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Research Article

D-Homothetic Deformations and Almost Paracontact Metric Manifolds

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Abstract: In this study, we determine some of the classes of almost paracontact metric structures which are invariant under D-homothetic deformations. We write the Riemannian curvature tensor, the Ricci tensor and the scalar curvature when the characteristic vector field is Killing. In addition, we give examples.

Keywords: Almost paracontact metric structure, D-homothetic deformation, Killing vector field.

1. Introduction

Differentiable manifolds having almost paracontact structures were introduced by [5] and after [11] many authors have made contribution, see [7, 9, 11–13] and references therein. Manifolds with almost paracontact metric structure were classified according to the Levi-Civita covariant derivative of the fundamental tensor. There are 2^{12} classes of almost paracontact metric manifolds. The defining relations and projections onto each subspace are given in [7, 13].

D-homothetic deformations of almost contact metric manifolds is extensively studied, see [1, 3] and references therein. For D-homothetic deformations of almost contact metric structures with B-metric, refer to [2]. D-homothetic deformations of almost paracontact metric structures were introduced in [11]. In [10], almost paracontact metric manifolds whose characteristic vector field is parallel are considered and their D-homothetic deformations are studied. Our aim is to investigate D-homothetic deformations of almost paracontact metric manifolds having arbitrary characteristic vector fields.

2. Preliminaries

Assume that M^{2n+1} is a smooth manifold having odd dimension. An ordered triple (φ, ξ, η) of an endomorphism, a vector field, a 1-form, respectively, with the properties below is called an almost

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paracontact structure on M

$$\varphi^2 = I - \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \varphi(\xi) = 0,$$

there is a distribution $\mathbb{D} : p \in M \longrightarrow \mathbb{D}_p = Ker\eta$. M together with the almost paracontact structure is said to be an almost paracontact manifold. In addition, if M carries a semi-Riemannian metric g satisfying

$$g(\varphi(x), \varphi(y)) = -g(x,y) + \eta(x)\eta(y),$$

where $\mathfrak{X}(M)$ is the set of smooth vector fields on M and $x, y \in \mathfrak{X}(M)$, then M is called an almost paracontact metric manifold. The fundamental 2-form of the almost paracontact metric structure is given as

$$\Phi(x,y) = g(\varphi x, y).$$

We denote the vector fields and tangent vectors by letters x, y, z.

Consider the tensor F defined by

$$F(x,y,z) = g((\nabla_x \varphi)(y), z), \tag{1}$$

for all $x, y, z \in T_pM$, where T_pM is the tangent space at p, ∇ is the Levi-Civita covariant derivative of g. Then F satisfies

$$F(x,y,z) = -F(x,z,y), \tag{2}$$

$$F(x,\varphi y,\varphi z) = F(x,y,z) + \eta(y)F(x,z,\xi) - \eta(z)F(x,y,\xi). \tag{3}$$

The forms below are defined for any almost paracontact metric structure.

$$\theta(x) = g^{ij}F(e_i, e_j, x), \quad \theta^*(x) = g^{ij}F(e_i, \varphi e_j, x), \quad \omega(x) = F(\xi, \xi, x),$$

where $u \in T_pM$, $\{e_i, \xi\}$ is a basis for T_pM and the inverse of the matrix g_{ij} is g^{ij}.

Let $\mathcal{F}$ be the set of $(0,3)$ tensors over T_pM having properties (2), (3). $\mathcal{F}$ is the direct sum of four subspaces W_i, $i = 1, \ldots, 4$, where projections F^{W_i} we use are

$$F^{W_1}(x,y,z) = F(\varphi^2 x, \varphi^2 y, \varphi^2 z), \tag{4}$$

$$F^{W_2}(x,y,z) = -\eta(y)F(\varphi^2 x, \varphi^2 z, \xi) + \eta(z)F(\varphi^2 x, \varphi^2 y, \xi). \tag{5}$$

In addition, W_1 is a direct sum of four subspaces $\mathbb{G}_i$, $i = 1, \ldots, 4$, $W_2 = \mathbb{G}_5 \oplus \ldots \oplus \mathbb{G}_{10}$, and denote W_3 and W_4 by $\mathbb{G}_{11}$ and $\mathbb{G}_{12}$, respectively. A manifold with almost paracontact metric structure is said to be in the class $\mathbb{G}_i \oplus \mathbb{G}_j$, etc. if F belongs to $\mathbb{G}_i \oplus \mathbb{G}_j$ over T_pM for all $p \in M$. The defining relations of $\mathbb{G}_i$ and projections F^i onto each $\mathbb{G}_i$ are given in [7, 13]. We only write the classes and projections we use:

$$\mathbb{G}_5: \ F(x,y,z) = \frac{\theta_F(\xi)}{2n}\{g(\varphi x,\varphi z)\eta(y) - g(\varphi x,\varphi y)\eta(z)\} \tag{6}$$

$$\begin{aligned} \mathbb{G}_8: \ F(x,y,z) &= -\eta(y)F(x,z,\xi) + \eta(z)F(x,y,\xi), \\ F(x,y,\xi) &= F(y,x,\xi) = -F(\varphi x,\varphi y,\xi), \quad \theta_F(\xi) = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} \mathbb{G}_9: \ F(x,y,z) &= -\eta(y)F(x,z,\xi) + \eta(z)F(x,y,\xi), \\ F(x,y,\xi) &= -F(y,x,\xi) = F(\varphi x,\varphi y,\xi) \end{aligned} \tag{8}$$

$$\begin{aligned} \mathbb{G}_{10}: \ F(x,y,z) &= -\eta(y)F(x,z,\xi) + \eta(z)F(x,y,\xi), \\ F(x,y,\xi) &= F(y,x,\xi) = F(\varphi x,\varphi y,\xi) \end{aligned} \tag{9}$$

$$\mathbb{G}_{11}: \ F(x,y,z) = \eta(x)F(\xi,\varphi y,\varphi z) \tag{10}$$

$$\mathbb{G}_{12}: \ F(x,y,z) = \eta(x)\{\eta(y)F(\xi,\xi,z) - \eta(z)F(\xi,\xi,y)\} \tag{11}$$

Some of the projections F^i onto each subspace $\mathbb{G}_i$ are

$$\begin{aligned} F^9(x,y,z) &= -\frac{1}{4}\eta(y)\left\{F(\varphi^2x,\varphi^2z,\xi) + F(\varphi x,\varphi z,\xi)\right. \\ &\left. -F(\varphi^2z,\varphi^2x,\xi) - F(\varphi z,\varphi x,\xi)\right\} + \frac{1}{4}\eta(z)\left\{F(\varphi^2x,\varphi^2y,\xi)\right. \\ &\left. +F(\varphi x,\varphi y,\xi) - F(\varphi^2y,\varphi^2x,\xi) - F(\varphi y,\varphi x,\xi)\right\}, \end{aligned} \tag{12}$$

$$\begin{aligned} F^{10}(x,y,z) &= -\frac{1}{4}\eta(y)\left\{F(\varphi^2x,\varphi^2z,\xi) + F(\varphi x,\varphi z,\xi)\right. \\ &\left. +F(\varphi^2z,\varphi^2x,\xi) + F(\varphi z,\varphi x,\xi)\right\} + \frac{1}{4}\eta(z)\left\{F(\varphi^2x,\varphi^2y,\xi)\right. \\ &\left. +F(\varphi x,\varphi y,\xi) + F(\varphi^2y,\varphi^2x,\xi) + F(\varphi y,\varphi x,\xi)\right\}, \end{aligned} \tag{13}$$

$$F^{11}(x,y,z) = \eta(x)F(\xi,\varphi^2y,\varphi^2z), \tag{14}$$

$$F^{12}(x,y,z) = \eta(x)\{\eta(y)F(\xi,\xi,\varphi^2z) - \eta(z)F(\xi,\xi,\varphi^2y)\}. \tag{15}$$

Note that ξ is Killing in any direct sum of $\mathbb{G}_1,\mathbb{G}_2,\mathbb{G}_3,\mathbb{G}_4,\mathbb{G}_5,\mathbb{G}_8,\mathbb{G}_9,\mathbb{G}_{11}$ and ξ is parallel in $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_3$, $\mathbb{G}_4$, $\mathbb{G}_{11}$ and also in any direct sum of these classes [10].

For any almost paracontact metric sructure (φ, ξ, η, g) on a manifold M, consider the quadruple $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ where

$$\tilde{\varphi} = \varphi, \quad \tilde{\xi} = \frac{1}{t}\xi, \quad \tilde{\eta} = t\eta, \quad \tilde{g} = -tg + t(t+1)\eta \otimes \eta \tag{16}$$

for a positive constant t [11]. The structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is called a D-homothetic deformation of (φ, ξ, η, g). In [10], the Levi-Civita covariant derivative $\tilde{\nabla}$ of metric $\tilde{g}$ is obtained as

$$\begin{aligned} g(\tilde{\nabla}_x y, z) \quad = \quad & g(\nabla_x y, z) + \frac{(t+1)^2}{2t}\eta(z)\left\{-\eta(x)g(\nabla_\xi \xi, y)\right. \\ & \left.-\eta(y)g(\nabla_\xi \xi, x) + g(\nabla_x \xi, y) + g(\nabla_y \xi, x)\right\} \\ & -\frac{(t+1)}{2}\left\{\eta(x)\left(g(\nabla_y \xi, z) - g(\nabla_z \xi, y)\right)\right. \\ & +\eta(y)\left(g(\nabla_x \xi, z) - g(\nabla_z \xi, x)\right) \\ & \left.+\eta(z)\left(g(\nabla_x \xi, y) + g(\nabla_y \xi, x)\right)\right\}. \end{aligned} \tag{17}$$

Also it is proved that the classes with parallel characteristic vector field does not change after D-homothetic deformations. Our aim is to study the invariance of remaining basic classes $\mathbb{G}_5$, $\mathbb{G}_6$, $\mathbb{G}_7$, $\mathbb{G}_8$, $\mathbb{G}_9$, $\mathbb{G}_{10}$, $\mathbb{G}_{12}$. We also write the curvature tensors of the deformed metric when ξ is Killing and we give examples.

3. Classes of Deformed Structures

Consider a D-homothetic deformation given by (16).

First let ξ be Killing. In this case (17) simplifies into

$$\begin{aligned} g(\tilde{\nabla}_x y, z) \quad = \quad & g(\nabla_x y, z) - (t+1)\left\{\eta(x)g(\nabla_y \xi, z)\right. \\ & \left.+\eta(y)g(\nabla_x \xi, z)\right\}, \end{aligned} \tag{18}$$

since g is non-degenerate, (18) gives

$$\tilde{\nabla}_x y = \nabla_x y - (t+1)\left\{\eta(x)\nabla_y \xi + \eta(y)\nabla_x \xi\right\}. \tag{19}$$

The Proposition 3.1 yields from (19).

Proposition 3.1 *Let ξ be g-Killing. Then $\tilde{\xi}$ is $\tilde{g}$-Killing.*

Now we write the curvature tensors of the deformed metric $\tilde{g}$ for an almost paracontact metric structure with Killing characteristic vector field. If $\{e_1, \ldots, e_n, \varphi e_1, \ldots, \varphi e_n, \xi\}$ is a g-orthonormal frame, then $\{f_1,\dots,f_{2n+1}\}=\{\frac{1}{\sqrt{t}}\varphi e_1,\dots,\frac{1}{\sqrt{t}}\varphi e_n,\frac{1}{\sqrt{t}}e_1,\dots,\frac{1}{\sqrt{t}}e_n,\frac{1}{t}\xi\}$ is $\tilde{g}$-orthonormal [10] and $\tilde{g}^{ij}=g^{ij}$. We use this basis in calculations.

If ξ is Killing, the Riemannian, the Ricci and the scalar curvatures of the deformed metric $\tilde{g}$ are evaluated by direct calculation.

$$\begin{aligned}\tilde{R}(x,y)z &= R(x,y)z-(t+1)\eta(z)R(x,y)\xi\\ &\quad -(t+1)\eta(x)\nabla_{\nabla_y z}\xi+(t+1)\eta(y)\nabla_{\nabla_x z}\xi\\ &\quad +(t+1)^2\eta(x)\eta(z)\nabla_{\nabla_y\xi}\xi-(t+1)^2\eta(y)\eta(z)\nabla_{\nabla_x\xi}\xi\\ &\quad +(t+1)g(\nabla_y\xi,z)\nabla_x\xi-(t+1)g(\nabla_x\xi,z)\nabla_y\xi\\ &\quad -2(t+1)g(\nabla_x\xi,y)\nabla_z\xi-(t+1)\eta(y)\nabla_x\nabla_z\xi\\ &\quad +(t+1)\eta(x)\nabla_y\nabla_z\xi,\end{aligned}\tag{20}$$

$$\begin{aligned}\tilde{Ric}(x,y) &= Ric(x,y)-(t+1)\eta(y)Ric(x,\xi)\\ &\quad +(t+1)\eta(x)\sum_{i=1}^{n}\{g(\nabla_{\nabla_{e_i}y}\xi,e_i)-g(\nabla_{\nabla_{\varphi e_i}y}\xi,\varphi e_i)\}\\ &\quad +(t+1)^2\eta(x)\eta(y)\sum_{i=1}^{n}\{-g(\nabla_{\nabla_{e_i}\xi}\xi,e_i)+g(\nabla_{\nabla_{\varphi e_i}\xi}\xi,\varphi e_i)\}\\ &\quad -(t+1)\eta(x)div(\nabla_y\xi)+2(t+1)g(\nabla_x\xi,\nabla_y\xi)\end{aligned}$$

and

$$\tilde{s} = \frac{1}{t}\{-s+(t+1)\sum_{i=1}^{n}\{g(\nabla_{\varphi e_i}\xi,\nabla_{\varphi e_i}\xi)-g(\nabla_{e_i}\xi,\nabla_{e_i}\xi)\}\}.$$

Now let ξ be any vector field which is not necessarily Killing. We write the tensor $\tilde{F}$ of the deformed structure in terms of F defined by (1). Since

$$(\tilde{\nabla}_x\tilde{\varphi})(y)=\tilde{\nabla}_x(\varphi y)-\varphi(\tilde{\nabla}_x y)\tag{21}$$

and

$$\begin{aligned}\tilde{F}(x,y,z) &= \tilde{g}\big((\tilde{\nabla}_x\tilde{\varphi})(y),z\big)\\ &= -tg\big((\tilde{\nabla}_x\tilde{\varphi})(y),z\big)\\ &\quad +t(t+1)\eta((\tilde{\nabla}_x\tilde{\varphi})(y))\eta(z),\end{aligned}\tag{22}$$

replacing (21) in (22) and using (17) and the identity $g(\nabla_x \xi, y) = -F(x, \varphi y, \xi)$ yields

$$\begin{aligned} \tilde{F}(x,y,z) &= -tF(x,y,z) \\ &+\frac{t(t+1)}{2}\{\eta(x)\{-F(\varphi y,\varphi z,\xi)+F(z,y,\xi) \\ &-F(y,z,\xi)+F(\varphi z,\varphi y,\xi)\} \\ &+\eta(z)\{F(x,y,\xi)-F(\varphi y,\varphi x,\xi)\} \\ &+\eta(y)\{-F(x,z,\xi)+F(\varphi z,\varphi x,\xi)\}\}. \end{aligned} \tag{23}$$

Now we study the invariance of classes W_i, $i = 1, \ldots, 4$ under a D-homothetic deformation. First note that for any almost paracontact metric structure in a direct sum of $W_1 \oplus W_3 = \mathbb{G}_1 \oplus \mathbb{G}_2 \oplus \mathbb{G}_3 \oplus \mathbb{G}_4 \oplus \mathbb{G}_{11}$, since ξ is parallel [10], the equation (23) implies $\tilde{F} = -tF$ and thus a D-homothetic deformation of any direct sum of $W_1 \oplus W_3$ is also in this class.

If ξ is any vector field, not necessarily parallel, from (4) and (23), we have

$$\tilde{F}^{W_1}(x,y,z) = \tilde{F}(\varphi^2 x, \varphi^2 y, \varphi^2 z) = -tF(\varphi^2 x, \varphi^2 y, \varphi^2 z) = -tF^{W_1}(x,y,z). \tag{24}$$

Thus $\tilde{F}^{W_1}$ is zero if and only if F^{W_1} is zero, that is, a deformed structure contains summands from the class W_1 if and only if the first structure has a summand from W_1.

By (5) and (23), we get

$$\begin{aligned} \tilde{F}^{W_2} &= \frac{t(t-1)}{2}F^{W_2}(x,y,z) \\ &+\frac{t(t+1)}{2}\{\eta(y)F(\varphi z,\varphi x,\xi)-\eta(z)F(\varphi y,\varphi x,\xi)\}. \end{aligned} \tag{25}$$

Define S as

$$S(x,y,z) = \frac{t(t+1)}{2}\{\eta(y)F(\varphi z,\varphi x,\xi)-\eta(z)F(\varphi y,\varphi x,\xi)\}. \tag{26}$$

Then it can be easily seen that $S^{W_2} = S$ and thus $S \in W_2$. In addition, we have $F^{W_2}(\varphi x, \varphi y, z) = \eta(z)F(\varphi x, \varphi y, \xi)$. So $F^{W_2} = 0$ if and only if $S = 0$. Thus a deformed structure has summands from the class W_2 if and only if the first structure has.

Consider the projection $F^{W_3} = F^{11}$. From (14) and (23), we have

$$\begin{aligned} \tilde{F}^{11}(x,y,z) &= -tF^{11}(x,y,z)+\frac{t(t+1)}{2}\eta(x)\{-F(\varphi y,\varphi z,\xi)+F(\varphi z,\varphi y,\xi) \\ &+F(\varphi^2 z,\varphi^2 y,\xi)-F(\varphi^2 y,\varphi^2 z,\xi)\}. \end{aligned} \tag{27}$$

Define

$$T(x,y,z) = \frac{t(t+1)}{2}\eta(x)\{-F(\varphi y,\varphi z,\xi)+F(\varphi z,\varphi y,\xi) +F(\varphi^2 z,\varphi^2 y,\xi)-F(\varphi^2 y,\varphi^2 z,\xi)\}. \quad (28)$$

It can be checked that T satisfies the defining relation (10) of $\mathbb{G}_{11}$, that is, $T^{11}=T$. Thus if $F^{11}=0$, or equivalently, if the first almost paracontact structure does not contain a summand from $\mathbb{G}_{11}$, and if $T\neq 0$, then the deformed structure contains a summand from $\mathbb{G}_{11}$ since $T\in\mathbb{G}_{11}$.

For the projection $F^{W_4}=F^{12}$, by using (23) and (15), we get

$$\tilde{F}^{12}(x,y,z) = t^2F^{12}(x,y,z). \quad (29)$$

Thus the deformed structure belongs to a direct sum containing $\mathbb{G}_{12}$ if and only if the first almost paracontact structure has summands from this class.

It is known that almost paracontact metric structures which belong to $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_3$, $\mathbb{G}_4$, $\mathbb{G}_{11}$ or one of their direct sums are invariant under D-homothetic deformations. These are structures with parallel characteristic vector fields [10]. We investigate the invariance of remaining basic classes $\mathbb{G}_5$, $\mathbb{G}_6$, $\mathbb{G}_7$, $\mathbb{G}_8$, $\mathbb{G}_9$, $\mathbb{G}_{10}$, $\mathbb{G}_{12}$.

Theorem 3.2 *The classes $\mathbb{G}_i$, where $i=5,6,7,8,10,12$ are invariant under a D-homothetic deformation, $\mathbb{G}_9$ is not invariant.*

Proof Assume that $\{e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n,\xi\}$ is a g-orthonormal frame. Then

$$\{f_1,\dots,f_{2n+1}\}=\{\tfrac{1}{\sqrt{t}}\varphi e_1,\dots,\tfrac{1}{\sqrt{t}}\varphi e_n,\tfrac{1}{\sqrt{t}}e_1,\dots,\tfrac{1}{\sqrt{t}}e_n,\tfrac{1}{t}\xi\}$$

is $\tilde{g}$-orthonormal and $\tilde{g}^{ij}=g^{ij}$.

Let $(\varphi,\xi,\eta,g)\in\mathbb{G}_5$. By (23), for $i=1,\dots,n$,

$$\tilde{F}(f_i,f_i,\tilde{\xi}) = \frac{1}{t^2}\tilde{F}(\varphi e_i,\varphi e_i,\xi) = \frac{t-1}{2t}F(\varphi e_i,\varphi e_i,\xi)-\frac{t+1}{2}F(e_i,e_i,\xi)$$

and for $i=n+1,\dots,2n$,

$$\tilde{F}(f_i,f_i,\tilde{\xi}) = \frac{1}{t^2}\tilde{F}(e_i,e_i,\xi) = \frac{t-1}{2t}F(e_i,e_i,\xi)-\frac{t+1}{2}F(\varphi e_i,\varphi e_i,\xi).$$

Thus

$$\begin{aligned}\tilde{\theta}_{\tilde{F}}(\tilde{\xi}) &= \tilde{g}^{ij}F(f_i, f_i, \tilde{\xi})\\ &= \sum_{i=1}^{n}\tilde{F}(\frac{1}{\sqrt{t}}\varphi e_i, \frac{1}{\sqrt{t}}\varphi e_i, \tilde{\xi}) - \sum_{i=1}^{n}\tilde{F}(\frac{1}{\sqrt{t}}e_i, \frac{1}{\sqrt{t}}e_i, \tilde{\xi})\\ &= -\theta_F(\xi).\end{aligned}$$

From (6) and (23), we get that $\tilde{F}$ satisfies the defining relation (6).

Similarly, the class $\mathbb{G}_6$ is invariant.

Let $(\varphi, \xi, \eta, g) \in \mathbb{G}_8$. Then the defining conditions (7) hold. First we evaluate $\tilde{\theta}_{\tilde{F}}(\tilde{\xi})$. If $\{e_1, \ldots, e_n, \varphi e_1, \ldots, \varphi e_n, \xi\}$ is a g-orthonormal frame, then

$\{f_1, \ldots, f_{2n+1}\} = \{\frac{1}{\sqrt{t}}\varphi e_1, \ldots, \frac{1}{\sqrt{t}}\varphi e_n, \frac{1}{\sqrt{t}}e_1, \ldots, \frac{1}{\sqrt{t}}e_n, \frac{1}{t}\xi\}$ is $\tilde{g}$-orthonormal and $\tilde{g}^{ij} = g^{ij}$. From (7) and (23), we have

$$\begin{aligned}\tilde{F}(\varphi e_i, \varphi e_i, \xi) &= -tF(\varphi e_i, \varphi e_i, \xi)\\ &\quad + \frac{t(t+1)}{2}\{F(\varphi e_i, \varphi e_i, \xi) - F(\varphi^2 e_i, \varphi^2 e_i, \xi)\}\\ &= -tF(\varphi e_i, \varphi e_i, \xi) + t(t+1)F(\varphi e_i, \varphi e_i, \xi)\\ &= t^2 F(\varphi e_i, \varphi e_i, \xi)\end{aligned}$$

and

$$\tilde{F}(e_i, e_i, \xi) = t^2 F(e_i, e_i, \xi),$$

thus

$$\begin{aligned}\tilde{\theta}_{\tilde{F}}(\tilde{\xi}) &= \tilde{g}^{ij}F(f_i, f_i, \tilde{\xi})\\ &= \sum_{i=1}^{n}\tilde{F}(\frac{1}{\sqrt{t}}\varphi e_i, \frac{1}{\sqrt{t}}\varphi e_i, \tilde{\xi}) - \sum_{i=1}^{n}\tilde{F}(\frac{1}{\sqrt{t}}e_i, \frac{1}{\sqrt{t}}e_i, \tilde{\xi})\\ &= \frac{1}{t^2}\left\{\sum_{i=1}^{n}t^2F(\varphi e_i, \varphi e_i, \xi) - \sum_{i=1}^{n}t^2F(e_i, e_i, \xi)\right\}\\ &= -\theta_F(\xi)\\ &= 0.\end{aligned}$$

In addition, from (7) and (23)

$$\begin{aligned}\tilde{F}(x,y,z) &= -tF(x,y,z)\\ &+\frac{t(t+1)}{2}\{2F(x,y,\xi)\eta(z)-2F(x,z,\xi)\eta(y)\}\\ &= -tF(x,y,z)+t(t+1)F(x,y,z)\\ &= t^2F(x,y,z)\end{aligned}$$

and

$$\begin{aligned}&-\tilde{\eta}(y)\tilde{F}(x,z,\tilde{\xi})+\tilde{\eta}(z)\tilde{F}(x,y,\tilde{\xi})\\ &= t^2F(x,y,z)\\ &= \tilde{F}(x,y,z).\end{aligned}$$

Also,

$$\tilde{F}(x,y,\tilde{\xi})=t^2F(x,y,\tilde{\xi})=t^2F(y,x,\tilde{\xi})=\tilde{F}(y,x,\tilde{\xi}),$$

$$\tilde{F}(x,y,\tilde{\xi})=t^2F(x,y,\tilde{\xi})=-t^2F(\varphi y,\varphi x,\tilde{\xi})=-\tilde{F}(\tilde{\varphi}y,\tilde{\varphi}x,\tilde{\xi}).$$

Thus the new structure satisfies (7).

A similar proof can be done for the class $\mathbb{G}_7$. In this case, $\tilde{\theta}^*_{\tilde{F}}(\tilde{\xi})=\frac{1}{t}\theta^*_F(\xi)$.

Let $(\varphi,\xi,\eta,g)\in\mathbb{G}_{10}$. Then the defining relations (9) hold. From (23), $\tilde{F}=-tF$ and (13) implies $\tilde{F}^{10}=-tF=-tF^{10}=\tilde{F}$.

Let $(\varphi,\xi,\eta,g)\in\mathbb{G}_{12}$. By using the defining relation (11) and (23), $\tilde{F}=t^2F$ and from (15), $\tilde{F}^{12}=t^2F^{12}=t^2F=\tilde{F}$. Since $\tilde{F}=\tilde{F}^{12}$, the deformed structure is in $\mathbb{G}_{12}$.

Now we show that the class $\mathbb{G}_9$ is not invariant.

For an arbitrary structure, using (23), we have

$$\tilde{F}(\varphi x,\varphi z,\xi)=\frac{t(t-1)}{2}\{F(\varphi x,\varphi z,\xi)\}+\frac{t(t+1)}{2}\left\{F(\varphi^2 z,\varphi^2 x,\xi)\right\} \tag{30}$$

and

$$\tilde{F}(\varphi^2 x,\varphi^2 z,\xi)=\frac{t(t-1)}{2}\left\{F(\varphi^2 x,\varphi^2 z,\xi)\right\}-\frac{t(t+1)}{2}\{F(\varphi z,\varphi x,\xi)\}. \tag{31}$$

By using equations (12), (30) and (31), we get $\tilde{F}^9=t^2F^9$.

Let $(\varphi,\xi,\eta,g)\in\mathbb{G}_9$. From (8), $\tilde{F}^9=t^2F^9=t^2F$ and also from (8) and (23),

$$\tilde{F}(x,y,z) = t^2F(x,y,z)-2t(t+1)\eta(x)F(y,z,\xi).$$

The structure is invariant if and only if $\tilde{F} = \tilde{F}^9$, that is

$$t^2F(x,y,z) = t^2F(x,y,z) - 2t(t+1)\eta(x)F(y,z,\xi)$$

holds. This implies $F(y,z,\xi) = 0$. Then the defining relation (8) of $\mathbb{G}_9$ implies $F = 0$. Thus a nontrivial structure in $\mathbb{G}_9$ is not in the same class after deformation. □

In addition, we determine the class of the deformed structure if the first structure is in $\mathbb{G}_9$.

Proposition 3.3 *Assume that the first almost paracontact metric structure belongs to the class $\mathbb{G}_9$. Then the deformed structure is in $\mathbb{G}_9 \oplus \mathbb{G}_{11}$.*

Proof Since $M \in \mathbb{G}_9$, we have $F^{W_1} = F^{W_3} = F^{11} = F^{W_4} = F^{12} = 0$ and $F^{W_2} = F^9$. From (24) and (29), we get $\tilde{F}^{W_1} = \tilde{F}^{W_4} = \tilde{F}^{12} = 0$. By using the defining relation (8), it can be seen that the tensor S defined in (26) also satisfies the defining relation of $\mathbb{G}_9$. Thus the equation (25) implies that $\tilde{F}^{W_2} = \frac{t(t-1)}{2}F^9 + S^9$, that is, the deformed structure contains a summand from $\mathbb{G}_9$ and no other summand from W_2. In addition, by using (8), the tensor T given in (28) is

$$T(x,y,z) = 2t(t+1)\eta(x)\{-F(\varphi y, \varphi z, \xi)\},$$

which is nonzero for a nontrivial structure in $\mathbb{G}_9$, otherwise (8) implies $F = 0$. From (27), $\tilde{F}^{11} = T \neq 0$.

To sum up, the deformed structure is in $\mathbb{G}_9 \oplus \mathbb{G}_{11}$. □

Proposition 3.4 *Normal almost paracontact manifolds are invariant under D-homothetic deformations.*

Proof Let the first almost paracontact metric structure be normal. Then

$$F(x,y,\varphi z) + F(\varphi x,y,z) + \eta(z)F(x,\varphi y,z) = 0. \tag{32}$$

(32) implies

$$F(x,\varphi y,\xi) = -F(\varphi x,y,\xi), \tag{33}$$

see [13]. Then by (23), (32) and (33), we get

$$\tilde{F}(x,y,\tilde{\varphi}z) + \tilde{F}(\tilde{\varphi}x,y,z) + \tilde{\eta}(z)\tilde{F}(x,\tilde{\varphi}y,z) = 0.$$

As a result, the deformed structure is also normal. □

Example 3.5 *Let L be Lie algebra having basis $\{e_1,e_2,e_3\}$ whose only nonzero bracket is*

$$[e_1,e_2] = \alpha e_3,$$

together with the semi-Riemannian metric satisfying $g(e_1, e_1) = -g(e_2, e_2) = g(e_3, e_3) = 1$ *and* $g(e_i, e_j) = 0$ *for* $i \neq j$. *Let* $\varphi(e_1) = e_2$, $\varphi(e_2) = e_1$, $\varphi(e_3) = 0$, $e_3 = \xi$ *and* $\eta = e^3$, *where* e^3 *is the metric dual of* e_3. *It is known that* $(L, \varphi, \xi, \eta, g)$ *is an almost paraconact metric manifold of class* $\mathbb{G}_5$. *The nonzero covariant derivatives are*

$$\nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = \frac{\alpha}{2} e_3, \quad \nabla_{e_1} e_3 = \nabla_{e_3} e_1 = \frac{\alpha}{2} e_2, \quad \nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \frac{\alpha}{2} e_1.$$

The Ricci tensor is

$$Ric(x, y) = sg(x, y) - 2s\eta(x)\eta(y),$$

where s *is the scalar curvature given by* $s = \alpha^2/2$, *that is,* L *is an* η*-Einstein manifold, see [13]. Then from (20),*

$$\begin{aligned} \tilde{Ric}(x, y) &= Ric(x, y) - (t+1)\eta(y) Ric(x, e_3) \\ &\quad -2(t+1)\frac{\alpha^2}{4}\{x_1 y_1 - x_2 y_2 - t\eta(x)\eta(y)\}, \end{aligned}$$

where $x = x_1 e_1 + x_2 e_2 + x_3 e_3$ *and* $y = y_1 e_1 + y_2 e_2 + y_3 e_3$. *It can be checked that*

$$\tilde{Ric}(x, y) = \frac{\alpha^2}{2} \tilde{g}(x, y) - \alpha^2 \tilde{\eta}(x) \tilde{\eta}(y),$$

that is the deformed manifold is also η*-Einstein.*

Example 3.6 *Consider the nilpotent Lie algebra* $\mathfrak{g}_1$ *given in [4] with basis* $\{e_1, \ldots, e_5\}$, *whose nonzero brackets are*

$$[e_1, e_2] = e_5, [e_3, e_4] = e_5.$$

Assume that g *is the metric such that* $\{e_1, \ldots, e_5\}$ *is orthonormal and* $\epsilon_i = g(e_i, e_i) = \pm 1$. *The nonzero covariant derivatives are evaluated in [8] by Kozsul's formula:*

$$\nabla_{e_1} e_2 = \frac{1}{2} e_5, \quad \nabla_{e_1} e_5 = -\frac{1}{2}\epsilon_2 \epsilon_5 e_2,$$

$$\nabla_{e_2} e_1 = -\frac{1}{2} e_5, \quad \nabla_{e_2} e_5 = \frac{1}{2}\epsilon_1 \epsilon_5 e_1,$$

$$\nabla_{e_3} e_4 = \frac{1}{2} e_5, \quad \nabla_{e_3} e_5 = -\frac{1}{2}\epsilon_4 \epsilon_5 e_4,$$

$$\nabla_{e_4} e_3 = -\frac{1}{2} e_5, \quad \nabla_{e_4} e_5 = \frac{1}{2}\epsilon_3 \epsilon_5 e_3,$$

$$\nabla_{e_5} e_1 = -\frac{1}{2}\epsilon_2 \epsilon_5 e_2, \quad \nabla_{e_5} e_2 = \frac{1}{2}\epsilon_1 \epsilon_5 e_1, \quad \nabla_{e_5} e_3 = -\frac{1}{2}\epsilon_4 \epsilon_5 e_4, \quad \nabla_{e_5} e_4 = \frac{1}{2}\epsilon_3 \epsilon_5 e_3.$$

Consider now the structure (φ, ξ, η, g) *defined by* $g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = -g(e_4, e_4) = g(e_5, e_5) = 1$, $\xi = e_5$, $\eta = e^5$, *whose endomorphism is given via basis elements as follows.*

$\varphi(e_1) = e_3$, $\varphi(e_2) = e_4$, $\varphi(e_3) = e_1$, $\varphi(e_4) = e_2$, $\varphi(e_5) = 0$. *Nonzero structure constants of* F *are*

$$F(e_1, e_4, e_5) = -F(e_1, e_5, e_4) = -F(e_2, e_3, e_5) = F(e_2, e_5, e_3) = 1/2,$$

$$-F(e_3, e_5, e_2) = F(e_3, e_2, e_5) = -F(e_4, e_1, e_5) = F(e_4, e_5, e_1) = 1/2,$$

$$-F(e_5, e_1, e_4) = F(e_5, e_4, e_1) = F(e_5, e_2, e_3) = -F(e_5, e_3, e_2) = 1.$$

Note that $\xi = e_5$ *is Killing [8] and this structure is in the class* $\mathbb{G}_9 \oplus \mathbb{G}_{11}$ *[6]. We determine the class of the deformed structure after a D-homothetic deformation. Proposition 3.1 implies that* $\tilde{\xi}$ *is Killing, so* $\tilde{F}^6 = \tilde{F}^7 = \tilde{F}^{10} = \tilde{F}^{12} = 0$. *Also since* $\tilde{F}^{W_1} = -tF^{W_1}$ *and* F^{W_1} *vanishes,* $\tilde{F}^{W_1}$ *also vanishes. It can be checked that this structure satisfies*

$$F(\varphi y, \varphi z, \xi) = -F(\varphi z, \varphi y, \xi) = F(\varphi^2 y, \varphi^2 z, \xi)$$

and thus

$$\begin{aligned}
\tilde{F}^{11}(x, y, z) &= -tF^{11}(x, y, z) + \frac{t(t+1)}{2}\eta(x)\{-F(\varphi y, \varphi z, \xi) + F(\varphi z, \varphi y, \xi) \\
&\quad + F(\varphi^2 z, \varphi^2 y, \xi) - F(\varphi^2 y, \varphi^2 z, \xi)\} \\
&= -2t(t+1)\eta(x)F(\varphi y, \varphi z, \xi) \\
&= t(t+1)x_5\{y_2 z_3 - y_3 z_2 + y_4 z_1 - y_1 z_4\} \neq 0.
\end{aligned}$$

In addition, by direct calculation

$$\begin{aligned}
F^9(x, y, z) &= \eta(y)F(\varphi z, \varphi x, \xi) - \eta(z)F(\varphi y, \varphi x, \xi) \\
&= -\frac{1}{2}y_5\{x_1 z_4 - x_2 z_3 + x_3 z_2 - x_4 z_1\} \\
&\quad + \frac{1}{2}z_5\{x_1 y_4 - x_2 y_3 + x_3 y_2 - x_4 y_1\}
\end{aligned}$$

and

$$
\begin{aligned}
\tilde{F}^{W_2} &= \frac{t(t-1)}{2}F^{W_2}(x,y,z) \\
&\quad + \frac{t(t+1)}{2}\left\{\eta(y)F(\varphi z,\varphi x,\xi)-\eta(z)F(\varphi y,\varphi x,\xi)\right\} \\
&= \frac{t(t-1)}{2}F^{9}(x,y,z) \\
&\quad + \frac{t(t+1)}{2}\left\{-\frac{1}{2}y_5\left\{x_1z_4-x_2z_3+x_3z_2-x_4z_1\right\}\right. \\
&\quad \left. +\frac{1}{2}z_5\left\{x_1y_4-x_2y_3+x_3y_2-x_4y_1\right\}\right\} \\
&= t^2F^9(x,y,z)\neq 0
\end{aligned}
$$

As a result the deformed structure is also in $\mathbb{G}_9\oplus\mathbb{G}_{11}$. *So we obtain infinitely many examples of structures of type* $\mathbb{G}_9\oplus\mathbb{G}_{11}$ *by D-homothetic deformation. Note that although an almost paracontact structure of class* $\mathbb{G}_9$ *is not invariant, a direct sum containing the class* $\mathbb{G}_9$ *may be invariant.*

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Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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Research Article

On Optimal Control of the Heat Flux at the Left-Hand Side in a Heat Conductivity System

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Abstract: We deal with an optimal boundary control problem in a 1-d heat equation with Neumann boundary conditions. We search for a boundary function which is the minimum element of a quadratic cost functional involving the H^1-norm of boundary controls. We prove that the cost functional has a unique minimum element and is Fréchet differentiable. We give a necessary condition for the optimal solution and construct a minimizing sequence using the gradient of the cost functional.

Keywords: Optimal control problems, heat equation, Fréchet differentiability, adjoint problem.

1. Introduction

Control problems are used to improve efficiency in many fields such as economics, biology, agriculture, robotics industry, chemical reactions, and gas dynamics. Mathematical modeling of many physical phenomena is known to lead to differential equations [1, 6–8, 21, 22, 24–26]. Therefore, it is important to study the control problems related to PDEs. Optimal control problems for parabolic equations arise in various areas of science including chemical reactions, heat transfer, and population dynamics and they have been widely studied due to their importance in the natural sciences and their applications. The boundary control problem for heat transfer systems is one of the most addressed control problems for PDEs. Some detailed works of problems in these areas can be found in [2, 3, 5, 9, 10, 14, 15, 17, 19, 20].

Lions [17] studied the optimal control problem in the parabolic system with the aim of finding a boundary condition that ensures the approach of the solution of the parabolic problem at the terminal time to the given desired function. He chose the Lebesque space L_2 as the space of bound-

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ary controls. Hasanoğlu [12] considered the problem of finding unknown pair $\{h(t,x), f(t)\}$ in the equation $y_t - (a(x)y_x)_x = h(t,x)$ with conditions $y_x(t,0) = 0, -a(L)y_x(t,L) = v[y(t,L) - f(t)]$ from the final overdetermination. Sadek and Bokhari [23] examined the controlling of Neumann boundary conditions for the heat conduction equation by minimizing the energy-based performance measure involving boundary controls.

Şener and Subaşi [27] analyzed the optimal control problem of the boundary function $s(t)$ in the system

$$\begin{cases} y_t = ay_{xx} + b(t,x), & (t,x) \in (0,T) \times (0,L), \\ y(0,x) = v(x), & 0 < x < L, \\ y_x(t,0) = 0,\ y_x(t,L) = s(t), & 0 < t < T. \end{cases}$$

They obtained the optimal solution as a minimum element of the cost functional

$$J_\alpha(s) = \int_0^L [y(T,x;s) - f(x)]^2 dx + \alpha \|s\|^2_{H^1(0,T)}$$

for the given target function $f(x) \in L_2(0,L)$ and $\alpha > 0$.

In this study, we consider the following mathematical model

$$\begin{cases} w_t = aw_{xx} + b(t,x), & (t,x) \in \Omega := (0,T) \times (0,L), \\ w(0,x) = w_0(x), & 0 < x < L, \\ w_x(t,0) = \mu(t),\ w_x(t,L) = 0, & 0 < t < T, \end{cases} \tag{1}$$

where T is a given final time, a is a positive constant, $b(t,x)$, $w_0(x)$ are given functions and $\mu(t)$ is an unknown function. Physically speaking, a is the heat conductivity, $b(t,x)$ is the heat source, $w_0(x)$ is the initial temperature, and $\mu(t)$ is the heat flux.

The aim of this study is to find a boundary function $\mu \in H^1(0,T)$ such that the corresponding solution to the system (1) approaches to the given desired $\nu(t,x) \in L_2(\Omega)$. More precisely, we want to minimize the cost functional

$$J_\alpha(\mu) = \int_0^T \int_0^L [w(t,x;\mu) - \nu(t,x)]^2 dx dt + \alpha \|\mu - \mu^+\|^2_{H^1(0,T)} \tag{2}$$

in the admissible controls set $M_{ad} \subset H^1(0,T)$. Here the function $\mu^+(t) \in H^1(0,T)$ is an initial guess for the optimal solution and $\alpha > 0$ is a regularization parameter. $w(t,x;\mu)$ stands for the dependence of the solution $w(t,x)$ of the system (1) on the boundary control $\mu(t)$.

This paper differs from existing works in the literature in view of the functional space of the controls and the choice of the cost functional. Previous studies propose the usage of the space L_2 as the control set [5, 12, 17, 23]. Moreover, this study investigates a different target than the study in [27]. With the choice of the functional in (2), we use $w(t,x;\mu)$ for the boundary control $\mu(t)$.

This paper is organized as follows: Firstly, we show that the conditions of the Goebel Theorem are valid for the optimal control problem considered. So, we prove that the optimal solution exists and is unique by this theorem. Then, we introduce an adjoint problem by the Lagrange multiplier method and calculate the Fréchet derivate of the cost functional via the adjoint approach. Finally, we state a necessary optimality condition and establish a minimizing sequence.

2. Existence and Uniqueness of a Minimizer for the Cost Functional

This section is dedicated to proving the conditions for the existence of the unique optimal solution to the optimal control problem (1)-(2). We denote the set of admissible boundary control functions with M_{ad}. Let M_{ad} be a non-empty subset of the space $H^1(0,T)$. Furthermore, we assume that M_{ad} is closed, convex, and bounded.

We know that for every $w_0(x) \in H^1(0,L)$, $b(t,x) \in L_2(\Omega)$ and $\mu(t) \in H^1(0,T)$, the parabolic system (1) has a unique solution $w \in H^{2,1}(\Omega)$ satisfies the following estimate:

$$\|w\|^2_{H^{2,1}(\Omega)} \le c_1\left(\|b\|^2_{L_2(\Omega)} + \|w_0\|^2_{H^1(0,L)} + \|\mu\|^2_{H^1(0,T)}\right), \tag{3}$$

where c_1 is a constant independent from b, w_0 and μ [18]. We refer to [16] for definitions of the spaces $H^{2,1}(\Omega)$, $H^1(0,L)$ and $L_2(\Omega)$.

Let $\delta\mu \in M_{ad}$ be an increment of the control at $\mu \in M_{ad}$ such that $\mu + \delta\mu \in M_{ad}$. Let us denote by $w_\delta = w(t,x;\mu+\delta\mu)$ the solution of the system (1) corresponding to the boundary condition $\mu + \delta\mu \in M_{ad}$. Then, the function $\delta w(t,x;\mu) = w(t,x;\mu+\delta\mu) - w(t,x;\mu) = w_\delta - w$ is the solution to the following difference problem

$$\begin{cases} \delta w_t = a\delta w_{xx}, & (t,x) \in \Omega, \\ \delta w(0,x) = 0, & 0 < x < L, \\ \delta w_x(t,0) = \delta\mu(t), \ \delta w_x(t,L) = 0, & 0 < t < T. \end{cases} \tag{4}$$

Furthermore, the difference problem is of the same type as the problem (1). So, it can be proven that the solution $\delta w(t,x;\mu)$ of the problem (4) satisfies the following inequality:

$$\|\delta w(t,x;\mu)\|^2_{L_2(\Omega)} \le c_2 \|\delta\mu\|^2_{H^1(0,T)}, \quad t \in [0,T]. \tag{5}$$

Here c_2 is independent from $\delta\mu$.

We can use the Goebel Theorem [11] widely referred to for the existence of a minimum element in optimal control problems. The following theorem states the existence and uniqueness of the solution to the optimal control problem under consideration.

Theorem 2.1 *Let $\mu^+ \in H^1(0,T)$ be a given element. There is a dense subset $G \in H^1(0,T)$ such that the cost functional $J_\alpha(\mu)$ has a unique minimum in the set M_{ad} for all $\mu^+ \in G$ and $\alpha > 0$.*

Proof We know that $H^1(0,T)$ is a uniformly convex Banach space [4] and the admissible set M_{ad} is a bounded, closed and convex subset of $H^1(0,T)$. Let's rewrite the cost functional as

$$J_\alpha(\mu) = J(\mu) + \alpha\|\mu\|^2_{H^1(0,T)},$$

where

$$J(\mu) = \int_0^T \int_0^L [w(t,x;\mu) - \nu(t,x)]^2 dxdt.$$

The functional $J(\mu)$ is bounded from below in the set M_{ad} since $J(\mu) \ge 0$ for any $\mu \in M_{ad}$. It is sufficient to show that the functional $J(\mu)$ is lower semi-continuous in the set M_{ad}. Let us evaluate the increment $\delta J(\mu) = J(\mu+\delta\mu) - J(\mu)$ for any $\mu \in M_{ad}$. We obtain

$$\begin{aligned}\delta J(\mu) &= \int_0^T \int_0^L [w(t,x;\mu+\delta\mu) - \nu(t,x)]^2 dxdt - \int_0^T \int_0^L [w(t,x;\mu) - \nu(t,x)]^2 dxdt \\ &= 2\int_0^T \int_0^L [w(t,x;\mu) - \nu(t,x)]\delta w(t,x;\mu) dxdt \\ &+ \int_0^T \int_0^L [\delta w(t,x;\mu)]^2 dxdt.\end{aligned} \tag{6}$$

Taking into account the inequalities (3) and (5), we can write that

$$|\delta J(\mu)| \le c_3(\|\delta\mu\|_{H^1(0,T)} + \|\delta\mu\|^2_{H^1(0,T)}). \tag{7}$$

Here c_3 is independent from $\delta\mu$.

(7) implies that the functional $J(\mu)$ is lower semi-continuous in the set M_{ad}. According to Goebel Theorem, there is a dense subset G of $H^1(0,T)$ such that the functional $J_\alpha(\mu)$ takes its minimum value at a unique point for every $\mu^+ \in G$. □

3. Fréchet Differentiability of the Cost Functional

In this section, we first apply the Lagrange multipliers method to obtain the adjoint problem and then find the Fréchet derivative of the functional $J_\alpha(\mu)$. In order to construct a minimizing sequence, it is important to prove that the cost functional is continuously differentiable.

Lagrange functional is defined by

$$L(w,\mu,\varphi) = J_\alpha(\mu) + \langle \varphi, w_t - a w_{xx} - b\rangle_{L_2(\Omega)}$$

, where the functional $J_\alpha(\mu)$ is the cost functional given in (2) and φ is called the Lagrange function.

It can be easily seen that the first variation for the Lagrange functional is:

$$\begin{aligned}\delta L &= \int_0^T \int_0^L 2[w(t,x;\mu) - \nu(t,x)]\delta w(t,x;\mu)dxdt \\ &- \int_0^T \int_0^L [\varphi_t + a\varphi_{xx}]\delta w(t,x;\mu)dxdt + \int_0^L \varphi(T,x)\delta w(T,x;\mu)dx \\ &+ \int_0^T \varphi_x(t,L)\delta w(t,L)dt - \int_0^T \varphi_x(t,0)\delta w(t,0)dt,\end{aligned} \tag{8}$$

where $\delta w(t,x;\mu)$ is the solution to the problem (4).

Using the $\delta L = 0$ stationarity condition, we get the following adjoint problem:

$$\begin{cases} \varphi_t + a\varphi_{xx} = 2[w(t,x;\mu) - \nu(t,x)], & (t,x) \in \Omega, \\ \varphi(T,x) = 0, & 0 < x < L, \\ \varphi_x(t,0) = 0, \;\; \varphi_x(t,L) = 0, & 0 < t < T. \end{cases} \tag{9}$$

If we replace t in (9) by new variable $\tau = T - t$, then we obtain a boundary value problem in the same type as the problem (1). The adjoint problem has a weak solution φ in $H^{2,1}(\Omega)$ since $w - \nu \in L_2(\Omega)$ [18].

Lemma 3.1 *Let $\mu, \mu + \delta\mu \in M_{ad}$ be given elements. If $w = w(t,x;\mu)$ is the solution to the problem (1) and $\varphi(t,x;\mu)$ is the solution to the adjoint problem (9), then the following identity holds:*

$$\int_0^T \int_0^L 2[w(t,x;\mu) - \nu(t,x)]\delta w(t,x;\mu)dxdt = a\int_0^T \delta\mu(t)\varphi(t,0)dt \tag{10}$$

for all $\mu \in M_{ad}$.

Proof Using the equation (9) and applying integration by parts, we write the left side of (10) as follows:

$$
\begin{aligned}
&2\int_0^T\int_0^L[w(t,x;\mu)-\nu(t,x)]\delta w(t,x;\mu)dxdt\\
&=\int_0^T\int_0^L[\varphi_t(t,x)+a\varphi_{xx}(t,x)]\delta w(t,x;\mu)dxdt\\
&=\int_0^L\Big\{[a\varphi(t,x)\delta w(t,x;\mu)]_{t=0}^{t=T}-\int_0^T\varphi(t,x)\delta w_t(t,x;\mu)dt\Big\}dx\\
&\quad+\int_0^T\Big\{[a\varphi_x(t,x)\delta w(t,x;\mu)]_{x=0}^{x=L}-\int_0^L a\varphi_x(t,x)\delta w_x(t,x;\mu)dx\Big\}dt.
\end{aligned}
$$

From (4) and (9), we get

$$
\begin{aligned}
&2\int_0^T\int_0^L[w(t,x;\mu)-\nu(t,x)]\delta w(t,x;\mu)dxdt\\
&=-\int_0^T\int_0^L\varphi(t,x)\delta w_t(t,x;\mu)dxdt\\
&\quad-\int_0^T\Big\{[a\varphi(t,x)\delta w_x(t,x;\mu)]_{x=0}^{x=L}-\int_0^L a\varphi(t,x)\delta w_{xx}(t,x;\mu)dx\Big\}dt\\
&=-\int_0^T\int_0^L[\delta w_t(t,x;\mu)-a\delta w_{xx}(t,x;\mu)]\varphi(t,x)dxdt\\
&\quad+\int_0^T a\varphi(t,0)\delta\mu(t)dt.
\end{aligned}
$$

Considering the equation (4), the integral identity (10) is obtained.

□

Let's evaluate the first variation of $J_\alpha(\mu)$. We write

$$
\begin{aligned}
\delta J_\alpha(\mu)&=J_\alpha(\mu+\delta\mu)-J_\alpha(\mu)\\
&=2\int_0^T\int_0^L[w(t,x;\mu)-\nu(t,x)]\delta w(t,x;\mu)dxdt\\
&\quad+\int_0^T\int_0^L[\delta w(t,x;\mu)]^2dxdt\\
&\quad+2\alpha\langle\mu-\mu^+,\delta\mu\rangle_{H^1(0,T)}+\|\delta\mu\|^2_{H^1(0,T)},
\end{aligned}
\tag{11}
$$

where $\mu+\delta\mu\in M_{ad}$ and $\delta w(t,x;\mu)$ is the solution to the problem (4).

Using the integral identity (10) on the formula (11) we can write the first variation of the cost functional $J_\alpha(\mu)$ as follows:

$$\delta J_\alpha(\mu) = \int_0^T a\varphi(t,0)\delta\mu(t)dt + \int_0^T \int_0^L [\delta w(t,x;\mu)]^2 dxdt$$
$$+ 2\alpha\langle \mu - \mu^+, \delta\mu\rangle_{H^1(0,T)} + \|\delta\mu\|^2_{H^1(0,T)}. \tag{12}$$

In order to get the Fréchet derivative of the cost functional, the first term on the right-hand side of (12) must be written as the inner product in the space $H^1(0,T)$. To do this we define the following problem

$$\begin{cases} \theta''(t) - \theta(t) = -a\varphi(t,0), \quad t \in (0,T), \\ \theta'(0) = 0, \ \theta'(T) = 0. \end{cases} \tag{13}$$

Using (13), the formula (12) can be written as

$$\delta J_\alpha(\mu) = \int_0^T (\theta(t)\delta\mu(t) + \theta'(t)\delta\mu'(t))dt + \int_0^T \int_0^L [\delta w(t,x;\mu)]^2 dxdt$$
$$+ 2\alpha\langle \mu - \mu^+, \delta\mu\rangle_{H^1(0,T)} + \|\delta\mu\|^2_{H^1(0,T)}. \tag{14}$$

The estimate (5) yields that the second term on the right-hand side of (14) is of the order $o(\|\delta\mu\|^2_{H^1(0,T)})$. The formula (14) becomes

$$\delta J_\alpha(\mu) = \langle \theta + 2\alpha(\mu - \mu^+), \delta\mu\rangle_{H^1(0,T)} + o(\|\delta\mu\|^2_{H^1(0,T)}).$$

So, the cost functional is Fréchet differentiable, that is $J_\alpha(\mu) \in C^1(M_{ad})$. The operator

$$J'_\alpha(\mu) = \theta + 2\alpha(\mu - \mu^+) \tag{15}$$

is the Fréchet derivative of the cost functional. Here $\theta(t)$ is the solution of (13).

4. Necessary Condition for the Optimal Solution and a Minimizing Sequence

We construct a minimizing sequence based on the gradient methods. According to the gradient method, a minimizer for the cost functional is chosen by the formula

$$\mu^{(j+1)} = \mu^{(j)} - \beta_j J'_\alpha\big(\mu^{(j)}\big), \quad j = 0,1,2,..., \tag{16}$$

where $\mu^{(0)} \in M_{ad}$ is a given initial element and $J'_\alpha\big(\mu^{(j)}\big)$ is the Fréchet derivative corresponding to $\mu^{(j)}$. The β_j is called the relaxation parameter. From the definition of Fréchet differentiability, we can obtain that

$$J_\alpha\big(\mu^{(j+1)}\big) - J_\alpha\big(\mu^{(j)}\big) = \beta_j\Big[- \|J'_\alpha\big(\mu^{(j)}\big)\|^2 + \frac{o(\beta_j)}{\beta_j}\Big] < 0$$

for sufficiently small $\beta_j > 0$ [13]. The choice of the relaxation parameter defines various gradient methods and this choice is very important.

To stop the iteration process, one of the following stopping criterion can be selected:

$$\|\mu^{(j+1)} - \mu^{(j)}\| < \epsilon_1, \quad \|J_\alpha\big(\mu^{(j+1)}\big) - J_\alpha\big(\mu^{(j)}\big)\| < \epsilon_2, \quad \|J'_\alpha\big(\mu^{(j)}\big)\| < \epsilon_3. \tag{17}$$

Now, we can state the optimality condition in view of [28]. Let $\mu^* \in M_{ad}$ be the optimal solution to the problem (1)-(2) and let us denote the solution of the adjoint problem corresponding to the optimal solution μ^* with $\varphi^*(t,x)$. We know that the cost functional $J_\alpha(\mu)$ is a continuously differentiable in the control set M_{ad}. In this case, the following inequality is provided for all $\mu \in M_{ad}$ [28]:

$$\langle J'_\alpha(\mu^*), \mu - \mu^* \rangle_{H^1(0,T)} \geq 0. \tag{18}$$

The following variational inequality states the necessary condition for the optimal solution:

$$\langle \theta^* + 2\alpha(\mu^* - \mu^+), \mu - \mu^* \rangle_{H^1(0,T)} \geq 0 \tag{19}$$

for all $\mu \in M_{ad}$, where $\theta^*(t)$ is the solution of the problem (13) corresponding to $\varphi^*(t,0)$.

5. Conclusions

In this study, we focus on investigating the optimality conditions in the optimal control problem governed by the parabolic system and obtaining a minimizer for the chosen cost functional. We prove that the boundary condition $w_x(t,0) = \mu(t)$ in the parabolic problem can be controlled from target $w(x,t) = \nu(x,t)$ using H^1-norm. The admissible control set is chosen as a bounded, convex, and closed subset of the space $H^1(0,T)$. Using Goebel Theorem, we prove that the optimal boundary control problem considered has a unique solution. Obtaining the explicit formula for the gradient of the cost functional allows the usage of the gradient method to construct a minimization sequence. Fréchet differentiable of the cost functional in the admissible controls set is proved and the explicit formula of this derivative is obtained by adjoint approach. The obtained results permit one to acquire the necessary optimality condition. This study provides some results for numerical research on obtaining the optimal solution.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Taha Koç]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%45).

Author [Yeşim Akbulut]: Thought and designed the research/problem, contributed to completing the research and solving the problem (%35).

Author [Seher Aslancı]: Contributed to completing the research and solving the problem (%20).

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

Generalized Kibria-Lukman Prediction Approximation in Linear Mixed Models

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Abstract: One of the new suggested prediction methods is the Kibria-Lukman's prediction approach under multicollinearity in linear mixed models and in this article, the generalized Kibria-Lukman estimator and predictor are introduced to combat multicollinearity problem. The comparisons between the proposed generalized Kibria-Lukman estimator/predictor and several other estimators/predictors, namely the best linear unbiased estimator/predictor and Kibria-Lukman estimator/predictor are done by using the matrix mean square error criterion. Lastly, the selection of the biasing parameter is given and to demonstrate the performance of our new defined prediction method, the greenhouse gases data analysis is made.

Keywords: Linear mixed model, mean square error, generalized Kibria-Lukman predictor, multicollinearity.

1. Introduction

The linear mixed model (LMM) is described the following form for $i = 1, \ldots, m$,

$$y_i = X_i\beta + Z_i u_i + \varepsilon_i,$$

where y_i is an $n_i \times 1$ vector of response variables measured on subject i, β is a $p \times 1$ parameter vector of fixed effects, X_i and Z_i are $n_i \times p$ and $n_i \times q$ known design matrices of the fixed and random effects, respectively, u_i is a $q \times 1$ random vector, the components of which are called random effects and ε_i is an $n_i \times 1$ random vector of errors. LMM mostly has the assumptions given below

$$u_i \overset{iid}{\sim} N_q\left(0, \sigma^2 F\right) \text{ and } \varepsilon_i \overset{iid}{\sim} N_{n_i}\left(0, \sigma^2 W_i\right), \quad i = 1, \ldots, m,$$

where u_i and ε_i are independent, F and W_i are $q \times q$ and $n_i \times n_i$ known positive definite (pd) matrices.

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$y = (y_1^T, \ldots, y_m^T)^T$, $X = (X_1^T, \ldots, X_m^T)^T$, $Z = \oplus_{i=1}^m Z_i$ ($\oplus$ is the direct sum), $u = (u_1^T, \ldots, u_m^T)^T$ and $\varepsilon = (\varepsilon_1^T, \ldots, \varepsilon_m^T)^T$ is taken. So, the more compact model can be written as

$$y = X\beta + Zu + \varepsilon, \tag{1}$$

this means $\begin{pmatrix} u \\ \varepsilon \end{pmatrix} \sim N_{qm+n}\left(\begin{pmatrix} 0_{qm} \\ 0_n \end{pmatrix}, \begin{pmatrix} \sigma^2 G & 0 \\ 0 & \sigma^2 W \end{pmatrix}\right)$, where $n = \sum_{i=1}^m n_i$, $G = I_m \otimes F$, $W = \oplus_{i=1}^m W_i$ ($\otimes$ is the Kronecker product) and I_m is the identity matrix of order m. $y \sim N\left(X\beta, \sigma^2 H\right)$ is written under model (1), where $H = ZGZ^T + W$. It is assumed that the G and W matrices are known for ease of theoretical calculations. But, if this assumption is not satisfied, we substitute their maximum likelihood (ML) or restricted maximum likelihood (REML) estimates for the G and W. $\hat{\beta}$ and $\hat{u}$ were obtained by [4, 5] as follows

$$\begin{aligned} \hat{\beta} &= (X^T H^{-1} X)^{-1} X^T H^{-1} y, \\ \hat{u} &= G Z^T H^{-1}(y - X\hat{\beta}), \end{aligned} \tag{2}$$

and they were, respectively, named as BLUE (the best linear unbiased estimator of β) and BLUP (the best linear unbiased predictor of u).

This article aims to reveal a new prediction method, which is an alternative to the existing estimators/predictors defined below in the LMM literature under multicollinearity and, for the sake of actualizing this aim, is to introduce a generalized form of Kibria-Lukman prediction method in LMMs by following [1] generalized Kibria-Lukman estimator in linear regression models. Thus, the rest of our study is configured as follows: We give our preliminaries in Section 2. We obtain the generalized Kibria-Lukman estimator and predictor in LMMs via [1] in linear regression models in Section 3. Matrix mean square error (MMSE) performances are evaluated in Section 4. We mention about biasing parameter selection in Section 5 and in Section 6, greenhouse gases data analysis is ensured to show our theoretical findings. Finally, in Section 7, we discuss some conclusions.

2. Preliminaries

Multicollinearity is defined as the linear dependence between the columns of X. The statistical consequences of this effect, such as the parameter estimates having large variances and being different from the true values, are well known in all linear regression models, including LMM. In order to eliminate the effects of this effect, there are many methods defined in both linear regression models and LMM, and ridge regression in the linear regression models recommended by [6] is the most well-known method among these methods. Under LMM, [11, 13] identified the ridge estimator and predictor with $k>0$ ridge biasing parameter as follows

$$\begin{aligned}\hat{\beta}_k &= (X^T H^{-1} X + kI_p)^{-1} X^T H^{-1} y,\\ \hat{u}_k &= GZ^T H^{-1}(y - X\hat{\beta}_k).\end{aligned} \tag{3}$$

In addition to ridge regression, [7, 10] suggested Liu's approach in linear regression models. By following [14, 15, 20] proposed the Liu estimator predictor via $0<d<1$ Liu biasing parameter under LMM as follows

$$\begin{aligned}\hat{\beta}_d &= (X^T H^{-1} X + I_p)^{-1}(X^T H^{-1} y + d\hat{\beta}),\\ \hat{u}_d &= GZ^T H^{-1}(y - X\hat{\beta}_d),\end{aligned} \tag{4}$$

where $\hat{\beta}$ is the BLUE in Equation (2).

In linear regression models, [9] proposed a new one-parameter estimator in the class of ridge and Liu estimators and they called their new estimator as the Kibria-Lukman (KL) estimator. By following [9] in linear regression models, [12] suggested respectively the KL estimator and the KL predictor in LMMs as

$$\begin{aligned}\hat{\beta}_{KL} &= (X^T H^{-1} X + kI_p)^{-1}(X^T H^{-1} y - k\hat{\beta}) = (X^T H^{-1} X + kI_p)^{-1}(X^T H^{-1} X - kI_p)\hat{\beta}\\ &= (I_p + k(X^T H^{-1} X)^{-1})^{-1}(I_p - k(X^T H^{-1} X)^{-1})\hat{\beta},\\ \hat{u}_{KL} &= GZ^T H^{-1}(y - X\hat{\beta}_{KL}).\end{aligned} \tag{5}$$

Now, we will introduce a new prediction approximation as an alternative to the estimators/predictors defined above under multicollinearity.

3. Introduced New Prediction Approximation

Via [1] in linear regression models, a new prediction approximation is handled in LMMs in this part. With model (1) assumptions, we have

$$\begin{pmatrix} u \\ y \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ X\beta \end{pmatrix}, \sigma^2 \begin{pmatrix} G & GZ^T \\ ZG & H \end{pmatrix}\right), y|u \sim N\left(X\beta + Zu, \sigma^2 W\right),$$

[5] maximize

$$\begin{aligned}f(y,u) &= f(y|u) f(u)\\ &= (2\pi\sigma^2)^{-(n+qm)/2} |W|^{-1/2} |G|^{-1/2}\\ &\quad \times \exp\{-\frac{1}{2\sigma^2}[(y - X\beta - Zu)^T W^{-1}(y - X\beta - Zu) + u^T G^{-1} u]\},\end{aligned}$$

where $|.|$ is a matrix determinant and thus, $\log f(y,u)$ is obtained

$$\begin{aligned}\log f(y,u) &= \log f(y|u) + \log f(u)\\ &= -\frac{1}{2}\{(n+qm)\log(2\pi) + (n+qm)\log\sigma^2 + \log|W| + \log|G|\\ &\quad +[(y-X\beta-Zu)^T W^{-1}(y-X\beta-Zu) + u^T G^{-1}u]/\sigma^2\}.\end{aligned}$$

Our goal is to describe a new prediction method which is resistant to multicollinearity alternative to ridge, Liu and KL prediction approaches in LMMs. Via [1], $\log f(y,u)$ is minimized under $(\beta+\hat{\beta})^T(\beta+\hat{\beta}) = c$ with $\delta = \frac{1}{2\sigma^2} \geq 0$ regularization parameter

$$\log f(y,u) - \frac{1}{2\sigma^2}K[(\beta+\hat{\beta})^T(\beta+\hat{\beta}) - c], \tag{6}$$

where $K = diag(k_1,\ldots,k_p)$ for $0 < k_i < 1$, $i = 1,\ldots,p$, as the ridge biasing parameters and c is a constant. Substituting the log function into Equation (6) and removing the constant term from the model,

$$\begin{aligned}&-\frac{1}{2\sigma^2}\{(y-X\beta)^T W^{-1}(y-X\beta) + K[(\beta+\hat{\beta})^T(\beta+\hat{\beta}) - c]\}\\ &-\frac{1}{2\sigma^2}\{u^T(Z^TW^{-1}Z+G^{-1})u - 2(y-X\beta)^T W^{-1}Zu\},\end{aligned} \tag{7}$$

is written. Initially, we get partial derivatives of Equation (7) corresponding to β and u. Later, we equalize these derivatives to zero. Thus, we derive the following equations

$$X^TW^{-1}(y-X\hat{\beta}_{GKL}) - K\hat{\beta} - K\hat{\beta}_{GKL} - X^TW^{-1}Z\hat{u}_{GKL} = 0, \tag{8}$$

$$Z^TW^{-1}(y-X\hat{\beta}_{GKL}) - (Z^TW^{-1}Z+G^{-1})\hat{u}_{GKL} = 0 \tag{9}$$

and we name as $\hat{\beta}_{GKL}$ and $\hat{u}_{GKL}$, respectively, as the generalized KL (GKL) estimator and predictor, respectively.

We present Equations (8) and (9) as

$$\begin{pmatrix} X^TW^{-1}X+K & X^TW^{-1}Z\\ Z^TW^{-1}X & Z^TW^{-1}Z+G^{-1}\end{pmatrix}\begin{pmatrix}\hat{\beta}_{GKL}\\ \hat{u}_{GKL}\end{pmatrix} = \begin{pmatrix}X^TW^{-1}y - K\hat{\beta}\\ Z^TW^{-1}y\end{pmatrix}. \tag{10}$$

We write Equation (10) via [3] as follows:

$$C\hat{\Psi} = \omega^TW^{-1}y + \kappa, \tag{11}$$

where $\hat{\Psi} = (\hat{\beta}^T_{GKL}, \hat{u}^T_{GKL})^T$, $\omega = (X, Z)$, $\kappa = (-K\hat{\beta}^T, 0^T)^T$ and $C = \omega^T W^{-1}\omega + Ģ^+$ is full rank with the Moore-Penrose inverse '+'

$$Ģ = \begin{pmatrix} \frac{I_p}{K} & 0 \\ 0 & G \end{pmatrix} \text{ and } Ģ^+ = \begin{pmatrix} K & 0 \\ 0 & G^{-1} \end{pmatrix}.$$

After Equation (11) is found, we obtain

$$\hat{\Psi} = C^{-1}\omega^T W^{-1} y + C^{-1}\kappa, \tag{12}$$

where C^{-1} is calculated from the inverse partitioned matrix [18] as

$$C^{-1} = \begin{pmatrix} \acute{N} & -\acute{N}X^T H^{-1} ZG \\ -GZ^T H^{-1} X\acute{N} & \Upsilon + GZ^T H^{-1} X\acute{N}X^T H^{-1} ZG \end{pmatrix},$$

where $\acute{N} = (X^T H^{-1} X + K)^{-1}$ and $\Upsilon = (Z^T W^{-1} Z + G^{-1})^{-1}$. Then, after C^{-1} puts in Equation (12), the GKL estimator and the GKL predictor are derived, respectively, as

$$\begin{aligned} \hat{\beta}_{GKL} &= (X^T H^{-1} X + K)^{-1}(X^T H^{-1} y - K\hat{\beta}) = (X^T H^{-1} X + K)^{-1}(X^T H^{-1} X - K)\hat{\beta} \\ &= (I_p + K(X^T H^{-1} X)^{-1})^{-1}(I_p - K(X^T H^{-1} X)^{-1})\hat{\beta}, \end{aligned} \tag{13}$$

$$\hat{u}_{GKL} = GZ^T H^{-1}(y - X\hat{\beta}_{GKL}). \tag{14}$$

4. Mean Square Error Performances

Prediction of linear combinations of β and u is explained as $\mu = L^T\beta + M^T u$ for specific $L \in \mathbb{R}^{p\times 1}$ and $M \in \mathbb{R}^{q\times 1}$ matrices (see [16, 17, 21]). With the help of [19], the MMSEs for $\hat{\mu}$, $\hat{\mu}_{KL}$ and $\hat{\mu}_{GKL}$ are written as

$$MMSE(\hat{\mu}) = \mathbb{Q}MMSE(\hat{\beta})\mathbb{Q}^T + \sigma^2 M^T(G - GZ^T H^{-1} ZG)M, \tag{15}$$

$$MMSE(\hat{\mu}_{KL}) = \mathbb{Q}MMSE(\hat{\beta}_{KL})\mathbb{Q}^T + \sigma^2 M^T(G - GZ^T H^{-1} ZG)M, \tag{16}$$

$$MMSE(\hat{\mu}_{GKL}) = \mathbb{Q}MMSE(\hat{\beta}_{GKL})\mathbb{Q}^T + \sigma^2 M^T(G - GZ^T H^{-1} ZG)M, \tag{17}$$

where $\hat{\mu} = L^T\hat{\beta} + M^T\hat{u} = \mathbb{Q}\hat{\beta} + M^TGZ^TH^{-1}y$, $\hat{\mu}_{KL} = L^T\hat{\beta}_{KL} + M^T\hat{u}_{KL} = \mathbb{Q}\hat{\beta}_{KL} + M^TGZ^TH^{-1}y$, $\hat{\mu}_{GKL} = L^T\hat{\beta}_{GKL} + M^T\hat{u}_{GKL} = \mathbb{Q}\hat{\beta}_{GKL} + M^TGZ^TH^{-1}y$, $\mathbb{Q} = L^T - M^TGZ^TH^{-1}X$,

$$MMSE(\hat{\beta}) = \sigma^2(X^TH^{-1}X)^{-1}, \tag{18}$$

$$\begin{aligned} MMSE(\hat{\beta}_{KL}) = {} & \sigma^2(I_p + k(X^TH^{-1}X)^{-1})^{-1}(I_p - k(X^TH^{-1}X)^{-1})(X^TH^{-1}X)^{-1} \\ & \times(I_p - k(X^TH^{-1}X)^{-1})(I_p + k(X^TH^{-1}X)^{-1})^{-1} \\ & +[(I_p + k(X^TH^{-1}X)^{-1})^{-1}(I_p - k(X^TH^{-1}X)^{-1}) - I_p] \\ & \times\beta\beta^T[(I_p + k(X^TH^{-1}X)^{-1})^{-1}(I_p - k(X^TH^{-1}X)^{-1}) - I_p]^T, \end{aligned} \tag{19}$$

$$\begin{aligned} MMSE(\hat{\beta}_{GKL}) = {} & \sigma^2(I_p + K(X^TH^{-1}X)^{-1})^{-1}(I_p - K(X^TH^{-1}X)^{-1})(X^TH^{-1}X)^{-1} \\ & \times(I_p - K(X^TH^{-1}X)^{-1})(I_p + K(X^TH^{-1}X)^{-1})^{-1} \\ & +[(I_p + K(X^TH^{-1}X)^{-1})^{-1}(I_p - K(X^TH^{-1}X)^{-1}) - I_p] \\ & \times\beta\beta^T[(I_p + K(X^TH^{-1}X)^{-1})^{-1}(I_p - K(X^TH^{-1}X)^{-1}) - I_p]^T. \end{aligned} \tag{20}$$

When we examine Equations (15), (16) and (17), it can be said that the superiority of $MMSE(\hat{\mu}_{GKL})$ over $MMSE(\hat{\mu})$ and $MMSE(\hat{\mu}_{KL})$ is equivalent to the superiority of $MMSE(\hat{\beta}_{GKL})$ over $MMSE(\hat{\beta})$ and $MMSE(\hat{\beta}_{KL})$ derived by, respectively, Equations (18), (19) and (20). Then, via orthogonal transformation, our model (1) is transformed to a canonical form. Because H is pd, there exists a nonsingular symmetric matrix N such that $H = N^TN$. Our new model is

$$y^* = X^*\beta + Z^*u + \varepsilon^*, \tag{21}$$

with $y^* = N^{-1}y$, $X^* = N^{-1}X$, $Z^* = N^{-1}Z$, $\varepsilon^* = N^{-1}\varepsilon$ and $Var(y^*) = \sigma^2 I$ is derived.

The spectral decomposition of the matrix $X^TH^{-1}X$ is $P^T\Lambda P$ with $\Lambda = diag(\lambda_i)$ the $p \times p$ orthogonal matrix of the eigenvalues of $X^TH^{-1}X$ ($\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \geq 0$) and $P = [P_1 \ldots P_p]$ the $p \times p$ orthogonal matrix of the standardized eigenvectors corresponding to the eigenvalues. Then, the model (21) can be written as $y^* = K^*\alpha + Z^*u + \varepsilon^*$, where $K^* = X^*P^T$ and $\alpha = P\beta$. In the transformed model, $MMSE(\tilde{\alpha}) = P[MMSE(\tilde{\beta})]P^T$ for any estimator $\tilde{\alpha}$ is derived. Hence, we have the following MMSE formulas via Equations (18), (19) and (20)

$$MMSE(\hat{\alpha}) = \sigma^2\Lambda^{-1}, \tag{22}$$

$$\begin{aligned} MMSE(\hat{\alpha}_{KL}) &= \sigma^2(I_p+k\Lambda^{-1})^{-1}(I_p-k\Lambda^{-1})\Lambda^{-1}(I_p-k\Lambda^{-1})(I_p+k\Lambda^{-1})^{-1} \\ &+[(I_p+k\Lambda^{-1})^{-1}(I_p-k\Lambda^{-1})-I_p]\alpha\alpha^T[(I_p+k\Lambda^{-1})^{-1}(I_p-k\Lambda^{-1})-I_p]^T, \end{aligned} \tag{23}$$

$$\begin{aligned} MMSE(\hat{\alpha}_{GKL}) &= \sigma^2(I_p+K\Lambda^{-1})^{-1}(I_p-K\Lambda^{-1})\Lambda^{-1}(I_p-K\Lambda^{-1})(I_p+K\Lambda^{-1})^{-1} \\ &+[(I_p+K\Lambda^{-1})^{-1}(I_p-K\Lambda^{-1})-I_p]\alpha\alpha^T[(I_p+K\Lambda^{-1})^{-1}(I_p-K\Lambda^{-1})-I_p]^T. \end{aligned} \tag{24}$$

We will define the two theorems given below, respectively, the GKL estimator vs the BLUE and the GKL estimator vs the KL estimator.

Theorem 4.1 *$MMSE(\hat{\alpha}) - MMSE(\hat{\alpha}_{GKL}) > 0$ iff*

$$\begin{aligned} &\alpha^T[(I_p+K\Lambda^{-1})^{-1}(I_p-K\Lambda^{-1})-I_p]^T \\ &\times[\sigma^2(\Lambda^{-1}-(I_p+K\Lambda^{-1})^{-1}(I_p-K\Lambda^{-1})\Lambda^{-1}(I_p-K\Lambda^{-1})(I_p+K\Lambda^{-1})^{-1})] \\ &\times[(I_p+K\Lambda^{-1})^{-1}(I_p-K\Lambda^{-1})-I_p]\alpha < 1. \end{aligned}$$

Theorem 4.2 *$MMSE(\hat{\alpha}_{KL}) - MMSE(\hat{\alpha}_{GKL}) > 0$ iff*

$$\begin{aligned} &\alpha^T[(I_p+K\Lambda^{-1})^{-1}(I_p-K\Lambda^{-1})-I_p]^T[\Omega+[(I_p+k\Lambda^{-1})^{-1}(I_p-k\Lambda^{-1})-I_p] \\ &\times\alpha\alpha^T[(I_p+k\Lambda^{-1})^{-1}(I_p-k\Lambda^{-1})-I_p]^T][(I_p+K\Lambda^{-1})^{-1}(I_p-K\Lambda^{-1})-I_p]\alpha < 1, \end{aligned}$$

where

$$\begin{aligned} \Omega &= \sigma^2((I_p+k\Lambda^{-1})^{-1}(I_p-k\Lambda^{-1})\Lambda^{-1}(I_p-k\Lambda^{-1})(I_p+k\Lambda^{-1})^{-1} \\ &-(I_p+K\Lambda^{-1})^{-1}(I_p-K\Lambda^{-1})\Lambda^{-1}(I_p-K\Lambda^{-1})(I_p+K\Lambda^{-1})^{-1}). \end{aligned}$$

[1] can be investigated for Theorems 4.1 and 4.2 proofs.

5. About Biasing Parameter Selection

Under our proposed new prediction approximation, an appropriate parameter k calculation is important. For this purpose, differentiating Equation (24) corresponding to k and then, equating to zero, we find

$$k_i = \frac{\sigma^2}{2\alpha_i^2 + (\sigma^2/\lambda_i)}, i = 1, \ldots, p, \tag{25}$$

Since the optimal value of k in Equation (25) depends on the unknown parameters σ^2 and α^2, we replace with their unbiased estimate and so, we have

$$\hat{k}_i = \frac{\hat{\sigma}^2}{2\hat{\alpha}_i^2 + (\hat{\sigma}^2/\lambda_i)}, i = 1, \ldots, p, \tag{26}$$

and then, we introduce the minimum version of Equation (26) as

$$\hat{k}_{\min} = \min\left[\frac{\hat{\sigma}^2}{2\hat{\alpha}_i^2 + (\hat{\sigma}^2/\lambda_i)}\right]. \tag{27}$$

6. Gases of Greenhouse Data Example

Greenhouse gases have increased greatly in the last 150 years and the most important reason for this increase is human activities. The burning of fossil fuels for heat, transportation and electricity is the largest cause of gas emissions from these human activities [2]. The transportation sector receives the largest portion of greenhouse gas emissions from these three sectors in the United States. In this data example, we employ data on 297 fuel combustion in transport from randomly selected 27 areas for the years including 2006-2016 (see [2]). To identify fuel combustion in transport (y), repeated measurements are taken from the cars (x_1), the light duty trucks (x_2), the heavy duty trucks-buses (x_3), the motorcycles (x_4) and railways (x_5). The areas factor effect is random effect. Thus, our model is yielded

$$y_{ij} = \beta_1 x_{ij1} + \beta_2 x_{ij2} + \beta_3 x_{ij3} + \beta_4 x_{ij4} + \beta_5 x_{ij5} + u_1 + u_2 t_{ij} + \varepsilon_{ij},\ i = 1, \ldots, 27,\ j = 1, \ldots, 11,$$

where y_{ij} shows the ith observation of the jth area of the response, x_{ijs} shows the ith observation of the jth area of the explanatory variable x_s, $s = 1, \ldots, 5$, t_{ij} denotes time corresponding to y_{ij}. In this example, we benefit from Matlab R2014a. Initially, we think covariance structures given below and then, for comparing these covariance models with ML and REML, we benefit from the Akaike's Information Criterion (AIC) and the Bayesian Information Criterion (BIC) (see Table 1).

Table 1: Covariance structures [1]

Cov. Struc.	Est. Met. for Cov. Par.	AIC	BIC
Unstructured (UN)	ML	**337.30**	374.24
	REML	**362.03**	398.76
Diagonal (UN(1))	ML	339.42	**372.67**
	REML	362.87	**395.93**
Variance Components (VC)	ML	391.56	421.11
	REML	416.72	446.11
Compound Symmetry (CS)	ML	393.42	426.67
	REML	418.60	451.66

The best models for modeling covariance matrix structure by response variable, which are the minimum values corresponding to AIC and BIC criteria, are the UN under AIC and UN(1) under BIC. By following [8] and [13]'s ideas, we choose UN(1) under ML and $\hat{G}_{ML} = \begin{bmatrix} 2.1913 & 0 \\ 0 & 0.0755 \end{bmatrix}$, $\hat{W}_{ML} = 0.25451 I_{297}$ are computed. Therefore, with $H = ZGZ^T + W$ formula, $\hat{H}_{ML}$ is derived. $X^T \hat{H}_{ML}^{-1} X$ matrix eigenvalues are computed as $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (1.4326 \times 10^{+7}, 1.5085 \times 10^{+4}, 4.7251 \times 10^{+3}, 247.7243, 41.5100)$. Since condition number $\lambda_{\max}/\lambda_{\min} = 345120 > 1000$ is obtained, one can say that severe multicollinearity is appeared.

To derive the GKL estimators/predictors, we get

$$K = diag(\hat{k}_i) = diag(1.03488, 5.56847, 6.80586, 9.04688, 0.10696), i = 1, \ldots, p,$$

by using Equation (26) and to get the KL estimators/predictors, we use $\hat{k} = \hat{k}_{\min} = 0.10696$ where $\hat{\sigma}^2$ is computed as 5.17298 given by Equation (27). In Table 2, fixed/random effects parameter estimates and scalar mean square error (SMSE) values are given. $\hat{\beta}_{GKL}$ outperforms $\hat{\beta}$ and $\hat{\beta}_{KL}$ in the sense of SMSE values under Table 2.

Table 2: Fixed/random effects parameter estimates and SMSE values

	β_1	β_2	β_3	β_4	β_5	$SMSE$		u_1	u_2
$\hat{\beta}$	1.02474	1.05007	0.93304	3.34361	3.67898	0.14693	$\hat{u}$	0.54883	−0.07806
$\hat{\beta}_{KL}$	1.02549	1.05044	0.93246	3.32847	3.65880	0.14599	$\hat{u}_{KL}$	0.54997	−0.07823
$\hat{\beta}_{GKL}$	1.03151	1.06769	0.89854	2.17688	3.65997	**0.05558**	$\hat{u}_{GKL}$	1.69062	−0.08354

Theorems 4.1 and 4.2 conditions are computed as, respectively, $0.01205 < 1$ and $0.01186 < 1$, hence $\hat{\beta}_{GKL}$ is also better than $\hat{\beta}$ and $\hat{\beta}_{KL}$ under the MMSE criterion.

Gases of greenhouse data example confirms that $\hat{\beta}_{GKL}$ is superior than $\hat{\beta}$ and $\hat{\beta}_{KL}$ when appropriate k values are employed.

[1]The abbreviations " Cov. Struc." and " Est. Met. for Cov. Par." refer to " Covariance Structures" and " Estimation Methods for Covariance Parameters".

7. Conclusion

The GKL prediction approach is extended to LMMs by using the method given in [1]. We also perform MMSE comparisons then, we give biasing parameter selection. Eventually, we support with our findings with gases of greenhouse data example.

This article presents that one can use the GKL estimator/predictor alternative to KL estimator/predictor in an LMM when multicollinearity problem exists and additionally, this article has affirmed that the GKL approach usage ensures a smaller MSE than the BLUE and KL estimator for appropriate selected ridge biasing parameter.

Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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Research Article

Dimodules

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Abstract: This paper introduces a new algebraic structure called dimodule. This structure is similar to a module. A dimodule occurs on a semigroup and a dimonoid in place of an additive abelian group and a ring, respectively. This paper presents some algebraic properties of the dimodules and supplies some of their examples. We suggest a definition of a distributive dimonoid. This paper includes examples of this notion that a distributive dimonoid does not have to be a commutative and idempotent dimonoid. We also have examples of dimonoids and dimonoid homomorphisms.

Keywords: Dimonoid, semigroup, dimodule.

1. Introduction

Jean-Louis Loday introduces the concept of dimonoid [4] as a tool to investigate Leibniz algebras. Dimonoids are nonempty sets with two associative operations providing some axioms. The dimonoid becomes a semigroup if the operations are the same.

Anatolii V. Zhuchok has made many contributions to the topics related to dimonoids. Some of these are to give some properties of commutative dimonoids and examples of commutative dimonoids, to introduce the notion of the diband of dimonoids, to construct different samples of dimonoids, to demonstrate that dimonoids are embedded into some dimonoid formed by a semigroup isomorphically, to set a free commutative dimonoid [5, 6, 8, 9].

This paper introduces a dimodule as a new algebraic structure on a semigroup and a dimonoid. This structure inspires by the algebraic form of modules. The dimodules are an algebraic expansion by processing with the dimonoid and semigroups under certain conditions. In this paper, there are studies of some algebraic properties of dimodule concepts and some dimodule examples. We have the definition of a distributive dimonoid. We show with examples that a distributive dimonoid does not have to be a commutative or an idempotent dimonoid. We also have some

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examples of dimonoid and dimonoid homomorphism.

2. Preliminaries

This section contains basic definitions of the semigroups and the modules [1–3]. In this section, there are definitions of the dimonoids and some concepts of them [4–6, 8, 9]. Moreover, this section includes the definition of a distributive dimonoid and some new examples of dimonoids.

2.1. Semigroups

Let S be a nonempty set and "$\cdot$" be a binary operation on S. Then the algebraic structure $(S,\cdot)$ is called a semigroup if and only if, for all $k,l,m \in S$, $k\cdot(l\cdot m)=(k\cdot l)\cdot m$. Let $P(S)$ denote the set of all the subsets of S and $K,M \in P(S)$. If $K=\varnothing$ or $M=\varnothing$, then $K\cdot M=\varnothing$. If otherwise, $K\cdot M$ is the set $\{k\cdot m \mid k\in K, m\in M\}$.

If, for all $s\in S$, $0\cdot s=0$ ($s\cdot 0=0$), then an element $0\in S$ is a left (right) zero element. If an element $0\in S$ is both the left and right zero elements, it is a zero element. A semigroup S in which each element is a left (right) zero element is a left (right) zero semigroup. Let there is an element $0\in S$ in a semigroup $(S,\cdot)$ such that $x\cdot y=0$ for all $x,y\in S$. Then the semigroup is a zero semigroup. Let $(S,\cdot)$ and $(Y,*)$ be semigroups. Then a mapping $f:S\to Y$ is a homomorphism of semigroups if, for all $k,l\in S$, $f(k\cdot l)=f(k)*f(l)$. Let $\{S_i \mid i\in I\}$ be a family of the semigroups. Then $\prod_{i\in I} S_i$ denotes the Cartesian product of the family $\{S_i \mid i\in I\}$ and $\prod_{i\in I} S_i$ is a semigroup.

2.2. Dimonoids

Jean-Louis Loday presented the concept of dimonoid in 2001.

Definition 2.1 *[4] An arbitrary set $D\neq\varnothing$ on which there are two associative operations "$*$" and "$\circ$" is a dimonoid if, for all $k,l,m\in D$, provide the axioms in below:*

(1) $(k*m)*l=k*(m\circ l)$,

(2) $(k\circ m)*l=k\circ(m*l)$,

(3) $(k*m)\circ l=k\circ(m\circ l)$.

Example 2.2 *[4] Let D be a nonempty set and let two binary operations "$*$" and "$\circ$" be defined by, respectively, $k*l=k$ and $k\circ l=l$ for all $k,l\in D$. Then $(D,*,\circ)$ is a dimonoid.*

Example 2.3 *Let* $D = \{k, l\}$. *Then* $(D, *, \circ)$ *is a dimonoid with the following binary operations* "$*$" *and* "$\circ$":

$*$	$\boldsymbol{k}$	$\boldsymbol{l}$
$\boldsymbol{k}$	k	k
$\boldsymbol{l}$	k	k

$\circ$	$\boldsymbol{k}$	$\boldsymbol{l}$
$\boldsymbol{k}$	k	l
$\boldsymbol{l}$	k	l

Definition 2.4 *[8] If, for all* $k \in D$, $k * k = k = k \circ k$, *then a dimonoid* $(D, *, \circ)$ *is an idempotent dimonoid (or diband).*

Example 2.5 *Let* $D = \{k, l\}$. *Then* $(D, *, *)$ *is an idempotent dimonoid with the* "$*$" *binary operation:*

$*$	$\boldsymbol{k}$	$\boldsymbol{l}$
$\boldsymbol{k}$	k	k
$\boldsymbol{l}$	l	l

Example 2.6 *Let* $D = \{k, l\}$. *Then* $(D, *, \circ)$ *is an idempotent dimonoid with the binary operations* "$*$" *and* "$\circ$" *which are defined by the following table:*

$*$	$\boldsymbol{k}$	$\boldsymbol{l}$
$\boldsymbol{k}$	k	k
$\boldsymbol{l}$	l	l

$\circ$	$\boldsymbol{k}$	$\boldsymbol{l}$
$\boldsymbol{k}$	k	l
$\boldsymbol{l}$	k	l

Example 2.7 *[5] Let* $(D, *)$ *be a zero semigroup including fixed elements with* $a \neq b, b \neq 0$ *and for all* $k, l \in D$, *a binary relation* "$\circ$" *on* D *be defined by*

$$k \circ l = \begin{cases} a, & k = l = b \\ 0, & otherwise. \end{cases}$$

Then $(D, *, \circ)$ *is a dimonoid.*

Example 2.8 *[9] Let* $(S, \cdot)$ *be a semigroup with zero and* A *be a nonempty set. Then* A *is both a left* S*-act and a right* S*-act with the following commutative actions:*

$$S \times A \longrightarrow A : (s, l) = s \odot l = l,$$

$$A \times S \longrightarrow A : (l, s) = l \circledcirc s = l.$$

Consider the S*-act morphism* $\psi : A \longrightarrow S, x \longmapsto 0$. *Then* $(A, *, \circ)$ *is a dimonoid with the following binary operations:*

$$m * n := m \circledcirc \psi(n),$$

$$m \circ n := \psi(m) \odot n.$$

Theorem 2.9 *Let* $(D, *, \circ)$ *be a dimonoid and* S *be a nonempty set. If* $\vartheta : D \to S$ *is a bijective function, then* $(S, *_1, \circ_1)$ *is a dimonoid with binary operations defined as follows:*

$$s *_1 v = \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(v)),$$

$$s \circ_1 v = \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(v))$$

for all $s, v \in S$.

Proof For all $s, p, z \in S$,

$$\begin{aligned}(s *_1 p) *_1 z &= \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p)) *_1 z = \vartheta(\vartheta^{-1}(\vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p))) * \vartheta^{-1}(z)) \\ &= \vartheta((\vartheta^{-1}(s) * \vartheta^{-1}(p)) * \vartheta^{-1}(z)) = \vartheta(\vartheta^{-1}(s) * (\vartheta^{-1}(p) \circ \vartheta^{-1}(z))) \\ &= \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(\vartheta(\vartheta^{-1}(p) \circ \vartheta^{-1}(z)))) = \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p \circ_1 z)) = s *_1 (p \circ_1 z),\end{aligned}$$

$$\begin{aligned}(s *_1 p) *_1 z &= \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(p)) *_1 z = \vartheta(\vartheta^{-1}(\vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(p))) * \vartheta^{-1}(z)) \\ &= \vartheta((\vartheta^{-1}(s) \circ \vartheta^{-1}(p)) * \vartheta^{-1}(z)) = \vartheta(\vartheta^{-1}(s) \circ (\vartheta^{-1}(p) * \vartheta^{-1}(z))) \\ &= \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(\vartheta(\vartheta^{-1}(p) * \vartheta^{-1}(z)))) = \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(p *_1 z)) = s \circ_1 (p *_1 z),\end{aligned}$$

$$\begin{aligned}(s *_1 p) \circ_1 z &= \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p)) \circ_1 z = \vartheta(\vartheta^{-1}(\vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p))) \circ \vartheta^{-1}(z)) \\ &= \vartheta((\vartheta^{-1}(s) * \vartheta^{-1}(p)) \circ \vartheta^{-1}(z)) = \vartheta(\vartheta^{-1}(s) \circ (\vartheta^{-1}(p) \circ \vartheta^{-1}(z))) \\ &= \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(\vartheta(\vartheta^{-1}(p) \circ \vartheta^{-1}(z)))) = \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(p \circ_1 z)) = s \circ_1 (p \circ_1 z),\end{aligned}$$

$$\begin{aligned}(s *_1 p) *_1 z &= \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p)) *_1 z = \vartheta(\vartheta^{-1}(\vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p))) * \vartheta^{-1}(z)) \\ &= \vartheta((\vartheta^{-1}(s) * \vartheta^{-1}(p)) * \vartheta^{-1}(z)) = \vartheta(\vartheta^{-1}(s) * (\vartheta^{-1}(p) * \vartheta^{-1}(z))) \\ &= \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(\vartheta(\vartheta^{-1}(p) * \vartheta^{-1}(z)))) = \vartheta(\vartheta^{-1}(s) * \vartheta^{-1}(p *_1 z)) = s *_1 (p *_1 z),\end{aligned}$$

$$\begin{aligned}(s \circ_1 p) \circ_1 z &= \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(p)) \circ_1 z = \vartheta(\vartheta^{-1}(\vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(p))) \circ \vartheta^{-1}(z)) \\ &= \vartheta((\vartheta^{-1}(s) \circ \vartheta^{-1}(p)) \circ \vartheta^{-1}(z)) = \vartheta(\vartheta^{-1}(s) \circ (\vartheta^{-1}(p) \circ \vartheta^{-1}(z))) \\ &= \vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(\vartheta(\vartheta^{-1}(p) \circ \vartheta^{-1}(z))))\vartheta(\vartheta^{-1}(s) \circ \vartheta^{-1}(p \circ_1 z)) = s \circ_1 (p \circ_1 z).\end{aligned}$$

□

Definition 2.10 *[5] Let* $(D_1, *_1, \circ_1), (D_2, *_2, \circ_2)$ *be dimonoids. Then a mapping* $f : D_1 \to D_2$ *is called a homomorphism of dimonoids if, for all* $k, l \in D_1$, $f(k *_1 l) = f(k) *_2 f(l)$ *and* $f(k \circ_1 l) = f(k) \circ_2 f(l)$.

Example 2.11 *Let D_1 and D_2 be dimonoids in Example 2.5 and Example 2.6, respectively. Then all the homomorphisms of dimonoids from D_1 to D_2 are the functions $f(a) = k$ and $g(a) = l$ for all $a \in D_1$.*

Definition 2.12 *[8] Let $\varnothing \neq T \subseteq D$. Then T is called a subdimonoid, if for all $k, l \in T$ implies $k * l \in T$, $k \circ l \in T$.*

Definition 2.13 *[5] Let $(D, *, \circ)$ be a dimonoid. Then D is called a commutative dimonoid if both operations are commutative.*

Example 2.14 *[5] Let A be an arbitrary set such that $0, k, l, m, n \in A$ and $k \neq l$, $l \neq m$, $m \neq n$, $n \neq k$. The operations "$*$" and "$\circ$" on the set A be defined as follows:*

$$x * y = \begin{cases} l, & x=y=k \\ 0, & otherwise \end{cases}, \qquad x \circ y = \begin{cases} n, & x=y=m \\ 0, & otherwise \end{cases}$$

for all $x, y \in A$. So $(A, *, \circ)$ is a commutative dimonoid.

Theorem 2.15 *[5] In a commutative dimonoid $(D, *, \circ)$, for all $k, l, m \in D$, the following equalities hold:*

$$(k * l) * m = k * (l \circ m) = (k \circ l) * m = k \circ (l * m) = (k * l) \circ m = k \circ (l \circ m).$$

Theorem 2.16 *[5] Let $(D, *, \circ)$ be a commutative dimonoid with an idempotent operation "$*$". Then its operations coincide.*

Definition 2.17 *$(D, *, \circ)$ is a distributive dimonoid if and only if*

$$k \circ (l * m) = (k \circ l) * (k \circ m),$$

$$(l * m) \circ k = (l \circ k) * (m \circ k)$$

for all $k, l, m \in D$.

Example 2.18 *Let $(D, *, \circ)$ be the dimonoid in Example 2.2. Then $(D, *, \circ)$ is a distributive dimonoid.*

Theorem 2.19 *If $(D, *, \circ)$ is a commutative idempotent dimonoid, then it is a distributive dimonoid.*

Proof Let $(D, *, \circ)$ is a commutative idempotent dimonoid. Then according to Theorem 2.16, "$*$" and "$\circ$" are the same operations. So $(k \circ l) * (k \circ m) = (k * l) * (k * m) = (k * k) * (l * m) =$

$k*(l*m)=k\circ(l*m)$ for all $k,l,m\in D$. Since $(D,*,\circ)$ is a commutative dimonoid, then $(D,*,\circ)$ is distributive dimonoid. □

The dimonoid $(D,*,\circ)$ in Example 2.2 is a distributive and non-commutative. In Example 2.7, the dimonoid $(D,*,\circ)$ is a distributive and commutative dimonoid but not idempotent since $b*b=0\neq b$.

Example 2.20 *Let $D=\{k,l,m\}$ be the commutative dimonoid with the operation "$*$" defined by the following table:*

$*$	k	l	m
k	k	k	k
l	k	l	m
m	k	m	l

Then $(D,*,*)$ is not distributive since $m*(l*l)\neq(m*l)*(m*l)$. Also $(D,*,*)$ is not idempotent since $m*m\neq m$.

Example 2.21 *Let $D=\{k,l,m\}$ be an arbitrary set. $(D,*,*)$ is a commutative with the operation "$*$"defined in table. Although $(D,*,*)$ commutative dimonoid is distributive dimonoid, it is not idempotent since $m*m=l\neq m$.*

$*$	k	l	m
k	k	k	k
l	k	l	l
m	k	l	l

Theorem 2.22 *Let $(D,*,\circ)$ be an arbitrary dimonoid, and let S be the dimonoid generated from D as in the Theorem 2.9. If $(D,*,\circ)$ is distributive, then S is so.*

Proof Let $k,l,m\in S$. Then $k\circ_1(l*_1m)=k\circ_1(\vartheta(\vartheta^{-1}(l)*\vartheta^{-1}(m)))=\vartheta(\vartheta^{-1}(k)\circ(\vartheta^{-1}(l)*\vartheta^{-1}(m)))=\vartheta((\vartheta^{-1}(k)\circ\vartheta^{-1}(l))*(\vartheta^{-1}(k)\circ\vartheta^{-1}(m)))$. Let $(\vartheta^{-1}(k)\circ\vartheta^{-1}(l)):=\vartheta^{-1}(a)$ and $(\vartheta^{-1}(k)\circ\vartheta^{-1}(m)):=\vartheta^{-1}(b)$. Then $k\circ_1(l*_1m)=\vartheta(\vartheta^{-1}(a)*\vartheta^{-1}(b))=a*_1b=\vartheta(\vartheta^{-1}(k)\circ\vartheta^{-1}(l))*_1\vartheta(\vartheta^{-1}(k)\circ\vartheta^{-1}(l))=(k\circ_1l)*_1(k\circ_1m)$. Thus S is left distributive since $k\circ_1(l*_1m)=(k\circ_1l)*_1(k\circ_1m)$. Similarly, S is right distributive. □

Theorem 2.23 *[7] Let $\{D_i\mid i\in I\}$ be a family of dimonoids. Then the Cartesian product of the family $\{D_i\mid i\in I\}$, $\prod_{i\in I}D_i$, is a dimonoid.*

3. Dimodules

Let $(D, *, \circ)$ be a dimonoid. A (left) D-dimodule is a semigroup $(S, \cdot)$ together with a function $D \times S \to S$ (the image of (u, x) being denoted by ux) such that for all $u, v \in D$ and for all $x, y \in S$:

(1) $u(x \cdot y) = ux \cdot uy$,

(2) $(u * v)x = ux \cdot vx$,

(3) $u(vx) = (u \circ v)x$.

A right D-dimodule is defined similarly via function $S \times D \to S$ denoted $(x, u) \mapsto xu$ and satisfying the obvious of $(1)-(3)$. In this paper, unless specified otherwise, a D-dimodule means a left D-dimodule. All theorems about left D-dimodules also hold for right D-dimodules.

Example 3.1 *Let $(D, *, \circ)$ be a dimonoid and $(S, \cdot)$ be a semigroup with an idempotent element a. Then S is a D-dimodule with the operation*

$$D \times S \longrightarrow S$$
$$(x, y) \longmapsto a$$

Example 3.2 *Let $D = S = \{a, b\}$. Then $(D, *, \circ)$ is dimonoid and $(S, \cdot)$ is a semigroup for the operations "$*, \circ, \cdot$" in the following tables:*

$*$	a	b
a	a	a
b	a	a

$\circ$	a	b
a	a	b
b	a	b

$\cdot$	a	b
a	a	b
b	b	b

(i) Let a function $D \times S \to S$ be defined as $(d, s) \to ds = s$. Then S is a D-dimodule.

(ii) Let a function $D \times S \to S$ be defined as $(d, s) \to ds = d$. Then S is not a D-dimodule since $(a * b)a = a \neq b = aa \cdot ba$.

Example 3.3 *Let $(D, *, \circ)$ be the dimonoid and let $(\mathbb{N}, \cdot)$ be the semigroup of natural numbers with the multiplication. Let a function $D \times \mathbb{N} \to \mathbb{N}$ be defined as follows:*

$$dn = \begin{cases} 0, & 2 \mid n \\ 1, & 2 \nmid n. \end{cases}$$

Then $\mathbb{N}$ is a D-dimodule.

Example 3.4 *Let $(D, *)$ be the semigroup in Example 2.20. If the function $D \times D \to D$ is defined as $(d, s) \longmapsto ds = d * s$, then D is not a D-dimodule since $(m * m)m = m$ and $mm * mm = l * l = l$.*

Example 3.5 *Let* $(D,*,\circ)$ *be a dimonoid in which* $(D,*)$ *is an idempotent semigroup and let a function* $D\times D\longrightarrow D$ *defined as* $(x,y)\longmapsto xy=y$. *Then* D *is a* D*-dimodule.*

Proposition 3.6 *Let* S_1,S_2 *be semigroups and* f *be a homomorphism of semigroup from* S_1 *to* S_2. *Then* S_1 *is a* D*-dimodule if* S_2 *is* D*-dimodule.*

Proof Let the semigroup S_2 be D-dimodule with the mapping $D\times S_2\to S_2$, $(u,y)\longmapsto uy$. Thus consider the mapping $D\times S_1\to S_1$, $(u,x)\longmapsto ux=uf(x)$. Then S_1 is a D-dimodule. □

Proposition 3.7 *Let* $(D,*,\circ)$ *be a distributive dimonoid and a function* $D\times D\longrightarrow D$ *be defined as* $(x,y)\longmapsto xy=x\circ y$. *Then* $(D,*)$ *is a* D*-dimodule.*

Proof Straightforward. □

Example 3.8 shows that the Proposition 3.7 may not be correct if $(D,*,\circ)$ is not a distributive dimonoid, in general.

Example 3.8 *Consider the dimonoid* D *in Example 2.20. Thus* $(D,*)$ *is not a* D*-dimodule since* $m*(l*l)=m\neq l=(m*l)*(m*l)$.

Proposition 3.9 *Let* $\{S_iD_i\text{-module}\,|\,i\in I\}$. *Then* $\prod_{i\in I}S_i$ *is a* $\prod_{i\in I}D_i$*-module.*

Proof Consider the mapping $\prod_{i\in I}D_i\times\prod_{i\in I}S_i\to\prod_{i\in I}S_i$, $((d_i)_{i\in I},(s_i)_{i\in I})\longmapsto(d_i)_{i\in I}.(s_i)_{i\in I}=(d_is_i)_{i\in I}$. Then $\prod_{i\in I}S_i$ is a $\prod_{i\in I}D_i$-module. □

Proposition 3.10 *Let* $(D,*,\circ)$ *be a dimonoid and a semigroup* S *be a* D*-dimodule with a bijective mapping* $D\times S\to S$. *Then* D *is a distributive dimodule.*

Proof Let $k,l,m\in D$ and $x\in S$. Thus $[k\circ(l*m)]x=k[(l*m)x]=k((lx)(mx))=(k(lx))(k(mx))=((k\circ l)x)((k\circ m)x)=[(k\circ l)*(k\circ m)]x$ and $[(l*m)\circ k]x=[(l\circ k)*(m\circ k)]x$ similarly. Hence D is distributive via bijectivity. □

Definition 3.11 *Let* $(S,\cdot)$ *be a* D*-dimodule and* $\varnothing\neq E\subseteq S$. *Then* E *is called a* D*-subdimodule of* S *if, for all* $x,y\in E$ *and* $u\in D$, $x\cdot y,ux\in E$.

Example 3.12 *Listed below are some examples of subdimodules:*

(i) Each dimodule is a subdimodule of itself.

(ii) Let D *be the* D*-dimodule in Example 3.5. Then each subsemigroup of* D *is a subdimodule of* D.

(iii) Let $(S,\cdot)$ be the D-dimodule in Example 3.2-(i) and $E=\{a\}$. Then E is a D-subdimodule of S.

Proposition 3.13 *Let S be a D-dimodule and $\{E_i \mid i\in I\}$ be a family of the D-subdimodules of S. Then $\bigcap_{i\in I} E_i$ is a D-subdimodule of S if $\bigcap_{i\in I} E_i \neq \varnothing$.*

Proof Let $x,y\in\bigcap_{i\in I} E_i$ and $u\in D$. Thus $x,y\in E_i$ for all $i\in I$. Hence, for all $i\in I$, $x\cdot y\in E_i$ and $ux\in E_i$ since E_i is a D-subdimodule. Then $x\cdot y, ux\in\bigcap_{i\in I} E_i$. Therefore $\bigcap_{i\in I} E_i$ is a D-dimodule of S. □

Example 3.14 shows that Proposition 3.13 may not be correct for the union of the families of subdimodules.

Example 3.14 *Let $D=\{a,b,c\}$ and $(D,*)$ be the semigroup with the table below. If the function $D\times D\longrightarrow D$ is defined as $(u,x)\longmapsto ux=x$, then D is a D-dimodule.*

$*$	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c

The subsets $A=\{b\}$ and $B=\{c\}$ of D are D-subdimodules. However, $A\cup B=\{b,c\}$ is not a D-subdimodule since $b*c=a\notin A\cup B$.

Proposition 3.15 *Let S be a D-dimodule and $A\subseteq S$.*

(i) Let $a\in A$ be idempotent element and $(A:^D S)_a$ be the set $\{u\in D \mid ux=a \text{ for all } x\in A\}$. Then $(A:^D S)_a$ is a subdimonoid of D if it is nonempty.

(ii) Let A is a subsemigroup of S and $(A:_D S)=\{u\in D\mid uS\subseteq A\}$. Then $(A:_D S)$ is a subdimonoid of D if it is nonempty.

Proof Straightforward. □

Proposition 3.16 *Let $\{S_i\mid i\in I\}$ be a family of the D-dimodules. Then $\prod_{i\in I} S_i$ is a D-dimodule and it is called direct product of the family $\{S_i\mid i\in I\}$.*

Proof Let the mapping $D\times\prod_{i\in I} S_i\to\prod_{i\in I} S_i$, $(d,(s_i)_{\in I})\longmapsto d(s_i)_{i\in I}=(ds_i)_{i\in I}$. Then $\prod_{i\in I} S_i$ is a D-dimodule. □

Definition 3.17 *Let S_1,S_2 be D-dimodules. A function $f:S_1\longrightarrow S_2$ is called a homomorphism of D-dimodules if, for all $x,y\in S_1$ and $u\in D$, $f(x\cdot y)=f(x)\cdot f(y)$ and $f(ux)=uf(x)$.*

Example 3.18 *Let S_1 be a D-dimodule and S_1 be the D-dimodule in Example 3.1. Let a function $f : S_1 \longrightarrow S_2$ be defined by $f(x) = a$ for all $x \in S_1$. Then f is a homomorphism of D-dimodules.*

Example 3.19 *Let two binary operations on $\mathbb{Z}_5$ be defined as follows:*

$$\bar{x} * \bar{y} = \begin{cases} \bar{2}, & \bar{x}=\bar{y}=\bar{1} \\ \bar{0}, & otherwise \end{cases}, \qquad \bar{x} \circ \bar{y} = \begin{cases} \bar{4}, & \bar{x}=\bar{y}=\bar{3} \\ \bar{0}, & otherwise. \end{cases}$$

$(\mathbb{Z}_5, *, \circ)$ is a dimonoid [5]. The semigroup $(\mathbb{Z}_2, \cdot)$ is a $\mathbb{Z}_5$-dimodule with the operation $\mathbb{Z}_5 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$, $(\bar{u}, \bar{x}) \longmapsto \bar{1}$ and the semigroup $(\mathbb{Z}_4, +)$ is a $\mathbb{Z}_5$-dimodule with the operation $\mathbb{Z}_5 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4$, $(\bar{u}, \bar{x}) \longmapsto \bar{0}$. Then a function $f : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$, $\bar{x} \longmapsto f(\bar{x}) = \bar{1}$ is a homomorphism of $\mathbb{Z}_5$-dimodules.

Example 3.20 *Let D be the dimonoid in Example 3.2 and S be the D-dimodule in the case (i). Then $\mathbb{N}$ is also a D-dimodule since D is an arbitrary dimonoid in Example 3.3. Consider $f : \mathbb{N} \to S$,*

$$f(n) = \begin{cases} b, & 2 \mid n \\ a, & 2 \nmid n. \end{cases}$$

Then f is a homomorphism of D-dimodules.

Theorem 3.21 *Let S and Y be D-dimodules, and $f : S \longrightarrow Y$ be a homomorphism of D-dimodules. If E is a subdimodule of S, then $f(E)$ is a subdimodule of Y.*

Proof $\varnothing \neq f(E) \subseteq Y$ since E is a subdimodule of S. Let $u \in D$ and $a, b \in f(E)$. There exist $x, y \in E$ such that $a = f(x), b = f(y)$ since $a, b \in f(E)$. $a \cdot b = f(x) \cdot f(y) = f(x \cdot y)$ and $ua = uf(x) = f(ux)$ since f is a homomorphism of D-dimodules. Hence $a \cdot b, ua \in f(E)$ since $x \cdot y, ux \in E$. Thus $f(E)$ is a subdimodule of Y. □

Theorem 3.22 *Let S and Y be D-dimodules, $f : S \longrightarrow Y$ be a homomorphism of D-dimodules and X be a subdimodule of Y. Then $f^{-1}(X)$ is a subdimodule of S if $f^{-1}(X) \neq \varnothing$.*

Proof $\varnothing \neq f^{-1}(X) \subseteq S$ since X is a subdimodule of Y. Let $u \in D$ and $x, y \in f^{-1}(X)$. Thus $f(x), f(y) \in X$. $f(x) \cdot f(y) = f(x \cdot y) \in X$ and $uf(x) = f(ux) \in X$ since f is a homomorphism of D-dimodule. Hence $x \cdot y, ux \in f^{-1}(X)$. Thus $f^{-1}(X)$ is a subdimodule of S. □

Corollary 3.23 *Let S and Y be D-dimodules, $f : S \longrightarrow Y$ be a surjective homomorphism of D-dimodule and X be a subdimodule of Y. Then $f^{-1}(X)$ is a subdimodule of S.*

Proof $X \neq \varnothing$ since X is a subdimodule of Y. Thus there exists $y \in X$. Hence there exists $x \in S$ such that $f(x) = y$ since f is a surjective function. Hence $f^{-1}(X) \neq \varnothing$. Thus $f^{-1}(X)$ is a subdimodule of S as per Theorem 3.22. □

Theorem 3.24 *Let S be a D-dimodule, $a \in S$ and $Da = \{da \mid d \in D\}$. Then Da is a subdimodule of S.*

Proof $\varnothing \neq Da \subseteq S$. Thus let $x, y \in Da$ and $u \in D$. Hence there exist $d_1, d_2 \in D$ such that $x = d_1 a, y = d_2 a$. $x \cdot y = (d_1 a) \cdot (d_2 a) = (d_1 * d_2)a \in Da$ since $d_1 * d_2 \in D$ and $ux = u(d_1 a) = (u \circ d_1)a \in Da$ since $u \circ d_1 \in D$. Therefore Da is a subdimodule of S. □

Theorem 3.25 *Let $(D, *, \circ)$ be a distributive dimonoid, S be a D-dimodule, $a \in S$ and $Da = \{da \mid d \in D\}$. Then the map $f : D \longrightarrow Da, f(d) = da$ is a surjective homomorphism of D-dimodule.*

Proof The surjective map f is a homomorphism of D-dimodule since $f(u) \cdot f(v) = (ua) \cdot (va) = (u * v)a = f(u * v)$ and $f(d \circ u) = (d \circ u)a = d(ua) = df(u)$ for all $u, v, d \in D$. □

Theorem 3.26 *Let D_1 and D_2 be two dimonoids and let $f : D_1 \to D_2$ be a dimonoid homomorphism. Then S is a D_1-dimodule if S is a D_2-dimodule.*

Proof Consider $D_1 \times S \to S$, $(u, x) \longmapsto f(u)x$. Let $u, v \in D_1$ and $x, y \in S$. Then

$$\begin{aligned} u(x \cdot y) &= f(u)(x \cdot y) = (f(u)x) \cdot (f(u)y) = (ux) \cdot (uy), \\ (u * v)x &= f(u * v)x = (f(u) * f(v))x = (f(u)x) \cdot (f(v)x) = (ux) \cdot (vx), \\ u(vx) &= u(f(v)x) = f(u)(f(v)x) = (f(u) \circ f(v))x = f(u \circ v) = (u \circ v)x \end{aligned}$$

since S is a D_2-dimodule and $f : D_1 \to D_2$ be a dimonoid homomorphism. □

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The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Ertuğrul Akçay]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%50).
Author [Canan Akın]: Thought and designed the research/problem, contributed to completing the research and solving the problem (%50).

Conflicts of Interest
The authors declare no conflict of interest.

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Research Article

A Robust Approach About Compact Operators on Some Generalized Fibonacci Difference Sequence Spaces

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Abstract: In this new study, which deals with the different properties of $\ell_p(\widehat{F}(r,s))$ $(1 \le p < \infty)$ and $\ell_\infty(\widehat{F}(r,s))$ spaces defined by Candan and Kara in 2015 by using Fibonacci numbers according to a certain rule, we have tried to review all the qualities and features that the authors of the previous editions have found most useful. This document provides everything needed to characterize the matrix class $(\ell_1, \ell_p(\widehat{F}(r,s)))$ $(1 \le p < \infty)$. Using the Hausdorff measure of non-compactness, we simultaneously provide estimates for the norms of the bounded linear operators L_A defined by these matrix transformations and identify requirements to derive the corresponding subclasses of compact matrix operators. The results of the current research can be regarded as to be more inclusive and broader when compared to the similar results available in the literature.

Keywords: Sequence spaces, Fibonacci numbers, compact operators, Hausdorff measure of noncompactness.

1. Elementary Classical Concepts

As always, our aim is to use the matrix domain and to remind readers about the information they will need to use calculus effectively in their work in later sections. To achieve this, we retained the paper's mathematical level, the orientation of the new sequence space to the Hausdorff measure, its concentration on previous works, and variety of the theorems, and continued to adapt some of the methods used in measurement theory. Although many of the presentations in this new paper are noticeably more general than those in earlier articles, the level of rigor is about the same. As part of the overall review plan, it is going to be helpful for beginners to review the five notable books given in [1–5] with accessible material, without sacrificing the standards or scope their users want to see. Let us start by trying to explain some of the essentials without exaggerating the obvious. The history of numbers is almost as old as the existence of humanity and was created to

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meet the mathematical needs of all people and scientists. This was true in the beginnings of the subject, and it is true today. In much of our work, the sequences will have domains and ranges that are sets of naturel numbers $\mathbb{N} = \{0, 1, 2, ...\}$ and real numbers $\mathbb{R}$, respectively. We will write $\lim_k$, $\sup_k$, $\inf_k$ and $\sum_k$ instead of $\lim_{k\to\infty}$, $\sup_{k\in\mathbb{N}}$, $\inf_{k\in\mathbb{N}}$ and $\sum_{k=0}^{\infty}$, respectively.

We will now consider two related topics that will be used in the next sections: infinite sequences and infinite series. An infinite sequence of numbers is a function whose domain is the set of natural numbers. The word series always implies an infinite number of term to be combined by adding in a definite order. The vector space of all real sequences is expressed by ω. We are quite familiar with that each subspace of ω is said a sequence space. In order to use in this work, a few additional notations concerning sequences are needed. The sets of all finite sequences, bounded sequences, convergent sequences, and null sequences, respectively, should be denoted by, φ, ℓ_∞, c and c_0. For any real number p with $1 \le p < \infty$, the sequence space $\left\{x \in \omega : \sum_k |x_k|^p < \infty\right\}$ is denoted by the notation ℓ_p. In addition to these, the sequence $(1, 1, ...)$ and for each natural number n, the sequence with 1 only in the n^{th} term and 0 in all other terms is denoted by the notations e and $e^{(n)}$, respectively. The sum $\sum_{k=0}^{n} x_k e^{(k)}$ is indicated by $x^{[n]}$ and is referred to as the n-section of any sequence x. Series whose partial sums sequence are convergent and bounded are also shown with cs and bs notations, respectively.

A complete normed space is referred to as a $B-space$. A $K-space$, on the other hand, is a topological sequence space in which all coordinate functionals π_k, given by $\pi_k(x) = x_k$, are continuous. A $BK-space$ is essentially a Banach space with continuous coordinates, meeting the requirements of both a $K-space$ and a $B-space$. If all sequences $x = (x_k) \in X$ share the same representation, then a $BK-space$ denoted as $X \supset \varphi$ is said to possess AK, where $x = \sum_k x_k e^{(k)}$. To provide an example, the sequence space ℓ_p $(1 \le p < \infty)$ can be regarded as a $BK-space$ with the norm $\|x\|_p = \left(\sum_k |x_k|^p\right)^{1/p}$. Furthermore, c_0, c, and ℓ_∞ also qualify as $BK-spaces$, possessing the norm $\|x\|_\infty = \sup_k |x_k|$. Additionally, the $BK-spaces$ c_0 and ℓ_p exhibit AK, where $1 \le p < \infty$.

If there exists a singular sequence (α_n) consisting of scalars such that $x = \sum_n \alpha_n b_n$, meaning that $\lim_m \|x - \sum_{n=0}^{m} \alpha_n b_n\| = 0$, then the sequence (b_n) in a normed space X is referred to as a Schauder basis for all $x \in X$.

The β-dual of a sequence space X is defined as follows:

$$X^\beta = \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X\}.$$

An infinite matrix of real numbers, denoted by $A = (a_{nk})_{n,k=0}^{\infty}$, where $n, k \in \mathbb{N}$, can be represented as A_n, which denotes the sequence in the n^{th} row of A. Furthermore, if $x = (x_k)_{k=0}^{\infty} \in$

ω, the A-transform of x is defined as the sequence $Ax = \{A_n(x)\}_{n=0}^{\infty}$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n \in \mathbb{N}), \tag{1}$$

provided that the series on the right-hand side converges for each $n \in \mathbb{N}$.

We denote (X, Y) as the class of all infinite matrices that map from X to Y, where X and Y are subsets of ω. In other words, $A \in (X, Y)$ if and only if $A_n \in X^{\beta}$ for every $n \in N$ and $Ax \in Y$ for every $x \in X$.

One way to create a new sequence space is by utilizing the matrix domain, and a thorough comprehension of it requires substantial expertise. Let X be any sequence space. Then the domain X_A of an infinite matrix A in X is defined by

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}. \tag{2}$$

Let us also mention here that X_A is also a sequence space. The reader can refer to the recent papers [6–10] on the domains of certain triangles in the classical sequence spaces and related topics.

The following results are fundamental and often used [11, 12].

Lemma 1.1 *Let $X \supset \phi$ and Y be a BK–space.*
(a) Therefore, for any matrix $A \in (X, Y)$, we get $(X, Y) \subset B(X, Y)$, so indicating that for any $x \in X$, $L_A(x) = Ax$ describes an operator $L_A \in B(X, Y)$.
(b) If X has AK, and after that $B(X, Y) \subset (X, Y)$, meaning that there is a $A \in (X, Y)$ with any operator having $L \in B(X, Y)$ and $L(x) = Ax$ for every $x \in X$.

2. The Hausdorff Measure of Non-Compactness

In this part, our aim is to describe the Hausdorff measure used in theory and practice that characterizes compact operators between Banach spaces. For this purpose, this section stars with clear expressions of relative definitions, guidelines and theorems together with explanatory and other demonstrative subject. It follows proven and supplementary theorems. The proven theorems give to demonstrate and magnify the theory, and to reiterate the fundamental principles that are crucial for effective learning. The concept of Hausdorff measure of non-compactness appears in some branches of mathematics. Recently, this concept has been used to characterize compact matrix operators between BK–spaces under certain conditions.

The Hausdorff measure of non-compactness χ concept stems largely from the investigations of Goldenštein, Gohberg and Markus [13] and in the following years this concept was taken up and studied by Goldenštein and Markus [14]. Yet some of its ideas date back to the time of Kuratowski

[15]. Later, Darbo [16] took this measure and generalized another concept besides the classical Schauder fixed point principle.

In the context of infinite-dimensional Banach spaces X and Y, it is important to restate the definition of a compact operator. A linear operator L that maps from X to Y is considered compact if it encompasses the entire domain of X and, in addition, if the sequence $(L(x_n))$ representing the images of all bounded sequences (x_n) in X under L has a convergent subsequence. In the field of functional analysis, the collection of all compact operators in $B(X,Y)$ is denoted by $C(X,Y)$.

Let (X,d) be a metric space. We define the open ball $B(x,r)$ as the set $\{x \in X : d(x,x_0) < r\}$, where r represents the radius and x_0 denotes the center. Furthermore, let $M(X)$ denote the collection of all bounded subsets of X. If $Q \in M(X)$, then the Hausdorff measure of non-compactness of the set Q, denoted by $\chi(Q)$, is defined as follows:

$$\chi(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{k=1}^{n} B(x_k,r_k), x_k \in X,\ r_k < \epsilon\ (k = 1,2,...),\ n \in \mathbb{N}\right\}.$$

The Hausdorff measure of non-compactness is defined as the function $\chi : M_X \to [0,\infty)$.

In previous works such as [11, 17–20], the applications of the Hausdorff measure theorems to condensing operators, compact matrix operators on some BK-spaces, and measures of non-compactness in Banach spaces are further explored.

The objective of this paragraph is to provide a concise description of the Hausdorff measure of non-compactness operators between Banach spaces. Let X and Y be Banach spaces, and let χ_1 and χ_2 be the Hausdorff measures of non-compactness on X and Y, respectively. If $L(Q) \in M(Y)$ for all $Q \in M(X)$, and if there exists $C \geq 0$ such that $\chi_2(L(Q)) \leq C\chi_1(Q)$ for all $Q \in M(X)$, then the operator $L : X \to Y$ is referred to as (χ_1,χ_2)-bounded. The quantity

$$|L|(\chi_1,\chi_2) = \inf\{C \geq 0 : \chi_2(L(Q)) \leq C\chi_1(Q) \text{ for all } Q \in \mathcal{M}X\}$$

is defined as the (χ_1,χ_2)–measure of non-compactness of L if the operator L is $(\chi_1,\ \chi_2)$-bounded.

It is important to note that if both χ_1 and χ_2 are denoted as χ, then $|L|(\chi_1,\chi_2) = |L|_\chi$.

Our primary objective in this context is to provide a comprehensive explanation of the applications of the Hausdorff measure of non-compactness in characterizing compact operators between Banach spaces. Let X and Y be Banach spaces, and let L be an element of $B(X,Y)$, indicating that L is a bounded linear operator from X to Y. If L is non-compact, the Hausdorff measure of non-compactness of L, denoted as $\|L\|_\chi$, is defined as follows ([20, Theorem 2.25]):

$$\|L\|_\chi = \chi(L(SX)). \tag{3}$$

Furthermore, L is characterized as a compact operator if and only if the Hausdorff measure of non-compactness $\|L\|_\chi$ is equal to zero, as expressed in ([20, Corollary 2.26]):

$$\|L\|_\chi = 0. \tag{4}$$

The determination of the Hausdorff measure of non-compactness, denoted as $\chi(Q)$, for bounded sets Q in a Banach space X is based on the identities presented in (3) and (4). These identities simplify the characterization of compact operators $L \in B(X,Y)$. Estimates, or even identities, for $\chi(Q)$ can be obtained when X possesses a Schauder basis.

Theorem 2.1 *([13] or [20, Theorem 2.23]) Let X be a Banach space with a Schauder basis $(b_k)_{k=0}^{\infty}$, $Q \in M_X$, $P_n : X \to X$ will be the projectors onto the linear span of $\{b_0, b_1, \ldots, b_n\}$ and $R_n = I - P_n$ for $n = 0, 1, \ldots$, in which I indicates the identity map on X. Under these conditions, the following inequality is satisfied*

$$\frac{1}{a} \cdot \limsup_{n\to\infty}\left(\sup_{x\in Q}\|\mathcal{R}_n(x)\|\right) \le \chi(Q) \le \limsup_{n\to\infty}\left(\sup_{x\in Q}\|\mathcal{R}_n(x)\|\right),$$

in which $a = \limsup_{n\to\infty}\|R_n\|$.

The following result, in especially, demonstrates how to calculate the Hausdorff measure of non-compactness in the BK-spaces with AK, c_0 and ℓ_p $(1 \le p < \infty)$.

Theorem 2.2 *([20, Theorem 2.15]) A bounded subset of the normed space X, in which X is ℓ_p for $1 \le p < \infty$ or c_0, is defined as Q. We can have*

$$\chi(Q) = \lim_{n\to\infty}\left(\sup_{x\in Q}\|\mathcal{R}_n(x)\|\right) \tag{5}$$

if $P_n : X \to X$ is the operator described by $P_n(x) = x^{[n]}$ for every $x = (x_k)_{k=0}^{\infty} \in X$ and $R_n = I - P_n$ for $n = 0, 1, \ldots$.

It is highly reasonable to deduce both necessary and sufficient criteria for matrix operators between a Schauder basis and a BK-space by employing the aforementioned discoveries, as well as the Hausdorff measure of non-compactness. Matrix mappings across BK-spaces give rise to bounded linear operators between these Banach spaces, rendering AK as compact operators. Presently, numerous researchers have embraced this approach in multiple research publications (see, for instance, [21–31]. The significance of these concepts will become evident in subsequent discussions. In this work, we provide a description of the matrix classes $(\ell_1, \ell_p(\widehat{F}(r,s)))$ $(1 \le p < \infty)$. Moreover, we establish conditions for deriving the relevant subclasses of compact matrix operators through the utilization of the Hausdorff measure of non-compactness. Additionally, we derive an identity for the norms of the bounded linear operators L_A that are determined by these matrix transformations.

3. The Fibonacci Difference Sequence Spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$

Although infinite sequences were used extensively in the early history of the calculus, especially, they have appeared in the history of mathematics since antiquity. In the middle ages the mathematician Fibonacci, in his work Liber Abaci (1202) used sequences of numbers $1,1,2,3,5,\ldots$. You may already be familiar with Fibonacci sequences, but if not, you will understand the following formula easy follow. For convenience, the steps in the sequence are usually labeled $1,1,2,3,5,\ldots$ and so on. In a much clearer way, the *Fibonacci sequences* $f=(f_n)$ starts with $f_0=f_1=1$ and uses the recursion formula

$$f_n = f_{n-1} + f_{n-2}; \quad n \geq 2.$$

The use of Fibonacci sequences is widely available and give opportunity for hands-on experience. When the most striking differences in art and architecture, plants and some living things in nature were carefully examined, it was seen that they were related to the Fibonacci numbers. Let me also point out here that, many applications of Fibonacci sequences are beyond the scope of this work, but the material in this section can prepare you for later study as well as provide knowledge that you can use as needed. Reference number [32] can be examined for a lot of information about Fibonacci numbers, including the Golden ratio.

Let $1 \leq p \leq \infty$ and q represent the conjugate of p throughout, that is, $q = p/(p-1)$ for $1<p<\infty$, that is, $q=p/(p-1)$ for $1<p<\infty$, $q=\infty$ for $p=1$ or $q=1$ for $p=\infty$.

In 2015, right after Kara [33], Candan and Kara [34] introduced the generalized Fibonacci difference sequence spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$, as follows;

$$\ell_p(\widehat{F}(r,s)) = \left\{ x=(x_n)\in\omega : \sum_n \left| r\frac{f_n}{f_{n+1}}x_n + s\frac{f_{n+1}}{f_n}x_{n-1}\right|^p < \infty \right\}; \ 1 \leq p < \infty$$

and

$$\ell_\infty(\widehat{F}(r,s)) = \left\{ x=(x_n)\in\omega : \sup_{n\in\mathbb{N}} \left| r\frac{f_n}{f_{n+1}}x_n + s\frac{f_{n+1}}{f_n}x_{n-1}\right| < \infty \right\}.$$

When we use the equivalent notation of (2) for the sequence spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$, related sequence spaces becomes

$$\ell_p(\widehat{F}(r,s)) = (\ell_p)_{\widehat{F}(r,s)} \ (1 \leq p < \infty) \ \text{ and also } \ \ell_\infty(\widehat{F}(r,s)) = (\ell_\infty)_{\widehat{F}(r,s)}, \tag{6}$$

in which the matrix $\widehat{F}(r,s) = (\widehat{f}_{nk}(r,s))$ is described by

$$\widehat{f}_{nk}(r,s) = \begin{cases} s\frac{f_{n+1}}{f_n} & (k = n-1) \\ r\frac{f_n}{f_{n+1}} & (k = n) \\ 0 & (0 \le k < n-1) \text{ or } (k > n) \end{cases} \quad (n,k \in \mathbb{N}). \tag{7}$$

To signal the fact that the sequence spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$ are BK-spaces according to the

$$\|x\|_{\ell_p(\widehat{F}(r,s))} = \left(\sum_n |y_n(x)|^p\right)^{1/p} \quad (1 \le p < \infty) \text{ and } \|x\|_{\ell_\infty(\widehat{F}(r,s))} = \sup_{n\in\mathbb{N}} |y_n(x)|, \tag{8}$$

norms defined on them, respectively, in which the sequence $y = (y_n) = (\widehat{F}(r,s)_n(x))$ which is the $\widehat{F}(r,s)$-transform of any sequence $x = (x_n)$, is used. That is

$$y_n = \widehat{F}(r,s)_n(x) = \begin{cases} r\frac{f_0}{f_1}x_0 = rx_0 & (n = 0) \\ r\frac{f_n}{f_{n+1}}x_n + s\frac{f_{n+1}}{f_n}x_{n-1} & (n \ge 1) \end{cases} \quad (n \in \mathbb{N}). \tag{9}$$

It should be emphasized that the findings of this study are more comprehensive than those of Alotaibi et al. [35] in 2015.

4. Main Results

Many applications of compact operators are beyond the scope of this paper, but the material in this section can prepare you to understand the subject and help you remember information you can use when needed. From a historical perspective, the current concept of the Hausdorff measure represents a culmination of the collective efforts of numerous individuals. However, the notion of non-compactness' Hausdorff measure was originally introduced in 1957 by Goldenštein, Gohberg, and Markus, and was subsequently further explored by Goldenštein and Markus. In the study [36], the sequence spaces Y, ℓ_∞, c_0 and c were considered, enabling the characterization of the classes $(\ell_p(\widehat{F}),Y)$, $(\ell_\infty(\widehat{F}),Y)$, $(\ell_1(\widehat{F}),Y)$, as well as the compact operators $(\ell_p(\widehat{F}),\ell_1)$ and $(\ell_1(\widehat{F}),\ell_p)$. In this study, we introduce the classes $B(\ell_1,\ell_p^\lambda)$ for $1 \le p < \infty$ and compute the operator norms in $B(\ell_1,\ell_p^\lambda)$. Furthermore, leveraging the findings from the previous section, we describe the classes $C(\ell_1,\ell_p)$ for $1 \le p < \infty$ and determine the Hausdorff measure of non-compactness for operators in $B(\ell_1,\ell_p^\lambda)$.

Let $1 \le p < \infty$. We now provide a characterization of $B(\ell_1,\ell_p(\widehat{F}(r,s)))$, along with the computation of the operator norms in $B(\ell_1,\ell_p(\widehat{F}(r,s)))$. Additionally, we can utilize the results presented in the previous section to both specify the Hausdorff measure of non-compactness for operators in $B(\ell_1,\ell_p(\widehat{F}(r,s)))$ and characterize the classes $C(\ell_1,\ell_p(\widehat{F}(r,s)))$ for $1 \le p < \infty$.

The following result is particularly advantageous in certain proofs.

Lemma 4.1 *([20, Theorem 3.8]) T is a triangular matrix and with it X and Y being any two sequence spaces; for the matrix A to be an element of the (X, Y_T) class, the necessary and sufficient condition is that $C = T \cdot A$ and the matrix C belongs to the class (X, Y). In addition, if the X and Y are BK-spaces, and also if the matrix A is an element of the class (X, Y_T), then*

$$\|L_A\| = \|L_C\|. \tag{10}$$

We then define the identities for the operator norm and the characterizations of the classes $B(\ell_1, \ell_p(\widehat{F}(r,s)))$ for $1 \le p < \infty$.

Theorem 4.2 *Let $1 \le p < \infty$.*

(a) We have $L \in B(\ell_1, \ell_p(\widehat{F}(r,s)))$ if and only if there exists an infinite matrix $A \in (\ell_1, \ell_p(\widehat{F}(r,s)))$ such that

$$\|A\| = \sup_k \left(\sum_n \left| r\frac{f_n}{f_{n+1}} a_{nk} + s\frac{f_{n+1}}{f_n} a_{n-1,k} \right|^p \right)^{1/p} < \infty \tag{11}$$

and

$$L(x) = Ax \text{ for all } x \in \ell_1. \tag{12}$$

(b) If $L \in B(\ell_1, \ell_p(\widehat{F}(r,s)))$, then

$$\|L\| = \|A\|. \tag{13}$$

Proof For (a), when we keep in mind that ℓ_1 is a BK-space with AK, for $L \in B(\ell_1, \ell_p(\widehat{F}(r,s)))$ from Lemma 1.1 under the condition $1 \le p < \infty$ hypothesis condition; the necessary and sufficient condition is that there is an infinite matrix A such that $A \in (\ell_1, \ell_p(\widehat{F}(r,s)))$ provided that the condition (12) is met. If we denote the product of the matrices $\widehat{F}(r,s) = (\widehat{f}_{nk}(r,s))$ and $A = (a_{nk})$ by $C = (c_{nk})$, that is, we can express it clearly as follows

$$c_{nk} = r\frac{f_n}{f_{n+1}} a_{nk} + s\frac{f_{n+1}}{f_n} a_{n-1,k}.$$

Now it is quiet easy to say that from Lemma 4.1 (a) that the necessary and sufficient condition $A \in (\ell_1, \ell_p(\widehat{F}(r,s)))$ is $C \in (\ell_1, \ell_p)$. If the Example 8.4.1D in the reference [12] is used at this stage of the proof, it is seen that the necessary and sufficient condition for $C \in (\ell_1, \ell_p)$ is

$$\|C\| = \sup_k \left(\sum_{n=0}^{\infty} |c_{nk}|^p \right)^{1/p} < \infty,$$

which proves the claim. □

(b) First, we show that $\|L\| \le \|A\|$. Let $L \in B(\ell_1, \ell_p^\lambda)$. It is seen from (10) that $\|L\| = \|L_C\|$ for $L_C \in B(\ell_1, \ell_p)$ is presented by the equation $L_C(x) = Cx$ for every $x \in \ell_1$. Now, we can write by the Minkowsky's inequality that, we can write the following expressions

$$\begin{aligned}\|L_C(x)\|_p &= \left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty} c_{nk}x_k\right|^p\right)^{1/p}\\ &\le \sum_{k=0}^{\infty}|x_k|\left(\sum_{n=0}^{\infty}|c_{nk}|^p\right)^{1/p}\\ &\le \|C\|\cdot\|x\|\\ &= \|A\|\cdot\|x\|\end{aligned}$$

and from here we can write the following inequality

$$\|L\| \le \|A\| \tag{14}$$

for the norms of L and A. Now, let us prove the other side of the inequality. For this, when $e^{(k)} \in S_{\ell_1}$ $(k \in N)$ is taken, it is seen that

$$\|L\| \ge \|A\| \tag{15}$$

from the equation below

$$\left\|L_C(e^{(k)})\right\| = \left(\sum_{n=0}^{\infty}|c_{nk}|^p\right)^p.$$

When (14) and (15) are considered together, it is proved that (13).

The Hausdorff measure of the non-compactness of operators in $B(\ell_1, \ell_p(\widehat{F}(r,s)))$ will be established in the expression below. Another closely related result to be used in the first come proof is given below.

Lemma 4.3 *([37, Theorem 4.2]) Let T be a triangle and χ and χ_T be the Hausdorff measures of non-compactness on M_X and M_{X_T}, respectively. Assume that X is a linear metric space with a translation invariant metric. If $Q \in M_{X_T}$, then $\chi_T(Q) = \chi(TQ)$.*

Theorem 4.4 *Let $L \in B(\ell_1, \ell_p(\widehat{F}(r,s)))$ with $1 \le p < \infty$ and A demonstrate the matrix which stands for L. In that case we get*

$$\|L\|_{\chi_{\ell_p(\widehat{F}(r,s))}} = \lim_{m\to\infty}\left(\sup_k \sum_{n=m}^{\infty}\left|r\frac{f_n}{f_{n+1}}a_{jk} + s\frac{f_{n+1}}{f_n}a_{j-1,k}\right|^p\right)^{1/p}.$$

Proof First of all, we briefly write $S = S_{\ell_1}$, also $C^{[m]}$ $(m \in \mathbb{N})$ for the matrix with the rows $C_n^{[m]} = 0$ for $0 \le n \le m$ and $C_n^{[m]} = C_n$ for $n \ge m+1$. In this case, if we use both Lemma 4.3 and together with (3), (5), (11) and (13) the following equations can easily be calculated

$$\begin{aligned}
\|L\|_{\chi_{\ell_p(\widehat{F}(r,s))}} &= \chi_{\ell_p(\widehat{F}(r,s))}(L(S)) \\
&= \chi_{\ell_p}(L_C(S)) \\
&= \lim_{m\to\infty}\left(\sup_{x\in S}\|\mathcal{R}_m(Cx)\|_p\right) \\
&= \lim_{m\to\infty}\left(\sup_{x\in S}\|C^{[m]}x\|_p\right) \\
&= \lim_{m\to\infty}\|C^{[m]}\| \\
&= \lim_{m\to\infty}\left(\sup_k \sum_{n=m}^{\infty}\left|r\frac{f_n}{f_{n+1}}a_{jk} + s\frac{f_{n+1}}{f_n}a_{j-1,k}\right|^p\right)^{1/p}.
\end{aligned}$$

□

This is the desired result.

We are now ready to give the following theorem, which obtains the characterization of $C(\ell_1, \ell_p(\widehat{F}(r,s)))$ by coordinating the condition given in (4) and Theorem 4.4.

Theorem 4.5 *If $L \in B(\ell_1, \ell_p(\widehat{F},\ 1 \le p < \infty(r,s)))$ and at the same time the matrix A is the matrix representing L, a necessary and sufficient condition for L to be compact is that the following limit is equal to zero, that is*

$$\lim_{m\to\infty}\left(\sup_k \sum_{n=m}^{\infty}\left|r\tfrac{f_n}{f_{n+1}}a_{jk} + s\tfrac{f_{n+1}}{f_n}a_{j-1,k}\right|^p\right) = 0.$$

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Declaration of Ethical Standards

The author declares that the materials and methods used in his study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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