Communications in Advanced Mathematical Sciences

VOLUME VI
ISSUE IV

ISSN 2651-4001

VOLUME 6 ISSUE 4
ISSN 2651-4001

December 2023
www.dergipark.org.tr/tr/pub/cams

COMMUNICATIONS IN ADVANCED MATHEMATICAL SCIENCES

Contents

Communications in Advanced Mathematical Sciences
Vol. 6, No. 4, 177-187, 2023
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1362694

Tensorial and Hadamard Product Inequalities for Synchronous Functions

Silvestru Sever Dragomir[1,2]

Abstract
Let H be a Hilbert space. In this paper we show among others that, if f, g are synchronous and continuous on I and A, B are selfadjoint with spectra $Sp(A)$, $Sp(B) \subset I$, then

$$(f(A)g(A)) \otimes 1 + 1 \otimes (f(B)g(B)) \geq f(A) \otimes g(B) + g(A) \otimes f(B)$$

and the inequality for Hadamard product

$$(f(A)g(A) + f(B)g(B)) \circ 1 \geq f(A) \circ g(B) + f(B) \circ g(A).$$

Let either $p, q \in (0, \infty)$ or $p, q \in (-\infty, 0)$. If A, $B > 0$, then

$$A^{p+q} \otimes 1 + 1 \otimes B^{p+q} \geq A^p \otimes B^q + A^q \otimes B^p,$$

and

$$\left(A^{p+q} + B^{p+q}\right) \circ 1 \geq A^p \circ B^q + A^q \circ B^p.$$

Keywords: Convex functions, Hadamard Product, Selfadjoint operators, Tensorial product
2010 AMS: 47A63, 47A99

[1] *Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.*
[2] *DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa, ORCID: 0000-0003-2902-6805*
sever.dragomir@vu.edu.au, http://rgmia.org/dragomir
Received: 19 September 2023, **Accepted:** 31 October 2023, **Available online:** 7 November 2023
How to cite this article: S. S. Dragomir, *Tensorial and Hadamard Product Inequalities for Synchronous Functions of Selfadjoint Operators in Hilbert Spaces*, Commun. Adv. Math. Sci., 6(4) (2023), 177-187.

1. Introduction

Let $I_1, \ldots, I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times \ldots \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, \ldots, A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, \ldots, H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, \ldots, k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1,\dots,k$; by following [1], we define

$$f(A_1,\dots,A_k) := \int_{I_1}\dots\int_{I_k} f(\lambda_1,\dots,\lambda_k)\,dE_1(\lambda_1)\otimes\dots\otimes dE_k(\lambda_k) \tag{1.1}$$

as a bounded selfadjoint operator on the tensorial product $H_1\otimes\dots\otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [1] extends the definition of Korányi [2] for functions of two variables and have the property that

$$f(A_1,\dots,A_k) = f_1(A_1)\otimes\dots\otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1,\dots,t_k) = f_1(t_1)\dots f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s,t\in[0,\infty)$$

and if f is continuous on $[0,\infty)$, then [3, p. 173]

$$f(A\otimes B) \geq (\leq) f(A)\otimes f(B) \text{ for all } A,\ B\geq 0. \tag{1.2}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t\,dE(t) \text{ and } B = \int_{[0,\infty)} s\,dF(s)$$

are the spectral resolutions of A and B, then

$$f(A\otimes B) = \int_{[0,\infty)}\int_{[0,\infty)} f(st)\,dE(t)\otimes dF(s) \tag{1.3}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B>0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

where $t\in[0,1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of # and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B)\otimes(B\#A) = (A\otimes B)\#(B\otimes A).$$

In 2007, S. Wada [4] obtained the following *Callebaut type inequalities* for tensorial product

$$(A\#B)\otimes(A\#B) \leq \frac{1}{2}\left[(A\#_\alpha B)\otimes(A\#_{1-\alpha}B) + (A\#_{1-\alpha}B)\otimes(A\#_\alpha B)\right] \leq \frac{1}{2}(A\otimes B + B\otimes A) \tag{1.4}$$

for $A,\ B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\left\langle (A\circ B)e_j, e_j\right\rangle = \left\langle Ae_j, e_j\right\rangle\left\langle Be_j, e_j\right\rangle$$

for all $j\in\mathbb{N}$, where $\left\{e_j\right\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H. It is known that, see [5], we have the representation

$$A\circ B = \mathscr{U}^*(A\otimes B)\mathscr{U} \tag{1.5}$$

where $\mathscr{U}: H\to H\otimes H$ is the isometry defined by $\mathscr{U}e_j = e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative and operator concave (sub-multiplicative and operator convex)* on $[0,\infty)$, then also [3, p. 173]

$$f(A\circ B) \geq (\leq) f(A)\circ f(B) \text{ for all } A,\ B\geq 0. \tag{1.6}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A+B}{2}\right) \circ 1 \text{ for } A,\ B \geq 0$$

and Fiedler inequality

$$A \circ A^{-1} \geq 1 \text{ for } A > 0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [6] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [7] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [8] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

For other inequalities concerning tensorial product, see [9] and [10].

Motivated by the above results, in this paper we show among others that if f, g are synchronous and continuous on I and A, B are selfadjoint with spectra $Sp(A)$, $Sp(B) \subset I$, then

$$(f(A)g(A)) \otimes 1 + 1 \otimes (f(B)g(B)) \geq f(A) \otimes g(B) + g(A) \otimes f(B)$$

and the inequality for Hadamard product

$$(f(A)g(A) + f(B)g(B)) \circ 1 \geq f(A) \circ g(B) + f(B) \circ g(A).$$

Let either $p, q \in (0, \infty)$ or $p, q \in (-\infty, 0)$. If A, $B > 0$, then

$$A^{p+q} \otimes 1 + 1 \otimes B^{p+q} \geq A^p \otimes B^q + A^q \otimes B^p,$$

and

$$\left(A^{p+q} + B^{p+q}\right) \circ 1 \geq A^p \circ B^q + A^q \circ B^p.$$

2. Main Results

We start with the following main result:

Theorem 2.1. *Assume that f, g are synchronous and continuous on I and h, k nonnegative and continuous on the same interval. If A, B are selfadjoint with spectra $Sp(A)$, $Sp(B) \subset I$, then*

$$[h(A) f(A) g(A)] \otimes k(B) + h(A) \otimes [k(B) f(B) g(B)] \geq [h(A) f(A)] \otimes [k(B) g(B)] + [h(A) g(A)] \otimes [k(B) f(B)] \tag{2.1}$$

or, equivalently

$$(h(A) \otimes k(B)) [(f(A) g(A)) \otimes 1 + 1 \otimes (f(B) g(B))] \geq (h(A) \otimes k(B)) [f(A) \otimes g(B) + g(A) \otimes f(B)]. \tag{2.2}$$

If f, g are asynchronous on I, then the inequality reverses in (2.1) and (2.2).

Proof. Assume that f and g are synchronous on I, then

$$f(t) g(t) + f(s) g(s) \geq f(t) g(s) + f(s) g(t)$$

for all $t, s \in I$. We multiply this inequality by $h(t) k(s) \geq 0$ to get

$$f(t) g(t) h(t) k(s) + h(t) f(s) g(s) k(s) \geq f(t) h(t) g(s) k(s) + f(s) k(s) g(t) h(t)$$

for all $t,s\in I$. If we take the double integral, then we get

$$\begin{aligned}&\int_I\int_I [f(t)g(t)h(t)k(s)+h(t)f(s)g(s)k(s)]\,dE(t)\otimes dF(s)\\&\geq \int_I\int_I [f(t)h(t)g(s)k(s)+f(s)k(s)g(t)h(t)]\,dE(t)\otimes dF(s).\end{aligned}\tag{2.3}$$

Observe that

$$\begin{aligned}\int_I\int_I [f(t)g(t)h(t)k(s)+h(t)f(s)g(s)k(s)]\,dE(t)\otimes dF(s) &= \int_I\int_I f(t)g(t)h(t)k(s)\,dE(t)\otimes dF(s)\\&\quad+\int_I\int_I h(t)f(s)g(s)k(s)\,dE(t)\otimes dF(s)\\&=[h(A)f(A)g(A)]\otimes k(B)+h(A)\otimes[k(B)f(B)g(B)]\end{aligned}$$

and

$$\begin{aligned}\int_I\int_I [f(t)h(t)g(s)k(s)+f(s)k(s)g(t)h(t)]\,dE(t)\otimes dF(s) &= \int_I\int_I f(t)h(t)g(s)k(s)\,dE(t)\otimes dF(s)\\&\quad+\int_I\int_I g(t)h(t)f(s)k(s)\,dE(t)\otimes dF(s)\\&=[h(A)f(A)]\otimes[k(B)g(B)]+[h(A)g(A)]\otimes[k(B)f(B)].\end{aligned}$$

By utilizing (2.3) we derive (2.2). Now, by making use of the tensorial property

$$(XU)\otimes(YV)=(X\otimes Y)(U\otimes V),$$

for any $X, U, Y, V\in B(H)$, we obtain

$$\begin{aligned}&[h(A)f(A)g(A)]\otimes k(B)+h(A)\otimes[k(B)f(B)g(B)]\\&=(h(A)\otimes k(B))[(f(A)g(A))\otimes 1]+(h(A)\otimes k(B))[1\otimes(f(B)g(B))]\\&=(h(A)\otimes k(B))[(f(A)g(A))\otimes 1+1\otimes(f(B)g(B))]\end{aligned}$$

and

$$\begin{aligned}&[h(A)f(A)]\otimes[k(B)g(B)]+[h(A)g(A)]\otimes[k(B)f(B)]\\&=(h(A)\otimes k(B))(f(A)\otimes g(B))+(h(A)\otimes k(B))(g(A)\otimes f(B))\\&=(h(A)\otimes k(B))[f(A)\otimes g(B)+g(A)\otimes f(B)],\end{aligned}$$

which proves (2.2). □

Remark 2.2. *With the assumptions of Theorem 2.1 and if we take $k=h$, then we get*

$$[h(A)f(A)g(A)]\otimes h(B)+h(A)\otimes[h(B)f(B)g(B)]\geq[h(A)f(A)]\otimes[h(B)g(B)]+[h(A)g(A)]\otimes[h(B)f(B)],\tag{2.4}$$

where f, g are synchronous and continuous on I and h is nonnegative and continuous on the same interval.
Moreover, if we take $h\equiv 1$ in (2.4), then we get

$$(f(A)g(A))\otimes 1+1\otimes(f(B)g(B))\geq f(A)\otimes g(B)+g(A)\otimes f(B),\tag{2.5}$$

where f, g are synchronous and continuous on I

Corollary 2.3. *Assume that f, g are synchronous and continuous on I and h, k nonnegative and continuous on the same interval. If A, B are selfadjoint with spectra $Sp(A)$, $Sp(B)\subset I$, then*

$$k(B)\circ[h(A)f(A)g(A)]+h(A)\circ[k(B)f(B)g(B)]\geq[h(A)f(A)]\circ[k(B)g(B)]+[k(B)f(B)]\circ[h(A)g(A)].\tag{2.6}$$

If f, g are asynchronous on I, then the inequality reverses in (2.6). In particular, we have

$$h(B)\circ[h(A)f(A)g(A)]+h(A)\circ[h(B)f(B)g(B)]\geq[h(A)f(A)]\circ[h(B)g(B)]+[h(B)f(B)]\circ[h(A)g(A)]\tag{2.7}$$

and

$$(f(A)g(A)+(f(B)g(B)))\circ 1\geq f(A)\circ g(B)+f(B)\circ g(A).\tag{2.8}$$

Proof. If we take $\mathscr{U}^*$ to the left and $\mathscr{U}$ to the right in the inequality (2.1), we get

$$\mathscr{U}^*([h(A)f(A)g(A)]\otimes k(B))\mathscr{U}+\mathscr{U}^*(h(A)\otimes[k(B)f(B)g(B)])\mathscr{U}\geq\mathscr{U}^*([h(A)f(A)]\otimes[k(B)g(B)])\mathscr{U}+\mathscr{U}^*([h(A)g(A)]\otimes[k(B)f(B)])\mathscr{U}$$

which is equivalent to (2.6). □

Corollary 2.4. *Assume that f, g are synchronous and continuous on I and h, k nonnegative and continuous on the same interval. If A_j, B_j are selfadjoint with spectra $Sp(A_j)$, $Sp(B_j)\subset I$ and $p_j,q_j\geq 0$, $j\in\{1,\ldots,n\}$, then*

$$\begin{aligned}&\left(\sum_{j=1}^n p_jh(A_j)f(A_j)g(A_j)\right)\otimes\left(\sum_{i=1}^n q_ik(B_i)\right)+\left(\sum_{j=1}^n p_jh(A_j)\right)\otimes\left(\sum_{i=1}^n q_ik(B_i)f(B_i)g(B_i)\right)\\&\geq\left(\sum_{j=1}^n p_jh(A_j)f(A_j)\right)\otimes\left(\sum_{i=1}^n q_ik(B_i)g(B_i)\right)+\left(\sum_{j=1}^n p_jh(A_j)g(A_j)\right)\otimes\left(\sum_{i=1}^n q_ik(B_i)f(B_i)\right).\end{aligned}\tag{2.9}$$

In particular,

$$\begin{aligned}&\left(\sum_{j=1}^n p_jh(A_j)f(A_j)g(A_j)\right)\otimes\left(\sum_{i=1}^n q_ih(B_i)\right)+\left(\sum_{j=1}^n p_jh(A_j)\right)\otimes\left(\sum_{i=1}^n q_ih(B_i)f(B_i)g(B_i)\right)\\&\geq\left(\sum_{j=1}^n p_jh(A_j)f(A_j)\right)\otimes\left(\sum_{i=1}^n q_ih(B_i)g(B_i)\right)+\left(\sum_{j=1}^n p_jh(A_j)g(A_j)\right)\otimes\left(\sum_{i=1}^n q_ih(B_i)f(B_i)\right)\end{aligned}\tag{2.10}$$

and, if $\sum_{j=1}^n p_j=\sum_{j=1}^n q_j=1$, then

$$\begin{aligned}\left(\sum_{j=1}^n p_jf(A_j)g(A_j)\right)\otimes 1+1\otimes\left(\sum_{i=1}^n q_if(B_i)g(B_i)\right)&\geq\left(\sum_{j=1}^n p_jf(A_j)\right)\otimes\left(\sum_{i=1}^n q_ig(B_i)\right)\\&\quad+\left(\sum_{j=1}^n p_jg(A_j)\right)\otimes\left(\sum_{i=1}^n q_if(B_i)\right).\end{aligned}\tag{2.11}$$

Proof. We have from (2.1) that

$$[h(A_j)f(A_j)g(A_j)]\otimes k(B_i)+h(A_j)\otimes[k(B_i)f(B_i)g(B_i)]\geq[h(A_j)f(A_j)]\otimes[k(B_i)g(B_i)]+[h(A_j)g(A_j)]\otimes[k(B_i)f(B_i)]$$

for all $i,j\in\{1,\ldots,n\}$. If we multiply by $p_jq_i\geq 0$ and sum over $j,i\in\{1,\ldots,n\}$, then we get

$$\begin{aligned}&\sum_{j,i=1}^n p_jq_i[h(A_j)f(A_j)g(A_j)]\otimes k(B_i)+\sum_{j,i=1}^n p_jq_ip_jq_ih(A_j)\otimes[k(B_i)f(B_i)g(B_i)]\\&\geq\sum_{j,i=1}^n p_jq_i[h(A_j)f(A_j)]\otimes[k(B_i)g(B_i)]+\sum_{j,i=1}^n p_jq_i[h(A_j)g(A_j)]\otimes[k(B_i)f(B_i)]\end{aligned}$$

and by using the properties of tensorial product we derive (2.9). □

Remark 2.5. *If we take $B_i=A_i$ and $p_i=q_i$, $i\in\{1,\ldots,n\}$, then we get*

$$\begin{aligned}\left(\sum_{i=1}^n p_if(A_i)g(A_i)\right)\otimes 1+1\otimes\left(\sum_{i=1}^n p_if(A_i)g(A_i)\right)&\geq\left(\sum_{i=1}^n p_if(A_i)\right)\otimes\left(\sum_{i=1}^n p_ig(A_i)\right)\\&\quad+\left(\sum_{i=1}^n p_ig(A_i)\right)\otimes\left(\sum_{i=1}^n p_if(A_i)\right),\end{aligned}\tag{2.12}$$

where f, g are synchronous and continuous on I and A_i are selfadjoint with spectra $Sp(A_i)\subset I$, $p_i\geq 0$ for $i\in\{1,\ldots,n\}$ and $\sum_{i=1}^n p_i=1$. By (2.12) we also have the inequality for the Hadamard product

$$\left(\sum_{i=1}^n p_if(A_i)g(A_i)\right)\circ 1\geq\left(\sum_{i=1}^n p_if(A_i)\right)\circ\left(\sum_{i=1}^n p_ig(A_i)\right),\tag{2.13}$$

where f, g are synchronous and continuous on I and A_i are selfadjoint with spectra $Sp(A_i) \subset I$, $p_i \geq 0$ for $i \in \{1,\ldots,n\}$ and $\sum_{i=1}^{n} p_i = 1$.

We also have:

Theorem 2.6. *Let $f, g : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be continuous on $[m,M]$ and differentiable on (m,M) with $g'(t) \neq 0$ for $t \in (m,M)$. Assume that*

$$-\infty < \gamma = \inf_{t\in(m,M)} \frac{f'(t)}{g'(t)}, \quad \sup_{t\in(m,M)} \frac{f'(t)}{g'(t)} = \Gamma < \infty,$$

and A, B selfadjoint operators with spectra $Sp(A)$, $Sp(B) \subseteq [m,M]$, then for any continuous and nonnegative function h defined on $[m,M]$,

$$\begin{aligned} &\gamma\left[\left(h(A)g^2(A)\right)\otimes h(B) + h(A)\otimes\left(h(B)g^2(B)\right) \ -2\left(g(A)h(A)\right)\otimes\left(h(B)g(B)\right)\right] \\ &\leq [h(A)f(A)g(A)]\otimes h(B) + h(A)\otimes[h(B)f(B)g(B)] - [h(A)f(A)]\otimes[h(B)g(B)] - [h(A)g(A)]\otimes[h(B)f(B)] \\ &\leq \Gamma\left[\left(h(A)g^2(A)\right)\otimes h(B) + h(A)\otimes\left(h(B)g^2(B)\right) \ -2\left(g(A)h(A)\right)\otimes\left(h(B)g(B)\right)\right]. \end{aligned} \tag{2.14}$$

In particular,

$$\begin{aligned} \gamma\left[g^2(A)\otimes 1 + 1\otimes g^2(B) - 2g(A)\otimes g(B)\right] &\leq [f(A)g(A)]\otimes 1 + 1\otimes[f(B)g(B)] - f(A)\otimes g(B) - g(A)\otimes f(B) \\ &\leq \Gamma\left[g^2(A)\otimes 1 + 1\otimes g^2(B) - 2g(A)\otimes g(B)\right]. \end{aligned} \tag{2.15}$$

Proof. Using the Cauchy mean value theorem, for all t, $s \in [m,M]$ with $t \neq s$ there exists ξ between t and s such that

$$\frac{f(t)-f(s)}{g(t)-g(s)} = \frac{f'(\xi)}{g'(\xi)} \in [\gamma,\Gamma].$$

Therefore

$$\gamma[g(t)-g(s)]^2 \leq [f(t)-f(s)][g(t)-g(s)] \leq \Gamma[g(t)-g(s)]^2$$

for all t, $s \in [m,M]$, which is equivalent to

$$\gamma\left[g^2(t) - 2g(t)g(s) + g^2(s)\right] \leq f(t)g(t) + f(s)g(s) - f(t)g(s) - f(s)g(t) \leq \Gamma\left[g^2(t) - 2g(t)g(s) + g^2(s)\right]$$

for all t, $s \in [m,M]$. If we multiply by $h(t)h(s) \geq 0$, then we get

$$\begin{aligned} \gamma\left[h(t)g^2(t)h(s) - 2g(t)h(t)h(s)g(s) + h(t)h(s)g^2(s)\right] &\leq h(t)f(t)g(t)h(s) + h(t)h(s)f(s)g(s) \\ &\quad - h(t)f(t)h(s)g(s) - h(t)g(t)h(s)f(s) \\ &\leq \Gamma\left[h(t)g^2(t)h(s) - 2g(t)h(t)h(s)g(s) + h(t)h(s)g^2(s)\right] \end{aligned}$$

for all t, $s \in [m,M]$.

This implies that

$$\begin{aligned} &\gamma\int_m^M\int_m^M \left[h(t)g^2(t)h(s) - 2g(t)h(t)h(s)g(s) + h(t)h(s)g^2(s)\right] \times dE(t)\otimes dF(s) \\ &\leq \int_m^M\int_m^M [h(t)f(t)g(t)h(s) + h(t)h(s)f(s)g(s) \ -h(t)f(t)h(s)g(s) - h(t)g(t)h(s)f(s)]\,dE(t)\otimes dF(s) \\ &\leq \Gamma\int_m^M\int_m^M \left[h(t)g^2(t)h(s) - 2g(t)h(t)h(s)g(s) + h(t)h(s)g^2(s)\right] \times dE(t)\otimes dF(s) \end{aligned}$$

and by performing the calculations as in the proof of Theorem 2.1, we derive (2.14). □

Corollary 2.7. *With the assumptions of Theorem 2.6 we have*

$$\begin{aligned} &\gamma\left[h(B)\circ\left(h(A)g^2(A)\right) + h(A)\circ\left(h(B)g^2(B)\right) \ -2\left(g(A)h(A)\right)\circ\left(h(B)g(B)\right)\right] \\ &\leq h(B)\circ[h(A)f(A)g(A)] + h(A)\circ[h(B)f(B)g(B)] - [h(A)f(A)]\circ[h(B)g(B)] - [h(A)g(A)]\circ[h(B)f(B)] \\ &\leq \Gamma\left[h(B)\circ\left(h(A)g^2(A)\right) + h(A)\circ\left(h(B)g^2(B)\right) \ -2\left(g(A)h(A)\right)\circ\left(h(B)g(B)\right)\right]. \end{aligned} \tag{2.16}$$

In particular,

$$\gamma\left[\left[g^2(A)+g^2(B)\right]\circ 1-2g(A)\circ g(B)\right]\leq [f(A)g(A)+[f(B)g(B)]]\circ 1-f(A)\circ g(B)-g(A)\circ f(B) \leq \Gamma\left[\left[g^2(A)+g^2(B)\right]\circ 1-2g(A)\circ g(B)\right]. \tag{2.17}$$

We also have:

Corollary 2.8. *With the assumptions of Theorem 2.6 and if A_j are selfadjoint with spectra $Sp(A_j)\subset I$ and $p_j\geq 0,\ j\in\{1,\ldots,n\}$, with $\sum_{j=1}^n p_j=1$, then*

$$\begin{aligned}&\gamma\left\{\left(\sum_{i=1}^n p_i g^2(A_i)\right)\otimes 1+1\otimes\left(\sum_{i=1}^n p_i g^2(A_i)\right)-2\left(\sum_{i=1}^n p_i g(A_i)\right)\otimes\left(\sum_{i=1}^n p_i g(A_i)\right)\right\}\\&\leq\left(\sum_{i=1}^n p_i f(A_i)g(A_i)\right)\otimes 1+1\otimes\left(\sum_{i=1}^n p_i f(A_i)g(A_i)\right)-\left(\sum_{i=1}^n p_i f(A_i)\right)\otimes\left(\sum_{i=1}^n p_i g(A_i)\right)\\&-\left(\sum_{i=1}^n p_i g(A_i)\right)\otimes\left(\sum_{i=1}^n p_i f(A_i)\right)\\&\leq\Gamma\left\{\left(\sum_{i=1}^n p_i g^2(A_i)\right)\otimes 1+1\otimes\left(\sum_{i=1}^n p_i g^2(A_i)\right)-2\left(\sum_{i=1}^n p_i g(A_i)\right)\otimes\left(\sum_{i=1}^n p_i g(A_i)\right)\right\}.\end{aligned}\tag{2.18}$$

Also,

$$\begin{aligned}&\gamma\left[\left(\sum_{i=1}^n p_i g^2(A_i)\right)\circ 1-\left(\sum_{i=1}^n p_i g(A_i)\right)\circ\left(\sum_{i=1}^n p_i g(A_i)\right)\right]\\&\leq\left(\sum_{i=1}^n p_i f(A_i)g(A_i)\right)\circ 1-\left(\sum_{i=1}^n p_i f(A_i)\right)\circ\left(\sum_{i=1}^n p_i g(A_i)\right)\\&\leq\Gamma\left[\left(\sum_{i=1}^n p_i g^2(A_i)\right)\circ 1-\left(\sum_{i=1}^n p_i g(A_i)\right)\circ\left(\sum_{i=1}^n p_i g(A_i)\right)\right].\end{aligned}\tag{2.19}$$

Proof. From (2.15) we get

$$\begin{aligned}\gamma\left[g^2(A_i)\otimes 1+1\otimes g^2(A_j)-2g(A_i)\otimes g(A_j)\right]&\leq[f(A_i)g(A_i)]\otimes 1+1\otimes[f(A_j)g(A_j)]\\&\quad-f(A_i)\otimes g(A_j)-g(A_i)\otimes f(A_j)\\&\leq\Gamma\left[g^2(A_i)\otimes 1+1\otimes g^2(A_j)-2g(A_i)\otimes g(A_j)\right]\end{aligned}$$

for all $i,j\in\{1,\ldots,n\}$. If we multiply by $p_ip_j\geq 0$ and sum, then we get

$$\begin{aligned}\gamma\sum_{i,j=1}^n p_ip_j\left[g^2(A_i)\otimes 1+1\otimes g^2(A_j)-2g(A_i)\otimes g(A_j)\right]&\leq\sum_{i,j=1}^n p_ip_j\left\{[f(A_i)g(A_i)]\otimes 1+1\otimes[f(A_j)g(A_j)]\right.\\&\quad\left.-f(A_i)\otimes g(A_j)-g(A_i)\otimes f(A_j)\right\}\\&\leq\Gamma\sum_{i,j=1}^n p_ip_j\left[g^2(A_i)\otimes 1+1\otimes g^2(A_j)-2g(A_i)\otimes g(A_j)\right],\end{aligned}$$

which gives (2.18). □

3. Some Examples

Let either $p,q\in(0,\infty)$ or $p,q\in(-\infty,0)$ and $r\in\mathbb{R}$. If $A,\ B>0$, then from (2.4) we get

$$A^{r+p+q}\otimes B^r+A^r\otimes B^{r+p+q}\geq A^{r+p}\otimes B^{r+q}+A^{r+q}\otimes B^{r+p}, \tag{3.1}$$

while from (2.6) we obtain

$$A^{r+p+q}\circ B^r+A^r\circ B^{r+p+q}\geq A^{r+p}\circ B^{r+q}+A^{r+q}\circ B^{r+p}. \tag{3.2}$$

If one of the parameters p, q is in $(-\infty, 0)$ while the other in $(0, \infty)$, then the inequality reverses in (3.1) and (3.2).
If we take $q = p$, then we get

$$A^{r+2p} \otimes B^{r} + A^{r} \otimes B^{r+2p} \geq 2A^{r+p} \otimes B^{r+p}, \tag{3.3}$$

and

$$A^{r+2p} \circ B^{r} + A^{r} \circ B^{r+2p} \geq 2A^{r+p} \circ B^{r+p} \tag{3.4}$$

for $p, r \in \mathbb{R}$ and $A, B > 0$.
If we take $q = -p$, then we get

$$2A^{r} \otimes B^{r} \geq A^{r+p} \otimes B^{r-p} + A^{r-p} \otimes B^{r+p}, \tag{3.5}$$

while from (2.6) we obtain

$$2A^{r} \circ B^{r} \geq A^{r+p} \circ B^{r-p} + A^{r-p} \circ B^{r+p}, \tag{3.6}$$

for $p, r \in \mathbb{R}$ and $A, B > 0$.
Assume that $A_j > 0,\ p_j \geq 0,\ j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$, then by (2.12) we get

$$\left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \geq \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{q}\right) + \left(\sum_{i=1}^{n} p_i A_i^{q}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{p}\right), \tag{3.7}$$

if either $p, q \in (0, \infty)$ or $p, q \in (-\infty, 0)$. If one of the parameters p, q is in $(-\infty, 0)$ while the other in $(0, \infty)$, then the inequality reverses in (3.7). In particular, we derive

$$\left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \geq \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \tag{3.8}$$

and

$$2 \geq \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{-p}\right) + \left(\sum_{i=1}^{n} p_i A_i^{-p}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{p}\right). \tag{3.9}$$

From (2.13) we obtain

$$\left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \circ 1 \geq \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{q}\right), \tag{3.10}$$

if either $p, q \in (0, \infty)$ or $p, q \in (-\infty, 0)$. If one of the parameters p, q is in $(-\infty, 0)$ while the other in $(0, \infty)$, then the inequality reverses in (3.10). In particular, we have

$$\left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \circ 1 \geq \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \tag{3.11}$$

and

$$1 \geq \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{-p}\right), \tag{3.12}$$

for $p \in \mathbb{R}, A_j > 0,\ p_j \geq 0,\ j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$.
Consider the functions $f(t) = t^{p},\ g(t) = t^{q}$ defined on $(0, \infty)$. Then $f'(t) = pt^{p-1},\ g'(t) = qt^{q-1}$ for $t > 0$ and

$$\frac{f'(t)}{g'(t)} = \frac{p}{q} t^{p-q},\ t > 0.$$

Assume that either $p,q\in(0,\infty)$ or $p,q\in(-\infty,0)$. Then $\frac{p}{q}>0$ and $\frac{f'(t)}{g'(t)}$ is increasing for $p>q$ and decreasing for $p<q$ and constant 1 for $p=q$.
Assume that $0<m\le A,\,B\le M$, then

$$\inf_{t\in[m,M]}\frac{f'(t)}{g'(t)}=\frac{p}{q}m^{p-q}\text{ and }\sup_{t\in[m,M]}\frac{f'(t)}{g'(t)}=\frac{p}{q}M^{p-q}\text{ for }p>q$$

and

$$\inf_{t\in[m,M]}\frac{f'(t)}{g'(t)}=\frac{p}{q}M^{p-q}\text{ and }\sup_{t\in[m,M]}\frac{f'(t)}{g'(t)}=\frac{p}{q}m^{p-q}\text{ for }p<q.$$

Assume that either $p,q\in(0,\infty)$ or $p,q\in(-\infty,0)$ and $0<m\le A,\,B\le M$. From (2.15) we get for $p>q$ that

$$\begin{aligned}0&\le\frac{p}{q}m^{p-q}\left(A^{2q}\otimes 1+1\otimes B^{2q}-2A^{q}\otimes B^{q}\right)\\&\le A^{p+q}\otimes 1+1\otimes B^{p+q}-A^{p}\otimes B^{q}-A^{q}\otimes B^{p}\\&\le\frac{p}{q}M^{p-q}\left(A^{2q}\otimes 1+1\otimes B^{2q}-2A^{q}\otimes B^{q}\right)\end{aligned}\tag{3.13}$$

and for $p<q$

$$\begin{aligned}0&\le\frac{p}{q}M^{p-q}\left(A^{2q}\otimes 1+1\otimes B^{2q}-2A^{q}\otimes B^{q}\right)\\&\le A^{p+q}\otimes 1+1\otimes B^{p+q}-A^{p}\otimes B^{q}-A^{q}\otimes B^{p}\\&\le\frac{p}{q}m^{p-q}\left(A^{2q}\otimes 1+1\otimes B^{2q}-2A^{q}\otimes B^{q}\right).\end{aligned}\tag{3.14}$$

From (2.17) we also have the inequalities for the Hadamard product for $p>q$ that

$$\begin{aligned}0&\le\frac{p}{q}m^{p-q}\left(\left(A^{2q}+B^{2q}\right)\circ 1-2A^{q}\circ B^{q}\right)\\&\le\left(A^{p+q}+B^{p+q}\right)\circ 1-A^{p}\circ B^{q}-A^{q}\circ B^{p}\\&\le\frac{p}{q}M^{p-q}\left(\left(A^{2q}+B^{2q}\right)\circ 1-2A^{q}\circ B^{q}\right)\end{aligned}\tag{3.15}$$

and for $p<q$

$$\begin{aligned}0&\le\frac{p}{q}M^{p-q}\left(\left(A^{2q}+B^{2q}\right)\circ 1-2A^{q}\circ B^{q}\right)\\&\le\left(A^{p+q}+B^{p+q}\right)\circ 1-A^{p}\circ B^{q}-A^{q}\circ B^{p}\\&\le\frac{p}{q}m^{p-q}\left(\left(A^{2q}+B^{2q}\right)\circ 1-2A^{q}\circ B^{q}\right).\end{aligned}\tag{3.16}$$

Assume that either $p,q\in(0,\infty)$ or $p,q\in(-\infty,0)$ and $0<m\le A_j\le M$, $p_j\ge 0$, $j\in\{1,\dots,n\}$ with $\sum_{j=1}^{n}p_j=1$. By (2.18) we get for $p>q$

$$\begin{aligned}0&\le\frac{p}{q}m^{p-q}\left\{\left(\sum_{i=1}^{n}p_iA_i^{2q}\right)\otimes 1+1\otimes\left(\sum_{i=1}^{n}p_iA_i^{2q}\right)-2\left(\sum_{i=1}^{n}p_iA_i^{q}\right)\otimes\left(\sum_{i=1}^{n}p_iA_i^{q}\right)\right\}\\&\le\left(\sum_{i=1}^{n}p_iA_i^{p+q}\right)\otimes 1+1\otimes\left(\sum_{i=1}^{n}p_iA_i^{p+q}\right)-\left(\sum_{i=1}^{n}p_iA_i^{p}\right)\otimes\left(\sum_{i=1}^{n}p_iA_i^{q}\right)-\left(\sum_{i=1}^{n}p_iA_i^{q}\right)\otimes\left(\sum_{i=1}^{n}p_iA_i^{p}\right)\\&\le\frac{p}{q}M^{p-q}\left\{\left(\sum_{i=1}^{n}p_iA_i^{2q}\right)\otimes 1+1\otimes\left(\sum_{i=1}^{n}p_iA_i^{2q}\right)-2\left(\sum_{i=1}^{n}p_iA_i^{q}\right)\otimes\left(\sum_{i=1}^{n}p_iA_i^{q}\right)\right\}\end{aligned}\tag{3.17}$$

and for $p < q$

$$0 \le \frac{p}{q} M^{p-q} \left\{ \left(\sum_{i=1}^{n} p_i A_i^{2q} \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i A_i^{2q} \right) - 2 \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \right\}$$
$$\le \left(\sum_{i=1}^{n} p_i A_i^{p+q} \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i A_i^{p+q} \right) - \left(\sum_{i=1}^{n} p_i A_i^{p} \right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{q} \right) - \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{p} \right) \tag{3.18}$$
$$\le \frac{p}{q} m^{p-q} \left\{ \left(\sum_{i=1}^{n} p_i A_i^{2q} \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i A_i^{2q} \right) - 2 \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \right\}.$$

Also, by (2.19) we get for $p > q$

$$0 \le \frac{p}{q} m^{p-q} \left[\left(\sum_{i=1}^{n} p_i A_i^{2q} \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \right]$$
$$\le \left(\sum_{i=1}^{n} p_i A_i^{p+q} \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{p} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \tag{3.19}$$
$$\le \frac{p}{q} M^{p-q} \left[\left(\sum_{i=1}^{n} p_i A_i^{2q} \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \right],$$

while for $p < q$

$$0 \le \frac{p}{q} M^{p-q} \left[\left(\sum_{i=1}^{n} p_i A_i^{2q} \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \right]$$
$$\le \left(\sum_{i=1}^{n} p_i A_i^{p+q} \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{p} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \tag{3.20}$$
$$\le \frac{p}{q} m^{p-q} \left[\left(\sum_{i=1}^{n} p_i A_i^{2q} \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \right].$$

Consider the exponential functions $f(t) = \exp(\alpha t)$, $g(t) = \exp(\beta t)$ with $\alpha, \beta \in \mathbb{R}$. If $\alpha\beta > 0$ then the functions have the same monotonicity. If $\alpha\beta < 0$ they have different monotonicity.
If $\alpha\beta > 0$ and A, B are selfadjoint operators, then by (2.5) we get

$$\exp[(\alpha+\beta)A] \otimes 1 + 1 \otimes \exp[(\alpha+\beta)B] \ge \exp(\alpha A) \otimes \exp(\beta B) + \exp(\beta A) \otimes \exp(\alpha B), \tag{3.21}$$

and

$$\exp[(\alpha+\beta)A] \circ 1 + 1 \circ \exp[(\alpha+\beta)B] \ge \exp(\alpha A) \circ \exp(\beta B) + \exp(\beta A) \circ \exp(\alpha B). \tag{3.22}$$

If $\alpha\beta < 0$, then the reverse inequality holds in (3.21) and (3.22).
If we take $f(t) = t^p$ and $g(t) = \ln t$, we also have the logarithmic inequalities

$$(A^p \ln A) \otimes 1 + 1 \otimes (B^p \ln B) \ge A^p \otimes \ln B + \ln A \otimes B^p, \tag{3.23}$$

and

$$(A^p \ln A + B^p \ln B) \circ 1 \ge A^p \circ \ln B + \ln A \circ B^p, \tag{3.24}$$

for A, $B > 0$ and $p > 0$. If $p < 0$, then the inequality reverses in (3.23) and (3.24).

Article Information

Acknowledgements: The author would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: The article has a single author. The author has read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private, or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism was detected.

Availability of data and materials: Not applicable.

References

[1] H. Araki, F. Hansen, *Jensen's operator inequality for functions of several variables*, Proc. Amer. Math. Soc., **128** (7) (2000), 2075-2084.

[2] A. Korányi, *On some classes of analytic functions of several variables*, Trans. Amer. Math. Soc., **101** (1961), 520-554.

[3] T. Furuta, J. Mićić Hot, J. Pečarić, Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[4] S. Wada, *On some refinement of the Cauchy-Schwarz inequality*, Lin. Alg. & Appl., **420** (2007), 433-440.

[5] J. I. Fujii, *The Marcus-Khan theorem for Hilbert space operators*, *Math. Jpn.*, **41** (1995), 531-535.

[6] T. Ando, *Concavity of certain maps on positive definite matrices and applications to Hadamard products*, Lin. Alg. & Appl., **26** (1979), 203-241.

[7] J. S. Aujila, H. L. Vasudeva, *Inequalities involving Hadamard product and operator means*, Math. Japon., **42** (1995), 265-272.

[8] K. Kitamura, Y. Seo, *Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities*, *Scient. Math.* **1**(2) (1998), 237-241.

[9] P. Bhunia, K. Paul, A. Sen, *Numerical radius inequalities for tensor product of operators*, Proc. Indian Acad. Sci. (Math. Sci.), **133**(3) (2023).

[10] H. L. Gau, K. Z. Wang, P. Y. Wu, *Numerical radii for tensor products of operators*, Integr. Equ. Oper. Theory, **78** (2014), 375–382.

Communications in Advanced Mathematical Sciences
Vol. 6, No. 4, 188-195, 2023
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1363596

On Lacunary $\mathscr{I}_2^*$-Convergence and Lacunary $\mathscr{I}_2^*$-Cauchy Sequence

Nimet Akın[1], Erdinç Dündar[2]*

Abstract
In the study conducted here, we have given some new concepts in summability. In this sense, firstly, we have given the concept of lacunary $\mathscr{I}_2^*$-convergence and we have investigated the relations between lacunary $\mathscr{I}_2$-convergence and lacunary $\mathscr{I}_2^*$-convergence. Also, we have given the concept of lacunary $\mathscr{I}_2^*$-Cauchy sequence and investigated the relations between lacunary $\mathscr{I}_2$-Cauchy sequence and lacunary $\mathscr{I}_2^*$-Cauchy sequence.

Keywords: Double sequence, Ideal convergence, Ideal Cauchy sequence, Lacunary sequence
2010 AMS:40A05, 40A35

[1] *Department of Mathematics Teaching, Afyon Kocatepe University, Afyonkarahisar, Türkiye, npancaroglu@aku.edu.tr, ORCID: 0000-0003-2886-3679*
[2] *Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Türkiye, edundar@aku.edu.tr, ORCID: 0000-0002-0545-7486*
***Corresponding author**
Received: 20 September 2023, **Accepted:** 1 November 2023, **Available online:** 7 November 2023
How to cite this article: N. Akın, E. Dündar, *On Lacunary $\mathscr{I}_2^*$-Convergence and Lacunary $\mathscr{I}_2^*$-Cauchy Sequence*, Commun. Adv. Math. Sci., 6(4) (2023), 188-195.

1. Introduction and Definitions

During this study, we take $\mathbb{N}$ as the set of all positive integers and $\mathbb{R}$ as the set of all real numbers. The convergence in sequences of real numbers is generalized to the concept of statistical convergence by Fast [1] and Schoenberg [2], independently. The concept of ideal convergence, which is a generalization of statistical convergence that would later inspire many researchers, was first defined by Kostyrko et al. [3]. Nabiev [4] studied on $\mathscr{I}$-Cauchy sequence and $\mathscr{I}^*$-Cauchy sequence with some characteristics. Using the ideal notion, ideal-statistical convergence and ideal lacunary statistical convergence were introduced by Das et al.[5] as new notions. In the topology induced by random n-normed spaces, the lacunary ideal convergence and lacunary ideal Cauchy with some important characteristics investigated by Yamanci and Gürdal [6]. The lacunary ideal convergence was studied by Debnath [7] in intuitionistic fuzzy normed linear spaces. The ideal lacunary convergence was introduced by Tripathy et al.[8]. In recent times, the concepts of the lacunary $\mathscr{I}^*$-convergence, strongly lacunary $\mathscr{I}^*$-convergence, lacunary $\mathscr{I}^*$-Cauchy sequence and strongly lacunary $\mathscr{I}^*$-Cauchy sequence were introduced by Akın and Dündar [9, 10]. Das et al. [11] studied $\mathscr{I}$ and $\mathscr{I}^*$-convergence for double sequences. Dündar and Altay [12, 13] introduced $\mathscr{I}_2$-ideal convergence and ideal Cauchy double sequences in the linear metric space and they investigated some characteristics and between relations. Dündar et al. [14] studied strongly $\mathscr{I}_2$-lacunary convergence and $\mathscr{I}_2$lacunary Cauchy double sequences of sets. Hazarika [15] studied the lacunary ideal convergence for double sequences.

In recently, the notions of convergence, statistical convergence and ideal convergence in some metric spaces and normed spaces were studied in summability theory by a lot of mathematicians. In the study conducted here, we defined the lacunary $\mathscr{I}_2^*$-convergence. We investigate the connections between lacunary $\mathscr{I}_2$-convergence and lacunary $\mathscr{I}_2^*$-convergence. Also, we defined the concept of lacunary $\mathscr{I}_2^*$-Cauchy sequence and investigate the relations between lacunary $\mathscr{I}_2$-Cauchy sequence and lacunary $\mathscr{I}_2^*$-Cauchy sequence.

Some basic definitions, concepts and characteristics that will be used throughout the study and are available in the literature will now be noted (see [3, 4], [6]-[10], [12, 13], [16]-[20])

For $\mathscr{I} \subseteq 2^{\mathbb{N}}$, if the following propositions
(i) $\emptyset \in \mathscr{I}$, (ii) If $G, H \in \mathscr{I}$, then $G \cup H \in \mathscr{I}$, (iii) If $G \in \mathscr{I}$ and $H \subseteq G$, then $H \in \mathscr{I}$
hold, then $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is named an ideal.

If $\mathbb{N} \notin \mathscr{I}$, then $\mathscr{I}$ is named a non-trivial ideal. Also, if $\{k\} \in \mathscr{I}$ for each $k \in \mathbb{N}$, then a non-trivial ideal is named an admissible ideal.

For $\mathscr{F} \subseteq 2^{\mathbb{N}}$, if the following propositions
(i) $\emptyset \notin \mathscr{F}$, (ii) If $G, H \in F$, then $G \cap H \in \mathscr{F}$, (iii) If $G \in \mathscr{F}$ and $H \supseteq G$, then $H \in \mathscr{F}$
hold, then $\mathscr{F} \subseteq 2^{\mathbb{N}}$ is named a filter.

For a non-trivial ideal $\mathscr{I} \subseteq 2^{\mathbb{N}}$

$$\mathscr{F}(\mathscr{I}) = \{G \subset \mathbb{N} : (\exists H \in \mathscr{I})(G = \mathbb{N} \backslash H)\}$$

is named the filter associated with $\mathscr{I}$.

By a lacunary sequence $\theta = \{k_r\}$, we mean an increasing integer sequence such that

$$k_0 = 0 \text{ and } h_r = k_r - k_{r-1} \to \infty, \text{ as } r \to \infty.$$

During this study, the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Then after this, we take $\theta = \{k_r\}$ be a lacunary sequence and $\mathscr{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal.

A sequence $(x_k) \subset \mathbb{R}$ is lacunary convergent to $\ell \in \mathbb{R}$, if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} x_k = \ell.$$

A sequence $(x_k) \subset \mathbb{R}$ is lacunary Cauchy sequence if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k,p \in I_r} (x_k - x_p) = 0.$$

If for each $\varepsilon > 0$

$$\left\{ r \in \mathbb{N} : \left| \frac{1}{h_r} \sum_{k \in I_r} x_k - \ell \right| \geq \varepsilon \right\} \in \mathscr{I}$$

holds, then the sequence $(x_k) \subset \mathbb{R}$ is lacunary $\mathscr{I}$-convergent to $\ell \in \mathbb{R}$ and we write $x_k \to \ell[\mathscr{I}_\theta]$.

A sequence $(x_k) \subset \mathbb{R}$ is lacunary $\mathscr{I}$-Cauchy if for every $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon)$ such that

$$\left\{ r \in \mathbb{N} : \left| \frac{1}{h_r} \sum_{k \in I_r} (x_k - x_{N_0}) \right| \geq \varepsilon \right\} \in \mathscr{I}.$$

A sequence $(x_k) \subset \mathbb{R}$ is lacunary $\mathscr{I}^*$-convergent to $\ell \in \mathbb{R}$ iff there exists any set $G = \{g_1 < g_2 < \cdots < g_k < \cdots\} \subset \mathbb{N}$ such that for the set $G' = \{r \in \mathbb{N} : g_k \in I_r\} \in \mathscr{F}(\mathscr{I})$, we have

$$\lim_{\substack{r\to\infty \\ (r \in G')}} \frac{1}{h_r} \sum_{k \in I_r} x_{g_k} = \ell$$

and we write $x_k \to \ell(\mathscr{I}_\theta^*)$.

A sequence $(x_k) \subset \mathbb{R}$ is lacunary $\mathscr{I}^*$-Cauchy sequence iff there exists any set $G = \{g_1 < g_2 < \cdots < g_k < \cdots\} \subset \mathbb{N}$ such that for the set $G' = \{r \in \mathbb{N} : g_k \in I_r\} \in \mathscr{F}(\mathscr{I})$, we have

$$\lim_{\substack{r\to\infty \\ (r \in G')}} \frac{1}{h_r} \sum_{k,p \in I_r} (x_{g_k} - x_{g_p}) = 0.$$

For a double sequence $\theta = \{(k_r, j_u)\}$, if there exist two increasing sequence of integers such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty \text{ and } j_0 = 0,\ \bar{h}_u = j_u - j_{u-1} \to \infty \text{ as } r,u \to \infty,$$

then $\theta = \{(k_r, j_u)\}$ is named a double lacunary sequence. We take the following screenings for double lacunary sequence:

$$k_{ru} = k_r j_u,\ h_{ru} = h_r \bar{h}_u,\ I_{ru} = \{(k,j) : k_{r-1} < k \leq k_r \text{ and } j_{u-1} < j \leq j_u\},\ \ q_r = \frac{k_r}{k_{r-1}} \text{ and } q_u = \frac{j_u}{j_{u-1}}.$$

Then after this, we think $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$ as a non-trivial admissible ideal.

For each $k \in \mathbb{N}$ and a non-trivial ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$, if $\{k\} \times \mathbb{N} \in \mathscr{I}_2$ and $\mathbb{N} \times \{k\} \in \mathscr{I}_2$, then we say that $\mathscr{I}_2$ is named strongly admissible ideal.

If $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$ is a strongly admissible ideal, then clearly $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$ is an admissible ideal.

Let $\mathscr{I}_2^0 = \{A \subset \mathbb{N}^2 : \big(\exists m(A) \in \mathbb{N}\big)\big(i,j \geq m(A) \Rightarrow (i,j) \notin A\big)\}$. Then, $\mathscr{I}_2^0$ is a non-trivial strongly admissible ideal and clearly $\mathscr{I}_2$ is a strongly admissible if and only if $\mathscr{I}_2^0 \subset \mathscr{I}_2$.

There is a filter $\mathscr{F}(\mathscr{I}_2)$ corresponding with $\mathscr{I}_2$ such that

$$\mathscr{F}(\mathscr{I}_2) = \left\{G \subset \mathbb{N}^2 : (\exists H \in \mathscr{I}_2)(G = \mathbb{N}^2 \backslash H)\right\}.$$

An admissible ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{G_1, G_2, ...\} \in \mathscr{I}_2$, there exists a countable family of sets $\{H_1, H_2, ...\}$ such that $G_k \Delta H_k \in \mathscr{I}_2^0$, i.e., $G_k \Delta H_k$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $k \in \mathbb{N}$ and $H = \bigcup_{k=1}^{\infty} H_k \in \mathscr{I}_2$ (hence $H_k \in \mathscr{I}_2$ for each $k \in \mathbb{N}$).

If for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - \ell| < \varepsilon$ whenever $k, j > n_\varepsilon$, then the double sequence $x = (x_{kj}) \subset \mathbb{R}$ is convergent to $\ell \in \mathbb{R}$ and denoted with

$$\lim_{k,j\to\infty} x_{kj} = \ell \ \text{ or } \ \lim_{k,j\to\infty} x_{kj} = \ell.$$

Then after this, we take $\theta = \{(k_r, j_u)\}$ as a double lacunary sequence and $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$ as a strongly admissible ideal.

For a double sequence $(x_{kj}) \subset \mathbb{R}$, if

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} = \ell$$

hold, then (x_{kj}) is lacunary convergent to $\ell \in \mathbb{R}$.

For a double sequence $(x_{kj}) \subset \mathbb{R}$, if

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) = 0$$

hold, then (x_{kj}) is lacunary Cauchy double sequence.

For a double sequence $(x_{kj}) \subset \mathbb{R}$, if for every $\varepsilon > 0$

$$\left\{(r,u) \in \mathbb{N}^2 : \left|\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell\right| \geq \varepsilon\right\} \in \mathscr{I}_2$$

hold, then (x_{kj}) is lacunary $\mathscr{I}_2$-convergent to $\ell \in \mathbb{R}$ and denoted with $x_{kj} \to \ell(\mathscr{I}_{\theta_2})$.

If for every $\varepsilon > 0$ there exist $N = N(\varepsilon)$ and $S = S(\varepsilon)$

$$\left\{(r,u) \in \mathbb{N}^2 : \left|\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} (x_{kj} - x_{NS})\right| \geq \varepsilon\right\} \in \mathscr{I}_2$$

hold, then (x_{kj}) is lacunary $\mathscr{I}_2$-Cauchy double sequence.

Lemma 1.1. *[12] Let $\{P_k\}_{k=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N}^2$ such that $P_k \in F(\mathscr{I}_2)$ for each k, where $\mathscr{F}(\mathscr{I}_2)$ is a filter associate with a strongly admissible ideal $\mathscr{I}_2$ by (AP2). Therefore, there exists a set $P \subset \mathbb{N}^2$ such that $P \in \mathscr{F}(\mathscr{I}_2)$ and the set $P \backslash P_k$ is finite for all k.*

2. Main Results

For double sequences, we first defined lacunary $\mathscr{I}_2^*$-convergence and gave theorems examining its relationship with lacunary $\mathscr{I}_2$-convergence.

Definition 2.1. *A double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2^*$-convergent to $\ell \in \mathbb{R}$ iff there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$, we have*

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} = \ell$$

and so we can write $x_{kj} \to \ell(\mathscr{I}_{\theta_2}^*)$.

Theorem 2.2. *If the double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2^*$-convergent to $\ell \in \mathbb{R}$, then it is lacunary $\mathscr{I}_2$-convergent to $\ell \in \mathbb{R}$.*

Proof. Let $x_{kj} \to L(\mathscr{I}_{\theta_2}^*)$. Then, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$ $(i.e., H = \mathbb{N}^2 \backslash G' \in \mathscr{I}_2)$ and for every $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon) \in \mathbb{N}$ such that for all $r,u > r_0$ we have

$$\left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| < \varepsilon, \ ((r,u) \in G').$$

Then,

$$A(\varepsilon) = \left\{ (r,u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| \geq \varepsilon \right\} \subset H \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\}) \right) \right].$$

Since $\mathscr{I}_2$ is a strongly admissible ideal, we have

$$H \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\}) \right) \right] \in \mathscr{I}_2$$

and so $A(\varepsilon) \in \mathscr{I}_2$. Hence, $x_{kj} \to \ell(\mathscr{I}_{\theta_2})$. □

Theorem 2.3. *Let $\mathscr{I}_2$ be a strongly admissible ideal by $(AP2)$. If the double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2$-convergent to $\ell \in \mathbb{R}$, then it is lacunary $\mathscr{I}_2^*$-convergent to $\ell \in \mathbb{R}$.*

Proof. Assume that $x_{kj} \to \ell(\mathscr{I}_{\theta_2})$. Then for each $\varepsilon > 0$,

$$T(\varepsilon) = \left\{ (r,u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| \geq \varepsilon \right\} \in \mathscr{I}_2.$$

Put

$$T_1 = \left\{ (r,u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| \geq 1 \right\}$$

and

$$T_p = \left\{ (r,u) \in \mathbb{N}^2 : \frac{1}{p} \leq \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| < \frac{1}{p-1} \right\},$$

for $p \geq 2$ and $p \in \mathbb{N}$. It is clear that $T_i \cap T_j = \emptyset$ for $i \neq j$ and $T_i \in \mathscr{I}_2$ for each $i \in \mathbb{N}$. By property $(AP2)$, there is a sequence $\{V_p\}_{p\in\mathbb{N}}$ such that $T_j \Delta V_j$ is included in finite union of rows and columns in $\mathbb{N}^2$ for each $j \in \mathbb{N}$ and

$$V = \bigcup_{j=1}^{\infty} V_j \in \mathscr{I}_2.$$

We prove that,

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} = \ell,$$

for $G' = \mathbb{N}^2 \backslash V \in \mathscr{F}(\mathscr{I}_2)$. Take $\delta > 0$. Select $q \in \mathbb{N}$ such that $\frac{1}{q} < \delta$. Therefore,

$$\left\{ (r,u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| \geq \delta \right\} \subset \bigcup_{j=1}^{q} T_j.$$

Since $T_j \Delta V_j$ is a finite set for $j \in \{1,2,\cdots,q\}$, there exists $r_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{q} T_j \right) \cap \left\{ (r,u) \in \mathbb{N}^2 : r \geq r_0 \wedge u \geq r_0 \right\} = \left(\bigcup_{j=1}^{q} V_j \right) \cap \left\{ (r,u) \in \mathbb{N}^2 : r \geq r_0 \wedge u \geq r_0 \right\}.$$

If $r,u \geq r_0$ and $(r,u) \notin V$, then

$$(r,u) \notin \bigcup_{j=1}^{q} V_j \text{ and so } (r,u) \notin \bigcup_{j=1}^{q} T_j.$$

We have

$$\left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| < \frac{1}{q} < \delta.$$

This implies that

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} = \ell,$$

Hence, we have $x_{kj} \to \ell(\mathscr{I}_{\theta_2}^*)$. This completes the proof. □

Now, for double sequences, we have defined lacunary $\mathscr{I}_2^*$-Cauchy sequence and given theorems examining its relationship with lacunary $\mathscr{I}_2$-Cauchy sequence.

Definition 2.4. *The double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2^*$-Cauchy sequence iff there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$, we have*

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) = 0.$$

Theorem 2.5. *If the double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2^*$-Cauchy sequence, then (x_{kj}) is lacunary $\mathscr{I}_2$-Cauchy sequence.*

Proof. Let $(x_{kj}) \subset \mathbb{R}$ is a lacunary $\mathscr{I}_2^*$-Cauchy double sequence. Then, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$ and for every $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon)$ such that

$$\left| \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) \right| < \varepsilon, \quad ((r,u) \in G')$$

for all $r,u > r_0$. Now, let $H = \mathbb{N} \setminus G'$. It is clear that $H \in \mathscr{I}_2$. Then, for $(r,u) \in G'$

$$A(\varepsilon) = \left\{ (r,u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) \right| \geq \varepsilon \right\} \subset H \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\}) \right) \right].$$

Since $\mathscr{I}_2$ is an admissible ideal, we have

$$H \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\}) \right) \right] \in \mathscr{I}_2$$

and so $A(\varepsilon) \in \mathscr{I}_2$. Hence, (x_{kj}) is lacunary $\mathscr{I}_2$-Cauchy double sequence. □

Theorem 2.6. *Let $\mathscr{I}_2$ be a strongly admissible ideal by $(AP2)$. If the double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2$-Cauchy double sequence, then (x_{kj}) is lacunary $\mathscr{I}_2^*$-Cauchy double sequence.*

Proof. Assume that (x_{kj}) is lacunary $\mathscr{I}_2$-Cauchy sequence. Then, for each $\varepsilon > 0$ there exist $N = N(\varepsilon)$ and $S = S(\varepsilon)$ such that

$$A(\varepsilon) = \left\{ (r,u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} (x_{kj} - x_{NS}) \right| \geq \varepsilon \right\} \in \mathscr{I}_2.$$

Let

$$P_i = \left\{ (r,u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} (x_{kj} - x_{s_i t_i}) \right| \geq \frac{1}{i} \right\},\ i = 1,2,\ldots,$$

where $s_i = N\left(\frac{1}{i}\right)$ and $t_i = S\left(\frac{1}{i}\right)$. It is clear that $P_i \in \mathscr{F}(\mathscr{I}_2)$ for $i = 1,2,\cdots$. Using the Lemma 1.1, since $\mathscr{I}_2$ has the $(AP2)$ so there exists a set $P \subset \mathbb{N}^2$ such that $P \in \mathscr{F}(\mathscr{I}_2)$ and $P \setminus P_i$ is finite for all i. At the moment, we demonstrate that

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in P)}} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) = 0.$$

For prove this let $\varepsilon > 0$, $m \in \mathbb{N}$ such that $m > \frac{2}{\varepsilon}$. If $(r,u) \in P$ then $P \setminus P_m$ is a finite set, so there exists $r_0 = r_0(m)$ such that $(r,u) \in P_m$ for all $r,u > r_0(m)$. Therefore, for all $r,u > r_0(m)$

$$\left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} (x_{kj} - x_{s_m t_m}) \right| < \frac{1}{m}$$

and

$$\left| \frac{1}{h_{ru}} \sum_{(s,t)\in I_{ru}} (x_{st} - x_{s_m t_m}) \right| < \frac{1}{m}.$$

Hence, for all $r,u > r_0(m)$ it follows that

$$\begin{aligned} \left| \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) \right| &\leq \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} (x_{kj} - x_{s_m t_m}) \right| + \left| \frac{1}{h_{ru}} \sum_{(s,t)\in I_{ru}} (x_{st} - x_{s_m t_m}) \right| \\ &< \frac{1}{m} + \frac{1}{m} < \varepsilon. \end{aligned}$$

Therefore, for any $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon)$ such that for $r,u > r_0(\varepsilon)$ and $(r,u) \in P \in \mathscr{F}(\mathscr{I}_2)$

$$\left| \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) \right| < \varepsilon.$$

This demonstrates that (x_{kj}) is lacunary $\mathscr{I}_2^*$-Cauchy double sequence. □

Theorem 2.7. *If the double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2^*$-convergent to $\ell \in \mathbb{R}$, so (x_{kj}) is lacunary $\mathscr{I}_2$-Cauchy double sequence.*

Proof. Let $x_{kj} \to \ell(\mathscr{I}_{\theta_2}^*)$. So, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$, we have

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| = 0.$$

It shows that there exist $r_0 = r_0(\varepsilon)$ such that

$$\left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| < \frac{\varepsilon}{2},\ ((r,u) \in G')$$

for every $\varepsilon > 0$ and all $r,u > r_0$. Since

$$\begin{aligned}\left|\frac{1}{h_{ru}}\sum_{(k,j),(s,t)\in I_{ru}}(x_{kj}-x_{st})\right| &\leq \left|\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}x_{kj}-\ell\right|+\left|\frac{1}{h_{ru}}\sum_{(s,t)\in I_{ru}}x_{st}-\ell\right| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\ ((r,u)\in G')\end{aligned}$$

for all $r,u > r_0$. Hence, we have

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}}\frac{1}{h_{ru}}\sum_{(k,j),(s,t)\in I_{ru}}(x_{kj}-x_{st})=0.$$

That is, (x_{kj}) is a lacunary $\mathscr{I}_2^*$-Cauchy double sequence. Therefore, by Theorem 2.5, (x_{kj}) is a lacunary $\mathscr{I}_2$-Cauchy double sequence. □

3. Conclusion

In summability theory, the notions of classical convergence, statistical and ideal convergence in some metric spaces and normed spaces were studied by a lot of mathematicians in recently. For double sequences, we investigated the lacunary $\mathscr{I}^*$-convergence and lacunary $\mathscr{I}^*$-Cauchy sequence in $\mathbb{R}$. In the future, for double sequences, the notions of strongly lacunary $\mathscr{I}^*$-convergence and strongly lacunary $\mathscr{I}^*$-Cauchy sequence in $\mathbb{R}$ are defined.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Authors contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2**(1951), 241–244.

[2] I.J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66**(1959), 361–375.

[3] P. Kostyrko, T. Šalat, W. Wilczynski, *$\mathscr{I}$-Convergence*, Real Anal. Exchange, **26**(2)(2000), 669–686.

[4] A. Nabiev, S. Pehlivan, M. Gürdal, *On $\mathscr{I}$-Cauchy sequence*, Taiwanese J. Math. **11**(2)(2007), 569–576.

[5] P. Das, E. Savaş, S.Kr. Ghosal, *On generalized of certain summability methods using ideals*, Appl. Math. Letter, **36**(2011), 1509–1514.

[6] U. Yamancı, M. Gürdal, *On lacunary ideal convergence in random n-normed space*, J. Math., Volume 2013, Article ID 868457, 8 pages http://dx.doi.org/10.1155/2013/868457

[7] P. Debnath, *Lacunary ideal convergence in intuitionistic fuzzy normed linear spaces*, Comput. Math. Appl., **63**(2012), 708–715.

[8] B.C. Tripathy, B. Hazarika, B. Choudhary, *Lacunary $\mathscr{I}$-convergent sequences*, Kyungpook Math. J. **52**(2012), 473–482.

[9] N.P. Akın, E. Dündar, Ş. Yalvaç, *Lacunary $\mathscr{I}^*$-convergence and lacunary $\mathscr{I}^*$-Cauchy sequence*, AKU J. Sci. Eng (in press).

[10] N.P. Akın, E. Dündar, *Strongly lacunary $\mathscr{I}^*$-convergence and strongly lacunary $\mathscr{I}^*$-Cauchy sequence*, Math. Sci. Appl. E-Notes, (in press).

[11] P. Das, P. Kostyrko, W. Wilczynski, P. Malik,*$\mathscr{I}$ and $\mathscr{I}^*$-convergence of double sequences*, Math. Slovaca, **58**(5)(2008), 605–620.

[12] E. Dündar, B. Altay, *$\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy of double sequences*, Acta Math. Sci., **34**B(2)(2014), 343–353.

[13] E. Dündar, B. Altay, On some properties of $\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy of double sequences, Gen. Math. Notes, **7**(1)(2011) 1–12.

[14] E. Dündar, U. Ulusu, N. Pancaroğlu, *Strongly $\mathscr{I}_2$-lacunary convergence and $\mathscr{I}_2$-lacunary Cauchy double sequences of sets*, Aligarh Bull. Math., **35**(1-2)(2016), 1–15.

[15] B. Hazarika, *Lacunary ideal convergence of multiple sequences*, J. Egyptian Math. Soc., **24**(2016), 54–59.

[16] E. Dündar, U. Ulusu, B. Aydın, *$\mathscr{I}_2$-lacunary statistical convergence of double sequences of sets*, Konuralp J. Math., **5**(1)(2017), 1–10.

[17] A.R. Freedman, J.J. Sember, M. Raphael, *Some Cesàro type summability spaces*, Proc. Lond. Math. Soc. **37**(1978), 508–520.

[18] Y. Sever, U. Ulusu, E. Dündar, *On strongly $\mathscr{I}$ and $\mathscr{I}^*$-lacunary convergence of sequences of sets*, AIP Conf. Proc., **1611**(357)(2014); doi: 10.1063/1.4893860, 7 pages.

[19] U. Ulusu, F. Nuray, *On strongly lacunary summability of sequences of sets*, J. Appl. Math. Inform., **3**(3)(2013), 75–88.

[20] U. Ulusu, E. Dündar, *$\mathscr{I}$-lacunary statistical convergence of sequences of sets*, Filomat, **28**(8)(2014), 1567–1574.

Communications in Advanced Mathematical Sciences
Vol. 6, No. 4, 196-210, 2023
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1365151

Quantify the Impact of Non-Response and Measurement Error of Sensitive Variable(s) under Two-Phase Sampling employing ORRT Models

Sunil Kumar[1], Sanam Preet Kour[2]*

Abstract
Throughout this article, a two-phase sampling (TPS) technique is employed to estimate the population mean of the sensitive variable. The current article endeavours to develop a chain ratio type estimator for the estimation of sensitive variable(s) in the presence of non-response and measurement error simultaneously by utilizing ORRT models under a two-phase sampling technique. The significant aspects associated with the suggested estimator characterized by bias and mean squared error have been evaluated. Besides this, the utterance for the minimum mean squared error for the optimal values has also been identified. The supremacy of the proposed estimator has been compared with the modified existing estimators under the TPS scheme by using two sensitive auxiliary variables. To clarify the theoretical findings, a simulation study along with a hypothetical generated population and a real population which is based on abortion rates from Statistical Abstract of the United States: 2011 are also addressed in this study.

Keywords: Measurement error, Non-response, Optional Randomized Response Models (ORRT), Sensitive variable(s), Two-phase Sampling (TPS)
2010 AMS: 62D05

[1] *Department of Statistics, University of Jammu, J&K, India, sunilbhougal06@gmail.com, ORCID: 0000-0003-0249-8415*
[2] *Department of Statistics, University of Jammu, J&K, India, sanamkour903@gmail.com, ORCID: 0000-0002-2894-2556*
***Corresponding author:** sanamkour903@gmail.com

Received: 23 September 2023, **Accepted:** 16 November 2023, **Available online:** 5 December 2023
How to cite this article: S. Kumar, S. P. Kour, *Quantify the Impact of Non-Response and Measurement Error of Sensitive Variable(s) under Two-Phase Sampling employing ORRT Models*, Commun. Adv. Math. Sci., 6(4) (2023), 196-210.

1. Introduction

Sympathetic or contentious issues that are raised in a brusque way may cause some respondents to feel anxious or insecure. As a consequence, they may hide the truth because they donot want their personal intentions to be revealed. Because of the perversion against negative behaviours, respondents may answer ‘No’ to questions like addiction of drugs, gambling, criminal conviction, domestic abuse, induced abortions, illegal income, tax evasion, even if they have. Such questionnaires encompassing sensitive characteristics necessarily entail the use of innovative techniques such as Randomized Response Technique (RRT) to evoke responses from the sampled units. Warner [1] is the first who posit an inventive RRT for estimating an unknown population prevalence of a sensitive criterion. Greenberg et al. [2] pioneered the estimation of the mean of quantitative sensitive variable by utilising RRT models. Afterwards, Pollack and Bek [3] developed the scrambling response technique for estimating the population mean of a sensitive variable. Gupta et al. [4] models are based on multiplicative scrambling whereas Gupta et al. [5] models are based on additive scrambling which works better than multiplicative scrambling as demonstrated by Gupta et al.

[6]. The notable authors include Zhang et al. [7], Kumar and Kour [8, 9], Kumar et al. [10, 11], Zaman et al. [12] and so forth developed estimation of mean of sensitive variables under non-response and measurement error using ORRT under simple random sampling and two-phase sampling.

In medical sciences, there are well documented instances where sensitive research must be surveilled over time in order to truly comprehend the problem. The evolution of these kind of varying variables may be analyzed by using two-phase sampling (TPS) technique which was first initiated by Neyman [13] and several researchers have since used it in varied incarnations. For illustration, in a survey to estimate the manufacturing of avocado crop predicated on orchards under the crop, only a sub-sampled of the orchards chosen for deciding land area is being used to ascertain the yield rate. Individual authors have been used TPS in varied incarnations including Sanullah et al. [14] who developed a generalized exponential chain ratio estimators under stratified two-phase random sampling, Zaman and Kadilar [15] introduced a new class of exponential estimators for estimating finite population mean in two-phase sampling, Khalil et al. [16] proposed an enhanced two-phase sampling ratio estimator for estimating population mean and among others.

A bulk of studies in a research presume that the data acquisition in a survey is error-free. Unfortunately this is not the reality; measurement error and non-response are very serious flaws in survey sampling. Measurement error (ME) is the difference between the observed value and the theoretical value of the target variable. Cognitive impairment, reputation bias, processing errors and erroneous respondent responses all contribute to measurement errors. Previously, Khalil et al. [17], Onyango et al. [18] deal with the problem of estimation of sensitive variable under measurement error in simple random sampling and double sampling. Withal, it is essential to tackle the issue of non-response in a sampling survey. Non-response (NR) happens when the analyst is unable to gather information from the estimated units of the population. Hansen and Hurwitz [19] is the first one who fix the problem of non-response by conducting a strategy that entails by collecting a sub-sample of non-respondents following the initial mail effort and then analyzing information through personal interview. Diana et al. [20], Gupta et al. [21], Zhang et al. [22], Mukhopadhhyay et al. [23] and so on addressed the problem of estimating the population mean to adjust non-response in varied sampling schemes.

Although we all aware that queries in a survey may have differing levels of sensitivity, and it may be important to quantify this sensitivity. Consequently, the accentuation of this article is exclusively on the chain ratio type estimator for the estimation of sensitive variable(s) in the presence of non-response and measurement error at the same time by making use of ORRT models when study and both two auxiliary variables are sensitive in nature under TPS technique. In section 2 and section 3 there are an ORR technique, an enhanced Hansen and Hurwitz [19] technique, some usual notations and some existing estimators. The Proposed estimator is described in section 4. In section 5, we have studied the efficiency comparisons of all considered estimator(s). To validate the theoretical findings an empirical study for both hypothetical and real population is performed in section 6. Finally, an ultimate conclusion is given in section 7.

2. The ORR Technique

Assume that $\Theta = \Theta_1, \Theta_2, ..., \Theta_N$ be a finite population of size N in which Y be the sensitive study variable and X and Z be two sensitive auxiliary variables with means $\bar{Y}$, $\bar{X}$ and $\bar{Z}$ and variances S_y^2, S_x^2 and S_z^2. Take S and S' be two scrambling variables with means $\bar{S}$ and $\bar{S}'$ and variances S_s^2 and $S_{s'}^2$, respectively. Let 'π' signifies the probability that the respondent will find the question sensitive. If the respondent consider the question is sensitive, then he or she is prompted to provide a scrambled response for the study Y as well as the auxiliary variables (X, Z), otherwise a legitimate response is recorded. Presuming simple random sampling without replacement (SRSWOR) at each phase, the TPS strategy works as follows

1. During the first phase, a large sample of fixed size n' is taken from Θ to examine X and Z in order to find estimates of $\bar{X}$ and $\bar{Z}$.

2. In the second phase sample, a fixed-size n sub-sample is taken from n' to observe Y only , so that $(n < n')$.

A conventional additive RRT model with $Y + S'$ as the scrambled response (as in Gupta et al. [6]) or a more comprehensive RRT model with $SY + S'$ as the scrambled response (as in Diana and Perri [24]) could be employed. The simple additive model is a particular case of the more general model if $E(S) = 1$ and with varying variances. The basic additive approach is more efficient, according to Khalil et al. [25], whereas the general model gives greater privacy. Even yet, the generalized randomized response model performs better when we utilize Gupta et al. [21] combined measure of efficiency and privacy, i.e. $\upsilon = \frac{Var(\mathcal{Z}_1)}{\Upsilon}$, where $\mathcal{Z}_1$ is the scrambled response and $\Upsilon = E(\mathcal{Z}_1 - Y)^2$ is the privacy level for the same model as given by Yan et al. [26]. It is important to note that the model with the lower value is preferable since it indicates either a higher level of privacy or a lower value of Var($\hat{y}$), or both. It is worth noting that

$$\upsilon_{additiveRRT} = 1 + \frac{S_y^2}{S_{s'}^2} > 1 + \frac{S_y^2}{S_{s'}^2 + S_s^2(\bar{y} + S_y^2)} = \upsilon_{generalRRT}$$

Under such circumstances, we will utilize the general scrambling model $\text{Ƶ}_1 = SY + S'$ as

$$\text{Ƶ}_1 = \begin{cases} Y & \text{with probability 1-}\pi \\ SY + S' & \text{with probability } \pi, \end{cases}$$

where S and S' follows normal distribution with mean $(1,0)$ and variances $(S_s^2, S_{s'}^2)$ i.e. $S \sim N(1, S_s^2)$ and $S' \sim N(0, S_{s'}^2)$. The mean and variance of Ƶ_1 are as

$E(\text{Ƶ}_1) = E(Y)(1-\pi) + E(SY + S')\pi = E(Y)$

and $Var(\text{Ƶ}_1) = E(\text{Ƶ}_1^2) - E^2(\text{Ƶ}_1) = S_y^2 + S_{s'}^2\pi + S_s^2(S_y^2 + \bar{Y}^2)\pi$.

We can write the randomized linear model as follows

$\text{Ƶ}_1 = (SY + S')J + Y(1-J)$, where $J \sim$ Bernoulli(π) with $E(J) = \pi, Var(J) = \pi(1-\pi)$ and $E(J^2) = Var(J) + E^2(J) = \pi$. And the expectation and variance of randomized mechanism is $E_R(\text{Ƶ}_1) = (\bar{S}\pi + 1 - \pi)Y + \bar{S}'\pi$ and $V_R(\text{Ƶ}_1) = (Y^2 S_s^2 + S_{s'}^2)\pi$.

In our research, we assume X and Z to be a sensitive variable(s) then first the general scrambling model for the auxiliary variable X is stated as follows

$$\text{Ƶ}_2 = \begin{cases} X & \text{with probability 1-}\pi \\ SX + S' & \text{with probability } \pi, \end{cases}$$

Now, The mean and variance of Ƶ_2 are given by

$E(\text{Ƶ}_2) = E(X)(1-\pi) + E(SX + S')\pi = E(X)$

and $Var(\text{Ƶ}_2) = E(\text{Ƶ}_2^2) - E^2(\text{Ƶ}_2) = S_x^2 + S_{s'}^2\pi + S_s^2(S_x^2 + \bar{X}^2)\pi$.

Likewise, we can write randomized linear model as $\text{Ƶ}_2 = (SX + S')J + X(1-J)$, where $J \sim$ Bernoulli(π) with $E(J) = \pi, Var(J) = \pi(1-\pi)$ and $E(J^2) = Var(J) + E^2(J) = \pi$. And the expectation and variance of randomized mechanism is $E_R(\text{Ƶ}_2) = (\bar{S}\pi + 1 - \pi)X + \bar{S}'\pi$ and $V_R(\text{Ƶ}_2) = (X^2 S_s^2 + S_{s'}^2)\pi$.

Similarly, for auxiliary variable Z, the general scrambling model is given as

$$\text{Ƶ}_3 = \begin{cases} Z & \text{with probability 1-}\pi \\ SZ + S' & \text{with probability } \pi, \end{cases}$$

Now, The mean and variance of Ƶ_3 are given by

$E(\text{Ƶ}_3) = E(Z)(1-\pi) + E(SZ + S')\pi = E(Z)$

and $Var(\text{Ƶ}_3) = E(\text{Ƶ}_3^2) - E^2(\text{Ƶ}_3) = S_z^2 + S_{s'}^2\pi + S_s^2(S_z^2 + \bar{Z}^2)\pi$.

As well, we can write randomized linear model as $\text{Ƶ}_3 = (SZ + S')J + Z(1-J)$, where $J \sim$ Bernoulli(π). The expectation and variance of randomized procedure is $E_R(\text{Ƶ}_3) = (\bar{S}\pi + 1 - \pi)Z + \bar{S}'\pi$ and $V_R(\text{Ƶ}_3) = (Z^2 S_s^2 + S_{s'}^2)\pi$.

The variance of Ƶ_1, Ƶ_2 and Ƶ_3 increases with increase in the probability π which demonstrates that the optional RRT model is definitely more efficient than the non-optional RRT model.

3. Enhanced Hansen and Hurwitz Technique [19]

From the population Θ, we suppose that only n_1 units respond on the first call and the residual $n_2 = n - n_1$ units do not respond. Out from n_2 non-responding units, a subsample of size $n_s = \frac{n_2}{k}$; $(k > 0)$ is selected. Also, (N_1, N_2) are the sizes of the respondent and non-respondent group. Let us suppose that $\bar{Y}_{(2)}$, $\bar{X}_{(2)}$ and $\bar{Z}_{(2)}$; $S^2_{y_{(2)}}$, $S^2_{x_{(2)}}$ and $S^2_{z_{(2)}}$ are the mean and variances of non-respondent group of size N_2, respectively. Hansen and Hurwitz [19] conducted a mail survey on the first conversation and then face-to-face interview on the second call.

The entire population mean of study variable is given by

$$\bar{Y} = W_1\bar{Y}_{(1)} + W_2\bar{Y}_{(2)},$$

where $W_1 = \frac{N_1}{N}$ and $W_2 = \frac{N_2}{N}$.

Let $\bar{y}_1 = \frac{\sum_{i=1}^{N_1} y_i}{n_1}$ be the sample mean for the response group, and $\bar{y}_2 = \frac{\sum_{i=1}^{N_2} y_i}{n_2}$ be the sample mean for non-response group. It is worth noting note that $\bar{y}_1$ and $\bar{y}_2$ are unbiased estimators of Y_1 and Y_2, respectively.

Hansen and Hurwitz [19] suggested an unbiased population mean estimator which is given by

$$\bar{y} = w_1\bar{y}_1 + w_2\bar{y}_{2s}$$

where $w_1 = \frac{n_1}{n}$ and $w_2 = \frac{n_2}{n}$.

The variance of $\bar{y}$ is given by

$$Var(\bar{y}) = \left(\frac{N-n}{Nn}\right)S_y^2 + \frac{W_2(k-1)}{n}S_{y(2)}^2$$

Within the second phase of the Hansen and Hurwitz [19] methodology, wherein face-to-face interviews of subsampled units of non-respondents are undertaken, we give respondents the opportunity to scramble their response using ORRT to incentivize them to answer a sensitive question honestly. In this scenario, we adapt Hansen and Hurwitz's technique by stating that the respondent group provides direct responses in the first phase, and then the ORRT model is being applied in the second phase to obtain answers from a sample of non-respondents.

Let $\hat{y}_i$ denote a transformation of the randomized response on the i^{th} unit, the expectation of which is the true response y_i under the randomization startegy is given by

$$\hat{y}_i = \frac{z_{1i} - \bar{S}'}{\bar{S}\pi + 1 - \pi}$$

with $E_R(\hat{y}_i) = y_i$ and $V_R(\hat{y}_i) = \frac{V_R(z_{1i})}{(\bar{S}\pi+1-\pi)^2} = \frac{(y_i^2 S_s^2 + S_{s'}^2)\pi}{(\bar{S}\pi+1-\pi)^2} = \delta_{1i}$

Contrastingly, assume that $\hat{x}_i$ and $\hat{z}_i$ denote a transformation of the randomized response on the i^{th} block, the expectation of which is the true response x_i and z_i, respectively under the mechanism and is given by

$$\hat{x}_i = \frac{z_{2i} - \bar{S}'}{\bar{S}\pi + 1 - \pi}$$

with $E_R(\hat{x}_i) = x_i$ and $V_R(\hat{x}_i) = \frac{V_R(z_{2i})}{(\bar{S}\pi+1-\pi)^2} = \frac{(x_i^2 S_s^2 + S_{s'}^2)\pi}{(\bar{S}\pi+1-\pi)^2} = \delta_{2i}$

Analogously

$$\hat{z}_i = \frac{z_{3i} - \bar{S}'}{\bar{S}\pi + 1 - \pi}$$

with $E_R(\hat{z}_i) = z_i$ and $V_R(\hat{z}_i) = \frac{V_R(z_{3i})}{(\bar{S}\pi+1-\pi)^2} = \frac{(z_i^2 S_s^2 + S_{s'}^2)\pi}{(\bar{S}\pi+1-\pi)^2} = \delta_{3i}$

From the previous discussions, we alter the Hansen and Hurwitz [19] estimator in the presence of non-response by utilizing ORRT.

$$\hat{\bar{y}} = w_1\bar{y}_1 + w_2\hat{\bar{y}}_2$$

$$\hat{\bar{x}} = w_1\bar{x}_1 + w_2\hat{\bar{x}}_2$$

$$\hat{\bar{z}} = w_1\bar{z}_1 + w_2\hat{\bar{z}}_2$$

where $\hat{\bar{y}}_2 = \sum_{i=1}^{n_s}(\frac{\hat{y}_i}{n_s})$, $\hat{\bar{x}}_2 = \sum_{i=1}^{n_s}(\frac{\hat{x}_i}{n_s})$ and $\hat{\bar{z}}_2 = \sum_{i=1}^{n_s}(\frac{\hat{z}_i}{n_s})$.

It is simple to illustrate that

$E(\hat{\bar{y}}) = \bar{Y}$; $E(\hat{\bar{x}}) = \bar{X}$; $E(\hat{\bar{z}}) = \bar{Z}$

and

The variance of $\hat{\bar{y}}$ is given by

$$Var(\hat{\bar{y}}) = \lambda S_y^2 + \lambda^* S_{y(2)}^2 + \frac{W_2 k}{n}\left[\frac{\{(S_{y(2)}^2 + \bar{y}_{(2)}^2)S_s^2 + S_{s'}^2\}\pi}{(\bar{S}\pi + 1 - \pi)^2}\right]$$

Similarly, the variance of $\hat{\bar{x}}$ is given by

$$Var(\hat{\bar{x}}) = \lambda S_x^2 + \lambda^* S_{x(2)}^2 + \frac{W_2 k}{n}\left[\frac{\{(S_{x(2)}^2 + \bar{x}_{(2)}^2)S_s^2 + S_{s'}^2\}\pi}{(\bar{S}\pi + 1 - \pi)^2}\right]$$

and

$$Var(\hat{\bar{z}}) = \lambda S_z^2 + \lambda^* S_{z(2)}^2 + \frac{W_2 k}{n}\left[\frac{\{(S_{z(2)}^2 + \bar{z}_{(2)}^2)S_s^2 + S_{s'}^2\}\pi}{(\bar{S}\pi + 1 - \pi)^2}\right]$$

where $\lambda = \frac{(N-n)}{Nn}$ and $\lambda^* = \frac{(k-1)W_2}{n}$.

Measurement error, additionally to non-response, is a prominent cause of non-sampling errors in a survey. Let $U_i = y_i - Y_i$, $V_i = x_i - X_i$ and $W_i = z_i - Z_i$ be the measurement error for the study variable (Y) and auxiliary variables (X, Z) in the population. Let $P_i = \mathfrak{z}_{1i} - \mathcal{Z}_{1i}$, $Q_i = \mathfrak{z}_{2i} - \mathcal{Z}_{2i}$ and $R_i = \mathfrak{z}_{3i} - \mathcal{Z}_{3i}$ indicate the respective measurement error associated with the sensitive variables ($\mathcal{Z}_1$, $\mathcal{Z}_2$ and $\mathcal{Z}_3$) in the face-to-face interview phase. These measurement errors are recognised to be random and uncorrelated, with mean zero and variances S_u^2, S_v^2, S_w^2, S_p^2, S_q^2 and S_r^2, respectively.

Numerous notations are presented here, supposing that the population mean of the sensitive auxiliary variable(s) are unknown and that non-response happens on both the study as well as on both the auxiliary variables i.e X, Y and Z.

$$\hat{\Delta}_y^* = \sum_{i=1}^{n}(y_i - \bar{Y});\ \hat{\Delta}_x^* = \sum_{i=1}^{n}(x_i - \bar{X});\ \hat{\Delta}_z^* = \sum_{i=1}^{n}(z_i - \bar{Z})$$

$$\hat{\Delta}_u^* = \sum_{i=1}^{n_1} U_i + \sum_{i=1}^{n_2} P_i;\ \hat{\Delta}_v^* = \sum_{i=1}^{n_1} V_i + \sum_{i=1}^{n_2} Q_i;\ \hat{\Delta}_w^* = \sum_{i=1}^{n_1} W_i + \sum_{i=1}^{n_2} R_i$$

where U_i, V_i, W_i, P_i, Q_i and R_i are measurement errors on Y, X, Z, $\mathcal{Z}_1$, $\mathcal{Z}_2$ and $\mathcal{Z}_3$ respectively.

Furthermore, in the presence of measurement error, the variance of $\hat{\bar{y}}$, $\hat{\bar{x}}$ and $\hat{\bar{z}}$ is given by

$$Var(\hat{\bar{y}}^*) = \lambda(S_y^2 + S_u^2) + \lambda^*(S_{y(2)}^2 + S_p^2) + \kappa_1;$$

$$Var(\hat{\bar{x}}^*) = \lambda(S_x^2 + S_v^2) + \lambda^*(S_{x(2)}^2 + S_q^2) + \kappa_2$$

and

$$Var(\hat{\bar{z}}^*) = \lambda(S_z^2 + S_w^2) + \lambda^*(S_{z(2)}^2 + S_r^2) + \kappa_3$$

where $\kappa_1 = \frac{W_2 k}{n}\left[\frac{\{(S_{y(2)}^2 + \bar{y}_{(2)}^2)S_s^2 + S_{s'}^2\}\pi}{(\bar{S}\pi + 1 - \pi)^2}\right]$; $\kappa_2 = \frac{W_2 k}{n}\left[\frac{\{(S_{x(2)}^2 + \bar{x}_{(2)}^2)S_s^2 + S_{s'}^2\}\pi}{(\bar{S}\pi + 1 - \pi)^2}\right]$ and $\kappa_3 = \frac{W_2 k}{n}\left[\frac{\{(S_{z(2)}^2 + \bar{z}_{(2)}^2)S_s^2 + S_{s'}^2\}\pi}{(\bar{S}\pi + 1 - \pi)^2}\right]$.

Taking $\hat{\bar{y}}^* = \bar{Y}(1 + \hat{e}_0^*)$, $\hat{\bar{x}}^* = \bar{X}(1 + \hat{e}_1^*)$, $\hat{\bar{z}}^* = \bar{Z}(1 + \hat{e}_2^*)$, $\bar{x}' = \bar{X}(1 + e_1')$ and $\bar{z}' = \bar{Z}(1 + e_2')$ such that $E(\hat{e}_0^*) = E(\hat{e}_1^*) = E(\hat{e}_2^*) = E(e_1') = E(e_2') = 0$

To acquire mean squared error, we will used the following notations

$E(\hat{e}_0^{*2}) = \frac{1}{\bar{Y}^2}\left[\lambda(S_y^2 + S_u^2) + \lambda^*(S_{y(2)}^2 + S_p^2) + \kappa_1\right] = \frac{1}{\bar{Y}^2}(A + \kappa_1) = A_1$;

$E(\hat{e}_0^2) = \frac{1}{\bar{Y}^2}\left[\lambda S_y^2 + \lambda^* S_{y(2)}^2 + \kappa_1\right] = \frac{1}{\bar{Y}^2}(\hat{A} + \kappa_1) = A_2$;

$E(e_0^{*2}) = \frac{1}{\bar{Y}^2}\left[\lambda(S_y^2 + S_u^2) + \lambda^*(S_{y(2)}^2 + S_{u(2)}^2)\right] = \frac{1}{\bar{Y}^2}A^* = A_3$;

$E(\hat{e}_1^{*2}) = \frac{1}{\bar{X}^2}\left[\lambda(S_x^2+S_v^2)+\lambda^*(S_{x(2)}^2+S_q^2)+\kappa_2\right] = \frac{1}{\bar{X}^2}(B+\kappa_2) = B_1;$

$E(\hat{e}_1^{2}) = \frac{1}{\bar{X}^2}\left[\lambda S_x^2+\lambda^* S_{x(2)}^2+\kappa_2\right] = \frac{1}{\bar{X}^2}(\hat{B}+\kappa_2) = B_2;$

$E(e_1^{*2}) = \frac{1}{\bar{X}^2}\left[\lambda(S_x^2+S_v^2)+\lambda^*(S_{x(2)}^2+S_{v(2)}^2)\right] = \frac{1}{\bar{X}^2}B^* = B_3;$

$E(\hat{e}_2^{*2}) = \frac{1}{\bar{Z}^2}\left[\lambda(S_z^2+S_w^2)+\lambda^*(S_{z(2)}^2+S_r^2)+\kappa_3\right] = \frac{1}{\bar{Z}^2}(C+\kappa_3) = C_1;$

$E(\hat{e}_2^{2}) = \frac{1}{\bar{Z}^2}\left[\lambda S_z^2+\lambda^* S_{z(2)}^2+\kappa_3\right] = \frac{1}{\bar{Z}^2}(\hat{C}+\kappa_3) = C_2;$

$E(e_2^{*2}) = \frac{1}{\bar{Z}^2}\left[\lambda(S_z^2+S_w^2)+\lambda^*(S_{z(2)}^2+S_{w(2)}^2)\right] = \frac{1}{\bar{Z}^2}A^* = C_3;$

$E(e_1') = \frac{1}{\bar{X}^2}\lambda S_x^2 = A_{11};\ E(e_2') = \frac{1}{\bar{Z}^2}\lambda S_z^2 = C_{11};$

$E(\hat{e}_0^*\hat{e}_1^*) = \frac{1}{\bar{Y}\bar{X}}\left[\lambda\rho_{yx}S_yS_x+\lambda^*\rho_{yx(2)}S_{y(2)}S_{x(2)}\right] = \frac{1}{\bar{Y}\bar{X}}D = D_1;$

$E(\hat{e}_0\hat{e}_1) = \frac{1}{\bar{Y}\bar{X}}\left(\lambda\rho_{yx}S_yS_x\right) = \frac{1}{\bar{Y}\bar{X}}\hat{D} = D_2;$

$E(\hat{e}_1^*\hat{e}_2^*) = \frac{1}{\bar{X}\bar{Z}}\left[\lambda\rho_{xz}S_xS_z+\lambda^*\rho_{xz(2)}S_{x(2)}S_{z(2)}\right] = \frac{1}{\bar{X}\bar{Z}}E = E_1;$

$E(\hat{e}_1\hat{e}_2) = \frac{1}{\bar{X}\bar{Z}}\left(\lambda\rho_{xz}S_xS_z\right) = \frac{1}{\bar{X}\bar{Z}}\hat{E} = E_2;$

$E(\hat{e}_0^*\hat{e}_2^*) = \frac{1}{\bar{Y}\bar{Z}}\left[\lambda\rho_{yz}S_yS_z+\lambda^*\rho_{yz(2)}S_{y(2)}S_{z(2)}\right] = \frac{1}{\bar{Y}\bar{Z}}F = F_1;$

$E(\hat{e}_0\hat{e}_2) = \frac{1}{\bar{Y}\bar{Z}}\left(\lambda\rho_{yz}S_yS_z\right) = \frac{1}{\bar{Y}\bar{Z}}\hat{F} = F_2;$

$E(\hat{e}_0^*e_1') = \frac{1}{\bar{Y}\bar{X}}\lambda'\rho_{yx}S_yS_x = \frac{1}{\bar{Y}\bar{X}}G = G_1;$

$E(\hat{e}_0^*e_2') = \frac{1}{\bar{Y}\bar{Z}}\lambda'\rho_{yz}S_yS_z = \frac{1}{\bar{Y}\bar{Z}}H = H_1;$

$E(\hat{e}_1^*e_1') = \frac{1}{\bar{X}^2}\lambda' S_x^2 = \frac{1}{\bar{X}^2}I = I_1;$

$E(\hat{e}_1^*e_2') = \frac{1}{\bar{X}\bar{Z}}\lambda'\rho_{xz}S_yS_z = \frac{1}{\bar{X}\bar{Z}}J = J_1;$

$E(\hat{e}_2^*e_1') = \frac{1}{\bar{Z}\bar{X}}\lambda'\rho_{zx}S_zS_x = \frac{1}{\bar{Z}\bar{X}}K = K_1;$

$E(\hat{e}_2^*e_2') = \frac{1}{\bar{Z}^2}\lambda' S_z^2 = \frac{1}{\bar{Z}^2}L = L_1;$

$E(e_1'e_2') = \frac{1}{\bar{X}\bar{Z}}\lambda'\rho_{xz}S_xS_z = \frac{1}{\bar{X}\bar{Z}}M = M_1.$

where $\lambda = \left(\frac{1}{n}-\frac{1}{N}\right)$; $\lambda' = \left(\frac{1}{n'}-\frac{1}{N}\right)$ and $\lambda^* = \frac{W_2(k-1)}{n}$.

Next we take the modified conventional estimators i.e. ratio and product estimators into six antithetic strategies depending upon the accessible sensitive auxiliary variables using ORRT models under two-phase sampling (TPS) scheme.

Strategies	*Conventional Estimator(s)*	*Bias*	*Mean Squared Error (MSE)*
Strategy 1: When $\hat{\bar{y}}^*$, $\hat{\bar{x}}^*$ and $\bar{x}'$ are used and NR and ME occurs on both sensitive study and auxiliary variable	$\hat{T}_r^* = \hat{\bar{y}}^*\left(\frac{\hat{\bar{x}}^*}{\bar{x}'}\right)$	$Bias(\hat{T}_r^*) = \varkappa\varphi B - \varkappa\varphi I - \varphi G + \varphi D + \varkappa\varphi\kappa_2$	$MSE(\hat{T}_r^*) = A + \varkappa^2 B + \varkappa^2 A_{11} - 2\varkappa^2 I + 2\varphi G + 2\varphi D + (\kappa_1 + \varkappa^2\kappa_2)$ where $\varkappa = \frac{\bar{Y}}{\bar{X}}$ and $\varphi = \frac{1}{\bar{X}}$
Strategy 2: When $\hat{\bar{y}}$, $\hat{\bar{x}}$ and $\bar{x}'$ are in use and there is absence of NR and ME	$\hat{T}_r = \hat{\bar{y}}\left(\frac{\hat{\bar{x}}}{\bar{x}'}\right)$	$Bias(\hat{T}_r) = \varkappa\varphi\hat{B} - \varkappa\varphi I - \varphi G + \varphi\hat{D} + \varkappa\varphi\kappa_2$	$MSE(\hat{T}_r) = \hat{A} + \varkappa^2\hat{B} + \varkappa^2 A_{11} - 2\varkappa^2 I + 2\varphi G + 2\varphi\hat{D} + (\kappa_1 + \varkappa^2\kappa_2)$
Strategy 3: When $\bar{y}^*$, $\bar{x}^*$ and $\bar{x}'$ are utilized and NR and ME both occurs on study as well as on auxiliary variable	$T_r^* = \bar{y}^*\left(\frac{\bar{x}^*}{\bar{x}'}\right)$	$Bias(T_r^*) = \varkappa\varphi B^* - \varkappa\varphi I - \varphi G + \varphi D + \varkappa\varphi\kappa_2$	$MSE(T_r^*) = A^* + \varkappa^2 B^* + \varkappa^2 A_{11} - 2\varkappa^2 I + +2\varphi G + 2\varphi D + (\kappa_1 + \varkappa^2\kappa_2)$
Strategy 4: When $\hat{\bar{y}}^*$, $\hat{\bar{x}}^*$, $\hat{\bar{z}}^*$, $\bar{x}'$ and $\bar{z}'$ are used and NR and ME happens on both the sensitive study as well as auxiliary variables	$\hat{T}_p^* = \hat{\bar{y}}^*\left(\frac{\hat{\bar{x}}^*}{\bar{x}'}\right)\left(\frac{\hat{\bar{z}}^*}{\bar{z}'}\right)$	$Bias(\hat{T}_r^*) = \varkappa\varphi A_{11} - \varpi\rho L + \varpi\rho C_{11} - \varkappa\rho K + \varkappa\rho E - \varkappa\varphi I - \rho H - \varphi G$	$MSE(\hat{T}_p^*) = A + \varkappa^2 B + \varpi^2 C + \rho^2 C_{11} + 4\varkappa^2 A_{11} + 4\varkappa\varpi M - 2\varpi^2 L - 2\varpi H - 4\varkappa^2 I - 4\varkappa\varpi K - 4\varkappa G + 2\varkappa\varpi E + 2\varkappa D + 2\varpi F + \kappa_1 + +\varkappa^2\kappa_2 + \varpi^2\kappa_3$
Strategy 5: When $\hat{\bar{y}}$, $\hat{\bar{x}}$, $\hat{\bar{z}}$, $\bar{x}'$ and $\bar{z}'$ are utilized and there is no NR and ME happens	$\hat{T}_p = \hat{\bar{y}}\left(\frac{\hat{\bar{x}}}{\bar{x}'}\right)\left(\frac{\hat{\bar{z}}}{\bar{z}'}\right)$	$Bias(\hat{T}_r) = \varkappa\varphi A_{11} - \varpi\rho L + \varpi\rho C_{11} - \varkappa\rho K + \varkappa\rho\hat{E} - \varkappa\varphi I - \rho H - \varphi G$	$MSE(\hat{T}_p) = \hat{A} + \varkappa^2\hat{B} + \varpi^2\hat{C} + \rho^2 C_{11} + 4\varkappa^2 A_{11} + 4\varkappa\varpi M - 2\varpi^2 L - 2\varpi H - 4\varkappa^2 I - 4\varkappa\varpi K - 4\varkappa G + 2\varkappa\varpi\hat{E} + 2\varkappa\hat{D} + 2\varpi\hat{F} + \kappa_1 + \varkappa^2\kappa_2 + \varpi^2\kappa_3$
Strategy 6: When $\bar{y}^*$, $\bar{x}^*$, $\bar{z}^*$, $\bar{x}'$ and $\bar{z}'$ are employed and NR and ME both occurs on study as well as on the auxiliary variables	$T_p^* = \bar{y}^*\left(\frac{\bar{x}^*}{\bar{x}'}\right)\left(\frac{\bar{z}^*}{\bar{z}'}\right)$	$Bias(T_r^*) = \varkappa\varphi A_{11} - \varpi\rho L + \varpi\rho C_{11} - \varkappa\rho K + \varkappa\rho E - \varkappa\varphi I - \rho H - \varphi G$	$MSE(T_p^*) = A^* + \varkappa^2 B^* + \varpi^2 C^* + \rho^2 C_{11} + 4\varkappa^2 A_{11} + 4\varkappa\varpi M - 2\varpi^2 L - 2\varpi H - 4\varkappa^2 I - 4\varkappa\varpi K - 4\varkappa G + 2\varkappa\varpi E + 2\varkappa D + 2\varpi F + \kappa_1 + \varkappa^2\kappa_2 + \varpi^2\kappa_3$

Table 1. Conventional estimators with their bias and mean square errors using ORRT

4. Proposed Chain Ratio Type Estimator

Grabbing inspiration from the existing evidences, an efforts have been made to propose an estimator to improve conventional estimators by multiplying a tuning constant term α whose optimum value is based on the coefficient of variation, which is relatively a stable variable. In addition, inspired by Kumar and Kour [8] and Zhang et al. [22], an information on more than one auxiliary variable can be utilized to suggest a more efficient chain ratio type estimator in the presence of non-response and measurement error simultaneously when the study as well as both the auxiliary variables are sensitive in its essence under three different strategies in two-phase sampling technique by utilizing ORRT models so that one could get a more precise estimate of the population mean.

Methodology 1: Assuming $\hat{\bar{y}}^*$, $\hat{\bar{x}}^*$, $\hat{\bar{z}}^*$, $\bar{x}'$ and $\bar{z}'$ are deployed and non-response and measurement error occured on both the sensitive study as well as auxiliary variables i.e. (X,Z) under TPS scheme then the chain ratio type estimator is given as

$$\hat{T}_s^* = \alpha\hat{\bar{y}}^*\left[\alpha_1\left(\frac{\hat{\bar{x}}^*}{\bar{x}'}\right)\left(\frac{\hat{\bar{z}}^*}{\bar{z}'}\right) + (1-\alpha_1)\left(\frac{\bar{x}'}{\hat{\bar{x}}^*}\right)\left(\frac{\bar{z}'}{\hat{\bar{z}}^*}\right)\right]$$

where $\alpha = \frac{\bar{Y}^2}{\bar{Y}^2 + \lambda S_y^2 + \lambda^* S_{y(2)}^2}$.

To evaluate the bias and mean squared error of $\hat{T}_s^*$ by reducing and eliminating terms to first order of approximation, one could verify that

$$(\hat{T}_s^* - \bar{Y}) = \bar{Y}(\alpha - 1) + \alpha\bar{Y}(\hat{e}_2^{*2} - \hat{e}_2^* + e_2' - \hat{e}_2^* e_2' + \hat{e}_1^*\hat{e}_2^* - \hat{e}_2^* e_2' + e_1' - \hat{e}_2^* e_1' + e_1' e_2' - \hat{e}_1^* e_1' - 2\alpha_1 e_2' + 2\alpha_1\hat{e}_2^* - 2\alpha_1 e_1' + \alpha_1 e_2'^2 + \alpha_1 e_1'^2 - \alpha_1\hat{e}_2^{*2} - \alpha_1\hat{e}_1^{*2} + 2\alpha_1\hat{e}_1^* + \hat{e}_0^* - \hat{e}_0^*\hat{e}_2^* + \hat{e}_0^* e_2' - \hat{e}_0^*\hat{e}_1^* + \hat{e}_0^* e_1' - 2\alpha_1\hat{e}_0^* e_2' + 2\alpha_1\hat{e}_0^*\hat{e}_2^* - 2\alpha_1\hat{e}_0^* e_1' + 2\alpha_1\hat{e}_0^*\hat{e}_1^*).$$

The bias and mean squared error of the chain ratio type estimator $\hat{\hat{T}}_s^*$ in the inclusion of non-response and measurement error at the same time, is given by

$$Bias(\hat{T}_s^*) = \alpha^*\{(2\alpha_1 - 1)(F_1 + D_1 - H_1 - G_1) + E_1 - (\alpha_1 - 1)C_1 - \alpha_1 B_1 - 2L_1 - K_1 + M_1 + J_1 + \alpha_1\zeta\} - \beta^*$$

where $\beta^* = \left(\frac{\bar{Y}^2 + \lambda S_y^2 + \lambda^* S_{y(2)}^2}{\bar{Y}} + \bar{Y}\right)$; $\alpha^* = \frac{\bar{Y}^3}{\bar{Y}^2 + \lambda S_y^2 + \lambda^* S_{y(2)}^2}$; $\zeta = \lambda'\left(\frac{1}{\bar{Z}^2}S_z^2 + \frac{1}{\bar{X}^2}S_x^2\right)$.

and

$$MSE(\hat{T}_s^*) = \gamma^2 + \alpha^{*2}\left[A_1 + \theta C_1 + \theta A_{11} + \theta C_{11} + 4\alpha_1^2 B_1 + 2\phi F_1 - 2\phi G_1 - 2\phi H_1 + 4\alpha_1 D_1 - 2\theta K_1 + 2\theta L_1 + 4\alpha_1\phi E_1 + 2\theta M_1 + 4\alpha_1\phi I_1 - 4\alpha_1\phi J_1\right] \tag{4.1}$$

where $\theta = 1 + 4\alpha_1^2 - 4\alpha_1$; $\phi = 2\alpha_1 - 1$ and $\gamma = \left(\frac{-(\lambda S_y^2 + \lambda^* S_{y(2)}^2)\bar{Y}^2}{\bar{Y}^2 + \lambda S_y^2 + \lambda^* S_{y(2)}^2}\right)$.

To get the optimum solution of the constant 'α_1' in $\hat{\hat{T}}_s^*$, we differentiate (4.1) with respect to 'α_1' and equating it to zero, we have

$$\hat{\alpha}_{1opt.}^* = \frac{-(\gamma^2 + 4\eta)}{8\alpha^{*2}\eta^*} \tag{4.2}$$

where $\eta = F_1 - G_1 - H_1 + D_1$;

and $\eta^* = B_1 + C_1 + A_{11} + C_{11} - 2K_1 + 2L_1 + 2M_1 + 2E_1 + 2I_1 - 2J_1$.

Substituting the optimum value from (4.2), the minimum mean squared error of $\hat{T}_s^*$ is given as

$$min.MSE(\hat{T}_s^*) = \gamma^2 + \alpha^{*2}(A_1 + C_1 + A_{11} + C_{11} - 2K_1 + 2L_1 + 2M_1 - 2F_1 + 2G_1 + 2H_1) - \psi(D_1 - F_1 + G_1 + H_1 - B_1) + \psi^*$$

where $\psi = \frac{-(\gamma^2 + 4\eta)}{2}$ and $\psi^* = \frac{\psi^2}{\alpha^{*2}\eta^*}$.

Methodology 2: Letting $\hat{\bar{y}}$, $\hat{\bar{x}}$, $\hat{\bar{z}}$, $\bar{x}'$ and $\bar{z}'$ are being used and there is absence of non-response and measurement error on the sensitive study as well as both the sensitive auxiliary variables i.e. (X, Z) under TPS technique then the chain ratio type estimator is given as

$$\hat{T}_s = \hat{\alpha}\hat{\bar{y}}\left[\alpha_1\left(\frac{\hat{\bar{x}}}{\bar{x}'}\right)\left(\frac{\hat{\bar{z}}}{\bar{z}'}\right) + (1 - \alpha_1)\left(\frac{\bar{x}'}{\hat{\bar{x}}}\right)\left(\frac{\bar{z}'}{\hat{\bar{z}}}\right)\right]$$

where $\hat{\alpha} = \frac{\bar{Y}^2}{\bar{Y}^2 + \lambda S_y^2}$.

The expressions for the bias as well as mean squared error are expressed as

$$Bias(\hat{T}_s) = \hat{\alpha}^*\{(2\hat{\alpha}_1 - 1)(F_2 + D_2 - H_1 - G_1) + E_2 - (\hat{\alpha}_1 - 1)C_2 - \hat{\alpha}_1 B_2 - 2L_1 - K_1 + M_1 + J_1 + \hat{\alpha}_1\zeta\} - \hat{\beta}^*$$

where $\hat{\beta}^* = \left(\frac{\bar{Y}^2 + \lambda S_y^2}{\bar{Y}} + \bar{Y}\right)$; $\hat{\alpha}^* = \frac{\bar{Y}^3}{\bar{Y}^2 + \lambda S_y^2}$.

$$MSE(\hat{T}_s) = \hat{\gamma}^2 + \alpha^{*2}\left[A_2 + \theta C_2 + \theta A_{11} + \theta C_{11} + 4\alpha_1^2 B_2 + 2\phi F_2 - 2\phi G_1 - 2\phi H_1 + 4\alpha_1 D_2 - 2\theta K_1 + 2\theta L_1 + 4\alpha_1\phi E_2 + 2\theta M_1 + 4\alpha_1\phi I_1 - 4\alpha_1\phi J_1\right]$$

where $\hat{\gamma} = \left(\frac{-(\lambda S_y^2)\bar{Y}^2}{\bar{Y}^2 + \lambda S_y^2}\right)$.

which is optimum when

$$\hat{\alpha}_{1opt.} = \frac{-(\hat{\gamma}^2 + 4\eta)}{8\alpha^{*2}\eta^*}$$

where $\hat{\eta} = F_2 - G_1 - H_1 + D_2$;

and $\hat{\eta}^* = B_2 + C_2 + A_{11} + C_{11} - 2K_1 + 2L_1 + 2M_1 + 2E_2 + 2I_1 - 2J_1$.

The minimum mean squared error for this methodology is given as

$$min.MSE(\hat{T}_s) = \hat{\gamma}^2 + \hat{\alpha}^{*2}(A_2 + C_2 + A_{11} + C_{11} - 2K_1 + 2L_1 + 2M_1 - 2F_2 + 2G_1 + 2H_1) - \hat{\psi}(D_2 - F_2 + G_1 + H_1 - B_2) + \hat{\psi}^*$$

where $\hat{\psi} = \frac{-(\hat{\gamma}^2 + 4\hat{\eta})}{2}$; $\hat{\psi}^* = \frac{\hat{\psi}^2}{\hat{\alpha}^{*2}\hat{\eta}^*}$.

Methodology 3: Suppose $\bar{y}^*$, $\bar{x}^*$, $\bar{z}^*$, $\bar{x}'$ and $\bar{z}'$ are employed and there is presence both non-response and measurement error on the sensitive study and auxiliary variables i.e. (X, Z) using TPS technique. For this strategy the chain ratio type estimator is given as

$$T_s^* = \alpha\bar{y}^*\left[\alpha_1\left(\frac{\bar{x}^*}{\bar{x}'}\right)\left(\frac{\bar{z}^*}{\bar{z}'}\right) + (1-\alpha_1)\left(\frac{\bar{x}'}{\bar{x}^*}\right)\left(\frac{\bar{z}'}{\bar{z}^*}\right)\right]$$

where $\alpha = \frac{\bar{Y}^2}{\bar{Y}^2 + \lambda S_{\bar{y}}^2 + \lambda^* S_{y(2)}^2}$.

The formulation of bias and MSE when there is a presence of non-response and measurement error are given as

$$Bias(T_s^*) = \alpha^*\{(2\alpha_1 - 1)(F_1 + D_1 - H_1 - G_1) + E_1 - (\alpha_1 - 1)C_3 - \alpha_1 B_3 - 2L_1 - K_1 + M_1 + J_1 + \alpha_1\zeta\} - \beta^*$$

and

$$MSE(T_s^*) = \gamma^2 + \alpha^{*2}\left[A_3 + \theta C_3 + \theta A_{11} + \theta C_{11} + 4\alpha_1^2 B_3 + 2\phi F_1 - 2\phi G_1 - 2\phi H_1 + 4\alpha_1 D_1 - 2\theta K_1 + 2\theta L_1 + 4\alpha_1\phi E_1 + 2\theta M_1 + 4\alpha_1\phi I_1 - 4\alpha_1\phi J_1\right]$$

which in itself is optimal when

$$\alpha^*_{1opt.} = \frac{-(\gamma^2 + 4\eta)}{8\alpha^{*2}\eta^{**}}$$

where $\eta^{**} = B_3 + C_3 + A_{11} + C_{11} - 2K_1 + 2L_1 + 2M_1 + 2E_1 + 2I_1 - 2J_1$.

Then, the min.MSE for this strategy is expressed as

$$min.MSE(T_s^*) = \gamma^2 + \alpha^{*2}(A_3 + C_3 + A_{11} + C_{11} - 2K_1 + 2L_1 + 2M_1 - 2F_1 + 2G_1 + 2H_1) - \psi(D_1 - F_1 + G_1 + H_1 - B_3) + \psi^{**}$$

where $\psi^{**} = \frac{\psi^2}{\alpha^{*2}\eta^{**}}$.

5. Efficiency Comparisons of Estimator(s)

To assess the effectiveness of the chain type proposed estimator, we relate it to the ratio and product estimator in different strategic plans as

(i) $\quad min.MSE(\hat{T}_s^*) - MSE(\hat{T}_r^*) < 0$

if $\quad \gamma^2 + \alpha^{*2}\hat{a}^* + \psi^* - \psi\hat{b}^* - \hat{c}^* < 0$

$(ii) \quad min.MSE(\hat{T}_s) - MSE(\hat{T}_r) < 0$

if $\quad \hat{\gamma}^2 + \hat{\alpha}^{*2}\hat{a} + \psi^* - \psi\hat{b} - \hat{c} < 0$

$(iii) \quad min.MSE(T_s^*) - MSE(T_r^*) < 0$

if $\quad \gamma^2 + \alpha^{*2}a^* + \psi^* - \psi b^* - c^* < 0$

$(iv) \quad min.MSE(\hat{T}_s^*) - MSE(\hat{T}_p^*) < 0$

if $\quad \gamma^2 + \alpha^{*2}\hat{a}^* + \psi^* - \psi\hat{b}^* - \hat{d}^* < 0$

$(v) \quad min.MSE(\hat{T}_s) - MSE(\hat{T}_p) < 0$

if $\quad \hat{\gamma}^2 + \hat{\alpha}^{*2}\hat{a} + \psi^* - \psi\hat{b} - \hat{d} < 0$

$(vi) \quad min.MSE(T_s^*) - MSE(T_p^*) < 0$

if $\quad \gamma^2 + \alpha^{*2}a^* + \psi^* - \psi b^* - d^* < 0$

where $\hat{a}^* = A_1 + C_1 + A_{11} + C_{11} - 2K_1 + 2L_1 + 2M_1 - 2F_1 + 2G_1 + 2H_1$;
$\hat{b}^* = D_1 - F_1 + G_1 + H_1 - B_1$;
$\hat{c}^* = A + \varkappa^2 B + \varkappa^2 A_{11} - 2\varkappa^2 I + 2\varphi G + 2\varphi D + (\kappa_1 + \varkappa^2\kappa_2)$;
$\hat{a} = A_2 + C_2 + A_{11} + C_{11} - 2K_1 + 2L_1 + 2M_1 - 2F_2 + 2G_1 + 2H_1$;
$\hat{b} = D_2 - F_2 + G_1 + H_1 - B_2$;
$\hat{c} = \hat{A} + \varkappa^2\hat{B} + \varkappa^2 A_{11} - 2\varkappa^2 I + 2\varphi G + 2\varphi\hat{D} + (\kappa_1 + \varkappa^2\kappa_2)$;
$a^* = A_3 + C_3 + A_{11} + C_{11} - 2K_1 + 2L_1 + 2M_1 - 2F_1 + 2G_1 + 2H_1$;
$b^* = D_1 - F_1 + G_1 + H_1 - B_3$;
$c^* = A + \varkappa^2 B + \varkappa^2 A_{11} - 2\varkappa^2 I + 2\varphi G + 2\varphi D + (\kappa_1 + \varkappa^2\kappa_2)$;

When the above conditions from $(i) - (vi)$ are met then it is evident that the suggested estimators i.e. $\hat{T}_s^*$, $\hat{T}_s$ and T_s^* are efficient than the existing one.

$min.MSE(\hat{T}_s) < min.MSE(T_s^*) < min.MSE(\hat{T}_s^*) < MSE(\hat{T}_p) < MSE(T_p^*) < MSE(\hat{T}_p^*) < MSE(\hat{T}_r) < MSE(T_r^*) < MSE(\hat{T}_p^*)$.

To verify the performance of the above relations, we execute a simulation study by using both hypothetical and real populations in R software which is relinquished in the next section.

6. Simulation Study

To gain a better understanding of the efficiency of the recommended estimators, we leverage R software to perform a simulation study to validate the effectiveness of our proposed estimator as compare to the ratio and the product type estimator(s). We generated a population of $N = 8000$ we take sample of size $n' = 6000$ and suppose that the response rate is 40% in the first phase. From n' we take sample of size $n = 2000$ using R software for different values of k and π sequentially. A variable $X \sim N(a,b)$; $Z \sim N(a,b)$ and variable Y which is related with X and Z is defined as $Y = X + Z + N(0,1)$ also generated from normal distribution where $a = 0.5$ and $b = 1.5$. The scrambling variables $S \sim N(1,a)$ and $S' \sim N(0,1)$, both taken from normal distribution and results are averaged over $8,000$ iterations.

The unified measure ω as described by Gupta et al. [21] are represented by

$$\hat{\omega}^* = \frac{MSE(\hat{T}_i^*)}{\Upsilon}; \tag{6.1}$$

where $\Upsilon = E(Z_1 - Y)^2$ is the privacy level of sensitive models and $T_i^* = \hat{T}_r^*$, $\hat{T}_p^*$ and $\hat{T}_s^*$.

$$\hat{\omega} = \frac{MSE(\hat{T}_i)}{\Upsilon}; \tag{6.2}$$

where $T_i^* = \hat{T}_r$, $\hat{T}_p$ and $\hat{T}_s$.

π	k	*Estimator(s)*			*Unified Measure*($\hat{\omega}^*$)		
		$MSE(\hat{T}_r^*)$	$MSE(\hat{T}_p^*)$	$MSE(\hat{T}_s^*)$	$\hat{\omega}^*(\hat{T}_r^*)$	$\hat{\omega}^*(\hat{T}_p^*)$	$\hat{\omega}^*(\hat{T}_s^*)$
0.2	2	0.0747	0.0601	0.0255	0.0254	0.0205	0.0087
	3	0.0963	0.0767	0.0326	0.0328	0.0261	0.0111
	4	0.1189	0.0974	0.0404	0.0406	0.0332	0.0138
	5	0.1403	0.1131	0.0475	0.0479	0.0385	0.0162
0.4	2	0.0755	0.0616	0.0262	0.0248	0.0202	0.0086
	3	0.0977	0.0790	0.0339	0.0321	0.0260	0.0111
	4	0.1210	0.1006	0.0423	0.0398	0.0331	0.0139
	5	0.1429	0.1159	0.0494	0.0470	0.0381	0.0162
0.6	2	0.0757	0.0656	0.0286	0.0246	0.0213	0.0092
	3	0.0983	0.0843	0.0371	0.0319	0.0274	0.0121
	4	0.1220	0.1050	0.0455	0.0396	0.0343	0.0148
	5	0.1440	0.1229	0.0537	0.0468	0.0399	0.0174
0.8	2	0.0764	0.0671	0.0294	0.0240	0.0210	0.0092
	3	0.0995	0.0867	0.0385	0.0312	0.0272	0.0121
	4	0.1230	0.1087	0.0474	0.0388	0.0341	0.0149
	5	0.1464	0.1257	0.0557	0.0459	0.0395	0.0174
1	2	0.0837	0.0709	0.0298	0.0256	0.0216	0.0091
	3	0.1109	0.0911	0.0386	0.0339	0.0278	0.0118
	4	0.1351	0.1106	0.0464	0.0413	0.0338	0.0141
	5	0.1613	0.1314	0.0557	0.0493	0.0401	0.0170

Table 2. Comparison of Mean squared error and privacy and efficiency ($\hat{\omega}^*$) of $\hat{T}_s^*$, $\hat{T}_r^*$ and $\hat{T}_p^*$ at varying values of k and π with non-response and measurement error.

Table 1 delineates the comparison of mean squared error of the suggested estimator $\hat{T}_s^*$ with other conventional estimators i.e. $\hat{T}_r^*$ and $\hat{T}_p^*$ and privacy protection measure suggested by Gupta et al. [21] which is represented in (6.1) at distinct values of k and π in the presence of non-response and measurement error at the same time under TPS technique. For increase in the value of π from 0.2 to 1 and k from 2 to 5, the mean squared error of each estimator grows and same behaviour is observed for the unified measure ($\hat{\omega}^*$).

Table 2 depicts the comparison of mean squared error of the suggested estimator $\hat{T}_s$ with other existing estimators i.e. $\hat{T}_r$ and $\hat{T}_p$ and privacy protection measure which is represented in (6.2) at distinct values of k and π in the absence of non-response and measurement error. The mean squared error of each estimator increases with increase in the value of π from 0.2 to 1 and k from 2 to 5, and same performance is detected for the privacy protection ($\hat{\omega}^*$).

It is also visualize from Tables 1 and Table 2 that the MSEs of ratio estimators ($\hat{T}_r^*$, $\hat{T}_r$) and product estimators ($\hat{T}_p^*$, $\hat{T}_p$) are the highest for all analyzed values of k, whereas our recommended estimators, ($\hat{T}_s^*$, $\hat{T}_s$) is the lowest among the ratio and the product type estimators. Also, the privacy measure is least for the proposed estimator ($\hat{T}_s^*$, $\hat{T}_s$) in the presence and absence of non-response and measurement error simultaneously. In both the scenario‘s, ($MSE(\hat{T}_s^*)$, $MSE(\hat{T}_s)$), i.e. the recommended estimator, is the most efficient amongst the alternatives. Furthermore, Table 1 and Table 2 indicates that the proposed estimator outperformed existing estimators also in terms of the unified measure ($\hat{\omega}^*$ and $\hat{\omega}$) of privacy and efficiency.

Table 3 illustrates the comparison of mean squared error of the suggested estimator T_s^* with other existing estimators i.e. T_r^* and T_p^* at specific values of k in the absence of non-response and measurement error entirely at the same time. When the value

π	k	*Estimator(s)*			*Unified Measure*($\hat{\omega}$)		
		$MSE(\hat{T}_r)$	$MSE(\hat{T}_p)$	$MSE(\hat{T}_s)$	$\hat{\omega}(\hat{T}_r)$	$\hat{\omega}(\hat{T}_p)$	$\hat{\omega}(\hat{T}_s)$
0.2	2	0.0209	0.0269	0.0080	0.0254	0.0092	0.0027
	3	0.0217	0.0274	0.0082	0.0328	0.0093	0.0028
	4	0.0226	0.0279	0.0086	0.0406	0.0095	0.0029
	5	0.0233	0.0283	0.0088	0.0479	0.0096	0.0030
0.4	2	0.0227	0.0285	0.0087	0.0248	0.0094	0.0028
	3	0.0243	0.0294	0.0093	0.0321	0.0097	0.0030
	4	0.0261	0.0306	0.0100	0.0398	0.0100	0.0033
	5	0.0275	0.0313	0.0105	0.0470	0.0103	0.0034
0.6	2	0.0240	0.0311	0.0099	0.0246	0.0101	0.0032
	3	0.0263	0.0325	0.0108	0.0319	0.0105	0.0035
	4	0.0289	0.0341	0.0118	0.0390	0.0110	0.0038
	5	0.0310	0.0354	0.0126	0.0468	0.0115	0.0040
0.8	2	0.0257	0.0328	0.0107	0.0240	0.0103	0.0033
	3	0.0287	0.0347	0.0120	0.0312	0.0109	0.0037
	4	0.0321	0.0368	0.0133	0.0388	0.0115	0.0041
	5	0.0348	0.0384	0.0143	0.0459	0.0120	0.0044
1	2	0.0294	0.0348	0.0116	0.0256	0.0106	0.0035
	3	0.0335	0.0371	0.0129	0.0339	0.0113	0.0039
	4	0.0372	0.0392	0.0142	0.0413	0.0119	0.0043
	5	0.0410	0.0416	0.0158	0.0493	0.0127	0.0048

Table 3. Comparison of Mean squared error and privacy and efficiency ($\hat{\omega}$) of $\hat{T}_r$, $\hat{T}_p$ and $\hat{T}_s$ at varying values of k and π without non-response and measurement error.

	Estimator(s)		
k	$MSE(T_r^*)$	$MSE(T_p^*)$	$MSE(T_s^*)$
2	0.0556	0.0529	0.0205
3	0.0692	0.0659	0.0254
4	0.0849	0.0815	0.0301
5	0.0960	0.0926	0.0347

Table 4. Comparison of Mean squared error of T_r^*, T_p^* and T_s^* at varying values of k with complete non-response and measurement error.

of k tends to increase, the mean squared error of each estimator also increases. The MSE of the suggested estimator i.e. T_s^* is minimal as the MSEs of the conventional one viz T_r^* and T_p^* are highest.

6.1 Natural population data set

The natural population dataset is based on abortion rates form Statistical Abstract of the United States: 2011 to elucidate the efficacious performance of our proposed estimator. The data is of $N = 51$ states and union territories of United States then a random sample is drawn from the population i.e., $n' = 20$. From n' we take sample of size $n = 12$. Let y, x, z be the number of abortions reported in the state of US during the years 2000, 2004, and 2005 respectively. The results are shown in Table 5 for different probability levels of sensitive variables, i.e. $\pi = 0.2, 0.4, 0.6, 0.8, 1$ when $k = 2$.

Estimator(s)	π				
(*Unified Measure*)	0.2	0.4	0.6	0.8	1
$MSE(\hat{T}_r^*)$ $(\hat{\omega}^*(\hat{T}_r^*))$	0.1500161 (0.0002191)	0.1500946 (0.0002192)	0.1501633 (0.0002193)	0.1502234 (0.0002194)	0.1502760 (0.0002195)
$MSE(\hat{T}_p^*)$ $(\hat{\omega}^*(\hat{T}_p^*))$	0.1550136 (0.0002264)	0.1551421 (0.0002266)	0.1552544 (0.0002267)	0.1553527 (0.0002269)	0.1554388 (0.0002270)
$MSE(\hat{T}_s^*)$ $(\hat{\omega}^*(\hat{T}_p^*))$	0.0346015 (0.0000505)	0.0345727 (0.0000505)	0.0345476 (0.0000504)	0.0345256 (0.0000505)	0.03450639 (0.0000504)
$MSE(\hat{T}_r)$ $(\hat{\omega}(\hat{T}_r))$	0.1492835 (0.0002180)	0.1493620 (0.0002181)	0.1494306 (0.0002182)	0.1494907 (0.0002183)	0.1495434 (0.0002184)
$MSE(\hat{T}_p)$ $(\hat{\omega}(\hat{T}_p))$	0.1540969 (0.0002251)	0.1542253 (0.0002252)	0.1543376 (0.0002254)	0.1544359 (0.0002256)	0.1545220 (0.0002257)
$MSE(\hat{T}_s)$ $(\hat{\omega}(\hat{T}_s))$	0.0348155 (0.0005085)	0.0347863 (0.0005081)	0.0347609 (0.0005077)	0.0347386 (0.0005074)	0.0347191 (0.0005071)

Table 5. Comparison of Mean squared error and unified measure at varying values of π when $k = 2$ and $(MSE(T_r^*) = 0.081953, MSE(T_p^*) = 0.088362 \;\&\; MSE(T_s^*) = 0.057562)$

Table 5 represents the comparison of mean squared error and unified measure of the proposed estimator i.e., ($\hat{T}_s^*$, $\hat{T}_s$ and T_s^*) with other existing estimators i.e. ($\hat{T}_r^*$, $\hat{T}_r$ and T_r^*) and ($\hat{T}_p^*$, $\hat{T}_p$ and T_p^*) at specific values of π in the presence and absence of non-response and measurement error simultaneously. When the value of π increases, the mean squared error and unified measure of existing estimators also increases but the mean squared error and unified measure of proposed estimator decreases. The MSE of proposed estimator is lowest and unified measure is highest which finds that the proposed estimator is better and each respondent privacy is protected as compared to the other existing estimators.

7. Conclusion

This study demonstrates a new chain ratio type estimator for estimating the population mean of the sensitive study as well as auxiliary variables in the presence of non-response and measurement error under two-phase sampling technique by utilizing ORRT models. The bias and mean squared errors of the proposed estimator are assessed up to the first order approximation. The efficiency of the proposed chain ratio type estimator has been compared with that of the existing one under TPS using two auxiliary variables. The condition by which the proposed estimator $\hat{T}_s^*$ proven to be more efficient than other existing estimators, notably $\hat{T}_r^*$ and $\hat{T}_p^*$ are also formed. The theoretical facts have been supported by conducting an empirical study. We executed a model-based simulation and a real dataset in R software to verify the theoretical results, and from the simulation results i.e., both hypothetical and real population shows that the suggested estimator outperform the other conventional estimators. As a result, if the requirements in Section 5 are satisfied, then the suggested estimators are encouraged for use in practice.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] S. L. Warner, *Randomized response: a survey technique for eliminating evasive answer bias*, J. Am. Stat. Assoc., **60**(309) (1965), 63–69.

[2] B. G. Greenberg, R. R. Jr. Kuebler, J. R. Abernathy, D. G. Hovertiz, *Application of the randomized response techniques in obtaining quantitative data*, J. Am. Stat. Assoc., **66**(334) (1971), 243-250.

[3] K. Pollock, Y. Bek, *A comparison of three randomized response models for quantitative data*, J. Am. Stat. Assoc., **71**(356) (1976), 884-886.

[4] S. Gupta, B. Gupta, S. Singh, *Estimation of sensitivity level of personal interview survey questions*, J. Stat. Plan. Inference, **100**(2) (2002), 239-247.

[5] S. Gupta, J. Shabbir, S. J. Sehra, *Mean and sensitivity estimation in optional randomized response models*, J. Stat. Plan. Inference, **140**(10) (2010), 2870-2874.

[6] S. Gupta, J. Shabbir, R. Sousa, P. Corte-Real, *Estimation of the mean of a sensitive variable in the presence of auxiliary information*, Commun. Stat. Theory Methods, **41**(13-14) (2012), 13-14.

[7] Q. Zhang, G. Kalucha, S. Gupta, S. Khalil, *Ratio estimation of the mean under RRT models*, Int. J. Stat. Manag. Syst., **22**(1) (2018), 97-113.

[8] S. Kumar, S. P. Kour, *Estimation of Sensitive Variable in Two-Phase Sampling under Measurement Error And Non-Response Using ORRT Models*, Sri Lankan J. Appl. Stat., **22**(3) (2021), 95-122.

[9] S. Kumar, S. P. Kour, *The joint influence of estimation of sensitive variable under measurement error and non-response using ORRT models*, J. Stat. Comput. Simul., **92**(17) (2022), 3583-3604.

[10] S. Kumar, S. P. Kour, Q. Zhang *An enhanced ratio-cum-product estimator with non-response and observational error by utilizing ORRT models: a sensitive estimation approach*, J. Stat. Comput. Simul., **93**(5) (2023), 818-836.

[11] S. Kumar, S. P. Kour, R. Gupta, J. P. S. Joorel *A Class of Logarithmic Type Estimator Under Non-Response and Measurement Error Using ORRT Models*, J. Indian Soc. Probab. Stat., (2023), doi:10.1007/s41096-023-00156-7.

[12] Q. Zaman, M. Ijaz, T. Zaman, *A randomization tool for obtaining efficient estimators through focus group discussion in sensitive surveys*, Commun. Stat. - Theory Methods, **52**(10) (2023), 3414-3428.

[13] J. Neyman, *Contribution to the theory of sampling human populations*, J Am Stat Assoc., **33**(201) (1938), 101-116.

[14] A. Sanaullah, H. Ali, M. Noor-ul-Amin, M. Hanif, *Generalized exponential chain ratio estimators under stratified two-phase random sampling*, Appl. Math. Comput., **226** (2014), 541-547.

[15] T. Zaman, C. Kadilar, *New class of exponential estimators for finite population mean in two-phase sampling*, Commun. Stat. Theory Methods, **50**(4) (2021), 874-889.

[16] S. Khalil, Q. Zhang, S. Gupta, *An enhanced two phase sampling ratio estimator for estimating population mean*, J. Sci. Res. **65**(3) (2021), 1-16.

[17] S. Khalil, Q. Zhang, S. Gupta, *Mean estimation of sensitive variables under measurement errors using optional RRT models*, Commun. Stat. Simul. Comput., **50**(5) (2021), 1417-1426.

[18] R. Onyango, B. Oduor, F. Odundol, *Joint influence of measurement errors and randomized response technique on mean estimation under stratified double sampling*, Open J. Math. Sci., **5**(1) (2021), 192-199.

[19] M. H. Hansen, W. N. Hurwitz, *The problem of non-response in sample surveys*, J. Am. Stat. Assoc., **41**(236) (1946), 517-529.

[20] G. Diana, S. Riaz, J. Shabbir, *Hansen and Hurwitz estimator with scrambled response on the second call*, J. Appl. Stat., **41**(3) (2014), 596-611.

[21] S. Gupta, S. Mehta, J. Shabbir, S. Khalil, *A unified measure of respondent privacy and model efficiency in quantitative RRT models*, J. Stat. Theory Pract., **12**(3) (2018), 506-511.

[22] Q. Zhang, S. Khalil, S. Gupta, *Ratio estimation of the mean under RRT models*, J. Stat. Theory Pract., **15**(3) (2021), 97-113.

[23] P. Mukhopadhyay, G. N. Singh, A. Bandyopadhyay, *A general estimation technique of population mean under stratified successive sampling in presence of random scrambled response and non-response*, Commun. Stat. Simul. Comput, **50**(5) (2021), 1417-1426.

[24] G. Diana, P. F. Perri, *A class of estimators for quantitative sensitive data*, Stat. Pap., **52**(3) (2011), 633-650.

[25] S. Khalil, M. Amin, M. Hanif, *Estimation of population mean for a sensitive variable in the presence of measurement error*, Int. J. Stat. Manag. Syst., **21**(1) (2018), 81-91.

[26] Z. Yan, J. Wang, J. Lai, *An efficiency and protection degree-based comparison among the quantitative randomized response strategies*, Commun. Stat. - Theory Methods, **38**(3) (2008), 400-408.

Communications in Advanced Mathematical Sciences
Vol. 6, No. 4, 211-225, 2023
Research Article
ISSN: 2651-4001
DOI:10.33434/cams.1313696

New Banach Sequence Spaces Defined by Jordan Totient Function

Uskan Devletli[1], Merve İlkhan Kara[2*]

Abstract
In this study, a special lower triangular matrix derived by combining Riesz matrix and Jordan totient matrix is used to construct new Banach spaces. $\alpha-,\beta-,\gamma-$duals of the resulting spaces are obtained and some matrix operators are characterized.

Keywords: Matrix mappings, Sequence space, $\alpha-,\beta-,\gamma-$duals
2010 AMS: 46B45, 47B07, 47H08

[1] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, ORCID: 0000-0001-9857-9513*
[2] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, ORCID: 0000-0002-0831-1474*
[*]**Corresponding author**: merveilkhan@gmail.com
Received: 13 June 2023, **Accepted:** 18 December 2023, **Available online:** 21 December 2023
How to cite this article: U. Devletli, M. İ. Kara, *New Banach Sequence Spaces Defined by Jordan Totient Function*, Commun. Adv. Math. Sci., 6(4) (2023), 211-225.

1. Introduction and Background

A sequence space is a vector subspace of the space ω of all sequences with real entries. Well known classical sequence spaces are the space of p-absolutely summable sequences ℓ_p, the space of bounded sequences ℓ_∞, the space of null sequences c_0, the space of convergent sequences c. Throughout the study, the notion ℓ is used instead of ℓ_1. Also bs, cs_0 and cs are the most frequently encountered spaces consisting of sequences generating bounded, null and convergent series, respectively. A Banach sequence space having continuous coordinates is called a BK space. Examples of BK spaces are c_0 and c endowed with the supremum norm $\|u\|_\infty = \sup_i |u_i|$, where $\mathbb{N} = \{1,2,3,...\}$.

By virtue of the fact that the matrix mappings between BK-spaces are continuous, the theory of matrix mappings plays an important role in the study of sequence spaces. Let U and V be two sequence spaces, $\Lambda = (\lambda_{ij})$ be an infinite matrix with real entries and Λ_i indicate the i^{th} row of Λ. If each term of the sequence $\Lambda u = ((\Lambda u)_i) = (\sum_j \lambda_{ij} u_j)$ is convergent, this sequence is called Λ-transform of $u = (u_i)$. Further, if $\Lambda u \in V$ for every sequence $u \in U$, then the matrix Λ defines a matrix mapping from U into V. (U,V) represents the collection of all matrices defined from U into V. Additionally, $B(U,V)$ is the set of all bounded (continuous) linear operators from U to V. A matrix $\Lambda = (\lambda_{ij})$ is called a triangle if $\lambda_{ii} \neq 0$ and $\lambda_{ij} = 0$ for $j > i$.

The matrix domain U_Λ of the matrix Λ in the space U is defined by

$$U_\Lambda = \{u \in \omega : \Lambda u \in U\}.$$

Since this space is also a sequnce space, the matrix domain has a crucial role to construct new sequence spaces. Moreover given any triangle Λ and a BK-space U, the sequence space U_Λ gives a new BK-space equipped with the norm $\|u\|_{U_\Lambda} = \|\Lambda u\|_U$. Several authors applied this technique to construct new Banach spaces with the help of special triangles. For relevant literature, the papers [1–17] can be referred.

The spaces

$$U^{\alpha}=\left\{t=(t_i)\in\omega:\sum_i|t_iu_i|<\infty \text{ for all } u=(u_i)\in U\right\},$$

$$U^{\beta}=\left\{t=(t_i)\in\omega:\sum_i t_iu_i \text{ converges for all } u=(u_i)\in U\right\},$$

$$U^{\gamma}=\left\{t=(t_i)\in\omega:\sup_i\left|\sum_i t_iu_i\right|<\infty \text{ for all } u=(u_i)\in U\right\},$$

are called the α-, β-, γ-duals of a sequence space U, respectively.

Note that $\frac{1}{p}+\frac{1}{q}=1$ and $\sup_i$, $\sum_i$, $\lim_i$ mean $\sup_{i\in\mathbb{N}}$, $\sum_{i=1}^{\infty}$, $\lim_{i\to\infty}$, respectively.

The Euler totient matrix $\Phi=(\phi_{ij})$ is defined as in [18]

$$\phi_{ij}=\begin{cases}\frac{\varphi(j)}{i} & , \quad \text{if } j\mid i\\ 0 & , \quad \text{if } j\nmid i,\end{cases}$$

where φ is the Euler totient function. In the recent time, by using this matrix, many new sequence and series spaces are defined and studied in the papers [19–27].

For $i\in\mathbb{N}$ with $i\neq 1$, $\varphi(i)$ gives the number of positive integers less than i which are coprime with i and $\varphi(1)=1$. Also, the equality

$$i=\sum_{j|i}\varphi(j)$$

holds for every $i\in\mathbb{N}$. For $i\in\mathbb{N}$ with $i\neq 1$, the Möbius function μ is defined as

$$\mu(i)=\begin{cases}(-1)^r & \text{if } i=p_1p_2...p_r, \text{ where } p_1,p_2,...,p_r \text{ are}\\ & \text{non-equivalent prime numbers}\\ 0 & \text{if } \tilde{p}^2\mid i \text{ for some prime number } \tilde{p}\end{cases}$$

and $\mu(1)=1$. The equality

$$\sum_{j|i}\mu(j)=0 \tag{1.1}$$

holds except for $i=1$.

The arithmetic function $J_r:\mathbb{N}\to\mathbb{N}$ with positive integer order r is called the Jordan totient function. This function generalizes the Euler totient function. If $r=1$, it is reduced to the Euler totient function. The value $J_r(i)$ gives the number of r-tuples of positive integers all less than or equal to i that form a coprime $(r+1)$-tuples together with i.

The Jordan function J_r is multiplicative, i.e. for $n_1,n_2\in\mathbb{N}$ with the greatest common divisor 1 the relation $J_r(n_1n_2)=J_r(n_1)J_r(n_2)$ holds.

Let $p_1^{a_1}p_2^{a_2}...p_k^{a_k}$ be the unique prime decomposition of $i\in\mathbb{N}$, then

$$J_r(i)=i^r(1-\frac{1}{p_1^r})(1-\frac{1}{p_2^r})...(1-\frac{1}{p_k^r}).$$

Also, the following equations hold:

$$\sum_{j|i}J_r(j)=i^r$$

and

$$\sum_{j|i}\frac{\mu(j)}{j^r}=\frac{J_r(i)}{i^r}.$$

In [28], the authors have defined a new matrix $\Upsilon^r = (\upsilon^r_{ij})$ as

$$\upsilon^r_{ij} = \begin{cases} \frac{J_r(j)}{i^r} &, \quad \text{if } j \mid i \\ 0 &, \quad \text{if } j \nmid i \end{cases}$$

for each $r \in \mathbb{N}$. It is observed that this matrix is regular; that is a limit preserving mapping c into c. By using this matrix they introduce a space consisting of sequences whose Υ^r-transforms are in the space ℓ_p for $1 \leq p < \infty$. Also, in [29], new Banach spaces are obtained by the aid of matrix domain of this matrix in the spaces ℓ_∞, c, c_0. In [30], the authors have studied the compact operators on the resulting spaces.

The Riesz matrix $E = (e_{ij})$ is defined as

$$e_{ij} = \begin{cases} \frac{q_j}{Q_i} &, \quad \text{if } 0 \leq j \leq i \\ 0 &, \quad \text{if } j > i, \end{cases}$$

where (q_j) is a sequence of positive numbers and $Q_i = \sum_{j=1}^{i} q_j$ for all $i \in \mathbb{N}$.

In a recent paper [31], the authors have constructed a new matrix called Riesz Euler totient matrix and study the domain of the matrix in the space ℓ_p. The Riesz Euler totient matrix $R_\Phi = (r_{ij})$ is defined as

$$r_{ij} = \begin{cases} \frac{q_j \varphi(j)}{Q_i} &, \quad \text{if } j \mid i \\ 0 &, \quad \text{if } j \nmid i. \end{cases}$$

The main purpose of this study is to construct new Banach spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$. The matrix R_{Υ^r} is obtained by combining Jordan totient matrix and Riesz matrix. After studying certain properties of the resulting spaces, α-, β- and γ-duals are computed. Finally some matrix mappings from the resulting spaces to the classical spaces are characterized.

2. The Sequence Spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$

In the present section, we introduce the sequence spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ by using the matrix R_{Υ^r}, where $1 < p < \infty$. Also, we present some theorems which give inclusion relations concerning these spaces.

The matrix $R_{\Upsilon^r} = (\nu_{ij})$ is defined as

$$\nu_{ij} = \begin{cases} \frac{q_j J_r(j)}{Q_i^r} &, \quad \text{if } j \mid i \\ 0 &, \quad \text{if } j \nmid i, \end{cases}$$

where $Q_i = q_1 + q_2 + ... + q_i$. We call this matrix as *Riesz Jordan totient matrix operator*.

Observe that in the special cases this matrix is reduced to the some matrices mentioned in the first section. If $r = 1$ and $q_j = 1$ for each j, it gives the Euler totient matrix. If $r = 1$, it gives the Riesz Euler totient matrix. If $q_j = 1$ for each j, it gives the Jorden totient matrix.

The inverse $R^{-1}_{\Upsilon^r} = (\nu^{-1}_{ij})$ of the matrix R_{Υ^r} is computed as

$$\nu^{-1}_{ij} = \begin{cases} \frac{\mu(\frac{i}{j})}{J_r(i)} \frac{Q_j^r}{q_i} &, \quad \text{if } j \mid i \\ 0 &, \quad \text{if } j \nmid i \end{cases}$$

for all $i, j \in \mathbb{N}$.

Now, we introduce the sequence spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ by

$$\ell_\infty(R_{\Upsilon^r}) = \left\{ u = (u_i) \in \omega : \sup_i \left| \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j \right| < \infty \right\},$$

$$\ell_p(R_{\Upsilon^r}) = \left\{ u = (u_i) \in \omega : \sum_i \left| \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j \right|^p < \infty \right\} \quad (1 < p < \infty),$$

$$\ell(R_{\Upsilon^r}) = \left\{ u = (u_i) \in \omega : \sum_i \left| \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j \right| < \infty \right\}.$$

Unless otherwise stated, $v = (v_i)$ will be the R_{Υ^r}-transform of a sequence $u = (u_i)$, that is, $v_i = (R_{\Upsilon^r} u)_i = \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j$ for all $i \in \mathbb{N}$.

Theorem 2.1. *The spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ are Banach spaces with the norms given by*

$$\|u\|_{\ell_\infty(R_{\Upsilon^r})} = \sup_i \left| \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j \right|,$$

$$\|u\|_{\ell_p(R_{\Upsilon^r})} = \left(\sum_i \left| \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j \right|^p \right)^{1/p} \quad (1 < p < \infty),$$

$$\|u\|_{\ell(R_{\Upsilon^r})} = \sum_i \left| \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j \right|.$$

Proof. We omit the proof which is straightforward. □

Corollary 2.2. *The spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ are BK-spaces, where $1 < p < \infty$.*

Theorem 2.3. *The space $U(R_{\Upsilon^r})$ is linearly isomorphic to U, where $U \in \{\ell_\infty, \ell_p, \ell\}$ and $1 < p < \infty$.*

Proof. Let f be a mapping defined from $U(R_{\Upsilon^r})$ to U such that $f(u) = R_{\Upsilon^r} u$ for all $u \in U(R_{\Upsilon^r})$. It is clear that f is linear. Also it is injective since the kernel of f consists of only zero. To prove that f is surjective consider the sequence $u = (u_i)$ whose terms are

$$u_i = \sum_{j|i} \frac{\mu(\frac{i}{j})}{J_r(i)} \frac{Q_j^r}{q_i} v_j$$

for all $i \in \mathbb{N}$, where $v = (v_j)$ is any sequence in U. It follows from (1.1) that

$$\begin{aligned}(R_{\Upsilon^r} u)_i &= \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j = \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) \sum_{k|j} \frac{\mu(\frac{j}{k})}{J_r(j)} \frac{Q_k^r}{q_j} v_k \\ &= \frac{1}{Q_i^r} \sum_{j|i} \sum_{k|j} \mu(\frac{j}{k}) Q_k^r v_k = \frac{1}{Q_i^r} \sum_{j|i} \left(\sum_{k|j} \mu(k) \right) Q_{\frac{i}{j}}^r v_{\frac{i}{j}} = \frac{1}{Q_i^r} \mu(1) Q_i^r v_i = v_i\end{aligned}$$

and so $u = (u_i) \in U(R_{\Upsilon^r})$. f preserves norms since the equality $\|u\|_{U(R_{\Upsilon^r})} = \|f(u)\|_U$ holds. □

Remark 2.4. *The space $\ell_2(R_{\Upsilon^r})$ is an inner product space with the inner product defined as $\langle u, \tilde{u} \rangle_{\ell_2(R_{\Upsilon^r})} = \langle R_{\Upsilon^r} u, R_{\Upsilon^r} \tilde{u} \rangle_{\ell_2}$, where $\langle .,. \rangle_{\ell_2}$ is the inner product on ℓ_2 which induces $\|.\|_{\ell_2}$.*

Theorem 2.5. *The space $\ell_p(R_{\Upsilon^r})$ is not an inner product space for $p \neq 2$.*

Proof. Consider the sequences $u = (u_i)$ and $\tilde{u} = (\tilde{u}_i)$, where

$$u_i = \begin{cases} \frac{\mu(i)}{J_r(i)} \frac{Q_1^r}{q_i} + \frac{\mu(\frac{i}{2})}{J_r(i)} \frac{Q_2^r}{q_i} & , \text{ if } i \text{ is even} \\ \frac{\mu(i)}{J_r(i)} \frac{Q_1^r}{q_i} & , \text{ if } i \text{ is odd} \end{cases}$$

and

$$\tilde{u}_i = \begin{cases} \frac{\mu(i)}{J_r(i)} \frac{Q_1^r}{q_i} - \frac{\mu(\frac{i}{2})}{J_r(i)} \frac{Q_2^r}{q_i} & , \text{ if } i \text{ is even} \\ \frac{\mu(i)}{J_r(i)} \frac{Q_1^r}{q_i} & , \text{ if } i \text{ is odd} \end{cases}$$

for all $i \in \mathbb{N}$. Then, we have $R_{\Upsilon^r} u = (1, 1, 0, ..., 0, ...) \in \ell_p$ and $R_{\Upsilon^r} \tilde{u} = (1, -1, 0, ..., 0, ...) \in \ell_p$. Hence, one can easily observe that

$$\|u + \tilde{u}\|_{\ell_p(R_{\Upsilon^r})}^2 + \|u - \tilde{u}\|_{\ell_p(R_{\Upsilon^r})}^2 \neq 2 \left(\|u\|_{\ell_p(R_{\Upsilon^r})}^2 + \|\tilde{u}\|_{\ell_p(R_{\Upsilon^r})}^2 \right).$$

□

Theorem 2.6. *The inclusion $\ell_p(R_{\Upsilon^r}) \subset \ell_q(R_{\Upsilon^r})$ strictly holds for $1 \le p < q < \infty$.*

Proof. It is clear that the inclusion $\ell_p(R_{\Upsilon^r}) \subset \ell_q(R_{\Upsilon^r})$ holds since $\ell_p \subset \ell_q$ for $1 \le p < q < \infty$. Also, $\ell_p \subset \ell_q$ is strict and so there exists a sequence $z = (z_i)$ in $\ell_q \backslash \ell_p$. By defining a sequence $u = (u_i)$ as

$$u_i = \sum_{j|i} \frac{\mu(\frac{i}{j})}{J_r(i)} \frac{Q_j^r}{q_i} z_j$$

for all $i \in \mathbb{N}$, we conclude that $u \in \ell_q(R_{\Upsilon^r}) \backslash \ell_p(R_{\Upsilon^r})$. Hence, the desired inclusion is strict. □

Theorem 2.7. *The inclusion $\ell_p(R_{\Upsilon^r}) \subset \ell_\infty(R_{\Upsilon^r})$ strictly holds for $1 \le p < \infty$.*

Proof. The inclusion is obvious since $\ell_p \subset \ell_\infty$ holds for $1 \le p < \infty$. Let $u = (u_i)$ be a sequence such that $u_i = \sum_{j|i} (-1)^j \frac{\mu(\frac{i}{j})}{J_r(i)} \frac{Q_j^r}{q_i}$ for all $i \in \mathbb{N}$. We obtain that $R_{\Upsilon^r} u = \left(\frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) \sum_{k|j} (-1)^k \frac{\mu(\frac{j}{k})}{J_r(j)} \frac{Q_k^r}{q_j} \right) = ((-1)^i) \in \ell_\infty \backslash \ell_p$ which implies that $u \in \ell_\infty(R_{\Upsilon^r}) \backslash \ell_p(R_{\Upsilon^r})$ for $1 \le p < \infty$. □

Lemma 2.8. *[32] The necessary and sufficient conditions for $\Lambda = (\lambda_{ij}) \in (U,V)$ with $U, V \in \{\ell_\infty, c, c_0, \ell_p, \ell\}$ and $p > 1$ can be read from Table 1. Here and in what follows, $\mathcal{N}$ denotes the family of all finite subsets of $\mathbb{N}$.*

To / From	ℓ_∞	c	c_0	ℓ_p	ℓ
ℓ_∞	**1.**	**4.**	**9.**	**14.**	**16.**
c	**1.**	**5.**	**10.**	**14.**	**16.**
c_0	**1.**	**6.**	**11.**	**14.**	**16.**
ℓ_p	**2.**	**7.**	**12.**	**-**	**17.**
ℓ	**3.**	**8.**	**13.**	**15**	**18.**

Table 1. The characterization of the class (U,V), where $U, V \in \{\ell_\infty, c, c_0, \ell_p, \ell\}$.

1.

$$\sup_i \sum_j |\lambda_{ij}| < \infty \tag{2.1}$$

2.

$$\sup_i \sum_j |\lambda_{ij}|^q < \infty \tag{2.2}$$

3.

$$\sup_{i,j} |\lambda_{ij}| < \infty \tag{2.3}$$

4.

$$\lim_i \lambda_{ij} \text{ exists for each } j \in \mathbb{N}, \tag{2.4}$$

$$\lim_i \sum_j |\lambda_{ij}| = \sum_j \left| \lim_i \lambda_{ij} \right|$$

5. (2.1), (2.4) *and*

$$\lim_i \sum_j \lambda_{ij} \text{ exists.}$$

6. (2.1) *and* (2.4)

7. (2.2) *and* (2.4)

8. (2.3) *and* (2.4)

9.

$$\lim_i \sum_j |\lambda_{ij}| = 0$$

10. (2.1) *and*

$$\lim_i \lambda_{ij} = 0 \text{ for each } j \in \mathbb{N}, \tag{2.5}$$

$$\lim_i \sum_j \lambda_{ij} = 0$$

11. (2.1) *and* (2.5)

12. (2.2) *and* (2.5)

13. (2.3) *and* (2.5)

14.

$$\sup_{K \in \mathscr{N}} \sum_i \left| \sum_{j \in K} \lambda_{ij} \right|^p < \infty$$

15.

$$\sup_j \sum_i |\lambda_{ij}|^p < \infty$$

16.

$$\sup_{N,K \in \mathscr{N}} \left| \sum_{i \in N} \sum_{j \in K} \lambda_{ij} \right| < \infty \Leftrightarrow \sup_{N \in \mathscr{N}} \sum_j \left| \sum_{i \in N} \lambda_{ij} \right| < \infty \Leftrightarrow \sup_{K \in \mathscr{N}} \sum_i \left| \sum_{j \in K} \lambda_{ij} \right| < \infty$$

17.

$$\sup_{N \in \mathscr{N}} \sum_j \left| \sum_{i \in N} \lambda_{ij} \right|^q < \infty$$

18.

$$\sup_j \sum_i |\lambda_{ij}| < \infty$$

3. The α-, β- and γ-duals

In this section, we determine the α-, β- and γ-duals of the sequence spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$, where $1<p<\infty$. In the following theorem, we determine the α-duals.

Theorem 3.1. *The α-duals of the spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ are as follows:*

$$(\ell_\infty(R_{\Upsilon^r}))^\alpha=\left\{t=(t_i)\in\omega:\sup_{N\in\mathscr{N}}\sum_j\left|\sum_{i\in N,j|i}\frac{\mu(\frac{i}{j})}{J_r(i)}\frac{Q_j^r}{q_i}t_i\right|<\infty\right\},$$

$$(\ell_p(R_{\Upsilon^r}))^\alpha=\left\{t=(t_i)\in\omega:\sup_{N\in\mathscr{N}}\sum_j\left|\sum_{i\in N,j|i}\frac{\mu(\frac{i}{j})}{J_r(i)}\frac{Q_j^r}{q_i}t_i\right|^q<\infty\right\},$$

$$(\ell(R_{\Upsilon^r}))^\alpha=\left\{t=(t_i)\in\omega:\sup_j\sum_{i\in\mathbb{N},j|i}\left|\frac{\mu(\frac{i}{j})}{J_r(i)}\frac{Q_j^r}{q_i}t_i\right|<\infty\right\}.$$

Proof. Consider the matrix $C=(c_{ij})$ defined by

$$c_{ij}=\begin{cases}\frac{\mu(\frac{i}{j})}{J_r(i)}\frac{Q_j^r}{q_i}t_i & , \quad j\mid i\\ 0 & , \quad j\nmid i\end{cases}$$

for any sequence $t=(t_i)\in\omega$. Let $U\in\{\ell_\infty,\ell_p,\ell\}$. Given any $u=(u_i)\in U(R_{\Upsilon^r})$, we have $t_iu_i=(Cv)_i$ for all $i\in\mathbb{N}$. This implies that $tu\in\ell$ with $u\in U(R_{\Upsilon^r})$ if and only if $Cv\in\ell$ with $v\in U$. It follows that $t\in(U(R_{\Upsilon^r}))^\alpha$ if and only if $C\in(U,\ell)$ which completes the proof in view of Lemma 2.8. ☐

Lemma 3.2. *[33, Theorem 3.1] Let $B=(b_{ij})$ be defined via a sequence $t=(t_k)\in\omega$ and the inverse matrix $\tilde{\Delta}=(\tilde{\delta}_{ij})$ of the triangle matrix $\Delta=(\delta_{ij})$ by*

$$b_{ij}=\sum_{k=j}^{i}t_k\tilde{\delta}_{kj}$$

for all $i,j\in\mathbb{N}$. Then,

$$U_\Delta^\beta=\{t=(t_k)\in\omega:B\in(U,c)\}$$

and

$$U_\Delta^\gamma=\{t=(t_k)\in\omega:B\in(U,\ell_\infty)\}.$$

Consequently, we have the following theorem.

Theorem 3.3. *Let define the following sets:*

$$A_1=\left\{t=(t_k)\in\omega:\lim_i\sum_{k=j,j|k}^{i}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}t_k \text{ exists for each } j\in\mathbb{N}\right\},$$

$$A_2=\left\{t=(t_k)\in\omega:\sup_i\sum_j\left|\sum_{k=j,j|k}^{i}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}t_k\right|^q<\infty\right\},$$

$$A_3=\left\{t=(t_k)\in\omega:\lim_i\sum_j\left|\sum_{k=j,j|k}^{i}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}t_k\right|=\sum_j\left|\sum_{k=j,j|k}^{\infty}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}t_k\right|\right\},$$

$$A_4=\left\{t=(t_k)\in\omega:\sup_{i,j}\left|\sum_{k=j,j|k}^{i}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}t_k\right|<\infty\right\}.$$

The β- and γ-duals of the spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ are as follows:
$(\ell_\infty(R_{\Upsilon^r}))^\beta=A_1\cap A_3$, $(\ell_p(R_{\Upsilon^r}))^\beta=A_1\cap A_2$, $(\ell(R_{\Upsilon^r}))^\beta=A_1\cap A_4$.
$(\ell_\infty(R_{\Upsilon^r}))^\gamma=A_2$ with $q=1$, $(\ell_p(R_{\Upsilon^r}))^\gamma=A_2$, $(\ell(R_{\Upsilon^r}))^\gamma=A_4$.

Proof. Let $t=(t_k)\in\omega$, $U\in\{\ell_\infty,\ell_p,\ell\}$ and $B=(b_{ij})$ be an infinite matrix with terms

$$b_{ij}=\begin{cases}\sum_{k=j,j|k}^{i} t_k\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k} & , \quad \text{if } 1\le j\le i\\ 0 & , \quad \text{if } j>i.\end{cases}$$

Hence it follows that

$$\sum_{j=1}^{i} t_j u_j=\sum_{j=1}^{j} t_j\left(\sum_{k|j}\frac{\mu(\frac{j}{k})}{J_r(j)}\frac{Q_k^r}{q_j}v_k\right)=\sum_{j=1}^{i}\left(\sum_{k=j,j|k}^{i} t_k\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}\right)v_j=(Bv)_i$$

for any $u=(u_i)\in U(R_{\Upsilon^r})$. This equality yields that $tu\in cs$ for $u\in U(R_{\Upsilon^r})$ if and only if $Bv\in c$ for $v\in U$. That is, $t\in(U(R_{\Upsilon^r}))^\beta$ if and only if $B\in(U,c)$. Hence, by Lemma 2.8, it is concluded that $(\ell_\infty(R_{\Upsilon^r}))^\beta=A_1\cap A_3$, $(\ell_p(R_{\Upsilon^r}))^\beta=A_1\cap A_2$, $(\ell(R_{\Upsilon^r}))^\beta=A_1\cap A_4$.

This equality also yields that $tu\in bs$ for $u\in U(R_{\Upsilon^r})$ if and only if $Bv\in\ell_\infty$ for $v\in U$. That is, $t\in(U(R_{\Upsilon^r}))^\gamma$ if and only if $B\in(U,\ell_\infty)$. Hence, by Lemma 2.8, it is concluded that $(\ell_\infty(R_{\Upsilon^r}))^\gamma=A_2$ with $q=1$, $(\ell_p(R_{\Upsilon^r}))^\gamma=A_2$, $(\ell(R_{\Upsilon^r}))^\gamma=A_4$. □

4. Certain Matrix Transformations

In this section, characterization of certain classes of matrices is given. The following result is obtained from Theorem 4.1 in [34] and this result is required to characterize the classes of matrices from $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ into ℓ_∞,c,c_0,ℓ.

Theorem 4.1. *Let $1<p<\infty$, $U\in\{\ell_\infty,\ell_p,\ell\}$ and $V\subset\omega$. Then, $\Lambda=(\lambda_{ij})\in(U_{R_{\Upsilon^r}},V)$ if and only if $\Theta^{(i)}=\left(\theta_{lj}^{(i)}\right)\in(U,c)$ for each fixed $i\in\mathbb{N}$ and $\Theta=(\theta_{ij})\in(U,V)$, where*

$$\theta_{lj}^{(i)}=\begin{cases}\sum_{k=j,j|k}^{l}\lambda_{ik}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k} & , \quad 1\le j\le l\\ 0 & , \quad j>l\end{cases}$$

and

$$\theta_{ij}=\sum_{k=j,j|k}^{\infty}\lambda_{ik}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}.$$

Proof. Let $\Lambda\in(U_{R_{\Upsilon^r}},V)$ and $u\in U_{R_{\Upsilon^r}}$. Then, the equality

$$\begin{aligned}\sum_{j=1}^{l}\lambda_{ij}u_j &= \sum_{j=1}^{l}\lambda_{ij}\left(\sum_{k|j}\frac{\mu(\frac{j}{k})}{J_r(j)}\frac{Q_k^r}{q_j}v_k\right) \\ &= \sum_{j=1}^{l}\left(\sum_{k=j,j|k}^{l}\lambda_{ik}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}\right)v_j=\sum_{j=1}^{l}\theta_{lj}^{(i)}v_j\end{aligned} \tag{4.1}$$

holds. Since Λu exists, it follows that $\Theta^{(i)}\in(U,c)$ for each fixed $i\in\mathbb{N}$. It is deduced that $\Lambda u=\Theta v$ as $l\to\infty$ in (4.1). Hence, $\Lambda u\in V$ implies that $\Theta v\in V$; that is $\Theta\in(U,V)$.

Conversely, suppose that $\Theta^{(i)}=\left(\theta_{lj}^{(i)}\right)\in(U,c)$ *for each fixed* $i\in\mathbb{N}$ and $\Theta=(\theta_{ij})\in(U,V)$. Let $u\in U_{R_{\Upsilon^r}}$. Then, $(\theta_{ij})\in U^\beta$ for each fixed $i\in\mathbb{N}$ implies that $(\lambda_{ij})\in U^\beta_{R_{\Upsilon^r}}$ for each fixed $i\in\mathbb{N}$. Hence, Λu exists. From equality (4.1), it follows that $\Lambda u=\Theta v$ as $l\to\infty$. This proves that $\Lambda\in(U_{R_{\Upsilon^r}},V)$. □

Theorem 4.2. *Let $\Lambda=(\lambda_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\Lambda\in(\ell_\infty(R_{\Upsilon^r}),\ell_\infty)$ if and only if

$$\lim_{l\to\infty}\sum_{k=j,j|k}^{l}\lambda_{ik}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k} \text{ exists for each fixed } i,j\in\mathbb{N}, \tag{4.2}$$

$$\lim_{l}\sum_{j}\left|\sum_{k=j,j|k}^{l}\lambda_{ik}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}\right|=\sum_{j}\left|\lim_{l}\sum_{k=j,j|k}^{l}\lambda_{ik}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}\right| \tag{4.3}$$

and

$$\sup_i \sum_j \left| \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| < \infty. \tag{4.4}$$

2. $\Lambda \in (\ell_\infty(R_{\Upsilon^r}), c)$ *if and only if* (4.2), (4.3),

$$\lim_i \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \text{ exists for each } j \in \mathbb{N}, \tag{4.5}$$

$$\lim_i \sum_j \left| \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| = \sum_j \left| \lim_i \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right|.$$

3. $\Lambda \in (\ell_\infty(R_{\Upsilon^r}), c_0)$ *if and only if* (4.2), (4.3),

$$\lim_i \sum_j \left| \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| = 0.$$

4. $\Lambda \in (\ell_\infty(R_{\Upsilon^r}), \ell)$ *if and only if* (4.2), (4.3) *and*

$$\sup_{N,K \in \mathscr{N}} \left| \sum_{i \in N} \sum_{j \in K} \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| < \infty. \tag{4.6}$$

Proof. The proof follows from Lemma 2.8 and Theorem 4.1. □

Theorem 4.3. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix and $p > 1$. Then, the following statements hold:*

1. $\Lambda \in (\ell_p(R_{\Upsilon^r}), \ell_\infty)$ *if and only if* (4.2),

$$\sup_{l \in \mathbb{N}} \sum_{j=1}^{l} \left| \sum_{k=j,j|k}^{l} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right|^q < \infty \text{ for each fixed } i \in \mathbb{N}, \tag{4.7}$$

$$\sup_i \sum_j \left| \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right|^q < \infty. \tag{4.8}$$

2. $\Lambda \in (\ell_p(R_{\Upsilon^r}), c)$ *if and only if* (4.2), (4.7), (4.5), (4.8).
3. $\Lambda \in (\ell_p(R_{\Upsilon^r}), c_0)$ *if and only if* (4.2), (4.7), (4.8),

$$\lim_i \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} = 0 \text{ for each } j \in \mathbb{N}. \tag{4.9}$$

4. $\Lambda \in (\ell_p(R_{\Upsilon^r}), \ell)$ *if and only if* (4.2), (4.7),

$$\sup_{N \in \mathscr{N}} \sum_j \left| \sum_{i \in N} \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right|^q < \infty.$$

Proof. The proof follows from Lemma 2.8 and Theorem 4.1. □

Theorem 4.4. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\Lambda \in (\ell(R_{\Upsilon^r}), \ell_\infty)$ *if and only if* (4.2),

$$\sup_{l,j} \left| \sum_{k=j, j|k}^{l} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| < \infty \text{ for each fixed } i \in \mathbb{N}, \tag{4.10}$$

$$\sup_{i,j} \left| \sum_{k=j, j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| < \infty. \tag{4.11}$$

2. $\Lambda \in (\ell(R_{\Upsilon^r}), c)$ *if and only if* (4.2), (4.10), (4.5), (4.11).
3. $\Lambda \in (\ell(R_{\Upsilon^r}), c_0)$ *if and only if* (4.2), (4.10), (4.9), (4.11).
4. $\Lambda \in (\ell(R_{\Upsilon^r}), \ell)$ *if and only if* (4.2), (4.10),

$$\sup_{j} \sum_{i} \left| \sum_{k=j, j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| < \infty.$$

Proof. The proof follows from Lemma 2.8 and Theorem 4.1. □

Corollary 4.5. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\Lambda \in (\ell_\infty(R_{\Upsilon^r}), bs)$ *if and only if* (4.2), (4.3),

$$\sup_{i} \sum_{j} \left| \sum_{l=1}^{i} \sum_{k=j, j|k}^{\infty} \lambda_{lk} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| < \infty. \tag{4.12}$$

2. $\Lambda \in (\ell_\infty(R_{\Upsilon^r}), cs)$ *if and only if* (4.2), (4.3),

$$\lim_{i} \sum_{l=1}^{i} \sum_{k=j, j|k}^{\infty} \lambda_{lk} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \text{ exists for each } j \in \mathbb{N}, \tag{4.13}$$

$$\lim_{i} \sum_{j} \left| \sum_{l=1}^{i} \sum_{k=j, j|k}^{\infty} \lambda_{lk} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| = \sum_{j} \left| \lim_{i} \sum_{l=1}^{i} \sum_{k=j, j|k}^{\infty} \lambda_{lk} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right|.$$

3. $\Lambda \in (\ell_\infty(R_{\Upsilon^r}), cs_0)$ *if and only if* (4.2), (4.3)

$$\lim_{i} \sum_{j} \left| \sum_{l=1}^{i} \sum_{k=j, j|k}^{\infty} \lambda_{lk} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| = 0.$$

Corollary 4.6. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\Lambda \in (\ell_p(R_{\Upsilon^r}), bs)$ *if and only if* (4.2), (4.7),

$$\sup_{i} \sum_{j} \left| \sum_{l=1}^{i} \sum_{k=j, j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right|^q < \infty. \tag{4.14}$$

2. $\Lambda \in (\ell_p(R_{\Upsilon^r}), cs)$ *if and only if* (4.2), (4.7), (4.13), (4.14).

3. $\Lambda \in (\ell_p(R_{\Upsilon^r}), cs_0)$ *if and only if* (4.2), (4.7), (4.14),

$$\lim_i \sum_{l=1}^{i} \sum_{k=j, j|k}^{\infty} \lambda_{lk} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} = 0 \text{ for each } j \in \mathbb{N}. \tag{4.15}$$

Corollary 4.7. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\Lambda \in (\ell(R_{\Upsilon^r}), bs)$ *if and only if* (4.2), (4.10),

$$\sup_{i,j} \left| \sum_{l=1}^{i} \sum_{k=j, j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| < \infty. \tag{4.16}$$

2. $\Lambda \in (\ell(R_{\Upsilon^r}), cs)$ *if and only if* (4.2), (4.10), (4.13), (4.16).

3. $\Lambda \in (\ell(R_{\Upsilon^r}), cs_0)$ *if and only if* (4.2), (4.10), (4.15), (4.16).

Theorem 4.8. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix and $p > 1$. Then, the following statements hold:*

(a) $\Lambda \in (\ell_\infty, \ell_p(R_{\Upsilon^r})) = (c, \ell_p(R_{\Upsilon^r})) = (c_0, \ell_p(R_{\Upsilon^r}))$ *if and only if*

$$\sup_{K \in \mathscr{N}} \sum_i \left| \sum_{j \in K} \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj} \right|^p < \infty.$$

(b) $\Lambda \in (\ell, \ell_p(R_{\Upsilon^r}))$ *if and only if*

$$\sup_j \sum_i \left| \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj} \right|^p < \infty.$$

Proof. The proof is given only for the matrix in $(\ell_\infty, \ell_p(R_{\Upsilon^r}))$ since the other case can be proven similarly. Given any infinite matrix $\Lambda = (\lambda_{ij}) \in (\ell_\infty, \ell_p(R_{\Upsilon^r}))$, define a new matrix $\hat{\Lambda} = (\hat{\lambda}_{ij})$ by

$$\hat{\lambda}_{ij} = \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj}$$

for all $i, j \in \mathbb{N}$. Then, for any $u = (u_j) \in \ell_\infty$, the equality

$$\sum_j \hat{\lambda}_{ij} u_j = \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \sum_j \lambda_{lj} u_j$$

means that $(\hat{\Lambda} u)_i = (R_{\Upsilon^r}(\Lambda u))_i$ for all $i \in \mathbb{N}$. This implies that $\Lambda u \in \ell_p(R_{\Upsilon^r})$ for $u = (u_j) \in \ell_\infty$ if and only if $\hat{\Lambda} u \in \ell_p$ for $u = (u_j) \in \ell_\infty$. Hence, we conclude from Lemma 2.8 that

$$\sup_{K \in \mathscr{N}} \sum_i \left| \sum_{j \in K} \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj} \right|^p < \infty.$$

□

Theorem 4.9. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix. Then, the following statements hold:*

(a) $\Lambda \in (\ell_\infty, \ell_\infty(R_{\Upsilon^r})) = (c, \ell_\infty(R_{\Upsilon^r})) = (c_0, \ell_\infty(R_{\Upsilon^r}))$ *if and only if*

$$\sup_i \sum_j \left| \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj} \right| < \infty.$$

(b) $\Lambda \in (\ell, \ell_\infty(R_{\Upsilon^r}))$ *if and only if*

$$\sup_{i,j} \left| \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj} \right| < \infty.$$

Proof. The proof follows with the same way in the proof of Theorem 4.8. □

Theorem 4.10. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix. Then, the following statements hold:*
(a) $\Lambda \in (\ell_\infty, \ell(R_{\Upsilon^r})) = (c, \ell(R_{\Upsilon^r})) = (c_0, \ell(R_{\Upsilon^r}))$ if and only if

$$\sup_{K\in\mathscr{N}} \sum_i \left| \sum_{j\in K} \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj} \right| < \infty.$$

(b) $\Lambda \in (\ell, \ell(R_{\Upsilon^r}))$ if and only if

$$\sup_j \sum_i \left| \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj} \right| < \infty.$$

Proof. The proof follows with the same way in the proof of Theorem 4.8. □

Now, we investigate the norm of the bounded linear matrix operators from $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ into $\ell_\infty(R_{\Upsilon^r})$ and $\ell(R_{\Upsilon^r})$. Firstly, we have a lemma which is essential for our investigation.

Lemma 4.11. *Given any infinite matrix $\Lambda = (\lambda_{ij})$, the norm of bounded linear operators is defined by*

$$\|\Lambda\|_{(\ell_\infty,\ell_\infty)} = \|\Lambda\|_{(\ell_p,\ell_\infty)} = \sup_i \sum_j |\lambda_{ij}|^q$$

$$\|\Lambda\|_{(\ell,\ell_\infty)} = \sup_{i,j} |\lambda_{ij}|$$

$$\|\Lambda\|_{(\ell_\infty,\ell)} = \|\Lambda\|_{(\ell_p,\ell)} = \sup_{K\in\mathscr{N}} \sum_j \left| \sum_{i\in K} \lambda_{ij} \right|^q$$

$$\|\Lambda\|_{(\ell,\ell)} = \sup_j \sum_i |\lambda_{ij}|.$$

Theorem 4.12. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix.*
(a) If $\Lambda \in B(\ell_\infty(R_{\Upsilon^r}), \ell_\infty(R_{\Upsilon^r}))$ or $\Lambda \in B(\ell_p(R_{\Upsilon^r}), \ell_\infty(R_{\Upsilon^r}))$, then

$$\sup_i \sum_j \left| \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right|^q < \infty$$

and

$$\|\Lambda\|_{(\ell_\infty(R_{\Upsilon^r}),\ell_\infty(R_{\Upsilon^r}))} = \|\Lambda\|_{(\ell_p(R_{\Upsilon^r}),\ell_\infty(R_{\Upsilon^r}))} = \sup_i \sum_j \left| \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right|^q.$$

(b) If $\Lambda \in B(\ell(R_{\Upsilon^r}), \ell_\infty(R_{\Upsilon^r}))$, then

$$\sup_{i,j} \left| \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right| < \infty$$

and

$$\|\Lambda\|_{(\ell(R_{\Upsilon^r}),\ell_\infty(R_{\Upsilon^r}))} = \sup_{i,j} \left| \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right|.$$

(c) If $\Lambda \in B(\ell_\infty(R_{\Upsilon^r}), \ell(R_{\Upsilon^r}))$ or $\Lambda \in B(\ell_p(R_{\Upsilon^r}), \ell(R_{\Upsilon^r}))$, then

$$\sup_{K\in\mathscr{N}} \sum_j \left| \sum_{i\in K} \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right|^q < \infty$$

and

$$\|\Lambda\|_{(\ell_\infty(R_{\Upsilon^r}),\ell(R_{\Upsilon^r}))} = \|\Lambda\|_{(\ell_p(R_{\Upsilon^r}),\ell(R_{\Upsilon^r}))} = \sup_{K\in\mathscr{N}} \sum_j \left| \sum_{i\in K} \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right|^q.$$

(d) If $\Lambda \in B(\ell(R_{\Upsilon^r}), \ell(R_{\Upsilon^r}))$, then

$$\sup_j \sum_i \left| \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right| < \infty$$

and

$$\|\Lambda\|_{(\ell(R_{\Upsilon^r}),\ell(R_{\Upsilon^r}))} = \sup_j \sum_i \left| \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right|.$$

Proof. Let $\tilde{\Lambda} = R_{\Upsilon^r} \Lambda R_{\Upsilon^r}^{-1}$. From Theorem 2.3, it is known that the spaces $U(R_{\Upsilon^r})$ and U are linearly isomorphic. Hence, we deduce from the following diagram

$$\begin{array}{ccc} U(R_{\Upsilon^r}) & \xrightarrow{\Lambda} & V(R_{\Upsilon^r}) \\ {\scriptstyle R_{\Upsilon^r}^{-1}} \uparrow & & \downarrow {\scriptstyle R_{\Upsilon^r}} \\ U & \xrightarrow[\tilde{\Lambda} = R_{\Upsilon^r} \Lambda R_{\Upsilon^r}^{-1}]{} & V \end{array}$$

that $\|\Lambda\|_{(U(R_{\Upsilon^r}),V(R_{\Upsilon^r}))} = \|\tilde{\Lambda}\|_{(U,V)}$, where $U \in \{\ell_\infty, \ell_p, \ell\}$ and $V \in \{\ell_\infty, \ell\}$. Thus, the desired results follows from Lemma 4.11. □

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] B. Altay, F. Başar, M. Mursaleen, *On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I*, Inform. Sci., **176**(10) (2006), 1450-1462.

[2] F. Başar, B. Altay, *On the space of sequences of p-bounded variation and related matrix mappings*, Ukrainian Math. J., **55**(1) (2003), 136-147.

[3] F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, İstanbul, 2012.

[4] E. E. Kara, *Some topological and geometrical properties of new Banach sequence spaces*, J. Inequal. Appl., **2013**(38) (2013), 15 pages.

[5] M. Kirişci, F. Başar, *Some new sequence spaces derived by the domain of generalized difference matrix*, Comput. Math. Appl., **60** (2010), 1299-1309.

[6] M. Kirişci, *Riesz type integrated and differentiated sequence spaces*, Bull. Math. Anal. Appl., **7**(2) (2015), 14-27.

[7] S. Demiriz, E. E. Kara, *On compact operators on some sequence spaces related to matrix $B(r,s,t)$*, Thai J. Math., **14**(3) (2016), 651-666.

[8] M. Mursaleen, A.K. Noman, *On some new difference sequence spaces of non-absolute type*, Math. Comput. Modelling, **52**(3-4) (2010), 603-617.

[9] T. Yaying, B. Hazarika, *On sequence spaces defined by the domain of a regular Tribonacci matrix*, Math. Slovaca, **70**(3) (2020), 697-706.

[10] P. Zengin Alp, *A new paranormed sequence space defined by Catalan conservative matrix*, Math. Methods Appl. Sci., **44**(9) (2021), 7651-7658.

[11] T. Yaying, B. Hazarika, M. Mursaleen, *On sequence space derived by the domain of q-Cesaro matrix in ℓ_p space and the associated operator ideal*, J. Math. Anal. Appl., **493**(1) (2021), 124453.

[12] M. İlkhan, E. E. Kara, *Matrix transformations and compact operators on Catalan sequence spaces*, J. Math. Anal. Appl., **498**(1) (2021), 124925.

[13] T. Yaying, *Paranormed Riesz difference sequence spaces of fractional order*, Kragujevac J. Math., **46**(2) (2022), 175-191.

[14] P. Zengin Alp, E. E. Kara, *New Banach spaces defined by the domain of Riesz-Fibonacci matrix*, Korean J. Math., **29**(4) (2021), 665-677.

[15] M. İlkhan Kara, H. Roopaei, *The Cesàro–Gamma operator and its associated sequence space*, Adv. Oper. Theory, **6**(3) (2021), 1-21.

[16] H. Roopaei, M. İlkhan Kara, *Negative difference operator and its associated sequence space*, Numer. Funct. Anal. Optim., **42**(4) (2021), 480-496.

[17] H. Roopaei, M. İlkhan, *Fractional Cesàro matrix and its associated sequence space*, Concr. Oper., **8**(1) (2021), 24-39.

[18] M. İlkhan, E. E. Kara, *A new Banach space defined by Euler totient matrix operator*, Oper. Matrices, **13**(2) (2019), 527-544.

[19] S. Demiriz, S. Erdem, *Domain of Euler-totient matrix operator in the space Li_p*, Korean J. Math., **28**(2) (2020), 361-378.

[20] S. Demiriz, M. İlkhan, E. E. Kara, *Almost convergence and Euler totient matrix*, Ann. Funct. Anal., **11**(3) (2020), 604-616.

[21] S. Erdem, S. Demiriz, *4-dimensional Euler-totient matrix operator and some double sequence spaces*, Math. Sci. Appl. E-Notes, **8**(2) (2020), 110-122.

[22] G. C. Hazar Güleç, Merve İlkhan, *A new paranormed series space using Euler totient means and some matrix transformations*, Korean J. Math., **28**(2) (2020), 205-221.

[23] G. C. Hazar Güleç, Merve İlkhan, *A new characterization of absolute summability factors*, Commun. Optim. Theory, **2020** (2020), Article ID 15, 11 pages.

[24] M. İlkhan, *Matrix domain of a regular matrix derived by Euler totient function in the spaces c_0 and c*, Mediterr. J. Math., **17** (2020), Article number 27.

[25] M. İlkhan, G. C. Hazar Güleç, *A study on absolute Euler totient series space and certain matrix transformations*, Mugla J. Sci. Technol., **6**(1) (2020), 112-119.

[26] M. İlkhan, S. Demiriz, E. E. Kara, *A new paranormed sequence space defined by Euler totient matrix*, Karaelmas Sci. Eng. J., **9**(2) (2019), 277-282.

[27] M. İlkhan, *Certain geometric properties and matrix transformations on a newly introduced Banach space*, Fundam. J. Math. Appl., **3**(1) (2020), 45-51.

[28] M. İlkhan, N. Şimşek, E. E. Kara, *A new regular infinite matrix defined by Jordan totient function and its matrix domain in ℓ_p*, Math. Methods Appl. Sci., **44**(9) (2021), 7622-7633.

[29] E. E. Kara, N. Şimşek, M. İlkhan Kara, *On new sequence spaces related to domain of the Jordan totient matrix*, S. A. Mohiuddine, B. Hazarika (editors), *Sequence space theory with applications*, New York, Chapman and Hall/CRC, 2022, pp. 1-19.

[30] M. İlkhan, E. E. Kara, F. Usta, *Compact operators on the Jordan totient sequence spaces*, Math. Methods Appl. Sci., **44**(9) (2021), 7666-7675.

[31] M. İlkhan, M. A. Bayrakdar, *A study on matrix domain of Riesz-Euler totient matrix in the space of p-absolutely summable sequences*, Commun. Adv. Math. Sci., **4**(1) (2021), 14-25.

[32] M. Stieglitz, H. Tietz, *Matrix transformationen von folgenraumen eine ergebnisübersicht*, Math. Z., **154** (1977), 1-16.

[33] B. Altay, F. Başar, *Certain topological properties and duals of the matrix domain of a triangle matrix in a sequence space*, J. Math. Anal. Appl., **336** (2007), 632–645.

[34] M. Kirişci, F. Başar, *Some new sequence spaces derived by the domain of generalized difference matrix*, Comput. Math. Appl., **60** (2010), 1299-1309.

Communications in Advanced Mathematical Sciences
Vol. 6, No. 4, 226-239, 2023
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1372245

On Some Properties of Bihyperbolic Numbers of The Lucas Type

Fügen Torunbalcı Aydın[1]

Abstract
To date, many authors in the literature have worked on special arrays in various computational systems. In this article, Lucas type bihyperbolic numbers were defined and their algebraic properties were examined. Bihyperbolic Lucas numbers were studied by Azak in 2021. Therefore, we only examined bihyperbolic Jacobsthal-Lucas and Pell-Lucas numbers. We also gave properties of bihyperbolic Jacobstal-Lucas and bihyperbolic Pell-Lucas numbers such as recursion relation, derivation function, Binet formula, D'Ocagne identity, Cassini identity and Catalan identity.

Keywords: Bihyperbolic number, Bihyperbolic Jacobsthal-Lucas number, Bihyperbolic Pell-Lucas number, Jacobsthal-Lucas number, Lucas number, Pell-Lucas number
2010 AMS: 05A15, 11B37,11B39

[1] *Department of Mathematical Engineering, Faculty of Chemical and Metallurgical Engineering, Yildiz Technical University, Davutpaşa Campus, 34220, Esenler, Istanbul, Turkey, ORCID: 0000-0001-9292-1832*

Received: 6 October 2023, **Accepted:** 22 December 2023, **Available online:** 25 December 2023
How to cite this article: F. Torunbalcı Aydın, *On Some Properties of Bihyperbolic Numbers of The Lucas Type*, Commun. Adv. Math. Sci., 6(4) (2023), 226-239.

1. Introduction

In 1843, Hamilton [1] discovered quaternions. After Hamilton's discovery of real quaternions Cockle [2] revealed the tessarine numbers in 1848. The difference between Tessarine numbers and quaternions is that they have the property of change. The quaternions are not commutative. After Cockle's work on Tessarines in 1892, Segre [3] obtained bicomplex numbers by replacing the quaternions found by Hamilton and Clifford with complex numbers with real coefficients and formed an isomorphic algebra with Tessarine numbers. With the discovery of bicomplex numbers, a new number system has been found which is called a system of real Tesssarines and defined as follows

$$\{a+jc \,|\, a,c\in\mathbb{R},\ j^2=1,\ j\notin\mathbb{R}\}$$

The real Tessarine numbers are called hyperbolic numbers [6]. These new numbers are called generalized commutative hypercomplex numbers as follows

$$\{q=q_0+iq_1+jq_2+kq_3 \,|\, q_0,q_1,q_2,q_3\in\mathbb{R}\}$$

where

$$i^2=k^2=\alpha,\ j^2=1,\ ij=ji=k.$$

These new numbers are called elliptic, parabolic, or hyperbolic commutative quaternion, respectively, according to which alpha is $\alpha < 0, \alpha = 0$ or $\alpha > 0$. In particular, bicomplex numbers are the $\alpha = -1$ case. The bicomplex numbers are generalized by Catoni et al.[4].
In [5], Price introduced the set of bicomplex numbers, which can be represented as

$$\mathbb{BC} = \{ q = (q_1 + i q_2) + j(q_3 + i q_4) \mid q_1, q_2, q_3, q_4 \in \mathbb{R} \}$$

where

$$i^2 = -1,\ j^2 = -1,\ ij = ji.$$

Recently, many authors have considered special number sequences with different number systems.
The bihyperbolic numbers are numbers that can be written as a linear combination of pairs of hyperbolic number. These numbers allow to establish a connection between bicomplex numbers and Euclidean 4-space. In 2008, Pogorui et al.[6] bihyperbolic numbers set is defined by as follows

$$\mathscr{B}h = \{ q = a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}; j_1, j_2, j_3 \notin \mathbb{R} \}$$

where j_1, j_2 and j_3 satisfy the conditions

$$j_1^2 = j_2^2 = j_3^2 = 1,\ j_1 j_2 = j_2 j_1 = j_3,\ j_1 j_3 = j_3 j_1 = j_2,\ j_2 j_3 = j_3 j_2 = j_1.$$

The addition and subtraction of two bihyperbolic numbers can be expressed as follows:

$$\begin{aligned} q \pm r = &\ (a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3) \pm (b_0 + b_1 j_1 + b_2 j_2 + b_3 j_3) \\ = &\ (a_0 \pm b_0) + j_1(a_1 \pm b_1) + j_2(a_2 \pm b_2) + j_3(a_3 \pm b_3) \end{aligned}$$

The multiplication of two bihyperbolic numbers can be expressed as follows:

$$\begin{aligned} q \times r = &\ (a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3) \times (b_0 + b_1 j_1 + b_2 j_2 + b_3 j_3) \\ = &\ (a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3) + j_1(a_0 b_1 + a_1 b_0 + a_2 b_3 + a_3 b_2) \\ &\ + j_2(a_0 b_2 + a_1 b_3 + a_2 b_0 + a_3 b_1) + j_3(a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) \end{aligned}$$

Bihyperbolic numbers have three different conjugations and represented as follows:

$$\bar{q}^{j_1} = a_0 + j_1 a_1 - j_2 a_2 - j_3 a_3,$$

$$\bar{q}^{j_2} = a_0 - j_1 a_1 + j_2 a_2 - j_3 a_3,$$

$$\bar{q}^{j_3} = a_0 - j_1 a_1 - j_2 a_2 + j_3 a_3.$$

In 2002, Olariu [7] introduced commutative hypercomplex numbers of different dimensions, and in his book he called these numbers in 4-dimensional circular fourcomplex numbers or hyperbolic fourcomplex numbers if $\alpha = -1$ or $\alpha = 1$, respectively. In 2008, hyperbolic fourcomplex numbers are called bihyperbolic numbers by Pogorui et al.[6] and they studied the roots of bihyperbolic polynomials. In 2020, the algebraic properties of these numbers were studied by Bilgin and Ersoy [8]. In 2021, Gürses et al. have studied dual-generalized complex and hyperbolic generalized complex numbers. Moreover, for $J = j$ and $p = 1$, they have obtained bihyperbolic numbers [9]. In 2021, [10] Brod et al. have introduced identities and summation formulas of bihyperbolic Fibonacci, Pell and Jacobsthal numbers as follows:

$$\mathscr{B}hF_n = F_n + F_{n+1} j_1 + F_{n+2} j_2 + F_{n+3} j_3,$$

$$\mathscr{B}hJ_n = J_n + J_{n+1} j_1 + J_{n+2} j_2 + J_{n+3} j_3,$$

$$\mathscr{B}hP_n = P_n + P_{n+1} j_1 + P_{n+2} j_2 + P_{n+3} j_3,$$

where j_1, j_2 and j_3 satisfy the conditions

$$j_1^2 = j_2^2 = j_3^2 = 1,\ j_1 j_2 = j_2 j_1 = j_3,\ j_1 j_3 = j_3 j_1 = j_2,\ j_2 j_3 = j_3 j_2 = j_1.$$

and in [11] Brod et al. were studied on a new generalization of bihyperbolic Pell numbers.
In 2021,[12] Azak defined bihyperbolic Lucas and bihyperbolic generalized Fibonacci numbers and given some new identities of these numbers as follows:

$$\mathscr{B}hL_n = L_n + L_{n+1}\, j_1 + L_{n+2}\, j_2 + L_{n+3}\, j_3$$

where j_1, j_2 and j_3 satisfy the conditions

$$j_1^2 = j_2^2 = j_3^2 = 1,\; j_1 j_2 = j_2 j_1 = j_3,\, j_1 j_3 = j_3 j_1 = j_2,\, j_2 j_3 = j_3 j_2 = j_1.$$

In 2022, Szynal-Liana et al.[13] introduced on certain bihypernomials related to Pell and Pell-Lucas numbers. In 2023, Gökbaş [24] introduced Gaussian-bihyperbolic numbers containing Pell and Pell-Lucas numbers.

In 1996, Horadam [14] introduced the Jacobsthal and Jacobsthal-Lucas sequences recurrence relation $\{J_n\}$ and $\{j_n\}$ are defined by the recurrence relations

$$J_0 = 0,\;\; J_1 = 1,\;\; J_n = J_{n-1} + 2J_{n-2},\;\; for\; n \geq 2,,$$

$$j_0 = 2,\;\; j_1 = 1,\;\; j_n = j_{n-1} + 2\, j_{n-2},\;\; for\; n \geq 2 \tag{1.1}$$

respectively.

In 1996, [14] Horadam studied on the Jacobsthal and Jacobsthal-Lucas sequences and in 1997, [15] he gave Cassini-like formulas as follows

$$J_{n+1}J_{n-1} - {J_n}^2 = (-1)^n.2^{n-1}, \tag{1.2}$$

$$j_{n+1}j_{n-1} - {j_n}^2 = 3^2.(-1)^{n+1}.2^{n-1}.$$

The first eleven terms of Jacobsthal sequence $\{J_n\}$ are $\{0,1,1,3,5,11,21,43,85,171,341\}$.
This sequence is given by the formula

$$J_n = \frac{2^n - (-1)^n}{3}. \tag{1.3}$$

The first eleven terms of Jacobsthal-Lucas sequence $\{j_n\}$ are $\{2,1,5,7,17,31,65,127,257,511,1025\}$.
This sequence is given by the formula

$$j_n = 2^n + (-1)^n.$$

Besides the n-th Jacobsthal and Jacobsthal-Lucas number are formulized as $J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $j_n = \alpha^n + \beta^n$, where $\alpha = 2$, $\beta = -1$.

Also, for Jacobsthal and Jacobsthal-Lucas numbers the following properties hold:

$$\begin{cases} J_n + j_n = 2J_{n+1}, \\ 3J_n + j_n = 2^{n+1}, \\ j_n J_n = J_{2n}, \\ J_m j_n + J_n j_m = 2J_{m+n}, \\ J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n}, \\ j_{n+1} + j_n = 3(J_{n+1} + J_n) = 3.2^n, \\ j_n J_{m+1} + 2 j_{n-1} J_m = j_{m+n}, \\ j_{n+1} - j_n = 3(J_{n+1} - J_n) + 4(-1)^{n+1} = 2^n + 2(-1)^{n+1}, \\ j_{n+r} - j_{n-r} = 3(J_{n+r} - J_{n-r}) = 2^{n-r}(2^{2r} - 1), \\ j_{n+r} + j_{n-r} = 3(J_{n+r} + J_{n-r}) + 4(-1)^{n-r}. \end{cases}$$

and summation formulas

$$\begin{cases} \sum\limits_{i=2}^{n} J_i = \frac{J_{n+2}-3}{2}, \\ \sum\limits_{i=1}^{n} j_i = \frac{j_{n+2}-5}{2}. \end{cases}$$

In 2018, [16] Torunbalcı Aydın were studied on the generalizations of the Jacobsthal sequence. In 2018, [17] gave a new generalization for Jacobsthal and Jacobsthal-Lucas sequences. In 2019, [18] Al-Kateeb gave a generalization of the Jacobsthal and Jacobsthal-Lucas numbers. In 2022, [19] Brod et al. were studied on generalized Jacobsthal and Jacobsthal-Lucas numbers.

In 1971, [20] Horadam studied on the Pell P_n and Pell-Lucas p_n sequences and Pell identities. The n-th Pell and n-th Pell-Lucas numbers is defined by respectively as follows

$$P_n = 2P_{n-1} + P_{n-2}\,,\ P_0 = 0,\ P_1 = 1,$$

$$p_n = 2p_{n-1} + p_{n-2}\,,\ p_0 = 2,\ p_1 = 2.$$

In 1985, Horadam and Mahon obtained some Pell P_n and Pell-Lucas p_n identities and summation formulas respectively as follows [21]

$$\begin{cases} P_{m-1}\,p_n + P_m\,p_{n+1} = p_{m+n}\,, \\ p_{n+1}\,p_{n-1} - p_n^2 = 8\,(-1)^{n+1}\,, \\ p_m\,p_n - p_{m+r}\,p_{n-r} = 8\,(-1)^{n-r+1}\,P_{m+r-n}\,P_r\,. \end{cases}$$

$$\begin{cases} \sum\limits_{r=1}^{n} p_r = \frac{(p_{n+1}+p_n-4)}{2}\,, \\ \sum\limits_{r=1}^{n} p_{2r} = \frac{(p_{2n+1}-2)}{2}\,, \\ \sum\limits_{r=1}^{n} p_{2r-1} = \frac{(p_{2n}-2)}{2}\,. \end{cases}$$

Besides the n-th Pell and Pell-Lucas number are formulized as $P_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$ and $p_n = \alpha^n + \beta^n$, where $\alpha = 1+\sqrt{2}$, $\beta = 1-\sqrt{2}$.

In 2006, some properties of sums involving Pell numbers were studied by Santana Falcon [22]. In 2018, Torunbalcı Aydın introduced bicomplex Pell and Pell-Lucas numbers [25].

Our subject of study is the combinatorial properties of bihyperbolic numbers of Lucas type, but since the article on bihyperbolic Lucas numbers was previously reviewed by Azak [6], only bihyperbolic Jacobsthal-Lucas and bihyperbolic Pell-Lucas numbers were examined in this study.

2. The Bihyperbolic Jacobsthal-Lucas Numbers

In this section, we define the bihyperbolic Jacobsthal-Lucas numbers. Then, we obtain the generating function, Binet's formula, d'Ocagne's identity, Cassini's identity, Catalan's identity and Honsberger identity.

Definition.2.1. For $n \geq 1$, the n-th bihyperbolic Jacobsthal-Lucas number $\mathscr{B}hj_n$ are defined by using the Jacobsthal-Lucas numbers as follows

$$\mathscr{B}hj_n = j_n + j_{n+1}\, j_1 + j_{n+2}\, j_2 + j_{n+3}\, j_3, \tag{2.1}$$

where j_1, j_2 and j_3 satisfy the conditions

$$j_1^2 = j_2^2 = j_3^2 = 1,\ \ j_1 j_2 = j_2 j_1 = j_3,\ j_1 j_3 = j_3 j_1 = j_2,\ j_2 j_3 = j_3 j_2 = j_1.$$

Theorem 2.1. *Let $\mathscr{B}hj_n$ be the n-th bihyperbolic Jacobsthal-Lucas number. For any integer $n \geq 0$,*

$$\mathscr{B}hj_n = \quad \mathscr{B}hj_{n-1} + 2\mathscr{B}hj_{n-2} \tag{2.2}$$

Proof. (2.2): By using Eq.(1.1) in Eq.(2.1) we obtain that,

$$\begin{aligned}
\mathscr{B}hj_n \quad &= j_n + j_{n+1}\, j_1 + j_{n+2}\, j_2 + j_{n+3}\, j_3 \\
&= (j_{n-1} + 2\, j_{n-2}) + j_1\,(j_n + 2\, j_{n-1}) \\
&\quad + j_2\,(j_{n+1} + 2\, j_n) + j_3\,(j_{n+2} + 2\, j_{n+1}) \\
&= (j_{n-1} + j_1\, j_n + j_2\, j_{n+1} + j_3\, j_{n+2}) \\
&\quad + 2\,(j_{n-2} + j_1\, j_{n-1} + j_2\, j_n + j_3\, j_{n+1}) \\
&= \mathscr{B}hj_{n-1} + 2\mathscr{B}hj_{n-2}
\end{aligned}$$

Also, initial values are $\mathscr{B}hj_0 = 2 + j_1 + 5\, j_2 + 7\, j_3,\ \mathscr{B}hj_1 = 1 + 5\, j_1 + 7\, j_2 + 17\, j_3$.

Let $\mathscr{B}hj_n$ and $\mathscr{B}hj_m$ be two bihyperbolic Jacobsthal-Lucas numbers such that

$$\mathscr{B}hj_n = j_n + j_1\, j_{n+1} + j_2\, j_{n+2} + j_3\, j_{n+3}$$

and

$$\mathscr{B}hj_m = j_m + j_1\, j_{m+1} + j_2\, j_{m+2} + j_3\, j\mathscr{B}hj_{m+3}$$

Then, the addition and subtraction of two bihyperbolic Jacobsthal-Lucas numbers are defined in the obvious way,

$$\begin{aligned}
\mathscr{B}hj_n \pm \mathscr{B}hj_m = &\quad (j_n + j_1\, j_{n+1} + j_2\, j_{n+2} + j_3\, j_{n+3}) \\
&\quad \pm (j_m + j_1\, j_{m+1} + j_2\, j_{m+2} + j_3\, j_{m+3}) \\
= &\quad (j_n \pm j_m) + j_1\,(j_{n+1} \pm j_{m+1}) \\
&\quad + j_2\,(j_{n+2} \pm j_{m+2}) + j_3\,(j_{n+3} \pm j_{m+3}).
\end{aligned}$$

The multiplication of two bihyperbolic Jacobsthal-Lucas numbers is defined by

$$\begin{aligned}
\mathscr{B}hj_n \times \mathscr{B}hj_m = &\quad (j_n + j_1\, j_{n+1} + j_2\, j_{n+2} + j_3\, j_{n+3}) \\
&\quad (j_m + j_1\, j_{m+1} + j_2\, j_{m+2} + j_3\, j_{m+3}) \\
= &\quad (j_n j_m + j_{n+1} j_{m+1} + j_{n+2} j_{m+2} + j_{n+3} j_{m+3}) \\
&\quad + j_1\,(j_{n+1} j_m + j_n j_{m+1} + j_{n+3} j_{m+2} + j_{n+2} j_{m+3}) \\
&\quad + j_2\,(j_{n+2} j_m + j_n j_{m+2} + j_{n+3} j_{m+1} + j_{n+1} j_{m+3}) \\
&\quad + j_3\,(j_{n+3} j_m + j_n j_{m+3} + j_{n+1} j_{m+2} + j_{n+2} j_{m+1}) \\
= &\quad \mathscr{B}hj_m \times \mathscr{B}hj_n.
\end{aligned} \tag{2.3}$$

Three kinds of conjugation can be defined for bihyperbolic numbers [6]. Therefore, conjugation of the bihyperbolic Jacobsthal-Lucas number is defined in three different ways as follows

$$\overline{\mathscr{B}hj}_n^{j_1} = j_n + j_1\, j_{n+1} - j_2\, j_{n+2} - j_3\, j_{n+3},$$

$$\overline{\mathscr{B}hj}_n^{j_2} = j_n - j_1\, j_{n+1} + j_2\, j_{n+2} - j_3\, j_{n+3}, \tag{2.4}$$

$$\overline{\mathscr{B}hj}_n^{j_3} = j_n - j_1\, j_{n+1} - j_2\, j_{n+2} + j_3\, j_{n+3}. \tag{2.5}$$

In the following theorem, some properties related to the conjugations of the bihyperbolic Jacobsthal-Lucas numbers are given.

Theorem 2.2. *Let $\overline{\mathscr{B}hj_n}^{j_1}$, $\overline{\mathscr{B}hj_n}^{j_2}$ and $\overline{\mathscr{B}hj_n}^{j_3}$, be three kinds of conjugation of the bihyperbolic Jacobsthal-Lucas number $\mathscr{B}hj_n$. In this case, we can give the following relations:*

$$\begin{aligned}
\mathscr{B}hj_n\overline{\mathscr{B}hj_n}^{j_1} &= j_n^2+j_{n+1}^2-j_{n+2}^2-j_{n+3}^2 \\
&\quad +2\,j_1\,(j_n j_{n+1}-j_{n+2}j_{n+3}),\\
\mathscr{B}hj_n\overline{\mathscr{B}hj_n}^{j_2} &= j_n^2-j_{n+1}^2-j_{n+2}^2+j_{n+3}^2 \\
&\quad +2\,j_2\,(j_n j_{n+2}-j_{n+1}j_{n+3}),\\
\mathscr{B}hj_n\overline{\mathscr{B}hj_n}^{j_3} &= j_n^2-j_{n+1}^2-j_{n+2}^2+j_{n+3}^2 \\
&\quad +2\,j_3\,(j_n j_{n+3}-j_{n+1}j_{n+2}).
\end{aligned}$$

Proof. The proof can be easily done using equations Eq.(2.4-2.5) .

In the following theorems, some properties related to the bihyperbolic Jacobsthal-Lucas numbers are given.

Theorem 2.3. *Let $\mathscr{B}hj_n$ be the n-th bihyperbolic Jacobsthal-Lucas number. For any integer $n\geq 0$, summation formula as follows:*

$$\sum_{k=0}^{n}\mathscr{B}hj_k=\frac{1}{2}\,(\mathscr{B}hj_{n+2}-\mathscr{B}hj_2)\,. \tag{2.6}$$

Proof. (2.6): Using the summation formula Eq.(1.3), we obtain

$$\begin{aligned}
\sum_{k=1}^{n}\mathscr{B}hj_k &= (\sum_{k=1}^{n}|_k+j_1\sum_{k=1}^{n}|_{k+1}+j_2\sum_{k=1}^{n}|_{k+2}+j_3\sum_{k=1}^{n}|_{k+3})\\
&= (\tfrac{j_{n+2}-5}{2})+j_1\,(\tfrac{j_{n+3}-7}{2})+j_2\,(\tfrac{j_{n+4}-17}{2})+j_3\,(\tfrac{j_{n+5}-31}{2})\\
&= \tfrac{1}{2}\,[\mathscr{B}hj_{n+2}-(5+7\,j_1+17\,j_2+31\,j_3)]\\
&= \tfrac{1}{2}\,[\mathscr{B}hj_{n+2}-(\mathscr{B}hj_2)]\,.
\end{aligned}$$

where $\mathscr{B}hj_2=(5+7\,j_1+17\,j_2+31\,j_3)\,.$

Theorem 2.4. ***(Generating function)***
Let $\mathscr{B}hj_n$ be the n-th bihyperbolic Jacobsthal-Lucas number. For the generating function of the bihyperbolic Jacobsthal-Lucas numbers is as follows:

$$\begin{aligned}
g_{\mathscr{B}hj_n}(t)=\sum_{n=0}^{\infty}\mathscr{B}hj_n t^n = &\quad \frac{\mathscr{B}hj_0+(\mathscr{B}hj_1-\mathscr{B}hj_0)\,t}{1-t-2t^2}\\
= &\quad \frac{(2+j_1+5\,j_2+7\,j_3)+t\,(-1+4\,j_1+2\,j_2+10\,j_3)}{1-t-2t^2}
\end{aligned}$$

Proof. (2.8): Using the definition of generating function, we obtain

$$g_{\mathscr{B}hj_n}(t)=\mathscr{B}hj_0+\mathscr{B}hj_1\,t+\ldots+\mathscr{B}hj_n\,t^n+\ldots\,. \tag{2.7}$$

Multiplying $(1-t-2t^2)$ both sides of Eq.(2.7) and using Eq.(2.2), we have

$$\begin{aligned}
(1-t-2t^2)\,g_{\mathscr{B}hj_n}(t) &= \mathscr{B}hj_0+(\mathscr{B}hj_1-\mathscr{B}hj_0)\,t\\
&\quad +(\mathscr{B}hj_2-\mathscr{B}hj_1-2\,\mathscr{B}hj_0)\,t^2\\
&\quad +(\mathscr{B}hj_3-\mathscr{B}hj_2-2\,\mathscr{B}hj_1)\,t^3+\ldots\\
&\quad +(\mathscr{B}hj_{k+1}-\mathscr{B}hj_k-2\,\mathscr{B}hj_{k-1})\,t^{k+1}+\ldots
\end{aligned}$$

where $\mathscr{B}hj_1-\mathscr{B}hj_0=-1+4\,j_1+2\,j_2+10\,j_3$, $\mathscr{B}hj_2-\mathscr{B}hj_1-2\,\mathscr{B}hj_0=0$ and $\mathscr{B}hj_3-\mathscr{B}hj_2-2\,\mathscr{B}hj_1=0\ldots=0.$

Theorem 2.5. ***(Binet's formula)*** *Let $\mathscr{B}hj_n$ be the n-th bihyperbolic Jacobsthal-Lucas number. For any integer $n \geq 0$, the Binet's formula for these numbers is as follows:*

$$\mathscr{B}hj_n = \hat{\alpha}\,\alpha^n + \hat{\beta}\,\beta^n\,. \tag{2.8}$$

where

$$\hat{\alpha} = 1 + j_1\,\alpha + j_2\,\alpha^2 + j_3\,\alpha^3,\;\; \alpha = 2\,,$$
$$\hat{\beta} = 1 + j_1\,\beta + j_2\,\beta^2 + j_3\,\beta^3,\;\; \beta = -1\,,$$
$$\hat{\alpha}\,\hat{\beta} = \hat{\beta}\,\hat{\alpha}\,.$$

Proof. Using the Binet's formula of Jacobsthal-Lucas number [15] and Eq.(2.1) we obtain that,

$$\begin{aligned}
\mathscr{B}hj_n = & \;\; j_n + j_1\,j_{n+1} + j_2\,j_{n+2} + j_3\,j_{n+3} \\
= & \;\; (\alpha^n + \beta^n) + j_1\,(\alpha^{n+1} + \beta^{n+1}) \\
& + j_2\,(\alpha^{n+2} + \beta^{n+2}) + j_3\,(\alpha^{n+3} + \beta^{n+3}) \\
= & \;\; \alpha^n(1 + j_1\,\alpha + j_2\,\alpha^2 + j_3\,\alpha^3) \\
& + \beta^n(1 + j_1\,\beta + j_2\,\beta^2 + i\,j_3\,\beta^3) \\
= & \;\; \hat{\alpha}\,\alpha^n + \hat{\beta}\,\beta^n\,.
\end{aligned}$$

Here, Binet's formula of the Jacobsthal-Lucas number sequence, $j_n = \alpha^n + \beta^n$ is used.

Theorem 2.6. ***(D'Ocagne's identity)*** *Let $\mathscr{B}hj_n$ be the n-th bihyperbolic Jacobsthal-Lucas number. For $m \geq n+1$, the following equality holds:*

$$\begin{aligned}
\mathscr{B}hj_m\,\mathscr{B}hj_{n+1} - \mathscr{B}hj_{m+1}\,\mathscr{B}hj_n = & \;\; (-2)^n\,(-9)\,J_{m-n}\,[-5 + 5\,j_1 - 5\,j_2 + 5\,j_3\,] \\
= & \;\; -3\,(\hat{\alpha}\,\hat{\beta}\,)\,(-2)^n\,(\alpha - \beta)\,J_{m-n}\,.
\end{aligned} \tag{2.9}$$

Proof. (2.9): Considering Eq.(2.3), using the commutative property of bihyperbolic numbers and d'Ocagne's identity of Jacobsthal-Lucas numbers [16], we obtain that

$$\begin{aligned}
\mathscr{B}hj_m\,\mathscr{B}hj_{n+1} - \mathscr{B}hj_{m+1}\,\mathscr{B}hj_n \;\; & = [\,(j_m j_{n+1} - j_{m+1} j_n) \\
& \quad + (j_{m+1} j_{n+2} - j_{m+2} j_{n+1}) \\
& \quad + (j_{m+2} j_{n+3} - j_{m+3} j_{n+2}) \\
& \quad + (j_{m+3} j_{n+4} - j_{m+4} j_{n+3})\,] \\
& + j_1\,[(j_m j_{n+2} - j_{m+1} j_{n+1}) \\
& \quad + (j_{m+1} j_{n+1} - j_{m+2} j_n) \\
& \quad + (j_{m+2} j_{n+4} - j_{m+3} j_{n+3}) \\
& \quad + (j_{m+3} j_{n+3} - j_{m+4} j_{n+2})\,] \\
& + j_2\,[(j_m j_{n+3} - j_{m+1} j_{n+2}) \\
& \quad + (j_{m+2} j_{n+1} - j_{m+3} j_n) \\
& \quad + (j_{m+1} j_{n+4} - j_{m+2} j_{n+3}) \\
& \quad + (j_{m+3} j_{n+2} - j_{m+4} j_{n+1})] \\
& + j_3\,[(j_m j_{n+4} - j_{m+1} j_{n+3}) \\
& \quad + (j_{m+3} j_{n+1} - j_{m+4} j_n) \\
& \quad + (j_{m+1} j_{n+3} - j_{m+2} j_{n+2}) \\
& \quad + (j_{m+2} j_{n+2} - j_{m+3} j_{n+1})] \\
= & \;\; (-2)^n\,(-9)\,J_{m-n} - 5(1 - j_1 + j_2 - j_3)\,.
\end{aligned}$$

where $\hat{\alpha}\,\hat{\beta} = \hat{\beta}\,\hat{\alpha}$ and the identities $j_m j_{n+1} - j_{m+1} j_n = (-2)^n\,(-9)\,j_{m-n}$, $-4\,j_{n-2} - j_{n+2} = -5\,j_n$ $-8\,j_{n-3} + j_{n+3} = 7\,j_n$ and $4\,j_{n-1} - 2\,j_{n+1} = -2\,j_n$ are used [16].

Theorem 2.6.A. *Now let's prove this identity using the Binet's formula:*

$$\begin{aligned}
\mathscr{B}hj_m\,\mathscr{B}hj_{n+1}-\mathscr{B}hj_{m+1}\,\mathscr{B}hj_n &= (\hat{\alpha}\,\alpha^m+\hat{\beta}\,\beta^m)\,(\hat{\alpha}\,\alpha^{n+1}+\hat{\beta}\,\beta^{n+1})\\
&\quad -(\hat{\alpha}\,\alpha^{m+1}+\hat{\beta}\,\beta^{m+1})\,(\hat{\alpha}\,\alpha^n+\hat{\beta}\,\beta^n)\\
&= \hat{\alpha}\,\hat{\beta}\,[\alpha^m\,\beta^n\,(-\alpha+\beta)+\alpha^n\,\beta^m\,(\alpha-\beta)]\\
&= \hat{\alpha}\,\hat{\beta}\,(\alpha\,\beta)^n\,(\alpha-\beta)\,[\beta^{m-n}-\alpha^{m-n}]\\
&= -(\hat{\alpha}\,\hat{\beta}\,)(-2)^n\,(\alpha-\beta)\,[\alpha^{m-n}-\beta^{m-n}]\\
&= -3\,(\hat{\alpha}\,\hat{\beta}\,)\,(-2)^n\,(\alpha-\beta)\,J_{m-n}\,.
\end{aligned}$$

where $\hat{\alpha}\,\hat{\beta}=-5(1-j_1+j_2-j_3)$ *and* $3\,J_{m-n}=\alpha^{m-n}-\beta^{m-n}$.

Theorem 2.7. ***(Cassini's identity)*** *Let* $\mathscr{B}hj_n$ *be the n-th bihyperbolic Jacobsthal-Lucas number. For* $n\geq 1$*, the following equality holds:*

$$\begin{aligned}
\mathscr{B}hj_{n-1}\,\mathscr{B}hj_{n+1}-\mathscr{B}hj_n\,\mathscr{B}hj_n &= 9\,(-2)^{n-1}\,[-5(1-j_1+j_2-j_3)]\\
&= 9\,(-2)^{n-1}\,(\,\hat{\alpha}\,\hat{\beta}\,)\,.
\end{aligned} \tag{2.10}$$

Proof. (2.10): By (2.3) and using the commutative property of bihyperbolic numbers and Cassini's identity of Jacobsthal-Lucas numbers [16], we obtain that

$$\begin{aligned}
\mathscr{B}hj_{n-1}\,\mathscr{B}hj_{n+1}-\mathscr{B}hj_n\,\mathscr{B}hj_n = &\quad [\,(j_{n-1}\,j_{n+1}-j_n\,j_n)\\
&\quad +(j_n\,j_{n+2}-j_{n+1}\,j_{n+1})\\
&\quad +(j_{n+1}\,j_{n+3}-j_{n+2}\,j_{n+2})\\
&\quad +(j_{n+2}\,j_{n+4}-j_{n+3}\,j_{n+3})]\\
&+j_1\,[(j_{n-1}\,j_{n+2}-j_n\,j_{n+1})\\
&\quad +(j_n\,j_{n+1}-j_{n+1}\,j_n)\\
&\quad +(j_{n+1}\,j_{n+4}-j_{n+2}\,j_{n+3})\\
&\quad +(j_{n+2}\,j_{n+3}-j_{n+3}\,j_{n+2})]\\
&+j_2\,[(j_{n-1}\,j_{n+3}-j_n\,j_{n+2})\\
&\quad +(j_{n+1}\,j_{n+1}-j_{n+2}\,j_n)\\
&\quad +(j_n\,j_{n+4}-j_{n+1}\,j_{n+3})\\
&\quad +(j_{n+2}\,j_{n+2}-j_{n+3}\,j_{n+1})]\\
&+j_3\,[(j_{n-1}\,j_{n+4}-j_n\,j_{n+3})\\
&\quad +(j_{n+2}\,j_{n+1}-j_{n+3}\,j_n)\\
&\quad +(j_n\,j_{n+3}-j_{n+1}\,j_{n+2})\\
&\quad +(j_{n+1}\,j_{n+2}-j_{n+2}\,j_{n+1})]\\
= &\quad 9\,(-2)^{n-1}\,[-5(1-j_1+j_2-j_3)]\,.
\end{aligned}$$

where the identity of the Jacobsthal-Lucas numbers $j_{n-1}\,j_{n+1}-j_n\,j_n=9\,(-2)^{n-1}$ is used [16].

Theorem 2.7.A. *Now let's prove this identity using the Binet's formula:*

$$\begin{aligned}
\mathscr{B}hj_{n-1}\,\mathscr{B}hj_{n+1}-\mathscr{B}hj_n\,\mathscr{B}hj_n &= (\hat{\alpha}\,\alpha^{n-1}+\hat{\beta}\,\beta^{n-1})\,(\hat{\alpha}\,\alpha^{n+1}+\hat{\beta}\,\beta^{n+1})\\
&\quad -(\hat{\alpha}\,\alpha^n+\hat{\beta}\,\beta^n)\,(\hat{\alpha}\,\alpha^n+\hat{\beta}\,\beta^n)\\
&= \hat{\alpha}\,\hat{\beta}\,\,(\alpha\,\beta)^n\,[\tfrac{\beta}{\alpha}+\tfrac{\alpha}{\beta}-2]\\
&= \hat{\alpha}\,\hat{\beta}\,\,(\alpha\,\beta)^n\,\tfrac{(\alpha^2+\beta^2-2\,\alpha\,\beta)}{\alpha\,\beta}\\
&= (-2)^{n-1}\,(\alpha-\beta)^2\,(\hat{\alpha}\,\hat{\beta}\,)\\
&= 9\,(-2)^{n-1}\,(\,\hat{\alpha}\,\hat{\beta}\,)\,.
\end{aligned}$$

where $\hat{\alpha}\,\hat{\beta}\, = -5\,(1-j_1+j_2-j_3)\,.$

Theorem 2.8. ***(Catalan's identity)*** *Let $\mathscr{B}hj_n$ be the n-th bihyperbolic Jacobsthal-Lucas number. For $n \geq 1$, the following equality holds:*

$$\begin{aligned} \mathscr{B}hj_n^2 - \mathscr{B}hj_{n-r}\mathscr{B}hj_{n+r} &= (-2)^{n-r}\,[j_r^2-(-2)^{r+2}]\,[-5(1-j_1+j_2-j_3)] \\ &= (-2)^{n-r}\,(\hat{\alpha}\,\hat{\beta}\,)\,[4\,(\alpha\beta)^2-(\alpha^r+\beta^r)^2]\,. \end{aligned} \tag{2.11}$$

Proof. (2.11): By (2.3) and using the commutative property of bihyperbolic numbers Catalan's identity of Jacobsthal-Lucas numbers [16], we obtain that

$$\begin{aligned} \mathscr{B}hj_n\mathscr{B}hj_n - \mathscr{B}hj_{n-r}\mathscr{B}hj_{n+r} = \; & [\,(j_nj_n - j_{n-r}j_{n+r}) \\ & \quad +(j_{n+1}j_{n+1} - j_{n-r+1}j_{n+r+1}) \\ & \quad +(j_{n+2}j_{n+3} - j_{n-r+2}j_{n+r+2}) \\ & \quad +(j_{n+2}j_{n+3} - j_{n-r+3}j_{n+r+3})\,] \\ & +j_1\,[(j_nj_{n+1} - j_{n-r}j_{n+r+1}) \\ & \quad +(j_{n+1}j_n - j_{n-r+1}j_{n+r}) \\ & \quad +(j_{n+2}j_{n+3} - j_{n-r+2}j_{n+r+3}) \\ & \quad +(j_{n+3}j_{n+2} - j_{n-r+3}j_{n+r+2})\,] \\ & +j_2\,[(j_nj_{n+2} - j_{n-r}j_{n+r+2}) \\ & \quad +(j_{n+2}j_n - j_{n-r+2}j_{n+r}) \\ & \quad +(j_{n+1}j_{n+3} - j_{n-r+1}j_{n+r+3}) \\ & \quad +(j_{n+3}j_{n+1} - j_{n-r+3}j_{n+r+1})] \\ & +j_3\,[(j_nj_{n+3} - j_{n-r}j_{n+r+3}) \\ & \quad +(j_{n+3}j_n - j_{n-r+3}j_{n+r}) \\ & \quad +(j_{n+1}j_{n+2} - j_{n-r+1}j_{n+r+2}) \\ & \quad +(j_{n+2}j_{n+1} - j_{n-r+2}j_{n+r+1})] \\ = \; & (-2)^{n-r}\,[j_r^2-(-2)^{r+2}]\,[-5(1-j_1+j_2-j_3)]\,. \end{aligned}$$

where the identities of the Jacobsthal-Lucas numbers

$$j_{n-r}\,j_{n+r} - j_n\,j_n = (-2)^{n-r}\,[\,j_r^2-(-2)^{r+2}]$$

is used [16].

Theorem 2.8.A. *Now let's prove this identity using the Binet's formula:*

$$\begin{aligned} \mathscr{B}hj_n\mathscr{B}hj_n - \mathscr{B}hj_{n-r}\mathscr{B}hj_{n+r} &= (\,\hat{\alpha}\;\alpha^n + \hat{\beta}\;\beta^n)\,(\hat{\alpha}\;\alpha^n + \hat{\beta}\;\beta^n) \\ &\quad -(\hat{\alpha}\;\alpha^{n-r} + \hat{\beta}\;\beta^{n-r})\,(\hat{\alpha}\;\alpha^{n+r} + \hat{\beta}\;\beta^{n+r}) \\ &= \hat{\alpha}\,\hat{\beta}\;(\alpha\,\beta)^n\,[2-(\tfrac{\beta}{\alpha})^r-(\tfrac{\alpha}{\beta})^r\,] \\ &= (-2)^{n-r}\,(\hat{\alpha}\,\hat{\beta}\,)\,[4\,(\alpha\beta)^2-(\alpha^r+\beta^r)^2]\,. \end{aligned}$$

where $\alpha^{2r}+\beta^{2r}-2(\alpha\,\beta)^r = j_r^2-(-2)^{2r}\,.$

3. The Bihyperbolic Pell-Lucas Numbers

In this section, we define the bihyperbolic Pell-Lucas numbers. Then, we obtain the generating function, Binet's formula, d'Ocagne's identity, Cassini's identity, Catalan's identity and Honsberger identity.

Definition 3.1. *For $n \geq 1$, the n-th bihyperbolic Pell-Lucas number $\mathscr{BHPL}_n$ are defined by using the Pell-Lucas numbers as follows*

$$\mathscr{B}hp_n = p_n + j_1\,p_{n+1} + j_2\,p_{n+2} + j_3\,p_{n+3}. \tag{3.1}$$

where j_1, j_2 *and* j_3 *satisfy the conditions*

$$j_1^2 = j_2^2 = j_3^2 = 1,\ j_1 j_2 = j_2 j_1 = j_3,\ j_1 j_3 = j_3 j_1 = j_2,\ j_2 j_3 = j_3 j_2 = j_1.$$

Theorem 3.1. *Let* $\mathscr{B}hp_n$ *be the n-th bihyperbolic Pell-Lucas number. For any integer* $n \geq 0$*,*

$$\mathscr{B}hp_n = 2\,\mathscr{B}hp_{n-1} + \mathscr{B}hp_{n-2} \tag{3.2}$$

Proof. (3.2): By placing Eq.(1.2) in Eq.(3.1) we obtain that,

$$\begin{aligned}
\mathscr{B}hp_n &= p_n + j_1\,p_{n+1} + j_2\,p_{n+2} + j_3\,p_{n+3} \\
&= (2\,p_{n-1} + p_{n-2}) + j_1\,(2\,p_n + p_{n-1}) \\
&\quad + j_2\,(2\,p_{n+1} + p_n) + j_3\,(2\,p_{n+2} + p_{n+1}) \\
&= 2\,(p_{n-1} + j_1\,q_n + j_2\,p_{n+1} + j_3\,p_{n+2}) \\
&\quad + (p_{n-2} + j_1\,p_{n-1} + j_2\,p_n + j_3\,p_{n+1}) \\
&= 2\,\mathscr{B}hp_{n-1} + \mathscr{B}hp_{n-2}
\end{aligned}$$

Also, initial values are $\mathscr{B}hp_0 = 2 + 2\,j_1 + 6\,j_2 + 14\,j_3$, $\mathscr{B}hp_1 = 2 + 6\,j_1 + 14\,j_2 + 34\,j_3$.
Let $\mathscr{B}hp_n$ and $\mathscr{B}hp_m$ be two bihyperbolic Pell-Lucas numbers such that

$$\mathscr{B}hp_n = p_n + j_1\,p_{n+1} + j_2\,p_{n+2} + j_3\,p_{n+3}$$

and

$$\mathscr{B}hp_m = p_m + j_1\,p_{m+1} + j_2\,p_{m+2} + j_3\,p_{m+3}$$

Then, the addition and subtraction of two bihyperbolic Pell numbers are defined in the obvious way,

$$\begin{aligned}
\mathscr{B}hp_n \pm \mathscr{B}hp_m = &\ (p_n \pm p_m) + j_1\,(p_{n+1} \pm p_{m+1}) \\
&+ j_2\,(p_{n+2} \pm p_{m+2}) + j_3\,(p_{n+3} \pm p_{m+3}).
\end{aligned}$$

Multiplication of two bihyperbolic Pell-Lucas numbers is defined by

$$\begin{aligned}
\mathscr{B}hp_n \times \mathscr{B}hp_m = &\ (p_n p_m + p_{n+1} p_{m+1} \\
&\quad + p_{n+2} p_{m+2} + p_{n+3} p_{m+3}) \\
&+ j_1\,(p_{n+1} p_m + p_n p_{m+1} \\
&\quad + p_{n+3} p_{m+2} + p_{n+2} p_{m+3}) \\
&+ j_2\,(p_{n+2} p_m + p_n p_{m+2} \\
&\quad + p_{n+3} p_{m+1} + p_{n+1} p_{m+3}) \\
&+ j_3\,(p_{n+3} p_m + p_n p_{m+3} \\
&\quad + p_{n+1} p_{m+2} + p_{n+2} p_{m+1}) \\
= &\ \mathscr{B}hp_m \times \mathscr{B}hp_n.
\end{aligned}$$

Three kinds of conjugation can be defined for bihyperbolic numbers [6]. Therefore, conjugation of the bihyperbolic Pell-Lucas number is defined in three different ways as follows

$$\overline{\mathscr{B}hp_n}^{j_1} = p_n + j_1\,p_{n+1} - j_2\,p_{n+2} - j_3\,p_{n+3},$$

$$\overline{\mathscr{B}hp_n}^{j_2} = p_n - j_1\,p_{n+1} + j_2\,p_{n+2} - j_3\,p_{n+3},$$

$$\overline{\mathscr{B}hp_n}^{j_3} = p_n - j_1\,p_{n+1} - j_2\,p_{n+2} + j_3\,p_{n+3}.$$

In the following theorem, some properties related to the conjugations of the bihyperbolic Pell-Lucas numbers are given.

Theorem 3.2. *Let* $\overline{\mathscr{B}hp_n}^{j_1}$*,* $\overline{\mathscr{B}hp_n}^{j_2}$ *and* $\overline{\mathscr{B}hp_n}^{j_3}$*, be three kinds of conjugation of the bihyperbolic Pell-Lucas number* $\mathscr{B}hp_n$*. In this case, we can give the following relations:*

$$\mathscr{B}hp_n\,\overline{\mathscr{B}hp_n}^{j_1} = p_n^2 + p_{n+1}^2 - p_{n+2}^2 - p_{n+3}^2 + 2\,j_1\,(p_n p_{n+1} - p_{n+2} p_{n+3}). \tag{3.3}$$

$$\mathscr{B}hp_n\,\overline{\mathscr{B}hp_n}^{j_2} = p_n^2 - p_{n+1}^2 - p_{n+2}^2 + p_{n+3}^2 + 2\,j_2\,(p_n p_{n+2} - p_{n+1} p_{n+3}),$$

$$\mathscr{B}hp_n\,\overline{\mathscr{B}hp_n}^{j_3} = p_n^2 - p_{n+1}^2 - p_{n+2}^2 + p_{n+3}^2 + 2\,j_3\,(p_n p_{n+3} - p_{n+1} p_{n+2}). \tag{3.4}$$

Proof. The proof can be easily done using equations Eq.(3.3-3.4).
In the following theorems, some properties related to the bihyperbolic Pell-Lucas numbers are given.

Theorem 3.3. *Let $\mathscr{B}hp_n$ be the bihyperbolic Pell-Lucas number. For any integer $n \geq 0$, summation formulas as follows:*

$$\sum_{k=0}^{n} \mathscr{B}hp_k = \frac{1}{2}\left[\mathscr{B}hp_{n+1} + \mathscr{B}hp_n + (\mathscr{B}hp_1 - \mathscr{B}hp_2)\right], \tag{3.5}$$

$$\sum_{k=0}^{n} \mathscr{B}hp_{2k} = \frac{1}{2}\left[\mathscr{B}hp_{2n+1} - \mathscr{B}hp_1\right], \tag{3.6}$$

$$\sum_{k=0}^{n} \mathscr{B}hp_{2k-1} = \frac{1}{2}\left[\mathscr{B}hp_{2n} - \mathscr{B}hp_0\right]. \tag{3.7}$$

Proof. (3.5): Using the summation formula $\sum\limits_{r=1}^{n} p_r = \frac{(p_{n+1}+p_n-4)}{2}$ in Eq.(1.3) , we obtain

$$\begin{aligned}
\sum_{k=1}^{n} \mathscr{B}hp_k &= (\sum_{k=1}^{n} p_k + j_1 \sum_{k=1}^{n} p_{k+1} + j_2 \sum_{k=1}^{n} p_{k+2} + j_3 \sum_{k=1}^{n} p_{k+3}) \\
&= (\tfrac{p_{n+1}+p_n-4}{2}) + j_1 (\tfrac{p_{n+2}+p_{n+1}-8}{2}) + j_2 (\tfrac{p_{n+3}+p_{n+2}-20}{2}) + j_3 (\tfrac{p_{n+4}+p_{n+3}-48}{2}) \\
&= \tfrac{1}{2}\left[\mathscr{B}hp_{n+1} + \mathscr{B}hp_n - (4 + 8 j_1 + 20 j_2 + 48 j_3)\right] \\
&= \tfrac{1}{2}\left[\mathscr{B}hp_{n+1} + \mathscr{B}hp_n + (\mathscr{B}hp_1 - \mathscr{B}hp_2)\right].
\end{aligned}$$

where $\mathscr{B}hp_2 = (6 + 14 j_1 + 34 j_2 + 82 j_3)$.

(3.6): Using the summation formula $\sum\limits_{r=1}^{n} p_{2r} = \frac{(p_{2n+1}-2)}{2}$ in Eq.(1.3) , we obtain

$$\begin{aligned}
\sum_{k=1}^{n} \mathscr{B}hp_{2k} &= (\sum_{k=1}^{n} p_{2k} + j_1 \sum_{k=1}^{n} p_{2k+1} + j_2 \sum_{k=1}^{n} p_{2k+2} + j_3 \sum_{k=1}^{n} p_{2k+3}) \\
&= (\tfrac{p_{2n+1}-2}{2}) + j_1 (\tfrac{p_{2n+2}-6}{2}) + j_2 (\tfrac{p_{2n+3}-14}{2}) + j_3 (\tfrac{p_{2n+4}-34}{2}) \\
&= \tfrac{1}{2}\left[\mathscr{B}hp_{2n+1} - (2 + 6 j_1 + 14 j_2 + 34 j_3)\right] \\
&= \tfrac{1}{2}\left[\mathscr{B}hp_{2n+1} - \mathscr{B}hp_1\right].
\end{aligned}$$

(3.7): Using the summation formula $\sum\limits_{r=1}^{n} p_{2r-1} = \frac{(p_{2n}-2)}{2}$ in Eq.(1.3) , we obtain

$$\begin{aligned}
\sum_{k=1}^{n} \mathscr{B}hp_{2k-1} &= (\sum_{k=1}^{n} p_{2k-1} + j_1 \sum_{k=1}^{n} p_{2k} + j_2 \sum_{k=1}^{n} p_{2k+1} + j_3 \sum_{k=1}^{n} p_{2k+2}) \\
&= (\tfrac{p_{2n}-2}{2}) + j_1 (\tfrac{p_{2n+1}-2}{2}) + j_2 (\tfrac{p_{2n+2}-6}{2}) + j_3 (\tfrac{p_{2n+3}-14}{2}) \\
&= \tfrac{1}{2}\left[\mathscr{B}hp_{2n} - (2 + 2 j_1 + 6 j_2 + 14 j_3)\right] \\
&= \tfrac{1}{2}\left[\mathscr{B}hp_{2n} - \mathscr{B}hp_0\right].
\end{aligned}$$

Theorem 3.4. ***(Generating function)***
Let $\mathscr{B}hp_n$ be the n-th bihyperbolic Pell-Lucas number. For the generating function for the bihyperbolic Pell-Lucas numbers is

as follows:

$$\begin{aligned} g_{\mathscr{B}hp_n}(t) = \sum_{n=1}^{\infty} \mathscr{B}hp_n t^n = & \frac{\mathscr{B}hp_0+(\mathscr{B}hp_1-2\mathscr{B}hp_0)t}{1-2t-t^2} \\ = & \frac{(2+2j_1+6j_2+14j_3)+t(-2+2j_1+2j_2+6j_3)}{1-2t-t^2}. \end{aligned} \tag{3.8}$$

where $\mathscr{B}hp_0 = 2+2j_1+6j_2+14j_3$, $\mathscr{B}hp_1 = 2+6j_1+14j_2+34j_3$ *and* $\mathscr{B}hp_2 = 6+14j_1+34j_2+82j_3$.

Proof. (3.8): Using the definition of generating function, we obtain

$$g_{\mathscr{B}hp_n}(t) = \mathscr{B}hp_0 + \mathscr{B}hp_1 t + \ldots + \mathscr{B}hp_n t^n + \ldots \tag{3.9}$$

Multiplying $(1-2t-t^2)$ both sides of Eq.(3.9) and using Eq.(3.2), we have

$$\begin{aligned} (1-2t-t^2)\, g_{\mathscr{B}hp_n}(t) &= \mathscr{B}hp_0 + (\mathscr{B}hp_1 - 2\mathscr{B}hp_0)\,t \\ &\quad +(\mathscr{B}hp_2 - 2\mathscr{B}hp_1 - \mathscr{B}hp_0)\,t^2 \\ &\quad +(\mathscr{B}hp_3 - 2\mathscr{B}hp_2 - \mathscr{B}hp_1)\,t^3 + \ldots \\ &\quad +(\mathscr{B}hp_{k+1} - 2\mathscr{B}hp_k - \mathscr{B}hp_{k-1})\,t^{k+1} + \ldots \end{aligned}$$

where $\mathscr{B}hp_1 - 2\mathscr{B}hp_0 = -2+2j_1+2j_2+6j_3$, $\mathscr{B}hp_2 - 2\mathscr{B}hp_1 - \mathscr{B}hp_0 = 0$, and $\mathscr{B}hp_3 - 2\mathscr{B}hp_2 - \mathscr{B}hp_1 = 0 \ldots = 0$.

Theorem 3.5. ***(Binet's formula)*** *Let $\mathscr{B}hp_n$ be the n-th bihyperbolic Pell-Lucas number. For any integer $n \geq 0$, the Binet's formula for these numbers is as follows:*

$$\mathscr{B}hp_n = \hat{\alpha}\,\alpha^n + \hat{\beta}\,\beta^n. \tag{3.10}$$

where

$$\hat{\alpha} = 1 + j_1\,\alpha + j_2\,\alpha^2 + j_3\,\alpha^3, \quad \alpha = 1+\sqrt{2},$$

$$\hat{\beta} = 1 + j_1\,\beta + j_2\,\beta^2 + j_3\,\beta^3, \quad \beta = 1-\sqrt{2}.$$

Proof. Using the Binet's formula of Pell-Lucas number [20, 21] and Eq.(3.1) we obtain that,

$$\begin{aligned} \mathscr{B}hp_n = &\ p_n + j_1\,p_{n+1} + j_2\,p_{n+2} + j_3\,p_{n+3} \\ = &\ (\alpha^n+\beta^n) + j_1\,(\alpha^{n+1}+\beta^{n+1}) \\ &+ j_2\,(\alpha^{n+2}+\beta^{n+2}) + j_3\,(\alpha^{n+3}+\beta^{n+3}) \\ = &\ \alpha^n(1+j_1\,\alpha + j_2\,\alpha^2 + j_3\,\alpha^3) \\ &+ \beta^n(1+j_1\,\beta + j_2\,\beta^2 + i\,j_3\,\beta^3) \\ = &\ \hat{\alpha}\,\alpha^n + \hat{\beta}\,\beta^n. \end{aligned}$$

Here, Binet's formula of the Pell-Lucas number sequence, $p_n = \alpha^n + \beta^n$ is used.

Theorem 3.6. ***(D'Ocagne's identity)*** *Let $\mathscr{B}hp_n$ be the n-th bihyperbolic Pell-Lucas number. For $m \geq n+1$, the following equality holds:*

$$\mathscr{B}hp_m\,\mathscr{B}hp_{n+1} - \mathscr{B}hp_{m+1}\,\mathscr{B}hp_n = (-1)^{n-1}\,\hat{\alpha}\,\hat{\beta}\,(\alpha-\beta)\,[\alpha^{m-n} - \beta^{m-n}]. \tag{3.11}$$

Proof. (3.11): let's prove this identity using the Binet's formula Eq.(3.10):

$$\begin{aligned} \mathscr{B}hp_m\,\mathscr{B}hp_{n+1} - \mathscr{B}hp_{m+1}\,\mathscr{B}hp_n &= (\hat{\alpha}\,\alpha^m + \hat{\beta}\,\beta^m)(\hat{\alpha}\,\alpha^{n+1} + \hat{\beta}\,\beta^{n+1}) \\ &\quad -(\hat{\alpha}\,\alpha^{m+1} + \hat{\beta}\,\beta^{m+1})(\hat{\alpha}\,\alpha^n + \hat{\beta}\,\beta^n) \\ &= \hat{\alpha}\,\hat{\beta}\,(\alpha\,\beta)^n\,[\alpha^{m-n}\,(\beta-\alpha) + \beta^{m-n}\,(\alpha-\beta)] \\ &= (-1)^{n-1}\,\hat{\alpha}\,\hat{\beta}\,(\alpha-\beta)\,[\alpha^{m-n} - \beta^{m-n}]. \end{aligned}$$

Theorem 3.7. ***(Cassini's identity)*** *Let $\mathscr{B}hp_n$ be the n-th bihyperbolic Pell-Lucas number. For $n \geq 1$, the following equality holds:*

$$\begin{aligned}\mathscr{B}hp_{n-1}\,\mathscr{B}hp_{n+1} - \mathscr{B}hp_n\,\mathscr{B}hp_n &= (-1)^{n-1}\,\hat{\alpha}\,\hat{\beta}\,(\alpha^2+\beta^2-2\,\alpha\,\beta) \\ &= 8\,(-1)^{n-1}\,\hat{\alpha}\,\hat{\beta}\,.\end{aligned} \tag{3.12}$$

Proof. (3.12): let's prove this identity using the Binet's formula Eq.(3.10):

$$\begin{aligned}\mathscr{B}hp_{n-1}\,\mathscr{B}hp_{n+1} - \mathscr{B}hp_n\,\mathscr{B}hp_n &= (\,\hat{\alpha}\;\alpha^{n-1} + \hat{\beta}\;\beta^{n-1})\,(\,\hat{\alpha}\;\alpha^{n+1} + \hat{\beta}\;\beta^{n+1}) \\ &\quad -(\,\hat{\alpha}\;\alpha^{n} + \hat{\beta}\;\beta^{n})\,(\,\hat{\alpha}\;\alpha^{n} + \hat{\beta}\;\beta^{n}) \\ &= \hat{\alpha}\,\hat{\beta}\,(\alpha\,\beta)^n\,[\,\tfrac{\beta}{\alpha} + \tfrac{\alpha}{\beta} - 2\,] \\ &= (-1)^{n-1}\,\hat{\alpha}\,\hat{\beta}\,(\alpha^2+\beta^2-2\,\alpha\,\beta) \\ &= 8\,(-1)^{n-1}\,\hat{\alpha}\,\hat{\beta}\,.\end{aligned}$$

Theorem 3.8. ***(Catalan's identity)*** *Let $\mathscr{B}hp_n$ be the n-th bihyperbolic Pell-Lucas number. For $n \geq 1$, the following equality holds:*

$$\mathscr{B}hp_n^2 - \mathscr{B}hp_{n-r}\,\mathscr{B}hp_{n+r} \quad = (-1)^n\,\hat{\alpha}\,\hat{\beta}\,[\,(\alpha-\beta)^2\,]^r\,. \tag{3.13}$$

Proof. (3.13): Let's prove this identity using the Binet's formula Eq.(3.10):

$$\begin{aligned}\mathscr{B}hp_n\,\mathscr{B}hp_n - \mathscr{B}hp_{n-r}\,\mathscr{B}hp_{n+r} &= (\,\hat{\alpha}\;\alpha^{n} + \hat{\beta}\;\beta^{n})\,(\,\hat{\alpha}\;\alpha^{n} + \hat{\beta}\;\beta^{n}) \\ &\quad -(\,\hat{\alpha}\;\alpha^{n-r} + \hat{\beta}\;\beta^{n-r})\,(\,\hat{\alpha}\;\alpha^{n+r} + \hat{\beta}\;\beta^{n+r}) \\ &= \hat{\alpha}\,\hat{\beta}\,(\alpha\,\beta)^n\,[\,2 - (\,\tfrac{\beta}{\alpha})^r - (\,\tfrac{\alpha}{\beta})^r\,] \\ &= (-1)^n\,\hat{\alpha}\,\hat{\beta}\,[\,(\alpha-\beta)^2\,]^r\,.\end{aligned}$$

4. Conclusion

In this paper, we introduced some properties of Lucas-type bihyperbolics. We gave the definition of bihyperbolic Jacobsthal-Lucas and bihyperbolic Pell-Lucas numbers and examined their algebraic properties. Additionally, by using the relationship of these numbers with Jacobsthal-Lucas and Pell-Lucas numbers, we obtained the Binet formula, generating function, d'Ocagne, Cassini and Catalan identities of bihyperbolic Jacobsthal-Lucas and bihyperbolic Pell-Lucas numbers.

Article Information

Acknowledgements: The author would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: The article has a single author. The author has read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private, or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism was detected.

Availability of data and materials: Not applicable.

References

[1] W. R. Hamilton, *Lectures on Quaternions*, Hodges and Smith. Dublin, 1853.

[2] J. Cockle, *On certain functions resembling quaternions and on a new imaginary in algebra.*, The London, Edinburg and Dublin Philosophical Mag. J. Sci., **33** (1848), 435-439.

[3] C. Segre, *Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici.*, Math. Ann., **40** (1892), 413-467.

[4] F. Catoni, D. Boccaletti, C. V. Cannata, Catoni, E. Nichelatti, F. Zampetti, *The Mathematics of Minkowski Space-Time with an Introduction to Commutative Hypercomplex Numbers*, Basel, Boston, Berlin: Birkhauser Verlag, 2008.

[5] G. B. Price, *An Introduction to Multicomplex Spaces and Functions*, M. Dekker New York, 1991.

[6] A. A. Pogorui, R. M. Rodrigez-Dagnino, R. D. Rodrigez-Said, *On the set of zeros of bihyperbolic polynomials.*, Complex Var. Elliptic Equ., **53**, (2008), 685-690.

[7] S. Olariu, *Complex Number in n-dimensions*, Nerth-Holland Mathematics Studies, **190**, Amsterdam, Boston: Elsevier, 51-148, 2002.

[8] M. Bilgin, S. Ersoy, *Algebraic properties of bihyperbolic numbers.*, Adv. Appl. Clifford Algebr., **30** (2020), 1-17.

[9] N. Gürses, G. Y. Sentürk, S. Yüce, *A study on dual-generalized complex and hyperbolic-generalized complex numbers.*, Gazi Univ. J. Sci., **34** (2021), 180-194.

[10] D. Brod, A. Syznal-Liana, I. Wloch, *On some combinatorial properties of bihyperbolic numbers of the Fibonacci type.*, Math. Meth. App. Sci., **44** (2021), 4607-4615.

[11] D. Brod, A. Syznal-Liana, I. Wloch, *On a new generalization of bihyperbolic Pell numbers.*, Ann. Alexandru Ioan Cuza Univ. Math., **67**(2) (2021).

[12] A. Z. Azak, *Some new identities with respect to bihyperbolic Fibonacci and Lucas numbers.*, Int. J. Sci.: Basic and App. Res., **60** (2021), 14-37.

[13] A. Szynal-Liana, I. Włoch, *On Jacobsthal and Jacobsthal-Lucas hybrid numbers.*, Ann. Math. Sil., **33** (2019), 276-283.

[14] A. F. Horadam, *Jacobsthal representation numbers.*, Fibonacci Quart., **34** (1996), 40-54.

[15] A. F. Horadam, *Jacobsthal representation polynomials.*, Fibonacci Quart., **35** (1997), 137-148.

[16] F. T. Aydın, *On generalizations of the Jacobsthal sequence.*, Notes Number Theory Discrete Math., **24**(1) (2018), 120-135.

[17] S. Uygun, *A new generalization for Jacobsthal and Jacobsthal-Lucas sequences.*, Asian J. Math., **2**(1) (2018), 14-21.

[18] A. Al-Kateeb, *A generalization of Jacobsthal and Jacobsthal-Lucas numbers.*, (2019), arXiv preprint:1911.11515.

[19] D. Brod, A. Michalski, *On generalized Jacobsthal and Jacobsthal-Lucas numbers.*, Ann. Math. Sil., **36**(2) (2022), 115-128.

[20] A.F. Horadam, *Pell identities.*, Fibonacci Quart., **9**(3) (1971), 245-252.

[21] A. F. Horadam, J. Mahon, *Pell and Pell-Lucas polynomials.*, Fibonacci Quart., **23**(1) (1985), 7-20.

[22] S. F. Santana, et al, *Some properties of sums involving Pell numbers*, Missouri J. Math. Sci. Uni. Central Missouri, Department of Mathematics and Computer Science, **18**(1) (2006), 33-40.

[23] A. Szynal-Liana, I. Włoch, *On certain bihypernomials related to Pell and Pell-Lucas numbers.*, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat., **71**(2) (2022), 422-433.

[24] H. Gökbaş, *Gaussian-bihyperbolic numbers containing Pell and Pell-Lucas numbers.*, J. Adv. Res. Nat. App. Sci., Çanakkale Onsekiz Mart Univ. **9**(1) (2023), 183-189.

[25] F. T. Aydın, *On bicomplex Pell and Pell-Lucas numbers.*, Comm. Adv. Math. Sci., **1**(2) (2018), 142-155.