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email: gcuvalcioglu@mersin.edu.tr

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Dear Scientists,

We have prepared the third (supplement) issue of the sixth year of our journal with your contributions and efforts.

We believe that the papers in this issue will contribute to researchers and scientists as in our other issues.

We believe that this issue of JUM will reach many universities and research institutions thanks to the painstaking work of our authors, referees and editors.

We thank all our colleagues for their contributions.

We look forward to your support from our esteemed researchers and authors in the next stages of our publication life.

We wish you a scientific life full of success..

Kind regards!

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MATRICES OF HYBRID NUMBERS

ÇAĞLA RAMİS AND YASIN YAZLIK

0000-0002-2809-8324 and 0000-0001-6369-540X

Abstract. In this study, we investigate the matrices over the new extension of the real numbers in four dimensional space $\mathbb{E}_2^4$ called the hybrid numbers. Since the hybrid multiplication is noncommutative, this leads to finding a linear transformation on the complex field. Thus we characterize the hybrid matrices and examine their algebraic properties with respect to their complex adjoint matrices. Moreover, we define the co-determinant of hybrid matrices which plays an important role to construct the Lie groups.

1. Introduction

The extension of the real number system raises by investigating the solutions of the quadratic equations given as follows:

$$x^2 + 1 = 0,\ x^2 - 1 = 0 \text{ and } x^2 = 0. \tag{1.1}$$

As a result, the new units called the imaginary $i^2 = -1$, the unipotent $h^2 = 1$ $(h \neq \mp 1)$ and the nilpotent $\varepsilon^2 = 0$ $(\varepsilon \neq 0)$ enter in the history of mathematics and yield the new number systems named by complex numbers, hyperbolic numbers and dual numbers, respectively [21, 23, 24]. All three number systems are two-dimensional vector spaces over the real numbers, this implies that the points of $\mathbb{R}^2$ can be identified by them with respect to their metric systems. These corresponding metrics yield two-dimensional Euclidean geometry, Lorentzian geometry and Galilean geometry, respectively. Then the identification of a point $A = (x, y)$ can be seen in the following planes with respect to the systems:

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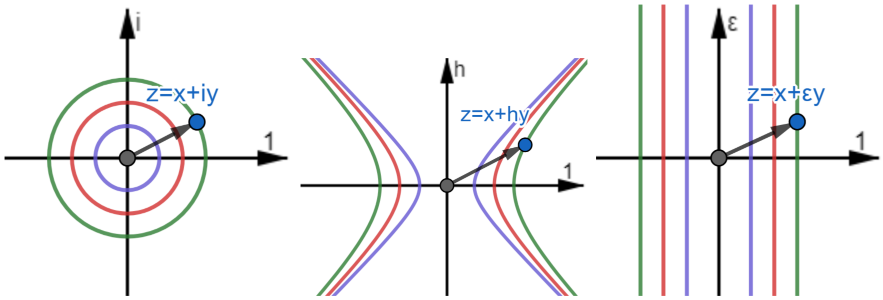

(A) Euclidean plane (B) Lorentzian plane (C) Galilean plane

FIGURE 1. Coordinate planes of metric systems in two-dimensional space

Moreover, Clifford algebras can be studied on the vector spaces of complex numbers, dual numbers and hyperbolic numbers via elliptic, parabolic and hyperbolic bilinear forms, respectively. It is also known as EPH-classification of these number systems. The EPH-classification is closely linked with the elliptic, hyperbolic and parabolic analytic function theories [4, 6, 16].

The historical evolution of the ideas on how to manage the extension of numbers gives us the quaternions introduced by Hamilton [12] as the most-known generalization of complex numbers. The set of quaternions is generally represented in the form:

$$\mathbb{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_s \in \mathbb{R}\ , 0 \leq s \leq 3\} \tag{1.2}$$

where i, j, k are quaternionic units and hold $i^2 = j^2 = k^2 = ijk = -1$. Since the set $\mathbb{H}$ is a non-commutative associative algebra over the real numbers, the matrices of quaternions becomes one of the interesting topics in the matrix theory. A brief survey on the quaternionic matrices given by Zhang [26] presents some methods for some basic functions for matrices such as determinant, computing the eigenvalue. The method is based on finding the complex adjoint matrix of any quaternionic matrix. After that, various studies are born about the matrices of quaternions and their applications [5, 8, 10, 14, 25].

Another well-known member of non-commutative algebras is the set of split quaternions introduced by Cockle [7] as follows:

$$\widehat{\mathbb{H}} = \{\hat{q} = \hat{q}_0 + \hat{q}_1 i + \hat{q}_2 j + \hat{q}_3 k : \hat{q}_t \in \mathbb{R}\ , 0 \leq t \leq 3\} \tag{1.3}$$

where $i^2 = -1$ and $j^2 = k^2 = ijk = 1$. The difference between $\widehat{\mathbb{H}}$ and $\mathbb{H}$ is the existence of zero divisors, nilpotent elements and nontrivial idempotents in $\widehat{\mathbb{H}}$. After work of Zhang, the quaternionic matrices and their properties are studied over $\widehat{\mathbb{H}}$ by the compatible methods [1, 11, 15, 19, 20].

In the system $\mathbb{R}^4$, we meet the new phenomenon named as hybrid numbers and given in the following form:

$$\mathbb{K} = \{X = x_0 + x_1 i + x_2 \varepsilon + x_3 h : x_j \in \mathbb{R}\ , 0 \leq j \leq 3\} \tag{1.4}$$

where i, ε and h are the complex, dual and hyperbolic units, respectively [17]. There are considerable differences between $\mathbb{K}$ and the two sets previously describe, out of the noncommutativity. Under this view, hybrid numbers firstly give the blood relativity of two different classes of vectors:

$$\overbrace{\left\{\begin{array}{l} \textit{Timelike} \\ \textit{Spacelike} \\ \textit{Lightlike} \end{array}\right\}}^{\textbf{Characters}} \underset{\textit{Hybrid Numbers}}{\longleftrightarrow} \overbrace{\left\{\begin{array}{l} \textit{Elliptic} \\ \textit{Parabolic} \\ \textit{Hyperbolic} \end{array}\right\}}^{\textbf{EPH-classification}}$$

Secondly, there is the isomorphism between 2×2 real matrices and hybrid numbers and thus a classification of 2×2 real matrices and an algebraic method to find their roots are obtained by the hybrid numbers [18]. The short history of the hybrid numbers reveals us their advantages on real matrix algebra and the sequences of special numbers [9, 22].

In this study, we will examine the hybrid matrices by improving the Zhang's method over $\mathbb{K}$. In the second section, we give some basic notions and properties of hybrid numbers, and more importantly, we change the spelling of the hybrid numbers. They are rewritten in the form named as the $\mathbb{C}-$type which will be used to built a linear transformation between $\mathbb{K}$ and the set of 2×2 complex matrices. This correspondence yields the second relationship between eigenvalues and types of hybrid numbers as follows:

$$\overbrace{\left\{\begin{array}{l} \lambda_{1,2} \in \mathbb{C} \\ \lambda_1 = \lambda_2 \\ \lambda_{1,2} \in \mathbb{R} \end{array}\right\}}^{\textbf{Eigenvalues}} \underset{\textit{Hybrid Numbers}}{\longleftrightarrow} \overbrace{\left\{\begin{array}{l} \textit{Elliptic} \\ \textit{Parabolic} \\ \textit{Hyperbolic} \end{array}\right\}}^{\textbf{EPH-classification}}$$

In the third section, the matrices of hybrid numbers are introduced and their properties are obtained. After that, in the fourth section, to prevent the disadvantages of the noncommutativity of hybrid numbers we define the complex adjoint of hybrid matrices. Hence the determinant of hybrid matrices could be characterized, and so they are analyzed in the theory of Lie groups.

2. Basic Concepts of Hybrid Numbers

In this section, we initially introduce hybrid numbers with fundamental features. Then we establish a new form called $\mathbb{C}-$type and give the properties of hybrid numbers in the new form.

A hybrid number occurs in the combination form of the three types of number systems, complex, dual and hyperbolic numbers, as the following:

$$X = x_0 + x_1 i + x_2 \varepsilon + x_3 h \tag{2.1}$$

where $x_j \in \mathbb{R}$, $0 \leq j \leq 3$ and the basis elements $\{1, i, \varepsilon, h\}$ are satisfying the multiplication rules given in the following table.

(2.2)

$\cdot$	1	i	ε	h
1	1	i	ε	h
i	i	-1	$1-h$	$i+\varepsilon$
ε	ε	$1+h$	0	$-\varepsilon$
h	h	$-i-\varepsilon$	ε	1

By the compotentwise addition and scalar multiplication, the set of hybrid numbers denoted by $\mathbb{K}$ becomes a 4-dimensional vector space over the real numbers.

Furthermore, the hybrid number algebra is an associative, noncommutative ring with respect to the addition and multiplication operations.

The hybrid number X is composed of the scalar part $S(X) = x_0$ and the vector part $V(X) = x_1 i + x_2\varepsilon + x_3 h$. The conjugate of X is the hybrid number defined by $\overline{X} = S(X) - V(X)$. If $x_2 = x_3 = 0$, then the conjugate of hybrid number means the conjugate of complex number, and vice versa. Moreover, there are two kinds of vectorial representation of X given by $\mathcal{V}(X) = (x_0, x_1 - x_2, x_2, x_3)$ and $\mathcal{V}_h(X) = (x_1 - x_2, x_2, x_3)$ which is specifically called the hybrid vector of X. Thus, there exist the following functions:

$$\begin{aligned} \mathcal{C}(X) &= x_0^2 + (x_1 - x_2)^2 - x_2^2 - x_3^2 \\ \mathcal{C}_h(X) &= -(x_1 - x_2)^2 + x_2^2 + x_3^2 \end{aligned} \tag{2.3}$$

where $\mathcal{C}(X) = -\langle \mathcal{V}(X), \mathcal{V}(X)\rangle$ and $\mathcal{C}_h(X) = \langle \mathcal{V}_h(X), \mathcal{V}_h(X)\rangle$ are equipped with the signature $(-,-,+,+)$ of $\mathbb{E}_2^4$ the four dimensional Minkowski space and the subspace $\mathbb{E}_1^3$, respectively. These functions yield the following classifications of the hybrid number X with respect to the corresponding Minkowski metrics:

A hybrid number $X \in \mathbb{K}$ is

- Spacelike if $\mathcal{C}(X) < 0$ or $X = 0$,
- Timelike if $\mathcal{C}(X) > 0$,
- Lightlike (null) if $\mathcal{C}(X) = 0$ and $X \neq 0$,

which are called the characters of the hybrid number X.

The types of the hybrid number X are given by

- If $\mathcal{C}_h(X) < 0$, X is elliptic,
- If $\mathcal{C}_h(X) > 0$, X is hyperbolic,
- If $\mathcal{C}_h(X) = 0$, X is parabolic.

Consequently, the following table is set to show the relation between the two characterizations of hybrid numbers.

(2.4)

Classification by Types	**Classification by Characters**
Elliptic	Timelike
Hyperbolic	Spacelike, Timelike, Lightlike
Parabolic	Timelike, Lightlike

Until now, we summarize briefly the basic algebraic properties of the noncommutative ring $\mathbb{K}$ for more details the reader is referred to [17].

Our first aim in the present paper is to find a linear transformation between hybrid numbers and express them via the matrix of the transformation thus we could explore the properties of hybrid numbers in another convenient way. For this inherent reason, the multiplication rule of the unit ε in (2.2) allows us to observe the hybrid numbers in terms of the basis $\{i, h\}$. Thus we can explain the hybrid number $X = x_0 + x_1 i + x_2\varepsilon + x_3 h$ as follows:

$$X = z_1 + z_2 h, \quad z_1, z_2 \in \mathbb{C} \tag{2.5}$$

where $z_1 = x_0 + (x_1 - x_2)\, i$, $z_2 = x_3 + x_2 i$. Since this appears, at first sight, to be a complex hyperbolic number, we call (2.5) as the $\mathbb{C}-$type of hybrid number X in order to avoid the confusion. Then we can obviously conclude the following.

Theorem 2.1. *Every hybrid number can be uniquely expressed in the form of* $\mathbb{C}-$*type.*

Corollary 2.2. *The $\mathbb{C}-$type of a hybrid number become equivalent to its open form if and only if the hybrid number is a complex number.*

The fundamental functions on the set of hybrid numbers of the $\mathbb{C}-$type are given as follows:
i) Addition:

$$X+Y=(z_1+w_1)+(z_2+w_2h),$$

ii) Multiplication:

$$XY=z_1w_1+z_2\overline{w_2}+(z_1w_2+z_2\overline{w_1})h,$$

iii) The hybrid conjugate:

$$\overline{X}=\overline{z_1}-z_2h,$$

iv) Functions of characteristics:

$$\mathcal{C}(X)=|z_1|^2-|z_2|^2 \text{ and } \mathcal{C}_h(X)=-V(z_1)^2+|z_2|^2,$$

v) The inverse of a hybrid number:

$$X^{-1}=\frac{\overline{z_1}}{\mathcal{C}(X)}-\frac{z_2}{\mathcal{C}(X)}h$$

vi) The two kind norms of a hybrid number:

$$\parallel X \parallel=\sqrt{|\mathcal{C}(X)|} \text{ and } \parallel X \parallel_h=\sqrt{|\mathcal{C}_h(X)|}$$

where $X=z_1+z_2h$, $Y=w_1+w_2h\in\mathbb{K}$ and $V(z_1)$ is the imaginary part of z_1.
The next theorem summarizes the properties of the hybrid conjugate.

Theorem 2.3. *For the hybrid numbers $X=z_1+z_2h$ and $Y=w_1+w_2h$, the properties listed below are true.*
i. $X=\overline{(\overline{X})}$,
ii. $X\overline{X}=\overline{X}X=z_1\overline{z_1}-z_2\overline{z_2}$,
iii. $\overline{X+Y}=\overline{X}+\overline{Y}$,
iv. $\overline{XY}=\overline{Y}\ \overline{X}$,
v. $\mathcal{C}(X)=\mathcal{C}(\overline{X})$ *and* $\mathcal{C}_h(X)=\mathcal{C}_h(\overline{X})$,
vi. $\overline{(X^{-1})}=\left(\overline{X}\right)^{-1}$,
vii. $X=\overline{X}$ *if and only if X is a real number,*
viii. $hz=\overline{z}h$ *or* $hzh=\overline{z}$ *for any complex number z.*

Proof. In general, the properties can be proved easily. Let's at least have confidence in the accuracy of (iv) and (vi).
The proof for (iv);

$$\begin{aligned}\overline{XY} &= \overline{z_1w_1+z_2\overline{w_2}-(z_1w_2+z_2\overline{w_1})\,h}\\ &= \overline{w_1}\ \overline{z_1}+w_2\overline{z_2}-(\overline{w_1}z_2+w_2z_1)\,h\\ &= \overline{Y}\ \overline{X}.\end{aligned}$$

The proof for (vi);

$$\begin{aligned}\left(\overline{X}\right)^{-1} &= \frac{\overline{(\overline{z_1})}}{\mathcal{C}(\overline{X})} - \frac{(-z_2)}{\mathcal{C}(\overline{X})}h \\ &= \frac{z_1}{\mathcal{C}(X)} + \frac{z_2}{\mathcal{C}(X)}h \\ &= \overline{(X^{-1})}.\end{aligned}$$

□

Now let us define the following bijective map,

$$(2.6)\qquad \begin{aligned}\psi_X :\ \ & \mathbb{K} && \to \mathbb{K} \\ & Y && \to \psi(Y) = YX\end{aligned}$$

where as a consequence of the ring structure of hybrid numbers we could see that ψ_X is a linear map. It is well known that every linear map can be represented by a matrix, so we get

$$\begin{aligned}\psi_X(1) &= z_1 + z_2 h, \\ \psi_X(h) &= \overline{z_2} + \overline{z_1} h,\end{aligned}$$

and then the matrix of the transformation ψ with respect to the standard bases is given as follows:

$$[\psi_X] = \begin{bmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{bmatrix}.$$

where $X = z_1 + z_2 h$.

Consequently, the following theorem is stated.

Theorem 2.4. *Every hybrid number can be represented by a 2×2 complex matrices.*

Notice that the subset of the matrix ring $M_2(\mathbb{C})$ given such as

$$(2.7)\qquad \mathcal{K} = \left\{ A = \begin{bmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{bmatrix} : z_1, z_2 \in \mathbb{C} \right\}$$

actually represents the set of hybrid numbers $\mathbb{K}$. Since the transformation between $\mathcal{K}$ and $\mathbb{K}$ is bijective and linear, then the operations are preserved. Moreover, let the corresponding matrix of $X = z_1 + z_2 h$ be $A = \begin{bmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{bmatrix}$, we have

$$(2.8)\qquad detA = \mathcal{C}(X),\ trA = 2\ funcRe(z_1) \text{ and } \lambda_{1,2} = \frac{trA}{2} \mp \sqrt{\mathcal{C}_h(X)}$$

where λ_1 and λ_2 are the eigenvalues of A.

Corollary 2.5. *The inverse of a hybrid number exists if and only if the determinant of the corresponding complex matrix of the hybrid number is different from zero.*

Definition 2.6. The characters of $A \in \mathcal{K}$ can be defined as
i. A is spacelike, if $\det A < 0$,
ii. A is timelike, if $\det A > 0$,
iii. A is lightlike, if $\det A = 0$.

Definition 2.7. The types of $A \in \mathcal{K}$ can be given in terms of its eigenvalues $\lambda_{1,2}$ as follows:
i. A is elliptic, if $\lambda_{1,2} \in \mathbb{C}$,
ii. A is hyperbolic, if $\lambda_{1,2} \in \mathbb{R}$,
iii. A is parabolic, if $\lambda_1 = \lambda_2$.

Corollary 2.8. *If $A \in \mathcal{K}$ is a Hermitian matrix, then its corresponding hybrid number must be hyperbolic or parabolic.*

Now, we observe the matrices of hybrid numbers according to the three different concepts of complex matrix theory which are unitary, Hermitian and skew-Hermitian matrices. Let $A \in \mathcal{K}$ be the corresponding complex matrix of the hybrid number $X = z_1 + z_2 h$, we can give the following statements.

- If A is the unitary matrix, then $A\overline{A}^T = I_2$ which yields

$$z_1\overline{z_1} + z_2\overline{z_2} = 1 \text{ and } z_1 z_2 = 0. \tag{2.9}$$

  For the case $z_1 = 0$, we have $z_2\overline{z_2} = 1$ means that $z_2 = \cos\theta + \sin\theta i$. Then the $\mathbb{C}-$type of X is

$$X = e^{i\theta} h \tag{2.10}$$

  where X is a spacelike hyperbolic hybrid number. On the other hand, if $z_2 = 0$, the $\mathbb{C}-$type forms of X meets the open form of it means that X is a complex number such that

$$X = e^{i\theta}. \tag{2.11}$$

- If A is the Hermitian matrix, then we obtain $z_1 = \overline{z_1}$ and $\mathcal{C}_h(X) = z_2\overline{z_2}$. From *Corollary 4*, we can distinguish two cases:
  i. If X is parabolic hybrid number, then $z_2 = 0$ and $X \in \mathbb{R}\backslash\{0\}$,
  ii. If X is hyperbolic hybrid number, X could have the three kinds of characters. In addition to, the null case will be appeared as the Pythagorean condition and therefore the components can be expressed as follows:

$$z_1 = w(u^2 + v^2) \text{ and } z_2 = w\left[2uv + \left(u^2 - v^2\right) i\right] \tag{2.12}$$

  where w is constant, u and v are relatively prime.
- A is the skew-Hermitian matrix if and only if X is a pure complex number, namely $X = xi$, $x \in \mathbb{R}$.

3. Introduction to Hybrid Matrices

In this section, our first results concern the matrices of hybrid numbers. After that we explain them in terms of the complex matrices by using the $\mathbb{C}-$type form of hybrid numbers. Hence we could analyze the properties of hybrid number matrices by using the algorithms of the complex matrix theory.

Let us introduce the set of $m \times n$ type matrices with the hybrid number components, denoted by $M_{m,n}(\mathbb{K})$. If $m = n$, then we briefly use the notation $M_n(\mathbb{K})$. With the ordinary matrix addition and multiplication the set $M_n(\mathbb{K})$ is going to become a noncommutative ring where the unit is I_n. If the equation $AB = BA = I_n$ exists for $B \in M_n(\mathbb{K})$, we call A is invertible and denote $B = A^{-1}$.

Moreover, the propositions of left vector space endowed by [26] can be satisfied by the following scalar multiplication:

$$XA = [Xa_{\alpha\beta}] \tag{3.1}$$

where $X \in \mathbb{K}$ and $A = [a_{\alpha\beta}] \in M_{m,n}(\mathbb{K})$. Hence we know that $M_{m,n}(\mathbb{K})$ is a left vector space over $\mathbb{K}$. Similarly, the definition of scalar multiplication $AX = [a_{\alpha\beta}X]$ yields the right vector space over $\mathbb{K}$.

If we use the $\mathbb{C}-$types of the components of the hybrid matrix $A = [a_{\alpha\beta}] \in M_{m,n}(\mathbb{K})$, then the components are written as $a_{\alpha\beta} = a^1_{\alpha\beta} + a^2_{\alpha\beta}h \in \mathbb{K}$ and we get

$$A = A_1 + A_2 h \tag{3.2}$$

where $A_1 = [a^1_{\alpha\beta}]$, $A_2 = [a^2_{\alpha\beta}] \in M_{m,n}(\mathbb{C})$. The transpose of A is $A^T = [a_{\beta\alpha}] = A_1^T + A_2^T h$ and the conjugate of A is $\overline{A} = [\overline{a_{\alpha\beta}}] = \overline{A_1} - A_2 h$.

Definition 3.1. The conjugate transpose of a hybrid matrix A, denoted by A^*, is

$$A^* = \overline{A}^T$$

where the entries of $\overline{A}$ are the hybrid conjugates of the corresponding entries of A.

Ideally, we shall consider specific square hybrid matrices in terms of the conjugate transpose as follows:

- $A = A^*$, A is Hermitian,
- $A = -A^*$, A is skew-Hermitian,
- $A^{-1} = A^*$, A is unitary,
- $AA^* = A^*A$, A is normal.

Definition 3.2. Let $\lambda \in \mathbb{K}$ and $A \in M_n(\mathbb{K})$. If λ satisfies the following equation

$$Ax = \lambda x, \tag{3.3}$$

then λ is called the left eigenvalue of A for some non-zero $x \in M_{n,1}(\mathbb{K})$. The set of the left eigenvalues of A is called the left spectrum of A.

Note that we can similarly define the right eigenvalue ($Ax = x\lambda$, $\lambda \in \mathbb{K}$) and the right spectrum of A because of the noncommutativity.

Example 3.3. (i) For the hybrid matrix $A \in M_2(\mathbb{K})$,

$$A = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{bmatrix}$$

$\{0, \varepsilon, -\varepsilon\}$ is the subset of the intersection of the left and the right spectrums of A.

(ii) For the hybrid matrix $B \in M_2(\mathbb{K})$,

$$B = \begin{bmatrix} 0 & h \\ -h & 0 \end{bmatrix}$$

some of the left eigenvalues of B are $\{\mp(i + \varepsilon)\}$ but none of them is the element of the right spectrum of B. Similarly, some of right eigenvalues of B are $\{\mp i\}$ but not the left eigenvalues of B.

The theorems below list several properties of hybrid matrices. The first theorem show the properties which are generally correct, therefore, we construct an example for an explicit proof. On the other hand, the direct proof method can be used for the consecutive theorem.

Theorem 3.4. *If $A \in M_{m,n}(\mathbb{K})$ and $B \in M_{n,s}(\mathbb{K})$, then the properties listed below are true in general.*

i. $\left(\overline{A}\right)^{-1} \neq \overline{(A^{-1})}$,

ii. $\left(A^T\right)^{-1} \neq \left(A^{-1}\right)^T$,

iii. $AB \neq BA$,
iv. $\overline{AB} \neq \overline{B}\ \overline{A}$,
v. $(AB)^T \neq B^T A^T$.

Example 3.5. Let the two hybrid matrices be $A = \begin{bmatrix} i & \varepsilon \\ 0 & h \end{bmatrix}$ and $B = \begin{bmatrix} \varepsilon & 0 \\ 0 & 0 \end{bmatrix}$. We obtain the following:

i. $\left(\overline{A}\right)^{-1} = \begin{bmatrix} i & 1-h \\ 0 & -h \end{bmatrix} \neq \begin{bmatrix} i & -1-h \\ 0 & -h \end{bmatrix} = \overline{(A^{-1})}$,

ii. $\left(A^T\right)^{-1} = \begin{bmatrix} -i & 0 \\ 1+h & h \end{bmatrix} \neq \begin{bmatrix} -i & 0 \\ -1+h & h \end{bmatrix} = \left(A^{-1}\right)^T$,

iii. $AB = \begin{bmatrix} 1-h & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1+h & 0 \\ 0 & 0 \end{bmatrix} = BA$

iv. $\overline{AB} = \begin{bmatrix} 1+h & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1-h & 0 \\ 0 & 0 \end{bmatrix} = \overline{B}\ \overline{A}$,

v. $(AB)^T = \begin{bmatrix} 1-h & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1+h & 0 \\ 0 & 0 \end{bmatrix} = B^T A^T$.

Remark 3.6. The sufficient condition for the existence of the third case of theorem 4 occur with the invertible matrices. Moreover, if $AB = I$ for any $A = A_1 + A_2 h$, $B = B_1 + B_2 h \in M_n(\mathbb{K})$, this provides $BA = I$. From the hypothesis we get

$$A_1 B_1 + A_2 \overline{B_2} + \left(A_1 B_2 + A_2 \overline{B_1}\right) h = I_n \tag{3.4}$$

and (3.3) is equal to the following matrix product

$$\begin{bmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ \overline{B_2} & \overline{B_1} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \tag{3.5}$$

which yields $BA = I$ since the left hand side of (3.4) is the product of $2n \times 2n$ complex matrices and the hypothesis is true for the complex matrices.

Theorem 3.7. *If $A \in M_{m,n}(\mathbb{K})$, $B \in M_{n,s}(\mathbb{K})$ and $X \in \mathbb{K}$, then the properties listed below are true.*
i. $\left(\overline{A}\right)^T = \overline{(A^T)}$,
ii. $\overline{(\overline{A})} = \left(A^T\right)^T = (A^*)^* = A$,
iii. $(XA)^* = A^* \overline{X}$,
iv. $(A+B)^* = A^* + B^*$,
v. $(AB)^* = B^* A^*$,
vi. $(AB)^{-1} = B^{-1}A^{-1}$ *if A and B are invertible,*
vii. $(A^*)^{-1} = \left(A^{-1}\right)^*$ *if A invertible.*

Proof. The proof of the first four properties and (vi) can be easily shown with using the properties of complex matrix theory and *Theorem 2* in the previous section. Let us prove (v) with $A = A_1 + A_2 h$ and $B = B_1 + B_2 h$, where A_1, A_2, B_1 and B_2 are process-compatible complex matrices, then we have

$$\begin{aligned}
(AB)^* &= \left[A_1 B_1 + A_2 \overline{B_2} + \left(A_1 B_2 + A_2 \overline{B_1}\right) h\right]^* \\
&= \left[A_1 B_1 + A_2 \overline{B_2}\right]^* - \left[A_1 B_2 + A_2 \overline{B_1}\right]^T h \\
&= (A_1 B_1)^* + \left(A_2 \overline{B_2}\right)^* - (A_1 B_2)^T h - \left(A_2 \overline{B_1}\right)^T h \\
&= B_1^* A_1^* + \left(\overline{B_2}\right)^* A_2^* - B_2^T A_1 h - \left(\overline{B_1}\right)^T A_2^T h \\
&= B^* A^*.
\end{aligned}$$

As a consequence of the fifth property, we can obtain the case (vii). □

4. Complex Matrix Equivalence For Hybrid Matrices

The linear map $\psi_X : \mathbb{K} \to \mathbb{K}$ defined in the second section gives us the opportunity to examine the properties of hybrid numbers over 2×2 complex matrices. Since the hybrid multiplication is noncommutative, there are also limitations in questioning the linear algebra over hybrid matrices. In this section, our aim is to turn a hybrid matrix into a complex matrix to use the several properties of linear algebra over the complex field.

In this way, we define a map, Ψ_n, that is between $M_n(\mathbb{K})$ and $M_{2n}(\mathbb{C})$, such as

$$\Psi_n(A) = \begin{bmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{bmatrix}$$

where $A = A_1 + A_2 h \in M_n(\mathbb{K})$ and $A_1, A_2 \in M_n(\mathbb{C})$.

Notice that Ψ_n is a continuous, injective ring homomorphism and described with respect to the linear map ψ_X. For $n = 1$, we can have the corresponding complex matrix of a hybrid number. We call that $\Psi_n(A)$ is the adjoint matrix of $A \in M_n(\mathbb{K})$, and denote by $\tilde{A} \in M_{2n}(\mathbb{C})$.

Example 4.1. Let $A = \begin{bmatrix} 1+\varepsilon & i+\varepsilon+h \\ 1+h & 1 \end{bmatrix}$ be a hybrid matrix. Then we can rewrite it by using the $\mathbb{C}-$types of the components as in following form:

$$A = \begin{bmatrix} 1-i & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} i & 1+i \\ 1 & 0 \end{bmatrix} h. \tag{4.1}$$

Thus the conjugate matrix of A is

$$\tilde{A} = \begin{bmatrix} 1-i & 0 & i & 1+i \\ 1 & 1 & 1 & 0 \\ -i & 1-i & 1+i & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}. \tag{4.2}$$

The following theorem summarizes the properties of adjoint matrices.

Theorem 4.2. *Let $A = A_1 + A_2 h$, $B = B_1 + B_2 h \in M_n(\mathbb{K})$ and their adjoint matrices be $\tilde{A}, \tilde{B} \in M_n(\mathbb{C})$, then the followings are true.*

i. If $A = I_n$, then $\tilde{A} = I_{2n}$,

ii. $\widetilde{A+B} = \tilde{A} + \tilde{B}$,

iii. $\widetilde{A\ B} = \tilde{A}\ \tilde{B}$,

iv. $\widetilde{A^{-1}} = \left(\tilde{A}\right)^{-1}$ if A^{-1} exists,

v. $\left(\tilde{A}\right)^T = \widetilde{(A^T)}$ if $A_2 \in M_n(\mathbb{R})$,

vi. $\widetilde{(\overline{A})} = \overline{\left(\tilde{A}\right)}$ if A_2 is a pure complex matrix,

vii. $\widetilde{(A^*)} = \left(\tilde{A}\right)^*$ if $A \in M_n(\mathbb{C})$.

Proof. Truth of (i) and (ii) are clear. Let us prove (iii). The adjoint matrices of A and B are

$$\tilde{A} = \begin{bmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{bmatrix} \text{ and } \tilde{B} = \begin{bmatrix} B_1 & B_2 \\ \overline{B_2} & \overline{B_1} \end{bmatrix}. \tag{4.3}$$

Multiplying the adjoints, we obtain the complex matrix

$$\tilde{A}\ \tilde{B}=\left[\begin{array}{cc} A_1B_1+A_2\overline{B_2} & A_1B_2+A_2\overline{B_1} \\ \overline{A_1}\ \overline{B_2}+\overline{A_2}B_1 & \overline{A_1}\ \overline{B_1}+\overline{A_2}B_2 \end{array}\right]. \tag{4.4}$$

Since the hybrid matrix form of (4.4) is

$$\begin{aligned} & A_1B_1+A_2\overline{B_2}+\left(A_1B_2+A_2\overline{B_1}\right)h \\ & \text{or }\ (A_1+A_2h)\,(B_1+B_2h) \end{aligned} \tag{4.5}$$

then we have $\widetilde{A\,B}=\tilde{A}\ \tilde{B}$.

Applying the third property for the matrices A and A^{-1} then we get (iv), furthermore, the properties given by *Theorem* 5 yields the rest. □

The fourth property of the previous theorem sets out that the image under Ψ_n of an invertible hybrid matrix is an invertible complex matrix. Hence we can talk about the determinant of a hybrid matrix by the combination of det and Ψ_n, then we can conclude the following.

Definition 4.3. Let $A\in M_n(\mathbb{K})$ and $\tilde{A}\in M_{2n}(\mathbb{C})$ be the adjoint matrix of A. The co-determinant of A is the complex determinant of $\tilde{A}$, denoted by $|A|_c$.

Theorem 4.4. *Let $A\in M_n(\mathbb{K})$. The following are equivalent:*
i. A is invertible,
ii. $Ax=0$ has a unique solution, $x=0$,
iii. The left (or right) eigenvalues of A do not vanish,
iv. $\tilde{A}$ is invertible.

Proof. (**i** ⇒ **ii**) This is a trivial outcome.

(**ii** ⟺ **iii**) Assume that A has zero eigenvalue. Then the equation (3.3) satisfies, such as $Ax=0$, for some non-zero values which is a contradiction.

(**iii** ⇒ **iv**) Consider the second case instead of (iii), if $Ax=0$ for $x=x_1+x_2h\in M_{n,1}(\mathbb{K})$ then we have

$$A_1x_1+A_2\overline{x_2}+\left(A_1x_2+A_2\overline{x_1}\right)h=0 \text{ or}$$

$$A_1x_1+A_2\overline{x_2}=0 \text{ and } \overline{A_2}x_1+\overline{A_1}\overline{x_2}=0.$$

This means that the determinant of $\left[\begin{array}{cc} A_1 & A_2 \\ \overline{A_2} & \overline{A_1}\end{array}\right]$ is different from zero due to the unique solution.

(**iv** ⇒ **i**) If $\tilde{A}$ is invertible, then there exist a complex matrix such that

$$\left[\begin{array}{cc} Z_1 & Z_2 \\ Z_3 & Z_4\end{array}\right]\left[\begin{array}{cc} A_1 & A_2 \\ \overline{A_2} & \overline{A_1}\end{array}\right]=I_{2n}.$$

It follows that

$$Z_1A_1+Z_2\overline{A_2}=I \text{ and } Z_1A_2+Z_2\overline{A_1}=0$$

which yields $\left(Z_1A_1+Z_2\overline{A_2}\right)+\left(Z_1A_2+Z_2\overline{A_1}\right)h=I$, then the hybrid matrix $Z=Z_1+Z_2h$ is the inverse of A from *Remark 2*. □

Note that the last case of the previous equivalence theorem implies that a hybrid matrix is invertible if and only if its co-determinant must be different from zero.

Thus we can state the concept of general linear, special linear, and symplectic groups to the hybrid numbers, respectively, as follows:

$$\begin{aligned} GL_n(\mathbb{K}) &= \{A \in M_n(\mathbb{K}) \mid |A|_c \neq 0\}, \\ SL_n(\mathbb{K}) &= \{A \in M_n(\mathbb{K}) \mid |A|_c = 1\}, \\ SP_n(\mathbb{K}) &= \left\{A \in GL_n(\mathbb{K}) \mid A^{-1} = A^*\right\}. \end{aligned} \tag{4.6}$$

Furthermore, since any closed subgroup of $GL_n(\mathbb{C})$ is a Lie group, these groups are Lie groups. Then we can obtain the Lie algebras along with the bracket operation over matrices such as $[A, B] = AB - BA,\ A, B \in M_n(\mathbb{K})$. For example, the Lie algebra of $GL_n(\mathbb{K})$ is the set of all $n \times n$ matrices with entries in $\mathbb{K}$, that is

$$gl_n(\mathbb{K}) = M_n(\mathbb{K}) \tag{4.7}$$

and the Lie algebra of $SP_n(\mathbb{K})$ is the set

$$sp_n(\mathbb{K}) = \{A \in M_n(\mathbb{K}) \mid A + A^* = 0\} \tag{4.8}$$

and finally the Lie algebra of $SL_n(\mathbb{K})$ is the set

$$sl_n(\mathbb{K}) = \left\{A \in M_n(\mathbb{K}) \mid tr\left(\tilde{A}\right) = 0\right\} \tag{4.9}$$

where it can be easily seen that $tr\left(\tilde{A}\right) = 0$ if and only if scalar part of $tr(A)$ is zero.

Theorem 4.5. *Let $A, B \in M_n(\mathbb{K})$, the co-determinants satisfy the following properties,*
i. $|A\ B|_c = |A|_c\ |B|_c$,
ii. $\left|A^{-1}\right|_c = |A|_c^{-1}$, *if* $A \in GL_n(\mathbb{K})$,
iii. $|P\ A\ Q|_c = |A|_c$, *for* $P, Q \in SL_n(\mathbb{K})$,
iv. Cayley-Hamilton Theorem for hybrid matrices: Let A be a square hybrid matrix and the characteristic polynomial of A be $P_A(\lambda) = \left|\lambda I_{2n} - \tilde{A}\right|$, $\lambda \in \mathbb{C}$, then $P_A(A) = 0$.

Proof. (i). From the third property of theorem 6, we get $|A\ B|_c = \left|\widetilde{A\ B}\right| = |\tilde{A}\ \tilde{B}| = |A|_c\ |B|_c$,
(ii). From the fourth property of theorem 6, if $A \in GL_n(\mathbb{K})$ then we have $\left|A^{-1}\right|_c = \left|\widetilde{A^{-1}}\right| = \left|\left(\tilde{A}\right)^{-1}\right| = |A|_c^{-1}$,
(iii). If the hybrid $P, Q \in SL_n(\mathbb{K})$ which means they are elementary matrices and $|P|_c = |Q|_c = 1$, this completes the proof.
(iv). The coefficients of the polynomial $P_A(\lambda)$ are real [26]. Then we have $\widetilde{p(A)} = P_A(\tilde{A})$ for any real coefficient polynomial p. On the other hand, Cayley-Hamilton theorem for the complex matrices proves that $P_A(\tilde{A}) = 0$ for $\tilde{A} \in M_{2n}(\mathbb{C})$. This implies that $\widetilde{p(A)} = 0$, namely $P_A(A) = 0$. □

5. Conclusion

In number theory, different studies are available in which the complex, dual and hyperbolic numbers systems are expressed in the one sentence [3, 13]. One of them has arisen recently and called the hybrid number. When the interdisciplinary applications of the constituent number systems are observed, we obviously see that the most effective results are obtained by their matrices. From a technique point of view, matrices can be taken into account as functional tools to organize the accumulated knowledge, accelerate the calculations and finally formulate the conclusions in various developed mathematical frameworks. The result of these motivations, the satisfactory concept of this study rises as the necessity of hybrid matrices.

The present paper is concerned with the linear algebra over hybrid matrices. However, we have to face the limitations on algebraic properties of the hybrid matrices by the fact that the hybrid multiplication is noncommutative. In this way, we firstly use an alternative form for hybrid numbers called the $\mathbb{C}-$type and obtain the subset of 2×2 complex matrices, $\mathcal{K}$ which represents the matrices corresponding to hybrid numbers by the transformation ψ_X. After describing and investigating the basic properties of hybrid matrices, we are aware of the need to rearrange them. Therefore, we define a continuous, injective ring homomorphism Ψ_n between $M_n(\mathbb{K})$ and $M_{2n}(\mathbb{C})$ by taking advantage of the effect of the transformation ψ_X on the hybrid numbers. Thus the adjoint matrices of hybrid matrices are obtained over complex numbers, this gives us the right to implement the properties of linear algebra over the complex field for hybrid matrices. Since Ψ_n turns an invertible hybrid matrix to an invertible complex matrix, we could have the one of the important result that is the calculation of determinant of hybrid matrices. This leads to describe general linear, special linear, and symplectic groups of the hybrid numbers and their corresponding Lie algebras, respectively. Finally, we state Cayley-Hamilton theorem for hybrid matrices.

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(Çağla RAMİS) Nevşehir Hacı Bektaş Veli University, Faculty of Science and Arts, Department of Mathematics, 50300, Nevşehir, Turkey
Email address: cramis@nevsehir.edu.tr, caglaramis@gmail.com

(Yasin YAZLIK) Nevşehir Hacı Bektaş Veli University, Faculty of Science and Arts, Department of Mathematics, 50300, Nevşehir, Turkey
Email address: yyazlik@nevsehir.edu.tr

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ON LEAP ZAGREB INDICES OF A SPECIAL GRAPH OBTAINED BY SEMIGROUPS

YAŞAR NACAROĞLU

0000-0001-7179-0490

Abstract. In 2013, Das et al. defined the monogenic semi-group graphs [10]. And, various topological indices of the monogenic semi-group graphs have been calculated so far [3, 21]. The aim of this study is to continue to create formulas for the topological indices of these special graphs. In this study, we give exact formulae for various the leap Zagreb indices of this special algebraic graph obtained from monogenic semigroups.

1. Introduction and Preliminaries

Let $\mathscr{F}$ be a simple connected graph with vertex set $V(\mathscr{F})$ and edge set $E(\mathscr{F})$, where $V(\mathscr{F}) = \{a^i : 1 < i < n\}$. In graph $\mathscr{F}$, the degree of a vertex a^i is defined as the number of vertices that incident to the vertex a^i and is denoted by $d_{\mathscr{F}}(a^i)$. The distance between any two vertices a^i and a^j in a graph $\mathscr{F}$, denoted as $d(a^i, a^j)$, is the length of the shortest path between these vertices. The eccentricity of the vertex a^i is the maximum distance from a^i to any vertex. That is, $ecc(a^i) = max\{d(a^i, a^j) : a^j \in V(\mathscr{F})\}$.

In a graph $\mathscr{F}$, the k-distance degree of a vertex a^i, denoted as $d_k(a^i/\mathscr{F})$, is defined as the number of vertices at a distance of k from a^i [25]. Clearly, $d_1(a^i/\mathscr{F}) = d_{\mathscr{F}}(a^i)$.

Topological indices are important tools used in the study of chemical and physical properties of molecules, especially in QSAR and QSPR researchs [12]. Many publications have been made about the Zagreb indices (especially the first Zagreb index $(M_1(\mathscr{F}))$[14] and the second Zagreb index $(M_2(\mathscr{F}))$[15]), which is one of the oldest topological indices. These indices are defined as follows :

$$M_1(\mathscr{F}) = \sum_{a^i \in V(\mathscr{F})} d^2_{\mathscr{F}}(a^i) \quad and \quad M_2(\mathscr{F}) = \sum_{a^i a^j \in E(\mathscr{F})} d_{\mathscr{F}}(a^i) d_{\mathscr{F}}(a^j).$$

Inspired by Zagreb indices, the leap Zagreb indices were defined by Naji et al. in 2017 [22]. For more information on the leap Zagreb indices, we prefer references

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[7, 9, 24, 28]. The first leap Zagreb index, the second leap Zagreb index and the third leap Zagreb index are defined as follows:

$$LM_1(\mathscr{F}) = \sum_{a^i \in V(\mathscr{F})} d_2^2(a^i)$$
$$LM_2(\mathscr{F}) = \sum_{a^i a^j \in E(\mathscr{F})} d_2(a^i) d_2(a^j)$$
$$LM_3(\mathscr{F}) = \sum_{a^i a^j \in E(\mathscr{F})} (d_2(a^i) + d_2(a^j)).$$

The F-leap index of a graph $\mathscr{F}$ is defined by Kulli [17] as follows:

$$LF(\mathscr{F}) = \sum_{a^i \in V(\mathscr{F})} d_2^3(a^i).$$

The leap eccentric connectivity index (LEC) is defined as follows in an unpublished work by Pawar et al. [23, 16, 18, 26, 27][1]

$$LEC(\mathscr{F}) = \sum_{a^i \in V(\mathscr{F})} d_2(a^i) ecc(a^i).$$

For more information on graph theory, we prefer references [13].

The concept of a zero-divisor graph defined on a commutative ring R was first introduced by Beck [8] in 1988. In his study, he presented results on the coloring of this graph. Anderson and Livingston [5] continued this study, and their definition of zero-divisor graphs has been widely accepted. Anderson and Livingston defined the zero-divisor graph as follows:

Let $\mathscr{R}$ be a commutative ring with identity. Let $\mathscr{Z}(\mathscr{R})$ denote its set of zero-divisors. The vertex set of the zero-divisor graph $\Gamma(\mathscr{R})$ consists of the elements of $\mathscr{Z}(\mathscr{R})$. Two distinct vertices x and y are adjacent if and only if their product is zero.

The concept of zero-divisor graphs defined on a commutative ring has been generalized by Demeyer [11] et al. to define zero-divisor graphs of a commutative semigroups. Many studies have been conducted and are still ongoing regarding the concept of zero-divisor graphs constructed on various algebraic structures, such as Cayley graphs, total graphs, annihilator graphs, power graphs, etc. [4, 6]

In this direction, Das et al. [10] introduced a new graph obtained from multiplicative semigroups in 2013. They defined this algebraic graph as follows:

Definition 1.1. [10] Let $\mathscr{S}_{\mathscr{A}} = \{a, a^2, a^3, \ldots, a^n\}$ be a monogenic semigroup (with zero). The vertex set of this graph consists of the elements of $\mathscr{S}_{\mathscr{A}}$, except for zero. Any two vertices a^i and a^j are adjacent if and only if $i + j \geq n$.

They investigated some graph parameters of this graph in [3]. For more properties of the monogenic semigroup graphs, we can refer to [1, 2, 19, 20]. In this paper, we present exact formulas for the leap Zagreb indices, F-leap Zagreb index, and leap eccentric connectivity index of monogenic semigroup graphs with given order.

2. Leap Zagreb Indices of Monogenic Semigroup Graphs

In this section, we will give our basic results. First of all, let's give some lemmas that we will use in the proofs of theorems. Here and later, we will prefer the notations $d_1(a^i)$ and $d_2(a^i)$ over $d_1(a^i/\Gamma(\mathscr{S}_\mathscr{A}))$ and $d_2(a^i/\Gamma(\mathscr{S}_\mathscr{A}))$ for $1 \leq i \leq n$, respectively.

The degree sequence of the monogenic semigroup graph is given in the following lemma.

Lemma 2.1. [10] *Let $\mathscr{S}_\mathscr{A} = \{a, a^2, a^3, \ldots, a^n\}$ be a monogenic semigroup (with zero). Then, the degree sequence of the monogenic semigroup graph is given as*

$$\left.\begin{array}{l} d_1(a) = 1, d_1(a^2) = 2, \cdots, d_1(a^{\lfloor \frac{n}{2} \rfloor}) = \left\lfloor \frac{n}{2} \right\rfloor, d_1(a^{\lfloor \frac{n}{2} \rfloor+1}) = \left\lfloor \frac{n}{2} \right\rfloor, \\ d_1(a^{\lfloor \frac{n}{2} \rfloor+2}) = \left\lfloor \frac{n}{2} \right\rfloor + 1, \cdots, d_1(a^{n-1}) = n-2, d_1(a^n) = n-1. \end{array}\right\}$$

Lemma 2.2. [10] *Let $\mathscr{S}_\mathscr{A} = \{a, a^2, a^3, \ldots, a^n\}$ be a monogenic semigroup (with zero). Then*

$$diam(\Gamma(\mathscr{S}_\mathscr{A})) = 2.$$

Lemma 2.3. [22] *Let $\mathscr{G}$ be a connected graph with n vertices. Then for any vertex $v \in V(\mathscr{G})$*

$$d_2(v) \leq n - 1 - d_1(v).$$

Equality holds if and only if G has diameter at most two.

With the help of Lemma 2.1, Lemma 2.2 and Lemma 2.3 we give the sequence of 2-distance degrees of vertices in $\Gamma(\mathscr{S}_\mathscr{A})$ in the following lemma.

Lemma 2.4. *Let $\mathscr{S}_\mathscr{A} = \{a, a^2, a^3, \ldots, a^n\}$ be a monogenic semigroup (with zero). Then, the 2-distance degree sequence of the monogenic semigroup graphs is given as*

$$\left.\begin{array}{l} d_2(a) = n-2, d_2(a^2) = n-3, \cdots, d_2(a^{\lfloor \frac{n}{2} \rfloor}) = n-1-\left\lfloor \frac{n}{2} \right\rfloor, d_2(a^{\lfloor \frac{n}{2} \rfloor+1}) = n-1-\left\lfloor \frac{n}{2} \right\rfloor, \\ d_2(a^{\lfloor \frac{n}{2} \rfloor+2}) = n-2-\left\lfloor \frac{n}{2} \right\rfloor, \cdots, d_2(a^{n-1}) = 1, d_2(a^n) = 0. \end{array}\right\}$$

Let us give the well-known equation that will appear here in the proofs of our theorems with the following lemma.

Lemma 2.5. *For the natural number n, we have*

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2}, & n \text{ even} \\ \frac{n-1}{2}, & n \text{ odd.} \end{cases} \tag{2.1}$$

We are now ready to calculate the first leap zagreb index of monogenic semigroup graphs.

Theorem 2.6. *Let $\Gamma(\mathscr{S}_\mathscr{A})$ be monogenic semigroup graphs. Then, we have*

$$LM_1(\Gamma(\mathscr{S}_\mathscr{A})) = \begin{cases} \frac{4n^3-15n^2+14n}{12}, & n \text{ even} \\ \frac{4n^3-15n^2+20n-9}{12}, & n \text{ odd.} \end{cases}$$

Proof. From definition of the first leap Zagreb index and by Lemma 2.4, we have

$$\begin{aligned}
LM_1(\Gamma(\mathscr{S}_{\mathscr{A}})) = & \sum_{a^i \in V(\Gamma(\mathscr{S}_{\mathscr{M}}))} d_2^2(a^i) \\
= & d_2^2(a) + d_2^2(a^2) + \cdots + d_2^2(a^{\lfloor \frac{n}{2} \rfloor}) + d_2^2(a^{\lfloor \frac{n}{2} \rfloor} + 1) + d_2^2(a^{\lfloor \frac{n}{2} \rfloor + 2}) + \\
& + \cdots + d_2^2(a^{n-1}) + d_2^2(a^n) \\
= & (n-2)^2 + (n-3)^2 + ... + \left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right)^2 + \left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right)^2 \\
& + \left(n - 2 - \left\lfloor \frac{n}{2} \right\rfloor\right)^2 + \cdots + 2^2 + 1^2 + 0^2 \\
= & \frac{(n-2)(n-1)(2n-3)}{6} + (n - 1 - \left\lfloor \frac{n}{2} \right\rfloor)^2 \qquad (2.2)
\end{aligned}$$

There are two possible situations from here. With (2.1) we get the desired result. Thus, the proof is completed. □

With the following theorem, we give the exact formula for the second leap Zagreb index of monogenic semigroup graphs.

Theorem 2.7. *Let $\Gamma(\mathscr{S}_{\mathscr{A}})$ be monogenic semigroup graphs. Then, we have*

$$LM_2(\Gamma(\mathscr{S}_{\mathscr{A}})) = \begin{cases} \frac{n^4 - 4n^3 + 2n^2 + 4n - 3}{48}, & n \text{ odd} \\ \frac{n^4 - 4n^3 + 2n^2 + 4n}{48}, & n \text{ even.} \end{cases}$$

Proof. There are two possible cases for the values n is odd or even.

Let n be odd. In this case, from definition of the second leap Zagreb index and by Lemma 2.4, we have

$$\begin{aligned}
LM_2(\Gamma(\mathscr{S}_{\mathscr{A}})) = & \sum_{a^i a^j \in E(\Gamma(\mathscr{S}_{\mathscr{M}}))} d_2(a^i) d_2(a^j) \\
= & d_2(a^n)d_2(a) + d_2(a^n)d_2(a^2) + \cdots + d_2(a^n)d_2(a^{n-2}) + d_2(a^n)d_2(a^{n-1}) + \\
& + d_2(a^{n-1})d_2(a^2) + \cdots + d_2(a^{n-1})d_2(a^{n-3}) + d_2(a^{n-1})d_2(a^{n-2}) + \\
& + \cdots + \\
& + d_2(a^{\frac{n+1}{2}+2})d_2(a^{\frac{n+1}{2}-2}) + d_2(a^{\frac{n+1}{2}+2})d_2(a^{\frac{n+1}{2}-1}) + d_2(a^{\frac{n+1}{2}+2})d_2(a^{\frac{n+1}{2}}) + \\
& + d_2(a^{\frac{n+1}{2}+2})d_2(a^{\frac{n+1}{2}+1}) + \\
& + d_2(a^{\frac{n+1}{2}+1})d_2(a^{\frac{n+1}{2}-1}) + d_2(a^{\frac{n+1}{2}+1})d_2(a^{\frac{n+1}{2}})
\end{aligned}$$

Consequently, the second leap Zagreb index of $\Gamma(\mathscr{S}_{\mathscr{A}})$ is written as given in the following

$$LM_2(\Gamma(\mathscr{S}_{\mathscr{A}})) = LM_{2,n} + LM_{2,n-1} + \cdots + LM_{2,\frac{n+1}{2}+2} + LM_{2,\frac{n+1}{2}+1}$$

When calculating these operations, we use $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ from (2.2) for n odd. Then, we have

$$\begin{aligned}
LM_{2,n} = & d_2(a^n)d_2(a) + d_2(a^n)d_2(a^2) + \cdots + d_2(a^n)d_2(a^{n-2}) + d_2(a^n)d_2(a^{n-1}) \\
= & 0.(n-2) + 0(n-3) + ... + 0.2 + 0.1 + 0(n - 1 - \frac{n-1}{2}) \\
= & 0.
\end{aligned}$$

In the case of applying operations similar to $LM_{2,n-1}$, $LM_{2,n-2}$, $\cdots$, $LM_{2,\frac{n+1}{2}+2}$ and $LM_{2,\frac{n+1}{2}+1}$; we get

$$\begin{aligned}
LM_{2,n-1} =& d_2(a^{n-1})d_2(a^2) + \cdots + d_2(a^{n-1})d_2(a^{n-3}) + d_2(a^{n-1})d_2(a^{n-2}) \\
=& 1.(n-3) + 1.(n-4) + \cdots + 1(n-1-\frac{n-1}{2}) + \cdots + 1.3 + 1.2 + \\
& + 1(n-1-\frac{n-1}{2}) \\
=& \sum_{q=2}^{n-3} 1.q + 1.\frac{n-1}{2},
\end{aligned}$$

$$\begin{aligned}
LM_{2,n-2} =& d_2(a^{n-2})d_2(a^3) + \cdots + d_2(a^{n-2})d_2(a^{n-2}) + d_2(a^{n-2})d_2(a^{n-3}) \\
=& 2.(n-4) + 2.(n-5) + \cdots + 2.(n-1-\frac{n-1}{2}) + \cdots + 2.4 + 2.3 + \\
& + 2.(n-1-\frac{n-1}{2}) \\
=& \sum_{q=3}^{n-4} 2.q + 2.\frac{n-1}{2},
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
LM_{2,\frac{n+1}{2}+2} =& d_2(a^{\frac{n+1}{2}+2})d_2(a^{\frac{n+1}{2}-2}) + d_2(a^{\frac{n+1}{2}+2})d_2(a^{\frac{n+1}{2}-1}) + d_2(a^{\frac{n+1}{2}+2})d_2(a^{\frac{n+1}{2}}) + \\
& + d_2(a^{\frac{n+1}{2}+2})d_2(a^{\frac{n+1}{2}+1}) \\
=& \frac{n-5}{2}.\frac{n+1}{2} + \frac{n-5}{2}.\frac{n-1}{2} + \frac{n-5}{2}.\frac{n-3}{2} + \frac{n-5}{2}.(n-1-\frac{n-1}{2}) \\
=& \sum_{q=\frac{n-5}{2}}^{\frac{n+1}{2}} \frac{n-5}{2}.q + \frac{n-5}{2}.\frac{n-1}{2},
\end{aligned}$$

and finally

$$\begin{aligned}
LM_{2,\frac{n+1}{2}+1} =& d_2(a^{\frac{n+1}{2}+1})d_2(a^{\frac{n+1}{2}-1}) + d_2(a^{\frac{n+1}{2}+1})d_2(a^{\frac{n+1}{2}}) \\
=& \frac{n-3}{2}.\frac{n-1}{2} + \frac{n-3}{2}.\frac{n-1}{2} \\
=& \sum_{q=\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{n-3}{2}.q + \frac{n-3}{2}.\frac{n-1}{2},
\end{aligned}$$

Hence

$$LM_{2,n} + LM_{2,n-1} + \cdots + LM_{2,\frac{n+1}{2}+2} + LM_{2,\frac{n+1}{2}+1} = \sum_{i=1}^{\frac{n-1}{2}-1} \sum_{q=r+1}^{n-2-r} r.q + \sum_{s=1}^{\frac{n-1}{2}-1} s.(\frac{n-1}{2}).$$

If similar operations are performed in case n is even, the following sum is obtained

$$LM_{2,n} + LM_{2,n-1} + \cdots + LM_{2,\frac{n}{2}+2} + LM_{2,\frac{n}{2}+1} = \sum_{r=1}^{\frac{n}{2}-1} \sum_{q=r+1}^{n-2-r} r.q + \sum_{s=1}^{\frac{n}{2}-1} s.(\frac{n}{2}-1)$$

So as desired. □

Theorem 2.8. *Let $\Gamma(\mathscr{S}_{\mathscr{A}})$ be monogenic semigroup graphs. Then, we have*

$$LM_3(\Gamma(\mathscr{S}_{\mathscr{A}})) = \begin{cases} \frac{2n^3-3n^2-2n+3}{12}, & n \text{ odd} \\ \frac{2n^3-3n^2-2n}{12}, & n \text{ even.} \end{cases}$$

Proof. There are two possible cases for the values n is odd or even.

Let n be odd. In this case, from definition of the third leap Zagreb index and by Lemma 2.4, we have

$$\begin{aligned} LM_3(\Gamma(\mathscr{S}_{\mathscr{A}})) = & \sum_{a^ia^j\in E(\Gamma(\mathscr{S}_{\mathscr{A}}))} [d_2(a^i)+d_2(a^j)] \\ =& (d_2(a^n)+d_2(a)) + (d_2(a^n)+d_2(a^2)) + \cdots + (d_2(a^n)+d_2(a^{n-2})) \\ &+(d_2(a^n)+d_2(a^{n-1})) + (d_2(a^{n-1})+d_2(a^2)) + \cdots + \\ &+(d_2(a^{n-1})+d_2(a^{n-3})) + (d_2(a^{n-1})+d_2(a^{n-2})) + \\ &+\cdots+ \\ &+(d_2(a^{\frac{n+1}{2}+2})+d_2(a^{\frac{n+1}{2}-2})) + (d_2(a^{\frac{n+1}{2}+2})+d_2(a^{\frac{n+1}{2}-1})) + \\ &+(d_2(a^{\frac{n+1}{2}+2})+d_2(a^{\frac{n+1}{2}})) + (d_2(a^{\frac{n+1}{2}+2})+d_2(a^{\frac{n+1}{2}+1})) + \\ &+(d_2(a^{\frac{n+1}{2}+1})+d_2(a^{\frac{n+1}{2}-1})) + (d_2(a^{\frac{n+1}{2}+1})+d_2(a^{\frac{n+1}{2}})). \end{aligned}$$

Consequently, the third leap Zagreb index of $\Gamma(\mathscr{S}_{\mathscr{A}})$ is written as given in the following

$$LM_3(\Gamma(\mathscr{S}_{\mathscr{A}})) = LM_{3,n} + LM_{3,n-1} + LM_{3,n-2} + \cdots + LM_{3,\frac{n+1}{2}+2} + LM_{3,\frac{n+1}{2}+1}$$

When calculating these operations, we use $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ from (2.2) for n odd. Then, we have

$$\begin{aligned} LM_{3,n} =& (d_2(a^n)+d_2(a)) + (d_2(a^n)+d_2(a^2)) + \cdots + (d_2(a^n)+d_2(a^{n-2})) \\ &+ (d_2(a^n)+d_2(a^{n-1})) \\ =& (0+n-2) + (0+n-3) + \cdots + (0+2) + (0+1) + (0+(n-1-\frac{n-1}{2})) \\ =& \sum_{q=1}^{n-2}(0+q) + (0+\frac{n-1}{2}). \end{aligned}$$

In the case of applying operations similar to $LM_{3,n-1}$,$LM_{3,n-2}$, $\cdots$, $LM_{3,\frac{n+1}{2}+2}$ and $LM_{3,\frac{n+1}{2}+1}$; we get

$$\begin{aligned} LM_{3,n-1} =& (d_2(a^{n-1})+d_2(a^2)) + \cdots + (d_2(a^{n-1})+d_2(a^{n-3})) + (d_2(a^{n-1})+d_2(a^{n-2})) \\ =& (1+n-3) + (1+n-4) + \cdots + (1+(n-1-\frac{n-1}{2})) + \cdots + (1+3) + \\ &+(1+2) + (1+(n-1-\frac{n-1}{2})) \\ =& \sum_{q=2}^{n-3}(1+q) + (1+\frac{n-1}{2}), \end{aligned}$$

$$\begin{aligned}LM_{3,n-2} =&(d_2(a^{n-2})+d_2(a^3))+\cdots+(d_2(a^{n-2})+d_2(a^{n-2}))+(d_2(a^{n-2})+d_2(a^{n-3}))\\ =&(2+(n-4))+(2+(n-5))+\cdots+(2+(n-1-\frac{n-1}{2}))+\cdots+(2+4)+\\ &+(2+3)+(2+(n-1-\frac{n-1}{2}))\\ =&\sum_{q=3}^{n-4}(2+q)+(2+\frac{n-1}{2}),\end{aligned}$$

$$\vdots$$

$$\begin{aligned}LM_{3,\frac{n+1}{2}+2} =&(d_2(a^{\frac{n+1}{2}+2})+d_2(a^{\frac{n+1}{2}-2}))+(d_2(a^{\frac{n+1}{2}+2})+d_2(a^{\frac{n+1}{2}-1}))+(d_2(a^{\frac{n+1}{2}+2})+d_2(a^{\frac{n+1}{2}}))+\\ &+(d_2(a^{\frac{n+1}{2}+2})+d_2(a^{\frac{n+1}{2}+1}))\\ =&(\frac{n-5}{2}+\frac{n+1}{2})+(\frac{n-5}{2}+\frac{n-1}{2})+(\frac{n-5}{2}+\frac{n-1}{2})+(\frac{n-5}{2}+\frac{n-3}{2})\\ =&\sum_{q=\frac{n-5}{2}}^{\frac{n+1}{2}}(\frac{n-5}{2}+q)+(\frac{n-5}{2}+\frac{n-1}{2}),\end{aligned}$$

and finally

$$\begin{aligned}LM_{3,\frac{n+1}{2}+1} =&(d_2(a^{\frac{n+1}{2}+1})+d_2(a^{\frac{n+1}{2}-1}))+(d_2(a^{\frac{n+1}{2}+1})d_2(a^{\frac{n+1}{2}}))\\ =&(\frac{n-3}{2}+\frac{n-1}{2})+(\frac{n-3}{2}+\frac{n-1}{2})\\ =&\sum_{q=\frac{n-1}{2}}^{\frac{n-1}{2}}(\frac{n-3}{2}+q)+(\frac{n-3}{2}+\frac{n-1}{2}),\end{aligned}$$

Hence

$$LM_{3,n}+LM_{3,n-1}+\cdots+LM_{3,\frac{n+1}{2}+2}+LM_{3,\frac{n+1}{2}+1}=\sum_{r=0}^{\frac{n-1}{2}-1}\sum_{q=r+1}^{n-2-r}(r+q)+\sum_{s=0}^{\frac{n-1}{2}-1}(s+\frac{n-1}{2})$$

If similar operations are performed in case n is even, the following sum is obtained

$$LM_{3,n}+LM_{3,n-1}+\cdots+LM_{3,\frac{n}{2}+2}+LM_{3,\frac{n}{2}+1}=\sum_{r=0}^{\frac{n}{2}-2}\sum_{q=r+1}^{n-2-r}(r+q)+\sum_{s=0}^{\frac{n}{2}-1}(s+\frac{n}{2}-1)$$

So as desired. □

Theorem 2.9. *Let $\Gamma(\mathscr{S}_{\mathscr{A}})$ be monogenic semigroup graphs. Then, we have*

$$LF(\Gamma(\mathscr{S}_{\mathscr{A}}))=\begin{cases}\frac{2n^4-11n^3+20n^2-12n}{8}, & n \text{ even}\\ \frac{2n^4-11n^3+23n^2-21n+7}{8}, & n \text{ odd.}\end{cases}$$

Proof. From definition of the first leap Zagreb index and by Lemma 2.4, we have

$$
\begin{aligned}
LF(\Gamma(\mathscr{S}_{\mathscr{A}})) = & \sum_{a^i \in V(\Gamma(\mathscr{S}_{\mathscr{A}}))} d_2^3(a^i) \\
=& d_2^3(a) + d_2^3(a^2) + \cdots + d_2^3(a^{\lfloor \frac{n}{2} \rfloor}) + d_2^3(a^{\lfloor \frac{n}{2} \rfloor} + 1) + d_2^3(a^{\lfloor \frac{n}{2} \rfloor + 2}) + \\
& + \cdots + d_2^3(a^{n-1}) + d_2^3(a^n) \\
=& (n-2)^3 + (n-3)^3 + ... + \left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right)^3 + \left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right)^3 \\
& + \left(n - 2 - \left\lfloor \frac{n}{2} \right\rfloor\right)^3 + \cdots + 2^3 + 1^3 + 0^3 \\
=& \left(\frac{(n-2)(n-1)}{2}\right)^2 + \left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right)^3 \qquad (2.3)
\end{aligned}
$$

There are two possible situations from here. With (2.1) we get the desired result. Thus, the proof is completed. □

Theorem 2.10. *Let $\Gamma(\mathscr{S}_{\mathscr{A}})$ be monogenic semigroup graphs. Then, we have*

$$
LEC(\Gamma(\mathscr{S}_{\mathscr{A}})) = \begin{cases} n^2 - 2n, & n \text{ odd} \\ n^2 - 2n + 1, & n \text{ even.} \end{cases}
$$

Proof. By definition of the monogenic semigroup graph, we see that $ecc(a^n) = 1$ and the eccentries of the other vertices are two the except of the vertex a^n. Thus, from definition of the leap eccentric connectivity index and Lemma 2.4, we have

$$
\begin{aligned}
LEC(\Gamma(\mathscr{S}_{\mathscr{A}})) = & \sum_{a^i \in V(\Gamma(\mathscr{S}_{\mathscr{A}}))} d_2(a^i).ecc(a^i) \\
=& d_2(a)ecc(a) + d_2(a^2)ecc(a^2) + \cdots + d_2(a^{\lfloor \frac{n}{2} \rfloor})ecc((a^{\lfloor \frac{n}{2} \rfloor})) \\
& + d_2(a^{\lfloor \frac{n}{2} \rfloor + 1})ecc(a^{\lfloor \frac{n}{2} \rfloor + 1}) + d_2(a^{\lfloor \frac{n}{2} \rfloor + 2})ecc(a^{\lfloor \frac{n}{2} \rfloor + 2}) + \\
& + \cdots + d_2(a^{n-1})ecc(a^{n-1}) + d_2(a^n)ecc(a^n) \\
=& (n-2).2 + (n-3).2 + ... + \left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right).2 + \left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right).2 \\
& + \left(n - 2 - \left\lfloor \frac{n}{2} \right\rfloor\right).2 + \cdots + 2.2 + 1.2 + 0.1 \\
=& (n-2)(n-1) + (n - 1 - \left\lfloor \frac{n}{2} \right\rfloor).2 \qquad (2.4)
\end{aligned}
$$

There are two possible situations from here. With (2.1) we get the desired result. Thus, the proof is completed. □

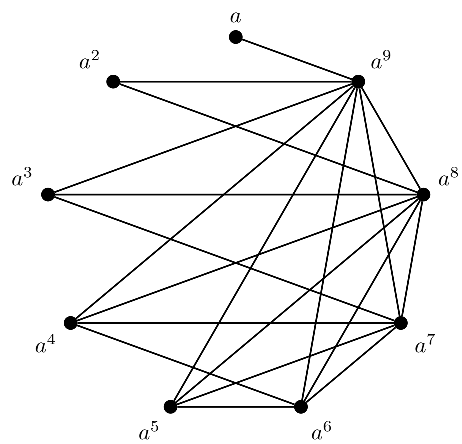

FIGURE 1. The graph of $\Gamma(\mathscr{S}_{\mathscr{A}})$

Example 2.11. Let us consider the monogenic semigroup graphs $\Gamma(\mathscr{S}_{\mathscr{A}})$ with nine vertices as Figure 1. Then we get

- $LM_1(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{4.9^3-15.9^2+20.9-9}{12} = 156$ (by Theorem 2.5)
- $LM_2(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{9^4-4.9^3+2.9^2+4.9-3}{48} = 80$ (by Theorem 2.6)
- $LM_3(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{2.9^3-3.9^2-2.9+3}{12} = 25$ (by Theorem 2.7)
- $LF(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{2.9^4-11.9^3+23.9^2-21.9+7}{8} = 848$ (by Theorem 2.8)
- $LEC(\Gamma(\mathscr{S}_{\mathscr{A}})) = 9^2 - 2.9 = 63$ (by Theorem 2.9)

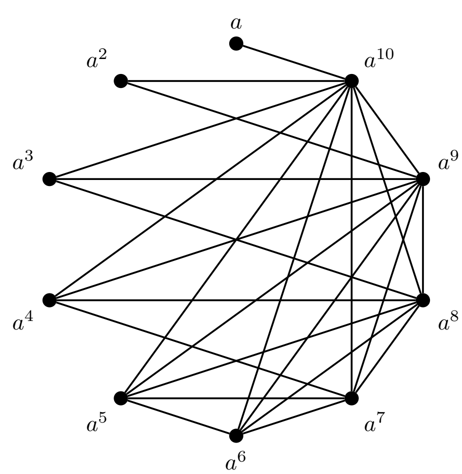

FIGURE 2. The graph of $\Gamma(\mathscr{S}_{\mathscr{A}})$

Example 2.12. Let us consider the monogenic semigroup graphs $\Gamma(\mathscr{S}_{\mathscr{A}})$ with ten vertices as Figure 2. Then we get

- $LM_1(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{4.10^3-15.10^2+14.10}{12} = 220$ (by Theorem 2.5)

- $LM_2(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{10^4-4.10^3+2.10^2+4.10}{48} = 130$ (by Theorem 2.6)
- $LM_3(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{2.10^3-3.10^2-2.10}{12} = 140$ (by Theorem 2.7)
- $LF(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{2.10^4-11.10^3+20.10^2-12.10}{8} = 1360$ (by Theorem 2.8)
- $LEC(\Gamma(\mathscr{S}_{\mathscr{A}})) = 10^2 - 2.10 + 1 = 81$ (by Theorem 2.9)

3. Conclusion

Topological indices are important tools that are widely used in revealing the chemical and physical properties of molecules, especially in QSAR and QSPR research. Leap indices have an important place among topological indices. In this study, we calculated the leap indices of monogenic semigroup graphs in terms of number of the vertices.

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Department of Mathematics, Faculty of Science, Kahramanmaras Sutcu Imam University, Kahramanmaras, Turkey

Email address: yasarnacaroglu@ksu.edu.tr

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DNA CODES FROM REVERSIBLE GROUP CODES BY A VIRUS OPTIMISATION ALGORITHM

ADRIAN KORBAN, SERAP ŞAHINKAYA, AND DENIZ USTUN

0000-0001-5206-6480, 0000-0002-2084-6260 and 0000-0002-5229-4018

Abstract. In this paper, we employ group rings and some known results on group codes to study reversible group DNA codes. We define and study reversible cyclic DNA codes from a group ring point of view and we also introduce the notion for self-reciprocal group ring elements. Moreover, we search for reversible group DNA codes with the use of a virus optimisation algorithm. We obtain many good DNA codes that satisfy the Hamming distance, the reverse, the reverse-complement and the fixed GC-content constraints.

The interest in studying and designing DNA codes has been started with Adleman when he solved a computationally difficult mathematical problem by introducing an algorithm using DNA strands and molecular biology tools [1] and it is still an ongoing research area. Some known methods for designing DNA codes that satisfy certain conditions include the study of reversible codes [15], reversible self-dual codes over $GF(4)$ [9], the study of cyclic and extended cyclic constructions or the study of linear constructions [7].

Recently in [4], linear codes derived from group ring elements are considered to construct reversible DNA codes that satisfy the Hamming distance constraint. This suggests that the study of group rings is an interesting research direction that may have some useful applications to DNA coding. In this work, we employ group rings and a matrix construction given in [4] to study reversible cyclic DNA codes. We also use group rings to define a self-reciprocal group ring element. Moreover, we construct reversible group codes of different lengths over the finite commutative Frobenius ring R, that satisfy the Hamming distance, the reverse, the reverse-complement and the fixed GC-content constraints.

The paper is organised as follows. In Section 2, we give the basic definitions and results on linear codes, DNA codes, group rings, group codes and reversible group codes. In Section 3, we define and study reversible cyclic DNA codes from a group ring point of view. In this section, we also define a self-reciprocal group ring element. In Section 4, we present two generator matrices for reversible group codes which we then use to search for DNA codes that satisfy the Hamming distance,

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the reverse, the reverse-complement and the fixed GC-content constraints. In our search scheme, we employ a virus optimisation algorithm which allows us to obtain numerical results in a reasonable quick time. We finish with concluding remarks and directions for possible future research.

1. Preliminaries

1.1. **Linear Codes and DNA Codes.** In this section, we recall basic definitions on linear codes, DNA codes and DNA constraints.

A linear code of length n over $\mathbb{F}_4$ is a subspace of $\mathbb{F}_4^n$, and we also call an element of a linear code a codeword. The Hamming distance $d(\mathbf{x}, \mathbf{y})$ between two codewords is the number of coordinates in which $\mathbf{x}$ and $\mathbf{y}$ are distinct. The minimum Hamming distance d_H of a linear code $\mathcal{C}$ is defined as

$$min\{d_H(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \neq \mathbf{y},\ \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{C}\}.$$

Let $S_{D_4} = \{A, C, G, T\}$ represents the four nucleotides in DNA, which are adenine (A), cytosine (C), guanine (G) and thymine (T) and let $\mathbf{x} = (x_1, x_2, \dots, x_n)$, where $x_i \in S_{D_4}$. A DNA code $\mathcal{D}$ of length n is defined as a set of codewords $(x_1, x_2, \dots, x_n)$ where $x_i \in S_{D_4} = \{A, T, C, G\}$. We use a hat to denote the Watson-Crick complement of a nucleotide, $\hat{A} = T, \hat{T} = A, \hat{C} = G$ and $\hat{G} = C$. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in S_{D_4}$, then $\mathbf{x}^r = (x_n, x_{n-1}, \dots, x_2, x_1)$ $\mathbf{x}^c = (x_1^c, x_2^c, \dots, x_n^c)$ and $\mathbf{x}^{rc} = (x_n^c, x_{n-1}^c, \dots, x_2^c, x_1^c)$ denote the reverse of a DNA codeword, the complement of a DNA codeword and the reverse complement of a DNA codeword respectively. In this paper, the fixed GC-content is simply half the length of the DNA code D.

A good DNA code $\mathcal{D}$ of length n is defined as a set of codewords $(x_1, x_2, \dots, x_n)$ where $x_i \in S_{D_4} = \{A, T, C, G\}$, such that $\mathcal{D}$ satisfies some or all of the following constraints [2]:

(i) The Hamming distance constraint (HD):

$$min\{d_H(\mathbf{x}, \mathbf{y}) :\ \ \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \text{ and } x \neq y\}$$

(ii) The reverse constraint (RV):

$$min\{d_H(\mathbf{x}^r, \mathbf{y}) : \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \text{ and } \mathbf{x}^r \neq \mathbf{y}\ \}$$

(iii) The reverse-complement constraint (RC):

$$min\{d_H(\mathbf{x}^{rc}, \mathbf{y}) :\ \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \text{ and } \mathbf{x}^{rc} \neq \mathbf{y}\}$$

(iv) The fixed GC-content constraint (GC): The set of codewords with length n, distance d and GC weight w, where w is the total number of Gs and Cs present in the DNA strand:

$$w_{\mathbf{x}_{DNA}} = |\{x_i : \mathbf{x}_{DNA} = (x_i), x_i \in \{C, G\}\}|.$$

A DNA code can be identified with a code over $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ by employing the standard bijective correspondence between $\mathbb{F}_4$ and the DNA alphabet $S_{D_4} = \{A, T, C, G\}$ given by

$$\eta : \mathbb{F}_4 \to S_{D_4},$$

with $\eta(0) = A$, $\eta(1) = T$, $\eta(\omega) = C$ and $\eta(\omega^2) = G$. The same correspondence has already been used in the literature, for example, please see [9]. We extend the bijection η so that $\eta(\mathcal{C})$ is regarded as a DNA code for some code $\mathcal{C}$ over $\mathbb{F}_4$.

We denote the complete weight enumerator of a code $\mathcal{C}$ over $\mathbb{F}_4$ by

$$CWE_{\mathcal{C}}(a,b,c,d) = \sum_{c \in \mathcal{C}} a^{n_0(c)} b^{n_1(c)} c^{n_\omega(c)} d^{n_{\omega^2}(c)},$$

where $n_s(c)$ denotes the number of occurrences of s in a codeword c. We identify the complete weight enumerator of a DNA code $\mathcal{D}$ with that of a code $\mathcal{C}$ over $\mathbb{F}_4$, where $\mathcal{D} = \eta(\mathcal{C})$. The GC-weight of a codeword $c \in \mathcal{C}$ is the sum of $n_\omega(c)$ and $n_{\omega^2}(c)$. Therefore, if we let

$$GCW_{\mathcal{C}}(a,b) = CWE_{\mathcal{C}}(a,a,b,b),$$

then $GCW_{\mathcal{C}}(a,b)$ is the GC-weight enumerator of a code $\mathcal{C}$, where the coefficient of b^i is the same as the number of codewords with GC-weight i.

Let $A_4^R(n,d)$ denote the maximum cardinality of a DNA code for a given distance d and length n that satisfies the Hamming distance and reverse constraints. Let $A_4^{RC}(n,d)$ be the maximum size of a DNA code of length n satisfying the HD and RC constraints for a given d, $A_4^{GC}(n,d,w)$ be the maximum size of a DNA code of length n satisfying the HD constraint for a given d with a constant GC-weight w, and $A_4^{RC,GC}(n,d,w)$ the maximum size of a DNA code of length n satisfying the HD and RC constraints for a given d with a constant GC-weight w. In [12], for an even n, the following equality is given;

$$A_4^{RC}(n,d) = A_4^R(n,d). \tag{1.1}$$

1.2. **Group Rings and Group Codes.** We shall now give the standard definition of group rings. Let G be a finite group of order n and let R be a finite ring. Then any element in RG is of the form $v = \sum_{i=1}^n \alpha_{g_i} g_i$, $\alpha_{g_i} \in R$, $g_i \in G$. Addition in RG is done by coordinate addition, namely

$$\sum_{i=1}^n \alpha_{g_i} g_i + \sum_{i=1}^n \beta_i g_i = \sum_{i=1}^n (\alpha_{g_i} + \beta_i) g_i.$$

The product of two elements in RG is given by

$$\left(\sum_{i=1}^n \alpha_{g_i} g_i \right) \left(\sum_{j=1}^n \beta_j g_j \right) = \sum_{i,j} \alpha_{g_i} \beta_j g_i g_j.$$

This gives that the coefficient of g_k in the product is $\sum_{g_i g_j = g_k} \alpha_{g_i} \beta_j$.

The following matrix construction was used to study group codes over Frobenius rings in [6]. Let R be a finite commutative Frobenius ring and let $G = \{g_1, g_2, \dots, g_n\}$ be a group of order n and let $v = \sum_{i=1}^n \alpha_{g_i} \in RG$. Define the matrix $\sigma(v) \in M_n(R)$ to be

$$\sigma(v) = \begin{pmatrix} \alpha_{g_1^{-1} g_1} & \alpha_{g_1^{-1} g_2} & \alpha_{g_1^{-1} g_3} & \dots & \alpha_{g_1^{-1} g_n} \\ \alpha_{g_2^{-1} g_1} & \alpha_{g_2^{-1} g_2} & \alpha_{g_2^{-1} g_3} & \dots & \alpha_{g_2^{-1} g_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{g_n^{-1} g_1} & \alpha_{g_n^{-1} g_2} & \alpha_{g_n^{-1} g_3} & \dots & \alpha_{g_n^{-1} g_n} \end{pmatrix}. \tag{1.2}$$

We note that the elements $g_1^{-1}, \dots, g_n^{-1}$ are simply the elements of the group G given in some order. This particular order is used because it aids in certain proofs and computations. In [6], the following code construction is given:

$$\mathcal{C}(v) = \langle \sigma(v) \rangle. \tag{1.3}$$

The code is formed by taking the row space of $\sigma(v)$ over the ring R. Such codes are refereed to as group codes or, for simplicity, G-codes. Moreover, in [6], it is shown that this matrix construction of G-codes corresponds to an ideal in the group ring RG and thus the resulting group code has the group G as a subgroup of its automorphism group. Please see [6] for more details on group codes generated from group rings. From now on, every time we refer to G-codes, we mean codes constructed as given above.

1.3. **Reversible Group Codes.** Here, we recall an interesting result from [4] on group codes. Namely, this result shows that for certain groups and for a specific ordering of the group elements, one can construct G-codes that are reversible. We first start with a definition from [4].

Definition 1.1. A code $\mathcal{C}$ is said to be reversible of index α if $\mathbf{a}_i$ is a vector of length α and $\mathbf{c}^\alpha = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{s-1}) \in \mathcal{C}$ implies that $(\mathbf{c}^\alpha)^r = (\mathbf{a}_{s-1}, \mathbf{a}_{s-2}, \ldots, \mathbf{a}_1, \mathbf{a}_0) \in \mathcal{C}$.

Let G be a finite group of order $n = 2l$ and let $H = \{e, h_1, h_2, \ldots, h_{\ell-1}\}$ be a subgroup of index 2 in G. Let $\beta \notin H$ be an element in G, with $\beta^{-1} = \beta$. We list the elements of $G = \{g_1, g_2, \ldots, g_n\}$ as follows:

$$\{e, h_1, \ldots, h_{\ell-1}, \beta h_{\ell-1}, \beta h_{\ell-2}, \beta h_2, \beta h_1, \beta\}. \tag{1.4}$$

The following result was proved in [4].

Theorem 1.2. *Let R be a finite ring. Let G be a finite group of order $n = 2\ell$ and let $H = \{e, h_1, h_2, \ldots, h_{\ell-1}\}$ be a subgroup of index 2 in G. Let $\beta \notin H$ be an element in G with $\beta^{-1} = \beta$. List the elements of G as in (1.4), then any linear G-code in R^n (a left ideal in RG) is a reversible code of index 1.*

In [4], the authors make a connection between reversible G-codes and DNA codes, this is because reversibility is a desirable property for DNA codes.

1.4. **Virus Optimization Algorithm.** A new bio-inspired optimization technique called as virus optimization algorithm (VOA) is proposed in [5] for difficult and complex mathematical and engineering problems. The VOA is a meta-heuristic optimization technique based on population and it mimics the behavior of viruses assaulting a living cell. In each replication step, the number of the viruses increases then antivirus applied to virus population to avoid the plosive growing of the virus population. Thus, the number of the virus in the population is controlled with help of the antivirus. In the VOA, the viruses in the population are separated into two groups as common and strong. In the initialization phase of the VOA, there are two steps; parameter setting and the generation of initial viruses. Parameter setting is a key for an effective search process in the search space. After the parameters have been set, the initial virus population is randomly produced and the viruses are classified. In the replication procedure, new viruses are produced by using strong and common viruses in the initial population. When the new viruses are generated by the replication procedure, the corresponding objective function values are evaluated. Then, the old and new viruses are then combined together. If the performance of the virus population is not improved, the antivirus procedure is applied to the population and it is followed by the verification of the termination criterion. If the termination criteria has not been met, the replication is repeated.For more details on this approach see [10].

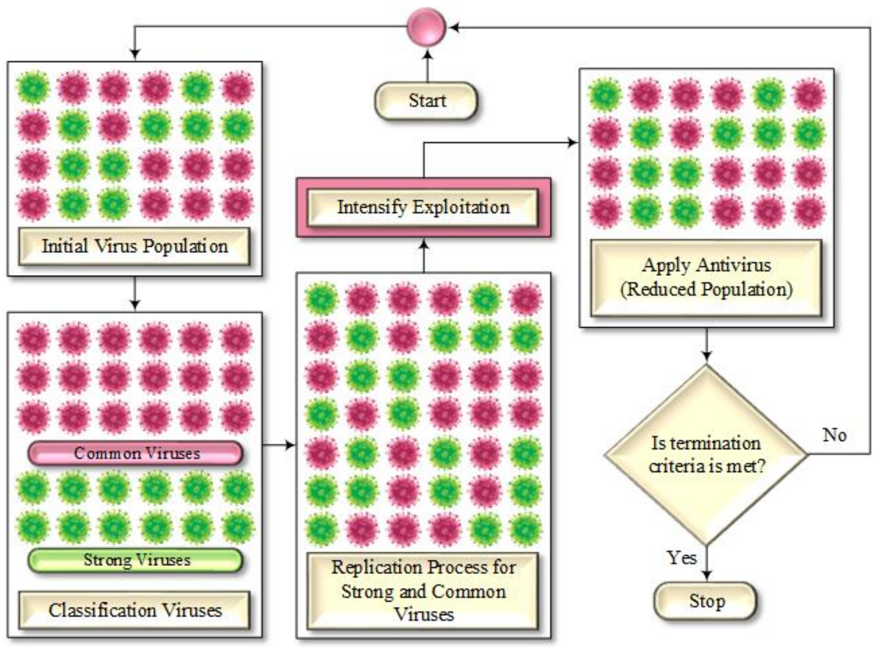

Figure 1. Flowchart of VOA

2. Reversible Group Codes and DNA Codes

In this section, we use the results from Section 1.3 to define and study reversible cyclic DNA codes. We also define self-reciprocal group ring elements.

2.1. **Reversible Cyclic DNA Codes as Ideal in Group Rings.** Cyclic codes have a canonical algebraic description as ideals in the polynomial ring $R[x]/\langle x^n - 1\rangle$, where R is a Frobenius ring and n is the length of the code. An alternate view of cyclic codes is to see them as ideals in the group ring RC_n where C_n is the cyclic group of order n.

For a cyclic code C, there exists a relationship between reversible codes and self-reciprocal polynomials. More precisely, in Theorem 1 in [13], the following is proven. The cyclic code over $\mathbb{F}_q$, generated by the monic polynomial $g(x)$, is reversible if and only if $g(x)$ is self-reciprocal. Therefore, in this setting the search for reversible codes coincides with the search for self-reciprocal polynomials that divide $x^n - 1$ over the field $\mathbb{F}_q$.

Often in the literature, reversible cyclic codes are studied over polynomial rings due to the fact that polynomial rings have a rich algebraic description. In this section, we intend to study reversible cyclic codes in a different setting - from a group ring point of view. We begin with a definition.

Definition 2.1. Let C_n be the cyclic group of order n and let

$$\{e = c^0, c, c^2, \dots, c^{n-1}\}$$

be a fixed listing of the elements of C_n. Let $v = \sum_{i=0}^{n-1} \alpha_{c^i} c^i \in RC_n$. The reciprocal of v is defined as

$$v^* = c^{n-1}(\sum_{i=0}^{n-1} \alpha_{c^i} (c^i)^{-1}) = \sum_{i=0}^{n-1} \alpha_{c^i} c^{n-(i+1)}.$$

We call the group ring element v self-reciprocal if and only if $v^* = v$.

For the cyclic group C_n, the matrices $\sigma(v)$ and $\sigma(v^*)$ can be written as follows:

$$\sigma(v) = \begin{pmatrix} \alpha_e & \alpha_c & \alpha_{c^2} & \cdots & \alpha_{c^{n-1}} \\ \alpha_{c^{n-1}} & \alpha_e & \alpha_c & \cdots & \alpha_{c^{n-2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_c & \alpha_{c^2} & \alpha_{c^3} & \cdots & \alpha_e \end{pmatrix},$$

$$\sigma(v^*) = \begin{pmatrix} \alpha_{c^{n-1}} & \alpha_{c^{n-2}} & \cdots & \alpha_c & \alpha_e \\ \alpha_e & \alpha_{c^{n-1}} & \cdots & \alpha_{c^2} & \alpha_c \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{c^{n-2}} & \alpha_{c^{n-3}} & \cdots & \alpha_e & \alpha_{c^{n-1}} \end{pmatrix}.$$

Theorem 2.2. *The cyclic code $C(v) = \langle \sigma(v) \rangle$ where $v \in RC_n$, is reversible of index 1 if and only if v is self-reciprocal.*

Proof. The proof follows from the fact that v is self-reciprocal if and only if $\sigma(v) = \sigma(v^*)$. The index 1 follows from the construction of the matrix $\sigma(v)$. □

We illustrate this theorem with an example.

Example 2.3. Let $v_1 = 1 + 2c + 2c^3 + c^4 \in \mathbb{Z}_3 C_5$, where $C_5 = \{e, c, c^2, c^3, c^4\}$. Here, $\alpha_e = 1$, $\alpha_c = 2$, $\alpha_{c^2} = 0$, $\alpha_{c^3} = 2$ and $\alpha_{c^4} = 1$. Then

$$\sigma(v_1) = \begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ 2 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 2 \\ 2 & 0 & 2 & 1 & 1 \end{pmatrix}.$$

Now, $v_1^* = c^4(1 + 2c^4 + 2c^2 + c) = 1 + 2c + 2c^3 + c^4 = v_1$, and

$$\sigma(v_1^*) = \begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ 2 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 2 \\ 2 & 0 & 2 & 1 & 1 \end{pmatrix}.$$

Thus $\sigma(v_1) = \sigma(v_1^*)$. Also, $\sigma(v) = \sigma(v^*)$ can be written as

$$\sigma(v_1) = \sigma(v_1^*) = \begin{pmatrix} 1 & 1 & 2 & 0 & 2 \\ 2 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 2 \\ 2 & 0 & 2 & 1 & 1 \end{pmatrix}.$$

Clearly, $C(v_1) = C(v_1^*) = \langle\sigma(v_1)\rangle = \langle\sigma(v_1^*)\rangle$ is the $[5,4,2]$ cyclic code. We also see that the code $C(v_1)$ is reversible since in the generator matrix the reverse of each row of $C(v_1)$ is also in $C(v_1)$.

We now give the group ring analogue of the notion of lifted polynomials which is defined in [16].

Definition 2.4. Let $C_n = \{e, c, \ldots, c^{n-1}\}$ be a cyclic group of order n and $v = \sum_{i=0}^{n-1} \alpha_{c^i} c^i \in \mathbb{F}_p C_n$ be a self-reciprocal element. A lifted element of v denoted by $\ell(v) \in \mathbb{F}_{p^s} C_n$ is defined as follows:

(1) if n is odd then

$$\ell(v) = \sum_{i=0}^{(n-1)/2} \theta_i;\ \theta_i = \begin{cases} \beta_i c^i + \beta_i c^{n-i}, & \alpha_{c^i} \neq 0, \\ 0, & \alpha_{c^i} = 0, \end{cases}$$

(2) if n is even then

$$\ell(v) = \sum_{i=0}^{n/2} \theta_i;\ \theta_i = \begin{cases} \beta_i c^i + \beta_i c^{n-i}, & \alpha_{c^i} \neq 0,\ i \neq \frac{n}{2}, \\ 0, & \alpha_{c^i} = 0, \\ \beta_{n/2} c^{n/2}, & \alpha_{c^i} \neq 0,\ i = \frac{n}{2}, \end{cases}$$

where $\beta_i \in \mathbb{F}_{p^s}^*$.

Lemma 2.5. *If the element v is self-reciprocal then $\ell(v)$ is self-reciprocal.*

Proof. The result follows from the definitions. □

Example 2.6. Let $v = 1 + 2c + 2c^3 + c^4 \in \mathbb{Z}_3 C_5$, where $C_5 = \{e, c, c^2, c^3, c^4\}$. Then, for $\beta_i \in \mathbb{F}_{3^4} = \mathbb{F}_{81}$,

$$\ell(v) = \sum_{i=0}^{2} \theta_i = \theta_0 + \theta_1 + \theta_2 = (\beta_0 1 + \beta_0 c^4) + (\beta_1 c + \beta_1 c^3) + 0,$$

$$\ell(v) = \beta_0 + \beta_1 c + \beta_1 c^3 + \beta_0 c^4.$$

For $\beta_0 = \alpha^4, \beta_1 = \alpha^6$, we have $\ell(v) = \alpha^4 + \alpha^6 c + \alpha^6 c^3 + \alpha^4 c^4$. Now,

$$\ell(v^*) = \alpha^4 + \alpha^6 c + \alpha^6 c^3 + \alpha^4 c^4 = \ell(v),$$

which gives that $\ell(v)$ is self-reciprocal.

Theorem 2.7. *Let R be a finite commutative Frobenius ring and let C_n be the cyclic group of order n. Let $\ell(v)$ be a lifted element of a self-reciprocal element of group ring RC_n. Then the cyclic code $C(\ell(v))$ is reversible.*

Proof. Follows from Theorem 2.2. □

The following definition is the group ring analogue of the notion of the co-term polynomial which is defined in [8].

Definition 2.8. Let C_n be the cyclic group of order n and let

$$\{e, c, c^2, \ldots, c^{n-1}\}$$

be a fixed listing of C_n. Let $v = \sum_{i=0}^{n-1} \alpha_{c^i} c^i \in RC_n$. Then v is called a co-term element if $\alpha_{c^i} = \alpha_{c^{n-i}}$ for all $1 \le i \le \lfloor \frac{n}{2} \rfloor$. Moreover, $v = \sum_{i=0}^{n-1} \alpha_{c^i} c^i \in RC_n$ is the co-term element if and only if $(\alpha_{c^1}, \alpha_{c^2} \ldots, \alpha_{c^{n-1}})$ is self-reversible.

Example 2.9. Consider the element v_1 from Example 2.3. We saw there that $v_1 = v_1^*$ and therefore v_1 is self-reciprocal. The element v_1 is not a co-term element since for instance, $\alpha_{c^1} \neq \alpha_{c^4}$, i.e., $\alpha_{c^1} = 2$ and $\alpha_{c^4} = 1$.

We denote the vector $\mathbf{v} = (\alpha_{c^0}, \alpha_{c^1}, \ldots, \alpha_{c^{n-1}}) \in R^n$ for $v = \sum_{i=0}^{n-1} \alpha_{c^i} c^i \in RC_n$. In [8], the following is proven.

Lemma 2.10. *Let $S \subseteq R^n$ be a non empty subset such that $\mathbf{v}^r \in S$ whenever $\mathbf{v} \in S$. Then the code generated by S is a linear reversible code of length n over R.*

Theorem 2.11. *Let $v = \sum_{i=0}^{n-1} \alpha_{c^i} c^i \in RC_n$ be a co-term element and let t be a specified positive integer. Suppose v corresponds to the vector $\mathbf{v} = (\alpha_{c^0}, \alpha_{c^1}, \ldots, \alpha_{c^{n-1}}) \in R^n$. For any length n and even dimension, define the $(2t+2) \times n$ matrix as:*

$$\kappa_t(v) = \begin{pmatrix} \alpha_{c^{n-(t+1)}} & \alpha_{c^{n-t}} & \cdots & \alpha_{c^{n-(t+3)}} & \alpha_{c^{n-(t+2)}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{c^1} & \alpha_{c^2} & \cdots & \alpha_{c^{n-1}} & \alpha_{c^0} \\ \alpha_{c^0} & \alpha_{c^1} & \cdots & \alpha_{c^{n-2}} & \alpha_{c^{n-1}} \\ \alpha_{c^{n-1}} & \alpha_{c^0} & \cdots & \alpha_{c^{n-3}} & \alpha_{c^{n-2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{c^{n-t}} & \alpha_{c^{n-(t-1)}} & \cdots & \alpha_{c^{n-(t+2)}} & \alpha_{c^{n-(t+1)}} \end{pmatrix},$$

and for odd length n and odd dimension, define the $(2t+2) \times n$ matrix as:

$$\kappa_t(v) = \begin{pmatrix} \alpha_{c^{n-(t+1)}} & \alpha_{c^{n-t}} & \cdots & \alpha_{c^{n-(t+3)}} & \alpha_{c^{n-(t+2)}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{c^1} & \alpha_{c^2} & \cdots & \alpha_{c^{n-1}} & \alpha_{c^0} \\ \alpha_{c^0} & \alpha_{c^1} & \cdots & \alpha_{c^{n-2}} & \alpha_{c^{n-1}} \\ \alpha_{c^{n-1}} & \alpha_{c^0} & \cdots & \alpha_{c^{n-3}} & \alpha_{c^{n-2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{c^{n-t}} & \alpha_{c^{n-(t-1)}} & \cdots & \alpha_{c^{n-(t+2)}} & \alpha_{c^{n-(t+1)}} \\ \alpha_{c^{n-(n-1)/2}} & \alpha_{c^{n-((n-1)/2-1)}} & \cdots & \alpha_{c^{n-((n-1)/2+2)}} & \alpha_{c^{n-((n-1)/2+1)}} \end{pmatrix},$$

where $t < \lfloor \frac{n}{2} \rfloor$. Then the code $C = \langle \kappa_t(v) \rangle$ is reversible.

Proof. Let $\mathbf{v} = (\alpha_{c^0}, \alpha_{c^1}, \ldots, \alpha_{c^{n-1}})$. Since $v \in RC_n$ is a co-term element, it follows that $(\alpha_{c^1}, \alpha_{c^2}, \ldots, \alpha_{c^{n-1}})$ is self-reversible. Also, since v is a co-term element, for a positive integer $i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$ the reverse of the i-th row of the matrix $\kappa_t(v)$ equals to the $(n+1) - i$-th row. We also have

$$(\alpha_{c^{n-(n-1)/2}}, \alpha_{c^{n-((n-1)/2-1)}}, \cdots, \alpha_{c^{n-((n-1)/2+2)}}, \alpha_{c^{n-((n-1)/2+1)}})^r = \\ (\alpha_{c^{n-(n-1)/2}}, \alpha_{c^{n-((n-1)/2-1)}}, \cdots, \alpha_{c^{n-((n-1)/2+2)}}, \alpha_{c^{n-((n-1)/2+1)}}).$$

In both cases of the theorem, the spanning sets, that is the rows of $\kappa_t(v)$, satisfy the conditions of Lemma 2.10. Thus the code is reversible. □

Example 2.12. Let $v = 1 + \omega c + c^2 + \omega^2 c^3 + \omega^2 c^6 + c^7 + \omega c^8 \in \mathbb{F}_4 C_9$ be a co-term element and

$$\mathbf{v} = (1, \omega, 1, \omega^2, 0, 0, \omega^2, 1, \omega)$$

be the corresponding vector. All the codes $\langle \kappa_t(v) \rangle$ are reversible. For $t = 0$,

$$\kappa_0(v) = \begin{pmatrix} \alpha_{c^1} & \alpha_{c^2} & \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} & \alpha_{c^7} & \alpha_{c^8} & \alpha_{c^0} \\ \alpha_{c^0} & \alpha_{c^1} & \alpha_{c^2} & \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} & \alpha_{c^7} & \alpha_{c^8} \end{pmatrix}$$
$$= \begin{pmatrix} \omega & 1 & \omega^2 & 0 & 0 & \omega^2 & 1 & \omega & 1 \\ 1 & \omega & 1 & \omega^2 & 0 & 0 & \omega^2 & 1 & \omega \end{pmatrix}.$$

For $t = 2$,

$$\kappa_2(v) = \begin{pmatrix} \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} & \alpha_{c^7} & \alpha_{c^8} & \alpha_{c^0} & \alpha_{c^1} & \alpha_{c^2} \\ \alpha_{c^2} & \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} & \alpha_{c^7} & \alpha_{c^8} & \alpha_{c^0} & \alpha_{c^1} \\ \alpha_{c^1} & \alpha_{c^2} & \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} & \alpha_{c^7} & \alpha_{c^8} & \alpha_{c^0} \\ \alpha_{c^0} & \alpha_{c^1} & \alpha_{c^2} & \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} & \alpha_{c^7} & \alpha_{c^8} \\ \alpha_{c^8} & \alpha_{c^0} & \alpha_{c^1} & \alpha_{c^2} & \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} & \alpha_{c^7} \\ \alpha_{c^7} & \alpha_{c^8} & \alpha_{c^0} & \alpha_{c^1} & \alpha_{c^2} & \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} \end{pmatrix}$$
$$= \begin{pmatrix} \omega^2 & 0 & 0 & \omega^2 & 1 & \omega & 1 & \omega & 1 \\ 1 & \omega^2 & 0 & 0 & \omega^2 & 1 & \omega & 1 & \omega \\ \omega & 1 & \omega^2 & 0 & 0 & \omega^2 & 1 & \omega & 1 \\ 1 & \omega & 1 & \omega^2 & 0 & 0 & \omega^2 & 1 & \omega \\ \omega & 1 & \omega & 1 & \omega^2 & 0 & 0 & \omega^2 & 1 \\ 1 & \omega & 1 & \omega & 1 & \omega^2 & 0 & 0 & \omega^2 \end{pmatrix}.$$

It can be easily seen that in each of the above matrices, the reverse of each row is contained in the same matrix.

2.2. **Self-Reciprocal Group Ring Elements.** In this section, we define a self-reciprocal group ring element which is the analogue notion of the notion of a self-reciprocal polynomial.

Definition 2.13. Let G be a finite group and let $\{g_1, g_2, \ldots, g_n\}$ be a fixed listing of the elements of G. Also, let $v = \sum_{i=1}^{n} \alpha_{g_i} g_i \in RG$. The reciprocal of v is defined as

$$v^* = \sum_{i=1}^{n} \alpha_{g_i} g_{n-(i-1)}.$$

We call v self-reciprocal if and only if $v^* = v$.

Lemma 2.14. *Let $v = \sum_{i=1}^{n} \alpha_{g_i} g_i \in RG$. Then $(v^*)^* = v$.*

Proof. We have that $v^* = \sum_{i=1}^{n} \alpha_{g_i} g_{n-(i-1)}$ by definition. Applying the reciprocal definition to the element v^* again:

$$(v^*)^* = \sum_{i=1}^{n} \alpha_{g_i} g_{n-[n-(i-1)-1]} = \sum_{i=1}^{n} \alpha_{g_i} g_i = v.$$

This gives the result. □

It is well known that a group ring is isomorphic to a well defined ring of matrices and thus every group ring element has an associated matrix. We now generalise the matrix representation of a reciprocal cyclic group ring element to a more general group ring.

The matrix representation of a reciprocal group ring element is as follows:

$$\sigma(v^*) = \begin{pmatrix} \alpha_{g_1 g_n} & \alpha_{g_1 g_{n-1}} & \alpha_{g_1 g_{n-2}} & \cdots & \alpha_{g_1 g_1} \\ \alpha_{g_2 g_n} & \alpha_{g_2 g_{n-1}} & \alpha_{g_2 g_{n-2}} & \cdots & \alpha_{g_2 g_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{g_n g_n} & \alpha_{g_n g_{n-1}} & \alpha_{g_n g_{n-2}} & \cdots & \alpha_{g_n g_1} \end{pmatrix}.$$

We now look at an example in which we give the matrix representations of a dihedral group ring element and its reciprocal.

Example 2.15. Consider $\mathbb{Z}_3 D_8$ where $\{e, a, a^2, a^3, ba^3, ba^2, ba, b\}$ is the fixed listing of elements of D_8. Let $v = 2 + a^2 + ba + 2ba^2 + ba^3 \in \mathbb{Z}_3 D_8$. Then

$$\sigma(v) = \begin{pmatrix} 2 & 0 & 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \end{pmatrix},$$

and $\sigma(v)$ can be written as following:

$$\sigma(v) = \begin{pmatrix} 2 & 0 & 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

Clearly $C(v) = \langle \sigma(v) \rangle$ is the $[8, 4, 4]$ code. It is also clear that $C(v)$ is reversible, that is, the reverse of each codeword of $C(v)$ is also in $C(v)$.

Next,

$$\sigma(v^*) = \begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 2 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 1 & 0 & 1 & 2 & 1 \\ 2 & 0 & 1 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

and $\sigma(v^*)$ can be written as follows

$$\sigma(v^*) = \begin{pmatrix} 2 & 0 & 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

We note here that although $\sigma(v) \neq \sigma(v^*)$, the codes $C(v)$ and $C(v^*)$ are the same.

An element v is said to be self-reversible if $v = v^*$.

Theorem 2.16. *Let $v = \sum_{i=1}^{n} \alpha_{g_i} g_i \in RG$ be a self-reversible element and let t be a specified positive integer where $t < \frac{n}{2}$. Suppose v corresponds to the vector $\mathbf{v} = (\alpha_{g_1}, \alpha_{g_2}, \alpha_{g_3}, \ldots, \alpha_{g_n}) \in R^n$. For any length n and even dimension, we define the $(2t+2) \times n$ matrix as*

$$\kappa_t(v) = \begin{pmatrix} \alpha_{g_{t+2}g_1} & \alpha_{g_{t+2}g_2} & \cdots & \alpha_{g_{t+2}g_{n-1}} & \alpha_{t+2g_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{g_3g_1} & \alpha_{g_3g_2} & \cdots & \alpha_{g_3g_{n-1}} & \alpha_{g_3g_n} \\ \alpha_{g_2g_1} & \alpha_{g_2g_2} & \cdots & \alpha_{g_2g_{n-1}} & \alpha_{g_2g_n} \\ \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \cdots & \alpha_{g_1^{-1}g_{n-1}} & \alpha_{g_1^{-1}g_n} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \cdots & \alpha_{g_2^{-1}g_{n-1}} & \alpha_{g_2^{-1}g_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{g_{t+1}^{-1}g_1} & \alpha_{g_{t+1}^{-1}g_2} & \cdots & \alpha_{g_{t+1}^{-1}g_{n-1}} & \alpha_{g_{t+1}^{-1}g_n} \end{pmatrix}.$$

Then the code $C = \langle \kappa_t(v) \rangle$ is reversible.

Proof. Since v is a self-reversible element, for a positive integer $i \in \{1, 2 \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$ the reverse of the i-th row of the matrix $\kappa_t(v)$ equals to the $(n+1) - i$-th row. Therefore, the spanning set, that is the rows of the matrix, satisfy Lemma 2.10. This completes the proof. □

Example 2.17. Let $v = 1 + ab \in \mathbb{F}_2 V_4$, be a self-reversible element, where $V_4 = \{1, b, ab, a\}$ is a Klein-4-group. We have that $\mathbf{v} = (1, 0, 1, 0)$. Then for $t = 0$

$$\kappa_0(v) = \begin{pmatrix} \alpha_{g_2g_1} & \alpha_{g_2g_2} & \alpha_{g_2g_3} & \alpha_{g_2g_4} \\ \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \alpha_{g_1^{-1}g_4} \end{pmatrix},$$

$$\kappa_0(v) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

so $\langle \kappa_0(v) \rangle$ is reversible. Also, for $t = 1$

$$\kappa_1(v) = \begin{pmatrix} \alpha_{g_3g_1} & \alpha_{g_3g_2} & \alpha_{g_3g_3} & \alpha_{g_3g_4} \\ \alpha_{g_2g_1} & \alpha_{g_2g_2} & \alpha_{g_2g_3} & \alpha_{g_2g_4} \\ \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \alpha_{g_1^{-1}g_4} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \alpha_{g_2^{-1}g_3} & \alpha_{g_2^{-1}g_4} \end{pmatrix},$$

$$\kappa_1(v) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

which is clear that $\langle \kappa_1(v) \rangle$ is reversible.

Example 2.18. For the quaternion group Q_8, the fixed listing of elements is

$$\{1, i, j, k, -k, -j, -i, -1\} = \{g_1, g_2, \ldots, g_8\}.$$

Let

$$v = 1 + 2j + k - k - 2j - 1 \in \mathbb{F}_3 Q_8,$$

then $\mathbf{v} = (1, 0, 2, 1, 1, 2, 0, 1)$ is the corresponding vector.

For $t=0$,

$$\kappa_0(v) = \begin{pmatrix} \alpha_{g_2g_1} & \alpha_{g_2g_2} & \alpha_{g_2g_3} & \alpha_{g_2g_4} & \alpha_{g_2g_5} & \alpha_{g_2g_6} & \alpha_{g_2g_7} & \alpha_{g_2g_8} \\ \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \alpha_{g_1^{-1}g_4} & \alpha_{g_1^{-1}g_5} & \alpha_{g_1^{-1}g_6} & \alpha_{g_1^{-1}g_7} & \alpha_{g_1^{-1}g_8} \end{pmatrix},$$

$$\kappa_0(v) = \begin{pmatrix} i & -1 & k & -j & j & -k & 1 & -i \\ 1 & i & j & k & -k & -j & -i & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 1 & 2 & 0 & 1 \end{pmatrix}.$$

For $t=2$,

$$\kappa_2(v) = \begin{pmatrix} \alpha_{g_4g_1} & \alpha_{g_4g_2} & \alpha_{g_4g_3} & \alpha_{g_4g_4} & \alpha_{g_4g_5} & \alpha_{g_4g_6} & \alpha_{g_4g_7} & \alpha_{g_4g_8} \\ \alpha_{g_3g_1} & \alpha_{g_3g_2} & \alpha_{g_3g_3} & \alpha_{g_3g_4} & \alpha_{g_3g_5} & \alpha_{g_3g_6} & \alpha_{g_3g_7} & \alpha_{g_3g_8} \\ \alpha_{g_2g_1} & \alpha_{g_2g_2} & \alpha_{g_2g_3} & \alpha_{g_2g_4} & \alpha_{g_2g_5} & \alpha_{g_2g_6} & \alpha_{g_2g_7} & \alpha_{g_2g_8} \\ \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \alpha_{g_1^{-1}g_4} & \alpha_{g_1^{-1}g_5} & \alpha_{g_1^{-1}g_6} & \alpha_{g_1^{-1}g_7} & \alpha_{g_1^{-1}g_8} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \alpha_{g_2^{-1}g_3} & \alpha_{g_2^{-1}g_4} & \alpha_{g_2^{-1}g_5} & \alpha_{g_2^{-1}g_6} & \alpha_{g_2^{-1}g_7} & \alpha_{g_2^{-1}g_8} \\ \alpha_{g_3^{-1}g_1} & \alpha_{g_3^{-1}g_2} & \alpha_{g_3^{-1}g_3} & \alpha_{g_3^{-1}g_4} & \alpha_{g_3^{-1}g_5} & \alpha_{g_3^{-1}g_6} & \alpha_{g_3^{-1}g_7} & \alpha_{g_3^{-1}g_8} \end{pmatrix},$$

$$\kappa_2(v) = \begin{pmatrix} k & j & -i & -1 & 1 & i & -j & -k \\ j & -k & -1 & i & -i & 1 & k & -j \\ i & -1 & k & -j & j & -k & 1 & -i \\ 1 & i & j & k & -k & -j & -i & -1 \\ -i & 1 & -k & j & -j & k & -1 & i \\ -j & k & 1 & -i & i & -1 & -k & j \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 2 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 & 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

3. Computational Results

In this section, we define two generator matrices using the map given in Equation (1.2) with a fixed listing of the group elements as given in Equation (1.4). We employ the cyclic group of even order and the dihedral group of order $2n$. We next use these generator matrices to search for DNA codes over $\mathbb{F}_4$. We perform our search in the software package MAGMA ([3]) using a heuristic search scheme called the virus optimization algorithm (VOA). This method, as shown in [10], allows us to obtain the computational results significantly faster then the standard linear search.

We obtain many DNA codes of up to and including length 32. Our DNA codes satisfy the Hamming distance, the reverse, the reverse-complement and the fixed GC-content constraints. We find the lower bounds on $A_4^{RC,GC}(n,d,k)$ by computing the complete weight enumerators of all DNA codes that we found. The generator matrices, weight enumerators, GC-weight enumerators for the codes constructed can be found at [11].

Let $w_1 \in R\mathcal{C}_{2n}$, where $\mathcal{C}_{2n}$ is the cyclic group of order $2n$ with its elements being listed as follows:

$$\{1, c^2, c^4, c^6, \ldots, c^{2n-2}, c^nc^{2n-2}, c^nc^{2n-4}, \ldots, c^nc^2, c^n\}. \tag{3.1}$$

Then the generator matrix $\sigma(w_1)$ has the following form:

$$\mathcal{G}_1 = \sigma(w_1) = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix},$$

where

$$A_1 = cir(\alpha_1, \alpha_{c^2}, \ldots, \alpha_{c^{2n-2}})$$

is a $n \times n$ circulant matrix,

$$B_1 = rcir(\alpha_{c^n c^{2n-2}}, \alpha_{c^n c^{2n-4}}, \ldots, \alpha_{c^n})$$

is a $n \times n$ reverse circulant matrix. We note that B_2 is a reverse circulant matrix in which the first row is obtained by reversing the last row of the matrix B_1. The matrix A_2 is a circulant matrix in which the first row is obtained by reversing the last row of the matrix A_1. More precisely, $A_2 = cir(\alpha_1, \alpha_{c^{2n-2}}, \ldots, \alpha_{c^4}, \alpha_{c^2})$ is an $n \times n$ circulant matrix and $B_2 = rcir(\alpha_{c^n c^2}, \ldots, \alpha_{c^n c^{2n-4}}, \alpha_{c^n c^{2n-2}}, \alpha_{c^n})$ is an $n \times n$ reverse circulant matrix.

Let $w_2 \in R\mathcal{D}_{2n}$, where $\mathcal{D}_{2n}$ is the dihedral group of order $2n$ with its elements being listed as follows:

$$\{e, a, a^2, \ldots, a^{n-1}, ba^{n-1}, ba^{n-2}, \ldots, ba, b\}. \tag{3.2}$$

Then the generator matrix $\sigma(w_2)$ has the following form:

$$\mathcal{G}_2 = \sigma(w_2) = \begin{pmatrix} A & B \\ B^T & A^T \end{pmatrix},$$

where

$$A = cir(\alpha_e, \alpha_a, \ldots, \alpha_{a^{n-1}})$$

is a $n \times n$ circulant matrix and

$$B = cir(\alpha_{ba^{n-1}}, \alpha_{ba^{n-2}}, \ldots, \alpha_b)$$

is a $n \times n$ circulant matrix.

We now present a small example of how we construct the DNA codes using our group ring approach.

Example 3.1. Let D_6 be a dihedral group of order 6 with the ordering of elements $\{e, a, a^2, ba^2, ba, b\}$, $v = w + wa + wa^2 \in \mathbb{F}_4 D_6$ then the generator matrix has the form

$$\sigma(v) = \mathcal{G}_2 = \begin{pmatrix} w & w & w & 0 & 0 & 0 \\ w & w & w & 0 & 0 & 0 \\ w & w & w & 0 & 0 & 0 \\ 0 & 0 & 0 & w & w & w \\ 0 & 0 & 0 & w & w & w \\ 0 & 0 & 0 & w & w & w \end{pmatrix}. \tag{3.3}$$

From the above generator matrix, we construct a DNA code $\mathcal{C}$ with 16 codewords satisfying R-constraint with $d = 3$ as follows;

(3.4)
$$\begin{aligned} A_4^R(6,3) = \{&AAAAAA, AAAGGG, AAATTT, GGGAAA, CCCAAA, AAACCC, \\ &TTTAAA, TTTTTT, GGGCCC, CCCCCC, GGGGGG, TTTCCC, \\ &CCCGGG, GGGTTT, TTTGGG, CCCTTT\}. \end{aligned}$$

We know by [12] that for an even n:

$$A_4^{RC}(n,d) = A_4^R(n,d). \tag{3.5}$$

Therefore $A_4^{RC}(6,3) = A_4^R(6,3)$. The GC-weight enumerator of $\mathcal{C}$ is

$$GCW(a,b) = 4a^6 + 8a^3b^3 + 4b^6.$$

Thus we construct a DNA code with 8 codewords satisfying hamming distance constraint 3, reversible complement constraint and fixed GC-content constraint with $k = 3$.

We now employ the generator matrices $\mathcal{G}_1$ and $\mathcal{G}_2$, to search for DNA codes that satisfy the Hamming distance, the reverse, the reverse-complement and the fixed GC-content constraints of lengths up to and including 32. We tabulate our findings in Table 2 and Table 1. The results that are equal to or better than the currently known best bounds are written in bold, and new results are also written in bold. Generator matrices, GC-weight enumerators and parameters of codes in Tables 2 and 1 can be found in [11].

Table 1. Lower bounds on $A_4^{RC}(n,d)$ and $A_4^{RC,GC}(n,d,k)$ from $\mathcal{G}_2$

n	d	$A_4^{RC}(n,d)$	$A_4^{RC,GC}(n,d,k)$
4	3	**16**	**12**
6	3	**64**	**30**
6	2	**1024**	**480**
8	4	**256**	**176**
8	3	**256**	**152**
8	2	**4096**	**2240**
12	6	**4096**	**1848**
12	4	**16384**	**6144**
14	5	**65536**	**13728**
14	4	**65536**	**13728**
16	6	**65536**	**25880**
16	4	**1048576**	**461824**
16	2	**268435456**	**105431040**
18	4	**4194304**	**1400256**
18	3	**16777216**	**3111680**
18	2	**4294967296**	**1429733376**
20	5	**16777216**	**2956096**
20	4	**1073741824**	**376832000**
20	3	**4294967296**	**756760576**
20	2	**68719476736**	**12108169216**
20	6	**1048576**	**369008**
20	7	**1048576**	**369512**
22	6	**16777216**	**2821728**
22	2	**1099511627776**	**339270959104**
24	4	**68719476736**	**22409117696**
24	3	**68719476736**	**11098587136**
24	2	**17592186044416**	**2841238306816**
24	6	**268435456**	**86739968**

4. Conclusion

In this work, we showed that one can construct good DNA codes from G-codes that are reversible- this is a crucial property for DNA codes. We defined and studied reversible cyclic DNA codes and we also defined self-reciprocal group ring elements. We presented two generator matrices that one can use to search for DNA codes. We employed these generator matrices with the use of only two groups, the cyclic group of even order and the dihedral group of order $2n$, to search for reversible cyclic and dihedral DNA codes that satisfy the Hamming distance, the reverse, the reverse complement and the GC-weight enumerator constraints. Our group ring approach proved to be successful as we constructed many DNA codes. A possible research direction is to consider reversible group ring approach and specifically Theorem 3.10 to construct, possibly, more DNA codes with better parameters.

TABLE 2. Lower bounds on $A_4^{RC}(n,d)$ and $A_4^{RC,GC}(n,d,k)$ from $\mathcal{G}_1$

n	d	$A_4^{RC}(n,d)$	$A_4^{RC,GC}(n,d,k)$
24	4	**4294967296**	**1387323392**
24	3	**68719476736**	**22160015360**
24	2	**17592186044416**	**2835513081856**
26	2	**281474976710656**	**81000264630272**
28	4	**1099511627776**	**328637349888**
28	3	**17592186044416**	**2630898155520**
28	2	**4503599627370496**	**1345974567960576**
30	4	**1125899906842624**	**30497345372160**0
30	3	**4503599627370496**	**650610034606080**
30	2	**1125899906842624**	**162652508651520**
32	2	**1152921504606846976**	**322709486693253120**
32	4	**17592186044416**	**4928618364928**

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(Adrian Korban) Department of Mathematical and Physical Sciences, University of Chester, Thornton Science Park, Pool Ln, Chester CH2 4NU, England,

Email address, Adrian Korban: `adrian3@windowslive.com`

(Serap Şahinkaya) Tarsus University, Faculty of Engineering, Department of Natural and Mathematical Sciences, Mersin, Turkey,

Email address, Serap Şahinkaya: `serap@tarsus.edu.tr`

(Deniz Ustun) Tarsus University, Faculty of Engineering, Department of Computer Engineering, Mersin, Turkey

Email address, Deniz Ustun: `denizustun@tarsus.edu.tr`

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EXACT SOLUTION OF THE SCHRÖDINGER EQUATION IN TOPOLOGICALLY MASSIVE SPACETIME

ALI TARSUSLU AND KENAN SOGUT

0009-0001-2839-7578 and 0000-0002-9682-2855

Abstract. We study exact solutions of the Schrödinger equation in a topologically massive space-time. Exact solutions are obtained in terms of the hypergeometric functions. We also obtained the momentum quantization with the help of the condition of the wave function to be bounded. The investigation is performed in the framework of rainbow formalism of the General Relativity Theory (RGT). The quantized momentum is evaluated for different choices of the rainbow functions.

1. Introduction

In last decades, it has been a crucial area to study exact solutions of the non-relativistic and relativistic wave equations that present precious data concerning to the quantum mechanical systems. In this circumstances, the Schrödinger (for spinless and non-relativistic massive particles), the Klein-Gordon (KG) (spin-0 particles, e.g., pions), the Dirac (for spin-1/2, e.g., electrons) and the Duffin-Kemmer-Petiau (DKP) (for spin-1, e.g., $W^{\pm}$, Z^0 bosons and photons) equations are the most examined equations [1, 2, 3, 4, 5, 6, 7, 8]. Except the Schrödinger equation, the rest are the fundamental single particle equations of the relativistic quantum mechanics.

The Schrödinger equation defines the non-relativistic quantum mechanical character of an isolated physical system by evolution of a wave function over the time. Its solutions given for the presence of external electromagnetical fields have fundamental applications used in technology, engineering, electro-mechanics, particle physics, medical physics, and so on. Compared to the other wave equations the Schrödinger equation has much less been studied in curved geometry. Examining the Schrödinger equation in curved space-time is a way of finding the effective low-energy characterization of a quantum particle in a curved geometry.

Recently, a new approach to the Einstein's General Relativity (EGG), which is called "Doubly General Relativity" (DGR) is introduced [9], thereafter called as Rainbow Gravity (RG) to study the quantum effects of the gravitation in the smallest accessible regions, namely the Planck scale. The idea behind the RG

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approach of gravitation is that at ultra-high energy regimes the geometry of classical space-time alters by the probing particles that have different energies [9, 10, 11]. Thus, the standard metric is deformed and this phenomenon is represented in space-time metric with rainbow functions. Because of this modified perspective, the rainbow version of a metric can be written by the replacements $dx^0 \to \frac{dx^0}{f}$ for the time coordinate and $dx^i \to \frac{dx^i}{g}$ for the spatial coordinates. As the particle moves in geometry, it will perceive gravity differently for each energy it has, as the way a prism acts on light.

The structure of the paper will be as follows. In section 2, we give a brief theoretical set-up of the problem and in section 3, we will solve the Schrödinger equation for the considered rainbow space-time. In section 4, by obtaining approximate solutions, the quantization condition of the momentum will be derived. Finally, section 5 is devoted to the discussion of results.

2. Preliminaries

For the investigation of our problem, we will study in the RG formalism and discuss the dynamics of particle by the topologically massive space-time given by,

$$ds^2 = d\theta^2 + d\phi^2 + 2\cos(\nu\theta)d\psi d\phi + d\psi^2 \tag{2.1}$$

which is basically a de Sitter space with the polar angle suffering a conic defect. This metric has been offered by Aliev et.al. [12] and it has obvious importance for the gauge field theories. Here, the term ν is the topological mass and it is related to the cosmological constant as $\lambda = \frac{\nu^2}{4}$. The metric can be diagonalized by introducing new variables as $\varphi = \psi + \phi$ and $\chi = \psi - \phi$. Therefore the above metric takes the following form,

$$ds^2 = d\theta^2 + \cos^2\frac{\nu\theta}{2}d\varphi^2 + \sin^2\frac{\nu\theta}{2}d\chi^2 \tag{2.2}$$

In the modified perspective, the rainbow counterpart of the above metric can be written as

$$ds^2 = \frac{1}{g^2(\varepsilon)}\left[d\theta^2 + \cos^2\frac{\nu\theta}{2}d\varphi^2 + \sin^2\frac{\nu\theta}{2}d\chi^2\right] \tag{2.3}$$

where $g(\varepsilon)$ is the energy-dependent rainbow function, $\varepsilon = \frac{E}{E_{Pl.}}$, E is the energy of the probing particle and $E_{Pl.}$ is the Planck energy.

3. Exact solution of the Schrödinger equation

The covariant form of the Schrödinger equation in curved space is given as follows [13],

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{-\hbar^2}{2m}\left[\frac{1}{\sqrt{detg_{\mu\nu}}}\frac{\partial}{\partial x^\mu}(\sqrt{detg_{\mu\nu}}g^{\mu\nu}\frac{\partial\Psi}{\partial x^\nu})\right] - \frac{\hbar^2}{6}R\Psi \tag{3.1}$$

where m is the mass of particle, $\hbar$ is the Planck constant, $g^{\mu\nu}$ is the metric given by Eq.(2.3) and R is the scalar curvature of the space which is calculated by the contradiction of the Ricci tensor and given as

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}\left(\partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\mu\Gamma^\alpha_{\alpha\nu} + \Gamma^\alpha_{\alpha\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\alpha_{\mu\rho}\Gamma^\rho_{\alpha\nu}\right) \tag{3.2}$$

where

$$\Gamma^{\alpha}_{\nu\lambda} = \frac{1}{2} g^{\alpha\beta} \left(\partial_{\nu} g_{\lambda\beta} + \partial_{\lambda} g_{\beta\nu} - \partial_{\beta} g_{\nu\lambda} \right) \tag{3.3}$$

are the Christoffel symbols [14] and obtained as follows

$$\Gamma^{1}_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{4}\sin(\nu\theta) & 0 \\ 0 & 0 & -\frac{\nu}{4}\sin(\nu\theta) \end{pmatrix}, \tag{3.4}$$

$$\Gamma^{2}_{ij} = \begin{pmatrix} 0 & -\frac{\nu}{2}\tan(\nu\theta) & 0 \\ -\frac{\nu}{2}\tan(\nu\theta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.5}$$

and

$$\Gamma^{2}_{ij} = \begin{pmatrix} 0 & 0 & \frac{\nu}{2}\cot(\nu\theta) \\ 0 & 0 & 0 \\ \frac{\nu}{2}\cot(\nu\theta) & 0 & 0 \end{pmatrix}. \tag{3.6}$$

By using the line element given by Eq.(2.3) and (3.4, 3.5, 3.6) in Eq.(3.2), the scalar curvature is obtained as $R = \frac{3g^2(\varepsilon)\nu^2}{2}$. With the help of these results and reminding that $\sqrt{detg_{\mu\nu}} = \frac{\sin(\nu\theta)}{2g^3}$, the Schrödinger equation (3.1) reduces to the following form,

$$f^{''}(\theta) + 2\cot(2y) f^{'}(\theta) + \left[c - \left(\frac{a}{\cos^2(y)} + \frac{b}{\sin^2(y)} \right) \right] f(\theta) = 0 \tag{3.7}$$

where the definitions $\Psi = e^{i(\alpha\varphi + \beta\chi - Et)} f(\theta)$, $a = \frac{4\alpha^2}{\nu^2}$, $b = \frac{4\beta^2}{\nu^2}$, $c = \frac{8m}{\nu^2 g^2}\left(E + \frac{\nu^2 g^2}{4}\right)$ and $y = \frac{\nu\theta}{2}$ were made. If the variable is changed as $\cos^2 y = \frac{1}{u}$, Eq.(3.7) is transformed to into the below form,

$$4u^2(u-1)^2 f^{''}(u) + 4u^2(u-1) f^{'}(u) - \left[au^2 - Bu + c \right] f(u) = 0 \tag{3.8}$$

where $B = a - b + c$.

If we define the wave function as $f(u) = u^{p-1}(u-1)^q \Omega(u)$, Eq.(3.8) can be written as

$$u(u-1)\Omega^{''} + [u(2p+2q+1) + 2(1-p)]\Omega^{'} + \left[2pq + \frac{p^2(u-1) - 2pu + u^3 p}{u} + \frac{qu(q-2) + 2q + \frac{uK}{4} + \frac{L}{4}}{u-1} + \frac{M}{4u(u-1)} \right] \Omega = 0 \tag{3.9}$$

where $K = 4 - a$, $L = B - 12$ and $M = 8 - c$.

For the choices of $p^2 - 3p + \frac{M}{4} = 0$, $4q^2 + K + L + M = 0$ and definitions $2(p-1) = \gamma$, $2(p+q) + 1 = P + Q + 1$, $2pq + p^2 - 2(p+q) - \frac{M+L}{4} = PQ$, we obtain

$$u(u-1)\Omega^{''}(u) + [(P+Q+1)u - \gamma]\Omega^{'}(u) + PQ\Omega(u) = 0 \tag{3.10}$$

which has the form of hypergeometric differential equation [15]. Solutions of this equation are given by

$$\Omega(u) = {}_2F_1\left(P, Q, \gamma; u\right) \tag{3.11}$$

and

$$\Omega(u) = u^{1-\gamma} {}_2F_1\left(P + 1 - \gamma, Q + 1 - \gamma, 2 - \gamma; u\right) \tag{3.12}$$

where ${}_2F_1$ are hypergeometric functions. Thence, exact solutions of the Schrödinger equation is obtained as

$$\Psi = e^{i(\alpha\varphi+\beta\chi-Et)}\left[\cos(\frac{\nu\theta}{2})\right]^{2(1-p)}\left[\cos(\frac{\nu\theta}{2})\right]^{-2q}{}_2F_1\left(P,Q,\gamma;\cos^{-2}(\frac{\nu\theta}{2})\right) \tag{3.13}$$

For the specific discussions of our general results, one can use various scenarios introduced in literature for the rainbow functions. We give a few well-known proposals of the rainbow functions in TABLE I.

TABLE 1. Most studied proposals of the rainbow functions. Here, c_1, c_2, c_3, c_4 and t are arbitrary parameters.

f	g	Reference
1	$\sqrt{1-c_1\chi^t}$	[16]
$(c_2\chi)^{-1}(\exp[c_2\chi]-1)$	1	[16]
$(1-c_3\chi)^{-1}$	$(1-c_3\chi)^{-1}$	[16]
$(1-c_4\chi)^{-1}$	1	[17]
$\exp[-\frac{\chi^2}{2}]$	1	[18]
1	$1+\frac{\chi}{2}$	[19]
$1+\frac{\chi}{2}$	$1+(2\chi)^{-1}$	[19]
1	$1+\chi^t$	[20]

4. Asymptotic solution of the Schrödinger equation

For the small value of the argument, namely $y=\frac{\nu\theta}{2}\ll 1$, Eq. (3.7) transforms into

$$y^2f^{''}(y)+2yf^{'}(y)+[(c-a)y^2-b]f(y)=0 \tag{4.1}$$

This is the Bessel differential equation and solution is given by [15]

$$f(y)=\frac{1}{\sqrt{y}}Z_{\sqrt{b+\frac{1}{4}}}(\sqrt{c-a}y) \tag{4.2}$$

where $Z_\mu(\zeta)$ are the cylindrical functions and can be written in terms of the Bessel functions as

$$Z_\mu = c_1J_\mu + c_2Y_\mu \tag{4.3}$$

where J_μ is first type and Y_μ is second type of Bessel functions that are related to Kummer functions, $M(a,b,z)$, as following [15]

$$J_\mu=\frac{(\frac{y}{2})^\mu}{\Gamma(\mu+1)}e^{-iy}M(\mu+\frac{1}{2},2\mu+1,2iy) \tag{4.4}$$

and

$$Y_\mu=\frac{J_\mu\cos(\mu\pi)-J_{-\mu}}{\sin(\mu\pi)} \tag{4.5}$$

In order Kummer functions to be finite, we require the bound condition of the Kummer functions as $\mu+\frac{1}{2}=-n$ [15]. Therefore, we obtain the quantized momentum of the Schrödinger particle in terms of the topological mass as in the follow,

$$\beta=\frac{\nu}{2}\sqrt{n(n+1)} \tag{4.6}$$

which is the momentum in the χ-direction.

5. Conclusion

In this study, we have analyzed the Schrödinger equation in a modified rainbow background. In the process of obtaining the solutions we used the separation of variables method. Both the wave function and momentum of the Schrödinger particle are obtained depending on the topological mass. The topological massive $(3+0)$-space is hard to study for the relativistic higher spinning particles. So, this study may have the potential of providing insights into the relativistic spinning particles as well. This is going to be a further study in the corresponding space. One of the interesting finding of this study is that although the dynamics of the particle depends on the angular variables, topological mass term and energy of the probing particle, the quantized momentum depends on only the topological mass.

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(Ali Tarsuslu) Mersin University, Physics Department, 33343, Mersin, TURKEY
Email address: alitarsuslu@gmail.com

(Kenan Sogut) Mersin University, Physics Department, 33343, Mersin, TURKEY
Email address: kenansogut@gmail.com

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BASIC BOUNDARY VALUE PROBLEM WITH RETARDED ARGUMENT CONTAINING AN EIGENPARAMETER IN THE TRANSMISSION CONDITION

ÖZGÜR MIZRAK

0000-0001-5961-6019

Abstract. In this paper basic boundary value problem with retarded argument that has a discontinuity point inside the interval will be studied. At the discontinuity point transmission conditions contain eigenparameter. Existence of eigenvalues and eigenfunctions will be studied. Asmyptotic properties of eigenvalues and eigenfunctions will be obtained.

1. Introduction

Many realistic system depend not only on current state but also the past. These systems can be modeled by using retarded argument equations. In detail these type of equations can be considered in two groups. Equations with constant delay is called equations with time lag and equations with functioanal delay is called equations with after affect.

After the development of control systems in engineering retarded equations become important. Before that scientists were aware of this type of delays in the control systems but there was not enough theory about this subject. Because of that this type of affects were ignored in the models. Delays have an important role to explain complex models mathematically and it also has important affects. Equations with retarded argument is used modeling problems in the fields of biology, chemistry, economics, mechanics, physics, physiology, population change, social networks, heat dissipation, interaction of species, microbiology and engineering. Unlike ordinary differantial equations, equations with retarded argument belong to functional diffrential equation class.

The fundamental study in this subject is made by Norkin in 1956 and 1958 [1, 2]. Şen - Bayramov [3], Yang [4], Akgün-Bayramov-Bayramoğlu [5], Şen - Seo- Arıcı [6], Bayramoğlu - Bayramov - Şen [7], Çetinkaya - Mamedov [8],F. Hira [9] have studied the retarded equation with discontinuity point in the interval.

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Freiling - Yurko [10], Mosazadeh [11], Bondarenko - Yurko [12] have studied the inverse problem.

In this work discontinuous equation on $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ with parameter in the transmission condition will be considered.

$$y''(x) + \lambda^2 y(x) + q(x)y(x - \Delta(x)) = 0 \tag{1.1}$$

$$y(0) = y(\pi) = 0 \tag{1.2}$$

$$y\left(\frac{\pi}{2} + 0\right) = \frac{\delta}{\lambda} y\left(\frac{\pi}{2} - 0\right) \tag{1.3}$$

$$y'\left(\frac{\pi}{2} + 0\right) = \frac{\delta}{\lambda} y'\left(\frac{\pi}{2} - 0\right) \tag{1.4}$$

here $q(x)$ and $\Delta(x) \geq 0$ are continuous functions on $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ and have finite left right limits at $\frac{\pi}{2}$, if $x \in [0, \frac{\pi}{2})$ then $x - \Delta(x) \geq 0$, if $x \in (\frac{\pi}{2}, \pi]$ then $x - \Delta(x) \geq \frac{\pi}{2}$, λ is a real eigenparameter and $\delta \neq 0$ is arbitrary real number.

Let $\omega_1(x, \lambda)$ be a solution of equation (1.1) on $[0, \frac{\pi}{2})$. After defining this solution, using the transmission conditions (1.3) and (1.4) we can define the solution of equation (1.1) on $(\frac{\pi}{2}, \pi]$ in terms of $\omega_1(x, \lambda)$ as follows:

$$\omega_2\left(\frac{\pi}{2}, \lambda\right) = \frac{\delta}{\lambda}\omega_1\left(\frac{\pi}{2}, \lambda\right), \quad \omega_2'\left(\frac{\pi}{2}, \lambda\right) = \frac{\delta}{\lambda}\omega_1'\left(\frac{\pi}{2}, \lambda\right) \tag{1.5}$$

Consequently, we can define $\omega(x, \lambda)$ on $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ as

$$\omega(x, \lambda) = \begin{cases} \omega_1(x, \lambda), & x \in [0, \frac{\pi}{2}) \\ \omega_2(x, \lambda), & x \in (\frac{\pi}{2}, \pi] \end{cases}$$

here $\omega(x, \lambda)$ solves equation (1.1) on $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ and satisfies left boundary condition and both transmission conditions (1.3) and (1.4).

Lemma 1.1. *Let $\omega(x, \lambda)$ be a solution of* (1.1) *and $\lambda > 0$. Then $\omega_1(x, \lambda)$ and $\omega_2(x, \lambda)$ are defined as:*

$$\omega_1(x, \lambda) = \sin \lambda x - \frac{1}{\lambda}\int_0^x \sin \lambda(x - \tau) q(\tau)\omega_1(\tau - \Delta(\tau), \lambda) d\tau \tag{1.6}$$

$$\begin{aligned} \omega_2(x, \lambda) = {} & \frac{\delta}{\lambda}\omega_1\left(\frac{\pi}{2}, \lambda\right)\cos\lambda\left(x - \frac{\pi}{2}\right) + \frac{\delta}{\lambda^2}\omega_1'\left(\frac{\pi}{2}, \lambda\right)\sin\lambda\left(x - \frac{\pi}{2}\right) \\ & - \frac{1}{\lambda}\int_{\frac{\pi}{2}}^x \sin\lambda(x - \tau) q(\tau)\omega_2(\tau - \Delta(\tau), \lambda) d\tau \end{aligned} \tag{1.7}$$

Theorem 1.2. *Eigenvalues of the problem* (1.1)-(1.4) *are simple.*

Proof. Let $\tilde{\lambda}$ be an eigenvalue of the problem (1.1)-(1.4) and

$$\tilde{u}(x, \tilde{\lambda}) = \begin{cases} \tilde{u}_1(x, \tilde{\lambda}), & x \in [0, \frac{\pi}{2}) \\ \tilde{u}_2(x, \tilde{\lambda}), & x \in (\frac{\pi}{2}, \pi] \end{cases}$$

be a corresponding eigenfunction. Then from (1.2), the wronskien becomes zero.

$$W[\tilde{u}_1(x, \tilde{\lambda}), \omega(x, \tilde{\lambda})] = \begin{vmatrix} \tilde{u}_1(0, \tilde{\lambda}) & 0 \\ \tilde{u}_1'(0, \tilde{\lambda}) & 1 \end{vmatrix} = 0$$

It means that these two functions corresponding to $\tilde{\lambda}$ are linearly deppendent. Similarly it can be shown that $\tilde{u}_2(x, \tilde{\lambda})$ and $\omega_2(x, \tilde{\lambda})$ are linearly dependent. Therefore eigenvalues are simple. □

Pluging $\omega(x,\lambda)$ into the other boundary condition, characteristic equation is obtained:

$$F(\lambda) = \frac{\delta}{\lambda}\sin\lambda\pi - \frac{\delta}{\lambda^2}\int_0^{\frac{\pi}{2}}\sin\lambda(\pi-\tau)q(\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau$$

$$- \frac{1}{\lambda}\int_{\frac{\pi}{2}}^{\pi}\sin\lambda(\pi-\tau)q(\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau = 0 \tag{1.8}$$

By Theorem 1.2 the the set of eigenvalues of the problem (1.1)-(1.4) and the set of real roots of equation (1.8) are same.

Lemma 1.3. *Let* $q_1 = \int\limits_0^{\frac{\pi}{2}}|q(\tau)|d\tau$ *and* $q_2 = \int\limits_{\frac{\pi}{2}}^{\pi}|q(\tau)|d\tau$

(1) *Let* $\lambda \geq 2q_1$*, then the solution of* (1.6) *satisfies*

$$|\omega_1(x,\lambda)| \leq 2 \tag{1.9}$$

(2) *Let* $\lambda \geq \max\{2q_1, 2q_2\}$*, then the solution of* (1.7) *satisfies*

$$|\omega_2(x,\lambda)| \leq \frac{8\delta}{q_1} \tag{1.10}$$

Proof. Let $B_{1,\lambda} = \max\limits_{x\in[0,\frac{\pi}{2})}|\omega_1(x,\lambda)|$. Then from (1.6),

$$B_{1,\lambda} \leq 1 + \frac{1}{\lambda}q_1 B_{1,\lambda}$$

for $\lambda \geq 2q_1$, it is obvious that $B_{1,\lambda} \leq 2$.

Differentiating (1.6) with respect to x, we obtain

$$\omega_1'(x,\lambda) = \lambda\cos\lambda x - \int\limits_0^x q(\tau)\cos\lambda(x-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau \tag{1.11}$$

From this we obtain

$$|\omega_1'(x,\lambda)| \leq \lambda + 2q_1 \leq 2\lambda \implies \frac{|\omega_1'(x,\lambda)|}{\lambda} \leq 2 \tag{1.12}$$

Let $B_{2,\lambda} = \max\limits_{x\in(\frac{\pi}{2},\pi]}|\omega_2(x,\lambda)|$. Then from (1.7), (1.9) and (1.12)

$$B_{2,\lambda} \leq \frac{4\delta}{\lambda} + \frac{1}{\lambda}q_2 B_{2,\lambda}$$

Therefore for $\lambda \geq \max\{2q_1, 2q_2\}$, (1.10) is obtained. □

Theorem 1.4. *The problem* (1.1)-(1.4) *has an infinite set of positive eigenvalues.*

Proof. Writing (1.6) and(1.11) into (1.8), we obtain:

$$\frac{\delta}{\lambda}\sin\lambda\pi - \frac{\delta}{\lambda^2}\int\limits_0^{\frac{\pi}{2}}q(\tau)\sin\lambda(\pi-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau$$

$$- \frac{1}{\lambda}\int\limits_{\frac{\pi}{2}}^{\pi}q(\tau)\sin\lambda(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau = 0 \tag{1.13}$$

Let λ be sufficiently large, from (1.9) and (1.10), equation (1.13) may be written as

Let λ be sufficiently large, then by (1.9) and (1.10), (1.8) may be written in the form:

$$\lambda \sin \lambda\pi + O(1) = 0 \tag{1.14}$$

Clearly, for large λ, equation (1.14) has infinite roots. □

2. Asymptotic Properties of Eigenvalues and Eigenfunctions

In this section we will investigate the asymptotic expressions of eigenvalues and eigenfunctions. From now on we will assume λ is sufficienly large. On $[0, \frac{\pi}{2})$, from (1.6) and (1.9)

$$\omega_1(x,\lambda) = O(1) \tag{2.1}$$

On $(\frac{\pi}{2}, \pi]$, from (1.7) and (1.10)

$$\omega_2(x,\lambda) = O\left(\frac{1}{\lambda}\right) \tag{2.2}$$

Derivatives of $\omega_1(x,\lambda)$ and $\omega_2(x,\lambda)$ with respect to λ exist and are continuous on $[0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi]$ respectively[Norkin 1972].

Lemma 2.1.

$$\omega'_{1\lambda}(x,\lambda) = O(1), \quad \textit{for } x \in [0, \frac{\pi}{2}) \tag{2.3}$$

$$\omega'_{2\lambda}(x,\lambda) = O\left(\frac{1}{\lambda}\right), \quad \textit{for } x \in (\frac{\pi}{2}, \pi] \tag{2.4}$$

Proof. Differentiating (1.6) with respect to λ and by (2.1)

$$\omega'_{1\lambda}(x,\lambda) = -\frac{1}{\lambda}\int_0^x q(\tau)\sin\lambda(x-\tau)\omega'_{1\lambda}(\tau-\Delta(\tau),\lambda)d\tau + K_1(x,\lambda), \quad |K_1(x,\lambda)| \le K_1$$

Let $C_{1,\lambda} = \max\limits_{x\in[0,\frac{\pi}{2})} |\omega'_{1\lambda}(x,\lambda)|$. Existence of $C_{1,\lambda}$ follows from continuity of the derivative of $\omega_1(x,\lambda)$. From the equation above we obtain

$$C_{1,\lambda} \le \frac{1}{\lambda} q_1 C_{1,\lambda} + K_1$$

Therefore for $\lambda \ge 2q_1$, we obtain $C_{1,\lambda} \le 2K_1$. Hence (2.3) is prooved. Similarly (2.4) can be proved.

□

Theorem 2.2. *Let $n \in \mathbb{N}$. For each sufficiently large n, there is only one eigenvalue of the problem* (1.1)-(1.4) *in the neighborhood of n.*

Proof. First multiply (1.13) with λ^2, then consider the $O(1)$ term

$$-\delta \int_0^{\frac{\pi}{2}} q(\tau)\sin\lambda(\pi-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau - \lambda\int_{\frac{\pi}{2}}^{\pi} q(\tau)\sin\lambda(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau$$

By (2.1)-(2.4), for large λ this expression has bounded derivative with respect to λ. Clearly (1.14) has infinitely many solutions. We need to show that these solutions are around natural numbers n for suffiently large n. Consider the function

$F(\lambda) = \lambda \sin \lambda\pi + O(1)$. Its derivative $F'(\lambda) = \sin \lambda\pi + \lambda\pi \cos \lambda\pi + O(1) \neq 0$ for λ close to n for sufficiently large n. Hence by Rolle's theorem proof is completed. □

From (1.14)

$$\lambda_n = n + O\left(\frac{1}{n}\right) \tag{2.5}$$

is obtained. Writing (2.5) into (1.6) and (1.7), eigenfunctions of the problem (1.1)-(1.4) are obtained.

$$u_{1n}(x) = \omega_1(x, \lambda_n) = \sin nx + O\left(\frac{1}{n}\right)$$

$$u_{2n}(x) = \omega_2(x, \lambda_n) = \frac{\delta}{n} \sin nx + O\left(\frac{1}{n^2}\right)$$

$$u_n(x) = \begin{cases} \sin nx + O\left(\frac{1}{n}\right), & x \in [0, \frac{\pi}{2}) \\ \frac{\delta}{n} \sin nx + O\left(\frac{1}{n^2}\right), & x \in (\frac{\pi}{2}, \pi] \end{cases} \tag{2.6}$$

3. Sharper Estimates for Eigenvalues and Eigenfunctions

Under additional hypotheses about the functions $q(x)$ and $\Delta(x)$, it is possible to improve the expressions given by (2.5), (2.6).

Lemma 3.1. *Suppose the derivatives $q'(x)$ and $\Delta''(x)$ exist and are bounded on $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, and have finite limits $q'\left(\frac{\pi}{2} \pm 0\right) = \lim_{x \to \frac{\pi}{2} \pm 0} q'(x)$, $\Delta''\left(\frac{\pi}{2} \pm 0\right) = \lim_{x \to \frac{\pi}{2} \pm 0} \Delta''(x)$, $\Delta'(x) \leq h < 2$ and $\Delta(0) = 0$, $\lim_{x \to \frac{\pi}{2} + 0} \Delta(x) = 0$. Then*

$$\int_0^x \cos \lambda(2\tau - \Delta(\tau)) q(\tau) d\tau = O\left(\frac{1}{\lambda}\right) \tag{3.1}$$

and

$$\int_0^x \sin \lambda(2\tau - \Delta(\tau)) q(\tau) d\tau = O\left(\frac{1}{\lambda}\right) \tag{3.2}$$

Proof. See Lemma III.3.3 in [13] □

Theorem 3.2. *Under the hypoteses of Lemma 3.1 eigenvalues of* (1.1)-(1.4) *problem can be improved as*

$$\lambda_n = n - \frac{B(\pi, n, \Delta(\tau))}{n\pi} + O\left(\frac{1}{n^2}\right) \tag{3.3}$$

Proof. From (2.6), we can write

$$\omega_1(\tau - \Delta(\tau), \lambda) = \sin \lambda(\tau - \Delta(\tau)) + O\left(\frac{1}{\lambda}\right) \tag{3.4}$$

$$\omega_2(\tau - \Delta(\tau), \lambda) = \frac{\delta}{\lambda} \sin \lambda(\tau - \Delta(\tau)) + O\left(\frac{1}{\lambda^2}\right) \tag{3.5}$$

Writing these into the characterisitc equation (1.8), and multiplying by λ^2 equation (1.8) turns into

$$\delta\lambda \sin\lambda\pi - \delta \int_0^{\frac{\pi}{2}} q(\tau)\sin\lambda(\pi-\tau)\left[\sin\lambda(\tau-\Delta(\tau)) + O\left(\frac{1}{\lambda}\right)\right]d\tau$$

$$- \lambda \int_{\frac{\pi}{2}}^{\pi} q(\tau)\sin\lambda(\pi-\tau)\left[\frac{\delta}{\lambda}\sin\lambda(\tau-\Delta(\tau)) + O\left(\frac{1}{\lambda^2}\right)\right]d\tau = 0 \tag{3.6}$$

defining

$$A(x,\lambda,\Delta(\tau)) = \frac{1}{2}\int_0^x q(\tau)\sin\lambda\Delta(\tau)d\tau \tag{3.7}$$

and

$$B(x,\lambda,\Delta(\tau)) = \frac{1}{2}\int_0^x q(\tau)\cos\lambda\Delta(\tau)d\tau \tag{3.8}$$

equation (3.6) simplifies as

$$\lambda\sin\lambda\pi + B(\pi,\lambda,\Delta(\tau))\cos\lambda\pi + A(\pi,\lambda,\Delta(\tau))\sin\lambda\pi + O\left(\frac{1}{\lambda}\right) = 0$$

writing $\lambda = \lambda_n = n + \delta_n$ and for large n

$$\tan\delta_n\pi = -\frac{B(\pi,n,\Delta(\tau))}{n} + O\left(\frac{1}{n^2}\right) \implies \delta_n = -\frac{B(\pi,n,\Delta(\tau))}{n\pi} + O\left(\frac{1}{n^2}\right)$$

Therefore the proof is complete. □

Theorem 3.3. *Under the hypoteses of Lemma 3.1 eigenfunctions u_{1n} and u_{2n} of* (1.1)-(1.4) *can be improved as*

$$u_{1n}(x) = \left(1 - \frac{A(x,n,\Delta(\tau))}{n}\right)\sin nx + \\ + \frac{xB(\pi,n,\Delta(\tau)) - \pi B(x,n,\Delta(\tau))}{n\pi}\cos nx + O\left(\frac{1}{n^2}\right) \tag{3.9}$$

$$u_{2n}(x) = \frac{\delta}{n}\left(1 - \frac{A(x,n,\Delta(\tau))}{n}\right)\sin nx + \\ + \frac{\delta(xB(\pi,n,\Delta(\tau)) - \pi B(x,n,\Delta(\tau)))}{n^2\pi}\cos nx + O\left(\frac{1}{n^3}\right) \tag{3.10}$$

Proof. First we write (3.4) into (1.6) and obtain

$$\omega_1(x,\lambda) = \sin\lambda x + \frac{1}{\lambda}\int_0^x q(\tau)\sin\lambda(x-\tau)\left[\sin\lambda(\tau-\Delta(\tau)) + O\left(\frac{1}{\lambda}\right)\right]d\tau$$

Then using (3.7) and (3.8), this expression becomes

$$\omega_1(x,\lambda) = \sin\lambda x + \frac{1}{\lambda}A(x,\lambda,\Delta(\tau))\sin\lambda x - \frac{1}{\lambda}B(x,\lambda,\Delta(\tau))\cos\lambda x + O\left(\frac{1}{\lambda^2}\right)$$

Now writing (3.3), we obtain the eigenfunction on $[0, \frac{\pi}{2})$ as

$$u_{1n}(x) = \omega_1(x, \lambda_n) = \left(1 - \frac{A(x, n, \Delta(\tau))}{n}\right) \sin nx +$$
$$+ \frac{xB(\pi, n, \Delta(\tau)) - \pi B(x, n, \Delta(\tau))}{n\pi} \cos nx + O\left(\frac{1}{n^2}\right)$$

Now we will improve the eigenfunction on $(\frac{\pi}{2}, \pi]$. In order to do that first we will write (1.9) and (1.12) into (1.10) and then we will use (3.4) and (3.5) together with (3.7) and (3.8) to obtain

$$\omega_2(x, \lambda) = \frac{\delta}{\lambda} \sin \lambda x + \frac{\delta}{\lambda^2} A(x, \lambda, \Delta(\tau)) \sin \lambda x - \frac{\delta}{\lambda^2} B(x, \lambda, \Delta(\tau)) \cos \lambda x + O\left(\frac{1}{\lambda^3}\right)$$

Now writing, (3.3) into this expression we obtain the eigenfunction on $(\frac{\pi}{2}, \pi]$.

$$u_{2n}(x) = \omega_2(x, \lambda_n) = \frac{\delta}{n}\left(1 - \frac{A(x, n, \Delta(\tau))}{n}\right) \sin nx +$$
$$+ \frac{\delta(xB(\pi, n, \Delta(\tau)) - \pi B(x, n, \Delta(\tau)))}{n^2\pi} \cos nx + O\left(\frac{1}{n^3}\right)$$

This completes the proof. □

4. Conclusion

In this paper discontinuous differential equation with retarded argument is studied. In the case of transmission condition that contains eigenparameter, eigenvalues and the corresponding eigenfunctions calculated asymptoticaly as follows:

$$\lambda_n = n - \frac{B(\pi, n, \Delta(\tau))}{n\pi} + O\left(\frac{1}{n^2}\right)$$

$$u_n(x) = \begin{cases} u_{1n}(x), & x \in [0, \frac{\pi}{2}) \\ u_{2n}(x), & x \in (\frac{\pi}{2}, \pi] \end{cases} \tag{4.1}$$

where $u_{1n}(x)$ and $u_{2n}(x)$ are defined by (3.9) and (3.10) respectively.

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Mersin university, Mathematics department, 33343, Mersin,Turkey
Email address: mizrak@mersin.edu.tr

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ABOUT GROUP OF POINTWISE INNER AUTOMORPHISMS FOR NILPOTENCY CLASS FOUR

ELA AYDIN

0000-0003-4867-0583

Abstract. Let $L_{m,c}$ stand for the free metabelian nilpotent Lie algebra of class c of rank m over a field K of characteristic zero. Automorphisms of the form $\varphi(x_i) = e^{adu_i}(x_i)$ are called pointwise inner, where e^{adu_i}, is the inner automorphism induced by the element $u_i \in L_{m,c}$ for each $i = 1, \ldots, m$. The descriptions of the groups $\mathrm{PInn}(L_{m,2})$ and $\mathrm{PInn}(L_{m,3})$ of pointwise inner automorphisms are well known. In the present study, we investigate the group structure of $\mathrm{PInn}(L_{m,4})$ of pointwise inner automorphisms of $L_{m,4}$ that can be considered as the next step in this direction.

1. Introduction

Pointwise inner automorphisms of the free metabelian nilpotent Lie algebra $L_{m,c}$ forms a group shown by the author [4]. A generating set for the group $\mathrm{PInn}(L_{m,c})$ was provided, as well, in the same study: Each automorphism φ in $\mathrm{PInn}(L_{m,c})$ is of the form

$$\varphi(x_i) = e^{\mathrm{ad}(u_i)}(x_i) = (u_1, \ldots, u_m)$$

for some $u_i \in L_{m,c}$, $i = 1, \ldots, m$. Let us define the set

$$I_i = \{\varphi_u = (0, \ldots, 0, u, 0, \ldots, 0) \mid u \in L_{m,c}\}, \quad i = 1, \ldots, m,$$

consisting of m-tuples where each coordinate except for i-th position is necessarily filled by zero.

Theorem 1.1. [4] *The set I_i is a group for every $i = 1, \ldots, m$.*

Theorem 1.2. [4] *The set $\mathrm{PInn}(L_{m,c})$ of pointwise inner automorphisms of the free metabelian nilpotent Lie algebra $L_{m,c}$ forms a group generated by the set $I_1 \cup \cdots \cup I_m$.*

In the following theorems, the desription of $\mathrm{PInn}(L_{m,2})$ and $\mathrm{PInn}(L_{m,3})$ were given.

Theorem 1.3. [5] *Let the nilpotency class $c = 2$. Then the group $\mathrm{PInn}(L_{m,2})$ of pointwise inner automorphisms of the free metabelian Lie algebra $L_{m,2}$ is abelian, and the composition of two pointwise inner automorphisms is given by $\phi_u \phi_v = \phi_{u+v}$.*

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Theorem 1.4. [5] *Let $c = 3$. Then the group* $\mathrm{PInn}(L_{m,3})$ *of pointwise inner automorphisms of the free metabelian Lie algebra* $L_{m,3}$ *is abelian by nilpotent of class 2. That is,* $[[\phi_u, \phi_v], \phi_w] = 0$, *where* $[\phi_u, \phi_v] = \phi_u\phi_v\phi_u^{-1}\phi_v^{-1}$. *Furthermore, the compositon of two pointwise inner automorphisms is given by*

$$\varphi_u\varphi_v = \varphi_{u+v+\sum_i d_i[x_i,u_{i1}]+\frac{1}{2}[u_{j1},v_{j1}]}$$

where u_{j1}, v_{j1} *are the linear parts of* u_j, v_j *in the expression of* $\varphi_u = (u_1, \dots, u_m)$, $\varphi_v = (v_1, \dots, v_m)$, *and*

$$d_1x_1 + \cdots + d_mx_m$$

is the linear part of v.

In the current study, we investigate an analogue of the Theorems 1.3 and 1.4 for the nilpotency class $c = 4$.

Note: One may easily observe that a pointwise inner automorphism

$$\varphi(x_i) = (u_1, \dots, u_m)$$

is inner if and only if $u_1 = \cdots = u_m$. In this respect, the group $\mathrm{Inn}(L_{m,c})$ of inner automorphisms is a normal subgroup of $\mathrm{PInn}(L_{m,c})$. We refer the reader for the structure of this group to the paper [3]. Additionally, each inner automorphism of $L_{m,c}$ preserves every ideal of the Lie algebra $L_{m,c}$, and by the paper [6] we have that such ideal preservative automorphisms are another generalization of inner automorphims.

2. Preliminaries

The free metabelian nilpotent Lie algebra $L_{m,c}$ over a field K of characteristic zero is the free algebra of rank n in the variety of the Lie algebras satisfying the identities

$$[[x, y], [z, t]] = 0, \qquad \text{and} \qquad [y_1, y_2, \dots, y_{c+1}] = 0$$

for all $x, y, z, t, y_1, y_2, \dots, y_{c+1} \in L_{m,c}$. For more information on the Lie algebra $L_{m,c}$ we refer to the books [1, 2]. In this paper, we use the left normed commutators as below.

$$[u_1, \dots, u_{n-1}, u_n] = [[u_1, \dots, u_{n-1}], u_n], \quad n = 3, 4, \dots$$

For each $v \in L_{m,c}$, the linear operator $\mathrm{ad}v : L_{m,c} \to L_{m,c}$ defined by

$$\mathrm{ad}v(u) = [u, v], \quad u \in L_{m,c},$$

is a derivation of $L_{m,c}$ which is nilpotent and $\mathrm{ad}^c v = (\mathrm{ad}v)^c = 0$ because $L_{m,c}^{c+1} = 0$, and thus the linear operator

$$e^{\mathrm{ad}(v)} = 1 + \frac{\mathrm{ad}v}{1!} + \frac{\mathrm{ad}^2 v}{2!} + \cdots + \frac{\mathrm{ad}^{c-1} v}{(c-1)!}$$

is well defined and is an automorphism of $L_{m,c}$. The set of all automorphisms are of the form $e^{\mathrm{ad}(v)}$, $v \in L_{m,c}$, is called the inner automorphism group of $L_{m,c}$ and is denoted by $\mathrm{Inn}(L_{m,c})$. The group $\mathrm{PInn}(L_{m,c})$ of pointwise inner automorphisms can be considered as a generalization of $\mathrm{Inn}(L_{m,c})$.

Our goal is to consider the group of pointwise inner automorphisms of $L_{m,4}$ and establish multiplication rule in this group for nilpotency class four.

3. Main Results

Theorem 3.1. *Let the nilpotency class $c = 4$. Then the group $\mathrm{PInn}(L_{m,4})$ of the free metabelian Lie algebra $L_{m,4}$ is metabelian. This means that*

$$[[\phi_u, \phi_v], [\phi_w, \phi_t]] = 0,$$

where $[\phi_u, \phi_v] = \phi_u \phi_v \phi_u^{-1} \phi_v^{-1}$.

Proof. In this case each element in $L_{m,4}$ is of the form

$$\sum_i c_i x_i + \sum_{i>j} c_{ij}[x_i, x_j] + \sum_{i>j\leq k} c_{ijk}[x_i, x_j, x_k].$$

Let's say

$$u_1 = \sum c_i x_i,\ u_2 = \sum_{i>j} c_{ij}[x_i, x_j],\ u_3 = \sum_{i>j\leq k} c_{ijk}[x_i, x_j, x_k] \text{ and}$$
$$v_1 = \sum_i d_i x_i,\ v_2 = \sum_{i>j} d_{ij}[x_i, x_j],\ v_3 = \sum_{i>j\leq k} d_{ijk}[x_i, x_j, x_k].$$

$$\phi_u(x) = x + [x, u] + \frac{1}{2}[x, u_1 + u_2, u_1] + \frac{1}{6}[x, u_1, u_1, u_1],$$

where $u = u_1 + u_2 + u_3$ and also let $v = v_1 + v_2 + v_3$. Hence we have

$$\phi_u \phi_v(x) = \phi_u(x + [x, v] + \frac{1}{2}[x, v_1 + v_2, v_1] + \frac{1}{6}[x, v_1, v_1, v_1]).$$

Consider the following elements:

$$w = w_1 + w_2 + w_3,\ \text{ where } w_1 = u_1 + v_1,$$
$$w_2 = u_2 + v_2 + d_1[x, u_1] + \frac{1}{2}[u_1, v_1] \text{ and}$$
$$w_3 = u_3 + v_3 + \sum_{1<i}[x, d_{i1}[x, u_1]] + \frac{1}{12}[v_1, u_1, u_1] + \frac{1}{12}[u_1, v_1, v_1].$$

Then we have

$$\phi_w(x) = x + [x, w_1] + [x, w_2] + \frac{1}{2}[x, w_1, w_1] + [x, w_3] + \frac{1}{2}[x, w_2, w_1] + \frac{1}{6}[x, w_1, w_1, w_1].$$

By some calculations we have the elements

$$w_3 = u_3 + v_3 + \sum_{i<j} d_{i1}[x_i, [x_1, u_1]] + \frac{1}{12}[v_1, u_1, u_1] + \frac{1}{12}[u_1, v_1, v_1] -$$
$$\frac{1}{2}[u_1, v_1, v_1] + d_1[x_1, u_2] - \frac{1}{2}[v_2, u_1] + \frac{1}{2}[u_2, v_1].$$

And consequently we obtain $\phi_u \phi_v = \phi_{w_1+w_2+w_3}$. □

4. Conclusion

In this study, group structure of the group $\mathrm{PInn}(L_{m,4})$ was provided via multiplication rule in it. The next step might be extending the nilpotency class $c \geq 5$, and obtain new results.

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Department of Mathematics, Çukurova University, 01330 Balcali, Adana, Turkey
Email address: eaydin@cu.edu.tr

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SOME QUALITATIVE PROPERTIES OF SOLUTIONS OF CERTAIN NONLINEAR THIRD-ORDER STOCHASTIC DIFFERENTIAL EQUATIONS WITH DELAY

R.O. BANIRE, O.O. FABELURIN, P.O. ARAWOMO, A.T. ADEMOLA, AND M.O. OMEIKE

0009-0004-4396-822X, 0000-0002-3129-5935, 0000-0003-0814-0342, 0000-0002-1036-1681 and 0000-0003-3616-2979

Abstract. This study considered certain nonlinear third-order stochastic differential equations with delay. The third-order equation is reduced to an equivalent system of first-order differential equations and used to construct the desired complete Lyapunov-Krasovskiĭ functional. Standard conditions guaranteeing stability when the forcing term is zero, boundedness of solutions when the forcing term is non-zero, and lastly the existence and uniqueness of solutions are derived. The obtained results indicated that the adopted technique is effective in studying the qualitative behaviour of solutions. The obtained results are not only new but extend the frontier of knowledge of the qualitative behaviour of solutions of nonlinear stochastic differential with delay. Finally, two special cases are given to illustrate the derived theoretical results.

1. Introduction

In recent years, the studies of stability, boundedness, existence and uniqueness of solutions of a nonlinear third-order stochastic differential equations with delay have been discussed and still under intensive investigations by researchers. Some outstanding works on deterministic model with and without delay using the technique of Lyapunov, we refer to the papers in [7–10, 12, 14, 18, 25].

In this paper, we shall consider the third-order nonlinear stochastic differential equation with delay defined as

$$\dddot{x}(t) + a\ddot{x}(t) + g(\cdot) + h(x(t-\tau(t))) + \sigma x(t-\tau(t))\dot{\omega}(t) = p(\cdot), \tag{1.1}$$

where $g(\cdot) = g(x(t-\tau(t)), \dot{x}(t-\tau(t)))$, $p(\cdot) = p(t, x(t), \dot{x}(t), \ddot{x}(t))$, for simplicity we shall write $x(t) = x$, $y(t) = y$, and $z(t) = z$. Assign $y = \dot{x}$ and $z = \ddot{x}$ equation (1.1)

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is equivalent to system of first order equations

$$\begin{aligned} &\dot{x} = y, \quad \dot{y} = z, \\ &\dot{z} = p(t,x,y,z) - h(x) - g(x,y) - az - \sigma\big[x - \int_{t-\tau(t)}^{t} y(s)ds\big]\dot{\omega}(t) \\ &+ \int_{t-\tau(t)}^{t} \big[g_x(x(s),y(s))y(s) + g_y(x(s),y(s))z(s) + h'(x(s))y(s)\big]ds, \end{aligned} \tag{1.2}$$

where the functions g, h, and p are continuous in their respective arguments on $\mathbb{R}^2$, $\mathbb{R}$, and $\mathbb{R}^+ \times \mathbb{R}^3$, respectively with $\mathbb{R}^+ = [0,\infty)$, $\mathbb{R} = (-\infty,\infty)$, and $\omega \in \mathbb{R}$ (a standard Wiener process, representing the noise) is defined on $\mathbb{R}^3$, $\tau(t)$ is a continuously differentiable function with $0 \le \tau(t) \le \tau_0$, τ_0, a, and σ are positive constants. The dots denote to differentiation with respect to the independent variable $t \in \mathbb{R}^+$, derivatives $h'(x), g_x(x,y)$, and $g_y(x,y)$ exist and are continuous. Moreover, the continuity of the functions g, h, and p is sufficient for the existences of the solutions and the local Lipschitz condition for system (1.2) to obtain a unique continuous solution represented by $(x(t), y(t), z(t))$.

Systematic investigations of differential equations of distinct orders, with and without delay and/or randomness, have been carried out by researchers. In particular, there are critical inspection on first order system of differential equations, we can mention the background books and papers in [15–17, 19–21, 24, 27–29]. In addition, researchers in [11] employed the direct method of Lyapunov to obtain standard criteria on stability and boundedness of solutions of a certain second-order non-autonomous stochastic differential equation

$$\ddot{x}(t) + f(x(t),\dot{x}(t))\dot{x}(t) + g(x(t)) + \gamma x(t)\dot{\omega}(t) = p(t,x(t),\dot{x}(t)),$$

where γ is a positive constant, $g \in C(\mathbb{R},\mathbb{R})$ $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, and $p \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R},\mathbb{R})$ are continuous functions. The function g is differentiable and continuous for all x.

Furthermore, authors in [2] considered stability of solutions of certain second-order stochastic delay differential equations

$$\ddot{x}(t)+b\dot{x}(t)+cx(t-\epsilon)+\gamma x(t)\dot{\omega}(t) = 0 \text{ and } \ddot{x}(t)+b\dot{x}(t)+f(x(t-\epsilon))+\gamma x(t-\psi_0)\dot{\omega}(t) = 0,$$

where b, c, γ are positive constants, ϵ and ψ_0 are positive constant delays, the function f is continuous with respect to x with $f(0) = 0$. What is more, article in [3] discussed new results on the stability and boundedness for solutions of second-order stochastic delay differential equation

$$\ddot{x}(t) + g(\dot{x}(t)) + bx(t-h) + \sigma x(t)\dot{\omega}(t) = p(t,x(t),\dot{x}(t),x(t-h)),$$

where b, σ are positive constants, h is a positive constant delay, g and p are continuous functions with $g(0) = 0$. In [5], a suitable Lyapunov functional is used to establish sufficient conditions guaranteeing the existence of stochastic asymptotic stability of the zero solution of the non-autonomous second-order stochastic delay differential equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)f(x(t-r)) + g(t,x)\dot{\omega}(t) = 0,$$

where $a(t)$ and $b(t)$ are two positive continuously differentiable functions on $[0,\infty)$, r is a positive constant delay, $f(x)$ and $g(t,x)$ are continuous functions defined on $\mathbb{R}$ and $\mathbb{R}^+ \times \mathbb{R}$ respectively with $f(0) = 0$. Researchers in [6] studied the stability and boundedness of solutions to certain nonlinear non autonomous second-order stochastic delay differential equations

$$\ddot{x}(t) + \psi(t)f(x(t), \dot{x}(t))\dot{x}(t) + g(x(t-\tau)) + \sigma x(t)\dot{\omega}(t) = p(t, x(t), \dot{x}(t), x(t-\tau)),$$

where ψ, f, g, p are continuous functions in their respective arguments on $\mathbb{R}^+, \mathbb{R}^2, \mathbb{R}$, $\mathbb{R}^+ \times \mathbb{R}^3$ respectively, $\sigma > 0$ is a constant, and τ is a positive constant delay. No doubt, articles [2, 3, 5, 6, 11] are special cases of equation (1.1).

When $\tau(t) \equiv 0$, $g(\cdot) \equiv b\dot{x}(t)$, $h(x(t-\tau(t))) \equiv cx(t)$, and $x(t-\tau(t)) \equiv x(t)$, equation (1.1) reduces to the third-order stochastic differential equation discussed in [1] namely

$$\dddot{x}(t) + a\ddot{x}(t) + b\dot{x} + cx(t) + \sigma x(t)\dot{\omega}(t) = p(t, x(t), \dot{x}(t), \ddot{x}(t)),$$

where $a > 0, b > 0, c > 0$, $\sigma > 0$ are constants, and $p(t, x, \dot{x}, \ddot{x})$ is a continuous function. Authors in [4] investigated the asymptotic stability of the zero solution for the third-order stochastic delay differential equations given by

$$\dddot{x}(t) + a_1\ddot{x}(t) + g_1(\dot{x}(t - r_1(t))) + f_1(x(t)) + \sigma_1 x(t)\dot{\omega}(t) = 0$$

and

$$\dddot{x}(t) + a_2\ddot{x}(t) + f_2(x(t))(\dot{x}(t) + f_3(x(t - r_2(t))) + \sigma_2 x(t - h(t))\dot{\omega}(t) = 0,$$

where a_1, a_2, σ_1, σ_2 are positive constants, γ_1, γ_2 are two positive constants such that $0 \le r_1(t) \le \gamma_1$, $0 \le r_2(t) \le \gamma_2$, $0 \le h(t)$, $\sup h(t) = H$; g_1, f_1, f_2, and f_3 are continuous functions with $g_1(0) = f_1(0) = f_3(0) = 0$. The two equations discussed in [4] are special cases of (1.1) since $g(\cdot) \equiv g_1(\dot{x}(t-\tau_1(t))$, $h(x(t-\tau(t))) \equiv f_1(x(t))$, $x(t-\tau(t)) \equiv x(t)$, and $p(\cdot) \equiv 0$ in the first equation and $g(\cdot) \equiv f_2(x(t))\dot{x}(t)$ and $p(\cdot) \equiv 0$ in the second equation. Whenever $g(\cdot), x(t-\tau(t))$, and $\tau(t)$ are equivalent to $b\dot{x}(t), x(t)$, and $\tau > 0$ a constant delay respectively then equation (1.1) is cut down to the third-order stochastic delay differential equations considered in [13] i.e.,

$$\dddot{x}(t) + a\ddot{x}(t) + b\dot{x}(t) + h(x(t-\tau)) + \sigma x(t)\dot{\omega}(t) = p(t, x(t), \dot{x}(t), \ddot{x}(t)),$$

where the constants a, b, σ are positive, h, p are nonlinear continuous functions in their respective arguments and $h(0) = 0, \tau > 0$ is a delay constant.

In the case $g(\cdot), x(t-\tau(t))$, and $p(\cdot)$ are equivalent to $\phi(\dot{x}(t - r(t))), x(t-h)$, and 0 respectively then equation (1.1) is trim down to the third-order stochastic differential equation

$$\dddot{x}(t) + a\ddot{x}(t) + \phi(\dot{x}(t - r(t))) + \psi(x(t - r(t))) + \sigma x(t-h)\dot{\omega}(t) = 0,$$

investigated in [22] where $a > 0$ and $\sigma > 0$ are constants, $h > 0$ is a constant delay, $r(t)$ is a continuously differentiable function satisfying $0 \le r(t) \le \beta_1$, $\beta_1 > 0$ a constant, ϕ and ψ are nonlinear continuous functions defined on $\mathbb{R}$ with $\phi(0) = \psi(0) = 0$. Motivation for this work comes from the works in [1, 4, 13, 22], where Lyapunov functionals are exploited to acquire asymptotic stability, boundedness, existence and uniqueness of solutions of the equations considered. Section 2 presents definitions of terms and basic results used in this paper, stability of the trivial solutions are stated and proved in Section 3, boundedness and existence results are communicated in Section 4, and special cases of the theoretical results discussed in Sections 3 and 4 are presented as examples in Section 5.

2. Preliminary Results

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t>0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t>0}$ satisfying the usual conditions (i.e., it is right continuous and $\{\mathfrak{F}_0\}$ contains all $\mathbb{P}-$null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an $m-$dimensional Brownian motion defined on the probability space. Let $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $\|A\| = \sqrt{\text{trace } (A^T A)}$. Details can be seen [15] and [23]. Consider a non autonomous $n-$dimensional stochastic delay differential equation

$$dx(t) = F(t, x(t), x(t-\tau))dt + G(t, x(t), x(t-\tau))dB(t) \tag{2.1}$$

on $t > 0$ with initial data $\{x(\theta) : -\tau \le \theta \le 0\}, x_0 \in C([-\tau, 0], \mathbb{R}^n)$. Here $F : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^n$ and $G : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^{n\times m}$ are measurable functions. Suppose that the functions F, G satisfy the local Lipschitz condition, given any $b > 0$, $p \ge 2$, $F(t, 0, 0) \in C^1([0, b], \mathbb{R}^n)$, and $G(t, 0, 0) \in C^p([0, b], \mathbb{R}^{m\times n})$. Then there must be a stopping time $\beta = \beta(\omega) > 0$ such that equation (2.1) with $x_0 \in C^p_{\mathfrak{F}_{t_0}}$ [class of $\mathfrak{F}_t$-measurable $C([-\tau, 0], \mathbb{R}^n)$-valued random variables ξ_t and $E\|\xi_t\|^p < \infty$] has a unique maximal solution on $t \in [t_0, \beta)$ which is denoted by $x(t, x_0)$. Assume further that $F(t, 0, 0) = G(t, 0, 0) = 0$ for all $t \ge 0$. Hence, the stochastic delay differential equation admits zero solution $x(t, 0) \equiv 0$ for any given initial value $x_0 \in C([-\tau, 0], \mathbb{R}^n)$.

Definition 2.1. The zero solution of the stochastic differential equation (2.1) is said to be stochastically stable or stable in probability, if for every pair $\epsilon \in (0, 1)$ and $r > 0$, there exists a $\delta_0 = \delta_0(\epsilon, r) > 0$ such that $Pr\{\|x(t; x_0)\| < r \text{ for all } t \ge 0\} \ge 1 - \epsilon$ whenever $\|x_0\| < \delta_0$. Otherwise, it is said to be stochastically unstable.

Definition 2.2. The zero solution of the stochastic differential equation (2.1) is said to be stochastically asymptotically stable if it is stochastically stable and in addition if for every $\epsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $Pr\{\lim_{t\to\infty} x(t; x_0) = 0\} \ge 1 - \epsilon$ whenever $\|x_0\| < \delta$.

Definition 2.3. A solution $x(t, x_0)$ of the stochastic delay differential equation (2.1) is said to be stochastically bounded or bounded in probability, if it satisfies

$$E^{x_0}\|x(t, x_0)\| \le N(t_0, \|x_0\|), \ \forall\ t \ge t_0 \tag{2.2}$$

where E^{x_0} denotes the expectation operator with respect to the probability law associated with x_0, $N : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a constant function depending on t_0 and x_0.

Definition 2.4. The solutions $x(t_0, x_0)$ of the stochastic delay differential equation (2.1) is said to be uniformly stochastically bounded if N in (2.2) is independent of t_0.

Let $\mathbb{K}$ denote the family of all continuous non-decreasing functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho(0) = 0$ and $\rho(r) > 0$ if $r \ne 0$. In addition, $\mathbb{K}_\infty$ denotes the family of all functions $\rho \in \mathbb{K}$ with

$$\lim_{r\to\infty} \rho(r) = \infty.$$

Suppose that $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, denotes the family of all non negative functions $V = V(t, x_t)$ (Lyapunov functional) defined on $\mathbb{R}^+ \times \mathbb{R}^n$ which are twice continuously differentiable in x and once in t. By Itô's formula we have

$$dV(t, x_t) = LV(t, x_t)dt + V_x(t, x_t)G(t, x_t)dB(t),$$

where

(2.3)
$$LV(t,x_t) \;=\frac{\partial V(t,x_t)}{\partial t}+\frac{\partial V(t,x_t)}{\partial x_i}F(t,x(t))+\frac{1}{2}\text{trace }[G^T(t,x_t)V_{xx}(t,x_t)G(t,x_t)]$$

with
$$V_{xx}(t,x_t)=\left(\frac{\partial^2 V(t,x_t)}{\partial x_i \partial x_j}\right)_{n\times n},\; i,j=1,\cdots,n$$

In this study we will use the diffusion operator $LV(t,x_t)$ defined in (2.3) to replace $V'(t,x(t))=\frac{d}{dt}V(t,x(t))$. We now present the basic results that will be used in the proofs of the main results.

Lemma 2.5. *(See [15]) Assume that there exist $V\in C^{1,2}(\mathbb{R}^+\times\mathbb{R}^n,\mathbb{R}^+)$, and $\eta\in\mathbb{K}$ such that*

(i) $V(t,0)=0$, *for all* $t\geq 0$;
(ii) $V(t,x_t)\geq \eta(\|x(t)\|)$, $\eta(r)\to\infty$ *as* $r\to\infty$; *and*
(iii) $LV(t,x_t)\leq 0$ *for all* $(t,x)\in\mathbb{R}^+\times\mathbb{R}^n$.

Then the zero solution of stochastic delay differential equation (2.1) *is* stochastically stable. *If conditions (ii) and (iii) hold then* (2.1) *with* $x_0\in C^p_{\mathfrak{F}_{t_0}}$ *has a unique global solution for* $t>0$ *denoted by* $x(t;x_0)$.

Lemma 2.6. *(See [15]) Suppose that there exist $V\in C^{1,2}(\mathbb{R}^+\times\mathbb{R}^n,\mathbb{R}^+)$, and $\eta_0,\eta_1,\eta_2\in\mathbb{K}$ such that*

(i) $V(t,0)=0$, *for all* $t\geq 0$;
(ii) $\eta_0(\|x(t)\|)\leq V(t,x_t)\leq \eta_1(\|x(t)\|)$, $\eta_0(r)\to\infty$ *as* $r\to\infty$; *and*
(iii) $LV(t,x_t)\leq -\eta_2(\|x(t)\|)$ *for all* $(t,x_t)\in\mathbb{R}^+\times\mathbb{R}^n$.

Then the zero solution of stochastic delay differential equation (2.1) *is* uniformly stochastically asymptotically stable in the large

Assumption 2.7. *(See [21, 26]) Let $V\in C^{1,2}(\mathbb{R}^+\times\mathbb{R}^n,\mathbb{R}^+)$, suppose that for any solutions $x(t_0,x_0)$ of stochastic delay differential equation* (2.1) *and for any fixed $0\leq t_0\leq T<\infty$, we have*

$$E^{x_0}\left\{\int_{t_0}^{T} V^2_{x_i}(t,x_t)G^2_{ik}(t,x_t)dt\right\}<\infty,\; 1\leq i\leq n,\; 1\leq k\leq m. \tag{2.4}$$

Assumption 2.8. *(See [21, 26]) A special case of the general condition* (2.4) *is the following condition. Assume that there exits a function $\rho(t)$ such that*

$$|V_{x_i}(t,x_t)G_{ik}(t,x_t)|<\rho(t),\; x\in\mathbb{R}^n,\; 1\leq i\leq n,\; 1\leq k\leq m, \tag{2.5}$$

for any fixed $0\leq t_0\leq T<\infty$,

$$\int_{t_0}^{T}\rho^2(t)dt<\infty. \tag{2.6}$$

Lemma 2.9. *(See [21, 26]) Assume there exists a Lyapunov function $V\in C^{1,2}(\mathbb{R}^+\times\mathbb{R}^n,\mathbb{R}^+)$, satisfying Assumption 2.7, such that for all $(t,x_t)\in\mathbb{R}^+\times\mathbb{R}^n$,*

(i) $\|x(t)\|^p\leq V(t,x_t)\leq\|x(t)\|^q$,
(ii) $LV(t,x_t)\leq -\alpha(t)\|x(t)\|^r+\psi(t)$,
(iii) $V(t,x_t)-V^{r/q}(t,x_t)\leq\mu$,

where $\alpha, \psi \in C(\mathbb{R}^+, \mathbb{R}^+)$, p, q, r are positive constants, $p \geq 1$, and μ is a non negative constant. Then all solutions of stochastic delay differential equation (2.1) *satisfy*

$$E^{x_0}\|x(t,x_0)\| \leq \left\{ V(t_0,x_0)e^{-\int_{t_0}^{t}\alpha(s)ds} + A \right\}^{1/p}, \tag{2.7}$$

for all $t \geq t_0$, where

$$A := \int_{t_0}^{t} \Big(\mu\alpha(u) + \psi(u)\Big)e^{-\int_{u}^{t}\alpha(s)ds}du.$$

Lemma 2.10. *(See [21, 26]) Assume there exists a Lyapunov function $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, satisfying Assumption 2.7, such that for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$,*

(i) $\|x(t)\|^p \leq V(t, x_t)$,
(ii) $LV(t, x_t) \leq -\alpha(t)V^q(t, x_t) + \psi(t)$,
(iii) $V(t, x_t) - V^q(t, x_t) \leq \mu$,

where $\alpha, \psi \in C(\mathbb{R}^+, \mathbb{R}^+)$, p, q are positive constants, $p \geq 1$, and μ is a non negative constant. Then all solutions of stochastic delay differential equation (2.1) *satisfy* (2.7) *for all $t \geq t_0$.*

Corollary 2.11. *(See [21, 26])*

(i) *Assume that hypotheses (i) to (iii) of Lemma 2.9 hold. In addition*

$$\int_{t_0}^{t} \Big(\mu\alpha(u) + \psi(u)\Big)e^{-\int_{u}^{t}\alpha(s)ds}du \leq M, \forall\ t \geq t_0 \geq 0, \tag{2.8}$$

for some positive constant M, then all solution of stochastic delay differential equation (2.1) *are* uniformly stochastically bounded.

(ii) *Assume the hypotheses (i) to (iii) of Lemma 2.10 hold. If condition* (2.8) *is satisfied, then all solutions of stochastic delay differential equation* (2.1) *are* stochastically bounded.

3. Stability of the Trivial Solution

We now present stability results of the trivial solution as follows. When $p(\cdot) \equiv 0$, (1.1) becomes

$$\dddot{x}(t)+a\ddot{x}(t)+g(x(t-\tau(t)),\ \dot{x}(t-\tau(t)))+h(x(t-\tau(t)))+\sigma x(t-\tau(t))\dot{\omega}(t) = 0. \tag{3.1}$$

As usual, by assigning $y = \dot{x}$ and $z = \ddot{x}$ equation (3.1) is stepped down to equivalent system of first order differential equations

$$\begin{aligned} &\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -h(x) - g(x,y) - az - \sigma\big[x - \int_{t-\tau(t)}^{t} y(s)ds\big]\dot{\omega}(t) \\ &+ \int_{t-\tau(t)}^{t} \big[g_x(x(s),\ y(s))y(s) + g_y(x(s),\ y(s))z(s) + h'(x(s))y(s)\big]ds, \end{aligned} \tag{3.2}$$

where the functions h and g are continuous in their respective arguments. For the purpose of this investigation, a continuously differential scalar functional constructed is defined as

$$\begin{aligned} V = V(t, X_t) &= \alpha \int_0^x h(s)ds + \frac{1}{2}\beta b x^2 + \frac{1}{2}(\alpha a + \gamma bc + c)y^2 + \frac{1}{2}(a + \gamma c)z^2 \\ &+ a^2 bxy + \gamma cyh(x) + \beta xz + \alpha yz + \int_{-\tau(t)}^{0} \int_{t+s}^{t} (\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta))d\theta ds, \end{aligned} \tag{3.3}$$

where $a > 0$, $b > 0$, $c > 0$, $\alpha := a^2 + ac + c^2$, $\beta := ab - c$, $\gamma := 1 + b$ are constants, $h,\ g$ are continuous functions, positive constants λ_i $(i = 1, 2)$ will be verified latter, the function $\tau(t) \leq \tau_0$ for $\tau_0 > 0$, and $X_t = x_t, y_t, z_t$. We have the following stability results.

Theorem 3.1. *In addition to the basic assumption on the functions g and h, suppose that $a, b, c, c_0, k_1, k_2, k_3, \beta_1$ are positive constants such that*

(i) $h(0) = 0,\ c_0 \leq \dfrac{h(x)}{x}$ *for all* $x \neq 0$;

(ii) $g(0, 0) = 0,\ b \leq \dfrac{g(x, y)}{y}$ *for all* x *and* $y \neq 0$;

(iii) $h'(x) \leq c$ *for all* x, $ab - c > 0$, $\sigma^2 < \dfrac{2(ab - c)c_0}{a + (b + 1)c}$;

(iv) $a^2 b(a + c\gamma) > \alpha\beta$, $b\beta(a\alpha + c) > a^4 b^2$, $(a\alpha + c)(a + c\gamma) > \alpha^2$, $a^2 b\alpha > \beta(a\alpha + c)$, $b\beta\big[(a\alpha + c)(a + c\gamma) - \alpha^2\big] + \beta\big[a^2 b\alpha - \beta(a\alpha + c)\big] > a^2 b\big[a^2 b(a + c\gamma) - \alpha\beta\big]$; *and*

(v) $|h'(x)| \leq k_1$, $|g_x(x, y)| \leq k_2$, $|g_y(x, y)| \leq k_3$.

Then the trivial solution of system (3.2) *is asymptotically stable, provided that*

$$\beta_1 < \min\left\{ \frac{2(ab - c)c_0 - (a + c\gamma)\sigma^2}{2B_3},\ \frac{(ab - c)c}{B_4},\ \frac{(ab - c)c}{B_5} \right\} \tag{3.4}$$

where

$$\begin{aligned} B_3 &:= k_0(k_1 + k_2 + k_3) - (a + \gamma c)\sigma^2, \\ B_4 &:= \big[3k_0(k_1 + k_2) + (a + c\gamma)\sigma^2 + k_0(k_1 + k_2 + k_3)(1 - \beta_0)\big]/(1 - \beta_0), \\ B_5 &:= \big(3k_0 k_3 + k_0(k_1 + k_2 + k_3)(1 - \beta_0)\big)/(1 - \beta_0), \text{ and} \\ k_0 &:= \max\{\alpha, \beta, (a + c\gamma)\}. \end{aligned}$$

Since asymptotic stability implies stability we have the following result.

Corollary 3.2. If all assumptions of Theorem 3.1 hold true, then the trivial solution of system (3.2) is stable if estimate (3.4) holds.

In what follows we present uniform asymptotic stability results.

Theorem 3.3. Further to the basic assumption on the functions g and h, suppose that $a,\ b,\ b_1,\ c,\ c_0,\ c_1,\ k_0,\ k_1,\ k_2$, and k_3 are positive constants such that

(i) $h(0) = 0,\ c_0 \leq \dfrac{h(x)}{x} \leq c_1$ for all $x \neq 0$;

(ii) $g(0, 0) = 0,\ b \leq \dfrac{g(x, y)}{y} \leq b_1$ for all x and $y \neq 0$;

(iii) $h'(x) \leq c$ for all x, $ab - c > 0$, $\sigma^2 < \dfrac{2(ab - c)c_0}{a + (b + 1)c}$;

(iv) $a^2b(a+c\gamma) > \alpha\beta$, $b\beta(a\alpha+c) > a^4b^2$, $(a\alpha+c)(a+c\gamma) > \alpha^2$, $a^2b\alpha > \beta(a\alpha+c)$, $b\beta\big[(a\alpha+c)(a+c\gamma)-\alpha^2\big]+\beta\big[a^2b\alpha-\beta(a\alpha+c)\big] > a^2b\big[a^2b(a+c\gamma)-\alpha\beta\big]$; and

(v) $|h'(x)| \le k_1$, $|g_x(x,y)| \le k_2$, $|g_y(x,y)| \le k_3$;

Then the trivial solution of system (3.2) is uniformly asymptotically stable provided that inequality (3.4) holds.

Next, the following corollary is immediate from Theorem 3.3.

Corollary 3.4. *If all assumptions of Theorem 3.3 hold, then the trivial solution of system* (3.2) *is uniformly stable provided that inequality* (3.4) *holds.*

To show that (3.3) is indeed a Lyapunov functional we need to state and prove two lemmas.

Lemma 3.5. Under the assumptions of Theorem 3.3 there exist positive constants E_1 and E_2 such that

$$E_1(x^2+y^2+z^2) \le V(t,X_t) \le E_2(x^2+y^2+z^2), \tag{3.5}$$

for all $t \ge 0$, x, y, and z. Moreover,

$$V(t,X_t) \to +\infty \text{ as } x^2+y^2+z^2 \to \infty. \tag{3.6}$$

Proof. To prove this lemma we shall show that $V(t,\mathbf{0}) = 0$ where $\mathbf{0} = (0,0,0)$, $V(t,X_t)$ is positive semi-definite, decrescent (or have an infinitesimal small upper-bound), and radially unbounded. To see these, equation (3.3) shows that

$$V(t,\mathbf{0}) = 0, \tag{3.7}$$

for all $t \ge 0$. Following, equation (3.3) can be represented in the form $V = \sum\limits_{j=1}^{3} V_j$ where

$$V_1 := \alpha\int_0^x h(s)ds + \frac{1}{2}bc\gamma y^2 + c\gamma y h(x);$$
$$V_2 := \frac{1}{2}b\beta x^2 + \frac{1}{2}\big[a\alpha+c\big]y^2 + \frac{1}{2}\big[a+c\gamma\big]z^2 + a^2bxy + \beta xz + \alpha yz; \text{ and}$$
$$V_3 := \int_{-\tau(t)}^{0}\int_{t+s}^{t}\big[\lambda_1 y^2(\theta)+\lambda_2 z^2(\theta)\big]d\theta ds.$$

Now the last two terms of V_1 can be represented as

$$\frac{1}{2}bc\gamma y^2 + c\gamma y h(x) = \frac{1}{2}bc\gamma\big[y+b^{-1}h(x)\big]^2 - \frac{1}{2}b^{-1}c\gamma h^2(x). \tag{3.8a}$$

Also, since $h^2(x) = 2\int_0^x h'(s)h(s)ds + h^2(0)$ and $h(0)=0$, it follows that

$$\begin{aligned}\alpha\int_0^x h(s)ds &= \alpha\int_0^x h(s)ds - \frac{1}{2}b^{-1}c\gamma h^2(x) + \frac{1}{2}b^{-1}c\gamma h^2(x)\\ &= \frac{1}{b}\int_0^x \big[b\alpha - c\gamma h'(s)\big]h(s)ds + \frac{1}{2}b^{-1}c\gamma h^2(x).\end{aligned} \tag{3.8b}$$

Adding equations (3.8a) and (3.8b) we have

$$V_1 = \frac{1}{b}\int_0^x \big[b\alpha - c\gamma h'(s)\big]h(s)ds + \frac{1}{2}bc\gamma\big[y+b^{-1}h(x)\big]^2.$$

Hypotheses (i) and (iii) of Theorem 3.3 result to

$$V_1 \geq \frac{1}{2b}c_0[b\alpha - c^2\gamma]x^2 + \frac{1}{2}bc\gamma[y + b^{-1}c_0x]^2 \geq \frac{1}{2b}c_0[b\alpha - c^2\gamma]x^2,$$

since $\frac{1}{2}bc\gamma[y + b^{-1}c_0x]^2 \geq 0$ for all x, y. The basic assumptions imply that

$$b\alpha - c^2\gamma = b(a^2 + ac + c^2) - (b+1)c^2 = a^2b + c\beta > 0.$$

Thus

$$V_1 \geq \frac{1}{2b}c_0[a^2b + c\beta]x^2, \text{ for all } x.$$

Next,

$$V_2 = \frac{1}{2}b\beta x^2 + \frac{1}{2}[a\alpha + c]y^2 + \frac{1}{2}[a + c\gamma]z^2 + a^2bxy + \beta xz + \alpha yz;$$

can be represented as $2V_2 := XAX^T$ where $X = \begin{pmatrix} x & y & z \end{pmatrix}$, X^T is the transpose of X, and

$$A := \begin{pmatrix} b\beta & a^2b & \beta \\ a^2b & a\alpha + c & \alpha \\ \beta & \alpha & a + c\gamma \end{pmatrix}.$$

We need to show that the determinant of the principal minors of matrix A (i.e., $|A_1|, |A_2|$, and $|A_3|$,) are positive. The basic assumptions indicate that

$$|A_1| := b\beta > 0.$$

Hypothesis (iv) gives raise to

$$|A_2| := \begin{vmatrix} b\beta & a^2b \\ a^2b & a\alpha + c \end{vmatrix} = b\beta(a\alpha + c) - a^4b^2 > 0,$$

and

$$|A_3| = |A| := \begin{vmatrix} b\beta & a^2b & \beta \\ a^2b & a\alpha + c & \alpha \\ \beta & \alpha & a + c\gamma \end{vmatrix} = b\beta\big[(a\alpha + c)(a + c\gamma) - \alpha^2\big] + \beta\big[a^2b\alpha - \beta(a\alpha + c)\big]$$
$$- a^2b\big[a^2b(a + c\gamma) - \alpha\beta\big] > 0.$$

Since all principal minors of matrix A are positive, then A is positive definite and a constant $\theta_1 = \theta_1(a, b, c) > 0$ exists such that

$$V_2 \geq \theta_1(x^2 + y^2 + z^2) \text{ for all } x, y, z.$$

Next, the double integrals in V_3 are obviously positive, thus there exist a constant $\mu > 0$ such that

$$V_3 = \int_{-\tau(t)}^{0}\int_{t+s}^{t}[\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)]d\theta ds \geq \mu(y^2 + z^2).$$

combining the $V_i (i = 1, 2, 3)$ there exists a positive constant θ_2 such that

$$V \geq \theta_2(x^2 + y^2 + z^2) \tag{3.9}$$

for all $t \geq 0,\ x,\ y$, and z where

$$\theta_2 = \theta_1 \cdot \min\left\{\frac{1}{2b}c_0[a^2b + c\beta],\ \mu\right\}.$$

Inequality (3.9) establishes the lower inequality in (3.5) with θ_2 equivalent to E_1, hence by inequality (3.9), the function $V(t, X_t)$ is positive semi-definite.

Moreover, from inequality (3.9), we have the following relations

$$V(t, X_t) = 0 \iff x^2 + y^2 + z^2 = 0, \tag{3.10a}$$

$$V(t, X_t) > 0 \iff x^2 + y^2 + z^2 \neq 0, \tag{3.10b}$$

it validly follows from equation (3.10a) and estimate (3.10b) that

$$V(t, X_t) \to +\infty \text{ as } x^2 + y^2 + z^2 \to \infty, \tag{3.10c}$$

so that the function $V(t, X_t)$ is radially unbounded. In addition, assumptions (i) and (ii) of Theorem 3.3, the obvious inequality $2ab \leq a^2+b^2$, the fact that $\dfrac{h(x)}{x} \leq c_1$ for all $x \neq 0$, and since $\tau(t) \leq \tau_0$, equation (3.3) becomes

$$\begin{aligned} V(t, X_t) \leq & \frac{1}{2}(c_1\alpha + b\beta + a^2b + cc_1\gamma + \beta)\|x\|^2 + \frac{1}{2}(a\alpha + bc\gamma + c + a^2b + cc_1\gamma + \alpha \\ & + \lambda_1\tau_0^2)\|y\|^2 + \frac{1}{2}(a + c\gamma + \beta + \alpha + \lambda_1\tau_0^2)\|z\|^2. \end{aligned}$$

In view of the last inequality, there exist a positive constant θ_3 such that

$$V(t, X_t) \leq \theta_3(x^2 + y^2 + z^2) \tag{3.11}$$

for all $t \geq 0, x, y,$ and z where

$$\begin{aligned} \theta_3 := \frac{1}{2}\max\{ & c_1\alpha + b\beta + a^2b + cc_1\gamma + \beta,\ a\alpha + bc\gamma + c + a^2b + cc_1\gamma + \alpha \\ & + \lambda_1\tau_0^2,\ a + c\gamma + \beta + \alpha + \lambda_1\tau_0^2\}. \end{aligned}$$

Inequality (3.11) fulfils the upper inequality in (3.5) with θ_3 equivalent to E_2, thus the functional $V(t, X_t)$ has an infinitesimal small upper bound. This completes the prove of Lemma 3.5. □

The following lemma establishes the derivative of the functional $V(t, X_t)$ defined by (3.3), using Itô's formula defined by equation (2.3).

Lemma 3.6. Under the assumption of Theorem 3.1 there exists a positive constant E_3 such that along the solution path of system (3.2)

$$LV(t, X_t) \leq -E_3(x^2 + y^2 + z^2),\ \forall\ x, y, z. \tag{3.12}$$

Proof. The first partial derivative of the functional $V(t, X_t)$ along the solution path of (3.2) is

$$\begin{aligned} LV_{(3.2)}(t, X_t) = & -\frac{1}{2}V_4 - V_5 + V_6 + (\lambda_1 y^2 + \lambda_2 z^2)\tau(t) \\ & - (1 - \tau'(t))\int_{t-\tau(t)}^{t} [\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)]d\theta, \end{aligned} \tag{3.13}$$

where

$$V_4 = \left(\beta\frac{h(x)}{x} - \frac{a+c\gamma}{2}\sigma^2\right)x^2 + \left(\alpha\frac{g(x,y)}{y} - (a^2b + c\gamma h'(x))\right)y^2 + \left(a(a+c\gamma) - \alpha\right)z^2;$$

$$V_5 = \frac{1}{2}\left(\beta\frac{h(x)}{x} - \frac{a+c\gamma}{2}\sigma^2\right)x^2 + \frac{1}{2}\left(\alpha\frac{g(x,y)}{y} - (a^2b + c\gamma h'(x))\right)y^2 + \frac{1}{2}\left(a(a+c\gamma) - \alpha\right)z^2$$
$$+ \beta\left(\frac{g(x,y)}{y} - b\right)xy + \left(a\beta + a - a^2b\right)xz + \left[(a + c\gamma\frac{g(x,y)}{y} - \beta - bc\gamma - c)\right]yz; \text{ and}$$

$$V_6 = \left[\beta x + \alpha y + (a+c\gamma)z\right]\int_{t-\tau(t)}^{t}[h'(\cdot)y(s) + g_x(\cdot)y(s) + g_y(\cdot)z(s)]ds$$
$$- \sigma^2(a+c\gamma)x\int_{t-\tau(t)}^{t} y(s)ds + \frac{1}{2}\sigma^2\int_{t-\tau(t)}^{t} y^2(s)ds.$$

Hypotheses (i) to (iii) of Theorem 3.1 the following inequalities hold:

$$\begin{aligned}
&\beta\frac{h(x)}{x} - \frac{a+c\gamma}{2}\sigma^2 \geq c_0\beta - \frac{a+c\gamma}{2}\sigma^2;\\
&\alpha\frac{g(x,y)}{y} - (a^2b + c\gamma h'(x)) \geq (a^2 + ac + c^2)b - a^2b - bc^2 - c^2 = abc - c^2;\\
&a(a+c\gamma) - \alpha \geq a^2 + ac(b+1) - (a^2 + ac + c^2) = abc - c^2.
\end{aligned} \tag{3.14}$$

Estimate (3.14) gives rise to

$$V_4 \geq \left[(ab-c)c_0 - \frac{a+c\gamma}{2}\sigma^2\right]x^2 + (abc - c^2)y^2 + (abc - c^2)z^2,$$

for all $t \geq 0, x, y, z$. Let $V_5 = \sum_{i=1}^{3} V_{5i}$ where

$$V_{51} := \frac{1}{4}\left(\beta\frac{h(x)}{x} - \frac{(a+c\gamma)}{2}\sigma^2\right)x^2 + \beta\left(\frac{g(x,y)}{y} - b\right)xy + \frac{1}{4}\left(\frac{\alpha g(x,y)}{y}\right.$$
$$\left. - (a^2b + c\gamma h'(x))\right)y^2;$$

$$V_{52} := \frac{1}{4}\left(\beta\frac{h(x)}{x} - \frac{(a+c\gamma)}{2}\sigma^2\right)x^2 + (a\beta + a - a^2b)xz + \frac{1}{4}\left(a(a+c\gamma) - \alpha\right)z^2; \text{ and}$$

$$V_{53} := \frac{1}{4}\left(\alpha\frac{g(x,y)}{y} - (a^2b + c\gamma h'(x))\right)y^2 + \left((a+c\gamma)\frac{g(x,y)}{y} - \beta - bc\gamma - c\right)yz$$
$$+ \frac{1}{4}\left(a(a+c\gamma) - \alpha\right)z^2.$$

Note that V_{5i} $(i = 1, 2, 3)$ is a quadratic function with coefficients of x^2, y^2, and z^2 positive, using Hessian matrix, we obtain

$$\left(\beta\frac{h(x)}{x} - \frac{(a+c\gamma)}{2}\sigma^2\right)\left(\alpha\frac{g(x,y)}{y} - (a^2b + c\gamma h'(x))\right) > 4\beta^2\left(\frac{g(x,y)}{y} - b\right)^2;$$

$$\left(\beta\frac{h(x)}{x} - \frac{(a+c\gamma)}{2}\sigma^2\right)\left(a(a+c\gamma) - \alpha\right) > 4\left(a\beta + a - a^2b\right)^2; \text{ and}$$

$$\left(\alpha\frac{g(x,y)}{y} - (a^2b + c\gamma h'(x))\right)\left(a(a+c\gamma) - \alpha\right) > 4\left((a+c\gamma)\frac{g(x,y)}{y} - \beta - bc\gamma - c\right)^2.$$

Applying these estimates in V_{5i} to give the following inequalities:

$$V_{51} \geq \left[\sqrt{\frac{1}{4}\big(\beta\frac{h(x)}{x} - \frac{(a+c\gamma)}{2}\sigma^2\big)}|x| + \sqrt{\frac{1}{4}\big(\alpha\frac{g(x,y)}{y} - (a^2b + c\gamma h'(x))\big)}|y|\right]^2$$
$$\geq 0, \forall\, t \geq 0, x, y;$$
$$V_{52} \geq \left[\sqrt{\frac{1}{4}\big(\beta\frac{h(x)}{x} - \frac{(a+c\gamma)}{2}\sigma^2\big)}|x| + \sqrt{\frac{1}{4}\big(a(a+c\gamma) - \alpha\big)}|z|\right]^2 \geq 0, \forall\, t \geq 0, x, z; \text{ and}$$
$$V_{53} \geq \left[\sqrt{\frac{1}{4}\big(\alpha\frac{g(x,y)}{y} - (a^2b + c\gamma h'(x))\big)}|y| + \sqrt{\frac{1}{4}\big(a(a+c\gamma) - \alpha\big)}|z|\right]^2 \geq 0, \forall\, t \geq 0, y, z.$$

These last three inequalities assure

$$V_5 \geq 0, \ \forall\, t \geq 0, x, y, z.$$

Apply the following inequality $2xy \leq 2|xy| \leq x^2+y^2$ and hypothesis (iv) of Theorem 3.1 give.

$$V_6 \leq \frac{k_0(k_1+k_2+k_3)}{2}(x^2+y^2+z^2)\tau(t) + \frac{1}{2}\int_{t-\tau(t)}^{t}\big[3k_0(k_1+k_2) + (a+c\gamma)\sigma^2\big]y^2(s)ds$$
$$+ \frac{3k_0k_3}{2}\int_{t-\tau(t)}^{t} z^2(s)ds + \frac{1}{2}(a+c\gamma)\sigma^2x^2\tau(t),$$

where $k_0 := \max\{\alpha, \beta, (a+c\gamma)\}$. Utilizing inequalities V_4, V_5, and V_6 in equation (3.12) we obtain

$$\begin{aligned}
LV_{(3.2)}(t,X_t) \leq& -\frac{1}{2}\Bigg[(ab-c)c_0 - \frac{a+c\gamma}{2}\sigma^2 - \Big(k_0(k_1+k_2+k_3)\\
&+ (a+c\gamma)\sigma^2\Big)\tau(t)\Bigg]x^2 - \frac{1}{2}\Big[(ab-c)c - \big(2\lambda_1 + k_0(k_1+k_2+k_3)\big)\tau(t)\Big]y^2\\
&- \frac{1}{2}\Big[(ab-c)c - \Big(2\lambda_2 + k_0(k_1+k_2+k_3)\Big)\tau(t)\Big]z^2 \qquad (3.15)\\
&- \Big(\lambda_1[1-\tau'(t)] - \frac{3}{2}k_0(k_1+k_2) - \frac{1}{2}(a+c\gamma)\sigma^2\Big)\int_{t-\tau(t)}^{t} y^2(s)ds\\
&- \Big(\lambda_2[1-\tau'(t)] - \frac{3}{2}k_0k_3\Big)\int_{t-\tau(t)}^{t} z^2(s)ds.
\end{aligned}$$

Let $\tau'(t) \leq \beta_0$, $\beta_0 \in (0,1)$, $\tau(t) \leq \beta_1$, suppose $\lambda_1 := \big[3k_0(k_1+k_2)+(a+c\gamma)\sigma^2\big]\big[2(1-\beta_0)\big]^{-1} > 0$, and $\lambda_2 := 3k_0k_3\big[2(1-\beta_0)\big]^{-1} > 0$ so estimate (3.15) becomes

$$\begin{aligned}
LV_{(3.2)}(t,X_t) \leq& -\frac{1}{2}\Bigg[(ab-c)c_0 - \frac{a+c\gamma}{2}\sigma^2\\
&- \Big(k_0(k_1+k_2+k_3) - (a+\gamma c)\sigma^2\Big)\beta_1\Bigg]x^2 \qquad (3.16)\\
&- \frac{1}{2}\Bigg[(ab-c)c - \Bigg(\bigg[\frac{3k_0(k_1+k_2)+(a+c\gamma)\sigma^2}{1-\beta_0}\bigg] + k_0(k_1+k_2+k_3)\Bigg)\beta_1\Bigg]y^2\\
&- \frac{1}{2}\Bigg[(ab-c)c - \Bigg(\bigg(\frac{3k_0k_3}{1-\beta_0}\bigg) + k_0(k_1+k_2+k_3)\Bigg)\beta_1\Bigg]z^2.
\end{aligned}$$

Inequalities (3.4) and (3.16) invoke the existence of a positive constant k_4 such that

$$LV_{(3.2)}(t, X_t) \leq -k_4(x^2 + y^2 + z^2) \tag{3.17}$$

for all $t \geq 0, x, y,$ and z where

$$\begin{aligned} k_4 :=& \frac{1}{2} \min \Bigg\{ (ab - c)c_0 - \frac{a + c\gamma}{2}\sigma^2 - \Big(k_0(k_1 + k_2 + k_3) - (a + \gamma c)\sigma^2 \Big)\beta_1, \\ & (ab - c)c - \left(\left[\frac{3k_0(k_1 + k_2) + (a + c\gamma)\sigma^2}{1 - \beta_0} \right] + k_0(k_1 + k_2 + k_3) \right)\beta_1, \\ & (ab - c)c - \left(\left(\frac{3k_0k_3}{1 - \beta_0} \right) + k_0(k_1 + k_2 + k_3) \right)\beta_1 \Bigg\}. \end{aligned}$$

Inequality (3.17) satisfies estimate (3.12) with k_4 equivalent to E_3, hence Lemma 3.6 is proved. □

Proof of Theorems 3.1. Suppose (X_t) is any solution of (3.2), the functional $V(t, X_t)$ defined in (3.3) satisfies equation (3.7), estimates (3.9), (3.10c), and (3.17), so that conditions (i), (ii), and (iii) of the Lemma 2.5 are satisfied, hence by Lemma 2.5 the solution of (3.2) is stochastically asymptotically stable. □

Proof of Theorems 3.3. Given that (X_t) is any solution of (3.2) and the functional $V(t, X_t)$ defined in (3.3) satisfies equation (3.7), estimates (3.9), (3.10c), (3.11), and (3.17), fulfil assumptions (i), (ii), and (iii) of the Lemma 2.6, hence by Lemma 2.6 the solution of (3.2) is uniformly stochastically asymptotically stable. □

4. Boundedness and Existence Results

Furthermore, if $p(t, x, y, z) \neq 0$ in system (1.2), we have the following boundedness and ultimate boundedness results

Theorem 4.1. Suppose conditions (i) to (iv) and inequality (3.4) of Theorem 3.1 hold and in addition, if $|p(t, x, y, z)| \leq P_0$ where P_0 is a finite constant, then the solutions (X_t) of system (1.2) are not only stochastically bounded but also stochastically ultimately bounded.

Proof. Let (X_t) be any solution of system (1.2), by applying the Itô's formula on the functional defined in (3.3), along the solution path of (1.2), results to

$$LV_{(1.2)}(t, X_t) = LV_{(3.2)}(t, X_t) + \big[\beta x + \alpha y + (a + c\gamma)z\big]p(t, x, y, z).$$

Now from estimate (3.17) we find that

$$LV_{(1.2)}(t, X_t) \leq -k_4(x^2 + y^2 + z^2) + k_5(|x| + |y| + |z|)|p(t, x, y, z)|$$

where $k_5 = \max\{\beta, \alpha, (a + c\gamma)\}$. Since $|p(t, x, y, z)| \leq P_0$ for all $t \geq 0, x, y,$ and z, it follows that

$$\begin{aligned} LV_{(1.2)}(t, X_t) \leq -\frac{1}{2}k_4(x^2 + y^2 + z^2) + P_0k_4^{-1}k_5^2 - \frac{1}{2}k_4P_0\Big[&(|x| - k_4^{-1}k_5)^2 \\ + (|y| - k_4^{-1}k_5)^2 + (|z| - k_4^{-1}k_5)^2 \Big] & \end{aligned}$$

$\forall\ t \geq 0, x, y,$ and z. Since k_4 and P_0 are positive constants and $(|x| - k_4^{-1}k_5)^2 + (|y| - k_4^{-1}k_5)^2 + (|z| - k_4^{-1}k_5)^2 \geq 0$ for all $x, y,$ and z. Therefore there exist positive constants k_6 and k_7 such that

$$LV_{(1.2)}(t, X_t) \leq -k_6(x^2 + y^2 + z^2) + k_7 \tag{4.1}$$

where $k_6 = \frac{1}{2}k_4$ and $k_7 = P_0 k_4^{-1}k_5^2$ for all $t \geq 0, x, y,$ and z. Estimate (3.11) implies that $\theta_3^{-1}V(t, X_t) \leq (x^2 + y^2 + z^2)$ for all $t \geq 0, x, y,$ and z. The last estimate and inequality (4.1) result to

$$LV_{(1.2)}(t, X_t) \leq -k_8 V(t, X_t) + k_7 \tag{4.2}$$

for all $t \geq 0, x, y,$ and z where $k_8 := k_6\theta_3^{-1}$. Inequality (4.2) fulfills condition (ii) of Lemma 2.10 with $\alpha(t) = k_8, \psi(t) = k_7, q = 1$.

Furthermore, the lower inequality (3.5) (or estimate (3.9)) satisfies hypothesis (i) of Lemma 2.10. Now by estimate (4.2), we have $q = 1$, this implies that $\mu = 0$, so that hypothesis (iii) of Lemma 2.10 holds. Substituting the values of $\alpha, \psi,$ and μ in (2.8), to find that

$$\int_{t_0}^{t} (\mu\alpha(u) + \psi(u))e^{-\int_u^t \alpha(s)ds}du = k_7 k_6^{-1}[1 - e^{-k_6(t-t_0)}] \leq k_7 k_6^{-1} \tag{4.3}$$

for all $t \geq t_0 \geq 0$, inequality (4.3) satisfies estimate (2.8) of Corollary 2.11 with $M = k_7 k_6^{-1} > 0$.

Also, to verify inequalities (2.5) and (2.6) of Assumption 2.8 (a special case of Assumption 2.7). System (1.2) and the Lyapunov functional (3.3) result to

$$|V_{xi}(t, X_t)G_{ik}(t, X_t)| \leq \frac{1}{2}\sigma\Big\{[2\beta + \alpha + (a + c\gamma) + k_5\beta_1]\|x\|^2 + [\alpha + k_5\beta_1]\|y\|^2 + [(a + c\gamma) + k_5\beta_1]\|z\|^2 + \frac{3}{4}k_5\sigma\beta_1^2\|y\|^2\Big\}.$$

In view of the above inequality there exists a positive constant k_9 such that

$$|V_{xi}(t, X_t)G_{ik}(t, X_t)| \leq k_9(x^2 + y^2 + z^2),$$

and for $0 \leq t_0 \leq T < \infty$ and

$$\int_{t_0}^{T} \rho^2(s)ds < \infty$$

where $\rho(t) := k_9(x^2 + y^2 + z^2)(t)$ and $k_9 := \frac{1}{2}\sigma \max\Big\{\alpha + 2\beta + a + c\gamma + k_5\beta_1, \alpha + k_5\beta_1 + \frac{3}{4}k_5\sigma\beta_1^2, a + c\gamma + k_5\beta_1\Big\}$. Thus, Assumption 2.7 is satisfied, i.e.,

$$E^{x_0}\Big\{\int_{t_0}^{T} V_{x_i}^2(t, X_t)G_{ik}^2(t, X_t)dt\Big\} < \infty. \tag{4.4}$$

Hypotheses (i) to (iii) of Lemma 2.10 and estimate (2.8) hold true so that Corollary 2.11 (ii) follows, hence by Corollary 2.11 (ii) all solutions of (1.2) are not only bounded but also ultimately stochastically bounded. □

Next theorem presents uniform stochastic boundedness and uniform ultimate stochastic boundedness of solutions of system (1.2).

Theorem 4.2. Suppose that conditions (i) to (v) of Theorem 3.3 and inequality (3.4) are satisfied and in addition $|p(t,x,y,z)| < P_0$ where P_0 is a finite constant, then the solutions (X_t) of system (1.2) are not only uniform stochastically bounded but also uniformly ultimately stochastically bounded.

Proof. Given that (X_t) is any solution of the system (1.2) and the functional (3.3) satisfy inequalities (3.9), (3.11), (4.1) so that hypotheses (i) and (ii) of Lemma 2.9 hold. Also with $p = q = r = 2$ we have $\mu = 0$ so that hypothesis (iii) of Lemma 2.9 holds. In addition, the inequalities (4.3) and (4.4) together with Lemma 2.9 satisfy the hypothesis of Corollary 2.11(i), hence by Corollary 2.11(i) the solutions of system (1.2) are not only uniform stochastically bounded, but also uniform ultimately stochastically bounded. □

Next, we shall state and prove an existence and uniqueness theorem as follows.

Theorem 4.3. If assumptions of Theorem 4.1 are satisfied, then there exists a unique solution of system (1.2).

Proof. Let (X_t) be any solution of (1.2), the functional defined in (3.3) satisfy the following estimates (3.9), (3.10c), and (3.17), these inequalities successfully satisfy all assumptions of Lemma 2.5 thus by Lemma 2.5 solution of system (1.2) exists and unique. Hence, the proof of Theorem 4.3 is completed. □

Next, we shall consider arbitrary third-order stochastic differential equations with delay and show that all assumptions of Theorems 3.1, 3.3, 4.1, 4.2, and 4.3 hold true.

5. Examples

Example 5.1. Consider the third-order stochastic differential equation

$$\begin{aligned}\dddot{x}(t) + a\ddot{x}(t) &+ \left[3x\dot{x}(t-\tau(t)) + \left(\frac{\dot{x}(t-\tau(t))}{2 + x^2(t-\tau(t)) + \dot{x}^2(t-\tau(t))}\right)\right] \\ &+ \left[x(t-\tau(t)) + \left(\frac{x(t-\tau(t))}{1 + x^2(t-\tau(t))}\right)\right] + \sigma x(t-\tau(t))\dot{\omega}(t) = 0.\end{aligned} \tag{5.1}$$

Assign $y = \dot{x}$ and $z = \ddot{x}$ equation (5.1) is equivalent to system of first order equations

$$\begin{aligned}\dot{x} &= y, \quad \dot{y} = z,\\ \dot{z} &= -\left(\frac{2x + x^3}{1 + x^2}\right) - \left[\frac{3(2y + x^2y + y^3) + y}{2 + x^2 + y^2}\right] - az - \sigma\left[x - \int_{t-\tau(t)}^{t} y(s)ds\right] \\ &+ \int_{t-\tau(t)}^{t}\left\{\left[1 + \frac{1}{1 + x^2(s)} - \frac{2x^2(s)}{(1 + x^2(s))^2}\right]y(s) - \frac{2x(s)y^2(s)}{(2 + x^2(s) + y^2(s))^2}\right. \\ &+ \left.\left[3 + \frac{1}{2 + x^2(s) + y^2(s)} - \frac{2y^2(s)}{(2 + x^2(s) + y^2(s))^2}\right]z(s)\right\}ds.\end{aligned} \tag{5.2}$$

Now, comparing equations (3.2) with (5.2) the following relations hold:

(i) The function $h(x) := \dfrac{x(2 + x^2)}{1 + x^2} = x + \dfrac{x}{1 + x^2}$, clearly $h(0) = 0$ and $\dfrac{h(x)}{x} = 1 + \dfrac{1}{1 + x^2}$. Since $1 + x^2 \geq 1$ for all x, it follows $0 < \dfrac{1}{1 + x^2} \leq 1$ for all x.

Further simplification of the last inequality gives

$$1 = c_0 \leq \frac{h(x)}{x} \leq c_1 = 2 \ \forall \ x \neq 0.$$

(ii) The derivative of $h = h(x)$ with respect to x is defined as $h'(x) := 1 + \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2}$. Since $2x^2(1+x^2)^{-2} \geq 0$ for all x and by (i) to find that

$$h'(x) \leq c = 2, \ \forall \ x. \tag{5.3}$$

Moreover,

$$|h'(x)| \leq k_1 = 2, \ \forall \ x. \tag{5.4}$$

See Figure 1 for the coincide bounds on $h'(x)$ and $|h'(x)|$. Inequalities (5.3) and (5.4) hold true for all $x \in \mathbb{R}$.

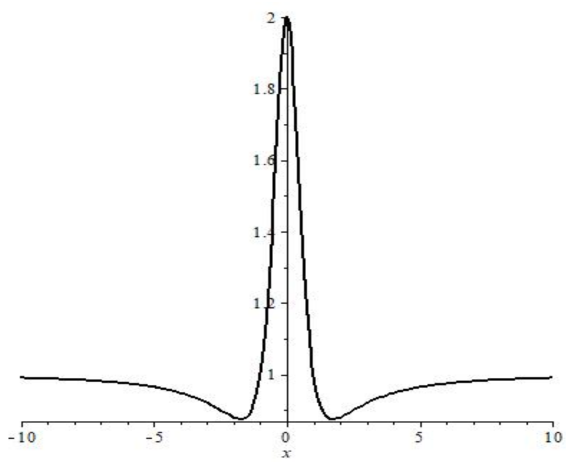

FIGURE 1. Upper bound on the functions $h'(x)$ and $|h'(x)|$ for $x \in [-10, 10]$

(iii) The function $g = g(x, y)$ is defined as $g(x, y) := 3y + \frac{y}{2+x^2+y^2}$. Obviously, $g(0, 0) = 0$ and that $\frac{g(x,y)}{y} = 3 + \frac{1}{2+x^2+y^2}$. It is not difficult to show that $3 = b \leq \frac{g(x,y)}{y} \leq b_1 = 3\frac{1}{2} \ \forall \ x, y \neq 0$.

(iv) The first partial derivatives of g with respect to x and y are given by $g_x(x, y) := \frac{-2xy}{(2+x^2+y^2)^2}$ and $g_y(x, y) := 3 + \frac{1}{2+x^2+y^2} - \frac{2y^2}{(2+x^2+y^2)^2}$ respectively, and is easy to see that

$$|g_x(x, y)| \leq k_2 = 0.12 \tag{5.5a}$$

for all x, y, and

$$|g_y(x, y)| \leq k_3 = 3.5 \tag{5.5b}$$

for all x, y. Figures 2 and 3 confirm estimates (5.5a) and (5.5b) respectively for $-2 \leq x, y \leq 2$.

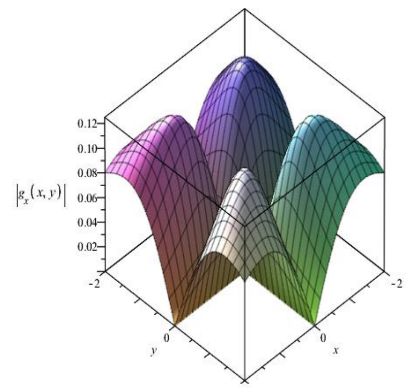

FIGURE 2. Bound on the function $|g_x(x,y)|$ for $x, y \in [-2,2]$

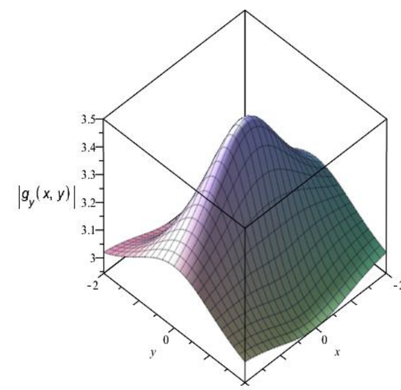

FIGURE 3. Bound on the function $|g_y(x,y)|$ for $x, y \in [-2,2]$

Since $b = 3, c = 2$, and $c_0 = 1$ it follows from the inequality $ab - c > 0$ that $a > 2/3 \approx 0.7$, we choose $a = 0.8$ and $\sigma^2 < \dfrac{2(ab-c)c_0}{a+(b+1)c}$ so that $\sigma < 0.3$ we choose $\sigma = 0.29$. The following assumptions are verified as

$$\alpha := a^2 + ac + c^2 = 6.24 > 0,\ \beta := ab - c = 0.4 > 0,\ \text{and } \gamma := 1 + b = 4 > 0,$$

$$k_0 = \max\{6.24,\ 0.4,\ 8.8\} = 8.8 > 0,\ a^2 b(a + c\gamma) - \alpha\beta = 14.4 > 0,$$

$$b\beta(a\alpha + c) - a^4 b^2 = 4.704 > 0, (a\alpha + c)(a + c\gamma) - \alpha^2 = 22.592 > 0,$$

$$a^2 b\alpha - \beta(a\alpha + c) = 9.184 > 0,\ b\beta\big[(a\alpha + c)(a + c\gamma) - \alpha^2\big] + \beta\big[a^2 b\alpha - \beta(a\alpha + c)\big]$$
$$- a^4 b^2(a + c\gamma) + a^2 b\alpha\beta = 3.136 > 0,\ B_3 := k_0(k_1 + k_2 + k_3) - (a + \gamma c)\sigma^2 = 48.7159 > 0.$$

Next, since $0 < \beta_0 < 1$ two cases are to be considered:

Case 1: When $\beta_0 = 0.001$, we have the following estimates:

$$B_4 := \big[3k_0(k_1 + k_2) + (a + c\gamma)\sigma^2 + k_0(k_1 + k_2 + k_3)(1 - \beta_0)\big]/(1 - \beta_0) = 106.2208 > 0,\ \text{and}$$
$$B_5 := \big(3k_0k_3 + k_0(k_1 + k_2 + k_3)(1 - \beta_0)\big)/(1 - \beta_0)141.9485.$$

In this case the inequality (3.4) yields

$$\begin{aligned}\beta_1 &< \min\left\{\frac{2(ab-c)c_0-(a+(b+1)c)\sigma^2}{2B_3},\ \frac{(ab-c)c}{B_4},\ \frac{(ab-c)c}{B_5}\right\}\\ &= \min\{6.1\times 10^{-4},\ 7.5\times 10^{-3},\ 5.6\times 10^{-3}\}\\ &= 6.1\times 10^{-4}\end{aligned} \tag{5.6}$$

Case 2: When $\beta_0 = 0.999$ we have the following estimates:

$B_4 := \big[3k_0(k_1+k_2)+(a+c\gamma)\sigma^2+k_0(k_1+k_2+k_3)(1-\beta_0)\big]/(1-\beta_0) = 56757.536 > 0,$ and
$B_5 := \big(3k_0k_3+k_0(k_1+k_2+k_3)(1-\beta_0)\big)/(1-\beta_0) = 92449.456 > 0.$

In this case inequality (3.4) yields

$$\beta_1 < \min\{6.1\times 10^{-4},\ 1.4\times 10^{-5},\ 8.6\times 10^{-6}\} = 8.6\times 10^{-6} \tag{5.7}$$

Thus in both cases β_1 is positive, hence for system (5.2) we have the following remark

Remark 5.2. If there exist positive constants $0.12, 1, 2, 3,$ and 3.5 such that

(i) $h(0) = 0$, $1 = c_0 \le \frac{h(x)}{x} \le c_1 = 2$ for all $x \ne 0$;
(ii) $g(0,0) = 0$, $3 = b \le \frac{g(x,y)}{y} \le b_1 = 3.5$ for all x and $y \ne 0$;
(iii) $h'(x) \le c = 2$ for all x, $ab - c = 0.4 > 0$, $\sigma = 0.29 > 0$;
(iv) $a^2b(a+c\gamma)-\alpha\beta = 14.4 > 0$, $b\beta(a\alpha+c)-a^4b^2 = 4.704 > 0$, $(a\alpha+c)(a+c\gamma)-\alpha^2 = 22.592 > 0$, $a^2b\alpha-\beta(a\alpha+c) = 9.184 > 0$, $b\beta\big[(a\alpha+c)(a+c\gamma)-\alpha^2\big]+\beta\big[a^2b\alpha-\beta(a\alpha+c)\big]-a^2b\big[a^2b(a+c\gamma)-\alpha\beta\big] = 3.136 > 0$; and
(v) $|h'(x)| \le k_1 = 2$, $|g_x(x,y)| \le k_2 = 0.12$, $|g_y(x,y)| \le k_3 = 3.5$;

Then the trivial solution of system (5.2) is stochastically stable, asymptotically stochastically stable, uniformly stochastically stable, and uniform asymptotically stochastically stable provided that $8.6\times 10^{-6} \le \beta_1 \le 6.1\times 10^{-4}$.

Finally, we shall consider the case $p(\cdot) \ne 0$.

Example 5.3. Consider the third-order stochastic differential equation

$$\begin{aligned}&\dddot{x}(t)+a\ddot{x}(t)+\left[3x\dot{x}(t-\tau(t))+\left(\frac{\dot{x}(t-\tau(t))}{2+x^2(t-\tau(t))+\dot{x}^2(t-\tau(t))}\right)\right]\\ &+\left[x(t-\tau(t))+\left(\frac{x(t-\tau(t))}{1+x^2(t-\tau(t))}\right)\right]+\sigma x(t-\tau(t))\dot{\omega}(t)\\ &=\frac{1}{10+t^2+x^2+\dot{x}^2+\ddot{x}^2},\end{aligned} \tag{5.8}$$

Assign $y = \dot{x}$ and $z = \ddot{x}$ equation (5.1) is equivalent to system of first order equations

$$\dot{x} = y, \quad \dot{y} = z,$$

$$\begin{aligned} \dot{z} = &-\left(\frac{2x + x^3}{1 + x^2}\right) - \left[\frac{3(2y + x^2y + y^3) + y}{2 + x^2 + y^2}\right] - az - \sigma\left[x - \int_{t-\tau(t)}^{t} y(s)ds\right] \\ &+ \int_{t-\tau(t)}^{t} \left\{\left[1 + \frac{1}{1 + x^2(s)} - \frac{2x^2(s)}{(1 + x^2(s))^2}\right]y(s) - \frac{2x(s)y^2(s)}{(2 + x^2(s) + y^2(s))^2}\right. \\ &+ \left.\left[3 + \frac{1}{2 + x^2(s) + y^2(s)} - \frac{2y^2(s)}{(2 + x^2(s) + y^2(s))^2}\right]z(s)\right\}ds \\ &+ \frac{1}{10 + t^2 + x^2 + y^2 + z^2}. \end{aligned} \tag{5.9}$$

Now comparing (1.2) with (5.9) items (i) to (v) of Remark 5.2 hold. In addition $p(t, x, y, z) := \dfrac{1}{10 + t^2 + x^2 + y^2 + z^2}$. Since $10 + t^2 + x^2 + y^2 + z^2 \geq 10$ for all $t \geq 0, x, y,$ and z there exists a finite constant P_0 such that $|p(t, x, y, z)| < P_0 = \frac{1}{10}$ for all $t \geq 0, x, y,$ and z.

Remark 5.4. If in addition to the hypotheses of Theorem 5.2, there exists a finite constant 1/10 such that $|p(t, x, y, z)| < P_0 = \frac{1}{10}$ for all $t \geq 0, x, y,$ and z, then the conclusions of Theorems 4.1, 4.2, and 4.3 hold true for all β_1 in the close interval $[8.6 \times 10^{-6},\ 6.1 \times 10^{-4}]$.

6. Conclusion

This paper presents some qualitative properties of solutions to certain third-order nonlinear nonautonomous stochastic differential equations with variable delay. Novel and outstanding results obtained in this paper compliment and extend many outstanding existing results in literature.

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(R.O. Banire) Department of Mathematics, Obafemi Awolowo University, Post Code 220005 Ile-Ife, Nigeria
Email address: rukayatbanire2018@gmail.com

(O.O. Fabelurin) Department of Mathematics, Obafemi Awolowo University, Post Code 220005 Ile-Ife, Nigeria
Email address: fabelurinoo@oauife.edu.ng

(P.O. Arawomo) Department of Mathematics, University of Ibadan, Ibadan, Nigeria
Email address: womopeter@gmail.com

(A.T. Ademola) Department of Mathematics, Obafemi Awolowo University, Post Code 220005 Ile-Ife, Nigeria
Email address: atademola@oauife.edu.ng

(M.O. Omeike) Department of Mathematics, University of Agriculture, Abeokuta, Nigeria
Email address: moomeike@yahoo.com