \in (F_2')^{S_2}$.

Theorem 3.6. *Let $\varphi \in \mathrm{Aut}(L_{2,c}^{S_2}) \cap \mathrm{Aut}(L_{2,c})$. Then φ is a product of a linear automorphism $\xi : x \to ax+by,\ y \to bx+ay$, and an automorphism $\phi \in \mathrm{IAut}(L_{2,c}^{S_2})$ of the form*

$$\begin{aligned}\phi : & x \to x + [x,y]f(x,y) \\ & y \to y - [x,y]f(y,x).\end{aligned}$$

Proof. It is sufficient to show that an automorphism $\phi \in \mathrm{IAut}(L_{2,c})$ preserving symmetric polynomials satisfies the condition of the theorem. In general ϕ is of the form

$$\begin{aligned}\phi : & x \to x + [x,y]f(x,y) \\ & y \to y + [x,y]g(x,y).\end{aligned}$$

Since $x+y \in L_{2,c}^{S_2}$, then $\phi(x+y) = x+y+[x,y](f(x,y)+g(x,y))$ is symmetric, and hence

$$x+y+[x,y](f(x,y)+g(x,y)) = y+x-[x,y](f(y,x)+g(y,x)).$$

This gives that

$$f(x,y)+g(x,y)+f(y,x)+g(y,x) = 0 \tag{3.1}$$

in the commutator ideal $L'_{2,c}$ of $L_{2,c}$, which is a $K[x,y]$-module freely generated by $[x,y]$. Now by the symmetric polynomial $[x,y](x-y)$, we have that

$$[x+[x,y]f(x,y), y+[x,y]g(x,y)](x-y) = [x,y](x-y)(1+f(x,y)y-xg(x,y))$$

is symmetric. Consequently $[x,y](x-y)(yf(x,y)-xg(x,y))$ is symmetric. The following computations complete the proof.

$$[x,y](x-y)(yf(x,y)-xg(x,y)) = [x,y](x-y)(xf(y,x)-yg(y,x)),$$

and thus using Equation 3.1 we have

$$\begin{aligned}0 &= yf(x,y)-xg(x,y)-xf(y,x)+yg(y,x) \\ &= yf(x,y)+xg(y,x)+xf(x,y)+yg(y,x) \\ &= (x+y)(f(x,y)+g(y,x)).\end{aligned}$$

□

4. Conclusion

In this study, inner automorphisms of algebras of symmetric polynomials of (relatively) free Lie algebras in the group of automorphisms of (relatively) free Lie algebras were determined. The next step might be sharpening the result by finding all inner automorphisms of those algebras. For this purpose, one needs to have generators for the algebras under consideration, and handle the automorphisms by their action on those generators.

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Department of Mathematics, Çukurova University, 01330 Balcali, Adana, Turkey
Email address: sfindik@cu.edu.tr, noguslu@cu.edu.tr

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AN INVESTIGATION OF THE SOLUTIONS AND THE DYNAMIC BEHAVIOR OF SOME RATIONAL DIFFERENCE EQUATIONS

NISREEN A. BUKHARY AND ELSAYED M. ELSAYED

0000-0001-7456-8545 and 0000-0003-0894-8472

Abstract. The main purpose of this work is to find the form of the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(\pm 1 \pm x_{n-2}x_{n-6})}, \quad n = 1, 2, ...,$$

where the initial conditions are arbitrary positive real numbers. Moreover, we gave the solutions of some special cases of this equation, and studied some dynamic behavior of these equations. At the end we illustrated our results by presenting some numerical examples to the equations are given.

1. Introduction

In the last few decades, there has been a major interest in studying a qualitative behavior of the solutions of rational difference equations. The reasons of this interest comes from the fact that these equations are powerful tool for applications since difference equations plays an important role in mathematics to describe and model a real life situations such as population dynamics, statistical problem, stochastic time series, number theory, biology, economic, probability theory, genetics, psychology, etc. [1]-[5]. It is well known that the field of difference equations is old and it has been developed incrementally, and the rational difference equations is important category of difference equations where they occupies a good place in applicable analysis, which has encouraged the mathematical researchers to continue investigating the qualitative properties of the solution of rational difference equations and the systems of difference equations.

Recently, Abo-Zeid [6] solved and studied the global behavior of the well defined solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-3}}{Ax_{n-2} + Bx_{n-3}},$$

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Elsayed [7] have obtained the solution and also he studied the behavior of the following rational difference equation

$$x_{n+1} = ax_n + \frac{bx_n x_{n-1}}{cx_n + dx_{n-1}}.$$

Cinar [8]-[10] have investigated the positive solutions of the following difference equations

$$x_{n+1} = \frac{\alpha x_{n-1}}{1 + b_n x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{1 + \alpha x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + \alpha x_n x_{n-1}}.$$

Ibrahim [11] got the solutions of the rational difference equation:

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + bx_n x_{n-2})}.$$

Bozkurt [12] was investigated the local and global behavior of the positive solutions of the following difference equation

$$y_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}}.$$

Simsek et. al. [13] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Xian and L. Wei [14] investigated the global asymptotic stability of the following difference equation

$$x_{n+1} = \frac{p + qx_n}{1 + rx_{n-k}}.$$

Karatas et. al. [15] studied study the positive solutions and attractivity of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-2} x_{n-5}}.$$

For other papers related to study the dynamic behavior of difference, we refer to [16]-[28].

Our goal is to study the dynamic behaviors of the solutions of the following difference equations.

$$x_{n+1} = \frac{x_{n-2} x_{n-6}}{x_{n-3}(\pm 1 \pm x_{n-2} x_{n-6})}, \tag{1.1}$$

where the initial conditions $x_{-6},\ x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary nonzero real numbers.

2. Preliminaries

Here, we review some results which will be useful in our investigation of the difference equation (1.1).

Definition 2.1. Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of the initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, x_{n-2} \dots, x_{n-k}), \qquad n = 0, 1, ..., \tag{2.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 2.2. A point $x^* \in I$ is called an **equilibrium point** of Eq. (2.1) if

$$x^* = F(x^*, x^*, x^*, ...).$$

That is,$x_n = x^*$, for $n \geq 0$, is a solution of Eq. (2.1), or equivalently, x^* is a fixed point of F.

Definition 2.3. Let x^* be an equilibrium point of (2.1).

(**i**) The equilibrium point x^* of Eq. (2.1) is called **locally stable** if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_0 \in I$ with

$$|x_{-k} - x^*| + |x_{-k+1} - x^*| + ... + |x_0 - x^*| < \delta,$$

we have,

$$|x_n - x^*| < \epsilon \ \ for\ all \qquad n \geq -k.$$

(**ii**) The equilibrium point x^* of Eq. (2.1) is called **locally asymptotically stable** if it is locally stable, and if there exists $\gamma > 0$ such that if $x_{-k}, x_{-k+1}, ..., x_0 \in I$ with

$$|x_{-k} - x^*| + |x_{-k+1} - x^*| + \cdots + |x_0 - x^*| < \gamma,$$

we have,

$$\lim_{n\to\infty} x_n = x^*.$$

(**iii**) The equilibrium point x^* of Eq. (2.1) is called **a global attractor** if for every solution $x_{-k}, x_{-k+1}, ..., x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = x^*.$$

(**iv**) The equilibrium point x^* of Eq. (2.1) is called **a global asymptotically stable** if it is locally stable and global attractor of Eq. (2.1).

(**v**) The equilibrium point x^* of Eq. (2.1) is called **unstable** if x^* is not locally stable.

The linearized equation of Eq. (2.1) about the equilibrium point x^* is the linear difference equation

$$z_{n+1} = \sum_{i=1}^{k} \frac{\partial F(\hat{x}, \dots, \hat{x})}{\partial x_{n-i}} z_{n-i}.$$

Definition 2.4. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with periodic p if $x_{n+p} = x_n$ for all $n \geq -k$.

Theorem 2.1. [30]. Assume that $p_0, p_1, \ldots, p_k \in R$, and $k \in \{0, 1, 2, \ldots\}$. Then

$$\sum_{i=1}^{k} |p_i| < 1, \tag{2.2}$$

is a sufficient condition for the asymptotic stability of the difference equation:

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \quad \text{n=0,1,}\ldots.$$

3. On the difference equation $x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(1+x_{n-2}x_{n-6})}$

In this section, we obtain a specific form of the solution of the first case of the equation (1.1):

$$x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(1 + x_{n-2}x_{n-6})}. \tag{3.1}$$

Theorem 3.1. Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of equation (1.1). Then for $n = 0, 1, \ldots,$

$$x_{24n-6} = g\prod_{i=0}^{n-1} \frac{(1+(8i+2)ae)(1+(8i+5)bf)(1+(8i)cg)}{(1+(8i+6)ae)(1+(8i+1)bf)(1+(8i+4)cg)},$$

$$x_{24n-5} = f\prod_{i=0}^{n-1} \frac{(1+(8i+5)ae)(1+(8i)bf)(1+(8i+3)cg)}{(1+(8i+1)ae)(1+(8i+4)bf)(1+(8i+7)cg)},$$

$$x_{24n-4} = e\prod_{i=0}^{n-1} \frac{(1+(8i)ae)(1+(8i+3)bf)(1+(8i+6)cg)}{(1+(8i+4)ae)(1+(8i+7)bf)(1+(8i+2)cg)},$$

$$x_{24n-3} = d\prod_{i=0}^{n-1} \frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)},$$

$$x_{24n-2} = c\prod_{i=0}^{n-1} \frac{(1+(8i+6)ae)(1+(8i+1)bf)(1+(8i+4)cg)}{(1+(8i+2)ae)(1+(8i+5)bf)(1+(8i+8)cg)},$$

$$x_{24n-1} = b\prod_{i=0}^{n-1} \frac{(1+(8i+1)ae)(1+(8i+4)bf)(1+(8i+7)cg)}{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+3)cg)},$$

$$x_{24n} = a\prod_{i=0}^{n-1} \frac{(1+(8i+4)ae)(1+(8i+7)bf)(1+(8i+2)cg)}{(1+(8i+8)ae)(1+(8i+3)bf)(1+(8i+6)cg)},$$

$$x_{24n+1} = \frac{cg}{d(1+cg)}\prod_{i=0}^{n-1} \frac{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+9)cg)},$$

$$x_{24n+2} = \frac{bf}{c(1+bf)} \prod_{i=0}^{n-1} \frac{(1+(8i+2)ae)(1+(8i+5)bf)(1+(8i+8)cg)}{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+4)cg)},$$

$$x_{24n+3} = \frac{ae}{b(1+ae)} \prod_{i=0}^{n-1} \frac{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+3)cg)}{(1+(8i+9)ae)(1+(8i+4)bf)(1+(8i+7)cg)},$$

$$x_{24n+4} = \frac{cg}{a(1+2cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+8)ae)(1+(8i+3)bf)(1+(8i+6)cg)}{(1+(8i+4)ae)(1+(8i+7)bf)(1+(8i+10)cg)},$$

$$x_{24n+5} = \frac{bdf(1+cg)}{cg(1+2bf)} \prod_{i=0}^{n-1} \frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+9)cg)}{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+5)cg)},$$

$$x_{24n+6} = \frac{ace(1+bf)}{bf(1+2ae)} \prod_{i=0}^{n-1} \frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+4)cg)}{(1+(8i+10)ae)(1+(8i+5)bf)(1+(8i+8)cg)},$$

$$x_{24n+7} = \frac{bcg(1+ae)}{ae(1+3cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+9)ae)(1+(8i+4)bf)(1+(8i+7)cg)}{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+11)cg)},$$

$$x_{24n+8} = \frac{abf(1+2cg)}{cg(1+3bf)} \prod_{i=0}^{n-1} \frac{(1+(8i+4)ae)(1+(8i+7)bf)(1+(8i+10)cg)}{(1+(8i+8)ae)(1+(8i+11)bf)(1+(8i+6)cg)},$$

$$x_{24n+9} = \frac{aceg(1+2bf)}{bdf(1+cg)(1+3ae)} \prod_{i=0}^{n-1} \frac{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+5)cg)}{(1+(8i+11)ae)(1+(8i+6)bf)(1+(8i+9)cg)},$$

$$x_{24n+10} = \frac{bfg(1+2ae)}{ae(1+bf)(1+4cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+10)ae)(1+(8i+5)bf)(1+(8i+8)cg)}{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)},$$

$$x_{24n+11} = \frac{aef(1+3cg)}{cg(1+ae)(1+4bf)} \prod_{i=0}^{n-1} \frac{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+11)cg)}{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)},$$

$$x_{24n+12} = \frac{ceg(1+3bf)}{bf(1+2cg)(1+4ae)} \prod_{i=0}^{n-1} \frac{(1+(8i+8)ae)(1+(8i+11)bf)(1+(8i+6)cg)}{(1+(8i+12)ae)(1+(8i+7)bf)(1+(8i+10)cg)},$$

$$x_{24n+13} = \frac{bdf(1+3ae)(1+cg)}{ae(1+2bf)(1+5cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+11)ae)(1+(8i+6)bf)(1+(8i+9)cg)}{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+13)cg)},$$

$$x_{24n+14} = \frac{ae(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)} \prod_{i=0}^{n-1} \frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)},$$

$$x_{24n+15} = \frac{cg(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)},$$

$$x_{24n+16} = \frac{bf(1+4ae)(1+2cg)}{e(1+3bf)(1+6cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+12)ae)(1+(8i+7)bf)(1+(8i+10)cg)}{(1+(8i+8)ae)(1+(8i+11)bf)(1+(8i+14)cg)},$$

$$x_{24n+17} = \frac{ae(1+2bf)(1+5cg)}{d(1+3ae)(1+6bf)(1+cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+13)cg)}{(1+(8i+11)ae)(1+(8i+14)bf)(1+(8i+9)cg)},$$

where $x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a$ are arbitrary nonzero real numbers.

Proof. The result holds for $n = 0$. Now, assume that $n > 0$ and our assumption holds for $n-1$. Then,

$$x_{24n-30} = g \prod_{i=0}^{n-2} \frac{(1+(8i+2)ae)(1+(8i+5)bf)(1+(8i)cg)}{(1+(8i+6)ae)(1+(8i+1)bf)(1+(8i+4)cg)},$$

$$x_{24n-29} = f \prod_{i=0}^{n-2} \frac{(1+(8i+5)ae)(1+(8i)bf)(1+(8i+3)cg)}{(1+(8i+1)ae)(1+(8i+4)bf)(1+(8i+7)cg)},$$

$$x_{24n-28} = e \prod_{i=0}^{n-2} \frac{(1+(8i)ae)(1+(8i+3)bf)(1+(8i+6)cg)}{(1+(8i+4)ae)(1+(8i+7)bf)(1+(8i+2)cg)},$$

$$x_{24n-27} = d \prod_{i=0}^{n-2} \frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)},$$

$$x_{24n-26} = c \prod_{i=0}^{n-2} \frac{(1+(8i+6)ae)(1+(8i+1)bf)(1+(8i+4)cg)}{(1+(8i+2)ae)(1+(8i+5)bf)(1+(8i+8)cg)},$$

$$x_{24n-25} = b \prod_{i=0}^{n-2} \frac{(1+(8i+1)ae)(1+(8i+4)bf)(1+(8i+7)cg)}{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+3)cg)},$$

$$x_{24n-24} = a \prod_{i=0}^{n-2} \frac{(1+(8i+4)ae)(1+(8i+7)bf)(1+(8i+2)cg)}{(1+(8i+8)ae)(1+(8i+3)bf)(1+(8i+6)cg)},$$

$$x_{24n-23} = \frac{cg}{d(1+cg)} \prod_{i=0}^{n-2} \frac{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+9)cg)},$$

$$x_{24n-22} = \frac{bf}{c(1+bf)} \prod_{i=0}^{n-2} \frac{(1+(8i+2)ae)(1+(8i+5)bf)(1+(8i+8)cg)}{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+4)cg)},$$

$$x_{24n-21} = \frac{ae}{b(1+ae)} \prod_{i=0}^{n-2} \frac{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+3)cg)}{(1+(8i+9)ae)(1+(8i+4)bf)(1+(8i+7)cg)},$$

$$x_{24n-20} = \frac{cg}{a(1+2cg)} \prod_{i=0}^{n-2} \frac{(1+(8i+8)ae)(1+(8i+3)bf)(1+(8i+6)cg)}{(1+(8i+4)ae)(1+(8i+7)bf)(1+(8i+10)cg)},$$

$$x_{24n-19} = \frac{bdf(1+cg)}{cg(1+2bf)} \prod_{i=0}^{n-2} \frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+9)cg)}{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+5)cg)},$$

$$x_{24n-18}=\frac{ace(1+bf)}{bf(1+2ae)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+4)cg)}{(1+(8i+10)ae)(1+(8i+5)bf)(1+(8i+8)cg)},$$

$$x_{24n-17}=\frac{bcg(1+ae)}{ae(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+4)bf)(1+(8i+7)cg)}{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+11)cg)},$$

$$x_{24n-16}=\frac{abf(1+2cg)}{cg(1+3bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+4)ae)(1+(8i+7)bf)(1+(8i+10)cg)}{(1+(8i+8)ae)(1+(8i+11)bf)(1+(8i+6)cg)},$$

$$x_{24n-15}=\frac{aceg(1+2bf)}{bdf(1+cg)(1+3ae)}\prod_{i=0}^{n-2}\frac{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+5)cg)}{(1+(8i+11)ae)(1+(8i+6)bf)(1+(8i+9)cg)},$$

$$x_{24n-14}=\frac{bfg(1+2ae)}{ae(1+bf)(1+4cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+10)ae)(1+(8i+5)bf)(1+(8i+8)cg)}{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)},$$

$$x_{24n-13}=\frac{aef(1+3cg)}{cg(1+ae)(1+4bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+11)cg)}{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)},$$

$$x_{24n-12}=\frac{ceg(1+3bf)}{bf(1+2cg)(1+4ae)}\prod_{i=0}^{n-2}\frac{(1+(8i+8)ae)(1+(8i+11)bf)(1+(8i+6)cg)}{(1+(8i+12)ae)(1+(8i+7)bf)(1+(8i+10)cg)},$$

$$x_{24n-11}=\frac{bdf(1+3ae)(1+cg)}{ae(1+2bf)(1+5cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+11)ae)(1+(8i+6)bf)(1+(8i+9)cg)}{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+13)cg)},$$

$$x_{24n-10}=\frac{ae(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)},$$

$$x_{24n-9}=\frac{cg(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)},$$

$$x_{24n-8}=\frac{bf(1+4ae)(1+2cg)}{e(1+3bf)(1+6cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+12)ae)(1+(8i+7)bf)(1+(8i+10)cg)}{(1+(8i+8)ae)(1+(8i+11)bf)(1+(8i+14)cg)},$$

$$x_{24n-7}=\frac{ae(1+2bf)(1+5cg)}{d(1+3ae)(1+6bf)(1+cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+13)cg)}{(1+(8i+11)ae)(1+(8i+14)bf)(1+(8i+9)cg)}.$$

Now, it follows from equation (3.1) that

$$x_{24n-6}=\frac{x_{24n-9}x_{24n-13}}{x_{24n-10}(1+x_{24n-9}x_{24n-13})}$$

$$=\frac{\frac{ae}{(1+5ae)}\prod_{i=0}^{n-2}\frac{1+(8i+5)ae}{1+(8i+13)ae}}{\frac{ae(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}\{1+\frac{ae}{1+5ae}\prod_{i=0}^{n-2}\frac{1+(8i+5)ae}{1+(8i+13)ae}\}}$$

$$=\frac{\frac{1}{(1+5ae)}\prod_{i=0}^{n-2}\frac{1+(8i+5)ae}{1+(8i+13)ae}}{\frac{(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}\{1+\frac{ae}{1+5ae}\prod_{i=0}^{n-2}\frac{1+(8i+5)ae}{1+(8i+13)ae}\}}$$

$$= \frac{\frac{1}{(1+5ae)}\{\frac{(1+5ae)(1+13ae)...(1+(8n-11)ae)}{(1+13ae)(1+21ae)...(1+(8n-3)ae)}\}}{\frac{(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}\{1+\frac{ae}{(1+5ae)}\{\frac{(1+5ae)(1+13ae)...(1+(8n-11)ae)}{(1+13ae)(1+21ae)...(1+(8n-3)ae)}\}\}}$$

$$= \frac{\frac{1}{(1+(8n-3)ae)}}{\frac{(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}\{1+\frac{ae}{(1+(8n-3)ae)}\}}$$

$$= \frac{1}{\frac{(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}(1+(8n-3)ae)\{1+\frac{ae}{(1+(8n-3)ae)}\}}$$

$$= \frac{1}{\frac{(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}\{(1+(8n-3)ae)+ae)\}}$$

$$= \frac{1}{\frac{(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}\{(1+(8n-2)ae\}}$$

$$= \frac{g(1+2ae)(1+5bf)}{(1+bf)(1+4cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)\{(1+(8n-2)ae\}}.$$

Hence,

$$x_{24n-6} = g\prod_{i=0}^{n-1}\frac{(1+(8i+2)ae)(1+(8i+5)bf)(1+(8i)cg)}{(1+(8i+6)ae)(1+(8i+1)bf)(1+(8i+4)cg)}.$$

Similarly, we have

$$x_{24n-5} = \frac{x_{24n-8}x_{24n-12}}{x_{24n-9}(1+x_{24n-8}x_{24n-12})}$$

$$= \frac{\frac{cg}{(1+6cg)}\prod_{i=0}^{n-2}\frac{1+(8i+6)cg}{1+(8i+4)cg}}{\frac{cg(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}\{1+\frac{cg}{(1+6cg)}\prod_{i=0}^{n-2}\frac{1+(8i+6)cg}{1+(8i+4)cg}\}}$$

$$= \frac{\frac{1}{(1+6cg)}\prod_{i=0}^{n-2}\frac{1+(8i+6)cg}{1+(8i+4)cg}}{\frac{(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}\{1+\frac{cg}{(1+6cg)}\prod_{i=0}^{n-2}\frac{1+(8i+6)cg}{1+(8i+4)cg}\}}$$

$$= \frac{\frac{1}{(1+6cg)}\{\frac{(1+6cg)(1+14cg)...(1+(8n-10)cg)}{(1+14cg)(1+22cg)...(1+(8n-2)cg)}\}}{\frac{(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}\{1+\frac{cg}{(1+6cg)}\{\frac{(1+6cg)(1+14cg)...(1+(8n-10)cg)}{(1+14cg)(1+22cg)...(1+(8n-2)cg)}\}\}}$$

$$= \frac{\frac{1}{(1+(8n-3)cg)}}{\frac{(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}\{1+\frac{cg}{(1+(8n-3)cg)}\}}$$

$$=\frac{1}{\frac{(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}(1+(8n-2)ae)\{1+\frac{cg}{(1+(8n-2)cg)}\}}$$

$$=\frac{1}{\frac{(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}\{(1+(8n-2)cg)+cg\}}$$

$$=\frac{1}{\frac{(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}\{(1+(8n-1)cg)\}}$$

$$=\frac{f(1+5ae)(1+3cg)}{(1+ae)(1+4bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}\frac{1}{\{(1+(8n-1)cg)\}}.$$

Then, we have

$$x_{24n-5}=f\prod_{i=0}^{n-1}\frac{(1+(8i+5)ae)(1+(8i)bf)(1+(8i+3)cg)}{(1+(8i+1)ae)(1+(8i+4)bf)(1+(8i+7)cg)}.$$

Again, applying the same steps,

$$x_{24n+1}=\frac{x_{24n-2}x_{24n-6}}{x_{24n-3}(1+x_{24n-2}x_{24n-6})}$$

$$=\frac{cg\prod_{i=0}^{n-1}\frac{1+(8i)cg}{1+(8i+8)cg}}{d\prod_{i=0}^{n-1}d\frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}\{1+cg\prod_{i=0}^{n-1}\frac{1+(8i)cg}{1+(8i+8)cg}\}}$$

$$=\frac{cg\{\frac{(1+8cg)(1+16cg)...(1+(8n-16)cg)(1+(8n-8)cg)}{(1+8cg)(1+16cg)...(1+(8n-8)cg)(1+(8n)cg)}\}}{d\prod_{i=0}^{n-1}\frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}\{1+cg\{\frac{(1+8cg)(1+16cg)...(1+(8n-16)cg)(1+(8n-8)cg)}{(1+8cg)(1+16cg)...(1+(8n-8)cg)(1+(8n)cg)}\}\}}$$

$$=\frac{\frac{cg}{(1+(8n)cg)}}{d\prod_{i=0}^{n-1}\frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}\{1+\frac{cg}{(1+(8n)cg)}\}}$$

$$=\frac{cg}{d\prod_{i=0}^{n-1}\frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}(1+(8n)cg)\{1+\frac{cg}{(1+(8n)cg)}\}}$$

$$=\frac{cg}{d\prod_{i=0}^{n-1}\frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}\{1+(8n)cg+cg\}}$$

$$=\frac{cg}{d\prod_{i=0}^{n-1}\frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}\{1+(8n)cg+cg\}}.$$

Hence,

$$x_{24n+1}=\frac{cg}{d(1+cg)}\prod_{i=0}^{n-1}\frac{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+9)cg)}.$$

Consequently, we can easily obtain the solutions of the other relations. Thus, the proof is completed. □

Theorem 3.2. Equation (3.1) has a unique equilibrium point $x^*=0$ which is not locally asymptotically stable.

Proof. For the equilibrium points of equation (3.1), we can write

$$x^*=\frac{x^{*2}}{x^*(1+x^{*2})},$$

$$\Rightarrow \quad x^{*2}(1+x^{*2})=x^{*2} \qquad \Rightarrow \quad 1+x^{*2}=1.$$

Thus, the equilibrium point of equation (3.1) is $x^*=0$.
Now, let F be a function define by

$$F(u,v,w)=\frac{uw}{v(1+uw)}.$$

Therefore,

$$F_u(u,v,w)=\frac{w}{v(1+uw)^2},\quad F_v(u,v,w)=\frac{-uw}{v^2(1+uw)},\quad F_w(u,v,w)=\frac{u}{v(1+uw)^2}.$$

Then,

$$F_u(x^*,x^*,x^*)=1,\quad F_v(x^*,x^*,x^*)=-1,\quad F_w(x^*,x^*,x^*)=1.$$

It follows from Theorem (2.1) that equation (3.1) is not asymptotically stable. □

Numerical Examples

To confirm the result of the first subsection, we assume the following numerical examples which illustrate difference types of solutions to equation (3.1).

Example 3.1. We put $x_{-6}=0.43,\ x_{-5}=0.22,\ x_{-4}=0.1,\ x_{-3}=0.4,\ x_{-2}=0.33,\ x_{-1}=0.7,\ x_0=0.5$ in equation (3.1). So from Figure 1, we can see the behavior of the solution of equation equation (3.1), where the solution dose not converge to zero which prove the fact that the equilibrium point 0 is not locally asymptotically stable .

Example 3.2. In Figure 2, since $x_{-6}=7,\ x_{-5}=6,\ x_{-4}=5,\ x_{-3}=4,\ x_{-2}=3,\ x_{-1}=2,\ x_0=1$, we assure the same result of the previous example.

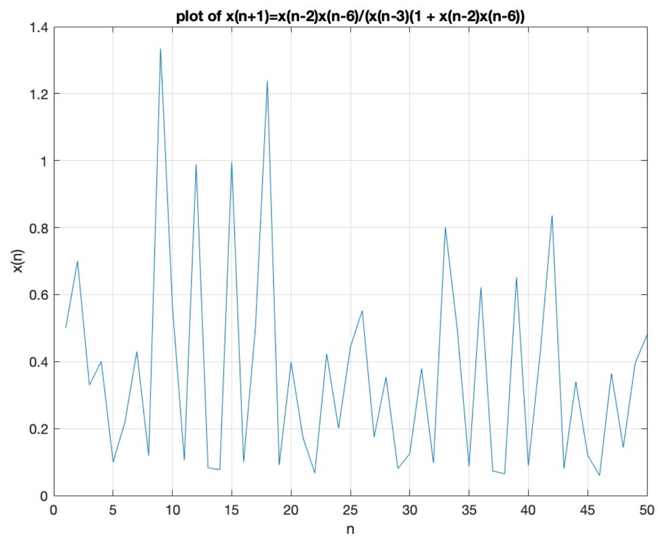

FIGURE 1

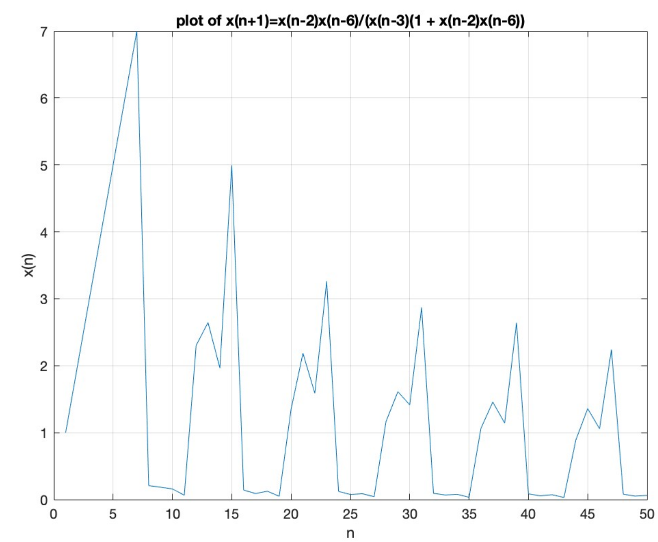

FIGURE 2

4. On the difference equation $x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(-1+x_{n-2}x_{n-6})}$

In this section, we study the second following case of the equation (1.1) in the form:

$$x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(-1 + x_{n-2}x_{n-6})}. \tag{4.1}$$

Theorem 4.1. Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of equation (4.1). Then the solutions of equation (4.1) are periodic of period 24 and given by:

$$x_{24n-6}=g, \qquad x_{24n-5}=f,$$

$$x_{24n-4}=e, \qquad x_{24n-3}=d,$$

$$x_{24n-2}=c, \qquad x_{24n-1}=b,$$

$$x_{24n}=a, \qquad x_{24n+1}=\frac{cg}{d(-1+cg)},$$

$$x_{24n+2}=\frac{bf}{c(-1+bf)}, \qquad x_{24n+3}=\frac{ae}{b(-1+ae)},$$

$$x_{24n+4}=\frac{cg}{a}, \qquad x_{24n+5}=\frac{bdf(-1+cg)}{cg},$$

$$x_{24n+6}=\frac{ace(-1+bf)}{bf}, \qquad x_{24n+7}=\frac{bcg(-1+ae)}{ae(-1+cg)},$$

$$x_{24n+8}=\frac{abf}{cg(-1+bf)}, \qquad x_{24n+9}=\frac{(ae)(cg)}{bdf(-1+ae)(-1+cg)},$$

$$x_{24n+10}=\frac{(bf)g}{(ae)(-1+bf)}, \qquad x_{24n+11}=\frac{(aef)(-1+cg)}{(cg)(-1+ae)},$$

$$x_{24n+12}=\frac{ceg(-1+bf)}{bf}, \qquad x_{24n+13}=\frac{bdf(-1+ae)}{ae},$$

$$x_{24n+14}=\frac{ae}{g}, \qquad \mathrm{x}_{24n+15}=\frac{cg}{f(-1+cg)},$$

$$x_{24n+16}=\frac{bf}{e(-1+bf)}, \qquad x_{24n+17}=\frac{ae}{d(-1+ae)},$$

where $x_{-6}=g,\ x_{-5}=f,\ x_{-4}=e,\ x_{-3}=d,\ x_{-2}=c,\ x_{-1}=b, x_0=a$ are arbitrary nonzero real numbers with initial conditions $x_{-2}x_{-6}\neq 1,\ \ x_{-1}x_{-5}\neq 1,$ $x_0x_{-4}\neq 1$.

Proof. For $n=0$ the conclusion holds. Now, suppose that $n>0$ and our assumption holds for $n-1$. Then,

$$x_{24n-30}=g, \qquad x_{24n-29}=f,$$

$$x_{24n-28}=e, \qquad x_{24n-27}=d,$$

$$x_{24n-26}=c, \qquad x_{24n-25}=b,$$

$$x_{24n-24}=a, \qquad x_{24n-23}=\frac{cg}{d(-1+cg)},$$

$$x_{24n-22}=\frac{bf}{c(-1+bf)}, \qquad x_{24n-21}=\frac{ae}{b(-1+ae)},$$

$$x_{24n-20}=\frac{cg}{a}, \qquad x_{24n-19}=\frac{bdf(-1+cg)}{cg},$$

$$x_{24n-18}=\frac{ace(-1+bf)}{bf}, \qquad x_{24n-17}=\frac{bcg(-1+ae)}{ae(-1+cg)},$$

$$x_{24n-16}=\frac{abf}{cg(-1+bf)}, \qquad x_{24n-15}=\frac{(ae)(cg)}{bdf(-1+ae)(-1+cg)},$$

$x_{24n-14} = \frac{bfg}{ae(-1+bf)}, \qquad x_{24n-13} = \frac{aef(-1+cg)}{cg(-1+ae)},$

$x_{24n-12} = \frac{ceg(-1+bf)}{bf}, \qquad x_{24n-11} = \frac{bdf(-1+ae)}{ae},$

$x_{24n-10} = \frac{ae}{g}, \qquad x_{24n-9} = \frac{cg}{f(-1+cg)},$

$x_{24n-8} = \frac{bf}{e(-1+bf)}, \qquad x_{24n-7} = \frac{ae}{d(-1+ae)}.$

Now, we proof some of the relations of equation (4.1).

$$x_{24n-6} = \frac{x_{24n-9}x_{24n-13}}{x_{24n-10}(-1+x_{24n-9}x_{24n-13})}$$

$$= \frac{\frac{cg}{f(-1+cg)}\frac{aef(-1+cg)}{cg(-1+ae)}}{\frac{ae}{g}\{-1+\{\frac{cg}{f(-1+cg)}\frac{aef(-1+cg)}{cg(-1+ae)}\}\}} = \frac{\frac{ae}{-1+ae}}{\frac{ae}{g}\{-1+\{\frac{ae}{-1+ae}\}\}}$$

$$= \frac{1}{\frac{1}{g}(-1+ae)\{-1+\{\frac{ae}{-1+ae}\}\}}v = \frac{g}{1-ae+ae} = g.$$

Similarly,

$$x_{24n+7} = \frac{x_{24n+4}x_{24n}}{x_{24n+3}(-1+x_{24n+4}x_{24n})} = \frac{\frac{cg}{a}(a)}{\frac{ae}{b(-1+ae)}\{-1+\frac{cg}{a}(a)\}}$$

$$= \frac{cg}{\frac{ae}{b(-1+ae)}\{-1+cg\}} = \frac{bcg(-1+ae)}{ae\{-1+cg\}}.$$

Also,

$$x_{24n+12} = \frac{x_{24n+9}x_{24n+5}}{x_{24n+8}(-1+x_{24n+9}x_{24n+5})}$$

$$= \frac{\frac{(ae)(cg)}{bdf(-1+ae)(-1+cg)}\frac{bdf(-1+cg)}{cg}}{\frac{abf}{cg(-1+bf)}\{-1+\{\frac{(ae)(cg)}{bdf(-1+ae)(-1+cg)}\frac{bdf(-1+cg)}{cg}\}}$$

$$= \frac{\frac{ae}{-1+ae}}{\frac{abf}{cg(-1+bf)}\{-1+\frac{ae}{-1+ae}\}} = \frac{ae}{\frac{abf}{cg(-1+bf)}} = \frac{ecg(-1+bf)}{bf}.$$

Hence, we can easily proof the other relations. Thus, the proof has been done. □

Theorem 4.2. Equation (4.1) has three equilibrium points which are 0 and $\pm\sqrt{2}$, where they are not locally asymptotically stable.

Proof. By using equation (4.1), and for the equilibrium points of (4.1) we can write

$$x^* = \frac{x^{*2}}{x^*(-1+x^{*2})}.$$

Then we have,

$$x^{*2}(-1+x^{*2}) = x^{*2},$$

or

$$x^{*2}(x^{*2}-2) = 0.$$

Thus, 0, $\pm\sqrt{2}$ are the equilibrium points.

Now, let F be a function define by

$$F(u,v,w) = \frac{uw}{v(-1+uw)}.$$

Therefore,

$$F_u(u,v,w) = \frac{-w}{v(-1+uw)^2}, \ F_v(u,v,w) = \frac{-uw}{v^2(-1+uw)}, \ F_w(u,v,w) = \frac{-u}{v(-1+uw)^2}.$$

Then,

$$F_u(x^*,x^*,x^*) = -1, \quad F_v(x^*,x^*,x^*) = \pm 1, \quad F_w(x^*,x^*,x^*) = -1.$$

Furthermore, we see from Theorem (2.1) that equation (4.1) is not asymptotically stable. □

Numerical Examples.

Conforming the result of the second subsection, we consider the following numerical examples which illustrate difference types of solutions to equation (4.1).

Example 4.1. In Figure 3 if we take the initial conditions as $x_{-6}=5,\ x_{-5}=3,\ x_4=4,\ x_{-3}=1,\ x_{-2}=1,\ x_{-1}=3,\ x_0=4$, then we see that the behavior of the solution of equation (4.1) doesn't converge to the equilibrium points zero or $\pm\sqrt{2}$, which confirm the result of Theorem (4.2.).

Example 4.2. Consider $x_{-6}=0.1,\ x_{-5}=0.2,\ x_{-4}=0.3,\ x_{-3}=0.4,\ x_{-2}=0.5,\ x_{-1}=0.6,\ x_0=0.7$. In Figure 4, we get the same result of Example 4.1.

5. On the difference equation $x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(1-x_{n-2}x_{n-6})}$

In this section, we get the expressions of the solution of the third case of the equation (1.1):

$$x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(1-x_{n-2}x_{n-6})}. \tag{5.1}$$

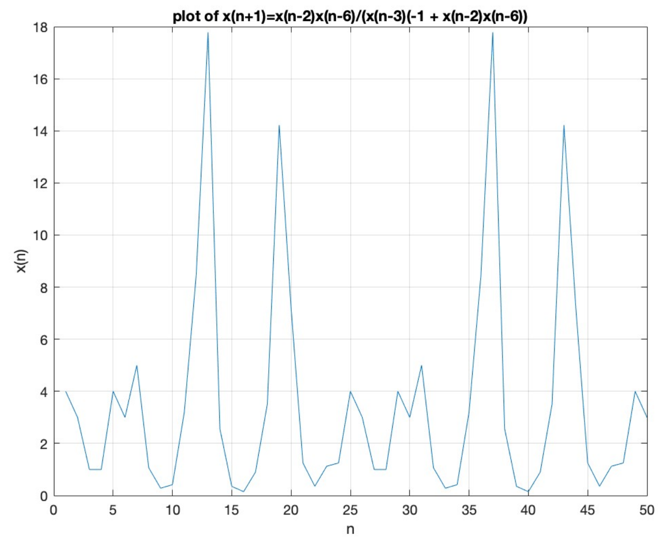

FIGURE 3

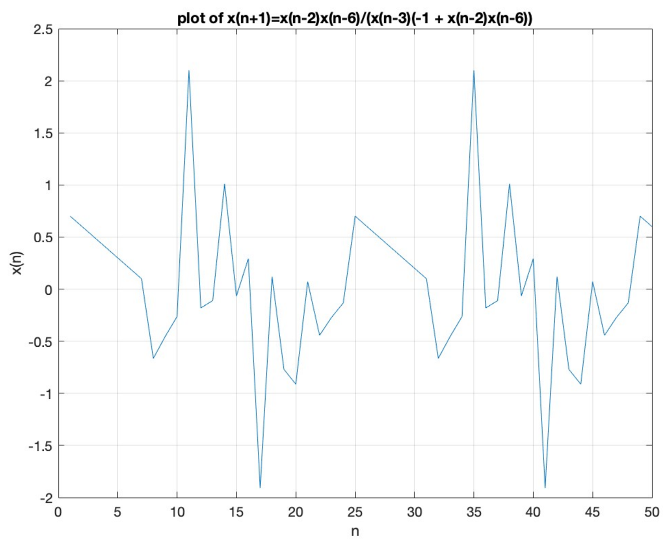

FIGURE 4

Theorem 5.1. Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of equation (5.1). Then

$$x_{24n-6} = g \prod_{i=0}^{n-1} \frac{(1-(8i+2)ae)(1-(8i+5)bf)(1-(8i)cg)}{(1-(8i+6)ae)(1-(8i+1)bf)(1-(8i+4)cg)},$$

$$x_{24n-5} = f\prod_{i=0}^{n-1}\frac{(1-(8i+5)ae)(1-(8i)bf)(1-(8i+3)cg)}{(1-(8i+1)ae)(1-(8i+4)bf)(1-(8i+7)cg)},$$

$$x_{24n-4} = e\prod_{i=0}^{n-1}\frac{(1-(8i)ae)(1-(8i+3)bf)(1-(8i+6)cg)}{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+2)cg)},$$

$$x_{24n-3} = d\prod_{i=0}^{n-1}\frac{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+1)cg)}{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)},$$

$$x_{24n-2} = c\prod_{i=0}^{n-1}\frac{(1-(8i+6)ae)(1-(8i+1)bf)(1-(8i+4)cg)}{(1-(8i+2)ae)(1-(8i+5)bf)(1-(8i+8)cg)},$$

$$x_{24n-1} = b\prod_{i=0}^{n-1}\frac{(1-(8i+1)ae)(1-(8i+4)bf)(1-(8i+7)cg)}{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+3)cg)},$$

$$x_{24n} = a\prod_{i=0}^{n-1}\frac{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+2)cg)}{(1-(8i+8)ae)(1-(8i+3)bf)(1-(8i+6)cg)},$$

$$x_{24n+1} = \frac{cg}{d(1-cg)}\prod_{i=0}^{n-1}\frac{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)}{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+9)cg)},$$

$$x_{24n+2} = \frac{bf}{c(1-bf)}\prod_{i=0}^{n-1}\frac{(1-(8i+2)ae)(1-(8i+5)bf)(1-(8i+8)cg)}{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+4)cg)},$$

$$x_{24n+3} = \frac{ae}{b(1-ae)}\prod_{i=0}^{n-1}\frac{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+3)cg)}{(1-(8i+9)ae)(1-(8i+4)bf)(1-(8i+7)cg)},$$

$$x_{24n+4} = \frac{cg}{a(1-2cg)}\prod_{i=0}^{n-1}\frac{(1-(8i+8)ae)(1-(8i+3)bf)(1-(8i+6)cg)}{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+10)cg)},$$

$$x_{24n+5} = \frac{bdf(1-cg)}{cg(1-2bf)}\prod_{i=0}^{n-1}\frac{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+9)cg)}{(1-(8i+7)ae)(1-(8i+10)bf)(1-(8i+5)cg)},$$

$$x_{24n+6} = \frac{ace(1-bf)}{bf(1-2ae)}\prod_{i=0}^{n-1}\frac{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+4)cg)}{(1-(8i+10)ae)(1-(8i+5)bf)(1-(8i+8)cg)},$$

$$x_{24n+7} = \frac{bcg(1-ae)}{ae(1-3cg)}\prod_{i=0}^{n-1}\frac{(1-(8i+9)ae)(1-(8i+4)bf)(1-(8i+7)cg)}{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+11)cg)},$$

$$x_{24n+8} = \frac{abf(1-2cg)}{cg(1-3bf)}\prod_{i=0}^{n-1}\frac{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+10)cg)}{(1-(8i+8)ae)(1-(8i+11)bf)(1-(8i+6)cg)},$$

$$x_{24n+9} = \frac{aceg(1-2bf)}{bdf(1-cg)(1-3ae)}\prod_{i=0}^{n-1}\frac{(1-(8i+7)ae)(1-(8i+10)bf)(1-(8i+5)cg)}{(1-(8i+11)ae)(1-(8i+6)bf)(1-(8i+9)cg)},$$

$$x_{24n+10} = \frac{bfg(1-2ae)}{ae(1-bf)(1-4cg)} \prod_{i=0}^{n-1} \frac{(1-(8i+10)ae)(1-(8i+5)bf)(1-(8i+8)cg)}{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+12)cg)},$$

$$x_{24n+11} = \frac{aef(1-3cg)}{cg(1-ae)(1-4bf)} \prod_{i=0}^{n-1} \frac{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+11)cg)}{(1-(8i+9)ae)(1-(8i+12)bf)(1-(8i+7)cg)},$$

$$x_{24n+12} = \frac{ceg(1-3bf)}{bf(1-2cg)(1-4ae)} \prod_{i=0}^{n-1} \frac{(1-(8i+8)ae)(1-(8i+11)bf)(1-(8i+6)cg)}{(1-(8i+12)ae)(1-(8i+7)bf)(1-(8i+10)cg)},$$

$$x_{24n+13} = \frac{bdf(1-3ae)(1-cg)}{ae(1-2bf)(1-5cg)} \prod_{i=0}^{n-1} \frac{(1-(8i+11)ae)(1-(8i+6)bf)(1-(8i+9)cg)}{(1-(8i+7)ae)(1-(8i+10)bf)(1-(8i+13)cg)},$$

$$x_{24n+14} = \frac{ae(1-bf)(1-4cg)}{g(1-2ae)(1-5bf)} \prod_{i=0}^{n-1} \frac{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+12)cg)}{(1-(8i+10)ae)(1-(8i+13)bf)(1-(8i+8)cg)},$$

$$x_{24n+15} = \frac{cg(1-ae)(1-4bf)}{f(1-5ae)(1-3cg)} \prod_{i=0}^{n-1} \frac{(1-(8i+9)ae)(1-(8i+12)bf)(1-(8i+7)cg)}{(1-(8i+13)ae)(1-(8i+8)bf)(1-(8i+11)cg)},$$

$$x_{24n+16} = \frac{bf(1-4ae)(1-2cg)}{e(1-3bf)(1-6cg)} \prod_{i=0}^{n-1} \frac{(1-(8i+12)ae)(1-(8i+7)bf)(1-(8i+10)cg)}{(1-(8i+8)ae)(1-(8i+11)bf)(1-(8i+14)cg)},$$

$$x_{24n+17} = \frac{ae(1-2bf)(1-5cg)}{d(1-3ae)(1-6bf)(1-cg)} \prod_{i=0}^{n-1} \frac{(1-(8i+7)ae)(1-(8i+10)bf)(1-(8i+13)cg)}{(1-(8i+11)ae)(1-(8i+14)bf)(1-(8i+9)cg)}.$$

where $x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a$ are arbitrary nonzero real numbers.

Proof. The result holds for $n = 0$. Now, assume that $n > 0$ and our assumption holds for $n-1$. Then,

$$x_{24n-30} = g \prod_{i=0}^{n-2} \frac{(1-(8i+2)ae)(1-(8i+5)bf)(1-(8i)cg)}{(1-(8i+6)ae)(1-(8i+1)bf)(1-(8i+4)cg)},$$

$$x_{24n-29} = f \prod_{i=0}^{n-2} \frac{(1-(8i+5)ae)(1-(8i)bf)(1-(8i+3)cg)}{(1-(8i+1)ae)(1-(8i+4)bf)(1-(8i+7)cg)},$$

$$x_{24n-28} = e \prod_{i=0}^{n-2} \frac{(1-(8i)ae)(1-(8i+3)bf)(1-(8i+6)cg)}{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+2)cg)},$$

$$x_{24n-27} = d \prod_{i=0}^{n-2} \frac{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+1)cg)}{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)},$$

$$x_{24n-26} = c\prod_{i=0}^{n-2}\frac{(1-(8i+6)ae)(1-(8i+1)bf)(1-(8i+4)cg)}{(1-(8i+2)ae)(1-(8i+5)bf)(1-(8i+8)cg)},$$

$$x_{24n-25} = b\prod_{i=0}^{n-2}\frac{(1-(8i+1)ae)(1-(8i+4)bf)(1-(8i+7)cg)}{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+3)cg)}$$

$$x_{24n-24} = a\prod_{i=0}^{n-2}\frac{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+2)cg)}{(1-(8i+8)ae)(1-(8i+3)bf)(1-(8i+6)cg)},$$

$$x_{24n-23} = \frac{cg}{d(1-cg)}\prod_{i=0}^{n-2}\frac{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)}{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+9)cg)},$$

$$x_{24n-22} = \frac{bf}{c(1-bf)}\prod_{i=0}^{n-2}\frac{(1-(8i+2)ae)(1-(8i+5)bf)(1-(8i+8)cg)}{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+4)cg)},$$

$$x_{24n-21} = \frac{ae}{b(1-ae)}\prod_{i=0}^{n-2}\frac{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+3)cg)}{(1-(8i+9)ae)(1-(8i+4)bf)(1-(8i+7)cg)},$$

$$x_{24n-20} = \frac{cg}{a(1-2cg)}\prod_{i=0}^{n-2}\frac{(1-(8i+8)ae)(1-(8i+3)bf)(1-(8i+6)cg)}{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+10)cg)},$$

$$x_{24n-19} = \frac{bdf(1-cg)}{cg(1-2bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+9)cg)}{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+5)cg)},$$

$$x_{24n-18} = \frac{ace(1-bf)}{bf(1-2ae)}\prod_{i=0}^{n-2}\frac{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+4)cg)}{(1-(8i+10)ae)(1-(8i+5)bf)(1-(8i+8)cg)},$$

$$x_{24n-17} = \frac{bcg(1-ae)}{ae(1-3cg)}\prod_{i=0}^{n-2}\frac{(1-(8i+9)ae)(1-(8i+4)bf)(1-(8i+7)cg)}{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+11)cg)},$$

$$x_{24n-16} = \frac{abf(1-2cg)}{cg(1-3bf)}\prod_{i=0}^{n-2}\frac{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+10)cg)}{(1-(8i+8)ae)(1-(8i+11)bf)(1-(8i+6)cg)},$$

$$x_{24n-15} = \frac{aceg(1-2bf)}{bdf(1-cg)(1-3ae)}\prod_{i=0}^{n-2}\frac{(1-(8i+7)ae)(1-(8i+10)bf)(1-(8i+5)cg)}{(1-(8i+11)ae)(1-(8i+6)bf)(1-(8i+9)cg)},$$

$$x_{24n-14} = \frac{bfg(1-2ae)}{ae(1-bf)(1-4cg)}\prod_{i=0}^{n-2}\frac{(1-(8i+10)ae)(1-(8i+5)bf)(1-(8i+8)cg)}{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+12)cg)},$$

$$x_{24n-13} = \frac{aef(1-3cg)}{cg(1-ae)(1-4bf)}\prod_{i=0}^{n-2}\frac{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+11)cg)}{(1-(8i+9)ae)(1-(8i+12)bf)(1-(8i+7)cg)},$$

$$x_{24n-12} = \frac{ceg(1-3bf)}{bf(1-2cg)(1-4ae)}\prod_{i=0}^{n-2}\frac{(1-(8i+8)ae)(1-(8i+11)bf)(1-(8i+6)cg)}{(1-(8i+12)ae)(1-(8i+7)bf)(1-(8i+10)cg)},$$

$$x_{24n-11} = \frac{bdf(1-3ae)(1-cg)}{ae(1-2bf)(1-5cg)}\prod_{i=0}^{n-2}\frac{(1-(8i+11)ae)(1-(8i+6)bf)(1-(8i+9)cg)}{(1-(8i+7)ae)(1-(8i+10)bf)(1-(8i+13)cg)},$$

$$x_{24n-10} = \frac{ae(1-bf)(1-4cg)}{g(1-2ae)(1-5bf)}\prod_{i=0}^{n-2}\frac{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+12)cg)}{(1-(8i+10)ae)(1-(8i+13)bf)(1-(8i+8)cg)},$$

$$x_{24n-9} = \frac{cg(1-ae)(1-4bf)}{f(1-5ae)(1-3cg)} \prod_{i=0}^{n-2} \frac{(1-(8i+9)ae)(1-(8i+12)bf)(1-(8i+7)cg)}{(1-(8i+13)ae)(1-(8i+8)bf)(1-(8i+11)cg)},$$

$$x_{24n-8} = \frac{bf(1-4ae)(1-2cg)}{e(1-3bf)(1-6cg)} \prod_{i=0}^{n-2} \frac{(1-(8i+12)ae)(1-(8i+7)bf)(1-(8i+10)cg)}{(1-(8i+8)ae)(1-(8i+11)bf)(1-(8i+14)cg)},$$

$$x_{24n-7} = \frac{ae(1-2bf)(1-5cg)}{d(1-3ae)(1-6bf)(1-cg)} \prod_{i=0}^{n-2} \frac{(1-(8i+7)ae)(1-(8i+10)bf)(1-(8i+13)cg)}{(1-(8i+11)ae)(1-(8i+14)bf)(1-(8i+9)cg)}.$$

Now, it follows from equation (5.1) that,

$$x_{24n+1} = \frac{x_{24n-2}x_{24n-6}}{x_{24n-3}(1-x_{24n-2}x_{24n-6})}$$

$$= \frac{cg\prod_{i=0}^{n-1}\frac{(1-(8i)cg)}{1-(8i+8)cg}}{d\prod_{i=0}^{n-1}\frac{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+1)cg)}{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)}\{1-cg\prod_{i=0}^{n-1}\frac{1-(8i)cg}{1-(8i+8)cg}\}}$$

$$= \frac{cg\{\frac{(1-8cg)(1-16cg)...(1-(8n-16)cg)(1-(8n-8)cg)}{(1-8cg)(1-16cg)...(1-(8n-8)cg)(1-(8n)cg)}\}}{d\prod_{i=0}^{n-1}\frac{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+1)cg)}{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)}\{1-cg\{\frac{(1-8cg)(1-16cg)...(1-(8n-16)cg)(1-(8n-8)cg)}{(1-8cg)(1-16cg)...(1-(8n-8)cg)(1-(8n)cg)}\}\}}$$

$$= \frac{\frac{cg}{(1-(8n)cg)}}{d\prod_{i=0}^{n-1}\frac{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+1)cg)}{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)}\{1-\frac{cg}{(1+(8n)cg)}\}}$$

$$= \frac{cg}{d(1-cg)} \prod_{i=0}^{n-1} \frac{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)}{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+9)cg)}.$$

We can easily proof the solutions of the other relations. Thus, the proof is completed. □

Theorem 5.2. Equation (5.1) has a unique equilibrium point that is number zero and this equilibrium point is not locally asymptotically stable.

Proof. As the proof of Theorem 3.2, and will be omitted. □

Numerical Examples.

In the next examples we can verify the result of Theorem (5.2.), that the solution does not converge to the equilibrium point 0.

Example 5.1. Assume the initial values of equation (5.1) are $x_{-6} = 2,\ x_{-5} = 1,\ x_{-4} = 2,\ x_{-3} = 3,\ x_{-2} = 4,\ x_{-1} = 2,\ x_0 = 5$. The behavior in Figure 5 shows that the solution of equation equation (5.1) dose not converge to zero which prove the result of Theorem (5.2.)

Example 5.2. See Figure 6 since (5.1) are $x_{-6} = -1,\ x_{-5} = 0.2,\ x_{-4} = -3,\ x_{-3} = 0.4,\ x_{-2} = 3,\ x_{-1} = -4,\ x_0 = -5.$, we got the same result of the previous example.

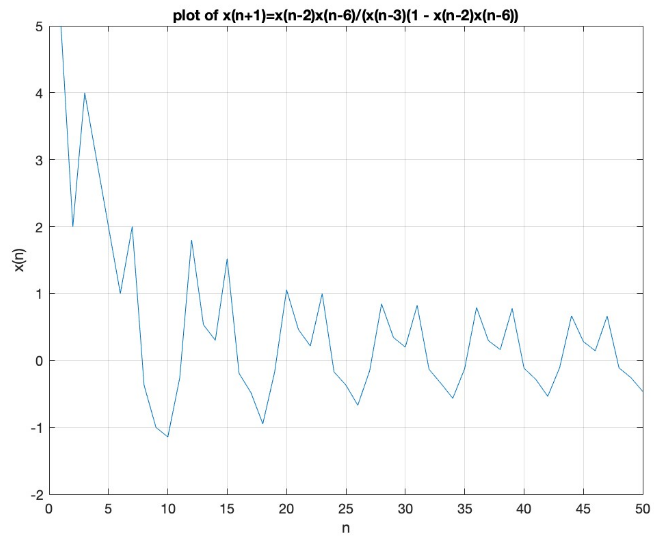

FIGURE 5

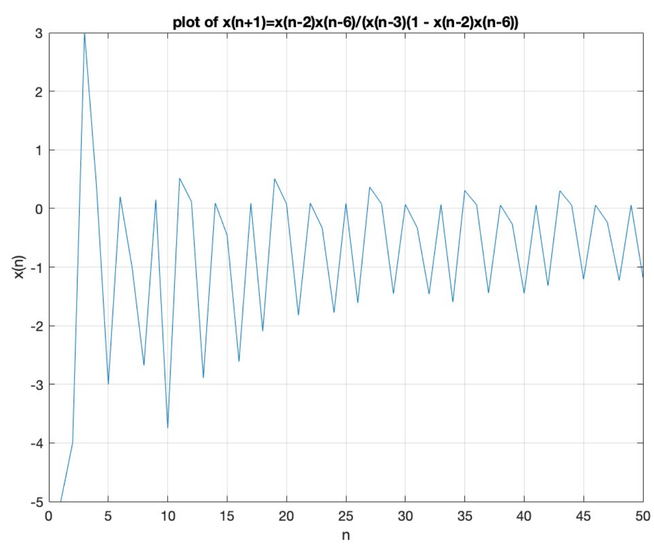

FIGURE 6

6. On the difference equation $x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(-1-x_{n-2}x_{n-6})}$

In this section, we study the last case of the equation (1.1) in the form:

$$x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(-1 - x_{n-2}x_{n-6})}. \tag{6.1}$$

Theorem 6.1. Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of equation (6.1). Then the solutions of equation (6.1) are periodic of period 24 and given by:

$x_{24n-6} = g,$ $\quad x_{24n-5} = f,$

$x_{24n-4} = e,$ $\quad x_{24n-3} = d,$

$x_{24n-2} = c,$ $\quad x_{24n-1} = b,$

$x_{24n} = a,$ $\quad x_{24n+1} = \frac{cg}{d(-1-cg)},$

$x_{24n+2} = \frac{bf}{c(-1-bf)},$ $\quad x_{24n+3} = \frac{ae}{b(-1-ae)},$

$x_{24n+4} = \frac{cg}{a},$ $\quad x_{24n+5} = \frac{bdf(-1-cg)}{cg},$

$x_{24n+6} = \frac{ace(-1-bf)}{bf},$ $\quad x_{24n+7} = \frac{bcg(-1-ae)}{ae(-1-cg)},$

$x_{24n+8} = \frac{abf}{cg(-1-bf)},$ $\quad x_{24n+9} = \frac{(ae)(cg)}{bdf(-1-ae)(-1-cg)},$

$x_{24n+10} = \frac{(bf)g}{(ae)(-1-bf)},$ $\quad x_{24n+11} = \frac{(aef)(-1-cg)}{(cg)(-1-ae)},$

$x_{24n+12} = \frac{ceg(-1-bf)}{bf},$ $\quad x_{24n+13} = \frac{bdf(-1-ae)}{ae},$

$x_{24n+14} = \frac{ae}{g},$ $\quad 24n+15 = \frac{cg}{f(-1-cg)},$

$x_{24n+16} = \frac{bf}{e(-1-bf)},$ $\quad x_{24n+17} = \frac{ae}{d(-1-ae)}.$

where $x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a$ are arbitrary nonzero real numbers with initial conditions $x_{-2}x_{-6} \neq -1,\ x_{-1}x_{-5} \neq -1,$ $x_0x_{-4} \neq -1$.

Proof. For $n = 0$ the conclusion holds. Now, suppose that $n > 0$ and our assumption holds for $n - 1$. Then,

$x_{24n-30} = g,$ $\quad x_{24n-29} = f,$

$x_{24n-28} = e,$ $\quad x_{24n-27} = d,$

$x_{24n-26} = c,$ $\quad x_{24n-25} = b,$

$x_{24n-24} = a,$ $\quad x_{24n-23} = \frac{cg}{d(-1-cg)},$

$x_{24n-22} = \frac{bf}{c(-1-bf)}$, $x_{24n-21} = \frac{ae}{b(-1-ae)}$,

$x_{24n-20} = \frac{cg}{a}$, $x_{24n-19} = \frac{bdf(-1-cg)}{cg}$,

$x_{24n-18} = \frac{ace(-1-bf)}{bf}$, $x_{24n-17} = \frac{bcg(-1-ae)}{ae(-1-cg)}$,

$x_{24n-16} = \frac{abf}{cg(-1-bf)}$, $x_{24n-15} = \frac{(ae)(cg)}{bdf(-1-ae)(-1-cg)}$,

$x_{24n-14} = \frac{bfg}{ae(-1-bf)}$, $x_{24n-13} = \frac{aef(-1-cg)}{cg(-1-ae)}$,

$x_{24n-12} = \frac{ceg(-1-bf)}{bf}$, $x_{24n-11} = \frac{bdf(-1-ae)}{ae}$,

$x_{24n-10} = \frac{ae}{g}$, $x_{24n-9} = \frac{cg}{f(-1-cg)}$,

$x_{24n-8} = \frac{bf}{e(-1-bf)}$, $x_{24n-7} = \frac{ae}{d(-1-ae)}$.

Now, we proof some of the relations of equation (6.1).

$$x_{24n+2} = \frac{x_{24n-1}x_{24n-5}}{x_{24n-2}(-1 - x_{24n-1}x_{24n-5})} = \frac{bf}{c(-1-bf)}.$$

Similarly,

$$x_{24n+9} = \frac{x_{24n+6}x_{24n+2}}{x_{24n+5}(-1 - x_{24n+6}x_{24n+2})} = \frac{\frac{ace(-1-bf)}{bf}\frac{bf}{c(-1-bf)}}{\frac{bdf(-1-cg)}{cg}(-1 - \frac{ace(-1-bf)}{bf}\frac{bf}{c(-1-bf)})}$$

$$= \frac{(ae)(cg)}{bdf(-1-ae)(-1-cg)}.$$

Hence, we can easily proof the other relations. Thus, the proof has been done. □

Theorem 6.2. Equation (6.1) has equilibrium point $x^* = 0$ and it is not locally asymptotically stable.

Proof. The proof is similar to the proof of Theorem 3.2, and will be omitted.

Numerical Examples.

Example 6.1. Figure 7 shows the periodic solution of equation (5.1) where the initial conditions are $x_{-6} = 9,\ x_{-5} = 4,\ x_{-4} = 3,\ x_{-3} = 4,\ x_{-2} = 10,\ x_{-1} = 7,\ x_0 = 9$. Also, it shows that the solution of equation (6.1) doesn't converge to the 0 and this confirms that the equation (6.1) is not asymptotically stable.

Example 6.2. Also in Figure 8 we assure the same results of Example 6.1. where the initial conditions are $x_{-6} = 1,\ x_{-5} = 0.22,\ x_{-4} = 0.3,\ x_{-3} = 7,\ x_{-2} = 1.0,\ x_{-1} = 0.7,\ x_0 = 0.9$.

7. Conclusion

In this article we presents the solution of the difference equation (1.1). First, we obtained the form of the solution of four special cases of the difference equation (1.1) and investigated the existence of the equilibrium point, the global asymptotic behavior and the existence of a periodic solutions of these equations. By the end, we gave some numerical examples of each case with different initial values by using the mathematical program MATLAB to confirm the obtained results.

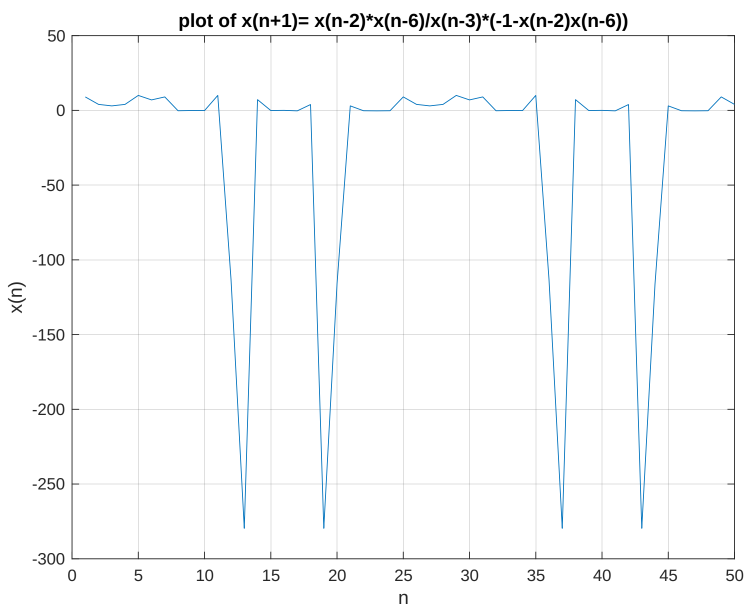

Figure 7

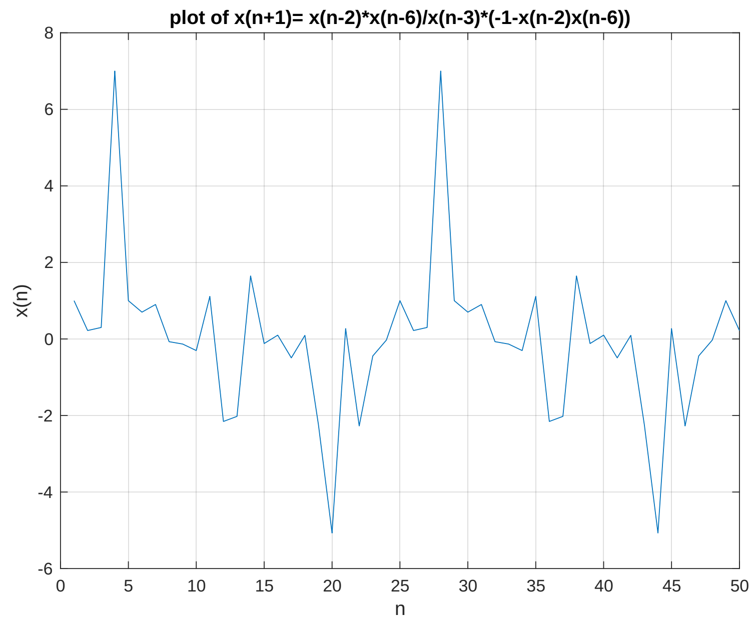

FIGURE 8

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The Declaration of Ethics Committee Approval

This study does not be necessary ethical committee permission or any special permission.

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The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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(NISREEN A. BUKHARY) 1-King Abdulaziz University, Faculty of Science, Mathematics Department, P.O.Box 80203, Jeddah 21589, Saudi Arabia., 2- Faculty of Science, Mathematics Department, Majmaah University, Saudi Arabia.

Email address: `nisreenbukhary@gmail.com`.

(ELSAYED M. ELSAYED) 1- King Abdulaziz University, Faculty of Science, Mathematics Department, P.O.Box 80203, Jeddah 21589, Saudi Arabia. 2-Faculty of Science, Mathematics Department, Mansoura University, Mansoura 35516, Egypt.

Email address: `emmelsayed@yahoo.com`.

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SOME CLASSES OF RICCATI EQUATIONS INTEGRABLE IN QUADRATURES

ANARKUL URDALETOVA, OLGA ZABINYAKOVA AND SYRGAK KYDYRALIEV

0000-0003-4420-3961, 0000-0002-6675-4265 and 0000-0001-6305-9251

Abstract. As it is known, the second-order ordinary linear differential equation with variable coefficients is solvable in case if related Riccati equation can be integrated by quadratures. This paper considers establishment of correspondence between such equations by the authors' method which means the second-order equation representation by a chain of the first-order equations. The algorithm of special Riccati equation solving is demonstrated (coefficients of these Riccati equations satisfy special conditions). One more peculiarity of this paper stands in consideration of exact applicational example – the Riccati equation which describes the magnetotellurics impedance behavior in geological media.

1. Introduction

Some mathematical problems have an amazing fate – for several centuries these have been attractive objects for the mathematics fans' attention. Of course, first of all, it is worth to mention the Fermat's Last Theorem. By the way, it is interesting to note that the theorem about the equation $x^3 + y^3 = z^3$ unsolvability in natural numbers appears in Abu-Mahmud Khojandi's (Xth century) investigations long before Pierre de Fermat.

In ordinary differential equations theory the same role belongs to the Riccati equations. Despite the fact that in 1841 Joseph Liouville proved that the general solutions of such equations are usually impossible to be express by quadratures in terms of elementary functions, a huge number of scientific papers are still devoted to the Riccati equations study till nowadays.

There are two main reasons for this popularity. Firstly, the Riccati equations are used in the mathematical description of a huge number of problems in the algebraic geometry and the theory of completely integrable Hamiltonian systems, in the calculus of variations and the conformal map theory, the quantum field theory, economics, biology, geophysics and etc. One of such application problems is presented

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in the current paper. Secondly, like in the case of the Fermat's Last Theorem, here is a simple formulation of the problem: it is needed to find solutions of the ordinary differential equation $y' = a(x) + b(x)y + c(x)y^2$. We hope the materials provided in this paper will help to make some contributions to these amazingly interesting and important equations study.

2. The linear differential equation of the second order with constant coefficients

The linear differential equation of the second order

$$y'' + ay' + by = f(x) \tag{2.1}$$

with constant coefficients a and b is representable like chain of differential equations of the first order $z' - pz = f(x)$ and $y' - qy = z$, where coefficients p and q are roots of the characteristic equation $k^2 + ak + b = 0$. To prove this statement it is needed to substitute the expression for z from the second equation to the first, equalize coefficients of this obtained equation and the basic one and find p and q values from outcome relations. It is important to note that this statement is true for linear equations of the higher order as well [1–3].

Example 1. Integrate the ordinary differential equation

$$y'' - 6y' + 9y = \frac{e^{3x}}{\sqrt[3]{x}}.$$

Using roots of a characteristic equation k^2–$6k + 9 = 0$ it is possible to expand the given equation in a view of the following chain of equations: $z' - 3z = \frac{e^{3x}}{\sqrt[3]{x}}$ and $y' - 3y = z$. Solving the first equation it can be found that $z = (1.5\sqrt[3]{x^2} + C)e^{3x}$, where $C \in \mathbb{R}$. Placing this result into the right side of the second equation, we can find that $y = (0.9\sqrt[3]{x^5} + Cx + C_1)e^{3x}$, where $C \in \mathbb{R}$ and $C_1 \in \mathbb{R}$.

3. The linear differential equation of the second order with variable coefficients

The chain expansion of the linear differential equation with constant coefficients shown above allows to suppose that the same approach is applicable to equations with variable coefficients, i.e. the linear differential equation of the second order with variable coefficients can be represented in the view of a chain of the liner differential equations of the first order with variable coefficients.

Let's suppose that the equation

$$y'' + a(x)y' + b(x)y = f(x), \tag{3.1}$$

where $a(x)$, $b(x)$ and $f(x)$ are some functions can be replaced by the chain of next equations:

$$z' - p(x)z = f(x),\ y' - q(x)y = z. \tag{3.2}$$

Substituting the expression for z from the second chain equation into the first, it can be found that the suggested chain leads to an equation

$$y'' + [-q(x) - p(x)]y' + [q(x)p(x) - q'(x)]y = f(x),$$

from which together with (2.1) it follows that

$$-p(x) - q(x) = a(x) \text{ and } q(x)p(x) - q'(x) = b(x).$$

Applying the expression for $p(x)$ function which follows from the first relation into the second one, the next equation of $q(x)$ can be obtained:

$$q'(x) + q(x)a(x) + q^2(x) = -b(x). \tag{3.3}$$

Solving the last equation we will have a chance to expand the basic equation (3.1) into the chain (3.2). However, (3.3) is Riccati equation which is unsolvable in general case in terms of elementary functions by quadratures [4]. Thus, from over here the next known result [4] follows once more: unfortunately, in general case, linear differential equations of the second order with variable coefficients cannot be integrated by quadratures. That's why we are suggesting to concentrate attention on some exact special cases.

4. The Euler differential equation

Chain approach usage allows to integrate not only equations with constant coefficients, but also some types of equations with variable coefficients as well. These equations include the Euler equations.

Theorem 4.1. *The Euler equation*

$$y'' + \frac{p}{x}y' + \frac{q}{x^2}y = f(x) \tag{4.1}$$

where p *and* q *are constant coefficients, can be represented in a view of the chain of linear differential equations of the first order*

$$z' - \frac{k}{x}z = f(x), \;\; y' - \frac{m}{x}y = z, \tag{4.2}$$

where coefficients k *and* m *are solutions of the algebraic equations system*

$$\begin{cases} k + m = -p; \\ (k+1)m = q. \end{cases} \tag{4.3}$$

Proof. It won't be complicated to proof this theorem's statement. Differentiating the second chain (4.2) equation it can be found that

$$z' = y'' + \frac{m}{x^2}y - \frac{m}{x}y'.$$

Using this result in the first equation of the chain (4.2) the next relation can be obtained:

$$y'' + \frac{-m-k}{x}y' + \frac{(k+1)m}{x^2}y = f(x).$$

Comparing coefficients of the last equation and the basic equation (4.1) it is easy to find that k and m coefficients are really solutions of the system (4.3).

$\square$

Example 2. To integrate an equation

$$y'' - \frac{5}{x}y' + \frac{8}{x^2}y = x^3e^x \tag{4.4}$$

let's consider an algebraic system

$$\begin{cases} k + m = 5; \\ (k+1)m = 8. \end{cases}$$

Using one pair of this system roots (such as $k = 3$ and $m = 2$) it is possible to represent the equation (4.4) in a chain form

$$z' - \frac{3}{x}z = x^3e^x, \ y' - \frac{2}{x}y = z.$$

From over here it follows that the first chain equation general solution takes the view of $z = x^3(e^x + C), \ C \in \mathbb{R}$, which means that solution of the equation (4.4) coincides with solution of an equation

$$y' - \frac{2}{x}y = x^3(e^x + C).$$

Thus, the general solution of the basic equation (4.4) is

$$y = x^2(xe^x - e^x + C_1x^2 + C_2), \ C_1 \in \mathbb{R}, \ C_2 \in \mathbb{R}.$$

5. The Euler-Riccati differential equation

In chapter 3 it was shown that integrability of linear differential equation of the second order by quadratures is defined by solvability of the related Riccati equation. We are going to call such equations by Euler-Riccati equations to consider these in the current section.

Theorem 5.1. *The Euler-Riccati equation*

$$y' = ay^2 + \frac{b}{x}y + \frac{c}{x^2} \tag{5.1}$$

where a, b and c are constant coefficients, is integrable by quadratures.

Proof. Following the traditional approach to find a general solution of the equation (5.1) it is possible to use its' particular solution $y = \frac{s}{x}$, where s is a root of equation $-s = as^2 + bs + c$. Substitution of $y = z + \frac{s}{x}$ allows to obtain the related Bernoulli Equation and, finally, deal with the linear differential equation of the first order. But we are going to discuss alternative approach to solve equation (5.1). Multiplying (5.1) by x^2, we have an equation $x^2y' = ax^2y^2 + bxy + c$, whose right side is represented by the square trinomial of unknown function xy. Adding xy to both sides of the last equation we will have a new one $x^2y' + xy = ax^2y^2 + (b+1)xy + c$, left side of it can be transformed by the next way: $x^2y' + xy = x(xy' + x'y) = x(xy)'$. Finally, usage of the substitution $u = xy$ leads to deal with a separable differential equation $xu' = au^2 + (b + 1)u + c$.

□

Example 3. Solve Euler-Riccati equation

$$y' = y^2 - \frac{9}{x}y + \frac{17}{x^2}.$$

Let's follow the algorithm demonstrated above. Multiplying the given equation by x^2 and applying the substitution $u = xy$, we have an equation $xu' = u^2 - 8u + 17$ which can be solved by variables separation:

$$\int \frac{du}{u^2 - 8u + 17} = \int \frac{dx}{x}.$$

As
$$\int \frac{du}{u^2-8u+17} = \int \frac{du}{(u-4)^2+1} = arctg(u-4)+A,\ A \in \mathbb{R}.$$
so the general solution of the basic equation is $arctg(xy-4) = C|lnx|,\ C \in \mathbb{R}.$.

It should be noted that the reasoning used in the process of the Euler-Riccati equations solving can be applied in some other cases.

6. The Riccati differential equation

Theorem 6.1. *The Riccati equation which has a view of*
$$y' = ax^q y^2 + \frac{by}{x} + \frac{c}{x^{q+2}} \tag{6.1}$$
where a, b, c and q are constant numbers, is integrable by quadratures.

Proof. Let's rewrite (6.1) to the view of $x^{q+2}y' = ax^{2q+2}y^2 + bx^{q+1}y + c$. Adding the addendum $(q+1)x^{q+1}y$ and applying substitution $z = x^{q+1}y$ we will have an separable variables equation $xz' = az^2 + (b+q+1)z + c$.

□

Example 4. Solve the differential equation
$$y' = x^{0.5}y^2 - 2.5\frac{y}{x} - \frac{6}{x^{2.5}}.$$

Multiplying the given equation by $x^{2.5}$, adding expression $1.5x^{1.5}y$ to both sides of equation, and using the substitution $z = x^{1.5}y$, we will have a new equation $xz' = z^2 - z - 6$. From the last one it follows that
$$\int \frac{dz}{(z+2)(z-3)} = \int \frac{dx}{x},$$
which means
$$\frac{5}{z+2} - 1 = x^5C \text{ or } x^{1.5}y + 2 = \frac{5}{x^5C+1}.$$

So, the general solution of the given equation is
$$y = \frac{1}{x^{1.5}}\left(\frac{5}{x^5C+1} - 2\right),\ C \in \mathbb{R}.$$

7. The inverse Euler-Riccati differential equation

Theorem 7.1. *The inverse Euler-Riccati differential equation*
$$y' = \frac{a_1y^2}{(qx+r)^2} + \frac{b_1y}{qx+r} + c \tag{7.1}$$
where a_1, b_1 and c are constant numbers, is integrable by quadratures.

Proof. It is easy to find that the equation (7.1) can be written as
$$y' = \frac{ay^2}{(x+p)^2} + \frac{by}{x+p} + c,$$
where $a = a_1/q^2,\ b = b_1/q$. Dividing the last equation by $(x+p)$ and subtracting $y/(x+p)^2$ from both sides of the last equation, we will have a new one:
$$\frac{y'}{x+p} - \frac{y}{(x+p)^2} = \frac{a}{x+p}\frac{y^2}{(x+p)^2} + \frac{(b-1)y}{(x+p)^2} + \frac{c}{x+p}.$$

Let's collapse the left side of the previous equation by the rule:

$$\frac{y'}{x+p} - \frac{y}{(x+p)^2} = \left(\frac{y}{x+p}\right)'$$

and apply the substitution $u = y/(x+p)$. Finally these allow to obtain a separable variables equation

$$u' = \frac{au^2 + (b-1)u + c}{x+p}.$$

□

Example 5. Let's consider the Riccati equation which describes a magnetotelluric impedance in a one-dimensional geological media:

$$\frac{dZ(z)}{dz} - \sigma(z)Z^2(z) = i\omega\mu_0, \tag{7.2}$$

where $Z(z)$ is an unknown impedance function depends on spatial coordinate z (z-axis directed into the depths of the Earth); i is an imaginary 1, i.e. $i^2 = -1$; ω is an electromagnetic field frequency; μ_0 is a vacuum permeability constant; $\sigma(z)$ is an electrical conductance of the media [5].

It is easy to solve the equation (7.2) in case if $\sigma(z) = \sigma \equiv const$, because in this case (7.2) becomes an equation with separable variables:

$$\frac{dZ}{\sigma Z^2 + i\omega\mu_0} = dz.$$

In case if $\sigma(z) = \sigma_0(1+pz)^{-2}$, where σ_0 and p are positive real numbers, the equation (7.2) takes the view $Z' = \sigma_0(1+pz)^{-2}Z^2 + i\omega\mu_0$. Dividing the last equation by $(1+pz)$ and subtracting fraction $\frac{pZ}{(1+pz)^2}$ from both sides of this equation we will have:

$$\frac{Z'}{1+pz} - \frac{pZ}{(1+pz)^2} = \frac{\sigma_0}{1+pz}\left(\frac{Z}{1+pz}\right)^2 + \frac{i\omega\mu_0}{1+pz} - \frac{pZ}{(1+pz)^2}$$

or, transforming to view:

$$\left(\frac{Z}{1+pz}\right)' = \frac{\sigma_0}{1+pz}\left(\frac{Z}{1+pz}\right)^2 + \frac{i\omega\mu_0}{1+pz} - \frac{p}{1+pz}\frac{Z}{(1+pz)^2}.$$

Let's rewrite the previous equation view denoting $u = \frac{Z}{1+pz}$:

$$u' = \frac{\sigma_0}{1+pz}u^2 - \frac{p}{1+pz}u + \frac{i\omega\mu_0}{1+pz}$$

or

$$(1+pz)u' = \sigma_0 u^2 - pu + i\omega\mu_0.$$

Obtained equation is an equation with separable variables

$$\frac{du}{\sigma_0 u^2 - pu + i\omega\mu_0} = \frac{dz}{1+pz},$$

general solution of which takes a form:

$$u = \frac{1}{2\sigma_0}\left(p - \nu - \frac{2\nu}{C(1+pz)^\nu - 1}\right),$$

where $\nu = \sqrt{p^2 - 4i\omega\mu_0\sigma_0}$ and C is a real number represents a constant of integration.

To turn back to the initial function Z and find the general solution of equation (7.2) let's use the relation $u = \frac{Z}{1+pz}$:

$$Z = \frac{1+pz}{2\sigma_0}\left(p - \nu - \frac{2\nu}{C(1+pz)^{\nu} - 1}\right), \ \nu = \sqrt{p^2 - 4i\omega\mu_0\sigma_0}, \ C \in \mathbb{R}.$$

8. Conclusion

The process of linear ordinary differential equations solving in analytical view is an essential element in teaching mathematics for future engineers, economists, chemist, etc. Very often, the corresponding mathematical courses are overloaded with long preliminary discussions about the linear independence of particular solutions, the basis and other special terms. The approach represented in this paper allows to join the world of differential equations solutions of which can be "touched" without requiring deep prior knowledge. Demonstrated in the work direct connection between the second-order differential equations and the Riccati equations can be served as a basis for students and junior scientists beginning their research activity.

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(ANARKUL URDALETOVA) Kyrgyz-Turkish Manas University, Department of Mathematics, Bishkek, Kyrgyz Republic
Email address: anarkul.urdaletova@manas.edu.kg

(OLGA ZABINYAKOVA) Research Station of RAS, Laboratory of Deep Magnetotelluric Studies, Bishkek, Kyrgyz Republic; American University of Central Asia, Applied Mathematics and Informatics Department, Bishkek, Kyrgyz Republic
Email address: perah.92@inbox.ru

(SYRGAK KYDYRALIEV) American University of Central Asia, Applied Mathematics and Informatics Department, Bishkek, Kyrgyz Republic
Email address: kydyraliev_s@auca.kg

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NEW DECOMPOSITIONS OF SOFT CONTINUITY IN SOFT TOPOLOGICAL SPACES

NAIME DEMIRTAŞ AND ZEHRA GÜZEL ERGÜL

0000-0003-4137-4810 and 0000-0001-5717-1626

Abstract. The aim of this paper is to produce new decompositions of soft continuity. For this reason, we first have introduced two new sets concepts named as soft AC-set with soft BC-set in soft topological spaces. Compairing with this two soft sets concepts and others soft set kinds, we have obtained two decompositions of soft open sets, supported by counterexamples. Utilizing soft AC-set and soft BC-set concepts, we have introduced two functions concepts named as soft AC-continuous with soft BC-continuous functions and we have obtained a new decomposition of soft continuity.

1. Introduction

The concept of soft sets was initiated by Molodtsov [11] as a new mathematical tool for dealing with uncertainty. In fact, a soft set is a parameterized family of subsets of a given universe set. The way of parameterization in problem solving makes soft set theory convenient and simple for application. Later Maji et al. [9] presented several operations in soft set theory. Shabir and Naz [12] introduced the soft topological spaces which are defined over an initial universe with a fixed set of parameters. They say that a soft topological space gives a parametrized family of topologies in the first universe, but the reverse is not true, i.e. if we are given some topology for each parameter, we cannot construct a soft topological space. As a result, we can say that soft topological spaces are more comprehensive and generalized than classical topological spaces. Many researchers [3, 6, 10, 17] studied some of basic concepts and properties of soft topological spaces. Recently, weak and strong forms of soft open sets were studied by many authors [1, 2, 4, 8, 13, 14, 15, 16]. The aim of this paper is to produce new decompositions of soft continuity. For this reason, we first give the concepts of soft AC-sets and soft BC-sets. We study the relationships between different types of soft sets in soft topological spaces. Later, we define soft AC-continuous and soft BC-continuous functions.

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2. Preliminaries

In this section, we present the basic definitions and results of soft set theory which may be found in earlier studies.

Definition 2.1. [11] Let X be an initial universe set and E be the set of all possible parameters with respect to X. Let $P(X)$ denote the power set of X. A pair (F, A) is called a soft set over X where $A \subseteq E$ and $F : A \to P(X)$ is a set valued mapping.

The set of all soft sets over X is denoted by $SS(X)_E$.

Definition 2.2. [9] A soft set (F, A) over X is said to be a null soft set denoted by Φ if for all $e \in A$, $F(e) = \emptyset$. A soft set (F, A) over X is said to be an absolute soft set denoted by $\widetilde{A}$ if for all $e \in A$, $F(e) = X$.

Definition 2.3. [12] Let Y be a nonempty subset of X, then $\widetilde{Y}$ denotes the soft set (Y, E) over X for which $Y(e) = Y$, for all $e \in E$. In particular, (X, E) will be denoted by $\widetilde{X}$.

Definition 2.4. [9] For two soft sets (F, A) and (G, B) over X, we say that (F, A) is a soft subset of (G, B) if $A \subseteq B$ and $F(e) \subseteq G(e)$ for all $e \in A$. We write $(F, A) \sqsubseteq (G, B)$. (F, A) is said to be a soft super set of (G, B), if (G, B) is a soft subset of (F, A). We denote it by $(G, B) \sqsubseteq (F, A)$. Then (F, A) and (G, B) are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.5. [9] The union of two soft sets (F, A) and (G, B) over X is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$, $H(e) = F(e)$ if $e \in A \backslash B$, $H(e) = G(e)$ if $e \in B \backslash A$, $H(e) = F(e) \cup G(e)$ if $e \in A \cap B$. We write $(F, A) \sqcup (G, B) = (H, C)$.

Definition 2.6. [5] The intersection (H, C) of two soft sets (F, A) and (G, B) over X, denoted $(F, A) \sqcap (G, B)$, is defined as $C = A \cap B$, and $H(e) = F(e) \cap G(e)$ for all $e \in C$.

Definition 2.7. [12] The difference (H, E) of two soft sets (F, E) and (G, E) over X, denoted by $(F, E) \backslash (G, E)$, is defined as $H(e) = F(e) \backslash G(e)$ for all $e \in E$.

Definition 2.8. [12] The relative complement of a soft set (F, E) is denoted by $(F, E)^c$ and is defined by $(F, E)^c = (F^c, E)$ where $F^c : E \longrightarrow P(X)$ is a mapping given by $F^c(e) = X \backslash F(e)$ for all $e \in E$.

Definition 2.9. [12] Let τ be the collection of soft sets over X, then τ is said to be a soft topology on X if

(1) Φ, $\widetilde{X} \in \tau$
(2) If (F, E), $(G, E) \in \tau$, then $(F, E) \sqcap (G, E) \in \tau$
(3) If $\{(F_i, E)\}_{i \in I} \in \tau$, $\forall i \in I$, then $\sqcup_{i \in I}(F_i, E) \in \tau$.

The triplet (X, τ, E) is called a soft topological space over X. Every member of τ is called a soft open set in X. A soft set (F, E) over X is called a soft closed set in X if its relative complement $(F, E)^c$ belongs to τ. We will denote the family of all soft open sets (resp., soft closed sets) of a soft topological space (X, τ, E) by SOS(X, τ, E) (resp., SCS(X, τ, E)).

Definition 2.10. Let (X, τ, E) be a soft topological space and (F, E) be a soft set over X.

(1) [12] The soft closure of (F,E) is the soft set $cl(F,E) = \sqcap\{(G,E) :\ (G,E)$ is soft closed and $(F,E) \sqsubseteq (G,E)\}$.

(2) [17] The soft interior of (F,E) is the soft set $int(F,E) = \sqcup\{(H,E) : (H,E)$ is soft open and $(H,E) \sqsubseteq (F,E)\}$.

Clearly, $cl(F,E)$ is the smallest soft closed set over X which contains (F,E) and $int(F,E)$ is the largest soft open set over X which is contained in (F,E).

Throughout the paper, the spaces X and Y (or (X,τ,E) and (Y,ν,K)) stand for soft topological spaces assumed unless stated otherwise.

Definition 2.11. *Let (X,τ,E) be a soft topological space.* A soft set (F,E) is called

(1) soft semi-open [4] in X if $(F,E) \sqsubseteq cl(int(F,E))$.
(2) soft α-open [1] in X if $(F,E) \sqsubseteq int(cl(int(F,E)))$.
(3) soft b-open [2] in X if $(F,E) \sqsubseteq int(cl(F,E)) \sqcup cl(int(F,E))$.

The relative complement of a soft semi-open (resp., soft α-open, soft b-open) set is called a soft semi-closed (resp., soft α-closed, soft b-closed) set.

Definition 2.12. *Let (X,τ,E) be a soft topological space. A soft set (F,E) is called*

(1) a soft regular open *(soft regular closed)* set [16] in X if $(F,E) = int(cl(F,E))$ $((F,E) = cl(int(F,E)))$.

(2) a soft A-set [14] in X if $(F,E) = (G,E)\backslash(H,E)$*,* where (G,E) is a soft open set and (H,E) is a soft regular open set in X.

(3) a soft t-set [14] in X if $int(cl(F,E)) = int(F,E)$.

(4) a soft B-set [14] in X if $(F,E) = (G,E) \sqcap (H,E)$, where (G,E) is a soft open set and (H,E) is a soft t-set in X.

(5) a soft α^*-set [13] in X if $int(cl(int(F,E))) = int(F,E)$.

(6) a soft C-set [13] in X if $(F,E) = (G,E) \sqcap (H,E)$, where (G,E) is a soft open set and (H,E) is a soft α^*-set in X.

(7) a soft semi-regular set [15] in X, if it is both soft semi-open and soft semi-closed.

(8) a soft AB-set [15] in X, if $(F,E) = (G,E) \sqcap (H,E)$, where (G,E) is a soft open set and (H,E) is a soft semi-regular set in X.

Definition 2.13. A soft set (F,E) is called a soft b-clopen set in a soft topological space X, if it is both soft b-open and soft b-closed.

Remark 2.14. In a soft topological space (X,τ,E);

(1) every soft open set is soft α-open [1],
(2) every soft regular open (closed) set is soft open (closed) [16],
(3) every soft open set is a soft A-set [14],
(4) every soft A-set is soft semi-open [14],
(5) every soft open set is a soft B-set [14],
(6) every soft A-set is a soft B-set [14].

Remark 2.15. [15] Since every soft regular closed set is soft semi-regular and since every soft semi-regular set is soft semi closed, then the following implications are obvious.

$$\text{soft A-set} \implies \text{soft AB-set} \implies \text{soft B-set}$$

Remark 2.16. [13, 14] Since every soft closed set is a soft t-set and since every soft t-set is a soft α^*-set, then the following implications are obvious.

$$\text{soft A-set} \implies \text{soft B-set} \implies \text{soft C-set}$$

Definition 2.17. [7] Let $SS(X)_E$ and $SS(Y)_K$ be families of soft sets, $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be mappings. Then the mapping $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ is defined as:

(1) Let $(F, E) \in SS(X)_E$. The image of (F, E) under f_{pu}, written as $f_{pu}(F, E) = (f_{pu}(F), p(E))$, is a soft set in $SS(Y)_K$ such that

$$f_{pu}(F)(y) = \begin{cases} \cup_{x \in p^{-1}(y) \cap A} u(F(x)) & , p^{-1}(y) \cap A \neq \emptyset \\ \emptyset & , otherwise \end{cases}$$

for all $y \in K$.

(2) Let $(G, K) \in SS(Y)_K$. The inverse image of (G, K) under f_{pu}, written as $f_{pu}^{-1}(G, K) = (f_{pu}^{-1}(G), p^{-1}(K))$, is a soft set in $SS(X)_E$ such that

$$f_{pu}^{-1}(G)(x) = \begin{cases} u^{-1}(G(p(x))) & , p(x) \in K \\ \emptyset & , otherwise \end{cases}$$

for all $x \in E$.

Definition 2.18. [17] Let (X, τ, E) and (Y, υ, K) be soft topological spaces and $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ be a function. Then f_{pu} is called a soft continuous function if for each $(G, K) \in \upsilon$ we have $f_{pu}^{-1}(G, K) \in \tau$.

Definition 2.19. Let (X, τ, E) and (Y, υ, K) be soft topological spaces and $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ be a function. Then f_{pu} is called

(1) a soft semi-continuous function [8] if for each $(G, K) \in SOS(Y)$, $f_{pu}^{-1}(G, K)$ is a soft semi-open set in X.

(2) a soft α-continuous function [1] if for each $(G, K) \in SOS(Y)$, $f_{pu}^{-1}(G, K)$ is a soft α-open set in X.

(3) a soft A-continuous function [14] if for each $(G, K) \in SOS(Y)$, $f_{pu}^{-1}(G, K)$ is a soft A-set in X.

(4) a soft B-continuous function [14] if for each $(G, K) \in SOS(Y)$, $f_{pu}^{-1}(G, K)$ is a soft B-set in X.

(5) a soft C-continuous function [13] if for each $(G, K) \in SOS(Y)$, $f_{pu}^{-1}(G, K)$ is a soft C-set in X.

(6) a soft AB-continuous function [15] if for each $(G, K) \in SOS(Y)$, $f_{pu}^{-1}(G, K)$ is a soft AB-set in X.

Remark 2.20. Let (X, τ, E) and (Y, υ, K) be soft topological spaces and $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ be a function. Then,

(1) every soft continuous function is soft α-continuous [1].
(2) every soft continuous function is soft A-continuous [14].
(3) every soft A-continuous function is soft semi-continuous [14].
(4) every soft continuous function is soft B-continuous [14].
(5) every soft A-continuous function is soft B-continuous [14].
(6) every soft B-continuous function is soft C-continuous [13].

Theorem 2.21. [15] *Let (X, τ, E) and (Y, υ, K) be soft topological spaces and $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ be a function. Then,*

(1) every soft A-continuous function is soft AB-continuous,
(2) every soft AB-continuous function is soft B-continuous.

3. Soft AC-sets and Soft BC-sets

Definition 3.1. A soft set (F, E) is called a soft BC-set in a soft topological space X if $(F, E) = (G, E) \sqcap (H, E)$ where (G, E) is soft open and (H, E) is soft b-closed.

Definition 3.2. A soft set (F, E) is called a soft AC-set in a soft topological space X if $(F, E) = (G, E) \sqcap (H, E)$ where (G, E) is soft open and (H, E) is soft b-clopen.

The family of all soft BC-sets (soft AC-sets) of a soft topological space (X, τ, E) is denoted by $SBCS(X)$ $(SACS(X))$.

Remark 3.3. The following implications hold and none of these implications is reversible as shown by examples given below.

$$\begin{array}{ccccccc}
\text{soft } A\text{-set} & \Longrightarrow & \text{soft } AB\text{-set} & \Longrightarrow & \text{soft } AC\text{-set} & & \\
 & & \Downarrow & & \Downarrow & & \\
 & & \text{soft } B\text{-set} & \Longrightarrow & \text{soft } BC\text{-set} & \Longrightarrow & \text{soft } C\text{-set}
\end{array}$$

Example 3.4. [15] Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ where $\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)$ are soft sets over X, defined as follows:

$$\begin{aligned}
(F_1, E) &= \{(e_1, \{x_2\}),\ (e_2, \{x_2\})\},\\
(F_2, E) &= \{(e_1, \{x_3\}),\ (e_2, \{x_3\})\},\\
(F_3, E) &= \{(e_1, \{x_2, x_3\}),\ (e_2, \{x_2, x_3\})\}.
\end{aligned}$$

Then τ defines a soft topology on X and thus (X, τ, E) is a soft topological space over X .Then, $(H, E) = \{(e_1, \{x_2\}),\ (e_2, \{x_1, x_2\})\}$ is a soft AB-set in X but not a soft A-set. Also $(G, E) = \{(e_1, \{x_1\}),\ (e_2, \{x_1\})\}$ is a soft B-set in X but not a soft AB-set.

Example 3.5. Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ where $\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)$ are soft sets over X, defined as follows:

$$\begin{aligned}
(F_1, E) &= \{(e_1, \{x_2\}),\ (e_2, \{x_2\})\},\\
(F_2, E) &= \{(e_1, \{x_1, x_2\}),\ (e_2, \{x_1, x_2\})\},\\
(F_3, E) &= \{(e_1, \{x_2, x_3\}),\ (e_2, \{x_2, x_3\})\}.
\end{aligned}$$

Then τ defines a soft topology on X and thus (X, τ, E) is a soft topological space over X. Clearly, $(G, E) = \{(e_1, \{x_2, x_3\}),\ (e_2, \{x_1, x_3\})\}$ is a soft AC-set in X but not a soft AB-set.

Example 3.6. Let $X = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$.Let us take the soft topology τ on X and the soft set $(G, E) = \{(e_1, \{x_1\}),\ (e_2, \{x_1\})\}$ in Example 3.4. Then, (G, E) is a soft BC-set in X but not a soft AC-set.

Example 3.7. Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ where $\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)$ are soft sets over X, defined as follows:

$$
\begin{aligned}
(F_1, E) &= \{(e_1, \{x_1\}),\ (e_2, \{x_1\})\},\\
(F_2, E) &= \{(e_1, \{x_2\}),\ (e_2, \{x_2\})\},\\
(F_3, E) &= \{(e_1, \{x_1, x_2\}),\ (e_2, \{x_1, x_2\})\}.
\end{aligned}
$$

Then τ defines a soft topology on X and thus (X, τ, E) is a soft topological space over X. Clearly, $(G, E) = \{(e_1, \{x_3\}),\ (e_2, \{x_1, x_3\})\}$ is a soft BC-set in X but not a soft B-set.

Example 3.8. Let $X = \{x_1, x_2, x_3, x_4\}$, $E = \{e_1, e_2\}$ and $\tau = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ where $\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)$ are soft sets over X, defined as follows:

$$
\begin{aligned}
(F_1, E) &= \{(e_1, \{x_4\}),\ (e_2, \{x_1, x_2\})\},\\
(F_2, E) &= \{(e_1, \{x_1, x_2\}),\ (e_2, \{x_4\})\},\\
(F_3, E) &= \{(e_1, \{x_1, x_2, x_4\}),\ (e_2, \{x_1, x_2, x_4\})\}.
\end{aligned}
$$

Then τ defines a soft topology on X and thus (X, τ, E) is a soft topological space over X. Clearly, $(G, E) = \{(e_1, \{x_2, x_3, x_4\}),\ (e_2, \{x_1, x_2, x_4\})\}$ is a soft C-set in X but not a soft BC-set.

Theorem 3.9. *In a soft topological space* (X, τ, E)*,*

(1) Every soft open set is a soft AC-set.
(2) Every soft b-clopen set is a soft AC-set.

Proof. (1) Let (F, E) be a soft open set in X. Then $(F, E) = (F, E) \sqcap \widetilde{X}$ such that $\widetilde{X}$ is soft b-clopen and hence it is a soft AC-set.

(2) It is shown in a similar way. □

Example 3.10. Let $X = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$. Let us take the soft topology τ on X and the soft set $(G, E) = \{(e_1, \{x_2, x_3\}),\ (e_2, \{x_1, x_3\})\}$ in Example 3.5. Then, (G, E) is a soft AC-set in X but not soft open.

Example 3.11. Let $X = \{x_1, x_2, x_3, x_4\}$, $E = \{e_1, e_2\}$ and $\tau = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (F_6, E), (F_7, E), (F_8, E), (F_9, E), (F_{10}, E), (F_{11}, E)\}$ where $\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (F_6, E), (F_7, E), (F_8, E), (F_9, E), (F_{10}, E), (F_{11}, E)$ are soft sets over X, defined as follows:

$$
\begin{aligned}
(F_1, E) &= \{(e_1, \{x_1\}),\ (e_2, \{x_1\})\},\\
(F_2, E) &= \{(e_1, \{x_2\}),\ (e_2, \{x_2\})\},\\
(F_3, E) &= \{(e_1, \{x_1, x_2\}),\ (e_2, \{x_1, x_2\})\},\\
(F_4, E) &= \{(e_1, \{x_1, x_2, x_3\}),\ (e_2, \{x_1, x_3\})\},\\
(F_5, E) &= \{(e_1, \{x_1, x_2, x_4\}),\ (e_2, \{x_1, x_2, x_3\})\},\\
(F_6, E) &= \{(e_1, \{x_2\}),\ (e_2, \emptyset)\},\\
(F_7, E) &= \{(e_1, \{x_1, x_2\}),\ (e_2, \{x_1\})\},\\
(F_8, E) &= \{(e_1, \{x_1, x_2, x_3\}),\ (e_2, \{x_1, x_2, x_3\})\},\\
(F_9, E) &= \{(e_1, X),\ (e_2, \{x_1, x_2, x_3\})\},\\
(F_{10}, E) &= \{(e_1, \{x_1, x_2\}),\ (e_2, \{x_1, x_2, x_3\})\},\\
(F_{11}, E) &= \{(e_1, \{x_1, x_2\}),\ (e_2, \{x_1, x_3\})\}.
\end{aligned}
$$

Then τ defines a soft topology on X and thus (X, τ, E) is a soft topological space over X [16]. Then, $(G, E) = \{(e_1, \{x_2, x_3\}),\ (e_2, \{x_3\})\}$ is a soft AC-set in X but not a soft b-clopen set.

Theorem 3.12. [2] *Let (X, τ, E) be a soft topological space. If (F, E) is soft open and (G, E) is soft b-open then $(F, E) \sqcap (G, E)$ is a soft b-open set in X.*

Theorem 3.13. *In a soft topological space (X, τ, E), every soft AC-set is a soft b-open set.*

Proof. Let (F, E) be a soft AC-set in X. Then $(F, E) = (G, E) \sqcap (K, E)$ where (G, E) is soft open and (K, E) is soft b-clopen. □

Theorem 3.14. *In a soft topological space (X, τ, E),*

(1) Every soft open set is a soft BC-set.
(2) Every soft b-closed set is a soft BC-set.

Proof. The proofs are similar with Theorem 3.9. □

Example 3.15. Let $X = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$. Let us take the soft topology τ on X and the soft set $(G, E) = \{(e_1, \{x_1\}),\ (e_2, \{x_1\})\}$ in Example 3.6. Then, (G, E) is a soft BC-set in X but not soft open.

Example 3.16. Let $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2\}$. Let us take the soft topology τ on X and the soft set $(G, E) = \{(e_1, \{x_2, x_3\}),\ (e_2, \{x_3\})\}$ in Example 3.11. Then, (G, E) is a soft BC-set in X but not a soft b-closed set.

Definition 3.17. [2] Let (X, τ, E) be a soft topological space and (F, E) be a soft set over X. Then soft b-closure of (F, E) ($sbcl(F, E)$) is the soft set $sbcl(F, E) = \sqcap\{(F, E) \sqsubseteq (G, E) : (G, E) \text{ is a soft } b\text{-closed set of } X\}$.

Theorem 3.18. *Let (X, τ, E) be a soft topological space. A soft set (F, E) over X is a soft BC-set if and only if $(F, E) = (G, E) \sqcap sbcl(F, E)$ for some soft open set (G, E).*

Proof. $\Longrightarrow$: Let (F, E) be a soft BC-set. We get $(F, E) = (G, E) \sqcap (H, E)$ where (G, E) is soft open and (H, E) is soft b-closed. Since $(F, E) \sqsubseteq (H, E)$, $sbcl(F, E) \sqsubseteq$

$sbcl(H,E) = (H,E)$. Thus $(G,E) \sqcap sbcl(F,E) \sqsubseteq (G,E) \sqcap (H,E) = (F,E) \sqsubseteq (G,E) \sqcap sbcl(F,E)$ and hence $(F,E) = (G,E) \sqcap sbcl(F,E)$.

$\Longleftarrow$: Suppose that $(F,E) = (G,E) \sqcap sbcl(F,E)$ for some soft open set (G,E). Since $sbcl(F,E)$ is soft b-closed, (F,E) is a soft BC-set in X. □

Theorem 3.19. *Let (X,τ,E) be a soft topological space. For a soft set (F,E) over X, the following are equivalent:*

(1) (F,E) is soft open in X,
(2) (F,E) is soft α-open and a soft AC-set in X,
(3) (F,E) is soft α-open and a soft BC-set in X,
(4) (F,E) is soft α-open and a soft C-set in X.

Proof. (1) $\Longrightarrow$ (2) : Since every soft open set is both soft α-open and a soft AC-set, the proof is completed.

(2) $\Longrightarrow$ (3) : The proof is obvious, since every soft AC-set is a soft BC-set.

(3) $\Longrightarrow$ (4) : Since every soft BC-set is a soft C-set, it is clear.

(4) $\Longrightarrow$ (1) : Follows from Proposition 3.8 of [13]. □

Theorem 3.20. *Let (X,τ,E) be a soft topological space and (F,E) be a soft set over X. If (F,E) is a soft BC-set, then*

(1) $sbcl(F,E) \setminus (F,E)$ is soft b-closed.
(2) $(F,E) \sqcup (sbcl(F,E))^c$ is soft b-open.

Proof. (1) Let (F,E) be a soft BC-set. From Theorem 3.18, $(F,E) = (G,E) \sqcap sbcl(F,E)$ for some soft open set (G,E). Thus $sbcl(F,E) \setminus (F,E) = sbcl(F,E) \setminus ((G,E) \sqcap sbcl(F,E)) = sbcl(F,E) \sqcap ((G,E) \sqcap sbcl(F,E))^c = sbcl(F,E) \sqcap ((G,E)^c \sqcup (sbcl(F,E))^c) = (sbcl(F,E) \sqcap (G,E)^c) \sqcup (sbcl(F,E) \sqcap (sbcl(F,E))^c)$
$= (sbcl(F,E) \sqcap (G,E)^c) \sqcup \Phi = sbcl(F,E) \sqcap (G,E)^c$. Hence we obtain $sbcl(F,E) \setminus (F,E)$ is soft b-closed.

(2) Since $sbcl(F,E) \setminus (F,E)$ is soft b-closed, $(sbcl(F,E) \setminus (F,E))^c$ is soft b-open. Thus $(sbcl(F,E) \setminus (F,E))^c = (sbcl(F,E) \sqcap (F,E)^c)^c = (sbcl(F,E))^c \sqcup (F,E)$. Hence we get $(F,E) \sqcup (sbcl(F,E))^c$ is soft b-open. □

4. Decompositions of Soft Continuity

Definition 4.1. Let (X,τ,E) and (Y,υ,K) be soft topological spaces. Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be mappings. Let $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ be a function. Then the function f_{pu}

(1) is called a soft AC-continuous if for each $(G,K) \in SOS(Y)$, $f_{pu}^{-1}(G,K)$ is a soft AC-set in X.

(2) is called a soft BC-continuous if for each $(G,K) \in SOS(Y)$, $f_{pu}^{-1}(G,K)$ is a soft BC-set in X.

Remark 4.2. We have the following diagram for f_{pu}:

$$\begin{array}{ccccccccc}
\text{soft } A\text{-continuous} & \Longrightarrow & \text{soft } AB\text{-continuous} & \Longrightarrow & \text{soft } AC\text{-continuous} & & \\
 & & \Downarrow & & \Downarrow & & \\
 & & \text{soft } B\text{-continuous} & \Longrightarrow & \text{soft } BC\text{-continuous} & \Longrightarrow & \text{soft } C\text{-continuous}
\end{array}$$

Theorem 4.3. *Let (X,τ,E) and (Y,υ,K) be soft topological spaces and $f_{pu} : SS(X)_E \longrightarrow SS(Y)_K$ be a function. Then the following are equivalent:*

(1) f_{pu} is soft continuous,
(2) f_{pu} is soft α-continuous and soft AC-continuous,
(3) f_{pu} is soft α-continuous and soft BC-continuous,
(4) f_{pu} is soft α-continuous and soft C-continuous.

Proof. The proof is an immediate consequence of Theorem 3.19. □

5. Conclusion

In this study, we have presented soft AC-sets and soft BC-sets in soft topological spaces which are defined over an initial universe with a fixed set of parameters. We have introduced their some basic properties with the help of counterexamples. Also, we have studied soft AC-continuos and soft BC-continuous functions and we have obtained the new decompositions of soft continuity. We hope that results in this paper will be heplfull for future studies in soft topological spaces.

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(Naime Demirtaş) Mersin University, Department of Mathematics, Mersin, Turkey.
Email address: `naimedemirtas@mersin.edu.tr`

(Zehra Güzel Ergül) Kirşehir Ahi Evran University, Department of Mathematics, Kirşehir, Turkey.
Email address: `zguzel@ahievran.edu.tr`

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ON D_4 INVARIANTS OF POLYNOMIAL ALGEBRAS

ŞEHMUS FINDIK AND NAZAR ŞAHİN ÖĞÜŞLÜ

0000-0001-5717-4413 and 0000-0001-7407-9178

Abstract. Let D_4 be the dihedral group of order 8. In the present study, we give generators of the algebra of D_4 invariants in the polynomial algebra with four generators over a field of characteristic zero.

1. Introduction

Let $K[X_n]$ be the polynomial algebra of rank n over a field K of characteristic zero, and let $GL_n(K)$ be the general linear group. Hilbert's fourteen problem (see [1, 2, 3, 4]) asks whether algebra $K[X_n]^G$ of constants of any subgroup G of $GL_n(K)$ is finitely generated. Although it was negated by Nagata [5] in general, Noether [6] showed that $K[X_n]^G$ is finitely generated for finite groups G. Our work was motivated by the approach above: What are the finite generators of $K[X_n]^G$ for some concrete groups G, in particular when G is a subgroup of the symmetric group S_n. The dihedral group

$$D_4 = \langle r, s \mid r^2 = s^4,\ rsr = s^3 \rangle = \{1, r, s, rs, s^2, rs^2, s^3, rs^3\}$$

of order 8 can be realized as a subgroup of the symmertic group S_4 by defining $r = (12)(34)$, $s = (1234)$. In this case we have that

$$D_4 = \{(1), (12)(34), (1234), (24), (13)(24), (14)(23), (1432), (13)\}.$$

Let $K[X_4] = K[x_1, x_2, x_3, x_4]$ be the commmutative unitary polynomial algebra of rank 4 over K. We define the invariant subalgebra

$$K[X_4]^{S_4} = \{f \in K[X_4] \mid f(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}) = f(x_1, x_2, x_3, x_4), \forall \pi \in S_4\}$$

induced by the group S_4. The algebra $K[X_4]^{S_4}$ is called the algebra of symmetric polynomials, and each polynomial in $K[X_4]^{S_4}$ is called a symmetric polynomial. It is well known that $K[X_4]^{S_4}$ is generated by elementary symmetric polynomials (see e.g. [7])

$$\alpha_1 = x_1 + x_2 + x_3 + x_4\ ,\quad \alpha_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4\ ,$$

$$\alpha_3 = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4\ ,\quad \alpha_4 = x_1x_2x_3x_4.$$

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Elementary symmetric polynomials are algebraically independent. There exists another set $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ of generators for $K[X_4]^{S_4}$, where

$$\beta_1 = x_1 + x_2 + x_3 + x_4\ , \quad \beta_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2\ ,$$
$$\beta_3 = x_1^3 + x_2^3 + x_3^3 + x_4^3\ , \quad \beta_4 = x_1^4 + x_2^4 + x_3^4 + x_4^4,$$

that is not algebraically independent. We extend this notation to

$$\beta_n = x_1^n + x_2^n + x_3^n + x_4^n$$

for all $1 \leq n$. Similar to $K[X_4]^{S_4}$, we have that

$$K[X_4]^{D_4} = \{f \in K[X_4] \mid f(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}) = f(x_1, x_2, x_3, x_4), \forall \pi \in D_4\}$$

Clearly, $K[X_4]^{S_4} \subsetneq K[X_4]^{D_4}$, since $p(x_1, x_2, x_3, x_4) = x_1x_3 + x_2x_4 \in K[X_4]^{D_4}$, while $p \notin K[X_4]^{S_4}$. In this paper, we aim to show that $K[X_4]^{D_4}$ is generated by polynomials $\alpha_1, \alpha_2, \alpha_3, \alpha_4, p$.

2. Preliminaries

We give some preliminary results, in this section. Let us define

$$p_{a,b} = x_1^a x_3^b + x_1^b x_3^a + x_2^a x_4^b + x_2^b x_4^a\ , \quad 0 \leq a, b.$$

One may easily check that $p_{a,b} \in K[X_4]^{D_4}$, and that $p = \frac{1}{2}p_{11}$. We give the next list of equations without proof which is straightforward.

$$p_{a,b} = p_{b,a}\ , \quad 0 \leq a, b \tag{2.1}$$

$$p_{a,b} = \frac{1}{2}p_{11}p_{a-1,b-1} - \alpha_4 p_{a-2,b-2}\ , \quad 2 \leq a, b \tag{2.2}$$

$$p_{2,2} = \frac{1}{2}p_{11}^2 - 4\alpha_4 \tag{2.3}$$

$$p_{1,b+3} = \frac{1}{2}\alpha_1 p_{1,b+2} - \frac{1}{2}p_{2,b+2} - \frac{1}{2}\alpha_3\beta_{b+1} + \frac{1}{2}\alpha_4\beta_b + \frac{1}{4}p_{1,1}\beta_{b+2}\ , \quad 0 \leq b \tag{2.4}$$

$$p_{1,3} = \frac{1}{2}\alpha_1 p_{1,2} - \frac{1}{2}p_{2,2} - \frac{1}{2}\alpha_3\beta_1 + 2\alpha_4 + \frac{1}{4}p_{1,1}\beta_2 \tag{2.5}$$

$$p_{1,2} = \frac{1}{2}\alpha_1 p_{1,1} - \alpha_3 \tag{2.6}$$

The next lemma is necessary in the proof of the main result.

Lemma 2.1. $K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{a,b} \mid 0 \leq a, b] = K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p]$.

Proof. Clearly $K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p] \subset K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{a,b} \mid 0 \leq a, b]$ because

$$p = \frac{1}{2}p_{1,1} \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{a,b} \mid 0 \leq a, b].$$

In order to complete the proof, it is sufficient to show that $p_{a,b}$ is included in the algebra generated by $\alpha_1, \alpha_2, \alpha_3, \alpha_4, p$ for every $0 \leq a, b$. Initially, using the equations (2.1), (2.2), (2.3) inductively, we obtain that every polynomial

$$p_{a,b} \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{1,1}]$$

for every $2 \leq a, b$. Now by (2.1), (2.4), (2.5), we have that

$$p_{1,b} \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{1,1}, p_{1,2}]$$

for all $3 \leq b$. Finally, we terminate the proof by (2.6) implying that

$$p_{1,2} \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{1,1}].$$

□

3. Main Results

The next theorem is the main result of the paper.

Theorem 3.1. *The algebra $K[X_4]^{D_4}$ is generated by $\alpha_1, \alpha_2, \alpha_3, \alpha_4, p$.*

Proof. It is sufficient to show that $K[X_4]^{D_4} \subset K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{ab} \mid 0 \leq a, b]$ by Lemma 2.1. Let

$$g = \sum_{0 \leq a,b,c,d} \varepsilon_{abcd} x_1^a x_2^b x_3^c x_4^d$$

be an arbitrary element of $K[X_4]^{D_4}$ of the form $g = g_1 + g_2 + g_3 + g_4$, where

$$g_1 = \sum_{0 \leq a,b,c,d} \varepsilon_{abcd} x_1^a x_2^b x_3^c x_4^d \ , \quad \mid \{a, b, c, d\} \mid = 1,$$

$$g_2 = \sum_{0 \leq a,b,c,d} \varepsilon_{abcd} x_1^a x_2^b x_3^c x_4^d \ , \quad \mid \{a, b, c, d\} \mid = 2,$$

$$g_3 = \sum_{0 \leq a,b,c,d} \varepsilon_{abcd} x_1^a x_2^b x_3^c x_4^d \ , \quad \mid \{a, b, c, d\} \mid = 3,$$

$$g_4 = \sum_{0 \leq a,b,c,d} \varepsilon_{abcd} x_1^a x_2^b x_3^c x_4^d \ , \quad \mid \{a, b, c, d\} \mid = 4.$$

It is clear that each g_i, $i = 1, 2, 3, 4$, is D_4 invariant; i.e.,

$$g_i(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}) = g_i(x_1, x_2, x_3, x_4), \forall \pi \in D_4.$$

Initially,

$$g_1 = \sum_{0 \leq a} \varepsilon_{aaaa} (x_1 x_2 x_3 x_4)^a = \sum_{0 \leq a} \varepsilon_{aaaa} \alpha_4^a \ \in K[\alpha_4].$$

On the other hand g_2 is of the form $g_2 = g_{21} + g_{22} + g_{23} + g_{24}$, where

$$g_{21} = \sum_{a<b} \varepsilon_{bbaa} x_1^b x_2^b x_3^a x_4^a + \varepsilon_{aabb} x_1^a x_2^a x_3^b x_4^b + \varepsilon_{abba} x_1^a x_2^b x_3^b x_4^a + \varepsilon_{baab} x_1^b x_2^a x_3^a x_4^b$$

$$g_{22} = \sum_{a<b} \varepsilon_{bbab} x_1^b x_2^b x_3^a x_4^b + \varepsilon_{bbba} x_1^b x_2^b x_3^b x_4^a + \varepsilon_{abbb} x_1^a x_2^b x_3^b x_4^b + \varepsilon_{babb} x_1^b x_2^a x_3^b x_4^b$$

$$g_{23} = \sum_{a<b} \varepsilon_{aaab} x_1^a x_2^a x_3^a x_4^b + \varepsilon_{aaba} x_1^a x_2^a x_3^b x_4^a + \varepsilon_{abaa} x_1^a x_2^b x_3^a x_4^a + \varepsilon_{baaa} x_1^b x_2^a x_3^a x_4^a$$

$$g_{24} = \sum_{a<b} \varepsilon_{abab} x_1^a x_2^b x_3^a x_4^b + \varepsilon_{baba} x_1^b x_2^a x_3^b x_4^a$$

One may easily show that no summand in a fixed g_{2i} turns into a summand in g_{2j}, $i \neq j$, under the action of D_4. Thus by

$$g_2(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}) = g_2(x_1, x_2, x_3, x_4), \forall \pi \in D_4,$$

we get that

$$g_{2i}(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}) = g_{2i}(x_1, x_2, x_3, x_4), \forall \pi \in D_4, \quad i = 1, 2, 3, 4.$$

Easy calculations computing the actions of permutations from D_4 gives that all coefficients in each g_{2i} for a fixed (a,b) are equal; i.e.,

$$g_{21}=\sum_{a<b}\varepsilon_{bbaa}(x_1^bx_2^bx_3^ax_4^a+x_1^ax_2^ax_3^bx_4^b+x_1^ax_2^bx_3^bx_4^a+x_1^bx_2^ax_3^ax_4^b)$$

$$g_{22}=\sum_{a<b}\varepsilon_{bbab}(x_1^bx_2^bx_3^ax_4^b+x_1^bx_2^bx_3^bx_4^a+x_1^ax_2^bx_3^bx_4^b+x_1^bx_2^ax_3^bx_4^b)$$

$$g_{23}=\sum_{a<b}\varepsilon_{aaab}(x_1^ax_2^ax_3^ax_4^b+x_1^ax_2^ax_3^bx_4^a+x_1^ax_2^bx_3^ax_4^a+x_1^bx_2^ax_3^ax_4^a)$$

$$g_{24}=\sum_{a<b}\varepsilon_{abab}(x_1^ax_2^bx_3^ax_4^b+x_1^bx_2^ax_3^bx_4^a)$$

which are in the form

$$g_{21}=\frac{1}{2}\sum_{a<b}\varepsilon_{bbaa}(p_{a,b}^2-p_{2a,2b}-p_{a+b,a+b})$$

$$g_{22}=\sum_{a<b}\varepsilon_{bbab}\alpha_4^a\left(\frac{1}{2}p_{b-a,b-a}\beta_{b-a}-p_{2(b-a),b-a}\right)$$

$$g_{23}=\sum_{a<b}\varepsilon_{aaab}\alpha_4^a\beta_{b-a}$$

$$g_{24}=\frac{1}{2}\sum_{a<b}\varepsilon_{abab}\alpha_4^ap_{b-a,b-a}$$

Therefore $g_2\in K[\alpha_1,\alpha_2,\alpha_3,\alpha_4,p_{ab}\mid 0\le a,b]$.

Next, g_3 is of the form $g_3=g_{31}+g_{32}+g_{33}+g_{34}+g_{35}+g_{36}$, where

$$g_{31}=\sum_{a<b<c}\begin{pmatrix}\varepsilon_{accb}x_1^ax_2^cx_3^cx_4^b+\varepsilon_{ccba}x_1^cx_2^cx_3^bx_4^a+\varepsilon_{cbac}x_1^cx_2^bx_3^ax_4^c+\varepsilon_{bacc}x_1^bx_2^ax_3^cx_4^c\\ \varepsilon_{cabc}x_1^cx_2^ax_3^bx_4^c+\varepsilon_{ccab}x_1^cx_2^cx_3^ax_4^b+\varepsilon_{bcca}x_1^bx_2^cx_3^cx_4^a+\varepsilon_{abcc}x_1^ax_2^bx_3^cx_4^c\end{pmatrix}$$

$$g_{32}=\sum_{a<b<c}\begin{pmatrix}\varepsilon_{acbb}x_1^ax_2^cx_3^bx_4^b+\varepsilon_{cbba}x_1^cx_2^bx_3^bx_4^a+\varepsilon_{bbac}x_1^bx_2^bx_3^ax_4^c+\varepsilon_{bacb}x_1^bx_2^ax_3^cx_4^b\\ \varepsilon_{cabb}x_1^cx_2^ax_3^bx_4^b+\varepsilon_{bcab}x_1^bx_2^cx_3^ax_4^b+\varepsilon_{bbca}x_1^bx_2^bx_3^cx_4^a+\varepsilon_{abbc}x_1^ax_2^bx_3^bx_4^c\end{pmatrix}$$

$$g_{33}=\sum_{a<b<c}\begin{pmatrix}\varepsilon_{aabc}x_1^ax_2^ax_3^bx_4^c+\varepsilon_{abca}x_1^ax_2^bx_3^cx_4^a+\varepsilon_{bcaa}x_1^bx_2^cx_3^ax_4^a+\varepsilon_{caab}x_1^cx_2^ax_3^ax_4^b\\ \varepsilon_{aacb}x_1^ax_2^ax_3^cx_4^b+\varepsilon_{baac}x_1^bx_2^ax_3^ax_4^c+\varepsilon_{cbaa}x_1^cx_2^bx_3^ax_4^a+\varepsilon_{acba}x_1^ax_2^cx_3^bx_4^a\end{pmatrix}$$

$$g_{34}=\sum_{a<b<c}\varepsilon_{acbc}x_1^ax_2^cx_3^bx_4^c+\varepsilon_{cbca}x_1^cx_2^bx_3^cx_4^a+\varepsilon_{bcac}x_1^bx_2^cx_3^ax_4^c+\varepsilon_{cacb}x_1^cx_2^ax_3^cx_4^b$$

$$g_{35}=\sum_{a<b<c}\varepsilon_{abcb}x_1^ax_2^bx_3^cx_4^b+\varepsilon_{bcba}x_1^bx_2^cx_3^bx_4^a+\varepsilon_{cbab}x_1^cx_2^bx_3^ax_4^b+\varepsilon_{babc}x_1^bx_2^ax_3^bx_4^c$$

$$g_{36}=\sum_{a<b<c}\varepsilon_{abac}x_1^ax_2^bx_3^ax_4^c+\varepsilon_{baca}x_1^bx_2^ax_3^cx_4^a+\varepsilon_{acab}x_1^ax_2^cx_3^ax_4^b+\varepsilon_{caba}x_1^cx_2^ax_3^bx_4^a$$

Similarly, g_{3i}, $i=1,2,3,4,5,6$, is D_4-invariant, and hence we have that

$$g_{31}=\sum_{a<b<c}\varepsilon_{accb}(p_{a,c}p_{b,c}-p_{a+b,2c}-p_{a+c,b+c})$$

$$g_{32}=\sum_{a<b<c}\varepsilon_{acbb}(p_{a,b}p_{b,c}-p_{a+c,2b}-p_{a+b,b+c})$$

$$g_{33}=\sum_{a<b<c}\varepsilon_{aabc}(p_{a,b}p_{a,c}-p_{2a,b+c}-p_{a+c,a+b})$$

$$g_{34} = \sum_{a<b<c} \varepsilon_{acbc}\alpha_4^a \left(\frac{1}{2}p_{c-a,c-a}\beta_{b-a} - p_{b+c-2a,c-a}\right)$$

$$g_{35} = \sum_{a<b<c} \varepsilon_{abcb}\alpha_4^a \left(\frac{1}{2}p_{b-a,b-a}\beta_{c-a} - p_{b+c-2a,b-a}\right)$$

$$g_{36} = \frac{1}{2}\sum_{a<b<c} \varepsilon_{abac}(p_{a,a}p_{b,c} - 2p_{a+b,a+c})$$

and that $g_3 \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{ab} \mid 0 \leq a, b]$.

Finally, g_3 is of the form $g_4 = g_{41} + g_{42} + g_{43}$, where

$$g_{41} = \sum_{a<b<c<d} \begin{pmatrix} \varepsilon_{abcd}x_1^a x_2^b x_3^c x_4^d + \varepsilon_{bcda}x_1^b x_2^c x_3^d x_4^a + \varepsilon_{cdab}x_1^c x_2^d x_3^a x_4^b + \varepsilon_{dabc}x_1^d x_2^a x_3^b x_4^c \\ \varepsilon_{badc}x_1^b x_2^a x_3^d x_4^c + \varepsilon_{cbad}x_1^c x_2^b x_3^a x_4^d + \varepsilon_{dcba}x_1^d x_2^c x_3^b x_4^a + \varepsilon_{adcb}x_1^a x_2^d x_3^c x_4^b \end{pmatrix}$$

$$g_{42} = \sum_{a<b<c<d} \begin{pmatrix} \varepsilon_{acbd}x_1^a x_2^c x_3^b x_4^d + \varepsilon_{cbda}x_1^c x_2^b x_3^d x_4^a + \varepsilon_{bdac}x_1^b x_2^d x_3^a x_4^c + \varepsilon_{dacb}x_1^d x_2^a x_3^c x_4^b \\ \varepsilon_{cadb}x_1^c x_2^a x_3^d x_4^b + \varepsilon_{bcad}x_1^b x_2^c x_3^a x_4^d + \varepsilon_{dbca}x_1^d x_2^b x_3^c x_4^a + \varepsilon_{adbc}x_1^a x_2^d x_3^b x_4^c \end{pmatrix}$$

$$g_{43} = \sum_{a<b<c<d} \begin{pmatrix} \varepsilon_{abdc}x_1^a x_2^b x_3^d x_4^c + \varepsilon_{bdca}x_1^b x_2^d x_3^c x_4^a + \varepsilon_{dcab}x_1^d x_2^c x_3^a x_4^b + \varepsilon_{cabd}x_1^c x_2^a x_3^b x_4^d \\ \varepsilon_{bacd}x_1^b x_2^a x_3^c x_4^d + \varepsilon_{dbac}x_1^d x_2^b x_3^a x_4^c + \varepsilon_{cdba}x_1^c x_2^d x_3^b x_4^a + \varepsilon_{acdb}x_1^a x_2^c x_3^d x_4^b \end{pmatrix}$$

Similar computations give that

$$g_{41} = \sum_{a<b<c<d} \varepsilon_{abcd}\alpha_4^a(\beta_{c-a}p_{d-a,b-a} - p_{d-a,b+c-2a} - p_{b-a,d+c-2a})$$

$$g_{42} = \sum_{a<b<c<d} \varepsilon_{acbd}\alpha_4^a(\beta_{b-a}p_{d-a,c-a} - p_{c-a,b+d-2a} - p_{d-a,b+c-2a})$$

$$g_{43} = \sum_{a<b<c<d} \varepsilon_{abdc}\alpha_4^a(\beta_{d-a}p_{c-a,b-a} - p_{c-a,b+d-2a} - p_{b-a,d+c-2a})$$

and that $g_4 \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{ab} \mid 0 \leq a, b]$. Consequently,

$$g = g_1 + g_2 + g_3 + g_4 \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{ab} \mid 0 \leq a, b].$$

□

4. Conclusion

In this study, generators for the algebra of D_4-invariants were provided. This might be an initial approach for determining generators for algebras of G-invariants, where G is a subgroup of S_n, $n \geq 4$.

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Department of Mathematics, Çukurova University, 01330 Balcali, Adana, Turkey
Email address: sfindik@cu.edu.tr, noguslu@cu.edu.tr

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A STUDY ON MODELING OF CONFLICT AND AGREEMENT WITH GAME THEORY

HAŞİM ÇAPIN AND ŞÜKRAN KONCA

0000-0001-8168-5360 and 0000-0003-4019-958X

ABSTRACT. Game theory is a mathematical approach to analyze the state of competition between players. The foundations of this theory go back about 170 years, and the main development of the subject is based on the last 55 years. In this study, the effect of game theory on political elections and political behaviors has been examined. The Nash equilibrium is investigated by creating a mathematical model of the gains and losses that two political parties obtain in the elections according to the coalition formation status of two political parties by using the Prisoners' Dilemma game model in cooperative and non cooperative games.

1. INTRODUCTION

In political science, there are several subjects in which game theory has a significant role. These include voting, political power affairs, diplomacy, negotiation and bargaining behavior, coalition formation among political groups, and political support [1].

The fact that those who participate in political life have different goals causes a rivalry and conflict between units and groups. As a result of conflicts and agreements in domestic and foreign politics, short-term or permanent political coalitions may occur [2]. Political coalitions are a common phenomenon in contemporary democratic systems [3].

Game theory is based on decision making for the most appropriate strategy selection, which can provide the best result in the face of problems. Considering the high importance of the decisions taken in politics, game theory can then be used appropriately on political issues. Through game theory, it is possible to learn how and why decision makers make mistakes in reasoning in political activities, which strategies provide the best results, why coalitions fail or persist, and what kinds of corrections can be made [2].

In this study, the gains of political parties or groups for two cases will be tried to be explained using game theory: if they enter the elections alone and if they enter the elections by forming a coalition. When the payoff matrices are examined, it will be revealed that it is necessary to decide on the formation of a political government with the best and most effective strategy.

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2. BASIC DEFINITIONS AND PRELIMINARIES

2.1. Game Theory

People have always been involved in situations where the most appropriate decisions need to be made. The decision taken may or may not affect other decision makers. The best decision may depend on one or more goals of the decision maker. The decision may relate to a static situation or a situation that evolves over time. Therefore, mathematical and algorithmic tools have been developed to model, analyze and solve such decision-making processes. Mathematical programming, multi-objective optimization, optimal control theory, and static and dynamic game theory provide the language and tools to achieve such goals.

With the development of optimization, control and game theory, it has been possible to fully achieve the analysis of many complex situations. The concepts of equilibrium and optimality are of great practical importance in policy and strategy decision-making problems, within the framework of understanding and predicting what will ultimately happen in systems across different application areas, from economics and engineering to military applications [4].

In game theory, a distinction can be made between non-cooperative games and cooperative ones. There are 2-player games and n-player games in each of these two categories.

Non-cooperative games are games in which its players are not allowed to cooperate with each other. Each player must individually decide what action to take and what strategy to follow. The best-known example of a two-player non-cooperative play is the prisoners' dilemma. In political science and other social sciences, prisoners' dilemma games have been applied to all kinds of problems.

The non-cooperative approach to political problems has some limitations. Most importantly, in most decision-making situations, actors actually cooperate to achieve certain outcomes. Decision making is often not just a conflict situation, so actors will try to find support for their views. This idea fits in with the concept of coalition, which is one of the core concepts of the cooperative game theory. Coalition is “an agreement of two or more actors who decide to cooperate in order to maximize their common return” [5].

Collaborative game theory deals with situations where at least two decision makers can increase their profits or reduce their costs by collaborating. For example, consider a situation where one person has the resources to make a particular product, someone else has the technical knowledge to make it, and a third party has the means to market it where he can sell it. While neither alone can make a profit, they can earn by working together [6].

2.2. Prisoners' Dilemma

One of the best-known examples of game theory is the prisoners' dilemma. It was revealed by Merrill Flood and Melvin Dresher between 1948-1950 [7].

The name of the Prisoners' Dilemma comes from the following scenario: Two thieves, Alice and Bob, are caught by the police and interrogated in separate cells with no communication between them. Unfortunately, the police lacks enough admissible evidence to convince the jury during a trial. Knowing this, the accused expect to be released with a sentence of less than one year. The prosecutors plan is to meet with the two prisoners and negotiate a bargain. One of the scenarios that may occur as a result of the bargaining is that both of them remain silent; the other one is that one of the detainees becomes the witness of the other friend's crime.

	Confess	Silent
Confess	(-5,-5)	(0,-10)
Silent	(-10,0)	(-1,-1)

TABLE 1. Prisoners' dilemma

Table 1 shows the returns of the prisoners according to their strategies. The most appropriate strategy for the accused is seen as keeping silent. However, if one of the prisoners confesses to the crime, the silent party will be sentenced to 10 years. Likewise, if both parties confess to the crime, each prisoner will be sentenced to 5 years each. These scenarios can happen because neither prisoner knows the other's strategy.

If the confession of prisoner 1 is held constant, the best strategy that prisoner 2 can employ would be to confess to the crime. Because if he confesses, he will be sentenced to 5 years, and if he does not, he will be sentenced to 10 years. With the silence of prisoner 2 held constant, prisoner 1's best choice would still be to confess. Because the 1st prisoner would prefer being released to a 1 year sentence. In this situation, confessing, regardless of what prisoner 2 does, is the best strategy for prisoner 1. The same is true for the prisoner 2. The result (confession, confession) that rational players reach while thinking about each other's behavior in the same rooms really gives the Nash balance of the game, because neither prisoner 1 nor prisoner 2 want to change their own confession strategy in the face of the other's confession strategy. However, both prefer to be sentenced to one year rather than five years each. Despite these preferences, they cannot obtain the result of cooperation (Red, Red) because they are rational and rationality is general knowledge. The word dilemma in the name of the game is derived from this [8].

If there was cooperation between the players, both players would choose the option to refuse, and they would choose the option to maximize the optimum benefit. However, it is not possible to cooperate in games played one time. Because mutual trust of the players cannot be fully established and if one player does not fulfill his obligation, the other player will not be sanctioned.

2.3. Nash Equilibrium Theorem

Each player wants to choose the strategy that will bring the highest payoff for himself. However, considering the strategies of other players, the same strategy may not always bring the best results [7]. Here, the Nash Equilibrium means that the most suitable combination of the player.

The main element in the Nash equilibrium is the existence of the equilibrium point [9]. When deciding the strategies that determine the Nash equilibrium, the value of any of the players that gives the maximum payoff against the strategy of the other players is determined. When we do the same for all players, the result will be the balance of the game [7].

The first studies on the Nash equilibrium were made by Cournot in 1838 and Bertrand in 1883 [9]. On the other hand, if a non-Nash equilibrium situation occurs, at least one player has made a mistake [10].

2.4. Repetitive Games

In static games, as in the prisoners' dilemma, players can influence each other one time. However, although the game is played in a certain period of the economic life, the effects of the decisions taken may last for the same period. The purpose of replay games is to compare short- and long-term wins. A short-term gain can be dispensed with for a strategy that yields more gains in the long run [11].

2.5. The Effect of Game Theory in Establishment of Political Power

Game theory is an effective model in both domestic and foreign policy of the countries. Formation of governments and distribution of ministries are balanced according to the voting power of the parties. Here game theory becomes an effective tool.

Again, in foreign policy, keeping the interests of a country against another or several countries at the highest level is the application of game theory.

The application of game theory on the academic studies of international relations started in 1960 with Thomas Schelling's "The Strategy of Conflict". The adaptation of the cowardly chicken game and the prisoner's dilemma game (which is still used today) to international relations started with Schelling. One of the greatest contributions of the game theory to the politics and international relations is to get rid of the idea based on states, and to use other decision-making actors for the analysis. In classical international relations, states are all considered as different sized structures with similar goals. In game theory, the characters, bureaucracies and decision-making processes of nations have gained importance [12].

Game theory in politics and international relations has arisen from the need to explain the observed issues and the questions asked. Revealing the problems in detail does not automatically create answers to the questions asked. For example, describing all the details of a negotiation does not indicate that the issue is intelligible. On the contrary, it shows how difficult a subject may be. However, this allows us to make explanations that go to the essence of the negotiation, when we treat the negotiation as a game, and the main variables and relations are handled, while other details are omitted.

Any game model models the outlines of a complex issue and tries to find its solution. In this solution, it is obvious that the game has equilibrium or equilibria [13].

Coalitions, which are a form of political unity and governance, refer to the communication and cooperation of more than one political party towards the achievement of common goals. Due to the heterogeneous structure of contemporary societies, in today's democracy, parties cannot reach the necessary majority in legislative assemblies for one party government, and coalitions are needed to form the government. In this framework, the place and importance of coalition in contemporary democracies becomes better understood [3].

Political coalition building games describe the formation and disintegration of nations, as well as the formation of coalition governments, the formation of political parties and other similar events [14].

Riker is among the first political scientists to use the collaborative gaming perspective to examine political coalitions with his 1962 work of "The Theory of Political Coalitions". Many other political theorists followed suit. The coalition has found an interesting application in cabinet formations. Significant work in applying n-person cooperative game theory to coalition cabinets includes Axelrod's "Conflict of Interest", published in 1970; “A Theory of Divergent Goals with Application To Politics”, De Swaan's 1973 “Coalition Theories and Cabinet Formations”, Taylor and Laver's 1973 “Government Coalitions in Western Europe” and Dodd "Coalitions in Parliamentary Government" published in 1976 [5].

The formation of a government also corresponds to the game of forming a political coalition, in which the political parties are players. Similarly, the parties themselves can be seen as the outcome of an underlying game of political coalition-building, this time with individual legislators as the players.

Holland	--------	--------	--------	--------	--------	--------	--------	--------	--------	--------
Luxembourg	--------	--------	--------	--------	--------	--------	--------	--------	--------	--------
Germany	XX---	--------	--------	--------	--------	--------	--------	--------	--------	--------
Belgium	XXXX	--------	--------	--------	--------	--------	--------	--------	--------	--------
Finland	XXXX	X------	--------	--------	--------	--------	--------	--------	--------	--------
France	XXXX	XXX--	--------	--------	--------	--------	--------	--------	--------	--------
Italy	XXXX	XXXX	--------	--------	--------	--------	--------	--------	--------	--------
Austria	XXXX	XXXX	XXXX	XX----	--------	--------	--------	--------	--------	--------
Denmark	XXXX	XXXX	XXXX	XXXX	X------	--------	--------	--------	--------	--------
Ireland	XXXX	XXXX	XXXX	XXXX	XXXX	XX----	--------	--------	--------	--------
Portugal	XXXX	XXXX	XXXX	XXXX	XXXX	XXXX	XXXX	X------	--------	--------
Sweden	XXXX	XXXX	XXXX	XXXX	XXXX	XXXX	XXXX	XX----	--------	--------
Norway	XXXX	XXXX	XXXX	XXXX	XXXX	XXXX	XXXX	XXX--	--------	--------
Greece	XXXX	XXXX	XXXX	XXXX	XXXX	XXXX	XXXX	XXXX	XXXX	XXX--

TABLE 2. Coalition and one-party governments in Europe 1945- 1999
(X): One-Party Governments (--): Coalition Governments [3]

The term of office of the governments established in Table 2 is taken as basis. When we look at the table, it is seen that coalitions are a management style that covers almost half of the democratic government. With the establishment of coalition governments, the effect of game theory is seen on ministries and the sharing of various public authorities. There are two basic approaches put forward to explain the tendency of political parties: to join or not to join coalitions. The first of these is the "office-seeking" approach, led by Anthony Down.

This approach, which is an adaptation of the Game Theory to the political science, assumes that the main purpose of politicians is to be in power and to hold administrative units under all circumstances. Therefore, all political parties seek ways to maximize their political interests and minimize their losses. Therefore, the way to win in this system is to be in power.
Otherwise, it results that all parties entering the coalition are faced with the second best choice, and therefore they will want to share their already diminished profits with as few partners as possible. At the same time, it is accepted that these coalitions can be established between parties that are ideologically closest to each other [3].

3. MAIN RESULTS

In this section, we consider two political parties, A and B. We aim to explain the conflict between these parties and define the situations for which we will model. Next, we will model these situations based on game theory. We will first build our model on the basis of the prisoner's dilemma in the first case and solve it. We will then expand the game for the second case, taking advantage of the prisoner's dilemma we created for the first case. In other words, we're going to make the game iterative. Finally, we will obtain the solution of the iterative game we created.

Let's assume that the political moves, interventions and similar situations of other parties or groups are ineffective in the situation that occurs between two political parties throughout the study.

Case I: The electoral base of political parties A and B is very close to each other. It is obvious that the votes will be divided when political parties A and B go to the elections alone, thinking only of their own interests. The call for a coalition or alliance of any of the parties receives positive reactions from the grassroots. The rest of the votes is distributed to other political parties in certain proportions.

Case II: The current government has been re-elected, and political parties A and B re-joined the election. In this case, let's examine what kind of attitude political parties A and B should display. Following, we model the first case with the help of game theory.

T: The situation of going to the election alone
K: State of coalition building

The row player will be the political party A, and the column player will be the political party B.
Case I:

$$G^1 = \begin{array}{c} \\ T \\ K \end{array} \begin{bmatrix} T & K \\ (-2,-2) & (-3,1) \\ (1,-3) & (2,2) \end{bmatrix} \tag{1}$$

The values in the payoff matrix express the percentage change in the votes of the political parties. For example, if both parties go to the elections alone and state that they are not open to the idea of coalition, this will cause a reaction from the grassroots and both parties will lose 2% of the votes.

We will find the solution of the first case with the Nash equilibrium point. To find the balance points, we will determine the largest element of the 1st component of each column element, and the largest element of the 2nd component of each row element. The marked sequential pair will be the balance point of the game.

$$G_1 = \begin{array}{c} \\ T \\ K \end{array} \begin{bmatrix} T & K \\ (-2,-2) & (-3,\mathbf{1}) \\ (\mathbf{1},-3) & (\mathbf{2},\mathbf{2}) \end{bmatrix} \tag{2}$$

It can be seen from the above marking that the game has a Nash equilibrium. This strategy is to form a coalition of two political parties.
Case II: Considering that there are some disagreements in the coalition in the future and that elections are held again, as it can be seen in the game tree of Figure 1. As the number of repetitions increases in the game, more strategies are obtained for the players.

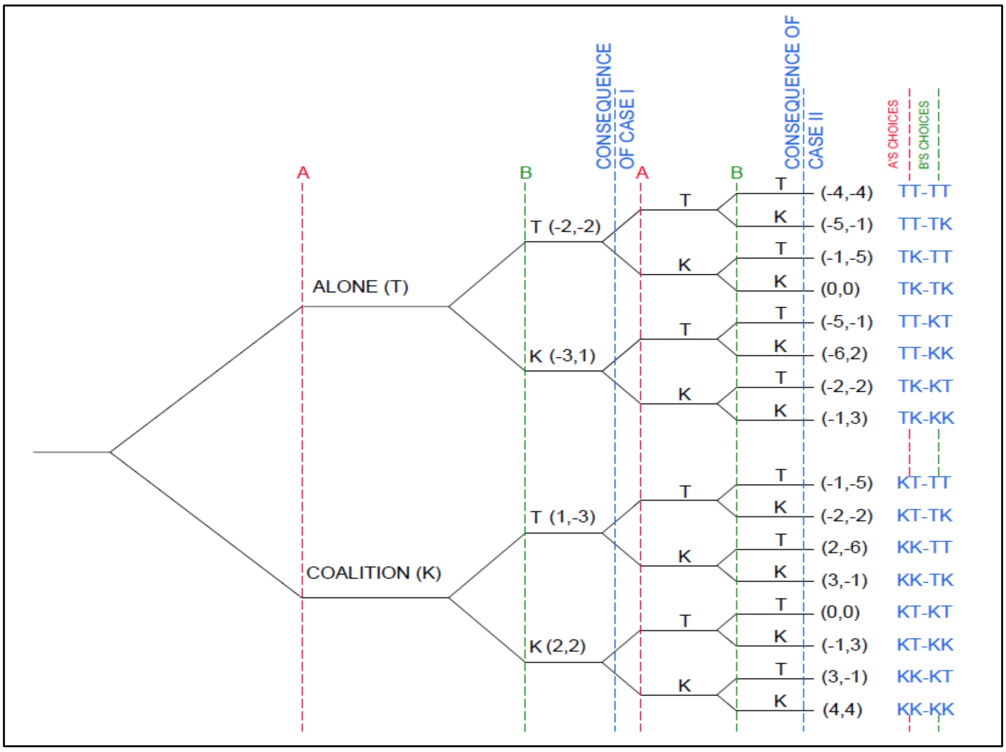

FIGURE 1. Demonstrating Case II with game tree

	TT	TK	KT	KK
TT	(-4,-4)	(-5,-1)	(-5,-1)	(-6,2)
TK	(-1,-5)	(0,0)	(-2,-2)	(-1,3)
KT	(-1,-5)	(-2,-2)	(0,0)	(-1,3)
KK	(2,-6)	(3,-1)	(3,-1)	(4,4)

TABLE 3. Strategies in the game tree

In Table 3, the expressions TT, TK, KT, KK in the row are the strategies chosen by party A in the 1st and 2nd elections. Similarly, the expressions TT, TK, KT, KK in the columns are the strategies of party B. For example, the expression TTKT (-5,1) in the 1st row, 3rd column element of the matrix means that party A wants to enter alone in both elections, while party B is open to coalition in the 1st election and wishes to enter alone in the 2nd election. The first two moves of the TTKT expression are for party A, and the other two moves are for party B.

$$G_2 = \begin{bmatrix} (-4,-4) & (-5,-1) & (-5,-1) & (-6,2) \\ (-1,-5) & (0,0) & (-2,-2) & (-1,3) \\ (-1,-5) & (-2,-2) & (0,0) & (-1,3) \\ (2,-6) & (3,-1) & (3,-1) & (4,4) \end{bmatrix} \tag{3}$$

Again, looking at the payoff matrix, if both parties decide to enter the elections on their own, and state that they are not open to the idea of coalition, they will see a reaction from the grassroots again. Both parties will lose 4% of the vote. When we look at the 2nd row and 1st column element, when parties A and B went to the election alone in the first election and declared that party A was open to the idea of coalition in the second election, there was a 1% gain in votes compared to the first election.

Then, there is the Nash equilibrium point. To find the balance points, let's determine the largest element of the 1st component of each column element and the largest element of the 2nd component of each row element. The marked ordered pairs will be the balance points of the game.

$$G_2 = \begin{bmatrix} (-4,-4) & (-5,-1) & (-5,-1) & (-6,\mathbf{2}) \\ (-1,-5) & (0,0) & (-2,-2) & (-1,\mathbf{3}) \\ (-1,-5) & (-2,-2) & (0,0) & (-1,\mathbf{3}) \\ (\mathbf{2},-6) & (\mathbf{3},-1) & (\mathbf{3},-1) & (\mathbf{4},\mathbf{4}) \end{bmatrix} \tag{4}$$

As a result, we analyzed the participation status of the two political parties in two stages with the Nash equilibrium. Increasing the stages will create more strategies. In both stages, forming a coalition of two parties is seen as the most optimal solution.

4. CONCLUSION

The problem of forming a coalition between two political parties has been handled with the help of the game theory. The moves that the political parties should take to keep their own interests at the highest level are examined with the Nash equilibrium. The gains and losses that they can achieve in case of cooperation and going to the elections alone have been modeled. In this cooperative game, the gains of the political parties according to the decisions they will take in the first election are shown in the yield matrix. It is seen that the balance point, that is, the most appropriate move for the two political parties, is to form a coalition. Afterwards, it has been researched which path would offer the most gains for the political parties if the government held elections again. In this repetitive cooperative game, it is seen that the political parties forming a coalition in both elections is the most lucrative strategy. The equilibrium concept of game theory can contribute to our understanding of political decision-making by predicting predictions that determine optimal strategies. Within the framework of these results, it is clear that cooperative games can be used in future international and political studies.

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(HAŞİM ÇAPIN), İZMİR BAKIRÇAY UNIVERSITY, GRADUATE EDUCATION INSTITUTE, SMART SYSTEMS ENGINEERING MASTER PROGRAM, DEPARTMENT OF APPLIED MATHEMATICS AND COMPUTATIONAL SCIENCES, İZMİR, TÜRKİYE
E-mail address: 6016008@bakircay.edu.tr

(ŞÜKRAN KONCA), İZMİR BAKIRÇAY UNIVERSITY, DEPARTMENT OF FUNDAMENTAL SCIENCES, FACULTY OF ENGINEERING AND ARCHITECTURE, İZMİR, TÜRKİYE
E-mail address: sukran.konca@bakircay.edu.tr

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(α, β)-INTERVAL VALUED INTUITIONISTIC FUZZY SETS DEFINED ON (α, β)-INTERVAL VALUED SET

ARİF BAL, GÖKHAN ÇUVALCIOĞLU AND CANSU ALTINCI

0000-0003-4386-7416, 0000-0001-5451-3336 and 0000-0003-0675-9555

ABSTRACT. In this paper, (α, β)-interval valued set is studied. The order relation on (α, β)-interval valued set is defined. It is shown that (α, β)-interval valued set is complete lattice by giving the definitions of infumum and supremum on these sets. Then, negation function on these sets is introduced. With the help of (α, β)-interval valued set ,(α, β)-interval valued intuitionistic fuzzy sets are defined. The fundamental algebraic properties of these sets are examined. The level subsets of (α, β)-interval valued intuitionistic fuzzy sets are given. Some propositions and examples are studied.

1. INTRODUCTION

Fuzzy set theory was introduced by Zadeh in 1965 [15]. The concept of interval valued fuzzy set was introduced by Zadeh [16-18]. The basic properties of interval valued fuzzy sets were studied by many authors [7-10,13,14,16-18]. It is crucial to analyze the properties of interval fuzzy sets on different structures in these sense, the topological properties of interval valued fuzzy sets were studied by Mondal and Samantha [11].

Interval valued intuitionistic fuzzy sets which is the generalization of intuitionistic fuzzy sets and interval valued fuzzy sets were introduced by Atanassov and Gargov in 1989 [2]. Membership and non-membership functions on interval valued intuitionistic fuzzy sets are closed intervals whose the sum of supremums is equal to 1 or less than 1 of unit interval $\mathrm{I} = [0,1]$ [2]. Other properties of these sets were studied and the concept of intuitionistic fuzzy sets was introduced by Atanassov [1-5]. The topological properties of interval valued intuitionistic fuzzy sets were studied by Mondal and Samantha [12]. α-interval valued fuzzy sets were introduced by Çuvalcıoğlu, Bal and Çitil in 2022 [6].

2. PRELIMINARIES

In this paper, D(I) represents all closed intervals of unit interval $\mathrm{I} = [0,1]$. The elements of D(I) set are shown with capital letters such as M,N... In this place, M^{L} and M^{U} are called respectively lower end point and upper end point for interval $\mathrm{M} = [\mathrm{M}^{\mathrm{L}}, \mathrm{M}^{\mathrm{U}}]$.

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Definition 1: [6] $D(I_\alpha) = \{[M^L, M^U; \alpha] | \alpha \in I\}$ is called α-interval valued set. In order to make easy, it is shown that

$$\{[M^L, M^U; \alpha] | \alpha \in I\} = \{[M; \alpha] | M \in D(I) \text{ and } \alpha \in M\}$$

Order relation on $D(I_\alpha)$ is defined below.

Definition 2: [6] $\forall [M; \alpha], [N; \alpha] \in D(I_\alpha)$,

$$[M; \alpha] \leq [N; \alpha] :\Leftrightarrow M^L \leq N^L \text{ and } M^U \geq N^U$$

It is easily seen from definition,

$$[M; \alpha] < [N; \alpha]$$
$$\Leftrightarrow M^L < N^L, M^U \geq N^U \text{ or } M^L \leq N^L, M^U > N^U \text{ or } M^L < N^L, M^U > N^U$$

Proposition 1: [6] $(D(I_\alpha), \leq)$ is partial ordered set.

By the help of order relation on $D(I_\alpha)$, the definitions of supremum and infimum on this set are given below.

Definition 3: [6] $\forall [M; \alpha], [N; \alpha] \in D(I_\alpha)$,

i. $\inf\{[M; \alpha], [N; \alpha]\} = [\inf\{M^L, N^L\}, \sup\{M^U, N^U\}; \alpha]$

ii. $\sup\{[M; \alpha], [N; \alpha]\} = [\sup\{M^L, N^L\}, \inf\{M^U, N^U\}; \alpha]$

Lemma 1: [6] $(D(I_\alpha), \wedge, \vee)$ is complete lattice with units $[0,1; \alpha]$ and $[\alpha, \alpha; \alpha]$.

Proposition 2: [6] $\forall \alpha \in I$,

$$\bigcup_{\alpha \in I} D(I_\alpha) = D(I)$$

The following function is a negation function on $D(I_\alpha)$.

Proposition 3: [6] $\forall [M; \alpha] \in D(I_\alpha)$ and $\mathcal{N}: D(I_\alpha) \to D(I_\alpha)$,

$$\mathcal{N}([M; \alpha]) = [\alpha - M^L, 1 + \alpha - M^U; \alpha]$$

Definition 4: [6] Let X be universal set and $[A; \alpha]: X \to D(I_\alpha)$ be function.

$$[A; \alpha] = \{[\langle x, [A^L(x), A^U(x)] \rangle; \alpha] | x \in X\}$$

where; $A^L: X \to [0,1]$ and $A^U: X \to [0,1]$ are fuzzy sets.

In order to make easy, it is shown that;

$$\{[\langle x, [A^L(x), A^U(x)] \rangle; \alpha] | x \in X\} = \{[\langle x, A(x) \rangle; \alpha] | x \in X\}$$

$[A; \alpha]$ is called $\alpha-$interval valued fuzzy set on X. The family of $\alpha-$interval valued fuzzy sets on X is shown by $\alpha-$IVFS(X).

Complement, inclusion, equation, intersection and union of $\alpha-$interval valued fuzzy sets are given below.

Definition 5: [6] Let X be universal set and $[A; \alpha], [B; \alpha] \in \alpha-$IVFS(X).

Λ is index set $\forall \lambda \in \Lambda$,

i. $[A^c; \alpha] = \{[< x, [\alpha - A^L(x), 1 + \alpha - A^U(x)] >; \alpha] \,|\, x \in X\}$

ii. $[A;\alpha] \sqsubseteq [B;\alpha] :\Leftrightarrow \forall x \in X, A^L(x) \leq B^L(x) \text{ and } A^U(x) \geq B^U(x)$

iii. $[A;\alpha] = [B;\alpha] :\Leftrightarrow \forall x \in X, A^L(x) = B^L(x) \text{ and } A^U(x) = B^U(x)$

iv. $[A \sqcap B;\alpha] = \{[< x, [\inf\{A^L(x), B^L(x)\}, \sup\{A^U(x), B^U(x)\}] >;\alpha] \,|\, x \in X\}$

v. $[A \sqcup B;\alpha] = \{[< x, [\sup\{A^L(x), B^L(x)\}, \inf\{A^U(x), B^U(x)\}] >;\alpha] \,|\, x \in X\}$

vi. $[\sqcap_{\lambda\in\Lambda} A_\lambda;\alpha] = \{[< x, [\wedge_{\lambda\in\Lambda} {A_\lambda}^L(x), \vee_{\lambda\in\Lambda} {A_\lambda}^U(x)] >;\alpha] \,|\, x \in X\}$

vii. $[\sqcup_{\lambda\in\Lambda} A_\lambda;\alpha] = \{[< x, [\vee_{\lambda\in\Lambda} {A_\lambda}^L(x), \wedge_{\lambda\in\Lambda} {A_\lambda}^U(x)] >;\alpha] \,|\, x \in X\}$

The algebraic properties of α −interval valued fuzzy sets are expressed below.

Proposition 4: [6] Let X be universal set. $\forall [A;\alpha], [B;\alpha], [C;\alpha] \in \alpha-\text{IVFS}(X)$ and Λ is index set $\forall \lambda \in \Lambda$,

i. $[A \sqcap B;\alpha] = [B \sqcap A;\alpha]$

ii. $[A \sqcup B;\alpha] = [B \sqcup A;\alpha]$

iii. $[A;\alpha] \sqcap ([B \sqcup C;\alpha]) = ([A \sqcap B;\alpha]) \sqcup ([A \sqcap C;\alpha])$

iv. $[A;\alpha] \sqcup ([B \sqcap C;\alpha]) = ([A \sqcup B;\alpha]) \sqcap ([A \sqcup C;\alpha])$

v. $[A;\alpha] \sqcap ([\sqcup_{\lambda\in\Lambda} B_\lambda;\alpha]) = [\sqcup_{\lambda\in\Lambda} (A \sqcap B_\lambda);\alpha]$

vi. $[A;\alpha] \sqcup ([\sqcap_{\lambda\in\Lambda} B_\lambda;\alpha]) = [\sqcap_{\lambda\in\Lambda} (A \sqcup B_\lambda);\alpha]$

Features about complement of α −interval valued fuzzy sets are stated following proposition.

Proposition 5: [6] Let X be universal set. $\forall [A;\alpha], [B;\alpha] \in \alpha-\text{IVFS}(X)$ and Λ is index set $\forall \lambda \in \Lambda$,

i. $[([A^c;\alpha])^c;\alpha] = [A;\alpha]$

ii. $([A \sqcap B;\alpha])^c = [A^c \sqcup B^c;\alpha]$

iii. $([A \sqcup B;\alpha])^c = [A^c \sqcap B^c;\alpha]$

iv. $([\sqcap_{\lambda\in\Lambda} A_\lambda;\alpha])^c = [\sqcup_{\lambda\in\Lambda} A_\lambda^c;\alpha]$

v. $([\sqcup_{\lambda\in\Lambda} A_\lambda;\alpha])^c = [\sqcap_{\lambda\in\Lambda} A_\lambda^c;\alpha]$

Proposition 6: [6] Let X be universal set. $\mathbf{0_X}: X \to [0,1;\alpha]$ and $\mathbf{1_X}: X \to [\alpha,\alpha;\alpha]$.

i. $(\mathbf{0_X})^c = \mathbf{1_X}$

ii. $(\mathbf{1_X})^c = \mathbf{0_X}$

Definition 6: [6] Let X be universal set and $[A;\alpha] \in \alpha-\text{IVFS}(X)$.

$$[A;\alpha] \text{ has sup} - \text{property}$$
$$:\Leftrightarrow \forall x \in X, \exists [\lambda_1,\lambda_2;\alpha] \in D(I_\alpha) \ni [A(x);\alpha] = [\lambda_1,\lambda_2;\alpha]$$

Definition 7: [6] Let X be universal set and $[A;\alpha] \in \alpha-\text{IVFS}(X)$.
$\forall [\lambda_1,\lambda_2;\alpha] \in D(I_\alpha)$,

$$[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} = \{x \in X | A^L(x) \geq \lambda_1 \text{and } A^U(x) \leq \lambda_2\}$$

The set $[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$ is called $[\lambda_1,\lambda_2;\alpha]$-level subset of $[A;\alpha]$. It is easily seen from definition, $[\lambda_1,\lambda_2;\alpha]$-level subsets of $[A;\alpha]$ are crisp sets.

Definition 8: [6] Let X be universal set and $[A;\alpha] \in \alpha-\mathrm{IVFS}(X)$.
$\forall[\lambda_1,\lambda_2;\alpha] \in D(I_\alpha)$,
$\forall[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$-level subsets of $[A;\alpha]$,

i. $A^L_{\lambda_1} = \{x \in X \mid A^L(x) \geq \lambda_1\}$

ii. $A^U_{\lambda_2} = \{x \in X \mid A^U(x) \leq \lambda_2\}$

iii. $B^L_{\lambda_1} = \{x \in X \mid B^L(x) \geq \lambda_1\}$

iv. $B^U_{\lambda_2} = \{x \in X \mid B^U(x) \leq \lambda_2\}$

The relations between level subsets of $\alpha-$interval valued fuzzy sets and crisp sets are given below.

Proposition 7: [6] Let X be universal set and $[A;\alpha],[B;\alpha] \in \alpha-\mathrm{IVFS}(X)$.
$\forall[\lambda_1,\lambda_2;\alpha] \in D(I_\alpha)$ and I is index set, $\forall i,j \in I$, $[\lambda_i,\lambda_j;\alpha] \in D(I_\alpha)$,

i. $x \in [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \Leftrightarrow [A(x);\alpha] \geq [\lambda_1,\lambda_2;\alpha]$

ii. $[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} = A^L_{\lambda_1} \cap A^U_{\lambda_2}$

iii. $([A \sqcup B;\alpha])_{[\lambda_1,\lambda_2;\alpha]}$

$$= [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup [B;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup \left(A^L_{\lambda_1} \cap B^U_{\lambda_2}\right) \cup \left(B^L_{\lambda_1} \cap A^U_{\lambda_2}\right)$$

iv. $([A \sqcap B;\alpha])_{[\lambda_1,\lambda_2;\alpha]} = [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cap [B;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$

v. $A^L_{\lambda_1} \supseteq A^L_{\lambda_2}$

vi. $A^U_{\lambda_1} \subseteq A^U_{\lambda_2}$

vii. $\bigcap_{i\in I} A^L{}_{\lambda_i} = A^L{}_{\wedge_{i\in I}\lambda_i}$

viii. $\bigcup_{j\in I} A^U{}_{\lambda_j} = A^U{}_{\vee_{j\in I}\lambda_j}$

Definition 9: [2] Let X be universal set.
M_A and $N_A: X \to D(I)$ such that $\forall x \in X, {M_A}^U(x) + {N_A}^U(x) \leq 1$,

$$A = \{< x, M_A(x), N_A(x) > | x \in X\}$$

is called interval valued intuitionistic fuzzy set. The family of interval valued intuitionistic fuzzy sets on X is shown by IVIFS(X).

Example 1: Let $X = \{a,b,c,d\}$.

$$A = \begin{Bmatrix} < a,[0.0,0.5],[0.2,0.4] >, < b,[0.1,0.3],[0.4,0.5] >, \\ < c,[0.2,0.7],[0.0,0.1] >, < d,[0.6,0.8],[0.1,0.2] > \end{Bmatrix}$$

is interval valued intuitionistic fuzzy set.

Definition 10: [2] Let X be universal set and A, B ∈ IVIFS(X).

i. $A \sqsubseteq_{\square,\inf} B :\Leftrightarrow \forall x \in X, {M_A}^L(x) \leq {M_B}^L(x)$

ii. $A \sqsubseteq_{\square,\sup} B :\Leftrightarrow \forall x \in X, {M_A}^U(x) \leq {M_B}^U(x)$

iii. $A \sqsubseteq_{\lozenge,\inf} B :\Leftrightarrow \forall x \in X, {N_A}^L(x) \geq {N_B}^L(x)$

iv. $A \sqsubseteq_{\lozenge,\sup} B :\Leftrightarrow \forall x \in X, {N_A}^U(x) \geq {N_B}^U(x)$

v. $A \sqsubseteq_{\square} B :\Leftrightarrow A \sqsubseteq_{\square,\inf} B$ and $A \sqsubseteq_{\square,\sup} B$

vi. $A \sqsubseteq_{\Diamond} B :\Leftrightarrow A \sqsubseteq_{\Diamond,\inf} B$ and $A \sqsubseteq_{\Diamond,\sup} B$

vii. $A \sqsubseteq B :\Leftrightarrow A \sqsubseteq_{\Box} B$ and $A \sqsubseteq_{\Diamond} B$

viii. $A = B :\Leftrightarrow A \sqsubseteq B$ and $B \sqsubseteq A$

It is easily seen that from definition,

i. $A \sqsubseteq_{\Box} B \Leftrightarrow \forall x \in X, {M_A}^L(x) \leq {M_B}^L(x)$ and ${M_A}^U(x) \leq {M_B}^U(x)$

ii. $A \sqsubseteq_{\Diamond} B \Leftrightarrow \forall x \in X, {N_A}^L(x) \geq {N_B}^L(x)$ and ${N_A}^U(x) \geq {N_B}^U(x)$

iii. $A \sqsubseteq B \Leftrightarrow \forall x \in X, {M_A}^L(x) \leq {M_B}^L(x), {M_A}^U(x) \leq {M_B}^U(x)$

and ${N_A}^L(x) \geq {N_B}^L(x), {N_A}^U(x) \geq {N_B}^U(x)$

iv. $A = B \Leftrightarrow \forall x \in X, {M_A}^L(x) = {M_B}^L(x)$ and ${M_A}^U(x) = {M_B}^U(x)$

and ${N_A}^L(x) = {N_B}^L(x)$ and ${N_A}^U(x) = {N_B}^U(x)$

Definition 11: [5] Let X be universal set and A, B ∈ IVIFS(X).

i. $A^c = \{< x, N_A(x), M_A(x) > | x \in X\}$

ii. $$A \sqcap B = \begin{Bmatrix} < x, \left[\min\{{M_A}^L(x), {M_B}^L(x)\}, \min\{{M_A}^U(x), {M_B}^U(x)\}\right], \\ \left[\max\{{N_A}^L(x), {N_B}^L(x)\}, \max\{{N_A}^U(x), {N_B}^U(x)\}\right] > | x \in X \end{Bmatrix}$$

iii. $$A \sqcup B = \begin{Bmatrix} < x, \left[\max\{{M_A}^L(x), {M_B}^L(x)\}, \max\{{M_A}^U(x), {M_B}^U(x)\}\right], \\ \left[\min\{{N_A}^L(x), {N_B}^L(x)\}, \min\{{N_A}^U(x), {N_B}^U(x)\}\right] > | x \in X \end{Bmatrix}$$

Theorem 1: [5] Let X be universal set and A, B, C ∈ IVIFS(X).

i. $A \sqcap B = B \sqcap A$

ii. $A \sqcup B = B \sqcup A$

iii. $(A \sqcap B) \sqcap C = A \sqcap (B \sqcap C)$

iv. $(A \sqcup B) \sqcup C = A \sqcup (B \sqcup C)$

v. $(A \sqcap B) \sqcup C = (A \sqcup C) \sqcap (B \sqcup C)$

vi. $(A \sqcup B) \sqcap C = (A \sqcap C) \sqcup (B \sqcap C)$

Theorem 2: [5] Let X be universal set and A, B ∈ IVIFS(X).

i. $(A^c)^c = A$

ii. $(A^c \sqcap B^c)^c = A \sqcup B$

iii. $(A^c \sqcup B^c)^c = A \sqcap B$

Definition 12: [5] There are some special sets on vague set theories. These special sets on the theory of crisp set are null set and universal set. The special sets on interval valued intuitionistic fuzzy sets are given below.

i. $O^* = \{< x, [0,0], [1,1] > | x \in X\}$

ii. $U^* = \{< x, [0,0], [0,0] > | x \in X\}$

iii. $X^* = \{< x, [1,1], [0,0] > | x \in X\}$

It is easily seen that;

$$O^* \sqsubseteq U^* \sqsubseteq X^*$$

$\forall A \in IVIFS(X)$,

i. $A \sqcap O^* = O^*$

ii. $A \sqcup O^* = A$

3. (α, β)-INTERVAL VALUED INTUITIONISTIC FUZZY SETS

$D(I_\alpha)$ is all closed sub-intervals of $I = [0,1]$ including $\alpha \in [0,1]$.

Definition 13:

$$D(I_\alpha) \times D(I_\beta) = \{([M^L, M^U; \alpha], [N^L, N^U; \beta]) \mid M^U + N^U \leq 1 \text{ and } M \in D(I_\alpha), N \in D(I_\beta)\}$$

is called (α, β)-interval valued set.

To make clear, it is shown below,

$$\{([M^L, M^U; \alpha], [N^L, N^U; \beta]) \mid M^U + N^U \leq 1 \text{ and } M \in D(I_\alpha), N \in D(I_\beta)\}$$
$$= \{([M; \alpha], [N; \beta]) \mid M^U + N^U \leq 1 \text{ and } M \in D(I_\alpha), N \in D(I_\beta)\}$$

The order relation on $D(I_\alpha) \times D(I_\beta)$ is defined below.

Definition 14: $\forall([M; \alpha], [N; \beta]), ([P; \alpha], [R; \beta]) \in D(I_\alpha) \times D(I_\beta)$,

$$([M; \alpha], [N; \beta]) \leq ([P; \alpha], [R; \beta]) :\Leftrightarrow [M; \alpha] \leq [P; \alpha] \text{ and } [N; \beta] \geq [R; \beta]$$

Here;

$$([M; \alpha], [N; \beta]) < ([P; \alpha], [R; \beta]) :\Leftrightarrow [M; \alpha] < [P; \alpha], [N; \beta] \geq [R; \beta] \text{ or}$$
$$[M; \alpha] \leq [P; \alpha], [N; \beta] > [R; \beta] \text{ or } [M; \alpha] < [P; \alpha], [N; \beta] > [R; \beta]$$

Proposition 8: $(D(I_\alpha) \times D(I_\beta), \leq)$ is partial ordered set.

Proof: $([M; \alpha], [N; \beta]), ([P; \alpha], [R; \beta]), ([S; \alpha], [T; \beta]) \in D(I_\alpha) \times D(I_\beta)$ are given arbitrary.

1. $M^L \leq M^L, M^U \geq M^U$ and $N^L \geq N^L, N^U \leq N^U$

$$\Rightarrow [M; \alpha] \leq [M; \alpha] \text{ and } [N; \beta] \geq [N; \beta] \Rightarrow ([M; \alpha], [N; \beta]) \leq ([M; \alpha], [N; \beta])$$

2. $([M; \alpha], [N; \beta]) \leq ([P; \alpha], [R; \beta])$ and $([M; \alpha], [N; \beta]) \geq ([P; \alpha], [R; \beta])$

$$\Rightarrow [M; \alpha] \leq [P; \alpha], [N; \beta] \geq [R; \beta] \text{ and } [M; \alpha] \geq [P; \alpha], [N; \beta] \leq [R; \beta]$$
$$\Rightarrow M^L \leq P^L, M^U \geq P^U, N^L \geq R^L, N^U \leq R^U \text{ and}$$
$$M^L \geq P^L, M^U \leq P^U, N^L \leq R^L, N^U \geq R^U$$
$$\Rightarrow M^L = P^L, M^U = P^U, N^L = R^L, N^U = R^U$$
$$\Rightarrow [M; \alpha] = [P; \alpha] \text{ and } [N; \beta] = [R; \beta] \Rightarrow ([M; \alpha], [N; \beta]) = ([P; \alpha], [R; \beta])$$

3. $([M; \alpha], [N; \beta]) \leq ([P; \alpha], [R; \beta])$ and $([P; \alpha], [R; \beta]) \leq ([S; \alpha], [T; \beta])$

$$\Rightarrow [M; \alpha] \leq [P; \alpha], [N; \beta] \geq [R; \beta] \text{ and } [P; \alpha] \leq [S; \alpha], [R; \beta] \geq [T; \beta]$$
$$\Rightarrow M^L \leq P^L, M^U \geq P^U, N^L \geq R^L, N^U \leq R^U \text{ and}$$
$$P^L \leq S^L, P^U \geq S^U, R^L \geq T^L, R^U \leq T^U$$
$$\Rightarrow M^L \leq S^L, M^U \geq S^U, N^L \geq T^L, N^U \leq T^U$$
$$\Rightarrow [M; \alpha] \leq [S; \alpha] \text{ and } [N; \beta] \geq [T; \beta] \Rightarrow ([M; \alpha], [N; \beta]) \leq ([S; \alpha], [T; \beta])$$

With the help of relation order on $D(I_\alpha) \times D(I_\beta)$, the definitions of supremum and infimum on this set are given below

Definition 15: $\forall([M; \alpha], [N; \beta]), ([P; \alpha], [R; \beta]) \in D(I_\alpha) \times D(I_\beta)$,

$$\inf\{([M; \alpha], [N; \beta]), ([P; \alpha], [R; \beta])\} = (\inf\{[M; \alpha], [P; \alpha]\}, \sup\{[N; \beta], [R; \beta]\})$$
$$\sup\{([M; \alpha], [N; \beta]), ([P; \alpha], [R; \beta])\} = (\sup\{[M; \alpha], [P; \alpha]\}, \inf\{[N; \beta], [R; \beta]\})$$

Lemma 2: $\left(D(I_\alpha) \times D(I_\beta), \wedge, \vee\right)$ is a complete lattice with units $([0,1-\beta;\alpha],[\beta,\beta;\beta])$ and $([\alpha,\alpha;\alpha],[0,1-\alpha;\beta])$.
Proof: It is clear from known order relation on $\mathbb{R}$.

Remark 1: The intersection and union of the family of (α,β)-interval valued sets are again (α,β)-interval valued sets. If any function satisfies below conditions, then it is called negation function.

Definition 16: L is complete lattice with units 0 and 1. $\mathcal{N}: L \to L$ and $\forall a,b \in L$,

i. $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$
ii. $\mathcal{N}(a) \leq \mathcal{N}(b) :\Leftrightarrow a \geq b$
iii. $\mathcal{N}(\mathcal{N}(a)) = a$

We try to define a negation function on $D(I_\alpha) \times D(I_\beta)$ by the help of following relation, $\forall([M;\alpha],[N;\beta]) \in D(I_\alpha) \times D(I_\beta)$,

$$\mathcal{N}(([M;\alpha],[N;\beta])) = ([\alpha - M^L, \alpha - \beta + N^U;\alpha],[\beta - N^L, \beta - \alpha + M^U;\beta])$$

This relation on $\in D(I_\alpha) \times D(I_\beta)$ is a function. Indeed,
$([M;\alpha],[N;\beta]) \in D(I_\alpha) \times D(I_\beta)$ is given arbitrary.

i. $M^L \leq \alpha \Rightarrow 0 \leq \alpha - M^L \leq \alpha$ and

$$N^U \geq \beta \Rightarrow \alpha - \beta + N^U \geq \alpha - \beta + \beta = \alpha$$

besides,

$$M^U + N^U \leq 1 \Rightarrow N^U \leq 1 - M^U \Rightarrow \alpha - \beta + N^U \leq \alpha - \beta + 1 - M^U \text{ and } M^U \geq \alpha \Rightarrow \alpha - \beta + N^U \leq \alpha - \beta + 1 - \alpha = 1 - \beta \leq 1$$

From above consequences, we get that $[\alpha - M^L, \alpha - \beta + N^U;\alpha]$

ii. $N^L \leq \beta \Rightarrow 0 \leq \beta - N^L \leq \beta$ and

$$M^U \geq \alpha \Rightarrow \beta - \alpha + M^U \geq \beta - \alpha + \alpha = \beta$$

besides,

$$M^U + N^U \leq 1 \Rightarrow M^U \leq 1 - N^U \Rightarrow \beta - \alpha + M^U \leq \beta - \alpha + 1 - N^U \text{ and } N^U \geq \beta \Rightarrow \beta - \alpha + M^U \leq \beta - \alpha + 1 - \beta = 1 - \alpha \leq 1$$

From above consequences, we get that $[\beta - N^L, \beta - \alpha + M^U;\beta]$

iii. $\alpha - \beta + N^U + \beta - \alpha + M^U = M^U + N^U \leq 1$

From above results,

$$([\alpha - M^L, \alpha - \beta + N^U;\alpha],[\beta - N^L, \beta - \alpha + M^U;\beta]) \in D(I_\alpha) \times D(I_\beta)$$

From previous discussions, we claim that $\mathcal{N}$ is negation function on $D(I_\alpha) \times D(I_\beta)$.

Proposition 9: $\forall([M;\alpha],[N;\beta]) \in D(I_\alpha) \times D(I_\beta)$,
$\mathcal{N}: D(I_\alpha) \times D(I_\beta) \to D(I_\alpha) \times D(I_\beta)$,

$$\mathcal{N}(([M;\alpha],[N;\beta])) = ([\alpha - M^L, \alpha - \beta + N^U;\alpha],[\beta - N^L, \beta - \alpha + M^U;\beta])$$

$\mathcal{N}$ satisfies conditions of Definition 16.
Proof: $([M;\alpha],[N;\beta]), ([P;\alpha],[R;\beta]) \in D(I_\alpha) \times D(I_\beta)$ are given arbitrary.

1. $([M;\alpha],[N;\beta]) = ([P;\alpha],[R;\beta]) \Rightarrow [M;\alpha] = [P;\alpha]$ and $[N;\beta] = [R;\beta]$

$$\Rightarrow M^L = P^L, M^U = P^U \text{ and } N^L = R^L, N^U = R^U$$
$$\Rightarrow \alpha - M^L = \alpha - P^L, \alpha - \beta + N^U = \alpha - \beta + R^U \text{ and}$$
$$\beta - N^L = \beta - R^L, \beta - \alpha + M^U = \beta - \alpha + P^U$$
$$([\alpha - M^L, \alpha - \beta + N^U;\alpha],[\beta - N^L, \beta - \alpha + M^U;\beta]) = ([\alpha - P^L, \alpha - \beta + R^U;\alpha],[\beta - R^L, \beta - \alpha + P^U;\beta])$$

$$\Rightarrow \mathcal{N}\big(([M;\alpha],[N;\beta])\big) = \mathcal{N}\big(([P;\alpha],[R;\beta])\big)$$

2. Now, it is shown that $\mathcal{N}$ satisfies the conditions of negation function,

i. $\mathcal{N}\big(([0,1-\beta;\alpha],[\beta,\beta;\beta])\big)$

$$= ([\alpha-0,\alpha-\beta+\beta;\alpha],[\beta-\beta,\beta-\alpha+1-\beta;\beta]) = ([\alpha,\alpha;\alpha],[0,1-\alpha;\beta])$$

$$\mathcal{N}\big(([\alpha,\alpha;\alpha],[0,1-\alpha;\beta])\big) = ([\alpha-\alpha,\alpha-\beta+1-\alpha;\alpha],[\beta-0,\beta-\alpha+\alpha;\beta])$$

$$= ([0,1-\beta;\alpha],[\beta,\beta;\beta])$$

ii. $\mathcal{N}\big(([M;\alpha],[N;\beta])\big) \le \mathcal{N}\big(([P;\alpha],[R;\beta])\big)$

$$\Leftrightarrow \begin{pmatrix}[\alpha-M^L,\alpha-\beta+N^U;\alpha],\\ [\beta-N^L,\beta-\alpha+M^U;\beta]\end{pmatrix} \le \begin{pmatrix}[\alpha-P^L,\alpha-\beta+R^U;\alpha],\\ [\beta-R^L,\beta-\alpha+P^U;\beta]\end{pmatrix}$$

$$\Leftrightarrow [\alpha-M^L,\alpha-\beta+N^U;\alpha] \le [\alpha-P^L,\alpha-\beta+R^U;\alpha] \text{ and}$$

$$[\beta-N^L,\beta-\alpha+M^U;\beta] \ge [\beta-R^L,\beta-\alpha+P^U;\beta]$$

$$\Leftrightarrow \alpha-M^L \le \alpha-P^L, \alpha-\beta+N^U \ge \alpha-\beta+R^U \text{ and}$$

$$\beta-N^L \ge \beta-R^L, \beta-\alpha+M^U \le \beta-\alpha+P^U$$

$$\Leftrightarrow M^L \ge P^L, N^U \ge R^U, N^L \le R^L, M^U \le P^U$$

$$\Leftrightarrow [M;\alpha] \ge [P;\alpha] \text{ and } [N;\beta] \le [R;\beta]$$

$$\Leftrightarrow ([M;\alpha],[N;\beta]) \ge ([P;\alpha],[R;\beta])$$

iii. $\mathcal{N}\Big(\mathcal{N}\big(([M;\alpha],[N;\beta])\big)\Big)$

$$= \mathcal{N}([\alpha-M^L,\alpha-\beta+N^U;\alpha],[\beta-N^L,\beta-\alpha+M^U;\beta])$$

$$= \begin{pmatrix}[\alpha-(\alpha-M^L),\alpha-\beta+\beta-\alpha+M^U;\alpha],\\ [\beta-(\beta-N^L),\beta-\alpha+\alpha-\beta+N^U;\beta]\end{pmatrix}$$

$$= ([M^L,M^U;\alpha],[N^L,N^U;\beta]) = ([M;\alpha],[N;\beta])$$

Definition 17: Let X be universal set.
For functions $[M_A;\alpha]: X \to D(I_\alpha)$ and $[N_A;\beta]: X \to D(I_\beta)$, $\forall x \in X$, ${M_A}^U(x) + {N_A}^U(x) \le 1$,

$$[A;\alpha;\beta] = \{\langle x,[M_A(x);\alpha],[N_A(x);\beta]\rangle | x \in X\}$$

To make clear, it is denoted by;

$$\{\langle x,[M_A(x);\alpha],[N_A(x);\beta]\rangle | x \in X\} = \{\langle x,[A(x);\alpha;\beta]\rangle | x \in X\}$$

is called (α,β) −interval valued intuitionistic fuzzy set. The family of (α,β) −interval valued intuitionistic fuzzy sets on X is shown by (α, β)-IVIFS(X).

Some algebraic operations on (α, β)-IVIFS(X) are defined below.

Definition 18: Let X be universal set. $[A;\alpha;\beta], [B;\alpha;\beta] \in$ (α, β)-IVIFS(X) and
Λ is index set $\forall\lambda \in \Lambda$,

i. $[A;\alpha;\beta]^c = \left\{\begin{matrix} < x,\big[\alpha-{M_A}^L(x),\alpha-\beta+{N_A}^U(x);\alpha\big], \\ \big[\beta-{N_A}^L(x),\beta-\alpha+{M_A}^U(x);\beta\big] > | x \in X\end{matrix}\right\}$

ii. $[A;\alpha;\beta] \sqsubseteq [B;\alpha;\beta] :\Leftrightarrow \forall x \in X, {M_A}^L(x) \le {M_B}^L(x), {M_A}^U(x) \ge {M_B}^U(x)$

$$\text{and } {N_A}^L(x) \ge {N_B}^L(x), {N_A}^U(x) \le {N_B}^U(x)$$

iii. $[A;\alpha;\beta] \sqcap [B;\alpha;\beta]$

$$= \left\{\begin{matrix} < x,\big[\inf\{{M_A}^L(x),{M_B}^L(x)\},\sup\{{M_A}^U(x),{M_B}^U(x)\};\alpha\big], \\ \big[\sup\{{N_A}^L(x),{N_B}^L(x)\},\inf\{{N_A}^U(x),{N_B}^U(x)\};\beta\big] > | x \in X\end{matrix}\right\}$$

iv. $[A;\alpha;\beta] \sqcup [B;\alpha;\beta]$

$$= \left\{\begin{matrix} < x,\big[\sup\{{M_A}^L(x),{M_B}^L(x)\},\inf\{{M_A}^U(x),{M_B}^U(x)\};\alpha\big], \\ \big[\inf\{{N_A}^L(x),{N_B}^L(x)\},\sup\{{N_A}^U(x),{N_B}^U(x)\};\beta\big] > | x \in X\end{matrix}\right\}$$

v. $\sqcap_{\lambda\in\Lambda}[A;\alpha;\beta]_\lambda = \left\{\left\langle \begin{matrix} x, \left[\wedge_{\lambda\in\Lambda} M_{A_\lambda}^{L}(x), \vee_{\lambda\in\Lambda} M_{A_\lambda}^{U}(x);\alpha\right], \\ \left[\vee_{\lambda\in\Lambda} N_{A_\lambda}^{L}(x), \wedge_{\lambda\in\Lambda} N_{A_\lambda}^{U}(x);\beta\right]\end{matrix}\right\rangle | x\in X\right\}$

vi. $\sqcup_{\lambda\in\Lambda}[A;\alpha;\beta]_\lambda = \left\{\left\langle \begin{matrix} x, \left[\vee_{\lambda\in\Lambda} M_{A_\lambda}^{L}(x), \wedge_{\lambda\in\Lambda} M_{A_\lambda}^{U}(x);\alpha\right], \\ \left[\wedge_{\lambda\in\Lambda} N_{A_\lambda}^{L}(x), \vee_{\lambda\in\Lambda} N_{A_\lambda}^{U}(x);\beta\right]\end{matrix}\right\rangle | x\in X\right\}$

Example 2: Let $X = \{a, b, c, d\}$.

$$[A;\alpha;\beta] = \left\{\begin{matrix} < a,[0.1,0.3;0.3],[0.4,0.6;0.4] >, < b,[0.0,0.4;0.3],[0.3,0.6;0.4] >, \\ < c,[0.2,0.5;0.3],[0.1,0.4;0.4] >, < d,[0.15,0.45;0.3],[0.3,0.5;0.4] > \end{matrix}\right\}$$

$$[B;\alpha;\beta] = \left\{\begin{matrix} < a,[0.05,0.35;0.3],[0.25,0.65;0.4] >, < b,[0.15,0.45;0.3],[0.2,0.4;0.4] >, \\ < c,[0.1,0.3;0.3],[0.3,0.7;0.4] >, < d,[0.1,0.4;0.3],[0.15,0.55;0.4] > \end{matrix}\right\}$$

For $\alpha = 0.3$ and $\beta = 0.4$, A and B are (α,β)-interval valued intuitionistic fuzzy sets,

$$[A;\alpha;\beta]^c = \left\{\begin{matrix} < a,[0.2,0.5;0.3],[0.0,0.4;0.4] >, < b,[0.3,0.5;0.3],[0.1,0.5;0.4] >, \\ < c,[0.1,0.3;0.3],[0.3,0.6;0.4] >, < d,[0.15,0.4;0.3],[0.1,0.55;0.4] > \end{matrix}\right\}$$

$$[A;\alpha;\beta] \sqcap [B;\alpha;\beta] = \left\{\begin{matrix} < a,[0.05,0.35;0.3],[0.4,0.6;0.4] >, < b,[0.0,0.45;0.3],[0.3,0.4;0.4] >, \\ < c,[0.1,0.5;0.3],[0.3,0.4;0.4] >, < d,[0.1,0.45;0.3],[0.3,0.5;0.4] > \end{matrix}\right\}$$

$$[A;\alpha;\beta] \sqcup [B;\alpha;\beta] = \left\{\begin{matrix} < a,[0.1,0.3;0.3],[0.25,0.65;0.4] >, < b,[0.15,0.4;0.3],[0.2,0.6;0.4] >, \\ < c,[0.2,0.3;0.3],[0.1,0.7;0.4] >, < d,[0.1,0.4;0.3],[0.15,0.55;0.4] > \end{matrix}\right\}$$

Proposition 10: Let X be universal set.

$[A;\alpha;\beta], [B;\alpha;\beta], [C;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X) and Λ is index set $\forall\lambda\in\Lambda$,

i. $[A;\alpha;\beta] \sqcap [B;\alpha;\beta] = [B;\alpha;\beta] \sqcap [A;\alpha;\beta]$

ii. $[A;\alpha;\beta] \sqcup [B;\alpha;\beta] = [B;\alpha;\beta] \sqcup [A;\alpha;\beta]$

iii. $[A;\alpha;\beta] \sqcap ([B;\alpha;\beta] \sqcup [C;\alpha;\beta])$

$$= ([A;\alpha;\beta] \sqcap [B;\alpha;\beta]) \sqcup ([A;\alpha;\beta] \sqcap [C;\alpha;\beta])$$

iv. $[A;\alpha;\beta] \sqcup ([B;\alpha;\beta] \sqcap [C;\alpha;\beta])$

$$= ([A;\alpha;\beta] \sqcup [B;\alpha;\beta]) \sqcap ([A;\alpha;\beta] \sqcup [C;\alpha;\beta])$$

v. $[A;\alpha;\beta] \sqcap (\sqcup_{\lambda\in\Lambda}[B;\alpha;\beta]_\lambda) = \sqcup_{\lambda\in\Lambda}([A;\alpha;\beta] \sqcap [B;\alpha;\beta]_\lambda)$

vi. $[A;\alpha;\beta] \sqcup (\sqcap_{\lambda\in\Lambda}[B;\alpha;\beta]_\lambda) = \sqcap_{\lambda\in\Lambda}([A;\alpha;\beta] \sqcup [B;\alpha;\beta]_\lambda)$

Proof: $[A;\alpha;\beta], [B;\alpha;\beta], [C;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X) are given arbitrary.

i. $[A;\alpha;\beta] \sqcap [B;\alpha;\beta]$

$$= \left\{\begin{matrix} < x, \left[\inf\{M_A{}^L(x), M_B{}^L(x)\}, \sup\{M_A{}^U(x), M_B{}^U(x)\};\alpha\right], \\ \left[\sup\{N_A{}^L(x), N_B{}^L(x)\}, \inf\{N_A{}^U(x), N_B{}^U(x)\};\beta\right] > | x\in X \end{matrix}\right\}$$

$$= \left\{\begin{matrix} < x, \left[\inf\{M_B{}^L(x), M_A{}^L(x)\}, \sup\{M_B{}^U(x), M_A{}^U(x)\};\alpha\right], \\ \left[\sup\{N_B{}^L(x), N_A{}^L(x)\}, \inf\{N_B{}^U(x), N_A{}^U(x)\};\beta\right] > | x\in X \end{matrix}\right\} = [B;\alpha;\beta] \sqcap [A;\alpha;\beta]$$

ii. $[A;\alpha;\beta] \sqcup [B;\alpha;\beta]$

$$= \left\{\begin{matrix} < x, \left[\sup\{M_A{}^L(x), M_B{}^L(x)\}, \inf\{M_A{}^U(x), M_B{}^U(x)\};\alpha\right], \\ \left[\inf\{N_A{}^L(x), N_B{}^L(x)\}, \sup\{N_A{}^U(x), N_B{}^U(x)\};\beta\right] > | x\in X \end{matrix}\right\}$$

$$= \left\{\begin{matrix} < x, \left[\sup\{M_B{}^L(x), M_A{}^L(x)\}, \inf\{M_B{}^U(x), M_A{}^U(x)\};\alpha\right], \\ \left[\inf\{N_B{}^L(x), N_A{}^L(x)\}, \sup\{N_B{}^U(x), N_A{}^U(x)\};\beta\right] > | x\in X \end{matrix}\right\} = [B;\alpha;\beta] \sqcup [A;\alpha;\beta]$$

iii. $[A;\alpha;\beta] \sqcap ([B;\alpha;\beta] \sqcup [C;\alpha;\beta])$

$$= [A;\alpha;\beta] \sqcap \left\{ \begin{array}{c} <\mathrm{x}, \begin{bmatrix} \sup\{M_B{}^{L}(x), M_C{}^{L}(x)\}, \\ \inf\{M_B{}^{U}(x), M_C{}^{U}(x)\}; \alpha \end{bmatrix}, \\ \begin{bmatrix} \inf\{N_B{}^{L}(x), N_C{}^{L}(x)\}, \\ \sup\{N_B{}^{U}(x), N_C{}^{U}(x)\}; \beta \end{bmatrix} > |x \in X \end{array} \right\}$$

$$= \left\{ \begin{array}{c} <\mathrm{x}, \begin{bmatrix} \inf\{M_A{}^{L}(x), \sup\{M_B{}^{L}(x), M_C{}^{L}(x)\}\}, \\ \sup\{M_A{}^{U}(x), \inf\{M_B{}^{U}(x), M_C{}^{U}(x)\}\}; \alpha \end{bmatrix}, \\ \begin{bmatrix} \sup\{N_A{}^{L}(x), \inf\{N_B{}^{L}(x), N_C{}^{L}(x)\}\}, \\ \inf\{N_A{}^{U}(x), \sup\{N_B{}^{U}(x), N_C{}^{U}(x)\}\}; \beta \end{bmatrix} > |x \in X \end{array} \right\}$$

$$= \left\{ \begin{array}{c} <\mathrm{x}, \begin{bmatrix} \sup\{\inf\{M_A{}^{L}(x), M_B{}^{L}(x)\}, \inf\{M_A{}^{L}(x), M_C{}^{L}(x)\}\}, \\ \inf\{\sup\{M_A{}^{U}(x), M_B{}^{U}(x)\}, \sup\{M_A{}^{U}(x), M_C{}^{U}(x)\}\}; \alpha \end{bmatrix}, \\ \begin{bmatrix} \inf\{\sup\{N_A{}^{L}(x), N_B{}^{L}(x)\}, \sup\{N_A{}^{L}(x), N_C{}^{L}(x)\}\}, \\ \sup\{\inf\{N_A{}^{U}(x), N_B{}^{U}(x)\}, \inf\{N_A{}^{U}(x), N_C{}^{U}(x)\}\}; \beta \end{bmatrix} > |x \in X \end{array} \right\}$$

$$= \left\{ \begin{array}{c} <\mathrm{x}, [\inf\{M_A{}^{L}(x), M_B{}^{L}(x)\}, \sup\{M_A{}^{U}(x), M_B{}^{U}(x)\}; \alpha], \\ [\sup\{N_A{}^{L}(x), N_B{}^{L}(x)\}, \inf\{N_A{}^{U}(x), N_B{}^{U}(x)\}; \beta] > |x \in X \end{array} \right\}$$

$$\sqcup \left\{ \begin{array}{c} <\mathrm{x}, [\inf\{M_A{}^{L}(x), M_C{}^{L}(x)\}, \alpha, \sup\{M_A{}^{U}(x), M_C{}^{U}(x)\}; \alpha], \\ [\sup\{N_A{}^{L}(x), N_C{}^{L}(x)\}, \beta, \inf\{N_A{}^{U}(x), N_C{}^{U}(x)\}; \beta] > |x \in X \end{array} \right\}$$

$$= ([A;\alpha;\beta] \sqcap [B;\alpha;\beta]) \sqcup ([A;\alpha;\beta] \sqcap [C;\alpha;\beta])$$

iv. $[A;\alpha;\beta] \sqcup ([B;\alpha;\beta] \sqcap [C;\alpha;\beta])$

$$= [A;\alpha;\beta] \sqcup \left\{ \begin{array}{c} <\mathrm{x}, \begin{bmatrix} \inf\{M_B{}^{L}(x), M_C{}^{L}(x)\}, \\ \sup\{M_B{}^{U}(x), M_C{}^{U}(x)\}; \alpha \end{bmatrix}, \\ \begin{bmatrix} \sup\{N_B{}^{L}(x), N_C{}^{L}(x)\}, \\ \inf\{N_B{}^{U}(x), N_C{}^{U}(x)\}; \beta \end{bmatrix} > |x \in X \end{array} \right\}$$

$$= \left\{ \begin{array}{c} <\mathrm{x}, \begin{bmatrix} \sup\{M_A{}^{L}(x), \inf\{M_B{}^{L}(x), M_C{}^{L}(x)\}\}, \\ \inf\{M_A{}^{U}(x), \sup\{M_B{}^{U}(x), M_C{}^{U}(x)\}\}; \alpha \end{bmatrix}, \\ \begin{bmatrix} \inf\{N_A{}^{L}(x), \sup\{N_B{}^{L}(x), N_C{}^{L}(x)\}\}, \\ \sup\{N_A{}^{U}(x), \inf\{N_B{}^{U}(x), N_C{}^{U}(x)\}\}; \beta \end{bmatrix} > |x \in X \end{array} \right\}$$

$$= \left\{ \begin{array}{c} <\mathrm{x}, \begin{bmatrix} \inf\{\sup\{M_A{}^{L}(x), M_B{}^{L}(x)\}, \sup\{M_A{}^{L}(x), M_C{}^{L}(x)\}\}, \\ \sup\{\inf\{M_A{}^{U}(x), M_B{}^{U}(x)\}, \inf\{M_A{}^{U}(x), M_C{}^{U}(x)\}\}; \alpha \end{bmatrix}, \\ \begin{bmatrix} \sup\{\inf\{N_A{}^{L}(x), N_B{}^{L}(x)\}, \inf\{N_A{}^{L}(x), N_C{}^{L}(x)\}\}, \\ \inf\{\sup\{N_A{}^{U}(x), N_B{}^{U}(x)\}, \sup\{N_A{}^{U}(x), N_C{}^{U}(x)\}\}; \beta \end{bmatrix} > |x \in X \end{array} \right\}$$

$$= \left\{ \begin{array}{c} <\mathrm{x}, [\sup\{M_A{}^{L}(x), M_B{}^{L}(x)\}, \inf\{M_A{}^{U}(x), M_B{}^{U}(x)\}; \alpha], \\ [\inf\{N_A{}^{L}(x), N_B{}^{L}(x)\}, \sup\{N_A{}^{U}(x), N_B{}^{U}(x)\}; \beta] > |x \in X \end{array} \right\}$$

$$\sqcap \left\{ \begin{array}{c} <\mathrm{x}, [\sup\{M_A{}^{L}(x), M_C{}^{L}(x)\}, \inf\{M_A{}^{U}(x), M_C{}^{U}(x)\}; \alpha], \\ [\inf\{N_A{}^{L}(x), N_C{}^{L}(x)\}, \sup\{N_A{}^{U}(x), N_C{}^{U}(x)\}; \beta] > |x \in X \end{array} \right\}$$

$$= ([A;\alpha;\beta] \sqcup [B;\alpha;\beta]) \sqcap ([A;\alpha;\beta] \sqcup [C;\alpha;\beta])$$

v. $[A;\alpha;\beta] \sqcap (\sqcup_{\lambda\in\Lambda} [B;\alpha;\beta]_{\lambda})$

$$= [A;\alpha;\beta] \sqcap \left\{ \left\langle \begin{matrix} x, \left[\bigvee_{\lambda\in\Lambda} M_{B_\lambda}^{L}(x), \bigwedge_{\lambda\in\Lambda} M_{B_\lambda}^{U}(x);\alpha\right], \\ \left[\bigwedge_{\lambda\in\Lambda} N_{B_\lambda}^{L}(x), \bigvee_{\lambda\in\Lambda} N_{B_\lambda}^{U}(x);\beta\right] \end{matrix} \right\rangle \middle| x \in X \right\}$$

$$= \left\{ \left\langle \begin{matrix} x, \left[M_A{}^{L}(x) \wedge \bigvee_{\lambda\in\Lambda} M_{B_\lambda}^{L}(x), M_A{}^{U}(x) \vee \bigwedge_{\lambda\in\Lambda} M_{B_\lambda}^{U}(x);\alpha\right], \\ \left[N_A{}^{L}(x) \vee \bigwedge_{\lambda\in\Lambda} N_{B_\lambda}^{L}(x), N_A{}^{U}(x) \wedge \bigvee_{\lambda\in\Lambda} N_{B_\lambda}^{U}(x);\beta\right] \end{matrix} \right\rangle \middle| x \in X \right\}$$

$$= \left\{ \left\langle \begin{matrix} x, \left[\bigvee_{\lambda\in\Lambda} M_A{}^{L}(x) \wedge M_{B_\lambda}^{L}(x), \bigwedge_{\lambda\in\Lambda} M_A{}^{U}(x) \vee M_{B_\lambda}^{U}(x);\alpha\right], \\ \left[\bigwedge_{\lambda\in\Lambda} N_A{}^{L}(x) \vee N_{B_\lambda}^{L}(x), \bigvee_{\lambda\in\Lambda} N_A{}^{U}(x) \wedge N_{B_\lambda}^{U}(x);\beta\right] \end{matrix} \right\rangle \middle| x \in X \right\}$$

$$= \sqcup_{\lambda\in\Lambda} \left\{ \left\langle \begin{matrix} x, \left[M_A{}^{L}(x) \wedge M_{B_\lambda}^{L}(x), M_A{}^{U}(x) \vee M_{B_\lambda}^{U}(x);\alpha\right], \\ \left[N_A{}^{L}(x) \vee N_{B_\lambda}^{L}(x), N_A{}^{U}(x) \wedge N_{B_\lambda}^{U}(x);\beta\right] \end{matrix} \right\rangle \middle| x \in X \right\} = \sqcup_{\lambda\in\Lambda} \left([A;\alpha;\beta] \sqcap [B;\alpha;\beta]_\lambda\right)$$

vi. $[A;\alpha;\beta] \sqcup (\sqcap_{\lambda\in\Lambda} [B;\alpha;\beta]_\lambda)$

$$= [A;\alpha;\beta] \sqcup \left\{ \left\langle \begin{matrix} x, \left[\bigwedge_{\lambda\in\Lambda} M_{B_\lambda}^{L}(x), \bigvee_{\lambda\in\Lambda} M_{B_\lambda}^{U}(x);\alpha\right], \\ \left[\bigvee_{\lambda\in\Lambda} N_{B_\lambda}^{L}(x), \bigwedge_{\lambda\in\Lambda} N_{B_\lambda}^{U}(x);\beta\right] \end{matrix} \right\rangle \middle| x \in X \right\}$$

$$= \left\{ \left\langle \begin{matrix} x, \left[M_A{}^{L}(x) \vee \bigwedge_{\lambda\in\Lambda} M_{B_\lambda}^{L}(x), M_A{}^{U}(x) \wedge \bigvee_{\lambda\in\Lambda} M_{B_\lambda}^{U}(x);\alpha\right], \\ \left[N_A{}^{L}(x) \wedge \bigvee_{\lambda\in\Lambda} N_{B_\lambda}^{L}(x), N_A{}^{U}(x) \vee \bigwedge_{\lambda\in\Lambda} N_{B_\lambda}^{U}(x);\beta\right] \end{matrix} \right\rangle \middle| x \in X \right\}$$

$$= \left\{ \left\langle \begin{matrix} x, \left[\bigwedge_{\lambda\in\Lambda} M_A{}^{L}(x) \vee M_{B_\lambda}^{L}(x), \bigvee_{\lambda\in\Lambda} M_A{}^{U}(x) \wedge M_{B_\lambda}^{U}(x);\alpha\right], \\ \left[\bigvee_{\lambda\in\Lambda} N_A{}^{L}(x) \wedge N_{B_\lambda}^{L}(x), \bigwedge_{\lambda\in\Lambda} N_A{}^{U}(x) \vee N_{B_\lambda}^{U}(x);\beta\right] \end{matrix} \right\rangle \middle| x \in X \right\}$$

$$= \sqcap_{\lambda\in\Lambda} \left\{ \left\langle \begin{matrix} x, \left[M_A{}^{L}(x) \vee M_{B_\lambda}^{L}(x), M_A{}^{U}(x) \wedge M_{B_\lambda}^{U}(x);\alpha\right], \\ \left[N_A{}^{L}(x) \wedge N_{B_\lambda}^{L}(x), N_A{}^{U}(x) \vee N_{B_\lambda}^{U}(x);\beta\right] \end{matrix} \right\rangle \middle| x \in X \right\} = \sqcap_{\lambda\in\Lambda} \left([A;\alpha;\beta] \sqcup [B;\alpha;\beta]_\lambda\right)$$

Proposition 11: Let X be universal set. $[A;\alpha;\beta], [B;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X) and Λ is index set $\forall\lambda \in \Lambda$,

i. $(([A;\alpha;\beta])^c)^c = [A;\alpha;\beta]$

ii. $([A;\alpha;\beta] \sqcap [B;\alpha;\beta])^c = ([A;\alpha;\beta])^c \sqcup ([B;\alpha;\beta])^c$

iii. $([A;\alpha;\beta] \sqcup [B;\alpha;\beta])^c = ([A;\alpha;\beta])^c \sqcap ([B;\alpha;\beta])^c$

iv. $(\sqcap_{\lambda\in\Lambda} [A;\alpha;\beta]_\lambda)^c = \sqcup_{\lambda\in\Lambda} ([A;\alpha;\beta]_\lambda)^c$

v. $(\sqcup_{\lambda\in\Lambda} [A;\alpha;\beta]_\lambda)^c = \sqcap_{\lambda\in\Lambda} ([A;\alpha;\beta]_\lambda)^c$

Proof: $[A;\alpha;\beta], [B;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X) are given arbitrary.

i. $([A;\alpha;\beta])^c = \left\{ \begin{matrix} < x, \left[\alpha - M_A{}^{L}(x), \alpha - \beta + N_A{}^{U}(x);\alpha\right], \\ \left[\beta - N_A{}^{L}(x), \beta - \alpha + M_A{}^{U}(x);\beta\right] > |x \in X \end{matrix} \right\}$

$$\Rightarrow (([A;\alpha;\beta])^c)^c$$

$$= \left\{ \begin{matrix} < x, \left[\alpha - \left(\alpha - {M_A}^L(x)\right), \alpha - \beta + \beta - \alpha + {M_A}^U(x); \alpha\right], \\ \left[\beta - \left(\beta - {N_A}^L(x)\right), \beta - \alpha + \alpha - \beta + {N_A}^U(x); \beta\right] > |x \in X \end{matrix} \right\}$$

$$= \left\{< x, \left[{M_A}^L(x), {M_A}^U(x); \alpha\right], \left[{N_A}^L(x), {N_A}^U(x); \beta\right] > |x \in X\right\} = [A;\alpha;\beta]$$

ii. $([A;\alpha;\beta] \sqcap [B;\alpha;\beta])^c$

$$= \left\{ \begin{matrix} < x, \begin{bmatrix} \alpha - \left(\inf\{{M_A}^L(x), {M_B}^L(x)\}\right), \\ \alpha - \beta + \left(\inf\{{N_A}^U(x), {N_B}^U(x)\}\right); \alpha \end{bmatrix}, \\ \begin{bmatrix} \beta - \left(\sup\{{N_A}^L(x), {N_B}^L(x)\}\right), \\ \beta - \alpha + \left(\sup\{{M_A}^U(x), {M_B}^U(x)\}\right); \beta \end{bmatrix} > |x \in X \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} < x, \begin{bmatrix} \sup\{\alpha - {M_A}^L(x), \alpha - {M_B}^L(x)\}, \\ \inf\{\alpha - \beta + {N_A}^U(x), \alpha - \beta + {N_B}^U(x)\}; \alpha \end{bmatrix}, \\ \begin{bmatrix} \inf\{\beta - {N_A}^L(x), \beta - {N_B}^L(x)\}, \\ \sup\{\beta - \alpha + {M_A}^U(x), \beta - \alpha + {M_B}^U(x)\}; \beta \end{bmatrix} > |x \in X \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} < x, \left[\alpha - {M_A}^L(x), \alpha - \beta + {N_A}^U(x); \alpha\right], \\ \left[\beta - {N_A}^L(x), \beta - \alpha + {M_A}^U(x); \beta\right] > |x \in X \end{matrix} \right\} \sqcup \left\{ \begin{matrix} < x, \left[\alpha - {M_B}^L(x), \alpha - \beta + {N_B}^U(x); \alpha\right], \\ \left[\beta - {N_B}^L(x), \beta - \alpha + {M_B}^U(x); \beta\right] > |x \in X \end{matrix} \right\}$$

$$= ([A;\alpha;\beta])^c \sqcup ([B;\alpha;\beta])^c$$

iii. $([A;\alpha;\beta] \sqcup [B;\alpha;\beta])^c$

$$= \left\{ \begin{matrix} < x, \begin{bmatrix} \alpha - \left(\sup\{{M_A}^L(x), {M_B}^L(x)\}\right), \\ \alpha - \beta + \left(\sup\{{N_A}^U(x), {N_B}^U(x)\}\right); \alpha \end{bmatrix}, \\ \begin{bmatrix} \beta - \left(\inf\{{N_A}^L(x), {N_B}^L(x)\}\right), \\ \beta - \alpha + \left(\inf\{{M_A}^U(x), {M_B}^U(x)\}\right); \beta \end{bmatrix} > |x \in X \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} < x, \begin{bmatrix} \inf\{\alpha - {M_A}^L(x), \alpha - {M_B}^L(x)\}, \\ \sup\{\alpha - \beta + {N_A}^U(x), \alpha - \beta + {N_B}^U(x)\}; \alpha \end{bmatrix}, \\ \begin{bmatrix} \sup\{\beta - {N_A}^L(x), \beta - {N_B}^L(x)\}, \\ \inf\{\beta - \alpha + {M_A}^U(x), \beta - \alpha + {M_B}^U(x)\}; \beta \end{bmatrix} > |x \in X \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} < x, \left[\alpha - {M_A}^L(x), \alpha - \beta + {N_A}^U(x); \alpha\right], \\ \left[\beta - {N_A}^L(x), \beta - \alpha + {M_A}^U(x); \beta\right] > |x \in X \end{matrix} \right\} \sqcap \left\{ \begin{matrix} < x, \left[\alpha - {M_B}^L(x), \alpha - \beta + {N_B}^U(x); \alpha\right], \\ \left[\beta - {N_B}^L(x), \beta - \alpha + {M_B}^U(x); \beta\right] > |x \in X \end{matrix} \right\}$$

$$= ([A;\alpha;\beta])^c \sqcap ([B;\alpha;\beta])^c$$

iv. $\sqcap_{\lambda\in\Lambda} [A;\alpha;\beta]_\lambda = \left\{ \left\langle \begin{matrix} x, \left[\wedge_{\lambda\in\Lambda} M_{A_\lambda}^L(x), \vee_{\lambda\in\Lambda} M_{A_\lambda}^U(x); \alpha\right], \\ \left[\vee_{\lambda\in\Lambda} N_{A_\lambda}^L(x), \wedge_{\lambda\in\Lambda} N_{A_\lambda}^U(x); \beta\right] \end{matrix} \right\rangle |x \in X \right\}$

$$\Rightarrow (\sqcap_{\lambda\in\Lambda} [A;\alpha;\beta]_\lambda)^c$$

$$= \left\{ \left\langle \begin{matrix} x, \left[\alpha - \bigwedge_{\lambda\in\Lambda} M_{A_\lambda}^L(x), \alpha - \beta + \bigwedge_{\lambda\in\Lambda} N_{A_\lambda}^U(x); \alpha\right], \\ \left[\beta - \bigvee_{\lambda\in\Lambda} N_{A_\lambda}^L(x), \beta - \alpha + \bigvee_{\lambda\in\Lambda} M_{A_\lambda}^U(x); \beta\right] \end{matrix} \right\rangle |x \in X \right\}$$

$$= \left\{ \left\langle \begin{matrix} x, \left[\bigvee_{\lambda\in\Lambda} \alpha - M_{A_\lambda}^L(x), \bigwedge_{\lambda\in\Lambda} \alpha - \beta + N_{A_\lambda}^U(x); \alpha\right], \\ \left[\bigwedge_{\lambda\in\Lambda} \beta - N_{A_\lambda}^L(x), \bigvee_{\lambda\in\Lambda} \beta - \alpha + M_{A_\lambda}^U(x); \beta\right] \end{matrix} \right\rangle |x \in X \right\} = \sqcup_{\lambda\in\Lambda} ([A;\alpha;\beta]_\lambda)^c$$

v. $\sqcup_{\lambda\in\Lambda} [A;\alpha;\beta]_\lambda = \left\{ \left\langle \begin{matrix} x, \left[\vee_{\lambda\in\Lambda} M_{A_\lambda}^L(x), \wedge_{\lambda\in\Lambda} M_{A_\lambda}^U(x); \alpha\right], \\ \left[\wedge_{\lambda\in\Lambda} N_{A_\lambda}^L(x), \vee_{\lambda\in\Lambda} N_{A_\lambda}^U(x); \beta\right] \end{matrix} \right\rangle |x \in X \right\}$

$$\Rightarrow (\sqcup_{\lambda\in\Lambda}[A;\alpha;\beta]_\lambda)^c$$

$$= \left\{ \left\langle \begin{matrix} x, \left[\alpha - \bigvee_{\lambda\in\Lambda} M_{A_\lambda}^{L}(x), \alpha - \beta + \bigvee_{\lambda\in\Lambda} N_{A_\lambda}^{U}(x); \alpha\right], \\ \left[\beta - \bigwedge_{\lambda\in\Lambda} N_{A_\lambda}^{L}(x), \beta - \alpha + \bigwedge_{\lambda\in\Lambda} M_{A_\lambda}^{U}(x); \beta\right] \end{matrix} \right\rangle | x \in X \right\}$$

$$= \left\{ \left\langle \begin{matrix} x, \left[\bigwedge_{\lambda\in\Lambda} \alpha - M_{A_\lambda}^{L}(x), \bigvee_{\lambda\in\Lambda} \alpha - \beta + N_{A_\lambda}^{U}(x); \alpha\right], \\ \left[\bigvee_{\lambda\in\Lambda} \beta - N_{A_\lambda}^{L}(x), \bigwedge_{\lambda\in\Lambda} \beta - \alpha + M_{A_\lambda}^{U}(x); \beta\right] \end{matrix} \right\rangle | x \in X \right\} = \sqcap_{\lambda\in\Lambda}([A;\alpha;\beta]_\lambda)^c$$

Proposition 12: Let X be universal set.

Functions $\mathbf{0_X}: X \to ([0,1-\beta;\alpha],[\beta,\beta;\beta])$ and $\mathbf{1_X}: X \to ([\alpha,\alpha;\alpha],[0,1-\alpha;\beta])$

i. $(\mathbf{0_X})^c = \mathbf{1_X}$

ii. $(\mathbf{1_X})^c = \mathbf{0_X}$

Proof:

i. $(\mathbf{0_X})^c = \big(([0,1-\beta;\alpha],[\beta,\beta;\beta])\big)^c$

$$= ([\alpha - 0, \alpha-\beta+\beta;\alpha],[\beta-\beta,\beta-\alpha+1-\beta;\beta])$$
$$= ([\alpha,\alpha;\alpha],[0,1-\alpha;\beta]) = \mathbf{1_X}$$

i. $(\mathbf{1_X})^c = \big(([\alpha,\alpha;\alpha],[0,1-\alpha;\beta])\big)^c$

$$= ([\alpha-\alpha,\alpha-\beta+1-\alpha;\alpha],[\beta-0,\beta-\alpha+\alpha;\beta])$$
$$= ([0,1-\beta;\alpha],[\beta,\beta;\beta]) = \mathbf{0_X}$$

Definition 19: Let X be universal set and $[A;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X).
$[A;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X) has sup-property: $\Leftrightarrow\ \forall x \in X$,
$\exists([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta]) \in D(I_\alpha)\times D(I_\beta) \ni [A;\alpha;\beta] = ([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$

Definition 20: Let X be universal set and $[A;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X).
$\forall([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta]) \in D(I_\alpha)\times D(I_\beta)$,

$$[A;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$$
$$= \{x \in X | [M_A(x);\alpha] \geq [\lambda_1,\lambda_2;\alpha] \text{ and } [N_A(x);\beta] \leq [\theta_1,\theta_2;\beta]\}$$

$[A;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$ is called $([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$-level subset of $[A;\alpha;\beta]$. It is easily seen that from definition,
$([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$-level subsets of $[A;\alpha;\beta]$ are crisp sets . Besides,

$$[M_A(x);\alpha] \geq [\lambda_1,\lambda_2;\alpha] \Rightarrow {M_A}^L(x) \geq \lambda_1 \text{ and } {M_A}^U(x) \leq \lambda_2$$
$$[N_A(x);\beta] \leq [\theta_1,\theta_2;\beta] \Rightarrow {N_A}^L(x) \leq \theta_1 \text{ and } {N_A}^U(x) \geq \theta_2$$

Proposition 13: Let X be universal set. $\forall[A;\alpha;\beta],[B;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X), $\forall([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta]) \in D(I_\alpha)\times D(I_\beta)$,

i. $x \in [A;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$

$$\Leftrightarrow ([M_A(x);\alpha],[N_A(x);\beta]) \geq ([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$$

ii. $[A;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])} = [M_A(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cap [N_A(x);\beta]_{[\theta_1,\theta_2;\beta]}$

iii. $([A;\alpha;\beta] \sqcup [B;\alpha;\beta])_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$

$$= \left([M_A(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup [M_B(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup \left({M_A}^L{}_{\lambda_1} \cap {M_B}^U{}_{\lambda_2}\right)\right) \cup \left({M_B}^L{}_{\lambda_1} \cap {M_A}^U{}_{\lambda_2}\right) \cap$$
$$\left([N_A(x);\beta]_{[\theta_1,\theta_2;\beta]} \cup [N_B(x);\beta]_{[\theta_1,\theta_2;\beta]} \cup \left({N_A}^L{}_{\theta_1} \cap {N_B}^U{}_{\theta_2}\right) \cup \left({N_B}^L{}_{\theta_1} \cap {N_A}^U{}_{\theta_2}\right)\right)$$

iv. $([A;\alpha;\beta] \sqcap [B;\alpha;\beta])_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$

$$= [A;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])} \cap [B;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$$

Proof: $[A;\alpha;\beta], [B;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X) and $([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta]) \in D(I_\alpha) \times D(I_\beta)$ are given arbitrary.

i. $x \in [A;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$

$$\Leftrightarrow [M_A(x);\alpha] \geq [\lambda_1,\lambda_2;\alpha] \text{ and } [N_A(x);\beta] \leq [\theta_1,\theta_2;\beta]$$
$$\Leftrightarrow ([M_A(x);\alpha],[N_A(x);\beta]) \geq ([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$$

ii. $x \in [A;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$ is given arbitrary.

$$([M_A(x);\alpha],[N_A(x);\beta]) \geq ([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$$
$$\Leftrightarrow [M_A(x);\alpha] \geq [\lambda_1,\lambda_2;\alpha] \text{and } [N_A(x);\beta] \leq [\theta_1,\theta_2;\beta]$$
$$\Leftrightarrow x \in [M_A(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \text{ and } x \in [N_A(x);\beta]_{[\theta_1,\theta_2;\beta]}$$
$$\Leftrightarrow x \in [M_A(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cap [N_A(x);\beta]_{[\theta_1,\theta_2;\beta]}$$

iii. $x \in ([A;\alpha;\beta] \sqcup [B;\alpha;\beta])_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$ is given arbitrary.

$$\left([M_{([A;\alpha;\beta]\sqcup[B;\alpha;\beta])}(x);\alpha],[N_{([A;\alpha;\beta]\sqcup[B;\alpha;\beta])}(x);\beta]\right) \geq ([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$$
$$\Leftrightarrow \begin{pmatrix} \begin{bmatrix} \sup\{{M_A}^L(x),{M_B}^L(x)\}, \\ \inf\{{M_A}^U(x),{M_B}^U(x)\};\alpha \end{bmatrix}, \\ \begin{bmatrix} \inf\{{N_A}^L(x),{N_B}^L(x)\}, \\ \sup\{{N_A}^U(x),{N_B}^U(x)\};\beta \end{bmatrix} \end{pmatrix} \geq ([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$$
$$\Leftrightarrow \begin{bmatrix} \sup\{{M_A}^L(x),{M_B}^L(x)\}, \\ \inf\{{M_A}^U(x),{M_B}^U(x)\};\alpha \end{bmatrix} \geq [\lambda_1,\lambda_2;\alpha]$$
$$\text{and } \begin{bmatrix} \inf\{{N_A}^L(x),{N_B}^L(x)\}, \\ \sup\{{N_A}^U(x),{N_B}^U(x)\};\beta \end{bmatrix} \leq [\theta_1,\theta_2;\beta]$$
$$\Leftrightarrow \sup\{{M_A}^L(x),{M_B}^L(x)\} \geq \lambda_1, \inf\{{M_A}^U(x),{M_B}^U(x)\} \leq \lambda_2 \text{ and}$$
$$\inf\{{N_A}^L(x),{N_B}^L(x)\} \leq \theta_1, \sup\{{N_A}^U(x),{N_B}^U(x)\} \geq \theta_2$$
$$\Leftrightarrow \{{M_A}^L(x) \geq \lambda_1 \text{ or } {M_B}^L(x) \geq \lambda_1\} \text{ and } \{{M_A}^U(x) \leq \lambda_2 \text{ or } {M_B}^U(x) \leq \lambda_2\}$$
$$\text{and } \{{N_A}^L(x) \leq \theta_1 \text{ or } {N_B}^L(x) \leq \theta_1\} \text{ and } \{{N_A}^U(x) \geq \theta_2 \text{ or } {N_B}^U(x) \geq \theta_2\}$$
$$\Leftrightarrow \{{M_A}^L(x) \geq \lambda_1 \text{ and } {M_A}^U(x) \leq \lambda_2\} \text{ or } \{{M_B}^L(x) \geq \lambda_1 \text{ and } {M_B}^U(x) \leq \lambda_2\}$$
$$\text{or } \{{M_A}^L(x) \geq \lambda_1 \text{ and } {M_B}^U(x) \leq \lambda_2\} \text{ or } \{{M_B}^L(x) \geq \lambda_1 \text{ and } {M_A}^U(x) \leq \lambda_2\} \text{ and}$$
$$\{{N_A}^L(x) \leq \theta_1 \text{ and } {N_A}^U(x) \geq \theta_2\} \text{ or } \{{N_B}^L(x) \leq \theta_1 \text{ and } {N_B}^U(x) \geq \theta_2\}$$
$$\text{or } \{{N_A}^L(x) \leq \theta_1 \text{ and } {N_B}^U(x) \geq \theta_2\} \text{ or } \{{N_B}^L(x) \leq \theta_1 \text{ and } {N_A}^U(x) \geq \theta_2\}$$
$$\Leftrightarrow x \in [M_A(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \text{ or } x \in [M_B(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \text{ or } \left\{x \in \left({{M_A}^L}_{\lambda_1} \cap {{M_B}^U}_{\lambda_2}\right)\right\}$$
$$\text{or } \left\{x \in \left({{M_B}^L}_{\lambda_1} \cap {{M_A}^U}_{\lambda_2}\right)\right\} \text{ and } x \in [N_A(x);\beta]_{[\theta_1,\theta_2;\beta]} \text{ or } x \in [N_B(x);\beta]_{[\theta_1,\theta_2;\beta]}$$
$$\text{or } \left\{x \in \left({{N_A}^L}_{\theta_1} \cap {{N_B}^U}_{\theta_2}\right)\right\} \text{ or } \left\{x \in \left({{N_B}^L}_{\theta_1} \cap {{N_A}^U}_{\theta_2}\right)\right\}$$
$$\Leftrightarrow x \in \left([M_A(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup [M_B(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup \left({{M_A}^L}_{\lambda_1} \cap {{M_B}^U}_{\lambda_2}\right)\right) \cup \left({{M_B}^L}_{\lambda_1} \cap {{M_A}^U}_{\lambda_2}\right) \cap$$
$$\left([N_A(x);\beta]_{[\theta_1,\theta_2;\beta]} \cup [N_B(x);\beta]_{[\theta_1,\theta_2;\beta]} \cup \left({{N_A}^L}_{\theta_1} \cap {{N_B}^U}_{\theta_2}\right) \cup \left({{N_B}^L}_{\theta_1} \cap {{N_A}^U}_{\theta_2}\right)\right)$$

iv. $x \in ([A;\alpha;\beta] \sqcap [B;\alpha;\beta])_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$ is given arbitrary.

$$\left(M_{([A;\alpha;\beta]\sqcap[B;\alpha;\beta])}(x),N_{([A;\alpha;\beta]\sqcap[B;\alpha;\beta])}(x)\right) \geq ([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$$

$$\Leftrightarrow \begin{pmatrix} \begin{bmatrix} \inf\{{M_A}^L(x), {M_B}^L(x)\}, \\ \sup\{{M_A}^U(x), {M_B}^U(x)\}; \alpha \end{bmatrix}, \\ \begin{bmatrix} \sup\{{N_A}^L(x), {N_B}^L(x)\}, \\ \inf\{{N_A}^U(x), {N_B}^U(x)\}; \beta \end{bmatrix} \end{pmatrix} \geq ([\lambda_1, \lambda_2; \alpha], [\theta_1, \theta_2; \beta])$$

$$\Leftrightarrow \begin{bmatrix} \inf\{{M_A}^L(x), {M_B}^L(x)\}, \\ \sup\{{M_A}^U(x), {M_B}^U(x)\}; \alpha \end{bmatrix} \geq [\lambda_1, \lambda_2; \alpha]$$

$$\text{and } \begin{bmatrix} \sup\{{N_A}^L(x), {N_B}^L(x)\}, \\ \inf\{{N_A}^U(x), {N_B}^U(x)\}; \beta \end{bmatrix} \leq [\theta_1, \theta_2; \beta]$$

$$\Leftrightarrow \inf\{{M_A}^L(x), {M_B}^L(x)\} \geq \lambda_1 \text{ and } \sup\{{M_A}^U(x), {M_B}^U(x)\} \leq \lambda_2$$
$$\text{and } \sup\{{N_A}^L(x), {N_B}^L(x)\} \leq \theta_1 \text{ and } \inf\{{N_A}^U(x), {N_B}^U(x)\} \geq \theta_2$$
$$\Leftrightarrow \{{M_A}^L(x) \geq \lambda_1 \text{ and } {M_B}^L(x) \geq \lambda_1\} \text{ and } \{{M_A}^U(x) \leq \lambda_2 \text{ and } {M_B}^U(x) \leq \lambda_2\}$$
$$\text{and } \{{N_A}^L(x) \leq \theta_1 \text{ and } {N_B}^L(x) \leq \theta_1\} \text{ and } \{{N_A}^U(x) \geq \theta_2 \text{ and } {N_B}^U(x) \geq \theta_2\}$$
$$\Leftrightarrow \{{M_A}^L(x) \geq \lambda_1 \text{ and } {M_A}^U(x) \leq \lambda_2\} \text{ and } \{{M_B}^L(x) \geq \lambda_1 \text{ and } {M_B}^U(x) \leq \lambda_2\}$$
$$\text{and } \{{N_A}^L(x) \leq \theta_1 \text{ and } {N_A}^U(x) \geq \theta_2\} \text{ and } \{{N_B}^L(x) \leq \theta_1 \text{ and } {N_B}^U(x) \geq \theta_2\}$$
$$\Leftrightarrow [M_A(x); \alpha] \geq [\lambda_1, \lambda_2; \alpha] \text{ and } [M_B(x); \alpha] \geq [\lambda_1, \lambda_2; \alpha]$$
$$\text{and } [N_A(x); \beta] \leq [\theta_1, \theta_2; \beta] \text{ and } [N_B(x); \beta] \leq [\theta_1, \theta_2; \beta]$$
$$\Leftrightarrow \{x \in [M_A(x); \alpha]_{[\lambda_1,\lambda_2;\alpha]} \text{ and } x \in [N_A(x); \beta]_{[\theta_1,\theta_2;\beta]}\}$$
$$\text{and } \{x \in [M_B(x); \alpha]_{[\lambda_1,\lambda_2;\alpha]} \text{ and } x \in [N_B(x); \beta]_{[\theta_1,\theta_2;\beta]}\}$$
$$\Leftrightarrow x \in [A; \alpha; \beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])} \cap [B; \alpha; \beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$$

Example 3: Let $X = \{a, b, c, d\}$.

$$A = \begin{Bmatrix} < a, [0.1, 0.6; 0.4], [0.1, 0.4; 0.3] >, < b, [0.2, 0.5; 0.4], [0.2, 0.4; 0.3] >, \\ < c, [0.3, 0.6; 0.4], [0.1, 0.3; 0.3] >, < d, [0.4, 0.6; 0.4], [0.1, 0.3; 0.3] > \end{Bmatrix},$$

For $\alpha = 0.4$ and $\beta = 0.3$, $[A; \alpha; \beta]$ is (α, β)-interval valued intuitionistic fuzzy set.

i. $([0.0, 0.5; 0.4], [0.2, 0.4; 0.3]) \in D(I_\alpha) \times D(I_\beta)$,
$$A_{([0.0,0.5;0.4],[0.2,0.4;0.3])} = \{b\}$$
ii. $([0.3, 0.6; 0.4], [0.1, 0.3; 0.3]) \in D(I_\alpha) \times D(I_\beta)$,
$$A_{([0.3,0.6;0.4],[0.1,0.3;0.3])} = \{c, d\}$$
iii. $([0.2, 0.7; 0.4], [0.2, 0.3; 0.3]) \in D(I_\alpha) \times D(I_\beta)$,
$$A_{([0.2,0.7;0.4],[0.2,0.3;0.3])} = \{b, c, d\}$$
iv. $([0.0, 0.7; 0.4], [0.3, 0.3; 0.3]) \in D(I_\alpha) \times D(I_\beta)$,
$$A_{([0.0,0.7;0.4],[0.3,0.3;0.3])} = \{a, b, c, d\} = X$$
v. $([0.1, 0.4; 0.4], [0.0, 0.3; 0.3]) \in D(I_\alpha) \times D(I_\beta)$, $A_{([0.1,0.4;0.4],[0.0,0.3;0.3])} = \emptyset$

4. CONCLUSION

In this study, the definition of (α, β)-interval set is given. It is shown that (α, β)-interval set is lattice by giving of definitions of order relation, infimum and supremum on this set. Afterwards, the definition of negation function on this set is given by the help of negation function on crisp sets and fuzzy sets.

In terms of above definitions and information, the definition of (α, β)-interval valued intuitionistic fuzzy set is introduced. The definitions of intersection, union and complement on this set are introduced and the fundamental algebraic properties of this set are studied. In addition, the level subset of (α, β)-interval valued intuitionistic fuzzy set is given.

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(Arif Bal) DEPARTMENT OF MOTOR VEHICLES AND TRANSPORTATION TECHNOLOGIES, VOCATIONAL SCHOOL OF TECHNICAL SCIENCES, MERSIN UNIVERSITY, MERSIN, TURKIYE
Email address: arif.bal.math@gmail.com

(Gökhan Çuvalcıoğlu) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, MERSIN UNIVERSITY, MERSIN, TURKIYE
Email address: gcuvalcioglu@gmail.com

(Cansu Altıncı) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, MERSIN UNIVERSITY, MERSIN, TURKIYE
Email address: cansu.altinci01@gmail.com