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We thank all our colleagues for their contributions.

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We wish you a scientific life full of success..

Kind regards!

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A NOTE OF THE COMBINATORIAL INTERPRETATION OF THE PERRIN AND TETRARRIN SEQUENCE

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Abstract. The present study carries out an investigation around the Perrin and Tetrarrin numbers, allowing a combinatorial interpretation for these sequences. Furthermore, it is possible to establish a study around the respective polynomial numbers of Perrin and Tetrarrin, using the bracelet method. With this, we have the definition of combinatorial models of these numbers, contributing to the evolution of these sequences with their respective combinatorial approaches. As a conclusion, there is a discussion of theorems referring to the combinatorial models of these sequences, allowing the study of the mathematical advancement of these numbers.

1. Introduction

The present word aims to introduce new interpretations for the Perrin sequence, its extension and polynomial forms. In fact, works in the literature containing the existence of recent works are identified, involving new combinatorial approaches of recurrent numerical sequences [1, 2, 3, 4, 8]. With this, it is possible to observe forms of visualization of the terms of these respective studied sequences.

Based on this, a combinatorial interpretation is performed for the sequence of Perrin, Tetrarrin and polynomial forms, based on the works of Tedford (2019) [5] and Vieira (2020) [7].

The Perrin sequence is closely related to the Padovan sequence. In a similar way as with the Fibonacci and Lucas sequence. With this, it is worth highlighting the work of Benjamin and Quinn (2003) [1] in which they carried out a study around the combinatorial model of Fibonacci and Lucas, investigating Lucas bracelets. So, the n-bracelet is defined as being a cover of a circular n-board. Lucas sequence has its combinatorial interpretation by means of bracelets, as l_n being the number of ways to tile a circular board composed of n cells marked with squares and 1 x 2

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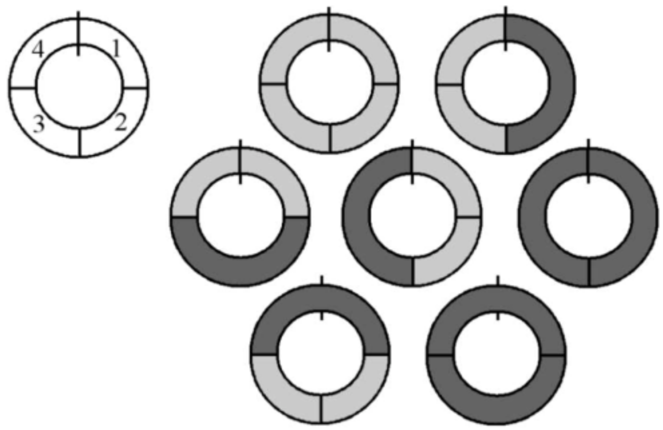
Figure 1. Size 4 Lucas bracelets. Source: Benjamin and Quinn[1]



dominoes. figure:lucas the number of tiles on Lucas bracelet of size 4, that is, l_4, obtaining a total of 7 ways to tile the bracelet.

Based on this, the work of Tedford (2019) [5] is presented, in which Padovan's combinatorial model is defined, based on a construction rule with the pieces: blue dominoes of size 1 x 2, gray triminoes of size 1 x 3 and green tetramino of size 1 x 4, all with weight 1. The particular rules mentioned are defined for the theorem concerning Padovan tiling [5].

In Figure 2, on the left side, some examples are provided in order to fill in the n-board corresponding to the Padovan sequence. On the right side are the terms corresponding to the Padovan numbers. With this, it is possible to perceive the term p_n as being the amount of tile shapes on the n-board, following the aforementioned rules, determines the relationship: $p_n = P_n, n \geqslant 0$.

In view of this, the bracelets of Perrin, Tetrarrin and their polynomial forms will be defined in a primordial way in this research, introducing the combinatorial model of Perrin, Tetrarrin, polynomial of Perrin and polynomial of Tetrarrin.

2. The Perrin sequence and its polynomial form

The Perrin sequence is a third-order, numerically recurring linear recurrent sequence given by the recurrence: $R_n = R_{n-2} + R_{n-3}, R_0 = 3, R_1 = 0, R_2 = 2, n \geq 3$ [9]. These numbers have a close relationship with the Padovan sequence, $\{P_n\}$, differing in their initial values, which are given by: $P_0 = P_1 = P_2 = 1$ [7]. The Tridovan sequence ($\{T_n\}$), for its part, was defined by Vieira (2020) [7], as being a fourth order sequence, derived from the Padovan sequence with recurrence $T_n = T_{n-2} + T_{n-3} + T_{n-4}$ and initial values given by $T_0 = 1, T_1 = 0, T_2 = T_3 = 1$. Thus, in this work an extension of the Perrin numbers is carried out, naming the Tetrarrin sequence, Te_n (fourth order) and its polynomial form, which will be discussed later.

Figure 2. Padovan tiling. Source: Adapted from [5].

Table 1. First terms of the Perrin sequence. Source: Prepared by the authors.

R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}
3	0	2	3	2	5	5	7	10	12	17

The Perrin polynomial numbers represent the polynomial form of the Perrin sequence.

Definition 2.1. For $x \in \mathbb{Z}$, the Perrin polynomial is defined, with $n \geqslant 3, n \in \mathbb{N}$, by recurrence:

$$R_n(x) = xR_{n-2}(x) + R_{n-3}(x),$$

with $R_0(x) = 3, R_1(x) = 0, R_2(x) = 2x$.

Thus, the first terms presented in Table 2.

Table 2. Perrin's first ten polynomial terms. Source: Prepared by the authors.

n	$R_n(x)$
0	3
1	0
2	$2x$
3	3
4	$2x^2$
5	$5x$
6	$2x^3+3$
7	$7x^2$
8	$2x^4+8x$
9	$9x^3+3x$

Yilmaz and Taskara (2013) [12] enabled an arithmetic relationship between the Padovan and Perrin sequences, through the equation: $R_n = 3P_{n-5} + 2P_{n-4}$.

3. The Tetrarrin sequence and its polynomial form

Based on the study by Vieira (2020) [7], who carried out an extension of the Padovan sequence, expanding the order of this sequence and defining new sequences arising from the Padovan numbers, we have the study for the Perrin numbers. With this, an extension of the Perrin sequence is performed, defining the Tetrarrin sequence.

The Tetrarrin sequence is a linear and recurrent sequence of the fourth order, primarily studied in this research.

Definition 3.1. The Tetrarrin sequence, represented by $Te_{(n)}$ with $n \geqslant 0$ and $n \in \mathbb{N}$, has the following recurrence formula:

$$Te_{(n)} = Te_{(n-2)} + Te_{(n-3)} + Te_{(n-4)},$$

with the following initial values: $Te_{(0)} = 3$, $Te_{(1)} = 0$, $Te_{(2)} = 2$ and $Te_{(3)} = 3$.

Thus, we have the first terms of this sequence as being in Table 3.

Table 3. First terms of the Tetrarrin sequence. Source: Prepared by the authors.

Te_0	Te_1	Te_2	Te_3	Te_4	Te_5	Te_6	Te_7	Te_8	Te_9	Te_{10}
3	0	2	3	5	5	10	13	20	28	43

Based on Yilmaz and Taskara (2013) [12], in which they presented a relationship between the Padovan sequence and Perrin, we then sought to obtain a linear combination of the terms of the Tetrarrin sequence ($Te_{(n)}$) and Tridovan ($T_{(n)}$). Taking as a premise that this linear combination is possible, the following system of equations was modeled: $Ax = y$, presenting the following definitions:

$A = \begin{bmatrix} T_{(0)} & T_{(1)} & T_{(2)} & T_{(3)} \\ T_{(1)} & T_{(2)} & T_{(3)} & T_{(4)} \\ T_{(2)} & T_{(3)} & T_{(4)} & T_{(5)} \end{bmatrix}$, $y = \begin{bmatrix} Te_{(4)} \\ Te_{(5)} \\ Te_{(6)} \\ Te_{(7)} \end{bmatrix}$ and x is a vector of coefficients satisfying the system. Thus, it was possible to obtain the relation:

$$Te_{(n)} = 3T_{(n-4)} + 3T_{(n-3)} + 4T_{(n-2)}. \tag{3.1}$$

Since other identities can be obtained, arising from arithmetic operations on the mathematical relation presented in Equation 3.1, we have: $Te_{(n)} = 2T_{(n-2)} + 3T_{(n-3)} + 3T_{(n-4)}$. From there, Tetrarrin's bracelet, $te_{(n)}$, will be defined in the next section.

So, based on the extension of the polynomial Padovan sequence, which is called the polynomial Tridovan, based on the definitions established by [10, 6, 11], thus defining the polynomial sequence of Tetrarrin.

Definition 3.2. The Tetrarrin polynomial sequence, $Te_{(n)}(x)$, satisfies the following recurrence formula, for $n \in \mathbb{N}$ and $n \geqslant 4$.

$$Te_{(n)}(x) = x^2 Te_{(n-2)}(x) + x Te_{(n-3)}(x) + Te_{(n-4)}(x),$$

with the initial terms: $Te_{(0)}(x) = 3, Te_{(1)}(x) = 0, Te_{(2)}(x) = 2x^2, Te_{(3)}(x) = 3x$.

Thus, we have the Table 4 with the first terms of the Tetrarrin polynomial sequence.

TABLE 4. First ten polynomial terms of Tetrarrin. Source: Prepared by the authors.

n	$Te_{(n)}(x)$
0	3
1	0
2	$2x^2$
3	$3x$
4	$2x^4 + 3$
5	$5x^3$
6	$2x^5 + 8x^2$
7	$7x^5 + 6x$
8	$2x^7 + 15x^4 + 3$
9	$7x^7 + 2x^6 + 13x^3 + 6x^2$

With this, the relationship between the polynomial sequences of Tridovan and Tetrarrin is investigated, through the resolution of linear systems, obtaining:

$$Te_{(n+2)}(x) = 2T_{(n-2)}(x) + 3T_{(n-3)}(x) + 3T_{(n-4)}(x) \tag{3.2}$$

Given this, the study of Tetrarrin's polynomial combinatorial model can be established.

4. The combinatorial approach

In view of the definitions and discussions of the sequences, a study of their combinatorial interpretations is carried out.

Let r_n be the amount of coverage of a circular board with n positions labeled clockwise, using blue curved dominoes and gray curved triminoes. It is called a n-bracelet, a covering of a circular n-board. It should be noted that a bracelet is said to be *out of phase* if there is a domino in position $(n, 1)$. Otherwise it is said to be *in phase*. Note that this definition can be extended in the case of a trimino in positions $(n-1, n, 1)$ or $(n, 1, 2)$.

Theorem 4.1. *For $n \geqslant 2$, the possible Perrin bracelets of size 1 x n with blue curved dominoes and gray curved triminoes, all weighing* 1 *is given by: $r_n = R_n$, where r_n is the number of $n-$Perrin bracelets and R_n is the nth term of the Perrin sequence.*

Proof. A simple count shows that for $n = 5$ we have exactly 5 3-bracelets (*in phase* and *out of phase*). The last piece of a $n-$ bracelet is defined by the one that occupies, even partially, the n position. Note that this tile ends with a domino in position $(n-3, n-2)$ or position $(n-2, 1)$, or it could be a trimino in position $(1, 2, 3)$, or position $(n-2, 1, 2)$ or at position $(n-3, n-2, n)$.

In the first case, there are $n-4$ positions left that must be covered in r_{n-2} ways. In case the last piece is a trimino, there are $n-5$ positions left that must be covered in r_{n-3} ways. So $r_n = r_{n-2} + r_{n-3}$. Or equivalently, $R_n = R_{n-2} + R_{n-3}$,

Combinatorial Identity Proof: $R_n = 3P_{n-5} + 2P_{n-4}$. Consider $n-$bracelets. These can be of two types, *in phase* and *out of phase*. A bracelet *in phase* can be stretched into a n cover, so there are $P_{n-2} = P_{n-4} + P_{n-5}$ n bracelets *in phase*. For the case of a bracelet out of phase, it is known that either there is a blue domino in position $(n-2, 1)$ or a gray trimino among positions $(n-3, n-2, 1)$ $(n-2, 1, 2)$. In the first case, the n bracelet can be stretched into a $(n-4)$ cover. In the second case, the n bracelet can be stretched into a $(n-5)-$ cover. Therefore, the number of bracelets out of phase is equal to $P_{n-4} + 2P_{n-5}$. The result follows, since every bracelet is either *in phase* or *out of phase*, soon: $R_n = P_{n-4} + P_{n-5} + P_{n-4} + 2P_{n-5} = 3P_{n-5} + 2P_{n-4}$. □

To exemplify the Theorem 4.1, we have Figure 3 with cases from r_2 to r_5.

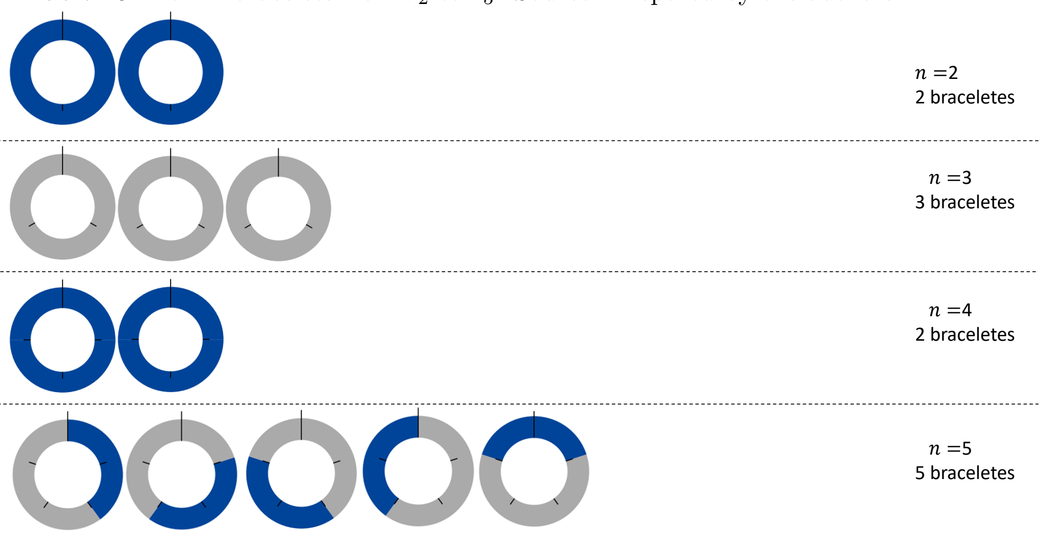
Note that for the initial case r_2, we have 2 bracelets, representing the R_2 term of the Perrin sequence. For r_3, there are 3 bracelets rotated, representing the R_3 of the sequence. For r_4, we have the amount of 2 bracelets, representing the R_4 of the sequence. For r_5, we have the amount of 5 bracelets, representing the R_5 of the sequence. The other cases can be made for the reader to follow the line of reasoning of the research.

In general, a combinatorial approach determines the size of a collection of objects in two different ways. This is done during the discussion of the theorem studied, referring to Perrin's combinatorial interpretation, conditioning the last element of each side by side, with the bias of demonstrating the recursive relationship

Next, there is Perrin's polynomial combinatorial interpretation.

Define $r_n(x)$ the covering amount of a circular polynomial Perrin board with n clockwise labeled positions, using blue curved dominoes and gray curved triminoes. It is called a n-bracelet, a covering of a circular n-board. It should be noted that a bracelet is said to be *out of phase* if there is a domino in position $(n, 1)$. Otherwise

FIGURE 3. Perrin bracelets from r_2 to r_5. Source: Prepared by the authors.



it is said to be *in phase*. Note that this definition can be extended in the case of a trimino in positions $(n-1, n, 1)$ or $(n, 1, 2)$.

Theorem 4.2. *For $n \geqslant 2$, the possible Perrin polynomial bracelets of size 1 x n with blue curved dominoes of weight x and gray curved triminoes of weight 1 is given by: $r_n(x) = R_n(x)$, where $r_n(x)$ is the number of $n-$Perrin polynomial bracelets and $R_n(x)$ is the nth term of the Perrin polynomial sequence.*

Proof. The proof is analogous to Theorem. 4.1. □

To exemplify, there is Figure 4 with cases from $r_2(x)$ to $r_5(x)$.

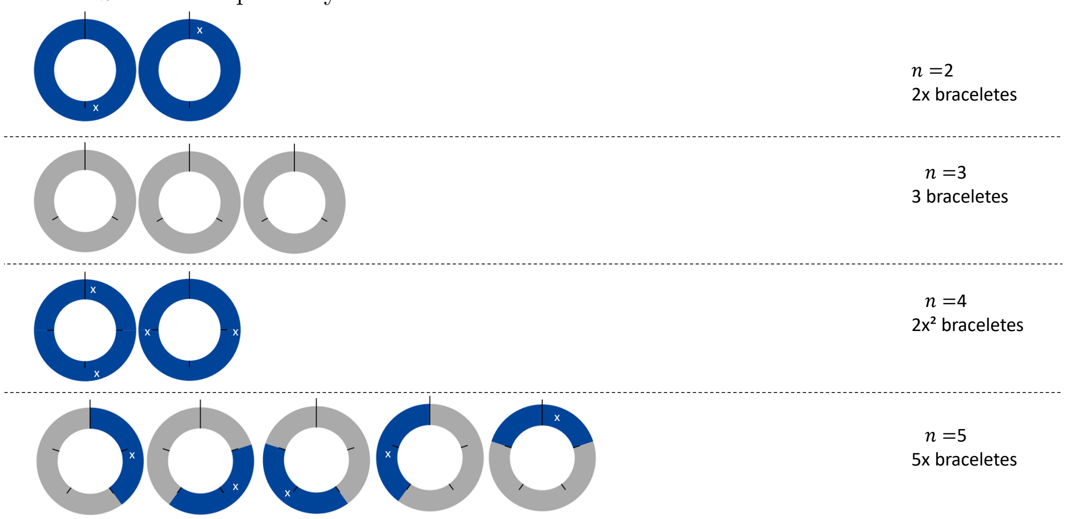
For the case $r_2(x)$, there are 2 bracelets of weight x, resulting in $2x$, representing the term $R_2(x)$. For the case $r_3(x)$, we have 3 bracelets weighing 1, resulting in 3 and representing the term $R_3(x)$. For the case $r_4(x)$, we have 2 bracelets of weight x^2, resulting in $2x^2$ and representing the term $R_4(x)$. For the case $r_5(x)$, we have 5 bracelets of weight x, resulting in $5x$ and representing the term $R_5(x)$.

Next, there is the combinatorial interpretation of Tetrarrin.

Set te_n the amount of coverage of a circular board with n positions labeled clockwise, using the pieces: blue curved dominoes, gray curved triminoes and green curved tetraminos. It is called a n-bracelet, a covering of a circular n-board. Note that a bracelet is said to be *out of phase* if there is a domino in position $(n, 1)$. Otherwise it is said to be *in phase*. The present definition is also valid for the cases of gray curved tetraminoes in positions $(n-1, n, 1)$ or $(n, 1, 2)$ and green curved tetraminoes in positions $(n-2, n-1, n, 1)$, $(n-1, n, 1, 2)$ or $(n, 1, 2, 3)$. The blue curved domino rotates only twice. The gray curved trimino and the green curved tetramino rotate only three times.

Theorem 4.3. *For $n \geqslant 2$, the possible bracelets of size 1 x n with blue curved dominoes, gray curved triminoes and green curved tetraminos, all with weight 1 is*

FIGURE 4. Polynomial Perrin bracelets from $r_2(x)$ to $r_5(x)$. Source: Prepared by the authors.



given by: $te_n = Te_n$, *where* te_n *is the number of* $n-$ *Tetrarrin bracelets and* Te_n *is the nth term of the Tetrarrin sequence.*

Proof. The proof is analogous to Theorem 4.1. □

To exemplify, there is Figure 5 with cases from te_2 to te_5.

For the case te_2, there are 2 bracelets, representing the term Te_2. For the case te_3, 3 bracelets are accounted for, representing the term Te_3. For the case te_4, 5 bracelets are accounted for, representing the term Te_4. For the case te_5, 5 bracelets are counted, representing the term Te_5.

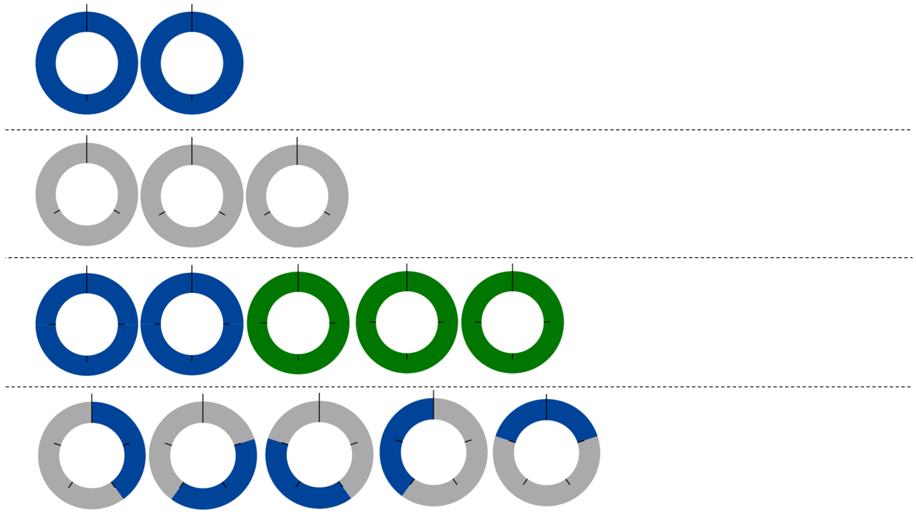
Next, there is the polynomial combinatorial interpretation of Tetrarrin.

To do this, define $te_n(x)$ the amount of coverage of a circular board with n positions labeled clockwise, using blue curved dominoes of weight x^2, gray curved triminoes of weight x and tetraminos weight green curves 1.

In this way, the previously mentioned denomination referring to the n-bracelet follows, bearing in mind that it has a covering of a circular n-tray. Similarly, one can say that a bracelet is said to be *out of phase* if there is a domino in position $(n, 1)$. Otherwise it is said to be *in phase*. It is noteworthy that this definition can be extended to the cases of gray curved tetraminos in positions $(n-1, n, 1)$ or $(n, 1, 2)$ and green curved tetraminos in positions $(n-2, n-1, n, 1)$, $(n-1, n, 1, 2)$ or $(n, 1, 2, 3)$. The blue curved domino rotates only twice. The gray curved trimino and the green curved tetramino rotate only three times.

Theorem 4.4. *For* $n \geqslant 2$, *the possible bracelets of size 1 x n with curved blue dominoes of weight* x^2, *curved gray triminoes of weight* x *and curved green tetraminoes*

FIGURE 5. Tetrarrin bracelets from te_2 to te_5. Source: Prepared by the authors.



of weight 1, is given by: $te_n(x) = Te_n(x)$*, where* $te_n(x)$ *is the number of* $n-$ *Tetrarrin polynomial bracelets and* $Te_n(x)$ *is the nth term of the polynomial sequence of Tetrarin.*

Proof. The proof follows analogous to the validation of the Theorem 4.3. □

An example of the model is Figure 6 for cases from $n = 2$ to $n = 5$.

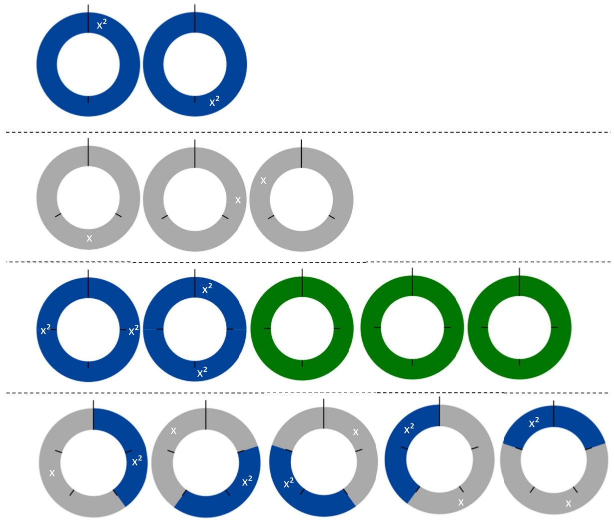
For the case $te_2(x)$, there are 2 bracelets of weights x^2, resulting in $2x^2$ and representing the term $Te_2(x)$. For the case $te_3(x)$, 3 weight bracelets x are accounted for, resulting in $3x$ and representing the term $Te_3(x)$. For the case $te_4(x)$, 5 bracelets are accounted for, resulting in $3x^4+3$ and representing the term $Te_4(x)$. For the case $te_5(x)$, 5 bracelets are accounted for, resulting in $5x^3$ and representing the term $Te_5(x)$.

5. Conclusions

From the relationship between the Padovan and Perrin sequences, it was possible to deepen Padovan's combinatorial approach in order to introduce Perrin's combinatorial interpretation. Thus, the present study introduced Perrin's combinatorial approaches and its Tetrarrin extension. Furthermore, the combinatorial approaches of Perrin and Tetrarrin polynomial sequences were obtained, contributing to the study of linear and recurrent sequences.

In fact, combinatorial models make it possible to integrate the study of sequences in the area of combinatorics, allowing a visualization of the numbers of these investigated sequences.

FIGURE 6. Tetrarrin polynomial bracelets from $te_2(x)$ to $te_5(x)$.
Source: Prepared by the authors.



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The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

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The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that

they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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ON $k-$CONFORMABLE FRACTIONAL OPERATORS

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Abstract. In this study, we define the left and right fractional $k-$conformable integrals and derivatives. Furthermore, we obtained the fractional $k-$conformable derivatives of functions associated with some spaces and express their properties.

1. Introduction

Fractional calculus was born in 1695. Moreover, the significant of fractional calculus gained more and more over the years. This field is substantial not only in the field of mathematics, but especially in terms of applied sciences. Application of fractional calculus were by way of majority utilized in numerous fields of science and engineering. The most widely utilized were Caputo and Riemann-Liouville derivatives. The most common use fields of Riemann-Liouville are physics, mechanics, electronics, chemistry, biology, engineering and other fields$[3-7]$. The Riemann-Liouville approach base upon iterating $n-$times the integral operator and is the fractional integral of noninteger order. The core of the standard fractional calculus can not be enough us for the required kernel . Furthermore, we need required kernel in order to obtain unification of fractional derivatives in their studies $[8-9]$. Additionally, Differentiation operator is the most appropriate operator for a starting point for the iteration method. In this circumstances, Abdeljawad described the left and right generalized conformable derivatives, respectively $[10]$,

$$ {}_aT^{\alpha} f(x) = (x-a)^{1-\alpha} f'(x), $$

$$ T_b^{\alpha} f(x) = (b-x)^{1-\alpha} f'(x). $$

In here, let f is a differentiable function, we have left and right integrals the following forms $[1]$,

$$ {}_a^{\beta}J^{\alpha} f(t) = \tfrac{1}{\Gamma(\beta)} \int_a^x \left(\tfrac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha} \right)^{\beta-1} f(t) \tfrac{dt}{(t-a)^{1-\alpha}} $$

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and

$$^{\beta}J_{b}^{\alpha}f\left(x\right) =\frac{1}{\Gamma(\beta)}\int_{a}^{x}\left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right) ^{\beta-1}f\left(t\right) \frac{dt}{(b-a)^{1-\alpha}}.$$

Respectively. In this point, Authors in [1] defined new fractional operators which have two parameters and also these operators have kernels different from usual kernels. In this article, we pay attention studies of depending on [1] and also we obtained new $k-$conformable fractional integrals and derivatives by way of new fractional operators in this paper. Additionally, we will give some basic definitions and tools related to classical fractional calculus.

Definition 1.1. $[16],[8]$ A real valued function $f(t)$, $t>0$ is said to be in the space C_{μ}, $\mu\in\mathbb{R}$ if there exists a complex number $p>\mu$ such that $f(t)=t^{p}f_{1}\left(t\right)$, where $f_{1}(t)\in C\left[0,\infty\right]$.

Definition 1.2. $[16],[8]$ A function $f(t)\in C_{\mu}$, $t>0$ is said to be in the $L_{p,k}\left(a,b\right)$ space if

$$L_{p,k}\left(a,b\right) =\left\{ f:\left\Vert f\right\Vert _{L_{p,k}(a,b)}=\left(\int_{a}^{b}\left\vert f(t)^{p}\right\vert t^{k}dt\right) ^{\frac{1}{p}}<\infty,\ 1\leq p<\infty,\ k\geq0\right\} .$$

Definition 1.3. $[16]$ Consider the space $X_{c}^{p}\left(a,b\right)$ $(c\in\mathbb{R},\ 1\leq p<\infty)$ of those real-valued Lebesgue measurable functions f on $[a,b]$ for which

$$\left\Vert f\right\Vert =\left(\int_{a}^{b}\left\vert t^{c}f(t)\right\vert ^{p}\frac{dt}{t}\right) ^{\frac{1}{p}}<\infty,\ (1\leq p<\infty,\ cp\geq1)$$

and for the case $p=\infty$

$$\left\Vert f\right\Vert _{X_{c}^{\infty}}=ess\sup_{a\leq t<b}\left[t^{c}f\left(t\right) \right] ,\ c\geq0.$$

Additionally, If we take $c=\frac{k+1}{p}$ $(1\leq p<\infty,\ k\geq0)$ the space $X_{c}^{p}\left(a,b\right)$, we have the $L_{p,k}\left(a,b\right) -$space. Moreover, If we take $c=\frac{1}{p}$ $(1\leq p<\infty)$ the space $X_{c}^{p}\left(a,b\right)$,we have the $L^{p}\left(a,b\right) -$space[16].

Katugampola obtained the generalized left and right fractional integrals for $\beta\in\mathbb{C}$ and $\operatorname{Re}\left(\beta\right) >0$ in [8] :

$$\left(_{a}I^{\beta,\alpha}f\right) \left(t\right) =\frac{1}{\Gamma(\beta)}\int_{a}^{t}\left(\frac{t^{\alpha}-y^{\alpha}}{\alpha}\right) ^{\beta-1}f\left(y\right) \frac{dy}{y^{1-\alpha}} \tag{1.1}$$

and

$$\left(I_{b}^{\beta,\alpha}f\right) \left(t\right) =\frac{1}{\Gamma(\beta)}\int_{t}^{b}\left(\frac{y^{\alpha}-t^{\alpha}}{\alpha}\right) ^{\beta-1}f\left(y\right) \frac{dy}{y^{1-\alpha}}, \tag{1.2}$$

respectively.

The following forms are left and right generalized fractional derivatives for $\beta\in\mathbb{C}$ and $\operatorname{Re}\left(\beta\right) \geq0$ in [9] :

$$\begin{aligned} \left(_{a}D^{\beta,\alpha}f\right) \left(t\right) & =\xi^{n}\left(_{a}I^{n-\beta,\alpha}f\right) \left(t\right) \\ & =\frac{\xi^{n}}{\Gamma(n-\beta)}\int_{a}^{t}\left(\frac{t^{\alpha}-y^{\alpha}}{\alpha}\right) ^{n-\beta-1}f\left(y\right) \frac{dy}{y^{1-\alpha}} \end{aligned} \tag{1.3}$$

and

$$\begin{aligned}\left(D_b^{\beta,\alpha} f\right)(t) &= (-\xi)^n \left({}_aI^{n-\beta,\alpha} f\right)(t) \\ &= \frac{(-\xi)^n}{\Gamma(n-\beta)} \int_t^b \left(\frac{y^\alpha - t^\alpha}{\alpha}\right)^{n-\beta-1} f(y) \frac{dy}{y^{1-\alpha}},\end{aligned} \tag{1.4}$$

respectively, where $\alpha > 0$ and where $\xi = t^{1-\alpha} \frac{d}{dt}$.

The following forms are the left and right generalized Caputo fractional derivatives which defined by the authors in [15] by using [9] ,

$$\begin{aligned}\left({}_a^C D^{\beta,\alpha} f\right)(t) &= \left({}_aI^{n-\beta,\alpha} (\xi)^n f\right)(t) \\ &= \frac{1}{\Gamma(n-\beta)} \int_a^t \left(\frac{t^\alpha - u^\alpha}{\alpha}\right)^{n-\beta-1} \frac{\xi^n f(y) dy}{y^{1-\alpha}}\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}\left({}^C D_b^{\beta,\alpha} f\right)(t) &= \left({}_aI^{n-\beta,\alpha} (-\xi)^n f\right)(t) \\ &= \frac{1}{\Gamma(n-\beta)} \int_t^b \left(\frac{y^\alpha - t^\alpha}{\alpha}\right)^{n-\beta-1} \frac{(-\xi)^n f(y) dy}{y^{1-\alpha}}.\end{aligned} \tag{1.6}$$

Respectively.

Now, after giving $k-$ conformable fractional integral and derivatives, respectively, we will demonstrate important consequences and some basic properties for these operators. Furthermore, we will obtain the properties of the defined $k-$conformable derivative and also we will acquire the $k-$conformable fractional derivatives on the Caputo setting. In conclusion, we will develop the previously obtained results for the generalized conformable derivatives and integrals.

2. THE $k-$CONFORMABLE FRACTIONAL OPERATORS

In this part, Abdeljawad defined the conformable integrals and we expanded to higher order in [10] . Furthermore, Jarad and et al. defined fractional integrals in [1] . Now, by considering these studies, we should give the following $k-$conformable derivative by using definitions of conformable derivative,

$${}_a^hT^\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f\left(t+\varepsilon \frac{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}{t^k}\right) - f(t)}{\varepsilon}. \tag{2.1}$$

We should consider (2.1). In here,

$$\Delta t = \varepsilon \frac{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}{t^k} \Rightarrow \varepsilon = \frac{\Delta t . t^k}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}. \tag{2.2}$$

We choose Δt in the form. Then,

$$\begin{aligned}{}_a^hT^\alpha f(t) &= \frac{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}{t^k} \lim_{\Delta t\to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t} \\ &= \frac{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}{t^k} f'(t).\end{aligned} \tag{2.3}$$

We can state the left and right $k-$conformable derivatives, respectively, as

$$\begin{aligned}{}_a^hT^\alpha f(x) &= \left(\frac{t^{k+1}-a^{k+1}}{t^k}\right)^{1-\alpha} f'(x), \\ {}^hT_b^\alpha f(x) &= \left(\frac{b^{k+1}-t^{k+1}}{t^k}\right)^{1-\alpha} f'(x).\end{aligned} \tag{2.4}$$

Moreover, we obtain $k-$conformable integral operator. For this,

$$\int_a^x \frac{t_1^k dt_1}{\left(t_1^{k+1}-a^{k+1}\right)^{1-\alpha}} \int_a^{t_1} \frac{t_2^k dt_2}{\left(t_2^{k+1}-a^{k+1}\right)^{1-\alpha}} \cdots \int_a^{t_{n-1}} \frac{t_n^k f(t_n) dt_n}{\left(t_n^{k+1}-a^{k+1}\right)^{1-\alpha}}, \tag{2.5}$$

we should get $n-$times repeated integral of the forms. In addition, if we apply a method as in classic fractional integral techniques,

$${}_a^k J^{n,\alpha} f(x) = \frac{1}{\Gamma(n)} \int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-1} \frac{t^k f(t) dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}. \tag{2.6}$$

We can write the equality. Furthermore, we can obtain definition of the following for $k-$conformable integrals with the help of this equality[2] .

Definition 2.1. Let $f \in X_c$. The left and right $k-$conformable fractional integrals of order $n \in \mathbb{C}$, $\mathrm{Re}(n) \geq 0$ and $\alpha > 0$,

$${}_a^k J^{n,\alpha} f(x) = \frac{1}{\Gamma(n)} \int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-1} \frac{t^k f(t) dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \tag{2.7}$$

and

$${}^k J_b^{n,\alpha} f(x) = \frac{1}{\Gamma(n)} \int_x^b \left[\frac{\left(b^{k+1}-x^{k+1}\right)^{\alpha}-\left(b^{k+1}-t^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-1} \frac{t^k f(t) dt}{\left(b^{k+1}-t^{k+1}\right)^{1-\alpha}}, \tag{2.8}$$

respectively.

In here, we will give the following new definition by considering the $k-$conformable derivative and integral operators.

Definition 2.2. Let $f \in X_c$. The left and right $k-$conformable fractional derivatives of order $\beta \in \mathbb{C}$ and $\mathrm{Re}(\beta) \geq 0$,

$$\begin{aligned} {}_a^k D^{\beta,\alpha} f(x) &= {}_a^k T^{n,\alpha} \left({}_a^k J^{n-\beta,\alpha}\right) f(x) \\ &= \frac{{}_a^k T^{n,\alpha}}{\Gamma(n-\beta)} \int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta-1} \frac{t^k f(t) dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} {}^k D_b^{\beta,\alpha} f(x) &= {}^k T_b^{n,\alpha} \left({}^k J_b^{n-\beta,\alpha}\right) f(x) \\ &= \frac{{}^k T_b^{n,\alpha} (-1)^n}{\Gamma(n-\beta)} \int_x^b \left[\frac{\left(b^{k+1}-x^{k+1}\right)^{\alpha}-\left(b^{k+1}-t^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta-1} \frac{t^k f(t) dt}{\left(b^{k+1}-t^{k+1}\right)^{1-\alpha}}, \end{aligned} \tag{2.10}$$

where $n = [\mathrm{Re}(\beta)] + 1$,

$$\begin{aligned} {}_a^k T^{n,\alpha} &= \underbrace{{}_a^k T^{\alpha} \;\; {}_a^k T^{\alpha} \cdots {}_a^k T^{\alpha}}_{n-times}, \\ {}^k T_b^{n,\alpha} &= \underbrace{{}^k T_b^{\alpha} \;\; {}^k T_b^{\alpha} \cdots {}^k T_b^{\alpha}}_{n-times}, \end{aligned} \tag{2.11}$$

and ${}_a^k T^{\alpha}$ and ${}^k T_b^{\alpha}$ are the left and right fractional $k-$conformable differential operators.

Theorem 2.3. *Let $f \in X_c$. Then, we get for fractional integrals for $Re(\beta) > 0$ and $Re(\gamma) > 0$,*

$$
\begin{aligned}
{}^{k}_{a}J^{\beta,\alpha}\left({}^{k}_{a}J^{\gamma,\alpha}\right) f(x) &= {}^{k}_{a}J^{(\beta+\gamma),\alpha} f(x), \\
{}^{k}J_{b}^{\beta,\alpha}\left({}^{k}J_{b}^{\gamma,\alpha}\right) f(x) &= {}^{k}J_{b}^{(\beta+\gamma),\alpha} f(x).
\end{aligned} \tag{2.12}
$$

Proof. We have with the aid of (2.7),

(2.13)

$$
\begin{aligned}
&{}^{k}_{a}J^{\beta,\alpha}\left({}^{k}_{a}J^{\gamma,\alpha}\right) f(x) \\
&= \frac{1}{\Gamma(\beta)}\int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-1} \frac{\left({}^{k}_{a}J^{\gamma,\alpha}\right)t^k dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\
&= \frac{1}{\Gamma(\beta)\Gamma(\gamma)}\int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-1} \\
&\times \left(\int_a^t \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(u^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\gamma-1} \frac{u^k f(u)du}{\left(u^{k+1}-a^{k+1}\right)^{1-\alpha}}\right) \frac{t^k f(t)dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\
&= \frac{1}{\Gamma(\beta)\Gamma(\gamma)}\int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(u^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta+\gamma-1} \left(\int_0^1 (1-z)^{\beta-1} z^{\gamma+1} dz\right) \frac{u^k f(u)du}{\left(u^{k+1}-a^{k+1}\right)^{1-\alpha}} \\
&= \frac{1}{\Gamma(\beta+\gamma)}\int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(u^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta+\gamma-1} \frac{u^k f(u)du}{\left(u^{k+1}-a^{k+1}\right)^{1-\alpha}} \\
&= {}^{k}_{a}J^{(\beta+\gamma),\alpha} f(x).
\end{aligned}
$$

In here, we have used the change of variable,

$$
\left(t^{k+1}-a^{k+1}\right)^{\alpha} = \left(u^{k+1}-a^{k+1}\right)^{\alpha} + z\left[\left(x^{k+1}-a^{k+1}\right)^{\alpha} - \left(u^{k+1}-a^{k+1}\right)^{\alpha}\right].
$$

The second formula can be demonstrated in the same manner. □

Lemma 2.4. *Let $f \in X_c$. We have for $Re(v) > 0$,*

$$
\begin{aligned}
{}^{k}_{a}J^{\beta,\alpha}\left(t^{k+1}-a^{k+1}\right)^{\alpha(v-1)}(x) &= \frac{\Gamma(v)}{\Gamma(\beta+v)} \frac{\left[\left(x^{k+1}-a^{k+1}\right)^{\alpha}\right]^{\beta+v-1}}{[\alpha(k+1)]^{\beta}}, \\
{}^{k}J_{b}^{\beta,\alpha}\left(b^{k+1}-t^{k+1}\right)^{\alpha(v-1)}(x) &= \frac{\Gamma(v)}{\Gamma(\beta+v)} \frac{\left[\left(b^{k+1}-x^{k+1}\right)^{\alpha}\right]^{\beta+v-1}}{[\alpha(k+1)]^{\beta}}.
\end{aligned} \tag{2.14}
$$

Proof. We have with the aid of (2.7),

$$
\begin{aligned}
&{}^{k}_{a}J^{\beta,\alpha}\left(t^{k+1}-a^{k+1}\right)^{\alpha(v-1)}(x) \\
&\quad= \frac{1}{\Gamma(\beta)}\int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-1} \frac{\left[\left(t^{k+1}-a^{k+1}\right)^{\alpha}\right]^{v-1} t^k dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\
&\quad= \frac{\left[\left(x^{k+1}-a^{k+1}\right)^{\alpha}\right]^{\beta+v-1}}{\Gamma(\beta)[\alpha(k+1)]^{\beta-1}} \int_0^1 (1-z)^{\beta-1} z^{v-1} dz \\
&\quad= \frac{\Gamma(v)}{\Gamma(\beta+v)} \frac{\left[\left(x^{k+1}-a^{k+1}\right)^{\alpha}\right]^{\beta+v-1}}{[\alpha(k+1)]^{\beta}}.
\end{aligned} \tag{2.15}
$$

In here, we have used the change of variable,

$$
\left(t^{k+1}-a^{k+1}\right)^{\alpha} = z\left(x^{k+1}-a^{k+1}\right)^{\alpha}.
$$

The second formula can be demonstrated in the same manner. □

Lemma 2.5. *Let $f \in X_c$. We have for $Re\,(n-\alpha) > 0$,*

$$\left[{}_a^kD^{\beta,\alpha}\left(t^{k+1}-a^{k+1}\right)^{\alpha(v-1)}\right](x) = \frac{[\alpha(k+1)]^{\beta}\Gamma(v)}{\Gamma(v-\beta)}\left[\left(x^{k+1}-a^{k+1}\right)^{\alpha}\right]^{v-\beta-1}, \tag{2.16}$$

$$\left[{}^kD_b^{\beta,\alpha}\left(b^{k+1}-t^{k+1}\right)^{\alpha(v-1)}\right](x) = \frac{[\alpha(k+1)]^{\beta}\Gamma(v)}{\Gamma(v-\beta)}\left[\left(b^{k+1}-x^{k+1}\right)^{\alpha}\right]^{v-\beta-1}.$$

Proof. We have with the aid of (2.9),

$$\begin{aligned}
&\left[{}_a^kD^{\beta,\alpha}\left(t^{k+1}-a^{k+1}\right)^{\alpha(v-1)}\right](x) \\
&= \frac{{}_a^kT^{n,\alpha}}{\Gamma(n-\beta)}\int_a^x\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta-1}\frac{\left[\left(t^{k+1}-a^{k+1}\right)^{\alpha}\right]^{v-1}t^kdt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\
&= \frac{{}_a^kT^{n,\alpha}\left[\left(x^{k+1}-a^{k+1}\right)^{\alpha}\right]^{n+v-\beta-1}}{\Gamma(n-\beta)[\alpha(k+1)]^{n-\beta}}\int_0^1(1-z)^{n-\beta-1}z^{v-1}dz \\
&= \frac{[\alpha(k+1)]^{\beta}\Gamma(v)}{\Gamma(v-\beta)}\left[\left(x^{k+1}-a^{k+1}\right)^{\alpha}\right]^{v-\beta-1}.
\end{aligned} \tag{2.17}$$

In here, we have used the change of variable

$$\left(t^{k+1}-a^{k+1}\right)^{\alpha} = z\left(x^{k+1}-a^{k+1}\right)^{\alpha}.$$

The second formula can be demonstrated in the same manner. □

Remark 2.6. It can be shown that

$$\begin{aligned}
{}_a^kD^{\beta,\alpha}f &= {}_a^kJ^{\beta,-\alpha}, \\
{}^kD_b^{\beta,\alpha}f &= {}^kJ_b^{\beta,-\alpha}.
\end{aligned} \tag{2.18}$$

3. $k-$CONFORMABLE FRACTIONAL DERIVATIVES ON THE CERTAIN SPACES

In this part, we will give some definitions with related to lemma and theorem. Moreover, we will demonstrate the substantial results of the $k-$conformable fractional derivatives on the space $C_{\alpha,a}^n$ and $C_{\alpha,b}^n$.

Definition 3.1. [10] For $0 < \alpha \leq 1$ and an interval $[a,b]$ define,

$$\begin{aligned}
{}^kI_{\alpha}([a,b]) \quad &= \Big\{f:[a,b]\to\mathbb{R}: f(x) = \left({}_a^kI^{\beta,\alpha}\varphi\right)(x)+f(a) \\
&\qquad \text{for some } \varphi\in {}^kL_{\alpha}(a)\Big\}
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
{}_{\alpha}^kI([a,b]) \quad &= \Big\{g:[a,b]\to\mathbb{R}: g(x) = \left({}^kI_b^{\beta,\alpha}\varphi\right)(x)+g(b) \\
&\qquad \text{for some } \varphi\in {}^kL_{\alpha}(b)\Big\}.
\end{aligned} \tag{3.2}$$

Where

$${}^kL_{\alpha}(a) = \left\{\varphi:[a,b]\to\mathbb{R},\ \left({}_a^kI^{\beta,\alpha}\varphi\right)(x) \text{ exists } \forall x\in[a,b]\right\} \tag{3.3}$$

and

$${}^kL_{\alpha}(b) = \left\{\varphi:[a,b]\to\mathbb{R},\ \left({}^kI_b^{\beta,\alpha}\varphi\right)(x) \text{ exists } \forall x\in[a,b]\right\}. \tag{3.4}$$

Definition 3.2. We can clearly describe for $\alpha \in (0,1]$ and $n = 1,2,3,...,$

$$C^n_{\alpha,a}([a,b]) = \left\{f : [a,b] \to \mathbb{R} \textit{ such that } {}^k_aT^{n-1,\alpha} f \in^k_a I^{\beta,\alpha}([a,b])\right\}, \tag{3.5}$$
$$C^n_{\alpha,b}([a,b]) = \left\{f : [a,b] \to \mathbb{R} \textit{ such that } {}^kT^{n-1,\alpha}_b f \in^k I^{\beta,\alpha}_b([a,b])\right\}.$$

Lemma 3.3. *Let $f \in C^n_{\alpha,a}([a,b])$ and $\alpha > 0$. Then, f is presented in form,*

$$f(x) = \frac{1}{(n-1)!}\int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-1} \frac{\varphi(t)t^k dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} + \sum_{s=0}^{n-1}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^s \frac{1}{s!}{}^k_aT^{s,\alpha} f(a). \tag{3.6}$$

In this place is $\varphi(t) = \left({}^k_aT^{s,\alpha} f\right)(t)$.

Proof. Since $f \in C^n_{\alpha,a}([a,b])$, ${}^k_aT^{n-1,\alpha} f \in {}^kI_\alpha([a,b])$ and φ is continuous function, we have,

$$\begin{aligned} {}^k_aT^{n-1,\alpha} f(x) &= \int_a^x \frac{\varphi(t)t^k dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} +^k_a T^{n-1,\alpha} f(a), \\ \frac{\left(x^{k+1}-a^{k+1}\right)^{1-\alpha}}{x^k}\frac{d}{dx}\,{}^k_aT^{n-2,\alpha} f(x) &= \int_a^x \frac{\varphi(t)t^k dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} +^k_a T^{n-1,\alpha} f(a) \\ \frac{d}{dx}\,{}^k_aT^{n-2,\alpha} f(x) &= \left[\frac{x^k}{\left(x^{k+1}-a^{k+1}\right)^{1-\alpha}}\int_a^x \frac{\varphi(t)t^k dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}\right. \\ &= \left. + \frac{x^k}{\left(x^{k+1}-a^{k+1}\right)^{1-\alpha}}.{}^k_aT^{n-1,\alpha} f(a)\right]. \end{aligned} \tag{3.7}$$

We integrate the both of side (3.7) from a to x by replacing $x \to t$ and $t \to s$ on the both side of the equation, then,

$$\begin{aligned} {}^k_aT^{n-2,\alpha} f(x) &= \int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(s^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right] \frac{\varphi(s)s^k ds}{\left(s^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ &+ \frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}.{}^k_aT^{n-1,\alpha} f(a) +^k_a T^{n-2,\alpha} f(a). \end{aligned} \tag{3.8}$$

By applying the equality same method once more, we get,

$$\begin{aligned} {}^k_aT^{n-3,\alpha} f(x) &= \int_a^x \frac{1}{2}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(s^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^2 \frac{\varphi(s)s^k ds}{\left(s^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ &+ \frac{1}{2}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^2 .{}^k_aT^{n-1,\alpha} f(a) \\ &+ \frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}.{}^k_aT^{n-2,\alpha} f(a) +^k_a T^{n-3,\alpha} f(a). \end{aligned} \tag{3.9}$$

If the same method is applied $n-3$ times, we have,

$$\begin{aligned} f(x) &= \frac{1}{(n-1)!}\int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-1} \frac{\varphi(t)t^k dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ &+ \sum_{s=0}^{n-1}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^s \frac{1}{s!}.{}^k_aT^{s,\alpha} f(a). \end{aligned} \tag{3.10}$$

For $\varphi(t) =^k_a T^{n,\alpha} f(t)$. It is clear that a similar lemma for right $k-$conformable fractional derivative. □

Lemma 3.4. *Let $f \in C^{n}_{\alpha,b}([a,b])$ for $\alpha>0$. Then, f is presented in form,*

$$
\begin{aligned}
f(x) &= \frac{1}{(n-1)!}\int_x^b \left[\frac{\left(b^{k+1}-x^{k+1}\right)^{\alpha}-\left(b^{k+1}-t^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-1} \frac{\varphi(t)t^k dt}{\left(b^{k+1}-t^{k+1}\right)^{1-\alpha}} \\
&+\sum_{s=0}^{n-1}\left[\frac{\left(b^{k+1}-x^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{s} \frac{(-1)^s}{s!}.{}^{k}T_b^{s,\alpha} f(a).
\end{aligned}
\tag{3.11}
$$

For $\varphi(t)=\left({}^{k}T_b^{s,\alpha}f\right)(t)$.

Proof. The proof is likewise as *Lemma* 3. □

Now we will give $k-$conformable fractional derivatives on $C^{n}_{\alpha,a}$ and $C^{n}_{\alpha,b}$ in the theorem 2.

Theorem 3.5. *Let $\beta\in\mathbb{C}$, $Re(\beta)>0$ and $n=[\beta]+1$. The left and right $k-$conformable fractional derivatives are demonstrated in the form for $f\in C^{n}_{\alpha,a}$ and $f \in C^{n}_{\alpha,b}$. Then,*

$$
\begin{aligned}
{}_{a}^{k}D^{\beta,\alpha}f(x) &= \left({}_{a}^{k}J^{n-\beta}\left({}_{a}^{k}T^{n,\alpha}f\right)\right)(x) \\
&+\sum_{m=0}^{n-1}\frac{{}_{a}^{k}T^{n,\alpha}f(a)}{\Gamma(m-\beta+1)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m-\beta}
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
{}^{k}D_b^{\beta,\alpha}f(x) &= \left({}^{k}J_b^{n-\beta}\left({}^{k}T_b^{n,\alpha}f\right)\right)(x) \\
&+\sum_{m=0}^{n-1}\frac{(-1)^m.{}^{k}T_b^{n,\alpha}f(b)}{\Gamma(m-\beta+1)}\left[\frac{\left(b^{k+1}-t^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m-\beta}.
\end{aligned}
\tag{3.13}
$$

Proof. By using $f\in C^{n}_{\alpha,a}([a,b])$, we should choose $f(x)$ in the *Lemma* 3 by replacing $x\to t$ and $t\to s$ that is as following form,

$$
\begin{aligned}
f(x) &= \frac{1}{(n-1)!}\int_a^t \left[\frac{\left(t^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-1} \frac{{}_{a}^{k}T^{n,\alpha}f(s)s^k ds}{\left(s^{k+1}-a^{k+1}\right)^{1-\alpha}} \\
&+\sum_{m=0}^{n-1}\left[\frac{\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m} \frac{1}{m!}.{}_{a}^{k}T^{m,\alpha}f(a).
\end{aligned}
\tag{3.14}
$$

In here, we can state the following equality by using (2.9) for (3.14),

$$
\begin{aligned}
{}_{a}^{k}D^{\beta,\alpha}f(x) &= \frac{{}_{a}^{k}T^{n,\alpha}}{\Gamma(n-\beta)}\int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta-1}\frac{t^k f(t)dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\
&= \frac{{}_{a}^{k}T^{n,\alpha}}{\Gamma(n-\beta)}\int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta-1} \\
&\times\left(\frac{1}{(n-1)!}\int_a^t \left[\frac{\left(t^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-1}\frac{{}_{a}^{k}T^{n,\alpha}f(s)s^k ds}{\left(s^{k+1}-a^{k+1}\right)^{1-\alpha}}\right)\frac{t^k f(t)dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\
&+\frac{{}_{a}^{k}T^{n,\alpha}}{\Gamma(n-\beta)}\int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta-1} \\
&\times\left(\sum_{m=0}^{n-1}\left[\frac{\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m}\frac{1}{m!}.{}_{a}^{k}T^{m,\alpha}f(a)\right)\frac{t^k f(t)dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}.
\end{aligned}
\tag{3.15}
$$

We used changing the order of integration and gamma and beta functions. Additionally, we use the following the equations,

$$
\left(t^{k+1}-a^{k+1}\right)^{\alpha}=\left(s^{k+1}-a^{k+1}\right)^{\alpha}+z\left[\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}\right]
\tag{3.16}
$$

and

$$\left(t^{k+1}-a^{k+1}\right)^{\alpha}=u\left(x^{k+1}-a^{k+1}\right)^{\alpha}.$$

We obtained following form,

(3.17)

$$\begin{aligned}{}_{a}^{k}D^{\beta,\alpha}f(x) &= \frac{{}_{a}^{k}T^{n,\alpha}}{\Gamma(n-\beta)(n-1)!}\int_{a}^{x}\frac{{}_{a}^{k}T^{n,\alpha}f(s)s^{k}ds}{(s^{k+1}-a^{k+1})^{1-\alpha}}\\ &\times\left(\int_{0}^{1}(1-z)^{n-\beta-1}(z)^{n-1}dz\right)\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{2n-\beta-1}\\ &+\sum_{m=0}^{n-1}\frac{{}_{a}^{k}T^{n,\alpha}.T^{n,\alpha}f(a)}{\Gamma(n-\beta).m!}\\ &\times\left(\int_{0}^{1}(1-u)^{n-\beta-1}(u)^{m}du\right)\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta+m}.\end{aligned}$$

In here, we obtain by means of operator ${}_{a}^{k}T^{n,\alpha}$,

(3.18)

$$\begin{aligned}{}_{a}^{k}D^{\beta,\alpha}f(x) &= \frac{1}{\Gamma(n-\beta)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta-1}\frac{{}_{a}^{k}T^{n,\alpha}f(s).s^{k}ds}{(s^{k+1}-a^{k+1})^{1-\alpha}}\\ &\quad+\sum_{m=0}^{n-1}\frac{{}_{a}^{k}T^{n,\alpha}f(a)}{\Gamma(m-\beta+1)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m-\beta}.\end{aligned}$$

We completed the proof. The proof of right $k-$conformable fractional derivative can be done by same way. □

Theorem 3.6. *We suppose that is* $Re(\beta)>m>0$ *for* $m\in\mathbb{N}$. *Then,*

(3.19)

$$\begin{aligned}{}_{a}^{k}T^{m,\alpha}\left({}_{a}^{k}J^{\beta,\alpha}f(x)\right) &= {}_{a}^{k}J^{\beta-m,\alpha}f(x),\\ {}^{k}T_{b}^{m,\alpha}\left({}^{k}J_{b}^{\beta,\alpha}f(x)\right) &= {}^{k}J_{b}^{\beta-m,\alpha}f(x).\end{aligned}$$

Proof. We have by using (2.7),

(3.20)

$${}_{a}^{k}T^{m,\alpha}\left({}_{a}^{k}J^{\beta,\alpha}f(x)\right) ={}_{a}^{k}T^{m,\alpha}\left[\frac{1}{\Gamma(\beta)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-1}\frac{t^{k}f(t)dt}{(t^{k+1}-a^{k+1})^{1-\alpha}}\right].$$

By using Leibniz rule for integrals,

(3.21)

$$\begin{aligned}&{}_{a}^{k}T^{m,\alpha}\left({}_{a}^{k}J^{\beta,\alpha}f(x)\right)\\ &={}_{a}^{k}T^{m-1,\alpha}\left[\frac{1}{\Gamma(\beta-1)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-2}\frac{t^{k}f(t)dt}{(t^{k+1}-a^{k+1})^{1-\alpha}}\right]\\ &={}_{a}^{k}T^{m-2,\alpha}\left[\frac{1}{\Gamma(\beta-2)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-3}\frac{t^{k}f(t)dt}{(t^{k+1}-a^{k+1})^{1-\alpha}}\right]\\ &\vdots\\ &=\left[\frac{1}{\Gamma(\beta-m)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-m-1}\frac{t^{k}f(t)dt}{(t^{k+1}-a^{k+1})^{1-\alpha}}\right]\\ &={}_{a}^{k}J^{\beta-m,\alpha}f(x).\end{aligned}$$

The poof is done. The second formula can be demonstrated similarly. □

Corollary 3.6.1. *If we take $Re(\gamma) < Re(\beta)$, Then,*

$$
{}_a^kD^{\gamma,\alpha}\left({}_a^kJ^{\beta,\alpha}f(x)\right) = {}_a^kJ^{\beta-\gamma,\alpha}f(x),
$$

(3.22)

$$
{}^kD_b^{\gamma,\alpha}\left({}^kJ_b^{\beta,\alpha}f(x)\right) = {}^kJ_b^{\beta-\gamma,\alpha}f(x).
$$

Proof. By using *Theorem* 1 and *Theorem* 3, we obtain,

(3.23)

$$
\begin{aligned}
{}_a^kD^{\gamma,\alpha}\left({}_a^kJ^{\beta,\alpha}f(x)\right) &= {}_a^kT^{m,\alpha}\left({}_a^kJ^{m-\gamma,\alpha}\left({}_a^kJ^{\beta,\alpha}f(x)\right)\right)\\
&= {}_a^kT^{m,\alpha}\left({}_a^kJ^{\beta+m-\gamma,\alpha}f(x)\right)\\
&= {}_a^kJ^{\beta-\gamma,\alpha}f(x).
\end{aligned}
$$

The proof is done. The second formula can be demonstrated likewise. □

Theorem 3.7. *Let $\beta > 0$ and $f \in C^n_{\alpha,a}[a,b]\left(f \in C^n_{\alpha,b}[a,b]\right)$. Then,*

$$
{}_a^kD^{\beta,\alpha}\left({}_a^kJ^{\beta,\alpha}f(x)\right) = f(x),
$$

(3.24)

$$
{}^kD_b^{\beta,\alpha}\left({}^kJ_b^{\beta,\alpha}f(x)\right) = f(x).
$$

Proof. If we possess by using (2.7) and (2.9),

(3.25)

$$
\begin{aligned}
&{}_a^kD^{\beta,\alpha}\left({}_a^kJ^{\beta,\alpha}f(x)\right)\\
&= \frac{{}_a^kT^{n,\alpha}}{\Gamma(n-\beta)\Gamma(\beta)}\int_a^x\int_a^t\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha-\left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-\beta-1}\\
&\times\left[\frac{\left(t^{k+1}-a^{k+1}\right)^\alpha-\left(u^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\beta-1}\frac{u^kf(u)du}{\left(u^{k+1}-a^{k+1}\right)^{1-\alpha}}\frac{t^kf(t)dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}\\
&= \frac{{}_a^kT^{n,\alpha}}{\Gamma(n-\beta)\Gamma(\beta)}\int_a^x\frac{u^kf(u)du}{\left(u^{k+1}-a^{k+1}\right)^{1-\alpha}}\\
&\times\left(\int_0^1(1-y)^{n-\beta-1}(y)^{\beta-1}dy\right)\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha-\left(u^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-1}\\
&= \frac{{}_a^kT^{n,\alpha}}{\Gamma(n-\beta)\Gamma(\beta)}\frac{\Gamma(n-\beta)\Gamma(\beta)}{\Gamma(n)}\int_a^x\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha-\left(u^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-1}\frac{f(u)u^kdu}{\left(u^{k+1}-a^{k+1}\right)^{1-\alpha}}\\
&={}_a^kT^{n,\alpha}\left({}_a^kJ^{n,\alpha}f\right)(x)\\
&= f(x).
\end{aligned}
$$

The proof is completed. □

Theorem 3.8. *Let $Re(\beta) > 0$, $n = Re(\beta)$, $f \in X_c$ and ${}_a^kJ^{\beta,\alpha}f \in C^n_{\alpha,a}[a,b]\left({}^kJ_b^{\beta,\alpha}f \in C^n_{\alpha,b}[a,b]\right)$. Then, we have,*

(3.26)

$$
{}_a^kJ^{\beta,\alpha}\left({}_a^kD^{\beta,\alpha}f(x)\right) = f(x) - \sum_{j=1}^n\frac{{}_a^kD^{\beta-j,\alpha}.f(a)}{\Gamma(\beta-j+1)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\beta-j}
$$

and

(3.27)

$$
{}^kJ_b^{\beta,\alpha}\left({}^kD_b^{\beta,\alpha}f(x)\right) = f(x) - \sum_{j=1}^n\frac{(-1)^n.{}^kD_b^{\beta-j,\alpha}f(b)}{\Gamma(\beta-j+1)}\left[\frac{\left(b^{k+1}-x^{k+1}\right)^\alpha}{\alpha}\right]^{\beta-j}.
$$

Proof. We can write by using (2.7) and (2.9),

(3.28)

$$
{}_a^kJ^{\beta,\alpha}\left({}_a^kD^{\beta,\alpha}f(x)\right) = \frac{1}{\Gamma(\beta)}\int_a^x\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha-\left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\beta-1}\frac{{}_a^kT^{n,\alpha}\left({}_a^kJ^{n-\beta,\alpha}f(t)\right)t^kdt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}.
$$

Using the integration by parts once, we have,

(3.29)

$$
\begin{aligned}
{}_{a}^{k}J^{\beta,\alpha}\left({}_{a}^{k}D^{\beta,\alpha}f(x)\right) &= \frac{{}_{a}^{k}T^{1,\alpha}}{\Gamma(\beta+1)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta}\frac{{}_{a}^{k}T^{n,\alpha}\left({}_{a}^{k}J^{n-\beta,\alpha}f(t)\right)t^{k}dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\
&\quad -\frac{1}{\Gamma(\beta+1)}\cdot{}_{a}^{k}T^{n,\alpha}\left({}_{a}^{k}J^{n-\beta,\alpha}f(t)\right).\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta}.
\end{aligned}
$$

Using the integration by parts $n-$times, we have,

(3.30)

$$
\begin{aligned}
&{}_{a}^{k}J^{\beta,\alpha}\left({}_{a}^{k}D^{\beta,\alpha}f(x)\right) \\
&= \frac{{}_{a}^{k}T^{1,\alpha}}{\Gamma(\beta-n+1)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-n}\frac{\left({}_{a}^{k}J^{n-\beta,\alpha}f(t)\right)t^{k}dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\
&\quad -\sum_{j=1}^{n}\frac{{}_{a}^{k}T^{n-j,\alpha}\left({}_{a}^{k}J^{n-\beta,\alpha}f(a)\right)}{\Gamma(\beta+2-j)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-j+1} \\
&= {}_{a}^{k}T^{1,\alpha}\left[{}_{a}^{k}J^{\beta-n+1,\alpha}\left({}_{a}^{k}J^{n-\beta,\alpha}f(x)\right)-\sum_{j=1}^{n}\frac{{}_{a}^{k}T^{n-j,\alpha}\left({}_{a}^{k}J^{n-\beta,\alpha}f(a)\right)}{\Gamma(\beta+2-j)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-j+1}\right] \\
&= {}_{a}^{k}T^{1,\alpha}\left[\left({}_{a}^{k}J^{1,\alpha}f(x)\right)-\sum_{j=1}^{n}\frac{{}_{a}^{k}T^{n-j,\alpha}\left({}_{a}^{k}J^{n-\beta,\alpha}f(a)\right)}{\Gamma(\beta+2-j)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-j+1}\right] \\
&= f(x)-\sum_{j=1}^{n}\frac{{}_{a}^{k}D^{\beta-j,\alpha}f(a)}{\Gamma(\beta+1-j)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-j}.
\end{aligned}
$$

Proof is done. The second formula can be demonstrated the same way. □

4. $k-$CONFORMABLE FRACTIONAL DERIVATIVES IN CAPUTO SETTING

At this stage, we will give some definitions concerned with the theorem and we will demonstrate some properties of the $k-$conformable derivative on Caputo setting.

Definition 4.1. Let $\alpha > 0$, $Re(\beta) \geq 0$ and $n = [Re(\beta)] + 1$. If we take $f \in C^{n}_{\alpha,a}\left(f \in C^{n}_{\alpha,b}\right)$,

(4.1)

$$\left({}_{a}^{k,C}D^{\beta,\alpha}f(x)\right) = {}_{a}^{k}D^{\beta,\alpha}\left[f(t)-\sum_{m=0}^{n-1}\frac{{}_{a}^{k}T^{m,\alpha}f(a)}{m!}\left(\frac{\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right)^{m}\right](x)$$

and

(4.2)

$$\left({}_{a}^{k,C}D^{\beta,\alpha}f(x)\right) = {}^{k}D_{b}^{\beta,\alpha}\left[f(t)-\sum_{m=0}^{n-1}\frac{(-1)^{m}.{}^{k}T_{b}^{m,\alpha}f(b)}{m!}\left(\frac{\left(b^{k+1}-t^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right)^{m}\right](x).$$

We acquire the left and right Caputo $k-$conformable fractional derivatives, respectively.

Theorem 4.2. *Let $Re(\beta) \geq 0$ and $n = [Re(\beta)]+1$. If we take $f \in C^{n}_{\alpha,a}\left(f \in C^{n}_{\alpha,b}\right)$,*

$$ {}_{a}^{k,C}D^{\beta,\alpha}f(x) = {}_{a}^{k}J^{n-\beta,\alpha}\left({}_{a}^{k}T^{n,\alpha}\right) \tag{4.3}$$

and

$$ {}^{k,C}D_{b}^{\beta,\alpha}f(x) = {}^{k}J_{b}^{n-\beta,\alpha}\left({}^{k}T_{b}^{n,\alpha}\right). \tag{4.4}$$

We acquire the left and right Caputo $k-$conformable fractional derivatives in Caputo setting, respectively.

Proof. By considering $Definition$ 5, we have,

(4.5)
$$\begin{aligned}&\left({}_{a}^{k,C}D^{\beta,\alpha}f\left(x\right)\right)\\&={}_{a}^{k}D^{\beta,\alpha}\left[f\left(t\right)-\sum_{m=0}^{n-1}\frac{{}_{a}^{k}T^{m,\alpha}f(a)}{m!}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m}\right](x)\\&={}_{a}^{k}D^{\beta,\alpha}f\left(x\right)-\sum_{m=0}^{n-1}\frac{{}_{a}^{k}T^{m,\alpha}f(a)}{m!}\frac{{}_{a}^{k}T^{n,\alpha}}{\Gamma(n-\beta)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta+m}\frac{\Gamma(n-\beta)\Gamma(m+1)}{\Gamma(n-\beta+m+1)}\\&={}_{a}^{k}D^{\beta,\alpha}f\left(x\right)-\sum_{m=0}^{n-1}\frac{{}_{a}^{k}T^{m,\alpha}f(a)}{\Gamma(m-\beta+1)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m-\beta}.\end{aligned}$$

Proof is done. □

Lemma 4.3. *Let $\alpha>0$, $Re\left(\beta\right)\geq 0$, $n=\left[Re\left(\beta\right)\right]+1$ and $\mathrm{Re}\left(\beta\right)\notin\mathbb{N}$. If $f\in C_{\alpha,a}^{n}\left[a,b\right]\left(f\in C_{\alpha,b}^{n}\left[a,b\right]\right)$, we have,*

$$\left.\begin{aligned}&{}_{a}^{k}J^{\beta-s,\alpha}f\left(a\right)=0,\\&{}_{a}^{k}J^{\beta-s,\alpha}f\left(b\right)=0\end{aligned}\right\}\ for\ s=0,1,...,n-1. \tag{4.6}$$

Proof. We obtain,

(4.7)
$$\begin{aligned}{}_{a}^{k}J^{\beta-s,\alpha}f\left(x\right)&={}_{a}^{k}D^{s,\alpha}\left({}_{a}^{k}J^{\beta,\alpha}f\left(x\right)\right)\\&=\frac{1}{\Gamma(\beta-s)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-s-1}\frac{f(t)t^{k}dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}.\end{aligned}$$

In here, we can state via Hölder's inequality,

(4.8)
$$\begin{aligned}&\left|{}_{a}^{k}J^{\beta-s,\alpha}f\left(x\right)\right|\\&\leq\frac{1}{\Gamma(\beta-s)}\left(\int_{a}^{x}\left|f\left(t\right)\right|^{p}t^{k}\right)^{\frac{1}{p}}\left(\int_{a}^{x}\left(\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-s-1}\frac{f(t)t^{k}dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}\right)^{q}\right)^{\frac{1}{q}}\\&\leq\frac{\left\|f\right\|_{X_{c}}}{(re(\beta)-s)\Gamma(\beta-s)}\left(\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right)^{(re(\beta)-s)}.\end{aligned}$$

For $x=a$, we can say that

$${}_{a}^{k}J^{\beta-s,\alpha}f\left(a\right)=0. \tag{4.9}$$

Proof is done. □

Lemma 4.4. *Let $\alpha>0$, $Re\left(\beta\right)\geq 0$ and $n=\left[Re\left(\beta\right)\right]+1$. If we take ${}_{a}^{k}T^{n,\alpha}\in C\left[a,b\right]\left({}^{k}T_{b}^{n,\alpha}\in C_{\alpha,b}^{n}\right)$,we obtain,*

$$\begin{aligned}&{}_{a}^{k,C}D^{\beta,\alpha}f\left(a\right)=0,\\&{}^{k,C}D_{b}^{\beta,\alpha}f\left(b\right)=0.\end{aligned} \tag{4.10}$$

Proof. It is clearly seen that

$$\left|{}_{a}^{k,C}D^{\beta,\alpha}f\left(x\right)\right|\leq\frac{\left\|{}_{a}^{k}T^{n,\alpha}\right\|_{X_{c}}}{(n-re(\beta))\Gamma(n-\beta)}\left(\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right)^{(n-re(\beta))} \tag{4.11}$$

and

$$\left|{}^{k,C}D_b^{\beta,\alpha}f(x)\right| \leq \frac{\left\|{}^{k}T_b^{n,\alpha}\right\|_{X_c}}{(n-re(\beta))\Gamma(n-\beta)}\left(\frac{\left(b^{k+1}-x^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right)^{(n-re(\beta))}. \tag{4.12}$$

Proof is done. □

Theorem 4.5. *Let* $Re(\beta)\geq 0$, $n=[Re(\beta)]+1$ *and* $f\in C^n_{\alpha,a}[a,b]$ $\left(f\in C^n_{\alpha,b}[a,b]\right)$,

(1) If we get $Re(\beta)\notin\mathbb{N}$ *or* $\beta\in\mathbb{N}$*, then,*

$$\begin{aligned}{}^{k,C}_{a}D^{\beta,\alpha}\left({}^{k}_{a}J^{\beta,\alpha}f(x)\right)&=f(x),\\ {}^{k,C}D_b^{\beta,\alpha}\left({}^{k}J_b^{\beta,\alpha}f(x)\right)&=f(x).\end{aligned} \tag{4.13}$$

(2) If we take $Re(\beta)\neq 0$ *or* $Re(\alpha)\in\mathbb{N}$*, then,*

$$\begin{aligned}{}^{k,C}_{a}D^{\beta,\alpha}\left({}^{k}_{a}J^{\beta,\alpha}f(x)\right)&=f(x)-\frac{{}^{k}_{a}J^{\beta-n+1,\alpha}f(a)}{\Gamma(n-\beta)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta},\\ {}^{k,C}D_b^{\beta,\alpha}\left({}^{k}J_b^{\beta,\alpha}f(x)\right)&=f(x)-\frac{{}^{k}J_b^{\beta-n+1,\alpha}f(a)}{\Gamma(n-\beta)}\left[\frac{\left(b^{k+1}-x^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta}.\end{aligned} \tag{4.14}$$

Proof. By using *Definition* 6, we have,

$$\begin{aligned}&{}^{k,C}_{a}D^{\beta,\alpha}\left({}^{k}_{a}J^{\beta,\alpha}f(x)\right)\\ &=f(x)-\frac{{}^{k}_{a}T^{n,\alpha}}{\Gamma(n-\beta)}\int_a^x\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta-1}\\ &\times\left(\sum_{m=0}^{n-1}\frac{{}^{k}_{a}J^{n+m-\beta,\alpha}f(a)}{m!}\left(\frac{\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right)^{m}\right)\frac{t^k.dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}\\ &=f(x)-\frac{{}^{h}_{a}T^{n,\alpha}}{\Gamma(n-\beta)}\left(\sum_{m=0}^{n-1}\frac{{}^{h}_{a}J^{n+m-\beta,\alpha}f(a)}{m!}\right)\\ &\times\int_a^x\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta-1}\left[\frac{\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m}\frac{t^k.dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}.\end{aligned} \tag{4.15}$$

In here, by using the following the change of variable,

$$\left(t^{k+1}-a^{k+1}\right)^{\alpha}=z\left(x^{k+1}-a^{k+1}\right)^{\alpha}, \tag{4.16}$$

we can write,

$${}^{k,C}_{a}D^{\beta,\alpha}\left({}^{k}_{a}J^{\beta,\alpha}f(x)\right)=f(x)-\sum_{m=0}^{n-1}\frac{{}^{k}_{a}J^{m-\beta,\alpha}f(a)}{\Gamma(m-\beta+1)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m-\beta}. \tag{4.17}$$

In here, we have ${}^{k}_{a}J^{\beta-s,\alpha}f(a)=0$ and ${}^{k}J_b^{\beta-s,\alpha}f(b)=0$ for $Re(\beta)\notin\mathbb{N}$ by using *Lemma* 4. The case $\beta\in\mathbb{N}$ is unimportant. Additionaly, if $Re(\beta)\in\mathbb{N}$, we state ${}^{k}_{a}J^{\beta-s,\alpha}f(a)=0$ and ${}^{k}J_b^{\beta-s,\alpha}f(b)=0$ for $s=0,1,...,n-2$ by using *Lemma* 4. □

Theorem 4.6. *Let $\beta \in \mathbb{C}$ and $f \in C^n_{\alpha,a}[a,b]$ $\left(f \in C^n_{\alpha,b}[a,b]\right)$. We have,*

$$
\begin{aligned}
{}^k_a J^{\beta,\alpha}\left({}^{k,C}_a D^{\beta,\alpha} f(x)\right) &= f(x) - \sum_{m=0}^{n-1} \frac{{}^h_a T^{m,\alpha} f(a)}{\Gamma(m+1)} \left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^m, \\
{}^k J^{\beta,\alpha}_b \left({}^{k,C} D^{\beta,\alpha}_b f(x)\right) &= f(x) - \sum_{m=0}^{n-1} \frac{{}^h T^{m,\alpha}_b f(a)}{\Gamma(m+1)} \left[\frac{\left(b^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^m.
\end{aligned} \tag{4.18}
$$

Proof. In here, we can write the following as,

$$
\begin{aligned}
{}^k_a J^{\beta,\alpha}\left({}^{k,C}_a D^{\beta,\alpha} f(x)\right) &= {}^k_a J^{\beta,\alpha}\left({}^k_a J^{n-\beta,\alpha}\left({}^k_a T^{n,\alpha} f(x)\right)\right) \\
&= {}^k_a J^{n,\alpha}\left({}^k_a T^{n,\alpha} f(x)\right) \\
&= f(x) - \frac{{}^k_a D^{\beta-j,\alpha} f(a)}{\Gamma(\beta-j+1)} \left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\beta-j} \\
&= f(x) - \frac{{}^k_a D^{m,\alpha} f(a)}{\Gamma(m+1)} \left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^m.
\end{aligned} \tag{4.19}
$$

Proof is done. □

Theorem 4.7. *Let $f \in C^{p+r}_{\alpha,a}[a,b]$ $\left(f \in C^{p+r}_{\alpha,b}[a,b]\right)$, $Re(\beta) \geq 0$, $Re(\mu) \geq 0$, $r-1 < [Re(\beta)] \leq r$ and $p-1 < [Re(\beta)] \leq p$. Then we get,*

$$
\begin{aligned}
{}^{k,C}_a D^{\beta,\alpha}\left({}^{k,C}_a D^{\mu,\alpha}(f(x))\right) &= {}^{k,C}_a D^{\beta+\mu,\alpha} f(x), \\
{}^{k,C} D^{\beta,\alpha}_b \left({}^{k,C} D^{\mu,\alpha}_b (f(x))\right) &= {}^{k,C} D^{\beta+\mu,\alpha}_b f(x).
\end{aligned} \tag{4.20}
$$

Proof. It is clear that the proof can complete by using $Theorem$ 1, $Theorem$ 4, $Theorem$ 6 and $Lemma$ 5. □

5. FRACTIONAL INTEGRALS AND DERIVATIVES CLASS

1. By considering $k = 0$ in $Definition$ 2,

$$ {}^k_a J^{\beta,\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left[\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right]^{\beta-1} \frac{f(t)dt}{(t-a)^{1-\alpha}}. $$

We obtain the left fractional conformable integrals in [1].

2. By considering $k = 0$ and $\alpha = 1$ in $Definition$ 2,

$$ {}^k_a J^{\beta,\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t)\, dt. $$

We obtain the left Riemann-Lioville fractional integrals.

3. By considering $k = 0$, $\alpha = 1$ and $a = -\infty$ in $Definition$ 2,

$$ {}^k_a J^{\beta,\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^x (x-t)^{\beta-1} f(t)\, dt. $$

We obtain the left Liouville fractional integrals.

4. By considering $k = 0$, $a = 0$ and $\alpha = 1$ in $Definition$ 2,

$$ {}^k_a J^{\beta,\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} f(t)\, dt. $$

We obtain the left Riemann fractional integrals.

5. By considering $\alpha = 1$, $k = 0$ and $g(x) = E^\gamma_{\alpha,\beta}\left(\omega (x-t)^\beta\right) f(x)$ in $Definition$ 2,

$$ \Gamma(\beta) {}^k_a J^{\beta,\alpha} g(x) = \int_a^x (x-t)^{\beta-1} E^\gamma_{\alpha,\beta}\left(\omega (x-t)^\beta\right) f(t)\, dt. $$

We obtain the left Prabhakar fractional integrals.

6. By considering $k=0$, $\alpha=1$ and $a=c$ in $Definition$ 2,

$$ {}_{c}^{k}J^{\beta,\alpha}f(x) = \frac{1}{\Gamma(\beta)}\int_{c}^{x}(x-t)^{\beta-1}f(t)\,dt. $$

We obtain the left Chen fractional integrals.

7. By considering $k=0$ in $Definition$ 3, we have the left fractional conformable derivatives in [1],

$$ \begin{aligned} {}_{a}^{k}D^{\beta,\alpha}f(x) &= {}_{a}^{k}T^{n,\alpha}\left({}_{a}^{k}J^{n-\beta,\alpha}\right)f(x) \\ &= \frac{{}_{a}^{k}T^{n,\alpha}}{\Gamma(n-\beta)}\int_{a}^{x}\left[\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right]^{n-\beta-1}\frac{f(t)dt}{(t-a)^{1-\alpha}}. \end{aligned} $$

8. By considering $k=0$, $\alpha=1$ in $Definition$ 3, we have Riemann-Liouville fractional derivative,

$$ \begin{aligned} {}_{0}^{k}D^{\beta,\alpha}f(x) &= {}_{0}^{k}T^{n}\left({}_{0}^{k}J^{n-\beta,\alpha}\right)f(x) \\ &= \frac{{}_{0}^{k}T^{n}}{\Gamma(n-\beta)}\int_{0}^{x}[x-t]^{n-\beta-1}f(t)\,dt. \end{aligned} $$

9. Taking $k=0$, $\alpha=1$ in $Definition$ 3, we have the left Caputo fractional derivative,

$$ \begin{aligned} {}_{a}^{k}D^{\beta,\alpha}f(x) &= \left({}_{a}^{k}J^{n-\beta,\alpha}\left({}_{a}^{k}T^{n}\right)\right)f(x) \\ &= \frac{1}{\Gamma(n-\beta)}\int_{a}^{x}[x-t]^{n-\beta-1}\left({}_{a}^{k}T^{n}\right)f(t)\,dt. \end{aligned} $$

10. Taking $k=0$, $\alpha=1$ and $a=0$ in $Definition$ 3, we have the Riemann fractional derivative,

$$ {}_{0}^{k}D^{\beta,\alpha}f(x) = \left(\frac{d}{dx}\right)^{n}\cdot{}_{0}^{k}J^{n-\beta}f(x). $$

12. Taking $k=0$, $\alpha=1$ and $a=c$ in $Definition$ 3, we have the Chen fractional derivative,

$$ {}_{c}^{k}D^{\beta,\alpha} = \left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\beta)}\int_{c}^{x}(x-t)^{n-\beta-1}f(t)\,dt. $$

13. Taking $k=0$, $a=0$, $\alpha=1$ and $g(x)=f(x)-f(0)$ in $Definition$ 3, we have Jumarie fractional derivative,

$$ {}_{0}^{k}D^{\beta,\alpha} = \left(\frac{d}{dx}\right)^{n}\cdot{}_{0}^{k}J^{n-\beta,\alpha}\left(f(x)-f(0)\right). $$

14. Taking $k=0$, $\alpha=1$,and $g(x)=E_{\rho,n-\beta}^{-\gamma}\left[\omega(x-t)^{\rho}\right]f(x)$ in $Definition$ 3, we have the Prabhakar fractional derivative,

$$ {}_{a}^{k}D^{\beta,\alpha}g(x) = \left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\beta)}\int_{a}^{x}(x-t)^{n-\beta-1}E_{\rho,n-\beta}^{-\gamma}\left[\omega(x-t)^{\rho}\right]f(t)\,dt. $$

15. Taking $k=0$, $\alpha=1$, $a=-\infty$ in $Definition$ 3, we have the Liouville fractional derivative,

$$ {}_{-\infty}^{k}D^{\beta,\alpha}f(x) = \left(-\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\beta)}\int_{-\infty}^{x}(x-t)^{n-\beta-1}f(t)\,dt. $$

16. Taking $k=0$, $\alpha=1$, $a=-\infty$ in $Definition$ 3, we have the Liouville-Caputo fractional derivative,

$$ {}_{-\infty}^{k}D^{\beta,\alpha}f(x) = \frac{1}{\Gamma(n-\beta)}\int_{-\infty}^{x}(x-t)^{n-\beta-1}\left(-\frac{d}{dx}\right)^{n}f(t)\,dt. $$

17. Taking $k=0$, $\alpha=1$, $b=\infty$ in $Definition$ 3, we have the Weyl fractional derivative,

$$ {}^{k}D_{\infty}^{\beta,\alpha}f(x) = (-1)^{n}\left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\beta)}\int_{x}^{\infty}(x-t)^{n-\beta-1}f(t)\,dt. $$

6. Conclusion

In this research, we defined the left and right $k-$ conformable fractional integral and derivatives, respectively, we demonstrated important consequences and some basic properties for these operators. Furthermore, we acquired the $k-$conformable fractional derivatives on the Caputo setting. In conclusion, we expressed the classical results for the generalized conformable derivatives and integrals.

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USING DECISION MAKING METHODS TO ASSESS STUDENT ACHIEVEMENT

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Abstract. In order to determine student success, considering other factors that affect success along with exam success is a very important step in terms of education. In this study, the decision-making method that evaluates student achievements with guidance counselors was used. Student achievements were estimated by using a combination of intuitionistic fuzzy sets and decision-making methods.

1. INTRODUCTION

The fuzzy logic was defined by Zadeh in 1965 [1]. In this process, while researches and applications on fuzzy logic continue, the intuitionistic fuzzy set theory, which is the generalization of fuzzy logic in which sensitivity is also activated in cases of uncertainty, was put forward by Atanassov in 1983 [3]. Multi-criteria decision making is interested in structuring and solving decision and planning problems involving multiple criteria. The goal is to support decision makers who face such problems. Typically, there is no unique optimal solution for such problems, and it is necessary to use the decision maker's preferences to distinguish between solutions. PROMETHEE was firstly developed by Brans in 1982 and continued to be developed thereafter [4, 5, 6]. The PROMETHEE method is in the outranking method class, one of the multi-criteria decision-making methods. Many researchers have developed applications using PROMETHEE methods and intuitionistic fuzzy sets [7, 8, 9, 10, 16, 17, 39, 26, 25]. The PROMETHEE method provides very effective benefits by combining intuitionistic fuzzy sets and decision-making methods. This system, which allows decision makers to make both objective and most accurate decisions, was used for educational application in our article. Student success and guidance can be made more accurately, thanks to a system that examines both students' course success and psychological characteristics and puts them all into action at the same time. In addition, thanks to this system, teachers who guide students can make more accurate determinations and be more successful in guiding. There are many studies in which decision-making methods are used together with

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intuitionistic fuzzy sets, and the authors have achieved effective results in their studies [12, 14, 15, 21, 22, 23, 24, 27, 28, 29, 30, 31, 32, 33, 37].

2. PRELIMINARIES

Definition 1. *([3, 2])Let $X \neq \emptyset$. An intuitionistic fuzzy set A in X;*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle |\ x \in X\},$$

$$\mu_A(x), \nu_A(x), \pi_A(x) : X \to [0,1]$$

defined membership, nonmembership and hesitation degree of the element $x \in X$ respectively.

$$\mu_A(x) + \nu_A(x) + \pi_A(x) = 1.$$

Intuitionistic fuzzy value (IFV) defined by Xu ([36]). Intuitionistic fuzzy value (IFV) is shown as follows: $\tilde{a} = (\mu_{\tilde{a}}, \nu_{\tilde{a}}, \pi_{\tilde{a}})$, where $\mu_{\tilde{a}}, \nu_{\tilde{a}}, \pi_{\tilde{a}} \in [0,1]$
For each IFS $\tilde{A}$;

$$\pi_{\tilde{A}} = 1 - \mu_{\tilde{A}} - \nu_{\tilde{A}} \tag{2.1}$$

For IFVs $\tilde{a} = (\mu_{\tilde{a}}, \nu_{\tilde{a}})$ and $\tilde{b} = (\mu_{\tilde{b}}, \nu_{\tilde{b}})$ the following operations have been carried out([36, 35]):

$$(1) \quad \tilde{a} \oplus \tilde{b} = (\mu_{\tilde{a}} + \mu_{\tilde{b}} - \mu_{\tilde{a}}\mu_{\tilde{b}}, \nu_{\tilde{a}}\nu_{\tilde{b}}) \tag{2.2}$$

$$(2) \quad \tilde{a} \otimes \tilde{b} = (\mu_{\tilde{a}}\mu_{\tilde{b}}, \nu_{\tilde{a}} + \nu_{\tilde{b}} - \nu_{\tilde{a}}\nu_{\tilde{b}}) \tag{2.3}$$

$$(3) \quad \oplus_{j=1}^{m}\tilde{a}_j = (1 - \prod_{j=1}^{m}(1-\mu_j), \prod_{j=1}^{m}\nu_j) \tag{2.4}$$

$$(4) \quad \otimes_{j=1}^{m}\tilde{a}_j = (\prod_{j=1}^{m}\mu_j, \prod_{j=1}^{m}(1-\nu_j)) \tag{2.5}$$

This function is used to rank IFVs:

$$\rho(\alpha) = 0.5(1 + \pi_\alpha)(1 - \mu_\alpha)) \tag{2.6}$$

As the $\rho(\alpha)$ value decreases, the preferred value α increases.

3. The Intuitionistic Fuzzy PROMETHEE Method

The criteria's weights could be depicted as IFVs: $\tilde{w}_j$ where $\mu_{\tilde{w}_j} \in [0,1], \nu_{\tilde{w}_j} \in [0,1], \mu_{\tilde{w}_j} + \nu_{\tilde{w}_j} \leq 1, j = 1,2,...,m$. According to the weights, $\mu_{\tilde{\omega_j}}$ and $\nu_{\tilde{\omega_j}}$ demonstrate the membership and non-membership degrees of the alternative x_i respectively. Some methods can help decision makers in determining intuitionistic fuzzy weights ([18, 19, 34, 38, 11, 13]). In this study, linguistic terms were used to make the evaluation more accurate. Also, V shape criterion type has been used:

$$P(d) = \begin{cases} 0, & d \leq q \\ \frac{d-q}{p-q}, & q < d \leq p \\ 1, & d > p \end{cases} \tag{3.1}$$

Parameter thresholds q and p are indicated as indifference and strict preference, respectively. Evaluate the alternatives $x_i(i = 1,2,...,n)$ with respect to the criteria $c_j(j = 1,2,...,m)$ and determine the deviations based on pairwise comparisons:

$$d_j(x,y) = c_j(x) - c_j(y) \tag{3.2}$$

where $d_j(x,y)$ shows the distinction between the alternatives' the assessments x and y on the criterion c_j.

Definition 2. *([37]) An intuitionistic fuzzy preference relation R on the set $X = x_1, x_2, ..., x_n$ is represented by a matrix $R = (r_{ik})_{n\times n}$, where $r_{ik} = \langle (x_i, x_k), \mu(x_i, x_k), \nu(x_i, x_k)\rangle$ for all $i, k = 1, 2, ..., n$. For convenience, we let $r_{ik} = (\mu_{ik}, \nu_{ik})$ where μ_{ik} denotes the degree to which the object x_i is preferred to the object x_k, ν_{ik} indicates the degree to which the object x_i is not preferred to the object x_k, and $\pi(x_i, x_k) = 1 - \mu(x_i, x_k) - \nu(x_i, x_k)$ is interpreted as an indeterminacy degree or a hesitancy degree, with the condition:*

$$\mu_{ik},\ \nu_{ik} \in [0,1],\ \mu_{ik} + \nu_{ik} \leq 1,\ \mu_{ik} = \nu_{ki},\ \mu_{ki} = \nu_{ik},$$

$$\mu_{ii} = \nu_{ii} = 0.5,\ \pi_{ik} = 1 - \mu_{ik} - \nu_{ik},$$

$$for\ all\ i, k = 1, 2, ..., n \tag{3.3}$$

The preferences μ_{ik} between the alternatives x_i and x_k according to the criterion c_j could be calculated by Equations (3.2) and (3.1), and then the preference matrix according to the criterion c_j is obtained as follows ([20]):

$$U^{(j)} = (\mu_{ik}^{(j)})_{n\times n} = \begin{bmatrix} - & \mu_{12}^{(j)} & \dots & \mu_{1n}^{(j)} \\ \mu_{21}^{(j)} & - & \dots & \mu_{2n}^{(j)} \\ \vdots & \vdots & - & \vdots \\ \mu_{n1}^{(j)} & \mu_{n2}^{(j)} & \dots & - \end{bmatrix} \tag{3.4}$$

Matrix of the intuitionistic fuzzy preference relation is obtained:

$$R^{(j)} = (r_{ik}^{(j)})_{n\times n} = \begin{bmatrix} - & (\mu_{12}^{(j)}, \nu_{12}^{(j)}) & \dots & (\mu_{1n}^{(j)}, \nu_{1n}^{(j)}) \\ (\mu_{21}^{(j)}, \nu_{21}^{(j)}) & - & \dots & (\mu_{2n}^{(j)}, \nu_{2n}^{(j)}) \\ \vdots & \vdots & - & \vdots \\ (\mu_{n1}^{(j)}, \nu_{n1}^{(j)}) & (\mu_{n2}^{(j)}, \nu_{n2}^{(j)}) & \dots & - \end{bmatrix} \tag{3.5}$$

The IFWA operator is used in this paper. The all intuitionistic fuzzy preference index of the alternative x_i to x_k on all criteria can be derived as:

$$r(x_i, x_k) = r_{ik} = \bigoplus_{j=1}^{m} \left(\tilde{w}_j \bigotimes r_{ik}^{(j)}\right) \tag{3.6}$$

where $r(x_i, x_k) = r_{ik}$ shows the degree to which the alternative x_i is preferred to the alternative x_k all criteria. Also, r_{ik} is an IFV. $\tilde{w}_j = (\mu_{\tilde{w}j}, \nu_{\tilde{w}j})$, then according to Equation (2.2), (2.3):

$$\tilde{w}_j \bigotimes r_{ik}^{(j)} = \left(\mu_{ik}^{(j)}\mu_{\tilde{w}j}, \nu_{ik}^{(j)} + \nu_{\tilde{w}j} - \nu_{ik}^{(j)}\nu_{\tilde{w}j}\right) \tag{3.7}$$

If Equations (2.4),(3.6) and (3.7) are combined;

$$\begin{aligned} r(x_i, x_k) &= \bigoplus_{j=1}^{m} \Big(\tilde{w}_j \bigotimes r_{ik}^{(j)}\Big) \\ &= \Big(1 - \prod_{j=1}^{m} (1 - \mu_{ik}^{(j)} \mu_{\tilde{w}j}), \\ &\quad \prod_{j=1}^{m} (\nu_{ik}^{(j)} + \nu_{\tilde{w}j} - \nu_{ik}^{(j)} \nu_{\tilde{w}j})\Big) \end{aligned} \tag{3.8}$$

Overall intuitionistic fuzzy preference relationship is obtained:

$$R = (r_{ik})_{n\times n} = \begin{bmatrix} - & (\mu_{12}, \nu_{12}) & \dots & (\mu_{1n}, \nu_{1n}) \\ (\mu_{21}, \nu_{21}) & - & \dots & (\mu_{2n}, \nu_{2n}) \\ \vdots & \vdots & - & \vdots \\ (\mu_{n1}, \nu_{n1}) & (\mu_{n2}, \nu_{n2}) & \dots & - \end{bmatrix} \tag{3.9}$$

Every alternative is compared to option $(n-1)$. As a result of intuitionistic fuzzy positive and negative outranking flow are achieved:

(1) The intuitionistic fuzzy positive outranking flow:

$$\tilde{\varphi}^{+}(x_i) = \frac{1}{n-1} \bigoplus_{k=1,k\neq i}^{n} r(x_i, x_k) = \frac{1}{n-1} \bigoplus_{k=1,k\neq i}^{n} r_{ik} \tag{3.10}$$

(2) The intuitionistic fuzzy negative outranking flow:

$$\tilde{\varphi}^{-}(x_i) = \frac{1}{n-1} \bigoplus_{k=1,k\neq i}^{n} r(x_k, x_i) = \frac{1}{n-1} \bigoplus_{k=1,k\neq i}^{n} r_{ki} \tag{3.11}$$

4. Application in Education

In this study, it is aimed to present a unique system that combines intuitionistic fuzzy sets and decision-making methods in the field of education. Both the exam success and psychological characteristics of the students were taken into consideration and evaluated. Along with the exam results, their psychological characteristics were evaluated by guidance teachers. For total 9 criteria; in addition to the success of the students in Turkish, Mathematics, Science and Social courses, their characteristics such as academic motivation, text anxiety, family relationship, sociability and technology approach were rated by their guidance counselors. The values in Table 1 were determined according to the exam success of the students and the evaluation results of the guidance teachers. Guidance teachers evaluated the students with questions they prepared themselves in order to measure the social, psychological and cultural aspects of the students. The values in the table were obtained accordingly.

The alternatives and criteria that form the basis of our algorithm are as follows: $A = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}\}$ being set of alternatives, each alternative represents a student. $K = \{K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8, K_9\}$ being set of criteria. The criteria are represented as follows: Turkish, Mathematics, Science, Social courses, Academic motivation, Text anxiety, Family relationship, Sociability, Technology approach.

The value of each criterion for each alternative is calculated as follows in Table 1:

	$K1$	$K2$	$K3$	$K4$	$K5$	$K6$	$K7$	$K8$	$K9$
$A1$	24	11	25	12	10	6	5	7	7
$A2$	25	12	11	1	11	11	5	8	5
$A3$	21	6, 5	19	2	6	5	5, 5	10	6
$A4$	18, 5	13	12	10	2	6	7, 5	6	6
$A5$	20	7	15	2	5	9, 33	8, 5	5	8
$A6$	18	0	9	9	2	7, 66	7	4	7
$A7$	25, 5	1	8	5	4	7	5	6	8
$A8$	19	4	15	11	7	8	5	7	4
$A9$	17, 5	10	2	10	10	9, 33	9	8	9
$A10$	22	6	1	6	5	7	8	9	7
$A11$	16, 5	8, 5	3	3	11	8, 33	8	10	9
$A12$	9	5	1	2	12	8, 33	7	9	9
$A13$	8	6	2	4	5	7	6	9	8
$A14$	8	1	5	5	7	5	5	8	5
$A15$	28	1	19	3	11	4	4	8	4

TABLE 1. Values of Alternatives by Criteria

In this study, criterion weights were calculated in linguistic terms. The weights of the criteria are as follows: Criterion 1 and Criterion 2 are very important, Criterion 3, Criterion 4, Criterion 5 are important, Criterion 6 and Criterion 7 are medium, Criterion 8 and Criterion 9 are important.
Net outranking flow values are specified as follows:

$\rho(\tilde{\varphi}(x_1)) =$	-0,00060639
$\rho(\tilde{\varphi}(x_2)) =$	-0,00002504
$\rho(\tilde{\varphi}(x_3)) =$	0,00000049
$\rho(\tilde{\varphi}(x_4)) =$	0,00000016
$\rho(\tilde{\varphi}(x_5)) =$	0,00000140
$\rho(\tilde{\varphi}(x_6)) =$	0,00013783
$\rho(\tilde{\varphi}(x_7)) =$	0,00008569
$\rho(\tilde{\varphi}(x_8)) =$	0,00000230
$\rho(\tilde{\varphi}(x_9)) =$	-0,00045314
$\rho(\tilde{\varphi}(x_{10})) =$	0,00000696
$\rho(\tilde{\varphi}(x_{11})) =$	-0,00009627
$\rho(\tilde{\varphi}(x_{12})) =$	-0,00000030
$\rho(\tilde{\varphi}(x_{13})) =$	0,00010351
$\rho(\tilde{\varphi}(x_{14})) =$	0,00155834
$\rho(\tilde{\varphi}(x_{15})) =$	0,00000004

TABLE 2. Intuitionistic Fuzzy Net Outranking Flow Values

When the students evaluated with the system created in our study are ranked according to their net flow values, the most successful student is A_1 and the least successful student is A_{14}. Students' achievements can be based on intuitionistic fuzzy net flow values. The lower the net flow value, the higher the student achievement.

5. CONCLUSION

In this study, an application of the intuitionistic fuzzy PROMETHEE method was developed to evaluate student achievements in education. A system has been created to evaluate student success in line with the students' course success and the opinions of their guidance counselors.

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PLANE KINEMATICS IN HOMOTHETIC MULTIPLICATIVE CALCULUS

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Abstract. In this study, pole points of motion, pole trajectories, velocities, accelerations and relations between velocities and accelerations are obtained. In addition we gave some new theorems

1. Introduction

Grossman and Katz introduced multiplicative calculus which is also called Non-Newtonian calculus. They defined derivative and integral in the multiplicative sense. We refer to Grossman and Katz [11], Stanley [18], Campbell [9], Grossman [12, 13], Jane Grossman [14, 15] for different kinds of Non-Newtonian calculus and its practices. Bashirov et al [3] given the entire mathematical definition of multiplicative calculus. An extension of multiplicative calculus to functions of complex variables can be found in [1, 2, 19, 20, 21]. Çakmak and Başar [8], characterized matrix transformations in sequence spaces based on multiplicative calculus. K. Boruah and B. Hazarika [5], have given the real number line and perpendicular axes system in multiplicative calculus. Gurefe [16], defined vector spaces, inner products and operations on matrices. K. Boruah and B. Hazarika [22] have given some conclusions about geometry. Selahattin Aslan et al. [23], gave geometric 3-space and multiplicative quaternions. Semra Kaya Nurkan et al. [24], gave vector properities of geometric calculus. Es [25], gave some basic concepts on the one-parameter motions witih multiplicative calculus.

2. BASIC CONCEPTS

Sentence $\mathbb{R}(G)$ can be defined as follows

$$\mathbb{R}(G) = \{\exp(p) = e^p : p \in \mathbb{R}\} \tag{2.1}$$

with the multiplicative addition

$$e^p \oplus e^r = e^{p+r} \tag{2.2}$$

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and the multiplicative multiplication

$$e^p \otimes e^r = e^{pr} \tag{2.3}$$

all $e^p, e^r \in \mathbb{R}(G)$. On the $\mathbb{R}(G)$ sentence, we can define addition $\oplus$ and multiplication $\otimes$, respectively (i.e.,$(\mathbb{R}(G), \oplus, \otimes)$), and it is a field with multiplicative zero $e^0 = 1$ and multiplicative identity $e^1 = e$. The connection between simple multiplicative operations and common arithmetic operations for each p, r elements of $\mathbb{R}(G)$ can be given as follows.

$$\begin{aligned}
p \oplus r &= p.r, \\
p \ominus r &= \frac{p}{r}, \\
p \otimes r &= p^{\ln r} = r^{\ln p}, \\
p \oslash r &= p^{\frac{1}{\ln r}},\ p \neq 1, \\
\sqrt{p}^G &= e^{(\ln p)^{\frac{1}{2}}}, \\
p^{-1_G} &= e^{\frac{1}{\log p}}, \\
\sqrt{p^{2_G}}^G &= |p|^G, \\
p^{2_G} &= p \otimes p = p^{\ln p}, \\
p \otimes e &= p,\ p \oplus 1 = p,
\end{aligned}$$

and thus we can write

$$\begin{gathered}
e^p \otimes e^r = e^{pr}, e^p \oplus e^r = e^{p+r}, \\
e^p \ominus e^r = e^{p-r}, e^p \oslash e^r = e^{\frac{p}{r}}, \\
\sqrt{e^p}^G = e^{\sqrt{p}}.
\end{gathered}$$

Positive and negative multiplicative real numbers can be defined as follows

$$\mathbb{R}^+(G) = \{m \in \mathbb{R}(G) : m > 1\}$$

and

$$\mathbb{R}^-(G) = \{m \in \mathbb{R}(G) : 0 < m < 1\},$$

respectively, [16, 20, 22].

The sentence $\mathbb{R}^2(G)$ is defined as follows

$$\mathbb{R}^2(G) = \{p^\circ = (e^{p_1}, e^{p_1}) : e^{p_1}, e^{p_1} \in \mathbb{R}(G)\} \subset \mathbb{R}^2$$

$$\begin{aligned}
p^\circ \oplus r^\circ &= (e^{p_1}, e^{p_2}) \oplus (e^{r_1}, e^{r_2}) \\
&= (e^{p_1} \oplus e^{r_1}, e^{p_2} \oplus e^{r_2}) \\
&= \left(e^{p_1+r_1}, e^{p_2+r_2}\right)
\end{aligned}$$

and the multiplicative scalar multiplication as

$$\begin{aligned}
e^c \otimes p^\circ &= e^c \otimes (e^{p_1}, e^{p_2}) \\
&= (e^c \otimes e^{p_1}, e^c \otimes e^{p_2}) \\
&= (e^{cp_1}, e^{cp_2}),
\end{aligned}$$

where $e^c \in \mathbb{R}(G),\ p^\circ,\ r^\circ \in \mathbb{R}^2(G)$.

Definition 1. *We can define multiplicative calculus absolute value as follows*

$$|p|^G = \begin{cases} p & , \quad p > 1, \\ 1 & , \quad p = 1, \\ p^{-1} & , \quad p < 1, \end{cases}$$

where $p \in \mathbb{R}(G)$ [20].

Definition 2. *The relationship between the multiplicative derivative and the classical derivative is as*

$$f^{*(n)}(x) = e^{(\ln f(x))^{(n)}}.$$

[1, 2, 3, 7, 10, 16].

Definition 3. *The relationship between trigonometry and multiplicative trigonometry is as* $\sin_g \theta = e^{\sin\theta}, \cos_g \theta = e^{\cos\theta}, \tan_g \theta = e^{\tan\theta} = \frac{\sin_g \theta}{\cos_g \theta}$ [4, 22, 23, 25].

Definition 4. *An* 2×2 *multiplicative matrix is defined by*

$$D = \begin{bmatrix} e^{d_{11}} & e^{d_{12}} \\ e^{d_{21}} & e^{d_{22}} \end{bmatrix}$$

where $e^{d_{11}}, e^{d_{12}}, e^{d_{21}}, e^{d_{22}} \in \mathbb{R}(G)$*. Let* D *and* G *be two multiplicative matrices and* $D \otimes G = E$ *be the multiplication of these matrices, where*

$$E = \begin{bmatrix} e^{d_{11}g_{11}+d_{12}g_{21}} & e^{d_{11}g_{12}+d_{12}g_{22}} \\ e^{d_{21}g_{11}+d_{22}g_{21}} & e^{d_{21}g_{12}+d_{22}g_{22}} \end{bmatrix}.$$

Definition 5. 2×2 *type identity matrix in multiplicative calculus is*

$$I = \begin{bmatrix} e & 1 \\ 1 & e \end{bmatrix}.$$

If matrix F *is a* 2×2 *type matrix and* $F^T \otimes F = F \otimes F^T = I$*, then* F *is called a multiplicative orthogonal matrix.*

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Definition 6. *The inner product of* $\mathbb{R}^2(G)$ *in multiplicative plane is*

$$\langle \alpha, \beta \rangle^G = e^{\alpha_1\beta_1+\alpha_2\beta_2}, \tag{3.1}$$

where $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2(G)$ [16, 23, 24, 25].

Definition 7. *The norm of a multiplicative vector* $\alpha = (\alpha_1, \alpha_2)$ *is*

$$\|\alpha\|^G = \sqrt{\langle \alpha, \alpha \rangle^G}^G = e^{\sqrt{\alpha_1^2+\alpha_2^2}} \tag{3.2}$$

[16, 23, 24, 25].

Definition 8. *The multiplicative unit circle* $S^1(G)$ *in* $\mathbb{R}^2(G)$ *can be defined as*

$$S^1(G) = \left\{ p^\circ = (e^{p_1}, e^{p_1}) \in \mathbb{R}^2(G) : \langle p^\circ, p^\circ \rangle^G = e \right\} \tag{3.3}$$
$$= (\cos_g \theta, \sin_g \theta) = \left(e^{\cos\theta}, e^{\sin\theta}\right).$$

Definition 9. *Let $u = (e^{u_1}, e^{u_2})$ and $v = (e^{v_1}, e^{v_2})$ be unit vectors in $\mathbb{R}^2(G)$. Then the equation*

$$\begin{bmatrix} e^{\cos\theta} & e^{-\sin\theta} \\ e^{\sin\theta} & e^{\cos\theta} \end{bmatrix} \otimes \begin{bmatrix} e^{u_1} \\ e^{u_2} \end{bmatrix} = \begin{bmatrix} e^{v_1} \\ e^{v_2} \end{bmatrix} \tag{3.4}$$

represents a rotation in $\mathbb{R}^2(G)$ of the multiplicative vector u by a multiplicative angle $\theta \in \mathbb{R}$ in positive direction around the origin $O = (1,1)$ of the Cartesian coordinate system of $\mathbb{R}^2(G)$. We will call this rotation as multiplicative planar rotation. After this rotation multiplicative vector u turns to the multiplicative vector v. Here $A(\theta) = \begin{bmatrix} e^{\cos\theta} & e^{-\sin\theta} \\ e^{\sin\theta} & e^{\cos\theta} \end{bmatrix}$ is a rotation matrix in multiplicative plane.

Definition 10. *In the multiplicative plane, a parameter homothetic multiplicative calculus motion is defined as*

$$\begin{bmatrix} Y \\ e \end{bmatrix} = \begin{bmatrix} h \otimes A & C \\ 1 & e \end{bmatrix} \otimes \begin{bmatrix} X \\ e \end{bmatrix} \tag{3.5}$$

where, $B = h \otimes A$, $A \in SO(2)_G$, A is a positive orthogonal matrix. Here $h = h(t)$, $A = A(t)$ and $C = C(t)$ are functions that can be differentiated with respect to the time parameter t to any order. Y, X and C are $2\times$ real matrices, and$Y$, X and $C \in \mathbb{R}^2_1(G)$. Equation 3.5 can be also given as

$$Y(t) = B(t) \otimes X(t) \oplus C(t) \tag{3.6}$$

$$Y = \begin{bmatrix} e^{v_1} \\ e^{v_2} \end{bmatrix}, X = \begin{bmatrix} e^{x_1} \\ e^{x_2} \end{bmatrix}, C = \begin{bmatrix} e^{a} \\ e^{b} \end{bmatrix},$$

where Y and X are the position vectors of the same point B, respectively, for the multiplicative fixed and multiplicative moving systems, and C is the multiplicative translation vector. By taking the derivatives with respect to t in 3.6, we get

$$Y^\star = B^* \otimes X \oplus B \otimes X^* \oplus C^* \tag{3.7}$$

Here $V_a = Y^\star, V_f = B^* \otimes X \oplus C^*$ and $V_r = B \otimes X^*$ are named absolute, sliding, and relative velocities of the multiplicative motion, respectively. These motions in multiplicative plane $R^2(G)$ are indicated by $B_1 = M/M'$ where M' and M are fixed and moving multiplicative planes, respectively. If the equation 3.7 is differentiated with respect to parameter t, we get

$$Y^{\star\star} = B^{**} \otimes X \oplus e^2 \otimes (B^* \otimes X^*) \oplus B \otimes X^{**} \oplus C^{**}, \tag{3.8}$$

$$b_a = b_r \oplus b_c \oplus b_f \tag{3.9}$$

where the velocities

$$b_a = Y^{\star\star}, b_f = B^{**} \otimes X \oplus C^{**}, b_r = B \otimes X^{**} \text{ and } b_c = e^2 \otimes (B^* \otimes X^*) \tag{3.10}$$

are named absolute acceleration, sliding acceleration, relative acceleration and Coriolis accelerations, respectively.

Definition 11. *The velocity vector V_r of the point X according to the moving plane M is called the relative velocity vector of X.*

Definition 12. *The velocity vector V_a of the point X according to the fixed plane M' is called the absolute velocity vector of X. Thus from equation 3.7 the relation between V_a, V_f, and V_r velocities is*

$$V_a = V_f \oplus V_r \tag{3.11}$$

If X is a fixed point in multiplicative moving plane M, then we have $V_a = V_f$, because $V_r = 1$. The equality 3.11 is said to be the velocity law of the motion $B_1 = M\,/M'$. Based on this information, we can state the following theorem.

Theorem 1. *In homothetic multiplicative motion, the absolute velocity vector is equal to the sum of the sliding velocity vector and the relative velocity vectors. So it is*

$$V_a = V_f \oplus V_r.$$

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Definition 13. *In a homothetic parameter motion in the Euclidean sense, the points (where the sliding velocity V_f at each moment t is multiplicative zero for a fixed point X in space) are moving and fixed points on the fixed plane. These points are the pole points of the motion.*

Theorem 2. *In a motion $B_1 = M\,/M'$ whose multiplicative angular velocity is not multiplicative zero, there is a single point that remains fixed in both multiplicative fixed plane and multiplicative moving plane at each time t.*

Proof. Since point X is fixed in both the moving and fixed planes, $V_r = 1$ and $V_f = 1$. Therefore, for such points, if $V_f = 1$, then,

$$B^* \otimes X \oplus C^* = 1, \tag{4.1}$$

and

$$X = e^{-1} \otimes (B^*)^{m-inv} \otimes C^*,$$

where $(B^*)^{m-inv}$ is the multiplicative inverse of B^*. Since

$$B = e^h \otimes \begin{bmatrix} e^{\cos\theta} & e^{-\sin\theta} \\ e^{\sin\theta} & e^{\cos\theta} \end{bmatrix} = \begin{bmatrix} e^{h\cos\theta} & e^{-h\sin\theta} \\ e^{h\sin\theta} & e^{h\cos\theta} \end{bmatrix}, C = \begin{bmatrix} e^a \\ e^b \end{bmatrix},$$

$$B^* = \begin{bmatrix} e^{h'\cos\theta - h\theta'\sin\theta} & e^{-h'\sin\theta - h\theta'\cos\theta} \\ e^{h'\sin\theta + h\theta'\cos\theta} & e^{h'\cos\theta - h\theta'\sin\theta} \end{bmatrix}, C^* = \begin{bmatrix} e^{a'} \\ e^{b'} \end{bmatrix},$$

we get $\det^G(B^*) = e^{(h')^2 + (h\theta')^2} \neq 1$. Thus B^* is regular and

$$(B^*)^{m-inv} = e^{\frac{1}{(h')^2+(h\theta')^2}} \otimes \begin{bmatrix} e^{h'\cos\theta - h\theta'\sin\theta} & e^{h'\sin\theta + h\theta'\cos\theta} \\ e^{-h'\sin\theta - h\theta'\cos\theta} & e^{h'\cos\theta - h\theta'\sin\theta} \end{bmatrix}. \tag{4.2}$$

□

Therefore, the equation $V_f = 1$ has a unique solution X. Point X is the pole point in plane M. Consequently from 3.1;

$$X = P = e^{\frac{1}{(h')^2+(h\theta')^2}} \otimes \begin{bmatrix} e^{(-a'h' - b'h\theta')\cos\theta + (a'h\theta' - b'h')\sin\theta} \\ e^{(a'h\theta' - b'h')\cos\theta + (a'h' + b'h\theta')\sin\theta} \end{bmatrix}, \tag{4.3}$$

and the pole point in the fixed plane is given as

$$(P)' = B \otimes P \oplus C \tag{4.4}$$

If the necessary calculations are carried out it can be obtained

$$(P)' = e^{\frac{1}{(h')^2+(h\theta')^2}} \otimes \begin{bmatrix} e^{-a'h'h - h^2 b'\theta'} \\ e^{h^2 a'\theta' - h'hb'} \end{bmatrix} \oplus \begin{bmatrix} e^a \\ e^b \end{bmatrix} \tag{4.5}$$

$$(P)' = \begin{bmatrix} e^{\frac{-a'h'h-h^2b'\theta'}{(h')^2+(h\theta')^2}+a} \\ e^{\frac{h^2a'\theta'-h'hb'}{(h')^2+(h\theta')^2}+b} \end{bmatrix}, \tag{4.6}$$

or as a vector

$$(P)' = \left(e^{\frac{-a'h'h-h^2b'\theta'}{(h')^2+(h\theta')^2}+a}, e^{\frac{h^2a'\theta'-h'hb'}{(h')^2+(h\theta')^2}+b} \right). \tag{4.7}$$

Here we assume that $\theta'(t) \neq 1$, for every t, i.e., non zero angular velocity. In this situation, there is only one pole point in each of the moving and fixed planes of each moment t.

Corollary 1. *If $\theta(t) = t$, then equation 4.3 will be obtained as*

$$X = P = e^{\frac{1}{(h')^2+h^2}} \otimes \begin{bmatrix} e^{(-a'h'-b'h)\cos\theta+(a'h-b'h')\sin\theta} \\ e^{(a'h-b'h')\cos\theta+(a'h'+b'h)\sin\theta} \end{bmatrix}.$$

Corollary 2. *For $\theta(t) = t$ and $h(t) = 1$, then equation 4.3 will be obtained as*

$$X = P = \begin{bmatrix} e^{a'\sin\theta-b'\cos\theta} \\ e^{a'\cos\theta+b'\cos\theta} \end{bmatrix}.$$

Corollary 3. *Let $\theta(t) = t$, then equation 4.7 will be obtained as*

$$P' = \left(e^{\frac{-a'h'h-h^2b'}{(h')^2+h^2}+a}, e^{\frac{h^2a'-h'hb'}{(h')^2+h^2}+b} \right).$$

Corollary 4. *For $\theta(t) = t$ and $h(t) = 1$, then equation 4.7 will be obtained as*

$$P' = \left(e^{-b'+a}, e^{a'+b} \right).$$

Definition 14. *In multiplicative plane motion, the point $P = (p_1, p_2)$ at time t is called the multiplicative pole of rotation or the center of sudden rotation.*

Theorem 3. *The relationship between the sliding velocity vector V_f and the pole passing from pole P to point X at every time t is as follows*

$$\|V_f\|^G \otimes \cos_g \theta = h^* \otimes \|P'Y\|^G$$

Proof. The pole point in multiplicative moving plane $Y = B \otimes X \oplus C$ implies that

$$X = (B)^{m-inv} \otimes \left(Y \oplus (e^{-1}) \otimes C)\right), \tag{4.8}$$

$$V_f = B^* \otimes X \oplus C^* \text{ and } B^* \otimes X \oplus C^* = 1$$

that leads to

$$X = P = e^{-1} \otimes (B^*)^{m-inv} \otimes C^*. \tag{4.9}$$

Now let us find pole points in multiplicative fixed plane. We have from equation

$$Y = B \otimes X \oplus C \tag{4.10}$$

$$Y' = P' = B \otimes \left(e^{-1} \otimes (B^*)^{m-inv} \otimes C^*) \oplus C \right),$$

Hence, we get

$$C^* = B^* \otimes (B)^{m-inv} \otimes \left(C \oplus (e^{-1} \otimes P')\right)$$

□

If we substitute this values in the equation $V_f = B^* \otimes X \oplus C^*$ we have $V_f = B^* \otimes (B)^{m-inv} \otimes P'Y$. Now let us calculate the value of $B^* \otimes (B)^{m-inv} \otimes P'Y$, where $P'Y = (e^{y_1-p_1}, e^{y_2-p_2})$, then

$$V_f = \begin{bmatrix} e^{\frac{h'}{h}(y_1-p_1)-\theta'(y_2-p_2)} \\ e^{\theta'(y_1-p_1)+\frac{h'}{h}(y_2-p_2)} \end{bmatrix} \tag{4.11}$$

or as a vector

$$V_f = \left(e^{\frac{h'}{h}(y_1-p_1)-\theta'(y_2-p_2)}, e^{\theta'(y_1-p_1)+\frac{h'}{h}(y_2-p_2)}\right), \tag{4.12}$$

hence we obtain

$$\begin{aligned} \langle V_f, P'Y\rangle^G &= \left\langle e^{\frac{h'}{h}(y_1-p_1)-\theta'(y_2-p_2)}, e^{\theta'(y_1-p_1)+\frac{h'}{h}(y_2-p_2)}, e^{y_1-p_1}, e^{y_2-p_2}\right\rangle^G \\ &= e^{\frac{h'}{h}\|P'Y\|^2} \end{aligned} \tag{4.13}$$

on the other hand we know that

$$\langle V_f, P'Y\rangle^G = \|V_f\|^G \otimes \|P'Y\|^G \otimes \cos_g \theta, \tag{4.14}$$

Thus, from the equalities in 4.13 and 4.14 we have that

$$\|V_f\|^G \otimes \cos_g \theta = h^* \otimes \|P'Y\|^G. \tag{4.15}$$

Corollary 5. *If the scalar matrix h is constant, the sliding velocity vector V_f is perpendicular to the pole ray passing from the pole P to vector X.*

Corollary 6. *In a $B_1 = M/M'$ multiplicative motion, the focus of the point X of M is an orbit, which it's normals pass through the rotation pole P.*

Theorem 4. *The norm of the sliding velocity vector is as*

$$\|V_f\|^G = \exp\left(\sqrt{\left(\frac{h'}{h}\right)^2 + (\theta')^2}\,\|P'Y\|\right). \tag{4.16}$$

Proof.

$$V_f = \left(e^{\frac{h'}{h}(y_1-p_1)-\theta'(y_2-p_2)}, e^{\theta'(y_1-p_1)+\frac{h'}{h}(y_2-p_2)}\right),$$

hence

$$\|V_f\|^G = \exp\left(\sqrt{\left(\frac{h'}{h}\right)^2 + (\theta')^2}\,\|P'Y\|\right).$$

□

Corollary 7. *If h is constant, the norm of the sliding velocity vector is*

$$\|V_f\|^G = \exp\left(|\theta'|\,\|P'Y\|\right). \tag{4.17}$$

Theorem 5. *The speed that occurs when drawing the curve (P) at point M at X is called V_r. At the same time, V_a is the speed that occurs when drawing the $(P)'$ curve of this point in the plane M'. These velocities are equal to each other at time t.*

Proof. $V_a = V_f \oplus V_r$, since $V_f = 1, V_a = V_r$. □

Definition 15. *The absolute acceleration vector of point X according to the plane M' is V_a. This vector V_a is determined by b_a. Since $V_a = Y^*$ then $b_a = V_a^* = Y^{**}$.*

Definition 16. *Let X be a fixed point on the moving plane M. This acceleration vector of the point X according to the fixed plane M' is called the sliding acceleration vector and is determined by b_f. Since acceleration of the multiplicative sliding acceleration X is a fixed point of M, then $b_f = V_f^* = B^{**} \otimes C^{**}$.*

5. ACCELERATIONS AND UNION OF ACCELERATIONS IN HOMOTHETIC MULTIPLICATIVE CALCULUS

Definition 17. *If the derivative of the vector $V_r = B \otimes X^*$ is taken, the vector $V_r^* = b_r = B \otimes X^{**}$ is obtained. The vector is called multiplicative relative acceleration vector and will be denoted by b_r.Considering point X as a moving point in M, matrix B is taken as constant*

Theorem 6. *Let X be a point moving in the plane M according to a parameter t. The relation between multiplicative acceleration formulas of this point is as*

$$b_a = b_r \oplus b_c \oplus b_f,$$

where $b_c = e^2 \otimes B^* \otimes X^*$ is denoted multiplicative Corilois acceleration.

Corollary 8. *If point X is a fixed point of multiplicative moving plane, multiplicative sliding acceleration of point X is equal to multiplicative absolute acceleration of that point.*

Proof. Note that

$$V_a = B^* \otimes X \oplus B \otimes X^* \oplus C^*,$$

Differentiating the both sides we have

$$V_a^* = B^{**} \otimes X \oplus e^2 \otimes (B^* \otimes X^*) \oplus B \otimes X^{**} \oplus C^{**},$$

since the point X is constant its derivative is 1. Hence

$$\begin{aligned} b_a &= V_a^* \\ &= B^{**} \otimes X \oplus C^{**} \\ &= b_f. \end{aligned}$$

□

Theorem 7. *The relationship between V_r and b_c can be given as*

$$\langle b_c, V_r \rangle^G = \exp\left(2hh'({x'_1}^2 + {x'_1}^2\right).$$

Proof.

$$\begin{aligned} V_r &= B \otimes X^*, \\ b_c &= e^2 \otimes (B^* \otimes X^*), \end{aligned}$$

So it is obvious that

$$\langle b_c, V_r \rangle^G = \exp\left(2hh'({x'_1}^2 + {x'_1}^2\right).$$

□

Corollary 9. *If h is a constant, then the Coriolis acceleration b_c is perpendicular to the relative velocity vector V_r at each instant moment t.*

6. THE ACCELERATION POLES ON THE MOTIONS

The solution of the equation $b_f = V_f^* = B^{**} \otimes X \oplus C^{**}$ gives us multiplicative acceleration pole of multiplicative motion.$V_f^* = B^{**} \otimes X \oplus C^{**}$ implies $X = e^{-1} \otimes (B^{**})^{m-inv} \otimes C^{**}$. Now calculating the matrices $e^{-1} \otimes (B^{**})^{m-inv}$ and C^{**}, and setting these in $X = P_1 = e^{-1} \otimes (B^{**})^{m-inv} \otimes C^{**}$, we obtain

$$X = P_1 = \begin{bmatrix} e^{\frac{1}{T}\left(a''(-r\cos\theta + z\sin\theta) - b''(r\sin\theta + z\cos\theta)\right)} \\ e^{\frac{1}{T}\left(a''(r\sin\theta + z\cos\theta) + b''(-r\cos\theta + z\sin\theta)\right)} \end{bmatrix}, \tag{6.1}$$

Here, the first-order pole curve of the plane M is denoted by P_1. If the pole curve of the plane M' plane is represented by P_1', then

$$P_1' = B \otimes P_1 \oplus C \tag{6.2}$$

Hence

$$P_1' = \begin{bmatrix} e^{\frac{1}{T}\left(-hra'' - hzb''\right) + a} \\ e^{\frac{1}{T}\left(hza'' - hrb''\right) + b} \end{bmatrix} \tag{6.3}$$

or as a vector

$$P_1' = \left(e^{\frac{1}{T}\left(-hra'' - hzb''\right) + a}, e^{\frac{1}{T}\left(hza'' - hrb''\right) + b}\right), \tag{6.4}$$

where $r = h'' - h(\theta')^2, z = 2h'\theta' + h\theta'', \quad T = r^2 + z^2$.

Corollary 10. *If $\theta(t) = t$, then equation 6.1 will be obtained as*

$$X = P_1 = \begin{bmatrix} e^{\frac{1}{(h''-h)^2 + 4(h')^2}\left(a''(-(h''-h)\cos\theta + 2h'\sin\theta) - b''((h''-h)\sin\theta + 2h'\cos\theta)\right)} \\ e^{\frac{1}{(h''-h)^2 + 4(h')^2}\left(a''((h''-h)\sin\theta + 2h'\cos\theta) + b''(-(h''-h)\cos\theta + 2h'\sin\theta)\right)} \end{bmatrix} \tag{6.5}$$

Corollary 11. *If $\theta(t) = t$ and $h(t) = 1$, then equation 6.1 will be obtained as*

$$X = P_1 = \begin{bmatrix} e^{a''\cos\theta + b''\sin\theta} \\ e^{-a''\sin\theta + b''\cos\theta} \end{bmatrix} \tag{6.6}$$

Corollary 12. *If $\theta(t) = t$, then equation 6.4 will be obtained as*

$$P_1' = \left(e^{\frac{1}{(h''-h)^2 + 4(h')^2}\left(-h(h''-h)a'' - 2hh'b''\right) + a}, e^{\frac{1}{(h''-h)^2 + 4(h')^2}\left(2hh'a'' - h(h''-h)b''\right) + b}\right). \tag{6.7}$$

Corollary 13. *If $\theta(t) = t$ and $h(t) = 1$, then equation 6.4,will be obtained as*

$$P_1' = \left(e^{-a'' + a}, e^{b'' + b}\right). \tag{6.8}$$

7. CONCLUSIONS

In multiplicative homothetic motions, velocities in plane motion, the relationship between velocities, pole points, and pole curves are given. Additionally, multiplicative accelerations and multiplicative acceleration combinations have been found.

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THE GRADIENT AND PARTIAL DERIVATIVES OF BICOMPLEX NUMBERS: A COMMUTATIVE-QUATERNION APPROACH

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Abstract. The study of bicomplex numbers, specifically commutative-quaternions, offers a fascinating exploration into the properties of complexified quaternions with commutative multiplication. Understanding the gradient and partial derivatives within this mathematical framework is crucial for analyzing the behavior of bicomplex functions. Real quaternions are not commutative but bicomplex numbers are commutative by multiplication. Bicomplex numbers are the special case of real quaternions. In this study, gradient and partial derivatives are obtained for bicomplex number valued functions.

1. Introduction

Commutative-quaternions, a specific subset of bicomplex numbers, have gained significant interest in mathematical research due to their commutative multiplication property. Unlike traditional quaternions, which are non-commutative, commutative-quaternions provide a unique algebraic structure for studying the behavior of bicomplex functions. For detailed information, see [5, 10, 11].

A real quaternion Q is defined by

$$Q = a + bi + cj + dk$$

where a, b, c, d are real numbers and

$$i^2 = j^2 = k^2 = ijk = -1$$

$$\begin{aligned} ij &= k,\ jk = i,\ ki = j, \\ ji &= -k,\ kj = -i,\ ik = -j. \end{aligned}$$

The conjugate of a real quaternion Q

$$\bar{Q} = a - bi - cj - dk$$

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and the norm of Q is

$$\begin{aligned} |Q| &= \sqrt{|Q|^2} \\ &= \sqrt{Q\overline{Q}} \\ &= \sqrt{a^2+b^2+c^2+d^2}. \end{aligned}$$

The set of quaternions is denoted by H [1, 6, 7].

A bicomplex number q is defined by

$$q = t + xi + yj + zk$$

where w, x, y, z are real numbers and

$$i^2 = j^2 = -1$$

$$\begin{aligned} ij &= ji = k, \\ ki &= ik = -j, \\ kj &= jk = -i. \\ k^2 &= ijij = iijj = i^2j^2 = 1 \end{aligned}$$

For detailed information about bicomplex numbers, we refer the reader to [5, 10].

The gradient of a scalar-valued function in bicomplex analysis allows us to determine the direction and magnitude of the steepest ascent or descent at any point. Similarly, partial derivatives provide a measure of how a function changes concerning each variable in a multidimensional space.

2. Preliminaries

Consider the bicomplex number function $f = f_1 + if_2 + jf_3 + kf_4$, whose components are bicomplex number valued functions. We can give the definition of derivative that

$$f'(q) = \frac{df}{dq} = \lim_{\Delta q \to 0} [f(q + \Delta q) - f(q)](\Delta q)^{(-1)}$$

where $q = t + xi + yj + zk$ is a bicomplex number. Then, $f(q) = f_1(q) + if_2(q) + jf_3(q) + kf_4(q)$.

In complex numbers algebra,

$$df/dz = \begin{bmatrix} \partial f_1/\partial x & \partial f_2/\partial x \\ \partial f_1/\partial y & \partial f_2/\partial y \end{bmatrix}$$

where $z=x+iy$ complex number and $f=f_1+if_2$ complex function. So, $f(z)=f_1(z)+if_2(z)$ is written.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}=\frac{\partial f}{\partial z}1=f\,'(z)\\ &\Longrightarrow f\,'(z)=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial x}i\\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}=\frac{\partial f}{\partial z}i=f\,'(z)i\\ &\Longrightarrow f\,'(z)=-\frac{\partial f}{\partial y}i=\frac{\partial f_2}{\partial y}-\frac{\partial f_1}{\partial y}i\end{aligned}$$

Real parts and the coefficient of i are equal. Also,

$$T_z=\begin{bmatrix} x & y\\ -y & x\end{bmatrix}$$

is the matrix representation of z complex number and

$$\frac{\partial f}{\partial x}=\frac{\partial f_1}{\partial x}+i\frac{\partial f_2}{\partial x}$$

is a complex derivative. We can write that

$$\begin{aligned}T_{z\,'} &= \begin{bmatrix} \partial f_1/\partial x & \partial f_2/\partial x\\ -\partial f_2/\partial x & \partial f_1/\partial x\end{bmatrix}\\ &= \begin{bmatrix} \partial f_1/\partial x & \partial f_2/\partial x\\ \partial f_1/\partial y & \partial f_2/\partial y\end{bmatrix}\end{aligned}$$

by considering the matrix representation of z. Here,

$$\partial f_1/\partial x=\partial f_2/\partial y,\ \partial f_2/\partial x=-\partial f_1/\partial y$$

are Cauchy-Riemann terms [3].

For real quaternions q_1 and q_2

$$q_1=\mu q_2\mu^{-1}$$

considering that the real quaternions q_1 and q_2 are similar if there is at least one μ real quaternion satisfying the equation. We can apply this feature for bicomplex numbers, which is the special case of real quaternion. Similar calculates are in [2, 8, 9] for quaternions. Hence,

$$\begin{aligned}q^i &= -iqi=-i(t+ix+jy+kz)i\\ &= -i(ti-x+ky-jz)\\ &= t+ix+jy+kz\\ q^j &= -jqj=-j(t+ix+jy+kz)j\\ &= -j(tj+kx-y-iz)\\ &= t+ix+jy+kz\\ q^k &= -kqk=-k(t+ix+jy+kz)k\\ &= -k(tk-jx-iy+z)\\ &= -t-ix-jy-kz\end{aligned}$$

involutions are obtained. Then, it is written

$$
\begin{aligned}
q &= t + ix + jy + kz \\
q^i &= t + ix + jy + kz \\
q^j &= t + ix + jy + kz \\
q^k &= -t - ix - jy - kz
\end{aligned}
$$

equation system. So,

$$
\begin{aligned}
t &= \frac{1}{4}(q + q^i + q^j - q^k) \\
x &= \frac{1}{4i}(q + q^i + q^j - q^k) \\
y &= \frac{1}{4j}(q + q^i + q^j - q^k) \\
z &= \frac{1}{4k}(q + q^i + q^j - q^k)
\end{aligned}
$$

are obtained. Hence,

$$
\begin{aligned}
dt &= \frac{1}{4}(dq + dq^i + dq^j - dq^k) \\
dx &= \frac{-i}{4}(dq + dq^i + dq^j - dq^k) \\
dy &= \frac{-j}{4}(dq + dq^i + dq^j - dq^k) \\
dz &= \frac{k}{4}(dq + dq^i + dq^j - dq^k)
\end{aligned}
$$

are written.

3. The Partial Derivatives of Bicomplex Functions

Partial derivatives in bicomplex analysis extend the concept from standard calculus to four dimensions. For a bicomplex function, the partial derivatives can be calculated by differentiating the function with respect to each variable while holding others constant.

We can give the following theorem similar to the case with complex numbers and by considering the theorem given in [4].

Theorem 3.1. *We can write that*

$$
\frac{\partial f}{\partial q} = \begin{bmatrix} \partial f_1/\partial t & \partial f_2/\partial t & \partial f_3/\partial t & \partial f_4/\partial t \\ \partial f_1/\partial x & \partial f_2/\partial x & -\partial f_3/\partial x & -\partial f_4/\partial x \\ -\partial f_1/\partial y & \partial f_2/\partial y & \partial f_3/\partial y & \partial f_4/\partial y \\ \partial f_1/\partial z & -\partial f_2/\partial z & \partial f_2/\partial z & \partial f_4/\partial z \end{bmatrix}
$$

where $f = f_1 + if_2 + jf_3 + kf_4$ *is a bicomplex function whose components are bicomplex number valued functions and* $q = t + ix + jy + kz$ *is a bicomplex number.*

Proof. We can write

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial q}\frac{\partial q}{\partial t} = \frac{\partial f}{\partial q}1 = f\,'(q) \\ &\Longrightarrow \ f\,'(q) = \frac{\partial f_1}{\partial t} + i\frac{\partial f_2}{\partial t} + j\frac{\partial f_3}{\partial t} + k\frac{\partial f_4}{\partial t}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = \frac{\partial f}{\partial q}i = f\,'(q)i \\ &\Longrightarrow \ f\,'(q) = \frac{-\partial f}{\partial x}i = \frac{\partial f_2}{\partial x} - i\frac{\partial f_1}{\partial x} - j\frac{\partial f_4}{\partial x} + k\frac{\partial f_3}{\partial x}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = \frac{\partial f}{\partial q}j = f\,'(q)j \\ &\Longrightarrow \ f\,'(q) = \frac{-\partial f}{\partial y}j = \frac{\partial f_3}{\partial y} + i\frac{\partial f_4}{\partial y} - j\frac{\partial f_1}{\partial y} - k\frac{\partial f_2}{\partial y}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial q}\frac{\partial q}{\partial z} = \frac{\partial f}{\partial q}k = f\,'(q)k \\ &\Longrightarrow \ f\,'(q) = \frac{\partial f}{\partial z}k = \frac{\partial f_4}{\partial z} - j\frac{\partial f_2}{\partial z} - i\frac{\partial f_3}{\partial z} + k\frac{\partial f_1}{\partial z}\end{aligned}$$

equations. Here, coefficients are equal. Also,

$$T_q = \begin{bmatrix} t & x & y & z \\ -x & t & -z & y \\ -y & -z & t & x \\ z & -y & -x & t \end{bmatrix}$$

is the matrix representation of bicomplex number and

$$\frac{\partial f}{\partial t} = \frac{\partial f_1}{\partial t} + i\frac{\partial f_2}{\partial t} + j\frac{\partial f_3}{\partial t} + k\frac{\partial f_4}{\partial t}$$

is bicomplex number derivative. We can write that

$$\begin{aligned}T_{f\,'} &= \begin{bmatrix} \partial f_1/\partial t & \partial f_2/\partial t & \partial f_3/\partial t & \partial f_4/\partial t \\ -\partial f_2/\partial t & \partial f_1/\partial t & -\partial f_4/\partial t & \partial f_3/\partial t \\ -\partial f_3/\partial t & -\partial f_4/\partial t & \partial f_1/\partial t & \partial f_2/\partial t \\ \partial f_4/\partial t & -\partial f_3/\partial t & -\partial f_2/\partial t & \partial f_1/\partial t \end{bmatrix} \\ &= \begin{bmatrix} \partial f_1/\partial t & \partial f_2/\partial t & \partial f_3/\partial t & \partial f_4/\partial t \\ \partial f_1/\partial x & \partial f_2/\partial x & -\partial f_3/\partial x & -\partial f_4/\partial x \\ -\partial f_1/\partial y & \partial f_2/\partial y & \partial f_3/\partial y & \partial f_4/\partial y \\ \partial f_1/\partial z & -\partial f_2/\partial z & \partial f_2/\partial z & \partial f_4/\partial z \end{bmatrix}\end{aligned}$$

(See [2] for similar operations). Thus, proof is complete. □

4. Gradient for Bicomplex Number Valued Functions

To calculate the gradient of a bicomplex function, we differentiate the function with respect to each variable (a, b, c, and d) independently. The resulting gradient vector provides the directional derivative along each axis.

Now let's replace these values in partial derivatives of the function f. Using these values in the partial derivatives of the function f,

$$\begin{aligned}
\frac{df}{dq} &= \frac{\partial f}{\partial t}\frac{\partial t}{\partial q} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial q} \\
&= \frac{\partial f}{\partial t}\frac{1}{4} + \frac{\partial f}{\partial x}\frac{(-i)}{4} + \frac{\partial f}{\partial y}\frac{(-j)}{4} + \frac{\partial f}{\partial z}\frac{k}{4} \\
&= \frac{1}{4}(\frac{\partial f}{\partial t} - i\frac{\partial f}{\partial x} - j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}) \\
\frac{df}{dq^i} &= \frac{\partial f}{\partial t}\frac{\partial t}{\partial q^i} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial q^i} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q^i} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial q^i} \\
&= \frac{\partial f}{\partial t}\frac{1}{4} + \frac{\partial f}{\partial x}\frac{1}{4i} + \frac{\partial f}{\partial y}\frac{1}{4j} + \frac{\partial f}{\partial z}\frac{k}{4} \\
&= \frac{1}{4}(\frac{\partial f}{\partial t} - i\frac{\partial f}{\partial x} - j\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}k) \\
\frac{df}{dq^j} &= \frac{\partial f}{\partial t}\frac{\partial t}{\partial q^j} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial q^j} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q^j} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial q^j} \\
&= \frac{\partial f}{\partial t}\frac{1}{4} + \frac{\partial f}{\partial x}\frac{1}{4i} + \frac{\partial f}{\partial y}\frac{1}{4j} + \frac{\partial f}{\partial z}\frac{k}{4} \\
&= \frac{1}{4}(\frac{\partial f}{\partial t} - i\frac{\partial f}{\partial x} - j\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}k) \\
\frac{df}{dq^k} &= \frac{\partial f}{\partial t}\frac{\partial t}{\partial q^k} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial q^k} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q^k} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial q^k} \\
&= \frac{\partial f}{\partial t}\frac{(-1)}{4} - \frac{\partial f}{\partial x}\frac{1}{4i} - \frac{\partial f}{\partial y}\frac{1}{4j} - \frac{\partial f}{\partial z}\frac{k}{4} \\
&= \frac{1}{4}(-\frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}k)
\end{aligned}$$

equations can be written.It is obtained that

$$\begin{bmatrix} \frac{\partial f(q,q^i,q^j,q^k)}{\partial q} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^i} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^j} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^k} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -i & -j & k \\ 1 & -i & -j & k \\ 1 & -i & -j & k \\ -1 & i & j & -k \end{bmatrix}\begin{bmatrix} \frac{df}{dt} \\ \frac{df}{dx} \\ \frac{df}{dy} \\ \frac{df}{dz} \end{bmatrix}$$

in matrix form. It can be written that

$$\begin{bmatrix} \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^*} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^{i*}} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^{j*}} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^{k*}} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & i & j & -k \\ 1 & i & j & -k \\ 1 & i & j & -k \\ -1 & -i & -j & k \end{bmatrix}\begin{bmatrix} \frac{df}{dt} \\ \frac{df}{dx} \\ \frac{df}{dy} \\ \frac{df}{dz} \end{bmatrix}$$

for conjugate. Here,

$$\nabla f = \begin{bmatrix} \frac{df}{dt} \\ \frac{df}{dx} \\ \frac{df}{dy} \\ \frac{df}{dz} \end{bmatrix}$$

is the gradient of f.

5. Conclusion

Bicomplex numbers extend the complex number system by introducing an additional imaginary unit, resulting in a four-dimensional algebraic structure. The commutative-quaternion algebra adds the property of commutativity to the quaternion algebra, allowing for a more versatile mathematical framework. The study of bicomplex numbers, specifically commutative-quaternions, offers a fascinating exploration into the properties of complexified quaternions with commutative multiplication. Understanding the gradient and partial derivatives within this mathematical framework is crucial for analyzing the behavior of bicomplex functions.

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PREDICTION OF STUDENT SUCCESS WITH DECISION-MAKING METHODS

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Abstract. Thanks to intuitionistic fuzzy sets, it has provided many benefits in application areas where the degree of precision is also taken into account. Education, one of these application areas, is an area where decision-making mechanisms play a very important role. Predicting student success and guiding the student in the future by taking into account every situation is an important step for decision makers and educators.

1. INTRODUCTION

Researchers who think that binary logic is insufficient in combating uncertainty have tried to find new ways over time. As a result of these efforts, the concept of fuzzy logic emerged [1]. Over time, fuzzy logic has become the basis of much research, and intuitionistic fuzzy logic, an expansion of fuzzy logic that is still up to date, has also introduced the degree of sensitivity [3]. With the degree of sensitivity also in play, the results of many studies have become much more objective. Decision makers made clearer decisions thanks to intuitionistic fuzzy sets, where they could also indicate uncertainty in their decisions. Efficient results have emerged thanks to the combination of decision-making methods with intuitionistic fuzzy sets. Thanks to the PROMETHEE method used in this study, positive and negative results will be evaluated simultaneously and a clear result will be obtained [4, 5, 6, 7, 8, 9, 10, 16, 17, 39, 26, 25]. Nowadays, intuitionistic fuzzy sets and decision-making methods attract the attention of many researchers [12, 14, 15, 21, 22, 23, 24, 27, 28, 29, 30, 31, 32, 33, 37].

2. PRELIMINARIES

Definition 1. *([3, 2])Let $X \neq \emptyset$. An intuitionistic fuzzy set A in X;*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle |\ x \in X\},$$

$$\mu_A(x), \nu_A(x), \pi_A(x) : X \to [0,1]$$

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defined membership, nonmembership and hesitation degree of the element $x \in X$ respectively.

$$\mu_A(x) + \nu_A(x) + \pi_A(x) = 1.$$

Intuitionistic fuzzy value (IFV) defined by Xu ([36]). Intuitionistic fuzzy value (IFV) is shown as follows: $\tilde{a} = (\mu_{\tilde{a}}, \nu_{\tilde{a}}, \pi_{\tilde{a}})$, where $\mu_{\tilde{a}}, \nu_{\tilde{a}}, \pi_{\tilde{a}} \in [0, 1]$
For each IFS $\tilde{A}$;

$$\pi_{\tilde{A}} = 1 - \mu_{\tilde{A}} - \nu_{\tilde{A}} \tag{2.1}$$

For IFVs $\tilde{a} = (\mu_{\tilde{a}}, \nu_{\tilde{a}})$ and $\tilde{b} = (\mu_{\tilde{b}}, \nu_{\tilde{b}})$ the following operations have been carried out([36, 35]):

$$(1) \quad \tilde{a} \oplus \tilde{b} = \quad (\mu_{\tilde{a}} + \mu_{\tilde{b}} - \mu_{\tilde{a}}\mu_{\tilde{b}}, \nu_{\tilde{a}}\nu_{\tilde{b}}) \tag{2.2}$$

$$(2) \quad \tilde{a} \otimes \tilde{b} = \quad (\mu_{\tilde{a}}\mu_{\tilde{b}}, \nu_{\tilde{a}} + \nu_{\tilde{b}} - \nu_{\tilde{a}}\nu_{\tilde{b}}) \tag{2.3}$$

$$(3) \quad \oplus_{j=1}^{m} \tilde{a}_j = \quad (1 - \prod_{j=1}^{m}(1 - \mu_j), \prod_{j=1}^{m} \nu_j) \tag{2.4}$$

$$(4) \quad \otimes_{j=1}^{m} \tilde{a}_j = \quad (\prod_{j=1}^{m} \mu_j, \prod_{j=1}^{m}(1 - \nu_j)) \tag{2.5}$$

This function is used to rank IFVs:

$$\rho(\alpha) = 0.5(1 + \pi_\alpha)(1 - \mu_\alpha)) \tag{2.6}$$

As the $\rho(\alpha)$ value decreases, the preferred value α increases.

3. The Intuitionistic Fuzzy PROMETHEE Method

The criteria's weights could be depicted as IFVs: $\tilde{w}_j$ where $\mu_{\tilde{w}_j} \in [0, 1], \nu_{\tilde{w}_j} \in [0, 1], \mu_{\tilde{w}_j} + \nu_{\tilde{w}_j} \leq 1, j = 1, 2, ..., m$. According to the weights, $\mu_{\tilde{\omega_j}}$ and $\nu_{\tilde{\omega_j}}$ demonstrate the membership and non-membership degrees of the alternative x_i respectively. Some methods can help decision makers in determining intuitionistic fuzzy weights ([18, 19, 34, 38, 11, 13]). In this study, linguistic terms were used to make the evaluation more accurate. Also, V shape criterion type has been used:

$$P(d) = \begin{cases} 0, & d \leq q \\ \frac{d-q}{p-q}, & q < d \leq p \\ 1, & d > p \end{cases} \tag{3.1}$$

Parameter thresholds q and p are indicated as indifference and strict preference, respectively. Evaluate the alternatives $x_i (i = 1, 2, ..., n)$ with respect to the criteria $c_j (j = 1, 2, ..., m)$ and determine the deviations based on pairwise comparisons:

$$d_j(x, y) = c_j(x) - c_j(y) \tag{3.2}$$

where $d_j(x, y)$ shows the distinction between the alternatives' the assessments x and y on the criterion c_j.

Definition 2. *(*[37]*) An intuitionistic fuzzy preference relation R on the set $X = x_1, x_2, ..., x_n$ is represented by a matrix $R = (r_{ik})_{n \times n}$, where $r_{ik} = \langle (x_i, x_k), \mu(x_i, x_k), \nu(x_i, x_k) \rangle$ for all $i, k = 1, 2, ..., n$. For convenience, we let $r_{ik} = (\mu_{ik}, \nu_{ik})$ where μ_{ik} denotes the degree to which the object x_i is preferred to the object x_k, ν_{ik} indicates the degree to which the object x_i is not preferred to the object*

x_k, and $\pi(x_i, x_k) = 1 - \mu(x_i, x_k) - \nu(x_i, x_k)$ is interpreted as an indeterminacy degree or a hesitancy degree, with the condition:

$$\begin{gathered}\mu_{ik},\ \nu_{ik} \in [0,1],\ \mu_{ik} + \nu_{ik} \leq 1,\ \mu_{ik} = \nu_{ki},\ \mu_{ki} = \nu_{ik},\\ \mu_{ii} = \nu_{ii} = 0.5,\ \pi_{ik} = 1 - \mu_{ik} - \nu_{ik},\\ for\ all\ i,k = 1,2,...,n \end{gathered} \tag{3.3}$$

The preferences μ_{ik} between the alternatives x_i and x_k according to the criterion c_j could be calculated by Equations (3.2) and (3.1), and then the preference matrix according to the criterion c_j is obtained as follows ([20]):

$$U^{(j)} = (\mu_{ik}^{(j)})_{n\times n} = \begin{bmatrix} - & \mu_{12}^{(j)} & \dots & \mu_{1n}^{(j)} \\ \mu_{21}^{(j)} & - & \dots & \mu_{2n}^{(j)} \\ \vdots & \vdots & - & \vdots \\ \mu_{n1}^{(j)} & \mu_{n2}^{(j)} & \dots & - \end{bmatrix} \tag{3.4}$$

Matrix of the intuitionistic fuzzy preference relation is obtained:

$$R^{(j)} = (r_{ik}^{(j)})_{n\times n} = \begin{bmatrix} - & (\mu_{12}^{(j)}, \nu_{12}^{(j)}) & \dots & (\mu_{1n}^{(j)}, \nu_{1n}^{(j)}) \\ (\mu_{21}^{(j)}, \nu_{21}^{(j)}) & - & \dots & (\mu_{2n}^{(j)}, \nu_{2n}^{(j)}) \\ \vdots & \vdots & - & \vdots \\ (\mu_{n1}^{(j)}, \nu_{n1}^{(j)}) & (\mu_{n2}^{(j)}, \nu_{n2}^{(j)}) & \dots & - \end{bmatrix} \tag{3.5}$$

The IFWA operator is used in this paper. The all intuitionistic fuzzy preference index of the alternative x_i to x_k on all criteria can be derived as:

$$r(x_i, x_k) = r_{ik} = \bigoplus_{j=1}^{m} \left(\tilde{w}_j \bigotimes r_{ik}^{(j)}\right) \tag{3.6}$$

where $r(x_i, x_k) = r_{ik}$ shows the degree to which the alternative x_i is preferred to the alternative x_k all criteria. Also, r_{ik} is an IFV. $\tilde{w}_j = (\mu_{\tilde{w}j}, \nu_{\tilde{w}j})$, then according to Equation (2.2), (2.3):

$$\tilde{w}_j \bigotimes r_{ik}^{(j)} = \left(\mu_{ik}^{(j)} \mu_{\tilde{w}j}, \nu_{ik}^{(j)} + \nu_{\tilde{w}j} - \nu_{ik}^{(j)} \nu_{\tilde{w}j}\right) \tag{3.7}$$

If Equations (2.4),(3.6) and (3.7) are combined;

$$\begin{aligned} r(x_i, x_k) &= \bigoplus_{j=1}^{m} \left(\tilde{w}_j \bigotimes r_{ik}^{(j)}\right) \\ &= \left(1 - \prod_{j=1}^{m} (1 - \mu_{ik}^{(j)} \mu_{\tilde{w}j}),\right. \\ &\quad \left.\prod_{j=1}^{m} (\nu_{ik}^{(j)} + \nu_{\tilde{w}j} - \nu_{ik}^{(j)} \nu_{\tilde{w}j})\right) \end{aligned} \tag{3.8}$$

Overall intuitionistic fuzzy preference relationship is obtained:

$$R = (r_{ik})_{n\times n} = \begin{bmatrix} - & (\mu_{12}, \nu_{12}) & \dots & (\mu_{1n}, \nu_{1n}) \\ (\mu_{21}, \nu_{21}) & - & \dots & (\mu_{2n}, \nu_{2n}) \\ \vdots & \vdots & - & \vdots \\ (\mu_{n1}, \nu_{n1}) & (\mu_{n2}, \nu_{n2}) & \dots & - \end{bmatrix} \tag{3.9}$$

Every alternative is compared to option $(n-1)$. As a result of intuitionistic fuzzy positive and negative outranking flow are achieved:
(1) The intuitionistic fuzzy positive outranking flow:

$$\tilde{\varphi}^{+}(x_i) = \frac{1}{n-1} \bigoplus_{k=1, k\neq i}^{n} r(x_i, x_k) = \frac{1}{n-1} \bigoplus_{k=1, k\neq i}^{n} r_{ik} \tag{3.10}$$

(2) The intuitionistic fuzzy negative outranking flow:

$$\tilde{\varphi}^{-}(x_i) = \frac{1}{n-1} \bigoplus_{k=1, k\neq i}^{n} r(x_k, x_i) = \frac{1}{n-1} \bigoplus_{k=1, k\neq i}^{n} r_{ki} \tag{3.11}$$

4. Student Success Prediction

A total of 15 students were evaluated based on 9 criteria to estimate student success. The first four of these criteria are Turkish, Mathematics, Science and Social courses, and the others are Anxiety, Attitude Toward Turkish, Attitude Toward Mathematics, Attitude Toward Science, Attitude Toward Social Studies, respectively. Student evaluations were graded with guidance counselors and Table 1 was created. Students were evaluated according to the scales determined by the guidance counselor. The values in Table 1 were determined according to the evaluation results.

The alternatives and criteria that form the basis of our algorithm are as follows: $A = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}\}$ being set of alternatives, each alternative represents a student. $K = \{K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8, K_9\}$ being set of criteria.

The value of each criterion for each alternative is calculated as follows in Table 1:

	$K1$	$K2$	$K3$	$K4$	$K5$	$K6$	$K7$	$K8$	$K9$
$A1$	10	11	9	9	11	5	4	5	7
$A2$	15	13	6	6	9	8	8	9	8
$A3$	11	6	9	8	5	4	7	7	7
$A4$	16	11	11	12	4	5	9	8	8
$A5$	18	8	14	5	3	9	9	7	9
$A6$	20	6	6	10	5	8	8	6	10
$A7$	15	5	8	7	6	9	6	5	10
$A8$	21	5	14	12	9	10	7	8	7
$A9$	16	5	3	13	10	5	5	7	9
$A10$	11	10	5	7	8	6	6	6	8
$A11$	12	2	4	5	7	7	9	10	10
$A12$	10	1	6	6	9	8	6	8	9
$A13$	15	1	3	6	4	7	7	8	8
$A14$	15	6	8	4	3	6	9	7	7
$A15$	29	5	20	7	8	5	8	6	6

Table 1. Values of Alternatives by Criteria

In this study, criterion weights were calculated in linguistic terms. The weights of the criteria are as follows: Criterion 1 and Criterion 2 are very important, Criterion 3, Criterion 4, Criterion 5 are important, Criterion 6 and Criterion 7 are medium, Criterion 8 and Criterion 9 are important.
Net outranking flow values are specified as follows:

$\rho(\tilde{\varphi}(x_1)) =$	-0,00000040
$\rho(\tilde{\varphi}(x_2)) =$	-0,00122824
$\rho(\tilde{\varphi}(x_3)) =$	0,00216354
$\rho(\tilde{\varphi}(x_4)) =$	-0,00005168
$\rho(\tilde{\varphi}(x_5)) =$	-0,00000404
$\rho(\tilde{\varphi}(x_6)) =$	-0,00004886
$\rho(\tilde{\varphi}(x_7)) =$	0,00007390
$\rho(\tilde{\varphi}(x_8)) =$	-0,00103610
$\rho(\tilde{\varphi}(x_9)) =$	-0,00002359
$\rho(\tilde{\varphi}(x_{10})) =$	0,00006432
$\rho(\tilde{\varphi}(x_{11})) =$	0,00000052
$\rho(\tilde{\varphi}(x_{12})) =$	0,00004843
$\rho(\tilde{\varphi}(x_{13})) =$	0,00252585
$\rho(\tilde{\varphi}(x_{14})) =$	0,00145190
$\rho(\tilde{\varphi}(x_{15})) =$	-0,00000940

TABLE 2. Intuitionistic Fuzzy Net Outranking Flow Values

To compare the values in Table 2, the order should be made from smallest to largest. It has been stated above that after sorting, the alternative with the lowest value is the best alternative. In addition, the alternative with the highest value in this table will be the last preferred alternative. When the students evaluated with the system created in our study are ranked according to their net flow values, the most successful student is A_2 and the least successful student is A_{13}. Students' achievements can be based on intuitionistic fuzzy net flow values. The lower the net flow value, the higher the student achievement.Thanks to the PROMETHEE method, which evaluates students by ranking them both positively and negatively, researchers are offered the opportunity to make bilateral observations, not one-sided.

5. Conclusion

The main goal of this study, in which intuitionistic fuzzy sets and decision-making methods are used together, is to create a system that takes both course success and psychological characteristics into consideration when evaluating student success. This study, conducted in the field of education where the decision-making mechanism plays an important role, will offer a new way to researchers who want to evaluate student achievements and guide students.

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