E-ISSN 2980-3020

ISTANBUL JOURNAL OF MATHEMATICS

VOLUME 2 • NUMBER 1 • JUNE 2024

İSTANBUL UNIVERSITY PRESS

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https://iupress.istanbul.edu.tr/en/journal/ijmath/home
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PUBLISHER
İstanbul University Press
İstanbul University Central Campus,
34452 Beyazıt, Fatih, İstanbul, Türkiye
Phone: +90 (212) 440 00 00

The publication language of the journal is English.

This is a scholarly, international, peer-reviewed and open-access journal published biannually in June and December.

Publication Type: Periodical

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EDITORIAL BOARD

Istanbul Journal of Mathematics
JUNE 2024,
VOLUME 2, NUMBER 1

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ISTANBUL UNIVERSITY PRESS

Istanbul Journal of Mathematics
ijmath 2 (1), 1–27, 2024
DOI: 10.26650/ijmath.2024.00011

RESEARCH ARTICLE

On the holonomic systems for the Gauss hypergeometric function and its confluent family of a matrix argument

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ABSTRACT
We investigate the several special functions defined by a matrix integral on the Hermitian matrix space of size n. They are the matrix argument analogues of the Gauss hypergeometric, Kummer's confluent hypergeometric, the Bessel, the Hermite-Weber and Airy functions which play important roles in the multivariate statistical analysis and the random matrix theory. We give the integral representations for them as functions of eigenvalues of the matrix argument by using the result of Harish-Chandra and Itzykson-Zuber, and give the systems of differential equations for them. We show that these system are holonomic and have the holonomic rank 2^n using the theory of Gröbner basis.

Mathematics Subject Classification (2020): 33C70, 33C80

Keywords: hypergeometric function, matrix integral, holonomic system, Gröbner basis

1. INTRODUCTION

In this paper, we are concerned with the special functions of a matrix argument defined by an integral on the space of complex Hermitian matrices or normal matrices. One of the most important classes of classical special functions may be the Gauss hypergeometric function (HGF) and its confluent family, namely, Kummer's confluent HGF, Bessel function, Hermite-Weber function and Airy function. For example, Gauss, Kummer and Bessel functions are given by the power series

$$
\begin{aligned}
{}_2F_1(a,b,c;x) &= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} x^m, \\
{}_1F_1(a,c;x) &= \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m m!} x^m, \\
{}_0F_1(c+1;-x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(c+1)_m m!} x^m,
\end{aligned}
$$

respectively, where $a, b, c \in \mathbb{C}$, x is the complex variable and $(a)_m = \Gamma(a+m)/\Gamma(a)$ is the so-called Pochhammer's symbol defined by the gamma function $\Gamma(a)$. In this paper we consider and study the matrix argument analogues of these classical HGF family. The matrix argument analogues of Gauss, Kummer and Bessel are studied in connection with the multivariate statistical analysis Muirhead (1982) and with the analysis on symmetric cones Faraut and A. Koranyi (1994). We also want to add in this list the matrix argument analogues of Hermite-Weber and Airy functions, which have been studied in Inamasu and Kimura, (2021).

Let us explain our motivation of our study. The above mentioned classical HGF family is sometimes displayed schematically as

$$
\begin{array}{ccccccc}
 & & & & \text{Bessel} & & \\
 & & & \nearrow & & \searrow & \\
\text{Gauss} & \longrightarrow & \text{Kummer} & & & & \text{Airy,} \\
 & & & \searrow & & \nearrow & \\
 & & & & \text{Hermite – Weber} & &
\end{array}
$$

where each arrow implies some kind of limiting process called confluence. These functions are studied by using various aspects

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Submitted: 08.04.2024 • **Last Revision Received:** 07.06.2024 • **Accepted:** 07.06.2024

of the functions: the power series expressions, the integral representations, the differential equations, the contiguity relations. Here we focus on the aspects of differential equations and integral representations. The differential equations and the integral representations for them are given as follows.

Differential equations:

$$\begin{aligned} \text{Gauss}: &\quad x(1-x)y'' + \{c-(a+b+1)x\}y' - aby = 0, \\ \text{Kummer}: &\quad xy'' + (c-x)y' - ay = 0, \\ \text{Bessel}: &\quad xy'' + (c+1)y' + y = 0, \\ \text{Hermite-Weber}: &\quad y'' - xy' + cy = 0, \\ \text{Airy}: &\quad y'' - xy = 0. \end{aligned}$$

Integral representations:

$$\begin{aligned} {}_2F_1(a,b,c;x) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}dt \\ {}_1F_1(a,c;x) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}e^{xt}dt, \\ {}_0F_1(c+1;-x) &= \int_C t^{c-1}e^{xt-\frac{1}{t}}dt, \\ H(c;x) &= \int_C t^{-c-1}e^{xt-\frac{1}{2}t^2}dt, \\ Ai(x) &= \int_C e^{xt-\frac{1}{3}t^3}dt, \end{aligned}$$

where C is an appropriate path in the complex t-plane. Note that we took the path $\overrightarrow{01}$ as the path of integration for the Gauss' case and the Kummer's case so that the integrals give the power series expressions. If one takes another appropriate paths of integration, we get various solutions to the differential equations (see Iwasaki et al. (1991)). We should comment on the Bessel equation. In many literatures, it has the form $z^2w'' + zw' + (z^2 - c^2)w = 0$. If one perform, for this equation, the change of unknown $w \mapsto y$ by $w = z^c y$ and then the change of independent variable $z \mapsto x$ by $x = z^2/4$, we get the differential equation we gave in the list.

The Gauss HGF and its confluent family appear in many research fields of mathematics and mathematical physics and play important roles. For example, it is known that the Gauss, Kummer, Hermite-Weber, Bessel and Airy functions appear as particular solutions of the Painlevé equations P_6, P_5, P_4, P_3 and P_2, respectively Iwasaki et al. (1991).

It is also known that they are understood as simple cases of Gelfand's HGF on the complex Grassmannian manifold $\mathrm{Gr}(r,N)$, the set of r-dimensional subspaces in $\mathbb{C}^N$. Roughly speaking, Gelfand's HGF on $\mathrm{Gr}(r,N)$ is defined as follows. First we consider the maximal abelian subgroup H_λ of $\mathrm{GL}(N)$ obtained as the centralizer of a regular element a of $\mathrm{GL}(N)$, where a is in the Jordan normal form and its cell structure is described by the partition λ of N. Then Gelfand's HGF of type λ on $\mathrm{Gr}(r,N)$ is defined as the Radon transform of a character of the universal covering group $\tilde{H}_\lambda$. In this context, the Gauss, Kummer, Bessel, Hermite-Weber and Airy functions are identified with Gelfand's HGFs on $\mathrm{Gr}(2,4)$ corresponding to the partitions $(1,1,1,1), (2,1,1), (2,2), (3,1)$ and (4), respectively.

Taking into account of these facts, we think it is natural to study the extension of classical HGF family to the functions of a matrix argument including those of the Hermite-Weber and Airy functions. It should be commented that the Airy function of a matrix argument, defined by a Hermitian matrix integral in Section 2.2, already played an important role in the resolution of Witten's conjecture on the 2-dimensional quantum gravity by M. Kontsevich Kontsevich (1992).

In Inamasu and Kimura, (2021), we discussed the relation of the HGFs of a matrix argument, defined by the integrals on the space $\mathscr{H}(n)$ of Hermitian matrices, to some semi-classical orthogonal polynomials and to the polynomial solutions to the quantum Painlevé systems (see also Nagoya (2011)). We stated in Inamasu and Kimura, (2021) a conjecture on the explict form of the systems of partial differential equations characterizing the Hermite-Weber and Airy functions of a matrix argument. We give the answer (Theorem 3.1) to this conjecture deriving the systems of differential equations for a matrix argument analogue of the Gauss and its confluent family defined by the matrix integrals (Definition 2.1). It should be mentioned that the differential equations for the matrix argument analogues of Gauss, Kummer and Bessel were obtained in Muirhead (1970) by J. Muirhead. He handled the functions given by the series expansion in terms of zonal polynomials and derived the differential equations characterizing them. Our approach is different from his. We treat the functions defined by the integrals with various possible choices of domain of integration in deriving the differential equations. On the other hand, the functions treated by Muirhead correspond to the integrals with a particular choice of domain of integration, see Proposition 2.8. Since we use the matrix integrals on $\mathscr{H}(n)$ or on the space of normal matrices to define the HGFs of a matrix argument, we call them the HGFs of matrix integral type.

Another main result of this paper is Theorem 5.1 on the holonomicity of the systems and on their holonomic ranks which give the dimension of the solution space for the systems at a generic point. This theorem is proved by computing a Gröbner basis for the ideal in the ring of differential operators generated by the differential operators characterizing the HGFs.

This paper is organized as follows. In Section 2, we introduce the HGFs defined by an integral on the Hermitian matrix or normal matrix space. We give the expressions of HGFs as the functions of eigenvalues of the variable matrix. The main tools are the Harish-Chandra and Itsykson-Zuber integral formulas. In Section 3, we give the systems of differential equations for the HGFs of matrix integral type as the functions of eigenvalues of the matrix argument (Theorem 3.1). Section 4 is devoted to the proof of this theorem. In Section 5, we discuss the holonomicity and the holonomic rank of the systems (Theorem 5.1).

2. HGF OF MATRIX INTEGRAL TYPE

2.1. Integrals on Hermitian matrix space

Let $\mathscr{H}(n)$ be the set of $n \times n$ complex Hermitian matrices. It is a real vector space of dimension n^2. For $Y = (Y_{ij}) \in \mathscr{H}(n)$, let dY denote the volume element on $\mathscr{H}(n)$, which is the usual Euclidean volume element

$$dY = \bigwedge_{i=1}^{n} dY_{ii} \bigwedge_{i<j} \left(d\mathrm{Re}(Y_{ij}) \wedge d\mathrm{Im}(Y_{ij})\right),$$

where we fix some order of indices in the right hand side.

The matrix integral version of the gamma function and the beta function are defined by

$$\Gamma_n(a) = \int_{Y>0} |Y|^{a-n}\mathrm{etr}(-Y)\, dY,$$
$$B_n(a,b) := \int_{0<Y<I} |Y|^{a-n}|I-Y|^{b-n}\, dY,$$

respectively, where $Y \in \mathscr{H}(n)$, $|Y|$ is the determinant of Y, $\mathrm{tr}\, Y$ is the trace of Y, $\mathrm{etr}(Y) := \exp(\,\mathrm{tr}\,(Y))$ and the integral is taken on the set of positive definite Hermitian matrices $Y > 0$ for the gamma function and on the subset of $\mathscr{H}(n)$ satisfying $Y > 0$ and $I - Y > 0$ for the beta function. The gamma integral converges for $\mathrm{Re}(a) > n-1$ and the beta integral for $\mathrm{Re}(a) > n-1, \mathrm{Re}(b) > n-1$, and they define holomorphic functions there.

Proposition. *(see Faraut and A. Koranyi (1994)) The following formulas hold.*
(i) $\Gamma_n(a) = \pi^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} \Gamma(a+i-1)$.
(ii) $B_n(a,b) = \frac{\Gamma_n(a)\Gamma_n(b)}{\Gamma_n(a+b)}$.

2.2. HGF of matrix integral type

We introduced the family of HGFs of matrix integral type in Inamasu and Kimura, (2021). We recall them.

Definition 2.1. For $X \in \mathscr{H}(n)$, put

$$I_G(a,b,c;X) = \int_C |Y|^{a-n}|I-Y|^{c-a-n}|I-XY|^{-b}\, dY,$$
$$I_K(a,c;X) = \int_C |Y|^{a-n}|I-Y|^{c-a-n}\mathrm{etr}(XY)\, dY$$
$$I_B(c;X) = \int_C |Y|^{c-n}\mathrm{etr}(XY - Y^{-1})\, dY,$$
$$I_{HW}(c;X) = \int_C |Y|^{-c-n}\mathrm{etr}(XY - \frac{1}{2}Y^2)\, dY,$$
$$I_A(X) = \int_C \mathrm{etr}(XY - \frac{1}{3}Y^3)\, dY,$$

where C is an appropriate domain of integration in $\mathscr{H}(n)$ or in the space of normal matrices of size n for which the differentiation with respect to the entries of X can be interchanged with the integration.

Comparing the above integrals with the integral representations for the classical hypergeometric family in the introduction, one may recognize that they are extensions of the classical HGF family to functions with a matrix argument. In fact, Muirhead treated in Muirhead (1970) the extension of Gauss and Kummer to the functions of a matrix argument expressed by the series in terms of zonal polynomials. They are denoted by ${}_2F_1(a,b,c;X)$ and ${}_1F_1(a,c;X)$ and have the integral representations:

$$
\begin{aligned}
{}_2F_1(a,b,c;X) &= \frac{\Gamma_n(c)}{\Gamma_n(a)\Gamma_n(c-a)} \int_{0<Y<I} |Y|^{a-n}|I-Y|^{c-a-n}|I-XY|^{-b}\, dY, \\
{}_1F_1(a,c;X) &= \frac{\Gamma_n(c)}{\Gamma_n(a)\Gamma_n(c-a)} \int_{0<Y<I} |Y|^{a-n}|I-Y|^{c-a-n}\mathrm{etr}(XY)\, dY.
\end{aligned}
$$

It should be mentioned on the choice of domains of integration C for the integrals in Definition 2.1. We required that C is chosen so that the differentiation with respect to the entries of X can be interchanged with the integration, and that we can apply the Stokes theorem. For example, to define the Airy function of matrix integral type, we consider the integral in the space of normal matrices. In this case, taking into account that a normal matrix is a matrix which is transformed to a diagonal matrix with complex eigenvalues by conjugating with a unitary matrix, we see in Proposition 2.6 that the matrix integral can be reduced to the integral on the space of eigenvalues. Then we may take the domain of integration C in the normal matrix space which, after a reduction of the integral, becomes an n-cycle of a locally finite homology group of the space of eigenvalues $y = (y_1, \dots, y_n) \in \mathbb{C}^n$ on which the integrand decreases to 0 exponentially when $|y| \to \infty$. See Hien (2007) for this kind of homology groups.

Remark 2.2. The matrix integrals in Definition 2.1 define functions of the eigenvalues $x_1, \dots, x_n$ of X, see the next subsection.

2.3. Integrals on the eigenvalues

For the HGFs of matrix integral type, we want to rewrite them to the integrals on the space of eigenvalues $y = (y_1, \dots, y_n)$ of $Y \in \mathscr{H}(n)$. To this end we need the following integral formulas . Let $\mathscr{U}(n)$ denote the group of unitary matrices of size n.

Proposition 2.3. *(Weyl integration formula) We have*

$$
\int f(Y)dY = \pi^{\frac{n(n-1)}{2}} (\prod_{p=1}^{n} p!)^{-1} \int f(gyg^*)\Delta(y)^2 dy dg,
$$

where $Y \sim y = \mathrm{diag}(y_1, \dots, y_n)$ *by* $Y = gyg^*$ *with* $g \in \mathscr{U}(n)$, $\Delta(y) = \prod_{i<j}(y_i - y_j)$, $dy = dy_1 \cdots dy_n$, *and* dg *is the normalized Haar measure on the unitary group* $\mathscr{U}(n)$.

We also need the following results due to Harish-Chandra and Itzykson-Zuber. We refer to Balantekin (2000); Bleher and Kuijlaars (2004); Deift, (2000); Harnad and Orlov (2007); Mehta (1991) for these formulas.

Proposition 2.4. *Let* A, B *be normal matrices of size* n *diagonalized as*

$$
A \sim \mathrm{diag}(a_1, \dots, a_n), \quad B \sim \mathrm{diag}(b_1, \dots, b_n),
$$

and assume that $a_i \neq a_j,\ b_i \neq b_j$ *for* $i \neq j$. *For* $t \in \mathbb{C}$, *we have*

$$
\int_{\mathscr{U}(n)} (\det(1 - tAgBg^*))^{-\alpha} dg = \prod_{p=1}^{n-1} \frac{p!}{(\alpha - n + 1)_p} \frac{\det[(1 - ta_i b_j)^{-\alpha+n-1}]}{\Delta(a)\Delta(b)}.
$$

Proposition 2.5. *Let* A, B *be as in Proposition 2.4. For* $t \in \mathbb{C}$, *we have*

$$
\int_{\mathscr{U}(n)} \exp[t \,\mathrm{tr}\,(AgBg^*)] dg = \left(\prod_{p=1}^{n-1} p!\right) \frac{\det(e^{ta_i b_j})}{\Delta(a)\Delta(b)}.
$$

By applying Propositions 2.3, 2.4 to the integrals in Definition 2.1, we obtain the following result.

Proposition 2.6. *Assume that* $X \in \mathscr{H}(n)$ *has distinct eigenvalues* $x_1, \dots, x_n$. *Then we have*

$$I_G(a,b,c;X) = C_0 \int_D \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n} \cdot \det\left((1-x_j y_k)^{-b+n-1}\right) \frac{\Delta(y)}{\Delta(x)} dy,$$

$$I_K(a,c;X) = C_1 \int_D \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n} \cdot \det(e^{x_j y_k}) \frac{\Delta(y)}{\Delta(x)} dy,$$

$$I_B(c;X) = C_1 \int_D \prod_{i=1}^n y_i^{c-n} e^{-1/y_i} \det\left(e^{x_j y_k}\right) \frac{\Delta(y)}{\Delta(x)} dy,$$

$$I_{HW}(c;X) = C_1 \int_D \prod_{i=1}^n y_i^{-c-n} e^{-\frac{1}{2}y_i^2} \det\left(e^{x_j y_k}\right) \frac{\Delta(y)}{\Delta(x)} dy,$$

$$I_A(X) = C_1 \int_D \prod_{i=1}^n e^{-\frac{1}{3}y_i^3} \det\left(e^{x_j y_k}\right) \frac{\Delta(y)}{\Delta(x)} dy,$$

where $C_0 = \pi^{\frac{n(n-1)}{2}}(n!\prod_{p=1}^{n-1}(b-n+1)_p)^{-1}, C_1 = \pi^{\frac{n(n-1)}{2}}(n!)^{-1}$*, and* D *is a twisted* n*-cycle of the homology group defined by the integrand.*

Proof. We show the assertion for $I_G(a,b,c;X)$ for the sake of completeness of presentation. We apply the Weyl integration formula to $f(Y) = |Y|^{c_1}|I-Y|^{c_2}|I-XY|^{-b}$ with $c_1 = a-n, c_2 = c-a-n$. Note that

$$\begin{aligned} f(gyg^*) &= |gyg^*|^{c_1}|I-gyg^*|^{c_2}|I-Xgyg^*|^{-b} \\ &= |y|^{c_1}|I-y|^{c_2}|I-Xgyg^*|^{-b} \\ &= \prod_{i=1}^n y_i^{c_1}(1-y_i)^{c_2} \cdot |I-Xgyg^*|^{-b}. \end{aligned}$$

Putting this in the Weyl formula and using Proposition 2.4 for $t=1$, we have

$$\begin{aligned} I_G(a,b,c;X) &= \pi^{\frac{n(n-1)}{2}}(\prod_{p=1}^n p!)^{-1} \int_D \left(\int_{\mathscr{U}(n)} |I-Xgyg^*|^{-b} dg\right) \prod_{i=1}^n y_i^{c_1}(1-y_i)^{c_2}\Delta(y)^2 dy \\ &= C_0 \int_D \frac{\det\left((1-x_j y_k)^{-b+n-1}\right)}{\Delta(x)\Delta(y)} \prod_{i=1}^n y_i^{c_1}(1-y_i)^{c_2}\Delta(y)^2 dy. \\ &= C_0 \int_D \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n} \cdot \det\left((1-x_j y_k)^{-b+n-1}\right) \frac{\Delta(y)}{\Delta(x)} dy. \end{aligned}$$

The expressions for the other HGFs can be obtained in a similar way by using Proposition 2.5.

Remark 2.7. For the Airy integral $I_A(X)$, we can take an n-cycle D in the rapidly decay homology group Hien (2007). Let γ_1, γ_2 be the paths in $\mathbb{C}$ as in Figure 1. Then $D_{i_1,\ldots,i_n} = \gamma_{i_1} \times \cdots \times \gamma_{i_n}$ for $i_1, \ldots, i_n \in \{1,2\}$ gives an n-cycle and there are 2^n choices.

Now the following statement is easily deduced from Proposition 2.6.

Proposition 2.8. *(1) For* ${}_2F_1(a,b,c;X)$*, we assume that* $X \in \mathscr{H}(n)$ *has distinct eigenvalues* $x_1,\ldots,x_n$*. Then we have*

$$\begin{aligned} {}_2F_1(a,b,c;X) &= C_2 \int_{(0,1)^n} \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n} \cdot \det\left((1-x_j y_k)^{-b+n-1}\right) \frac{\Delta(y)}{\Delta(x)} dy \\ &= n!C_2 \int_{(0,1)^n} \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n}(1-x_i y_i)^{-b+n-1} \frac{\Delta(y)}{\Delta(x)} dy, \end{aligned}$$

where $C_2 = \frac{\Gamma_n(c)}{\Gamma_n(a)\Gamma_n(c-a)}C_0$.

(2) For ${}_1F_1(a,b,c;X)$*, we assume that* $X \in \mathscr{H}(n)$ *has distinct eigenvalues* $x_1,\ldots,x_n$*. Then we have*

$$\begin{aligned} {}_1F_1(a,c;X) &= C_3 \int_{(0,1)^n} \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n} \cdot \det(e^{x_j y_k}) \frac{\Delta(y)}{\Delta(x)} dy \\ &= n!C_3 \int_{(0,1)^n} \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n} e^{x_i y_i} \frac{\Delta(y)}{\Delta(x)} dy, \end{aligned}$$

where $C_3 = \frac{\Gamma_n(c)}{\Gamma_n(a)\Gamma_n(c-a)}C_1$.

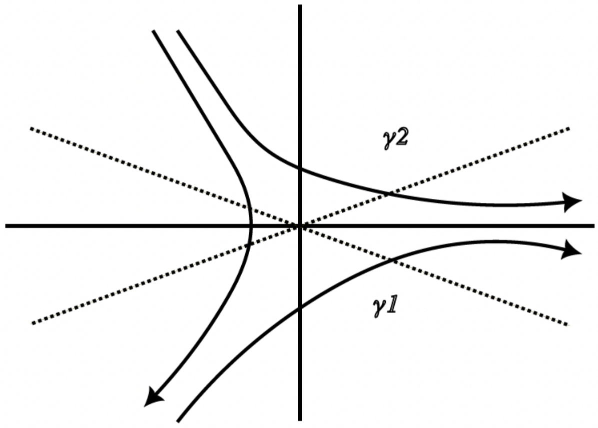

Figure 1. Figure 1

Proof. We show (2). The first representation for ${}_1F_1(a,c;X)$ is obvious by Propostion 2.6. We show the second one for ${}_1F_1$. Put $U(y) := \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n}$ and consider the expansion

$$\det(e^{x_j y_k}) = \sum_{\sigma\in\mathfrak{S}_n} (\operatorname{sgn}\sigma) e^{x_1 y_{\sigma(1)}} \cdots e^{x_n y_{\sigma(n)}},$$

where $\mathfrak{S}_n$ is the symmetric group of degree n. Then we have

$$\frac{\Delta(x)}{C_3}\,{}_1F_1(a,c;X) = \sum_{\sigma\in\mathfrak{S}_n} \int_{(0,1)^n} (\operatorname{sgn}\sigma) U(y) \prod_{i=1}^n e^{x_i y_{\sigma(i)}} \Delta(y) dy.$$

Consider the integral in the right hand side for any fixed $\sigma \in \mathfrak{S}_n$ and make a change of variables $y \to y'$ defined by $y'_i = y_{\sigma(i)}$ $(i = 1,\dots,n)$. Note that, by the change $y \to y'$, the function $U(y)$ and the domain of integration $(0,1)^n$ are invariant, and $\Delta(y') = (\operatorname{sgn}\sigma)\Delta(y)$. Hence the integrals in the right hand side are all equal to

$$\int_{(0,1)^n} \prod_{i=1}^n y_i^{a-n}(1-y_i)^{c-a-n} e^{x_i y_i} \Delta(y) dy.$$

This establishes the second representation for ${}_1F_1$.

3. SYSTEM OF DIFFERENTIAL EQUATIONS FOR HGF

We give the systems of differential equations satisfied by the family of HGFs of matrix integral type given in Definition 2.1. We assume that the domain of integration C for these integrals is chosen so that the interchange of derivation with respect to X and the integration with respect to Y is allowed and the Stokes theorem can be applied. Let ∂_i denote the partial derivation $\frac{\partial}{\partial x_i}$.

Theorem 3.1. *The HGF $I_*(X)$ $(* = G, K, B, HW, A)$ satisfies, as a function of eigenvalues of X, the following system of differential equations $\mathcal{S}_*$.*

Gauss $\mathcal{S}_G$:

$$x_i(1-x_i)\partial_i^2 F + \{c-(n-1)-(a+b+1-(n-1))x_i\}\partial_i F + \sum_{j(\neq i)} \frac{x_i(1-x_i)\partial_i F - x_j(1-x_j)\partial_j F}{x_i - x_j} - abF = 0, \quad 1 \le i \le n.$$

Kummer $\mathcal{S}_K$:

$$x_i\partial_i^2 F + \{c-(n-1)-x_i\}\partial_i F + \sum_{j(\neq i)} \frac{x_i\partial_i F - x_j\partial_j F}{x_i - x_j} - aF = 0, \quad 1 \le i \le n.$$

Bessel $\mathcal{S}_B$:

$$x_i\partial_i^2 F + \{c+1\}\partial_i F + \sum_{j(\neq i)} \frac{x_i\partial_i F - x_j\partial_j F}{x_i - x_j} + F = 0, \quad 1 \le i \le n.$$

Hermite-Weber $\mathcal{S}_{HW}$:

$$\partial_i^2 F - x_i \partial_i F + \sum_{j(\neq i)} \frac{\partial_i F - \partial_j F}{x_i - x_j} + cF = 0, \quad 1 \leq i \leq n. \tag{1}$$

Airy $\mathcal{S}_A$:

$$\partial_i^2 F + \sum_{j(\neq i)} \frac{\partial_i F - \partial_j F}{x_i - x_j} - x_i F = 0, \quad 1 \leq i \leq n. \tag{2}$$

The proof of the theorem is given in the next section.

As a particular case of Theorem 3.1, we have the following result, which was given by Muirhead in Muirhead (1970).

Proposition 3.2. *(1) ${}_2F_1(a,b,c;X)$, as a function of eigenvalues of X, is characterized as the holomorphic solution F to the system $\mathcal{S}_G$ which is symmetric in the variables and satisfies $F(0) = 1$.*

(2) ${}_1F_1(a,c;X)$, as a function of eigenvalues of X, is characterized as the holomorphic solution F to the system $\mathcal{S}_K$, which is symmetric in the variables and satisfies $F(0) = 1$.

Once we get the system of differential equations $\mathcal{S}_*$ $(* = G, K, B, HW, A)$, we can consider it as defined on $\mathbb{C}^n$. In Theorem 5.1 of the last section, we show that these systems are holonomic on the Zariski open set $\Omega_* \subset \mathbb{C}^n$ and their holonomic rank is 2^n, namely the systems are equivalent to the completely integrable Pfaffian systems of rank 2^n.

4. PROOF OF THEOREM 3.1

In this section, we use Y_{ij} $(1 \leq i, j \leq n)$, the entries of matrix integration variable Y, as the independent variables of the real space $\mathscr{H}(n)$ instead of $Y_{ii}, \mathrm{Re}(Y_{ij}), \mathrm{Im}(Y_{ij})$ $(1 \leq i < j \leq n)$. Note that, since

$$\mathrm{Re}(Y_{ij}) = \frac{Y_{ij} + Y_{ji}}{2}, \quad \mathrm{Im}(Y_{ij}) = \frac{Y_{ij} - Y_{ji}}{2\sqrt{-1}}, \quad (1 \leq i \leq j \leq n),$$

we have

$$dY = \bigwedge_{i=1}^{n} dY_{ii} \bigwedge_{i<j} \left(\frac{\sqrt{-1}}{2} Y_{ij} \wedge Y_{ji} \right).$$

4.1. Lemmas

Let $X = (X_{ij}) \in \mathscr{H}(n)$ be diagonalized as $x = UXU^\dagger, x = \mathrm{diag}(x_1, \ldots, x_n)$ by a unitary matrix U, where $U^\dagger$ is the hermitian conjugate of U, namely $U^\dagger = {}^t\bar{U}$. Assume that $x_1, \ldots, x_n$ are distinct. Note that x and U depends on X. The following lemmata are known (Adler and Moerbeke (1992), p50). For the sake of completeness of presentation, we give their proof.

Lemma 4.1. *The following equalities hold.*

$$\frac{\partial x_\alpha}{\partial X_{ij}} = U_{\alpha i} U^\dagger_{j\alpha}, \tag{3}$$

$$(x_\alpha - x_\beta) \left(\frac{\partial U}{\partial X_{ij}} U^\dagger \right)_{\alpha\beta} = U_{\alpha i} U^\dagger_{j\beta}, \quad \text{if } \alpha \neq \beta. \tag{4}$$

Proof. Differentiate the both sides of $x = UXU^\dagger$ with respect to X_{ij}. Using the identity

$$\frac{\partial U}{\partial X_{ij}} U^\dagger + U \frac{\partial U^\dagger}{\partial X_{ij}} = 0, \tag{5}$$

which comes from $UU^\dagger = I$, we have

$$\begin{aligned} \frac{\partial x}{\partial X_{ij}} &= \left(\frac{\partial U}{\partial X_{ij}} U^\dagger \right) x + UE_{ij}U^\dagger + x \left(U \frac{\partial U^\dagger}{\partial X_{ij}} \right) \\ &= \left(\frac{\partial U}{\partial X_{ij}} U^\dagger \right) x + UE_{ij}U^\dagger - x \left(\frac{\partial U}{\partial X_{ij}} U^\dagger \right), \end{aligned} \tag{6}$$

where E_{ij} is the (i,j) matrix unit, namely the $n \times n$ matrix whose only non-zero entry is 1 at the (i,j)-entry. Comparing the (α,α)-entry of both sides of (6), we get (3) and comparing the (α,β)-entry with $\alpha \neq \beta$, we get (4).

Lemma 4.2. *For (α,β) with $1 \le \alpha,\beta \le n$, we have the equalities:*

$$\sum_{i,j} U^\dagger_{i\beta} \frac{\partial x_\alpha}{\partial X_{ij}} U_{\beta j} = \delta_{\alpha\beta}. \tag{7}$$

$$\sum_{i,j,k} U^\dagger_{i\beta} \frac{\partial^2 x_\alpha}{\partial X_{ij}\partial X_{jk}} U_{\beta k} = \begin{cases} \frac{1}{x_\alpha - x_\beta}, & \text{if } \alpha \neq \beta, \\ \sum_{\gamma\neq\alpha} \frac{1}{x_\alpha - x_\gamma}, & \text{if } \alpha = \beta. \end{cases} \tag{8}$$

$$\sum_{a,b,p,q} U^\dagger_{al} U^\dagger_{ps} U_{sb} U_{lq} \frac{\partial^2 x_\alpha}{\partial X_{ab}\partial X_{pq}} = \begin{cases} \frac{1}{x_l - x_s}, & \text{if } \alpha = l, l \neq s, \\ -\frac{1}{x_l - x_s}, & \text{if } \alpha = s, l \neq s, \\ 0, & \text{otherwise}. \end{cases} \tag{9}$$

Proof. From the equality (3) of Lemma 4.1, we have

$$\sum_{i,j} U^\dagger_{i\beta} \frac{\partial x_\alpha}{\partial X_{ij}} U_{\beta j} = \sum_{i,j} U^\dagger_{i\beta} U_{\alpha i} U^\dagger_{j\alpha} U_{\beta j} = \delta_{\alpha\beta}\delta_{\beta\alpha} = \delta_{\alpha\beta}.$$

To show the second equality, differentiate the both sides of $\frac{\partial x_\alpha}{\partial X_{ij}} = U_{\alpha i} U^\dagger_{j\alpha}$ with respect to X_{jk} and obtain $\frac{\partial^2 x_\alpha}{\partial X_{ij}\partial X_{jk}} = \frac{\partial U_{\alpha i}}{\partial X_{jk}} U^\dagger_{j\alpha} + U_{\alpha i} \frac{\partial U^\dagger_{j\alpha}}{\partial X_{jk}}$. Denote the left hand side of (8) as $A(\alpha,\beta)$. Then

$$\begin{aligned} A(\alpha,\beta) &= \sum_{i,j,k} U^\dagger_{i\beta} \frac{\partial U_{\alpha i}}{X_{jk}} U^\dagger_{j\alpha} U_{\beta k} + \sum_{i,j,k} U^\dagger_{i\beta} U_{\alpha i} \frac{\partial U^\dagger_{j\alpha}}{\partial X_{jk}} U_{\beta k} \\ &= \sum_{j,k} \left(\frac{\partial U}{\partial X_{jk}} U^\dagger \right)_{\alpha\beta} U^\dagger_{j\alpha} U_{\beta k} + \delta_{\alpha\beta} \sum_{j,k} \frac{\partial U^\dagger_{j\alpha}}{\partial X_{jk}} U_{\beta k} =: A_1(\alpha,\beta) + A_2(\alpha,\beta). \end{aligned}$$

In the case $\alpha \neq \beta$, the contribution to A comes only from A_1. Using the equality (4) of Lemma 4.1, we have

$$A_1(\alpha,\beta) = \sum_{j,k} \frac{1}{x_\alpha - x_\beta} U_{\alpha j} U^\dagger_{k\beta} U^\dagger_{j\alpha} U_{\beta k} = \frac{1}{x_\alpha - x_\beta}.$$

In the case $\alpha = \beta$, using (5) and $U^\dagger_{j\alpha} U_{\alpha i} = \delta_{ij} - \sum_{\gamma\neq\alpha} U^\dagger_{j\gamma} U_{\gamma i}$, we have

$$\begin{aligned} A_1(\alpha,\alpha) &= -\sum_{j,k} \left(U \frac{\partial U^\dagger}{\partial X_{jk}} \right)_{\alpha\alpha} U^\dagger_{j\alpha} U_{\alpha k} = -\sum_{i,j,k} U_{\alpha i} \frac{\partial U^\dagger_{i\alpha}}{\partial X_{jk}} U^\dagger_{j\alpha} U_{\alpha k} \\ &= \sum_{\gamma\neq\alpha} \sum_{i,j,k} \frac{\partial U^\dagger_{i\alpha}}{\partial X_{jk}} U^\dagger_{j\gamma} U_{\gamma i} U_{\alpha k} - A_2(\alpha,\alpha). \end{aligned}$$

Hence using the identity (5) and Lemma 4.1, we have

$$\begin{aligned} A(\alpha,\alpha) &= A_1(\alpha,\alpha) + A_2(\alpha,\alpha) \\ &= \sum_{\gamma\neq\alpha} \sum_{j,k} \left(U \frac{\partial U^\dagger}{\partial X_{jk}} \right)_{\gamma\alpha} U^\dagger_{j\gamma} U_{\alpha k} = -\sum_{\gamma\neq\alpha} \sum_{j,k} \left(\frac{\partial U}{\partial X_{jk}} U^\dagger \right)_{\gamma\alpha} U^\dagger_{j\gamma} U_{\alpha k} \\ &= -\sum_{\gamma\neq\alpha} \frac{1}{x_\gamma - x_\alpha} \sum_{j,k} U_{\gamma j} U^\dagger_{k\alpha} U^\dagger_{j\gamma} U_{\alpha k} = -\sum_{\gamma\neq\alpha} \frac{1}{x_\gamma - x_\alpha}. \end{aligned}$$

To obtain the equality (9), differentiate the both sides of $\frac{\partial x_\alpha}{\partial X_{pq}} = U_{\alpha p} U^\dagger_{q\alpha}$ with respect to X_{ab} to get

$$\frac{\partial^2 x_\alpha}{\partial X_{ab}\partial X_{pq}} = \frac{\partial U_{\alpha p}}{\partial X_{ab}} U^\dagger_{q\alpha} + U_{\alpha p} \frac{\partial U^\dagger_{q\alpha}}{\partial X_{ab}}.$$

Denote the left hand side of (9) as $A(\alpha, l, s)$. Then

$$\begin{aligned}
A(\alpha, l, s) &= \sum_{a,b,p,q} U^{\dagger}_{al} U^{\dagger}_{ps} U_{sb} U_{lq} \left(\frac{\partial U_{\alpha p}}{\partial X_{ab}} U^{\dagger}_{q\alpha} + U_{\alpha p} \frac{\partial U^{\dagger}_{q\alpha}}{\partial X_{ab}} \right) \\
&= \sum_{a,b,p,q} U^{\dagger}_{al} U^{\dagger}_{ps} U_{sb} U_{lq} \frac{\partial U_{\alpha p}}{\partial X_{ab}} U^{\dagger}_{q\alpha} + \sum_{a,b,p,q} U^{\dagger}_{al} U^{\dagger}_{ps} U_{sb} U_{lq} U_{\alpha p} \frac{\partial U^{\dagger}_{q\alpha}}{\partial X_{ab}} \\
&= \sum_{a,b,q} U^{\dagger}_{al} U_{sb} U_{lq} \left(\frac{\partial U}{\partial X_{ab}} U^{\dagger} \right)_{\alpha s} U^{\dagger}_{q\alpha} + \sum_{a,b,p} U^{\dagger}_{al} U^{\dagger}_{ps} U_{sb} U_{\alpha p} \left(U \frac{\partial U^{\dagger}}{\partial X_{ab}} \right)_{l\alpha} \\
&= \sum_{a,b} U^{\dagger}_{al} U_{sb} \delta_{l\alpha} \left(\frac{\partial U}{\partial X_{ab}} U^{\dagger} \right)_{\alpha s} + \sum_{a,b} U^{\dagger}_{al} U_{sb} \delta_{\alpha s} \left(U \frac{\partial U^{\dagger}}{\partial X_{ab}} \right)_{l\alpha} \\
&= A_1(\alpha, l, s) + A_2(\alpha, l, s).
\end{aligned}$$

Let us compute the first term A_1. In the case $\alpha \neq l$, this term vanishes. So assume $\alpha = l$. When $l \neq s$, we have

$$A_1(l, l, s) = \sum_{a,b} U^{\dagger}_{al} U_{sb} \left(\frac{\partial U}{\partial X_{ab}} U^{\dagger} \right)_{ls} = \sum_{a,b} U^{\dagger}_{al} U_{sb} \frac{1}{x_l - x_s} U_{la} U^{\dagger}_{bs} = \frac{1}{x_l - x_s}.$$

Let us compute the term A_2 . In the case $\alpha \neq s$, this term vanishes. So assume $\alpha = s$. When $l \neq s$, we have

$$A_2(s, l, s) = \sum_{a,b} U^{\dagger}_{al} U_{sb} \delta_{\alpha s} \left(U \frac{\partial U^{\dagger}}{\partial X_{ab}} \right)_{l\alpha} = -\sum_{a,b} U^{\dagger}_{al} U_{sb} \left(\frac{\partial U}{\partial X_{ab}} U^{\dagger} \right)_{ls} = -\frac{1}{x_l - x_s}.$$

When $\alpha = l = s$, we have

$$\begin{aligned}
A(l, l, l) &= \sum_{a,b} U^{\dagger}_{al} U_{lb} \left(\frac{\partial U}{\partial X_{ab}} U^{\dagger} \right)_{ll} + \sum_{a,b} U^{\dagger}_{al} U_{lb} \left(U \frac{\partial U^{\dagger}}{\partial X_{ab}} \right)_{ll} \\
&= \sum_{a,b} U^{\dagger}_{al} U_{lb} \left(\frac{\partial U}{\partial X_{ab}} U^{\dagger} + U \frac{\partial U^{\dagger}}{\partial X_{ab}} \right)_{ll} = 0.
\end{aligned}$$

Thus we have proved the equality (9).

4.2. Gauss case

In the Gauss case, we put

$$F(X) = \int_C |Y|^{c_1} |I - Y|^{c_2} |I - XY|^{c_3} dY = \int_C \exp f(Y) dY, \quad X, Y \in \mathscr{H}(n), \tag{10}$$

where $c_1 = a - n, c_2 = c - a - n, c_3 = -b$,

$$f(Y) = c_1 \log |Y| + c_2 \log |I - Y| + c_3 \log |I - XY|,$$

and C is the domain of integration explained in the last paragraph of Section 2.2. By virtue of this choice of C, we can interchange the operations of differentiation with respect to X_{ij} and integration with respect to Y. In the following we will not write C in the integrals for the sake of simplicity. For a function $g(Y)$ of Y, we use the notation:

$$\langle g \rangle := \int g(Y) \exp f \, dY.$$

Lemma 4.3. *For any $1 \leq i, j \leq n$, we have*

$$\frac{\partial f}{\partial Y_{ji}} = c_1 (Y^{-1})_{ij} - c_2 \left((I - Y)^{-1} \right)_{ij} - c_3 \left((I - XY)^{-1} X \right)_{ij}, \tag{11}$$

$$\frac{\partial f}{\partial X_{ji}} = -c_3 \left(Y (I - XY)^{-1} \right)_{ij}. \tag{12}$$

Proof. We see that

$$\begin{aligned}\frac{\partial f}{\partial Y_{ji}} &= \frac{\partial}{\partial Y_{ji}}(c_1 \log|Y| + c_2 \log|I-Y| + c_3 \log|I-XY|)\\ &= c_1 \frac{1}{|Y|}\frac{\partial |Y|}{\partial Y_{ji}} + c_2 \frac{1}{|I-Y|}\frac{\partial|I-Y|}{\partial Y_{ji}} + c_3 \frac{1}{|I-XY|}\frac{\partial |I-XY|}{\partial Y_{ji}}\\ &= c_1 \frac{1}{|Y|}C_{ji}(Y) - c_2\frac{1}{|I-Y|}C_{ji}(I-Y) + c_3 \frac{1}{|I-XY|}\sum_{k=1}^{n}(-X_{kj})C_{ki}(I-XY),\end{aligned}$$

where $C_{ji}(Y)$ is the (j,i)-cofactor of $|Y|$, and we used $\frac{\partial}{\partial Y_{ji}}(XY)_{ki} = X_{kj}$ to compute the last term. Then noting that $\frac{1}{|Y|}C_{ji}(Y) = (Y^{-1})_{ij}$, we get (11). The equality (12) is shown in a similar way.

Lemma 4.4. *For any $1 \le i,j \le n$, we have*

$$-c_3 \left\langle \left(Y(I-XY)^{-1}\right)_{ij}\right\rangle = \frac{\partial F}{\partial X_{ji}}, \tag{13}$$

$$-c_3 \left\langle \left((I-XY)^{-1}\right)_{ij}\right\rangle = \sum_a X_{ia}\frac{\partial F}{\partial X_{ja}} - \delta_{ij}c_3 F. \tag{14}$$

Proof. Differentiate the both sides of (10) with respect to X_{ji} and use (12) to obtain

$$\frac{\partial F}{\partial X_{ji}} = \int \frac{\partial f}{\partial X_{ji}} \exp f(Y) dY = \left\langle \frac{\partial f}{\partial X_{ji}}\right\rangle = -c_3\left\langle \left(Y(I-XY)^{-1}\right)_{ij}\right\rangle.$$

The second equality follows from

$$\begin{aligned}\sum_a X_{ia}\frac{\partial F}{\partial X_{ja}} &= \sum_a X_{ia}(-c_3)\left\langle\left(Y(I-XY)^{-1}\right)_{aj}\right\rangle = -c_3\sum_a\left\langle X_{ia}\left(Y(I-XY)^{-1}\right)_{aj}\right\rangle\\ &= -c_3\left\langle\left(XY(I-XY)^{-1}\right)_{ij}\right\rangle = -c_3\left\langle\left(\left(I-(I-XY)\right)(I-XY)^{-1}\right)_{ij}\right\rangle\\ &= -c_3\left\langle (I-XY)^{-1}_{ij}\right\rangle + c_3\delta_{ij}F.\end{aligned}$$

Put

$$\omega = \exp f(Y)\, dY, \quad \omega_{ij} = i_{\partial/\partial Y_{ij}} dY, \quad 1 \le i,j \le n, \tag{15}$$

where $i_{\partial/\partial Y_{ij}}$ is the inner derivation with the vector field $\partial/\partial Y_{ij}$.

Lemma 4.5. *For any $1 \le i,j \le n$, we have*

$$\left\langle c_2\left((I-Y)^{-1}\right)_{ij}\right\rangle = \sum_a X_{aj}\frac{\partial F}{\partial X_{ai}} + \delta_{ij}(c_1+c_2+n)F \tag{16}$$

and

$$\begin{aligned}\Big\langle \left((I-XY)^{-1}X\right)_{ij} &\operatorname{tr}\left((I-XY)^{-1}(I-X)Y\right)\\ &+ (c_1+n)\left((I-XY)^{-1}(I-X)\right)_{ij} - c_2\left((I-XY)^{-1}(I-X)Y(I-Y)^{-1}\right)_{ij}\\ &-c_3\left((I-XY)^{-1}(I-X)Y(I-XY)^{-1}X\right)_{ij}\Big\rangle = 0.\end{aligned} \tag{17}$$

Proof. To obtain the equality (16), consider $\eta_{ij} = \sum_{k=1}^{n} Y_{ik}\exp f(Y)\,\omega_{jk}$ for $1 \le i,j \le n$. Then using Lemma 4.3,

$$\begin{aligned}d\eta_{ij} &= \left(\sum_{k=1}^{n}\frac{\partial Y_{ik}}{\partial Y_{jk}} + \sum_{k=1}^{n} Y_{ik}\frac{\partial f}{\partial Y_{jk}}\right)\omega\\ &= \left\{n\delta_{ij} + \sum_{k=1}^{n} Y_{ik}\left(c_1(Y^{-1})_{kj} - c_2((I-Y)^{-1})_{kj} - c_3\left((I-XY)^{-1}X\right)_{kj}\right)\right\}\omega\\ &= \left\{(c_1+c_2+n)\delta_{ij} - c_2((I-Y)^{-1})_{ij} - c_3\left(Y(I-XY)^{-1}X\right)_{ij}\right\}\omega.\end{aligned}$$

Since $\int d\eta_{ij} = 0$ by virtue of the Stokes theorem, using (13) we have

$$\left\langle c_2\left((I-Y)^{-1}\right)_{ij}\right\rangle = -c_3\left\langle\left(Y(I-XY)^{-1}X\right)_{ij}\right\rangle + \delta_{ij}(c_1+c_2+n)F$$
$$= \sum_a X_{aj}\frac{\partial F}{\partial X_{ai}} + \delta_{ij}(c_1+c_2+n)F.$$

To obtain the equality (17), put $\eta_{ij} = \sum_{k=1}^n \left((I-XY)^{-1}(I-X)Y\right)_{ik} \exp f(Y)\,\omega_{jk}$ and compute its exterior derivative. We have

$$d\eta_{ij} = \left(\sum_{k=1}^n \frac{\partial}{\partial Y_{jk}}\left((I-XY)^{-1}(I-X)Y\right)_{ik} + \sum_{k=1}^n\left((I-XY)^{-1}(I-X)Y\right)_{ik}\frac{\partial f}{\partial Y_{jk}}\right)\omega. \tag{18}$$

Noting that

$$\frac{\partial (I-XY)^{-1}}{\partial Y_{jk}} = (I-XY)^{-1}\left(\sum_{a=1}^n X_{aj}E_{ak}\right)(I-XY)^{-1},$$

the first terms of (18) are computed as

$$\sum_{k=1}^n \frac{\partial}{\partial Y_{jk}}\left((I-XY)^{-1}(I-X)Y\right)_{ik}$$
$$= \left((I-XY)^{-1}X\right)_{ij}\operatorname{tr}\left((I-XY)^{-1}(I-X)Y\right) + n\left((I-XY)^{-1}(I-X)\right)_{ij}. \tag{19}$$

Using (11), the second terms of (18) are computed as

$$\sum_{k=1}^n\left((I-XY)^{-1}(I-X)Y\right)_{ik}\frac{\partial f}{\partial Y_{jk}}$$
$$= c_1\left((I-XY)^{-1}(I-X)\right)_{ij} - c_2\left((I-XY)^{-1}(I-X)Y(I-Y)^{-1}\right)_{ij}$$
$$- c_3\left((I-XY)^{-1}(I-X)Y(I-XY)^{-1}X\right)_{ij}. \tag{20}$$

Then the equality (17) follows from (18),(19),(20) and $\int d\eta_{ij} = 0$.

We shall derive the system of differential equations for F from (17). So it is necessary to compute

$$A_{ij} := c_2\left\langle\left((I-XY)^{-1}(I-X)Y(I-Y)^{-1}\right)_{ij}\right\rangle,$$
$$B_{ij} := c_3\left\langle\left((I-XY)^{-1}(I-X)Y(I-XY)^{-1}X\right)_{ij}\right\rangle.$$

To compute A_{ij}, note that

$$(I-XY)^{-1}(I-X)Y(I-Y)^{-1} = -(I-XY)^{-1} + (I-Y)^{-1}.$$

Then from (14) and (16) we have

$$c_3A_{ij} = \left\langle -c_2c_3\left((I-XY)^{-1}\right)_{ij}\right\rangle + \left\langle c_2c_3\left((I-Y)^{-1}\right)_{ij}\right\rangle$$
$$= c_2\left(\sum_a X_{ia}\frac{\partial F}{\partial X_{ja}} - \delta_{ij}c_3F\right) + c_3\left(\sum_a X_{aj}\frac{\partial F}{\partial X_{ai}} + \delta_{ij}(c_1+c_2+n)F\right)$$
$$= c_2\sum_a X_{ia}\frac{\partial F}{\partial X_{ja}} + c_3\sum_a X_{aj}\frac{\partial F}{\partial X_{ai}} + \delta_{ij}(c_1+n)c_3F. \tag{21}$$

To compute B_{ij}, note that

$$\left\langle\left((I-XY)^{-1}(I-X)Y(I-XY)^{-1}X\right)_{ij}\right\rangle$$
$$= \sum_a X_{aj}\sum_b\left\langle\left((I-XY)^{-1}(I-X)\right)_{ib}\left(Y(I-XY)^{-1}\right)_{ba}\right\rangle.$$

Taking this into account, we differentiate $\left\langle\left((I-XY)^{-1}(I-X)\right)_{ib}\right\rangle$ with respect to X_{ab} and get

$$\begin{aligned}
&\frac{\partial}{\partial X_{ab}}\left((I-XY)^{-1}(I-X)\right)_{ib}\\
&=\left((I-XY)^{-1}(\sum_k Y_{bk}E_{ak})(I-XY)^{-1}(I-X)\right)_{ib}-\left((I-XY)^{-1}E_{ab}\right)_{ib}\\
&=(I-XY)^{-1}_{ia}\left(Y(I-XY)^{-1}(I-X)\right)_{bb}-(I-XY)^{-1}_{ia}.
\end{aligned}$$

Using (12) we have

$$\begin{aligned}
\frac{\partial}{\partial X_{ab}}&\left\langle\left((I-XY)^{-1}(I-X)\right)_{ib}\right\rangle\\
&=\left\langle(I-XY)^{-1}_{ia}\left(Y(I-XY)^{-1}(I-X)\right)_{bb}-(I-XY)^{-1}_{ia}\right.\\
&\qquad\left.-c_3\left((I-XY)^{-1}(I-X)\right)_{ib}\left(Y(I-XY)^{-1}\right)_{ba}\right\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_a X_{aj}\sum_b\frac{\partial}{\partial X_{ab}}\left\langle\left((I-XY)^{-1}(I-X)\right)_{ib}\right\rangle\\
&=\sum_a X_{aj}\left\langle(I-XY)^{-1}_{ia}\operatorname{tr}\left(Y(I-XY)^{-1}(I-X)\right)-n(I-XY)^{-1}_{ia}\right.\\
&\qquad\left.-c_3\left((I-XY)^{-1}(I-X)Y(I-XY)^{-1}\right)_{ia}\right\rangle\\
&=\left\langle\left((I-XY)^{-1}X\right)_{ij}\operatorname{tr}\left(Y(I-XY)^{-1}(I-X)\right)-n\left((I-XY)^{-1}X\right)_{ij}\right.\\
&\qquad\left.-c_3\left((I-XY)^{-1}(I-X)Y(I-XY)^{-1}X\right)_{ij}\right\rangle.
\end{aligned}$$

Thus we have

$$\begin{aligned}
B_{ij}=&-\sum_a X_{aj}\sum_b\frac{\partial}{\partial X_{ab}}\left\langle\left((I-XY)^{-1}(I-X)\right)_{ib}\right\rangle-n\left\langle\left((I-XY)^{-1}X\right)_{ij}\right\rangle\\
&+\left\langle\left((I-XY)^{-1}X\right)_{ij}\operatorname{tr}\left(Y(I-XY)^{-1}(I-X)\right)\right\rangle.
\end{aligned}$$

Hence the relation (17) becomes

$$\begin{aligned}
&\sum_a X_{aj}\sum_b\frac{\partial}{\partial X_{ab}}\left\langle c_3\left((I-XY)^{-1}(I-X)\right)_{ib}\right\rangle\\
&\qquad+\left\langle(c_1+n)c_3\left((I-XY)^{-1}\right)_{ij}\right\rangle-\left\langle c_1c_3\left((I-XY)^{-1}X\right)_{ij}\right\rangle-c_3A_{ij}=0. \qquad (22)
\end{aligned}$$

We assert that this relation gives the differential equations for F.

Lemma 4.6. *The function F, defined by (10), satisfies the differential equations*

$$\begin{aligned}
&\sum_{a,b,p,q}X_{aj}(I-X)_{pb}X_{iq}\frac{\partial^2F}{\partial X_{ab}\partial X_{pq}}+\sum_{b,p}X_{ij}(I-X)_{pb}\frac{\partial F}{\partial X_{pb}}\\
&\qquad-c_3\sum_{a,b}X_{aj}(I-X)_{ib}\frac{\partial F}{\partial X_{ab}}-(c_1+n)\sum_{p,q}X_{pj}X_{iq}\frac{\partial F}{\partial X_{pq}}\\
&\qquad+(c_1+c_2+n)\sum_a X_{ia}\frac{\partial F}{\partial X_{ja}}+c_3\sum_a X_{aj}\frac{\partial F}{\partial X_{ai}}+(c_1+n)c_3X_{ij}F=0,\ 1\le i,j\le n. \qquad (23)
\end{aligned}$$

Proof. We express all the terms in (22) in terms of F and its derivatives. The first term in (22) is computed as follow. From (14), we have

$$\left\langle c_3\left((I-XY)^{-1}(I-X)\right)_{ib}\right\rangle=-\sum_p(I-X)_{pb}\left(\sum_q X_{iq}\frac{\partial F}{\partial X_{pq}}-\delta_{ip}c_3F\right).$$

Differentiate the both sides with respect to X_{ab}. Then, from the right hand side, we have

$$-\sum_p (I-X)_{pb}\left(\sum_q X_{iq}\frac{\partial^2 F}{\partial X_{ab}\partial X_{pq}}+\delta_{ia}\frac{\partial F}{\partial X_{pb}}-\delta_{ip}c_3\frac{\partial F}{\partial X_{ab}}\right)+\left(\sum_q X_{iq}\frac{\partial F}{\partial X_{aq}}-\delta_{ia}c_3 F\right).$$

Hence we have

$$\begin{aligned}\sum_a X_{aj}\sum_b \frac{\partial}{\partial X_{ab}}\left\langle c_3\left((I-XY)^{-1}(I-X)\right)_{ib}\right\rangle \\ &= -\sum_{a,b,p,q} X_{aj}(I-X)_{pb}X_{iq}\frac{\partial^2 F}{\partial X_{ab}\partial X_{pq}}-\sum_{b,p}X_{ij}(I-X)_{pb}\frac{\partial F}{\partial X_{pb}} \\ &\quad + c_3\sum_{a,b}X_{aj}(I-X)_{ib}\frac{\partial F}{\partial X_{ab}}+\sum_{a,b,q}X_{aj}X_{iq}\frac{\partial F}{\partial X_{aq}}-nX_{ij}c_3 F.\end{aligned}$$

Then using (14), (21) and

$$\begin{aligned}\left\langle c_3\left((I-XY)^{-1}X\right)_{ij}\right\rangle &= -\sum_p X_{pj}\left\langle -c_3\left((I-XY)^{-1}\right)_{ip}\right\rangle \\ &= -\sum_{p,q}X_{pj}X_{iq}\frac{\partial F}{\partial X_{pq}}+X_{ij}c_3F,\end{aligned}$$

we obtain the differential equations (23) from (22).

Theorem 3.1 for the Gauss HGF of matrix integral type is the following.

Proposition 4.7. *As a function of* eigenvalues $x_1,\dots,x_n$ *of* X, $I_G(a,b,c;X)$ *satisfies the system*

$$\begin{aligned}x_l(1-x_l)\frac{\partial^2 F}{\partial x_l^2}+\sum_{\alpha\neq l}\frac{x_l(1-x_l)\frac{\partial F}{\partial x_l}-x_\alpha(1-x_\alpha)\frac{\partial F}{\partial x_\alpha}}{x_l-x_\alpha} \\ +\{(c-(n-1))-(a+b+1-(n-1))x_l\}\frac{\partial F}{\partial x_l}-abF=0,\quad 1\le l\le n.\end{aligned}\tag{24}$$

We give the proof of this proposition. Take any $1\le l\le n$ and fix it. Multiply the both sides of (23) by $U^\dagger_{jl}U_{li}$ and take a sum for $i,j=1,\dots,n$. We compute the term which comes from the first term of the left hand side of (23):

$$I:=\sum_{i,j}U^\dagger_{jl}\cdot\left(\sum_{a,b,p,q}X_{aj}(I-X)_{pb}X_{iq}\frac{\partial^2 F}{\partial X_{ab}\partial X_{pq}}\right)\cdot U_{li}.$$

Noting that

$$\begin{aligned}\frac{\partial F}{\partial X_{pq}} &= \sum_\alpha \frac{\partial x_\alpha}{\partial X_{pq}}\frac{\partial F}{\partial x_\alpha}, \\ \frac{\partial^2 F}{\partial X_{ab}\partial X_{pq}} &= \sum_\alpha\frac{\partial^2 x_\alpha}{\partial X_{ab}\partial X_{pq}}\frac{\partial F}{\partial x_\alpha}+\sum_{\alpha,\beta}\frac{\partial x_\alpha}{\partial X_{pq}}\frac{\partial x_\beta}{\partial X_{ab}}\frac{\partial^2 F}{\partial x_\alpha\partial x_\beta},\end{aligned}$$

we write I as $I=I_1+I_2$ with

$$\begin{aligned}I_1 &= \sum_{\alpha,\beta}\frac{\partial^2 F}{\partial x_\alpha\partial x_\beta}\sum_{i,j,a,b,p,q}U^\dagger_{jl}X_{aj}(I-X)_{pb}X_{iq}\frac{\partial x_\alpha}{\partial X_{pq}}\frac{\partial x_\beta}{\partial X_{ab}}U_{li}, \\ I_2 &= \sum_\alpha\frac{\partial F}{\partial x_\alpha}\sum_{a,b,p,q}U^\dagger_{jl}X_{aj}(I-X)_{pb}X_{iq}\frac{\partial^2 x_\alpha}{\partial X_{ab}\partial X_{pq}}U_{li}.\end{aligned}$$

For I_1, using the equality (3) of Lemma 4.1 and $x = UXU^\dagger$, we have

$$\begin{aligned}
I_1 &= \sum_{\alpha,\beta} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \sum_{i,j,a,b,p,q} U^\dagger_{jl} X_{aj} (I-X)_{pb} X_{iq} U_{\alpha p} U^\dagger_{q\alpha} U_{\beta a} U^\dagger_{b\beta} U_{li} \\
&= \sum_{\alpha,\beta} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \left(\sum_{j,a} U_{\beta a} X_{aj} U^\dagger_{jl} \right) \left(\sum_{p,b} U_{\alpha p} (I-X)_{pb} U^\dagger_{b\beta} \right) \left(\sum_{i,q} U_{li} X_{iq} U^\dagger_{q\alpha} \right) \\
&= \sum_{\alpha,\beta} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} (\delta_{\beta l} x_l)(\delta_{\alpha\beta}(1-x_\alpha))(\delta_{l\alpha} x_l) \\
&= x_l^2 (1-x_l) \frac{\partial^2 F}{\partial x_l^2}.
\end{aligned}$$

Next we compute I_2. Note that, from $X = U^\dagger x U$, we have $X_{aj} = \sum_r U^\dagger_{ar} x_r U_{rj}$, etc. By virtue of (9) of Lemma 4.2 we have

$$\begin{aligned}
I_2 &= \sum_{\alpha} \frac{\partial F}{\partial x_\alpha} \sum_{i,j,a,b,p,q} U^\dagger_{jl} X_{aj} (I-X)_{pb} X_{iq} \frac{\partial^2 x_\alpha}{\partial X_{ab} \partial X_{pq}} U_{li} \\
&= \sum_{\alpha} \frac{\partial F}{\partial x_\alpha} \sum_{i,j,a,b,p,q} \sum_{r,s,u} U^\dagger_{jl} U^\dagger_{ar} x_r U_{rj} U^\dagger_{ps} (1-x_s) U_{sb} U^\dagger_{iu} x_u U_{uq} \frac{\partial^2 x_\alpha}{\partial X_{ab} \partial X_{pq}} U_{li} \\
&= x_l^2 \sum_{\alpha,s} \frac{\partial F}{\partial x_\alpha} (1-x_s) \sum_{a,b,p,q} U^\dagger_{al} U^\dagger_{ps} U_{sb} U_{lq} \frac{\partial^2 x_\alpha}{\partial X_{ab} \partial X_{pq}} \\
&= x_l^2 \left\{ \sum_{\alpha \neq l} \frac{\partial F}{\partial x_s} (1-x_\alpha) \frac{-1}{x_l - x_\alpha} + \frac{\partial F}{\partial x_l} \sum_{s \neq l} (1-x_s) \frac{1}{x_l - x_s} \right\} \\
&= x_l^2 \sum_{\alpha \neq l} \frac{(1-x_\alpha) \frac{\partial F}{\partial x_l} - (1-x_\alpha) \frac{\partial F}{\partial x_\alpha}}{x_l - x_\alpha}. \\
&= x_l^2 \sum_{\alpha \neq l} \frac{(1-x_l) \frac{\partial F}{\partial x_l} - (1-x_\alpha) \frac{\partial F}{\partial x_\alpha}}{x_l - x_\alpha} + (n-1) x_l^2 \frac{\partial F}{\partial x_l}
\end{aligned}$$

Thus we have

$$I = x_l \left\{ x_l (1-x_l) \frac{\partial^2 F}{\partial x_l^2} + x_l \sum_{\alpha \neq l} \frac{(1-x_l) \frac{\partial F}{\partial x_l} - (1-x_\alpha) \frac{\partial F}{\partial x_\alpha}}{x_l - x_\alpha} + (n-1) x_l \frac{\partial F}{\partial x_l} \right\}. \tag{25}$$

To compute the contribution, which comes from the other terms of the left hand side of (23), we need the following lemma, which can be shown in a similar way as above using Lemma 4.2.

Lemma 4.8. *We have*

$$\sum_{i,j} U^\dagger_{jl} \cdot \left(\sum_{b,p} X_{ij} (I-X)_{pb} \frac{\partial F}{\partial X_{pb}} \right) \cdot U_{li} = x_l \cdot \sum_{\alpha} (I - x_\alpha) \frac{\partial F}{\partial x_\alpha}, \tag{26}$$

$$\sum_{i,j} U^\dagger_{jl} \cdot \left(\sum_{a,b} X_{aj} (I-X)_{ib} \frac{\partial F}{\partial X_{ab}} \right) \cdot U_{li} = x_l (1-x_l) \frac{\partial F}{\partial x_l}. \tag{27}$$

$$\sum_{i,j} U^\dagger_{jl} \cdot \left(\sum_{a,b} X_{aj} X_{ib} \frac{\partial F}{\partial X_{ab}} \right) \cdot U_{li} = x_l^2 \frac{\partial F}{\partial x_l} \tag{28}$$

$$\sum_{i,j} U^\dagger_{jl} \cdot \left(\sum_{a} X_{ia} \frac{\partial F}{\partial X_{ja}} \right) \cdot U_{li} = x_l \frac{\partial F}{\partial x_l}, \tag{29}$$

$$\sum_{i,j} U^\dagger_{jl} \cdot \left(\sum_{a} X_{aj} \frac{\partial F}{\partial X_{ai}} \right) \cdot U_{li} = x_l \frac{\partial F}{\partial x_l}, \tag{30}$$

By the help of (25) and Lemma 4.8, we can derive from (23) the equation

$$x_l(1-x_l)\frac{\partial^2 F}{\partial x_l^2}+x_l\sum_{\alpha\neq l}\frac{(1-x_l)\frac{\partial F}{\partial x_l}-(1-x_\alpha)\frac{\partial F}{\partial x_\alpha}}{x_l-x_\alpha}+(n-1)x_l\frac{\partial F}{\partial x_l}$$
$$+\sum_{\alpha}(1-x_\alpha)\frac{\partial F}{\partial x_\alpha}-c_3(1-x_l)\frac{\partial F}{\partial x_l}-(c_1+n)x_l\frac{\partial F}{\partial x_l}$$
$$+(c_1+c_2+c_3+n)\frac{\partial F}{\partial x_l}+(c_1+n)c_3F=0.$$

Using

$$\sum_{\alpha}(1-x_\alpha)\frac{\partial F}{\partial x_\alpha}=\sum_{\alpha\neq l}\frac{x_l-x_\alpha}{x_l-x_\alpha}(1-x_\alpha)\frac{\partial F}{\partial x_\alpha}+(1-x_l)\frac{\partial F}{\partial x_l},$$
$$=\sum_{\alpha\neq l}\frac{x_l(1-x_\alpha)}{x_l-x_\alpha}\frac{\partial F}{\partial x_\alpha}-\sum_{\alpha\neq l}\frac{x_\alpha(1-x_\alpha)}{x_l-x_\alpha}\frac{\partial F}{\partial x_\alpha}+(1-x_l)\frac{\partial F}{\partial x_l},$$

we obtain the differential equation

$$x_l(1-x_l)\frac{\partial^2 F}{\partial x_l^2}+\sum_{\alpha\neq l}\frac{x_l(1-x_l)\frac{\partial F}{\partial x_l}-x_\alpha(1-x_\alpha)\frac{\partial F}{\partial x_\alpha}}{x_l-x_\alpha}$$
$$+\{(c_1+c_2+n-1)-(c_1-c_3-2)x_l\}\frac{\partial F}{\partial x_l}+(c_1+n)c_3F=0.$$

Recovering the original parameters $c_1=a-n, c_2=c-a-n, c_3=-b$, we obtain the desired differential equations (24) and finish the proof of Proposition 4.7.

4.3. Kummer case

We prove Theorem 3.1 for Kummer's HGF of matrix integral type following the same line of thought as in the Gauss case. Put

$$F(X)=\int_C \mathrm{etr}\,(XY)\,|Y|^{c_1}|I-Y|^{c_2}dY=\int_C \exp f(Y)dY,\quad X,Y\in\mathscr{H}(n), \tag{31}$$

where $c_1=a-n, c_2=c-a-n$, C is a domain of integration which allows us to apply the Stokes theorem, and

$$f(Y)=\mathrm{tr}\,(XY)+c_1\log|Y|+c_2\log|I-Y|.$$

The usage of the symbol $\langle g\rangle$ for a function $g(Y)$ is the same as in the Gauss case. A simple computation shows the following.

Lemma 4.9. *For any $1\le i,j\le n$, we have*

$$\frac{\partial f}{\partial X_{ij}}=Y_{ji},\quad \frac{\partial f}{\partial Y_{ij}}=X_{ji}+c_1(Y^{-1})_{ji}+c_2((I-Y)^{-1})_{ji}.$$

Lemma 4.10. *The function F, defined by (31), satisfies the differential equations*

$$\sum_{k,m}X_{kj}\frac{\partial^2 F}{\partial X_{km}\partial X_{mi}}-\sum_k X_{kj}\frac{\partial F}{\partial X_{ki}}+(c_1+c_2+n)\frac{\partial F}{\partial X_{ji}}$$
$$+\delta_{ij}\left\{\sum_k\frac{\partial F}{\partial X_{kk}}-(c_1+n)F\right\}=0.\quad 1\le i,j\le n. \tag{32}$$

Proof. Let ω_{jk},ω be those by (15) with $f(Y)$ in (31). Consider (n^2-1)-form

$$\eta_{ij}=\sum_{k=1}^{n}(Y(I-Y))_{ik}\exp f(Y)\,\omega_{jk},\quad 1\le i,j\le n$$

and compute $d\eta_{ij}$. Using Lemma 4.9, we have

$$\begin{aligned} d\eta_{ij} &= \sum_{k=1}^{n} \frac{\partial}{\partial Y_{jk}} (Y(I-Y))_{ik}\, \omega + \sum_{k=1}^{n} (Y(I-Y))_{ik} \frac{\partial f}{\partial Y_{jk}} \omega \\ &= \sum_{k=1}^{n} (\delta_{ij} - \delta_{ij} Y_{kk} - Y_{ij}) \cdot \omega \\ &\quad + \sum_{k=1}^{n} (Y(I-Y))_{ik} \left(c_1 (Y^{-1})_{kj} - c_2 ((I-Y)^{-1})_{kj} + X_{kj} \right) \omega \\ &= \left\{ n\delta_{ij} - \delta_{ij} \operatorname{tr} Y - nY_{ij} + c_1 (I-Y)_{ij} - c_2 Y_{ij} + (Y(I-Y)X)_{ij} \right\} \omega \\ &= \left\{ (c_1+n)\delta_{ij} - \delta_{ij} \operatorname{tr} Y - (c_1+c_2+n) Y_{ij} + \sum_k Y_{ik} X_{kj} - \sum_{k,m} Y_{im} Y_{mk} X_{kj} \right\} \omega. \end{aligned}$$

Then the Stokes theorem implies

$$(c_1+n)\delta_{ij}\langle 1 \rangle - \delta_{ij} \sum_k \langle Y_{kk} \rangle - (c_1+c_2+n) \langle Y_{ij} \rangle + \sum_k X_{kj} \langle Y_{ik} \rangle - \sum_{k,m} X_{kj} \langle Y_{im} Y_{mk} \rangle = 0. \quad (33)$$

Since $\langle 1 \rangle = F$ by definition and $\langle Y_{ab} \rangle = \partial F / \partial X_{ba}$ by virtue of Lemma 4.9, the equality (33) implies the differential equation (32).

Theorem 3.1 for the Kummer's case is the following.

Proposition 4.11. *As a function of* eigenvalues $x_1, \ldots, x_n$ *of* X, $I_K(a, c; X)$ *satisfies the differential equations*

$$x_l \frac{\partial^2 F}{\partial x_l^2} + (c - n + 1 - x_l) \frac{\partial F}{\partial x_l} + \sum_{\alpha \neq l} \frac{x_l \frac{\partial F}{\partial x_l} - x_\alpha \frac{\partial F}{\partial x_\alpha}}{x_l - x_\alpha} - aF = 0, \quad 1 \leq l \leq n. \quad (34)$$

Proof. For a fixed $1 \leq l \leq n$, multiply the both sides of (32) by $U_{jl}^\dagger U_{li}$ and take a sum over $i, j = 1, \ldots, n$. The terms containing the second order derivatives are computed as follows. Since

$$\frac{\partial^2 F}{\partial X_{km} \partial X_{mi}} = \sum_\alpha \frac{\partial^2 x_\alpha}{\partial X_{km} \partial X_{mi}} \frac{\partial F}{\partial x_\alpha} + \sum_{\alpha,\beta} \frac{\partial x_\alpha}{\partial X_{mi}} \frac{\partial x_\beta}{\partial X_{km}} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta},$$

we have $\sum_{i,j} U_{jl}^\dagger \left(\sum_{k,m} X_{kj} \frac{\partial^2 F}{\partial X_{km} \partial X_{mi}} \right) U_{li} = I_1 + I_2$ with

$$\begin{aligned} I_1 &= \sum_{\alpha,\beta} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \sum_{i,j} U_{jl}^\dagger \left(\sum_{k,m} X_{kj} \frac{\partial x_\alpha}{\partial X_{mi}} \frac{\partial x_\beta}{\partial X_{km}} \right) U_{li}, \\ I_2 &= \sum_\alpha \frac{\partial F}{\partial x_\alpha} \sum_{i,j} U_{jl}^\dagger \left(\sum_{k,m} X_{kj} \frac{\partial^2 x_\alpha}{\partial X_{km} \partial X_{mi}} \right) U_{li}. \end{aligned}$$

Using Lemma 4.1 and $x = UXU^\dagger$, I_1 is computed as

$$\begin{aligned} I_1 &= \sum_{\alpha,\beta} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \sum_{i,j,k,m} X_{kj} U_{jl}^\dagger U_{\alpha m} U_{i\alpha}^\dagger U_{\beta k} U_{m\beta}^\dagger U_{li} \\ &= \sum_{\alpha,\beta} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \delta_{l\alpha} \delta_{\beta\alpha} \sum_{j,k} U_{\beta k} X_{kj} U_{jl}^\dagger = x_l \frac{\partial^2 F}{\partial x_l^2}. \end{aligned}$$

Noting that $X_{kj} = \sum_p U_{kp}^\dagger x_p U_{pj}$ and using (8) of Lemma 4.2, I_2 is computed as

$$\begin{aligned} I_2 &= \sum_\alpha \frac{\partial F}{\partial x_\alpha} \sum_{i,j,k,m} \sum_p U_{jl}^\dagger U_{kp}^\dagger x_p U_{pj} \frac{\partial^2 x_\alpha}{\partial X_{km} \partial X_{mi}} U_{li} \\ &= x_l \sum_\alpha \frac{\partial F}{\partial x_\alpha} \sum_{i,k,m} U_{kl}^\dagger \frac{\partial^2 x_\alpha}{\partial X_{km} \partial X_{mi}} U_{li} = x_l \sum_{\alpha \neq l} \frac{1}{x_\alpha - x_l} \left(\frac{\partial F}{\partial x_\alpha} - \frac{\partial F}{\partial x_l} \right). \end{aligned}$$

Thus we have

$$\sum_{i,j} U_{jl}^{\dagger}\cdot\left(\sum_{k,m} X_{kj}\frac{\partial^2 F}{\partial X_{km}\partial X_{mi}}\right)\cdot U_{li} = x_l\frac{\partial^2 F}{\partial x_l^2} + x_l\sum_{\alpha\neq l}\frac{1}{x_\alpha - x_l}\left(\frac{\partial F}{\partial x_\alpha} - \frac{\partial F}{\partial x_l}\right). \tag{35}$$

For the other terms of the first derivatives, contribution from the second term in (32) is already computed in (30), and that from the rest is computed by using as

$$\sum_{i,j} U_{jl}^{\dagger}\cdot\left(\frac{\partial F}{\partial X_{ji}}\right)\cdot U_{li} = \frac{\partial F}{\partial x_l}, \tag{36}$$

$$\sum_{i,j}\delta_{ij}U_{jl}^{\dagger}\cdot\left(\sum_k\frac{\partial F}{\partial X_{kk}}\right)\cdot U_{li} = \sum_\alpha\frac{\partial F}{\partial x_\alpha}. \tag{37}$$

Noting that $\sum_{i,j}\delta_{ij}U_{jl}^{\dagger}U_{li} = 1$, from the differential equation (32), we have

$$x_l\frac{\partial^2 F}{\partial x_l^2} + x_l\sum_{\alpha\neq l}\frac{1}{x_l - x_\alpha}\left(\frac{\partial F}{\partial x_l} - \frac{\partial F}{\partial x_\alpha}\right) + \sum_\alpha\frac{\partial F}{\partial x_\alpha} + (c_1 + c_2 + n - x_l)\frac{\partial F}{\partial x_l} - (c_1 + n)F = 0.$$

Using $c_1 = a - n, c_2 = c - a - n$ and

$$\sum_\alpha\frac{\partial F}{\partial x_\alpha} = \sum_{\alpha\neq l}\frac{x_l - x_\alpha}{x_l - x_\alpha}\frac{\partial F}{\partial x_\alpha} + \frac{\partial F}{\partial x_l},$$

we have the differential equation (34).

4.4. Bessel case

We prove Theorem 3.1 for the Bessel integral $I_B(c; X)$. Put

$$F(X) = \int \operatorname{etr}\left(XY - Y^{-1}\right)|Y|^{c-n}dY = \int \exp f(Y)dY, \quad X, Y\in\mathscr{H}(n), \tag{38}$$

where

$$f(Y) = \operatorname{tr}(XY - Y^{-1}) + (c - n)\log|Y|.$$

The usage of the symbol $\langle g\rangle$ is same as above. The following lemma is now easy to show.

Lemma 4.12. *For any $1\le i, j\le n$, we have*

$$\frac{\partial f}{\partial X_{ij}} = \langle Y_{ji}\rangle, \quad \frac{\partial f}{\partial Y_{ij}} = X_{ji} + (Y^{-2})_{ji} + (c - n)(Y^{-1})_{ji}.$$

Lemma 4.13. *The function F, defined by (38), satisfies the differential equations*

$$\sum_{k,m} X_{kj}\frac{\partial^2 F}{\partial X_{km}\partial X_{mi}} + c\frac{\partial F}{\partial X_{ji}} + \delta_{ij}\left\{\sum_k\frac{\partial F}{\partial X_{kk}} + F\right\} = 0, \quad 1\le i, j\le n. \tag{39}$$

Proof. For $1\le i, j\le n$, consider the $(n^2 - 1)$-form

$$\eta_{ij} = \sum_{k=1}^n\left(Y^2\right)_{ik}\exp f(Y)\,\omega_{jk},$$

and compute $d\eta_{ij}$:

$$\begin{aligned} d\eta_{ij} &= \left\{\sum_{k=1}^n\frac{\partial}{\partial Y_{jk}}\left(Y^2\right)_{ik} + \sum_{k=1}^n\left(Y^2\right)_{ik}\frac{\partial f}{\partial Y_{jk}}\right\}\omega \\ &= \left\{\sum_{k=1}^n\frac{\partial}{\partial Y_{jk}}\left(\sum_{m=1}^n Y_{im}Y_{mk}\right) + \sum_{k=1}^n\left(Y^2\right)_{ik}\left(X_{kj} + (Y^{-2})_{kj} + (c - n)(Y^{-1})_{kj}\right)\right\}\omega \\ &= \left\{\delta_{ij}\left(1 + \sum_k Y_{kk}\right) + cY_{ij} + \left(Y^2X\right)_{ij}\right\}\omega, \end{aligned}$$

where the usage of ω and ω_{jk} are similar as in the Kummer's case. Then the Stokes theorem implies that

$$\sum_{k,m} X_{kj} \langle Y_{im} Y_{mk} \rangle + c \left\langle Y_{ij} \right\rangle + \delta_{ij} \sum_{k} \langle Y_{kk} \rangle + \delta_{ij} \langle 1 \rangle = 0. \tag{40}$$

Since $\langle Y_{ab} \rangle = \partial F / \partial X_{ba}$ by virtue of Lemma 4.12, the equality (40) implies the differential equation (39).

Theorem 3.1 for the Bessel function of matrix integral type is the following.

Proposition 4.14. *As a function of* eigenvalues $x_1, \ldots, x_n$ *of* X*, the Bessel integral* $I_B(X)$ *satisfies the differential equations*

$$x_l \frac{\partial^2 F}{\partial x_l^2} + (c+1) \frac{\partial F}{\partial x_l} + \sum_{\alpha \neq l} \frac{x_l \frac{\partial F}{\partial x_l} - x_\alpha \frac{\partial F}{\partial x_\alpha}}{x_l - x_\alpha} + F = 0, \quad 1 \leq l \leq n. \tag{41}$$

Proof. For a fixed $1 \leq l \leq n$, multiply the both sides of (39) by $U^{\dagger}_{jl} U_{li}$ and take a sum over $i, j = 1, \ldots, n$. Using the identities (35), (36) and (37), we have from (39) the equation

$$x_l \frac{\partial^2 F}{\partial x_l^2} + c \frac{\partial F}{\partial x_l} + x_l \sum_{\alpha \neq l} \frac{1}{x_l - x_\alpha} \left(\frac{\partial F}{\partial x_l} - \frac{\partial F}{\partial x_\alpha} \right) + \sum_{\alpha} \frac{\partial F}{\partial x_\alpha} + F = 0.$$

Rewriting it using

$$\sum_{\alpha} \frac{\partial F}{\partial x_\alpha} = \sum_{\alpha \neq l} \frac{x_l - x_\alpha}{x_l - x_\alpha} \frac{\partial F}{\partial x_\alpha} + \frac{\partial F}{\partial x_l},$$

we obtain the differential equation (41).

4.5. Hermite-Weber case

We prove Theorem 3.1 for the Hermite-Weber matrix integral $I_{HW}(c; X)$. Put

$$F(X) = \int_C |Y|^{-c-n} \mathrm{etr} \left(XY - \frac{1}{2} Y^2 \right) dY = \int_C \exp f(Y) dY, \quad X \in \mathscr{H}(n), \tag{42}$$

where C is a domain of integration as in the previous cases and

$$f(Y) = (-c-n) \log |Y| + \mathrm{tr}\,(XY - \frac{1}{2} Y^2).$$

The usage of the symbol $\langle g \rangle$ is the same as in the previous cases. The following lemma is shown easily.

Lemma 4.15. *For any* $1 \leq i, j \leq n$*, we have*

$$\frac{\partial f}{\partial X_{ij}} = Y_{ji}, \quad \frac{\partial f}{\partial Y_{ij}} = (-c-n)(Y^{-1})_{ji} + X_{ji} - Y_{ji}. \tag{43}$$

Lemma 4.16. *The function* F*, defined by the integral (42), satisfies the differential equations*

$$\sum_{k} \frac{\partial^2 F}{\partial X_{jk} \partial X_{ki}} - \sum_{k} X_{kj} \frac{\partial}{\partial X_{ki}} F + c \delta_{ij} F = 0, \quad 1 \leq i, j \leq n. \tag{44}$$

Proof. For any pair (i, j), define $\eta_{ij} = \sum_{k=1}^{n} Y_{ik} \exp f(Y) \, \omega_{jk}$ as in the previous cases. Then using Lemma 4.15, we have

$$\begin{aligned} d\eta_{ij} &= \left\{ \sum_{k=1}^{n} \frac{\partial Y_{ik}}{\partial Y_{jk}} + \sum_{k=1}^{n} Y_{ik} \frac{\partial f}{\partial Y_{jk}} \right\} \omega \\ &= \left\{ n\delta_{ij} + \sum_{k=1}^{n} Y_{ik} \left((-c-n)(Y^{-1})_{kj} + X_{kj} - Y_{kj} \right) \right\} \omega \\ &= \left\{ -c\delta_{ij} + \sum_{k=1}^{n} Y_{ik} X_{kj} - \sum_{k=1}^{n} Y_{ik} Y_{kj} \right\} \omega. \end{aligned}$$

Since $\int d\eta_{ij} = 0$ by the Stokes theorem, we have

$$\sum_{k} \left\langle Y_{ik} Y_{kj} \right\rangle - \sum_{k=1}^{n} X_{kj} \left\langle Y_{ik} \right\rangle + c\delta_{ij} \left\langle 1 \right\rangle = 0. \tag{45}$$

Then we see that the identities (45) lead to the differential equations (44) since Lemma 4.15 implies $\partial F / \partial X_{ab} = \langle Y_{ba} \cdot 1 \rangle$.

Theorem 3.1 for the Hermite-Weber case is the following.

Proposition 4.17. *As a function of* eigenvalues $x_1, \dots, x_n$*, the Hermite-Weber integral* $I_{HW}(X)$ *satisfies*

$$\frac{\partial^2 F}{\partial x_l^2} - x_l \frac{\partial F}{\partial x_l} + \sum_{\alpha \neq l} \frac{1}{x_l - x_\alpha} \left(\frac{\partial F}{\partial x_l} - \frac{\partial F}{\partial x_\alpha} \right) + cF = 0, \quad 1 \leq i, j \leq n. \tag{46}$$

Proof. We proceed as in the previous case. For a fixed $1 \leq l \leq n$, multiply the both sides of (44) by $U^\dagger_{jl} U_{li}$ and take a sum over $i, j = 1, \dots, n$. Then we easily see that

$$\sum_{i,j,k} U^\dagger_{jl} \cdot \frac{\partial^2 F}{\partial X_{jk} \partial X_{ki}} \cdot U_{li} = \frac{\partial^2 F}{\partial x_l^2} + \sum_{\alpha \neq l} \frac{1}{x_l - x_\alpha} \left(\frac{\partial F}{\partial x_l} - \frac{\partial F}{\partial x_\alpha} \right).$$

For the second term in (44), we use (30). Then we obtain the differential equation (46) from (44).

4.6. Airy case

We prove Theorem 3.1 for the Airy integral $I_A(X)$. Put

$$F(X) = \int_C \operatorname{etr} \left(XY - \frac{1}{3} Y^3 \right) dY = \int_C \exp f(Y) dY, \quad X \in \mathscr{H}(n), \tag{47}$$

where C is a domain of integration explained in the last paragraph of Section 2.2 and

$$f(Y) = \operatorname{tr} \left(XY - \frac{1}{3} Y^3 \right).$$

By virtue of this choice of C, we can interchange the operations of differentiation with respect to X_{ij} integration with respect to Y. See also Remark 2.7. The usage of the symbol $\langle g \rangle$ is the same as in the previous cases. The following lemma is easy.

Lemma 4.18. *For any* $1 \leq i, j \leq n$*, we have*

$$\frac{\partial f}{\partial X_{ij}} = Y_{ji}, \quad \langle (Y^2)_{ji} \rangle - \langle X_{ji} \rangle = 0. \tag{48}$$

Lemma 4.19. *The function* F *satisfies the differential equations*

$$\sum_k \frac{\partial^2 F}{\partial X_{ik} \partial X_{kj}} - X_{ji} F = 0, \quad 1 \leq i, j \leq n. \tag{49}$$

Proof. The equation (49) follows from Lemma 4.18 and $\left(Y^2\right)_{ji} = \sum_k Y_{jk} Y_{ki}$.

Theorem 3.1 for the Airy integral is the following, whose proof is similar to that for Proposition 4.17 and is omitted.

Proposition 4.20. *As a function of* eigenvalues $x_1, \dots, x_n$ *of* X*, the Airy integral* $I_A(X)$ *satisfies the differential equation*

$$\frac{\partial^2 F}{\partial x_l^2} + \sum_{\alpha \neq l} \frac{1}{x_l - x_\alpha} \left(\frac{\partial F}{\partial x_l} - \frac{\partial F}{\partial x_\alpha} \right) - x_l F = 0, \quad 1 \leq l \leq n.$$

5. HOLONOMICITY OF THE SYSTEM FOR HGF

Theorem 5.1. *The system* $\mathcal{S}_*$ $(* = G, K, B, HW, A)$ *is holonomic in* $\Omega_* \subset \mathbb{C}^n$ *and is equivalent to the completely integrable Pfaffian system of rank* 2^n*, where*

$$\Omega_G = \{x \in \mathbb{C}^n \mid \prod_{i=1}^n x_i (x_i - 1) \cdot \Delta(x) \neq 0\},$$

$$\Omega_K = \Omega_B = \{x \in \Omega \mid \prod_{i=1}^n x_i \cdot \Delta(x) \neq 0\},$$

$$\Omega_{HW} = \Omega_A = \{x \in \Omega \mid \Delta(x) \neq 0\}.$$

We prove the theorem in detail for the systems $\mathcal{S}_{HW}, \mathcal{S}_A$ by using the theory of Gröbner basis.

Let $\mathbb{C}[x]$ be the ring of polynomials in $x = (x_1, \dots, x_n)$ and let R be the localization of $\mathbb{C}[x]$ by $\Delta = \prod_{i<j}(x_i - x_j)$, namely $R = \{f/\Delta^m \mid f \in \mathbb{C}[x], m \in \mathbb{Z}_{\geq 0}\}$ which is also denoted as $\mathbb{C}[x]_\Delta$. We denote by D the ring of differential operators in x with coefficients in R. Any $P \in D$ can be expressed uniquely in the so-called normal form

$$P = \sum_\alpha a_\alpha(x)\partial^\alpha = \sum_\alpha a_\alpha(x)\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}, \quad a_\alpha(x) \in R,$$

where $\sum_\alpha$ is a finite sum with respect to multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. To this $P \in D$ we associate its symbol:

$$\sigma(P) = \sum_\alpha a_\alpha(x)\xi^\alpha = \sum_\alpha a_\alpha(x)\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n} \in R[\xi].$$

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, let $|\alpha|$ denote the sum $\alpha_1 + \cdots + \alpha_n$.

Let us fix an order in the set of monomials $\{a_\alpha(x)\partial^\alpha\}$ in D as follows. Firstly, we use the lexicographic order as a monomial order $\prec_{lex}$ in $\mathbb{C}[\xi]$, namely, $\xi^\alpha \prec_{lex} \xi^\beta$ means that either $|\alpha| < |\beta|$ holds or $|\alpha| = |\beta|$ and the most left nonzero member of $(\beta_1 - \alpha_1, \dots, \beta_n - \alpha_n)$ is > 0. Using $\prec_{lex}$, define the order in D as

$$a_\alpha(x)\partial^\alpha \prec b_\beta(x)\partial^\beta \Leftrightarrow \quad \xi^\alpha \prec_{lex} \xi^\beta.$$

For $P \in D$, the initial term $\mathrm{in}_\prec(P)$ is the symbol of the greatest monomial in P with respect to the order $\prec$. For $P, Q \in R$ with $\mathrm{in}_\prec(P) = a(x)\xi^\alpha, \mathrm{in}_\prec(Q) = b(x)\xi^\beta$, let $\gamma = (\max(\alpha_1, \beta_1), \dots, \max(\alpha_n, \beta_n)) \in \mathbb{Z}^n_{\geq 0}$. Then S-pair $\mathrm{sp}(P, Q)$ for P, Q is defined by

$$\mathrm{sp}(P, Q) = b(x)\partial^{\gamma-\alpha}P - a(x)\partial^{\gamma-\beta}Q.$$

Let $\mathcal{I}$ be a left ideal of D. By $\mathrm{in}_\prec(\mathcal{I})$ we denote the ideal of $R[\xi]$ generated by $\{\mathrm{in}_\prec(P) \mid P \in \mathcal{I}\}$. Let $\{f_1, \dots, f_d\}$ be a generator of the ideal $\mathcal{I}$. It should be noted that $\{\mathrm{in}_\prec(f_1), \dots, \mathrm{in}_\prec(f_d)\}$ does not necessarily generate $\mathrm{in}_\prec(\mathcal{I})$, in general. A generator $G = \{g_1, \dots, g_m\}$ of $\mathcal{I}$ is said to be a Gröbner basis for $\mathcal{I}$ if $(\mathrm{in}_\prec(g_1), \dots, \mathrm{in}_\prec(g_m))$ generates $\mathrm{in}_\prec(\mathcal{I})$, namely $\mathrm{in}_\prec(\mathcal{I}) = \langle \mathrm{in}_\prec(g_1), \dots, \mathrm{in}_\prec(g_m)\rangle$. We can apply the Buchberger's algorithm to find a Gröbner basis for a given left ideal $\mathcal{I}$ of D.

5.1. Hermite-Weber

Consider the system of differential equations $\mathcal{S}_{HW}$ for the Hermite-Weber function $I_{HW}(c, X)$ and put

$$L_i = \partial_i^2 - x_i\partial_i + \sum_{k(\neq i)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) + c, \quad 1 \leq i \leq n.$$

Let $\mathcal{I}_{HW} \subset D$ be the left ideal with the generator $G_{HW} = \{L_1, \dots, L_n\}$.

Proposition 5.2. *G_{WH} is a Gröbner basis of the left ideal $\mathcal{I}_{HW}$.*

Proof. It is enough to show that, for any pair L_i, L_j $(i \neq j)$, the S-pair $\mathrm{sp}(L_i, L_j) \equiv 0$ after applying the division algorithm of Buchberger using G_{WH}. Since the largest term of L_i is ∂_i^2, we have

$$\begin{aligned}
\mathrm{sp}(L_i, L_j) &= \partial_j^2 L_i - \partial_i^2 L_j \\
&= \partial_j^2\left(\partial_i^2 - x_i\partial_i + \sum_{k(\neq i)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) + c\right) \\
&\quad - \partial_i^2\left(\partial_j^2 - x_j\partial_j + \sum_{k(\neq j)} \frac{1}{x_j - x_k}(\partial_j - \partial_k) + c\right) \\
&= A + B + C + D,
\end{aligned}$$

where

$$\begin{aligned}
A &= -x_i\partial_i\partial_j^2 + x_j\partial_j\partial_i^2, \\
B &= \partial_j^2 \cdot \frac{1}{x_i - x_j}(\partial_i - \partial_j) - \partial_i^2 \cdot \frac{1}{x_j - x_i}(\partial_j - \partial_i), \\
C &= \sum_{k(\neq i,j)} \left\{\frac{1}{x_i - x_k}(\partial_i - \partial_k)\partial_j^2 - \frac{1}{x_j - x_k}(\partial_j - \partial_k)\partial_i^2\right\}, \\
D &= c(\partial_j^2 - \partial_i^2).
\end{aligned}$$

We carry out a reduction of A, B, C, D by the division algorithm using the generator G_{WH}. Noting that

$$\partial_i^2 = L_i + x_i\partial_i - \sum_{k(\neq i)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) - c, \tag{50}$$

we have

$$\begin{aligned}
A &\equiv -x_i\partial_i \left\{ x_j\partial_j - \sum_{k(\neq j)} \frac{1}{x_j - x_k}(\partial_j - \partial_k) - c \right\} + x_j\partial_j \left\{ x_i\partial_i - \sum_{k(\neq i)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) - c \right\} \\
&= \sum_{k(\neq i,j)} \left\{ \frac{x_i}{x_j - x_k}\partial_i(\partial_j - \partial_k) - \frac{x_j}{x_i - x_k}\partial_j(\partial_i - \partial_k) \right\} \\
&+ x_i \left\{ \frac{1}{x_j - x_i}\partial_i(\partial_j - \partial_i) + \frac{1}{(x_j - x_i)^2}(\partial_j - \partial_i) \right\} - x_j \left\{ \frac{1}{x_i - x_j}\partial_j(\partial_i - \partial_j) + \frac{1}{(x_i - x_j)^2}(\partial_i - \partial_j) \right\}, \\
&+ c(x_i\partial_i - x_j\partial_j),
\end{aligned}$$

and

$$\begin{aligned}
B &\equiv \frac{1}{x_i - x_j}(\partial_i - \partial_j)\partial_j^2 + \frac{2}{(x_i - x_j)^2}(\partial_i - \partial_j)\partial_j + \frac{2}{(x_i - x_j)^3}(\partial_i - \partial_j) \\
&- \frac{1}{x_j - x_i}(\partial_j - \partial_i)\partial_i^2 - \frac{2}{(x_j - x_i)^2}(\partial_j - \partial_i)\partial_i - \frac{2}{(x_j - x_i)^3}(\partial_j - \partial_i) \\
&= \frac{1}{x_i - x_j}(\partial_i - \partial_j)(\partial_j^2 - \partial_i^2) - \frac{2}{(x_i - x_j)^2}(\partial_j^2 - \partial_i^2) \\
&=: \frac{1}{x_i - x_j}B_1 - \frac{2}{(x_i - x_j)^2}(\partial_j^2 - \partial_i^2).
\end{aligned}$$

To compute B_1, we use

$$\partial_j^2 - \partial_i^2 = L_j - L_i + x_j\partial_j - x_i\partial_i + \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right\}$$

and we have

$$\begin{aligned}
B_1 &\equiv (\partial_i - \partial_j) \left\{ x_j\partial_j - x_i\partial_i + \sum_{k(\neq i,j)} \left(\frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right) \right\} \\
&= x_j(\partial_i - \partial_j)\partial_j - x_i(\partial_j - \partial_i)\partial_i - (\partial_j + \partial_i) \\
&+ \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_j)(\partial_i - \partial_k) - \frac{1}{(x_i - x_k)^2}(\partial_i - \partial_k) \right\} \\
&- \sum_{k(\neq i,j)} \left\{ \frac{1}{x_j - x_k}(\partial_i - \partial_j)(\partial_j - \partial_k) + \frac{1}{(x_j - x_k)^2}(\partial_j - \partial_k) \right\}.
\end{aligned}$$

Similarly, we compute C. Using (50) we have

$$\begin{aligned}
C &\equiv \sum_{k(\neq i,j)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) \left\{ x_j\partial_j - \sum_{\ell(\neq j)} \frac{1}{x_j - x_\ell}(\partial_j - \partial_\ell) - c \right\} \\
&- \sum_{k(\neq i,j)} \frac{1}{x_j - x_k}(\partial_j - \partial_k) \left\{ x_i\partial_i - \sum_{\ell(\neq i)} \frac{1}{x_i - x_\ell}(\partial_i - \partial_\ell) - c \right\} \\
&= \sum_{k(\neq i,j)} \left\{ \frac{x_j}{x_i - x_k}(\partial_i - \partial_k)\partial_j - \frac{x_i}{x_j - x_k}(\partial_j - \partial_k)\partial_i \right\} \\
&- c \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right\} + C_1,
\end{aligned}$$

where

$$C_1 = -\sum_{k(\neq i,j)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) \sum_{\ell(\neq j)} \frac{1}{x_j - x_\ell}(\partial_j - \partial_\ell) + \sum_{k(\neq i,j)} \frac{1}{x_j - x_k}(\partial_j - \partial_k) \sum_{\ell(\neq i)} \frac{1}{x_i - x_\ell}(\partial_i - \partial_\ell).$$

In C_1, we consider separately the cases $\ell = i, k$ in the first part and the cases $\ell = j, k$ in the second part. Thus

$$\begin{aligned} C_1 = &- \sum_{k(\neq i,j)} \frac{1}{x_i - x_k}(\partial_i - \partial_k)\frac{1}{x_j - x_i}(\partial_j - \partial_i) \\ &- \sum_{k(\neq i,j)} \frac{1}{x_i - x_k}(\partial_i - \partial_k)\frac{1}{x_j - x_k}(\partial_j - \partial_k) \\ &- \sum_{k(\neq i,j)} \sum_{\ell(\neq i,j,k)} \frac{1}{(x_i - x_k)(x_j - x_\ell)}(\partial_i - \partial_k)(\partial_j - \partial_\ell) \\ &+ \sum_{k(\neq i,j)} \frac{1}{x_j - x_k}(\partial_j - \partial_k)\frac{1}{x_i - x_j}(\partial_i - \partial_j) \\ &+ \sum_{k(\neq i,j)} \frac{1}{x_j - x_k}(\partial_j - \partial_k)\frac{1}{x_i - x_k}(\partial_i - \partial_k) \\ &+ \sum_{k(\neq i,j)} \sum_{\ell(\neq i,j,k)} \frac{1}{(x_j - x_k)(x_i - x_\ell)}(\partial_j - \partial_k)(\partial_i - \partial_\ell). \end{aligned}$$

Reducing C_1 to the normal form we have

$$\begin{aligned} C_1 = &-\frac{1}{x_i - x_j} \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right\} (\partial_i - \partial_j) \\ &+ \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k} + \frac{1}{x_j - x_k} \right\} \frac{1}{(x_i - x_j)^2}(\partial_i - \partial_j) \\ &- \sum_{k(\neq i,j)} \left\{ \frac{1}{(x_i - x_k)^2(x_j - x_k)}(\partial_i - \partial_k) - \frac{1}{(x_i - x_k)(x_j - x_k)^2}(\partial_j - \partial_k) \right\}, \end{aligned} \tag{51}$$

and we get the normal form of C. Collecting the terms in A, C, D containing c as a coefficient, we see that they are equal to

$c(L_j - L_i)$ and is 0 after applying the division algorithm. Also summing up all the other terms in A, B, C, we have

$$\begin{aligned}
\mathrm{sp}(L_i, L_j) \equiv & \sum_{k(\neq i,j)} \left\{ \frac{x_i}{x_j - x_k}\partial_i(\partial_j - \partial_k) - \frac{x_j}{x_i - x_k}\partial_j(\partial_i - \partial_k) \right\} \\
& + x_i \left\{ \frac{1}{x_j - x_i}\partial_i(\partial_j - \partial_i) + \frac{1}{(x_j - x_i)^2}(\partial_j - \partial_i) \right\} \\
& - x_j \left\{ \frac{1}{x_i - x_j}\partial_j(\partial_i - \partial_j) + \frac{1}{(x_i - x_j)^2}(\partial_i - \partial_j) \right\} \\
& + \frac{1}{x_i - x_j} \left\{ x_j(\partial_i - \partial_j)\partial_j - x_i(\partial_j - \partial_i)\partial_i - (\partial_j + \partial_i) \right\} \\
& + \frac{1}{x_i - x_j} \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_j)(\partial_i - \partial_k) - \frac{1}{(x_i - x_k)^2}(\partial_i - \partial_k) \right\} \\
& - \frac{1}{x_i - x_j} \sum_{k(\neq i,j)} \left\{ \frac{1}{x_j - x_k}(\partial_i - \partial_j)(\partial_j - \partial_k) + \frac{1}{(x_j - x_k)^2}(\partial_j - \partial_k) \right\} \\
& + \frac{2}{(x_i - x_j)^2}(\partial_i^2 - \partial_j^2) \\
& + \sum_{k(\neq i,j)} \left\{ \frac{x_j}{x_i - x_k}(\partial_i - \partial_k)\partial_j - \frac{x_i}{x_j - x_k}(\partial_j - \partial_k)\partial_i \right\} \\
& - \frac{1}{x_i - x_j} \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right\} (\partial_i - \partial_j) \\
& + \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k} + \frac{1}{x_j - x_k} \right\} \frac{1}{(x_i - x_j)^2}(\partial_i - \partial_j) \\
& - \sum_{k(\neq i,j)} \left\{ \frac{1}{(x_i - x_k)^2(x_j - x_k)}(\partial_i - \partial_k) - \frac{1}{(x_i - x_k)(x_j - x_k)^2}(\partial_j - \partial_k) \right\}.
\end{aligned}$$

This reduces to

$$\begin{aligned}
\mathrm{sp}(L_i, L_j) \equiv & \frac{1}{(x_i - x_j)^2} \left\{ 2(\partial_i^2 - \partial_j^2) - 2(x_i\partial_i - x_j\partial_j) + \sum_{k(\neq i,j)} \left(\frac{1}{x_i - x_k} + \frac{1}{x_j - x_k} \right)(\partial_i - \partial_j) \right\} \\
& + \sum_{k(\neq i,j)} \frac{1}{(x_i - x_j)(x_j - x_k)(x_k - x_i)} \left\{ (\partial_i - \partial_k) + (\partial_j - \partial_k) \right\} \\
& = \frac{2}{(x_i - x_j)^2}(L_i - L_j) \\
& \equiv 0.
\end{aligned}$$

by applying the division algorithm using G_{HW}. Thus we have shown that G_{HW} is a Gröbner basis for the ideal I_{HW}.

Since G_{HW} is the Gröbner basis of the ideal $\mathscr{I}_{HW}$ and $\mathrm{in}_<(L_i) = \xi_i^2$, we see that $\mathscr{I}_{HW}$ is a zero-dimensional ideal of D, $\mathrm{rank}_R(D/\mathscr{I}_{HW}) = \mathrm{rank}_R(R[\xi]/\langle \xi_1^2, \dots, \xi_n^2 \rangle) = 2^n$ and

$$\{\partial_1^{k_1}\partial_2^{k_2}\cdots\partial_n^{k_n} \mid k_1, \dots, k_n = 0, 1\}$$

gives a basis of R-free module $D/\mathscr{I}_{HW}$, where when $k_1 = \cdots = k_n = 0$, above element is understood to be 1. Thus we have shown the following proposition and completed the proof of Theorem 5.1 for $\mathcal{S}_{HW}$.

Proposition 5.3. *The system $\mathcal{S}_{HW}$ is holonomic on $\mathbb{C}^n \setminus S$, $S = \cup_{i<j}\{x_i - x_j = 0\}$ and the holonomic rank is 2^n.*

5.2. Airy

We show the similar result for the system $\mathcal{S}_A$ for the Airy integral $I_A(X)$ of matrix integral type. Put

$$L_i = \partial_i^2 + \sum_{k(\neq i)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) - x_i, \quad 1 \le i \le n,$$

and let $\mathscr{I}_A \subset D$ be the left ideal of D with the generator $G_A = \{L_1, \dots, L_n\}$.

Proposition 5.4. *The generator G_A is a Gröbner basis of the left ideal $\mathcal{I}_A$.*

Proof. For $i \neq j$, let us compute the S-pair $\mathrm{sp}(L_i, L_j)$ and show that $\mathrm{sp}(L_i, L_j) \equiv 0$ after carrying out the division algorithm using G_A.

$$\begin{aligned}
\mathrm{sp}(L_i, L_j) &= \partial_j^2 L_i - \partial_i^2 L_j \\
&= \partial_j^2 \left(\partial_i^2 + \sum_{k(\neq i)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) - x_i \right) \\
&\quad - \partial_i^2 \left(\partial_j^2 + \sum_{k(\neq j)} \frac{1}{x_j - x_k}(\partial_j - \partial_k) - x_j \right) \\
&= B + C + D,
\end{aligned}$$

where

$$\begin{aligned}
B &= \partial_j^2 \cdot \frac{1}{x_i - x_j}(\partial_i - \partial_j) - \partial_i^2 \cdot \frac{1}{x_j - x_i}(\partial_j - \partial_i), \\
C &= \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_k)\partial_j^2 - \frac{1}{x_j - x_k}(\partial_j - \partial_k)\partial_i^2 \right\}, \\
D &= -x_i \partial_j^2 + x_j \partial_i^2.
\end{aligned}$$

Note that B, C has the same form as in the proof of Proposition 5.2. We make a reduction of B, C in a similar way. B can be written as

$$B = \frac{1}{x_i - x_j} B_1 + \frac{2}{(x_i - x_j)^2}(\partial_i^2 - \partial_j^2),$$

where $B_1 = (\partial_i - \partial_j)(\partial_j^2 - \partial_i^2)$. Note that

$$\partial_i^2 = L_i - \sum_{k(\neq i)} \frac{1}{x_i - x_k}(\partial_i - \partial_k) + x_i, \tag{52}$$

we have

$$\partial_j^2 - \partial_i^2 = L_j - L_i + \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right\} - (x_i - x_j),$$

and see that

$$\begin{aligned}
B_1 &\equiv (\partial_i - \partial_j) \left\{ \sum_{k(\neq i,j)} \left(\frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right) - (x_i - x_j) \right\} \\
&= \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_j)(\partial_i - \partial_k) - \frac{1}{(x_i - x_k)^2}(\partial_i - \partial_k) \right\} \\
&\quad - \sum_{k(\neq i,j)} \left\{ \frac{1}{x_j - x_k}(\partial_i - \partial_j)(\partial_j - \partial_k) + \frac{1}{(x_j - x_k)^2}(\partial_j - \partial_k) \right\} \\
&\quad - (x_i - x_j)(\partial_i - \partial_j) - 2.
\end{aligned}$$

Similarly we compute C using (52) and get

$$C \equiv C_1 + \sum_{k(\neq i,j)} \left(\frac{x_j}{x_i - x_k}(\partial_i - \partial_k) - \frac{x_i}{x_j - x_k}(\partial_j - \partial_k) \right),$$

where C_1 is the same as in (51). For D we have

$$D \equiv (\partial_i - \partial_j) - \sum_{k(\neq i,j)} \left(\frac{x_j}{x_i - x_k}(\partial_i - \partial_k) - \frac{x_i}{x_j - x_k}(\partial_j - \partial_k) \right).$$

Summing up B, C, D, we have

$$
\begin{aligned}
\operatorname{sp}(L_i, L_j) \equiv & \frac{2}{(x_i - x_j)^2}(\partial_i^2 - \partial_j^2) - (\partial_i - \partial_j) - \frac{2}{x_i - x_j} \\
& + \frac{1}{x_i - x_j} \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_j)(\partial_i - \partial_k) - \frac{1}{(x_i - x_k)^2}(\partial_i - \partial_k) \right\} \\
& - \frac{1}{x_i - x_j} \sum_{k(\neq i,j)} \left\{ \frac{1}{x_j - x_k}(\partial_i - \partial_j)(\partial_j - \partial_k) + \frac{1}{(x_j - x_k)^2}(\partial_j - \partial_k) \right\} \\
& - \frac{1}{x_i - x_j} \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k}(\partial_i - \partial_k) - \frac{1}{x_j - x_k}(\partial_j - \partial_k) \right\} (\partial_i - \partial_j) \\
& + \sum_{k(\neq i,j)} \left\{ \frac{1}{x_i - x_k} + \frac{1}{x_j - x_k} \right\} \frac{1}{(x_i - x_j)^2}(\partial_i - \partial_j) \\
& - \sum_{k(\neq i,j)} \left\{ \frac{1}{(x_i - x_k)^2 (x_j - x_k)}(\partial_i - \partial_k) - \frac{1}{(x_i - x_k)(x_j - x_k)^2}(\partial_j - \partial_k) \right\}, \\
& + (\partial_i - \partial_j) - \sum_{k(\neq i,j)} \left(\frac{x_j}{x_i - x_k}(\partial_i - \partial_k) - \frac{x_i}{x_j - x_k}(\partial_j - \partial_k) \right).
\end{aligned}
$$

This reduces to

$$
\begin{aligned}
\operatorname{sp}(L_i, L_j) \equiv & \frac{1}{(x_i - x_j)^2} \left\{ 2(\partial_i^2 - \partial_j^2) - 2(x_i - x_j) + \sum_{k(\neq i,j)} \left(\frac{1}{x_i - x_k} + \frac{1}{x_j - x_k} \right) (\partial_i - \partial_j) \right\} \\
& + \sum_{k(\neq i,j)} \frac{1}{(x_i - x_j)(x_j - x_k)(x_k - x_i)} \{(\partial_i - \partial_k) + (\partial_j - \partial_k)\} \\
= & \frac{2}{(x_i - x_j)^2}(L_i - L_j) \\
\equiv & \, 0
\end{aligned}
$$

by the division algorithm using G_A. Thus we have shown that G_A is a Gröbner basis for the ideal $\mathcal{I}_A$.

Since G_A is the Gröbner basis of the ideal $\mathcal{I}_A$ and $\operatorname{in}_\prec(L_i) = \xi_i^2$, we see that $\mathcal{I}_A$ is a zero-dimensional ideal of D, $\operatorname{rank}_R(D/\mathcal{I}_A) = \operatorname{rank}_R(R[\xi]/\langle \xi_1^2, \dots, \xi_n^2 \rangle) = 2^n$ and

$$
\{\partial_1^{k_1} \partial_2^{k_2} \cdots \partial_n^{k_n} \mid k_1, \dots, k_n = 0, 1\}
$$

gives a basis of R-free module $D/\mathcal{I}_A$, where when $k_1 = \cdots = k_n = 0$, above element is understood to be 1. Thus we have shown the following proposition and completed the proof of Theorem 5.1 for $\mathcal{S}_A$.

Proposition 5.5. *The Airy system $\mathcal{S}_A$ is holonomic on $\mathbb{C}^n \setminus S$, $S = \cup_{i<j}\{x_i - x_j = 0\}$ and its holonomic rank is 2^n.*

5.3. Gauss, Kummer, Bessel

In this section, we give the reduced form of the S-polynomial $\operatorname{sp}(L_i, L_j)$ for the systems $\mathcal{S}_G, \mathcal{S}_K, \mathcal{S}_B$ for Gauss, Kummer and Bessel without explicit computation. For the proof of these cases, we must modify the ring $R = \mathbb{C}[x]_\Delta$, which is used in the cases $\mathcal{S}_{HW}, \mathcal{S}_A$, as

$$
R = \{f/g^m \mid f \in \mathbb{C}[x], m \in \mathbb{Z}_{\geq 0}\}
$$

with $g = \prod_{i=1}^n x_i(x_i - 1) \cdot \Delta(x)$ for the case $\mathcal{S}_G$ and $g = \prod_{i=1}^n x_i \cdot \Delta(x)$ for the cases $\mathcal{S}_K, \mathcal{S}_B$.

5.3.1. System $\mathcal{S}_G$

Put

$$
\begin{aligned}
L_i = & x_i(1 - x_i)\partial_i^2 + \{c - (n-1) - (a + b + 1 - (n-1))x_i\}\partial_i \\
& + \sum_{j(\neq i)} \frac{x_i(1 - x_i)\partial_i - x_j(1 - x_j)\partial_j}{x_i - x_j} - ab
\end{aligned}
$$

and $G = \{L_1, \dots, L_n\}$. Then

$$\mathrm{sp}(L_i, L_j) \equiv \frac{2x_i(x_i-1)x_j(x_j-1)}{(x_i-x_j)^2}(L_i - L_j)$$

after applying the division algorithm using G.

5.3.2. System $\mathcal{S}_K$

Put

$$L_i = x_i\partial_i^2 + \{c-(n-1)-x_i\}\partial_i + \sum_{j(\neq i)} \frac{x_i\partial_i - x_j\partial_j}{x_i - x_j} - a$$

and $G_K = \{L_1, \dots, L_n\}$. Then

$$\mathrm{sp}(L_i, L_j) \equiv \frac{2x_ix_j}{(x_i-x_j)^2}(L_i - L_j)$$

after applying the division algorithm using G_K.

5.3.3. System $\mathcal{S}_B$

Put

$$L_i = x_i\partial_i^2 + \{c-n+1\}\partial_i + \sum_{j(\neq i)} \frac{x_i\partial_i - x_j\partial_j}{x_i - x_j} + 1$$

and $G_B = \{L_1, \dots, L_n\}$. Then

$$\mathrm{sp}(L_i, L_j) \equiv \frac{2x_ix_j}{(x_i-x_j)^2}(L_i - L_j)$$

after applying the division algorithm using G_B.

Peer Review: Externally peer-reviewed.
Conflict of Interest: Author declared no conflict of interest.
Financial Disclosure: This work was supported by JSPS KAKENHI Grant Number JP19K03521.

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Istanbul Journal of Mathematics
ijmath 2 (1), 28–32, 2024
DOI: 10.26650/ijmath.2024.00012

RESEARCH ARTICLE

Some characterizations of hyperbolic Ricci solitons on nearly cosymplectic manifolds with respect to the Tanaka-Webster connection

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ABSTRACT

It is known that a hyperbolic Ricci soliton is one of the generalization of the Ricci solitons and it is a Riemannian manifold (M, g) furnished with a differentiable vector field U on M and two real numbers λ and μ ensuring $Ric + \lambda L_U g + \frac{1}{2} L_U(L_U g) = \mu g$, where L_U denotes the Lie derivative with respect to the vector field X on M. Furthermore, hyperbolic Ricci solitons yield similar solutions to hyperbolic Ricci flow. In this paper, we study hyperbolic Ricci solitons on nearly cosymplectic manifolds endowed with the Tanaka-Webster connection. We give some results for these manifolds when the potential vector field is a pointwise collinear with the Reeb vector field and a concircular vector field.

Mathematics Subject Classification (2020): 53B21 53C25

Keywords: hyperbolic Ricci soliton, nearly cosymplectic manifold, Tanaka, Webster connection

1. INTRODUCTION

The notion of hyperbolic Ricci flow was introduced in Kong and Liu (2007). Let $g_{ij}(t)$ be a family of Riemannian metrics on a Riemannian manifold (M_n, g_0). The hyperbolic Ricci flow is defined by

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}$$

with $g(0) = g_0$, $\frac{\partial g_{ij}}{\partial t} = k_{ij}$, where k_{ij} is a symmetric$(0, 2)-$type tensor field. A self-similar solution $g(t)$ of the hyperbolic Ricci flow on M_n is a hyperbolic Ricci soliton if there exists a 1-parameter family of diffeomorphisms $\rho(t) : M \to M$ and a positive function $\sigma(t)$ such that

$$g(t) = \sigma(t)\rho(t)^*(g_0).$$

If we differentiate above equation twice, we get

$$-2Ric(g(t)) = \sigma''(t)\rho(t)^*(g_0) + 2\sigma'(t)\rho(t)^*(L_X g_0) + \sigma(t)\rho(t)^*(L_X L_X g_0)),$$

where Ric is the Ricci curvature on M, X is the time-dependent vector field and L is the Lie derivative. The family of metrics are said to be expanding, steady or shrinking if σ' is positive, zero or negative, respectively. Substituting $\sigma''(0) = -2\mu$, $\sigma(0) = 1$ and $\sigma'(0) = \lambda$ in the above equation, we get

$$Ric(g_0) + \lambda L_X g_0 + \frac{1}{2} L_X L_X g_0 = \mu g_0$$

for some real constants λ and μ. According to this equation, a hyperbolic Ricci soliton on a Riemannian manifold (M, g) is defined by

$$Ric + \lambda L_X g + \frac{1}{2} L_X(L_X g) = \mu g. \tag{1}$$

A hyperbolic Ricci soliton is called expanding, steady or shrinking if μ is negative, zero or positive, respectively. For recent papers about hyperbolic Ricci solitons see Azami and Fasihi (2023), Azami and Fasihi (2024), Blaga and Özgür (2023), Faraji et al. (2023).

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Submitted: 29.04.2024 • **Last Revision Received:** 14.05.2024 • **Accepted:** 15.05.2024

In this paper, we investigate hyperbolic Ricci solitons on nearly cosymplectic manifolds. The manifolds will be considered with the Tanaka-Webster connection. The paper is organized as follows: In Section 2, we give some fundamental information about nearly cosymplectic manifolds. In Section 3, we express some properties of cosymplectic manifolds satisfying Tanaka-Webster connection. In the final section, we give our main results.

2. NEARLY COSYMPLECTIC MANIFOLDS

An $n = (2k+1)-$dimensional smooth manifold M is called an almost contact metric manifold if it admits a $(1,1)-$tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Riemannian metric g which fulfill, Blair (1976)

$$\phi^2(U) = -U + \eta(U)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta(\phi U) = 0, \tag{2}$$

$$\begin{aligned} g(\phi U, \phi V) &= g(U,V) - \eta(U)\eta(V),\ g(\phi U, V) = -g(U, \phi V), \\ g(U,\xi) &= \eta(U),\ \forall U, V \in \chi(M). \end{aligned} \tag{3}$$

An almost contact metric manifold (M, g, η, ξ, ϕ) is called a contact metric manifold if

$$g(U, \phi V) = d\eta(U, V).$$

An almost contact metric manifold (M, g, η, ξ, ϕ) is said to be a nearly cosymplectic manifold if

$$(\nabla_U \phi)V + (\nabla_V \phi)U = 0,\ \forall U, V \in \chi(M).$$

For a nearly cosymplectic manifold, we have

$$\nabla_\xi \xi = 0 \text{ and } \nabla_\xi \eta = 0.$$

On the other hand, for a $(1,1)-$type tensor field H which is defined as

$$\nabla_U \xi = HU. \tag{4}$$

It is known that H is skew symmetric and anti-commutative with ϕ. Moreover, H satisfies $H\xi = 0$ and $\eta \circ H = 0$ and fulfills the following situations, Nicola et al. (2018):

$$(\nabla_\xi \phi)U = \phi H U = \frac{1}{3}(\nabla_\xi \phi)U\ ,$$

$$g((\nabla_U \phi)V, HW) = \eta(V)g(H^2U, \phi W) - \eta(U)g(H^2V, \phi W),$$

$$(\nabla_U H)V = g(H^2U\ , V)\xi - \eta(V)H^2U,$$

$$tr(H^2) = \text{constant},$$

$$R(V,W)\xi = \eta(V)H^2W - \eta(W)H^2V,$$

$$S(\xi, W) = -\eta(W)tr(H^2),$$

$$S(\phi V, W) = S(V, \phi W),\ \ \phi Q = Q\phi,$$

$$S(\phi V, \phi W) = S(V,W) + \eta(V)\eta(W)tr(H^2).$$

3. NEARLY COSYMPLECTIC MANIFOLDS ADMITTING TANAKA-WEBSTER CONNECTION

Let (M, g, η, ξ, ϕ) be an almost contact metric manifold. The Tanaka-Webster connection $\bar{\nabla}$ with respect to the Levi-Civita connection ∇ is defined by

$$\bar{\nabla}_U V = \nabla_U V + (\nabla_U \eta)(V)\xi - \eta(V)\nabla_U \xi - \eta(U)\phi V, \tag{5}$$

for all $U, V \in \chi(M)$, Tanno (1969). Using (3) and (4), we rewrite equation (5) as

$$\bar{\nabla}_U V = \nabla_U V + g(\nabla_U \xi, V)\xi - \eta(V)HU - \eta(U)\phi V. \tag{6}$$

Putting $V = \xi$ in (6) and using (2) and (4), we obtain

$$\bar{\nabla}_U \xi = 0. \tag{7}$$

Using (6), the Riemannian curvature tensor $\bar{R}$ of the connection $\bar{\nabla}$ is given by

$$\begin{aligned}\bar{R}(U,V)W &= R(U,V)W - g(W,HU)HV - g(H^2V,W)\eta(U)\xi - 2g(V,HU)\phi W\eta(U)\eta(W)\phi HV \\ &+ g(H^2U,W)\eta(V)\xi - \eta(V)(\nabla_U\phi)W - \eta(V)g(HU,\phi W)\xi + g(W,HV)HU \\ &+ \eta(W)\eta(U)H^2V - \eta(W)\eta(V)H^2U - \eta(V)\eta(W)\phi HU \\ &+ \eta(U)(\nabla_V\phi)W + \eta(U)g(HV,\phi W)\xi. \end{aligned} \tag{8}$$

Taking contraction in (8), the Ricci tensor $\overline{Ric}$ of the connection $\bar{\nabla}$ is given by

$$\begin{aligned}\overline{Ric}(V,W) &= Ric(V,W) + 2g(HV,\phi W) - \eta(V)div(\phi)W + g(W,HV)tr(H) \\ &- \eta(W)\eta(V)tr(H^2) - \eta(V)\eta(W)tr(\phi H) + 2g(HW,HV),\end{aligned} \tag{9}$$

where Ric denotes the Ricci tensor of the Levi-Civita connection ∇. Contracting in (9), the scalar curvature $\bar{r}$ is obtained as

$$\bar{r} = r - tr(H^2)(2k+1),$$

where r is the scalar curvature of the Levi-Civita connection ∇, Ayar (2022).

4. MAIN RESULTS

Before expressing our main results, we should remind definitions of the nearly quasi-Einstein manifolds and Einstein manifolds.

Definition 4.1. Let (M,g) be a Riemannian manifold. If $Ric = \alpha g + \beta E$ for some functions α and β on M, where E is a non-zero tensor of type $(0,2)$, then the manifold (M,g) is called a nearly quasi-Einstein manifold. If $\beta = 0$, then the manifold (M,g) is said to be an Einstein manifold. Here, Ric denotes the Ricci tensor of the Levi-Civita connection ∇.

Now, we can give our findings.

Theorem 4.2. *Let M be a nearly cosymplectic manifold with the Tanaka-Webster connection admitting a hyperbolic Ricci soliton. If the potential vector field X is a pointwise collinear with ξ, then M is a nearly-quasi Einstein manifold.*

Proof. If the potential vector field X is a pointwise collinear with ξ, then there exists a smooth function b such that $X = b\xi$. Using (7), we have

$$\begin{aligned}(\bar{L}_X g)(U,V) &= g(\bar{\nabla}_U X, V) + g(\bar{\nabla}_V X, U) \\ &= g(U(b)\xi + b\bar{\nabla}_U\xi, V) + g(V(b)\xi + b\bar{\nabla}_V\xi, U) \\ &= U(b)\eta(V) + V(b)\eta(U) \\ &= g(\nabla b, U)\eta(V) + g(\nabla b, V)\eta(U)\end{aligned} \tag{10}$$

for all $U, V \in \chi(M)$, where ∇ denotes the gradient operator. The Lie derivative of (7) is given by

$$\begin{aligned}(\bar{L}_X \circ \bar{L}_X)g(U,V) &= X\bar{L}_X g(U,V) - \bar{L}_X g(\bar{L}_X U, V) - \bar{L}_X g(U, \bar{L}_X V) \\ &= X[g(\nabla b, U)\eta(V) + g(\nabla b, V)\eta(U)] \\ &- [g(\nabla b, \bar{L}_X U)\eta(V) + g(\nabla b, V)\eta(\bar{L}_X U)] \\ &- [g(\nabla b, \bar{L}_X V)\eta(U) + g(\nabla b, U)\eta(\bar{L}_X V)] \\ &= Xg(\nabla b, U)\eta(V) + g(\nabla b, U)X\eta(V) + Xg(\nabla b, V)\eta(U) \\ &+ g(\nabla b, V)X\eta(U) - g(\nabla b, \bar{L}_X U)\eta(V) - g(\nabla b, V)\eta(\bar{L}_X U) \\ &- g(\nabla b, \bar{L}_X V)\eta(U) - g(\nabla b, U)\eta(\bar{L}_X V).\end{aligned} \tag{11}$$

Putting (10) and (11) in (1), we occur

$$\begin{aligned}\overline{Ric}(U,V) &= \mu g(U,V) - \lambda(\bar{L}_X g)(U,V) - \frac{1}{2}(\bar{L}_X \circ \bar{L}_X)g(U,V) \\ &= \mu g(U,V) - \lambda g(\nabla b, U)\eta(V) - \lambda g(\nabla b, V)\eta(U) \\ &- \frac{1}{2}Xg(\nabla b, U)\eta(V) - \frac{1}{2}g(\nabla b, U)X\eta(V) - \frac{1}{2}Xg(\nabla b, V)\eta(U) \\ &- \frac{1}{2}g(\nabla b, V)X\eta(U) + \frac{1}{2}g(\nabla b, \bar{L}_X U)\eta(V) + \frac{1}{2}g(\nabla b, V)\eta(\bar{L}_X U) \\ &+ \frac{1}{2}g(\nabla b, \bar{L}_X V)\eta(U) + \frac{1}{2}g(\nabla b, U)\eta(\bar{L}_X V).\end{aligned} \tag{12}$$

Taking a non-vanishing $(0,2)-$type tensor E as

$$\begin{aligned} E(U,V) &= -\lambda g(\nabla b,U)\eta(V) - \lambda g(\nabla b,V)\eta(U) \\ &\quad -\frac{1}{2}[Hess(b)(X,U)\eta(V) - Hess(b)(X,V)\eta(U) + (\overline{\nabla}_U X)(b)\eta(V) \\ &\quad +(\overline{\nabla}_V X)(b)\eta(U) + V(b)g(\overline{\nabla}_U X,\xi) + U(b)g(\overline{\nabla}_U X,\xi)]. \end{aligned} \tag{13}$$

Equation (12) becomes

$$\overline{Ric}(U,V) = \mu g(U,V) + E(U,V).$$

This shows that M is a nearly quasi-Einstein manifold with respect to the Tanaka-Webster connection $\bar{\nabla}$.

Proposition 4.3. *Let M be a nearly cosymplectic manifold with the Tanaka-Webster connection admitting a hyperbolic Ricci soliton. If the potential vector field is the Reeb vector field ξ, then M is an Einstein manifold.*

Proof. Taking $b = 1$ in (13) shows that $\overline{Ric}(U,V) = \mu g(U,V)$. This gives us M is an Einstein manifold.

Theorem 4.4. *Let M be a nearly cosymplectic manifold with the Tanaka-Webster connection admitting a hyperbolic Ricci soliton. If the potential vector field is a concircular vector field X, then*

$$\mu = -2tr(H^2) - tr(H) + 2f^2 + 2\lambda f.$$

Proof. It is known that if X is concircular vector field on M, then there exists a smooth function f such that

$$\nabla_U X = fU \tag{14}$$

for all $U \in \chi(M)$. Using (14), we obtain

$$\begin{aligned} (\bar{L}_X g)(U,V) &= g(\bar{\nabla}_U X,V) + g(\bar{\nabla}_V X,U) \\ &= g(fU,V) + g(U,fV) \\ &= 2fg(U,V). \end{aligned} \tag{15}$$

Using equation (15), we get

$$\begin{aligned} (\bar{L}_X \circ \bar{L}_X)g(U,V) &= X\bar{L}_X g(U,V) - \bar{L}_X g(\bar{L}_X U,V) - \bar{L}_X g(U,\bar{L}_X V) \\ &= X(2fg(U,V)) - 2fg(\bar{L}_X U,V) - 2fg(U,\bar{L}_X V) \\ &= 2(Xf)g(U,V) + 2fg(\bar{\nabla}_X U,V) + 2fg(U,\bar{\nabla}_X V) \\ &\quad -2fg(\bar{\nabla}_U X,V) + 2fg(\bar{\nabla}_U X,V) - 2fg(U,\bar{\nabla}_X V) + 2fg(U,\bar{\nabla}_V X) \\ &= 2(Xf)g(U,V) + 2fg(\bar{\nabla}_U X,V) + 2fg(U,\bar{\nabla}_V X) \\ &= 2(Xf)g(U,V) + 4f^2 g(U,V). \end{aligned} \tag{16}$$

Putting (15) and (16) in (1), we deduce

$$Ric(U,V) + (Xf)g(U,V) + 2f^2 g(U,V) + 2\lambda fg(U,V) = \mu g(U,V).$$

Substituting $U = V = \xi$ in (9), we obtain $\mu = -2tr(H^2) - tr(H) + 2f^2 + 2\lambda f$.

5. CONCLUSION

In this paper, we study hyperbolic Ricci solitons on nearly cosymplectic manifolds with respect to the Tanaka-Webster connection by considering the potential vector field as a pointwise collinear with the Reeb vector field and a concircular vector field. Our results in the present work may provide an insight for further studies on hyperbolic Ricci solitons with respect to some other connections.

Peer Review: Externally peer-reviewed.
Conflict of Interest: Author declared no conflict of interest.
Financial Disclosure: Author declared no financial support.

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Istanbul Journal of Mathematics
ijmath 2 (1), 33–38, 2024
DOI: 10.26650/ijmath.2024.00013

RESEARCH ARTICLE

Parseval-Goldstein type theorems for integral transforms in a general setting

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ABSTRACT
This research paper explores Parseval-Goldstein type relations concerning general integral operators. It investigates the continuity properties of these operators and their adjoints over Lebesgue spaces. Through rigorous analysis, the study elucidates the intricate connections between these operators and sheds light on their behaviour within functional spaces. By exploring the convergence and stability of these relations, the paper contributes to a deeper understanding of integral operators behaviour and their implications in various mathematical contexts. The paper also examines specific cases of the main index transforms, including the Kontorovich-Lebedev transform, the Mehler-Fock transform of general order, the index ${}_2F_1$-transform, the Lebedev-Skalskaya transforms and the index Whittaker transform, as well as operators with complex Gaussian kernels, contributing valuable insights into their behaviour and applications.

Mathematics Subject Classification (2020): 44A15, 46E30, 47G10

Keywords: Integral operators, weighted Lebesgue spaces, Parseval-Goldstein relations, index transforms, Gaussian kernels

1. INTRODUCTION

We consider the integral operator given by

$$(\mathcal{F}f)(y) = \int_I f(x)K(x,y)dx, \ y \in I, \tag{1}$$

where K is a measurable complex-valued function $K : I \times I \longrightarrow \mathbb{C}$ (I denoting some open interval in $\mathbb{R}$, possibly unbounded) over the spaces $L^p(I, \tilde{K}(x)dx)$, $1 \leq p < \infty$, and $L^\infty(I)$, being $\tilde{K}(x)$ a measurable function on I which satisfies $|K(x,y)| \leq \tilde{K}(x)$, for all $x, y \in I$.
We also consider the integral operator

$$(\mathcal{F}^* g)(x) = \int_I g(y)K(x,y)dy, \ x \in I, \tag{2}$$

over the space $L^1(I)$. In 1989, Yürekli Yürekli (1989) introduced a Parseval-Goldstein type theorem, elucidating the interconnection between Laplace and Stieltjes transforms, and subsequently explored its ramifications. In 1992, Yürekli extended this investigation to encompass the generalized Stieltjes transform Yürekli (1992). Building upon this foundation, various researchers have delved into analogous connections amonxxg diverse integral transforms, leveraging Parseval-Goldstein type theorems, as evidenced by works from several authors Albayrak and Dernek (2021); Albayrak (2024); Karataş et al. (2020). Parseval's and Plancherel's theorems stand as cornerstone results in mathematics, establishing pivotal relationships between original functions and their transforms, showcasing the preservation of energy or inner products under transformation Dernek et al. (2008, 2007); Yürekli (1989).

The Parseval-Goldstein relations for integral transforms establish a crucial link between norms in the original domain and their transformed counterparts, shedding light on the energy-preserving characteristics and inter-domain consistency of these transforms. This profound analysis significantly contributes to understanding the fundamental properties and applications of integral transforms in mathematical analysis Yürekli (1989, 1992); Albayrak and Dernek (2021); Albayrak (2024); Karataş et al. (2020); Srivastava and Yürekli (1995). The present article delves into the study of Parseval-Goldstein type relations for integral

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Submitted: 14.04.2024 • **Last Revision Received:** 05.06.2024 • **Accepted:** 07.06.2024

operators over Lebesgue spaces.
The $C_c^k(\mathbb{R}_+)$, $k \in \mathbb{N}$, denotes as it is usual the space of compactly supported functions on $\mathbb{R}_+$ which are k-times differentiable with continuity. The article is structured as follows: Section 1 presents an introduction about the general setting. Sections 2 and 3 delve into the continuity properties over Lebesgue spaces of both the integral operators and their adjoints. Section 4 establishes Parseval-Goldstein type relations for these operators. Section 5 examines integral transforms characterized by kernels satisfying specific conditions. Finally, Section 6 offers concluding remarks.

2. THE $\mathcal{F}$ TRANSFORM OVER THE SPACES $L^p(I, \tilde{K}(x)dx)$, $1 \le p < \infty$

Proposition 2.1. *The next results hold*
(i) *The integral operator $\mathcal{F}$ given by (1) is a bounded linear operator from $L^1(I, \tilde{K}(x)dx)$ into $L^\infty(I)$. If $f \in L^1(I, \tilde{K}(x)dx)$ then*

$$\|\mathcal{F}f\|_{L^\infty(I)} \le \|f\|_{L^1(I,\tilde{K}(x)dx)},$$

furthermore if $K(x,\cdot)$ is continuous for each $x \in I$ then $\mathcal{F}f$ is a continuous function on I. Moreover, the operator $\mathcal{F}$ is a continuous map from $L^1(I, \tilde{K}(x)dx)$ to the Banach space of bounded continuous functions on I.
(ii) *The integral operator $\mathcal{F}$ given by (1) is a bounded linear operator from $L^p(I, \tilde{K}(x)dx)$ into $L^\infty(I)$, $1 < p < \infty$, whenever $\int_I \tilde{K}(x)dx < \infty$. Also if $f \in L^p(I, \tilde{K}(x)dx)$, $1 < p < \infty$, then*

$$\|\mathcal{F}f\|_{L^\infty(I)} \le M\|f\|_{L^p(I,\tilde{K}(x)dx)}, \text{ for some } M > 0,$$

furthermore if $K(x,\cdot)$ is continuous for each $x \in I$ then $\mathcal{F}f$ is a continuous function on I. Moreover, the operator $\mathcal{F}$ is a continuous map from $L^p(I, \tilde{K}(x)dx)$ to the Banach space of bounded continuous functions on I.
(iii) *The integral operator $\mathcal{F}$ given by (1) is a bounded linear operator from $L^\infty(I)$ into $L^\infty(I)$ whenever $\int_I \tilde{K}(x)dx < \infty$. Also if $f \in L^\infty(I)$ then*

$$\|\mathcal{F}f\|_{L^\infty(I)} \le M\|f\|_{L^\infty(I)}, \text{ for some } M > 0,$$

furthermore if $K(x,\cdot)$ is continuous for each $x \in I$ then $\mathcal{F}f$ is a continuous function on I. Moreover, the operator $\mathcal{F}$ is a continuous map from $L^\infty(I)$ to the Banach space of bounded continuous functions on I.

Proof. (i) Let $y_0 \in I$ be arbitrary. Since the map $y \to K(x,y)$ is continuous for each fixed $x \in I$, we have

$$K(x,y) \to K(x,y_0) \text{ as } y \to y_0.$$

Further, we have that $|K(x,y) - K(x,y_0)|\,|f(x)|$ is dominated by the integrable function $2\tilde{K}(x)|f(x)|$. Therefore, by using dominated convergence theorem, we get

$$|(\mathcal{F}f)(y) - (\mathcal{F}f)(y_0)| \le \int_I |f(x)|\,|K(x,y) - K(x,y_0)|\,dx \to 0, \text{ as } y \to y_0.$$

Thus, $\mathcal{F}f$ is a continuous function on I.
Since for each $y \in I$

$$\begin{aligned} |(\mathcal{F}f)(y)| &\le \int_I |f(x)|\,|K(x,y)|\,dx \\ &\le \int_I |f(x)|\tilde{K}(x)dx = \|f\|_{L^1(I,\tilde{K}(x)dx)}, \end{aligned} \tag{3}$$

one has that $\mathcal{F}f$ is a bounded function.
The linearity of the integral operator implies that the $\mathcal{F}$ integral operator is linear. Also from (3) we get that $\|\mathcal{F}f\|_{L^\infty(I)} \le \|f\|_{L^1(I,\tilde{K}(x)dx)}$ and hence $\mathcal{F} : L^1(I, \tilde{K}(x)dx) \to L^\infty(I)$ is a continuous linear map.
(ii) Observe that using Hölder's inequality we have for $y \in I$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p < \infty$,

$$\begin{aligned} |(\mathcal{F}f)(y)| &\le \int_I |f(x)|\,|K(x,y)|\,dx \\ &\le \int_I |f(x)|\tilde{K}(x)dx \\ &= \int_I |f(x)|\tilde{K}(x)^{\frac{1}{p}}\tilde{K}(x)^{\frac{1}{p'}}\,dx \\ &\le \left(\int_I |f(x)|^p\tilde{K}(x)dx\right)^{\frac{1}{p}}\left(\int_I \tilde{K}(x)dx\right)^{\frac{1}{p'}} \\ &= \|f\|_{L^p(I,\tilde{K}(x)dx)}\left(\int_I \tilde{K}(x)dx\right)^{\frac{1}{p'}}. \end{aligned} \tag{4}$$

Proceeding as in (i) one obtains (ii).
(iii) Observe that for $y \in I$

$$\begin{aligned} |(\mathcal{F}f)(y)| &\leq \int_I |f(x)|\tilde{K}(x)dx \\ &\leq \operatorname{esssup}_{x\in I}|f(x)| \cdot \int_I \tilde{K}(x)dx \\ &= \|f\|_{L^\infty(I)} \cdot \int_I \tilde{K}(x)dx. \end{aligned} \tag{5}$$

Thus similar to (i) one obtains (iii).

Proposition 2.2. *The next results hold*
(i) *The integral operator $\mathcal{F}$ given by (1) is a bounded linear operator from $L^1(I, \tilde{K}(x)dx)$ into $L^q(I, w(x)dx)$, $0 < q < \infty$, when w is a measurable function on I such that $w > 0$ a.e. on I and $\int_I w(x)dx < \infty$.*
(ii) *The integral operator $\mathcal{F}$ given by (1) is a bounded linear operator from $L^p(I, \tilde{K}(x)dx)$, $1 < p < \infty$, into $L^q(I, w(x)dx)$, $0 < q < \infty$, when $\int_I \tilde{K}(x)dx < \infty$, $\int_I w(x)dx < \infty$, being w a measurable function on I such that $w > 0$ a.e. on I.*
(iii) *The integral operator $\mathcal{F}$ given by (1) is a bounded linear operator from $L^\infty(I)$ into $L^q(I, w(x)dx)$, $0 < q < \infty$, when $\int_I \tilde{K}(x)dx < \infty$, $\int_I w(x)dx < \infty$, being w a measurable function on I such that $w > 0$ a.e. on I.*

Proof. (i) Observe that from (3) for each $y \in I$ one has

$$|(\mathcal{F}f)(y)| \leq \|f\|_{L^1(I,\tilde{K}(x)dx)}.$$

Then, for $0 < q < \infty$, one has

$$\left(\int_I |(\mathcal{F}f)(x)|^q w(x)dx\right)^{\frac{1}{q}} \leq \|f\|_{L^1(I,\tilde{K}(x)dx)} \left(\int_I w(x)dx\right)^{\frac{1}{q}} < \infty.$$

(ii) The proof is similar to (i) when one make use of (4) instead of (3).
(iii) The proof is similar to (i) when one make use of (5) instead of (3).

3. THE TRANSFORM $\mathcal{F}^*$ OVER THE SPACES $L^1(I)$

Proposition 3.1. *The integral operator $\mathcal{F}^*$ given by (2) is a bounded linear operator from $L^1(I)$ into $L^q(I, w(x)dx)$, $0 < q < \infty$, when w is a measurable function on I such that $w > 0$ a.e. on I and $\tilde{K} \in L^q(I, w(x)dx)$.*

Proof. Observe that for each $x \in I$

$$\begin{aligned} |(\mathcal{F}^* f)(x)| &\leq \int_I |f(y)|\,|K(x,y)|\,dy \\ &\leq \int_I |f(y)|dy \cdot \tilde{K}(x). \end{aligned}$$

Then, for $0 < q < \infty$, one has

$$\left(\int_I |(\mathcal{F}^* f)(x)|^q\, w(x)dx\right)^{\frac{1}{q}} \leq \|f\|_{L^1(I)} \left(\int_I (\tilde{K}(x))^q\, w(x)dx\right)^{\frac{1}{q}} < \infty.$$

4. PARSEVAL-GOLDSTEIN TYPE THEOREMS

Theorem 4.1. *For $\mathcal{F}$ and $\mathcal{F}^*$ given by (1) and (2), respectively, and $g \in L^1(I)$, then the following Parseval-Goldstein type relation holds*

$$\int_I (\mathcal{F}f)(x)g(x)dx = \int_I f(x)(\mathcal{F}^* g)(x)dx, \tag{6}$$

whenever
(i) *$f \in L^1(I, \tilde{K}(x)dx)$,*
or
(ii) *$f \in L^p(I, \tilde{K}(x)dx)$, $1 < p < \infty$, and $\int_I \tilde{K}(x)dx < \infty$,*
or
(iii) *$f \in L^\infty(I)$, and $\int_I \tilde{K}(x)dx < \infty$,*

where for all cases $\tilde{K}(x)$ satisfies $|K(x,y)| \leq \tilde{K}(x)$, for all $x, y \in I$.

Proof. (i) In fact, from (3) and for each $y \in I$ one has

$$|(\mathcal{F}f)(y)| \leq \|f\|_{L^1(I,\tilde{K}(x)dx)}.$$

Therefore,

$$\int_I |(\mathcal{F}f)(y)||g(y)|dy \leq \|f\|_{L^1(I,\tilde{K}(x)dx)}\|g\|_{L^1(I)}.$$

Also, for each $x \in I$

$$|(\mathcal{F}^* g)(x)| \leq \int_I |g(y)|\,|K(x,y)|\,dy \leq \tilde{K}(x)\|g\|_{L^1(I)}.$$

Then

$$\begin{aligned}\int_I |f(x)||(\mathcal{F}^* g)(x)|dx &\leq \int_I |f(x)|\tilde{K}(x)dx\|g\|_{L^1(I)} \\ &= \|f\|_{L^1(I,\tilde{K}(x)dx)}\|g\|_{L^1(I)}.\end{aligned}$$

Thus, by using Fubini's theorem one obtains the result (6).
(ii) The proof is similar to (i) making use of (4) instead of (3).
(iii) The proof is similar to (i) making use of (5) instead of (3).

Remark 4.2. From this result the operator $\mathcal{F}^*$ becomes the adjoint of the operator $\mathcal{F}$ over $L^p(I,\tilde{K}(x)dx)$, $1 \leq p < \infty$, and $L^\infty(I)$.

Assume that $K(\cdot,y) \in C^n(I)$ for each $y \in I$ and A_x is a n-th differential operator such that

$$A_x\,(K(x,y)) = P(y)K(x,y), \tag{7}$$

for all $x, y \in I$, where P is a polynomial.
For $k \in \mathbb{N}$ and $K(\cdot,y) \in C^{nk}(I)$ for each $y \in I$, then

$$A_x^k\,(K(x,y)) = [P(y)]^k K(x,y),$$

where A_x^k denotes the k-th power of the operator A_x.
Denote $A_x^{'}$ be the adjoint of A_x.
For $f \in C_c^{nk}(I)$ and $K(\cdot,y) \in C^{nk}(I)$ for each $y \in I$, $k \in \mathbb{N}$, one has

$$\left(\mathcal{F}\left(A_x^{'\,k} f\right)\right)(y) = [P(y)]^k(\mathcal{F}f)(y),\;\; y \in I.$$

Thus for Q being a polynomial of degree m and $f \in C_c^{nm}(I)$ and $K(\cdot,y) \in C^{nm}(I)$ for each $y \in I$, $m \in \mathbb{N}$, then

$$\left(\mathcal{F}\left(Q\left(A_x^{'} f\right)\right)\right)(y) = Q\left(P(y)\right)(\mathcal{F}f)(y),\;\; y \in I. \tag{8}$$

Theorem 4.3. *Set A_x a n-th differential operator satisfying the equality (7) and denote by $A_x^{'}$ its adjoint. Let Q be a polynomial of degree m and $f \in C_c^{nm}(I)$, $K(\cdot,y) \in C^{nm}(I)$ for each $y \in I$. Then, for any $g \in L^1(I)$, the following Parseval-Goldstein relation holds*

$$\int_I (\mathcal{F}f)(x)g(x)Q\left(P(x)\right)dx = \int_I \left(Q\left(A_x^{'} f\right)\right)(x)(\mathcal{F}^* g)(x)dx.$$

Proof. The proof is an immediate consequence of relation (8) and (i) of Theorem 4.1, having into account that $C_c^{nm}(I) \subseteq L^1(I,\tilde{K}(x)dx)$.

5. PARTICULAR CASES: THE MAIN INDEX TRANSFORMS AND THE OPERATORS WITH COMPLEX GAUSSIAN KERNELS

In this section, we explore a range of integral transforms characterized by kernels that fulfill specific conditions. These conditions play a crucial role in the properties and applications of these transforms, making them particularly noteworthy in studying Parseval-Goldstein type relations in mathematical analysis. Below, we present examples of integral transforms with kernels satisfying the condition $|K(x,y)| \leq \tilde{K}(x)$, for all $x, y \in I$ and $\int_I \tilde{K}(x)dx < \infty$.
(i) For the Kontorovich-Lebedev transform González and Negrín (2019); Naylor (1990); Prasad A. and Mandal (2018); Srivastava

et al. (2016); Yakubovich (2012); Maan and Negrín (2024) $I = (0, \infty)$, $\tilde{K}(x) = K_0(x)$, where $K_0(x)$ is the modified Bessel function of the third kind (or the Macdonald function) defined by (Erdélyi et al. 1953, p. 5, section 7.2.2., Entry 13) one has that $\int_I \tilde{K}(x)dx < \infty$ and the differential operator is given by $A_x = x^2 D_x^2 + xD_x - x^2$.
(ii) For the Mehler-Fock transform of general order μ with $\Re(\mu) > \frac{-1}{2}$ González and Negrín (2019, 2017); Lebedev (1949); Srivastava et al. (2016); Yakubovich (2012); Maan and Negrín (2024) $I = (0, \infty)$, $\tilde{K}(x) = P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)$, where $P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)$ is the associated Legendre function of the first kind (Erdélyi et al. 1953, p. 122, section 3.2., Entry 3) one has that $\int_I \tilde{K}(x)dx < \infty$ and $A_x = (\sinh x)^{-\mu-1} D_x (\sinh x)^{2\mu+1} D_x (\sinh x)^{-\mu}$ as the differential operator.
(iii) For the index ${}_2F_1$-transform Hayek et al. (1992); Hayek and González (1993, 1994, 1997); Maan and Negrín (2024); Maan et al. (2023) $I = (0, \infty)$, $\Re(\mu) > -1/2$, $\alpha \in \mathbb{C}$, $\tilde{K}(x) = x^{\Re(\alpha)} {}_2F_1\left(\Re(\mu) + \frac{1}{2}, \Re(\mu) + \frac{1}{2}; \Re(\mu) + 1; -x\right)$, where ${}_2F_1\,(a, b; c; z)$ represents the Gauss hypergeometric function (Erdélyi et al. 1953, p. 56, section 2.1.1., Entry 2). Observe that $\int_I \tilde{K}(x)dx < \infty$ for $-1 < \Re(\alpha) < -1 + \Re(\mu) + \frac{1}{2}$ and the differential operator as $A_x = x^{\alpha-\mu}(x+1)^{-\mu} D_x x^{\mu+1}(x+1)^{\mu+1} D_x x^{-\alpha}$.
(iv) For the Lebedev-Skalskaya transforms Mandal and Prasad (2022); Mandal et al. (2022); Maan and Negrín (2024) $I = (0, \infty)$, $\tilde{K}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}}$ which $\int_I \tilde{K}(x)dx < \infty$ and $A_x = x^2 D_x^2 + 2xD_x - x(x-1)$ as the differential operator for the Lebedev-Skalskaya transform $(\Re f)(y) = \int_0^\infty \Re K_{\frac{1}{2}+iy}(x) f(x)dx$, and $A_x = x^2 D_x^2 + 2xD_x - x(x+1)$ is the differential operator for the Lebedev-Skalskaya transform $(\Im f)(y) = \int_0^\infty \Im K_{\frac{1}{2}+iy}(x) f(x)dx$. Here $\Re K_{\frac{1}{2}+iy}(x)$ and $\Im K_{\frac{1}{2}+iy}(x)$ are the real and imaginary parts of the Macdonald function $K_{\frac{1}{2}+iy}(x)$, respectively (Erdélyi et al. 1953, p. 5, section 7.2.2., Entry 13).
(v) The operators with complex Gaussian kernels González and Negrín (2019, 2018); Negrín (1995) are given by

$$(\mathcal{F} f)(y) = \int_{-\infty}^{\infty} f(x) \exp\left\{-\epsilon x^2 - \beta y^2 + 2\delta xy + \gamma x + \zeta y\right\} dx, \ \ y \in \mathbb{R}, \epsilon, \beta, \delta, \gamma, \zeta \in \mathbb{C}.$$

In this case $I = (-\infty, \infty)$ and $K(x, y) = \exp\left\{-\epsilon x^2 - \beta y^2 + 2\delta xy + \gamma x + \zeta y\right\}$. Observe that: $|K(x, y)| \leq \exp\left\{-\Re\epsilon x^2 - \Re\beta y^2 + 2\Re\delta xy + \Re\gamma\right.$
And so, for $\Re\beta \geq 0$ and $\Re\delta = \Re\zeta = 0$ one has $|K(x, y)| \leq \exp\left\{-\Re\epsilon x^2 + \Re\gamma x\right\} = \tilde{K}(x)$.
Thus for (i) one works in $L^1(I, \tilde{K}(x)dx)$ for $\Re\beta \geq 0$, $\Re\delta = \Re\zeta = 0$.
For the cases (ii) and (iii) and being $\Re\beta \geq 0$, $\Re\delta = \Re\zeta = 0$, one also needs $\int_{-\infty}^{\infty} \tilde{K}(x)dx < \infty$ which holds for $\Re\epsilon > 0$.
Concerning the differential operator for the operators with complex Gaussian kernels observe that:

$$\begin{aligned} D_x\left(K(x, y)\right) &= D_x\left(\exp\left\{-\epsilon x^2 - \beta y^2 + 2\delta xy + \gamma x + \zeta y\right\}\right) \\ &= \left(-2\epsilon x + 2\delta y + \gamma\right) \exp\left\{-\epsilon x^2 - \beta y^2 + 2\delta xy + \gamma x + \zeta y\right\} \end{aligned}$$

Then

$$D_x\left(K(x, y)\right) + 2\epsilon x K(x, y) = (2\delta y + \gamma)\, K(x, y)$$

So, for this case we take the differential operator as $A_x = D_x + 2\epsilon x$.

Remark 5.1. In the case of index Whittaker transform Maan and Prasad (2022, 2024); Sousa et al. (2020, 2019) $I = (0, \infty)$, $\tilde{K}(x) = x^a \Psi(a, 1; x) x^{-2a-1} e^{-x}$, $a > 0$, where $\Psi(a, 1; x)$ is known as the Tricomi function Sousa et al. (2019). The convergence of the integral $\int_I \tilde{K}(x)dx$ is not assured for $a > 0$.

6. CONCLUSIONS

The current research article extensively explores the continuity properties across Lebesgue spaces for integral transforms within a general framework, including their adjoints. By placing a significant emphasis on Parseval-Goldstein relations, this study unveils the energy-preserving characteristics and inter-domain consistency inherent in these transforms. Such a comprehensive analysis greatly contributes to our comprehension of the fundamental properties and applications of these integral transforms within mathematical analysis. The findings presented herein offer a systematic examination of various index integral transforms, such as the Kontorovich-Lebedev transform, the Mehler-Fock transform of general order, the ${}_2F_1$-transform, the Lebedev-Skalskaya transforms, and also the operators with complex Gaussian kernels.

Peer Review: Externally peer-reviewed.
Conflict of Interest: Authors declared no conflict of interest.
Financial Disclosure: Authors declared no financial support.

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Istanbul Journal of Mathematics
ijmath 2 (1), 39–43, 2024
DOI: 10.26650/ijmath.2024.00014

RESEARCH ARTICLE

A note on vanishing elements and co-degrees of strongly monolithic characters of finite groups

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ABSTRACT

Character theory of finite groups have an important role in understanding the structure of finite groups. A number of previously unresolved problems related to the structure of finite groups have been solved with the development of representation and character theory. There are many articles in the literature on the relationships between the structure of finite groups and their irreducible characters. Today, many researchers continue to study these relationships. Our purpose in this paper is to prove that for determining some properties of the structure of a finite group G, it is enough to consider only strongly monolithic characters of G instead of all irreducible characters of G. We give relationships between the structure of G and the vanishing elements, co-degrees of strongly monolithic characters of G.

Mathematics Subject Classification (2020): 20C15

Keywords: strongly monolithic characters – vanishing elements – co-degree – solvable groups

1. INTRODUCTION

Let G be a finite group and $\chi \in \mathrm{Irr}(G)$, where $\mathrm{Irr}(G)$ denotes the set of irreducible complex characters of G. An irreducible character χ of G is called a monolithic character of G if $G/\ker\chi$ has only one minimal normal subgroup. Also, an irreducible character χ of G is said to be monomial if it is induced from a linear character of some subgroup of G. An element $g \in G$ is called a vanishing element if there exists an irreducible character χ of G such that $\chi(g) = 0$. We know from Burnside's theorem (Theorem 3.15) in Isaacs (1976) that a nonlinear irreducible character of a finite group G always vanishes on some conjugacy class of G. An element $g \in G$ is non-vanishing if $\chi(g) \neq 0$ for every irreducible character χ of G. It is known from Isaacs et all. (1999) that if G is solvable and a non-vanishing element x has odd order, then x must lie in the Fitting subgroup $\mathrm{F}(G)$. Later, Dolfi et all. proved in Dolfi et all. (2010) that if x is a non-vanishing element and the order of x is coprime to 6, then $x \in \mathrm{F}(G)$. Erkoç et all. consider in Erkoç et all. (2023) a smaller subset named the set of SM-vanishing conjugacy classes instead of the set of vanishing conjugacy classes of G.

Firstly the co-degree of an irreducible character χ of G was defined as $|G|/\chi(1)$ in Chillag and Herzog (1990). Then it has been given in Qian et all. (2007) as the number $\mathrm{cod}(\chi) = \frac{|G:\ker\chi|}{\chi(1)}$ because it is very useful for inductive proofs of theorems giving information about the structure of G. In Chen and Yang (2020), authors consider the co-degrees of monolithic, monomial irreducible characters.

Motivated by above papers, we give some results about the relationships between the structure of a finite group and its strongly monolitic characters.

2. PRELIMINARIES

In this paper, all groups under consideration are finite and all characters are complex characters. We use the standard notations such as in Isaacs (1976). The definition of strongly monolithic character of a group have been first given in Erkoç et all. (2023).

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Submitted: 22.05.2024 • **Last Revision Received:** 14.06.2024 • **Accepted:** 15.06.2024

It is known from Proposition 2.3 in Erkoç et all. (2023) that linear characters of a group are not strongly monolithic. Thus, abelian groups do not have strongly monolithic characters. However, a nonabelian group have at least one strongly monolithic character. Also, every nonabelian solvable group has at least one monomial strongly monolithic character. The definition of a strongly monolithic character of a group G is the following:

Definition 2.1. *(Erkoç et all. 2023, Definition 2.2)* Let G be a group. An irreducible character χ of G is called a monolithic character if $G/\ker\chi$ has only one minimal normal subgroup. A monolithic character χ of G is called a strongly monolithic character if one of the following conditions is satisfied:

(i) $Z(\chi) = \ker\chi$, where $Z(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\}$,
(ii) $G/\ker\chi$ is a p-group whose commutator subgroup is its unique minimal normal subgroup.

Definition 2.2. *(Erkoç et all. 2023, Definition 2.2)* Let G be a group. An element g in G is called an SM-vanishing element of G if there exists a strongly monolithic character χ of G such that $\chi(g) = 0$. The conjugacy class of such an element is called an SM-vanishing conjugacy class of G. If χ is a monomial strongly monolithic character of G, then the conjugacy class of such an element is called an MSM-vanishing conjugacy class of G.

Let $\mathrm{Van}_{\mathrm{sm}}(G)$ be the set of SM-vanishing elements of G, that is,

$$\mathrm{Van}_{\mathrm{sm}}(G) = \{g \in G \mid \chi(g) = 0 \text{ for some } \chi \in \mathrm{Irr}_{\mathrm{sm}}(G)\},$$

where $\mathrm{Irr}_{\mathrm{sm}}(G)$ is the set of all strongly monolithic characters of G.

Let g be an element of a finite group G. If $\chi(g) \neq 0$ for every strongly monolithic character χ of G, then the element g is called an SM-nonvanishing element. If $\chi(g) \neq 0$ for every monomial strongly monolithic character χ of G, then the element g is called an MSM-nonvanishing element.

The following lemma and Theorem 2.4 will be useful when we prove Theorem 3.2. Actually, we know from Lemma 2.3 of Isaacs et all. (1999) that if x is a nonvanishing element in a finite group G, then x fixes some member of each orbit of the action of G on $\mathrm{Irr}(N)$ where $N \lhd G$.

Lemma 2.3. *Let G be a solvable group with a unique minimal normal subgroup M and $\Phi(G) = 1$. Assume that $x \in G$ is an MSM-nonvanishing element of G. Then x fixes an element in every G-orbit on $Irr(M)$.*

Proof. $1_M \neq \lambda \in \mathrm{Irr}(M)$ and $T = I_G(\lambda)$, where $I_G(\lambda)$ is the inertia group of λ in G. Since $\Phi(G) = 1$, there is a subgroup H of G such that $G = MH$ and $M \cap H = 1$. We know from Problem 6.18 in Isaacs (1976) that there exists a linear character $\theta \in \mathrm{Irr}(T)$ such that $\theta_M = \lambda$. Let $\chi = \theta^G$. Then χ is a faithful irreducible character of G. Otherwise, we would have that $M \leq \ker\chi = \cap_{g\in G}(\ker\theta)^g \leq \ker\theta$, which is a contradiction that $\theta_M = \lambda = 1$. On the other hand, it is clear that $\mathrm{Z}(G) = 1$ since $\Phi(G) = 1$. This implies that $\chi \in \mathrm{Irr}(G)$ is a monomial strongly monolithic character of G. Since $x \in G$ is an MSM-nonvanishing element of G, we get that $\chi(x) \neq 0$. By the definition of the induced character θ^G, there exists an element g of G such that $x^g \in T$. Then x stabilizes $\lambda^{g^{-1}}$, and the proof is complete.

Theorem 2.4. (Isaacs et all. 1999, Theorem 4.2) *Let G act faithfully and irreducibly on a finite vector space V. Let $x \in F(G)$ fix an element in each orbit of G on V. Then $x^2 = 1$.*

3. MAIN RESULTS

It is known that an irreducible character of a group G is called to be of q-defect zero if q does not divide $|G|/\chi(1)$, where q is a prime number. We know from Theorem 8.17 in Isaacs (1976) that if χ is an irreducible character of q-defect zero of G, then $\chi(g) = 0$ whenever q divides the order of $g \in G$.

Let $N \lhd G$ and $\chi \in \mathrm{Irr}(G)$ such that $N \leq \ker\chi$. It is well-known that there exists a one-to-one correspondence between irreducible characters of G/N and irreducible characters of G with kernel containing N. Thus, it is easy to see that χ is a strongly monolithic character of G if and only if χ is a strongly monolithic character of G/N. In the following theorem, we use the notation x^G to denote the conjugacy class of G containing $x \in G$.

Theorem 3.1. *Let G be a finite group. If the set of SM-vanishing elements of G are the union of at most three conjugacy classes of G, then G is solvable.*

Proof. Let G be a counterexample to the theorem with minimum possible order. Suppose that G has two distinct minimal normal subgroups M_1 and M_2. It is easy to see that the hypotheses of theorem are inherited by factor groups. Thus, both of G/M_1 and

G/M_2 are solvable groups by induction. Since G is isomorphic to a subgroup of $G/M_1 \times G/M_2$, we have a contradiction that G is solvable. This implies that G cannot have two distinct minimal normal subgroups. Now, let M be the unique minimal normal subgroup of G. Since G is a counterexample, M must be nonabelian and $Z(G) = 1$. Therefore, there exists a nonabelian simple group S such that $M = S_1 \times \cdots \times S_k$ where $k \geq 1$ and $S_i \cong S$ for every i. First assume that S has irreducible characters of q-defect zero for every prime q dividing the order of S. Thus, if θ is an irreducible character of q-defect zero of S, then $\psi := \theta \times \cdots \times \theta$ is an irreducible character of q-defect zero of M. It follows from Lemma 2.4 in Erkoç et all. (2023) that every element of M of order divisible by q is an SM-vanishing element of G. We know that $2 \| M|$, because M is a nonsolvable group. Also, there exist distinct primes p and q such that $p, q \geq 3$ and $p, q \in \pi(M)$. Hence, there exist x, y and z elements of S such that $|x| = 2$, $|y| = p$ and $|z| = q$. Since $x, y, z \in \mathrm{Van}_{\mathrm{sm}}(G)$ and the set of SM-vanishing elements of G are the union of at most three conjugacy classes of G, we get that $\pi(M) = \{2, p, q\}$ and $\mathrm{Van}_{\mathrm{sm}}(G) = x^G \cup y^G \cup z^G$. Then, M must be a simple group. Otherwise, we would have $k \geq 2$. Without loss of generality, we may assume that $y \in S_1$ and $z \in S_2$. Thus, we would have that $|yz| = pq$. But this contradicts with the hypothesis of theorem because $(yz)^G \notin \{x^G, y^G, z^G\}$ and $yz \in \mathrm{Van}_{\mathrm{sm}}(G)$. Since M is non-cyclic simple group of order divisible by exactly three primes, we obtain from Theorem 1 in Herzog (1968) that
$M \in \{PSL(2,5),\ PSL(2,8),\ PSL(2,17),\ PSL(2,7),\ PSL(2,9),\ PSL(3,3),\ U_3(3),\ U_4(2)\}$. Using the Atlas Conway et all. (1985), we obtain the following table containing $x_i \in M$ of distinct orders for $1 \leq i \leq 4$.

M	$\lvert x_1\rvert$	$\lvert x_2\rvert$	$\lvert x_3\rvert$	$\lvert x_4\rvert$
$PSL(2,7)$	$2A$	$3A$	$4A$	$7A$
$PSL(2,9)$	$2A$	$3A$	$4A$	$5A$
$PSL(2,8)$	$2A$	$3A$	$7A$	$9A$
$PSL(2,17)$	$2A$	$3A$	$4A$	$17A$
$PSL(3,3)$	$2A$	$3A$	$4A$	$13A$
$U_3(3)$	$2A$	$3A$	$4A$	$7A$
$U_4(2)$	$2A$	$3C$	$4A$	$5A$

Therefore, M cannot be groups in the list. Since $C_G(M) = 1$, we know that G is almost simple group. Therefore, we get that $G \cong A_5$ or $G \cong S_5$. But this is a contradiction because the set of SM-vanishing elements of A_5 or S_5 are union of more than three conjugacy classes of the group. Therefore, there exists a prime number q dividing the order of S such that S does not have any irreducible character of q-defect zero. It follows from Lemma 2.3 in Robati (2019) that there exist irreducible characters θ_1, θ_2, θ_3, θ_4 of S which extend to $\mathrm{Aut}(S)$ and elements x_1, x_2, x_3, x_4 of distinct order such that $\theta_i(x_i) = 0$ for $1 \leq i \leq 4$. Also, we have from Lemma 5 in Bianchi et all. (2007) that $\theta_i \times \cdots \times \theta_i \in \mathrm{Irr}(M)$ extends to G for $1 \leq i \leq 4$. Now, let $\psi_i \in \mathrm{Irr}(G)$ such that $(\psi_i)_M = \theta_i \times \cdots \times \theta_i$ for $1 \leq i \leq 4$. It is clear that ψ_i is a faithful irreducible character of G for $1 \leq i \leq 4$. Otherwise, we would have that $M \leq \ker\psi_i \cap M = \ker(\psi_i)_M = \ker(\theta_i \times \cdots \times \theta_i) = 1$ for $1 \leq i \leq 4$, which is a contradiction. Therefore, ψ_i is a strongly monolithic character of G and $\psi_i(x_i) = 0$ for $1 \leq i \leq 4$. Since the elements x_i are of distinct orders, x_i elements lie in distinct conjugacy classes of G for $1 \leq i \leq 4$ and so, the set of SM-vanishing elements of G are the union of at least four conjugacy classes of G, which is a contradiction. This contradiction completes the proof. □

Now, we consider the semidirect product $G := He_3 \rtimes C_2$ (SmallGroup $(54, 8)$ in GAP) where C_2 acts faithfully on He_3. The notations C_2 and He_3 denote a cyclic group of order 2 and a nonabelian group of order 27 of exponent 3, respectively. Since $1 < Z(G)$, all faithful irreducible characters of G are not strongly monolithic. G has only four strongly monolithic characters of degree 2. While the set of SM-vanishing elements of G are the union of three conjugacy classes of G, the set of vanishing elements of G are the union of seven conjugacy classes of G. Thus, Theorem 3.1 generalizes [Robati (2019), Theorem 2.8].

Theorem 3.2. *Let G be a solvable group and x be an element of odd order of G. If $\chi(x) \neq 0$ for all monomial strongly monolithic character χ of G, then $x \in F(G)$.*

Proof. Let G be a counterexample to the theorem with minimum possible order. By induction, $xN \in \mathrm{F}(G/N)$ for every nontrivial normal subgroup N of G because $2 \nmid |xN|$ and $\theta(xN) \neq 0$ for every monomial strongly monolithic character θ of G/N. Suppose that G has two distinct minimal normal subgroups M_1 and M_2. Then we know that $\varphi : G \longrightarrow G/M_1 \times G/M_2$, defined by $\varphi(g) = (gM_1, gM_2)$ for $g \in G$, is an injective homomorphism. Hence, we get that $\varphi(x) \in \mathrm{F}(G/M_1) \times \mathrm{F}(G/M_2) = \mathrm{F}(G/M_1 \times G/M_2)$ and so, $\varphi(x) \in \varphi(G) \cap \mathrm{F}(G/M_1 \times G/M_2) \leq \mathrm{F}(\varphi(G))$. Thus, we obtain that $x \in \mathrm{F}(G)$, which is a contradiction. This implies that G cannot have two distinct minimal normal subgroups. Let M be be the unique minimal normal subgroup of G. It is clear that $\Phi(G) = 1$ because $\mathrm{F}(G/\Phi(G)) = \mathrm{F}(G)/\Phi(G)$. It follows from Gaschütz Theorem (III, 4.5 in Huppert (1967)) that $\mathrm{F}(G) = M$ and so $C_G(M) = M$. Now, let V be the group of irreducible characters of M. Then, V is faithful and irreducible G/M-module. Also, we know from Lemma 2.3 that the element xM fixes some element of each orbit of G/M on V. On the other hand, we see that $xM \in \mathrm{F}(G/M)$ by the induction. Hence, we have from Teorem 2.4 that $(xM)^2 = x^2M = M$ and so, we obtain that $x^2 \in M$.

Therefore, we conlude that $x \in M = \mathrm{F}(G)$ because x is an element of odd order of G, which is a contradiction. This contradiction completes the proof. □

Let G be a finite group and $g \in G$. In Pang et all. (2016), authors prove that if the order of $gG' \in G/G'$ does not divide $|\mathrm{Irr}_m(G)|$, then there exists χ in $\mathrm{Irr}_m(G)$ such that $\chi(g) = 0$ where $\mathrm{Irr}_m(G)$ is the set of all irreducible monomial characters of G. Similarly, we give the following theorem.

Theorem 3.3. *Let G be a finite group, χ be a nonlinear irreducible character of G whose kernel is maximal among the kernels of all nonlinear irreducible characters of G and $g \in G$. If the order of gN in G/N does not divide $|Irr_{sm}(G/ker\chi)|$ where $N = G'ker\chi$, then g is an SM-vanishing element of G.*

Proof. Let χ be a nonlinear irreducible character of G whose kernel is maximal among the kernels of all nonlinear irreducible characters of G. We know from Corollary 2.6 in Erkoç et all. (2023) χ is a strongly monolithic character of $G/\ker\chi$. Furthermore for any linear character λ of $G/\ker\chi$, $\chi\lambda$ is a strongly monolithic character of $G/\ker\chi$. Hence, λ permutes $\mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi)$. We get that

$$\mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi) = \{\, \theta\lambda \mid \theta \in \mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi)\,\}.$$

This implies that

$$\prod_{\theta\in\mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi)} \theta(g) = \prod_{\theta\in\mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi)} (\theta\lambda)(g) = \left(\prod_{\theta\in\mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi)} \theta(g)\right)\lambda(g)^n,$$

where $n = |\mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi)|$. If g is an SM-nonvanishing element of G, then by the above equality, $\lambda(g)^n = 1$ for any linear character λ of $G/\ker\chi$. It follows that $g^n\ker\chi \in G'\ker\chi/\ker\chi$. Then, we have that $|gN|$ divides $|\mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi)|$, which contradicts with our hypothesis. This contradiction completes the proof. □

Theorem 3.4. *Let G be a solvable group and let p be a prime divisor of $|G|$. If $cod(\chi)$ is a p'-number for every monomial strongly monolithic character χ of G, then G has a normal p-complement.*

Proof. Let G be a counterexample to the assertion with the minimal possible order. Since the hypotheses of the theorem are inherited by factor groups, G has a unique minimal normal subgroup M. It follows that G/M has a normal p-complement by induction. Since G does not have a normal p-complement, p must divide $|M|$. Thus, M is elementary abelian p-subgroup. Furthermore, we have $\mathrm{Z}(G) = 1$. Otherwise, a Hall p'-subgroup H of G would be normal since $MH \trianglelefteq G$ and H is a characteristic subgroup of MH. Moreover, we have from Lemma 1 (a) in Berkovich and Zhmud' (1997) that $\Phi(G) = 1$. Then, there exists a subgroup K of G such that $G = MK$ and $M \cap K = 1$. Let λ be a nonprincipal character in $\mathrm{Irr}(M)$. Write $T = \mathrm{I}_G(\lambda)$ as the inertia group of λ in G. Notice that M is complemented in G and so is in T. We get that $T = M\mathrm{I}_K(\lambda)$. It follows from Problem 6.18 in Isaacs (1976) that λ extends to T and so there exists a linear character $\theta \in \mathrm{Irr}(T)$ such that $\theta_M = \lambda$. This implies that $\chi = \theta^G$ is a monomial irreducible character of G. Thus, χ is a faithful irreducible character of G. Otherwise, we get that $M \leq \ker\chi = \bigcap_{g\in G} (\ker\theta)^g \leq \ker\theta$. But this contradicts with $\theta_M = \lambda \neq 1$. Hence χ is a monomial strongly monolithic character of G, since $\mathrm{Z}(G) = 1$. By the assumption, we have that

$$\mathrm{cod}(\chi) = \frac{|G : \ker\chi|}{\chi(1)} = \frac{|G|}{\theta^G(1)} = \frac{|G|}{|G : T|} = |T| = |M|.|\mathrm{I}_K(\lambda)|$$

is a p'-number. This contradicts with the fact that M is a p-group. The proof is complete. □

Peer Review: Externally peer-reviewed.
Conflict of Interest: Authors declared no conflict of interest.
Financial Disclosure: The work of the authors was supported by the Scientific and Technological Research Council of Türkiye (TÜBİTAK). The project number is 123F260.

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Istanbul Journal of Mathematics
ijmath 2 (1), 44–54, 2024
DOI: 10.26650/ijmath.2024.00015

RESEARCH ARTICLE

Approximating Higher Order Linear Fredholm Integro-Differential Equations by an Efficient Adomian Decomposition Method

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ABSTRACT
This work presents a unique technique for the precise and efficient solution of Linear Fredholm integro-differential equations (LFDEs), the technique is based on the Modification of Adomian Decomposition Method (MADM). The MADM extends the well-known Adomian Decomposition Method (ADM) by integrating novel changes that improve convergence and computing efficiency. The LFDEs are essential for simulating a wide range of phenomena in science and engineering. Because their analytical solutions are frequently difficult to achieve, the development of efficient and trustworthy numerical approaches is required. We present an introduction of the MADM method and its important characteristics emphasizing its capacity to handle a wide range of LFDEs seen in scientific and engineering applications. We demonstrate the method's usefulness in contrast to the true approach, stressing its computational benefits and precision.

Mathematics Subject Classification (2020): 65R20, 45G15, 45B99, 45D99

Keywords: Fredholm Integro, differential Equations, Numerical Solutions, Computational Efficiency.

1. INTRODUCTION

Linear Fredholm integro-differential equations (IFDEs) are a type of mathematical model used to describe complicated events including both differential and integral elements in a variety of scientific and engineering areas. These equations are critical for understanding and forecasting real-world phenomena including heat conduction, diffusion, population dynamics, and electromagnetic fields. Regardless of their importance, analytical solutions for linear Fredholm IFDEs are frequently elusive, necessitating the development of strong numerical approaches. The Modified Adomian Decomposition Method builds upon the strengths of the original ADM while incorporating innovative adjustments to overcome limitations in convergence behavior and stability. The method involves decomposing the unknown function into a series of auxiliary functions and using a recursive scheme to obtain successive approximations. According to Abdella and Ross (2020); Acar and Dascioglu (2019); Akyuz (2006); Amin et all. (2020), integral equations are categorized into two primary categories based on the limits of integration: Fredholm and Volterra integral equations. Ayinde, James, Ishaq and Oyedepo (2022); Bogdan and Madalina (2021); Buranay, Ozarslan and Falahesar (2021), integro-differential equations are essential in both pure and practical mathematics, having numerous applications in mechanics, engineering, physics, and other fields. The behavior and evolution of many physical systems in many fields of science and engineering, including viscoelasticity, evolutionary problems, fluid dynamics, population dynamics, and many others, can be successfully modeled using Fredholm and Volterra type integrodifferential equations. Davaeeifar and Rashidainia (2017); El-Hawary and El-Sheshtawy (2010); Hosry, Nakad and Bhalekar (2020); Lofti and Alipanah (2020); Kabiru et all. (2023); Kamoh, Gyemang and Soomiyol (2019); Kurkou, Aslan, and Sezer (2017); Kabiru, Morufu and Muideen (2023); Maturi and Simbawa (2020) derived the classical operational matrices and the unknown to be approximated by First Boubaker Polynomials, with Newton-Cotes points serving as collocation points. Ming and Huanga (2017); Mishra et all. (2017) examine the existence, uniqueness, and regularity features of solutions to generic Volterra functional integral equations with non-vanishing delalys, focusing on the local representation. Ogunniran et all. (2022) developed a discrete hybrid block approach and used relevant existing concepts to test its stability, consistency, and convergence. Ogunrinde, Obayomi and Olayemi (2023); Ogunrinde et all. (2020) discussed how the Fredholm integro-differential equation has numerous applications in science, engineering, and all aspects of

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Submitted: 04.02.2024 • **Last Revision Received:** 06.06.2024 • **Accepted:** 10.06.2024

human endeavors, including kinetic theory of gases, geophysics, communication theory, mathematical economics, queuing theory, and hereditary phenomena in physics and biology. Oyedepo et all. (2023, 2022); Ramadan et all. (2016); Sabzevari (2019) proposed a collocation computing approach for solving Volterra-Fredholm integro-differential equations using fourth kind chebyshev polynomials as basis functions. According to Scathar et all. (2020); Shang and Han (2010), integral equations have applications in a variety of domains, including mathematics, physics, and engineering. The analytical solution of integral equations is quite complex, especially for application applications. Tunc (2021) investigated a linear system of integro-delay differential equations with constant time retardation. Wazwaz (2011); Yuksel et all. (2012) defined an integral equation as one in which the unknown function occurs within an integral sign.

2. DEFINITION OF TERMS

Integro-Differential Equations (IDEs): Integro-differential equations are a form of mathematical problem in which the derivatives and integrals of an unknown function are both involved. Integral terms alter the connection between a function and its rate of change in these equations, which are used to simulate a wide range of phenomena in numerous scientific and technical fields.
Fredholm Integro-Differential Equations (FIDEs): Fredholm integro-differential equations are a type of integro-differential equation that involves the use of derivatives and integrals in a mathematical formulation. These equations are named after Erik Ivar Fredholm, a Swedish mathematician who made substantial contributions to integral equations.
Linear Fredholm Integro-Differential Equations (LFIDEs): Linear Fredholm integro-differential equations are a type of integro-differential equation in which the dependent variable and its derivatives are linear. These equations, which combine differential and integral operators in a linear framework, are critical in describing a wide range of phenomena in numerous scientific and engineering fields.
Adomian Decomposition Method (ADM): The Adomian Decomposition Method is a strong analytical approach for solving nonlinear ordinary, partial differential, and integral problems. This approach, named after its originator, George Adomian, seeks approximate solutions by decomposing a given nonlinear differential equation into an endless sequence of smaller terms that may then be solved systematically.
Modified Adomian Decomposition Method (MADM): The Modified Adomian Decomposition Method improves and modifies the original Adomian Decomposition Method (ADM). It is intended to overcome some constraints and improve the ADM's convergence behavior when used to specific sorts of problems. The MADM modifies the algorithm in order to improve its efficiency and reliability for solving nonlinear ordinary and partial differential equations, as well as integral equations.

3. METHOD

To improve on the accuracies and subsequently the convergence of these approaches, we shall based our assumption on the decomposition of the source term $h(x)$ in Taylors series of the form

$$s(x) = \sum_{j=0}^{+\infty} h_i(x) \tag{1}$$

and the new recursive relation obtained as:

$$y_0(x) = k_0(x), \tag{2}$$

$$y_1(x) = k_1(x) + k_2(x) + \lambda \int_a^x h(x,t)\,(L(u_0(x)) + A_0))\,dt, \tag{3}$$

$$y_2(x) = k_3(x) + k_4(x) + \lambda \int_a^x \mathrm{h}(x,t)\,(L(u_1(x)) + A_1))\,dt, \tag{4}$$

$$\vdots$$

$$y_{j+1}(x) = k_{2(j+1)}(x) + k_{2(j+1)-1}(x) + \lambda \int_a^x \mathrm{h}(x,t)\left(L(u_j(x)) + A_\mathrm{j})\right)dt. \tag{5}$$

And subsequently the function $u(x)$ is obtained as

$$y(x) = \sum_{j=0}^{+\infty} y_j(x). \tag{6}$$

Assuming that the nonlinear function is $P(y(x))$ therefore, below are few of the Adomian polynomials.

$$A{=}P(y_0), \tag{7}$$

$$A_1{=}y_1P'(y_0), \tag{8}$$

$$A_2{=}y_2P'(y_0)+\frac{1}{2!}y_1^2P''(y_0) \tag{9}$$

$$A_3{=}y_3P'(y_0)+y_1y_2P''(y_0)+\frac{1}{3!}y_1^3P'''(y_0), \tag{10}$$

$$A_4{=}y_4P'(y_0)+\left(\frac{1}{2!}y_2^2+y_1y_3\right)P''(y_0)+\frac{1}{2!}y_1^2y_2P'''(y_0)+\frac{1}{4}y_1^4P^{(iv)}(y_0). \tag{11}$$

Two important observations can be made here. First, A_0 depends only on y_0, A_1 depends only on y_0 and y_1, A_2 depends on y_0, y_1 and y_2, and so on. Secondly, substituting these $A_{j's}$ gives:

$$P(u)=A_0+A_1+A_2+A_3+\cdots$$

$$=P(y_0)+(y_1+y_2+y_3+\cdots)P'(y_0)+\frac{1}{2!}(y_1^2+2y_1y_2+2y_1y_3+y_2^2)P''(y_0)$$

$$+\frac{1}{3!}(y_1^3+3y_1^2y_3+6y_1y_2y_3+\cdots)P'''(y_0)+\cdots$$

$$=P(y_0)+(y-y_0)P'(y_0)+\frac{1}{2!}(y-y_0)^2P''(y_0)+\cdots$$

4. NUMERICAL EXAMPLES

Example 1: Bogdan and Madalina (2021) Consider the eighth-order linear Fredholm integro-differential equation

$$y^{(8)}(x)=y(x)-8e^x+x^2+\int_0^1 x^2y'(t)\,dt \tag{12}$$

Subject to the conditions $y(0)=1$, $y'(0)=0$, $y''(0)=-1$, $y'''(0)=-2$, $y^{(4)}(0)=-3$, $y^{(5)}(0)=-4$, $y^{(6)}(0)=-5$, and $y^{(7)}(0)=-6$.
The exact solution is $y(x)=(1-x)e^x$
Using the new Modified Adomian Decomposition Method (MADM),
We transform each term in (12) to have the following

$$\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x y^{(8)}(x)\,dxdxdxdxdxdxdxdx$$

$$=y(x)+\frac{1}{840}x^7+\frac{1}{144}x^6+\frac{1}{30}x^5+\frac{1}{8}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2-1 \tag{13}$$

$$\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x y(x)\,dxdxdxdxdxdxdxdx=$$

$$\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x y(t)\,dtdxdxdxdxdxdxdx \tag{14}$$

$$\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x 8e^x\,dxdxdxdxdxdxdxdx$$

$$=-8-8x-4x^2-\frac{4}{3}x^3-\frac{1}{3}x^4-\frac{1}{15}x^5-\frac{1}{90}x^6-\frac{1}{630}x^7+8e^x \tag{15}$$

$$\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x x^2\,dxdxdxdxdxdxdxdx=\frac{1}{1814400}x^{10} \tag{16}$$

$$\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^1 x^2y'(t)\,dtdxdxdxdxdxdxdxdx =$$

$$\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^1 x^2y'(t)\,dtdxdxdxdxdxdxdxdx$$

Substitute the results of (13) – (4) into (12)
We have

$$y(x) = 9+8x+\tfrac{7}{2}x^2+x^3+\tfrac{5}{24}x^4+\tfrac{1}{30}x^5+\tfrac{1}{240}x^6+\tfrac{1}{2520}x^7+\tfrac{1}{1814400}x^{10}-8e^x$$
$$+\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x y(t)\,dtdxdxdxdxdxdxdx$$

$$+\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^1 x^2y'(t)\,dtdxdxdxdxdxdxdxdx \quad (17)$$

Let

$$r = 9+8x+\frac{7}{2}x^2+x^3+\frac{5}{24}x^4+\frac{1}{30}x^5+\frac{1}{240}x^6+\frac{1}{2520}x^7+\frac{1}{1814400}x^{10}-8e^x$$

Then
Expand taylor (r, x, 10)

$$= 1-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{8}x^4-\frac{1}{30}x^5-\frac{1}{144}x^6-\frac{1}{840}x^7-\frac{1}{5040}x^8-\frac{1}{45360}x^9$$

And

$$a_0 = 1$$

$$y_0(t) = 1$$

$$y_0'(t) = 0$$

$$g_0 = -\frac{1}{2}x^2-\frac{1}{3}x^3$$

$$a_1 = g_0+\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x y_0(t)\,dtdxdxdxdxdxdxdx$$

$$+\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^1 x^2y_0'(t)\,dtdxdxdxdxdxdxdxdx \quad (18)$$

$$a_1 = -\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{1}{40320}x^8$$

$$y_1(t) = -\frac{1}{2}t^2-\frac{1}{3}t^3+\frac{1}{40320}t^8$$

$$y_1'(t) = -t-t^2+\frac{1}{5040}t^7$$

$$g_1 = -\frac{1}{8}x^4-\frac{1}{30}x^5$$

$$a_2 = g_1+\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x y_1(t)\,dtdxdxdxdxdxdxdx$$

$$+\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^1 x^2y_1'(t)\,dtdxdxdxdxdxdxdxdx \quad (19)$$

$$a_2 = -\frac{1}{8}x^4 - \frac{1}{30}x^5 - \frac{53759}{73156608000}x^{10} - \frac{1}{19958400}x^{11} + \frac{1}{20922789888000}x^{16}$$

$$y_2(t) = -\frac{1}{8}t^4 - \frac{1}{30}t^5 - \frac{53759}{73156608000}t^{10} - \frac{1}{19958400}t^{11} + \frac{1}{20922789888000}t^{16}$$

$$y_2'(t) = -\frac{1}{2}t^3 - \frac{1}{6}t^4 - \frac{53759}{7315660800}t^9 - \frac{1}{1814400}t^{10} + \frac{1}{1307674368000}t^{15}$$

$$g_2 = -\frac{1}{144}x^6 - \frac{1}{840}x^7$$

$$\begin{aligned} a_3 &= g_2 + \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x y_2(t)\,dtdxdxdxdxdxdxdx \\ &+ \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^1 x^2 y_2'(t)\,dtdxdxdxdxdxdxdxdx \end{aligned} \tag{20}$$

$$a_3 = -\tfrac{1}{144}x^6 - \tfrac{1}{840}x^7 - \tfrac{1}{159667200}x^{12} - \tfrac{1}{1556755200}x^{13} - \tfrac{53759}{1290718539074764800000}x^{18} - \tfrac{1}{60822550204416000}x^{19} + \tfrac{1}{620448401733239439360000}x^{24} - \tfrac{473255926999}{542318713896960000}x^{10}$$

$$y_3(t) = -\tfrac{1}{144}t^6 - \tfrac{1}{840}t^7 - \tfrac{1}{159667200}t^{12} - \tfrac{1}{1556755200}t^{13} - \tfrac{53759}{1290718539074764800000}t^{18} - \tfrac{1}{60822550204416000}t^{19} + \tfrac{1}{620448401733239439360000}t^{24} - \tfrac{473255926999}{542318713896960000}t^{10}$$

$$y_3'(t) = -\tfrac{1}{24}t^5 - \tfrac{1}{120}t^6 - \tfrac{1}{13305600}t^{11} - \tfrac{1}{119750400}t^{12} - \tfrac{53759}{71706585504153600 0}t^{17} - \tfrac{1}{3201186852864000}t^{18} + \tfrac{1}{25852016738884976640000}t^{23} - \tfrac{473255926999}{54231871389696000}t^0$$

$$g_3 = -\frac{1}{5040}x^8 - \frac{1}{45360}x^9$$

$$\begin{aligned} a_4 &= g_3 + \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x y_3(t)\,dtdxdxdxdxdxdxdx \\ &+ \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^1 x^2 y_3'(t)\,dtdxdxdxdxdxdxdxdx \end{aligned} \tag{21}$$

$$a_4 = -\tfrac{1}{5040}x^8 - \tfrac{1}{45360}x^9 - \tfrac{1}{17435658240}x^{14} - \tfrac{1}{217945728000}x^{15} - \tfrac{1}{810967336058880000}x^{20} - \tfrac{1}{12772735542927360000}x^{21} - \tfrac{53759}{813035585631236961337344 0000000}x^{26} - \tfrac{1}{5444434725209176080384000000}x^{27} + \tfrac{1}{263130836933693530167218012160000000}x^{32} - \tfrac{473255926999}{9568251416385920434176000000}x^{18} - \tfrac{25236784671012694969327}{5628707900523948193873920000000}x^{10}$$

$$\begin{aligned} y_4(t) = &-\tfrac{1}{5040}t^8 - \tfrac{1}{45360}t^9 - \tfrac{1}{17435658240}t^{14} - \tfrac{1}{217945728000}t^{15} - \tfrac{1}{810967336058880000}t^{20} \\ &- \tfrac{1}{12772735542927360000}t^{21} - \tfrac{53759}{8130355856312369613373440000000}t^{26} \\ &- \tfrac{1}{5444434725209176080384000000}t^{27} + \tfrac{1}{263130836933693530167218012160000000}t^{32} \\ &- \tfrac{473255926999}{9568251416385920434176000000}t^{18} - \tfrac{25236784671012694969327}{5628707900523948193873920000000}t^{10} \end{aligned} \tag{22}$$

$$y_n(x) = \sum_{j=0}^{4} y_j(x)$$

$$\begin{aligned} y_n(x) = 1 &- \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{8}x^4 - \tfrac{1}{30}x^5 - \tfrac{1}{144}x^6 - \tfrac{1}{840}x^7 - \tfrac{1}{5760}x^8 - \tfrac{1}{45360}x^9 \\ &- \tfrac{4652673275517556384854127}{5628707900523948193873920000000}x^{10} - \tfrac{1}{19958400}x^{11} - \tfrac{1}{159667200}x^{12} - \tfrac{1}{1556755200}x^{13} \\ &- \tfrac{1}{17435658240}x^{14} - \tfrac{1}{217945728000}x^{15} + \tfrac{1}{20922789888000}x^{16} - \tfrac{4458475107799}{9568251416385920434176000000}x^{18} \\ &- \tfrac{1}{60822550204416000}x^{19} - \tfrac{1}{810967336058880000}x^{20} - \tfrac{1}{12772735542927360000}x^{21} \\ &+ \tfrac{1}{620448401733239439360000}x^{24} - \tfrac{53759}{8130355856312369613373440000000}x^{26} \\ &- \tfrac{1}{5444434725209176080384000000}x^{27} + \tfrac{1}{263130836933693530167218012160000000}x^{32} \end{aligned} \tag{23}$$

Example 2: Oyedepo et all. (2023) Consider the fifth-order linear Fredholm integro-differential equation

$$y^{v}(x) - x^2 y'''(x) - y'(x) - wy(x) = w^2\cos(x) - x\sin(x) + \int_{-1}^{1} y(t)\,dt \tag{24}$$

Subject to the conditions $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = -1$ and $y^{iv}(0) = -1$.
The exact solution is $y(x) = \sin(x)$
Using the new Modified Adomian Decomposition Method (MADM),
We obtained the following

$$y_0(x) = x \tag{25}$$

$$y_1(x) = -\frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}wx^6 \tag{26}$$

$$\begin{aligned} y_2(x) = {} & \tfrac{1}{120}w^2x^5 + \left(-\tfrac{1}{5040}w^2 - \tfrac{1}{2520}\right)x^7 + \tfrac{1}{113400}wx^{10} + \tfrac{1}{17280}x^9 - \tfrac{13}{40320}x^8 - \tfrac{1}{720}x^7 \\ & + \tfrac{1}{39916800}w^2x^{11} - \tfrac{1}{362880}wx^9 - \tfrac{1}{40320}wx^8 + \tfrac{1}{5}\left(-\tfrac{1}{1440} + \tfrac{1}{60480}w\right)x^5 \end{aligned} \tag{27}$$

Then,

$$y_n(x) = \sum_{j=0}^{2} y_j(x)$$

$$\begin{aligned} y_n(x) = {} & x - \tfrac{1}{6}x^3 - \tfrac{1}{24}x^4 + \tfrac{1}{120}x^5 + \tfrac{1}{720}wx^6 + \tfrac{1}{120}w^2x^5 + \left(-\tfrac{1}{5040}w^2 - \tfrac{1}{2520}\right)x^7 + \tfrac{1}{113400}wx^{10} \\ & + \tfrac{1}{17280}x^9 - \tfrac{13}{40320}x^8 - \tfrac{1}{720}x^7 + \tfrac{1}{39916800}w^2x^{11} - \tfrac{1}{362880}wx^9 - \tfrac{1}{40320}wx^8 + \tfrac{1}{5}\left(-\tfrac{1}{1440} + \tfrac{1}{60480}w\right)x^5 \end{aligned} \tag{28}$$

When $w = 0$,
We have

$$y_n(x) = x - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{59}{7200}x^5 - \frac{1}{560}x^7 - \frac{13}{40320}x^8 + \frac{1}{17280}x^9 \tag{29}$$

Example 3: Ogunrinde, Obayomi and Olayemi (2023) Consider the third-order linear Fredholm integro-differential equation

$$y'''(x) = 6 + x - \int_{0}^{1} xy'(t)\,dt \tag{30}$$

Subject to the conditions $y(0) = -1$, $y'(0) = 1$, and $y''(0) = -2$.
The exact solution is $y(x) = x^3 - x^2 + x - 1$
Using the new Modified Adomian Decomposition Method (MADM),
We obtained the following

$$y_0(x) = -1 \tag{31}$$

$$y_1(x) = x - x^2 \tag{32}$$

$$y_2(x) = x^3 + \frac{1}{8}x^4 \tag{33}$$

$$y_3(x) = -\frac{7}{48}x^4 \tag{34}$$

Then,

$$y_n(x) = \sum_{j=0}^{3} y_j(x)$$

$$y_n(x) = -1 + x - x^2 + x^3 - \frac{1}{48}x^4 \tag{35}$$

5. TABLES OF RESULTS

Table 1: Numerical Results for Example 1

x	Exact	MADM	MADM_Error
0	1.000000	1.000000	0.00 E 00
0.1	0.994654	0.994654	0.00 E 00
0.2	0.977122	0.977122	1.73 E-13
0.3	0.944901	0.944901	1.01 E-11
0.4	0.895095	0.895095	1.82 E-10
0.5	0.824361	0.824361	1.72 E-09
0.6	0.728848	0.728848	1.08 E-08
0.7	0.604126	0.604126	5.09 E-08
0.8	0.445108	0.445108	1.96 E-07
0.9	0.245960	0.245961	6.45 E-07
1	0.000000	1.87 E-06	1.87 E-06

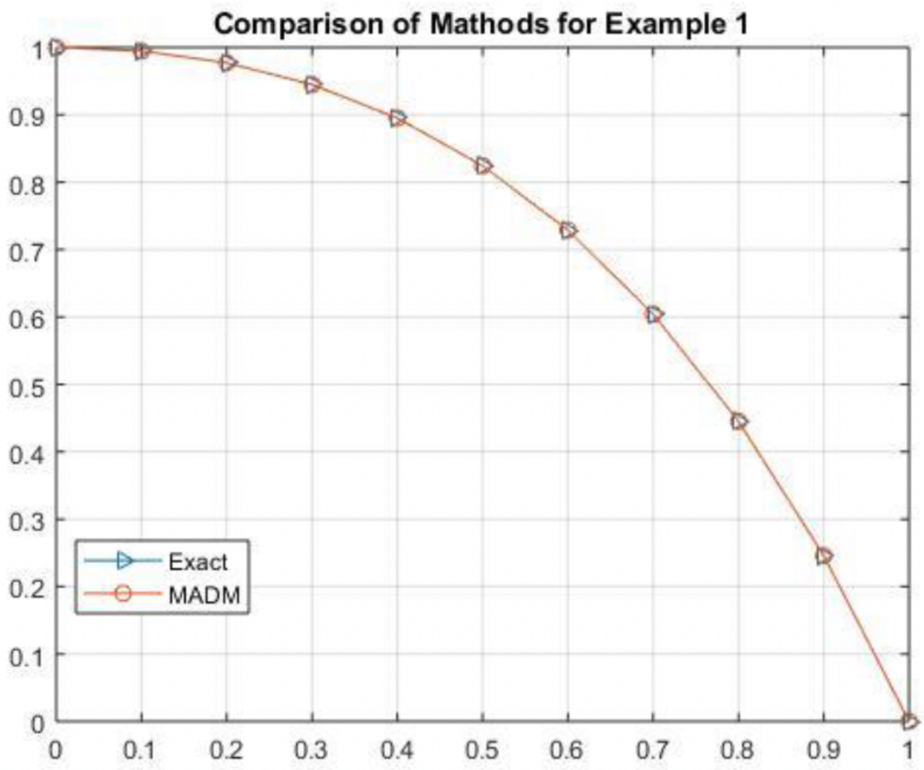

Figure 1. Graph of Comparison for Example 1

Table 2: Numerical Results for Example 2

x	Exact	MADM	MADM_Error
0	0.000000	0.000000	0.0000 E 00
0.1	0.099833	0.099829	4.1700 E-06
0.2	0.198669	0.198603	6.6700 E-05
0.3	0.295520	0.295183	3.3800 E-04
0.4	0.389418	0.388350	1.0680 E-03
0.5	0.479426	0.476812	2.6130 E-03
0.6	0.564642	0.559204	5.4380 E-03
0.7	0.644218	0.634090	1.0128 E-02
0.8	0.717356	0.699955	1.7401 E-02
0.9	0.783327	0.755195	2.8132 E-02
1	0.841471	0.798089	4.3382 E-02

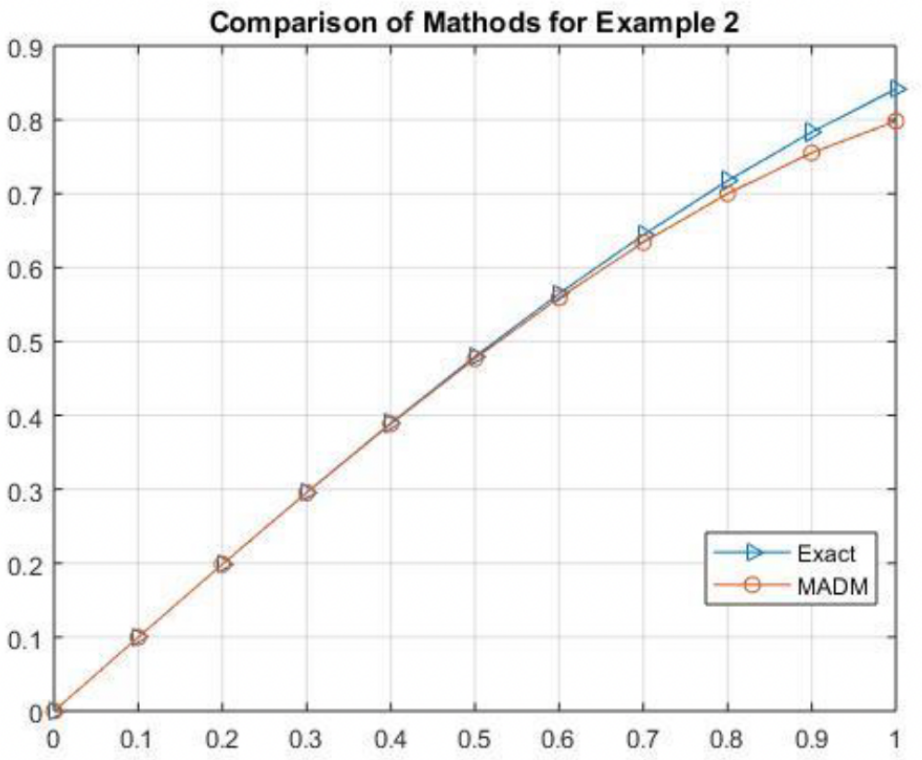

Figure 2. Graph of Comparison for Example 2

Table 3: Numerical Results for Example 3

X	Exact	MADM	MADM_Error
0	-1.000	-1.00000	0.0000 E 00
0.1	-0.909	-0.90900	2.0800 E-06
0.2	-0.832	-0.83203	3.3300 E-05
0.3	-0.763	-0.76317	1.6900 E-04
0.4	-0.696	-0.69653	5.3300 E-04
0.5	-0.625	-0.62630	1.3020 E-03
0.6	-0.544	-0.54670	2.7000 E-03
0.7	-0.447	-0.45200	5.0020 E-03
0.8	-0.328	-0.33653	8.5330 E-03
0.9	-0.181	-0.19467	1.3669 E-02
1	0.000	-0.02083	2.0833 E-02

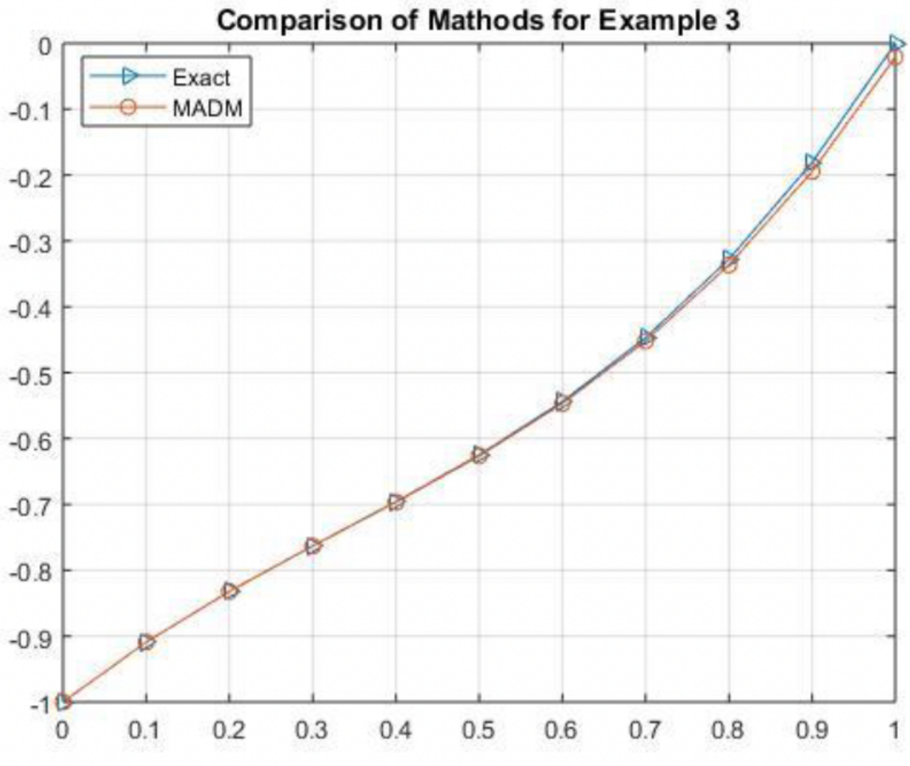

Figure 3. Graph of Comparison for Example 3

6. DISCUSSION OF RESULTS

The study of LFIDE findings using the Modified Adomian Decomposition Method demonstrates its efficacy in resolving problems associated with older methodologies. The method's improved convergence, stability, and adaptability make it an appealing tool for academics and practitioners working on linear Fredholm integro-differential equation issues. The findings given here add to the expanding body of knowledge on appropriate numerical strategies for solving complicated mathematical models in a variety of scientific and engineering disciplines.

7. CONCLUSION

The use of MADM to LFIDEs yields encouraging results, with enhanced accuracy and stability over standard approaches. The method's capacity to manage a wide spectrum of LFIDEs seen in scientific and engineering applications is emphasized, highlighting its adaptability and dependability. This study's numerical studies and comparisons give solid proof of the MADM's effectiveness. The convergence evaluations validate the method's resilience in addressing LFIDEs, giving it a viable option for academics and practitioners looking for accurate and efficient solutions to complicated issues in a variety of domains.

Peer Review: Externally peer-reviewed.
Conflict of Interest: Authors declared no conflict of interest.
Financial Disclosure: Authors declared no financial support.

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