ISSN: 2667-419X

ESKİŞEHİR TECHNICAL UNIVERSITY JOURNAL OF SCIENCE AND TECHNOLOGY
B– Theoretical Sciences

ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ
B– Teorik Bilimler

Volume/Cilt 12 Number/Sayı 1 – February / Şubat 2024

ESKİŞEHİR TECHNICAL UNIVERSITY JOURNAL OF SCIENCE AND TECHNOLOGY
B- Theoretical Sciences

Eskişehir Teknik Üniversitesi Bilim ve Teknoloji Dergisi B – Teorik Bilimler

Volume: 12 / Number: 1 – February / 2024

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ISSN: 2667-419X

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B- Theoretical Sciences

Eskişehir Teknik Üniversitesi Bilim ve Teknoloji Dergisi B – Teorik Bilimler

Volume: 12 / Number: 1 – February / 2024

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e-mail: skurama@eskisehir.edu.tr

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Eskişehir Technical University, Institute of Graduate Programs, 26470 Eskişehir, TURKEY
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CONTACT / İLETİŞİM

Eskişehir Technical University, Graduate School of Sciences
26470 Eskişehir, TURKEY
Phone: +90-222-213 7485
e-mail : btdb@eskisehir.edu.tr

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Volume/Cilt**: 12 / Number/**Sayı**: 1 – February /** Şubat **2024**

CONTENTS / İÇİNDEKİLER

Sayfa / Page

ARAŞTIRMA MAKALESİ / RESEARCH ARTICLE

ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ
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Eskişehir Technical University Journal of Science and Technology B- Theoretical Sciences

`2024, 12(1), pp.1 - 6, DOI: 10.20290/estubtdb.1315254`

RESEARCH ARTICLE

INVESTIGATION OF STRUCTURAL AND REACTIVITY PROPERTIES OF SOME HALOGENATED PIPERAZINES

Fulden DEMİRCİOĞLU OVEZOV [1], **Cemal PARLAK** [2], **Özgür ALVER** [3, *]

[1] Department of Physics, Graduate School of Natural and Applied Sciences, Ege University, Izmir, Turkey
[2] Department of Physics, Faculty of Science, Ege University, Izmir, Turkey
[3] Department of Physics, Faculty of Science, Eskişehir Technical University, Eskişehir, Turkey

ABSTRACT

The chemical and biological activities are strongly related to the molecular structures and the media to which the molecules were subjected. Therefore, it is crucial to know the behaviors of molecular systems in the examined solvent media and their structural and reactivity properties. In the framework of this study, molecular and reactivity properties of some halogenated piperazine derivatives were undertaken by using the density functional theory. It was observed that halogen atoms have considerable effects on the behavior of piperazine molecule. Further, the atypical characteristic of fluorine affecting conformational preference is observed for di-halogenated piperazines with the same halogen atoms.

Keywords: DFT, Halogen effect, Molecular structure, Reactivity properties

1. INTRODUCTION

Piperazine derivatives are considered special types of compounds which were found in large scale applications possibilities in various fields of science. For example, piperazine and its derivatives have been found as a component in different marketed drugs [1]. Piperazines were also used as a starting material to synthesize many different compounds [2, 3]. Recently, Lipin et. al reported the effect of piperazine substituted Favipiravir drug molecule against the Nipah virus [4]. Further, many piperazine derivatives have been also known for their antihelmintic effect [5].

Density functional theory (DFT) has been extensively used to probe the molecular structures, spectroscopic and reactivity properties of many different compounds ranging from small to macromolecules depending on the computational source available [6-8]. In the essence of DFT, by taking the electron density as a major parameter, it is possible to understand the ground and also excited states of molecular structures and many other electronic properties of the investigated systems [9].

Stable molecular structures are closely related to chemical or biological activities. Therefore, in continuation with our interests in the investigation of piperazine derivatives [10-12], the prime objective of this search is to predict the stable molecular structures of the examined halogenated piperazines. Within this objective, we have examined the conformational and reactivity properties of the compounds together with the halogen and solvent effects. The compounds studied, their names and the abbreviations used are summarized as follows; $C_4H_{10}N_2$: piperazine (p), $C_4H_9FN_2$: 1-fluoropiperazine (1-fp), $C_4H_9ClN_2$: 1-chloropiperazine (1-cp), $C_4H_9BrN_2$: 1-bromopiperazine (1-bp), $C_4H_9ClFN_2$:1-chloro-4-fluoropiperazine (1-cfp), $C_4H_9BrFN_2$:1-fluoro-4-bromopiperazine (1-fbp), $C_4H_9BrClN_2$:1-bromo-4-chloropiperazine (1-bcp), $C_4H_8F_2N_2$: 1,4-difluoropiperazine (1,4-fp), $C_4H_8Cl_2N_2$: 1,4-dichloropiperazine (1,4-cp), $C_4H_8Br_2N_2$: 1,4-dibromopiperazine (1,4-bp).

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*Corresponding Author:ozguralver@eskisehir.edu.tr
Received: 15.06.2023 Published: 27.02.2024

2. CALCULATIONS

As seen in Figure 1, the calculations were focused on every four conformers (ee, ae, ea, aa, e: equatorial and a: axial) of the ten compounds. These conformational notations are also in line with the order of the abbreviations of the molecules. The first character of the ae and ea symbols represents the position of halogen atoms for 1-fp, 1-cp and 1-bp. Similarly, for 1-fbp, 1-cfp and 1-bcp, the first character of ae and ea shows fluorine, chlorine and bromine halogen atoms correspondingly.

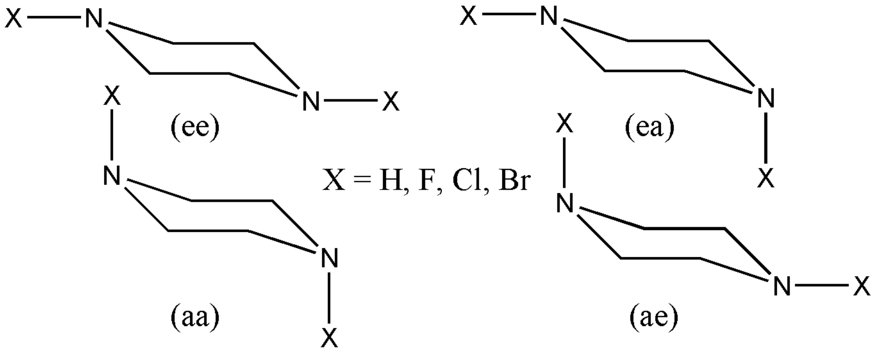

Figure 1. Conformational isomer and notation for the compounds.

Calculations were carried out in the gas phase, benzene and water media using the B3LYP functional with the cc-pvdz basis set. The polarizable continuum model in which the solvent is considered as a continuous medium was used to understand the solvation effect [13]. Computations were performed using the Gaussian 16 program package [14]. GaussView was used for building molecules and visualization of molecular orbitals [15].

3. RESULTS AND DISCUSSIONS

3.1. Conformational Studies

Piperazine in the chair form is more stable than the skewed boat by ~8 kcal/mol and the most stable conformation is ee chair conformer. This form is predominant in pure solids whereas the strongly dipolar ea form is most stabilized in an aqueous solution [16]. Relative stabilities and mole fractions for the optimized structures of the conformers of the compounds in different environments are given in Table 1. Regarding the calculations by B3LYP/cc-pvdz in the gas phase, benzene and water, the ee form is more stable than ea (ae) and aa 0.36 / 1.16, 0.29 / 0.93 and 0.18 / 0.60 kcal/mol, respectively. For the gas phase, piperazine prefers ee, ea (ae) and aa conformers with probabilities of 59.28%, 32.38% and 8.34% correspondingly. These results are also suitable with MP2 and B3LYP calculations with the aug-cc-pvdz basis set [16]. Similarly, for benzene and water, the molecule prefers the ee form with approximate probabilities of 55% and 48% respectively and the percentage of the most stable form has decreased. It is observed that the most stable form of piperazine is independent of the medium and methods.

Table 1. Relative energy (kcal/mol) and mole fraction (%) for conformers of the compounds.

Molecule	Relative Energy	Mole Fraction	Relative Energy	Mole Fraction	Relative Energy	Mole Fraction
p	Gas		Benzene		Water	
aa	1.16	8.34	0.93	11.46	0.60	17.23
ea (ae)	0.36	32.38	0.29	33.75	0.18	35.15
ee	-	59.28	-	54.79	-	47.62
1-fp						
aa	1.55	3.43	1.50	4.13	1.33	6.42
ae	0.78	12.60	0.96	10.36	1.27	7.04
ea	-	47.25	-	52.33	-	60.66
ee	0.15	36.72	0.27	33.18	0.50	25.88
1-cp						
aa	2.58	0.63	2.51	0.78	2.32	1.26
ae	1.79	2.36	1.93	2.06	2.20	1.53
ea	-	48.81	-	53.96	-	63.54
ee	0.01	48.20	0.13	43.20	0.38	33.67
1-bp						
aa	2.45	0.81	2.39	0.99	2.22	1.55
ae	1.73	2.75	1.87	2.39	2.15	1.75
ea	-	51.05	-	56.47	-	65.97
ee	0.07	45.39	0.20	40.15	0.45	30.73
1-cfp						
aa	0.98	10.63	1.04	10.16	1.08	10.04
ae	-	55.60	-	58.62	-	62.11
ea	1.20	7.30	1.18	7.97	1.16	8.75
ee	0.44	26.47	0.55	23.25	0.70	19.09
1-fbp						
aa	0.95	11.17	1.02	10.40	1.07	10.10
ae	-	55.37	-	58.46	-	61.88
ea	1.22	7.04	1.21	7.54	1.20	8.20
ee	0.44	26.42	0.54	23.60	0.67	19.82
1-bcp						
aa	2.02	2.37	1.97	2.44	1.86	2.72
ae	1.04	12.39	0.93	14.05	0.80	16.32
ea	0.97	13.94	0.87	15.65	0.74	18.07
ee	-	71.30	-	67.86	-	62.89
1,4-fp						
aa	-	54.67	-	51.64	-	48.44
ae (ea)	0.25	35.78	0.15	39.83	0.05	44.60
ee	1.03	9.55	1.07	8.53	1.15	6.96
1,4-cp						
aa	2.05	2.56	1.99	2.77	1.85	3.30
ae (ea)	0.99	15.30	0.88	17.82	0.75	21.38
ee	-	82.14	-	79.41	-	75.32
1,4-bp						
aa	1.99	2.88	1.96	2.96	1.87	3.30
ae (ea)	1.03	14.48	0.94	16.42	0.82	19.31
ee	-	82.64	-	80.62	-	77.39

Turning to single halogenated piperazines as 1-fp, 1-cp and 1-bp, the most stable conformation is ea with a probability of 47-66% in the considered environments. Further, the percentage of the most stable form of the compounds has increased as one goes to the polar water environment (Table 1). The most stable form of the compounds is independent of the solvent and halogen atom.

According to the calculated energies of di-halogenated piperazines with different halogen atoms, 1-cfp and 1-fbp prefer ae form with approximate probabilities of 55-61% for the used media whereas the most stable form of 1-bcp is ee by the probabilities of 63-71%. As going to the polar water environment, the percentage of preference for the most stable form of 1-cfp and 1-fbp increases whereas it decreases for 1-bcp. The most stable conformer of these compounds is dependent on the halogen atom.

Moving to the optimization energy values of di-halogenated piperazines with the same halogens in the gas phase, benzene and water, the ee form of 1,4-cp and 1,4-bp is also more stable than other conformers with the approximate preference of 75-83% probabilities. When all compounds are considered, the data are the highest percentages. 1,4-fp, however, prefers aa form with the probabilities of 48-55%. The percentage of preference for the most stable form of the compounds decreases as going to the polar water environment. The most stable conformer of these di-halogenated piperazines is also dependent on the halogen atom.

3.2. Reactivity Properties

The frontier orbitals of a molecular system can be used to determine which part of the molecule can accept and donate electrons [17]. Therefore, distributions of the highest occupied molecular orbital (HOMO) and the lowest unoccupied molecular orbital (LUMO) as the ionization potential and electron affinity correspondingly are the indicators of molecular activity. A more negative LUMO value is reported as the descriptor of the potent biological activities of a given molecule [18]. The energy gap between these frontier orbitals is known as the electrical band gap (E_g) or transport gap. As an example, the reactivity parameters such as band gap, chemical hardness (η), Fermi energy (E_F) and electrophilicity index (ω) of the compounds in water are presented in Table 2.

Table 2. Reactivity parameters (eV) of the compounds in water.

Molecule	E_{HOMO}	E_{LUMO}	E_g	η	μ (E_F)	ω
p (ee)	-6.068	1.538	7.606	3.803	-2.265	0.674
1-fp (ea)	-6.393	0.566	6.959	3.480	-2.914	1.220
1-cp (ea)	-6.325	-1.118	5.207	2.604	-3.722	2.660
1-bp (ea)	-6.272	-1.693	4.579	2.290	-3.983	3.464
1-cfp (ae)	-6.745	-1.172	5.573	2.787	-3.959	2.812
1-fbp (ae)	-6.630	-1.755	4.875	2.438	-4.193	3.606
1-bcp (ee)	-6.733	-1.741	4.992	2.496	-4.237	3.596
1,4-fp (aa)	-6.389	0.470	6.859	3.430	-2.960	1.277
1,4-cp (ee)	-6.811	-1.190	5.621	2.811	-4.001	2.847
1,4-bp (ee)	-6.694	-1.748	4.946	2.473	-4.221	3.602

The band gap of piperazine in the gas phase, benzene and water is the highest value by 7.385, 7.479 and 7.606 eV correspondingly and it implies high kinetic stability and low chemical reactivity. However, the lowest value of the band gap belongs to 1-bp with 4.768, 4.688 and 4.579 eV respectively, and it is the most conductive. Further, these energy gaps of the compounds are sufficiently large to meet the viability criterion [19]. Similarly, 1-bp has the lowest value of chemical hardness for all media. 1,4-bp, 1-fbp and 1-bcp have more negative LUMO values than others in the gas phase, benzene and water, respectively. Similarly, these compounds have the highest value of the electrophilicity index. The density plots for HOMO and LUMO of 1-fbp in water are shown in Figure 2 as an example. HOMO is delocalized on almost all atoms except some H atoms, but LUMO is delocalized on almost all atoms except some H and halogen (F) atoms.

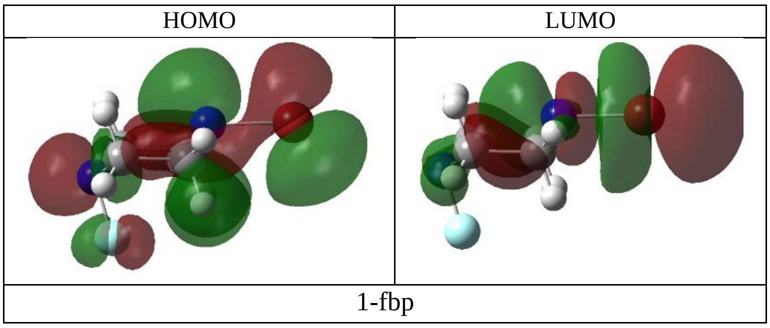

Figure 2. Frontier molecular orbitals of 1-fbp.

3. CONCLUSIONS

In the scope of this study, very detailed theoretical conformational analysis and reactivity parameters on halogenated piperazines were carried out. The main conclusions of this work are as follows: Conformational preference of all compounds is independent of the solvent but there is a halogen effect for di-halogenated piperazines with both different and same halogen atoms. Further, by comparing the energetic data of 1,4-fp with those of 1,4-cp and 1,4-bp, it is confirmed that fluorine has an atypical characteristic affecting conformational preference. The lowest value of band gap and chemical hardness belongs to 1-bp and it is the most conductive compound. The lowest band gap and chemical hardness values are independent of the solvent but there are both halogen and solvent effects on more negative LUMO and the highest electrophilicity index values. It can be said that 1,4-bp, 1-fbp and 1-bcp are more biologically active than others in the gas, benzene and water media.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

AUTHORSHIP CONTRIBUTIONS

All authors contributed equally.

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ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ
B- TEORİK BİLİMLER

Eskişehir Technical University Journal of Science and Technology B- Theoretical Sciences

`2024, 12(1), pp.7 - 21, DOI: 10.20290/estubtdb.1334360`

RESEARCH ARTICLE

DETERMINATION OF POLAR PESTICIDES IN PLANT-BASED FOODS BY LC-MS/MS

Deniz ARSLAN BURAK [1, *], **Yücel ŞAHİN** [2], **Muhammet BURAK** [2]

[1] Occupational Health and Safety, Vocational School, Istanbul Arel University, Istanbul, Turkey.
[2] Department of Chemistry, Faculty of Science and Arts, Yildiz Technical University, Istanbul, Turkey.

ABSTRACT

Pesticides in plant-derived foods are becoming an important problem due to their intensive use in plant cultivation. Today, specific pesticides are used in agriculture. Ever since pesticides are potentially hazardous to the environment and thus to human health through the consumption of pesticide-contaminated food. The European Community, as well as the Ministry of Agriculture and Forestry of the Republic of Turkey, have set maximum residue levels (MRLs) for pesticide residues in plant-based foods, based on the assumption that good agricultural practices are applied in the use of pesticides in agriculture. As a result, food products must be checked to ensure that MRLs are not violated. Therefore, an appropriate control of its residues in the samples should be carried out.

In this study, polar pesticides in food based on plant origin were determined by LC-MS/MS after extraction with methanol according to modified Quick Polar Pesticides method which released and implemented by EURL. Optimization of 13 different polar pesticides was performed with LC-MS/MS Q-Trap. Once the optimization process was completed, the samples to be initialized according to the SANTE (Guidance document on analytical quality control and method validation procedures for pesticide residues and analysis in food and feed) guidelines have been carefully selected, including high water, high acid/high water, high sugar/low water, high oil/low water and high starch/low water content and difficult or unique commodities. Tomato, lemon, dried fig, red lentil, walnut and sage samples carefully selected according to SANTE guidelines, were run at two different concentration levels, at the detection limit of 10 µg/L and 100 µg/L. In the light of the studies, it was determined that the RSD (Relative Standard Deviation) criteria for reproducibility for all substances were below 20%. Furthermore, the recovery value for all substances was in the range of 70 – 120%. As a consequence, 13 different high-polar pesticide substances can be analyzed in plant-derived product groups with the LC-MS/MS method developed using a hypercarb column.

Keywords: Polar pesticide, SANTE, LC-MS/MS, MRL, QuPPe.

1. INTRODUCTION

Pesticides are chemicals used to kill pests that live or live on humans, animals and plants, and also reduce and damage their nutritional value during the production, preparation, storage and consumption of nutrients. These pests are parasites that carry various diseases, insects that are harmful to agriculture and plants, weeds and fungi, flying and walking creatures such as flies, lice, fleas, ticks, scabies, cockroaches in humans, animals, environment and shelters [1-3]. If pesticides are not applied, approximately 65% of product loss occurs [4].

Pesticides may leave a large amount of residue in foodstuffs in cases where pesticides are used above the recommended dose, applied more than necessary, mixed with more than one pesticide when necessary, or when the time required to be left between the last spraying and the harvest period is not respected.

Pesticides not only increase agricultural productivity, but also pose a serious threat to human and environmental health with the residues they leave when they are used unconsciously and incorrectly.

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*Corresponding Author: denizarslanburak@arel.edu.tr
Received: 01.08.2023 Published: 27.02.2024

Humans and other living things can be exposed to acute or chronic poisoning by consuming foods containing high doses of pesticide residues [5-11].

In addition to the obligation to use pesticides in agriculture, a tolerance value has been determined for pesticide residues due to poisoning and other toxic effects.

Pesticides in terms of food safety since 1962, European Food Safety Authority (EFSA) from 1976, Food Codex Commission (CAC) established under the World Health Organization (WHO), Food and Agriculture Organization (FAO) working under the United Nations at the international level. It evaluates the residues and determines the maximum residue limits that can be found in foods. In Turkey, pesticide residue limits are determined by the Ministry of Agriculture and Forestry. Applications are made according to the Turkish Food Codex Maximum Residue Limits Regulation of Pesticides, which was issued taking into account the European Union Directive No. 91/414/EEC and the relevant provisions of the European Union Parliament and Council Regulation No. 396/2005/EC.

Determination of high polarity pesticides by liquid chromatography mass spectrometry (LC-MS/MS) method is carried out successfully, especially in order to determine whether tolerance limits are exceeded in vegetable original foods [12-17].

The aim of this study is to analyze AMPA, glyphosate, glufosinate, ethephon, ethephon hydroxy, N-acetyl glufosinate, fosethyl aluminum, chlorate, perchlorate, MPPA, N-acetyl AMPA, maleic hydrazide and phosphonic acid to determine a new LC-MS/MS method for the qualitative and quantitative analysis of highly polar pesticides such as phosphonic acid to be performed in a short time, reliably and accurately.

The study consists of four stages. Determination of the most suitable conditions for the successful analysis of the pesticide components to be analyzed in the LC-MS/MS device in the first stage; In the second stage, pesticide components are extracted from plant-based food products using acidified methanol solution, in the third stage, standard solutions of different concentrations are prepared for the quantitative determination of each pesticide component, calibration curves are drawn in the LC-MS/MS device, and in the last stage, recovery studies are carried out using real samples and analysis is carried out. The performance of the method was measured.

LC-MS/MS device and extra optimization conditions were determined in order to perform the analysis method in the shortest time, with high efficiency and in the most accurate way.

2. MATERIALS AND METHODS

2.1. Chemicals

1 % (v/v) acidified methanol solution was prepared by adding 10 mL of formic acid to a 1000 mL flask and completing with methanol. A stock solution was prepared by dissolving the internal standard (Cyanuric acid) solution in water (LC-MS grade) at 1000 mg/kg. It was used in studies by diluting it with methanol as 10 mg/kg and 1 mg/kg. Calibration solutions were prepared as a mixture of 3 different concentrations, with the main stock standards being 10 mg/L. 10 mg/L intermediate stock mix solutions and 500 µg/L and 25 µg/L calibration mix solutions were prepared with LC-MS purity methanol.

2.2. Methods

To obtain a representative analytical portion from the laboratory sample, the sample is homogenized, from the homogenized sample to a 50 mL centrifuge tube with a cap, 10±0.1 g for wet products, 5±0.05 g for dry products (such as cereal, legumes, dried fruit, etc.) or 13.5±0.1 g (if the product is rehydrated with water, that is, if 500 g of sample is added with 850 g of water and ground) and difficult products (such as spices, fruit tea, tea, etc.) are weighed 2.5±0.05 g.

Water was added to the tared centrifuge tube at the rate appropriate to the content of the product. Dry products were kept in water for 20 minutes for wetting after adding water. For 10 g sample, 100 µL of 10 mg/L cyanuric acid (internal standard) standard solution to 100 µg/L; 50 µL for 5 g sample; 25 µL was added for 2.5 g sample. 10 mL of acidified methanol was added to the prepared sample solution. The centrifuge tube should be shaken vigorously for 1 minute to effectively break up the crystalline aggregation. After the prepared solution was mixed in vortex, it was centrifuged at 4100 rpm at 10 ºC for 10 minutes. 3 mL of the centrifuged supernatant was taken and filtered through a 0.20 µ PET syringe filter into a 2 mL polypropylene vial. The filtered sample was diluted five times with water and analyzed by LC-MS/MS

2.2.1. LC-MS/MS Conditions

The mass spectrometry conditions for LC-MS/MS, AB Sciex 5500 Q-Trap, determined for the analysis method are given in Table 1.

Table 1. LC-MS/MS conditions

LC-MS/MS Parameters	Conditions
Ion Source (ESI, Turbo Ion Spray Mode)	Negative
Carrier Gas (Nitrogen) pressure	30psi
Collision Gas Level	Medium
Ion Spray Voltage	-4500V
Gas 1 (Zero Degree Air) Pressure	55psi
Gas 2 (Nitrogen) pressure	65psi
Gas 2 temperature	650 °C
Min/Max Waiting Time	80/300 minute

2.3. Validation Parameters and Criteria

For pesticide analysis, the SANTE [18] document titled “Guidance document on analytical quality control and method validation procedures for pesticide residues and analysis in food and feed.” numbered SANTE/11813/2017 is the main guide.

According to SANTE and UGRL [19] (National Food Reference Laboratory) guideline; linearity, sample effect (matrix effect), limit of detection (LOD) and limit of determination (LOQ), specificity, precision, robustness and bias are the parameters controlled during the validation of this study.

In the validation study, sample compatible calibrations were drawn at 7 different concentrations for 6 samples. Evaluation of sample-matched calibrations was done with a correlation value above a minimum of 0.95.

In this study, “matrix-matched calibration” was used to compensate for the matrix effect. Most remarkable way to eliminate the matrix effects of sample is to use matrix-matched calibration. Matrix-matched calibration works to compansate the effects of sample on calibration curve. To prepare the calibration curve level by level is way adding the standards as amounts by calculation of specified concentrations. Spiked samples for level of matrix-matched calibration curve is followed up in accordance with analysis procedure. Afterwards. each levels are injected to instrument to draw curve by concentration levels.

Within the scope of the analysis, 10 µg/L contamination studies were performed for each sample, with 10 replications in each product group, and the standard deviation of the results was taken.

In the SANTE document, specificity is defined as the detector's ability to provide the signal that will effectively identify the analyte (supported by selective extraction, cleaning, derivatization or separation where necessary). Retention times and signal heights in the LC-MS/MS device were considered distinctive for 13 different active substances. Retention times of high polarity pesticide active ingredients are given in Table 2.

Table 2. Retention times of pesticide agents

Pesticide	Retention time/min.
Ethephone	5.25
Glyphosate	2.34
Focetyl-Aluminium	4.52
Glufosinate	1.96
Maleic hydrazide	9.88
Chlorate	4.70
Perchlorate	9.47
N-Acetyl-AMPA	5.52
AMPA	1.14
HEPA	2.51
N-acetyl-Glufosinate	10.41
MPPA	5.55
Phosphonic acid	2.74

The SANTE document envisages the control of reality over the recovery values. Recovery values at both contamination concentrations should be in the range of 70-120%.

In the reality calculations, 10 and 100 µg/L contamination studies were carried out separately for tomato, lemon, dried fig, red lentil, walnut and sage samples, with a total of 10 repetitions, and the standard deviation of the results was taken. Accordingly, for the accuracy control, which constitutes a part of the accuracy parameter, 5 re-runs at 10 µg/L (at LOQ level) and 100 µg/L concentrations were performed to check the compliance of the recovery values to the range of 70-120%. The results were found to be appropriate.

Within the scope of this study, reproducibility and reproducibility studies were carried out. According to the SANTE document, the repeatability and reproducibility checks are made over the RSD values, and the compliance of the relevant RSD values with the ≤ 20% condition is checked. According to this for the repeatability control, which is one of the precision components, 5 re-runs were made at 10 µg/L (LOQ level) and 100 µg/L concentrations, and the conformity of the recovery values to the range of 70-120% was checked. The results were found to be appropriate and RSDr values of the measured values were calculated separately for each concentration and it was checked whether these values met the ≤20% condition.

According to the SANTE document, recovery studies should be carried out in "at least five repetitions" in contaminated samples at "at least two different concentrations" in method validation.

The first contamination concentration was chosen as a level below the maximum residue limit, and the second contamination concentration was chosen as a higher concentration.

According to the SANTE document, the reporting limit may be equal to or higher than the LOQ. Accordingly, in accordance with the SANTE document, validation studies were conducted in at least two concentrations, one of which must be at the LOQ level, and with at least five replications.

3. RESULTS and DISCUSSIONS

This analysis method, as indicated in Table 3, high water content, high acid and high water content, high sugar and low water content, high oil and very low water content, high oil and medium water content, high starch and/or protein content and low water and oil content products. The products include the analysis of 13 polar pesticides found in difficult or unique products. Pesticide agents: Ethephon, glyphosate, focetyl-aluminium, glufosinate, maleic hydrazide, chlorate, perchlorate, N-Acetyl-AMPA; AMPA, ethephon-hydroxy (HEPA), N-acetyl glufosinate, MPPA and phosphonic acid.

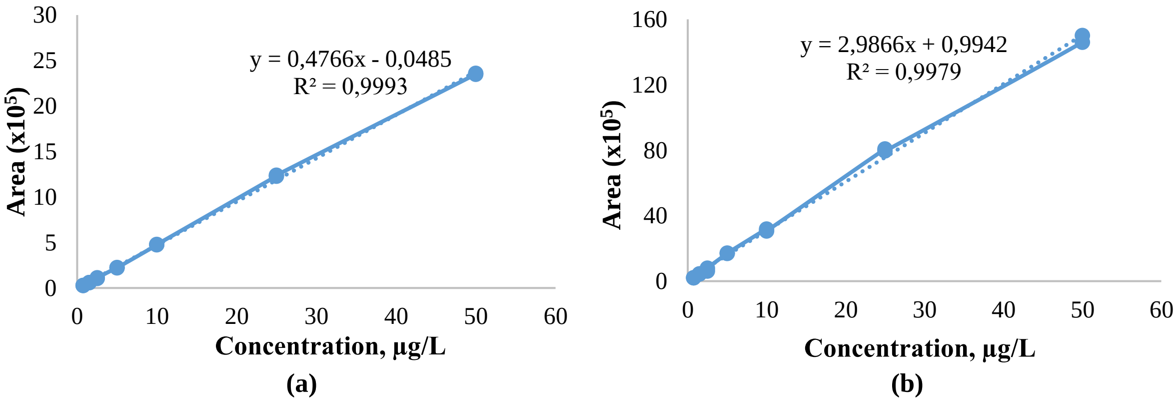
Table 3. Representative products used in the experimental study

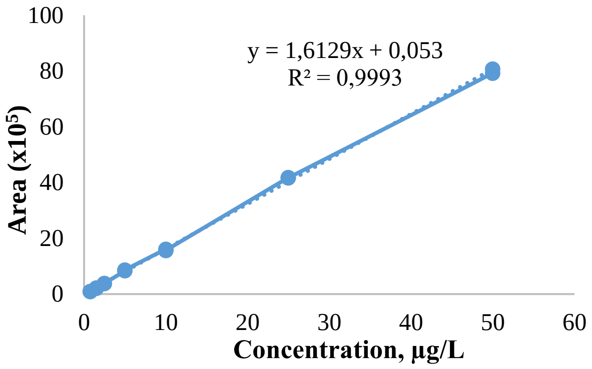
Product groups	Typical product categories	Selected representative products
Products with high water content	Seed vegetables / zucchini	Tomatoes
Products with high acid and high water content	Citrus	Lemon
Products with high sugar and low water content	Honey and dried fruits	Dry fig
Products with high oil and very low water content	Tree nuts	Walnut
Products with high starch and/or protein and low water and fat content	Dried beans/Legumes	Red lentil
Difficult or unique products		Sage tea

3.1. Verification

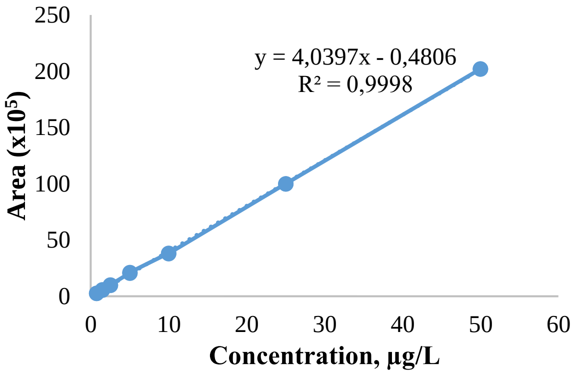
3.1.1. Linear calibration curve

As an example, as given in Figure 1 for the 13 polar pesticide agents, the linear measurement range, correlation values and sample-matched calibration curve studies performed with the blank sample showed the linearity, and the correlation curve value was obtained above the value of 0.95 as stated in the SANTE document. All relevant study results were evaluated according to these calibration curves.

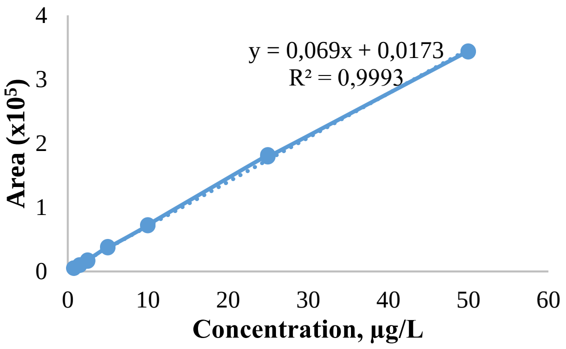



y = 1,6129x + 0,053
R² = 0,9993
Area (x10⁵)
100
80
60
40
20
0
0
10
20
30
40
50
60
Concentration, µg/L

(c)


y = 4,0397x - 0,4806
R² = 0,9998
Area (x10⁵)
250
200
150
100
50
0
0
10
20
30
40
50
60
Concentration, µg/L

(d)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R² = 0,9993
Area (x10⁵)
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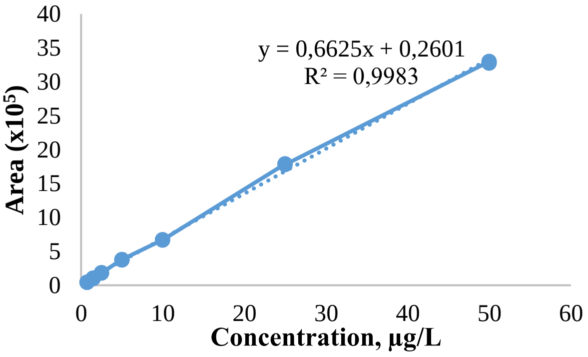
3
2
1
0
0
10
20
30
40
50
60
Concentration, µg/L

(e)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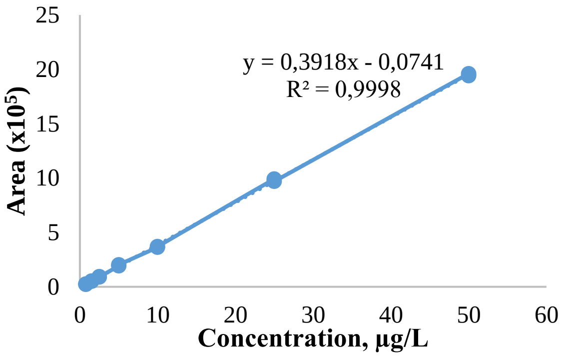
35
30
25
20
15
10
5
0
0
10
20
30
40
50
60
Concentration, µg/L

(f)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R² = 0,9998
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20
15
10
5
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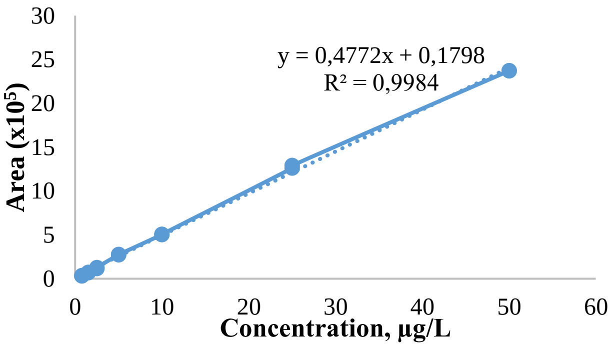
0
10
20
30
40
50
60
Concentration, µg/L

(g)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R² = 0,9984
Area (x10⁵)
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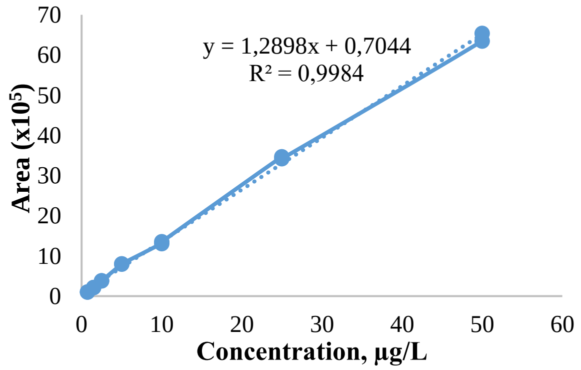
25
20
15
10
5
0
0
10
20
30
40
50
60
Concentration, µg/L

(h)


y = 1,2898x + 0,7044
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Area (x10⁵)
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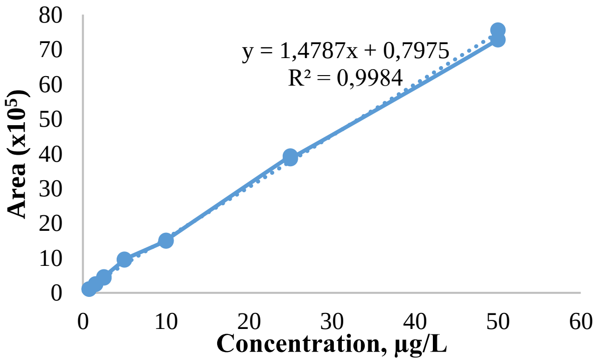
60
50
40
30
20
10
0
0
10
20
30
40
50
60
Concentration, µg/L

(i)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Area (x10⁵)
80
70
60
50
40
30
20
10
0
0
10
20
30
40
50
60
Concentration, µg/L

(j)

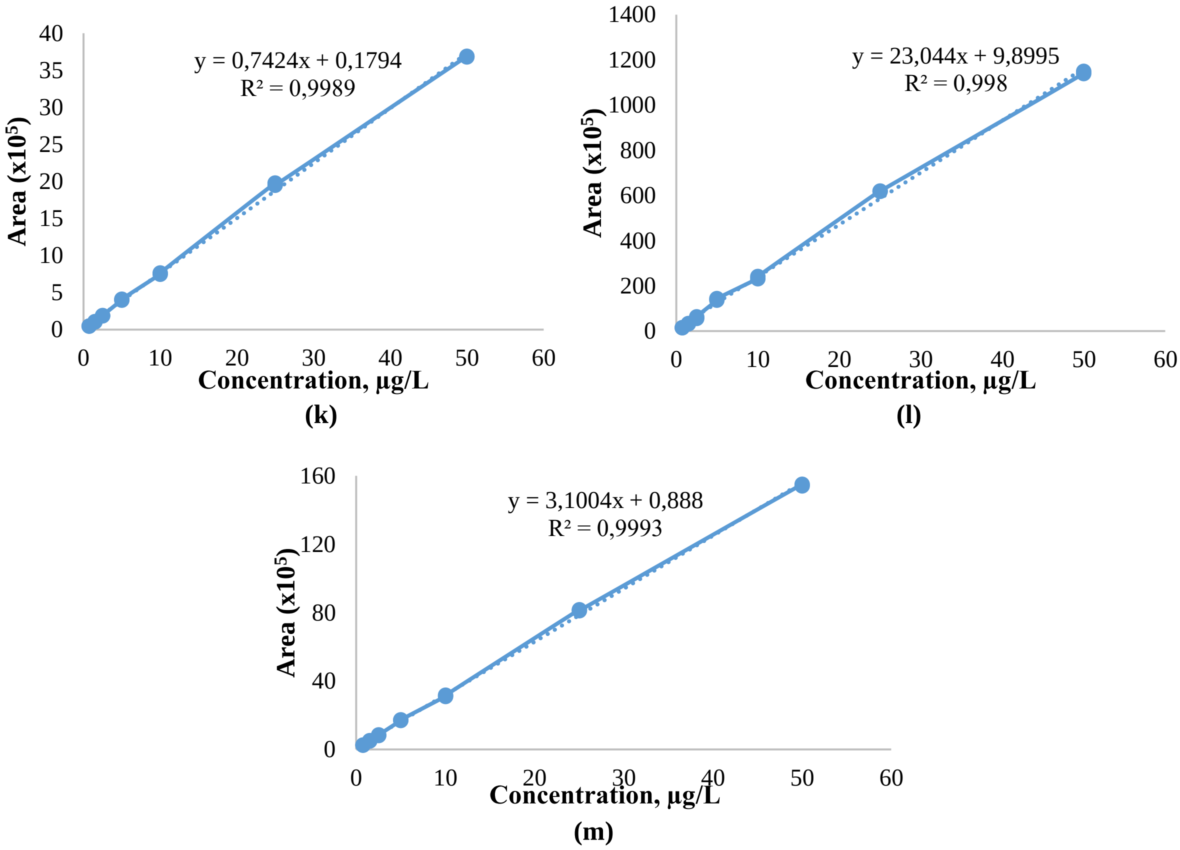

Figure 1. Calibration curve of (a) AMPA, (b) Chlorate, (c) Ethephone, (d) Focetyl-Aluminium, (e) Glufosinate, (f) Glyphosate, (g) HEPA, (h) Maleic hydrazide, (i) MPPA, (j) N-Acetyl-AMPA, (k) N-acetyl-Glufosinate, (l) Perchlorate, (m) Phosphonic acid

3.1.2. Limit of detection (LOD) and limit of quantification (LOQ)

Limit of detection (LOD) is the verified lowest residue concentration which can be quantified and reported by routine monitoring with verified methods. LOD is calculated as 3 times the standard deviation [18].

Limit of quantification (LOQ) is the lowest concentration or mass of the analyte that has been verified with approved accuracy by applying the complete analytical method. LOQ is calculated as 10 times the standard deviation [18].

Limit of detection and limit of determination values of 13 pesticide active ingredients in µg/L are given in Table 4.

3.1.3. Recovery studies

The recovery values calculated as a result of the studies carried out for reality were calculated separately for 13 different high polarity pesticides for each product group in studies belonging to two different levels. The recovery rates of the studies carried out are given in Table 5.

Table 4. LOD and LOQ values

Pesticides	**Tomatoes**		**Lemon**		**Red lentil**		**Dry fig**		**Walnut**		**Sage tea**	
	LOD	**LOQ**	**LOD**	**LOQ**	**LOD**	**LOQ**	**LOD**	**LOQ**	**LOD**	**LOQ**	**LOD**	**LOQ**
Ethephone	2.29	7.62	2.41	8.02	1.70	5.67	2.31	7.69	1.87	6.24	0.98	3.26
Glyphosate	1.31	4.35	1.43	4.78	0.56	1.86	2.81	9.38	1.52	5.06	2.35	7.84
Focetyl-Aluminium	1.13	3.76	1.77	5.90	1.22	4.05	1.06	3.52	1.39	4.65	1.32	4.40
Glufosinate	1.57	5.23	1.47	4.90	0.93	3.09	2.45	8.16	2.09	6.96	0.53	1.78
Maleic hydrazide	1.46	4.88	1.56	5.19	1.31	4.37	1.05	3.51	1.08	3.59	0.75	2.48
Chlorate	1.81	6.03	0.84	2.81	1.26	4.20	1.20	3.99	1.53	5.11	1.92	6.40
Perchlorate	2.65	8.85	1.51	5.04	1.43	4.78	1.41	4.69	0.68	2.27	0.91	3.03
N-Acetyl-AMPA	2.50	8.35	1.02	3.40	1.17	3.91	2.18	7.27	1.30	4.34	1.76	5.85
AMPA	1.35	4.50	1.76	5.88	1.40	4.65	1.59	5.32	1.38	4.61	0.79	2.62
HEPA	1.58	5.27	1.18	3.94	0.97	3.22	0.58	1.94	0.73	2.42	0.70	2.33
N-acetyl-Glufosinate	1.41	4.71	0.60	2.01	0.70	2.34	1.66	5.55	1.54	5.12	0.66	2.19
MPPA	2.86	9.52	0.81	2.71	0.83	2.76	2.68	8.92	2.77	9.23	2.99	9.95
Phosphonic acid	1.86	6.20	2.18	7.26	0.66	2.20	1.88	6.25	1.20	4.02	0.67	2.22

Table 5. Recovery rates of different products

Pesticide	**Recovery, %**					
	Tomatoes	**Lemon**	**Red lentil**	**Dry fig**	**Walnut**	**Sage tea**
Ethephone	87.0	92.0	91.9	89.3	87.4	83.1
Glyphosate	88.3	86.6	87.1	86.6	105.1	96.3
Focetyl-Aluminium	83.0	88.2	89.6	82.4	84.3	77.8
Glufosinate	83.1	82.7	95.7	85.2	87.8	79.3
Maleic hydrazide	82.8	85.1	87.1	79.9	81.8	77.1
Chlorate	89.0	97.9	96.2	84.9	108.7	101.5
Perchlorate	84.8	87.1	80.7	89.1	74.3	80.2
N-Acetyl-AMPA	88.6	86.4	84.7	94.2	91.9	86.5
AMPA	84.0	90.2	92.4	84.7	86.1	81.7
HEPA	81.9	83.3	85.4	78.8	84.6	77.6
N-acetyl-Glufosinate	89.6	85.0	102.2	92.7	99.0	93.0
MPPA	93.1	84.4	87.7	87.4	95.5	98.4
Phosphonic acid	86.3	85.8	93.6	79.4	79.6	78.5

3.1.4. Repeatability studies

The relative standard deviation (RSD) values of the studies performed meet the criteria of ≤20%, as stated in the SANTE document. The values of the studies carried out are given in Table 6 and Table 7.

Table 6. 10 µg/L RSD values

Pesticide	**RSD,%**					
	Tomatoes	**Lemon**	**Red lentil**	**Dry fig**	**Walnut**	**Sage tea**
Ethephone	8.79	7.96	5.50	7.76	7.23	3.85
Glyphosate	4.95	5.66	2.11	10.85	4.65	8.10
Focetyl-Aluminium	4.92	6.23	4.36	4.45	6.10	5.91
Glufosinate	6.16	5.63	2.64	8.55	8.14	2.40
Maleic hydrazide	5.97	5.92	4.79	4.54	4.65	3.38
Chlorate	7.21	2.46	3.62	5.17	4.46	6.00
Perchlorate	10.41	5.07	5.84	5.42	3.17	3.87
N-Acetyl-AMPA	10.46	4.60	5.25	8.43	5.84	7.01
AMPA	5.41	6.28	4.55	6.04	5.62	3.45
HEPA	6.75	4.76	3.88	2.67	3.27	3.19
N-acetyl-Glufosinate	5.80	2.81	2.03	6.31	6.14	2.47
MPPA	11.74	3.68	3.68	10.03	10.37	10.20
Phosphonic acid	7.35	8.98	1.91	7.92	5.25	2.98

Table 7. 100 µg/L RSD values

Pesticide	RSD,%					
	Tomatoes	**Lemon**	**Red lentil**	**Dry fig**	**Walnut**	**Sage tea**
Ethephone	4.06	5.31	6.44	4.27	5.09	4.68
Glyphosate	3.58	9.13	6.09	4.47	5.82	6.45
Focetyl-Aluminium	4.66	5.31	5.48	5.25	3.87	4.42
Glufosinate	3,78	5.86	5.36	3.48	2.31	4.92
Maleic hydrazide	4.40	5.22	5.43	3.87	2.70	4.41
Chlorate	3.72	5.88	4.45	5.15	4.01	8.84
Perchlorate	3.96	6.90	5.49	6.03	3.53	5.48
N-Acetyl-AMPA	3.77	7.46	12.69	5.76	2.36	5.30
AMPA	4.14	9.40	7.31	4.19	3.30	5.91
HEPA	2.37	4.19	6.37	4.32	2.35	4.21
N-acetyl-Glufosinate	2.89	10.57	8.02	4.83	3.02	4.54
MPPA	6.08	2.53	9.77	10.42	19.75	4.82
Phosphonic acid	3.46	3.51	4.49	4.44	4.57	3.19

3.1.5. Reproducibility studies

The RSD values of the studies carried out at low and high levels in six product groups on five different days meet the criteria of ≤20% as stated in the SANTE document. The values of the studies carried out are given in Table 8 and Table 9.

Table 8. 10 µg/L RSD values

Pesticide agent	RSD,%					
	Tomatoes	**Lemon**	**Red lentil**	**Dry fig**	**Walnut**	**Sage tea**
Ethephone	12.02	9.03	16.09	7.53	12.14	3.85
Glyphosate	10.69	7.28	18.82	9.52	20.35	8.10
Focetyl-Aluminium	15.58	10.63	11.71	17.22	16.19	5.91
Glufosinate	14.49	6.99	18.90	10.96	15.27	2.40
Maleic hydrazide	12.65	8.30	8.66	14.59	14.33	3.38
Chlorate	10.91	7.24	17.45	8.13	17.65	6.00
Perchlorate	8.97	7.41	6.04	6.27	4.99	3.87
N-Acetyl-AMPA	16.15	14.59	17.58	12.07	19.37	7.01
AMPA	11.38	6.15	12.03	7.24	12.48	3.45
HEPA	12.67	8.47	9.91	15.02	12.68	3.19
N-acetyl-Glufosinate	14.93	17.18	15.99	7.04	16.41	2.47
MPPA	17.21	6.60	19.91	10.79	12.03	10.20
Phosphonic acid	10.14	17.02	16.73	13.10	9.29	2.98

Table 9. 100 µg/L RSD values

Pesticide	RSD,%					
	Tomatoes	**Lemon**	**Red lentil**	**Dry fig**	**Walnut**	**Sage tea**
Ethephone	11.52	9.45	8.12	12.57	8.75	9,.0
Glyphosate	16.79	10.79	16.17	12.91	13.81	12.83
Focetyl-Aluminium	7.79	12.74	13.68	7.23	17.66	14.02
Glufosinate	15.45	18.49	9.73	16.04	8.94	17.61
Maleic hydrazide	12.70	7.92	11.40	9.55	14.23	10.21
Chlorate	13.89	11.28	15.52	15.91	11.34	12.79
Perchlorate	15.63	13.02	8.18	13.96	7.27	8.63
N-Acetyl-AMPA	14.99	10.87	13.24	15.26	18.07	9.39
AMPA	12.83	9.58	9.38	11.54	9.34	13.07
HEPA	14.84	16.77	8.85	17.10	10.66	18.60
N-acetyl-Glufosinate	9.23	4.28	8.05	6.47	10.46	6.88
MPPA	16.73	13.19	16.20	14.98	16.26	11.39
Phosphonic acid	15.83	19.94	10.22	15.68	13.76	16.26

3.2. Investigation of Updates to the Analysis Procedure

3.2.1. Comparison of injection volumes

As a result of the studies carried out by considering the original method source of QuPPe [20], injections at LOQ level were performed in 2 µL, 5 µL, 10 µL and 15 µL volumes to examine the effect of the injection amount on the analysis result. In order to examine the effect of injection amount on polar pesticide active substances, signal fields for each active substance are given in Table 9.

As shown in Figure 2 for the AMPA agent as an example, these injection volumes were not preferred due to the weak signals for 2 µL and 5 µL injection amounts. When the signals of 10 µL and 15 µL injection volumes were examined, it was observed that the peak shapes and areas of the peaks were higher. As a result of the recovery control of 10 µL and 15 µL volumes, it was determined that the values for all factors were more appropriate. In addition to all these examinations, a volume of 10 µL was accepted as the method injection amount in order to prevent contamination of the MS detector and LC system.

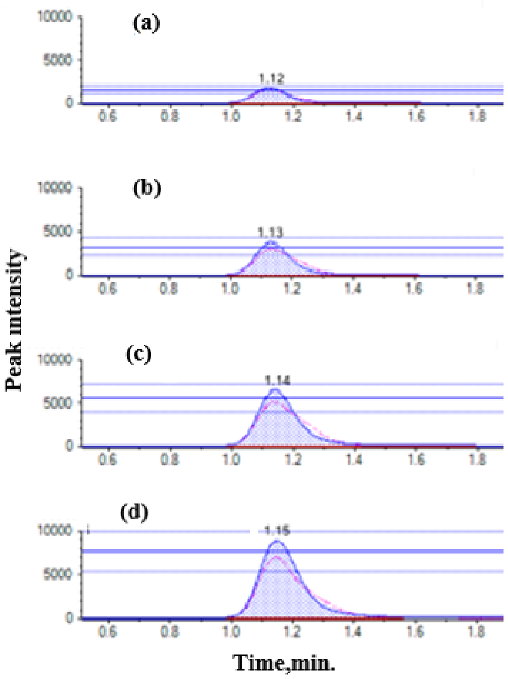

Figure 2. AMPA agent obtained in different injections (a) 2µL (b) 5µL (c) 10µL (d) 15µL

3.2.2. Comparison of dilution coefficients

After the extraction phase was completed, the peak shapes formed between the injection of the sample directly into the device after filtration and the injection as a result of different dilutions were examined. Based on the QuPPe method, it is recommended that the sample be injected directly into the device after extraction. As a result of the studies, it has been observed that the part of the extracted sample is directly injected into the device, resulting in distortion in the peak shapes and signals are obtained in a sparse manner. At the same time, it was observed that the signal heights obtained as a result of direct injection were lower than the signals obtained by dilution. In order to examine the dilution step after extraction, the sample was diluted 2, 5 and 10 times with ultrapure water in the last step.

As a result of multiple replication studies, it was seen that the signals of the samples diluted 10 times were sharper and the peak heights were better than the other diluted samples, as in the AMPA factor in Figure 3. At the same time, dilution prevents contamination that may occur in the device. As a result of these studies, the method dilution coefficient was accepted as 10.

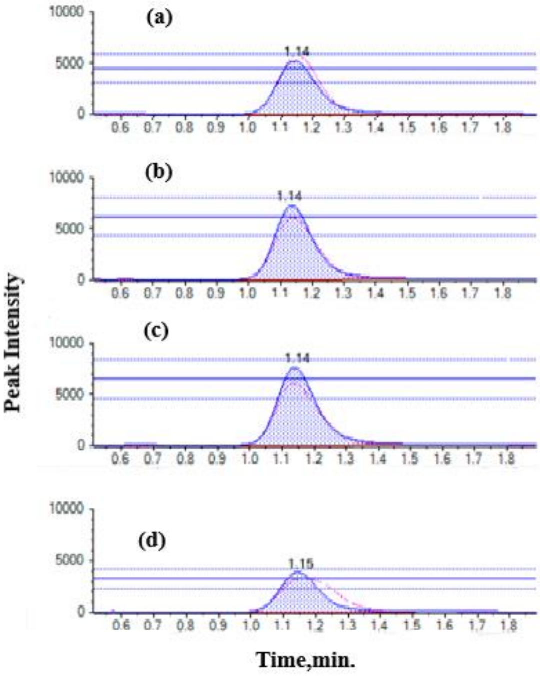

Figure 3. AMPA agent obtained at different dilution coefficients (a) 2 times dilution (b) 5 times dilution (c) 10 times dilution (d) Direct injection

3.2.3. Comparison of run time

There are two screening and quantitative methods published in QuPPe method version 10 and defined for 13 active substances. In the first of these methods, the ethephone; glyphosate; focetyl-aluminum; glufosinate; maleic hydrazide; N-Acetyl-AMPA; AMPA; HEPA; N-acetyl-glufosinate and MPPA agents; chlorate in the other; It was aimed to analyze perchlorate and phosphonic acid agents. The total time required for two different injections of these methods for a sample analysis is 45 minutes. In our study, as a result of combining two different methods, it was aimed to analyze 13 active substances in one study period and in the shortest time. By keeping the chromatographic column and mobile phases used in the relevant analysis method constant, the pump flow and the UPLC flow program were changed, making it possible to analyze all factors in a shorter time.

As a result of the injections, the method run time was carried out in as little as 20 minutes, and at the same time, 13 active substances were detected and determined together. For the AMPA agent, the signals of the peaks obtained as a result of 20 minutes (a) and 30 minutes (b) run times are given in Figure 4. As a result of the studies, the comparisons of the retention times of the QuPPe method flow program and the thesis study flow programs are given in Table 10.

Furthermore**,** the results in Table 11 were performed according to the technical data sheet of the analysis for each blank matrix. Verification of the analysis was performed with blank samples deemed 'Not Detected' for each component.

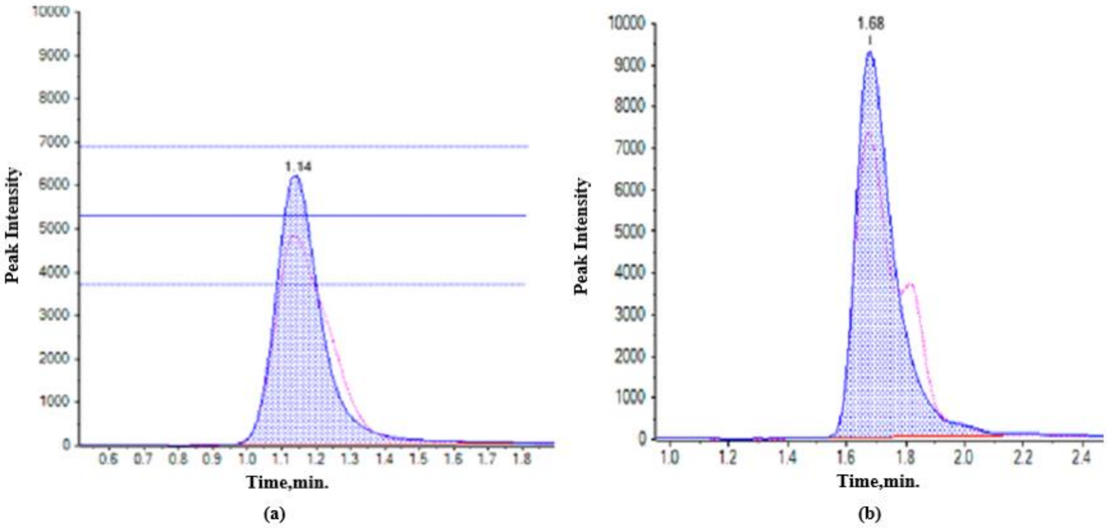

Figure 4. Signals and retention times related to the run time of AMPA agent (a) 20 minutes (b) 30 minutes

Table 10. This method and QuPPe method flow programs retention times

Pesticide	**Retention time/min.**	
	Improved Method	**QuPPe Method**
Ethephone	1.14	1.68
Glyphosate	4.72	7.16
Focetyl-Aluminium	5.33	8.26
Glufosinate	2.54	3.81
Maleic hydrazide	4.61	7.04
Chlorate	1.98	2.97
Perchlorate	2.36	3.54
N-Acetyl-AMPA	9.82	11.43
AMPA	5.60	8.02
HEPA	5.57	8.11
N-acetyl-Glufosinate	10.36	11.96
MPPA	9.37	12.74
Phosphonic acid	2.74	4.11

Table 11. Results of blank samples for each substance

Pesticide	**Blank Results, µg/L**					
	Tomatoes	**Lemon**	**Red lentil**	**Dry fig**	**Walnut**	**Sage tea**
Ethephone	0.52	0.15	0.12	0.27	0.75	0.60
Glyphosate	0.49	0.19	0.17	0.21	0.81	0.63
Focetyl-Aluminium	0.50	0.14	0.18	0.23	0.76	0.62
Glufosinate	0.45	0.19	0.13	0.24	0.74	0.61
Maleic hydrazide	0.51	0.12	0.10	0.25	0.73	0.61
Chlorate	0.43	0.18	0.12	0.31	0.74	0.69
Perchlorate	0.43	0.12	0.18	0.26	0.67	0.63
N-Acetyl-AMPA	0.49	0.17	0.14	0.26	0.77	0.69
AMPA	0.41	0.18	0.18	0.24	0.74	0.67
HEPA	0.55	0.17	0.15	0.20	0.66	0.60
N-acetyl-Glufosinate	0.43	0.18	0.15	0.27	0.76	0.68
MPPA	0.53	0.19	0.10	0.28	0.76	0.69
Phosphonic acid	0.50	0.14	0.12	0.28	0.76	0.66

4. CONCLUSION

As a result, an analytical method has been developed that enables the simultaneous quantitative determination of 13 different pesticide active ingredients. With the help of this method, it is possible to analyze pesticides with different high polarity, which cannot be analyzed together at the same time. According to the QuPPe method, 13 pesticides, which were analyzed with two different analytical methods, were analyzed with a single method.

The most suitable conditions were determined by analyzing 13 pesticide active ingredients in different injection volumes and using different dilution factors, and the precision of the analytical method was increased by enabling analysis in a short time like 20 minutes.

Reproducibility and reproducibility studies were carried out with the analytical method determined for each pesticide agent, and some statistical calculations were made as a result of these studies. As a result of these calculations, the relative standard deviation value for each active substance of the analysis method was below the 20% criterion and the recovery value was between 70-120%. At the same time, the expanded resultant uncertainty value calculated as a result of the repeatability and reproducibility studies was obtained below the 50% criterion as expected for each pesticide agent.

In this study, 13 different polar pesticides were determined simultaneously by the developed QuPPe method. Compared to similar studies in the literature, method superiority in LOQ, RSD and Recovery values according to the commodity of samples was compared and this study was found to be more advantageous. In a study [12] in which the active ingredient glufosinate was performed using the QuPPe method, it was observed that the RSD values obtained at 10 and 100 µg/L levels were higher than the RSD values obtained in this study. In the literature, in a different combined analysis method [13], it was observed that the values obtained as a result of LOQ studies carried out in the oily product group were higher than the values obtained in this study in similar product group as well. In a study where the QuPPe method was not used in the literature, but the parameters of maleic hydrazide, glyphosate, fosetyl-Al, and ethephon were performed, which was carried out as a similar rapid method [14], it was understood that the LOQ and recovery values obtained in this study were more advantageous compared to the values obtained in a similar product group.

As a continuation of the study, it is aimed to analyze more pesticide agents with the same method by increasing the number of pesticide active ingredients with high polarity.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

AUTHORSHIP CONTRIBUTIONS

The authors contributed equally to this work.

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ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ
B- TEORİK BİLİMLER

Eskişehir Technical University Journal of Science and Technology B- Theoretical Sciences

`2024, 12(1), pp.22 - 31, DOI: 10.20290/estubtdb.1346741`

RESEARCH ARTICLE

INTERACTIONS OF ANTIDEPRESSANT ESCITALOPRAM WITH ACETIC ACID AND LACTIC ACID: DFT AND MOLECULAR DOCKING ANALYSES

Cemal PARLAK [1, *] and İlkin YÜCEL ŞENGÜN [2]

[1] Department of Physics, Science Faculty, Ege University, Izmir, 35040, Turkey
[2] Food Engineering Department, Engineering Faculty, Ege University, Izmir 35040, Turkey

ABSTRACT

Many possible drugs have taken their places in the world market for the treatment of various medical diseases. Drug interactions involve combinations with drugs or other substances that alter the effect of a drug on the body. In this research, by using density functional theory, quantum theory of atoms in molecules, and in silico molecular docking against the receptor for antidepressant, we have investigated possible outcomes if antidepressant escitalopram comes across with an organic acid as acetic or lactic acids. The results suggest that escitalopram and acetic or lactic acids can interact spontaneously without requiring extra energy and depending on the interaction site different stabilities and reactivities are possible. Further, the findings show potential improvement in the effectiveness of antidepressant after interacting.

Keywords: Escitalopram, Drug interactions, DFT, Molecular docking

1. INTRODUCTION

A drug interaction is known as a reaction between a drug and a food, herb, beverage, supplement or between two or more drugs. These kinds of interactions can either produce desired or undesired results [1-3]. Escitalopram (ESCI), approved for medical use in 2002, is an antidepressant of the selective serotonin reuptake inhibitor. ESCI is mainly employed for the treatment of depression, general anxiety and panic disorder [4,5]. The investigations showed that ESCI may be more successful compared to many other antidepressants for the acute phase of the treatment, particularly with higher tolerance and acceptability [6]. In 2020, with more than 30 million prescriptions, it was the fifteenth most commonly prescribed drug in the US [7].

Lactic acid (LA) which is a chiral organic acid has biological significance owing to its function as a metabolite. LA, a normal intermediate in sugar fermentation, is made after the anaerobic metabolism of glucose in eukaryotic cells [8-9]. LA has been used in the production of solvents, metal pickling, and food additives, as an additive in various pharmaceutical products and as an emollient and keratolytic agent in various cosmetic products [10-12]. Acetic acid (AA) is produced and excreted by acetic acid bacteria, acetobacter or clostridium acetobutylicum found universally in foodstuffs, water and soil [13]. AA which gives vinegar its characteristic odor is produced by the acetification of alcohol and is an important chemical reagent and industrial chemical [14] as well it is also a biologically important metabolic intermediate and is found naturally in body fluids and plant juices.

All 5-hydroxytryptamine (5HT) receptor is the only pentameric ligand-gated ion channel found in the central and peripheral nervous systems whereas 5-hydroxytryptamine type-3 ($5HT_3$) differs markedly in structure and mechanism from the other 5HT receptors which are all G-protein-coupled. This family,

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*Corresponding Author: cemal.parlak@ege.edu.tr
Received: 20.08.2023 Published: 27.02.2024

in mammals, contains serotonin, glycine, acetylcholine, γ-aminobutyric acid receptors and zinc-activated ion channels [15]. $5HT_3$ receptor has played an important role in the pathophysiology of neuropsychiatric disorders including anxiety, depression, schizophrenia and drug use and is linked to the improvement of health problems such as depression [16].

Density functional theory (DFT) has been widely used for the examination of molecular structure and physical, chemical, or biological properties of different types of compounds [17-19]. Furthermore, the quantum theory of atoms in molecules (QTAIM) has been extensively employed to enlighten the covalent and non-covalent atom-atom interactions [20, 21]. For drug discovery molecular docking technique has become an increasingly important tool. It can be used to model the interaction between small molecules (ligands, drugs) and proteins (receptors) at the atomic level. It characterizes the behavior of the ligand molecules in the binding site of target receptors as well as elucidates fundamental biochemical processes [22].

In 2021, the effect of ESCI on the AA-induced ulcerative colitis model was experimentally investigated in rats [23]. The results were in support of the beneficial anti-inflammatory and anti-oxidant effects of ESCI in ulcerative colitis regardless of depressive conditions. Further, in 2022, the effect of the LA bacteria capsules on small intestinal bacterial overgrowth in patients with depression and diabetes was explored [24]. The results of this study used ESCI for depression states suggested that the compound LA bacteria capsule is a boon for patients of small intestinal bacterial overgrowth with depression and diabetes and can improve immune function and ease depression symptoms and reduce fasting plasma glucose and level of inflammatory factors. Some parts of the experimental and medical studies mentioned above are related to the interaction of AA and LA molecules with the antidepressant ESCI. This study, by taking into account the interaction at the molecular level with physical and chemical parameters by using DFT and molecular docking, has aimed to create a theoretical approach in terms of drug interactions and to be a model for similar drug types. In continuation of our interest in drug interactions and molecular docking research[25, 26], the effects of the AA and LA molecules on the antidepressant ESCI have been examined by using DFT and QTAIM. Regarding the DFT calculations, the structure – activity relationship was also investigated. To gain deep insights into the interactions and to compare the interactions of ESCI with AA and LA, the compounds interacted were docked against the potential protein receptor 6HIS for antidepressants. All of the results obtained were discussed in brief.

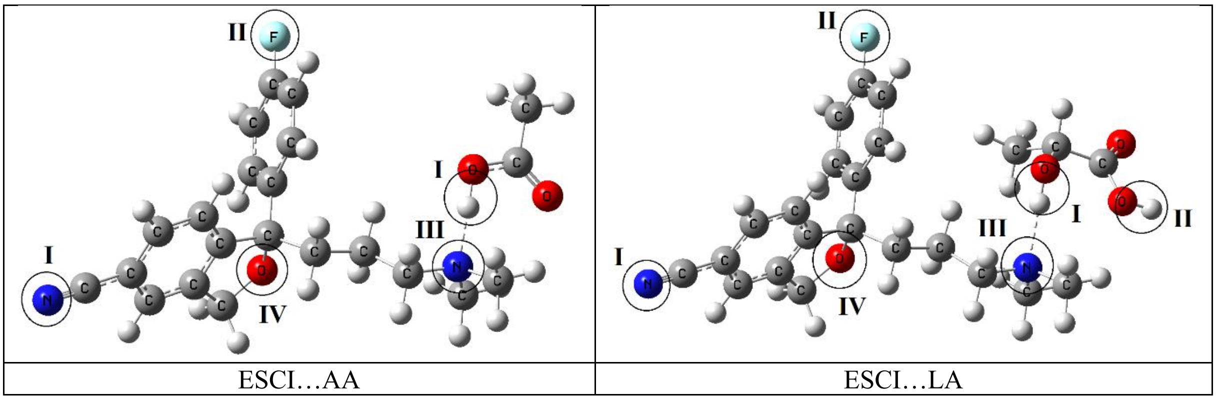
2. CALCULATIONS

All compounds and interactions were optimized without any molecular constraints in water with the polarizable continuum model [27]. Vibrational frequency calculations were performed to see that the optimized structures converge to a certain minimum on the potential energy surface. Computations were carried out to B3LYP functional [28, 29] and 6-31G(d) basis set. B3LYP/6-31G(d) can be used to investigate the binding energy and reactivity properties for such interactions of the compounds compared to the highly demanding B3LYP-D3 functional or cc-pvdz basis set [30]. Binding energies (E_b) and reactivity parameters were calculated as in previous studies [17, 25, 31, 32]. For the E_b computations, the basis set superposition errors (BSSE) were considered by the counterpoise correction method [33]. Band gap energies (E_g) were taken as magnitudes of differences between the highest occupied and lowest unoccupied molecular orbitals (HOMO and LUMO). Gaussian and Multiwfn programs were used for the DFT and QTAIM calculations correspondingly whereas GaussView was used to construct the examined structures [34-36].

In order to analyse the protein–ligand interactions, docking computations were performed by AutoDock software [37]. X-ray crystal structure of the protein was obtained from RCSB-PDB [38] encoded with 6HIS for $5HT_3$ [39]. After removing water molecules from the protein, polar hydrogen atoms and Kollman charges were evaluated whereas for the compounds randomized starting positions, Gasteiger charges, optimizations and torsions were performed. Lamarkian genetic algorithms were used in the docking process. Docking graphics were evaluated by the discovery studio visualizer [40].

3. RESULTS and DISCUSSIONS

Investigated configurations of the couples ESCI…AA and ESCI…LA were presented in Figure 1. OH…C≡N, OH…F, OH…N and OH…O interactions between the atoms of ESCI and AA or LA. Figure 1 also gives the optimized structures of OH…N site interactions encoded with III…I of both couples. All the interactions were examined and the important results were discussed and presented in brief.



ESCI…AA | ESCI…LA

Figure 1. Encoding of interactions investigated and examples of some optimized structures.

3.1. ESCI…AA interactions

Binding energies of the interacted systems between C≡N, F, N, O atoms of ESCI and OH group of AA were found as -4.44, -1.62, -8.40 and -4.05 kcal/mol (Table 1) respectively and the OH…N (III…I) interaction is stronger than others. Following this interaction between the compounds, the OH stretching band of AA shifted from 3679 cm^{-1} to 2497 cm^{-1} and the intensity of it increased from 74.7 to 3724.2. The red-shift around 1182 cm^{-1} indicates a strong intermolecular hydrogen bond (HB) interaction. In this case, the C=O stretching vibration of AA at the interaction site shifted from 1819 cm^{-1} to 1764 cm^{-1}. The related intensity decreased from 447.6 to 441.8. The OH bond length increased from 0.977 Å to 1.038 Å whereas the C=O bond length shifted from 1.215 Å to 1.225 Å correspondingly. In the OH…N interaction region, it is obvious that the O atom of the C=O group of AA interacts with the H atom of the CH_3 group attached to the N atom of ESCI. The magnitudes of the COH angle and COH…N dihedral angle for OH…N interaction are found to be 110.35° and 179.68° respectively.

Table 1. Binding energies (kcal/mol) of the investigated interactions.

Interaction	**Type**	**E_b**	**Corrected E_b**
ESCI…AA			
I…I	OH…C≡N	-5.70	-4.44
II…I	OH…F	-4.94	-1.62
III…I	OH…N	-12.39	-8.40
IV…I	OH…O	-7.60	-4.05
ESCI…LA (1)			
I…II	OH…C≡N	-4.36	-2.81
II…II	OH…F	-4.12	-0.15
III…II	OH…N	-8.90	-4.75
IV…II	OH…O	-6.05	-1.83
ESCI…LA (2)			
I…I	OH…C≡N	-6.36	-5.04
II…I	OH…F	-3.99	-1.30
III…I	OH…N	-17.45	-13.07
IV…I	OH…O	-8.60	-4.17

In this configuration, QTAIM results suggest three possible HBs occurred between OH…N, H (CH_3)…O=C and H (another CH_3)…O=C and the strongest interaction takes place at the OH…N site with a value of E_{HB} = -16.56 kcal/mol. At this interaction site, the Laplacian of electron density ($\nabla^2\rho$: 0.1278 a.u.) was found as positive while the value of electronic energy density (H: -0.010 a.u.) showed a negative value showing a partially covalent interaction [25].

The reactivity parameters of the compounds and interactions as electron affinity ($-E_{LUMO}$), ionization energy ($-E_{HOMO}$), band gap (E_g), electronegativity (χ), hardness (η), chemical potential (μ) or Fermi energy and electrophilicity index (ω) are given in Table 2. The magnitudes of HOMO-LUMO energy differences were calculated as 7.817 and 4.327 eV for AA and ESCI correspondingly. For the OH…N site interaction, the E_g value of the coupled system was computed as 5.122 eV which is higher than the E_g value of ESCI and lower than the E_g value of AA. Therefore, it can be concluded that the reactivity of the interacted system is higher than single isolated AA whereas it is lower than single isolated ESCI. However, for the OH…C≡N, OH…F and OH…O interactions, E_g values reduced to 4.172, 4.320 and 4.288 eV, respectively, indicating an increase in the reactivity of ESCI…AA.

Table 2. Reactivity parameters of the compounds and interactions.

Molecule	**E_{LUMO}**	**E_{HOMO}**	**E_g**	**χ**	**η**	**μ (E_F)**	**ω**
ESCI	-5.788	-1.461	4.327	3.625	2.164	-3.625	3.036
AA	-7.609	0.208	7.817	3.701	3.909	-3.701	1.752
LA	-7.221	0.021	7.242	3.600	3.621	-3.600	1.790
Interaction	**ESCI…AA**						
OH…C≡N	-5.790	-1.618	4.172	3.704	2.086	-3.704	3.288
OH…F	-5.789	-1.469	4.320	3.629	2.160	-3.629	3.049
OH…N	-6.594	-1.472	5.122	4.033	2.561	-4.033	3.176
OH…O	-5.803	-1.515	4.288	3.659	2.144	-3.659	3.122
	ESCI…LA (1)						
OH…C≡N	-5.791	-1.641	4.150	3.716	2.075	-3.716	3.327
OH…F	-5.790	-1.478	4.312	3.634	2.156	-3.634	3.063
OH…N	-6.246	-1.493	4.753	3.870	2.377	-3.870	3.150
OH…O	-5.790	-1.524	4.266	3.657	2.133	-3.657	3.135
	ESCI…LA (2)						
OH…C≡N	-5.791	-1.591	4.200	3.691	2.100	-3.691	3.244
OH…F	-5.788	-1.475	4.313	3.632	2.157	-3.632	3.058
OH…N	-6.314	-1.467	4.847	3.891	2.424	-3.891	3.123
OH…O	-5.781	-1.503	4.278	3.642	2.139	-3.642	3.101

3.2. ESCI…LA Interactions

As encoded in Figure 1, LA has two OH groups for interaction. Binding energies of the interacted couples between C≡N, F, N, O atoms of ESCI and OH (1) group of LA were found as -2.81, -0.15, -4.75 and -1.83 kcal/mol, respectively, while E_b values between ESCI and OH (2) of LA were -5.04, -1.30, -13.07 and -4.17 kcal/mol (Table 1). According to the results, the OH…N interaction for both OH groups is stronger than other interactions. This interaction occurred with ESCI and the OH (2) group of LA is the strongest. For this interaction, the OH stretching band of LA shifted from 3677 cm^{-1} to 2473 cm^{-1} and its intensity increased from 96.3 to 2921.8. The change in the wavenumber is 1204 cm^{-1} and the change in infrared intensity is around 2825 addressing a very strong hydrogen bonding at the OH edge. Following the interaction OH bond length increased from 0.977 Å to 1.585 Å which confirms the red-shift in the IR spectrum. Furthermore, it is noteworthy that the hydrogen atom of the OH group moves to the nitrogen atom of ESCI after interaction and the NH bond length was found as 1.082 Å. In this case, the C=O stretching vibration of LA shifted from 1817 cm^{-1} to 1683 cm^{-1}. The related intensity decreased from 385.8 to 447.5. The carbonyl bond length shifted from 1.214 Å to 1.247 Å. In the OH (2)…N interaction region, it is obvious that O atom of the carbonyl group of LA interacted with the H atom of the CH_3 group attached to the N atom of ESCI. The magnitudes of the COH angle and COH…N dihedral angle for OH (2)…N interaction are found to be 108.75^o and -8.15^o respectively.

In this configuration, QTAIM results suggest that three possible HBs occurred as ESCI…LA and the strongest interaction takes place at the OH (2)…N site with a value of E_{HB} = -17.64 kcal/mol. For this interaction, $\nabla^2\rho$ (0.1772 a.u.) was found as positive while the value of H (-0.0059 a.u.) showed a negative value addressing a partially covalent interaction [20, 25].

As seen from Table 2, band gap values were calculated as 7.242 and 4.327 eV for LA and ESCI, respectively. For the OH (2)…N site interaction, the value of band gap energy for the coupled system was calculated as 4.847 eV which is higher than the E_g value of ESCI and lower than the E_g value of LA. Therefore, it can be concluded that the reactivity of the coupled system is higher than single isolated LA while it is lower than single isolated ESCI. For the other interactions, however, E_g values reduced to 4.200, 4.313 and 4.278 eV indicating an increase in the reactivity of ESCI…LA. A similar trend is observed for electronegativity, hardness and Fermi energy as well.

3.3. Molecular Docking

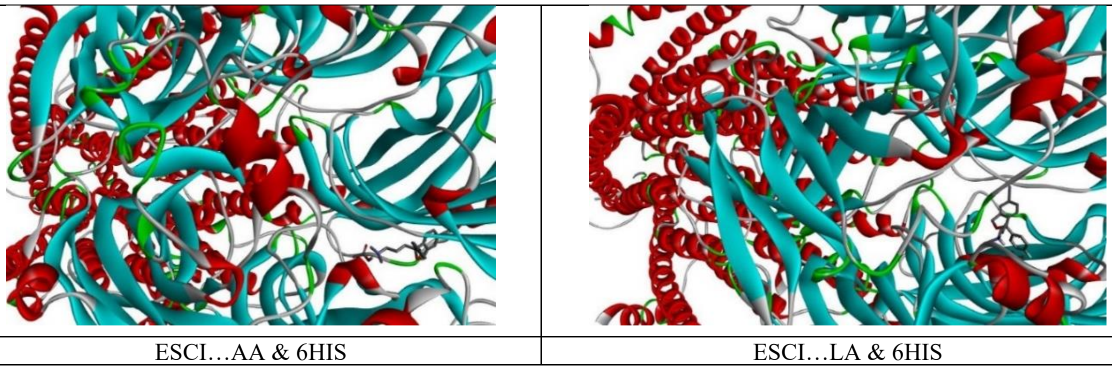
Molecular docking results on the couples ESCI…AA and ESCI…LA with the protein 6HIS for the $5HT_3$ receptor are herein reported together with the data for the drug ESCI [25]. Docking results for ESCI & 6HIS, ESCI…AA & 6HIS and ESCI…LA & 6HIS interactions are given ten binding poses ranked from the lowest to the highest energetic conformation and the related energy ranges are –6.19 to –5.51, –7.38 to –5.97 and –7.07 to –4.95 kcal/mol, respectively. The best conformer for each docking was chosen based on the most favourable docking score and the presence of non-covalent bonding interactions.

The results of the lowest binding and decomposed energies are given in Table 3. Approximate binding free energy (ΔG) is the sum of the terms; dispersion/repulsion (ΔG_{vdW}), hydrogen bonding (ΔG_{Hbond}), electrostatic energy (ΔG_{elec}), deviation from the covalent geometry ($\Delta G_{conform}$), restriction of internal rotors, global rotation and translation (ΔG_{tor}), desolvation and hydrophobic effect (ΔG_{desolv}). The best poses docked in receptors are given in Figure 2.

Table 3. Lowest binding energy (kcal/mol) and the decomposed component terms.

Structure	ΔG	$\Delta G_{vdW}+\Delta G_{Hbond}+\Delta G_{desolv}$	ΔG_{elec}	$\Delta G_{intermol}$	$\Delta G_{tot\ int}$	ΔG_{tor}	$\Delta G_{unbound}$
ESCI & 6HIS [25]	–6.19	–6.43	–1.25	–7.68	–1.19	+1.49	–1.19
ESCI…AA & 6HIS	–7.38	–9.71	–0.06	–9.77	–1.65	+2.39	–1.65
ESCI…LA & 6HIS	–7.07	–8.79	–1.26	–10.05	–2.66	+2.98	–2.66

The values of free lowest binding energies for docking ESCI…AA and ESCI…LA against the protein 6HIS were calculated as –7.38 and –7.07 kcal/mol correspondingly. All the compounds show very good binding for the receptor 6HIS. However, the magnitudes of free binding energies of the complexes are higher than the drug ESCI alone which is referring to a stronger interaction for the complexes.

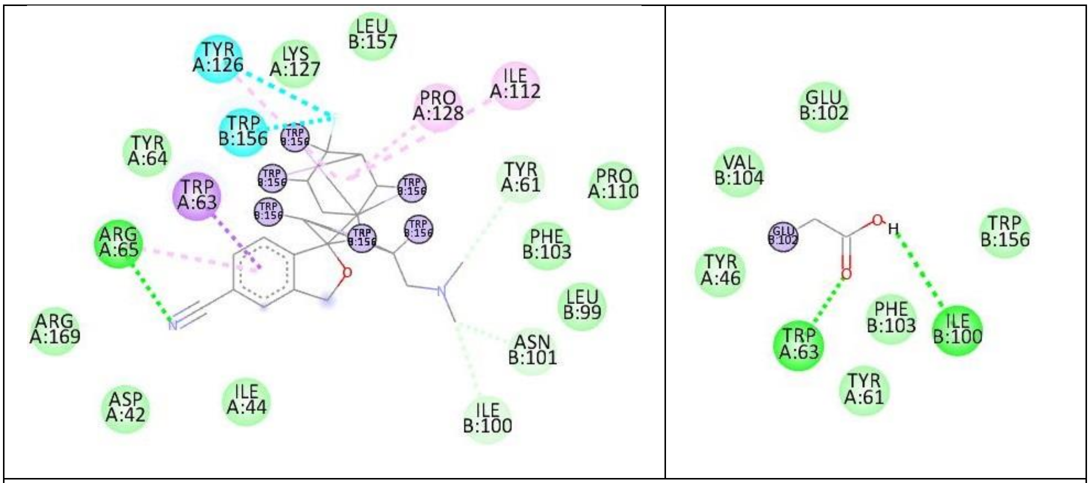


ESCI…AA & 6HIS

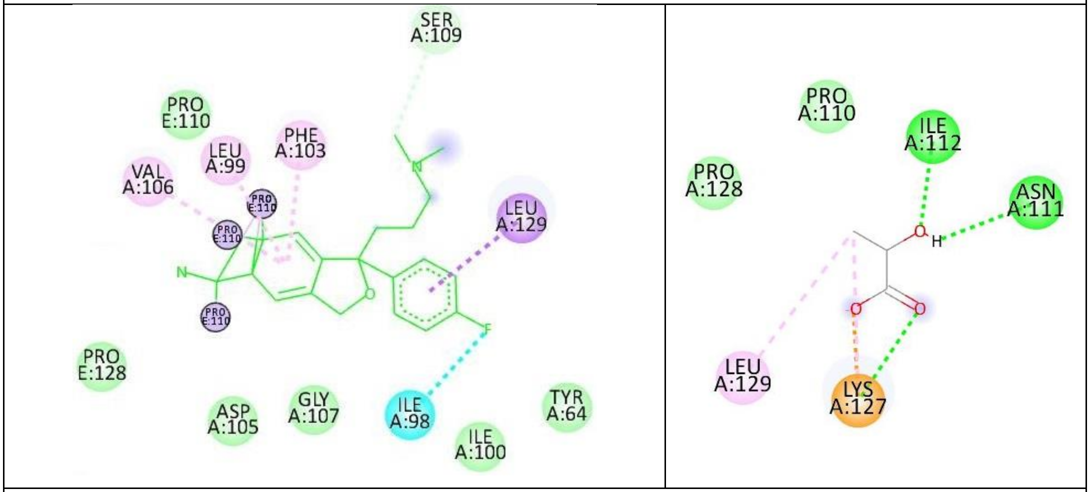
ESCI…LA & 6HIS

Figure 2. Best poses of the complexes docked in 6HIS.

In our previous study, the docking of the drug ESCI alone indicated three hydrogen bonding interactions with the amino acid residues of 6HIS as Asn$_{A50}$, Val$_{A51}$ and Tyr$_{A223}$ as well as some hydrophobic interactions [25]. Due to the successful results in binding energy and hydrogen bonds in the docking of ESCI with 6HIS, the same grid parameters in the receptor were considered for the complexes. Under the same docking conditions, as for ESCI…AA docked in 6HIS, the results have shown three hydrogen bonding interactions through both its drug ESCI and AA, namely one hydrogen bond of 3.89 Å linking the drug ESCI to Arg$_{A65}$ amino acid, as well as two interactions formed between AA and Trp$_{A63}$ or Ile$_{B100}$ (of 5.08 and 6.20 Å correspondingly).



ESCI…AA & 6HIS



ESCI…LA & 6HIS

Figure 3. 2D interactions of ESCI, AA and LA with the amino acids of 6HIS.

ESCI is involved a π–π stacking with Trp$_{A63}$. Hydrophobic interactions as π, alkyl and mixed π/alkyl are showed around ESCI as Tpr$_{A63}$, Pro$_{A128}$, Ile$_{A112}$, Tyr$_{A126}$ and Arg$_{A65}$ residues in the ESCI…AA case as displayed in Figure 3. Furthermore, van der Waals (Asp$_{A42}$, Ile$_{A44}$, Tyr$_{A64}$, Lys$_{A127}$, Leu$_{B157}$, Phe$_{B103}$, Leu$_{B99}$, Pro$_{A110}$ and Arg$_{A169}$), non-classical carbon hydrogen bonds (Tyr$_{A61}$, Asn$_{B101}$ and Ile$_{B100}$), covalent bond (Trp$_{B156}$) and halogen interactions (Tyr$_{A126}$ and Trp$_{B156}$) have been seen between the ESCI and residues of 6HIS whereas some interactions such as van der Waals (Tyr$_{A46}$, Val$_{B104}$, Glu$_{B102}$, Phe$_{B103}$, Trp$_{B156}$ and Tyr$_{A61}$) and covalent bond (Glu$_{B102}$) interactions are found for AA and residues.

For ESCI…LA docked in 6HIS, three hydrogen bonding interactions between LA and amino acids Asn$_{A111}$, Ile$_{A112}$ and Lys$_{A127}$ (of 3.81, 4.24 and 4.39 Å, respectively) are shown whereas the results have suggested no formation of the classical hydrogen bonding between ESCI and 6HIS. ESCI is involved in a π–σ stacking with Leu$_{A129}$. Hydrophobic interactions as π, alkyl and mixed π/alkyl are shown around ESCI as Leu$_{A99}$, Phe$_{A103}$, Leu$_{A129}$, and Val$_{A106}$ amino acids (Figure 3). Moreover, van der Waals interactions (Asp$_{A105}$, Gly$_{A107}$, Ile$_{A100}$, Try$_{A64}$, Pro$_{E110}$ and Pro$_{E128}$), non-classical carbon hydrogen bond (Ser$_{A109}$), covalent bond (Pro$_{E110}$) and halogen interaction (Ile$_{A98}$) are showed between the ESCI and residues of 6HIS while van der Waals (Pro$_{A110}$ and Pro$_{A128}$), alkyl hydrophobic (Leu$_{A129}$ and Lys$_{A127}$) and electrostatic attractive charge (Lys$_{A127}$) interactions are found for LA and residues.

When evaluating all docking results, ESCI…AA and ESCI…LA couples show very good binding for the protein 6HIS. It indicates that the associations of AA and LA with the drug ESCI might enhance their antidepressant-like action. In addition, these results agree with existing experimental results [23, 24]. Further, the ESCI…AA interaction is found theoretically to be performing better than the drug ESCI alone and ESCI…LA couple docked in 6HIS.

4. CONCLUSION

In the present research, the most possible interaction sites and mechanisms of ESCI with AA and LA and some important structural parameters, and diagnostic vibrational bands and chemical reactivities of interacted couples were investigated by DFT and QTAIM. Further, the protein – ligand binding interactions between 6HIS and the interacted couples were simulated by the molecular docking method. The results suggest that OH…N site interaction for both ESCI…AA and ESCI…LA is the most stronger based on the binding energy calculations whereas OH…N≡C sites are found to be more reactive than others. The most reactive and the most stable interactions would be preferable for the purpose of use. Molecular docking results indicate that both associations might enhance the ESCI antidepressant-like action. Further, ESCI…AA & 6HIS association has performed better than the drug ESCI alone and ESCI…LA docked in the binding pocket of 6HIS. In addition, it seems that the theoretical procedures carried out at the molecular level with physical and chemical parameters can complement clinical trials or medical experiments, or even venture with some confidence into experimentally unexplored territory.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

AUTHORSHIP CONTRIBUTIONS

All authors contributed equally.

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ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ
B- TEORİK BİLİMLER

Eskişehir Technical University Journal of Science and Technology B- Theoretical Sciences

2024, 12(1), pp. 32-54, DOI: 10.20290/estubtdb.1356881

RESEARCH ARTICLE

A COMPREHENSIVE STUDY ON SOFT BINARY PIECEWISE DIFFERENCE OPERATION

Aslıhan SEZGİN [1,*], **Hamza ÇALIŞICI** [2]

[1] Department of Mathematics and Science Education, Faculty of Education, Amasya University, Amasya, Turkey
[2] Department of Mathematics and Science Education, Faculty of Education, Ondokuz Mayıs University, Samsun, Turkey

ABSTRACT

Soft set theory, developed by Molodtsov, has been applied both theoretically and practically in many fields. It is a useful mathematical tool for handling uncertainty. Numerous variations of soft set operations, the crucial concept for the theory, have been described and used since its introduction. In this paper, we explore more about soft binary piecewise difference operation (defined first as "difference of soft sets") and its whole properties are examined especially in comparison with the basic properties of difference operation in classical set theory. Several striking properties of soft binary piecewise operations are obtained as analogous to the characteristic of difference operation in classical set theory. Also, we show that the collection of all soft sets with a fixed parameter set together with the soft binary piecewise difference operation is a bounded BCK-algebra.

Keywords: Soft sets, Soft set operations, Soft binary piecewise difference operation

1. INTRODUCTION

Due to the existence of some types of uncertainty, we are unable to employ traditional ways effectively to address issues in many domains, including engineering, environmental and health sciences, and economics. Molodtsov [1], in 1999, proposed Soft Set Theory as a mathematical method to deal with these uncertainties. Since then, this theory has been applied to a variety of fields, including information systems, decision-making, optimization theory, game theory, operations research, measurement theory, and some algebraic structures. The initial contributions to soft set operations were stated by Maji et al. [2] and Pei and Miao [3]. Following this, Ali et al. [4] introduced and discussed several soft set operations, including restricted and extended soft set operations. Sezgin and Atagün [5] discussed the basic properties of soft set operations and the connections between them. They also investigated and defined the idea of restricted symmetric difference of soft sets. A brand-new soft set operation called "extended difference of soft sets" was presented by Sezgin et al. [6]. Stojanovic [7] introduced the term "extended symmetric difference of soft sets" and its characteristics were investigated. The two main categories into which the operations of soft set theory fall, according to the soft set literature, are restricted soft set operations and extended soft set operations. Soft binary piecewise operations were defined by Yavuz [8], who also carefully analyzed their core characteristics. Since the creation of new soft set operations and derivation of their algebraic properties as well as the introduction of new soft set operations and their implemantations offer new perspectives for solving parametric data problems, the operations of soft sets are the fundemantal concepts of soft set theory, and thus soft set operations have been extensively studied since 2003. For more details, we refer to [9-36].

There is a lot of algebra related to logic. Boolean algebra is related to traditional two valued Aristotelean logic. MV algebra is suitable for multi-valued logic. BCI/BCK algebra generalizes the concept of set algebra of sets with the set subtraction as the only non-nullary operation, while these algebras generalize the algebra of implication. The concept of BCI/BCK algebra was introduced by Imai and Iseki [37] to

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*Corresponding Author: aslihan.sezgin@amasya.edu.tr

Received: 12.09.2023 Published: 27.02.2024

study non-classical propositional logic. Although soft sets over algebraic structures have been studied extensively with the initial study of soft groups, the soft set algebras themselves have been studied extensively by [10,18,38,39].

There are five different types of difference operation defined in the soft set literature; one of which is restricted difference of soft sets defined in [4], the other is extended difference of soft sets defined in [6], the other is complementary extended difference of soft sets defined in [9], the other is complementary soft binary piecewise difference operation defined in [28]. Eren [17], in 2019, introduced a brand-new class of soft difference operations, which we call here as "soft binary piecewise difference" to avoid confusion, and the core characteristics of the operation together with the distribution laws were analyzed. This paper is a following study of [17] and we aim to enrich the paper [17] by investigating more on soft binary piecewise difference operation.

In this study, we especially examine the full algebaric properties of this soft set operation comparatively with the basic properties of difference operation existing in classical set theory and we obtain many interesting similarities. Moreover, we prove that the set of all the soft sets with a fixed parameter set together with the soft binary piecewise difference operation is a BCK-algebra. This paper is arranged in the following manner. In Section 2, we recall the preliminary concepts in soft set theory together with BCK-algebras. In Section 3, definition and the example of soft binary piecewise difference operation defined in [17] are reminded. The full analysis of the algebraic properties of the new operation, including closure, associativity, unit, inverse element, and abelian property, are then examined. Besides, the properties of this soft set opertaion are handled comparatively with the difference operation in classical set theory and we obtain stunning analogies. In the same section, it is proved that the set of all the soft sets with a fixed parameter set with respect to soft binary piecewise difference operation is a BCK algebra. In the conclusion section, we put into focus the meaning of the study's findings and its potential influence on the field.

2. PRELIMINARIES

Definition 2.1. [1] Let U be the universal set, E be the parameter set, $\mathrm{P(U)}$ be the power set of U and $\mathrm{A} \subseteq \mathrm{E}$. A pair (F, A) is called a soft set over U where F is a set-valued function such that $\mathrm{F}: \mathrm{A} \rightarrow \mathrm{P(U)}$.

Throughout this paper, the collections of all the soft sets defined over U is designated by $\mathrm{S_E(U)}$. Let A be a fixed subset of E and $\mathrm{S_A(U)}$ be the collection of all those soft sets over U with the fixed parameter set A. Clearly, $\mathrm{S_A(U)}$ is a subset of $\mathrm{S_E(U)}$.

Definition 2.2. [4] (K, W) is called a relative null soft set (with regard to W), denoted by $\emptyset_{\mathrm{W}}$, if $\mathrm{K}(\vartheta) = \emptyset$ for all $\vartheta \in \mathrm{W}$ and (K, W) is called a relative whole soft set (with regard to W), denoted by $\mathrm{U_W}$ if $\mathrm{K}(\vartheta) = \mathrm{U}$ for all $\vartheta \in \mathrm{W}$. The relative whole soft set $\mathrm{U_E}$ with regard to E is called the absolute soft set over U. We shall denote by $\emptyset_{\emptyset}$ the unique soft set over U with an empty parameter set, which is called the empty soft set over U. Note that by $\emptyset_{\emptyset}$ and by $\emptyset_{\mathrm{A}}$ are different soft sets over U [10].

Definition 2.3. [3] For two soft sets (K, W) and $(\mathrm{T}, \text{Ş})$, (K, W) is a soft subset of $(\mathrm{T}, \text{Ş})$ and it is denoted by $(\mathrm{K}, \mathrm{W}) \tilde{\subseteq} (\mathrm{T}, \text{Ş})$, if $\mathrm{W} \subseteq \text{Ş}$ and $\mathrm{K}(\vartheta) \subseteq \mathrm{T}(\vartheta)$, $\forall \vartheta \in \mathrm{W}$. Two soft sets (K, W) and $(\mathrm{T}, \text{Ş})$ are said to be soft equal if (K, W) is a soft subset of $(\mathrm{T}, \text{Ş})$ and $(\mathrm{T}, \text{Ş})$ is a soft subset of (K, W).

Definition 2.4. [4] The relative complement of a soft set (K, W), denoted by $(\mathrm{K}, \mathrm{W})^{\mathrm{r}}$, is defined by $(\mathrm{K}, \mathrm{W})^{\mathrm{r}} = (\mathrm{K}^{\mathrm{r}}, \mathrm{W})$, where $\mathrm{K}^{\mathrm{r}}: \mathrm{W} \rightarrow \mathrm{P(U)}$ is a mapping given by $(\mathrm{K}, \mathrm{W})^{\mathrm{r}} = \mathrm{U} \setminus \mathrm{W}(\vartheta)$ for all $\vartheta \in \mathrm{W}$. From now on, $\mathrm{U} \setminus \mathrm{K}(\vartheta) = [\mathrm{K}(\vartheta)]'$ will be designated by $\mathrm{K}'(\vartheta)$ for the sake of ease.

Soft set operations can be grouped into the following categories as a summary: If " Θ " is used to denote the set operations (Namely, Θ here can be $\cap$, $\cup$, $\setminus$, Δ), then the following soft set operations are defined as follows:

Definition 2.5. [4,5] Let (K, W) and $(T, \text{Ş})$ be soft sets over U. The restricted Θ operation of (K, W) and $(T, \text{Ş})$ is the soft set (B,X), denoted by, $(K, W)\Theta_R(T, \text{Ş}) = (B, X)$, where $X = W \cap \text{Ş} \neq \emptyset$ and $\forall\vartheta \in X$, $B(\vartheta) = K(\vartheta)\ \Theta\ T(\vartheta)$. Here note that if $W \cap \text{Ş} = \emptyset$, then $(K, W)\Theta_R(T, \text{Ş}) = \emptyset_\emptyset$ [10].

Definition 2.6. [2,4,6,7,11] Let (K, W) and $(T, \text{Ş})$ be soft sets over U. The extended Θ operation of (K, W) and $(T, \text{Ş})$ is the soft set (B,X), denoted by, $(K, W)\Theta_\varepsilon(T, \text{Ş}) = (B, X)$, where $X = W \cup \text{Ş}$ and $\forall\vartheta \in X$,

$$B(\vartheta) = \begin{cases} K(\vartheta), & \vartheta \in W \setminus \text{Ş}, \\ T(\vartheta), & \vartheta \in \text{Ş} \setminus W, \\ K(\vartheta)\Theta T(\vartheta), & \vartheta \in W \cap \text{Ş}. \end{cases}$$

Definition 2.7. [9,12,23] Let (K, W) and $(T, \text{Ş})$ be soft sets over U. The complementary extended Θ operation of (K, W) and $(T, \text{Ş})$ is the soft set (B,X), denoted by, $(K, W)\Theta_\varepsilon(T, \text{Ş}) = (B, X)$, where $X = W \cup \text{Ş}$ and $\forall\vartheta \in X$,

$$B(\vartheta) = \begin{cases} K'(\vartheta), & \vartheta \in W \setminus \text{Ş}, \\ T'(\vartheta), & \vartheta \in \text{Ş} \setminus W, \\ K(\vartheta)\Theta T(\vartheta), & \vartheta \in W \cap \text{Ş}. \end{cases}$$

Definition 2.8. [8,17] Let (K, W) and $(T, \text{Ş})$ be soft sets over U. The soft binary piecewise Θ operation of (K, W) and $(T, \text{Ş})$ is the soft set, (B,W), denoted by, $(K, W)\widetilde{\Theta}(T, \text{Ş}) = (B, W)$, where $\forall\vartheta \in W$,

$$B(\vartheta) = \begin{cases} K(\vartheta), & \vartheta \in W \setminus \text{Ş} \\ K(\vartheta)\ \Theta\ T(\vartheta), & \vartheta \in W \cap \text{Ş} \end{cases}$$

A set X containing a binary operation ζ and a constant 0 is called a BCI algebra if it satisfies

BCI-1 $((a\ \zeta\ b)\ \zeta\ (a\ \zeta\ c))\ \zeta\ (c\ \zeta\ b) = 0$, BCI-2 $(a\ \zeta\ (a\ \zeta\ b))\ \zeta\ b = 0$, BCI-3 $a\ \zeta\ a = 0$, BCI-4 $a\ \zeta\ b = 0$ and $b\ \zeta\ a = 0$ imply $a = b$. A BCI algebra is called a BCK algebra if it additionally satisfies: BCK-5 $0\ \zeta\ a = 0$.A BCK algebra X is called bounded if there exists some element $1 \in X$ such that $a * \zeta\ 1 = 0$ for all $x \in X$. For a bounded BCK algebra X, if an element $a \in X$ satisfies $1\ \zeta\ (1\ \zeta\ a) = a$, then a is called an involution.

3. MORE ON THE PROPERTIES OF SOFT BINARY PIECEWISE DIFFERENCE OPERATION

Definition 3.1. [17] Let $(V, \aleph)$ and (Y, I) be soft sets over U. The complementary soft binary piecewise difference operation of $(V, \aleph)$ and (Y, I) is the soft set $(Q, \aleph)$, denoted by $(V, \aleph)\tilde{\setminus}(Y, I) = (Q, \aleph)$, where $\forall\vartheta \in \aleph$,

$$Q(\vartheta) = \begin{cases} V(\vartheta), & \vartheta \in \aleph \setminus I \\ V(\vartheta) \setminus Y(\vartheta), & \vartheta \in \aleph \cap I \end{cases}$$

Here note that, in [17], the above definition was given as "difference of soft sets"; however since there are five types of difference of soft sets operations in the literature, in order to avoid confusion, we prefer to use "soft binary piecewise operation" for the above definition.

Example 3.2. Let $E=\{e_1,e_2,e_3,e_4\}$ be the parameter set $Q=\{e_1, e_3\}$ and $I=\{e_2, e_3, e_4\}$ be the subsets of E and $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the initial universe set. Assume that $(V,\aleph)$ and (Y,I) are the soft sets over U defined as follows:

$(V,\aleph) = \{(e_1,\{h_2,h_5\}), (e_3,\{h_1,h_2,h_5\})\}$

$(Y,I)=\{(e_2,\{h_1,h_4,h_5\}), (e_3,\{h_2,h_3,h_4\}),(e_4,\{h_3,h_5\})\}$. Let $(V,\aleph)\ \tilde{\setminus}\ (Y,I)=(Q,\aleph)$. Then,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Since $\aleph=\{e_1, e_3\}$ and $\aleph\setminus I=\{e_1\}$, so $Q(e_1) = V(e_1)=\{h_2,h_5\}$. And since $\aleph\cap I=\{e_3\}$, so $Q(e_3)= V(e_3)\setminus Y(e_3)=\{h_1, h_5\}$. Thus, $(V,\aleph)\ \tilde{\setminus}(Y,I)=\{(e_1,\{h_1,h_3,h_4\}), (e_3, \{h_1, h_5\})\}$.

In classical set theory, $V\setminus Y=V\cap Y'$. Now, we have the following analogy.

Theorem 3.3 $(V,\aleph)\ \tilde{\setminus}(Y,R) =(V,\aleph)\cap_{\varepsilon} (Y, \aleph\cap R)^r$ [17].

Theorem 3.4.$(V,\aleph)\ \tilde{\setminus}(Y,R) =(V,\aleph)\ \tilde{\cap}\ (Y, R)^r$.

Proof: Let $(V,\aleph)\ \tilde{\cap}\ (Y, R)^r=(Q,\aleph)$, where $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus R \\ V(\vartheta)\cap Y'(\vartheta), & \vartheta\in\aleph\cap R \end{cases}$$

Thus, $(Q,\aleph)= (V,\aleph)\ \tilde{\setminus}(Y,R)$.

Theroem 3.5. (Algebraic properties of the operation)

1) The set $S_E(U)$ is closed under the operation $\tilde{\setminus}$.

Proof: It is clear that $\tilde{\setminus}$ is a binary operation in $S_E(U)$. That is,

$$\tilde{\setminus}: S_E(U)\times S_E(U)\to S_E(U)$$
$$((V,\aleph) , (Y,I)) \to(V,\aleph)\ \tilde{\setminus}\ (Y,I)= (Q,\aleph)$$

In classical set theory, difference operation does not have associative property. Now, we have the following analogy:

2) $[(V,\aleph)\tilde{\setminus}(Y,\aleph)]\ \tilde{\setminus}(Q,\aleph) \neq (V,\aleph)\ \tilde{\setminus}[(Y,\aleph)\ \tilde{\setminus}(Q,\aleph)]$

Proof: Let $(V,\aleph)\ \tilde{\setminus}\ (Y,\aleph)=(T,\aleph)$, where $\forall\vartheta\in\aleph$;

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(T, \aleph)\ \tilde{\setminus}(Q, \aleph)=(M, \aleph)$, where $\forall\vartheta\in\aleph$;

$$M(\vartheta)=\begin{cases} T(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ T(\vartheta)\setminus Q(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus,

$$M(\vartheta)=\begin{cases} T(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta)]\setminus Q(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(Y,\aleph)\tilde{\setminus}\ (Q,\aleph) =(L,\aleph)$, where $\forall\vartheta\in\aleph$;

$$L(\vartheta)=\begin{cases} Y(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ Y(\vartheta)\setminus Q(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(V,\aleph)\ \tilde{\setminus}(L,\aleph) =(D,\aleph)$, where $\forall\vartheta\in\aleph$;

$$D(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\backslash\aleph=\emptyset \\ V(\vartheta)\backslash L(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus,

$$D(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\backslash\aleph=\emptyset \\ V(\vartheta)\backslash[Y(\vartheta)\backslash Q(\vartheta)], & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

It is seen that $(M,\aleph)$ and $(D,\aleph)$ are not the same soft sets.

That is, on the soft sets whose parameter set are the same, the operation $\tilde{\backslash}$ does not have associativity property. Moreover, we have the following:

3) $[(V,\aleph)\,\tilde{\backslash}\,(Y,I)]\,\tilde{\backslash}(Q,Z)\neq(V,\aleph)\,\tilde{\backslash}[(Y,I)\,\tilde{\backslash}(Q,Z)]$

Proof: Let $(V,\aleph)\,\tilde{\backslash}\,(Y,I)=(T,\aleph)$, where $\forall\vartheta\in\aleph$;

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\backslash I \\ V(\vartheta)\backslash Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Let $(T,\aleph)\,\tilde{\backslash}(Q,Z)=(M,\aleph)$, where $\forall\vartheta\in\aleph$;

$$M(\vartheta)=\begin{cases} T(\vartheta), & \vartheta\in\aleph\backslash Z \\ T(\vartheta)\backslash Q(\vartheta), & \vartheta\in\aleph\cap Z \end{cases}$$

Thus,

$$M(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in(\aleph\backslash I)\backslash Z=\aleph\cap I'\cap Z' \\ V(\vartheta)\backslash Y(\vartheta), & \vartheta\in(\aleph\cap I)\backslash Z=\aleph\cap I\cap Z' \\ V(\vartheta)\backslash Q(\vartheta), & \vartheta\in(\aleph\backslash I)\cap Z=\aleph\cap I'\cap Z \\ [V(\vartheta)\backslash Y(\vartheta)]\backslash Q(\vartheta), & \vartheta\in(\aleph\cap I)\cap Z=\aleph\cap I\cap Z \end{cases}$$

Let $(Y,I)\,\tilde{\backslash}\,(Q,Z)=(K,I)$, where $\forall\vartheta\in I$;

$$K(\vartheta)=\begin{cases} Y(\vartheta), & \vartheta\in I\backslash Z \\ Y(\vartheta)\backslash Q(\vartheta), & \vartheta\in I\cap Z \end{cases}$$

Let $(V,\aleph)\,\tilde{\backslash}\,(K,I)=(S,\aleph)$, where $\forall\vartheta\in\aleph$;

$$S(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\backslash I \\ V(\vartheta)\backslash K(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Thus,

$$S(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\backslash I \\ V(\vartheta)\backslash Y(\vartheta), & \vartheta\in\aleph\cap(I\backslash Z)=\aleph\cap I\cap Z' \\ V(\vartheta)\backslash[Y(\vartheta)\backslash Q(\vartheta)], & \vartheta\in\aleph\cap(I\cap Z)=\aleph\cap I\cap Z \end{cases}$$

Here let handle $\vartheta\in\aleph\backslash I$ in the first line of $S(\vartheta)$. Since $\aleph\backslash I=\aleph\cap I'$, if $\vartheta\in I'$, then $\vartheta\in Z\backslash I$ or $\vartheta\in(I\cup Z)'$. Hence, if $\vartheta\in\aleph\backslash I$, then $\vartheta\in\aleph\cap I'\cap Z'$ or $\vartheta\in\aleph\cap I'\cap Z$. Thus, it is seen that $(M,\aleph)$ and $(S,\aleph)$ are not the same soft set. That is, for the soft sets whose parameter set are not the same, the operation $\tilde{\backslash}$ does not have associativity property in the set $S_E(U)$.

In classical set theory, difference operation does not have commutative property. Now, we have the following analogy:

4) $(V,\aleph)\,\tilde{\setminus}\,(Y,I)\neq(Y,I)\,\tilde{\setminus}\,(V,\aleph)$

Proof: Let $(V,\aleph)\,\tilde{\setminus}(Y,I)=(Q,\aleph)$. Then, $\forall\vartheta\in\aleph$;

,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Let $(Y,I)\,\tilde{\setminus}\,(V,\aleph)=(T,I)$. Then $\forall\vartheta\in I$;

$$T(\vartheta)=\begin{cases} Y(\vartheta), & \vartheta\in I\setminus\aleph \\ Y(\vartheta)\setminus\aleph(\vartheta), & \vartheta\in I\cap\aleph \end{cases}$$

Here, while the parameter set of the soft set of the left hand side is $\aleph$; the parameter set of the soft set of the right hand side is I. Thus, by the definition of soft equality

$$(V,\aleph)\,\tilde{\setminus}\,(Y,I)\neq(Y,I)\,\tilde{\setminus}(V,\aleph).$$

Hence, the operation $\tilde{\setminus}$ does not have commutative property in the set $S_E(U)$, where the parameter sets of the soft sets are different. Moreover, it is easy to see that $(V,\aleph)\,\tilde{\setminus}\,(Y,\aleph)\neq(Y,\aleph)\,\tilde{\setminus}(V,\aleph)$ since $V(\vartheta)\setminus Y(\vartheta)\neq Y(\vartheta)\setminus\aleph(\vartheta)$. That is, the operation $\tilde{\setminus}$ does not have commutative property when the parameter sets of the soft sets are the same.

5) $(V,\aleph)\tilde{\setminus}\,(V,\aleph)=\emptyset_{\aleph}$.

Proof: Let $(V,\aleph)\,\tilde{\setminus}\,(V,\aleph)=(Q,\aleph)$, where $\forall\vartheta\in\aleph$;

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus V(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Here $\forall\vartheta\in\aleph$; $Q(\vartheta)=V(\vartheta)\setminus V(\vartheta)=\emptyset$, thus $(Q,\aleph)=\emptyset_{\aleph}$.

That is, the operation $\tilde{\setminus}$ does not have idempotency property in the set $S_E(U)$.

6) $(V,\aleph)\,\tilde{\setminus}\,\emptyset_K=(V,\aleph)$ [17].

Here note that, for the soft sets (no matter what the parameter set is), null soft sets with respect to any parameter set (Here, K may be E, $\aleph$, $\emptyset$, or any set) is the right identity element for the operation $\tilde{\setminus}$ in the set $S_E(U)$.

7) $\emptyset_K\tilde{\setminus}\,(V,\aleph)=\emptyset_K$ [17].

Here note that, for the soft sets (no matter what the parameter set is), null soft sets with respect to any parameter (Here, K may be E, $\aleph$, $\emptyset$, or any set) is the left-absorbing element for the operation $\tilde{\setminus}$ in the set $S_E(U)$.

8) $(V,\aleph)\,\tilde{\setminus}\,U_{\aleph}=(V,\aleph)\,\tilde{\setminus}U_E=\emptyset_{\aleph}$ [17].

9) $U_{\aleph}\tilde{\setminus}(V,\aleph)=(V,\aleph)^r$ and $U_E\tilde{\setminus}(V,\aleph)\neq(V,\aleph)^r$ [17].

10) $(V,\aleph)\,\tilde{\setminus}(V,\aleph)^r=(V,\aleph)$.

Proof: Let $(V,\aleph)^r=(Q,\aleph)$. Hence, $\forall\vartheta\in\aleph$; $Q(\vartheta)=V'(\vartheta)$. Let $(V,\aleph)\,\tilde{\setminus}(Q,\aleph)=(T,\aleph)$, where $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\cap Q'(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Hence, $\forall\vartheta\in\aleph$; $T(\vartheta)= V(\vartheta)\cap Q'(\vartheta)=V(\vartheta)\cap V(\vartheta)= V(\vartheta)$, thus $(T,\aleph)=(V,\aleph)$.

Note that, relative complement of every soft set is its own right identity element for the operation $\tilde{\setminus}$ in the set $S_E(U)$.

11) $(V,\aleph)^r\,\tilde{\setminus}(V,\aleph) = (V,\aleph)^r$.

Proof: Let $(V,\aleph)^r=(Q,\aleph)$. Hence, $\forall\vartheta\in\aleph$; $Q(\vartheta)=V'(\vartheta)$. Let $(Q,\aleph)\tilde{\setminus}(V,\aleph)=(T,\aleph)$, where $\forall\vartheta\in\aleph$;

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ Q(\vartheta)\cap V'(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Hence, $\forall\vartheta\in\aleph$; $T(\vartheta)= Q(\vartheta)\cap V'(\vartheta)= V'(\vartheta)\cap V'(\vartheta)= V'(\vartheta)$, thus $(T,\aleph) = (V,\aleph)^r$.

Note that, relative complement of a soft set is its own left-absorbing element for the operation $\tilde{\setminus}$ in the set $S_E(U)$.

12) $[(V,\aleph)\tilde{\setminus}(Y,I)]^r=(V,\aleph)\overset{*}{\underset{+}{\sim}}(Y,I)$.

Proof: Let $(V,\aleph)\tilde{\setminus}(Y,I)=(Q,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\cap Y'(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Let $(Q,\aleph)^r=(T,\aleph)$, so $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} V'(\vartheta), & \vartheta\in\aleph\setminus I \\ V'(\vartheta)\cup Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Thus, $(T,\aleph)=(V,\aleph)\overset{*}{\underset{+}{\sim}}(Y,I)$.

In classical set theory, $V\cap Y=U \Leftrightarrow V=U$ and $Y=U$. Now, we have the following:

13) $(V,\aleph)\tilde{\setminus}(Y,\aleph)= U_\aleph \Leftrightarrow (V,\aleph)=U_\aleph$ and $(Y,\aleph)=\emptyset_\aleph$.

Proof: Let $(V,\aleph)\tilde{\setminus}(Y,\aleph)=(T,\aleph)$. Hence, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph-\aleph=\emptyset \\ V(\vartheta)\cap Y'(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Since $(T,\aleph)=U_\aleph$, $\forall\vartheta\in\aleph$, $T(\vartheta)=U$. Hence, $\forall\vartheta\in\aleph$, $T(\vartheta)= V(\vartheta)\cap Y'(\vartheta)=U \Leftrightarrow \forall\vartheta\in\aleph$, $V(\vartheta)=U$ and $Y'(\vartheta)=U \Leftrightarrow \forall\vartheta\in\aleph$, $V(\vartheta)=U$ and $Y(\vartheta)=\emptyset \Leftrightarrow (V,\aleph)=U_\aleph$ and $(Y,\aleph)=\emptyset_\aleph$.

In classical set theory, for all A, $\emptyset\subseteq A$. Now, we have the following:

14) $\emptyset_\aleph\,\tilde{\subseteq}(V,\aleph)\tilde{\setminus}(Y,I)$ and $\emptyset_I\,\tilde{\subseteq}(Y,I)\tilde{\setminus}(V,\aleph)$.

In classical set theory, for all A, $A \subseteq U$. Now, we have the following:

15) $(V,\aleph)\,\tilde{\setminus}\,(Y,I) \,\tilde{\subseteq}\, U_{\aleph}$ and $(Y,I)\,\tilde{\setminus}\,(V,\aleph) \,\tilde{\subseteq}\, U_{I}$.

In classical set theory, $V\setminus Y \subseteq V$ and $Y\setminus V\subseteq Y$. Moreover, $V\setminus Y \subseteq Y'$ and $Y\setminus V \subseteq V'$ Now, we have the following analogy:

16) $(V,\aleph)\,\tilde{\setminus}\,(Y,I) \,\tilde{\subseteq}\,(V,\aleph)$ and $(Y,I)\,\tilde{\setminus}\,(V,\aleph) \,\tilde{\subseteq}\,(Y,I)$. Moreover, $(V,\aleph)\,\tilde{\setminus}\,(Y,\aleph) \,\tilde{\subseteq}\,(Y,\aleph)^{r}$ and $(Y,\aleph)\,\tilde{\setminus}\,(V,\aleph) \,\tilde{\subseteq}\,(V,\aleph)^{r}$.

Proof: Let $(V,\aleph)\,\tilde{\setminus}\,(Y,I) =(Q,\aleph)$. First of all, $\aleph \subseteq \aleph$. Moreover, $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\cap Y'(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Since $\forall\vartheta\in\aleph\setminus I$; $Q(\vartheta)\subseteq V(\vartheta)$ and $\forall\vartheta\in\aleph\cap I$; $Q(\vartheta)=V(\vartheta)\cap Y'(\vartheta) \subseteq V(\vartheta)$, thus $\forall\vartheta\in\aleph$; $Q(\vartheta)\subseteq V(\vartheta)$. This shows that $(Q,\aleph) = (V,\aleph)\tilde{\setminus}(Y,I)\tilde{\subseteq}(V,\aleph)$. $(Y,I)\tilde{\setminus}(V,\aleph)\tilde{\subseteq}(Y,I)$ can be shown similarly.

Let $(V,\aleph)\,\tilde{\setminus}\,(Y,\aleph) =(K,\aleph)$. First of all, $\aleph \subseteq \aleph$. Moreover, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\cap Y'(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Since $\forall\vartheta\in\aleph$; $K(\vartheta)=V(\vartheta)\cap Y'(\vartheta)\subseteq Y'(\vartheta)$, this shows that $(K,\aleph) = (V,\aleph)\tilde{\setminus}(Y,\aleph)\tilde{\subseteq}(Y,\aleph)^{r}$. $(Y,\aleph)\tilde{\setminus}(V,\aleph) \tilde{\subseteq}(V,\aleph)^{r}$ can be shown similarly.

In classial set theory; $V=(V\setminus Y) \cup (V\cap Y)$ and $Y=(Y\setminus V) \cup (Y\cap V)$. Now, we have the following analogy:

17) $(V,\aleph) =[(V,\aleph)\,\tilde{\setminus}\,(Y,I)\,]\,\tilde{\cup}\,[(V,\aleph)\,\tilde{\cap}\,(Y,I)\,]$ and $(Y,I) =[(Y,I)\,\tilde{\setminus}\,(V,\aleph)\,]\,\tilde{\cup}\,[(Y,I)\,\tilde{\cap}\,(V,\aleph)]$.

Proof: Let $(V,\aleph)\,\tilde{\setminus}\,(Y,I) =(Q,\aleph)$, where $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

and $(V,\aleph)\,\tilde{\cap}\,(Y,I) =(K,\aleph)$, where $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\cap Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Let $(Q,\aleph)\,\tilde{\cup}\,(K,\aleph) =(T,\aleph)$, where $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ Q(\vartheta)\cup K(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus,

$$T(\vartheta)=\begin{cases} V(\vartheta)\cup V(\vartheta), & \vartheta\in(\aleph\setminus I)\cap(\aleph\setminus I)=(\aleph\setminus I) \\ V(\vartheta)\cup[V(\vartheta)\cap Y(\vartheta)], & \vartheta\in(\aleph\setminus I)\cap(\aleph\cap I)=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta)]\cup V(\vartheta), & \vartheta\in(\aleph\cap I)\cap(\aleph\setminus I)=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta)]\cup[V(\vartheta)\cap Y(\vartheta)], & \vartheta\in(\aleph\cap I)\cap(\aleph\cap I)=(\aleph\cap I) \end{cases}$$

Hence,

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I\\ [V(\vartheta)\setminus Y(\vartheta)]\ \cup [V(\vartheta)\cap Y(\vartheta)], & \vartheta\in\aleph\cap I\end{cases}$$

Since, $\forall\vartheta\in\aleph$, $[V(\vartheta)\setminus Y(\vartheta)]\cup[V(\vartheta)\cap Y(\vartheta)]=V(\vartheta)$, $T(\vartheta)=V(\vartheta)$. Thus, $(T,\aleph)=(V,\aleph)$. The other can be shown similarly.

In classical set theory, $V\cup Y=(V\setminus Y)\cup Y$ and $Y\cup V=(Y\setminus V)\cup V$. Now, we have the following analogy.

18) $(V,\aleph)\ \tilde{\cup}\ (Y,I)=[(V,\aleph)\ \tilde{\setminus}(Y,I)\ ]\ \tilde{\cup}(Y,I)$ and $(Y,I)\ \tilde{\cup}\ (V,\aleph)=[(Y,I)\ \tilde{\setminus}\ (V,\aleph)\ ]\ \tilde{\cup}(V,\aleph)$.

Proof: Let $(V,\aleph)\ \tilde{\setminus}(Y,I)=(Q,\aleph)$, where $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I\\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I\end{cases}$$

and $(Q,\aleph)\ \tilde{\cup}(Y,I)=(K,\aleph)$, where $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus I\\ Q(\vartheta)\cup Y(\vartheta), & \vartheta\in\aleph\cap I\end{cases}$$

Thus,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in(\aleph\setminus I)\setminus I=\aleph\setminus I\\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in(\aleph\cap I)\setminus I=\emptyset\\ V(\vartheta)\cup Y(\vartheta), & \vartheta\in(\aleph\setminus I)\cap I=\emptyset\\ [V(\vartheta)\setminus Y(\vartheta)]\cup Y(\vartheta) & \vartheta\in(\aleph\cap I)\cap(\aleph\cap I)=(\aleph\cap I)\end{cases}$$

Since $[V(\vartheta)\setminus Y(\vartheta)]\cup Y(\vartheta)=V(\vartheta)\cup Y(\vartheta)$, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I\\ V(\vartheta)\cup Y(\vartheta), & \vartheta\in\aleph\cap I\end{cases}$$

Thus, $(K,\aleph)=(V,\aleph)\ \tilde{\cup}(Y,\aleph)$. The other can be shown similarly.

In classical set theory, $V\subseteq Y\Leftrightarrow V\setminus Y=\emptyset$. In [17], it was shown that if $(V,\aleph)\ \tilde{\subseteq}(Y,I)$, then $(V,\aleph)\ \tilde{\setminus}(Y,I)=\emptyset_{\aleph}$. For satisfying also the necessity, we have the following:

19) $(V,\aleph)\ \tilde{\subseteq}(Y,\aleph)\ \Leftrightarrow(V,\aleph)\ \tilde{\setminus}(Y,\aleph)=\emptyset_{\aleph}$.

Proof: Let $(V,\aleph)\ \tilde{\subseteq}(Y,\aleph)$. Then, $\forall\vartheta\in\aleph, V(\vartheta)\subseteq Y(\vartheta)$. And let $(V,\aleph)\ \tilde{\setminus}(Y,\aleph)=(Q,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset\\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph\end{cases}$$

Since $\forall\vartheta\in\aleph$, $V(\vartheta)\subseteq Y(\vartheta)$, then $V(\vartheta)\setminus Y(\vartheta)=\emptyset$, and hence $(Q,\aleph)=(V,\aleph)\ \tilde{\setminus}(Y,\aleph)=\emptyset_{\aleph}$. For the converse, we need to show that when $(V,\aleph)\tilde{\setminus}(Y,\aleph)=\emptyset_{\aleph}$, then $(V,\aleph)\ \tilde{\subseteq}(Y,\aleph)$. In order to show this, let $(V,\aleph)\ \tilde{\setminus}(Y,\aleph)=(T,\aleph)$. Then,

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset\\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph\end{cases}$$

Since, $(T,\aleph)=\emptyset_{\aleph}$, $\forall\vartheta\in\aleph$, $V(\vartheta)\setminus Y(\vartheta)=\emptyset$. Thus, $\forall\vartheta\in\aleph$, $V(\vartheta)\subseteq Y(\vartheta)$. Hence, $(V,\aleph)\,\tilde{\subseteq}(Y,\aleph)$.

In classical set theory, $V\setminus(V\setminus Y)=V\cap Y$ and $Y\setminus(Y\setminus V)=Y\cap V$. Now, we have the following:

20) $(V,\aleph)\ \tilde{\setminus}\,[(V,\aleph)\ \tilde{\setminus}(Y,\aleph)]=(V,\aleph)\ \tilde{\cap}\,(Y,\aleph)$ and $(Y,\aleph)\,\tilde{\setminus}\,[(Y,\aleph)\,\tilde{\setminus}(V,\aleph)]=(Y,\aleph)\ \tilde{\cap}\,(V,\aleph)$.

Proof: Let $(V,\aleph)\,\tilde{\setminus}(Y,\aleph)=(Q,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(V,\aleph)\,\tilde{\setminus}(Q,\aleph)=(K,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus Q(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus(V(\vartheta)\setminus Y(\vartheta)), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Hence, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\cap Y(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus, $\forall\vartheta\in\aleph$, $(K,\aleph)=(V,\aleph)\,\tilde{\setminus}\,(Y,\aleph)$. Moreover $(Y,\aleph)\,\tilde{\setminus}\,[(Y,\aleph)\,\tilde{\setminus}(V,\aleph)]=(Y,\aleph)\,\tilde{\cap}(V,\aleph)$ can be shown similarly.

In classical set theory, $V\setminus(Y\cap V)=V\setminus Y$ and $Y\setminus(V\cap Y)=Y\setminus V$. Now we have the following:

21) $(V,\aleph)\tilde{\setminus}\,[(Y,I)\tilde{\cap}(V,\aleph)]=(V,\aleph)\tilde{\setminus}(Y,I)$ and $(Y,I)\,\tilde{\setminus}\,[(V,\aleph)\ \tilde{\cap}\,(Y,I)]=(Y,I)\,\tilde{\setminus}(V,\aleph)$.

Proof: Let $(Y,I)\,\tilde{\cap}(V,\aleph)=(Q,\aleph)$. Then, $\forall\vartheta\in I$,

$$Q(\vartheta)=\begin{cases} Y(\vartheta), & \vartheta\in I\setminus\aleph \\ Y(\vartheta)\cap V(\vartheta), & \vartheta\in I\cap\aleph \end{cases}$$

Let $(V,\aleph)\,\tilde{\setminus}(Q,I)=(K,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Q(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta) & \vartheta\in\aleph\cap(I\setminus\aleph)=\emptyset \\ V(\vartheta)\setminus[(Y(\vartheta)\cap V(\vartheta)], & \vartheta\in\aleph\cap(I\cap\aleph)=\aleph\cap I \end{cases}$$

Since $V(\vartheta)\setminus[(Y(\vartheta)\cap V(\vartheta)]=V(\vartheta)\setminus Y(\vartheta)$, hence $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Thus, $(K,\aleph) = (V,\aleph)\tilde{\setminus}(Y,I)$. Moreover $(Y,I)\ \tilde{\setminus}\ [(V,\aleph)\tilde{\cap}\ (Y,I)\ ] = (Y,I)\ \tilde{\setminus}(V,\aleph)$ can be shown similarly.

In classical set theory, $V\setminus(V\cap Y)=V\setminus Y$ and $Y\setminus(Y\cap V)=Y\setminus V$. Now, we have the following:

22) $(V,\aleph)\tilde{\setminus}[(V,\aleph)\ \tilde{\cap}(Y,\aleph)\ ] = (V,\aleph)\tilde{\setminus}(Y,\aleph)$ and $(Y,\aleph)\tilde{\setminus}[(Y,\aleph)\tilde{\cap}(V,\aleph)\ ] = (Y,\aleph)\tilde{\setminus}(V,\aleph)$.

Proof: Let $(V,\aleph)\ \tilde{\cap}(Y,\aleph) = (Q,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\cap Y(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(V,\aleph)\ \tilde{\setminus}\ (Q,\aleph) = (K,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus Q(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus[(V(\vartheta)\cap Y(\vartheta)], & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Hence, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus, $\forall\vartheta\in\aleph$, $(K,\aleph)=(V,\aleph)\tilde{\setminus}(Y,\aleph)$. Moreover $(Y,\aleph)\tilde{\setminus}[(Y,\aleph)\tilde{\cap}(V,\aleph)] = (Y,\aleph)\tilde{\setminus}(V,\aleph)$ can be shown similarly.

NOTE: In classical set theory, $V\cap Y=Y\cap V$, hence $V\setminus(Y\cap V)=V\setminus(V\cap Y)=V\setminus Y$ and $Y\setminus(V\cap Y)= Y\setminus(Y\cap V)=Y\setminus V$. However, since $(V,\aleph)\ \tilde{\cap}\ (Y,I)\neq (Y,I)\ \tilde{\cap}(V,\aleph)$; while Theorem 3.4 (21) is satisfied when the parameter sets of soft sets are different; Theorem 3.4. (22) is satisfied only when the parameter sets of soft sets are the same.

In classical set theory, if $V\cap Y=\emptyset$, then $V\setminus Y=V$. Now, we have the following analogy:

23) If $(V,\aleph)\ \tilde{\cap}\ (Y,I) = \emptyset_{\aleph}$, then $(V,\aleph)\ \tilde{\setminus}(Y,I) = (V,\aleph)$.

Proof: Let $(V,\aleph)\ \tilde{\cap}(Y,I) = (Q,\aleph)$. Then, for all $\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\cap Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Since, $(Q,\aleph) = \emptyset_{\aleph}$, then for all $\vartheta\in\aleph$. $Q(\vartheta)=\emptyset$. Thus, for all $\vartheta\in\aleph\setminus I$; $Q(\vartheta)=V(\vartheta)=\emptyset$, and for all $\vartheta\in\aleph\cap I$; $Q(\vartheta)=V(\vartheta)\cap Y(\vartheta)=\emptyset$. Let $(V,\aleph)\ \tilde{\setminus}(Y,I) = (S,\aleph)$. Then, for all $\vartheta\in\aleph$,

$$S(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Since, for all $\vartheta\in\aleph\cap I$; $V(\vartheta\cap Y(\vartheta)=\emptyset$, $V(\vartheta)\setminus Y(\vartheta)=V(\vartheta)$. Therefore,

$$S(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Thus, $(S,\aleph)=(V,\aleph)\,\tilde{\setminus}(Y,I)=(V,\aleph)$.

In classical set theory, $(V\setminus Y)\cap Y=\emptyset$ and $(Y\setminus V)\cap V=\emptyset$. Now, we have the similar analogy:

24) $[(V,\aleph)\,\tilde{\setminus}\,(Y,\aleph)\,]\tilde{\cap}(Y,\aleph)=\emptyset_{\aleph}$ and $[(Y,\aleph)\,\tilde{\setminus}(V,\aleph)\,]\tilde{\cap}(V,\aleph)=\emptyset_{\aleph}$.

Proof: Let $(V,\aleph)\,\tilde{\setminus}\,(Y,\aleph)=(Q,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

And let $(Q,\aleph)\tilde{\cap}(Y,\aleph)=(T,\aleph)$, where $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ Q(\vartheta)\cap Y(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta)]\cap Y(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Hence, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ \emptyset, & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Since $\forall\vartheta\in\aleph$, $T(\vartheta)=\emptyset$, thus $(T,\aleph)=\emptyset_{\aleph}$. Moreover, $[(Y,\aleph)\,\tilde{\setminus}\,(V,\aleph)\,]\,\tilde{\cap}(V,\aleph)=\emptyset_{\aleph}$ can be shown similarly.

NOTE: In classical set theory, $(V\setminus Y)\setminus Y=V\setminus Y$ (as $(V\setminus Y)\cap Y=\emptyset$) and $(Y\setminus V)\setminus V=Y\setminus V$ (as $(Y\setminus V)\cap V=\emptyset$).

As an analogy, we have the following:

25) $[(V,\aleph)\tilde{\setminus}(Y,I)\,]\,\tilde{\setminus}(Y,I)=(V,\aleph)\tilde{\setminus}(Y,I)$ and $[(Y,I)\,\tilde{\setminus}(V,\aleph)\,]\,\tilde{\setminus}(V,\aleph)=(Y,I)\tilde{\setminus}(V,\aleph)$.

Proof: Let $(V,\aleph)\,\tilde{\setminus}\,(Y,I)=(Q,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

And let $(Q,\aleph)\,\tilde{\setminus}\,(Y,I)=(T,\aleph)$, where $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus I \\ Q(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in(\aleph\setminus I)\setminus I=\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in(\aleph\cap I)\setminus I=\emptyset \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in(\aleph\setminus I)\cap I=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta)]\setminus Y(\vartheta), & \vartheta\in(\aleph\cap I)\cap I=\aleph\cap I \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta)], & \vartheta\in\aleph\cap I \end{cases}$$

Thus, $(T,\aleph)=(V,\aleph)\tilde{\setminus}(Y,I)$. Also, $[(Y,I)\tilde{\setminus}(V,\aleph)]\,\tilde{\setminus}(V,\aleph)=(Y,I)\tilde{\setminus}(V,\aleph)$ can be shown simillarly.

In classical set theory, $(V\setminus Y)\cap(Y\setminus V)=\emptyset$. Now, we have the following analogy.

26) $[(V,\aleph)\tilde{\setminus}(Y,\aleph)]\,\tilde{\cap}\,[(Y,\aleph)\tilde{\setminus}(V,\aleph)]=\emptyset_{\aleph}$ and $[(Y,\aleph)\tilde{\setminus}(V,\aleph)]\,\tilde{\cap}\,[(V,\aleph)\tilde{\setminus}(Y,\aleph)]=\emptyset_{\aleph}$.

Proof: Let $(V,\aleph)\tilde{\setminus}(Y,\aleph)=(Q,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(Y,\aleph)\tilde{\setminus}(V,\aleph)=(K,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} Y(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ Y(\vartheta)\setminus V(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

And let $(Q,\aleph)\tilde{\cap}(K,\aleph)=(T,\aleph)$, where $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ Q(\vartheta)\cap K(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta)]\cap[Y(\vartheta)\setminus V(\vartheta)], & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Hence, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ \emptyset, & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Since $\forall\vartheta\in\aleph$, $T(\vartheta)=\emptyset$, $(T,\aleph)=\emptyset_{\aleph}$. Moreover $[(Y,\aleph)\tilde{\setminus}(V,\aleph)]\,\tilde{\cap}\,[(V,\aleph)\,\tilde{\setminus}(Y,\aleph)]=\emptyset_{\aleph}$ can be shown similarly.

NOTE: In classical set theory, $(V\setminus Y)\setminus(Y\setminus V)=V\setminus Y$ (as $(V\setminus Y)\cap(Y\setminus V)=\emptyset$) and $(Y\setminus V)\setminus(V\setminus Y)=Y\setminus V$ (as $(Y\setminus V)\cap(V\setminus Y)=\emptyset$). As an analogy, we have the following:

27) $[(V,\aleph)\,\tilde{\setminus}(Y,I)]\,\tilde{\setminus}\,[(Y,I)\,\tilde{\setminus}(V,\aleph)]=(V,\aleph)\,\tilde{\setminus}(Y,I)$ and $[(Y,I)\,\tilde{\setminus}(V,\aleph)]\,\tilde{\setminus}\,[(V,\aleph)\tilde{\setminus}(Y,I)]=(Y,I)\tilde{\setminus}(V,\aleph)$

Proof: Let $(V,\aleph)\tilde{\setminus}(Y,I)=(Q,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Let $(Y,I)\tilde{\setminus}(V,\aleph)=(K,I)$. Then, $\forall\vartheta\in I$,

$$K(\vartheta)=\begin{cases} Y(\vartheta), & \vartheta\in I\setminus\aleph \\ Y(\vartheta)\setminus V(\vartheta), & \vartheta\in I\cap\aleph \end{cases}$$

And let $(Q,\aleph)\ \tilde{\setminus}\ (K,I)=(T,\aleph)$, where $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus I \\ Q(\vartheta)\setminus K(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in(\aleph\setminus I)\setminus I=\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in(\aleph\cap I)\setminus I=\emptyset \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in(\aleph\setminus I)\cap(I\setminus\aleph)=\emptyset \\ V(\vartheta)\setminus[Y(\vartheta)\setminus V(\vartheta)], & \vartheta\in(\aleph\setminus I)\cap(I\cap\aleph)=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta)]\setminus Y(\vartheta), & \vartheta\in(\aleph\cap I)\cap(I\setminus\aleph)=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta)]\setminus[Y(\vartheta)\setminus V(\vartheta)], & \vartheta\in(\aleph\cap I)\cap(I\cap\aleph)=I\cap\aleph \end{cases}$$

Hence, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Thus, $(T,\aleph)=(V,\aleph)\ \tilde{\setminus}((Y,I)$. Moreover, $[(Y,I)\tilde{\setminus}(V,\aleph)]\tilde{\setminus}[(V,\aleph)\tilde{\setminus}((Y,I)]=(Y,I)\tilde{\setminus}(V,\aleph)$ can be shown similarly.

In classical set theory, $(V\setminus Y)\cap(Y\cap V)=\emptyset$ and $(Y\setminus V)\cap(V\cap Y)=\emptyset$. Now, we have the following analogy.

28) $[(V,\aleph)\ \tilde{\setminus}\ (Y,\aleph)]\ \tilde{\cap}\ [(Y,\aleph)\ \tilde{\cap}\ (V,\aleph)]=\emptyset_\aleph$ and $[(Y,\aleph)\ \tilde{\setminus}\ (V,\aleph)]\ \tilde{\cap}\ [(V,\aleph)\ \tilde{\cap}\ (Y,\aleph)]=\emptyset_\aleph$.

Proof: Let $(V,\aleph)\ \tilde{\setminus}(Y,\aleph)=(Q,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(Y,\aleph)\tilde{\cap}\ (V,\aleph)=(K,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} Y(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ Y(\vartheta)\cap V(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

And let $(Q,\aleph)\ \tilde{\cap}\ (K,\aleph)=(T,\aleph)$, where $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ Q(\vartheta)\cap K(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta)]\cap[Y(\vartheta)\cap V(\vartheta)], & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Hence, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ \emptyset, & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Since $\forall\vartheta\in\aleph$, $T(\vartheta)=\emptyset$, $(T,\aleph)=\emptyset_{\aleph}$. Moreover, $[(Y,\aleph)\,\tilde{\setminus}(V,\aleph)\,]\,\tilde{\cap}\,[(V,\aleph)\,\tilde{\cap}\,(Y,\aleph)\,]=\emptyset_{\aleph}$ can be shown similarly.

NOTE: In classical set theory, $(V\setminus Y)\setminus(Y\cap V)=(V\setminus Y)$ (since $(V\setminus Y)\cap(Y\cap V)=\emptyset$) and $(Y\setminus V)\setminus(V\cap Y)=Y\setminus V$ (since $(Y\setminus V)\cap(V\cap Y)=\emptyset$). As an analogy, we have the following:

29) $[(V,\aleph)\,\tilde{\setminus}(Y,I)\,]\,\tilde{\setminus}\,[(Y,I)\,\tilde{\cap}(V,\aleph)\,]=(V,\aleph)\,\tilde{\setminus}(Y,I)$ and $[(Y,I)\,\tilde{\setminus}(V,\aleph)\,]\,\tilde{\setminus}[(V,\aleph)\,\tilde{\cap}(Y,I)\,]=(Y,I)\,\tilde{\setminus}\,(V,\aleph)$.

Proof: Let $(V,\aleph)\tilde{\setminus}(Y,I)=(Q,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Let $(Y,I)\,\tilde{\cap}\,(V,\aleph)=(K,I)$. Then, $\forall\vartheta\in I$,

$$K(\vartheta)=\begin{cases} Y(\vartheta), & \vartheta\in I\setminus\aleph \\ Y(\vartheta)\cap V(\vartheta), & \vartheta\in I\cap\aleph \end{cases}$$

And let $(Q,\aleph)\,\tilde{\setminus}\,(K,I)=(T,\aleph)$, where $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus I \\ Q(\vartheta)\setminus K(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in(\aleph\setminus I)\setminus I=\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta)\,, & \vartheta\in(\aleph\cap I)\setminus I=\emptyset \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in(\aleph\setminus I)\cap(I\setminus\aleph)=\emptyset \\ V(\vartheta)\setminus[Y(\vartheta)\cap V(\vartheta)], & \vartheta\in(\aleph\setminus I)\cap(I\cap\aleph)=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta)]\setminus Y(\vartheta), & \vartheta\in(\aleph\cap I)\cap(I\setminus\aleph)=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta)]\setminus[Y(\vartheta)\cap V(\vartheta)], & \vartheta\in(\aleph\cap I)\cap(I\cap\aleph)=I\cap\aleph \end{cases}$$

Hence, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Thus, $(T,\aleph)=(V,\aleph)\,\tilde{\setminus}((Y,I)$. Moreover, $[(Y,I)\,\tilde{\setminus}(V,\aleph)]\tilde{\setminus}[(V,\aleph)\,\tilde{\cap}(Y,I)]=(Y,I)\,\tilde{\setminus}(V,\aleph)$ can be shown similarly.

In classical set theory, $V\cap(Y\setminus V)=\emptyset$ and $Y\cap(V\setminus Y)=\emptyset$. Now, we have the following analogy:

30) $(V,\aleph)\,\tilde{\cap}\,[(Y,\aleph)\tilde{\setminus}(V,\aleph)]=\emptyset_{\aleph}$ and $(Y,\aleph)\,\tilde{\cap}\,[(V,\aleph)\,\tilde{\setminus}\,(Y,\aleph)\,]=\emptyset_{\aleph}$.

Proof: Let $(Y,\aleph)\,\tilde{\setminus}(V,\aleph)=(Q,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} Y(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ Y(\vartheta)\setminus V(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(V,\aleph)\tilde{\cap}\,(Q,\aleph)=(K,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\cap Q(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\cap[Y(\vartheta)\setminus V(\vartheta)], & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Hence, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ \emptyset, & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Since $\forall\vartheta\in\aleph$, $K(\vartheta)=\emptyset$, $(K,\aleph)=\emptyset_{\aleph}$. Moreover $(Y,\aleph)\,\tilde{\cap}\,[(V,\aleph)\tilde{\setminus}(Y,\aleph)]=\emptyset_{\aleph}$ can be shown similarly.

NOTE: In classical set theory, $V\setminus(Y\setminus V)=V$ (since $V\cap(Y\setminus V)=\emptyset$) and $Y\setminus(V\setminus Y)=Y$ (since $Y\cap(V\setminus Y)=\emptyset$).

Now, we have the followin analogy:

31) $(V,\aleph)\,\tilde{\setminus}\,[(Y,I)\tilde{\setminus}(V,\aleph)]=(V,\aleph)$ and $(Y,I)\,\tilde{\setminus}\,[(V,\aleph)\,\tilde{\setminus}(Y,I)]=(Y,I)$.

Proof: Let $(Y,I)\,\tilde{\setminus}(V,\aleph)=(Q,I)$. Then, $\forall\vartheta\in I$,

$$Q(\vartheta)=\begin{cases} Y(\vartheta), & \vartheta\in I\setminus\aleph \\ Y(\vartheta)\setminus V(\vartheta), & \vartheta\in I\cap\aleph \end{cases}$$

Let $(V,\aleph)\tilde{\setminus}(Q,I)=(K,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Q(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap(I\setminus\aleph)=\emptyset \\ V(\vartheta)\setminus[Y(\vartheta)\setminus V(\vartheta)], & \vartheta\in\aleph\cap(I\cap\aleph)=\aleph\cap I \end{cases}$$

Hence, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Since $\forall\vartheta\in\aleph$, $K(\vartheta)=V(\vartheta)$, $(K,\aleph)=(V,\aleph)$. Moreover, $(Y,I)\,\tilde{\setminus}\,[(V,\aleph)\,\tilde{\setminus}(Y,I)]=(Y,I)$ can be shown similarly.

In classical set theory, $V\cup Y=(V\setminus Y)\cup(Y\setminus V)\cup(V\cap Y)$. Now, we have the following analogy:

32) $(V,\aleph)\,\tilde{\cup}\,(Y,I)=[(V,\aleph)\,\tilde{\setminus}\,(Y,I)]\,\tilde{\cup}\,[(Y,I)\tilde{\setminus}\,(V,\aleph)]\,\tilde{\cup}\,[(V,\aleph)\,\tilde{\cap}\,(Y,I)]$.

Proof: Let $(V,\aleph)\tilde{\setminus}(Y,I)=(Q,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$Q(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Let $(Y,I)\,\tilde{\setminus}\,(V,\aleph)=(K,I)$. Then, $\forall\vartheta\in I$,

$$K(\vartheta)=\begin{cases} Y(\vartheta), & \vartheta\in I\setminus\aleph \\ Y(\vartheta)\setminus V(\vartheta), & \vartheta\in I\cap\aleph \end{cases}$$

And let $(Q,\aleph)\ \widetilde{\cup}\ (K,I)=(T,\aleph)$, where $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in\aleph\setminus I \\ Q(\vartheta)\cup K(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in(\aleph\setminus I)\setminus I=\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in(\aleph\cap I)\setminus I=\emptyset \\ V(\vartheta)\cup Y(\vartheta), & \vartheta\in(\aleph\setminus I)\cap(I\setminus\aleph)=\emptyset \\ V(\vartheta)\cup[Y(\vartheta)\setminus V(\vartheta)], & \vartheta\in(\aleph\setminus I)\cap(I\cap\aleph)=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta)]\cup Y(\vartheta), & \vartheta\in(\aleph\cap I)\cap(I\setminus\aleph)=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta)]\cup[Y(\vartheta)\setminus V(\vartheta)], & \vartheta\in(\aleph\cap I)\cap(I\cap\aleph)=I\cap\aleph \end{cases}$$

Hence, $\forall\vartheta\in\aleph$,

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ [V(\vartheta)\setminus Y(\vartheta)]\cup[Y(\vartheta)\setminus V(\vartheta)], & \vartheta\in\aleph\cap I \end{cases}$$

Let $(V,\aleph)\ \widetilde{\cap}(Y,I)=(W,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$W(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\cap Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Let $(T,\aleph)\ \widetilde{\cup}(W,\aleph)=(R,\aleph)$. Thus, for all $\vartheta\in\aleph$;

$$R(\vartheta)=\begin{cases} T(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ T(\vartheta)\cup W(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$R(\vartheta)=\begin{cases} V(\vartheta)\cup V(\vartheta), & \vartheta\in(\aleph\setminus I)\cap(\aleph\setminus I)=\aleph\setminus I \\ V(\vartheta)\cup[V(\vartheta)\cap Y(\vartheta)], & \vartheta\in(\aleph\setminus I)\cap(\aleph\cap I)=\emptyset \\ [[V(\vartheta)\setminus Y(\vartheta)]\cup[Y(\vartheta)\setminus V(\vartheta)]]\cup V(\vartheta), & \vartheta\in(\aleph\cap I)\cap(\aleph\setminus I)=\emptyset \\ [[V(\vartheta)\setminus Y(\vartheta)]\cup[Y(\vartheta)\setminus V(\vartheta)]]\cup[V(\vartheta)\cap Y(\vartheta)], & \vartheta\in(\aleph\cap I)\cap(\aleph\cap I)=\aleph\cap I \end{cases}$$

Since $[[V(\vartheta)\setminus Y(\vartheta)]\cup[Y(\vartheta)\setminus V(\vartheta)]]\cup[V(\vartheta)\cap Y(\vartheta)]=V(\vartheta)\cup Y(\vartheta)$, thus,

$$R(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\cup Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Therefore, $(R,\aleph)=(V,\aleph)\ \widetilde{\cup}\ (Y,I)$.

NOTE: Since $[(V,\aleph)\ \widetilde{\cup}\ (Y,I)]\ \widetilde{\cup}\ (Z,K)\neq(V,\aleph)\ \widetilde{\cup}\ [(Y,I))\ \widetilde{\cup}\ (Z,K)]$, we should also have looked the following:

33) $(V,\aleph)\ \widetilde{\cup}\ (Y,I)=[(V,\aleph)\ \widetilde{\setminus}\ (Y,I)\ ]\ \widetilde{\cup}\ \ [[(Y,I)\widetilde{\setminus}\ (V,\aleph)]\ \widetilde{\cup}\ [(V,\aleph)\ \widetilde{\cap}\ (Y,I)]]$.

Proof: Let $(Y,I)\widetilde{\setminus}(V,\aleph)=(Q,I)$. Then, $\forall\vartheta\in I$,

$$Q(\vartheta)=\begin{cases} Y(\vartheta), & \vartheta\in I\setminus\aleph \\ Y(\vartheta)\setminus V(\vartheta), & \vartheta\in I\cap\aleph \end{cases}$$

Let $(V,\aleph)\ \tilde{\setminus}\ (Y,I)=(K,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$K(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

And let $(Q,I)\ \tilde{\cup}\ (K,\aleph)=(T,I)$, where $\forall\vartheta\in I$,

$$T(\vartheta)=\begin{cases} Q(\vartheta), & \vartheta\in I\setminus\aleph \\ Q(\vartheta)\cup K(\vartheta), & \vartheta\in I\cap\aleph \end{cases}$$

Thus, $\forall\vartheta\in I$, hence,

$$T(\vartheta)=\begin{cases} Y(\vartheta), & \vartheta\in(I\setminus\aleph)\setminus\aleph=I\setminus\aleph \\ Y(\vartheta)\setminus V(\vartheta), & \vartheta\in(I\cap\aleph)\setminus\aleph=\emptyset \\ Y(\vartheta)\cup V(\vartheta), & \vartheta\in(I\setminus\aleph)\cap(\aleph\setminus I)=\emptyset \\ Y(\vartheta)\cup[V(\vartheta)\setminus Y(\vartheta)], & \vartheta\in(I\setminus\aleph)\cap(\aleph\cap I)=\emptyset \\ [Y(\vartheta)\setminus V(\vartheta)]\cup V(\vartheta), & \vartheta\in(I\cap\aleph)\cap(\aleph\setminus I)=\emptyset \\ [Y(\vartheta)\setminus V(\vartheta)]\cup[V(\vartheta)\setminus Y(\vartheta)], & \vartheta\in(I\cap\aleph)\cap(\aleph\cap I)=\aleph\cap I \end{cases}$$

Hence, $\forall\vartheta\in I$,

$$T(\vartheta)=\begin{cases} Y(\vartheta), & \vartheta\in I\setminus\aleph \\ [Y(\vartheta)\setminus V(\vartheta)]\cup[V(\vartheta)\setminus Y(\vartheta)], & \vartheta\in I\cap\aleph \end{cases}$$

Let $(V,\aleph)\ \tilde{\setminus}(Y,I)=(W,\aleph)$. Then, $\forall\vartheta\in\aleph$,

$$W(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Let $(W,\aleph)\tilde{\cup}(T,I)=(R,\aleph)$. Thus, for all $\vartheta\in\aleph$;

$$R(\vartheta)=\begin{cases} W(\vartheta), & \vartheta\in\aleph\setminus I \\ W(\vartheta)\cup T(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$R(\vartheta)=\begin{cases} V(\vartheta) & \vartheta\in(\aleph\setminus I)\setminus I=\aleph\setminus I \\ V(\vartheta)\setminus Y(\vartheta), & \vartheta\in(\aleph\cap I)\setminus I=\emptyset \\ V(\vartheta)\cup Y(\vartheta), & \vartheta\in(\aleph\setminus I)\cap(I\setminus\aleph)=\emptyset \\ V(\vartheta)\cup[[Y(\vartheta)\setminus V(\vartheta)]\cup[V(\vartheta)\setminus Y(\vartheta)]] & \vartheta\in(\aleph\setminus I)\cap(I\cap\aleph)=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta)]\cup Y(\vartheta), & \vartheta\in(\aleph\cap I)\cap(I\setminus\aleph)=\emptyset \\ [V(\vartheta)\setminus Y(\vartheta))]\cup\{[Y(\vartheta)\setminus V(\vartheta)]\cup[V(\vartheta)\setminus Y(\vartheta)]\} & \vartheta\in(\aleph\cap I)\cap(I\cap\aleph)=\aleph\cap I \end{cases}$$

Since $[V(\vartheta)\setminus Y(\vartheta)]\cup[[Y(\vartheta)\setminus V(\vartheta)]\cup[V(\vartheta)\cap Y(\vartheta)]]=V(\vartheta)\cup Y(\vartheta)$,

$$R(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus I \\ V(\vartheta)\cup Y(\vartheta), & \vartheta\in\aleph\cap I \end{cases}$$

Therefore, $(R,\aleph)=(V,\aleph)\ \tilde{\cup}\ (Y,I)$.

In [10], it was shown that the collection of soft sets with a fixed set of parameter set associated respect to restricted difference becomes a BCK algebra. Now, we have the following:

Theorem 3.6. $(S_{\aleph}(U), \tilde{\setminus}, \emptyset_{\aleph})$ is a bounded BCK algebra whose every element is an involution.
Proof: Let $(V,\aleph), (Q,\aleph), (Z,\aleph)\in S_{\aleph}(U)$. Then,
BCI-1 $[((V,\aleph)\ \tilde{\setminus}\ (Q,\aleph))\ \tilde{\setminus}\ ((V,\aleph)\ \tilde{\setminus}\ (Z,\aleph))]\ \tilde{\setminus}((Z,\aleph)\ \tilde{\setminus}\ (Q,\aleph))=\emptyset_{\aleph}$. In fact,
Let $(V,\aleph)\ \tilde{\setminus}\ (Q,\aleph)=(T,\aleph)$, where $\forall\vartheta\in\aleph$;

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus Q(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(V,\aleph)\ \tilde{\setminus}(Z,\aleph)=(M,\aleph)$, where $\forall\vartheta\in\aleph$;

$$M(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus Z(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(T,\aleph)\ \tilde{\setminus}(M,\aleph)=(W,\aleph)$, where $\forall\vartheta\in\aleph$;

$$W(\vartheta)=\begin{cases} T(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ T(\vartheta)\setminus M(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Hence, $\forall\vartheta\in\aleph$;

$$W(\vartheta)=\begin{cases} T(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ [V(\vartheta)\setminus Q(\vartheta)]\setminus[V(\vartheta)\setminus Z(\vartheta)], & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(Z,\aleph)\ \tilde{\setminus}(Q,\aleph)=(S,\aleph)$, where $\forall\vartheta\in\aleph$;

$$S(\vartheta)=\begin{cases} Z(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ Z(\vartheta)\setminus Q(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(W,\aleph)\ \tilde{\setminus}(S,\aleph)=(X,\aleph)$, where $\forall\vartheta\in\aleph$;

$$X(\vartheta)=\begin{cases} W(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ W(\vartheta)\setminus S(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$X(\vartheta)=\begin{cases} W(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ \{[V(\vartheta)\setminus Q(\vartheta)]\setminus[V(\vartheta)\setminus Z(\vartheta)]\}\setminus[Z(\vartheta)\setminus Q(\vartheta)], & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus, $\forall\vartheta\in\aleph$;

$$X(\vartheta)=\begin{cases} W(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ \emptyset\,, & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

It is seen that $(X,\aleph)=\emptyset_{\aleph}$.

BCI-2 $[(V,\aleph)\,\tilde{\setminus}\,((V,\aleph)\,\tilde{\setminus}(Q,\aleph))]\,\tilde{\setminus}\,(Q,\aleph)=\emptyset_{\aleph}$. In fact, let $(V,\aleph)\,\tilde{\setminus}\,(Q,\aleph)=(T,\aleph)$, where $\forall\vartheta\in\aleph$;

$$T(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus Q(\vartheta)\,, & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(V,\aleph)\,\tilde{\setminus}(T,\aleph)=(M,\aleph)$, where $\forall\vartheta\in\aleph$;

$$M(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus T(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus, $\forall\vartheta\in\aleph$

$$M(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\setminus[V(\vartheta)\setminus Q(\vartheta)], & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus, $\forall\vartheta\in\aleph$,

$$M(\vartheta)=\begin{cases} V(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ V(\vartheta)\cap Q(\vartheta), & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(M,\aleph)\tilde{\setminus}(Q,\aleph)=(L,\aleph)$, where $\forall\vartheta\in\aleph$;

$$L(\vartheta)=\begin{cases} M(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ M(\vartheta)\setminus Q(\vartheta)\,, & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Let $(V,\aleph)\,\tilde{\setminus}(L,\aleph)=(D,\aleph)$, where $\forall\vartheta\in\aleph$;

$$D(\vartheta)=\begin{cases} M(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ [V(\vartheta)\cap Q(\vartheta)]\setminus Q(\vartheta)\,, & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

Thus,

$$D(\vartheta)=\begin{cases} M(\vartheta), & \vartheta\in\aleph\setminus\aleph=\emptyset \\ \emptyset, & \vartheta\in\aleph\cap\aleph=\aleph \end{cases}$$

It is seen that $(D,\aleph)=\emptyset_{\aleph}$.

BCI-3 By Theorem 3.5. (5), $(V,\aleph)\,\tilde{\setminus}(V,\aleph)=\emptyset_{\aleph}$.

BCI-4 By Theorem 3.5. (19), $(V,\aleph)\tilde{\setminus}(Y,\aleph)=\emptyset_{\aleph}\Rightarrow(V,\aleph)\tilde{\subseteq}(Y,\aleph)$ and $(Y,\aleph)\tilde{\setminus}(V,\aleph)=\emptyset_{Q}\Rightarrow(Y,\aleph)\,\tilde{\subseteq}(V,\aleph)$ and thus, $(V,\aleph)=(Y,\aleph)$.

BCK-5 By Theorem 3.5. (7), $\emptyset_{\aleph}\tilde{\setminus}(V,\aleph)=\emptyset_{\aleph}$.

Thus, $(S_{\aleph}(U), \tilde{\setminus}, \emptyset_{\aleph})$ is a BCK-algebra. Since, $(V, \aleph)\tilde{\setminus}\, U_{\aleph}=\emptyset_{\aleph}$ for all $(V, \aleph)\in S_{\aleph}(U)$ (b)y Theorem 3.5. (8), $(S_{\aleph}(U), \tilde{\setminus}, \emptyset_{\aleph})$ is a bounded BCK-algebra, where $\aleph$ is a fixed set of parameter set. Moreover since $U_{\aleph}\tilde{\setminus}[U_{\aleph}\tilde{\setminus}(V, \aleph)]=(V, \aleph)$ for all $(V, \aleph)\in S_{\aleph}(U)$, (As $U_{\aleph}\tilde{\setminus}(V, \aleph)= (V, \aleph)^r$ by Theorem 3.5. (9)), and $U_{\aleph}\tilde{\setminus}(V, \aleph)^r =[(V, \aleph)^r]^r=(V, \aleph)$), every element of $S_{\aleph}(U)$ is an involution.

In fact, since restricted difference soft set operation coincides with soft complementary difference operation in the collection of soft sets with a fixed parameter set, the BCK algebra in [10] and the BCK-algebra in this paper are in fact the same.

4. CONCLUSION

Since the inception of soft set theory by Molodtsov, numerous variations of soft set operations have been described and used. In this article, in order to improve the soft binary piecewise difference operation, we have explored its overall properties, especially in comparison with the fundamental properties of the difference operation in classical set theory and we have obtained very interesting analogies. Furthermore, we have proved that the set of all soft sets with a fixed parameter set is a bounded BCK-algebra together with the soft binary piecewise difference operation. Since studying the algebraic structure of soft sets from the perspective of operations provides deep insight into the algebraic structure of soft sets and its application and soft set algebra shows the potential applications of soft sets in classical and nonclassical logic, we hope this paper contibutes to the literature of soft set in this regard.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

AUTHORSHIP CONTRIBUTIONS

The authors contributed equally to this work.

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ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ
B- TEORİK BİLİMLER

Eskişehir Technical University Journal of Science and Technology B- Theoretical Sciences

`2024, 12(1), pp. 55-61, DOI: 10.20290/estubtdb.1357676`

RESEARCH ARTICLE

ROGUE WAVES IN DISCRETE KDV EQUATION

Semiha TOMBULOĞLU

Sağlık Hizmetleri MYO, Kırklareli Üniversitesi, Kırklareli, Türkiye

ABSTRACT

This study considers an array of waveguides described by a discrete KdV equation. Rogue wave solutions are numerically derived for the dKdV equation under periodic and non-vanishing boundary conditions. When the dKdV equation is solved numerically under periodic boundary conditions, a discrete rogue wave occurs due to shock front breaking. Furthermore, the dKdV equation has been solved numerically under non-vanishing boundary conditions, and it has been found that the rogue wave amplitude depends on the ρ_0 parameter.

Keywords: Rogue wave, Discrete KdV equation

1. INTRODUCTION

Huge amplitude waves, generally called 'rogue waves' or 'freak waves,' are a hot topic because of their exciting feature [1], [2]. One of its features is unpredictability; it appears without warning and disappears suddenly [3], [4]. In addition, their amplitude exceeds at least twice the surrounding background [5].

Scientists are considering the nonlinearity of the freak waves. They found that modulational instability (MI) (often called Benjamin-Feir instability) plays an essential role in explaining rogue wave structure [7], [8]. Because of periodic perturbation on a plane wave, MI has occurred and is critical in the nonlinear evaluation process. MI occurs in different physical systems such as fiber optic [32], supercontinuum generation [31], and plasmas [33], not only water waves [1]. Although researchers mostly used a numerical approach to investigate MI, the nonlinear Schrödinger (NLS) equation was solved analytically and called Akhmediev breather [10]. Tulin and Waseda conducted another study about MI in 1999 [11]. As can be seen in many studies, rogue waves show nonlinear features [12]. The dynamic becomes more complex when MI develops, and energy exchange is involved. Besides focusing on the NLS equation [13], [14], classical nonlinear evolution equations like the Korteweg de Vries equation [15], [16], [17], Gardner equation [18], Ablowitz Ladik and Hirota equations [19], and similar nonlinear equations can also describe rogue waves [20], [21].

J.S. Russell found the first solitary wave as a nonlinear coherent structure in 1834 [22]. The analysis was made theoretically by Rayleigh in 1876 [23] and Boussinesq in 1871 [24]. Korteweg De Vries formulated the KdV equation for shallow waters in 1895 [25]. It was not easy to find a physical application of the KdV equation until the 1960s. In 1965, Zabusky and Kruskal solved the KdV equation numerically for nonlinear mesh points and determined how long after the system returns to its initial state [26]. It is the pioneering work that reveals the presence of soliton solutions in this equation. We note that there is a connection between the KdV equation and the NLS equation [35]. NLS equation is derived from the KdV equation, or the inverse of this is possible. The general thought that the KdV equation was not able to explain rogue

*Corresponding Author: semihatombuloglu@klu.edu.tr
Received: 09.09.2023 Published: 27.02.2024

wave solutions until recently [36]. This assumption is valid only if the wave described by the KdV equation is purely real. However, if we consider complex-valued solutions to the KdV equation, it becomes possible to obtain rogue wave solutions [35]. The KdV equation has a variety of solutions; indeed, as well as the classical KdV equation, it is possible to get rogue wave solutions in discrete systems.

Suppose that there is an array of waveguides. All the waveguide components are identical and also an equal distance from each other. This study considers such an array of waveguides that a discrete KdV equation can define. We study the dKdV equation numerically under periodic boundary conditions and non-vanishing (constant) boundary conditions to show that rogue wave(s) can occur in an array of waveguides. Firstly, we will numerically derive discrete rogue wave solutions for the dKdV equation under non-vanishing boundary conditions. To our knowledge, it is the first time that the dKdV equation is numerically solved for discrete rogue wave solutions. Then, we numerically solve the dKdV equation under periodic boundary conditions. We show that rogue waves occur not because of MI but because of shock front breaking. This is important since there is a general thought that rogue waves have nonlinear characteristics and occur primarily because of MI. This study can also enable the comparison of discrete rogue wave amplitudes derived from the dKdV and the discrete Schrödinger equations. The results of this comparison can shed light on rogue wave evolution.

2. DISCRETE ROGUE WAVES

The discrete KdV equation was given in [34] in the form

$$\frac{dw_j}{d1 + w_j} = w_{j-{}^1/_2} + w_{j+{}^1/_2}$$

where w_j is the field variable at site j, and j is the dimensionless variable, t is time. As studied in [34], the equation reduces to the KdV equation in a weakly nonlinear limit. When we perform the differentiation of this equation and denote $\psi = 1 + w$, we get the discrete KdV equation, which was already derived by [29]

$$\frac{d\psi_j}{dt} = \left(\psi_{j-1} - \psi_{j+1}\right)\psi_j^2 \tag{1}$$

where ψ_j is the field amplitude at the jth waveguide and j=1,2,..., N, takes positive integers. Here, N is a number of lattice sites. If we assume that the wave amplitude is too small, we will obtain the KdV equation for which the absence of MI has been proved. Now, we numerically solve equation (1) with the properly given initial condition. Let us now start with the initial wave,

$$\psi_j(t = 0) = A(1 + \xi j)e^{-iA^2L} \tag{2}$$

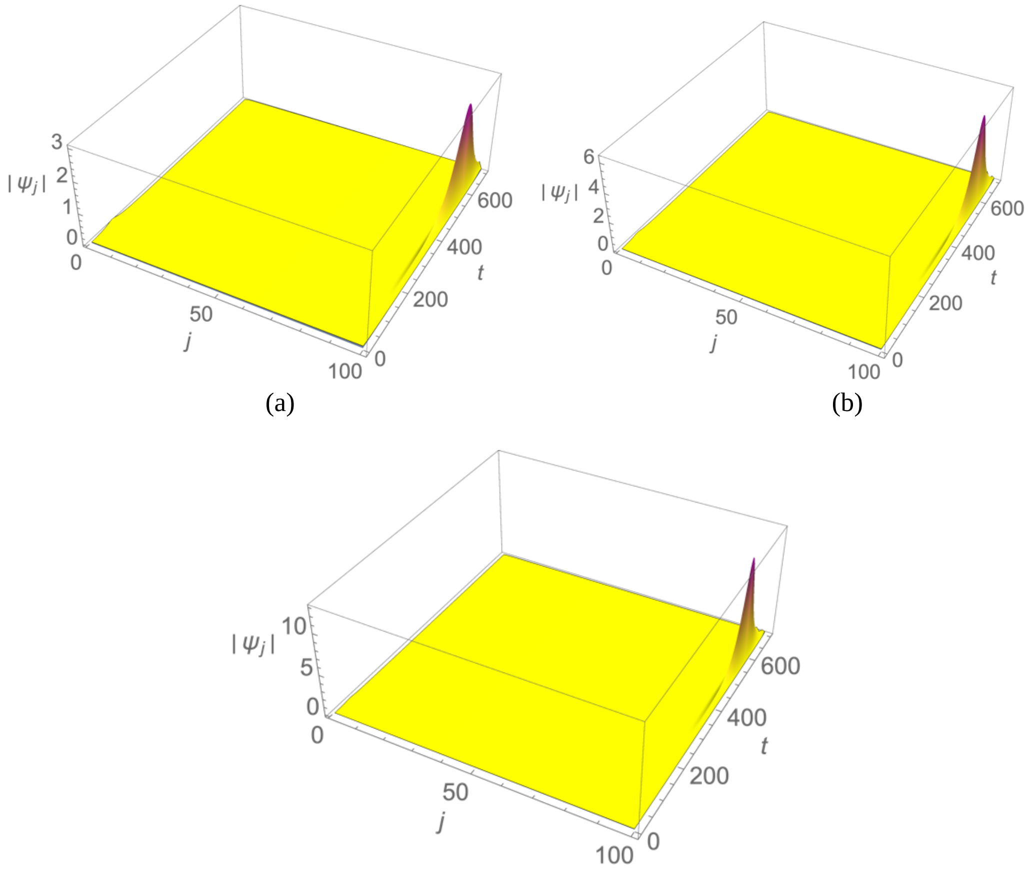
where ξ is the very small parameter, $\xi << 1$ is the perturbation term, L is the length of each waveguide in the array, and A is the background intensity. In the case of $\xi \longrightarrow 0$ in equation (2), it defines a constant background whose amplitude equals A. Since these waves arise and recede to the constant background, we can solve equation (1) with non-vanishing boundary conditions. We perform numerical solutions to find the system's time evolution under non-vanishing boundary conditions. We take the non-vanishing boundary conditions as

$$\lim_{j\to\pm\infty} \psi(j, t) = \rho_0 \tag{3}$$

where $\rho_0$is a real constant here. Assume that the initial condition in equation (2) is subject to those non-vanishing boundary conditions. We solve the numerically discrete KdV equation to show that rogue wave occurs under non-vanishing boundary conditions.

Figure 1 shows us the spatial evaluation of the initial wave. The total energy is highly localized in a few waveguides located at the output end of the system. This concentration leads to the amplitude of the maximum value being nearly thirty times higher than the background amplitude when ρ_0= 0,4, accepting that background amplitude, κ, nearly equals 0,1, as seen in Figure 1. a. We see that even small perturbation affects the stability of this uniform solution. The perturbation is very small, beginning from the left side and getting larger while going on the right side. The distribution is perturbated, beginning from the t = 0. It grows in time until its mean peak reaches its maximum value. Then, its amplitude decreases, and finally, we see that this wave disappears at around t = 700. These features are compatible with rogue waves. We show that discrete rogue waves appear in discrete KdV equation under non-vanishing boundary conditions for the first time. Even non-vanishing boundary conditions do not describe the physical system; mathematically, discrete rogue wave evaluation can be found.

When $\rho_0$increases, rogue wave amplitude rises, as resulted in Figure 1. The absolute of the maximum amplitude is nearly sixty times higher than the background amplitude, κ=0,1, when ρ_0= 0,5 as shown in Figure 1. b. In comparison, the absolute of the maximum amplitude is nearly over a hundred times higher than the background amplitude, κ=0,1, when ρ_0= 0,6 as shown in Figure 1. c. We show that the real constant ρ_0, is important in terms of the rogue wave's amplitude, and even a small change in constant ρ_0 leads to a significant effect on it.



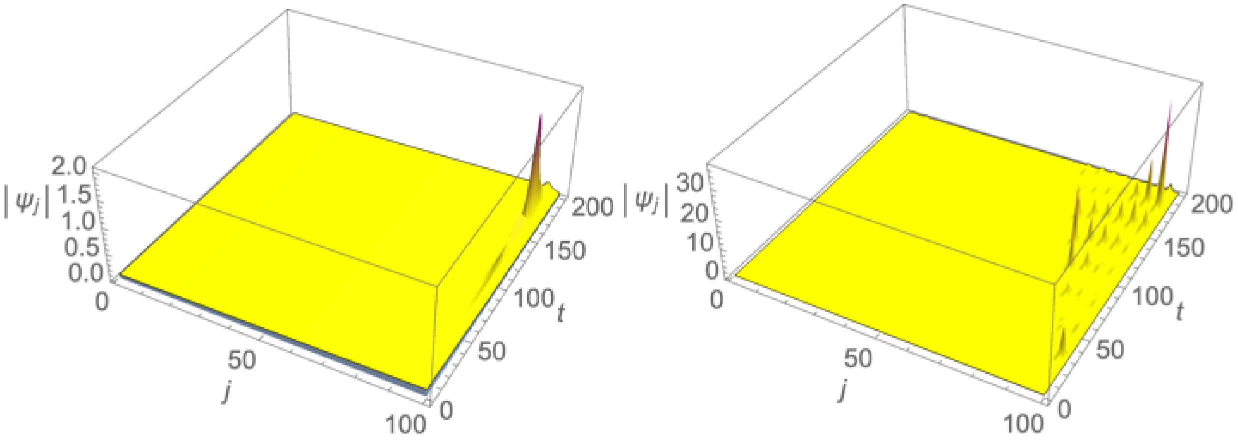
(a) (b)

(c)

Figure 1: The absolute field amplitude for N=100, A=0.05, ξ= 0,007176, and L=100. The initial wave is given by equation (2), and boundary conditions are provided by equation (3). In Figure 1. a ρ_0= 0,4, Figure 1. b ρ_0= 0,5, Figure 1. c ρ_0= 0,6. The amplitude of discrete rogue waves increases according to ρ_0.

We assume the system is subject to periodic boundary conditions, i.e., $\psi(N+1) = \psi 1$. We aim to solve equation (1) numerically under periodic boundary conditions to show that discrete rogue waves occur in a discrete KdV equation. We will use the complex solution of the initial condition mentioned in equation (2). The perturbation is given very small on the left side (j=1) and gets larger when going on the right side (j=100). As a result of the periodic boundary conditions, the discontinuity inevitably breaks up into waves due to dispersion, causing the waves to arise on the right edge. Note that the waves occur not because of MI but because of the shock front breaking.

As seen from Figure 2. a, the amplitude of a wave is nearly twenty times higher than the average surrounding background, κ=0,1. We note that such a huge amplitude growth is much higher than that of a discrete rogue wave for the discrete nonlinear Schrödinger equation [28]. The exciting point of view is that the study allows comparing the amplitude of discrete rogue waves calculated by the discrete KdV equation and the discrete Schrödinger equation. Also, when the discrete KdV equation is solved numerically under periodic boundary conditions, the rogue wave occurs at about t=180; instead, the equation is solved numerically under non-vanishing boundary conditions, and the rogue wave occurs at about t=600. We observe that such a different boundary condition results in a shift of freak wave peak. Indeed, it is well known that rogue waves sensitively depend on the initial condition. Even small changes can make all the features of rogue waves different and even unstable [28], [30]. If the constant A increases, the discrete rogue waves become more chaotic, as seen in Figure 2. b. We numerically observe that if A gets more significant, more rogue waves occur, and their amplitude increases. Figures 2. a and 2. b show us that discrete rogue waves can happen in a discrete KdV equation, and the numerousness of discrete rogue waves and their amplitude depends on the A parameter. One question came up. What is the long-time behavior of the system after the rogue wave disappears? We see that the discrete system fluctuations begin, leading to other random rogue waves for very long times.



(a)

(b)

Figure 2. The absolute field amplitude for N=100, A=0.1, ξ = 0,0065, and L=100. We study equation (1), and the initial form (for t=0) is given by equation (2). A wave has occurred, and its amplitude is nearly twenty times higher than the average background, as shown in Figure 2. a. In this study, Figure 2. b shows the absolute field amplitude for N=100, A=0.2, ξ = 0,0065, and L=100. We study equation (1), and the initial form (for t=0) is given by equation (2). We see that more rogue waves have occurred. Their amplitude increases as well.

In general relief, rogue wave evolution depends on only initial condition or small experimental imperfections [28], [30]. However, this study discusses that it can also depend on assumed boundary conditions in theoretical studies. We study the discrete KdV equation under periodic and non-vanishing boundary conditions with the initial wave given in equation (2). We see that a very small perturbation leads to strong localization, and a large amplitude discrete rogue wave has occurred. Moreover, we see that the localization of discrete rogue waves under non-vanishing boundary conditions differs from under periodic boundary conditions, as seen in Figure 1. a and Figure 2. Also, the amplitude of these waves depends on parameters.

3. CONCLUSION

As rogue waves are rare, it is crucial to understand the mathematical explanation of rogue waves. In this study, we numerically solve the discrete KdV equation for the first time with the proper initial condition under non-vanishing and periodic boundary conditions. We show that discrete rogue wave(s) occur in these types of boundary conditions, and we examine discrete rogue wave features under these boundary conditions. Even in the NLS equation, MI is mostly one step ahead of explaining rogue waves; we have discussed that when we solve the dKdV equation under periodic boundary conditions, discrete rogue wave occurs because of shock front breaking. We observe that rogue wave amplitude rises when the A parameter increases; more freak waves can also occur. We have solved the dKdV equation numerically under non-vanishing boundary conditions. We show that ρ_0 parameter is also important in enhancing the rogue wave's amplitude.

ACKNOWLEDGEMENT

This study is supported by Kırklareli University Scientific Research Projects Coordination Unit under grant no: KLÜBAP-208

CONFLICT OF INTEREST

The author have no conflicts of interest to declare that they are relevant to the content of this article.

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B- TEORİK BİLİMLER

Eskişehir Technical University Journal of Science and Technology B- Theoretical Sciences

2024, 12(1), pp. 62-74, DOI: 10.20290/estubtdb.1403701

RESEARCH ARTICLE

SOLVABILITY OF A FOUR DIMENSIONAL SYSTEM OF DIFFERENCE EQUATIONS

Şule DEVECİOGLU [1], **Merve KARA** [2, *]

[1,2] Department of Mathematics, Kamil Ozdag Science Faculty, Karamanoglu Mehmetbey University, Karaman, Turkey

ABSTRACT

In this study, we investigate the following four-dimensional difference equations system

$$\begin{cases} u_n = \frac{\alpha u_{n-3} t_{n-2} + \beta}{\gamma v_{n-1} t_{n-2} u_{n-3}}, \\ v_n = \frac{\alpha v_{n-3} u_{n-2} + \beta}{\gamma w_{n-1} u_{n-2} v_{n-3}}, n \in \mathbb{N}_0, \\ w_n = \frac{\alpha w_{n-3} v_{n-2} + \beta}{\gamma t_{n-1} v_{n-2} w_{n-3}}, \\ t_n = \frac{\alpha t_{n-3} w_{n-2} + \beta}{\gamma u_{n-1} w_{n-2} t_{n-3}}, \end{cases}$$

where the initial values $u_{-d}, v_{-d}, w_{-d}, t_{-d}$, $d \in \{1,2,3\}$ are non-zero real numbers and the parameters α, β are real numbers, γ is non-zero real number. Then, we obtain the solutions of system of third-order difference equations in explicit form. In addition, the solutions according to some special cases of the parameters are examined. Finally, numerical examples are given to demonstrate the theoretical results.

Keywords: Periodicity, System of difference equation, Solution

1. INTRODUCTION

First of all, recall that $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ sembolize natural, non-negative integer, integer, real and complex numbers, respectively. The notation of $\lfloor x \rfloor$ stands for $\mathrm{m} \leq \mathrm{x} < \mathrm{m} + 1, \mathrm{m} \in \mathbb{Z}$. If $\mathrm{a}, \mathrm{b} \in \mathbb{Z}, a \leq b$, the notation $c = \overline{a, \mathrm{b}}$ means to come $\{c \in \mathbb{Z}: a \leq c \leq b\}$.

Difference equations come into view the study of the evolution of naturally occurring events. The theory of system of difference equations improved until today. Recently, there has been great interest in studying difference equation or difference equations systems [1-3,5-7,9,12-14].

One of the important difference equation is

$$x_{n+1} = \frac{a x_n + b}{c x_n + d}, \quad n \in \mathbb{N}_0, \tag{1}$$

for $c \neq 0$, $ad \neq bc$ where the initial condition x_0 and the parameters a, b, c, d, are real numbers, which called Riccati difference equation. Further, the general solution of equation (1) can be obtained in the following form

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*Corresponding Author: mervekara@kmu.edu.tr
This study is a part of the first author's Master Thesis.
Received: 12.12.2023 Published: 27.02.2024

$$x_n = \frac{x_0(bc-ad)s_{n-1} + (ax_0+b)s_n}{(cx_0-a)s_n + s_{n+1}}, n \in \mathbb{N}, \tag{2}$$

where the sequence $(s_n)_{n\in\mathbb{N}_0}$is satisfying

$$s_{n+1} - (a+d)s_n - (bc-ad)s_{n-1} = 0, n \in \mathbb{N}, \tag{3}$$

where $s_0 = 0, s_1 = 1$ in [11].

In [15], authors acquired the solutions of the following two difference equations systems

$$x_{n+1} = \frac{x_{n-1} \pm 1}{y_n x_{n-1}}, y_{n+1} = \frac{y_{n-1} \pm 1}{x_n y_{n-1}},\ n \in \mathbb{N}_0. \tag{4}$$

Then, the solution of the following rational difference equation is obtained

$$x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma x_n x_{n-1}}, n \in \mathbb{N}_0, \tag{5}$$

where the initial values x_{-1}, x_0 are non zero real numbers and $\alpha, \beta, \gamma \in \mathbb{R}^+$ [8]. In addition, same authors investigated the two-dimensional case of equation (5) given by

$$x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma y_n x_{n-1}}, y_{n+1} = \frac{\alpha y_{n-1} + \beta}{\gamma x_n y_{n-1}}, n \in \mathbb{N}_0. \tag{6}$$

Finally, the solutions of the following system of difference equations investigated

$$\begin{cases} x_n = \dfrac{a x_{n-3} z_{n-2} + b}{c y_{n-1} z_{n-2} x_{n-3}}, \\ y_n = \dfrac{a y_{n-3} x_{n-2} + b}{c z_{n-1} x_{n-2} y_{n-3}},\ n \in \mathbb{N}_0, \\ z_n = \dfrac{a z_{n-3} y_{n-2} + b}{c x_{n-1} y_{n-2} z_{n-3}}, \end{cases} \tag{7}$$

where the initial values $x_{-j}, y_{-j}, z_{-j},\ j \in \{1,2,3\}$, and the parameters, $a, b, c,$ are real numbers [10]. Based on the above mentioned studies, we investigate the following system of difference equations

$$\begin{cases} u_n = \dfrac{\alpha u_{n-3} t_{n-2} + \beta}{\gamma v_{n-1} t_{n-2} u_{n-3}}, \\ v_n = \dfrac{\alpha v_{n-3} u_{n-2} + \beta}{\gamma w_{n-1} u_{n-2} v_{n-3}},\ n \in \mathbb{N}_0, \\ w_n = \dfrac{\alpha w_{n-3} v_{n-2} + \beta}{\gamma t_{n-1} v_{n-2} w_{n-3}}, \\ t_n = \dfrac{\alpha t_{n-3} w_{n-2} + \beta}{\gamma u_{n-1} w_{n-2} t_{n-3}}, \end{cases} \tag{8}$$

where the parameters α,β are real numbers, the parameter γ and the initial values $u_{-f},v_{-f},w_{-f},t_{-f},\ f\in\{1,2,3\}$, are non-zero real numbers. We solve system (8) in explicit form. Then, we examine the solutions according to some special cases of the parameters. Further, numerical examples are given to demonstrate the theoretical results.

Definition 1. [4] (Periodicity) A sequence $(z_n)_{n=-k}^{\infty}$ is said to be eventually periodic with period p if there exits $n_0\geq -k$ such that $z_{n+p}=z_n$ for all $n\geq n_0$.If $n_0=-k$ then the sequence $(z_n)_{n=-k}^{\infty}$ is said to be periodic with period p.

2. EXPLICIT SOLUTIONS OF THE SYSTEM (8)

Let $(u_n,v_n,w_n,t_n)_{n\geq -3}$ be a solution of system (8). System (8) can be written in the following form

$$\begin{cases} u_nv_{n-1}=\dfrac{\alpha u_{n-3}t_{n-2}+\beta}{\gamma t_{n-2}u_{n-3}}=\dfrac{\alpha}{\gamma}+\dfrac{\beta}{\gamma t_{n-2}u_{n-3}}, \\ v_nw_{n-1}=\dfrac{\alpha v_{n-3}u_{n-2}+\beta}{\gamma u_{n-2}v_{n-3}}=\dfrac{\alpha}{\gamma}+\dfrac{\beta}{\gamma u_{n-2}v_{n-3}}, & n\in N_0. \\ w_nt_{n-1}=\dfrac{\alpha w_{n-3}v_{n-2}+\beta}{\gamma v_{n-2}w_{n-3}}=\dfrac{\alpha}{\gamma}+\dfrac{\beta}{\gamma v_{n-2}w_{n-3}}, \\ t_nu_{n-1}=\dfrac{\alpha t_{n-3}w_{n-2}+\beta}{\gamma w_{n-2}t_{n-3}}=\dfrac{\alpha}{\gamma}+\dfrac{\beta}{\gamma w_{n-2}t_{n-3}}, \end{cases} \tag{9}$$

By using the following transformations

$$\begin{cases} u_nv_{n-1}=x_n, \\ v_nw_{n-1}=y_n, & n\geq -2, \\ w_nt_{n-1}=z_n, \\ t_nu_{n-1}=r_n, \end{cases} \tag{10}$$

system (9) is transformed into the following system

$$\begin{cases} x_n=\dfrac{\alpha}{\gamma}+\dfrac{\beta}{\gamma r_{n-2}}, \\ y_n=\dfrac{\alpha}{\gamma}+\dfrac{\beta}{\gamma x_{n-2}}, & n\in N_0, \\ z_n=\dfrac{\alpha}{\gamma}+\dfrac{\beta}{\gamma y_{n-2}}, \\ r_n=\dfrac{\alpha}{\gamma}+\dfrac{\beta}{\gamma z_{n-2}}, \end{cases} \tag{11}$$

which can be written as

$$\begin{cases} x_n = \dfrac{(3\alpha^2\beta\gamma + \alpha^4 + \beta^2\gamma^2)x_{n-8} + \alpha^3\beta + 2\alpha\beta^2\gamma}{(\alpha^3\gamma + 2\alpha\beta\gamma^2)x_{n-8} + \alpha^2\gamma\beta + \beta^2\gamma^2}, \\ y_n = \dfrac{(3\alpha^2\beta\gamma + \alpha^4 + \beta^2\gamma^2)y_{n-8} + \alpha^3\beta + 2\alpha\beta^2\gamma}{(\alpha^3\gamma + 2\alpha\beta\gamma^2)y_{n-8} + \alpha^2\gamma\beta + \beta^2\gamma^2}, \qquad n \geq 6. \\ z_n = \dfrac{(3\alpha^2\beta\gamma + \alpha^4 + \beta^2\gamma^2)z_{n-8} + \alpha^3\beta + 2\alpha\beta^2\gamma}{(\alpha^3\gamma + 2\alpha\beta\gamma^2)z_{n-8} + \alpha^2\gamma\beta + \beta^2\gamma^2}, \\ r_n = \dfrac{(3\alpha^2\beta\gamma + \alpha^4 + \beta^2\gamma^2)r_{n-8} + \alpha^3\beta + 2\alpha\beta^2\gamma}{(\alpha^3\gamma + 2\alpha\beta\gamma^2)r_{n-8} + \alpha^2\gamma\beta + \beta^2\gamma^2}, \end{cases} \tag{12}$$

Then, we consider the following equation

$$\delta_n = \frac{(3\alpha^2\beta\gamma + \alpha^4 + \beta^2\gamma^2)\delta_{n-8} + \alpha^3\beta + 2\alpha\beta^2\gamma}{(\alpha^3\gamma + 2\alpha\beta\gamma^2)\delta_{n-8} + \alpha^2\gamma\beta + \beta^2\gamma^2}, \qquad n \geq 6, \tag{13}$$

instead of equations in (12).

If we apply the decomposition of indices $n \to 8(m+1) + k$, $k = \overline{-2,5}$ and $m \geq -1$ to (13), then it can be written as follows

$$\delta_{m+1}^{(k)} = \frac{(3\alpha^2\beta\gamma + \alpha^4 + \beta^2\gamma^2)\delta_m^{(k)} + \alpha^3\beta + 2\alpha\beta^2\gamma}{(\alpha^3\gamma + 2\alpha\beta\gamma^2)\delta_m^{(k)} + \alpha^2\gamma\beta + \beta^2\gamma^2}, \tag{14}$$

where $\delta_m^{(k)} = \delta_{8m+k}, m \in \mathbb{N}_0, k = \overline{-2,5}$.

Let

$$\begin{cases} A_1 := 3\alpha^2\beta\gamma + \alpha^4 + \beta^2\gamma^2, \\ B_1 := \alpha^3\beta + 2\alpha\beta^2\gamma, \\ C_1 := \alpha^3\gamma + 2\alpha\beta\gamma^2, \\ D_1 := \alpha^2\gamma\beta + \beta^2\gamma^2. \end{cases}$$

From equation (2), the general solution of (14) follows straightforwardly as

$$\delta_m^{(k)} = \frac{-\beta^4\gamma^4\delta_0^{(k)} s_{m-1} + \left(A_1\delta_0^{(k)} + B_1\right)s_m}{\left(C_1\delta_0^{(k)} - A_1\right)s_m + s_{m+1}}, \ m \in \mathbb{N}_0, \tag{15}$$

for $k = \overline{-2,5}$, where the sequence $(s_m)_{m\in\mathbb{N}_0}$ is satisfying

$s_{m+1} - (A_1 + D_1)s_m - (B_1C_1 - A_1D_1)s_{m-1} = 0.$

We use (10) in (12) and from (15), equations in (12) are expressed as

$$\begin{cases} x_{8m+k} = \dfrac{-\beta^4\gamma^4 u_k v_{k-1} s_{m-1} + (A_1 u_k v_{k-1} + B_1) s_m}{(C_1 u_k v_{k-1} - A_1) s_m + s_{m+1}}, \\ y_{8m+k} = \dfrac{-\beta^4\gamma^4 v_k w_{k-1} s_{m-1} + (A_1 v_k w_{k-1} + B_1) s_m}{(C_1 v_k w_{k-1} - A_1) s_m + s_{m+1}}, \qquad m \in N_0, \\ z_{8m+k} = \dfrac{-\beta^4\gamma^4 w_k t_{k-1} s_{m-1} + (A_1 w_k t_{k-1} + B_1) s_m}{(C_1 w_k t_{k-1} - A_1) s_m + s_{m+1}}, \\ r_{8m+k} = \dfrac{-\beta^4\gamma^4 t_k u_{k-1} s_{m-1} + (A_1 t_k u_{k-1} + B_1) s_m}{(C_1 t_k u_{k-1} - A_1) s_m + s_{m+1}}, \end{cases} \tag{16}$$

for $k = \overline{-2,5}$. From (10), we get

$$\begin{cases} u_n = \dfrac{x_n}{v_{n-1}} = \dfrac{x_n w_{n-2}}{y_{n-1}} = \dfrac{x_n z_{n-2}}{y_{n-1} t_{n-3}} = \dfrac{x_n z_{n-2} u_{n-4}}{y_{n-1} r_{n-3}} = \dfrac{x_n z_{n-2} x_{n-4}}{y_{n-1} r_{n-3} v_{n-5}} \\ \quad = \dfrac{x_n z_{n-2} x_{n-4} w_{n-6}}{y_{n-1} r_{n-3} y_{n-5}} = \dfrac{x_n z_{n-2} x_{n-4} z_{n-6}}{y_{n-1} r_{n-3} y_{n-5} t_{n-7}} = \dfrac{x_n z_{n-2} x_{n-4} z_{n-6}}{y_{n-1} r_{n-3} y_{n-5} r_{n-7}} u_{n-8}, \\ v_n = \dfrac{y_n}{w_{n-1}} = \dfrac{y_n t_{n-2}}{z_{n-1}} = \dfrac{y_n r_{n-2}}{z_{n-1} u_{n-3}} = \dfrac{y_n r_{n-2} v_{n-4}}{z_{n-1} x_{n-3}} = \dfrac{y_n r_{n-2} y_{n-4}}{z_{n-1} x_{n-3} w_{n-5}} \\ \quad = \dfrac{y_n r_{n-2} y_{n-4} t_{n-6}}{z_{n-1} x_{n-3} z_{n-5}} = \dfrac{y_n r_{n-2} y_{n-4} r_{n-6}}{z_{n-1} x_{n-3} z_{n-5} u_{n-7}} = \dfrac{y_n r_{n-2} y_{n-4} r_{n-6}}{z_{n-1} x_{n-3} z_{n-5} x_{n-7}} v_{n-8}, \\ w_n = \dfrac{z_n}{t_{n-1}} = \dfrac{z_n u_{n-2}}{r_{n-1}} = \dfrac{z_n x_{n-2}}{r_{n-1} v_{n-3}} = \dfrac{z_n x_{n-2} w_{n-4}}{r_{n-1} y_{n-3}} = \dfrac{z_n x_{n-2} z_{n-4}}{r_{n-1} y_{n-3} t_{n-5}} \\ \quad = \dfrac{z_n x_{n-2} z_{n-4} u_{n-6}}{r_{n-1} y_{n-3} r_{n-5}} = \dfrac{z_n x_{n-2} z_{n-4} x_{n-6}}{r_{n-1} y_{n-3} r_{n-5} v_{n-7}} = \dfrac{z_n x_{n-2} z_{n-4} x_{n-6}}{r_{n-1} y_{n-3} r_{n-5} y_{n-7}} w_{n-8}, \\ t_n = \dfrac{r_n}{u_{n-1}} = \dfrac{r_n v_{n-2}}{x_{n-1}} = \dfrac{r_n y_{n-2}}{x_{n-1} w_{n-3}} = \dfrac{r_n y_{n-2} t_{n-4}}{x_{n-1} z_{n-3}} = \dfrac{r_n y_{n-2} r_{n-4}}{x_{n-1} z_{n-3} u_{n-5}} \\ \quad = \dfrac{r_n y_{n-2} r_{n-4} v_{n-6}}{x_{n-1} z_{n-3} x_{n-5}} = \dfrac{r_n y_{n-2} r_{n-4} y_{n-6}}{x_{n-1} z_{n-3} x_{n-5} w_{n-7}} = \dfrac{r_n y_{n-2} r_{n-4} y_{n-6}}{x_{n-1} z_{n-3} x_{n-5} z_{n-7}} t_{n-8}, \end{cases} \tag{17}$$

for $n \geq 5$. From (17), we have

$$\begin{cases} u_{8m+h} = \dfrac{x_{8m+h} z_{8m+h-2} x_{8m+h-4} z_{8m+h-6}}{y_{8m+h-1} r_{8m+h-3} y_{8m+h-5} r_{8m+h-7}} u_{8(m-1)+h}, \\ v_{8m+h} = \dfrac{y_{8m+h} r_{8m+h-2} y_{8m+h-4} r_{8m+h-6}}{z_{8m+h-1} x_{8m+h-3} z_{8m+h-5} x_{8m+h-7}} v_{8(m-1)+h}, \qquad m \in N_0, \\ w_{8m+h} = \dfrac{z_{8m+h} x_{8m+h-2} z_{8m+h-4} x_{8m+h-6}}{r_{8m+h-1} y_{8m+h-3} r_{8m+h-5} y_{8m+h-7}} w_{8(m-1)+h}, \\ t_{8m+h} = \dfrac{r_{8m+h} y_{8m+h-2} r_{8m+h-4} y_{8m+h-6}}{x_{8m+h-1} z_{8m+h-3} x_{8m+h-5} z_{8m+h-7}} t_{8(m-1)+h}, \end{cases} \tag{18}$$

for $h = \overline{5,12}$.

Multiplying the equalities which are obtained from (18), from 0 to m, it follows that

$$\left\{\begin{aligned}
u_{8m+k+7} &= u_{k-1}\prod_{l=0}^{m}\left(\frac{x_{8\left(l+\left\lfloor\frac{k+9}{8}\right\rfloor\right)+k+7-8\left\lfloor\frac{k+9}{8}\right\rfloor}z_{8\left(l+\left\lfloor\frac{k+7}{8}\right\rfloor\right)+k+5-8\left\lfloor\frac{k+7}{8}\right\rfloor}}{y_{8\left(l+\left\lfloor\frac{k+8}{8}\right\rfloor\right)+k+6-8\left\lfloor\frac{k+8}{8}\right\rfloor}r_{8\left(l+\left\lfloor\frac{k+6}{8}\right\rfloor\right)+k+4-8\left\lfloor\frac{k+6}{8}\right\rfloor}}\right.\\
&\qquad\left.\times\frac{x_{8\left(l+\left\lfloor\frac{k+5}{8}\right\rfloor\right)+k+3-8\left\lfloor\frac{k+5}{8}\right\rfloor}z_{8\left(l+\left\lfloor\frac{k+3}{8}\right\rfloor\right)+k+1-8\left\lfloor\frac{k+3}{8}\right\rfloor}}{y_{8\left(l+\left\lfloor\frac{k+4}{8}\right\rfloor\right)+k+2-8\left\lfloor\frac{k+4}{8}\right\rfloor}r_{8\left(l+\left\lfloor\frac{k+2}{8}\right\rfloor\right)+k-8\left\lfloor\frac{k+2}{8}\right\rfloor}}\right),\\
v_{8m+k+7} &= v_{k-1}\prod_{l=0}^{m}\left(\frac{y_{8\left(l+\left\lfloor\frac{k+9}{8}\right\rfloor\right)+k+7-8\left\lfloor\frac{k+9}{8}\right\rfloor}r_{8\left(l+\left\lfloor\frac{k+7}{8}\right\rfloor\right)+k+5-8\left\lfloor\frac{k+7}{8}\right\rfloor}}{z_{8\left(l+\left\lfloor\frac{k+8}{8}\right\rfloor\right)+k+6-8\left\lfloor\frac{k+8}{8}\right\rfloor}x_{8\left(l+\left\lfloor\frac{k+6}{8}\right\rfloor\right)+k+4-8\left\lfloor\frac{k+6}{8}\right\rfloor}}\right.\\
&\qquad\left.\times\frac{y_{8\left(l+\left\lfloor\frac{k+5}{8}\right\rfloor\right)+k+3-8\left\lfloor\frac{k+5}{8}\right\rfloor}r_{8\left(l+\left\lfloor\frac{k+3}{8}\right\rfloor\right)+k+1-8\left\lfloor\frac{k+3}{8}\right\rfloor}}{z_{8\left(l+\left\lfloor\frac{k+4}{8}\right\rfloor\right)+k+2-8\left\lfloor\frac{k+4}{8}\right\rfloor}x_{8\left(l+\left\lfloor\frac{k+2}{8}\right\rfloor\right)+k-8\left\lfloor\frac{k+2}{8}\right\rfloor}}\right),\\
w_{8m+k+7} &= w_{k-1}\prod_{l=0}^{m}\left(\frac{z_{8\left(l+\left\lfloor\frac{k+9}{8}\right\rfloor\right)+k+7-8\left\lfloor\frac{k+9}{8}\right\rfloor}x_{8\left(l+\left\lfloor\frac{k+7}{8}\right\rfloor\right)+k+5-8\left\lfloor\frac{k+7}{8}\right\rfloor}}{r_{8\left(l+\left\lfloor\frac{k+8}{8}\right\rfloor\right)+k+6-8\left\lfloor\frac{k+8}{8}\right\rfloor}y_{8\left(l+\left\lfloor\frac{k+6}{8}\right\rfloor\right)+k+4-8\left\lfloor\frac{k+6}{8}\right\rfloor}}\right.\\
&\qquad\left.\times\frac{z_{8\left(l+\left\lfloor\frac{k+5}{8}\right\rfloor\right)+k+3-8\left\lfloor\frac{k+5}{8}\right\rfloor}x_{8\left(l+\left\lfloor\frac{k+3}{8}\right\rfloor\right)+k+1-8\left\lfloor\frac{k+3}{8}\right\rfloor}}{r_{8\left(l+\left\lfloor\frac{k+4}{8}\right\rfloor\right)+k+2-8\left\lfloor\frac{k+4}{8}\right\rfloor}y_{8\left(l+\left\lfloor\frac{k+2}{8}\right\rfloor\right)+k-8\left\lfloor\frac{k+2}{8}\right\rfloor}}\right),\\
t_{8m+k+7} &= t_{k-1}\prod_{l=0}^{m}\left(\frac{r_{8\left(l+\left\lfloor\frac{k+9}{8}\right\rfloor\right)+k+7-8\left\lfloor\frac{k+9}{8}\right\rfloor}y_{8\left(l+\left\lfloor\frac{k+7}{8}\right\rfloor\right)+k+5-8\left\lfloor\frac{k+7}{8}\right\rfloor}}{x_{8\left(l+\left\lfloor\frac{k+8}{8}\right\rfloor\right)+k+6-8\left\lfloor\frac{k+8}{8}\right\rfloor}z_{8\left(l+\left\lfloor\frac{k+6}{8}\right\rfloor\right)+k+4-8\left\lfloor\frac{k+6}{8}\right\rfloor}}\right.\\
&\qquad\left.\times\frac{r_{8\left(l+\left\lfloor\frac{k+5}{8}\right\rfloor\right)+k+3-8\left\lfloor\frac{k+5}{8}\right\rfloor}y_{8\left(l+\left\lfloor\frac{k+3}{8}\right\rfloor\right)+k+1-8\left\lfloor\frac{k+3}{8}\right\rfloor}}{x_{8\left(l+\left\lfloor\frac{k+4}{8}\right\rfloor\right)+k+2-8\left\lfloor\frac{k+4}{8}\right\rfloor}z_{8\left(l+\left\lfloor\frac{k+2}{8}\right\rfloor\right)+k-8\left\lfloor\frac{k+2}{8}\right\rfloor}}\right),
\end{aligned}\right. \tag{19}$$

for $h = k+7,\ k = \overline{-2,5}$.

By substituting the formulas in (16) into (19), we obtain

$$\left\{\begin{array}{l} u_{8m+k+7} = u_{k-1}\displaystyle\prod_{l=0}^{m}\left(\frac{\dfrac{-\beta^4\gamma^4 u_{k+7-8\lfloor\frac{k+9}{8}\rfloor} v_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} s_{l+\lfloor\frac{k+9}{8}\rfloor-1} + \left(A_1 u_{k+7-8\lfloor\frac{k+9}{8}\rfloor} v_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor}}{\left(C_1 u_{k+7-8\lfloor\frac{k+9}{8}\rfloor} v_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor} + s_{l+\lfloor\frac{k+9}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 v_{k+6-8\lfloor\frac{k+8}{8}\rfloor} w_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} s_{l+\lfloor\frac{k+8}{8}\rfloor-1} + \left(A_1 v_{k+6-8\lfloor\frac{k+8}{8}\rfloor} w_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor}}{\left(C_1 v_{k+6-8\lfloor\frac{k+8}{8}\rfloor} w_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor} + s_{l+\lfloor\frac{k+8}{8}\rfloor+1}}}\right. \\
\times\dfrac{\dfrac{-\beta^4\gamma^4 w_{k+5-8\lfloor\frac{k+7}{8}\rfloor} t_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} s_{l+\lfloor\frac{k+7}{8}\rfloor-1} + \left(A_1 w_{k+5-8\lfloor\frac{k+7}{8}\rfloor} t_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor}}{\left(C_1 w_{k+5-8\lfloor\frac{k+7}{8}\rfloor} t_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor} + s_{l+\lfloor\frac{k+7}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 t_{k+4-8\lfloor\frac{k+6}{8}\rfloor} u_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} s_{l+\lfloor\frac{k+6}{8}\rfloor-1} + \left(A_1 t_{k+4-8\lfloor\frac{k+6}{8}\rfloor} u_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor}}{\left(C_1 t_{k+4-8\lfloor\frac{k+6}{8}\rfloor} u_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor} + s_{l+\lfloor\frac{k+6}{8}\rfloor+1}}} \\
\times\dfrac{\dfrac{-\beta^4\gamma^4 u_{k+3-8\lfloor\frac{k+5}{8}\rfloor} v_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} s_{l+\lfloor\frac{k+5}{8}\rfloor-1} + \left(A_1 u_{k+3-8\lfloor\frac{k+5}{8}\rfloor} v_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor}}{\left(C_1 u_{k+3-8\lfloor\frac{k+5}{8}\rfloor} v_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor} + s_{l+\lfloor\frac{k+5}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 v_{k+2-8\lfloor\frac{k+4}{8}\rfloor} w_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} s_{l+\lfloor\frac{k+4}{8}\rfloor-1} + \left(A_1 v_{k+2-8\lfloor\frac{k+4}{8}\rfloor} w_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor}}{\left(C_1 v_{k+2-8\lfloor\frac{k+4}{8}\rfloor} w_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor} + s_{l+\lfloor\frac{k+4}{8}\rfloor+1}}} \\
\left.\times\dfrac{\dfrac{-\beta^4\gamma^4 w_{k+1-8\lfloor\frac{k+3}{8}\rfloor} t_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} s_{l+\lfloor\frac{k+3}{8}\rfloor-1} + \left(A_1 w_{k+1-8\lfloor\frac{k+3}{8}\rfloor} t_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor}}{\left(C_1 w_{k+1-8\lfloor\frac{k+3}{8}\rfloor} t_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor} + s_{l+\lfloor\frac{k+3}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 t_{k-8\lfloor\frac{k+2}{8}\rfloor} u_{k-8\lfloor\frac{k+2}{8}\rfloor-1} s_{l+\lfloor\frac{k+2}{8}\rfloor-1} + \left(A_1 t_{k-8\lfloor\frac{k+2}{8}\rfloor} u_{k-8\lfloor\frac{k+2}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor}}{\left(C_1 t_{k-8\lfloor\frac{k+2}{8}\rfloor} u_{k-8\lfloor\frac{k+2}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor} + s_{l+\lfloor\frac{k+2}{8}\rfloor+1}}}\right),
\end{array}\right. \tag{20}$$

$$\left\{\begin{array}{l} v_{8m+k+7} = v_{k-1}\displaystyle\prod_{l=0}^{m}\left(\frac{\dfrac{-\beta^4\gamma^4 v_{k+7-8\lfloor\frac{k+9}{8}\rfloor} w_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} s_{l+\lfloor\frac{k+9}{8}\rfloor-1} + \left(A_1 v_{k+7-8\lfloor\frac{k+9}{8}\rfloor} w_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor}}{\left(C_1 v_{k+7-8\lfloor\frac{k+9}{8}\rfloor} w_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor} + s_{l+\lfloor\frac{k+9}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 w_{k+6-8\lfloor\frac{k+8}{8}\rfloor} t_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} s_{l+\lfloor\frac{k+8}{8}\rfloor-1} + \left(A_1 w_{k+6-8\lfloor\frac{k+8}{8}\rfloor} t_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor}}{\left(C_1 w_{k+6-8\lfloor\frac{k+8}{8}\rfloor} t_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor} + s_{l+\lfloor\frac{k+8}{8}\rfloor+1}}}\right. \\
\times\dfrac{\dfrac{-\beta^4\gamma^4 t_{k+5-8\lfloor\frac{k+7}{8}\rfloor} u_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} s_{l+\lfloor\frac{k+7}{8}\rfloor-1} + \left(A_1 t_{k+5-8\lfloor\frac{k+7}{8}\rfloor} u_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor}}{\left(C_1 t_{k+5-8\lfloor\frac{k+7}{8}\rfloor} u_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor} + s_{l+\lfloor\frac{k+7}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 u_{k+4-8\lfloor\frac{k+6}{8}\rfloor} v_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} s_{l+\lfloor\frac{k+6}{8}\rfloor-1} + \left(A_1 u_{k+4-8\lfloor\frac{k+6}{8}\rfloor} v_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor}}{\left(C_1 u_{k+4-8\lfloor\frac{k+6}{8}\rfloor} v_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor} + s_{l+\lfloor\frac{k+6}{8}\rfloor+1}}} \\
\times\dfrac{\dfrac{-\beta^4\gamma^4 v_{k+3-8\lfloor\frac{k+5}{8}\rfloor} w_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} s_{l+\lfloor\frac{k+5}{8}\rfloor-1} + \left(A_1 v_{k+3-8\lfloor\frac{k+5}{8}\rfloor} w_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor}}{\left(C_1 v_{k+3-8\lfloor\frac{k+5}{8}\rfloor} w_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor} + s_{l+\lfloor\frac{k+5}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 w_{k+2-8\lfloor\frac{k+4}{8}\rfloor} t_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} s_{l+\lfloor\frac{k+4}{8}\rfloor-1} + \left(A_1 w_{k+2-8\lfloor\frac{k+4}{8}\rfloor} t_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor}}{\left(C_1 w_{k+2-8\lfloor\frac{k+4}{8}\rfloor} t_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor} + s_{l+\lfloor\frac{k+4}{8}\rfloor+1}}} \\
\left.\times\dfrac{\dfrac{-\beta^4\gamma^4 t_{k+1-8\lfloor\frac{k+3}{8}\rfloor} u_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} s_{l+\lfloor\frac{k+3}{8}\rfloor-1} + \left(A_1 t_{k+1-8\lfloor\frac{k+3}{8}\rfloor} u_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor}}{\left(C_1 t_{k+1-8\lfloor\frac{k+3}{8}\rfloor} u_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor} + s_{l+\lfloor\frac{k+3}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 u_{k-8\lfloor\frac{k+2}{8}\rfloor} v_{k-8\lfloor\frac{k+2}{8}\rfloor-1} s_{l+\lfloor\frac{k+2}{8}\rfloor-1} + \left(A_1 u_{k-8\lfloor\frac{k+2}{8}\rfloor} v_{k-8\lfloor\frac{k+2}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor}}{\left(C_1 u_{k-8\lfloor\frac{k+2}{8}\rfloor} v_{k-8\lfloor\frac{k+2}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor} + s_{l+\lfloor\frac{k+2}{8}\rfloor+1}}}\right),
\end{array}\right. \tag{21}$$

$$
\left\{
\begin{aligned}
w_{8m+k+7} = w_{k-1}\prod_{l=0}^{m} & \left(\frac{\dfrac{-\beta^4\gamma^4 w_{k+7-8\lfloor\frac{k+9}{8}\rfloor} t_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} s_{l+\lfloor\frac{k+9}{8}\rfloor-1} + \left(A_1 w_{k+7-8\lfloor\frac{k+9}{8}\rfloor} t_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor}}{\left(C_1 w_{k+7-8\lfloor\frac{k+9}{8}\rfloor} t_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor} + s_{l+\lfloor\frac{k+9}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 t_{k+6-8\lfloor\frac{k+8}{8}\rfloor} u_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} s_{l+\lfloor\frac{k+8}{8}\rfloor-1} + \left(A_1 t_{k+6-8\lfloor\frac{k+8}{8}\rfloor} u_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor}}{\left(C_1 t_{k+6-8\lfloor\frac{k+8}{8}\rfloor} u_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor} + s_{l+\lfloor\frac{k+8}{8}\rfloor+1}}}\right. \\
& \times\frac{\dfrac{-\beta^4\gamma^4 u_{k+5-8\lfloor\frac{k+7}{8}\rfloor} v_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} s_{l+\lfloor\frac{k+7}{8}\rfloor-1} + \left(A_1 u_{k+5-8\lfloor\frac{k+7}{8}\rfloor} v_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor}}{\left(C_1 u_{k+5-8\lfloor\frac{k+7}{8}\rfloor} v_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor} + s_{l+\lfloor\frac{k+7}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 v_{k+4-8\lfloor\frac{k+6}{8}\rfloor} w_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} s_{l+\lfloor\frac{k+6}{8}\rfloor-1} + \left(A_1 v_{k+4-8\lfloor\frac{k+6}{8}\rfloor} w_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor}}{\left(C_1 v_{k+4-8\lfloor\frac{k+6}{8}\rfloor} w_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor} + s_{l+\lfloor\frac{k+6}{8}\rfloor+1}}} \\
& \times\frac{\dfrac{-\beta^4\gamma^4 w_{k+3-8\lfloor\frac{k+5}{8}\rfloor} t_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} s_{l+\lfloor\frac{k+5}{8}\rfloor-1} + \left(A_1 w_{k+3-8\lfloor\frac{k+5}{8}\rfloor} t_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor}}{\left(C_1 w_{k+3-8\lfloor\frac{k+5}{8}\rfloor} t_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor} + s_{l+\lfloor\frac{k+5}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 t_{k+2-8\lfloor\frac{k+4}{8}\rfloor} u_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} s_{l+\lfloor\frac{k+4}{8}\rfloor-1} + \left(A_1 t_{k+2-8\lfloor\frac{k+4}{8}\rfloor} u_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor}}{\left(C_1 t_{k+2-8\lfloor\frac{k+4}{8}\rfloor} u_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor} + s_{l+\lfloor\frac{k+4}{8}\rfloor+1}}} \\
& \left.\times\frac{\dfrac{-\beta^4\gamma^4 u_{k+1-8\lfloor\frac{k+3}{8}\rfloor} v_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} s_{l+\lfloor\frac{k+3}{8}\rfloor-1} + \left(A_1 u_{k+1-8\lfloor\frac{k+3}{8}\rfloor} v_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor}}{\left(C_1 u_{k+1-8\lfloor\frac{k+3}{8}\rfloor} v_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor} + s_{l+\lfloor\frac{k+3}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 v_{k-8\lfloor\frac{k+2}{8}\rfloor} w_{k-8\lfloor\frac{k+2}{8}\rfloor-1} s_{l+\lfloor\frac{k+2}{8}\rfloor-1} + \left(A_1 v_{k-8\lfloor\frac{k+2}{8}\rfloor} w_{k-8\lfloor\frac{k+2}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor}}{\left(C_1 v_{k-8\lfloor\frac{k+2}{8}\rfloor} w_{k-8\lfloor\frac{k+2}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor} + s_{l+\lfloor\frac{k+2}{8}\rfloor+1}}}\right),
\end{aligned}
\right.
\tag{22}
$$

$$
\left\{
\begin{aligned}
t_{8m+k+7} = t_{k-1}\prod_{l=0}^{m} & \left(\frac{\dfrac{-\beta^4\gamma^4 t_{k+7-8\lfloor\frac{k+9}{8}\rfloor} u_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} s_{l+\lfloor\frac{k+9}{8}\rfloor-1} + \left(A_1 t_{k+7-8\lfloor\frac{k+9}{8}\rfloor} u_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor}}{\left(C_1 t_{k+7-8\lfloor\frac{k+9}{8}\rfloor} u_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor} + s_{l+\lfloor\frac{k+9}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 u_{k+6-8\lfloor\frac{k+8}{8}\rfloor} v_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} s_{l+\lfloor\frac{k+8}{8}\rfloor-1} + \left(A_1 u_{k+6-8\lfloor\frac{k+8}{8}\rfloor} v_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor}}{\left(C_1 u_{k+6-8\lfloor\frac{k+8}{8}\rfloor} v_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor} + s_{l+\lfloor\frac{k+8}{8}\rfloor+1}}}\right. \\
& \times\frac{\dfrac{-\beta^4\gamma^4 v_{k+5-8\lfloor\frac{k+7}{8}\rfloor} w_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} s_{l+\lfloor\frac{k+7}{8}\rfloor-1} + \left(A_1 v_{k+5-8\lfloor\frac{k+7}{8}\rfloor} w_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor}}{\left(C_1 v_{k+5-8\lfloor\frac{k+7}{8}\rfloor} w_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor} + s_{l+\lfloor\frac{k+7}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 w_{k+4-8\lfloor\frac{k+6}{8}\rfloor} t_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} s_{l+\lfloor\frac{k+6}{8}\rfloor-1} + \left(A_1 w_{k+4-8\lfloor\frac{k+6}{8}\rfloor} t_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor}}{\left(C_1 w_{k+4-8\lfloor\frac{k+6}{8}\rfloor} t_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor} + s_{l+\lfloor\frac{k+6}{8}\rfloor+1}}} \\
& \times\frac{\dfrac{-\beta^4\gamma^4 t_{k+3-8\lfloor\frac{k+5}{8}\rfloor} u_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} s_{l+\lfloor\frac{k+5}{8}\rfloor-1} + \left(A_1 t_{k+3-8\lfloor\frac{k+5}{8}\rfloor} u_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor}}{\left(C_1 t_{k+3-8\lfloor\frac{k+5}{8}\rfloor} u_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor} + s_{l+\lfloor\frac{k+5}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 u_{k+2-8\lfloor\frac{k+4}{8}\rfloor} v_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} s_{l+\lfloor\frac{k+4}{8}\rfloor-1} + \left(A_1 u_{k+2-8\lfloor\frac{k+4}{8}\rfloor} v_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor}}{\left(C_1 u_{k+2-8\lfloor\frac{k+4}{8}\rfloor} v_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor} + s_{l+\lfloor\frac{k+4}{8}\rfloor+1}}} \\
& \left.\times\frac{\dfrac{-\beta^4\gamma^4 v_{k+1-8\lfloor\frac{k+3}{8}\rfloor} w_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} s_{l+\lfloor\frac{k+3}{8}\rfloor-1} + \left(A_1 v_{k+1-8\lfloor\frac{k+3}{8}\rfloor} w_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor}}{\left(C_1 v_{k+1-8\lfloor\frac{k+3}{8}\rfloor} w_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor} + s_{l+\lfloor\frac{k+3}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 w_{k-8\lfloor\frac{k+2}{8}\rfloor} t_{k-8\lfloor\frac{k+2}{8}\rfloor-1} s_{l+\lfloor\frac{k+2}{8}\rfloor-1} + \left(A_1 w_{k-8\lfloor\frac{k+2}{8}\rfloor} t_{k-8\lfloor\frac{k+2}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor}}{\left(C_1 w_{k-8\lfloor\frac{k+2}{8}\rfloor} t_{k-8\lfloor\frac{k+2}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor} + s_{l+\lfloor\frac{k+2}{8}\rfloor+1}}}\right),
\end{aligned}
\right.
\tag{23}
$$

for $m \in N_0, k = \overline{-2,5}$.

3. PARTICULAR CASES OF SYSTEM (8)

In this section, we will consider the solutions according to some special cases of the parameters.

3.1. Case 1 $\alpha = \gamma = 1, \beta = 0$.

In this case, system (8) can be written to the following form

$$\begin{cases} u_n = \dfrac{u_{n-3}t_{n-2}}{v_{n-1}t_{n-2}u_{n-3}} = \dfrac{1}{v_{n-1}} = w_{n-2} = \dfrac{1}{t_{n-3}} = u_{n-4}, \\ v_n = \dfrac{v_{n-3}u_{n-2}}{w_{n-1}u_{n-2}v_{n-3}} = \dfrac{1}{w_{n-1}} = t_{n-2} = \dfrac{1}{u_{n-3}} = v_{n-4}, \\ w_n = \dfrac{w_{n-3}v_{n-2}}{t_{n-1}v_{n-2}w_{n-3}} = \dfrac{1}{t_{n-1}} = u_{n-2} = \dfrac{1}{v_{n-3}} = w_{n-4}, \\ t_n = \dfrac{t_{n-3}w_{n-2}}{u_{n-1}w_{n-2}t_{n-3}} = \dfrac{1}{u_{n-1}} = v_{n-2} = \dfrac{1}{w_{n-3}} = t_{n-4}, \end{cases} \tag{24}$$

for $n \geq 3$.

We obtain the solutions of equations in (24) as in the following form

$$u_{4m+j} = u_{j-4}, \quad v_{4m+j} = v_{j-4}, \quad w_{4m+j} = w_{j-4}, \quad t_{4m+j} = t_{j-4}, \quad m \in N_0, \quad j = \overline{3,6}.$$

Now, we give numerical example that represent the solutions of system (8) when $\alpha = \gamma = 1, \beta = 0$.

Example 3.1. Consider the system (8) with the parameters $\alpha = \gamma = 1, \beta = 0$ and the initial conditions $u_{-3} = 0.1, u_{-2} = 2.4, u_{-1} = 3, v_{-3} = 4.06, v_{-2} = 0.05, v_{-1} = 0.6, w_{-3} = 70.54, w_{-2} = 0.86, w_{-1} = 9.05, t_{-3} = 0.3, t_{-2} = 1.7, t_{-1} = 3.2$, the solutions are given as in Figure 1.

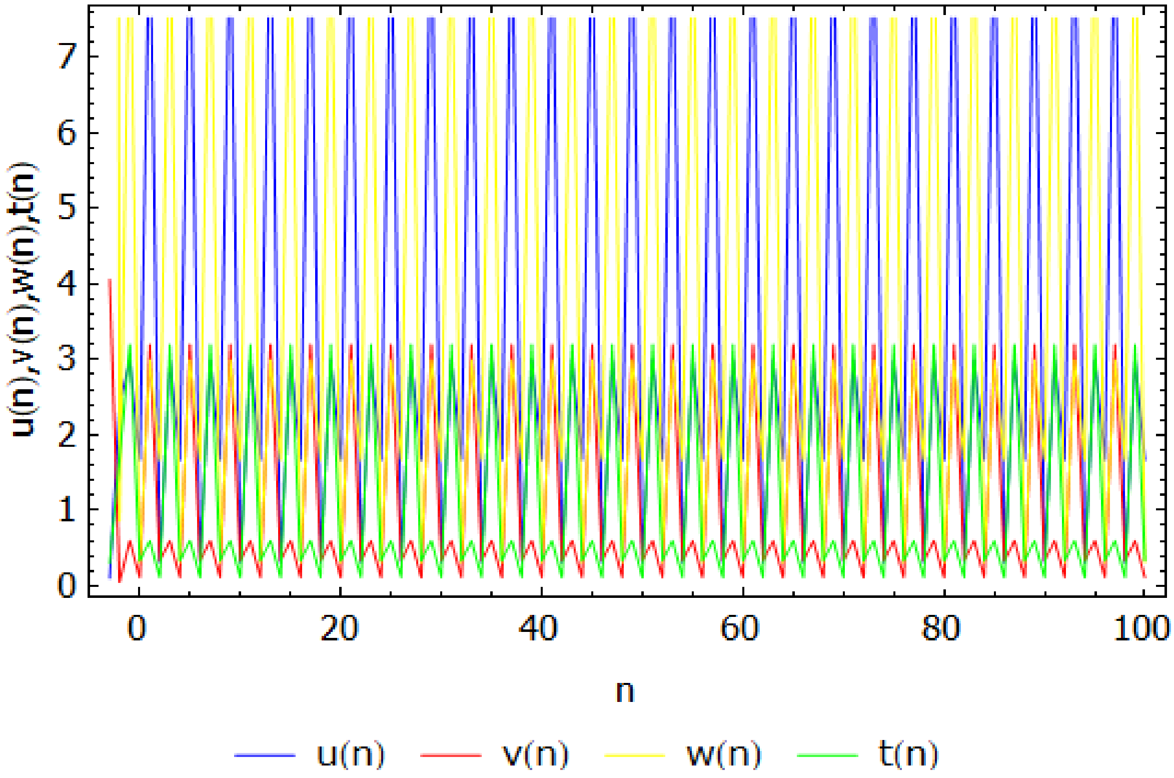

Figure 1. Plots of u_n, v_n, w_n, t_n in case $\alpha = \gamma = 1, \beta = 0$

Therefore, the solutions of system (8) are eventually periodic with period 4.

3.2. Case 2 $\alpha = \gamma = -1, \beta = 0$.

In this case, system (8) becomes

$$\begin{cases} u_n = \dfrac{-u_{n-3}t_{n-2}}{-v_{n-1}t_{n-2}u_{n-3}} = \dfrac{1}{v_{n-1}} = w_{n-2} = \dfrac{1}{t_{n-3}} = u_{n-4}, \\ v_n = \dfrac{-v_{n-3}u_{n-2}}{-w_{n-1}u_{n-2}v_{n-3}} = \dfrac{1}{w_{n-1}} = t_{n-2} = \dfrac{1}{u_{n-3}} = v_{n-4}\,, \\ w_n = \dfrac{-w_{n-3}v_{n-2}}{-t_{n-1}v_{n-2}w_{n-3}} = \dfrac{1}{t_{n-1}} = u_{n-2} = \dfrac{1}{v_{n-3}} = w_{n-4}, \\ t_n = \dfrac{-t_{n-3}w_{n-2}}{-u_{n-1}w_{n-2}t_{n-3}} = \dfrac{1}{u_{n-1}} = v_{n-2} = \dfrac{1}{w_{n-3}} = t_{n-4}, \end{cases} \tag{25}$$

for $n \geq 3$.

We obtain the solutions of equations in (25) as in the following form

$$u_{4m+j} = u_{j-4}, \qquad v_{4m+j} = v_{j-4}, \qquad w_{4m+j} = w_{j-4}, \qquad t_{4m+j} = t_{j-4}, \qquad m \in N_0, \qquad j = \overline{3,6}.$$

Now, we give numerical example that represent the solutions of system (8) when $\alpha = \gamma = -1, \beta = 0$.

Example 3.2. Consider the system (8) with the initial values $u_{-3} = 0.05$, $u_{-2} = 6$, $u_{-1} = 0.63$, $v_{-3} = 7$, $v_{-2} = 0.6$, $v_{-1} = 0.68$, $w_{-3} = 7.5$, $w_{-2} = 0.6$, $w_{-1} = 7.5$, $t_{-3} = 5.3$, $t_{-2} = 8.6$, $t_{-1} = 2.1$, and the parameters $\alpha = \gamma = -1, \beta = 0$ the solutions are represented as in Figure 2.

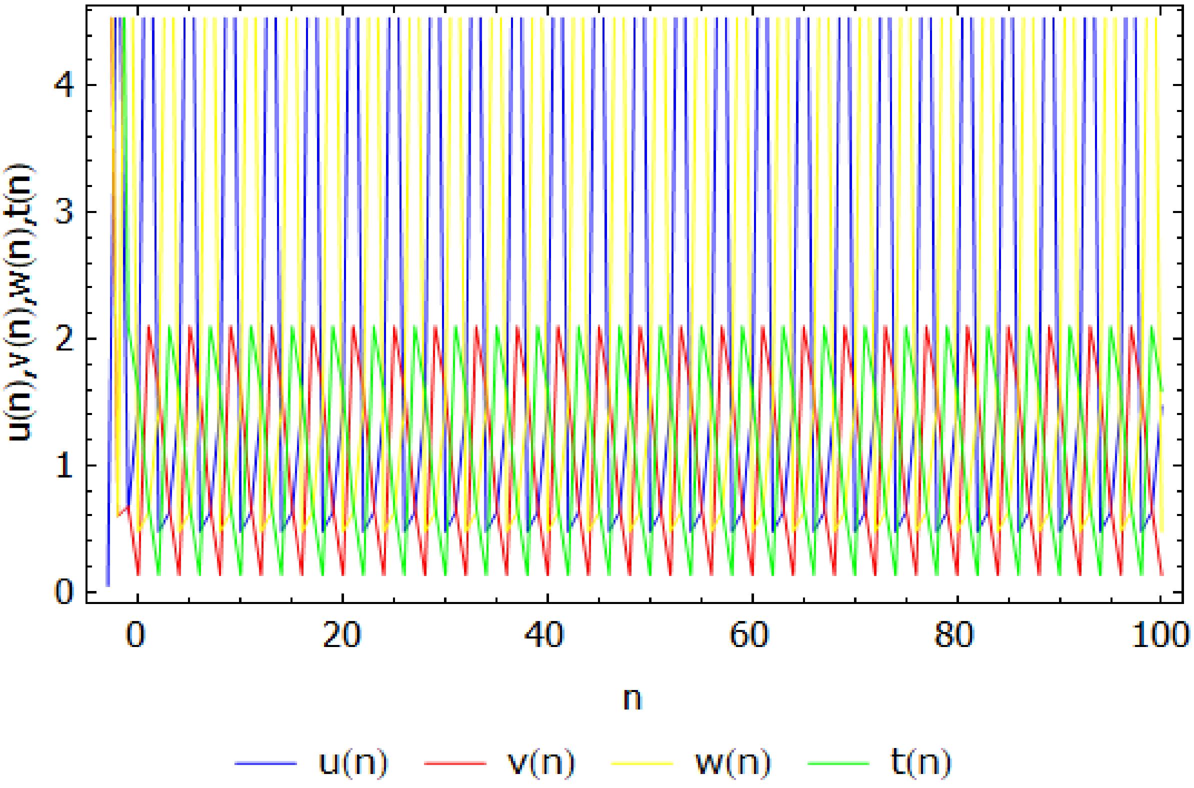

Figure 2. Plots of u_n, v_n, w_n, t_n in case $\alpha = \gamma = -1, \beta = 0$

Therefore, the solutions of system (8) are eventually periodic with period 4.

3.3. Case 3 $\alpha = 1, \gamma = -1, \beta = 0$.

In this case, system (8) becomes

$$\begin{cases} u_n = \dfrac{u_{n-3}t_{n-2}}{-v_{n-1}t_{n-2}u_{n-3}} = -\dfrac{1}{v_{n-1}} = w_{n-2} = -\dfrac{1}{t_{n-3}} = u_{n-4}, \\ v_n = \dfrac{v_{n-3}u_{n-2}}{-w_{n-1}u_{n-2}v_{n-3}} = -\dfrac{1}{w_{n-1}} = t_{n-2} = -\dfrac{1}{u_{n-3}} = v_{n-4}\,, \\ w_n = \dfrac{w_{n-3}v_{n-2}}{-t_{n-1}v_{n-2}w_{n-3}} = -\dfrac{1}{t_{n-1}} = u_{n-2} = -\dfrac{1}{v_{n-3}} = w_{n-4}, \\ t_n = \dfrac{t_{n-3}w_{n-2}}{-u_{n-1}w_{n-2}t_{n-3}} = -\dfrac{1}{u_{n-1}} = v_{n-2} = -\dfrac{1}{w_{n-3}} = t_{n-4}, \end{cases} \tag{26}$$

for $n \geq 3$.

We obtain the solutions of equations in (26) as in the following form

$$u_{4m+j} = u_{j-4}, \quad v_{4m+j} = v_{j-4}, \quad w_{4m+j} = w_{j-4}, \quad t_{4m+j} = t_{j-4}, \quad m \in N_0, \quad j = \overline{3,6}.$$

Now, we give numerical example that represent the solutions of system (8) when $\alpha = 1, \gamma = -1, \beta = 0$.

Example 3.3. Consider the system (8) with the initial values $u_{-3} = 4\,, u_{-2} = 6.2,\ u_{-1} = 0.2\,, v_{-3} = 0.09\,, v_{-2} = 1.7,\ v_{-1} = 0.6\,, w_{-3} = -7.5\,, w_{-2} = -0.63,\ w_{-1} = 2.6\,, t_{-3} = -3.2, t_{-2} - 6.8,\ t_{-1} = 2.1$ and the parameters $\alpha = 1, \gamma = -1, \beta = 0$ the solutions are represented as in Figure 3.

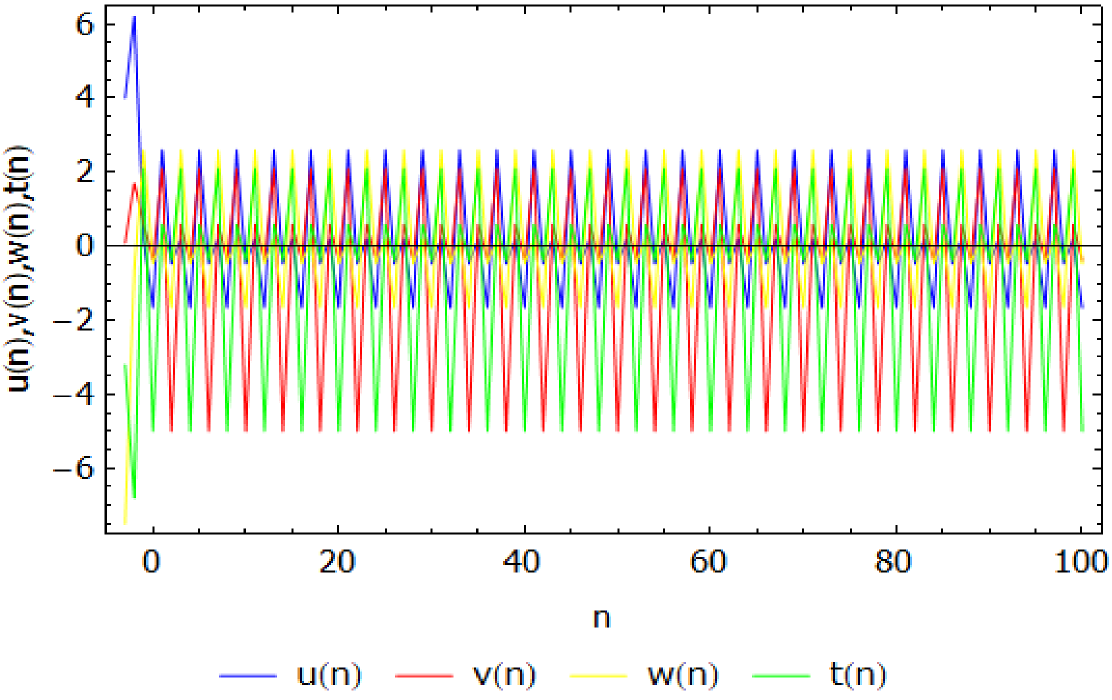

Figure 3. Plots of u_n, v_n, w_n, t_n in case $\alpha = 1, \gamma = -1, \beta = 0$

Therefore, the solutions of system (8) are eventually periodic with period 4.

4. CONCLUSION

In this paper, four-dimesional system of difference equations is solved in explicit form by using convenient transformation. In addition, the periodic solutions of aforementioned system of difference equations are obtained according to some special cases of the parameters. Finally, to support obtained results, we give numerical examples.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

AUTHORSHIP CONTRIBUTIONS

The authors contributed equally to this work.

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