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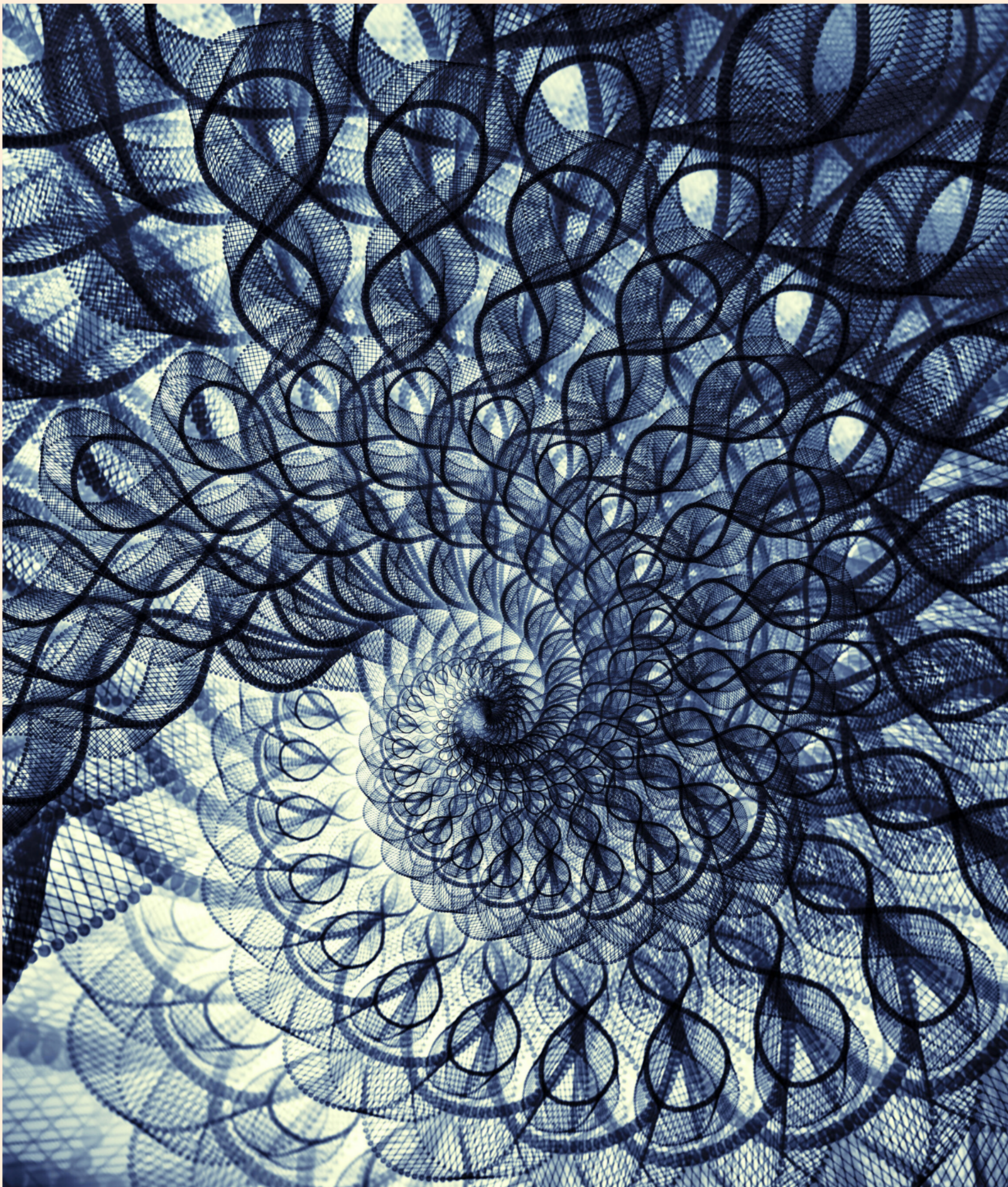

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Generalized Turán-type Inequalities for Polar Derivative of a Polynomial

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Abstract
Let $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, be a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$. We obtain an improvement and a generalization of an inequality in polar derivative proved by Somsuwan and Nakprasit [1]. Further, we also extend a result proved by Chanam and Dewan [2] to its polar version. Besides, our results are also found to generalize and improve some known inequalities.

Keywords: *Turán-type inequality; polynomial; polar derivative; maximum modulus*

AMS Subject Classification (2020): *Primary: 30C15: 30C10 ; Secondary: 30A10.*

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1. Introduction and statement of results

The study of geometric relationship between the maximum moduli of a complex polynomial and its derivative on the same circle or different circles by taking into account the position of zeros of the polynomial inside or outside the same or a different circle has been drawing great interest among researchers for many decades. One of the pioneering works in this area is due to S. Bernstein.

If $P(z)$ is a polynomial of degree n, Bernstein [3] proved

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The above inequality is the famous Bernstein's Inequality. Equality holds in (1.1) if all zeros of $P(z)$ are found at the origin.

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If we restrict ourselves to the class of polynomials $P(z)$ of degree n having no zero in $|z| < 1$, then inequality (1.1) can be refined and substituted by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

Inequality (1.2) was conjectured by Erdös and later proved by Lax [4]. The result is sharp and equality holds for the polynomial $P(z) = \lambda + \mu z^n$, where $|\lambda| = |\mu|$.

On the other hand, if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, Turán [5] proved

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Inequality (1.3), which is often referred to as Turán's Inequality, is best possible and equality occurs if $P(z)$ has all its zeros on $|z| = 1$.

It was asked by Professor R.P. Boas that if $P(z)$ is a polynomial of degree n not vanishing in $|z| < k$, $k > 0$, then how large can $\left\{ \max_{|z|=1} |P'(z)| / \max_{|z|=1} |P(z)| \right\}$ be. A partial answer to this problem was given by Malik [6], who proved that if $P(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.4}$$

The result is sharp and equality is attained for $P(z) = (z + k)^n$. Whereas, for the polynomial $P(z)$ having all its zeros in $|z| \leq k$, $k \leq 1$, by applying the above inequality (1.4) to the polynomial $q(z)$, where $q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)}$, Malik [6] further obtained a generalization of (1.3) as

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.5}$$

Inequality (1.5) is sharp and equality holds for $P(z) = (z + k)^n$.

In inequalities (1.2) and (1.3) the boundaries of zero-free regions and the circle on which the estimates of $P(z)$ and its derivative are compared is the unit circle, which is not the case in inequalities (1.4) and (1.5) where the two circles are not same. It is of interest to obtain generalization of the above inequalities by considering the maximum moduli of the polynomial and its derivative on different circles other than the unit circle. In this direction the following result was proved by Aziz and Zargar [7].

Theorem 1.1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then for real numbers r and R such that $rR \geq k^2$ and $r \leq R$,*

$$\max_{|z|=R} |P'(z)| \geq \frac{n\,(R+k)^{n-1}}{(r+k)^n} \left\{ \max_{|z|=r} |P(z)| + \min_{|z|=k} |P(z)| \right\}. \tag{1.6}$$

Equality in (1.6) holds for the polynomial $P(z) = (z + k)^n$.

Chanam and Dewan [2] generalized and improved Theorem 1.1 by involving certain coefficients of $P(z)$. They proved

Theorem 1.2. *Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $a_0 \neq 0$ and $1 \leq \mu < n$, be a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq k$, where $k > 0$, then for $rR \geq k^2$ and $r \leq R$,*

$$\max_{|z|=R} |P'(z)| \geq n \left\{ \frac{n|a_n|R^\mu k^{\mu-1} + \mu|a_{n-\mu}|R^{\mu-1}}{n|a_n|R^{\mu+1}k^{\mu-1} + n|a_n|k^{2\mu} + \mu|a_{n-\mu}|\left(Rk^{\mu-1} + R^\mu\right)} \right\} \times \left(\frac{R+k}{r+k}\right)^n \left\{ \max_{|z|=r} |P(z)| + \min_{|z|=k} |P(z)| \right\}. \tag{1.7}$$

Let $P(z)$ be a polynomial of degree n and let α be any complex number. Then, the polar derivative of $P(z)$ with respect to α, denoted by $D_\alpha P(z)$, is defined as

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z). \tag{1.8}$$

The polynomial $D_\alpha P(z)$ is of degree at most $(n-1)$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

The following result, proved by Aziz and Rather [8], generalizes and extends Turán's inequality (1.3) to its polar version.

Theorem 1.3. *Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every real or complex number α with $|\alpha| \ge k$,*

$$\max_{|z|=1} |D_\alpha P(z)| \ge \frac{n(|\alpha|-k)}{1+k^n} \max_{|z|=1} |P(z)|. \tag{1.9}$$

Further, Dewan and Upadhye [9] improved Theorem 1.3 by involving $\min\limits_{|z|=k} |P(z)|$. They proved

Theorem 1.4. *Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every real or complex number α with $|\alpha| \ge k$,*

$$\max_{|z|=1} |D_\alpha P(z)| \ge n(|\alpha|-k)\left\{\frac{1}{1+k^n}\max_{|z|=1}|P(z)| + \frac{1}{2k^n}\left(\frac{k^n-1}{k^n+1}\right)\min_{|z|=k}|P(z)|\right\}. \tag{1.10}$$

Nakprasit and Somsuwan [1] generalized Theorem 1.4 by proving the following result.

Theorem 1.5. *Let $P(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every real or complex number α with $|\alpha| \ge k$ and $1 \le R \le k$,*

$$\begin{aligned}\max_{|z|=R} |D_\alpha P(z)| \ge nR^{n-1}(|\alpha|-k)\Bigg[&\frac{R^n}{R^n+k^n}\left(\frac{k^\mu+1}{k^\mu+R^\mu}\right)^{\frac{n}{\mu}}\max_{|z|=1}|P(z)| \\ &+ \left\{\frac{k^n}{R^n+k^n}\left(1-\left(\frac{k^\mu+1}{k^\mu+R^\mu}\right)^{\frac{n}{\mu}}\right) + \frac{1}{2k^n}\left(\frac{k^n-R^n}{k^n+R^n}\right)\right\}\min_{|z|=k}|P(z)|\Bigg].\end{aligned} \tag{1.11}$$

In this paper, we first obtain an improvement and a generalization of Theorem 1.5. Theorem 1.5 is generalized in the sense that inequality (1.11) is extended to circles with smaller radii, viz., for $0 < r \le 1$ when the estimate of $\max |D_\alpha P(z)|$ is considered. More precisely, we prove

Theorem 1.6. *Let $P(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every real or complex number α with $|\alpha| \ge k$ and $0 < r \le 1 \le R \le k$,*

$$\begin{aligned}\max_{|z|=R} |D_\alpha P(z)| \ge nR^{n-1}(|\alpha|-k)\Bigg[&\frac{R^n}{k^n+R^n} B \max_{|z|=r}|P(z)| \\ &+ \left\{\frac{k^n}{k^n+R^n}(1-B) + \frac{1}{2k^n}\left(\frac{k^n-R^n}{k^n+R^n}\right)\right\}\min_{|z|=k}|P(z)|\Bigg],\end{aligned} \tag{1.12}$$

where

$$B = \exp\left\{-n\int_r^R \frac{\dfrac{\mu}{n}\dfrac{\frac{|a_{n-\mu}|}{k^{2\mu}}}{|a_n|-\frac{m}{k^n}}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1}+k^{\mu+1}+\dfrac{\mu}{n}\dfrac{\frac{|a_{n-\mu}|}{k^{2\mu}}}{|a_n|-\frac{m}{k^n}}\left(k^{\mu+1}t^\mu + k^{2\mu}t\right)}dt\right\} \tag{1.13}$$

and $m = \min\limits_{|z|=k} |P(z)|$.

The following result is obtained by taking $r = 1$ in Theorem 1.6.

Corollary 1.1. *Let $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every real or complex number α with $|\alpha| \ge k$ and $1 \le R \le k$,*

$$\begin{aligned}\max_{|z|=R} |D_\alpha P(z)| \ge nR^{n-1}\left(|\alpha| - k\right) \Bigg[& \frac{R^n}{k^n + R^n} B_1 \max_{|z|=1} |P(z)| \\ & + \left\{ \frac{k^n}{k^n + R^n}(1 - B_1) + \frac{1}{2k^n}\left(\frac{k^n - R^n}{k^n + R^n}\right)\right\} \min_{|z|=k} |P(z)| \Bigg],\end{aligned} \tag{1.14}$$

where

$$B_1 = \left\{ -n \int_1^R \frac{\dfrac{\mu}{n} \dfrac{\frac{|a_{n-\mu}|}{k^{2\mu}}}{|a_n| - \frac{m}{k^n}} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \dfrac{\mu}{n} \dfrac{\frac{|a_{n-\mu}|}{k^{2\mu}}}{|a_n| - \frac{m}{k^n}} \left(k^{\mu+1} t^\mu + k^{2\mu} t\right)} dt \right\} \tag{1.15}$$

and $m = \min_{|z|=k} |P(z)|$.

Remark 1.1. Corollary 1.1 is an improvement of Theorem 1.5. It is sufficient to show that the bound given by inequality (1.14) is bigger than the bound given by inequality (1.11) concerning the estimate of $\max |D_\alpha P(z)|$, i.e.,

$$\begin{aligned}&\left[\frac{R^n}{k^n + R^n} B_1 \max_{|z|=1} |P(z)| + \left\{ \frac{k^n}{k^n + R^n}(1 - B_1) + \frac{1}{2k^n}\left(\frac{k^n - R^n}{k^n + R^n}\right)\right\} \min_{|z|=k} |P(z)| \right] \\ &\ge \left[\frac{R^n}{R^n + k^n} \left(\frac{k^\mu + 1}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}} \max_{|z|=1} |P(z)| + \left\{ \frac{k^n}{R^n + k^n}\left(1 - \left(\frac{k^\mu + 1}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}}\right) + \frac{1}{2k^n}\left(\frac{k^n - R^n}{k^n + R^n}\right)\right\} \min_{|z|=k} |P(z)| \right].\end{aligned}$$

From (2.1) of Lemma 2.2, we have

$$\max_{|z|=1} |P(z)| \ge k^n \min_{|z|=k} |P(z)|.$$

Since $R \ge 1$, it follows that

$$R^n \max_{|z|=1} |P(z)| - k^n \min_{|z|=k} |P(z)| \ge 0. \tag{1.16}$$

Putting $r = 1$ in (2.9) of Lemma 2.5, we get

$$B_1 \ge \left(\frac{k^\mu + 1}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}}, \tag{1.17}$$

where B_1 is given by (1.15).
In view of inequality (1.17), it is sufficient to show that the function f such that

$$f(x) = R^n \max_{|z|=1} |P(z)| x + k^n (1 - x) \min_{|z|=k} |P(z)|$$

is a non-decreasing function of x. Now as

$$\begin{aligned} f'(x) &= R^n \max_{|z|=1} |P(z)| - k^n \min_{|z|=k} |P(z)| \\ &\ge 0 \quad (by \ (1.16)), \end{aligned}$$

f is a non-decreasing function of x, which proves our claim.

Further, for $r = R = 1$ in Theorem 1.6, we have the following result.

Corollary 1.2. *Let $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every real or complex number α with $|\alpha| \ge k$,*

$$\max_{|z|=1} |D_\alpha P(z)| \ge n(|\alpha| - k) \left\{ \frac{1}{k^n+1} \max_{|z|=1} |P(z)| + \frac{1}{2k^n} \left(\frac{k^n-1}{k^n+1} \right) \min_{|z|=k} |P(z)| \right\}. \tag{1.18}$$

Remark 1.2. It is clear from Corollary 1.1 that Theorem 1.6 is a generalization of Theorem 1.4, as taking $\mu = 1$ along with $r = R = 1$, inequality (1.12) of Theorem 1.6 reduces to (1.10) of Theorem 1.4.

Dividing both sides of (1.12) by $|\alpha|$ and letting $|\alpha| \to \infty$ and putting $R = k$, we have the next result.

Corollary 1.3. *If $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for $0 < r \le k$,*

$$\max_{|z|=k} |P'(z)| \ge \frac{nk^{n-1}}{2} \left\{ B \max_{|z|=r} |P(z)| + (1-B) \min_{|z|=k} |P(z)| \right\}, \tag{1.19}$$

where B is given by (1.13).

Remark 1.3. In particular, if we let $r = k = 1$ and $\mu = 1$, (1.19) reduces to Turán's inequality (1.3).

Next, we extend Theorem 1.2 due to Chanam and Dewan [2] to its polar version in which the assumption $a_0 \neq 0$ in the constant term of the polynomial $P(z)$ is also dropped. The result also improves as well as generalizes other well known inequalities.

Theorem 1.7. *Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in $|z| \le k$, $k > 0$, then for every real or complex number α with $\frac{|\alpha|}{R} \ge A_{\mu,n}$ and for $rR \ge k^2$ and $r \le R$,*

$$\max_{|z|=R} |D_\alpha P(z)| + mn \ge \frac{n}{1 + A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n} \right) \left(\frac{R+k}{r+k} \right)^n \left[\max_{|z|=r} |P(z)| + \min_{|z|=k} |P(z)| \right], \tag{1.20}$$

where $m = \min_{|z|=k} |P(z)|$ *and*

$$A_{\mu,n} = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|Rk^{\mu-1}}{\mu|a_{n-\mu}|R^\mu + n|a_n|R^{\mu+1}k^{\mu-1}}. \tag{1.21}$$

Dividing on both sides of (1.20) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following result.

Corollary 1.4. *Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in $|z| \le k$, $k > 0$, then for $rR \ge k^2$ and $r \le R$,*

$$\max_{|z|=R} |P'(z)| \ge \frac{n}{R(1 + A_{\mu,n})} \left(\frac{R+k}{r+k} \right)^n \left[\max_{|z|=r} |P(z)| + \min_{|z|=k} |P(z)| \right], \tag{1.22}$$

where $A_{\mu,n}$ is given by (1.21).

Remark 1.4. In view of Corollary 1.4, Theorem 1.7 is the polar derivative version of Theorem 1.2 in a richer form for restrictions concerning the polynomial $P(z)$, namely $a_0 \neq 0$, $\mu \neq n$ and $n \neq 1$ in the hypotheses of Theorem 1.2 have all been dropped in Theorem 1.7 and hence consequently in Corollary 1.4. In other words, Corollary 1.4 is a better version of Theorem 1.2.

Further, taking $k = R = r = 1$ in Corollary 1.4, we have the following result.

Corollary 1.5. *Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in $|z| \le 1$, then*

$$\max_{|z|=R} |P'(z)| \ge \frac{n}{2} \left\{ \max_{|z|=r} |P(z)| + \min_{|z|=k} |P(z)| \right\}. \tag{1.23}$$

Inequality (1.23) verifies that Corollary 1.4 is a generalization as well as an improvement of inequality (1.3) due to Turán [5].

Remark 1.5. Corollary 1.4 is also an improvement and a generalization of Theorem 1.1 as explained by Chanam and Dewan [2, Remark 2].

2. Lemmas

We need the following lemmas to prove our theorems.

The following lemma was proved by Gardner et al.[10].

Lemma 2.1. *If $P(z)=\sum\limits_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n, $P(z)\neq 0$ in $|z|<k$, $k>0$, then $|P(z)|\geq m$ for $|z|\leq k$, where $m=\min\limits_{|z|=k}|P(z)|$.*

Lemma 2.2. *If $P(z)=a_nz^n+\sum\limits_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu}$, $1\leq\mu\leq n$, is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\geq 1$, then*

$$\max_{|z|=1}|P(z)|\geq k^n\min_{|z|=k}|P(z)|. \tag{2.1}$$

Proof. Let $q(z)=z^n\overline{P\left(\frac{1}{\overline{z}}\right)}=\overline{a}_0z^n+\sum\limits_{\nu=\mu}^{n}\overline{a}_\nu z^{n-\nu}$, $1\leq\mu\leq n$. Since $P(z)$ has all its zeros in $|z|\leq k$, $k\geq 1$, therefore $q(z)$ has no zero in $|z|<\frac{1}{k}$, $\frac{1}{k}\leq 1$. Let $Q(z)=q\left(\frac{z}{k^2}\right)=\frac{\overline{a}_0}{k^{2n}}z^n+\sum\limits_{\nu=\mu}^{n}\frac{\overline{a}_\nu}{k^{2(n-\nu)}}z^{n-\nu}=\overline{a}_n+\sum\limits_{\nu=\mu}^{n}\frac{\overline{a}_{n-\nu}}{k^{2\nu}}z^\nu$, then $Q(z)\neq 0$ in $|z|<k$, $k\geq 1$.
Therefore, by applying Lemma 2.1 to $Q(z)$, we get for $|z|=k$

$$\begin{aligned}|Q(z)|&\geq\min_{|z|=k}|Q(z)|\\&=\min_{|z|=k}\left|q\left(\frac{z}{k^2}\right)\right|\\&=\min_{|z|=k}\left|\left(\frac{z}{k^2}\right)^n\overline{P\left(\frac{1}{\overline{z}/k^2}\right)}\right|\\&=\frac{1}{k^n}\min_{|z|=k}\left|P\left(\frac{k^2}{\overline{z}}\right)\right|\\&=\frac{1}{k^n}\min_{|z|=k}|P(z)|.\end{aligned} \tag{2.2}$$

Now as $1\leq k$ and hence in particular inequality (2.2) gives for $|z|=1$

$$|Q(z)|\geq\frac{1}{k^n}\min_{|z|=k}|P(z)|,\qquad\text{from which it is implied that}$$

$$\max_{|z|=1}|Q(z)|\geq\frac{1}{k^n}\min_{|z|=k}|P(z)|,\qquad\text{which is equivalent to}$$

$$\max_{|z|=1}\left|q\left(\frac{z}{k^2}\right)\right|\geq\frac{1}{k^n}\min_{|z|=k}|P(z)|,$$

which implies

$$\max_{|z|=1}\left|\left(\frac{z}{k^2}\right)^n\overline{P\left(\frac{1}{\overline{z}/k^2}\right)}\right|\geq\frac{1}{k^n}\min_{|z|=k}|P(z)|,$$

which is equivalent to

$$\frac{1}{k^{2n}}\max_{|z|=1}\left|P\left(\frac{k^2}{\overline{z}}\right)\right|\geq\frac{1}{k^n}\min_{|z|=k}|P(z)|. \tag{2.3}$$

Since $k \geq 1$, it is obvious that $k^2 \geq k \geq 1$ and hence by Maximum Modulus Principle [11]

$$\max_{|z|=k^2}\left|P\left(\frac{k^2}{\overline{z}}\right)\right| \geq \max_{|z|=1}\left|P\left(\frac{k^2}{\overline{z}}\right)\right|, \quad \text{which is equivalent to}$$
$$\frac{1}{(k^2)^n}\max_{|z|=k^2}\left|P\left(\frac{k^2}{\overline{z}}\right)\right| \geq \frac{1}{(k^2)^n}\max_{|z|=1}\left|P\left(\frac{k^2}{\overline{z}}\right)\right|, \quad \text{which simplifies to}$$
$$\frac{1}{(k^n)^2}\max_{|z|=1}|P(z)| \geq \frac{1}{k^{2n}}\max_{|z|=1}\left|P\left(\frac{k^2}{\overline{z}}\right)\right|. \tag{2.4}$$

Combining (2.3) and (2.4) we get

$$\frac{1}{(k^n)^2}\max_{|z|=1}|P(z)| \geq \frac{1}{k^n}\min_{|z|=k}|P(z)|. \tag{2.5}$$

Hence,

$$\max_{|z|=1}|P(z)| \geq k^n \min_{|z|=k}|P(z)|.$$

□

Lemma 2.3. *If $P(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$,*

$$\max_{|z|=r}|P(z)| \geq B' \max_{|z|=R}|P(z)| + (1 - B')\min_{|z|=k}|P(z)|, \tag{2.6}$$

where

$$B' = \exp\left\{-n\int_r^R \frac{\dfrac{\mu}{n}\dfrac{|a_\mu|}{|a_0| - m}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \dfrac{\mu}{n}\dfrac{|a_\mu|}{|a_0| - m}\left(k^{\mu+1}t^\mu + k^{2\mu}t\right)}dt\right\} \tag{2.7}$$

and $m = \min\limits_{|z|=k}|P(z)|$. Equality holds in (2.6) for $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

Lemma 2.4. *If $P(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\mu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, where $k > 0$, then for $0 < r \leq R \leq k$,*

$$B' \geq \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}}, \tag{2.8}$$

where B' is given by (2.7).

Lemma 2.3 and Lemma 2.4 are due to Chanam and Dewan [12].

Lemma 2.5. *Let $P(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$,*

$$B \geq \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}}, \tag{2.9}$$

where B is given by (1.13).

Proof. Let $q(z) = z^n\overline{P\left(\frac{1}{\overline{z}}\right)}$ and $Q(z) = q\left(\frac{z}{k^2}\right)$. Then, $Q(z) = \frac{\overline{a}_0}{k^{2n}}z^n + \sum_{\nu=\mu}^{n}\frac{\overline{a}_\nu}{k^{2(n-\nu)}}z^{n-\nu} = \overline{a}_n + \sum_{\nu=\mu}^{n}\frac{\overline{a}_{n-\nu}}{k^{2\nu}}z^\nu$, where $1 \le \mu \le n$. Since $P(z) \ne 0$ in $|z| < k$, $k > 0$, we have $Q(z) \ne 0$ in $|z| < k$, $k > 0$. Hence, applying Lemma 2.4 to $Q(z)$, we get

$$B \ge \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}},$$

where B is given by (1.13). □

The next lemma is due to Qazi [13, Proof and Remark of Lemma 1].

Lemma 2.6. *If $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, then*

$$|q'(z)| \ge k^{\mu+1}\frac{\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu-1} + 1}{1 + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}|P'(z)| \quad \text{on} \quad |z| = 1 \tag{2.10}$$

and

$$\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^\mu \le 1, \tag{2.11}$$

where $q(z) = z^n\overline{P\left(\frac{1}{\overline{z}}\right)}$.

Lemma 2.7. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}$, $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$, then for every real or complex number β with $|\beta| \ge A$,*

$$\max_{|z|=1}|D_\beta P(z)| \ge \frac{n}{1+A}\left(|\beta| - A\right)\max_{|z|=1}|P(z)|, \tag{2.12}$$

where

$$A = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{\mu|a_{n-\mu}| + n|a_n|k^{\mu-1}}.$$

Inequality (2.12) is best possible for $\mu = 1$ and equality occurs for $P(z) = (z-k)^n$ with $|\beta| \ge A = k$.

Proof. Let $q(z) = z^n\overline{P\left(\frac{1}{\overline{z}}\right)}$. Then it can be easily verified that

$$|q'(z)| = |nP(z) - zP'(z)|, \quad \text{for} \quad |z| = 1. \tag{2.13}$$

Since the polynomial $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}$, $1 \le \mu \le n$, has all its zeros in $|z| \le k$, $k \le 1$, $q(z) = \overline{a}_n + \sum_{\nu=\mu}^{n}\overline{a}_{n-\nu}z^\nu$ has no zero in $|z| < \frac{1}{k}$, $\frac{1}{k} \ge 1$, therefore, by applying Lemma 2.6 to $q(z)$, we have from (2.10)

$$\begin{aligned}|P'(z)| &\ge \frac{1}{k^{\mu+1}}\left(\frac{\frac{\mu}{n}\frac{|a_{n-\mu}|}{|a_n|}\frac{1}{k^{\mu-1}} + 1}{1 + \frac{\mu}{n}\frac{|a_{n-\mu}|}{|a_n|}\frac{1}{k^{\mu+1}}}\right)|q'(z)| \\ &= \frac{\mu|a_{n-\mu}| + n|a_n|k^{\mu-1}}{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}|q'(z)|.\end{aligned}$$

Therefore,

$$|q'(z)| \leq \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{\mu|a_{n-\mu}| + n|a_n|k^{\mu-1}}|P'(z)| \\ = A|P'(z)|. \tag{2.14}$$

From (2.14), we have

$$|P'(z)| + |q'(z)| \leq (1+A)|q'(z)|. \tag{2.15}$$

Also, for $|z|=1$, with the help of (2.13), we have

$$\begin{aligned} n|P(z)| &= |nP(z) - zP'(z) + zP'(z)| \\ &\leq |nP(z) - zP'(z)| + |P'(z)| \\ &= |q'(z)| + |P'(z)|. \end{aligned} \tag{2.16}$$

Combining (2.15) and (2.16), we get

$$n|P(z)| \leq (1+A)|P'(z)|$$

i.e.,

$$|P'(z)| \geq \frac{n}{1+A}|P(z)|, \quad \text{for} \quad |z|=1. \tag{2.17}$$

For every real or complex number β, by definition, we have

$$D_\beta P(z) = nP(z) + (\beta - z)P'(z),$$

from which for $|z|=1$, we have

$$\begin{aligned} |D_\beta P(z)| &\geq ||\beta||P'(z)| - |nP(z) - zP'(z)|| \\ &= ||\beta||P'(z)| - |q'(z)|| \quad \text{(by (2.13))}. \end{aligned} \tag{2.18}$$

Further, by (2.14)

$$\begin{aligned} |\beta||P'(z)| - |q'(z)| &\geq |\beta||P'(z)| - A|P'(z)| \\ &= (|\beta| - A)|P'(z)|, \end{aligned} \tag{2.19}$$

which is non-negative, since $|\beta| \geq A$.
Combining (2.18) and (2.19), we get

$$|D_\beta P(z)| \geq (|\beta| - A)\,|P'(z)|,$$

which on using (2.17), gives

$$|D_\beta P(z)| \geq (|\beta| - A)\,\frac{n}{1+A}|P(z)|.$$

□

The following lemma is due to Aziz and Zargar [7].

Lemma 2.8. *If $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k>0$, then for $rR \geq k^2$ and $r \leq R$, we have for $|z|=1$,*

$$|P(Rz)| \geq \left(\frac{R+k}{r+k}\right)^n |P(rz)|. \tag{2.20}$$

Equality holds in (2.20) for $P(z) = (z+k)^n$.

3. Proof of the theorems

Proof of Theorem 1.6. Let $F(z) = P(Rz)$. Then $F(z)$ has all its zeros in the closed disk $|z| \leq \frac{k}{R}$, $\frac{k}{R} \geq 1$. Applying Theorem 1.5 to $F(z)$, we have

$$\max_{|z|=1} |D_{\alpha/R}F(z)| \geq n\left(\frac{|\alpha|-k}{R}\right)\left[\frac{1}{1+\frac{k^n}{R^n}}\max_{|z|=1}|F(z)| + \frac{1}{2\frac{k^n}{R^n}}\left(\frac{\frac{k^n}{R^n}-1}{\frac{k^n}{R^n}+1}\right)\min_{|z|=\frac{k}{R}}|F(z)|\right]$$
$$= n\left(|\alpha|-k\right)R^{n-1}\left[\frac{1}{k^n+R^n}\max_{|z|=1}|F(z)| + \frac{1}{2k^n}\left(\frac{k^n-R^n}{k^n+R^n}\right)\min_{|z|=\frac{k}{R}}|F(z)|\right]. \tag{3.1}$$

Using the relations,

$$\max_{|z|=1}|D_{\alpha/R}F(z)| = \max_{|z|=R}|D_\alpha P(z)|,$$
$$\max_{|z|=1}|F(z)| = \max_{|z|=R}|P(z)|$$
$$\text{and} \quad \min_{|z|=\frac{k}{R}}|F(z)| = \min_{|z|=k}|P(z)|$$

in inequality (3.1), we get

$$\max_{|z|=R}|D_\alpha P(z)| \geq nR^{n-1}\left(|\alpha|-k\right)\left[\frac{1}{k^n+R^n}\max_{|z|=R}|P(z)| + \frac{1}{2k^n}\left(\frac{k^n-R^n}{k^n+R^n}\right)\min_{|z|=k}|P(z)|\right]. \tag{3.2}$$

Let $q(z) = z^n\overline{P\left(\frac{1}{\overline{z}}\right)}$ and $Q(z) = q\left(\frac{z}{k^2}\right)$, then

$$Q(z) = \frac{z^n}{k^{2n}}\overline{P\left(\frac{k^2}{\overline{z}}\right)}. \tag{3.3}$$

Therefore, $q(z)$ has no zero in $|z| < \frac{1}{k}$ and $Q(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$. Thus, applying Lemma 2.3 to $Q(z)$, we have

$$\max_{|z|=r}|Q(z)| \geq B\max_{|z|=R}|Q(z)| + (1-B)\min_{|z|=k}|Q(z)|, \tag{3.4}$$

where B is given by (1.13).
From (3.3) we have for $r > 0$

$$\max_{|z|=r}|Q(z)| = \frac{r^n}{k^{2n}}\max_{|z|=r}\left|P\left(\frac{k^2}{\overline{z}}\right)\right|. \tag{3.5}$$

Since $0 < r \leq 1 \leq R \leq k$, we have by Maximum Modulus Principle [11],

$$\max_{|z|=k^2}\left|P\left(\frac{k^2}{\overline{z}}\right)\right| \geq \max_{|z|=1}\left|P\left(\frac{k^2}{\overline{z}}\right)\right| \quad (\because\ k^2 \geq 1)$$
$$i.e.\quad \frac{1}{k^{2n}}\max_{|z|=k^2}\left|P\left(\frac{k^2}{\overline{z}}\right)\right| \geq \frac{1}{k^{2n}}\max_{|z|=1}\left|P\left(\frac{k^2}{\overline{z}}\right)\right|$$
$$i.e.\quad \frac{1}{k^{2n}}\max_{|z|=1}|P(z)| \geq \frac{1}{k^{2n}}\max_{|z|=1}\left|P\left(\frac{k^2}{\overline{z}}\right)\right|$$
$$\geq \frac{r^n}{k^{2n}}\max_{|z|=1}\left|P\left(\frac{k^2}{\overline{z}}\right)\right|$$
$$\geq \frac{r^n}{k^{2n}}\max_{|z|=r}\left|P\left(\frac{k^2}{\overline{z}}\right)\right|$$
$$= \max_{|z|=r}|Q(z)| \quad (by\ (3.5)).$$

Hence

$$\max_{|z|=r}|Q(z)| \leq \frac{1}{k^{2n}}\max_{|z|=1}|P(z)| \\ \leq \frac{1}{k^{2n}}\max_{|z|=R}|P(z)| \quad (\because R \geq 1). \tag{3.6}$$

Again from (3.3) we have

$$\begin{aligned} \max_{|z|=R}|Q(z)| &= \max_{|z|=R}\left|\frac{z^n}{k^{2n}}\overline{P\left(\frac{k^2}{\bar{z}}\right)}\right| \\ &= \frac{R^n}{k^{2n}}\max_{|z|=R}\left|\overline{P\left(\frac{k^2}{\bar{z}}\right)}\right| \\ &= \frac{R^n}{k^{2n}}\max_{|z|=k^2/R}|P(z)| \\ &\geq \frac{R^n}{k^{2n}}\max_{|z|=r}|P(z)| \quad \left(\because \frac{k^2}{R} \geq r\right). \end{aligned} \tag{3.7}$$

Also, we know that

$$\min_{|z|=k}|Q(z)| = \frac{1}{k^n}\min_{|z|=k}|P(z)|. \tag{3.8}$$

Using (3.6), (3.7) and (3.8) in inequality (3.4), we get

$$\frac{1}{k^{2n}}\max_{|z|=R}|P(z)| \geq \frac{R^n}{k^{2n}}B\max_{|z|=r}|P(z)| + (1-B)\frac{1}{k^n}\min_{|z|=k}|P(z)|$$

i.e.,

$$\max_{|z|=R}|P(z)| \geq R^nB\max_{|z|=r}|P(z)| + k^n(1-B)\min_{|z|=k}|P(z)|. \tag{3.9}$$

Combining inequalities (3.2) and (3.9), we obtain

$$\begin{aligned} \max_{|z|=R}|D_\alpha P(z)| \geq nR^{n-1}(|\alpha|-k)\Bigg[&\frac{1}{k^n+R^n}\left(R^nB\max_{|z|=r}|P(z)| + k^n(1-B)\min_{|z|=k}|P(z)|\right) \\ &+ \frac{1}{2k^n}\left(\frac{k^n-R^n}{k^n+R^n}\right)\min_{|z|=k}|P(z)|\Bigg], \end{aligned}$$

which is equivalent to

$$\begin{aligned} \max_{|z|=R}|D_\alpha P(z)| \geq nR^{n-1}(|\alpha|-k)\Bigg[&\frac{R^n}{k^n+R^n}B\max_{|z|=r}|P(z)| \\ &+ \left\{\frac{k^n}{k^n+R^n}(1-B) + \frac{1}{2k^n}\left(\frac{k^n-R^n}{k^n+R^n}\right)\right\}\min_{|z|=k}|P(z)|\Bigg]. \end{aligned}$$

This completes the proof of Theorem 1.6. □

Proof of Theorem 1.7. Let $m = \min_{|z|=k}|P(z)|$, then $m \leq |P(z)|$ for $|z| = k$. Since all the zeros of $P(z)$ lie in $|z| \leq k$, $k > 0$, therefore, for every complex number λ with $|\lambda| < 1$, it follows from Rouche's Theorem that for $m > 0$, the polynomial $G(z) = P(z) + \lambda m$ has all its zeros in $|z| \leq k$, $k > 0$.
Let

$$\begin{aligned} H(z) &= G(Rz) \\ &= P(Rz) + \lambda m \\ &= a_nR^nz^n + a_{n-\mu}R^{n-\mu}z^{n-\mu} + a_{n-\mu-1}R^{n-\mu-1}z^{n-\mu-1} + \ldots + a_1Rz + a_0 + \lambda m. \end{aligned}$$

Therefore, $H(z)$ has all its zeros in $|z| \leq \frac{k}{R}$, $\frac{k}{R} \leq 1$. Hence applying Lemma 2.7 to $H(z)$, we get from (2.12)

$$\max_{|z|=1} \left|D_{\alpha/R} H(z)\right| \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \max_{|z|=1} |H(z)|, \tag{3.10}$$

where $A_{\mu,n}$ is given by (1.21). Therefore,

$$\max_{|z|=1} \left|D_{\alpha/R} G(Rz)\right| \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \max_{|z|=1} |G(Rz)|,$$

which is equivalent to

$$\max_{|z|=R} |D_\alpha G(z)| \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \max_{|z|=R} |G(z)|. \tag{3.11}$$

Applying (2.20) of Lemma 2.8 to $G(z)$, we have

$$\max_{|z|=R} |G(z)| \geq \left(\frac{R+k}{r+k}\right)^n \max_{|z|=r} |G(z)| \quad \text{for } r \leq R \text{ and } rR \geq k^2. \tag{3.12}$$

Combining (3.11) and (3.12), we get

$$\begin{aligned} &\max_{|z|=R} |D_\alpha G(z)| \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \left(\frac{R+k}{r+k}\right)^n \max_{|z|=r} |G(z)| \\ i.e. \quad &\max_{|z|=R} |D_\alpha P(z) + \lambda mn| \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \left(\frac{R+k}{r+k}\right)^n \max_{|z|=r} |P(z) + \lambda m| \\ &\text{for } r \leq R \text{ and } rR \geq k^2. \end{aligned} \tag{3.13}$$

Let z_0 on the circle $|z| = r$ be such that $\max_{|z|=r} |P(z)| = |P(z_0)|$. Then, in particular,

$$\max_{|z|=r} |P(z) + \lambda m| \geq |P(z_0) + \lambda m|. \tag{3.14}$$

Combining (3.13) and (3.14), we get

$$\max_{|z|=R} |D_\alpha P(z) + \lambda mn| \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \left(\frac{R+k}{r+k}\right)^n |P(z_0) + \lambda m| \quad \text{for } r \leq R \text{ and } rR \geq k^2.$$

Choosing the argument of λ on the right hand side of (3.14) such that $|P(z_0) + \lambda m| = |P(z_0)| + |\lambda| m$, we get

$$\begin{aligned} \max_{|z|=R} |D_\alpha P(z) + \lambda mn| \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \left(\frac{R+k}{r+k}\right)^n \left\{\max_{|z|=r} |P(z)| + |\lambda| \min_{|z|=k} |P(z)|\right\} \\ \text{for } r \leq R \text{ and } rR \geq k^2. \end{aligned} \tag{3.15}$$

Using the simple fact that

$$|D_\alpha P(z) + \lambda mn| \leq |D_\alpha P(z)| + |\lambda| mn$$

in (3.15) and letting $|\lambda| \to 1$, we get

$$\begin{aligned} \max_{|z|=R} |D_\alpha P(z)| + mn \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \left(\frac{R+k}{r+k}\right)^n \left\{\max_{|z|=r} |P(z)| + \min_{|z|=k} |P(z)|\right\} \\ \text{for } r \leq R \text{ and } rR \geq k^2. \end{aligned}$$

This completes the proof of Theorem 1.7. □

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A Note on Rough Abel Convergence

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Abstract
In this paper, we define a new type of Abel convergence by using the rough convergence of a sequence. We also obtained some results for this convergence.

Keywords: *Abel convergence; Abel summability; rough convergence; rough limits.*

AMS Subject Classification (2020): *Primary 40A05; Secondary 40D09.*

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1. Introduction and Background

The idea of rough convergence of a sequence was first given by Phu [1] in normed linear spaces as follows:

Let (a_n) be a sequence in the normed linear space X, and r be a nonnegative real number. The sequence (a_n) is said to be *rough convergent* to a with the roughness degree r, denoted by $a_n \xrightarrow{r} a$, if for every $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that $\|a_n - a\| < r + \varepsilon$ for all $n \geq N(\varepsilon)$ [1].

The r-limit set of the sequence (a_n) is denoted by

$$LIM^r a_n = \left\{ a \in X : a_n \xrightarrow{r} a \right\} \text{ [1].}$$

The sequence (a_n) is said to be rough convergent if $LIM^r a_n \neq \varnothing$.

If a sequence is convergent, then it is rough convergent to the same value for each r. The converse of this claim is false, as shown in Example 1.1.

Example 1.1. Let $X = \mathbb{R}^2$ and define a sequence (a_n) as follows:

$$a_n := \left(\frac{(-1)^n}{2}, 0 \right)$$

This sequence is rough convergent to $a = \{(0,0)\}$ for $r \geq \frac{1}{2}$. But it is not convergent to $a = \{(0,0)\}$.

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A sequence (a_n) is said to be rough Cauchy sequence (or ρ-Cauchy sequence) with roughness degree ρ if for every $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$\|a_m - a_n\| < \rho + \varepsilon \text{ for } m, n \geq N(\varepsilon) \text{ [1].}$$

ρ is also called a Cauchy degree of (a_n).

Proposition 1.1. *([2]) Let (a_n) be rough convergent, i.e., $LIM^r a_n \neq \varnothing$. Then (a_n) is a ρ-Cauchy sequence for every $\rho \geq 2r$. This bound for the Cauchy degree cannot be generally decreased.*

We note that a convergent (or non-convergent) sequence can have different rough limits with a certain degree of roughness. This theory has been generalized by many authors with different theories. Aytar [3] gave the definition of rough statistical convergence of a sequence. The rough ideal convergence of a sequence is given in [4] and [5]. Malik and Maity [6] introduced the rough statistical convergence for double sequences. Laterly, Kişi and Dündar [7] defined the rough I_2-lacunary statistical convergence of double sequences. The concept of rough convergence is expressed in general metric spaces by Debnath and Rakshit [8]. Moreover, Arslan and Dündar [9] extended this concept to 2-normed spaces. On the other hand, Dündar and Ulusu [10] studied on the rough convergence of a sequence of functions defined on amenable semigroups. Kişi and Dündar [11] investigated the rough ΔI-statistical convergence for difference sequences. Recently, the rough convergence of a sequence of sets has also been studied (see [12], [13]).

Our aim is to show that the rough convergence theory can be applied on many types of convergence in the summability theory, such as the Abel convergence. In this way, we think that new research topics can be obtained.

Throughout this paper, we suppose that (a_n) be a sequence of complex numbers. Now let's remind the definition of Abel convergence.

We say that a sequence (a_n) is *Abel convergent* to ℓ if the limit

$$\lim_{t \to 1^-} (1-t) \sum_{n=0}^{\infty} a_n t^n = \ell \text{ for each } t \in (0,1) \text{ [14].}$$

Note that any convergent sequence is Abel convergent to the same value but not conversely ([14]).

Finally, let's give the series formulas that we will use throughout this paper:

$$(1-t) \sum_{n=0}^{\infty} t^n = 1 \text{ and } (1-t) \sum_{n=0}^{\infty} \ell t^n = \ell \text{ for each } t \in (0,1).$$

Hence, we have

$$(1-t) \sum_{n=0}^{\infty} a_n t^n - \ell = (1-t) \sum_{n=0}^{\infty} (a_n - \ell) t^n.$$

In this paper, we first give the definition of the rough Abel convergence. We have proved that they are equivalent by giving an alternative representation of this convergence (see Proposition 2.1). We also show that every rough convergent sequence is rough Abel convergent (see Theorem 2.1). Lastly, we expressed the relationship between rough Abel convergence and Abel convergence (see Theorem 2.2).

2. Main Results

Definition 2.1. A sequence (a_n) is said to be rough *Abel convergent* to ℓ if for every $\varepsilon > 0$ and each $t \in (0,1)$ there is an $N(\varepsilon) \in \mathbb{N}$ such that

$$\left\| (1-t) \sum_{n=0}^{\infty} a_n t^n - \ell \right\| < r + \varepsilon \text{ for all } n \geq N(\varepsilon).$$

In this case, we write $a_n \overset{r-A}{\to} \ell$ as $n \to \infty$.

The r-Abel limit set of the sequence (a_n) is denoted by

$$A - LIM^r a_n = \left\{ \ell \in X : a_n \overset{r-A}{\to} \ell \right\}.$$

Let us now give an alternative representation of the rough Abel convergence of a sequence.

Proposition 2.1. *For every $\varepsilon > 0$ and each $t \in (0,1)$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that*

$$\left\| (1-t) \sum_{n=0}^{\infty} a_n t^n - \ell \right\| < r + \varepsilon \text{ for all } n \geq N(\varepsilon)$$

if and only if the following condition holds:

$$\limsup_{n\to\infty} \left\| (1-t) \sum_{n=0}^{\infty} a_n t^n - \ell \right\| \leq r.$$

Its proof can be given in a similar way by taking

$$f_n(t) = \left\| (1-t) \sum_{n=0}^{\infty} a_n t^n - \ell \right\|$$

instead of the f function in the proof of [12, Proposition 2.2].

Theorem 2.1. *If $a_n \xrightarrow{r} \ell$, then $a_n \xrightarrow{r-A} \ell$.*

Proof. Given $0 < \varepsilon < 1$. Since $a_n \xrightarrow{r} \ell$, for every $\varepsilon > 0$ and each $t \in (0,1)$ there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\|a_n - \ell\| < r + \frac{\varepsilon}{2}$$

for all $n \geq N$. Let $M = \max\{\|a_0 - \ell\|, \|a_1 - \ell\|, \ldots \|a_N - \ell\|\}$. Take $\delta = \delta(\varepsilon) = \frac{\varepsilon}{2(N+1)(M+1)}$. If $t \in (1-\delta, 1)$ then

$$\begin{aligned}
\left\| (1-t) \sum_{n=0}^{\infty} a_n t^n - \ell \right\| &= \left\| (1-t) \sum_{n=0}^{\infty} (a_n - \ell) t^n \right\| \\
&\leq \left\| (1-t) \sum_{n=0}^{N} (a_n - \ell) t^n \right\| + \left\| (1-t) \sum_{n=N+1}^{\infty} (a_n - \ell) t^n \right\| \\
&< (1-t)(N+1) M + r + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2(N+1)(M+1)} (N+1) M + r + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + r + \frac{\varepsilon}{2} = r + \varepsilon.
\end{aligned}$$

Consequently, we have $a_n \xrightarrow{r-A} \ell$. □

The next theorem shows the relationship between rough Abel convergence and Abel convergence.

Theorem 2.2. *The sequence (a_n) is rough Abel convergent to ℓ if and only if there exists a sequence (b_n) in $\mathbb{C}$ such that $b_n \xrightarrow{A} \ell$ and $\|a_n - b_n\| \leq r$ for every $n \in \mathbb{N}$.*

Proof. ($\Leftarrow$) Since $b_n \xrightarrow{A} \ell$, for every $\varepsilon > 0$ and each $t \in (0,1)$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$\left\| (1-t) \sum_{n=0}^{\infty} b_n t^n - \ell \right\| < \varepsilon \text{ for all } n \geq N(\varepsilon). \tag{2.1}$$

By assumption $\|a_n - b_n\| \leq r$, we can write for each $t \in (0,1)$

$$\begin{aligned}
\left\| (1-t) \sum_{n=0}^{\infty} (a_n - b_n) t^n \right\| &= |1-t| \left\| \sum_{n=0}^{\infty} (a_n - b_n) t^n \right\| \\
&\leq |1-t| \sum_{n=0}^{\infty} \|a_n - b_n\| |t^n| \\
&\leq (1-t) r \sum_{n=0}^{\infty} t^n = r.
\end{aligned}$$

Thus we have
$$\left\|(1-t)\sum_{n=0}^{\infty}(a_n-b_n)\,t^n\right\|\leq r. \tag{2.2}$$
On the other hand,
$$\begin{aligned}\left\|(1-t)\sum_{n=0}^{\infty}a_nt^n-\ell\right\| &= \left\|(1-t)\sum_{n=0}^{\infty}(a_n-b_n+b_n)\,t^n-\ell\right\| \\ &= \left\|(1-t)\sum_{n=0}^{\infty}(a_n-b_n)\,t^n+(1-t)\sum_{n=0}^{\infty}b_nt^n-\ell\right\| \\ &\leq \left\|(1-t)\sum_{n=0}^{\infty}(a_n-b_n)\,t^n\right\|+\left\|(1-t)\sum_{n=0}^{\infty}b_nt^n-\ell\right\|.\end{aligned}$$
From (2.1) and (2.2), we see immediately that
$$\left\|(1-t)\sum_{n=0}^{\infty}a_nt^n-\ell\right\|<r+\varepsilon.$$
This shows that the sequence (a_n) rough Abel converges to ℓ.

($\Rightarrow$) Let $a_n \overset{r-A}{\to} \ell$ and define the sequence (b_n) by
$$b_n:=\begin{cases}\ell & ,\ \|a_n-\ell\|\leq r\\ a_n+r\frac{\ell-a_n}{\|\ell-a_n\|} & ,\ \|a_n-\ell\|>r\end{cases}.$$
It follows that the inequality $\|a_n-b_n\|\leq r$ holds for every $n\in\mathbb{N}$. We also obtain
$$\|b_n-\ell\|\leq\begin{cases}0 & ,\ \|a_n-\ell\|\leq r\\ \|a_n-\ell\|-r & ,\ \|a_n-\ell\|>r\end{cases}.$$
Let us show that $b_n \overset{A}{\to} \ell$.
$$\begin{aligned}\left\|(1-t)\sum_{n=0}^{\infty}b_nt^n-\ell\right\| &= \left\|(1-t)\sum_{n=0}^{\infty}(b_n-\ell)\,t^n\right\| \\ &\leq (1-t)\sum_{n=0}^{\infty}\|b_n-\ell\|\,t^n \\ &\leq (1-t)\sum_{n=0}^{\infty}(\|a_n-\ell\|-r)\,t^n \\ &= (1-t)\sum_{n=0}^{\infty}\|a_n-\ell\|\,t^n-(1-t)r\sum_{n=0}^{\infty}t^n\end{aligned}$$
and thus we have
$$\left\|(1-t)\sum_{n=0}^{\infty}b_nt^n-\ell\right\|\leq(1-t)\sum_{n=0}^{\infty}\|a_n-\ell\|\,t^n-r. \tag{2.3}$$
Since $a_n \overset{r-A}{\to} \ell$, we can write
$$\limsup_{n\to\infty}\left\|(1-t)\sum_{n=0}^{\infty}a_nt^n-\ell\right\|=\limsup_{n\to\infty}\left\|(1-t)\sum_{n=0}^{\infty}(a_n-\ell)\,t^n\right\|\leq r.$$
Taking the limit superior both sides in (2.3), we obtain
$$\limsup_{n\to\infty}\left\|(1-t)\sum_{n=0}^{\infty}b_nt^n-\ell\right\|\leq\limsup_{n\to\infty}\left[(1-t)\sum_{n=0}^{\infty}\|a_n-\ell\|\,t^n\right]-r.$$
This implies that $b_n \overset{A}{\to} \ell$. □

Proposition 2.2. *(i) If $a_n \overset{r-A}{\to} \ell_1$ and $b_n \overset{r-A}{\to} \ell_2$, then $a_n+b_n \overset{2r-A}{\to} \ell_1+\ell_2$.*
(ii) If $a_n \overset{r-A}{\to} \ell$, then $\lambda a_n \overset{|\lambda|r-A}{\to} \lambda\ell$ for each $\lambda\in\mathbb{R}$.

Proof. (i) Suppose that $a_n \overset{r-A}{\to} \ell_1$ and $b_n \overset{r-A}{\to} \ell_2$. Let $\varepsilon > 0$ and $t \in (0,1)$. Then there exist $N_1(\varepsilon), N_2(\varepsilon) \in \mathbb{N}$ such that

$$\left\| (1-t) \sum_{n=0}^{\infty} a_n t^n - \ell_1 \right\| < r + \frac{\varepsilon}{2} \text{ for all } n \geq N_1(\varepsilon)$$

and

$$\left\| (1-t) \sum_{n=0}^{\infty} b_n t^n - \ell_2 \right\| < r + \frac{\varepsilon}{2} \text{ for all } n \geq N_2(\varepsilon).$$

Let $N(\varepsilon) = \max\{N_1(\varepsilon), N_2(\varepsilon)\}$. Hence, we have

$$\begin{aligned} \left\| (1-t) \sum_{n=0}^{\infty} (a_n + b_n) t^n - (\ell_1 + \ell_2) \right\| &\leq \left\| (1-t) \sum_{n=0}^{\infty} a_n t^n + (1-t) \sum_{n=0}^{\infty} b_n t^n - \ell_1 - \ell_2 \right\| \\ &\leq \left\| (1-t) \sum_{n=0}^{\infty} a_n t^n - \ell_1 \right\| + \left\| (1-t) \sum_{n=0}^{\infty} b_n t^n - \ell_2 \right\| \\ &< r + \varepsilon/2 + r + \varepsilon/2 = 2r + \varepsilon \end{aligned}$$

for all $n \geq N(\varepsilon)$, which completes the proof.

(ii) For $\lambda = 0$ the statement is trivial. Let $\lambda \neq 0$. Since $a_n \overset{r-A}{\to} \ell$, for every $\varepsilon > 0$ and each $t \in (0,1)$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$\left\| (1-t) \sum_{n=0}^{\infty} a_n t^n - \ell \right\| < r + \frac{\varepsilon}{|\lambda|} \text{ for all } n \geq N(\varepsilon).$$

Then we have

$$\begin{aligned} \left\| (1-t) \sum_{n=0}^{\infty} \lambda a_n t^n - \lambda \ell \right\| &= |\lambda| \left\| (1-t) \sum_{n=0}^{\infty} a_n t^n - \ell \right\| \\ &\leq |\lambda| \left(r + \frac{\varepsilon}{|\lambda|} \right) \\ &= |\lambda| r + \varepsilon \end{aligned}$$

for all $n \geq N(\varepsilon)$. Thus, we obtain $\lambda a_n \overset{|\lambda| r-A}{\to} \lambda \ell$ for each $\lambda \in \mathbb{R}$. □

3. Conclusion

The converse of Theorem 2.1 is not true. That is, if a sequence is rough Abel convergent, it may not rough convergent to the same point. The Proposition 2.2 shows that the sum of rough Abel convergent sequences with the same degree of roughness converges with a different degree of roughness. In other words, the roughness degree $2r$ cannot be decreased. It also states that the scalar product of a rough Abel convergent sequence converges with a different degree of roughness. After that, we can examine some properties of the set of rough Abel limit points of a sequence.

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A New Sequence of Bernstein-Durrmeyer Operators and Their L_p-Approximation Behaviour

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Abstract
The purpose of the present manuscript is to present a new sequence of Bernstein-Durrmeyer operators. First, we investigate approximation behaviour for these sequences of operators in Lebesgue Measurable space. Further, we discuss rate of convergence and order of approximation with the aid of Korovkin theorem, modulus of continuity and Peetre K-functional in l_p space. Moreover, Voronovskaja type theorem is introduced to approximate a class of functions which has first and second order continuous derivatives. In the last section, numerical and graphical analysis are investigated to show better approximation behaviour for these sequences of operators.

Keywords: *Rate of convergence; order of approximation; modulus of continuity; Bernstein-Durrmeyer operators.*

AMS Subject Classification (2020): *41A10, 41A25, 41A28, 41A35, 41A36.*

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1. Introduction

Operators theory is a fascinating field of research for the last two decades due to the advent of computer. It contributes important role in applied and pure mathematics, viz, fixed point theory, numerical analysis etc. In computational aspects of mathematics and shape of geometric objects, CAGD (Computer-aided Geometric design) plays an interesting role with the mathematical description. It focuses on mathematics which is compatible with computers in shape designing. To investigate the behavior of parametric surfaces and curves, control nets and control points has a significant role respectively. CAGD is widely used as an application in applied mathematics and industries. It has several applications in other branches of sciences, e.g., approximation theory, computer graphics, data structures, numerical analysis, computer algebra etc. In 1912, Bernstein [1] was the first who introduced a sequence of polynomials to present a smallest and easiest proof of celebrated theorem named as Weierstrass

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approximation theorem with the aid of binomial distribution as follows:

$$B_l(g;x) = \sum_{\nu=0}^{l} g\left(\frac{\nu}{l}\right)\left(\begin{array}{c} l \\ \nu \end{array}\right)\mu^{\nu}(1-\mu)^{l-\nu}, \quad \mu \in [0,1], \tag{1.1}$$

where g is a bounded function defined on $[0,1]$. The basis $\binom{l}{\nu}\mu^{\nu}(1-\mu)^{l-\nu}$ of Bernstein polynomials (1.1) has significant role in preserving the shape of the surfaces or curves (see [2]-[4]). Graphic design programs, viz, photoshop inkspaces and Adobe's illustrator deals with Bernstein polynomials in the form of Bèzier curves. To preserve the shape of the parametric surface or curve, it depends on basis $\binom{l}{\nu}\mu^{\nu}(1-\mu)^{l-\nu}$ which is used to design the curves.

In 1962, Schurer [5] presented the following modification of Bernstein operators (1.1) is denoted as $B_{m,l}$: $C[0,1+l] \to C[0,1]$ and given by:

$$B_{m,l}(g;\mu) = \sum_{i=0}^{m+l} g\left(\frac{j}{m}\right)\left(\begin{array}{c} m+l \\ j \end{array}\right)\mu^{k}(1-\mu)^{m+l-j}, \mu \in [0,1],$$

for $l \in \mathbb{N} \cup \{0\}$ and $g \in C[0,1+l]$. In the recent past, Several modifications have studied in various functional spaces to achieve better approximation results (see Acar et al. [6], Acu et al. [7], Braha et al. ([8], [9]), Cai et al. [10], Cetin et al. [11], Kajla et al. [12], Mohiuddine et al. [13]). Izgi [14] introduced a new sequence of Bernstein polynomials as:

$$A_n(h;u) = \sum_{k=0}^{n} q_{n,k,a,b}(u) h\left(\frac{k}{n}\frac{n+a}{n+b}\right),$$

where $q_{n,k,a,b}(u) = \left(\frac{n+b}{n+a}\right)^n \binom{n}{k} u^k \left(\frac{n+a}{n+b} - u\right)^{n-k}$, $0 \le a \le b$, $u \in \left[0, \frac{n+a}{n+b}\right]$ and $h \in C\,[0,1]$. Further, he constructed two dimentional sequences of operators to approximate a class of bivariate continuous functions on square and triangular domain. Moreover, he investigated rate of convergence and order of approximation in different functional spaces with the aid of modulus of continuity, In the last, he presented another variant of these sequences to approximate a wider class, i.e., Lebesgue measurable class as:

$$T_n\,(h;u) = \frac{(n+b)(n+1)}{n+a} \sum_{k=0}^{n} q_{n,k}^{\sim}(u) \int_{0}^{\frac{n+a}{n+b}} q_{n,k}^{\sim}(t) h(t) dt, \tag{1.2}$$

where $q_{n,k}^{\sim}(u) =: q_{n,k,a,b}(u)$ and $I_n = \left[0, \frac{n+a}{n+b}\right]$.

and

$$T_n^*\,(h;u) = \begin{cases} T_n\,(h;u) & u \in I_n \\ h(u) & u \in [0,1] \ / \ I_n \end{cases} \tag{1.3}$$

Remark 1.1. Bernstein-Durrmeyer type operators defined by (1.2) are linear and positive.

2. Preliminaries

In this section, let's calculate the values of our operator $1, t, t^2, t^3$ and t^4 to examine the convergence states and show that our operator satisfies the Korovkin conditions. After that, with the help of these values, let's calculate their central moments.

Lemma 2.1. *Let $f_p(t) = t^p, p \in N \bigcup \{0\}$ be the test functions. Then, we have*

$$T_n\,(t^p;u) = \frac{(n+1)!}{(n+p+1)!} \sum_{s=0}^{p} \binom{p}{s} \frac{p!n!}{s!\,(n-s)!} \left(\frac{n+a}{n+b}\right)^{p-s} u^s.$$

Proof. We know

$$\begin{aligned}
\int_0^{\frac{n+a}{n+b}} q_{n,k}^{\sim}(t)t^p dt &= \left(\frac{n+b}{n+a}\right)^n \binom{n}{k} \int_0^{\frac{n+a}{n+b}} t^k \left(\frac{n+a}{n+b}-t\right)^{n-k} t^p dt \\
&= \left(\frac{n+b}{n+a}\right)^n \binom{n}{k} \int_0^1 \left(\frac{n+a}{n+b}\right)^{n+p+1} x^{k+p}(1-x)^{n-k} dx \\
&= \left(\frac{n+a}{n+b}\right)^{p+1} \frac{n!}{k!(n-k)!} \frac{(k+p)!}{(n+p+1)!}.
\end{aligned}$$

In view of (1.2), we have

$$\begin{aligned}
T_n\left(t^p;u\right) &= \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q_{n,k}^{\sim}(u) \int_0^{\frac{n+a}{n+b}} q_{n,k}^{\sim}(t)t^p dt \\
&= \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q_{n,k}^{\sim}(u) \left(\frac{n+a}{n+b}\right)^{p+1} \frac{n!}{k!(n-k)!} \frac{(k+p)!}{(n+p+1)!} \\
&= \left(\frac{n+a}{n+b}\right)^{p+1} \frac{(n+1)!}{(n+p+1)!} \sum_{k=0}^{n} q_{n,k}^{\sim}(u) \frac{k!}{(k+p)!}.
\end{aligned}$$

Now, the p^{th} order derivative of the $x^p(x+y)^n$ expression is as:

$$\begin{aligned}
\frac{\partial^p}{\partial u^p}\left[u^p(u+v)^n\right] &= \frac{\partial^p}{\partial u^p}\sum_{k=0}^{n}\binom{n}{k} u^{k+p} v^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{(k+p)!}{k!} u^k v^{n-k}, \qquad (2.1)
\end{aligned}$$

or

$$\frac{\partial^p}{\partial u^p}\left[u^p(u+v)^n\right] = \sum_{s=0}^{p}\binom{p}{s}\frac{p!n!}{s!(n-s)!} u^s (u+v)^{n-s}. \qquad (2.2)$$

Combining equation (2.1) and equation (2.2), we obtain

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(k+p)!}{k!} u^k v^{n-k} = \sum_{s=0}^{p}\binom{p}{s}\frac{p!n!}{s!(n-s)!} u^s (u+v)^{n-s}.$$

Choosing $u+v=\frac{n+a}{n+b}$ and multiply both the sides in the above equation with $\left(\frac{n+b}{n+a}\right)^n$, we have

$$\left(\frac{n+b}{n+a}\right)^n \sum_{k=0}^{n}\binom{n}{k}\frac{(k+p)!}{k!} u^k \left(\frac{n+a}{n+b}-u\right)^{n-k} = \left(\frac{n+b}{n+a}\right)^n \sum_{s=0}^{p}\binom{p}{s}\frac{p!n!}{s!(n-s)!} u^s \left(\frac{n+a}{n+b}\right)^{n-s}. \qquad (2.3)$$

In the light of equation (2.1) and (2.3), we arrive at the required result. □

Lemma 2.2. *Let $f_p(t) = t^p, p \in \{0,1,2,3,4\}$ be the test function. Then*

$$\begin{aligned}
T_n(1;u) &= 1, \\
T_n(t;u) &= u - \frac{2}{n+2}u + \frac{n+a}{(n+2)(n+b)}, \\
T_n(t^2;u) &= u^2 - \frac{6(n+1)}{(n+2)(n+3)}u^2 + \frac{4n(n+a)}{(n+2)(n+3)(n+b)}u + \frac{2}{(n+2)(n+3)}\left(\frac{n+a}{n+b}\right)^2, \\
T_n(t^3;u) &= u^3 - 12\frac{(n^2+2n+2)}{(n+2)(n+3)(n+4)}u^3 + \frac{9n(n-1)(n+a)}{(n+2)(n+3)(n+4)(n+b)}u^2 \\
&\quad + \frac{18n}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^2 u + \frac{6}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^3, \\
T_n(t^4;u) &= u^4 - 20\frac{(n^3+3n^2+8n+6)}{(n+2)(n+3)(n+4)(n+5)}u^4 + \frac{16n(n-1)(n-2)(n+a)}{(n+2)(n+3)(n+4)(n+5)(n+b)}u^3 \\
&\quad + \frac{72n(n-1)}{(n+2)(n+3)(n+4)(n+5)}\left(\frac{n+a}{n+b}\right)^2 u^2 + \frac{96}{(n+2)(n+3)(n+4)(n+5)}\left(\frac{n+a}{n+b}\right)^3 u \\
&\quad + \frac{24}{(n+2)(n+3)(n+4)(n+5)}\left(\frac{n+a}{n+b}\right)^4.
\end{aligned}$$

Proof. In the direction of Lemma 2.1, one can easily arrive at the proof of Lemma 2.2. □

Consider $\delta_{n,p}(u) = T_n((t-u)^p; u)$, $p \in \{0,1,2...\}$. Then, we obtain the central moments in the following Lemma 2.3:

Lemma 2.3. *For the operators given by $\delta_{n,p}(u)$, we have*

$$\begin{aligned}
\delta_{n,0}(u) &= 1, \\
\delta_{n,1}(u) &= -\frac{2}{n+2}u + \frac{2}{n+2}\frac{n+a}{n+b}, \\
\delta_{n,2}(u) &= \frac{2(n-3)}{(n+2)(n+3)}u\left(\frac{n+a}{n+b} - u\right) + \frac{2}{(n+2)(n+3)}\left(\frac{n+a}{n+b}\right)^2, \\
\delta_{n,3}(u) &= \frac{24(n-1)}{(n+2)(n+3)(n+4)}u^3 - \frac{36(n-1)}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)u^2 \\
&\quad + \frac{12(n-2)}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^2 u + \frac{6}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^3, \\
\delta_{n,4}(u) &= \frac{12(n^2-21n+10)}{(n+2)(n+3)(n+4)(n+5)}u^4 - \frac{2(5n^3-3n^2-242n+120)}{(n+2)(n+3)(n+4)(n+5)}\left(\frac{n+a}{n+b}\right)u^3 \\
&\quad + \frac{12(n^2-27n+20)}{(n+2)(n+3)(n+4)(n+5)}\left(\frac{n+a}{n+b}\right)^2 u^2 - \frac{24(n+1)}{(n+2)(n+3)(n+4)(n+5)}\left(\frac{n+a}{n+b}\right)^3 u \\
&\quad + \frac{24}{(n+2)(n+3)(n+4)(n+5)}\left(\frac{n+a}{n+b}\right)^4.
\end{aligned}$$

Proof. In view of Lemma 2.2, we can easily proof Lemma 2.3. □

Now, we consider

$$\delta = \max_{0 \le u \le \frac{n+a}{n+b}} \delta_{n,2}(u) = \frac{(n+1)}{2(n+2)(n+3)}\left(\frac{n+a}{n+b}\right)^2 \le \frac{1}{2(n+2)} < \frac{1}{n} \tag{2.4}$$

and

$$\mu = \max_{0 \le u \le \frac{n+a}{n+b}} \delta_{n,4}(u) \le \frac{24}{(n+2)(n+3)} \text{ and } \mu < \frac{1}{n} \text{ for } n > 20.$$

Let $C_{M_u}[0,1] = \left\{h \in C[0,1] : |h(u)| \le M\left(1+u^2\right) \text{ for all } u \in \mathbb{R},\, M > 0\right\}$ and for $1 \le p < \infty$,

$$L_p[0,1] = \left\{ h \text{ is measurable} : \int_0^1 |h(u)|^p \, du < \infty \right\}.$$

Lemma 2.4. *For $h \in C_{M_u}[0,1]$ endowed with the norm $\|h(u)\|_\infty = \sup_{u\in[0,1]} |h(u)|$, we have*

$$\|T_n(h)\|_\infty \le \|h\|_\infty ,$$

i.e., the operator given by (1.2) is bounded.

Proof. In terms of the definition (1.2) and Lemma 2.2, we get

$$\begin{aligned}
\|T_n(h)\|_\infty &\le \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u) \int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)\,|h(t)|\,dt \\
&\le \|h\|_\infty T_n(1;u) \\
&= \|h\|_\infty .
\end{aligned}$$

Since the operators introduced by (1.2) is linear and bounded. Therefore, it is continuous.

Let

$$W_n(u,t) = \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u) q^{\sim}_{n,k}(t),$$

then we can write (1.2) as:

$$T_n(h;u) = \int_{I_n} W_n(u,t)h(t)dt.$$

It is easy to see that

$$\int_{I_n} W_n(u,t)dt = 1 < \infty,$$

$$\int_{I_n} W_n(u,t)du = 1 < \infty,$$

for all $n = 0,1,2...$ (see [15], page 31-32), for $h \in L_p(I_n)$, $T_n(h;u)$ exist for almost all u and belongs to $L_p(I_n)$. Due to Orlicz theorem, there exist a $K > 0$ such that

$$\int_{I_n} |T_n(h;u)|^p \, du \le K \|h\|_\infty . \tag{2.5}$$

□

3. Direct approximation results

Theorem 3.1. *Let $h \in C_{M_u}[0,1]$. Then, one has*

$$\lim_{n\to\infty} T_n(h;u) = h(u),$$

uniformly on $[0,1]$.

Proof. In view of Lemma 2.2, it is easy to check

$$\lim_{n\to\infty} T_n(f_p(t);u) = f_p(u),$$

for $p = 0,1,2$ uniformly on $[0,1]$. Applying Bohman-Korovkin Theorem, the result follows. □

The first modulus of continuity is given by

$$\omega_1(h,\delta)=\sup_{\substack{|t-u|<\delta\\ t,u\in[0,1]}}|h(t)-h(u)|.$$

Theorem 3.2. *Let $h\in C_{M_u}[0,1]$. Then, we have*

$$|T_n(h;u)-h(u)|\leq 2\omega_1\left(h,\frac{1}{\sqrt{n}}\right).$$

Proof. In view of Lemma 2.3, (2.4) and Cauchy-Schwartz inequality, we get

$$\begin{aligned}
|T_n(h;u)-h(u)| &\leq T_n(|h(t)-h(u)|;u)\\
&\leq T_n(\left(1+\frac{|t-u|}{\delta}\right)\omega_1(h,\delta);u)\\
&= \omega_1(h,\delta)\left[1+\frac{1}{\delta}T_n(|t-u|;u)\right]\\
&\leq \omega_1(h,\delta)\left[1+\frac{1}{\delta}\sqrt{T_n((t-u)^2;u)}\right]\\
&\leq \omega_1(h,\delta)\left[1+\frac{1}{\delta}\sqrt{\frac{1}{n}}\right].
\end{aligned}$$

Choosing $\delta=\frac{1}{\sqrt{n}}$, we arrive at the desired result. □

For each $0\leq\alpha\leq 1$ and $M>0$, let $Lip_m\alpha$ denote the set of all functions h on $[0,1]$ such that

$$|h(u)-h(v)|\leq M|u-v|^{\alpha}. \tag{3.1}$$

Theorem 3.3. *If h satisfy condition (3.1), then we have*

$$|T_n(h;u)-h(u)|\leq M\left(\frac{1}{n}\right)^{\frac{\alpha}{2}}.$$

Proof. Use Cauchy-Schwartz inequality, (2.4) and (3.1), we have

$$\begin{aligned}
|T_n(h;u)-h(u)| &= \left|\frac{n+b}{n+a}(n+1)\sum_{k=0}^{n}\widetilde{q_{n,k}}(u)\int_0^{\frac{n+a}{n+b}}\widetilde{q_{n,k}}(t)h(t)dt-h(u)\right|\\
&\leq \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n}\widetilde{q_{n,k}}(u)\int_0^{\frac{n+a}{n+b}}\widetilde{q_{n,k}}(t)\,|h(t)-h(u)|\,dt\\
&\leq \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n}\widetilde{q_{n,k}}(u)\int_0^{\frac{n+a}{n+b}}\widetilde{q_{n,k}}(t)M\,|u-v|^{\alpha}\,dt\\
&\leq M\left(\frac{n+b}{n+a}(n+1)\sum_{k=0}^{n}\widetilde{q_{n,k}}(u)\int_0^{\frac{n+a}{n+b}}\widetilde{q_{n,k}}(t)\,(u-v)^2\,dt\right)^{\frac{\alpha}{2}}\\
&\leq M\left(\frac{1}{n}\right)^{\frac{\alpha}{2}}
\end{aligned}$$

the proof is completed. □

Theorem 3.4. *If* $h \in L_1[0,1]$, $u \in (0,1)$ *and* h *endowed with a continuous derivative on the interval* $[0,1]$, *then*

$$|T_n(h;u)-h(u)| \leq \left|-\frac{2}{n+2}u+\frac{1}{n+2}\frac{n+a}{n+b}\right| + 2\sqrt{\delta_{n,2}(u)}\omega_1\left(h^{'},\sqrt{\delta_{n,2}(u)}\right).$$

Proof. Since h is differentiable on $[0,1]$ therefore by mean value theorem of differential calculus we have

$$h(t)-h(u)=(t-u)h^{'}(\theta)=(t-u)h^{'}(u)+(t-u)\left(h^{'}(\theta)-h^{'}(u)\right), \tag{3.2}$$

where $\theta := \theta(u,t)$ belongs to the interval obtained by u and t. Then, on combining (1.3) to (3.2), we have

$$\begin{aligned}
T_n(h;u)-h(u) &= \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u)\int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)(t-u)h^{'}(u)dt \\
&\quad +\frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u)\int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)(t-u)\left(h^{'}(\theta)-h^{'}(u)\right)dt \\
&= h^{'}(u)T_n((t-u);u) \\
&\quad +\frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u)\int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)(t-u)\left(h^{'}(\theta)-h^{'}(u)\right)dt,
\end{aligned}$$

$$\begin{aligned}
|T_n(h;u)-h(u)| &\leq \left|-\frac{2}{n+2}u+\frac{1}{n+2}\frac{n+a}{n+b}\right| \\
&\quad +\frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u)\int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)\left|t-u\right|\left|h^{'}(\theta)-h^{'}(u)\right|dt.
\end{aligned}$$

Now, we use properties of modulus of continuity

$$\begin{aligned}
\left|h^{'}(\theta)-h^{'}(u)\right| &\leq \omega_1\left(h^{'},|\theta-u|\right) \leq \left(1+\frac{|\theta-u|}{\beta}\right)\omega\left(h^{'},\beta\right) \\
&\leq \left(1+\frac{|t-u|}{\beta}\right)\omega\left(h^{'},\beta\right).
\end{aligned}$$

Since $u \leq \theta \leq t$. Therefore, we have

$$\begin{aligned}
|T_n(h;u)-h(u)| &\leq \left|-\frac{2}{n+2}u+\frac{1}{n+2}\frac{n+a}{n+b}\right| \\
&\quad +\omega\left(h^{'},\beta\right)\frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u)\int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)\left|t-u\right|\left(1+\frac{|t-u|}{\beta}\right)dt.
\end{aligned}$$

Let's examine the last term of last inequality;

$$\begin{aligned}
&\omega\left(h^{'},\beta\right)\frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u)\int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)\left|t-u\right|\left(1+\frac{|t-u|}{\beta}\right)dt \\
=\ &\omega\left(h^{'},\beta\right)\left(T_n(\left|t-u\right|;u)+\frac{1}{\beta}T_n((t-u)^2;u)\right) \\
=\ &\omega\left(h^{'},\beta\right)\left(\sqrt{T_n((t-u)^2;u)}+\frac{1}{\beta}T_n((t-u)^2;u)\right) \\
=\ &\omega\left(h^{'},\beta\right)\left(\sqrt{\delta_{n,2}(u)}\left[1+\frac{1}{\beta}\delta_{n,2}(u)\right]\right).
\end{aligned}$$

Then, on choosing $\beta = \sqrt{\delta_{n,2}(u)}$, we prove the desired result. □

4. Voronovskaya-type theorem

In this section, we prove Voronvoskaya-type asymptotic theorem for the operators $T_n(h;u)$ to approximate a class of functions which has first and second order continuous derivatives.

Theorem 4.1. *Let $h \in C_{M_u}[0,1]$. If $h^{'}, h^{''}$ exists at a fixed point $u \in [0,1]$ then we have*

$$\lim_{n\to\infty} n\left\{T_n(h;u) - h(u)\right\} = (-2u+1)\, h^{'}(u) + u(1-u)h^{''}(u).$$

Proof. Let $u \in [0,1]$ be fixed. By Taylor's expansion of h, we can write

$$h(t) = h(u) + (t-u)h^{'}(u) + \frac{1}{2}(t-u)^2 h^{''}(u) + \varphi(t,u)(t-u)^2. \tag{4.1}$$

Where the function $\varphi(t,u)$ is the Peano form of remainder, $\varphi(t,u) \in C_{M_u}[0,1]$ and

$$\lim_{n\to\infty} \varphi(t,u) = 0.$$

Applying $T_n(h;u)$ both the sides of (4.1) and Lemma 2.3, we have

$$\begin{aligned} n\left\{T_n(h;u) - h(u)\right\} &= n\Bigg\{\left(-\frac{2}{n+2}u + \frac{1}{n+2}\frac{n+a}{n+b}\right)h^{'}(u) \\ &+\frac{1}{2}\left(\frac{2(n-3)}{(n+2)(n+3)}u\left(\frac{n+a}{n+b} - u\right) + \frac{2}{(n+2)(n+3)}\left(\frac{n+a}{n+b}\right)^2\right)h^{''}(u)\Bigg\} \\ &+nT_n(\varphi(t,u)(t-u)^2;u). \end{aligned}$$

Using Cauchy-Schwarz inequality, we have

$$nT_n(\varphi(t,u)(t-u)^2;u) \le \left(T_n(\varphi^2(t,u);u)\right)^{\frac{1}{2}} \left(T_n((t-u)^4;u)\right)^{\frac{1}{2}}. \tag{4.2}$$

One can observe that $\varphi^2(u,u) = 0$ and $\varphi^2(.,u) \in C_{M_u}[0,1]$. Then, it follows that

$$\lim_{n\to\infty} T_n(\varphi^2(t,u);u) = \varphi^2(u,u) = 0. \tag{4.3}$$

Now, from (4.2) and (4.3), we obtain

$$\lim_{n\to\infty} nT_n(\varphi(t,u)(t-u)^2;u) = 0. \tag{4.4}$$

From (4.4), we get the required result. □

5. Local approximation

The K-functional is given by :

$$K_2(h,\delta) = \inf_{g\in W^2}\left\{\|h-g\|_\infty + \delta\left\|g^{''}\right\|\right\},$$

where $\delta > 0$, $W^2 = \left\{g : g^{'},\ g^{''} \in C[0,1]\right\}$ and by [13] there exists a positive constant $M > 0$ such that

$$K_2(h,\delta) \le M\,\omega_2(h,\delta).$$

Where the second order modulus of continuity for $h \in C_{M_u}[0,1]$ is defined as:

$$\omega_2(h,\delta) = \sup_{\substack{|t-u|<\delta \\ t,u\in[0,1]}} |h(t+2x) - 2h(u+x) + h(u)|.$$

Theorem 5.1. *For the operators introduced by $T_n(.;.)$ and $h \in C_{M_u}[0,1]$, we have*

$$\left\|T_n(h;u)-h(u)\right\|_\infty \le 2K_2\left(h,\frac{\delta_n^1}{2}\right)+\delta_n^2\left\|g^{'}\right\|_\infty,$$

here $\delta_n^1=\max_{u\in[0,1]}\left\{\frac{8u^3+u+1}{(n+3)(n+4)}\right\}=\frac{10}{(n+3)(n+4)}$ *and* $\delta_n^2=\max\left\{\inf_{u\in[0,1]}\left\{\frac{|1-2u|}{n+2}\right\}\right\}=\frac{1}{(n+2)}$.

Proof. Let $g\in W^2$ and $t\in[0,1]$. By Taylor's expansion, we have

$$g(t)=g(u)+(t-u)g^{'}(u)+\int_u^t (t-v)g^{''}(v)dv.$$

Applying (1.2) on both the sides of above relation and using Lemma 2.3, we have

$$T_n(g;u)=g(u)+\left(-\frac{2}{n+2}u+\frac{1}{n+2}\frac{n+a}{n+b}\right)g^{'}(u)+T_n\left(\int_u^t (t-v)g^{''}(v)dv;u\right).$$

Further

$$\begin{aligned}\left|T_n(g;u)-g(u)\right| &\le \left|-\frac{2}{n+2}u+\frac{1}{n+2}\frac{n+a}{n+b}\right|\left|g^{'}(u)\right|+T_n\left(\int_u^t |t-v|\left|g^{''}(v)\right|dv;u\right)\\ &\le \left|-\frac{2}{n+2}u+\frac{1}{n+2}\frac{n+a}{n+b}\right|\left\|g^{'}\right\|_\infty+\left\|g^{''}\right\|_\infty\left[T_n\left(\int_u^t (t-v)^2dv;u\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{5.1}$$

In the light of Lemma 2.2

$$\begin{aligned}T_n\left(\int_u^t (t-v)^2dv;u\right) &= T_n\left(\frac{1}{3}(t-u)^3;u\right)\\ &= 8\frac{n-1}{(n+2)(n+3)(n+4)}u^3-12\frac{n-1}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)u^2\\ &\quad +4\frac{n-2}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^2u\\ &\quad +\frac{2}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^3. \end{aligned} \tag{5.2}$$

Combining equation (5.2) and (5.1), we obtain

$$\begin{aligned}\left|T_n(g;u)-g(u)\right| &\le \left|-\frac{2}{n+2}u+\frac{1}{n+2}\frac{n+a}{n+b}\right|\left\|g^{'}\right\|_\infty\\ &\quad +\left\|g^{''}\right\|_\infty\left\{8\frac{n-1}{(n+2)(n+3)(n+4)}u^3-12\frac{n-1}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)u^2\right|\\ &\quad \left|+4\frac{n-2}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^2u+\frac{2}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^3\right\}^{\frac{1}{2}}\\ &\le \frac{|1-2u|}{n+2}\left\|g^{'}\right\|_\infty+\left\|g^{''}\right\|_\infty\left[\frac{8u^3+u+1}{(n+3)(n+4)}\right]^{\frac{1}{2}}. \end{aligned}$$

With the aid of Lemma 2.4

$$\begin{aligned}
|T_n(h;u)-h(u)| &= |T_n(h;u)-T_n(g;u)+T_n(g;u)-g(u)+g(u)-h(u)| \\
&\leq |T_n(h-g;u)|+|T_n(g;u)-g(u)|+|g(u)-h(u)| \\
&\leq 2\|h-g\|_\infty+\frac{|1-2u|}{n+2}\left\|g^{'}\right\|_\infty+\left\|g^{''}\right\|_\infty\left[\frac{8u^3+u+1}{(n+3)(n+4)}\right]^{\frac{1}{2}},
\end{aligned}$$

for $u\in[0,1]$. If the right side of the last inequality is taken as the maximum. The proof is completed. □

Here. We introduce the direct estimate of the operators (1.2) with the aid of Lipschitz-type maximal function of order $\beta\in(0,1]$ defined by Lenze [16] as follows:

$$\omega_\beta^*(h,u)=\sup_{\substack{t\neq u\\ u,t\in[0,1]}}\frac{|h(t)-h(u)|}{|t-u|^\beta}. \tag{5.3}$$

Using (5.3), the following inequality is achieved.

$$|h(t)-h(u)|\leq\omega_\beta^*(h,u)\,|t-u|^\beta\,\omega_\beta^*(h,u)\,\delta^{\frac{\beta}{2}}. \tag{5.4}$$

Theorem 5.2. *Let $h\in C_{M_u}[0,1]$ and $\beta\in(0,1]$. Then, we have*

$$|T_n(h;u)-h(u)|\leq\omega_\beta^*(h,u)\,\delta^{\frac{\beta}{2}}.$$

Proof. If we use (5.4), (2.4) $\left(\delta=\max_{0\leq u\leq\frac{n+a}{n+b}}\delta\,(t-u)^2\right)$ and use Cauchy-Schwartz-Bunyakowsky inequlity, then by using the operators (1.2) we have

$$\begin{aligned}
|T_n(h;u)-h(u)| &\leq T_n(|h(t)-h(u)|\,;u) \\
&\leq \omega_\beta^*(h,u)\,T_n\left(|t-u|^\beta;u\right) \\
&\leq \omega_\beta^*(h,u)\,T_n\left((t-u)^2;u\right)^{\frac{\beta}{2}} \\
&\leq \omega_\beta^*(h,u)\,\delta^{\frac{\beta}{2}}.
\end{aligned}$$

□

6. L_p approximation

Theorem 6.1. *Let $h\in L_p[0,1]$ for $0\leq p<\infty$. Then*

$$\lim_{n\to\infty}\|T_n(h)-h\|_{L_p(I_n)}=0,$$

is available.

Proof. First, we need to show that there exist a $K>0$ such that $\|T_n\|_{L_p(I_n)}\leq K$ for any $n\in\mathbb{N}$. For this purpose, if we use (2.5) we have $\|T_n\|_{L_p(I_n)}\leq K$. We consider the operator (1.3).

Let's remember the Luzin theorem for a given $\varepsilon>0$, there exists $f\in C[0,1]$ such that

$$\|h-f\|_{L_p[0,1]}<\frac{\varepsilon}{2(K+1)}.$$

By using Theorem 3.1 for the same ε there exist n_0 such that for all $n>n_0$

$$\|T_n(f;u)-f(u)\|_{L_p(I_n)}\leq\frac{\varepsilon}{2}.$$

Based on this information, the following result is obtained

$$\begin{aligned}
\|T_n(h)-h\|_{L_p(I_n)} &\leq \|T_n(h)-T_n(f)\|_{L_p(I_n)}+\|T_n(f)-f\|_{C(I_n)}+\|h-f\|_{L_p(I_n)} \\
&= (K+1)\|h-f\|_{L_p(I_n)}+\|T_n(f)-f\|_{C(I_n)} \\
&< \varepsilon.
\end{aligned}$$

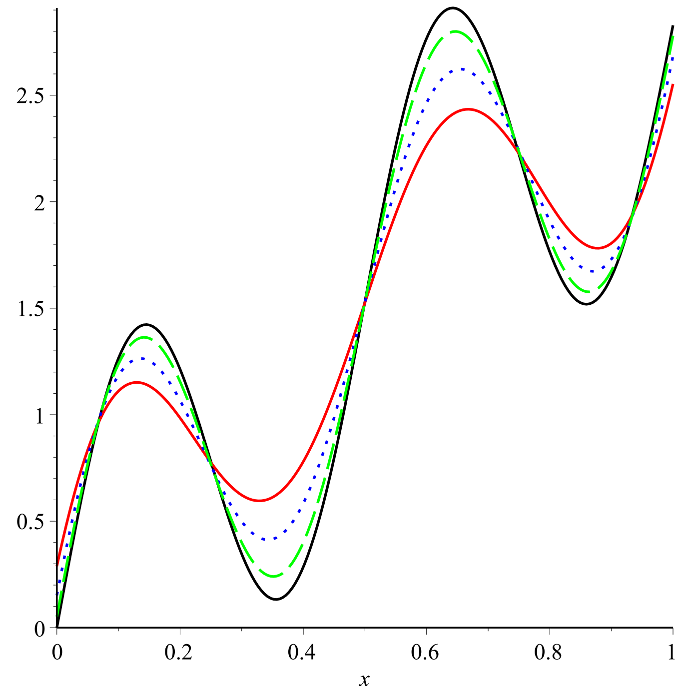
Then, the prof is completed. □

7. Some plots

In this section, we discuss the approximation behaviour of the sequence of operator defined by (1.2) for different functions with the help of graphs. In addition, margins of error is shown with tables of numerical values.

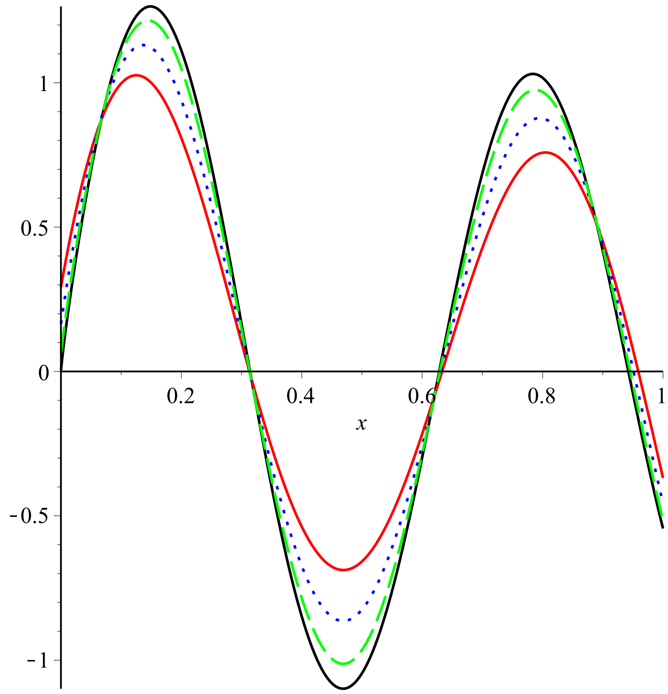
Example 7.1. Let $a = 0.4$, $b = 0.5$ and $h(u) = \sin(4\pi u) + 4\sin(\frac{1}{4}\pi u)$. Fig. 1 shows the $T_n(h;u)$ operator's approximation to the $h(u)$ (black) function for the values $n = 50$ (red), $n = 100$ (blue) and $n = 300$ (green).

Figure 1. $T_n(h;u)$ Operator's approximation to the function $h(u) = \sin(4\pi u) + 4\sin(\frac{1}{4}\pi u)$ for different n values.



Example 7.2. Let be $a = 0.9$, $b = 0.8$ and $h(u) = u^{\frac{-1}{8}}\sin(10u)$. Fig. 2 shows the $T_n(h;u)$ operator's approximation to the $h(u)$ (black) function for the values $n = 50$ (red), $n = 100$ (blue) and $n = 300$ (green).

Figure 2. $T_n\,(h;u)$ Operator's approximation to the function $h(u)=u^{\frac{-1}{8}}\sin(10u)$ for different n values.



Now let's compare the classical Bernstein -Durrmeyer operator defined below with our operator defined in (1.2) with a graph;

$$S_n(h;u)=(n+1)\sum_{k=0}^{n}\varphi_{n,k}(u)\int_0^1\varphi_{n,k}(t)h(t)dt$$

here $\varphi_{n,k}(u)=\binom{n}{k}u^k\,(1-u)^{n-k}$, $h\in C\,[0,1]\,,u\in[0,1]\,.$

Example 7.3. Let be $a=10$, $b=50$ and $h(u)=u^{\frac{-1}{8}}\sin(10u)$. Fig. 3 shows the $T_n\,(h;u)$ (blue) and $S_n(h;u)$ (red) operators are approximation to the $h(u)$ (black) function for the value $n=100$.

Table 1 shows the numerical values obtained with the maximum value of the statement $|T_n\,(h;u)-h(u)|$, in order to examine how the $T_n\,(h;u)$ operator approximation the function $h(u)=\sin(4\pi u)+4\sin(\frac{1}{4}\pi u)$ for $a=0.4$, $b=0.5$ and different n, u values.

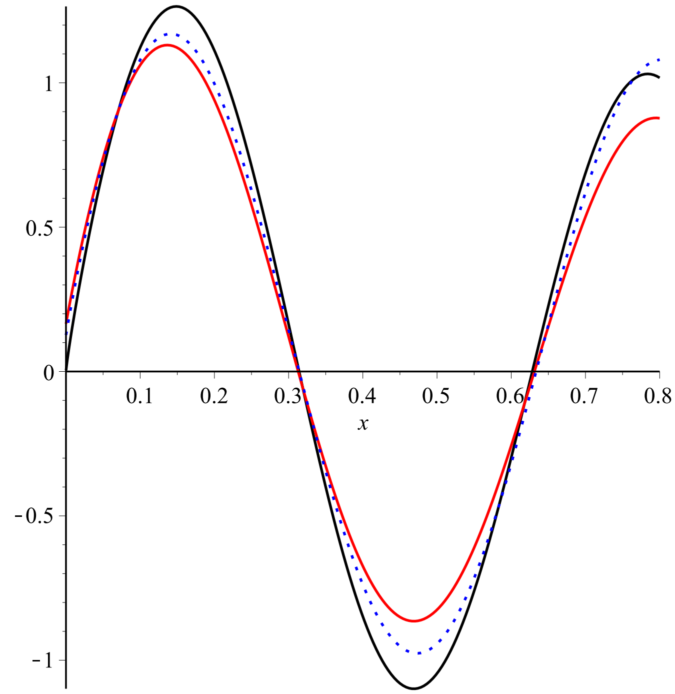
Table 1. Error margins between the $T_n\,(h;u)$ operator and $h(u)=\sin(4\pi u)+4\sin(\frac{1}{4}\pi u)$

n	$u=0.2$	$u=0.4$	$u=0.6$	$u=0.8$
150	0.1065373586	0.213548394	0.216177741	0.106684568
250	0.0686472096	0.135810327	0.137416949	0.068763059
500	0.0362438876	0.071056815	0.071872379	0.036314182
1000	0.0186274446	0.036364660	0.036775374	0.018666478

The definition Izgi[14] provided to compare the approaches of different operators can be given as it comprises statements that can be simplified as the numerator and the denominator. L_n and T_n are operators defined in the same range:

$$\lim_{n\to\infty}\frac{\sup_{0\le u\le\frac{n+a}{n+b}}|L_n(h;u)-h(u)|}{\sup_{0\le u\le\frac{n+a}{n+b}}|T_n(h;u)-h(u)|}=\begin{cases}0, & T_n,\ \textit{faster}\\ \infty, & L_n,\ \textit{faster}\\ c(\text{constant}), & \textit{equally fast}\end{cases}$$

Figure 3. $T_n\,(h;u)$ and $S_n(h;u)$ Operators are approximation to the function $h(u) = u^{\frac{-1}{8}}\sin(10u)$ for $n = 100$.



Based on this definition, it is possible to examine the rate of approximation of the operators defined by

$$E_n(h;u) = \frac{\sup_{0\le u\le \frac{n+a}{n+b}} |T_n(h;u) - h(u)|}{\sup_{0\le u\le \frac{n+a}{n+b}} |S_n(h;u) - h(u)|}.$$

The E_n operator was defined by Aydın Izgi [17] in 2013 and this ratio is used as a measurement in many articles.

As shown in Fig 3, the operators will be compared in Table 2 for $a = 0.1$, $b = 0.8$ and different n values in order to use the u points where the difference is seen more clearly.

Table 2. Error margins between the $E_n\,(h;u)$ operator and $h(u) = u^{\frac{-1}{8}}\sin(10u)$

n	$u = 0.15$	$u = 0.45$	$u = 0.6$	$u = 0.75$
150	0.9946389397	0.9924182950	0.9755127166	0.9850499535
250	0.9967498131	0.9953122198	0.9842312426	0.9907327967
500	0.9983634502	0.9976010256	0.9916588067	0.9952461550
1000	0.9991709201	0.9987821715	0.9956978930	0.9975866425

Table 2, shows that $T_n\,(h;u)$ operators approximation the $h(u) = u^{\frac{-1}{8}}\sin(10u)$ function better than the operators of $S_n\,(h;u)$.

8. Conclusion

In general, a new sequence of Bernstein-Durrmeyer operators was defined in our study. First, the approximation behaviors for the defined operator sequences in the Lebesgue Measurable space were investigated in the article. Then, with the help of Korovkin's theorem, the modulus of continuity and the Peetre K-function on the space l_p, the convergence rate and the order of approximation are discussed. Also, the Voronovskaja type theorem was proved to approximate a class of functions with continuous derivatives of the first and second order. Finally, numerical and graphical analyses were examined to show better approximation behavior for these operator sequences, and it was seen that the operator we have just defined works more efficiently than the Bernstein-Durrmeyer operator defined earlier.

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On Generalizations of Hölder's and Minkowski's Inequalities

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Abstract
We present the generalizations of Hölder's inequality and Minkowski's inequality along with the generalizations of Aczél's, Popoviciu's, Lyapunov's and Bellman's inequalities. Some applications for the metric spaces, normed spaces, Banach spaces, sequence spaces and integral inequalities are further specified. It is shown that $(\mathbb{R}^n, d)$ and $(l_p, d_{m,p})$ are complete metric spaces and $(\mathbb{R}^n, \|x\|_m)$ and $\left(l_p, \|x\|_{m,p}\right)$ are $\frac{1}{m}-$Banach spaces. Also, it is deduced that $\left(b_{p,1}^{r,s}, \|x\|_{r,s,m}\right)$ is a $\frac{1}{m}-$normed space.

Keywords: *Aczél's inequality; Bellman's inequality; Hölder's inequality; Lyapunov's inequality; Minkowski's inequality; Popoviciu's inequality.*

AMS Subject Classification (2020): *Primary:26D15; Secondary:26A51; 26D10; 26A42; 46B20; 46B25.*

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1. Introduction

We shall use $\mathbb{N}$ to denote the set of positive integers, $\mathbb{C}$ for the set of complex numbers, $\mathbb{R}$ for the set of real numbers and $\mathbb{R}^n$ for the set of all ordered n-tuples $x = (x_1, x_2, \ldots, x_n)$ of real numbers x_i.

In [1], the following extensions of the inequalities of Hölder and Minkowski are given respectively:
If $x_{i,j} > 0 \;\; for\; i = 1, 2, \ldots, n \; and \; j = 1, 2, \ldots, m, \; and \; if \; p_j > 0 \; with \; \sum_{j=1}^{m} \frac{1}{p_j} = 1$, then

$$\sum_{i=1}^{n} \prod_{j=1}^{m} x_{ij} \leq \prod_{j=1}^{m} \left(\sum_{i=1}^{n} x_{ij}^{p_j} \right)^{1/p_j}, \tag{1.1}$$

the sign of equality holding if and only if the m sets $(x_{i1}^{p_1}), (x_{i2}^{p_2}), \ldots, (x_{im}^{p_m})$ are proportional, that is, if and only if there are numbers $\lambda_i, \; not \; all \; 0$, such that $\sum_{j=i}^{m} \lambda_j x_{ij}^{p_j} = 0 \;\; for \; i = 1, 2, \ldots, n.$

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If $x_{i,j} \geq 0 \ \ for\ i = 1, 2, \ldots, n\ and\ j = 1, 2, \ldots, k,\ and\ if\ p > 1$, then

$$\left(\sum_{i=1}^{n} \left(\sum_{j=1}^{k} x_{ij} \right)^{p} \right)^{1/p} \leq \sum_{j=1}^{k} \left(\sum_{i=1}^{n} x_{ij}^{p} \right)^{1/p}. \tag{1.2}$$

The inequality is reversed for $p < 1\ (p \neq 0)$. $(For\ p < 0,\ \ we\ assume\ that\ x_{i,j} > 0)$. In each case, the sign of equality holds if and only if the k sets $(x_{i1}), (x_{i2}), \ldots, (x_{ik})$ are proportional.

Similarly, the integral form of the Hölder inequality is

$$\int_a^b \left(\prod_{j=1}^{m} f_j(x) \right) dx \leq \prod_{j=1}^{m} \left(\int_a^b f_j^{p_j}(x)\, dx \right)^{1/p_j}, \tag{1.3}$$

where $f_j(x) > 0\ (j = 1, 2, \ldots, m),\ x \in [a, b],\ -\infty < a < b < +\infty,\ p_j > 0,\ \sum_{j=1}^{m} \frac{1}{p_j} = 1$ and $f_j \in L^{p_j}[a, b]$.

Furthermore, the integral form of the Minkowski inequality is

$$\left(\int_a^b \left(\sum_{j=1}^{k} f_j(x) \right)^{p} dx \right)^{1/p} \leq \sum_{j=1}^{k} \left(\int_a^b f_j^{p}(x) dx \right)^{1/p},$$

where $f_j(x) > 0\ (j = 1, 2, \ldots, k),\ x \in [a, b],\ -\infty < a < b < +\infty,\ p > 0$ and $f_j \in L^p[a, b]$.

A normed linear space is called complete if every Cauchy sequence in the space converges, that is, if for each Cauchy sequence (f_n) in the space there is an element f in the space such that $f_n \to f$. A complete normed linear space is called a Banach space. [2](p. 115).

For $1 \leq p < \infty$, we denote by l_p the space of all sequences $x = (x_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. The space l_p is a Banach space by the norm

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p},$$

which is given by Yosida in [3] (p. 55).

In [4, 5], the sequence space $b_p^{r,s}$ is given by

$$b_p^{r,s} = \left\{ x = (x_k) \in w : \sum_{n} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right|^p < \infty \right\}.$$

Where $1 \leq p < \infty,$ r and s are nonzero real numbers with $r + s \neq 0$. The binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is defined as follows:

$$b_{nk}^{r,s} = \begin{cases} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k & , 0 \leq k \leq n \\ 0 & , k > n \end{cases},$$

for all $k, n \in N$. For $sr > 0$, one can easily check that the following properties hold for the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$:

(i) $\|B^{r,s}\| < \infty,\quad (ii)\ \lim_{n\to\infty} b_{nk}^{r,s} = 0\quad (each\ k \in N),\quad (iii) \lim_{n\to\infty} \sum_k b_{nk}^{r,s} = 1$.

Thus, the binomial matrix is regular whenever $sr > 0$.

Young's inequality asserts that,

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab,\quad for\ all\ a, b \geq 0,$$

whenever $p, q \in (1, \infty)\ and\ \frac{1}{p} + \frac{1}{q} = 1$; the equality holds if and only if $a^p = b^q$.

W.H. Young actually proved a much more general inequality which yields the aforementioned one for $f(x) = x^{p-1}$:

Theorem 1.1 (Young's inequality). *Suppose that $f:[0,\infty)\to[0,\infty)$ is an increasing continuous function such that $f(0)=0$ and $\lim_{x\to\infty} f(x)=\infty$. Then*

$$ab\le\int_0^a f(x)dx+\int_0^b f^{-1}(x)\,dx,$$

$for\ all\ a,b\ge 0,\ and\ equality\ occurs\ if\ and\ only\ if\ b=f(a)$. *[6] (p. 15).*

In the next section, we consider the following form of Young's inequality, for $s\ge 1$

$$\frac{a^{sp}}{sp}+\frac{b^{sq}}{sq}\ge ab,$$

where $a,b\ge 0,\quad sp,sq\in(1,\infty)\ \ and\ \frac{1}{sp}+\frac{1}{sq}=1.$

In [7], H. Agahi et al. gave some generalizations of Hölder's and Minkowski's inequalities for the pseudo-integral. In [8], C.J. Zhao and W.S. Cheung gave an improvement of Minkowski's inequality. In [9], X. Zhou established some functional generalizations and refinements of Aczél's inequality and of Bellman's inequality. In [10], S.I. Butt et al. gave refinements of the discrete Hölder's and Minkowski's inequalities for finite and infinite sequences by using cyclic refinements of the discrete Jensen's inequality. In [11], S. Rashid et al. established Minkowski and reverse Hölder inequalities by employing weighted $AB\mathscr{A}\mathscr{B}-$fractional integral. In [12], S. Rashid et al. gave new fractional behavior of Minkowski inequality and several other related generalizations in the frame of the newly proposed fractional operators. In [13], S. Rashid et al. presented the major consequences of the certain novel versions of reverse Minkowski and related Hölder-type inequalities via discrete $\hbar\hbar$-proportional fractional sums. In [14], S. Rashid et al. gave the certain novel versions of reverse Minkowski and related Hölder-type inequalities via discrete-fractional operators having $\hbar\hbar$-discrete generalized Mittag-Leffler kernels. In [15], S. Rafeeq et al. presented the explicit bounds for three generalized delay dynamic Gronwall–Bellman type integral inequalities on time scales, which are the unification of continuous and discrete results. In [16], Z. Zong et al. investigated the n-dimensional $(n\ \ge\ 1)$ Jensen inequality, Hölder inequality, and Minkowski inequality for dynamically consistent nonlinear evaluations in $L^1\left(\Omega,F,(F_t)_{t\ge 0},P\right)$. Furthermore, they gave four equivalent conditions on the n-dimensional Jensen inequality for g-evaluations induced by backward stochastic differential equations with non-uniform Lipschitz coefficients in $L^p\left(\Omega,F,(F_t)_{0\le t\le T},P\right)$ $(1\ <\ p\ \le\ 2)$. Finally, they gave a sufficient condition on g that satisfies the non-uniform Lipschitz condition under which Hölder's inequality and Minkowski's inequality for the corresponding g-evaluation hold true.

Hölder's inequality, power-mean inequality and Jensen's inequality are used to obtain Hermite-Hadamard type inequalities and Ostrowski's type inequalities for different kinds of convexity which are used in the fields of integral inequalities, approximation theory, special means theory, optimization theory, information theory and numerical analysis. Furthermore, both the Hölder inequality and the Minkowski inequality play an important role in many areas of pure and applied mathematics. These inequalities have been used in several areas of mathematics, especially in functional analysis and generalized in various directions.

The main aim of this paper is to give generalizations of Hölder's, Minkowski's, Aczél's, Popoviciu's, Lyapunov's and Bellman's inequalities.

For several recent results concerning Hölder's inequality, Minkowski's inequality, Hermite-Hadamard type inequalities and Banach spaces, we refer to [1, 6, 7, 10, 16–40]. For Aczél's, Popoviciu's, Bellman's inequalities and the related results, we refer to [3, 9, 36].

2. Main results

First, we give a generalization of Hölder's inequality.

Theorem 2.1. *If $a_k,\ b_k\ge 0$ for $k=1,2,\dots,n$ and $\frac{1}{sp}+\frac{1}{sq}=1$ with $p>1,\ \ s\ge 1$, then*

$$\sum_{k=1}^{n}(a_kb_k)^{1/s}\le\left(\sum_{k=1}^{n}a_k^p\right)^{1/sp}\left(\sum_{k=1}^{n}b_k^q\right)^{1/sq}, \tag{2.1}$$

with equality holding if and only if $\alpha a_k^p=\beta b_k^q$ for $k=1,2,\dots n$, where α and β are real nonnegative constants such that $\alpha^2+\beta^2>0$.

Proof. If $\sum_{k=1}^{n} a_k^p = 0 \;\; or \;\; \sum_{k=1}^{n} b_k^q = 0$, then equality holds in (2.1). Let $\sum_{k=1}^{n} a_k^p > 0 \; and \; \sum_{k=1}^{n} b_k^q > 0$. Substituting

$$a = a_\nu^{1/s}\left(\sum_{k=1}^{n} a_k^p\right)^{-1/sp}, \; b = b_\nu^{1/s}\left(\sum_{k=1}^{n} b_k^q\right)^{-1/sq}, \tag{2.2}$$

into the inequality

$$\frac{a^{sp}}{sp} + \frac{b^{sq}}{sq} \geq ab, \tag{2.3}$$

we get

$$\frac{a_\nu^p}{sp}\left(\sum_{k=1}^{n} a_k^p\right)^{-1} + \frac{b_\nu^q}{sq}\left(\sum_{k=1}^{n} b_k^q\right)^{-1} \geq \frac{a_\nu^{1/s} b_\nu^{1/s}}{(\sum_{k=1}^{n} a_k^p)^{1/sp}(\sum_{k=1}^{n} b_k^q)^{1/sq}}$$

and

$$\frac{1}{sp}\frac{a_\nu^p}{\sum_{k=1}^{n} a_k^p} + \frac{1}{sq}\frac{b_\nu^q}{\sum_{k=1}^{n} b_k^q} \geq \frac{a_\nu^{1/s} b_\nu^{1/s}}{(\sum_{k=1}^{n} a_k^p)^{1/sp}(\sum_{k=1}^{n} b_k^q)^{1/sq}}.$$

Adding together these inequalities for $\nu = 1, 2, \ldots, n$, we have

$$\frac{1}{sp} + \frac{1}{sq} \geq \frac{\sum_{k=1}^{n} a_k^{1/s} b_k^{1/s}}{(\sum_{k=1}^{n} a_k^p)^{1/sp}(\sum_{k=1}^{n} b_k^q)^{1/sq}}.$$

For $\frac{1}{sp} + \frac{1}{sq} = 1$, we obtain the inequality (2.1).
Since equality holds in (2.3) if and only if $a^{sp} = b^{sq}$, we conclude, in virtue of (2.2), that there is equality in (2.1) if and only if $a_k^p(\sum_{k=1}^{n} a_k^p)^{-1} = b_k^q(\sum_{k=1}^{n} b_k^q)^{-1} \; for \; k = 1, 2, \ldots, n$, i.e., if and only if $\alpha a_k^p = \beta b_k^q \; for \; k = 1, 2, \ldots, n$. This completes the proof. □

Remark 2.1. a) If we put $s = 1$ in (2.1), we get Hölder's inequality.
b) If we put $s = 1$ and $p = q = 2$ in (2.1), we get Cauchy-Schwarz inequality.
c) From (1.1), the extension of (2.1) becomes, for $\sum_{j=1}^{m} \frac{1}{sp_j} = 1$

$$\sum_{i=1}^{n}\left(\prod_{j=1}^{m} x_{ij}\right)^{1/s} \leq \prod_{j=1}^{m}\left(\sum_{i=1}^{n} x_{ij}^{p_j}\right)^{1/sp_j}.$$

d) By (1.3), the integral form of inequality (2.1) becomes

$$\int_a^b \left(\prod_{j=1}^{m} f_j(x)\right)^{1/s} dx \leq \prod_{j=1}^{m}\left(\int_a^b f_j^{p_j}(x)\, dx\right)^{1/sp_j}. \tag{2.4}$$

e) By the inequality (2.3) in [28], we have for $s > 1 \; and \; x_i, y_i > 0$,

$$i = 1, 2, \ldots, n,$$

$$\left(\sum_{i=1}^{n} x_i^s\right)^{1/s} \leq \sum_{i=1}^{n} x_i.$$

Using the inequality above, from the Hölder's inequality and taking $x_i = a_k b_k$, we get, for $s > 1$

$$\left(\sum_{k=1}^{n} (a_k b_k)^s\right)^{1/s} \leq \sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p}\left(\sum_{k=1}^{n} b_k^q\right)^{1/q},$$

and from this inequality we obtain

$$\sum_{k=1}^{n} (a_k b_k)^s \leq (\sum_{k=1}^{n} a_k b_k)^s \leq \left(\sum_{k=1}^{n} a_k^p\right)^{s/p}\left(\sum_{k=1}^{n} b_k^q\right)^{s/q},$$

which is another generalization of Hölder's inequality.
f) Let $\sum_{k=1}^{\infty} a_k^p$ and $\sum_{k=1}^{\infty} b_k^q$ be convergent series. Then, from the last inequalities in e), we have

$$\sum_{k=1}^{\infty}(a_k b_k)^s \leq \left(\sum_{k=1}^{\infty} a_k b_k\right)^s \leq \left(\sum_{k=1}^{\infty} a_k^p\right)^{\frac{s}{p}} \left(\sum_{k=1}^{\infty} b_k^q\right)^{\frac{s}{q}}.$$

Now, we give a generalization of Minkowski's inequality.

Theorem 2.2. *If* $a_k,\ b_k \geq 0,$ *for* $k = 1, 2, \ldots, n$ *and* $p > 1,$ *then*

$$(\sum_{k=1}^{n}(a_k + b_k)^p)^{1/mp} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/mp} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/mp}, \tag{2.5}$$

with equality holding if and only if the n-tuples $a = (a_1, a_2, \ldots, a_n)$ *and* $b = (b_1, b_2, \ldots, b_n)$ *are proportional, where* $m \in \mathbb{N}$.

Proof. We consider the identity

$$(a_k + b_k)^p = (a_k + b_k)^{1/m}(a_k + b_k)^{p-1/m}.$$

Using the inequality, for $a > 0,\ b > 0$ *and* $m \in \mathbb{N}$,

$$\sqrt[m]{a+b} \leq \sqrt[m]{a} + \sqrt[m]{b},$$

we obtain

$$(a_k + b_k)^p \leq (\sqrt[m]{a_k} + \sqrt[m]{b_k})(a_k + b_k)^{p-1/m}$$
$$\leq \sqrt[m]{a_k}(a_k + b_k)^{p-\frac{1}{m}} + \sqrt[m]{b_k}(a_k + b_k)^{p-1/m}.$$

Summing over $k = 1, 2, \ldots, n$, we get

$$\sum_{k=1}^{n}(a_k + b_k)^p \leq \sum_{k=1}^{n} \sqrt[m]{a_k}(a_k + b_k)^{p-\frac{1}{m}} + \sum_{k=1}^{n} \sqrt[m]{b_k}(a_k + b_k)^{p-1/m}.$$

By the inequality (2.1), for $\frac{1}{mp} + \frac{1}{mq} = 1$ *and* $p > 1$, we have

$$\sum_{k=1}^{n} \sqrt[m]{a_k}(a_k + b_k)^{p-\frac{1}{m}} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/mp} \left(\sum_{k=1}^{n}(a_k + b_k)^{q(mp-1)}\right)^{1/mq}$$

and

$$\sum_{k=1}^{n} \sqrt[m]{b_k}(a_k + b_k)^{p-\frac{1}{m}} \leq \left(\sum_{k=1}^{n} b_k^p\right)^{1/mp} \left(\sum_{k=1}^{n}(a_k + b_k)^{q(mp-1)}\right)^{1/mq}.$$

Adding the last two relations, we obtain,

$$\sum_{k=1}^{n}(a_k + b_k)^p \leq \left[\left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{mp}} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/mp}\right] \left(\sum_{k=1}^{n}(a_k + b_k)^{q(mp-1)}\right)^{1/mq}.$$

Since $\frac{1}{mp}+\frac{1}{mq} = 1,$ *we get* $p = q(mp-1)$. Also, by dividing both sides of the inequality above by $(\sum_{k=1}^{n}(a_k + b_k)^p)^{1/mq}$, we obtain

$$(\sum_{k=1}^{n}(a_k + b_k)^p)^{1/mp} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/mp} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/mp},$$

which is required. □

Remark 2.2. a) If $m = 1$ is substituted into (2.5), we get Minkowski's inequality.
b) If $p = 2$ is substituted into (2.5), we get

$$\left(\sum_{k=1}^{n}(a_k+b_k)^2\right)^{1/2m} \leq \left(\sum_{k=1}^{n}a_k^2\right)^{1/2m} + \left(\sum_{k=1}^{n}b_k^2\right)^{1/2m}. \tag{2.6}$$

c) From (1.2), the extension of (2.5) becomes

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{k}x_{ij}\right)^{p}\right)^{1/mp} \leq \sum_{j=1}^{k}\left(\sum_{i=1}^{n}x_{ij}^{p}\right)^{1/mp}.$$

In the following theorems, we give the generalizations of the reverse Hölder inequality, Popoviciu's inequality, Lyapunov's inequality and Bellman's inequality respectively.

Theorem 2.3. *If $a_k,\ b_k > 0$ for $k = 1, 2, \ldots, n$ and $\frac{1}{sp} + \frac{1}{sq} = 1$ with $sp < 0$ or $sq < 0$, for $s \geq 1$, then*

$$\sum_{k=1}^{n}(a_k b_k)^{1/s} \geq \left(\sum_{k=1}^{n}a_k^p\right)^{1/sp}\left(\sum_{k=1}^{n}b_k^q\right)^{1/sq}, \tag{2.7}$$

with equality holding if and only if $\alpha a_k^p = \beta b_k^q$ for $k = 1, 2, \ldots n$, where α and β are real nonnegative constants such that $\alpha^2 + \beta^2 > 0$.

Proof. Let $sp < 0$ and put $P = -\frac{p}{sq},\ Q = \frac{1}{s^2 q}. Then\ \frac{1}{sP} + \frac{1}{sQ} = 1$ with $sP > 0\ and\ sQ > 0$. Therefore, according to (2.1), we obtain

$$\left(\sum_{k=1}^{n}A_k^P\right)^{1/sP}\left(\sum_{k=1}^{n}B_k^Q\right)^{1/sQ} \geq \sum_{k=1}^{n}(A_k B_k)^{1/s},$$

where $A_k > 0\ and\ B_k > 0\ for\ k = 1, 2, \ldots, n.$ The last inequality for $A_k = a_k^{-sq}$ and $B_k = a_k^{sq}b_k^{sq}$ becomes

$$\left(\sum_{k=1}^{n}a_k^p\right)^{-\frac{sq}{sp}}\left(\sum_{k=1}^{n}(a_k b_k)^{1/s}\right)^{sq} \geq \sum_{k=1}^{n}(b_k^{sq})^{1/s}.$$

Hence, we have

$$\sum_{k=1}^{n}(a_k b_k)^{1/s} \geq \left(\sum_{k=1}^{n}a_k^p\right)^{1/sp}\left(\sum_{k=1}^{n}b_k^q\right)^{1/sq},$$

which is (2.7). □

Theorem 2.4. *Let a and b be two nonnegative n-tuples, p and q are real numbers such that $p, q \neq 0,\ s \geq 1$ and $\frac{1}{sp} + \frac{1}{sq} = 1$ and let $a_1^p - a_2^p - \cdots - a_n^p > 0$ and $b_1^q - b_2^q - \cdots - b_n^q > 0$. Then, we have for $p > 1$,*

$$(a_1^p - a_2^p - \cdots - a_n^p)^{1/s}(b_1^q - b_2^q - \cdots - b_n^q)^{1/s} \leq a_1^{\frac{1}{s}}b_1^{\frac{1}{s}} - a_2^{\frac{1}{s}}b_2^{\frac{1}{s}} - .. - a_n^{\frac{1}{s}}b_n^{\frac{1}{s}}. \tag{2.8}$$

If $p < 1$ $(p \neq 0)$, we have the reverse inequality.

Proof. Replacing a_1^p and b_1^q by $a_1^p - a_2^p - \cdots - a_n^p$ and $b_1^q - b_2^q - \cdots - b_n^q$, respectively, in (2.1), we have

$$(a_1 b_1)^{1/s} \leq (a_1 b_1)^{\frac{1}{s}} - (a_2 b_2)^{\frac{1}{s}} - \cdots - (a_n b_n)^{\frac{1}{s}}.$$

Resubstituting, the last inequality becomes

$$(a_1^p - a_2^p - \cdots - a_n^p)^{1/s}(b_1^q - b_2^q - \cdots - b_n^q)^{1/s} \leq a_1^{\frac{1}{s}}b_1^{\frac{1}{s}} - a_2^{\frac{1}{s}}b_2^{\frac{1}{s}} - .. - a_n^{\frac{1}{s}}b_n^{\frac{1}{s}}.$$

which is (2.8). □

Remark 2.3. a) If we put $s = 1$ in (2.8), we get Popoviciu's inequality.
b) If we put $s = 1$ *and* $p = q = 2$ in (2.8), we get Aczél's inequality.

Remark 2.4. From (2.1) for $u \geq 1$, we get

$$\sum_{k=1}^{n} (a_k b_k)^{1/u} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/up} \left(\sum_{k=1}^{n} b_k^q\right)^{1/uq},$$

where $\frac{1}{up} + \frac{1}{uq} = 1$. Substituting $p = \frac{r-t}{r-s}$, $q = \frac{r-t}{s-t}$ $(r > s > t > 0)$, $a_k^p = p_k x_k^t$ *and* $b_k^q = p_k x_k^r$, $(p_k \geq 0,\ x_k \geq 0\ for\ k = 1, 2, \ldots, n\ )$ into the inequality above, we have

$$\sum_{k=1}^{n} (p_k x_k^t)^{\frac{r-s}{u(r-t)}} (p_k x_k^r)^{\frac{s-t}{u(r-t)}} \leq (\sum_{k=1}^{n} p_k x_k^t)^{\frac{r-s}{u(r-t)}} (\sum_{k=1}^{n} p_k x_k^r)^{\frac{s-t}{u(r-t)}}.$$

From the last inequality, we get

$$\sum_{k=1}^{n} \left(p_k x_k^{\frac{s}{u}}\right)^{u(r-t)} \leq (\sum_{k=1}^{n} p_k x_k^t)^{r-s} (\sum_{k=1}^{n} p_k x_k^r)^{s-t}, \tag{2.9}$$

which is the generalization of Lyapunov's inequality. Letting $u = 1$ in (2.9), we obtain Lyapunov's inequality.

Theorem 2.5. *Let a and b be n-tuples of nonnegative numbers such that* $a_1^{mp} - a_2^p - \cdots - a_n^p > 0$ *and* $b_1^{mp} - b_2^p - \cdots - b_n^p > 0$. *If* $p \geq 1$ $(or\ p < 0)$, *then*

$$\left[(a_1^{mp} - a_2^p - \cdots - a_n^p)^{\frac{1}{p}} + (b_1^{mp} - b_2^p - \cdots - b_n^p)^{\frac{1}{p}}\right]^p$$

$$\leq (a_1 + b_1)^{mp} - (a_2 + b_2)^p - .. - (a_n + b_n)^p. \tag{2.10}$$

If $0 < p < 1$, *then the reverse inequality in* (2.10) *holds, where* $m \in \mathbb{N}$.

Proof. Replacing a_1^p and b_1^p by $a_1^{mp} - a_2^p - \cdots - a_n^p$ and $b_1^{mp} - b_2^p - \cdots - b_n^p$, respectively, in (2.5), we have

$$(a_1 + b_1)^p \leq (a_1 + b_1)^{mp} - (a_2 + b_2)^p - \cdots - (a_n + b_n)^p.$$

Resubstituting, the last inequality becomes

$$\left[(a_1^{mp} - a_2^p - \cdots - a_n^p)^{\frac{1}{p}} + (b_1^{mp} - b_2^p - \cdots - b_n^p)^{\frac{1}{p}}\right]^p$$

$$\leq (a_1 + b_1)^{mp} - (a_2 + b_2)^p - \cdots - (a_n + b_n)^p,$$

which is (2.10). □

Remark 2.5. If $m = 1$ is substituted into (2.10), we get Bellman's inequality.

3. Applications

Now, using inequalities (2.5) and (2.6), we give some applications for the metric spaces, normed spaces, Banach spaces and sequence spaces. Furthermore, using inequality (2.4), we give an integral inequality.

Corollary 3.1. *Let* $d : \mathbb{R}^n x \mathbb{R}^n \to \mathbb{R}$ *be the function such that*

$$d(x, y) = \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2\right]^{1/2m}. \tag{3.1}$$

Then $(\mathbb{R}^n, d)$ *is a metric space for* $m \in \mathbb{N}$.

Proof. The properties (M1) and (M2) of the metric are obvious. Applying the inequality (2.6), we obtain for $x, y, z \in \mathbb{R}^n$

$$d(x,y) = \Big(\sum_{k=1}^{n} |x_k - y_k|^2\Big)^{1/2m} = \Big(\sum_{k=1}^{n} |x_k - z_k + z_k - y_k|^2\Big)^{1/2m}$$
$$\leq \left(\sum_{k=1}^{n} |x_k - z_k|^2\right)^{\frac{1}{2m}} + \Big(\sum_{k=1}^{n} |z_k - y_k|^2\Big)^{1/2m}$$
$$\leq d(x,z) + d(z,y),$$

which is (M3). □

Corollary 3.2. *The space $\mathbb{R}^n$ with the norm defined for $x = (x_1, x_2, \ldots, x_n)$ and $m \in \mathbb{N}$ by*

$$\|x\|_m = \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2m},$$

is a $\frac{1}{m}-$normed vector space.

Proof. The space $\mathbb{R}^n$ is an n-dimensional vector space, so we need to verify the properties of the norm. We have
(N1). $\|x\|_m = 0 \Leftrightarrow x = \theta$.
(N2). For $\alpha \epsilon \mathbb{R}$,

$$\|\alpha x\|_m = \Big(\sum_{i=1}^{n} |\alpha x_i|^2\Big)^{1/2m} = \alpha^{1/m}\Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2m} = \alpha^{1/m}\|x\|_m.$$

(N3). Applying the inequality (2.6), we get

$$\|x+y\|_m = \Big(\sum_{i=1}^{n} |x_i + y_i|^2\Big)^{1/2m} \leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2m}} + \left(\sum_{i=1}^{n} |y_i|^2\right)^{\frac{1}{2m}} = \|x\|_m + \|y\|_m.$$

Thus, $(\mathbb{R}^n, \|x\|_m)$ is a $\frac{1}{m}-$normed vector space. □

Corollary 3.3. *The metric space $(\mathbb{R}^n, d)$ is complete.*

Proof. Suppose that (x_m) is a Cauchy sequence in $\mathbb{R}^n$. Then, we have

$$d(x_m, x_k) \to 0 \qquad (m, k \to \infty).$$

Note that each member of the sequence $(x^{(m)})$ is itself a sequence
$x_m = \left(x_i^{(m)}\right) = \left(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}\right)$, for $m = 1, 2, 3 \ldots$
Now, for each $\varepsilon > 0$ there exists $n_o \in \mathbb{N}$ such that $d(x_m, x_k) < \varepsilon,\ \forall m, k \geq n_o$. By (3.1), we have

$$d(x_m, x_k) = \left(\sum_{i=1}^{n} \left(x_i^{(m)} - x_i^{(k)}\right)^2\right)^{1/2t} < \varepsilon\,, \quad for\ \forall m, k \geq n_o\ and\ t \in \mathbb{N}.$$

Since each term in the above inequality is positive,

$$\left|x_i^{(m)} - x_i^{(k)}\right| < \varepsilon^t \ \ for\ i = 1, 2, \ldots, n\ and\ \forall m, k \geq n_o.$$

Hence $\left(x_i^{(m)}\right) = \left(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}\right)$ is a Cauchy sequence in $\mathbb{R}$, $for\ i = 1, 2, \ldots, n$. Since $\mathbb{R}$ is complete, $\left(x_i^{(m)}\right)$ converges to x_i in $\mathbb{R}$ for $i = 1, 2, \ldots, n$. So,

$$\lim_{m\to\infty} x_i^{(m)} = x_i \quad for\ i = 1, 2, \ldots, n.$$

Let $x = (x_1, x_2, \dots, x_n)$, *then* $x \in \mathbb{R}^n$. We now prove that (x_m) converges to x.

$$d(x_m, x) = \left(\sum_{i=1}^{n}\left(x_i^{(m)} - x_i\right)^2\right)^{1/2t} = \lim_{k\to\infty}\left(\sum_{i=1}^{n}\left(x_i^{(m)} - x_i^{(k)}\right)^2\right)^{1/2t} < \varepsilon,\ \forall m \geq n_0 \,.$$

Hence the Cauchy sequence (x_m) converges to $x \in \mathbb{R}^n$. Thus, $(\mathbb{R}^n, d)$ is a complete metric space. □

Corollary 3.4. *The vector space $\mathbb{R}^n$ with the norm defined for $x = (x_1, x_2, \dots, x_n)$ and $m \in \mathbb{N}$ by*

$$\|x\|_m = (\sum_{i=1}^{n} |x_i|^2)^{1/2m},$$

is a $\frac{1}{m}$−Banach space.

Proof. Note that a Banach space is a normed linear space that is a complete metric space with respect to the metric derived from its norm. For this reason, the claim follows from Corollaries 3.2 and 3.3. □

Corollary 3.5. *Let $d_{m,p}:\ l_p x l_p \to \mathbb{R}$ be the function such that*

$$d_{m,p} = (\sum_{i=1}^{\infty} |x_i - y_i|^p)^{1/mp},$$

for $1 \leq p < \infty$, $m \in \mathbb{N}$ and $x = (x_1, x_2, \dots)$. Then $(l_p, d_{m,p})$ is a metric space.

Proof. The properties (M1) and (M2) of the metric are obvious. The property (M3) follows from the inequality (2.5). □

Corollary 3.6. *The space l_p with the norm defined for $1 \leq p < \infty$, $m \in \mathbb{N}$ and $x = (x_1, x_2, \dots)$ by*

$$\|x\|_{m,p} = (\sum_{i=1}^{\infty} |x_i|^p)^{1/mp},$$

is a $\frac{1}{m}$−normed vector space.

Proof. The properties (N1) and (N2) of the norm are obvious. The property (N3) follows from the inequality (2.5). □

Corollary 3.7. *The metric space $(l_p, d_{m,p})$ is complete.*

Proof. Let (x_n) be a Cauchy sequence in the space l_p, where $x_n = \left(x_i^{(n)}\right) = \left(x_1^{(n)}, x_2^{(n)}, \dots\right)$. Let $\varepsilon > 0$ be a real number. Then, there exists a positive integer n_o such that

$$d_{m,p}(x_n, x_t) = \left(\sum_{i=1}^{n}\left(x_i^{(n)} - x_i^{(t)}\right)^p\right)^{1/mp} < \varepsilon \;, \tag{3.2}$$

$for\ all\ n, t \geq n_o\ and\ m \in \mathbb{N}$. This shows that $\left|x_i^{(n)} - x_i^{(t)}\right| < \varepsilon^m$, for all $n, t \geq n_o$ and consequently $\left(x_i^{(n)}\right) = \left(x_1^{(n)}, x_2^{(n)}, \dots\right)$ is a Cauchy sequence in $\mathbb{K}$ ($\mathbb{R}$ *or* $\mathbb{C}$). Since these spaces are complete, $\left(x_i^{(n)}\right)$ converges to a point $x_i \in \mathbb{K}$. Also, for each $k \in \mathbb{N}$, the statement (3.2) gives

$$\sum_{i=1}^{k}\left|x_i^{(n)} - x_i^{(t)}\right|^p < \varepsilon^{mp}\ for\ all\ n, t \geq n_o. \tag{3.3}$$

From (3.3) with $t \to \infty$, we get

$$\sum_{i=1}^{k}\left|x_i^{(n)} - x_i\right|^p < \varepsilon^{mp}. \tag{3.4}$$

We need to prove that $x = (x_1, x_2, \dots)$ *is in* l_p. The inequalities (3.4) and (2.5) show that

$$\begin{aligned}
(\sum_{i=1}^{k} |x_i|^p)^{1/mp} &= (\sum_{i=1}^{k} \left| x_i - x_i^{(n)} + x_i^{(n)} \right|^p)^{1/mp} \\
&\leq \left(\sum_{i=1}^{k} \left| x_i - x_i^{(n)} \right|^p \right)^{\frac{1}{mp}} + \left(\sum_{i=1}^{k} \left| x_i^{(n)} \right|^p \right)^{1/mp} \\
&< \varepsilon + \left(\sum_{i=1}^{k} \left| x_i^{(n)} \right|^p \right)^{1/mp}.
\end{aligned}$$

Since $\left(x_i^{(n)} \right)$ is in l_p, the above inequality shows that $(\sum_{i=1}^{k} |x_i|^p)^{1/mp}$ is bounded and monotonically increasing, therefore the series $\sum_{i=1}^{k} |x_i|^p$ is convergent. Thus, x *is in* l_p. Also, it is obvious from (3.4) that (x_n) *converges to* x. Therefore, $(l_p, d_{m,p})$ is a complete metric space. □

Corollary 3.8. *The space* l_p *, with the norm defined for* $1 \leq p < \infty,\ m \in \mathbb{N}$ *and* $x = (x_1, x_2, \dots)$ *by*

$$\|x\|_{m,p} = (\sum_{i=1}^{\infty} |x_i|^p)^{1/mp},$$

is a $\frac{1}{m}$*−Banach space.*

Proof. Note that a Banach space is a normed linear space that is a complete metric space with respect to the metric derived from its norm. For this reason, the claim follows from Corollaries 3.6 and 3.7. □

Let $b_{p,1}^{r,s}$ be the binomial sequence space such that

$$b_{p,1}^{r,s} = \left\{ x = (x_k) \in w : \sum_n \left| \frac{1}{(s+r)^n} \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k} s^{i-k} r^k x_k \right|^p < \infty \right\} \quad , 1 \leq p < \infty.$$

The space $b_{p,1}^{r,s}$ includes the spaces l_p and $b_p^{r,s}$. Hence we may give the following corollary.

Corollary 3.9. *The space* $b_{p,1}^{r,s}$ *with the norm defined for* $m \in \mathbb{N}$ *by*

$$\|x\|_{r,s,m} = \left(\sum_{n=0}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k} s^{i-k} r^k x_k \right|^p \right)^{1/mp},$$

is a $\frac{1}{m}$*−normed space.*

Proof. So, we need to verify the conditions (N1)-(N3) of the norm. We have
(N1). $\|x\|_{r,s,m} = 0 \Leftrightarrow x = \theta$.
(N2). For $\alpha \epsilon \mathbb{R}$,

$$\|\alpha x\|_{r,s,m} = \left(\sum_{n=0}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k} s^{i-k} r^k (\alpha x_k) \right|^p \right)^{1/mp} = \alpha^{1/m} \|x\|_{r,s,m}.$$

(N3). Applying the inequality (2.5), we get

$$\begin{aligned}
\|x+y\|_{r,s,m} &= \left(\sum_{n=0}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k} s^{i-k} r^k (x_k + y_k) \right|^p \right)^{1/mp} \\
&\leq \left(\sum_{n=0}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k} s^{i-k} r^k x_k \right|^p \right)^{\frac{1}{mp}} +
\end{aligned}$$

$$+\left(\sum_{n=0}^{\infty}\left|\frac{1}{(s+r)^n}\sum_{i=0}^{n}\sum_{k=0}^{i}\left(\begin{array}{c} i \\ k \end{array}\right)s^{i-k}r^k y_k\right|^p\right)^{1/mp}$$

$$\leq \|x\|_{r,s,m}+\|y\|_{r,s,m}.$$

Thus, $\left(b_{p,1}^{r,s},\|x\|_{r,s,m}\right)$ is a $\frac{1}{m}-$normed space. □

Finally, we give an integral inequality:

Corollary 3.10. *Let f be a real valued function defined on $[a,b]\subset\mathbb{R}^+$ such that the functions $|f|^p$ and $|f|^q$ are integrable on $[a,b]$ and let*

$$I_{n/s}=\int_a^b f(x)^{n/s}dx,$$

then we have for $n>1$ and $s\in\mathbb{N}$

$$I_{2(n-1)/s}^{s}= \ \leq I_{np}^{\frac{1}{p}}I_{(n-2)q}^{\frac{1}{q}}.$$

Proof. Applying the inequality (2.4) for $j=1,2$, we obtain

$$I_{2(n-1)/s}^{s}=\left(\int_a^b f(x)^{2(n-1)/s}dx\right)^s=\left(\int_a^b f(x)^{n/s}f(x)^{(n-2)/s}dx\right)^s$$

$$\leq\left(\int_a^b f(x)^{np}dx\right)^{1/p}\left(\int_a^b f(x)^{(n-2)q}dx\right)^{1/q}\quad\leq I_{np}^{\frac{1}{p}}I_{(n-2)q}^{\frac{1}{q}},$$

which is required. □

4. Conclusion

In this paper, the generalizations of Hölder's inequality and Minkowski's inequality have been presented. Furthermore, the generalizations of Aczél's, Popoviciu's, Lyapunov's and Bellman's inequalities have been given. Finally, some applications for the metric spaces, normed spaces, Banach spaces, sequence spaces and integral inequalities have been provided.

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New Exact Soliton Solutions and Multistability for the Modified Zakharov-Kuznetsov Equation with Higher Order Dispersion

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Abstract
The aim of the present paper is to obtain and analyze new exact travelling wave solutions and bifurcation behavior of modified Zakharov-Kuznetsov (mZK) equation with higher-order dispersion term. For this purpose, the first and second simplest methods are used to build soliton solutions of travelling wave solutions. Furthermore, the bifurcation behavior of traveling waves including new types of quasiperiodic and multi-periodic traveling wave motions have been examined depending on the physical parameters. Multistability for the nonlinear mZK equation has been investigated depending on fixed values of physical parameters with various initial conditions. The suggested methods for the analytical solutions are powerful and beneficial tools to obtain the exact travelling wave solutions of nonlinear evolution equations (NLEEs). Two and three-dimensional plots are also provided to illustrate the new solutions. Bifurcation and multistability behaviors of traveling wave solution of the nonlinear mZK equation with higher-order dispersion will add some value to the literature of mathematical and plasma physics.

Keywords: *Bifurcation; First simplest method; Modified Zakharov-Kuznetsov equation; Second simplest method; Quasiperiodic motion.*

AMS Subject Classification (2020): *35C08 ; 65P30; 35E05; 35C07*

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1. Introduction

NLEEs involve nonlinear complex physical phenomena and play an outstanding role while characterizing complicated phenomena rooting in different branches of science for example fluid flow, wave propagations, fluid mechanics, nonlinear optics, optical fibres, chemical kinematics, chemical physics, plasma physics, solid-state physics, hydrodynamic, nonlinear transmission lines, plasma physics, geochemistry, biology and soil consolidations. Therefore, obtaining exact solutions of such nonlinear equations are a rich area of research for the scientists because

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the resulting solutions can describe physical behaviour of concerned problems in the best way [1–3]. These solutions define various phenomena in nature, such as vibrations, solitons and propagation with a finite speed [4]. In the recent past, many researcher developed a wide range of methods and still trying to construct new methods to establish analytical and solitary traveling wave solutions of the NLEEs. Some of these methods are: inverse scattering method [5, 6], Backlund transformation method [7], modified simplest equation method [8], homogeneous balance method [9], direct algebraic method [10, 11], Hirota bilinear transformation method [12], tanh–sech method [13, 14], extended tanh method [15–17], Jacobi elliptic function expansion method [18–20] , generalized Riccati equation method [21], sine–cosine method [22], F-expansion method [23–25], homogeneous balance method [26], Exp function method [27–30], Cole-Hopf transformation method [31], Adomian decomposition method [32], homotopy analysis method [33, 34], homotopy perturbation method [35], first and second simplest method [36, 37], bifurcation method [38, 39] and first integral method [40].

The nonlinear Zakharov-Kuznetsov (NZK) equation is an another alternative version of nonlinear model describing (2+1)-dimensional modulation of a KdV soliton equation in fluid mechanics [41–43]. In two-and three-dimensional spaces, the NZK equation is given by

$$u_t + auu_x + bu_{xxx} + cu_{xyy} = 0, \tag{1.1}$$

and

$$u_t + auu_x + bu_{xxx} + c(u_{xyy} + u_{xzz}) = 0, \tag{1.2}$$

respectively, where a is known as the coefficient of nonlinear term and $b,\ c$ are called the coefficients of dispersion terms. Here x, y, z are space variables, t is time and u is acoustic wave potential. This equation was first derived to model the propagation of weakly nonlinear ion-acoustic waves in plasma, which involves cold ions and hot-isothermal electrons in a medium with a uniform magnetic field. The equation is also used to define different types of acoustic waves in magnetized plasmas [44]. It has been shown the equation is not integrable by means of the inverse scattering transform method. It was found that the solitary-wave solutions of the ZK equation are inelastic. Hesam et al. [45] developed differential transform method for Zakharov equation. The nonlinear ZK equation with higher order dispersion term is given by

$$u_t + auu_x + bu_{xxx} + c(u_{xyy} + u_{xzz}) + du_{xxxxx} = 0, \tag{1.3}$$

where a, b, c are same as equation (1.2) and d is the coefficient of fifth order dispersion. With an appropriately modified form of the electron number density given in [46], Munro and Parkes [41] demonstrated that reductive perturbation can induce following modified Zakharov–Kuznetsov (mZK) equation

$$16(u_t - ku_x) + 30u^{1/2}u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0, \tag{1.4}$$

where k is a positive constant. The mZK equation have solutions that symbolize plane-periodic and solitary traveling waves propagating. It is noted that the mZK equation is a high dimensional nonlinear evolution equation and, thus, the study of its reduction problem is of theoretical interest [47]. Park et al. [48] applied modified Khater method to equation. The extended mapping method is developed to study the traveling wave solution for a mZK equation by Peng [49]. In our manuscript, we study the following nonlinear modified ZK equation with higher order dispersion term as

$$u_t + au^2u_x + bu_{xxx} + c(u_{xyy} + u_{xzz}) + du_{xxxxx} = 0. \tag{1.5}$$

Multistability alludes to an interesting phenomenon where a dynamical system provides more than one numerical solution for a fixed values of the parameters at various initial conditions [50, 51]. Arecchi et al. [52] performed experimental observation of multistability behavior in a Q-switch laser system. Natiq et al. [53] experienced coexisting features involving chaotic and quasi-periodic phenomena and the coexistence of symmetric Hopf bifurcations. Morfu et al. [54] reported multistability in Cellular Nonlinear Network in image processing. Rahim et al. [55] investigated multistability behavior in a hyperchaotic system. Li and Sprott [56] studied multistability phenomenon in the famous Lorenz system in a special parametric range space. In various fields of plasmas, multistability behavior also known as coexisting features were extensively investigated in discharge plasmas [57], plasma diodes [58], solar wind plasma [59], electron-ion plasma [60], and in various quantum plasmas [61, 62].

The aims of this study are twofold and will take place for the first time in the literature. Firstly, we introduce the soliton solutions of the mZK equation with higher order dispersion term using different typies of two simplest methods:

· First simplest method was suggested by Nikolay A. Kudryashov [63], its applications have also been shown in [36, 64] and

· Second simplest method was suggested by Khalid K. Ali [37].

Secondly, we examined the bifurcation behavior of traveling waves including quasiperiodic, multi-periodic, and multistability motion for the mZK equation with higher-order dispersion depending on the physical parameters. Thus, we construct new exact and travelling wave solutions in soliton.

The remnant rest of this paper is systematized as follows: An introduction is given in Section 1. The main steps of the first and second simplest methods are specified in Section 2. In the next section, in Section 3, we apply these methods in detail with finding the exact travelling wave solutions of the mZK equation. In Section 4, some figures are presented in two and three-dimensional to display the solutions given in Section 3. Bifurcation behavior of travelling wave solution containing: the Quasiperiodic, multi-periodic, and multistability wave motion of the mZK equation is investigated in Section 5. A discussion about the equation is given in Section 6. Finally, the paper ends with a conclusion in Section 7.

2. Overview of the methods

2.1 First simplest method [63]

Let's consider the

$$F(u, u_t, u_x, u_y, u_z, u_{tt}, u_{xx}, u_{yy}, u_{zz}, ...) = 0, \tag{2.1}$$

nonlinear partial differential equation where $u = u(x, y, z, t)$ is the unknown function.

Step 1: Use the following wave transformation:

$$u(x, y, z, t) = u(\xi),\ \ \xi = lx + my + nz - vt, \tag{2.2}$$

where l, m, n are constants and v is velocity of the traveling wave.

$$P(u', u'', u''', ...) = 0, u' = \frac{du}{d\xi}. \tag{2.3}$$

By using above terms, equation (2.1) is reduced to a non-linear ordinary differential equation.

Step 2: Assume solution of (2.3) takes form of a finite series

$$u(\xi) = \sum_{i=0}^{N}(A_i(Q(\xi))^i, \tag{2.4}$$

$A_i(i = 0, 1, 2, ..., N), A_N \neq 0$, are unknowns with $(A_i \neq 0)$ to be calculated. N is a positive integer and will be computed by homogeneous balance algorithm.

Step 3: The function $Q(\xi)$ satisfies auxiliary differential equation:

$$(Q'(\xi))^2 = \alpha^2 Q(\xi)^2(1 - \Omega Q(\xi)^2), \tag{2.5}$$

(2.5) gives the following solution:

$$Q(\xi) = \frac{4\sigma \exp(-\alpha\xi)}{4\sigma^2 + \Omega \exp(-2\alpha\xi)}. \tag{2.6}$$

Step 4: By substituting (2.4) and (2.5) into (2.3) and collecting all terms with the same power of $Q(\xi)$ together, (2.3) turn into a polynomial, taking each coefficient equal to zero, a system of algebraic equations are obtained.

Step 5: By using the Mathematica 11 program, we can obtain the exact solution of (2.3).

2.2 Second simplest method [37]

We illustrate modified Kudryashov method in this section as follows:

Step 1: Assume a solution of (2.3) given in a series form:

$$u(\xi)=\sum_{i=0}^{N}(A_i(Q(\xi))^i, \tag{2.7}$$

where A_i is the same as in First simple method.

Step 2: Function $Q(\xi)$ fulfills the differential equation:

$$(Q'(\xi))^2=\alpha^2(\log(C))^2Q(\xi)^2(1-\Omega Q(\xi)^2), \tag{2.8}$$

the solution of (2.8) is introduced by:

$$Q(\xi)=\frac{4\sigma C^{(-\alpha\xi)}}{4\sigma^2+\Omega C^{(-2\alpha\xi)}}. \tag{2.9}$$

Step 3: Putting (2.7) and (2.8) into (2.3), we procure a polynomial of $Q(\xi)$. Setting all the coefficients of the like powers of $Q(\xi)$ to zero, a system of algebraic equations are obtained.

Step 4: System of equations are solved by Mathematica 11 program. Consequently, we can obtain exact solution of (2.3).

3. Implementations of the methods

We employ the transformation (2.2) with $l^2+m^2+n^2=1$. Then, the equation (1.5) becomes

$$-(v-alu^2)u_\xi+(bl^3+clm^2+cln^2)u_{\xi\xi\xi}+dl^5u_{\xi\xi\xi\xi\xi}=0. \tag{3.1}$$

Integrating equation (3.1) according to ξ,

$$-(v-\frac{alu^2}{3})u+(bl^3+clm^2+cln^2)u_{\xi\xi}+dl^5u_{\xi\xi\xi\xi}=c_1, \tag{3.2}$$

is obtained and here c_1 is an integrating constant. Applying the boundary conditions $u\to 0,\ u_\xi\to 0,\ u_{\xi\xi}\to 0,$ $u_{\xi\xi\xi}\to 0,\ u_{\xi\xi\xi\xi}\to 0$ as $\xi\to\pm\infty$ in equation (3.2), one can obtain $c_1=0$. Then equation (3.2) becomes

$$-(v-\frac{alu^2}{3})u+(bl^3+cl(1-l^2))u_{\xi\xi}+dl^5u_{\xi\xi\xi\xi}=0. \tag{3.3}$$

Balancing u^3 with $u_{\xi\xi\xi\xi}$ in (3.3), following relation is obtained:

$$3N=N+4\Rightarrow N=2. \tag{3.4}$$

3.1 First simplest method

From (2.4) and (3.4), the solution of (3.3) is written in the form:

$$u(\xi)=A_0+A_1Q(\xi)+A_2Q^2(\xi). \tag{3.5}$$

By setting above solution in equation (3.3) and equating factors of each power of $Q(\xi)$ in resulting equation to zero, we reach following nonlinear algebraic system:

$$\frac{1}{3}aA_0^3l-A_0v=0,$$

$$aA_0^2A_1l+\alpha^2A_1l\left(bl^2-cl^2+c+\alpha^2dl^4\right)-A_1v=0,$$

$$aA_0A_1^2l+aA_0^2A_2l+4\alpha^2A_2l\left(bl^2-cl^2+c+4\alpha^2dl^4\right)-A_2v=0,$$

$$
\begin{gathered}
\frac{1}{3}A_1 l\left(aA_1^2-6\alpha^2\Omega\left(bl^2-cl^2+c+10\alpha^2 dl^4\right)\right)+2aA_0A_1A_2l=0,\\
aA_0A_2^2l+aA_1^2A_2l-6\alpha^2A_2bl^3\Omega+6\alpha^2A_2cl^3\Omega-6\alpha^2A_2cl\Omega-120\alpha^4A_2dl^5\Omega=0,\\
aA_1A_2^2l+24\alpha^4A_1dl^5\Omega^2=0,\\
\frac{1}{3}aA_2^3l+120\alpha^4A_2dl^5\Omega^2=0.
\end{gathered}
$$

Solving the previous system, we obtain the following solutions:

$$
\begin{aligned}
&A_0=0,\;\; A_1=0,\;\; A_2=\mp\frac{3\sqrt{\frac{5}{2}}\sqrt{v}\Omega}{2\sqrt{a}\sqrt{l}},\\
&d=-\frac{v}{64\alpha^4l^5},\;\; b=-\frac{c}{l^2}+c+\frac{5v}{16\alpha^2l^3}.
\end{aligned}
\tag{3.6}
$$

Substituting (3.6) in (3.5) with (2.6) and (2.2), we get the following solutions of (1.5):

$$
u_{1,2}(x,y,z,t)=\mp\frac{3\sqrt{\frac{5}{2}}\sqrt{v}\Omega}{2\sqrt{a}\sqrt{l}}\left(\frac{4\sigma\exp(-\alpha(lx+my+nz-vt))}{4\sigma^2+\Omega\exp(-2\alpha(lx+my+nz-vt))}\right)^2.
\tag{3.7}
$$

3.2 Second simplest method

From (2.7) and (3.4), the solution of (3.3) is written in the form:

$$
u(\xi)=A_0+A_1Q(\xi)+A_2Q^2(\xi).
\tag{3.8}
$$

By setting above solution (3.8) in (3.3) and equating coefficients of like powers of $Q(\xi)$, we obtain following set of non-linear algebraic equations:

$$
\begin{gathered}
\frac{1}{3}aA_0^3l-A_0v=0,\\
aA_0^2A_1l+\alpha^2A_1l\log^2(C)\left(bl^2-cl^2+c+\alpha^2dl^4\log^2(C)\right)-A_1v=0,\\
aA_0A_1^2l+aA_0^2A_2l+4\alpha^2A_2l\log^2(C)\left(bl^2-cl^2+c+4\alpha^2dl^4\log^2(C)\right)-A_2v=0,\\
\frac{1}{3}aA_1^3l+2aA_0A_1A_2l+2\alpha^2A_1l^3\Omega(c-b)\log^2(C)\\
-2\alpha^2A_1cl\Omega\log^2(C)-20\alpha^4A_1dl^5\Omega\log^4(C)=0,\\
aA_0A_2^2l+aA_1^2A_2l-6\alpha^2A_2bl^3\Omega\log^2(C)+6\alpha^2A_2cl^3\Omega\log^2(C)\\
-6\alpha^2A_2cl\Omega\log^2(C)-120\alpha^4A_2dl^5\Omega\log^4(C)=0,\\
aA_1A_2^2l+24\alpha^4A_1dl^5\Omega^2\log^4(C)=0,\\
\frac{1}{3}aA_2^3l+120\alpha^4A_2dl^5\Omega^2\log^4(C)=0.
\end{gathered}
$$

Now, the following new exact solutions for (1.5) will be produced:

$$
\begin{aligned}
&A_0=0,\;\; A_1=0,\;\; A_2=-\frac{3\sqrt{\frac{5}{2}}\sqrt{v}\Omega}{2\sqrt{a}\sqrt{l}},\\
&d=-\frac{v}{64\alpha^4l^5\log^4(C)},\;\; b=\frac{16\alpha^2cl^3\log^2(C)-16\alpha^2cl\log^2(C)+5v}{16\alpha^2l^3\log^2(C)}.
\end{aligned}
\tag{3.9}
$$

Substituting (3.9) in (3.8) with (2.9) and (2.2), we get the following solutions of (1.5):

$$
u_{1,2}(x,y,z,t)=-\frac{3\sqrt{\frac{5}{2}}\sqrt{v}\Omega}{2\sqrt{a}\sqrt{l}}\left(\frac{4\sigma C^{(-\alpha(lx+my+nz-vt))}}{4\sigma^2+\Omega C^{(-2\alpha(lx+my+nz-vt))}}\right)^2.
\tag{3.10}
$$

4. Graphical illustrations

Now, some figures in two and three dimensional have been drawn to exemplify solutions given above. The graph of (3.7) using the first simplest method at $c = 0.2, \sigma = 5, a = 4, \Omega = 6, v = 0.5, l = 0.55, m = 0.35, n = 0.1, y = z = 2$ is introduced in Fig. (1). Finally, we shown the graph of (3.10) using the second simplest method at $c = 0.2, a = 4, \sigma = 5, k = 0.001, \Omega = 6, v = 0.5, l = 0.55, m = 0.35, n = 0.1, C = 0.4, y = z = 2$ in Fig. (2).

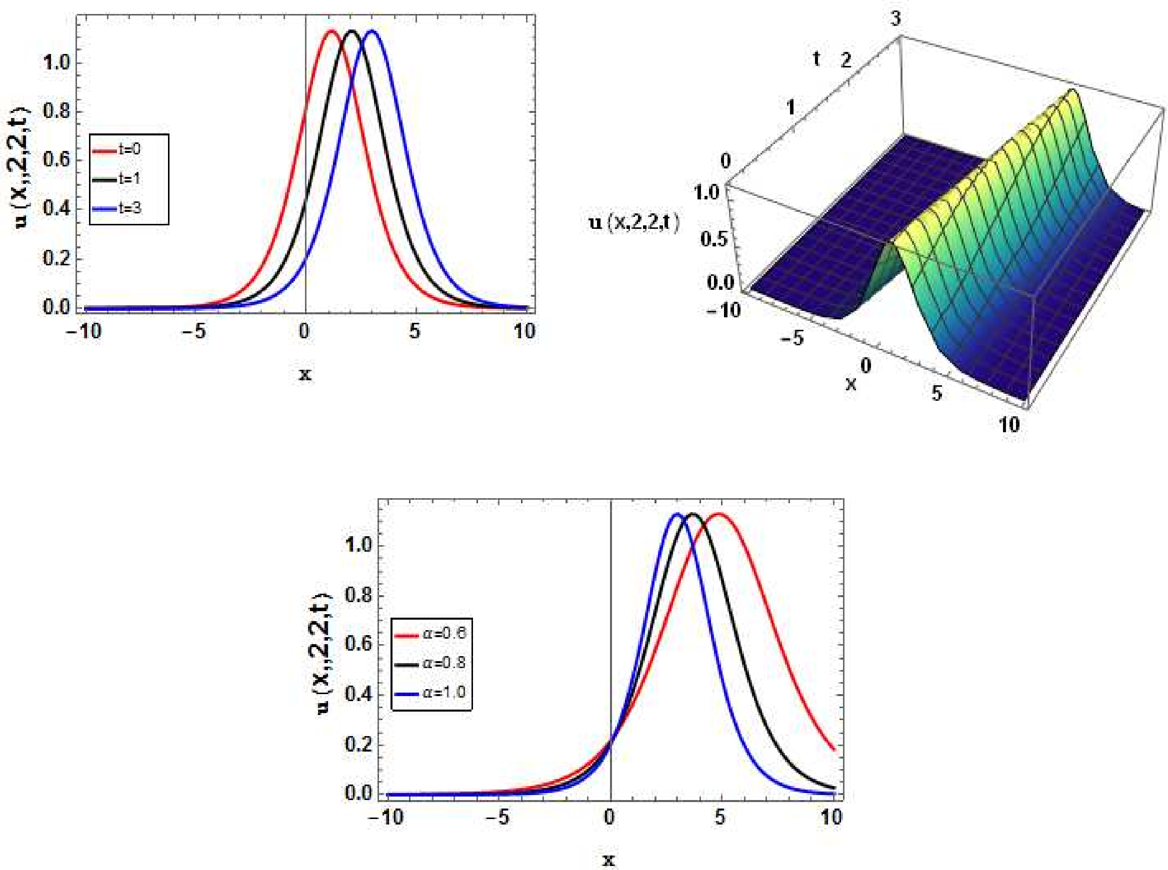

Figure 1. Profile of (3.7) using the first simplest method at c=0.2, $\sigma = 5$, a=4,$\Omega = 6$, v=0.5, l=0.55, m=0.35, n=0.1.

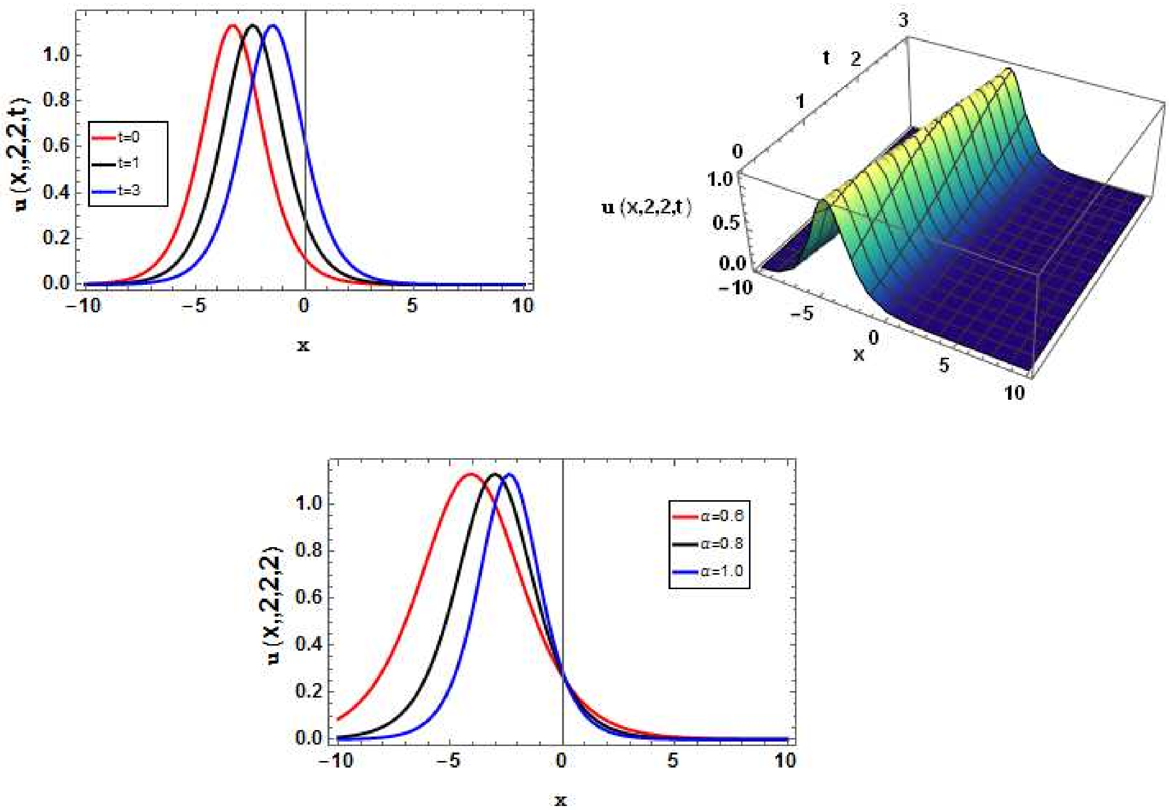

Figure 2. Profile of (3.10) using the second simplest method at c=0.2, a=4, $\sigma = 5$, k=0.001, $\Omega = 6$, v=0.5, l=0.55, m=0.35, n=0.1, C=0.4.

5. Bifurcation analysis

We investigate bifurcation behavior of traveling wave solution of the nonlinear modified ZK equation with higher order dispersion (1.5) for the first time in the literature. To discover all possible traveling wave solutions of nonlinear modified ZK equation (1.5), we form the following dynamical system [65–70] (with parameters $a,\ b,\ c,\ d,\ l$ and v) from equation (3.3):

$$\begin{cases} u_\xi = X, \\ X_\xi = Y, \\ Y_\xi = Z, \\ Z_\xi = (v - \frac{al}{3}u^2)\frac{u}{dl^5} - \frac{(bl^2+c(1-l^2))Y}{dl^4}. \end{cases} \tag{5.1}$$

Let $\overrightarrow{F} = \big(X,\ Y,\ Z,\ (v - \frac{al}{3}u^2)\frac{u}{dl^5} - \frac{(bl^2+c(1-l^2))Y}{dl^4}\big)$. Then divergence of $\overrightarrow{F}$ is:

$$\overrightarrow{\nabla} \cdot \overrightarrow{F} = \frac{\partial u_\xi}{\partial u} + \frac{\partial X_\xi}{\partial X} + \frac{\partial Y_\xi}{\partial Y} + \frac{\partial Z_\xi}{\partial Z} = 0. \tag{5.2}$$

Thus one can make a conclusion on the conservativeness of the dynamical system (5.1). The singular points of the system (5.1) are given by solutions of the following set of equations:

$$\begin{cases} X = 0, \\ Y = 0, \\ Z = 0, \\ (v - \frac{al}{3}u^2)\frac{u}{dl^5} - \frac{(bl^2+c(1-l^2))Y}{dl^4} = 0. \end{cases} \tag{5.3}$$

The dynamical system (5.1) has three equilibrium points at $P_1(u_1, 0, 0, 0)$, $P_2(u_2, 0, 0, 0)$ and $P_3(u_3, 0, 0, 0)$, where $u_1 = 0$, $u_2 = \sqrt{\frac{3v}{al}}$, and $u_3 = -\sqrt{\frac{3v}{al}}$.

The stability of the singular point based on the character of eigenvalues of the Jacobian matrix J_P. After making linearisation of the dynamical system (5.1) at the singular point $P(u,\ X,\ Y,\ Z)$, the Jacobian matrix J_P can be written as:

$$J_P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{dl^5}(v - alu^2) & 0 & -\frac{1}{dl^4}(bl^2 + c(1 - l^2)) & 0 \end{pmatrix}. \tag{5.4}$$

One can acquire eigenvalues of the system (5.1) at $P(u,\ X,\ Y,\ Z)$ by making solution of the following equation:

$$|\lambda I - J_P| = 0. \tag{5.5}$$

Then one can obtain the following characteristic equation as:

$$\lambda^4 + M_1\lambda^3 + M_2\lambda^2 + M_3\lambda + M_4 = 0, \tag{5.6}$$

where

$$\begin{cases} M_1 = 0, \\ M_2 = -\frac{1}{dl^4}(bl^2 + c(1 - l^2)), \\ M_3 = 0, \\ M_4 = \frac{1}{dl^5}(alu^2 - v). \end{cases}$$

The singular point $P(u,\ X,\ Y,\ Z)$ is considered as stable if all possible solutions of equation (5.6) having real parts less than zero for the singular point or it will be considered as unstable.

5.1 Quasiperiodic and multi-periodic traveling wave motions

Some new types of quasiperiodic and multi-periodic motions for the travelling wave solutions of the modified ZK equation (1.5) are investigated through the conservative dynamical system (5.1) based on suitable values of the parameters a, b, c, d, l, and v in Figures (3-7).

In Figure (3), we present phase space and variation of wave profile u for a quasiperiodic motion of the modified ZK equation (1.5) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.3$ and $v = 6$ with initial condition $(3.1, 1.1, -0.1, -0.2)$. In this case, the phase space forms a torus connected with two leafs faced to each other. In Figure (4), we present phase space and variation of wave profile u for a quasiperiodic motion of the modified ZK equation (1.5) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.4$ and $v = 6$ with initial condition $(3.1, 1.1, -0.1, -0.2)$. In this case, the phase space looks like a heart-shape. In Figure (5), we present phase space and variation of wave profile u for a quasiperiodic motion of the modified ZK equation (1.5) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.48$ and $v = 6$ with initial condition $(3.1, 1.1, -0.1, -0.2)$. In this case, the phase space forms a torus connected with two leafs faced to each other with multi-bends. In Figure (6), we present phase space and variation of wave profile u for a quasiperiodic motion of the modified ZK equation (1.5) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.494$ and $v = 6.$ with initial condition $(3.1, 1.1, -0.1, -0.2)$. In this case, the phase space forms a torus connected with two sets of multi-torus structures faced to each other.

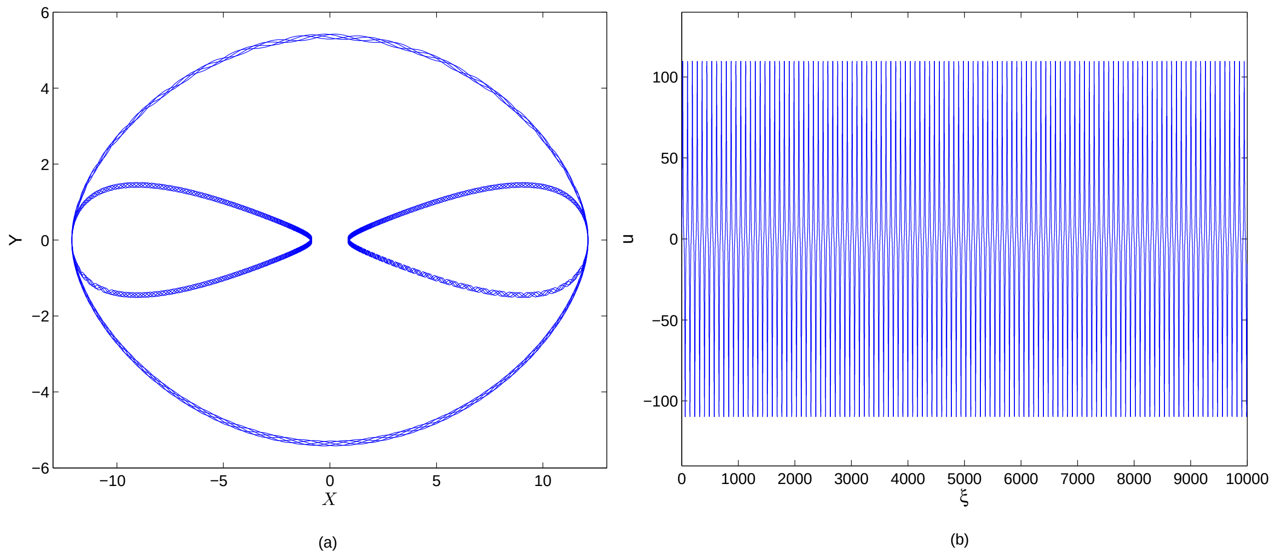

Figure 3. Phase space of the system (5.1) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.3$ and $v = 6$.

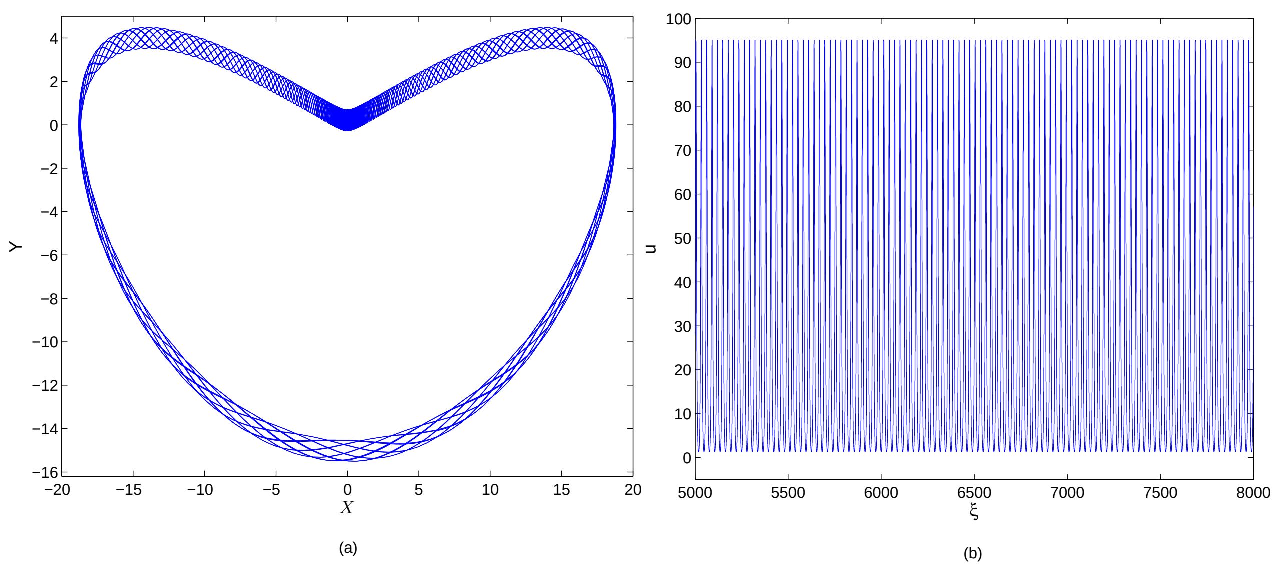

Figure 4. Phase space of the system (5.1) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.4$ and $v = 6$.

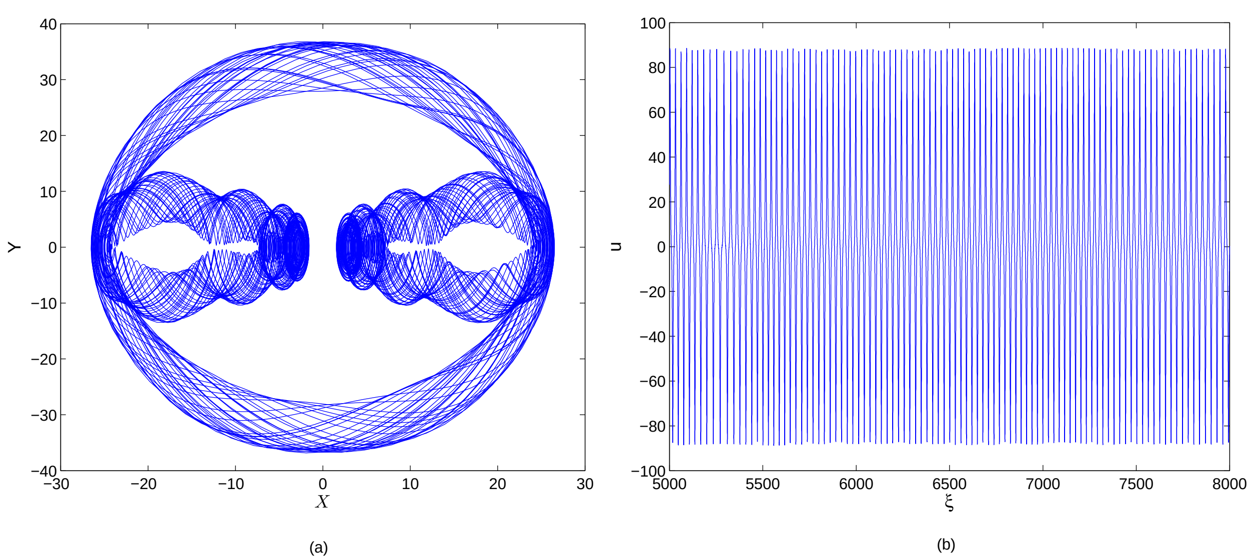

Figure 5. Phase space of the system (5.1) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.48$ and $v = 6$.

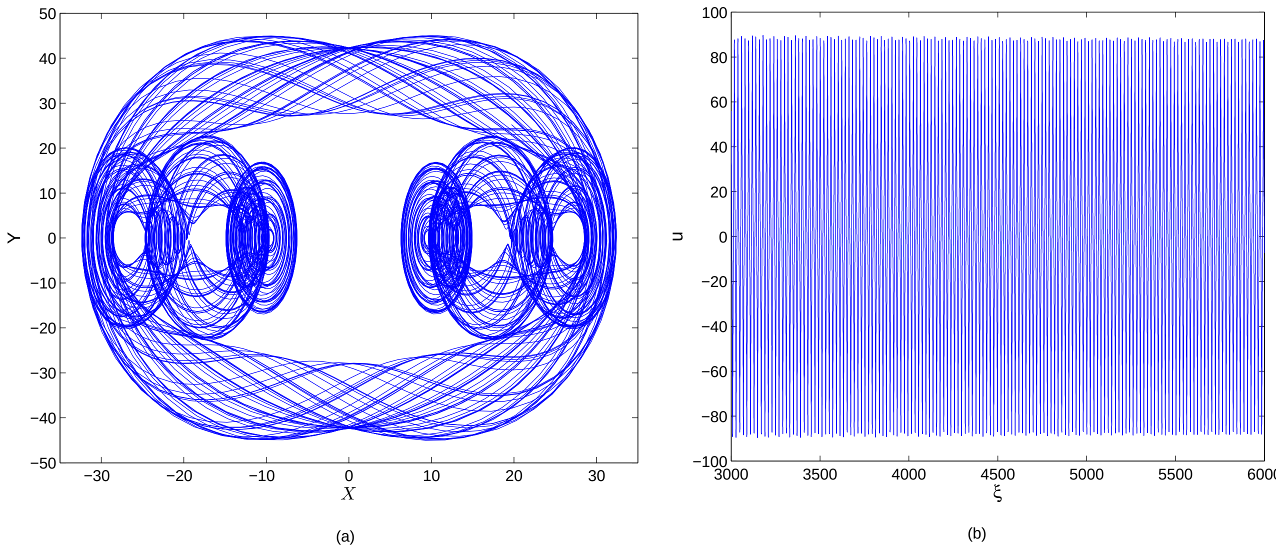

Figure 6. Phase space of the system (5.1) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.494$ and $v = 6$.

There exists a period-9 motion of the dynamical system (5.1) and corresponding phase portrait is shown in Figure (7) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.5$ and $v = 6$. with initial condition $(3.1, 1.1, -0.1, -0.2)$. It is important to note that all phase spaces, presented in Figures (3-7), are symmetric with respect to Y-axis. Such phase spaces, presented in Figures (3-7), are observed for the first time in the literature of nonlinear modified ZK equation (1.5) with higher order dispersion term.

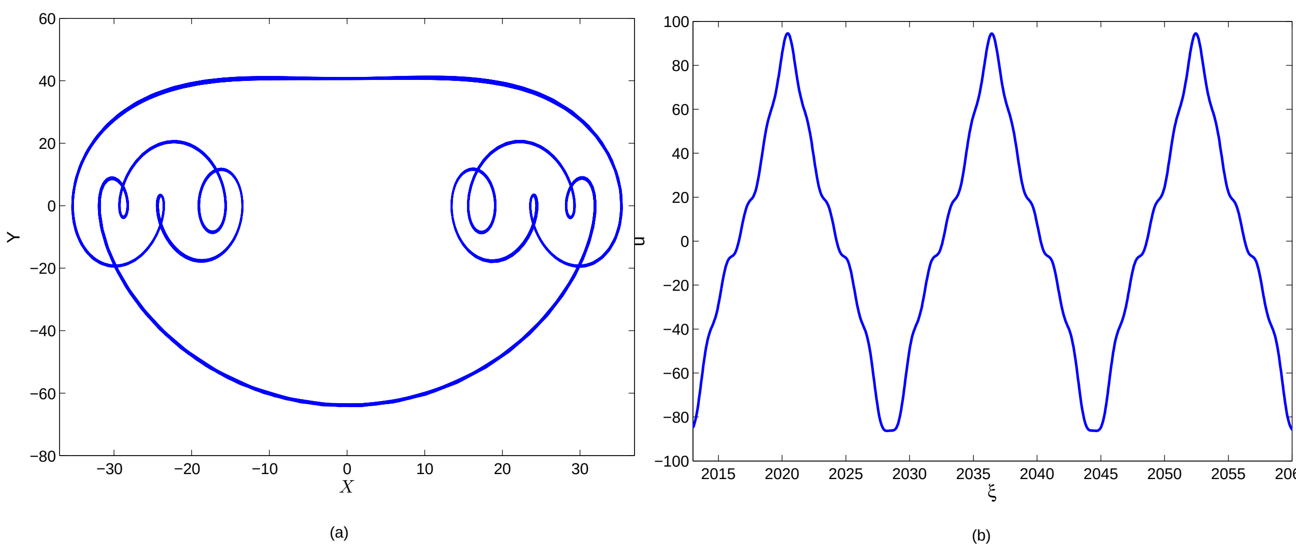

Figure 7. Phase space of the system (5.1) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.5$ and $v = 6$.

5.2 Multistability of traveling wave motion

Multistability behaviors for the travelling wave solutions of modified ZK equation (1.5) are examined through conservative dynamical system (5.1) based on fixed values of the parameters $a, b, c, d, l,$ and v in Figure (8) with different initial conditions: (a) $(0.1, 1.1, -0.1, -0.2)$, (b) $(0.5, 1.1, -0.1, -0.2)$, (c) $(0.1, 0.1, -0.1, -0.2)$, (d) $(0.1, 0.1, 0.1, -0.2)$, (e) $(0.1, 0.1, 0.1, 0.2)$, and (f) $(0.1, 1.1, 0.9, 0.2)$. All these phase spaces are qualitatively different from each other. It is important to note that all phase spaces of Figure (8) are symmetric in nature with respect to Y-axis. This kind of multistability behaviors for the travelling wave solutions of the modified ZK equation (1.5) with higher order dispersion term are reported for the first time in the literature.

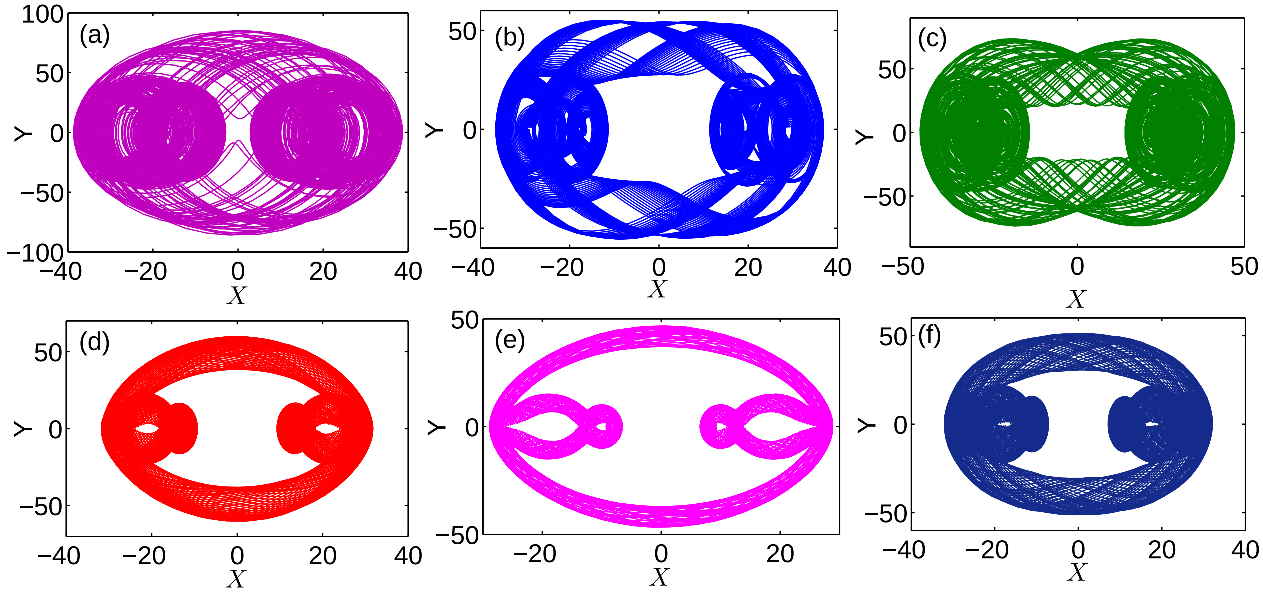

Figure 8. Phase spaces of the system (5.1) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.494$ and $v = 6$. with different initial conditions: (a) $(0.1, 1.1, -0.1, -0.2)$, (b) $(0.5, 1.1, -0.1, -0.2)$, (c) $(0.1, 0.1, -0.1, -0.2)$, (d) $(0.1, 0.1, 0.1, -0.2)$, (e) $(0.1, 0.1, 0.1, 0.2)$, and (f) $(0.1, 1.1, 0.9, 0.2)$.

6. Discussion

The (3+1)-dimensional modified Zakharov-Kuznetsov equation is a mathematical model that describes the propagation of nonlinear waves in a medium. It is an extension of the Zakharov-Kuznetsov equation, which is a well-known model for the propagation of ion-acoustic waves in plasma. The modified Zakharov-Kuznetsov equation takes into account the effects of both dispersion and nonlinearity, which can lead to the formation of solitary waves. The physical meaning of the equation is that it describes the evolution of these solitary waves as they propagate through the medium. Overall, the (3+1)-dimensional modified Zakharov-Kuznetsov equation is a useful tool for studying the behavior of waves in a variety of physical systems, including plasmas, fluids, and optical fibers. Its solutions can help researchers better understand the dynamics of these systems and predict how they will behave under different conditions.

7. Conclusions

In this paper, by successfully implementing the two simplest methods, traveling wave solutions for the nonlinear (3+1) dimensional mZK equation have been obtained. New soliton solutions are derived. For a clear understanding, solutions are illustrated with details in 2D and 3D. These solutions have many applications and can supply a beneficial contribution for researchers to examine and discover wave features in several areas of physics and applied sciences. The bifurcation behavior of travelling wave solutions of the mZK equation was also analyzed. A collection of new types of quasiperiodic motions was reported for the first time in the literature on the mZK equation with higher-order dispersion terms. Considering fixed values of the parameters, the multistability behavior of the mZK equation was shown at different initial conditions. In a conclusion, it can be easily seen that the methods used in this paper may further be improved to solve and analyze the qualitative behavior of nonlinear traveling wave solutions for other NLEEs in mathematical and plasma physics.

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