CONSTRUCTIVE MATHEMATICAL ANALYSIS

Volume VII

Issue III

CMA
CONSTRUCTIVE MATHEMATICAL ANALYSIS

ISSN 2651-2939

https://dergipark.org.tr/en/pub/cma

VOLUME VII ISSUE III
ISSN 2651-2939

September 2024
https://dergipark.org.tr/en/pub/cma

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CONSTRUCTIVE MATHEMATICAL ANALYSIS
7 (2024), No. 3, pp. 90-97
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

The relationship between modular metrics and fuzzy metrics revisited

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ABSTRACT. In a famous article published in 1975, Kramosil and Michálek introduced a notion of fuzzy metric that was the origin of numerous researches and publications in several frameworks and fields. In 2010, Chistyakov introduced and discussed in detail the concept of modular metric. Since then, some authors have investigated the problem of establishing connections between the notions of fuzzy metric and modular metric, obtaining positive partial solutions. In this paper, we are interested in determining the precise relationship between these two concepts. To achieve this goal, we examine a proof, based on the use of uniformities, of the important result that the topology induced by a fuzzy metric is metrizable. As a consequence of that analysis, we introduce the notion of a weak fuzzy metric and show that every weak fuzzy metric, with continuous t-norm the minimum t-norm, generates a modular metric and, conversely, we show that every modular metric generates a weak fuzzy metric, with continuous t-norm the product t-norm. It follows that every modular metric can be generated from a suitable weak fuzzy metric, and that several examples and properties of modular metrics can be directly deduced from those previously obtained in the field of fuzzy metrics.

Keywords: fuzzy metric, weak fuzzy metric, modular metric

2020 Mathematics Subject Classification: 54E35, 54A40.

1. INTRODUCTION

With the aim in offering a fuzzy approach of statistical metric spaces and Menger spaces, Kramosil and Michálek introduced in [13] the fruitful notions of fuzzy metric and fuzzy metric space. Modifications of these concepts were proposed by Grabiec [8], and George and Veeramani [6]. These structures have been extensively explored both from the point of view of their topological and metric properties, as well as the development of a fixed point theory for them, and their application to various fields. There are obviously numerous relevant publications on fuzzy metric spaces and related structures. In order not to make the reference section too long, we will limit ourselves to cite the books [3, 10] and the very recent [7] joint with the references therein.

On the other hand and partly motivated by the studies about modulars on vector spaces ([14, 15, 16, 17]) Chistyakov introduced and discussed in [4] (see also [5]) the concepts of modular metric and modular metric space. Looking at Chistyakov's definition, it can be well intuited that there is a strong connection between modular metrics and fuzzy metrics. In fact, some authors have explored such a connection when working in the construction of a fixed point theory for modular metric spaces, obtaining various positive partial solutions (see, e.g., [11, 20]). In this paper, we are interested in determining the precise relationship between these

Received: 17.06.2024; Accepted: 07.08.2024; Published Online: 09.08.2024
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DOI: 10.33205/cma.1502096

two concepts. To achieve this goal, we examine a proof, based on the use of uniformities, of the important result that the topology induced by a fuzzy metric is metrizable. As a consequence of that analysis, we introduce the notion of a weak fuzzy metric and show that every weak fuzzy metric, with continuous t-norm the minimum t-norm, generates a modular metric such that the induced topologies agree; and, conversely, we show that every modular metric generates a weak fuzzy metric, with continuous t-norm the product t-norm, such that the induced topologies agree. It follows that every modular metric can be generated from a suitable weak fuzzy metric, and that several examples and properties of modular metrics can be directly deduced from those previously obtained in the field of fuzzy metrics.

2. Remarks on the notion of fuzzy metric

First, we emphasize that our notation and terminology will be standard. By $\mathbb{R}^+$, we design the set of non-negative real numbers and by $\mathbb{N}$ the set of natural numbers.

Now, we recall the notions of fuzzy metric and fuzzy metric space in the aforementioned senses. To this end, the following well-known concept will play a fundamental role.

Definition 2.1. *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is said to be a continuous triangular norm (continuous t-norm in short) if* $([0,1], *)$ *is a topological Abelian monoid with neutral 1 such that* $a * c \leq b * c$ *if* $a \leq b$*, with* $a, b, c \in [0,1]$.

As distinguished examples of continuous t-norm that will be used throughout this paper, we have the minimum t-norm $\wedge$, the product t-norm $*_P$ and the Łukasiewicz t-norm $*_L$, which are defined as follows: $a \wedge b = \min\{a, b\}$, $a *_P b = ab$, and $a *_L b = \max\{a + b - 1, 0\}$, for all $a, b \in [0,1]$. Recall that $\wedge \geq *_P \geq *_L$. In fact, $\wedge \geq *$ for any continuous t-norm $*$.

The books [10, 12] provide suitable sources to the study of continuous t-norms.

Now, consider the following axioms for a set X, a fuzzy set M in $X \times X \times \mathbb{R}^+$, and $x, y, z \in X$:

(KM1) $M(x, y, 0) = 0$;
(GV1) $M(x, y, t) > 0$ for all $t > 0$;
(KM2) $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$;
(GV2) $M(x, x, t) = 1$ for all $t > 0$, and $M(x, y, t) < 1$ whenever $y \neq x$;
(KM3) $M(x, y, t) = M(y, x, t)$ for all $t > 0$;
(KM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s > 0$;
(KM5) the function $t \to M(x, y, t)$ is left continuous on $\mathbb{R}^+$;
(GV5) the function $t \to M(x, y, t)$ is continuous on $\mathbb{R}^+$;
(KM6) $\lim_{t \to \infty} M(x, y, t) = 1$.

The triple $(X, M, *)$ is a fuzzy metric space in the sense of Kramosil and Michálek [13] provided that axioms (KM1), (KM2), (KM3), (KM4), (KM5) and (KM6) are fulfilled. In that case, we will say that the pair $(M, *)$, or simply M, is a KM-fuzzy metric.

In [8], Grabiec removed axiom (KM6) in the definition of fuzzy metric space because it was not necessary for his research about fixed point theory on fuzzy metric spaces. Then, a fuzzy metric $(M, *)$, or simply M, in Grabiec's sense will be called a Gr-fuzzy metric.

Later, George and Veeramani [6] defined a fuzzy metric space as a triple $(X, M, *)$, where X is a set, M is a fuzzy set in $X \times X \times (0, \infty)$ and $*$ is a continuous t-norm such that axioms (GV1), (GV2), (KM3), (KM4) and (GV5) are fulfilled. In that case, we will say that the pair $(M, *)$, or simply M, is a GV-fuzzy metric.

Note that every GV-fuzzy metric $(M, *)$ can be considered as a Gr-fuzzy metric simply defining $M(x, y, 0) = 0$ for all $x, y \in X$.

In [8, Lemma 4], Grabiec stated assertion (1) of Lemma 2.1 below in the framework of Gr-fuzzy metrics.

Lemma 2.1. *Let X be a set, M be a fuzzy set in $X \times X \times \mathbb{R}^+$ and $*$ be a continuous t-norm for which axioms (KM1), (KM2) and (KM4) are fulfilled. Then, for each $x, y \in X$, we get*

(1) *The function $t \to M(x, y, t)$ is nondecreasing on $\mathbb{R}^+$,*
(2) *$x = y$ if and only if $M(x, y, t) \geq 1 - t$ for all $t > 0$.*

Proof.

(1) Fix $x, y \in X$. Let $s, t \in \mathbb{R}^+$ such that $0 \leq s < t$. If $s = 0$, we get $M(x, y, s) = 0$ by (KM1). If $s > 0$ we get $M(x, y, t) \geq M(x, x, t-s) * M(x, y, s) = M(x, y, s)$ by (KM2) and (KM4).
(2) Suppose that $x = y$. Then, $M(x, y, t) \geq 1 - t$ for all $t > 0$ by (KM2). Conversely, suppose that $x \neq y$. By (KM2), there are $t > 0$ and $\delta \in (0, 1)$ such that $M(x, y, t) < 1 - \delta$. It follows from hypothesis and assertion (1) that, for any $s \in (0, t)$, $1 - s \leq M(x, y, s) \leq M(x, y, t) < 1 - \delta$, a contradiction.

□

Every Gr-fuzzy metric M on a set X induces in a natural way a topology $\mathfrak{T}_M$ on X. We show that axioms (KM1), (KM2) and (KM4) are sufficient to construct such a topology.

Lemma 2.2. *Let X be a set, M be a fuzzy set in $X \times X \times \mathbb{R}^+$ and $*$ be a continuous t-norm for which axioms (KM1), (KM2) and (KM4) are fulfilled. For each $x \in X, \varepsilon \in (0, 1)$ and $t > 0$, set $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$. Then, the family*

$$\mathfrak{T}_M = \{A \subseteq X : \text{ for each } x \in A \text{ there is } \varepsilon \in (0,1) \text{ and } t > 0 \text{ such that } B_M(x, \varepsilon, t) \subseteq A\},$$

is a topology on X.

Proof. It is obvious that the union of any family of members of $\mathfrak{T}_M$ belongs to $\mathfrak{T}_M$. Now, let $A_1, ..., A_n$, be a finite family of members of $\mathfrak{T}_M$. Let $x \in \bigcap_{k=1}^n A_k$. For each $k \in \{1, ..., n\}$ there is $\varepsilon_k \in (0, 1)$ and $t_k > 0$ such that $B_M(x, \varepsilon_k, t_k) \subseteq A_k$. Put $\varepsilon = \min\{\varepsilon_k : k \in \{1, ..., n\}\}$ and $t = \min\{t_k : k \in \{1, ..., n\}\}$. It follows from Lemma 2.1 that $B_M(x, \varepsilon, t) \subseteq B_M(x, \varepsilon_k, t_k)$ for all $k \in \{1, ..., n\}$. Therefore, $B_M(x, \varepsilon, t) \subseteq \bigcap_{k=1}^n A_k$. We conclude that $\mathfrak{T}_M$ is a topology on X. □

Remark 2.1. *It is well known that the topology $\mathfrak{T}_M$ is metrizable, i.e., there is a metric on X such that its induced topology agrees with $\mathfrak{T}_M$. Next, we present an outline of a proof of this fundamental result based on the construction of a suitable uniformity (see, e.g., [9, 19]) and emphasizing about those axioms that are really used. The conclusions derived from this examination will be key later on.*

Indeed, let X be a set, M be a fuzzy set in $X \times X \times \mathbb{R}^+$ and $*$ be a continuous t-norm for which axioms (KM1), (KM2), (KM3) and (KM4) are fulfilled. For each $n \in \mathbb{N}$ put

$$U_n = \{(x, y) \in X \times X : M(x, y, 1/n) > 1 - 1/n\}.$$

Then, we obtain

- *$\{(x, x) : x \in X\} = \bigcap_{n=1}^\infty U_n$ by (KM1), (KM2) and (KM4) (Lemma 2.1),*
- *$U_n = U_n^{-1}$ for all $n \in \mathbb{N}$ by (KM3),*
- *for each $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $U_m^2 \subseteq U_n$ by continuity of $*$ and (KM4).*

Therefore, the family $\{U_n : n \in \mathbb{N}\}$ is a (countable) base for a uniformity on X whose induced topology agrees with $\mathfrak{T}_M$, which implies that $\mathfrak{T}_M$ is a metrizable topology on X.

Next, we remind some typical and well-known examples of KM, Gr and GV fuzzy metric spaces. In all cases we will assume, without explicit mention, that axiom (KM1) is satisfied.

Example 2.1. *Let (X, d) be a metric space and $*$ be a continuous t-norm. Let $M_d : X \times X \times \mathbb{R}^+ \to [0, 1]$ given by*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $x, y \in X$ and $t > 0$. Then, the pair $(M_d, *)$ is a GV-fuzzy metric on X, whose induced topology coincides with the topology induced by d.

Example 2.2. *Let (X, d) be a metric space and $*$ be a continuous t-norm. Let $M_{01} : X \times X \times \mathbb{R}^+ \to [0, 1]$ given by $M_{01}(x, y, t) = 1$ if $d(x, y) < t$, and $M_{01}(x, y, t) = 0$ if $d(x, y) \geq t$, for all $x, y \in X$ and $t > 0$. Then, the pair $(M_{01}, *)$ is a KM-fuzzy metric on X whose induced topology coincides with the topology induced by d. Clearly, $(M_{01}, *)$ is not a GV-fuzzy metric whenever $|X| \geq 2$ because axiom (GV1) is not satisfied.*

Example 2.3. *Let $X = [0, 1]$ and let $*$ be a continuous t-norm. Then, the pair $(M_*, *)$ is a Gr-fuzzy metric on X, where $M_*(x, x, t) = 1$ for all $x \in X$ and $t > 0$, and $M(x, y, t) = x * y$ for all $x, y \in X$ with $x \neq y$, and $t > 0$. Clearly, $(M_*, *)$ is not a KM-fuzzy metric because axiom (KM6) is not satisfied. Moreover, it is not a GV-fuzzy metric because $M_*(0, y, t) = 0$ whenever $y \neq 0$.*

Example 2.4. *Let (X, d) be a metric space such that $d(x, y) \leq 1$ for all $x, y \in X$. Then, the pair $(M_1, *_L)$ is a Gr-fuzzy metric on X, where $M_1(x, y, t) = 1 - d(x, y)$ for all $x, y \in X$ and $t > 0$. Moreover, the topologies induced by $(M_1, *_L)$ and d coincide. Note also that if there are $x, y \in X$ such that $d(x, y) = 1$, then $(M_1, *_L)$ is not a GV-fuzzy metric because axiom (GV1) is not be satisfied. Moreover, $(M_1, *_L)$ is not a KM-fuzzy metric whenever $|X| \geq 2$ because axiom (KM6) is not be satisfied.*

Remark 2.2. *It is well known (see, e.g.,* [6, Result 3.2]*) that for every Gr-fuzzy metric M on a set X the balls $B_M(x, \varepsilon, t)$ are $\mathfrak{T}_M$-open sets. To show it axiom (KM5) is essential. Fortunately, this result will not be relevant for our targets.*

By virtue of Remark 2.1, we propose the following notion.

Definition 2.2. *A weak fuzzy metric space is a triple $(X, M, *)$, where X is a set, M is a fuzzy set in $X \times X \times \mathbb{R}^+$ and $*$ is a continuous t-norm for which axioms (KM1), (KM2), (KM3) and (KM4) are fulfilled. In this case, we say that the pair $(M, *)$, or simply M, is a weak fuzzy metric on X.*

Remark 2.3. *Let $(X, M, *)$ be a weak fuzzy metric space. It follows from Remark 2.1 that a sequence $(x_n)_n$ in X is $\mathfrak{T}_M$-convergent to a $x \in X$ if and only if, for each $t > 0$, $\lim_{n \to \infty} M(x, x_n, t) = 1$.*

Remark 2.4. *Recall that Kramosil and Michálek added axioms (KM5) and (KM6) in their definition with the aim of having that, for each $x, y \in X$, the function $t \to M(x, y, t)$ be a (generalized) distribution function, as occurs for statistical metric spaces. Note also that axiom (KM6) is crucial in the realm of fuzzy normed spaces, concretely, to show that every fuzzy normed space is a topological vector space (see* [1, 3, 18]*).*

We conclude this section with three examples of weak fuzzy metrics that are not Gr-fuzzy metric, obtained by suitable modifications in Examples 2.1, 2.2 and 2.3, respectively.

Example 2.5. *Let (X, d) be a metric space and $*$ be a continuous t-norm. Let $M_{d,w} : X \times X \times \mathbb{R}^+ \to [0, 1]$ given by*

$$M_{d,w}(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $x, y \in X$ and $t \in (0, 1)$, and $M_{d,w}(x, y, t) = 1$ for all $x, y \in X$ and $t \geq 1$. Then, the pair $(M_{d,w}, *)$ is a weak fuzzy metric on X whose induced topology coincides with the topology induced by d. Note that $(M_{d,w}, *)$ is not a Gr-fuzzy metric if $|X| \geq 2$ because, for $x \neq y$, the function $t \to M_{d,w}(x, y, t)$ is not left continuous at $t = 1$.

Example 2.6. *Let (X, d) be a metric space and $*$ be a continuous t-norm. Let $M_{01,w} : X \times X \times \mathbb{R}^+ \to [0, 1]$ given by $M_{01,w}(x, y, t) = 1$ if $d(x, y) < t$, $M_{01,w}(x, y, t) = 1/2$ if $d(x, y) = t$, and*

$M_{01}(x,y,t)=0$ if $d(x,y)>t$, for all $x,y\in X$ and $t>0$. Then, the pair $(M_{01,w},*)$ is a weak fuzzy metric on X whose induced topology coincides with the topology induced by d. Note that $(M_{01,w},*)$ is not a Gr-fuzzy metric if $|X|\geq 2$ because, for $x\neq y$, the function $t\to M_{01,w}(x,y,t)$ is not left continuous at $t=d(x,y)$. Note also that the "open" ball $B_{01,w}(x,1,1)$ is not necessarily a $\mathfrak{T}_M$-open set because $B_{01,w}(x,1,1)=\{y\in X: d(x,y)\leq 1\}$.

Example 2.7. *Let $X=[0,1]$. Then, the pair $(M_\wedge,\wedge)$ is a weak fuzzy metric on X, where $M_\wedge(x,x,t)=1$ for all $x\in X$ and $t>0$, $M(x,y,t)=x\wedge y$ for all $x,y\in X$ with $x\neq y$ and $t\in(0,1)$, and $M(x,y,t)=1$ for all $x,y\in X$ and $t\geq 1$. Clearly, $(M_\wedge,\wedge)$ is not a Gr-fuzzy metric because the function $t\to M(x,y,t)$ is not left continuous at $t=1$ for $x\neq y$.*

3. Modular metrics and their relation with fuzzy metrics

We start this section recalling the notion of modular metric as given by Chistyakov ([4, 5]).

Definition 3.3. *A modular metric on a set X is a function $w:(0,\infty)\times X\times X\to[0,\infty]$ that fulfills the following axioms for all $x,y,z\in X$:*

(MM1) $x=y$ *if and only if* $w(t,x,y)=0$ *for all* $t>0$;
(MM2) $w(t,x,y)=w(t,y,x)$ *for all* $t>0$;
(MM3) $w(t+s,x,y)\leq w(t,x,z)+w(s,z,y)$ *for all* $t,s>0$.

A modular metric space is a pair (X,w) such that X is a set and w is a modular metric on X.

Let w be a modular metric on a set X. For each $x\in X$, and $\varepsilon,t>0$, set $B_w(x,\varepsilon,t)=\{y\in X: w(t,x,y)<\varepsilon\}$. Since, for each $x,y\in X$, the function $t\to w(t,x,y)$ is nonincreasing on $(0,\infty)$ ([4, p. 3]), we deduce, similarly to the proof of Lemma 2.2, that the family

$$\mathfrak{T}_w=\{A\subseteq X: \text{ for each } x\in A \text{ there exist } \varepsilon,t>0 \text{ such that } B_w(x,\varepsilon,t)\subseteq A\},$$

is a topology on X.

Remark 3.5. *If, for each $x,y\in X$, the function $t\to w(t,x,y)$ is left continuous on $(0,\infty)$, then each ball $B_w(x,\varepsilon,t)$ is a $\mathfrak{T}_w$-open set. Indeed, suppose that $y\in B_w(x,\varepsilon,t)$ for some $x\in X$ and $\varepsilon,t>0$. Put $\delta=\varepsilon-w(t,x,y)$ and choose $r\in(0,t)$ such that $w(t-r,x,y)<w(t,x,y)+\delta/2$. Thus, for each $z\in B_w(y,\delta/2,r)$, we have*

$$w(t,x,z)\leq w(t-r,x,y)+w(r,y,z)<w(t,x,y)+\delta/2+\delta/2=\varepsilon.$$

We conclude that $B_w(y,\delta/2,r)\subseteq B_w(x,\varepsilon,t)$, and hence, $B_w(x,\varepsilon,t)$ is a $\mathfrak{T}_w$-open set.

In the light of the preceding remark, and by analogy with the fuzzy metric case, by a Gr-modular metric on a set X we will mean a modular metric w on X satisfying that, for each $x,y\in X$, the function $t\to w(t,x,y)$ is left continuous on $(0,\infty)$. If, in addition, w satisfies that $\lim_{t\to\infty}w(t,x,y)=0$ for all $x,y\in X$, we will say that w is a KM-modular metric on X.

Proposition 3.1. *Let $(M,*)$ be a weak fuzzy metric on a set X such that $*=\wedge$. Then, the function $w_M: X\times X\times(0,\infty)\to[0,\infty]$ defined by*

$$w_M(t,x,y)=\frac{1-M(x,y,t)}{M(x,y,t)} \tag{3.1}$$

for all $x,y\in X$ and $t>0$, is a modular metric on X such that the topologies $\mathfrak{T}_M$ and $\mathfrak{T}_{w_M}$ coincide on X. Furthermore, if $(M,\wedge)$ is a Gr-fuzzy metric (resp. a KM-fuzzy metric) on X, then w_M is a Gr-modular metric (resp. a KM-modular metric) on X.

Proof. We first note that w_M satisfies axioms (MM1) and (MM2) as a direct consequence of (KM2) and (KM3), respectively.

In order to verify that w_M satisfies (MM3) we shall distinguish two cases.

Case 1. $\min\{M(x,z,t), M(z,y,s)\} = 0$. Then, $\max\{w_M(t,x,z), w_M(s,z,y)\} = \infty$.
Case 2. $\min\{M(x,z,t), M(z,y,s)\} > 0$. Then, $M(x,y,t+s) > 0$ by (KM4).
Put $a = M(x,z,t)$, $b = M(z,y,s)$ and $c = M(x,y,t+s)$. Then, $a+b-ab \geq a \vee b$, so

$$c(a+b-ab) \geq (a \wedge b)(a+b-ab) \geq ab.$$

Hence, $c(a+b-2ab) \geq (1-c)ab$, and, thus

$$w_M(t+s,x,y) = \frac{1-c}{c} \leq \frac{1-a}{a} + \frac{1-b}{b} = w_M(t,x,z) + w_M(s,z,y).$$

We conclude that w_M is a modular metric on X.

The fact that the topologies $\mathfrak{T}_M$ and $\mathfrak{T}_{w_M}$ coincide on X is a consequence from the following easy relations:

$$B_M(x,\varepsilon/L,t) \subseteq B_{w_M}(x,\varepsilon,t) \subseteq B_M(x,\varepsilon,t)$$

for all $x \in X$, $\varepsilon \in (0,1)$ and $t > 0$, where $L > 1+\varepsilon$.

Finally, it is obvious that if $(M,\wedge)$ is a Gr-fuzzy metric (resp. a KM-fuzzy metric) on X, then w_M is a Gr-modular metric (resp. a KM-modular metric) on X. □

Conversely, we have:

Proposition 3.2. *Let w be a modular metric on a set X and let M_w the fuzzy set in $X \times X \times \mathbb{R}^+$ defined by $M_w(x,y,0) = 0$ for all $x,y \in X$ and*

$$M_w(x,y,t) = \frac{1}{1+w(t,x,y)} \tag{3.2}$$

for all $x,y \in X$ and $t > 0$. Then, the pair $(M_w, *_P)$ is a weak fuzzy metric on X such that the topologies $\mathfrak{T}_w$ and $\mathfrak{T}_{M_w}$ coincide on X. Furthermore, if w is a Gr-modular metric (resp. a KM-modular metric) on X, then $(M_w, *_P)$ is a Gr-fuzzy metric (resp. a KM-fuzzy metric) on X.

Proof. We first note that M_w satisfies axioms (KM2) and (KM3) as a direct consequence of (MM1) and (MM2), respectively.

To show that $(M_w, *_P)$ is a weak fuzzy metric on X it remains to verify that w_M satisfies (KM4). To this end, let $x,y,z \in X$ and $t,s \geq 0$. If $\min\{t,s\} = 0$, we have $M_w(x,y,t+s) \geq 0 = M_w(x,z,t) *_P M(z,y,s) = 0$ by the definition of M_w. So, we will assume that $t > 0$ and $s > 0$. Then, we obtain

$$\begin{aligned} M_w(x,y,t+s) = \frac{1}{1+w(t+s,x,y)} &\geq \frac{1}{1+w(t,x,z)+w(s,z,y)} \\ &\geq \frac{1}{1+w(t,x,z)} \frac{1}{1+w(s,z,y)} = M_w(x,z,t) *_P M_w(z,y,s). \end{aligned}$$

Similarly to the proof of Proposition 3.1, the fact that the topologies $\mathfrak{T}_w$ and $\mathfrak{T}_{M_w}$ coincide on X is a consequence from the following easy inclusions:

$$B_{M_w}(x,\varepsilon/2,t) \subseteq B_w(x,\varepsilon/2,t) \subseteq B_{M_w}(x,\varepsilon,t) \tag{3.3}$$

for all $x \in X$, $\varepsilon \in (0,1)$ and $t > 0$. □

Remark 3.6. *Remark 2.1 and relations (3.3) imply that the topology $\mathfrak{T}_w$ is metrizable. Moreover, by Remark 2.3, we get that a sequence $(x_n)_n$ in X is $\mathfrak{T}_w$-convergent to a $x \in X$ if and only if, for each $t > 0$, $\lim_{n\to\infty} w(t,x,x_n) = 0$, thus recovering an important result by Chystiakov (compare [4,* Theorem 2.13]*).*

With the help of Proposition 3.1 and examples given in Section 2, it is easy to obtain several instances of modular metrics (compare [4, Examples 2.4]).

Example 3.8.

(A) *Let (X, d) be a metric space. By applying Example 2.1 and Proposition 3.1 (formula (3.1)), we immediately deduce that the function $w : (0, \infty) \times X \times X \to [0, \infty]$ given by $w(t, x, y) = d(x, y)/t$ for all $x, y \in X$ and $t > 0$, is a KM-modular metric on X. If we apply Example 2.5 instead of Example 2.1, we obtain that the function $w : (0, \infty) \times X \times X \to [0, \infty]$ given by $w(t, x, y) = d(x, y)/t$ for all $x, y \in X$ and $t \in (0, 1)$, and $w(t, x, y) = 0$ for all $x, y \in X$ and $t \geq 1$, is a modular metric on X that is not a Gr-modular metric whenever $|X| \geq 2$.*

(B) *Let (X, d) be a metric space. By applying Example 2.2 and Proposition 3.1 (formula (3.1)), we immediately deduce that the function $w : (0, \infty) \times X \times X \to [0, \infty]$ given by $w(t, x, y) = 0$ if $d(x, y) < t$, and $w(t, x, y) = \infty$ if $d(x, y) \geq t$, is a KM-modular metric on X. If we apply Example 2.6 instead of Example 2.2, we obtain that the function $w : (0, \infty) \times X \times X \to [0, \infty]$ given by $w(t, x, y) = 0$ if $d(x, y) < t$, $w(t, x, y) = 1$ if $d(x, y) = t$, and $w(t, x, y) = \infty$ if $d(x, y) > t$, is a modular metric on X that is not a Gr-modular metric whenever $|X| \geq 2$.*

(C) *Let $X = [0, 1]$. By applying Example 2.3, with $* = \wedge$, and Proposition 3.1 (formula (3.1)) we immediately deduce that the function $w : (0, \infty) \times X \times X \to [0, \infty]$ given by $w(t, x, x) = 0$ for all $x \in X$ and $t > 0$, and $w(t, x, y) = (1/(x \wedge y)) - 1$ whenever $x \neq y$ and $t > 0$, is a Gr-modular metric on X that is not a KM-modular metric on X. If we apply Example 2.7 instead of Example 2.3, we obtain that the function $w : (0, \infty) \times X \times X \to [0, \infty]$ given by $w(t, x, x) = 0$ for all $x \in X$ and $t > 0$, $w(t, x, y) = (1/(x \wedge y)) - 1$ whenever $x \neq y$ and $t \in (0, 1)$, and $w(t, x, y) = 0$ whenever $x \neq y$ and $t \geq 1$, is a modular metric on X that is not a Gr-modular metric on X.*

Note that Propositions 3.1 and 3.2 imply the following statements:

(s1) If $(M, \wedge)$ is a weak fuzzy metric on a set X, then $M = M_{w_M}$,

(s2) If w is a modular metric on a set X, then $w = w_{M_w}$.

In turn, statements (s1) and (s2) suggest the next notions.

Definition 3.4.

(i) *A weak fuzzy metric $(M, *)$ on a set X is called modulable if there is a modular metric w on X such that $M = M_w$.*

(ii) *A modular metric w on a set X is called fuzziable if there is a weak fuzzy metric $(M, *)$ on X such that $w = w_M$.*

Therefore, we obtain:

Proposition 3.3.

(a) *Every fuzzy metric $(M, *)$ on a set X such that $* = \wedge$ is modulable.*

(b) *Every modular metric on a set X is fuzziable.*

We finish the paper by recalling that precedents of Propositions 3.1 and 3.2 can be found in [11] and [20], respectively. Thus, in [11] it was proved Proposition 3.1 for the case that $(M, *)$ is a triangular GV-fuzzy metric in the sense of [2], while that in [20] is was proved that under the assumption that w is a non-Archimedean modular metric on a set X such that, for each $x, y \in X$, the function $t \to w(t, x, y)$ is continuous on $(0, \infty)$, and condition $w(t, x, y) > 0$ whenever $x \neq y$ is also satisfied, then equality (3.2) defines a non-Archimedean triangular GV-fuzzy metric on X for the product t-norm.

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CONSTRUCTIVE MATHEMATICAL ANALYSIS
7 (2024), No. 3, pp. 98-113
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

New ideals of Bloch mappings which are $\mathcal{I}$-factorizable and Möbius-invariant

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ABSTRACT. In this paper, we introduce a unified method for generating ideals of Möbius-invariant Banach-valued Bloch mappings on the complex open unit disc $\mathbb{D}$, through the composition with the members of a Banach operator ideal $\mathcal{I}$. Using the linearization of derivatives of Banach-valued normalized Bloch mappings on $\mathbb{D}$, this composition method yields the so-called ideals of $\mathcal{I}$-factorizable normalized Bloch mappings $\mathcal{I} \circ \hat{\mathcal{B}}$, where $\hat{\mathcal{B}}$ denotes the class of normalized Bloch mappings on $\mathbb{D}$. We present new examples of them as ideals of separable (Rosenthal, Asplund) normalized Bloch mappings and p-integral (strictly p-integral, p-nuclear) normalized Bloch mappings for any $p \in [1, \infty)$. Moreover, the Bloch dual ideal $\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}$ of an operator ideal $\mathcal{I}$ is introduced and shown that it coincides with the composition ideal $\mathcal{I}^{\text{dual}} \circ \hat{\mathcal{B}}$.

Keywords: Bloch mapping, linearization, factorization theorems, operator ideal.

2020 Mathematics Subject Classification: 30H30, 46E15, 46E40, 47B38.

1. INTRODUCTION

Let $\mathbb{D}$ be the complex open unit disc, let X be a complex Banach space and let $\mathcal{H}(\mathbb{D}, X)$ be the space of all holomorphic mappings from $\mathbb{D}$ into X. The Bloch space $\mathcal{B}(\mathbb{D}, X)$ is the linear space of all mappings $f \in \mathcal{H}(\mathbb{D}, X)$ such that

$$\rho_{\mathcal{B}}(f) := \sup\{(1 - |z|^2)\|f'(z)\|\colon z \in \mathbb{D}\} < \infty,$$

under the Bloch seminorm $\rho_{\mathcal{B}}$, and the normalized Bloch space $\hat{\mathcal{B}}(\mathbb{D}, X)$ is the closed subspace of $\mathcal{B}(\mathbb{D}, X)$ formed by all those maps f for which $f(0) = 0$, under the Bloch norm $\rho_{\mathcal{B}}$. For simplicity, it is usual to write $\mathcal{B}(\mathbb{D}) := \mathcal{B}(\mathbb{D}, \mathbb{C})$ and $\hat{\mathcal{B}}(\mathbb{D}) := \hat{\mathcal{B}}(\mathbb{D}, \mathbb{C})$. These spaces of Bloch mappings have been studied by some authors and we know a lot of their properties (see, e.g., [2]).

A useful procedure for constructing new X-valued Bloch mappings on $\mathbb{D}$ consists of composing the X-valued Bloch mappings on $\mathbb{D}$ with operators of some distinguished Banach operator ideal $\mathcal{I}$. This process was used in [13] to characterize X-valued Bloch mappings on $\mathbb{D}$ that have a Bloch range which is relatively (weakly) compact in X or such that the linear hull of its Bloch range is a finite-dimensional subspace of X. Furthermore, these latter function spaces enjoy the property of being invariant under the action of Möbius transformations of the unit disc.

Motivated by these results, our main purpose here is to present a unified method of composition for generating ideals of Möbius-invariant $\mathcal{I}$-factorizable Bloch mappings $\mathcal{I} \circ \mathcal{B}$, where

Received: 18.07.2024; Accepted: 07.08.2024; Published Online: 09.08.2024
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DOI: 10.33205/cma.1518651

$\mathcal{I}$ is a Banach operator ideal and $\mathcal{B}$ is the class of Bloch mappings on $\mathbb{D}$. For a first study of Möbius invariant function spaces, we refer the reader to [3].

In the literature, we can find some interesting papers where this composition method has been applied for constructing new classes of functions in different contexts as, for example, the Lipschitz setting in [1] by Achour, Rueda, Sánchez-Pérez and Yahi and [18] by Saadi; the polynomial and holomorphic contexts in [4] by Aron, Botelho, Pellegrino and Rueda; the polynomial and multilinear settings in [5] by Belaada, Saadi and Tiaiba and [7] by Botelho, Pellegrino and Rueda; and the bounded holomorphic setting in [8] by Cabrera-Padilla and the same authors of this paper.

This note is organized as follows: Section 2 presents the composition method for generating Banach (normalized) Bloch ideals for a given Banach operator ideal $\mathcal{I}$. In the normalized case, our approach is based on the characterization of normalized $\mathcal{I}$-factorizable Bloch mappings in terms of their linearizations on a suitable Banach predual space of $\hat{\mathcal{B}}(\mathbb{D})$ and, towards this end, we will first recall some definitions and results of the theory of Bloch maps.

Section 3 contains a complete study on some new ideals of Banach (normalized) Bloch ideals which can be generated by composition with a Banach operator ideal. This is the case of separable (Rosenthal, Asplund) Bloch mappings and p-integral (strictly p-integral, p-nuclear) Bloch mappings where $p \in [1, \infty)$.

Section 4 deals the notion of Bloch dual ideal $\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}$ of an operator ideal $\mathcal{I}$ with the aid of the transpose of a Bloch mapping. To this end, let us recall that the dual $\mathcal{I}^{\text{dual}}$ of an operator ideal $\mathcal{I}$ is an operator ideal, and it consists for any normed spaces X, Y of those $T \in \mathcal{L}(X, Y)$ such that $T^* \in \mathcal{I}(Y^*, X^*)$. Thus, we will be able to prove that $\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}$ is justly the composition ideal generated by $\mathcal{I}^{\text{dual}}$. Analogous studies have been done in some other nonlinear contexts such as the Lipschitz setting in [1, 18], or the polynomial and multilinear settings in [6, 11, 16].

Notation. Throughout this paper, X and Y will denote complex Banach spaces. As usual, B_X and X^* stand for the closed unit ball and the dual space of X, respectively. κ_X denotes the canonical isometric linear embedding from X into X^{**}. $\mathbb{T}$ represents the set of all complex numbers with modulus 1. For a set $A \subseteq X$, $\text{lin}(A)$, $\overline{\text{lin}}(A)$ and $\overline{\text{aco}}(A)$ denote the linear hull, the norm-closed linear hull and the norm-closed absolutely convex hull of A in X, respectively. We denote by $\mathcal{L}(X, Y)$ the Banach space of all continuous linear operators from X to Y endowed with the operator canonical norm, and by $\mathcal{F}(X, Y)$ the linear space of all finite-rank bounded linear operators from X to Y. We denote by $\mathcal{B}(\mathbb{D}, \mathbb{D})$ the set of all Bloch functions $h\colon \mathbb{D} \to \mathbb{D}$, and by $\hat{\mathcal{B}}(\mathbb{D}, \mathbb{D})$ its subset formed by all those h so that $h(0) = 0$.

2. $\mathcal{I}$-FACTORIZABLE BLOCH MAPPINGS

We begin by introducing these types of Bloch mappings. A complete information on operator ideals may be seen in the monograph [15] by Pietsch.

Definition 2.1. *Let X be a complex Banach space and let $\mathcal{I}$ be an operator ideal. A mapping $f \in \mathcal{H}(\mathbb{D}, X)$ is said to be $\mathcal{I}$-factorizable Bloch if there exist a complex Banach space Y, an operator $T \in \mathcal{I}(Y, X)$ and a mapping $g \in \mathcal{B}(\mathbb{D}, Y)$ such that $f = T \circ g$, that is, the following diagram commutes*

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & \nearrow_{T} & \\ Y & & \end{array}$$

The set of all $\mathcal{I}$-factorizable Bloch mappings from $\mathbb{D}$ into X is denoted by $\mathcal{I}\circ\mathcal{B}(\mathbb{D},X)$. If in the preceding factorization of f, the mapping g satisfies in addition that $g(0)=0$, then the set of all such mappings f is denoted by $\mathcal{I}\circ\hat{\mathcal{B}}(\mathbb{D},X)$.

If $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ is a normed operator ideal and $f\in\mathcal{I}\circ\mathcal{B}(\mathbb{D},X)$, we set

$$\gamma_{\mathcal{I}\circ\mathcal{B}}(f)=\inf\left\{\|T\|_{\mathcal{I}}\,\rho_{\mathcal{B}}(g)\right\},$$

where the infimum extends over all possible factorizations of f as above. In particular, we will write $\|f\|_{\mathcal{I}\circ\mathcal{B}}$ instead of $\gamma_{\mathcal{I}\circ\mathcal{B}}$ whenever $f\in\mathcal{I}\circ\hat{\mathcal{B}}(\mathbb{D},X)$.

An easy argument will show that the functions introduced are in fact Bloch and by the way we will see that they are invariant by Möbius transformations of $\mathbb{D}$.

The Möbius group of $\mathbb{D}$ is formed by all one-to-one holomorphic maps ϕ that send $\mathbb{D}$ onto $\mathbb{D}$. This set is denoted by $\mathrm{Aut}(\mathbb{D})$, and each $\phi\in\mathrm{Aut}(\mathbb{D})$ has the form $\phi=\eta\phi_a$ for some $\eta\in\mathbb{T}$ and $a\in\mathbb{D}$, where $\phi_a(z)=(a-z)/(1-\overline{a}z)$ for all $z\in\mathbb{D}$.

Let us recall that a seminormed space $(\mathcal{A}(\mathbb{D},X),\rho_{\mathcal{A}})$ of holomorphic maps from $\mathbb{D}$ into X is said to be Möbius-invariant if the following two conditions are satisfied:

(1) $\mathcal{A}(\mathbb{D},X)\subseteq\mathcal{B}(\mathbb{D},X)$ and there exists $k>0$ such that $\rho_{\mathcal{B}}(f)\leq k\rho_{\mathcal{A}}(f)$ for all $f\in\mathcal{A}(\mathbb{D},X)$,
(2) $f\circ\phi\in\mathcal{A}(\mathbb{D},X)$ with $\rho_{\mathcal{A}}(f\circ\phi)=\rho_{\mathcal{A}}(f)$ for every $f\in\mathcal{A}(\mathbb{D},X)$ and $\phi\in\mathrm{Aut}(\mathbb{D})$.

Proposition 2.1. *Let X be a complex Banach space and let $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ be a normed operator ideal. Then $(\mathcal{I}\circ\mathcal{B}(\mathbb{D},X),\gamma_{\mathcal{I}\circ\mathcal{B}})$ is a Möbius-invariant space.*

Proof. Let $f\in\mathcal{I}\circ\mathcal{B}(\mathbb{D},X)$ and assume that $f=T\circ g$ for some complex Banach space Y, an operator $T\in\mathcal{I}(Y,X)$ and a mapping $g\in\mathcal{B}(\mathbb{D},Y)$.

(1) Then $f'=T\circ g'$. Therefore

$$(1-|z|^2)\,\|f'(z)\|=(1-|z|^2)\,\|T(g'(z))\|\leq(1-|z|^2)\,\|T\|\,\|g'(z)\|\leq\|T\|\,\rho_{\mathcal{B}}(g)$$

for all $z\in\mathbb{D}$, and so $f\in\mathcal{B}(\mathbb{D},X)$ with $\rho_{\mathcal{B}}(f)\leq\|T\|\,\rho_{\mathcal{B}}(g)$. Taking the infimum over all such factorizations of f, we conclude that $\rho_{\mathcal{B}}(f)\leq\gamma_{\mathcal{I}\circ\mathcal{B}}(f)$. If $\lambda\in\mathbb{C}$, then $\lambda f=\lambda T\circ g$, hence $\lambda f\in\mathcal{I}\circ\mathcal{B}(\mathbb{D},X)$ with $\gamma_{\mathcal{I}\circ\mathcal{B}}(\lambda f)\leq\|\lambda T\|\,\rho_{\mathcal{B}}(g)=|\lambda|\,\|T\|\,\rho_{\mathcal{B}}(g)$ and taking the infimum over all the factorizations of f yields $\gamma_{\mathcal{I}\circ\mathcal{B}}(\lambda f)\leq|\lambda|\,\gamma_{\mathcal{I}\circ\mathcal{B}}(f)$. Conversely, if $\lambda\neq0$, this implies that $\gamma_{\mathcal{I}\circ\mathcal{B}}(f)=\gamma_{\mathcal{I}\circ\mathcal{B}}(\lambda^{-1}(\lambda f))\leq|\lambda|^{-1}\gamma_{\mathcal{I}\circ\mathcal{B}}(\lambda f)$ and thus $|\lambda|\,\gamma_{\mathcal{I}\circ\mathcal{B}}(f)\leq\gamma_{\mathcal{I}\circ\mathcal{B}}(\lambda f)$, while if $\lambda=0$, it is clear that $\gamma_{\mathcal{I}\circ\mathcal{B}}(\lambda f)=0=|\lambda|\,\gamma_{\mathcal{I}\circ\mathcal{B}}(f)$.

If $f_1,f_2\in\mathcal{I}\circ\mathcal{B}(\mathbb{D},X)$, given $\varepsilon>0$, for each $i=1,2$, we can find a complex Banach space Y_i, an operator $T_i\in\mathcal{I}(Y_i,X)$ and a mapping $g_i\in\mathcal{B}(\mathbb{D},Y_i)$ with $\rho_{\mathcal{B}}(g_i)=1$ and $\|T_i\|_{\mathcal{I}}\leq\gamma_{\mathcal{I}\circ\mathcal{B}}(f_i)+\varepsilon/2$ such that $f_i=T_i\circ g_i$. Consider the Banach space $Y=Y_1\oplus_\infty Y_2$ and define the mappings $T\colon Y\to X$ and $g\colon\mathbb{D}\to Y$ by $T(y_1,y_2)=T_1(y_1)+T_2(y_2)$ for all $(y_1,y_2)\in Y$ and $g(z)=(g_1(z),g_2(z))$ for all $z\in\mathbb{D}$, respectively. An easy calculation shows that $T \in \mathcal{I}(Y, X)$ with $\|T\|_{\mathcal{I}} \leq \|T_1\|_{\mathcal{I}} + \|T_2\|_{\mathcal{I}}$ and $g \in \mathcal{B}(\mathbb{D}, Y)$ with $\rho_{\mathcal{B}}(g) \leq 1$. Clearly, $T \circ g = f_1 + f_2$, and so $f_1 + f_2 \in \mathcal{I} \circ \mathcal{B}(\mathbb{D}, X)$ with

$$\gamma_{\mathcal{I}\circ\mathcal{B}}(f_1 + f_2) \leq \|T\|_{\mathcal{I}} \, \rho_{\mathcal{B}}(g) \leq \|T_1\|_{\mathcal{I}} + \|T_2\|_{\mathcal{I}} \leq \gamma_{\mathcal{I}\circ\mathcal{B}}(f_1) + \gamma_{\mathcal{I}\circ\mathcal{B}}(f_2) + \varepsilon.$$

The arbitrariness of $\varepsilon > 0$ ensures that $\gamma_{\mathcal{I}\circ\mathcal{B}}(f_1 + f_2) \leq \gamma_{\mathcal{I}\circ\mathcal{B}}(f_1) + \gamma_{\mathcal{I}\circ\mathcal{B}}(f_2)$. So we have proved that $(\mathcal{I} \circ \mathcal{B}(\mathbb{D}, X), \gamma_{\mathcal{I}\circ\mathcal{B}})$ is a seminormed space.

(2) Let $\phi \in \mathrm{Aut}(\mathbb{D})$. Then $f \circ \phi = T \circ g \circ \phi$, where $g \circ \phi \in \mathcal{B}(\mathbb{D}, Y)$ with $\rho_{\mathcal{B}}(g \circ \phi) = \rho_{\mathcal{B}}(g)$. Hence $f \circ \phi \in \mathcal{I} \circ \mathcal{B}(\mathbb{D}, X)$ and $\gamma_{\mathcal{I}\circ\mathcal{B}}(f \circ \phi) \leq \|T\| \, \rho_{\mathcal{B}}(g)$. Passing to the infimum over all the factorizations of f yields $\gamma_{\mathcal{I}\circ\mathcal{B}}(f \circ \phi) \leq \gamma_{\mathcal{I}\circ\mathcal{B}}(f)$. The converse inequality follows from what we have proved.

□

Motivated by [13, Definition 5.11] (see the definition that follows Theorem 2.2), we say that a seminormed ideal of Bloch mappings (or simply, a seminormed Bloch ideal) is a subclass $\mathcal{I}^{\mathcal{B}}$ of the class of all Bloch mappings $\mathcal{B}$, equipped with a function $\rho_{\mathcal{I}^{\mathcal{B}}} \colon \mathcal{I}^{\mathcal{B}} \to \mathbb{R}$, such that for each complex Banach space X, the components

$$\mathcal{I}^{\mathcal{B}}(\mathbb{D}, X) := \mathcal{I}^{\mathcal{B}} \cap \mathcal{B}(\mathbb{D}, X)$$

satisfy the properties:

(P1) $(\mathcal{I}^{\mathcal{B}}(\mathbb{D}, X), \rho_{\mathcal{I}^{\mathcal{B}}})$ is a seminormed space with $p_{\mathcal{B}}(f) \leq \rho_{\mathcal{I}^{\mathcal{B}}}(f)$ for all $f \in \mathcal{I}^{\mathcal{B}}(\mathbb{D}, X)$,

(P2) for any $g \in \mathcal{B}(\mathbb{D})$ and $x \in X$, the mapping $g \cdot x \colon z \mapsto g(z)x$ from $\mathbb{D}$ to X is in $\mathcal{I}^{\mathcal{B}}(\mathbb{D}, X)$ with $\rho_{\mathcal{I}^{\mathcal{B}}}(g \cdot x) = p_{\mathcal{B}}(g) \|x\|$,

(P3) the ideal property: if $f \in \mathcal{I}^{\mathcal{B}}(\mathbb{D}, X)$, $h \in \mathcal{B}(\mathbb{D}, \mathbb{D})$ and $T \in \mathcal{L}(X, Y)$ where Y is a complex Banach space, then $T \circ f \circ h$ is in $\mathcal{I}^{\mathcal{B}}(\mathbb{D}, Y)$ with $\rho_{\mathcal{I}^{\mathcal{B}}}(T \circ f \circ h) \leq \|T\| \, \rho_{\mathcal{I}^{\mathcal{B}}}(f)$.

Proposition 2.2. *Let $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ be a normed operator ideal. Then $[\mathcal{I} \circ \mathcal{B}, \gamma_{\mathcal{I}\circ\mathcal{B}}]$ is a seminormed Bloch ideal.*

Proof. (P1): It has been proved in Proposition 2.1.

(P2): Take g and x as in (P2), note that $g \cdot x = M_x \circ g$, where $M_x \in \mathcal{F}(\mathbb{C}, X) \subseteq \mathcal{I}(\mathbb{C}, X)$ is the operator defined by $M_x(\lambda) = \lambda x$ for all $\lambda \in \mathbb{C}$, and so $g \cdot x \in \mathcal{I} \circ \mathcal{B}(\mathbb{D}, X)$ with $\gamma_{\mathcal{I}\circ\mathcal{B}}(g \cdot x) \leq \|M_x\|_{\mathcal{I}} \, \rho_{\mathcal{B}}(g) = \|M_x\| \, \rho_{\mathcal{B}}(g) = \|x\| \, \rho_{\mathcal{B}}(g)$ and, conversely, $\rho_{\mathcal{B}}(g) \, \|x\| = \rho_{\mathcal{B}}(g \cdot x) \leq \gamma_{\mathcal{I}\circ\mathcal{B}}(g \cdot x)$ by the inequality in (P1).

(P3): Take $f \in \mathcal{I} \circ \mathcal{B}(\mathbb{D}, X)$, and h and T as in (P3). We can write $f = S \circ g$ for some complex Banach space Z, an operator $S \in \mathcal{I}(Z, X)$ and a mapping $g \in \mathcal{B}(\mathbb{D}, Z)$. Hence $T \circ f \circ h = T \circ S \circ g \circ h$. Therefore $T \circ f \circ h \in \mathcal{I} \circ \mathcal{B}(\mathbb{D}, Y)$ with $\gamma_{\mathcal{I}\circ\mathcal{B}}(T \circ f \circ h) \leq \|T \circ S\|_{\mathcal{I}} \, \rho_{\mathcal{B}}(g \circ h) \leq \|T\| \, \|S\|_{\mathcal{I}} \, \rho_{\mathcal{B}}(g)$. Note that $\rho_{\mathcal{B}}(g \circ h) \leq \rho_{\mathcal{B}}(g)$ since $(1 - |z|^2)|h'(z)| \leq 1 - |h(z)|^2$ for all $z \in \mathbb{D}$ by the Pick–Schwarz Lemma. Taking the infimum over all the factorizations of f gives $\gamma_{\mathcal{I}\circ\mathcal{B}}(T \circ f \circ h) \leq \|T\| \, \gamma_{\mathcal{I}\circ\mathcal{B}}(f)$. □

Our next goal is to characterize $\mathcal{I}$-factorizable normalized Bloch mappings by means of their linearizations on a strongly unique predual of $\hat{\mathcal{B}}(\mathbb{D})$, called Bloch-free Banach space over $\mathbb{D}$. With this purpose, we now recall some basic concepts of the theory initiated in [13] on this predual. A similar technique of linearization of Bloch mappings has been applied in the recent paper [17] by Quang.

For each $z \in \mathbb{D}$, a Bloch atom of $\mathbb{D}$ is the functional $\gamma_z \in \hat{\mathcal{B}}(\mathbb{D})^*$ defined by $\gamma_z(f) = f'(z)$ for all $f \in \hat{\mathcal{B}}(\mathbb{D})$. The elements of $\mathrm{lin}(\{\gamma_z \colon z \in \mathbb{D}\}) \subseteq \hat{\mathcal{B}}(\mathbb{D})^*$ are referred to as Bloch molecules of $\mathbb{D}$, and the Bloch-free Banach space over $\mathbb{D}$ is the space $\mathcal{G}(\mathbb{D}) := \overline{\mathrm{lin}}(\{\gamma_z \colon z \in \mathbb{D}\}) \subseteq \hat{\mathcal{B}}(\mathbb{D})^*$.

We collect some basic properties of $\mathcal{G}(\mathbb{D})$ in the following result.

Theorem 2.1 ([13]).

(i) *The mapping* $\Gamma\colon \mathbb{D} \to \mathcal{G}(\mathbb{D})$, *defined by* $\Gamma(z) = \gamma_z$ *for all* $z \in \mathbb{D}$, *is holomorphic with* $\|\gamma_z\| = 1/(1 - |z|^2)$.

(ii) *The space* $\hat{\mathcal{B}}(\mathbb{D})$ *is isometrically isomorphic to* $\mathcal{G}(\mathbb{D})^*$, *via* $\Lambda\colon \hat{\mathcal{B}}(\mathbb{D}) \to \mathcal{G}(\mathbb{D})^*$ *given by*

$$\Lambda(f)(\gamma) = \sum_{k=1}^{n} \lambda_k f'(z_k) \qquad \left(f \in \hat{\mathcal{B}}(\mathbb{D}),\ \gamma = \sum_{k=1}^{n} \lambda_k \gamma_{z_k} \in \operatorname{lin}(\Gamma(\mathbb{D})) \right).$$

(iii) *The closed unit ball of* $\mathcal{G}(\mathbb{D})$ *coincides with* $\overline{\operatorname{aco}}(\mathcal{M}(\mathbb{D}))$, *where* $\mathcal{M}(\mathbb{D}) = \{(1 - |z|^2)\gamma_z \colon z \in \mathbb{D}\}$.

(iv) *For each function* $h \in \hat{\mathcal{B}}(\mathbb{D}, \mathbb{D})$, *the composition operator* $C_h\colon \hat{\mathcal{B}}(\mathbb{D}) \to \hat{\mathcal{B}}(\mathbb{D})$, *defined by* $C_h(f) = f \circ h$ *for all* $f \in \hat{\mathcal{B}}(\mathbb{D})$, *is linear and continuous with* $||C_h|| \leq 1$.

(v) *For each function* $h \in \hat{\mathcal{B}}(\mathbb{D}, \mathbb{D})$, *there exists a unique operator* $\hat{h} \in \mathcal{L}(\mathcal{G}(\mathbb{D}), \mathcal{G}(\mathbb{D}))$ *such that* $\hat{h} \circ \Gamma = h' \cdot (\Gamma \circ h)$. *Furthermore,* $||\hat{h}|| = ||C_h||$.

(vi) *For every complex Banach space* X *and every mapping* $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$, *there exists a unique operator* $S_f \in \mathcal{L}(\mathcal{G}(\mathbb{D}), X)$ *such that* $S_f \circ \Gamma = f'$. *Furthermore,* $\|S_f\| = \rho_{\mathcal{B}}(f)$.

(vii) *The mapping* $f \mapsto S_f$ *is an isometric isomorphism from* $\hat{\mathcal{B}}(\mathbb{D}, X)$ *onto* $\mathcal{L}(\mathcal{G}(\mathbb{D}), X)$.

(viii) *Given* $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$, *the mapping* $f^t\colon X^* \to \hat{\mathcal{B}}(\mathbb{D})$, *defined by* $f^t(x^*) = x^* \circ f$ *for all* $x^* \in X^*$ *and called Bloch transpose of* f, *is linear and continuous with* $\|f^t\| = \rho_{\mathcal{B}}(f)$. *Furthermore,* $f^t = \Lambda^{-1} \circ (S_f)^*$, *where* $(S_f)^*\colon X^* \to \mathcal{G}(\mathbb{D})^*$ *is the adjoint operator of* S_f. □

We are now prepared to establish the announced result.

Theorem 2.2. *Let* X *be a complex Banach space and* $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$. *Given an operator ideal* $\mathcal{I}$, *the following conditions are equivalent:*

(1) f *belongs to* $\mathcal{I} \circ \hat{\mathcal{B}}(\mathbb{D}, X)$,

(2) S_f *belongs to* $\mathcal{I}(\mathcal{G}(\mathbb{D}), X)$.

If, in addition, $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ *is a normed operator ideal, we have* $\|f\|_{\mathcal{I}\circ\mathcal{B}} = ||S_f||_{\mathcal{I}}$, *where the infimum* $\|f\|_{\mathcal{I}\circ\mathcal{B}}$ *is attained at* $S_f \circ \Gamma$. *Furthermore, the mapping* $f \mapsto S_f$ *is an isometric isomorphism from* $(\mathcal{I} \circ \hat{\mathcal{B}}(\mathbb{D}, X), \|\cdot\|_{\mathcal{I}\circ\mathcal{B}})$ *onto* $(\mathcal{I}(\mathcal{G}(\mathbb{D}), X), \|\cdot\|_{\mathcal{I}})$.

Proof. (1) ⇒ (2): If $f \in \mathcal{I} \circ \hat{\mathcal{B}}(\mathbb{D}, X)$, then we can find a complex Banach space Y, an operator $T \in \mathcal{I}(Y, X)$ and a mapping $g \in \hat{\mathcal{B}}(\mathbb{D}, Y)$ such that $f = T \circ g$. Since $g' = S_g \circ \Gamma$ by Theorem 2.1, it follows that $f' = T \circ g' = (T \circ S_g) \circ \Gamma$, and since $T \circ S_g \in \mathcal{L}(\mathcal{G}(\mathbb{D}), X)$, we have that $S_f = T \circ S_g$ by Theorem 2.1, and thus $S_f \in \mathcal{I}(\mathcal{G}(\mathbb{D}), X)$ by the ideal property of $\mathcal{I}$. Further, if the ideal $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ is normed, we have

$$\|S_f\|_{\mathcal{I}} = \|T \circ S_g\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}} \|S_g\| = \|T\|_{\mathcal{I}} \rho_{\mathcal{B}}(g),$$

and taking the infimum over all factorizations of f as above, we deduce that $||S_f||_{\mathcal{I}} \leq \|f\|_{\mathcal{I}\circ\mathcal{B}}$.

(2) ⇒ (1): Assume that $S_f \in \mathcal{I}(\mathcal{G}(\mathbb{D}), X)$. We can write $f' = S_f \circ \Gamma$ by Theorem 2.1. Since $\Gamma \in \mathcal{H}(\mathbb{D}, \mathcal{G}(\mathbb{D}))$ by Theorem 2.1, an application of [13, Lemma 2.9] provides a mapping $\Upsilon \in \mathcal{H}(\mathbb{D}, \mathcal{G}(\mathbb{D}))$ with $\Upsilon(0) = 0$ such that $\Upsilon' = \Gamma$. Furthermore, $(1-|z|^2)\|\Upsilon'(z)\| = (1-|z|^2)\|\Gamma(z)\| = 1$ for all $z \in \mathbb{D}$, and thus $\Upsilon \in \hat{\mathcal{B}}(\mathbb{D}, \mathcal{G}(\mathbb{D}))$ with $\rho_{\mathcal{B}}(\Upsilon) = 1$. Hence $f' = S_f \circ \Upsilon'$. We claim that $f = S_f \circ \Upsilon$. Indeed, since $f' = (S_f \circ \Upsilon)'$, we have $(x^* \circ f)' = x^* \circ f' = x^* \circ (S_f \circ \Upsilon)' = (x^* \circ S_f \circ \Upsilon)'$ for all $x^* \in X^*$, this implies that $x^* \circ f = x^* \circ S_f \circ \Upsilon$ for all $x^* \in X^*$ since $x^* \circ f, x^* \circ S_f \circ \Upsilon \in \mathcal{H}(\mathbb{D})$ and $x^* \circ f(0) = x^* \circ S_f \circ \Upsilon(0) = 0$, and our claim follows because X^* separates the point of X. Hence $f \in \mathcal{I} \circ \hat{\mathcal{B}}(\mathbb{D}, X)$. If now $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ is normed, we have

$$\|f\|_{\mathcal{I}\circ\mathcal{B}} \leq \|S_f\|_{\mathcal{I}} \rho_{\mathcal{B}}(\Upsilon) = \|S_f\|_{\mathcal{I}}.$$

The last assertion of the statement follows readily from Theorem 2.1 and the proof above. □

We will now show that the ideal property of $\mathcal{I}$ is inherited by $\mathcal{I}\circ\hat{\mathcal{B}}$. Towards this end, we first recall the concept of normalized Bloch ideal introduced in [13] and some of its properties.

A normed (Banach) normalized Bloch ideal is a subclass $\mathcal{I}^{\hat{\mathcal{B}}}$ of the class of all normalized Bloch mappings $\hat{\mathcal{B}}$, endowed with a function $\|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}}}\colon \mathcal{I}^{\hat{\mathcal{B}}} \to \mathbb{R}$, such that for every complex Banach space X, the components

$$\mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X) := \mathcal{I}^{\hat{\mathcal{B}}} \cap \hat{\mathcal{B}}(\mathbb{D},X),$$

satisfy the following properties:

(Q1) $(\mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X), \|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}}})$ is a normed (Banach) space with $\rho_{\mathcal{B}}(f) \leq \|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}}$ for all $f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X)$.

(Q2) $g\cdot x \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X)$ with $\|g\cdot x\|_{\mathcal{I}^{\hat{\mathcal{B}}}} = \rho_{\mathcal{B}}(g)\,\|x\|$ for every $g \in \hat{\mathcal{B}}(\mathbb{D})$ and $x \in X$.

(Q3) The ideal property: $T\circ f\circ h \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},Y)$ with $\|T\circ f\circ h\|_{\mathcal{I}^{\hat{\mathcal{B}}}} \leq \|T\|\,\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}}$ whenever $f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X)$, $h \in \hat{\mathcal{B}}(\mathbb{D},\mathbb{D})$ and $T \in \mathcal{L}(X,Y)$ where Y is a complex Banach space.

Let us recall (see [15, Chapter B.3]) that if X and Y are Banach spaces and $T \in \mathcal{L}(X,Y)$, then T is called a metric injection if $\|T(x)\| = \|x\|$ for all $x \in X$, and T is called a metric surjection if T is surjective and $\|T(x)\| = \inf\{\|y\| : T(y) = T(x)\}$ for all $x \in X$.

A normed normalized Bloch ideal $(\mathcal{I}^{\hat{\mathcal{B}}}, \|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}}})$ is said to be:

(I) Injective if for any $f \in \hat{\mathcal{B}}(\mathbb{D},X)$, any complex Banach space Y and any metric injection $\iota\colon X \to Y$, one has that $f \in \mathcal{I}^{\hat{\mathcal{B}}}(D,X)$ with $\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}} = \|\iota\circ f\|_{\mathcal{I}^{\hat{\mathcal{B}}}}$ whenever $\iota\circ f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},Y)$.

(S) Surjective if for any complex Banach space X, any $f \in \hat{\mathcal{B}}(\mathbb{D},X)$ and any $\pi \in \hat{\mathcal{B}}(\mathbb{D},\mathbb{D})$ such that $\hat{\pi}$ is a metric surjection, it holds that $f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X)$ with $\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}} = \|f\circ\pi\|_{\mathcal{I}^{\hat{\mathcal{B}}}}$ if $f\circ\pi \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X)$.

(R) Regular if for any $f \in \hat{\mathcal{B}}(\mathbb{D},X)$, we have that $f \in \mathcal{I}^{\hat{\mathcal{B}}}(D,X)$ with $\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}} = \|\kappa_X\circ f\|_{\mathcal{I}^{\hat{\mathcal{B}}}}$ whenever $\kappa_X\circ f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D},X^{**})$.

Corollary 2.1. *We have:*

(1) If $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ is a normed (Banach) operator ideal, then $[\mathcal{I}\circ\hat{\mathcal{B}},\|\cdot\|_{\mathcal{I}\circ\mathcal{B}}]$ is a normed (Banach) normalized Bloch ideal.

(2) If $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ is an injective (surjective, regular) normed operator ideal, then $[\mathcal{I}\circ\hat{\mathcal{B}},\|\cdot\|_{\mathcal{I}\circ\mathcal{B}}]$ is an injective (surjective, regular) normed normalized Bloch ideal.

Proof.

(1) Let $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ be a normed operator ideal. Then $[\mathcal{I}\circ\hat{\mathcal{B}},\|\cdot\|_{\mathcal{I}\circ\mathcal{B}}]$ is a seminormed Bloch ideal by Proposition 2.2. Furthermore, if $f \in \mathcal{I}\circ\hat{\mathcal{B}}(\mathbb{D},X)$ and $\|f\|_{\mathcal{I}\circ\mathcal{B}} = 0$, the inequality $\rho_{\mathcal{B}}(f) \leq \|f\|_{\mathcal{I}\circ\mathcal{B}}$ implies that $f = 0$, and thus $\|\cdot\|_{\mathcal{I}\circ\mathcal{B}}$ is a norm on $\mathcal{I}\circ\hat{\mathcal{B}}(\mathbb{D},X)$.

Since $(\mathcal{I}\circ\hat{\mathcal{B}}(\mathbb{D},X),\|\cdot\|_{\mathcal{I}\circ\mathcal{B}})$ is isometrically isomorphic to $(\mathcal{I}(\mathcal{G}(\mathbb{D}),X),\|\cdot\|_{\mathcal{I}})$ by Theorem 2.2, then $[\mathcal{I}\circ\hat{\mathcal{B}},\|\cdot\|_{\mathcal{I}\circ\mathcal{B}}]$ is a Banach space whenever $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ is so.

(2) Suppose that the normed operator ideal $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ is injective (surjective, regular). We have:

(I) Assume that $\iota\circ f \in \mathcal{I}\circ\hat{\mathcal{B}}(\mathbb{D},Y)$, where Y is a complex Banach space and $\iota\colon X \to Y$ is a metric injection. Since $\iota\circ S_f = S_{\iota\circ f} \in \mathcal{I}(\mathcal{G}(\mathbb{D}),Y)$ by Theorems 2.1 and 2.2, and the operator ideal $\mathcal{I}$ is injective, we deduce that $S_f \in \mathcal{I}(\mathcal{G}(\mathbb{D}),X)$ with $||S_f||_{\mathcal{I}} = ||\iota\circ S_f||_{\mathcal{I}}$, thus $f \in \mathcal{I}\circ\hat{\mathcal{B}}(\mathbb{D},X)$ by Theorem 2.2 with

$$\|f\|_{\mathcal{I}\circ\mathcal{B}} = \|S_f\|_{\mathcal{I}} = \|\iota\circ S_f\|_{\mathcal{I}} = \|S_{\iota\circ f}\|_{\mathcal{I}} = \|\iota\circ f\|_{\mathcal{I}\circ\mathcal{B}},$$

and this proves that $[\mathcal{I} \circ \hat{\mathcal{B}}, \|\cdot\|_{\mathcal{I}\circ\mathcal{B}}]$ is injective.

(S) Suppose that $f \circ \pi \in \mathcal{I} \circ \hat{\mathcal{B}}(\mathbb{D}, X)$, where $\pi \in \hat{\mathcal{B}}(\mathbb{D}, \mathbb{D})$ such that $\hat{\pi}$ is a metric surjection. Since $S_f \circ \hat{\pi} = S_{f\circ\pi} \in \mathcal{I}(\mathcal{G}(\mathbb{D}), X)$ by Theorems 2.1 and 2.2, and the operator ideal $\mathcal{I}$ is surjective, we have that $S_f \in \mathcal{I}(\mathcal{G}(\mathbb{D}), X)$ with $||S_f||_{\mathcal{I}} = ||S_f \circ \hat{\pi}||_{\mathcal{I}}$, hence $f \in \mathcal{I} \circ \hat{\mathcal{B}}(\mathbb{D}, X)$ by Theorem 2.2 with

$$\|f\|_{\mathcal{I}\circ\mathcal{B}} = \|S_f\|_{\mathcal{I}} = \|S_f \circ \hat{\pi}\|_{\mathcal{I}} = \|S_{f\circ\pi}\|_{\mathcal{I}} = \|f \circ \pi\|_{\mathcal{I}\circ\mathcal{B}},$$

and thus $[\mathcal{I} \circ \hat{\mathcal{B}}, \|\cdot\|_{\mathcal{I}\circ\mathcal{B}}]$ is surjective.

(R) It follows with a proof similar to that of (I).

□

3. New examples of ideals of $\mathcal{I}$-factorizable Bloch mappings

In this section, we will present some notable subclasses of Bloch mappings: separable (Rosenthal, Asplund) Bloch mappings and p-integral (strictly p-integral, p-nuclear) Bloch mappings for any $p \in [1, \infty)$. We will show their most important properties and demonstrate that they correspond to $\mathcal{I}$-factorizable Bloch mappings when $\mathcal{I}$ is the corresponding operator ideal.

Given two normed normalized Bloch ideals $[\mathcal{I}^{\hat{\mathcal{B}}}, \|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}}}]$ and $[\mathcal{J}, \|\cdot\|_{\mathcal{J}^{\hat{\mathcal{B}}}}]$, we will write

$$[\mathcal{I}^{\hat{\mathcal{B}}}, \|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}}}] \leq [\mathcal{J}, \|\cdot\|_{\mathcal{J}^{\hat{\mathcal{B}}}}]$$

to indicate that for any Banach space X, we have $\mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D}, X) \subseteq \mathcal{J}^{\hat{\mathcal{B}}}(\mathbb{D}, X)$ and $\|f\|_{\mathcal{J}^{\hat{\mathcal{B}}}} \leq \|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}}$ for all $f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D}, X)$.

3.1. Mappings whose Bloch range is separable, Rosenthal or Asplund.

Given a Banach space X, let us recall that a set $A \subseteq X$ is called:

- Rosenthal if every sequence in A admits a weak Cauchy subsequence,
- Asplund if A is bounded and for any countable set $D \subseteq A$, the seminormed space $(X^*, \|\cdot\|_D)$ is separable, where $||x^*||_D = \sup_{x\in D} |x^*(x)|$.

Let us recall that an operator $T \in \mathcal{L}(X, Y)$ is said to be compact (resp., weakly compact, separable, Rosenthal, Asplund) if $T(B_X)$ is a relatively compact (resp., relatively weakly compact, separable, Rosenthal, Asplund) subset of Y.

For $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}, \mathcal{S}, \mathcal{R}, \mathcal{AS}$, we will denote by $\mathcal{I}(X, Y)$ the linear space of all finite-rank (approximable, compact, weakly compact, separable, Rosenthal, Asplund) bounded linear operators from X to Y, respectively. We refer to the monograph [15] for a complete study of the ideal structure of such operators. The following inclusions are known:

$$\mathcal{F}(X,Y) \subseteq \overline{\mathcal{F}}(X,Y) \subseteq \mathcal{K}(X,Y) \subseteq \mathcal{W}(X,Y) \subseteq \mathcal{R}(X,Y) \cap \mathcal{AS}(X,Y),$$
$$\mathcal{K}(X,Y) \subseteq \mathcal{S}(X,Y).$$

Our aim is to study some Bloch variants of these classes of operators introduced with the aid of the following set. Given a complex Banach space X and a mapping $f \in \mathcal{H}(\mathbb{D}, X)$, the Bloch range of f is defined as

$$\mathrm{rang}_{\mathcal{B}}(f) := \left\{(1 - |z|^2)f'(z) \colon z \in \mathbb{D}\right\} \subseteq X.$$

Note that $f \in \mathcal{H}(\mathbb{D}, X)$ is Bloch if and only if $\mathrm{rang}_{\mathcal{B}}(f)$ is a norm-bounded subset of X. Some of the following concepts were introduced in [13, Definitions 5.1 and 5.2].

Definition 3.2. *Let X be a complex Banach space. A mapping $f \in \mathcal{H}(\mathbb{D}, X)$ is said to be compact (resp., weakly compact, separable, Rosenthal, Asplund) if $\mathrm{rang}_{\mathcal{B}}(f)$ is a relatively compact (resp., relatively weakly compact, separable, Rosenthal, Asplund) subset of X.*

A mapping $f \in \mathcal{H}(\mathbb{D}, X)$ is said to have finite dimensional Bloch rank if $\mathrm{lin}(\mathrm{rang}_{\mathcal{B}}(f))$ is a finite dimensional subspace of X, and f is said to be approximable Bloch if it is the limit in the Bloch seminorm $\rho_{\mathcal{B}}$ of a sequence of finite-rank Bloch mappings of $\mathcal{B}(\mathbb{D}, X)$.

For $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}, \mathcal{S}, \mathcal{R}, \mathcal{AS}$, we denote by $\mathcal{B}_{\mathcal{I}}(\mathbb{D}, X)$ the linear space of all finite-rank (resp., approximable, compact, weakly compact, separable, Rosenthal, Asplund) Bloch mappings from $\mathbb{D}$ into X. We write $\hat{\mathcal{B}}_{\mathcal{I}}(\mathbb{D}, X)$ to represent the subspace consisting of all functions $f \in \mathcal{B}_{\mathcal{I}}(\mathbb{D}, X)$ so that $f(0) = 0$.

The following two results were established in [13] for the cases $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}$. We now complete it here for $\mathcal{I} = \mathcal{S}, \mathcal{R}, \mathcal{AS}$ with similar proofs.

Proposition 3.3. *Let X be a complex Banach space. For $\mathcal{I} = S, \mathcal{R}, \mathcal{AS}$, the space $(\mathcal{B}_{\mathcal{I}}(\mathbb{D}, X), \rho_{\mathcal{B}})$ is Möbius-invariant.*

Proof. Given $f \in \mathcal{H}(\mathbb{D}, X)$ and $\phi \in \mathrm{Aut}(\mathbb{D})$, for all $z \in \mathbb{D}$ we have

$$(1 - |z|^2)(f \circ \phi)'(z) = (1 - |z|^2) f'(\phi(z))\phi'(z) = (1 - |\phi(z)|^2) f'(\phi(z)) \frac{\phi'(z)}{|\phi'(z)|}.$$

Hence $\mathrm{rang}_{\mathcal{B}}(f \circ \phi) \subseteq \mathbb{T}\,\mathrm{rang}_{\mathcal{B}}(f)$, and thus if $\mathrm{rang}_{\mathcal{B}}(f)$ has the $\mathcal{I}$-property (the terminology should be self-explanatory), it is readily seen that $f \circ \phi$ has the $\mathcal{I}$-property with $p_{\mathcal{B}}(f \circ \phi) = p_{\mathcal{B}}(f)$. □

We next analyse the relationship of a mapping f in $\hat{\mathcal{B}}_{\mathcal{I}}(\mathbb{D}, X)$ with its linearization S_f in $\mathcal{L}(\mathcal{G}(\mathbb{D}), X)$.

Theorem 3.3. *Let X be a complex Banach space and $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$. For the operator ideal $\mathcal{I} = \mathcal{S}, \mathcal{R}, \mathcal{AS}$, the following conditions are equivalent:*

(1) f belongs to $\hat{\mathcal{B}}_{\mathcal{I}}(\mathbb{D}, X)$,
(2) S_f belongs to $\mathcal{I}(\mathcal{G}(\mathbb{D}), X)$.

Furthermore, $f \mapsto S_f$ is an isometric isomorphism from $(\hat{\mathcal{B}}_{\mathcal{I}}(\mathbb{D}, X), \rho_{\mathcal{B}})$ onto $(\mathcal{I}(\mathcal{G}(\mathbb{D}), X), \|\cdot\|)$.

Proof. First, using Theorem 2.1, we obtain the relations:

$$\begin{aligned}\mathrm{rang}_{\mathcal{B}}(f) = S_f(\mathcal{M}_{\mathcal{B}}(\mathbb{D})) &\subseteq S_f(B_{\mathcal{G}(\mathbb{D})}) = S_f(\overline{\mathrm{aco}}(\mathcal{M}_{\mathcal{B}}(\mathbb{D}))) \\ &\subseteq \overline{\mathrm{aco}}(S_f(\mathcal{M}_{\mathcal{B}}(\mathbb{D}))) = \overline{\mathrm{aco}}(\mathrm{rang}_{\mathcal{B}}(f)).\end{aligned}$$

(1) ⇒ (2): Assume $\mathcal{I} = \mathcal{S}, \mathcal{R}, \mathcal{AS}$. If $f \in \hat{\mathcal{B}}_{\mathcal{I}}(\mathbb{D}, X)$, then $\mathrm{rang}_{\mathcal{B}}(f)$ has the $\mathcal{I}$-property. It is known that $\overline{\mathrm{aco}}(\mathrm{rang}_{\mathcal{B}}(f))$ has the $\mathcal{I}$-property and since the $\mathcal{I}$-property is hereditary, the second inclusion above tells us that $S_f(B_{\mathcal{G}(\mathbb{D})})$ has the $\mathcal{I}$-property. This means that $S_f \in \mathcal{I}(\mathcal{G}(\mathbb{D}), X)$.

(2) ⇒ (1): If $S_f \in \mathcal{I}(\mathcal{G}(\mathbb{D}), X)$, then $S_f(B_{\mathcal{G}(\mathbb{D})})$ has the $\mathcal{I}$-property, hence $\mathrm{rang}_{\mathcal{B}}(f)$ has the $\mathcal{I}$-property by the first inclusion above, and this means that $f \in \hat{\mathcal{B}}_{\mathcal{I}}(\mathbb{D}, X)$.

The last assertion of the statement follows using Theorem 2.1 and what was proved above. □

It is known that the Banach operator ideals $\mathcal{S}, \mathcal{R}, \mathcal{AS}$ are injective and surjective (see [15] or a list of such examples in [12]). In view of Theorem 3.3, the combination of Theorem 2.2 and Corollary 2.1 yield the following.

Proposition 3.4. *For $\mathcal{I} = S, \mathcal{R}, \mathcal{AS}$, we have $[\hat{\mathcal{B}}_{\mathcal{I}}, \rho_{\mathcal{B}}] = [\mathcal{I} \circ \hat{\mathcal{B}}, \|\cdot\|_{\mathcal{I}\circ\mathcal{B}}]$ and, in particular, $[\hat{\mathcal{B}}_{\mathcal{I}}, \rho_{\mathcal{B}}]$ is a surjective and injective Banach normalized Bloch ideal.* □

3.2. **p-Integral Bloch mappings.** Following [15, Section 19.2], given two Banach spaces X, Y and $p \in [1, \infty)$, an operator $T \in \mathcal{L}(X, Y)$ is said to be p-integral if there exist a probability measure μ and two operators $R \in \mathcal{L}(L_p(\mu), Y^{**})$ and $S \in \mathcal{L}(X, L_\infty(\mu))$ such that

$$\kappa_Y \circ T = R \circ I^{\mu}_{\infty,p} \circ S,$$

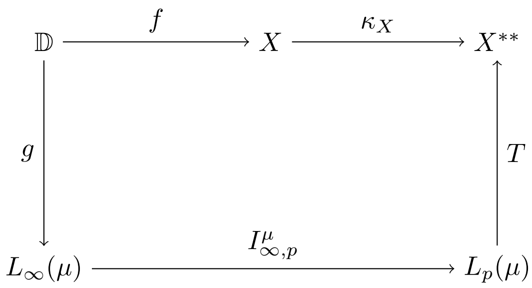
being $I^{\mu}_{\infty,p} \colon L_\infty(\mu) \to L_p(\mu)$ the formal identity. The set of all p-integral operators from X into Y is denoted by $\mathcal{I}_p(X, Y)$, and the p-integral norm of $T \in \mathcal{I}_p(X, Y)$ is

$$\iota_p(T) = \inf\{\|R\| \, \|S\|\},$$

where the infimum is taken over all such factorizations of $\kappa_Y \circ T$ as above. It is well known that $[\mathcal{I}_p, \iota_p]$ is a Banach operator ideal.

We now introduce a Bloch variant of this concept.

Definition 3.3. *Let X be a complex Banach space and $p \in [1, \infty)$. A mapping $f \in \mathcal{H}(\mathbb{D}, X)$ is said to be p-integral Bloch if there exists a probability measure μ, an operator $T \in \mathcal{L}(L_p(\mu), X^{**})$ and a Bloch mapping $g \in \mathcal{B}(\mathbb{D}, L_\infty(\mu))$ such that the following diagram commutes*



The triple $(T, I^{\mu}_{\infty,p}, g)$ is termed a p-integral Bloch representation of f. We define

$$\iota^{\mathcal{B}}_p(f) = \inf\{\|T\|\rho_{\mathcal{B}}(g)\},$$

where the infimum is taken over all p-integral Bloch representations of f. The set of all p-integral Bloch mappings from $\mathbb{D}$ into X is denoted by $\mathcal{I}^{\mathcal{B}}_p(\mathbb{D}, X)$. If in the factorization of $\kappa_X \circ f$, g verifies also that $g(0) = 0$, then the set of such mappings f is denoted by $\mathcal{I}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$.

A proof similar to that of Proposition 2.1 shows the following fact.

Proposition 3.5. *Let X be a complex Banach space and $p \in [1, \infty)$. Then $(\mathcal{I}^{\mathcal{B}}_p(\mathbb{D}, X), \iota^{\mathcal{B}}_p)$ is Möbius-invariant.* □

Proof. Let $f \in \mathcal{I}^{\mathcal{B}}_p(\mathbb{D}, X)$ and $(T, I^{\mu}_{\infty,p}, g)$ be a p-integral Bloch representation of f. Hence $\kappa_X \circ f = T \circ I^{\mu}_{\infty,p} \circ g$.

(1) Then $\kappa_X \circ f' = T \circ I^{\mu}_{\infty,p} \circ g'$. Therefore, for all $z \in \mathbb{D}$, we have

$$\begin{aligned}(1 - |z|^2)\,\|f'(z)\| = (1 - |z|^2)\,\|\kappa_X(f'(z))\| &= (1 - |z|^2)\,\left\|T(I^{\mu}_{\infty,p}(g'(z)))\right\| \\ &\le (1 - |z|^2)\,\|T\|\,\|g'(z)\| \le \|T\|\,\rho_{\mathcal{B}}(g),\end{aligned}$$

and so $f \in \mathcal{B}(\mathbb{D}, X)$ with $\rho_{\mathcal{B}}(f) \le \|T\|\,\rho_{\mathcal{B}}(g)$. Taking the infimum over all such $(T, I^{\mu}_{\infty,p}, g)$, we have $\rho_{\mathcal{B}}(f) \le \iota^{\mathcal{B}}_p(f)$.

(2) Let $\phi \in \mathrm{Aut}(\mathbb{D})$. Then $\kappa_X \circ f \circ \phi = T \circ I^{\mu}_{\infty,p} \circ g \circ \phi$, where $g \circ \phi \in \mathcal{B}(\mathbb{D}, Y)$ with $\rho_{\mathcal{B}}(g \circ \phi) = \rho_{\mathcal{B}}(g)$. Hence $f \circ \phi \in \mathcal{I}^{\mathcal{B}}_p(\mathbb{D}, X)$ and $\iota^{\mathcal{B}}_p(f \circ \phi) \leq \|T\|\, \rho_{\mathcal{B}}(g)$. Taking the infimum over all p-integral Bloch representations of f, we obtain $\iota^{\mathcal{B}}_p(f \circ \phi) \leq \iota^{\mathcal{B}}_p(f)$. The converse inequality follows from what we have proved.

□

We now study the linearization of p-integral normalized Bloch mappings.

Theorem 3.4. *Let X be a complex Banach space, $p \in [1, \infty)$ and $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$. Then $f \in \mathcal{I}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$ if and only if $S_f \in \mathcal{I}_p(\mathcal{G}(\mathbb{D}), X)$. In this case, $\iota^{\mathcal{B}}_p(f) = \iota_p(S_f)$. Moreover, $(\mathcal{I}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X), \iota^{\mathcal{B}}_p)$ and $(\mathcal{I}_p(\mathcal{G}(\mathbb{D}), X), \iota_p)$ are isometrically isomorphic through the map $f \mapsto S_f$.*

Proof. Assume that $f \in \mathcal{I}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$ and let $\varepsilon > 0$. Then there exist a probability measure μ, a mapping $g \in \hat{\mathcal{B}}(\mathbb{D}, L_\infty(\mu))$ and an operator $T \in \mathcal{L}(L_p(\mu), X^{**})$ such that

$$\kappa_X \circ f = T \circ I^{\mu}_{\infty,p} \circ g \colon \mathbb{D} \xrightarrow{g} L_\infty(\mu) \xrightarrow{I^{\mu}_{\infty,p}} L_p(\mu) \xrightarrow{T} X^{**},$$

with $\|T\| \rho_{\mathcal{B}}(g) \leq \iota^{\mathcal{B}}_p(f) + \varepsilon$. By Theorem 2.1, from the equality

$$\kappa_X \circ S_f \circ \Gamma = \kappa_X \circ f' = T \circ I^{\mu}_{\infty,p} \circ g' = T \circ I^{\mu}_{\infty,p} \circ S_g \circ \Gamma,$$

we infer that $\kappa_X \circ S_f = T \circ I^{\mu}_{\infty,p} \circ S_g$, where $S_g \in \mathcal{L}(\mathcal{G}(\mathbb{D}), L_\infty(\mu))$. Hence $S_f \in \mathcal{I}_p(\mathcal{G}(\mathbb{D}), X)$ with

$$\iota_p(S_f) \leq \|T\| \|S_g\| = \|T\| \rho_{\mathcal{B}}(g) \leq \iota^{\mathcal{B}}_p(f) + \varepsilon.$$

Letting $\varepsilon \to 0$ yields $\iota_p(S_f) \leq \iota^{\mathcal{B}}_p(f)$.

Conversely, suppose that $S_f \in \mathcal{I}_p(\mathcal{G}(\mathbb{D}), X)$. Then, for each $\varepsilon > 0$ there exist a probability measure μ and two operators $T \in \mathcal{L}(L_p(\mu), X^{**})$ and $S \in \mathcal{L}(\mathcal{G}(\mathbb{D}), L_\infty(\mu))$ such that

$$\kappa_X \circ S_f = T \circ I^{\mu}_{\infty,p} \circ S : \mathcal{G}(\mathbb{D}) \xrightarrow{S} L_\infty(\mu) \xrightarrow{I^{\mu}_{\infty,p}} L_p(\mu) \xrightarrow{T} X^{**},$$

with $\|T\| \|S\| \leq \iota_p(S_f) + \varepsilon$. As $S \circ \Gamma \in \mathcal{H}(\mathbb{D}, L_\infty(\mu))$, [13, Lemma 2.9] provides a mapping $g \in \mathcal{H}(\mathbb{D}, L_\infty(\mu))$ with $g(0) = 0$ such that $g' = S \circ \Gamma$, and hence

$$(1 - |z|^2)\|g'(z)\| = (1 - |z|^2)\|(S \circ \Gamma)(z)\| \leq \|S\| \qquad (z \in \mathbb{D}).$$

Thus, $g \in \hat{\mathcal{B}}(\mathbb{D}, L_\infty(\mu))$ with $\rho_{\mathcal{B}}(g) \leq \|S\|$. Moreover

$$(\kappa_X \circ f)' = \kappa_X \circ f' = \kappa_X \circ S_f \circ \Gamma = T \circ I^{\mu}_{\infty,p} \circ S \circ \Gamma = T \circ I^{\mu}_{\infty,p} \circ g' = (T \circ I^{\mu}_{\infty,p} \circ g)',$$

which implies that $\kappa_X \circ f = T \circ I^{\mu}_{\infty,p} \circ g$, and then $f \in \mathcal{I}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$ with

$$\iota^{\mathcal{B}}_p(f) \leq \|T\| \rho_{\mathcal{B}}(g) \leq \|T\|\, \|S\| \leq \iota_p(S_f) + \varepsilon.$$

Just letting $\varepsilon \to 0$, the proof can be concluded. □

Since $[\mathcal{I}_p, \iota_p]$ is a Banach operator ideal, Theorems 2.2 and 3.4 and Corollary 2.1 yield the following result.

Corollary 3.2. *Let $p \in [1, \infty)$. Then $[\mathcal{I}^{\hat{\mathcal{B}}}_p, \iota^{\mathcal{B}}_p] = [\mathcal{I}_p \circ \hat{\mathcal{B}}, \|\cdot\|_{\mathcal{I}_p \circ \hat{\mathcal{B}}}]$. In particular, $[\mathcal{I}^{\hat{\mathcal{B}}}_p, \iota^{\mathcal{B}}_p]$ is a Banach ideal of normalized Bloch mappings.* □

Applying Theorem 3.4 and [15, Proposition 19.2.10], we deduce immediately the following.

Corollary 3.3. *If $1 \leq p \leq q < \infty$, then $[\mathcal{I}^{\hat{\mathcal{B}}}_p, \iota^{\mathcal{B}}_p] \leq [\mathcal{I}^{\hat{\mathcal{B}}}_q, \iota^{\mathcal{B}}_q]$.* □

Let us recall from [9] that a mapping $f \in \mathcal{H}(\mathbb{D}, X)$ is said to be p-summing Bloch for $p \in [1, \infty)$ if there is a constant $C \geq 0$ such that for any $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $z_1, \ldots, z_n \in \mathbb{D}$, we have

$$\left(\sum_{i=1}^{n} |\lambda_i|^p \|f'(z_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{g \in B_{\hat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} |\lambda_i|^p |g'(z_i)|^p \right)^{\frac{1}{p}}.$$

The least of all the constants C for which such an inequality holds, denoted $\pi_p^{\mathcal{B}}(f)$, defines a norm on the linear space of all p-summing Bloch map $f\colon \mathbb{D} \to X$ so that $f(0) = 0$, denoted $\Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$.

We can connect p-integral Bloch mappings with this class of Bloch maps. Compare the following result with [10, Proposition 5.5] and [15, Proposition 19.2.12].

Corollary 3.4. *Let $p \in [1, \infty)$. Then:*

(1) $[\mathcal{I}_p^{\hat{\mathcal{B}}}, \iota_p^{\mathcal{B}}] \leq [\Pi_p^{\hat{\mathcal{B}}}, \pi_p^{\mathcal{B}}]$.

(2) $[\mathcal{I}_p^{\hat{\mathcal{B}}}, \iota_p^{\mathcal{B}}] \leq [\hat{\mathcal{B}}_{\mathcal{W}}, \rho_{\mathcal{B}}]$.

Proof. Let $f \in \mathcal{I}_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$. Then $S_f \in \mathcal{I}_p(\mathcal{G}(\mathbb{D}), X)$ with $\iota_p(S_f) = \iota_p^{\mathcal{B}}(f)$ by Theorem 3.4:

(1) Then $S_f \in \Pi_p(\mathcal{G}(\mathbb{D}), X)$ with $\pi_p(S_f) \leq \iota_p(S_f)$ by [10, Proposition 5.5]. It is immediate to prove that $f \in \Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$ and $\pi_p^{\mathcal{B}}(f) \leq \pi_p(S_f)$, and this completes the proof.

(2) Now, $S_f \in \mathcal{W}(\mathcal{G}(\mathbb{D}), X)$ with $\|S_f\| \leq \iota_p(S_f)$ by [15, Proposition 19.2.12]. Hence $f \in \hat{\mathcal{B}}_{\mathcal{W}}(\mathbb{D}, X)$ with $\rho_{\mathcal{B}}(f) = \|S_f\|$ by [13, Theorem 5.6], and we have finished.

□

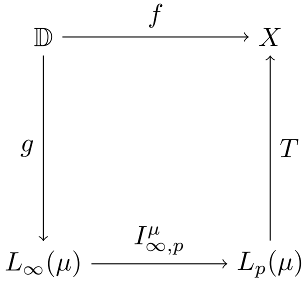
3.3. **Strictly p-integral Bloch mappings.** In the linear context, an important subclass of p-integral operators appears in [14] when the passage to the bidual in their definition is superfluous. Given two Banach spaces X, Y and $p \in [1, \infty)$, an operator $T \in \mathcal{L}(X, Y)$ is said to be strictly p-integral (or Pietsch p-integral) if there exist a probability measure μ and two operators $R \in \mathcal{L}(L_p(\mu), Y)$ and $S \in \mathcal{L}(X, L_\infty(\mu))$ such that $T = R \circ I_{\infty,p}^{\mu} \circ S$. We set

$$\upsilon_p(T) = \inf\{\|R\| \, \|S\|\},$$

where the infimum is taken over all such factorizations of T as above. The set of all strictly p-integral operators from X into Y is denoted by $\mathcal{PI}_p(X, Y)$, and it is known that $[\mathcal{PI}_p, \upsilon_p]$ is a Banach operator ideal.

The corresponding Bloch version of this concept introduces a new class of Bloch mappings.

Definition 3.4. *Given $p \in [1, \infty)$, a mapping $f \in \mathcal{H}(\mathbb{D}, X)$ is said to be strictly p-integral Bloch or Pietsch p-integral Bloch if there exist a probability measure μ, an operator $T \in \mathcal{L}(L_p(\mu), X)$ and a Bloch mapping $g \in \mathcal{B}(\mathbb{D}, L_\infty(\mu))$ giving rise the commutative diagram*



The triple $(T, I^{\mu}_{\infty,p}, g)$ *is called a strictly* p*-integral Bloch representation of* f. *We denote by* $\mathcal{PI}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$ *the set of all strictly* p*-integral Bloch maps from* $\mathbb{D}$ *to* X. *If in the factorization of* f, *we also demand that* $g(0) = 0$, *then the set of such maps* f *is represented by* $\mathcal{PI}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$. *We set*

$$\upsilon^{\mathcal{B}}_p(f) = \inf\{\|T\| \rho_{\mathcal{B}}(g)\},$$

where the infimum is taken over all strictly p*-integral Bloch representations of* f.

From Definitions 3.3 and 3.4, it is clear that $\mathcal{PI}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X) \subseteq \mathcal{I}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$ and $\iota^{\mathcal{B}}_p(f) \leq \upsilon^{\mathcal{B}}_p(f)$ for all $f \in \mathcal{PI}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$.

With proofs similar to those of Subsection 3.2, we establish the following results.

Proposition 3.6. *Let* X *be a complex Banach space and* $p \in [1, \infty)$. *Then* $(\mathcal{PI}^{\mathcal{B}}_p(\mathbb{D}, X), \upsilon^{\mathcal{B}}_p)$ *is Möbius-invariant.* □

Theorem 3.5. *Let* X *be a complex Banach space,* $p \in [1, \infty)$ *and* $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$. *Then* $f \in \mathcal{PI}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$ *if and only if* $S_f \in \mathcal{PI}_p(\mathcal{G}(\mathbb{D}), X)$, *in whose case* $\upsilon^{\mathcal{B}}_p(f) = \upsilon_p(S_f)$. *As a consequence,*

(1) $f \mapsto S_f$ *is an isometric isomorphism from* $(\mathcal{PI}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X), \upsilon^{\mathcal{B}}_p)$ *onto* $(\mathcal{PI}_p(\mathcal{G}(\mathbb{D}), X), \upsilon_p)$,

(2) $[\mathcal{PI}^{\hat{\mathcal{B}}}_p, \upsilon^{\mathcal{B}}_p] = [\mathcal{PI} \circ \hat{\mathcal{B}}, \|\cdot\|_{\mathcal{PI}\circ\hat{\mathcal{B}}}]$, *and thus* $[\mathcal{PI}^{\hat{\mathcal{B}}}_p, \upsilon^{\mathcal{B}}_p]$ *is a Banach normalized Bloch ideal.* □

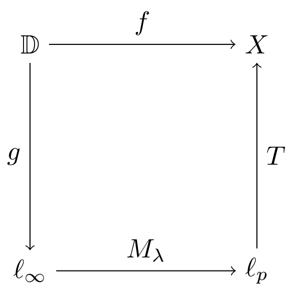
3.4. **p-Nuclear Bloch mappings.** Let us recall from [15, Definition 18.1.1] that an operator $T \in \mathcal{L}(X, Y)$ is said to be p-nuclear for $p \in [1, \infty)$ if $T = R \circ M_\lambda \circ S$, where $S \in \mathcal{L}(X, \ell_\infty)$, $R \in \mathcal{L}(\ell_p, Y)$ and M_λ denotes the diagonal operator from ℓ_∞ to ℓ_p defined by $M_\lambda((x_n)) = (\lambda_n x_n)$ for all $(x_n) \in \ell_\infty$, being $\lambda = (\lambda_n) \in \ell_p$. We denote $\mathcal{N}_p(X, Y)$ to the set of all p-nuclear operators from X into Y. It is known that $\mathcal{N}_p$ is a Banach operator ideal under the norm

$$\nu_p(T) = \inf\{\|R\| \, \|\lambda\|_p \, \|S\|\},$$

where the infimum is taken over all representations of T as above.

Next, we introduce the analogue to the concept of p-nuclear operator in the Bloch setting.

Definition 3.5. *Let* $p \in [1, \infty)$, *let* X *be a complex Banach space and* $f \in \mathcal{H}(\mathbb{D}, X)$. *We say that* f *is* p*-nuclear Bloch if the following diagram commutes*



where $g \in \mathcal{B}(\mathbb{D}, \ell_\infty)$, $T \in \mathcal{L}(\ell_p, X)$ *and* $\lambda = (\lambda_n) \in \ell_p$. *We say that the triple* (T, M_λ, g) *is a* p*-nuclear Bloch representation of* f. *The set of all* p*-nuclear Bloch mappings from* $\mathbb{D}$ *to* X *is denoted by* $\mathcal{N}^{\mathcal{B}}_p(\mathbb{D}, X)$. *If in the factorization of* f, *we also require that* $g(0) = 0$, *then* $\mathcal{N}^{\hat{\mathcal{B}}}_p(\mathbb{D}, X)$ *stands the set of all such mappings* f. *Define*

$$\nu^{\mathcal{B}}_p(f) = \inf\{\|T\| \, \|\lambda\|_p \, \rho_{\mathcal{B}}(g)\},$$

where the infimum is taken over all the above factorizations.

As in the preceding sections, we can prove the following.

Proposition 3.7. *Let X be a complex Banach space and $p \in [1, \infty)$. Then $(\mathcal{N}_p^{\mathcal{B}}(\mathbb{D}, X), \nu_p^{\mathcal{B}})$ is Möbius-invariant.* □

We first study the relationship between a p-nuclear Bloch mapping and its linearization.

Theorem 3.6. *Let $p \in [1, \infty)$, let X be a complex Banach space and $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$. Then $f \in \mathcal{N}_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$ if and only if $S_f \in \mathcal{N}_p(\mathcal{G}(\mathbb{D}), X)$. In this case, $\nu_p^{\mathcal{B}}(f) = \nu_p(S_f)$. Furthermore, $(\mathcal{N}_p^{\hat{\mathcal{B}}}(\mathbb{D}, X), \nu_p^{\mathcal{B}})$ and $(\mathcal{N}_p(\mathcal{G}(\mathbb{D}), X), \nu_p)$ are isometrically isomorphic via the mapping $f \mapsto S_f$.*

Proof. Let $f \in \mathcal{N}_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$. We can write $f = T \circ M_\lambda \circ g$, where $T \in \mathcal{L}(\ell_p, X)$, $\lambda \in \ell_p$ and $g \in \hat{\mathcal{B}}(\mathbb{D}, \ell_\infty)$. An application of Theorem 2.1 shows that $S_f = T \circ M_\lambda \circ S_g$ where $S_g \in \mathcal{L}(\mathcal{G}(\mathbb{D}), \ell_\infty)$. Hence $S_f \in \mathcal{N}_p(\mathcal{G}(\mathbb{D}), X)$ with

$$\nu_p(S_f) \leq \|T\| \|M_\lambda\|_p \|S_g\| = \|T\| \|M_\lambda\|_p \, \rho_{\mathcal{B}}(g).$$

Taking the infimum over all p-nuclear Bloch representations of f, we conclude that $\nu_p(S_f) \leq \nu_p^{\mathcal{B}}(f)$.

Conversely, suppose that $S_f \in \mathcal{N}_p(\mathcal{G}(\mathbb{D}), X)$ and let $\varepsilon > 0$. Hence we can assure the existence of $\lambda \in \ell_p$, $R \in \mathcal{L}(\mathcal{G}(\mathbb{D}), \ell_\infty)$ and $S \in \mathcal{L}(\ell_p, X)$ such that $S_f = S \circ M_\lambda \circ R$ and $\|S\| \|\lambda\|_p \|R\| \leq (1 + \varepsilon)\nu_p(S_f)$. Thus $f' = S_f \circ \Gamma = S \circ M_\lambda \circ R \circ \Gamma$.

Since $R \circ \Gamma \in \mathcal{H}(\mathbb{D}, \ell_\infty)$, by [13, Lemma 2.9], we can find $h \in \mathcal{H}(\mathbb{D}, \ell_\infty)$ with $h(0) = 0$ such that $h' = R \circ \Gamma$, and in fact $h \in \hat{\mathcal{B}}(\mathbb{D}, \ell_\infty)$ with $\rho_{\mathcal{B}}(h) \leq \|R\|$. Hence we have

$$f' = S \circ M_\lambda \circ R \circ \Gamma = S \circ M_\lambda \circ h' = (S \circ M_\lambda \circ h)'.$$

It follows that $f = S \circ M_\lambda \circ h$, and therefore $f \in \mathcal{N}_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$ with

$$\nu_p^{\mathcal{B}}(f) \leq \|S\| \|\lambda\|_p \, \rho_{\mathcal{B}}(h) \leq \|S\| \|\lambda\|_p \|R\| \leq (1 + \varepsilon)\nu_p(S_f).$$

Just letting $\varepsilon \to 0$, we obtain $\nu_p^{\mathcal{B}}(f) \leq \nu_p(S_f)$. □

In view of Theorem 3.6, both Theorems 2.2 and Corollary 2.1 yield the following.

Corollary 3.5. *Let $p \in [1, \infty)$. Then $[\mathcal{N}_p^{\hat{\mathcal{B}}}, \nu_p^{\mathcal{B}}] = [\mathcal{N}_p \circ \hat{\mathcal{B}}, \|\cdot\|_{\mathcal{N}_p \circ \hat{\mathcal{B}}}]$. In particular, $[\mathcal{N}_p^{\hat{\mathcal{B}}}, \nu_p^{\mathcal{B}}]$ is a Banach ideal of normalized Bloch mappings.* □

Applying Theorem 3.6, [10, Corollary 5.24] and [13, Theorem 5.9] gives some relations.

Corollary 3.6. *(1) $[\mathcal{N}_p^{\hat{\mathcal{B}}}, \nu_p^{\mathcal{B}}] \leq [\mathcal{N}_q^{\hat{\mathcal{B}}}, \nu_q^{\mathcal{B}}]$ whenever $1 \leq p \leq q < \infty$,*

(2) $[\mathcal{N}_p^{\hat{\mathcal{B}}}, \nu_p^{\mathcal{B}}] \leq [\mathcal{PI}_p^{\hat{\mathcal{B}}}, \upsilon_p^{\mathcal{B}}]$ and $[\mathcal{N}_p^{\hat{\mathcal{B}}}, \nu_p^{\mathcal{B}}] \leq [\hat{\mathcal{B}}_{\overline{\mathcal{F}}}, \rho_{\mathcal{B}}]$ whenever $1 \leq p < \infty$.

□

With Theorems 5.27 and 5.28 of [10] in mind, it is natural to expect some composition results.

Corollary 3.7. *Let $p \in [1, \infty)$, let X be a complex Banach space and $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$. The following are equivalent:*

(1) $f \in \mathcal{N}_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$.

(2) There exist a complex Banach space Y, $T \in \mathcal{K}(Y, X)$ and $g \in \mathcal{I}_p^{\hat{\mathcal{B}}}(\mathbb{D}, Y)$ such that $f = T \circ g$.

(3) There is a complex Banach space Y, $T \in \mathcal{PI}_p(Y, X)$ and $g \in \hat{\mathcal{B}}_{\mathcal{K}}(\mathbb{D}, Y)$ such that $f = T \circ g$.

In this case, taking the infimum extended over all such factorizations yields

$$\nu_p^{\mathcal{B}}(f) = \inf\left\{\|T\| \, \iota_p^{\mathcal{B}}(g)\right\} = \inf\left\{\upsilon_p(T)\rho_{\mathcal{B}}(g)\right\}.$$

Proof.

(1) ⇒ (2): If $f \in \mathcal{N}_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$, then $S_f \in \mathcal{N}_p(\mathcal{G}(\mathbb{D}), X)$ with $\nu_p^{\mathcal{B}}(f) = \nu_p(S_f)$ by Theorem 3.6. Given $\varepsilon > 0$, by [10, Theorem 5.27], we can take a complex Banach space Y, $T \in \mathcal{K}(Y, X)$ and $S \in \mathcal{I}_p(\mathcal{G}(\mathbb{D}), Y)$ such that $S_f = T \circ S$ and $\|T\| \, \iota_p(S) \leq \nu_p(S_f) + \varepsilon$. By Theorem 3.4, there is a $g \in \mathcal{I}_p^{\hat{\mathcal{B}}}(\mathbb{D}, Y)$ such that $S_g = S$ and $\iota_p^{\mathcal{B}}(g) = \iota_p(S)$. Hence $S_f = T \circ S_g$, therefore $f' = S_f \circ \Gamma = T \circ S_g \circ \Gamma = T \circ g' = (T \circ g)'$ and this implies that $f = T \circ g$. Further, $\|T\| \, \iota_p^{\mathcal{B}}(g) = \|T\| \, \iota_p(S) \leq \nu_p(S_f) + \varepsilon = \nu_p^{\mathcal{B}}(f) + \varepsilon$, and the arbitrariness of ε gives $\|T\| \, \iota_p^{\mathcal{B}}(g) \leq \nu_p^{\mathcal{B}}(f)$.

(2) ⇒ (1): Assume that $f = T \circ g$ with $T \in \mathcal{K}(Y, X)$ and $g \in \mathcal{I}_p^{\hat{\mathcal{B}}}(\mathbb{D}, Y)$ for some complex Banach space Y. Hence $S_f = T \circ S_g$ with $S_g \in \mathcal{I}_p(\mathcal{G}(\mathbb{D}), Y)$ and $\iota_p(S_g) = \iota_p^{\mathcal{B}}(g)$ by Theorem 3.4. Then $S_f \in \mathcal{N}_p(\mathcal{G}(\mathbb{D}), X)$ with $\nu_p(S_f) \leq \|T\| \, \iota_p(S_g)$ by [10, Theorem 5.27]. We conclude that $f \in \mathcal{N}_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$ with $\nu_p^{\mathcal{B}}(f) = \nu_p(S_f)$ by Theorem 3.6. Moreover, $\nu_p^{\mathcal{B}}(f) \leq \|T\| \, \iota_p^{\mathcal{B}}(g)$ and since we are working with an arbitrary factorization $T \circ g$ for f, we get that $\nu_p^{\mathcal{B}}(f) \leq \inf \left\{ \|T\| \, \iota_p^{\mathcal{B}}(g) \right\}$.

(1) ⇔ (3): This can be proved as the preceding implications by using now [10, Theorem 5.28] and [13, Theorem 5.4] instead of [10, Theorem 5.27] and Theorem 3.4, respectively.

□

4. Bloch dual ideal of an operator ideal

Following [15, Section 4.4], given a normed operator ideal $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$, recall that for any normed spaces X and Y, the components

$$\mathcal{I}^{\text{dual}}(X, Y) := \{T \in \mathcal{L}(X, Y) \colon T^* \in \mathcal{I}(Y^*, X^*)\},$$

endowed with the norm

$$\|T\|_{\mathcal{I}^{\text{dual}}} = \|T^*\|_{\mathcal{I}} \qquad (T \in \mathcal{I}^{\text{dual}}(X, Y)),$$

define a normed operator ideal $[\mathcal{I}^{\text{dual}}, \|\cdot\|_{\mathcal{I}^d}]$ called dual ideal of $\mathcal{I}$. Moreover, $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ is called symmetric if $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}] \leq [\mathcal{I}^{\text{dual}}, \|\cdot\|_{\mathcal{I}^{\text{dual}}}]$. In the case $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}] = [\mathcal{I}^{\text{dual}}, \|\cdot\|_{\mathcal{I}^{\text{dual}}}]$, the operator ideal is said to be completely symmetric.

With the aid of the notion of transpose of a Bloch mapping, we now introduce the next concept.

Definition 4.6. *Let $\mathcal{I}$ be an operator ideal. For any complex Banach space X, we define*

$$\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}(\mathbb{D}, X) = \{f \in \hat{\mathcal{B}}(\mathbb{D}, X) \colon f^t \in \mathcal{I}(X^*, \hat{\mathcal{B}}(\mathbb{D}))\}.$$

If $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ is a normed operator ideal, we set

$$\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}} = \|f^t\|_{\mathcal{I}} \qquad (f \in \mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}(\mathbb{D}, X)).$$

We now show that $[\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}, \| \cdot \|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}}]$ is really an ideal of normalized Bloch mappings.

Theorem 4.7. *Let X be a complex Banach space and $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$. If $\mathcal{I}$ is an operator ideal, then $f \in \mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}(\mathbb{D}, X)$ if and only if $f \in \mathcal{I}^{\text{dual}} \circ \hat{\mathcal{B}}(\mathbb{D}, X)$. Moreover, if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed operator ideal, then $\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}} = \|f\|_{\mathcal{I}^{dual} \circ \hat{\mathcal{B}}}$ for all $f \in \mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}(\mathbb{D}, X)$.*

Proof. Let us assume that $f \in \mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}(\mathbb{D}, X)$. Then $f^t \in \mathcal{I}(X^*, \hat{\mathcal{B}}(\mathbb{D}))$. By Theorem 2.1, there exists $S_f \in \mathcal{L}(\mathcal{G}(\mathbb{D}), X)$ such that $S_f \circ \Gamma = f'$ and also $(S_f)^* = \Lambda \circ f^t$. Hence $(S_f)^* \in \mathcal{I}(X^*, \mathcal{G}(\mathbb{D})^*)$ and therefore $S_f \in \mathcal{I}^{\text{dual}}(\mathcal{G}(\mathbb{D}), X)$. Thus, by Theorem 2.2, we have $f \in \mathcal{I}^{\text{dual}} \circ \hat{\mathcal{B}}(\mathbb{D}, X)$ with $\|f\|_{\mathcal{I}^{\text{dual}} \circ \hat{\mathcal{B}}} = \|S_f\|_{\mathcal{I}^{\text{dual}}}$. Further,

$$\|f\|_{\mathcal{I}^{\text{dual}} \circ \hat{\mathcal{B}}} = \|S_f\|_{\mathcal{I}^{\text{dual}}} = \|(S_f)^*\|_{\mathcal{I}} = \|\Lambda \circ f^t\|_{\mathcal{I}} \leq \|\Lambda\| \|f^t\|_{\mathcal{I}} = \|f\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}}.$$

Conversely, let $f \in \mathcal{I}^{\mathrm{dual}} \circ \hat{\mathcal{B}}(\mathbb{D}, X)$. Then there are a complex Banach space Y, a mapping $g \in \hat{\mathcal{B}}(\mathbb{D}, Y)$ and an operator $T \in \mathcal{I}^{\mathrm{dual}}(Y, X)$ such that $f = T \circ g$. Given $x^* \in X^*$, we have

$$f^t(x^*) = (T \circ g)^t(x^*) = x^* \circ (T \circ g) = (x^* \circ T) \circ g = T^*(x^*) \circ g = g^t(T^*(x^*)) = (g^t \circ T^*)(x^*),$$

and thus $f^t = g^t \circ T^*$. Since $T^* \in \mathcal{I}(X^*, Y^*)$ and $g^t \in \mathcal{L}(Y^*, \hat{\mathcal{B}}(\mathbb{D}))$, we obtain that $f^t \in \mathcal{I}(X^*, \hat{\mathcal{B}}(\mathbb{D}))$. Hence $f \in \mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}(\mathbb{D}, X)$ and moreover, we have

$$\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}} = \left\|f^t\right\|_{\mathcal{I}} = \left\|g^t \circ T^*\right\|_{\mathcal{I}} \leq \left\|g^t\right\| \left\|T^*\right\|_{\mathcal{I}} = \rho_{\mathcal{B}}(g) \left\|T\right\|_{\mathcal{I}^{\mathrm{dual}}},$$

and taking the infimum over all representations $T \circ g$ of f, we conclude that $\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}} \leq \|f\|_{\mathcal{I}^{\mathrm{dual}} \circ \hat{\mathcal{B}}}$. □

An immediate consequence of Theorem 4.7 is the following.

Corollary 4.8. $[\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}, \|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}}] = [\mathcal{I} \circ \hat{\mathcal{B}}, \|\cdot\|_{\mathcal{I} \circ \mathcal{B}}]$ *whenever* $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ *is a completely symmetric normed operator ideal.* □

Since the ideal $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}$ is completely symmetric by [15, Proposition 4.4.7], Corollary 4.8 combined with Theorem 2.2 and [13, Theorems 5.4, 5.6, 5.7 and 5.9] yield the following identifications.

Corollary 4.9. $[\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}, \|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}\text{-dual}}}] = [\hat{\mathcal{B}}_{\mathcal{I}}, \rho_{\mathcal{B}}]$ *for* $\mathcal{I} = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{K}, \mathcal{W}$. □

Since the normed operator ideal $[\mathcal{I}_1, \iota_1]$ is completely symmetric by [10, Theorem 5.15], Corollaries 4.8 and 3.2 give the following result.

Corollary 4.10. $[(\mathcal{I}_1)^{\hat{\mathcal{B}}\text{-dual}}, \|\cdot\|_{(\mathcal{I}_1)^{\hat{\mathcal{B}}\text{-dual}}}] = [\mathcal{I}_1^{\hat{\mathcal{B}}}, \iota_1^{\mathcal{B}}]$. □

5. Conclusions

This study has presented a unified method for generating ideals of Möbius-invariant Bloch mappings by composition of a member of a distinguished Banach operator ideal and a Bloch mapping on the complex unit open disc. Our approach is based on the application of a known technique of linearization of Bloch mappings. The aforementioned method permits an extensive exploration into new classes of Bloch mappings in connection with known Banach operator ideals. This highlights the close interconnection between the linear setting and the holomorphic setting. Notably, the study has drawn meaningful results which contributes to understanding of richness of Bloch mappings.

Funding. This research has been supported in part by grant PID2021-122126NB-C31 funded by MCIN/AEI/ 10.13039/501100011033 and by "ERDF A way of making Europe", and by Junta de Andalucía grant FQM194.

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CONSTRUCTIVE MATHEMATICAL ANALYSIS
7 (2024), No. 3, pp. 114-125
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

Weighted approximations by sampling type operators: recent and new results

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ABSTRACT. In this paper, we collect some recent results on the approximation properties of generalized sampling operators and Kantorovich operators, focusing on pointwise and uniform convergence, rate of convergence, and Voronovskaya-type theorems in weighted spaces of functions. In the second part of the paper, we introduce a new generalization of sampling Durrmeyer operators including a special function ρ which satisfies certain assumptions. For the family of newly constructed operators, we obtain pointwise convergence, uniform convergence and rate of convergence for functions belonging to weighted spaces of functions.

Keywords: Sampling series, generalized sampling operator, Kantorovich operator, Durrmeyer type sampling operator, weighted approximation.

2020 Mathematics Subject Classification: 41A25, 41A35.

1. INTRODUCTION

The reconstruction of a function from its sample values is an extensively studied problem in approximation theory. Butzer and his school extended the approximation to the entire real axis (see [9, 10, 11, 12]) by defining the family of generalized sampling operators:

$$(S_w^\phi f)(t) = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\phi(wt-k), \quad x\in\mathbb{R}, w>0, \tag{1.1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a kernel that meets specific approximate identities, and $f : \mathbb{R} \to \mathbb{R}$ is a bounded, continuous function on $\mathbb{R}$.

The series given by (1.1) is meaningful for functions that make the series converge and provides an approximation method in the case, where the function f is continuous. However, the reconstruction problem of functions that do not need to be continuous was solved in [7], by replacing the data (sample) points $\frac{k}{w}$, for $k \in \mathbb{Z}$ and $w > 0$, with the integral mean value $w\int_{\frac{k}{w}}^{\frac{k+1}{w}} f(u)\,du$ and defining the generalized sampling Kantorovich operators, which are the L^1 version of the generalized sampling operators

$$(K_w^\chi f)(x) = \sum_{k\in\mathbb{Z}} \left\{ w\int_{\frac{k}{w}}^{\frac{k+1}{w}} f(u)\,du \right\} \chi(wx-k), \quad x\in\mathbb{R}. \tag{1.2}$$

Here, $f : \mathbb{R} \to \mathbb{R}$ is a locally integrable function and $\chi : \mathbb{R} \to \mathbb{R}$ is a kernel satisfying certain suitable conditions.

Received: 04.06.2024; Accepted: 13.08.2024; Published Online: 19.08.2024
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DOI: 10.33205/cma.1528004

The generalized sampling Kantorovich operators, represented by (1.2), have been effectively utilized in the engineering fields. Significant numerical results have been obtained, particularly in the study of thermal bridges and the behavior of buildings under seismic actions using thermographic images (see, [6, 13, 18]).

While the sampling Kantorovich series offers an approximation for functions belonging to the L^1 space, it does not provide an approximation for functions in L^p spaces. To solve this problem, C. Bardaro and I. Mantellini [14] introduced the sampling Durrmeyer series, meaningful for L^p, $1 \leq p < \infty$, by taking the convolution of function f with a kernel function instead of the mean values of f. This is given by

$$(S_w^{\varphi,\psi} f)(x) := \sum_{k\in\mathbb{Z}} \varphi(wx-k)w \int_{\mathbb{R}} \psi(wu-k)f(u)\, du, \quad x \in \mathbb{R},\, w > 0. \tag{1.3}$$

For more recent papers about sampling type series see [16, 17].

2. Preliminaries

Throughout this paper, we denote the sets of all positive integers, integers, and real numbers by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$, respectively. The space of all continuous functions on $\mathbb{R}$ (not necessarily bounded) is represented by $C(\mathbb{R})$. The space of all bounded continuous functions on $\mathbb{R}$, denoted by $C_B(\mathbb{R})$, is equipped with the norm $\|f\| := \sup_{x\in\mathbb{R}} |f(x)|$. Additionally, $UC(\mathbb{R})$ refers to the subspace of $C_B(\mathbb{R})$ that includes all uniformly continuous functions and for $r \in \mathbb{R}$, we denote the space of $C^r(\mathbb{R})$ which consists of all r-times continuously differentiable functions on $\mathbb{R}$.

A function $\chi : \mathbb{R} \to \mathbb{R}$ is called a kernel function if it satisfies the following assumptions:

(χ1) χ is continuous on $\mathbb{R}$.

(χ2) The discrete algebraic moment of order 0

$$m_0(\chi, u) = \sum_{k\in\mathbb{Z}} \chi(u-k) = 1 \quad \text{for every } u \in \mathbb{R}.$$

(χ3) There exists $\beta > 0$ such that the discrete absolute moment of order β is finite, i.e.,

$$M_\beta(\chi) = \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u-k)||u-k|^\beta < +\infty.$$

Lemma 2.1 ([7]). *Let χ be a kernel satisfying (χ1) and (χ3). For every $\delta > 0$ there holds:*

$$\lim_{w\to+\infty} \sum_{|k-wx|>w\delta} |\chi(wx-k)| = 0$$

uniformly with respect to $x \in \mathbb{R}$.

From [15, Lemma 2.1 (i)], if χ satisfies the assumptions (χ1) and (χ3), it follows that

$$M_\gamma(\chi) = \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u-k)||u-k|^\gamma < +\infty$$

for every $0 \leq \gamma \leq \beta$.

Now, we recall the weighted spaces of continuous functions. A function ω is said to be a weight function if it is a positive continuous function on the whole real axis $\mathbb{R}$. Here, we consider the weight function

$$\omega(x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}.$$

By $B_\omega(\mathbb{R})$, we denote the space

$$B_\omega(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} : \sup_{x\in\mathbb{R}} \omega(x)|f(x)| \in \mathbb{R} \right\}.$$

The following natural subspaces of $B_w(\mathbb{R})$ will be used in the rest of the paper

$$C_\omega(\mathbb{R}) := C(\mathbb{R}) \cap B_\omega(\mathbb{R}),$$

$$C^*_\omega(\mathbb{R}) := \left\{ f \in C_\omega(\mathbb{R}) : \lim_{x\to\pm\infty} \omega(x)f(x) \in \mathbb{R} \right\},$$

$$U_\omega(\mathbb{R}) := \{f \in C_\omega(\mathbb{R}) : \omega f \text{ is uniformly continuous}\}.$$

The linear space of functions $B_\omega(\mathbb{R})$, and its above subspaces are normed spaces with the norm

$$\|f\|_\omega := \sup_{x\in\mathbb{R}} \omega(x)|f(x)|$$

(see [3, 4, 5, 8, 19, 20]).

The weighted modulus of continuity, considered in [22] and denoted by $\Omega(f;\cdot)$ is defined for $f \in C_\omega(\mathbb{R})$ by

$$\Omega(f;\delta) = \sup_{|h|<\delta, x\in\mathbb{R}} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)} \quad \text{for } \delta > 0. \tag{2.4}$$

Some elementary properties of $\Omega(f;\delta)$ are collected in the following lemma.

Lemma 2.2 ([22]). *Let $\delta > 0$, $x \in \mathbb{R}$. Then,*

(i) *$\Omega(f;\delta)$ is an increasing function of δ,*
(ii) *$\lim_{\delta\to0^+} \Omega(f;\delta) = 0$ when $f \in C^*_\omega(\mathbb{R})$,*
(iii) *For each $\lambda > 0$ and $f \in C_\omega(\mathbb{R})$,*

$$\Omega(f;\lambda\delta) \le 2(1+\lambda)\left(1+\delta^2\right)\Omega(f;\delta). \tag{2.5}$$

Remark 2.1 ([1]). *Using the inequality (2.5) with $\lambda = \frac{|y-x|}{\delta}$, $x, y \in \mathbb{R}$, $\delta > 0$ and choosing $0 < \delta \le 1$, we get*

$$|f(y) - f(x)| \le 16(1+x^2)\Omega(f;\delta)\left(1 + \frac{|y-x|^3}{\delta^3}\right)$$

for every $f \in C_\omega(\mathbb{R})$, $x, y \in \mathbb{R}$.

In a very recent paper [23], Turgay and Acar introduced a new generalization of generalized sampling operators (1.1) by considering a special function ρ.

Let $\rho : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function that satisfies the following conditions:

(ρ_1) $\rho \in C(\mathbb{R})$;
(ρ_2) $\rho(0) = 0$, $\lim_{x\to\pm\infty} \rho(x) = \pm\infty$.

Let $\tau \in C(\mathbb{R})$ and $\varphi \in L^1(\mathbb{R})$ be functions such that for every $u, x \in \mathbb{R}$,

$$m_0^\rho(\tau, x) = \sum_{k\in\mathbb{Z}} \tau(\rho(x) - k) = 1, \quad \mathsf{m}_0(\varphi, u) = \int_{\mathbb{R}} \varphi(u)du = 1. \tag{2.6}$$

For any $\beta \in \mathbb{N}_0$, let us define the ρ-algebraic moment of order β of τ and algebric moment of order β of φ, respectively, by

$$m_\beta^\rho(\tau, x) = \sum_{k\in\mathbb{Z}} \tau(\rho(x) - k)(k - \rho(x))^\beta$$

$$\mathsf{m}_\beta(\varphi, u) = \int_{\mathbb{R}} \varphi(u)u^\beta du$$

and for $\alpha \geq 0$ the ρ-absolute moment of order α of τ and absolute moment of order α of φ, respectively, by

$$M_\alpha^\rho(\tau) = \sup_{x\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\tau(\rho(x)-k)||(k-\rho(x))|^\alpha$$

$$\mathcal{M}_\alpha(\varphi) = \int_{\mathbb{R}} |\varphi(u)||u|^\alpha du.$$

From now on, τ will be called a ρ-kernel and φ will be called a kernel, if they satisfy the condition (2.6) such that there exists $\eta, \nu > 0$ with $M_\eta^\tau(\tau) < +\infty$ and $\mathcal{M}_\nu(\varphi) < +\infty$.

Lemma 2.3 ([1]). *Let τ be a kernel satisfying the conditions*

(1) τ is continuous on $\mathbb{R}$,

(2) there exists $\alpha \geq 0$, such that

$$M_\alpha^\rho(\tau) = \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\tau(\rho(u)-k)||k-\rho(u)|^\beta$$

is finite.

For every $\delta > 0$ there holds:

$$\lim_{w\to\infty} \sum_{|k-w\rho(x)|>w\delta} |\tau(w\rho(x)-k)| = 0$$

uniformly with respect to $x \in \mathbb{R}$.

Now, we introduce the modified Durrmeyer type sampling operators as follows

$$(S_w^{\tau,\varphi} f)(x) = \sum_{k\in\mathbb{Z}} \tau(w\rho(x)-k) w \int_{\mathbb{R}} \varphi(wu-k)(fo\rho^{-1})(u)du. \tag{2.7}$$

Remark 2.2. *The operator (2.7) is well-defined if, for example, f is bounded. Indeed, if $|f(x)| \leq L$ for every $x \in \mathbb{R}$, then $fo\rho^{-1}$ is also bounded function. Then*

$$\begin{aligned}|(S_w^{\tau,\varphi} f)(x)| =& \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)||(fo\rho^{-1})(u)|du \\ \leq& L \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)|du \\ \leq& L M_0^\rho(\tau)\mathcal{M}_0(\varphi) < \infty.\end{aligned}$$

Remark 2.3. *In the special case of $\rho(x) = x$ (it is clear that (ρ_1) and (ρ_2) are satisfied), the operators (2.7) reduce to the classical sampling Durrmeyer operators*

$$(S_w^{\tau,\varphi} f)(x) = \sum_{k\in\mathbb{Z}} \tau(wx-k) w \int_{\mathbb{R}} \varphi(wu-k) f(u)du$$

which was introduced in [14].

3. Recent Results

In this section, we present some recent results on the convergence theorems of generalized sampling operators and Kantorovich forms in weighted spaces of continuous functions. The proofs of these theorems are omitted here. For further details, readers are referred to the original sources (see [1, 2]).

3.1. Pointwise and uniform convergences of G_w^χ and K_w^χ in weighted spaces.

Theorem 3.1 ([1]). *Let χ be a kernel satisfying the assumptions (χ_1), (χ_2), and (χ_3) for $\beta = 2$. Moreover, let $f \in C_\omega(\mathbb{R})$ be fixed. Then,*

$$\lim_{w\to\infty} (G_w^\chi f)(x) = f(x)$$

holds for every $x \in \mathbb{R}$. Moreover, if $f \in U_\omega(\mathbb{R})$, then

$$\lim_{w\to\infty} \|G_w^\chi f - f\|_\omega = 0.$$

Theorem 3.2 ([2]). *Let χ be a kernel satisfying (χ_1), (χ_2), and $(\chi 3)$ for $\beta = 2$ and $f \in C_\omega(\mathbb{R})$ be fixed. Then,*

$$\lim_{w\to+\infty} (K_w^\chi f)(x) = f(x)$$

holds for every $x \in \mathbb{R}$. Moreover, if $f \in U_\omega(\mathbb{R})$, then

$$\lim_{w\to+\infty} \|K_w^\chi f - f\|_\omega = 0.$$

3.2. Rate of convergences of G_w^χ and K_w^χ in weighted spaces.

Theorem 3.3 ([1]). *Let χ be a kernel satisfying the assumptions (χ_1), (χ_2), and (χ_3) for $\beta = 3$. Then, for $f \in C_\omega^*(\mathbb{R})$, we have*

$$\|G_w^\chi f - f\|_\omega \le 16\Omega(f; w^{-1})(M_0(\chi) + M_3(\chi)), \quad \text{for } w \ge 1.$$

Theorem 3.4 ([2]). *Let χ be a kernel satisfying the assumptions (χ_1), (χ_2) and $(\chi 3)$ with $\beta = 3$. For $f \in C_\omega^*(\mathbb{R})$, we have*

$$\|K_w^\chi f - f\|_\omega \le 32\Omega(f; w^{-1})[M_0(\chi) + 2M_3(\chi)]$$

for every $w \ge 1$.

3.3. Voronovskaja type formulae for G_w^χ and K_w^χ.

A quantitative form of Voronovskaja theorem for the operators (1.1) was obtained as following.

Theorem 3.5 ([1]). *Let χ be a kernel satisfying the assumptions (χ_1), (χ_2) and (χ_3) for $\beta = 4$. Furthermore, we assume in addition that the first-order algebraic moment of χ is constant, i.e.:*

$$m_1(\chi, x) = m_1(\chi) \in \mathbb{R} \setminus \{0\} \text{ for every } x \in \mathbb{R}.$$

If $f \in C_\omega^*(\mathbb{R})$, then we have for $x \in \mathbb{R}$ that

$$|w(G_w^\chi f)(x) - f(x) - f'(x)m_1(\chi)| \le 16(1 + x^2)\Omega(f'; w^{-1})\,(M_1(\chi) + M_4(\chi)).$$

If we suppose in addition $m_j(\chi, x) = 0$, for every $x \in \mathbb{R}$, for $j = 1, \dots, r-1$, $r \in \mathbb{N}$, that (χ_3) is satisfied for $\beta = r + 3$, and $m_r(\chi, x) = m_r(\chi) \in \mathbb{R} \setminus \{0\}$, for every $x \in \mathbb{R}$, then we have for $f^{(r)} \in C_\omega^*(\mathbb{R})$ that

$$\left| w^r (G_w^\chi f)(x) - f(x) - f^{(r)}(x)\frac{m_r(\chi)}{r!} \right| \le \frac{16}{r!}(1 + x^2)\Omega(f^{(r)}; w^{-1})\,(M_r(\chi) + M_{r+3}(\chi)).$$

Theorem 3.6 ([2]). *Let χ be a kernel satisfying the assumptions (χ_1), (χ_2), and (χ_3) for $\beta = r + 3$, $r \in \mathbb{N}$. Then, for $f \in C^r(\mathbb{R})$ such that $f^{(r)} \in C_\omega^*(\mathbb{R})$, there holds*

$$\left| w\,[(K_w^\chi f)(x) - f(x)] - \sum_{j=1}^{r} \frac{f^{(j)}(x)}{j! w^{j-1}} \sum_{l=0}^{j} \binom{j}{l} \frac{m_l(\chi)}{(j-l+1)} \right|$$
$$\le \frac{2^{r+3}}{w^r r!}(1 + x^2)\Omega\left(f^{(r)}, w^{-1}\right)\left[M_r(\chi) + \frac{M_0(\chi)}{r+1} + 8M_{r+3}(\chi) + \frac{8M_0(\chi)}{r+4}\right].$$

4. New Results

In [23], Turgay and Acar studied the approximation properties of the modified generalized sampling operators in weighted spaces of continuous functions. In this section, we present the approximation properties of the modified Durrmeyer type sampling operators in weighted spaces of continuous functions. For the weight function $\psi : \mathbb{R} \to \mathbb{R}$, defined by $\psi(x) = 1+\rho^2(x)$, we consider the following classes of functions:

$$B_\psi(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \mid \text{for every } x \in \mathbb{R},\ \frac{|f(x)|}{\psi(x)} \leq M_f \right\},$$

$$C_\psi(\mathbb{R}) = C(\mathbb{R}) \cap B_\psi(\mathbb{R}),$$

$$U_\psi(\mathbb{R}) = \left\{ f \in C_\psi(\mathbb{R}) \mid \frac{|f(x)|}{\psi(x)} \text{ is uniformly continuous on } \mathbb{R} \right\},$$

where M_f is a constant depending only on f. These spaces are normed linear spaces with the norm

$$\|f\|_\psi = \sup_{x\in\mathbb{R}} \frac{|f(x)|}{\psi(x)}.$$

The weighted modulus of continuity, defined in [21], is given by

$$\omega_\psi(f;\delta) = \sup_{x,t\in\mathbb{R}, |\rho(t)-\rho(x)|\leq\delta} \frac{|f(t)-f(x)|}{\psi(t)+\psi(x)}$$

for each $f \in C_\psi(\mathbb{R})$ and for every $\delta > 0$. We observe that

$$\omega_\psi(f;0) = 0$$

for every $f \in C_\psi(\mathbb{R})$, and the function $\omega_\psi(f;\delta)$ is nonnegative and nondecreasing with respect to δ for $f \in C_\varphi(\mathbb{R})$. Additionally,

$$\lim_{\delta\to 0} \omega_\psi(f;\delta) = 0$$

for every $f \in U_\psi(\mathbb{R})$ (for more details, see [21]).

We recall the following auxiliary lemma to obtain an estimate for $|f(u) - f(x)|$.

Lemma 4.4 ([21]). *For every $f \in C_\psi(\mathbb{R})$ and $\delta > 0$*

$$|f(u) - f(x)| \leq (\psi(u) + \psi(x)) \left(2 + \frac{|\rho(u)-\rho(x)|}{\delta}\right) \omega_\psi(f,\delta) \tag{4.8}$$

holds for all $x, u \in \mathbb{R}$.

Remark 4.4 ([23]). *If we consider inequality (4.8), since*

$$\psi(u) + \psi(x) \leq \delta^2 + 2\rho^2(x) + 2|\rho(x)|\delta \quad \textit{whenever} \quad |\rho(u)-\rho(x)| \leq \delta,$$

and

$$\psi(u) + \psi(x) \leq \left(\delta^2 + 2\rho^2(x) + 2|\rho(x)|\delta\right) \left(\frac{|\rho(u)-\rho(x)|}{\delta}\right)^2 \quad \textit{whenever} \quad |\rho(u)-\rho(x)| > \delta,$$

by choosing $\delta \leq 1$, it turns out that

$$|f(u) - f(x)| \leq 9(1 + |\rho(x)|)^2 \omega_\varphi(f;\delta) \left(1 + \frac{|\rho(u)-\rho(x)|^3}{\delta^3}\right). \tag{4.9}$$

As a first main result of this section, we present the well-definiteness of the family of operators $(S_w^{\tau;\varphi})$ in weighted spaces of continuous functions. To prove this, we need the following proposition.

Lemma 4.5. *Let τ be a ρ- kernel and φ be a kernel with $\beta = 2$. Further we denote by $v(x) := 1+\rho^2(x)$, $x \in \mathbb{R}$ and for any fixed $w > 0$. Then the following inequality holds:*

$$|(S_w^{\tau,\varphi}v)(x)| \leq M_0^\rho(\tau)\Big(\frac{1}{w}\mathcal{M}_0(\varphi) + \frac{2}{w^2}\mathcal{M}_2(\varphi) + 4\rho^2(x)\mathcal{M}_0(\varphi)\Big) + \frac{4}{w^2}M_2^\rho(\tau)\mathcal{M}_0(\varphi). \tag{4.10}$$

Proof. By using the definition and linearity of the operators, we get

$$\begin{aligned}
|(S_w^{\tau,\varphi}v)(x)| \leq & \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)|(1+\rho^2(\rho^{-1}(u))du \\
\leq & \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)|du + \sum_{k\in\mathbb{Z}} |\tau(w\rho-k)| w \int_{\mathbb{R}} |\varphi(wu-k)|u^2 du \\
\leq & \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)|du \\
& + \frac{1}{w}\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)|(wu-k+k)^2 du \\
\leq & M_0^\rho(\tau)\mathcal{M}_0(\varphi) + \frac{2}{w}\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)|([wu-k]^2+k^2)du \\
\leq & M_0^\rho(\tau)\mathcal{M}_0(\varphi) + \frac{2}{w}\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)|(wu-k)^2 du \\
& + \frac{2}{w}\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)|(k-w\rho(x)+w\rho(x))^2 du \\
\leq & M_0^\rho(\tau)\mathcal{M}_0(\varphi) + \frac{2}{w^2}M_0^\rho(\tau)\mathcal{M}_2(\varphi) \\
& + \frac{4}{w}\Big(\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)|(k-w\rho(x))^2 \int_{\mathbb{R}} |\varphi(wu-k)|du \\
& + w^2\rho^2(x)\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)|du\Big) \\
\leq & M_0^\rho(\tau)\mathcal{M}_0(\varphi) + \frac{2}{w^2}M_0^\rho(\tau)\mathcal{M}_2(\varphi) + \frac{4}{w^2}M_2^\rho(\tau)\mathcal{M}_0(\varphi) + 4\rho^2(x)M_0^\rho(\tau)\mathcal{M}_0(\varphi) \\
\leq & M_0^\rho(\tau)\Big(\mathcal{M}_0(\varphi) + \frac{2}{w^2}\mathcal{M}_2(\varphi) + 4\rho^2(x)\mathcal{M}_0(\varphi)\Big) + \frac{4}{w^2}M_2^\rho(\tau)\mathcal{M}_0(\varphi).
\end{aligned}$$

This completes the proof. □

Now, we give the well definiteness of modified sampling Durrmeyer type operator and some convergence results.

Theorem 4.7. *Let τ be a ρ- kernel and φ be a kernel with $\beta = 2$. For any fixed $w > 0$ the operator $S_w^{\tau,\varphi}$ is a linear operator from $B_\psi(\mathbb{R})$ to $B_\psi(\mathbb{R})$ and the inequality*

$$\|S_w^{\tau,\varphi}f\|_{B_\psi(\mathbb{R})\to B_\psi(\mathbb{R})} \leq M_0^\rho(\tau)\Big(\mathcal{M}_0(\varphi) + \frac{2}{w^2}\mathcal{M}_2(\varphi) + 4\mathcal{M}_0(\varphi)\Big) + \frac{4}{w^2}M_2^\rho(\tau)\mathcal{M}_0(\varphi)$$

holds.

Proof. By using the Lemma (4.5), we can easily obtain the inequality

$$\begin{aligned}
|(S_w^{\tau,\varphi}f)(x)| \leq & \sum_{k\in\mathbb{Z}}|\tau(w\rho(x)-k)|w\int_{\mathbb{R}}|\varphi(wu-k)|f(\rho^{-1}(u))du \\
= & \sum_{k\in\mathbb{Z}}|\tau(w\rho(x)-k)|w\int_{\mathbb{R}}|\varphi(wu-k)|\frac{f(\rho^{-1}(u))}{1+\rho^2(\rho^{-1}(u))}(1+\rho^2(\rho^{-1}(u))du \\
\leq & \|f\|_{\psi}\sum_{k\in\mathbb{Z}}|\tau(w\rho(x)-k)|w\int_{\mathbb{R}}|\varphi(wu-k)|(1+\rho^2(\rho^{-1}(u))du \\
\leq & \|f\|_{\psi}\Big[M_0^{\rho}(\tau)\Big(\mathcal{M}_0(\varphi)+\frac{2}{w^2}\mathcal{M}_2(\varphi)+4\rho^2(x)\mathcal{M}_0(\varphi)\Big)+\frac{4}{w^2}M_2^{\rho}(\tau)\mathcal{M}_0(\varphi)\Big].
\end{aligned}$$

Now if we multiply both sides with $\frac{1}{1+\rho^2(x)}$, we get

$$\frac{|(S_w^{\tau,\varphi}f)(x)|}{1+\rho^2(x)} \leq \|f\|_{\psi}\Big[M_0^{\rho}(\tau)\Big(\mathcal{M}_0(\varphi)+\frac{2}{w^2}\mathcal{M}_2(\varphi)+4\mathcal{M}_0(\varphi)\Big)+\frac{4}{w^2}M_2^{\rho}(\tau)\mathcal{M}_0(\varphi)\Big]$$

for every $x\in\mathbb{R}$. By assumptions, we conclude that $\|S_w^{\tau,\varphi}f\|_{\psi}<+\infty$ that is $S_w^{\tau,\varphi}\in B_{\psi}(\mathbb{R})$. Now taking supremum over $x\in\mathbb{R}$ and the supremum with respect to $f\in B_{\psi}(\mathbb{R})$ with $\|f\|\leq 1$, it turns out

$$\|S_w^{\tau,\varphi}\|_{B_{\psi}(\mathbb{R})\to B_{\psi}(\mathbb{R})} \leq M_0^{\rho}(\tau)\Big(\mathcal{M}_0(\varphi)+\frac{2}{w^2}\mathcal{M}_2(\varphi)+4\mathcal{M}_0(\varphi)\Big)+\frac{4}{w^2}M_2^{\rho}(\tau)\mathcal{M}_0(\varphi).$$

□

Theorem 4.8. *Let τ be a ρ- kernel and φ be a kernel with $\beta=2$. and $f\in C_{\psi}(\mathbb{R})$. Then, we have*

$$\lim_{w\to\infty}(S_w^{\tau,\varphi}f)(x)=f(x). \tag{4.11}$$

Proof. By straightforward calculations, we have

$$\begin{aligned}
|(S_w^{\tau,\varphi}f)(x)-f(x)| \leq & \sum_{k\in\mathbb{Z}}|\tau(w\rho(x)-k)|w\int_{\mathbb{R}}|\varphi(wu-k)|\Big[(fo\rho^{-1})(u)-f(x)\Big]du \\
\leq & \sum_{k\in\mathbb{Z}}|\tau(w\rho(x)-k)|w\int_{\mathbb{R}}|\varphi(wu-k)|\Big\{\frac{|(fo\rho^{-1})(u)|}{(\psi o\rho^{-1})(u)}|(\psi o\rho^{-1})(u)-\psi(x)| \\
& +\psi(x)\Big|\frac{|(fo\rho^{-1})(u)|}{(\psi o\rho^{-1})(u)}-\frac{f(x)}{\psi(x)}\Big|\Big\}du \\
:= & I_1+I_2. \qquad (4.12)
\end{aligned}$$

Let's first estimate I_1. Since $f\in C_{\psi}(\mathbb{R})$, we have

$$
\begin{aligned}
I_1 \leq & \|f\|_\psi \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)||u^2-\rho^2(x)| du \\
\leq & \|f\|_\psi \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)| \Big[|u-\rho(x)||u+\rho(x)| \Big] du \\
\leq & \frac{\|f\|_\psi}{w} \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)||wu-w\rho(x)||wu+w\rho(x)| du \\
\leq & \frac{\|f\|_\psi}{w} \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)| \Big\{ (|wu-k|+|k-w\rho(x)|)(|wu-k|+|k+w\rho(x)|) \Big\} du \\
\leq & \frac{\|f\|_\psi}{w} \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)| \Big\{ |wu-k|^2 + |wu-k||k+w\rho(x)| \\
& + |k-w\rho(x)||wu-k| + |k-w\rho(x)||k+w\rho(x)| \Big\} du \\
\leq & \frac{\|f\|_\psi}{w} \Big[\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)||wu-k|^2 du \\
& + \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)||k+w\rho(x)| \int_{\mathbb{R}} |\varphi(wu-k)||wu-k| du \\
& + \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)||k-w\rho(x)| \int_{\mathbb{R}} |\varphi(wu-k)||wu-k| du \\
& + \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)||k-w\rho(x)||k+w\rho(x)| \int_{\mathbb{R}} |\varphi(wu-k)| du
\end{aligned}
\tag{4.13}
$$

$$
:= I_{1.1} + I_{1.2} + I_{1.3} + I_{1.4}.
$$

Since $|k+w\rho(x)| = |k-w\rho(x)+2w\rho(x)| \leq |k-w\rho(x)| + |2\rho(x)|$, it is clear to see that the following statements hold

$$
\begin{aligned}
I_{1.1} = & \frac{\|f\|_\psi}{w^2} M_0^\rho(\tau)\mathcal{M}_2(\varphi), \\
I_{1.2} = & \frac{\|f\|_\psi}{w^2} \Big[M_1^\rho(\tau)\mathcal{M}_1(\varphi) + 2|w\rho(x)| M_0^\rho(\tau)\mathcal{M}_1(\varphi) \Big], \\
I_{1.3} = & \frac{\|f\|_\psi}{w^2} M_1^\rho(\tau)\mathcal{M}_1(\varphi), \\
I_{1.4} = & \frac{\|f\|_\psi}{w^2} \Big[M_2^\rho(\tau)\mathcal{M}_0(\varphi) + 2|w\rho(x)| M_1^\rho(\tau)\mathcal{M}_0(\varphi) \Big].
\end{aligned}
$$

If we substitute $I_{1.1}, I_{1.2}, I_{1.3}$ and $I_{1.4}$ in (4.13), we can get

$$
\begin{aligned}
I_1 \leq & \frac{\|f\|_\psi}{w^2} \Big[M_0^\rho(\tau)\mathcal{M}_2(\varphi) + M_1^\rho(\tau)\mathcal{M}_1(\varphi) + 2w|\rho(x)| M_0^\rho(\tau)\mathcal{M}_1(\varphi) + M_1^\rho(\tau)\mathcal{M}_1(\varphi) + M_2^\rho(\tau)\mathcal{M}_0(\varphi) \\
& + 2w|\rho(x)| M_1^\rho(\tau)\mathcal{M}_0(\varphi) \Big] \\
= & \frac{\|f\|_\psi}{w^2} \Big[M_0^\rho(\tau)\mathcal{M}_0(\varphi) + 2M_1^\rho(\tau)\mathcal{M}_1(\varphi) + 2w|\rho(x)| \Big(M_0^\rho(\tau)\mathcal{M}_1(\varphi) + M_1^\rho(\tau)\mathcal{M}_0(\varphi) \Big) \\
& + M_2^\rho(\tau)\mathcal{M}_0(\varphi) \Big).
\end{aligned}
$$

Now lets consider I_2. Let $x \in \mathbb{R}$ and $\epsilon > 0$ be fixed. Then there exists $\delta > 0$ such that

$$\Big|\frac{(fo\rho^{-1}(u)}{(\psi o\rho^{-1}(u)} - \frac{f(x)}{\psi(x)}\Big| < \epsilon$$

when $|\rho^{-1}(u) - x| < \delta$. Hence we can write

$$\begin{aligned}
I_2 =& \psi(x) \sum_{k\in\mathbb{Z}} |\tau(w\rho(x) - k)| w \int_{\mathbb{R}} |\varphi(wu - k)| \Big|\frac{(fo\rho^{-1}(u)}{(\psi o\rho^{-1}(u)} - \frac{f(x)}{\psi(x)}\Big| du \\
=& w\psi(x) \Big[\sum_{|k-w\rho(x)|<\frac{w\delta}{2}} |\tau(w\rho(x) - k)| \int_{|wu-k|\leq\frac{w\delta}{2}} |\varphi(wu - k)| \Big|\frac{(fo\rho^{-1}(u)}{(\psi o\rho^{-1}(u)} - \frac{f(x)}{\psi(x)}\Big| du \\
&+ \sum_{|k-w\rho(x)|\leq\frac{w\delta}{2}} |\tau(w\rho(x) - k)| \int_{|wu-k|>\frac{w\delta}{2}} |\varphi(wu - k)| \Big|\frac{(fo\rho^{-1}(u)}{(\psi o\rho^{-1}(u)} - \frac{f(x)}{\psi(x)}\Big| du \\
&+ \sum_{|k-w\rho(x)|>\frac{w\delta}{2}} |\tau(w\rho(x) - k)| \int_{\mathbb{R}} |\varphi(wu - k)| \Big|\frac{(fo\rho^{-1}(u)}{(\psi o\rho^{-1}(u)} - \frac{f(x)}{\psi(x)}\Big| du \Big] \\
:=& w\psi(x) \Big[I_{2.1} + I_{2.2} + I_{2.3} \Big].
\end{aligned}$$

For $|k - w\rho(x)| \leq \dfrac{w\delta}{2}$, if $|wu - k| \leq \dfrac{w\delta}{2}$, we have

$$|u - \rho(x)| \leq |u - \frac{k}{w}| + |\frac{k}{w} - \rho(x)| < \delta.$$

Since $f \in C_\psi(\mathbb{R})$, we get

$$I_{2.1} \leq \epsilon M_0^\rho(\tau)\mathcal{M}_0(\varphi).$$

Taking supremum for $u \in \mathbb{R}$, we can write

$$I_{2.2} \leq 2\|f\|_\psi \sum_{|k-w\rho(x)|\leq\frac{w\delta}{2}} |\tau(w\rho(x) - k| \int_{|wu-k|>\frac{w\delta}{2}} |\varphi(wu - k)| du$$

and $\int_{|wu-k|>\frac{w\delta}{2}} |\varphi(wu - k)| du = \int_{|t|>\frac{w\delta}{2}} |\varphi(t)| dt \to 0$ as $w \to \infty$ for sufficiently large w. Hence, we get

$$I_{2.2} \leq \frac{2}{w}\|f\|_\psi M_0^\rho(\tau)\epsilon.$$

Finally, by Lemma (2.3), since

$$\lim_{w\to+\infty} \sum_{|k-w\rho(x)|>\frac{w\delta}{2}} |\tau(w\rho(x) - k)| = 0,$$

then we get

$$I_{2.3} \leq \frac{2}{w}\|f\|_\psi \mathcal{M}_0(\varphi)\epsilon$$

for sufficiently large w. Combining the above estimates, we have

$$\begin{aligned}
|(S_w^\tau, \varphi f)(x) - f(x)| \leq& I_1 + I_{2.1} + I_{2.2} + I_{2.3} \\
\leq& \frac{\|f\|_\psi}{w^2} \Big[M_0^\rho(\tau)\mathcal{M}_2(\varphi) + 3M_1^\rho(\tau)\mathcal{M}_1(\varphi) + |4\rho(x)| M_0^\rho(\tau)\mathcal{M}_1(\varphi) \Big] \\
&+ \psi(x) \Big[\epsilon \Big(M_0^\rho(\tau)\mathcal{M}_0(\varphi) + 2\|f\|_\psi M_0^\rho(\tau) + 2\|f\|_\psi \mathcal{M}_0(\varphi) \Big) \Big]. \qquad (4.14)
\end{aligned}$$

By taking limit as $w \to \infty$, we get the desired result. □

Theorem 4.9. *Let τ be a ρ- kernel and φ be a kernel with $\beta = 2$ and $\dfrac{fo\rho^{-1}}{\varphi o\rho^{-1}} \in U_\psi(\mathbb{R})$. Then*

$$\lim_{w\to\infty} \|S_w^{\tau,\varphi} f - f\|_\psi = 0$$

holds.

Proof. For functions $f \in U_\psi(\mathbb{R})$, let us follow the same steps with the proof of Theorem (4.11) and replace δ with corresponding parameter of the uniform continuity of $\dfrac{fo\rho^{-1}}{\psi o\rho^{-1}} \in U_\psi(\mathbb{R})$ also considering the inequality (4.14), we have

$$\begin{aligned}\frac{|(S_w^{\tau,\varphi} f(x) - f(x)|}{\psi(x)} \leq & \frac{\|f\|_\varphi}{w^2\psi(x)}\Big[M_0^\rho(\tau)\mathcal{M}_2(\varphi) + 3M_1^\rho(\tau)\mathcal{M}_1(\varphi) + |4\rho(x)|M_0^\rho(\tau)\mathcal{M}_1(\varphi)\Big] \\ & + \epsilon\Big(M_0^\rho(\tau)\mathcal{M}_0(\varphi) + 2\|f\|_\psi M_0^\rho(\tau) + 2\|f\|_\psi \mathcal{M}_0(\varphi)\Big)\end{aligned}$$

and taking supremum over $x \in \mathbb{R}$ we obtain the desired result. □

Theorem 4.10. *Let τ be a ρ- kernel and φ be a kernel with $\beta = 3$. Then for $f \in C_\psi(\mathbb{R})$, we get*

$$|(S_w^{\tau,\varphi} f)(x) - f(x)| \leq 9(1+|\rho(x)|)^2\omega_\varphi(fo\rho^{-1}; w^{-1})\Big(M_0^\rho(\tau)\mathcal{M}_0(\varphi) + 4(M_0^\rho(\tau)\mathcal{M}_3(\varphi) + M_3^\rho(\tau)\mathcal{M}_0(\varphi)\Big).$$

Proof. Using the definition of the operators $S_w^{\tau,\varphi}$ and (4.9), we have

$$\begin{aligned}|(S_w^{\tau,\varphi} f)(x) - f(x)| \leq & \sum_{k\in\mathbb{Z}} |\tau(w\rho(x) - k)| w \int_{\mathbb{R}} |\varphi(wu-k)| |(fo\rho^{-1}(u) - f(x)| du \\ \leq & 9w(1+|\rho(x)|)^2\omega_\varphi(f;\delta) \sum_{k\in\mathbb{Z}} |w\rho(x) - k| \int_{\mathbb{R}} |\varphi(wu-k)| \Big(1 + \frac{|u-\rho(x)|^3}{\delta^3}\Big) du \\ = & 9w(1+|\rho(x)|)^2\omega_\varphi(f;\delta)\Big[\sum_{k\in\mathbb{Z}} |w\rho(x) - k| \int_{\mathbb{R}} |\varphi(wu-k)| du \\ & + \frac{1}{\delta^3}\sum_{k\in\mathbb{Z}} |w\rho(x) - k| \int_{\mathbb{R}} |\varphi(wu-k)| |u-\rho(x)|^3 du \\ = & 9w(1+|\rho(x)|)^2\omega_\varphi(fo\rho^{-1};\delta)(I_1 + I_2).\end{aligned}$$

It can be easily seen that

$$I_1 \leq \frac{1}{w} M_0^\rho(\tau)\mathcal{M}_0(\varphi).$$

Now, we need to estimate I_2. By elementary calculations, we have

$$\begin{aligned}I_2 \leq & \frac{1}{\delta^3}\sum_{k\in\mathbb{Z}} |w\rho(x) - k| \int_{\mathbb{R}} |\varphi(wu-k)\Big(|u - \frac{k}{w}| + |\frac{k}{w} - \rho(x)|\Big)^3 du \\ \leq & \frac{4}{\delta^3 w^3}\sum_{k\in\mathbb{Z}} |w\rho(x) - k| \int_{\mathbb{R}} |\varphi(wu-k)\Big(|wu-k|^3 + |k - w\rho(x)|^3\Big) du \\ = & \frac{4}{\delta^3 w^4}\Big(M_0^\rho(\tau)\mathcal{M}_3(\varphi) + M_3^\rho(\tau)\mathcal{M}_0(\varphi)\Big).\end{aligned}$$

Substituting I_1 and I_2 and choosing $\delta = w^{-1}$, we immediately obtain the result.

□

Acknowledgements This study was supported by Scientific and Technological Research Council of Turkey (TUBITAK) under the Grant Number 119F263. The author thank to TUBITAK for their supports.

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CONSTRUCTIVE MATHEMATICAL ANALYSIS
7 (2024), No. 3, pp. 126-133
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

C-symmetric Toeplitz operators on Hardy spaces

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ABSTRACT. We characterize all the Toeplitz operators that are complex symmetric with respect to a class of conjugations induced by a permutation. Our results provide an affirmative answer to a conjecture from a paper of Chattopadhyay et al. (2023) [1].

Keywords: Toeplitz operators, conjugations, complex symmetric, Hardy spaces.

2020 Mathematics Subject Classification: 47B35, 47A05.

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and $\mathbb{T}$ be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. The Hardy space H^2 of $\mathbb{D}$ consists of all analytic functions f on $\mathbb{D}$ such that

$$\sup_{0\le r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 dm < \infty,$$

where m is the normalized Lebesgue measure on $\mathbb{T}$, i.e., $dm := d\theta/2\pi$. If $f \in H^2$, its radial limit

$$f^*(e^{i\theta}) := \lim_{r\to 1^-} f(re^{i\theta})$$

exists m-a.e. on $\mathbb{T}$ and the mapping $f \mapsto f^*$ is an isometry of H^2 onto a closed subspace of $L^2(\mathbb{T}, dm)$. The extension of f to $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$, also denoted by f, is defined such that $f|_{\mathbb{T}} = f^*$. It is known that H^2 is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g\rangle_{H^2} := \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, dm \quad \text{for all } f, g \in H^2.$$

The standard orthonormal basis for H^2 is $\{1, z, z^2, \dots\}$. Given $\phi \in L^\infty(\mathbb{T})$, the Toeplitz operator $T_\phi : H^2 \to H^2$ is defined by

$$T_\phi f = P(\phi f) \quad \text{for every } f \in H^2,$$

where P is the orthogonal projection from $L^2(\mathbb{T}, dm)$ onto H^2. A more detailed introduction of Toeplitz operators is available in [2, 18]. These operators have also been studied extensively in the literature, for example in [3, 4, 10, 11].

Let $\mathcal{H}$ be a separable complex Hilbert space. A mapping $A : \mathcal{H} \to \mathcal{H}$ is said to be a conjugation if it satisfies the following conditions:

(i) anti-linear (or conjugate-linear), i.e., $A(ax + by) = \overline{a}Ax + \overline{b}Ay$ for all $x, y \in \mathcal{H}$ and $a, b \in \mathbb{C}$,

Received: 23.06.2024; Accepted: 22.08.2024; Published Online: 27.08.2024
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DOI: 10.33205/cma.1503726

(ii) involutive, i.e., $A^2 = I$, the identity operator, and
(iii) isometric, i.e., $\|Ax\| = \|x\|$ for each $x \in \mathcal{H}$.

The adjoint A^* of a bounded and anti-linear operator A is defined to satisfy the property that

$$\langle Ax, y\rangle = \overline{\langle x, A^*y\rangle} = \langle A^*y, x\rangle \quad \text{for all } x, y \in \mathcal{H}.$$

In view of (ii) and (iii), we also have $A^* = A$, i.e., A is self-adjoint.

A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is said to be complex symmetric if there exists a conjugation $C : \mathcal{H} \to \mathcal{H}$ such that $CTC = T^*$ (or equivalently, $CT^* = TC$). In this case, we say T is C-symmetric (or complex symmetric with respect to C). The study of complex symmetric operators was initiated by Garcia et al. in [5, 6, 7, 8]. These operators play a significant role in control theory, signal processing and non-Hermitian quantum mechanics. Examples of complex symmetric maps include normal operators, Hankel operators, Volterra operators and truncated Toeplitz operators.

Investigation of the complex symmetry of Toeplitz operators on Hilbert spaces of analytic functions was motivated by [9], in which the question of characterizing complex symmetric Toeplitz operators on H^2 was posed. Ko and Lee provided a necessary and sufficient condition for T_ϕ to be complex symmetric with respect to a special class of conjugations on H^2 [12, Theorem 2.4]. This result, together with [13, Theorem 2.11(a)] and [16, Theorem 3.6], was generalized by the authors in [17, Theorem 3.4]. Complex symmetric Toeplitz operators on Bergman and Dirichlet spaces have been studied in [14] and [15], respectively.

Let p be a fixed positive integer. In [1], Chattopadhyay et al. introduced the conjugation $C_\sigma : H^2 \to H^2$ defined as

$$C_\sigma\left(\sum_{l=0}^{\infty}\sum_{r=0}^{p-1} a_{pl+r}z^{pl+r}\right) = \sum_{l=0}^{\infty}\sum_{r=0}^{p-1} \overline{\sigma(a_{pl+r})}z^{pl+r},$$

where

(i) $\sum_{l=0}^{\infty}\sum_{r=0}^{p-1} a_{pl+r}z^{pl+r} \in H^2$,
(ii) σ is a permutation on the set $\{a_{pl}, a_{pl+1}, \ldots, a_{pl+p-1}\}$ for $l = 0, 1, \ldots,$ and
(iii) the order of σ is 2.

A special case of C_σ is the operator $C_p^{i,j} : H^2 \to H^2$ given by

$$C_p^{i,j}\left(\sum_{l=0}^{\infty}\sum_{r=0}^{p-1} a_{pl+r}z^{pl+r}\right) = \sum_{l=0}^{\infty} \overline{a_{pl+j}}z^{pl+i} + \sum_{l=0}^{\infty} \overline{a_{pl+i}}z^{pl+j} + \sum_{l=0}^{\infty}\sum_{\substack{r=0\\ r\neq i,j}}^{p-1} \overline{a_{pl+r}}z^{pl+r},$$

where i, j are any fixed integers such that $0 \le i < j < p$. They characterized all $C_p^{i,j}$-symmetric Toeplitz operators on H^2 with additional assumptions on i, j and p [1, Theorem 2.2]. In the next section, we provide an affirmative answer to Remark 2.3 therein that these characterizations are valid whenever $0 \le i < j < p$.

2. Main Results

We will characterize all the Toeplitz operators T_ϕ on H^2 that are complex symmetric with respect to $C_p^{i,j}$ in terms of the Fourier coefficients of ϕ.

Theorem 2.1. *Let $\phi(z) = \sum_{n=-\infty}^{\infty} \hat{\phi}(n)z^n \in L^\infty(\mathbb{T})$ and i, j, p be integers such that $0 \le i < j < p$. Then the operator T_ϕ is $C_p^{i,j}$-symmetric if and only if*

$$\hat{\phi}(pl) = \hat{\phi}(-pl)$$

and

$$\hat{\phi}(pl+r)=0$$

for every integer l and $r=1,2,\ldots,p-1$.

Proof. Suppose $\hat{\phi}(pl)=\hat{\phi}(-pl)$ and $\hat{\phi}(pl+r)=0$ for all integers l and $r=1,2,\ldots,p-1$. Let l' be any fixed non-negative integer. Then,

$$\begin{aligned}
T_\phi C_p^{i,j} z^{pl'+i} &= T_\phi z^{pl'+j}\\
&= P\left(\hat{\phi}(0)z^{pl'+j}+\sum_{l=1}^{\infty}\hat{\phi}(pl)(z^{p(l+l')+j}+\overline{z}^{pl}z^{pl'+j})\right)\\
&= \begin{cases} \hat{\phi}(0)z^{pl'+j}+\sum_{l=1}^{\infty}\hat{\phi}(pl)z^{p(l+l')+j} & \text{if } l'=0;\\ \hat{\phi}(0)z^{pl'+j}+\sum_{l=1}^{\infty}\hat{\phi}(pl)z^{p(l+l')+j}+\sum_{l=1}^{l'}\hat{\phi}(pl)z^{p(l'-l)+j} & \text{if } l'\geq 1;\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
C_p^{i,j}T_\phi^* z^{pl'+i} &= C_p^{i,j}T_{\overline{\phi}}z^{pl'+i}\\
&= C_p^{i,j}P\left(\overline{\hat{\phi}(0)}z^{pl'+i}+\sum_{l=1}^{\infty}\overline{\hat{\phi}(pl)}(z^{p(l+l')+i}+\overline{z}^{pl}z^{pl'+i})\right)\\
&= \begin{cases} C_p^{i,j}\left(\overline{\hat{\phi}(0)}z^{pl'+i}+\sum_{l=1}^{\infty}\overline{\hat{\phi}(pl)}z^{p(l+l')+i}\right) & \text{if } l'=0;\\ C_p^{i,j}\left(\overline{\hat{\phi}(0)}z^{pl'+i}+\sum_{l=1}^{\infty}\overline{\hat{\phi}(pl)}z^{p(l+l')+i}+\sum_{l=1}^{l'}\overline{\hat{\phi}(pl)}z^{p(l'-l)+i}\right) & \text{if } l'\geq 1;\end{cases}\\
&= \begin{cases} \hat{\phi}(0)z^{pl'+j}+\sum_{l=1}^{\infty}\hat{\phi}(pl)z^{p(l+l')+j} & \text{if } l'=0;\\ \hat{\phi}(0)z^{pl'+j}+\sum_{l=1}^{\infty}\hat{\phi}(pl)z^{p(l+l')+j}+\sum_{l=1}^{l'}\hat{\phi}(pl)z^{p(l'-l)+j} & \text{if } l'\geq 1.\end{cases}
\end{aligned}$$

Similarly,

$$T_\phi C_p^{i,j}z^{pl'+j}=C_p^{i,j}T_\phi^* z^{pl'+j}.$$

When $r=0,1,\ldots,p-1$ with $r\neq i,j$, we have

$$\begin{aligned}
T_\phi C_p^{i,j}z^{pl'+r} &= C_p^{i,j}T_\phi^* z^{pl'+r}\\
&= \begin{cases} \hat{\phi}(0)z^{pl'+r}+\sum_{l=1}^{\infty}\hat{\phi}(pl)z^{p(l+l')+r} & \text{if } l'=0;\\ \hat{\phi}(0)z^{pl'+r}+\sum_{l=1}^{\infty}\hat{\phi}(pl)z^{p(l+l')+r}+\sum_{l=1}^{l'}\hat{\phi}(pl)z^{p(l'-l)+r} & \text{if } l'\geq 1.\end{cases}
\end{aligned}$$

Thus, $T_\phi C_p^{i,j}=C_p^{i,j}T_\phi^*$.

Conversely, assume T_ϕ is $C_p^{i,j}$-symmetric. Let l,m be any non-negative integers. We first show that $\hat{\phi}(n)=0$ for $|n|=pl+i,\ldots,pl+j$. Since

$$\begin{aligned}
\left\langle P(\overline{\phi}z^{pl+i}),C_p^{i,j}z^m\right\rangle_{H^2} &= \left\langle T_{\overline{\phi}}z^{pl+i},C_p^{i,j}z^m\right\rangle_{H^2}=\left\langle z^m,C_p^{i,j}T_\phi^* z^{pl+i}\right\rangle_{H^2}\\
&= \left\langle z^m,T_\phi C_p^{i,j}z^{pl+i}\right\rangle_{H^2}=\left\langle T_{\overline{\phi}}z^m,C_p^{i,j}z^{pl+i}\right\rangle_{H^2}\\
&= \langle\overline{\phi}z^m,z^{pl+j}\rangle_{H^2}=\left\langle\sum_{n=-\infty}^{\infty}\overline{\hat{\phi}(n)}z^{m-n},z^{pl+j}\right\rangle_{H^2}\\
&= \overline{\phi(m-pl-j)}
\end{aligned}$$

and $\{C_p^{i,j} z^m\}_{m=0}^{\infty}$ is an orthonormal basis for H^2, it follows that

$$\left\|P(\overline{\phi} z^{pl+i})\right\|^2 = \sum_{m=0}^{\infty} \left|\left\langle P(\overline{\phi} z^{pl+i}), C_p^{i,j} z^m \right\rangle_{H^2}\right|^2 = \sum_{m=0}^{\infty} |\hat{\phi}(m-pl-j)|^2 = \sum_{n=-pl-j}^{\infty} |\hat{\phi}(n)|^2.$$

On the other hand,

$$P(\overline{\phi} z^{pl+i}) = P\left(\sum_{n=-\infty}^{\infty} \overline{\hat{\phi}(n)} z^{pl+i-n}\right) = \sum_{n=-\infty}^{pl+i} \overline{\hat{\phi}(n)} z^{pl+i-n}$$

and so,

$$\left\|P(\overline{\phi} z^{pl+i})\right\|^2 = \sum_{n=-\infty}^{pl+i} |\hat{\phi}(n)|^2.$$

Thus,

$$\sum_{n=-pl-j}^{\infty} |\hat{\phi}(n)|^2 = \sum_{n=-\infty}^{pl+i} |\hat{\phi}(n)|^2. \tag{2.1}$$

By considering $P(\overline{\phi} z^{pl+j})$, we obtain

$$\sum_{n=-pl-i}^{\infty} |\hat{\phi}(n)|^2 = \sum_{n=-\infty}^{pl+j} |\hat{\phi}(n)|^2 \tag{2.2}$$

in a similar fashion. From (2.1) and (2.2),

$$\sum_{n=-pl-j}^{-pl-i-1} |\hat{\phi}(n)|^2 = -\sum_{n=pl+i+1}^{pl+j} |\hat{\phi}(n)|^2$$

which implies

$$\hat{\phi}(n) = 0 \quad \text{for } |n| = pl+i+1, \ldots, pl+j. \tag{2.3}$$

Note that

$$\begin{aligned} T_\phi C_p^{i,j} z^{pl+i} = T_\phi z^{pl+j} &= P\left(\sum_{n=-\infty}^{\infty} \hat{\phi}(n) z^{pl+j+n}\right) \\ &= \sum_{n=-pl-j}^{\infty} \hat{\phi}(n) z^{pl+j+n} = \sum_{n=0}^{\infty} \hat{\phi}(n-pl-j) z^n \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} C_p^{i,j} T_\phi^* z^{pl+i} &= C_p^{i,j} P\left(\sum_{n=-\infty}^{\infty} \overline{\hat{\phi}(n)} z^{pl+i-n}\right) \\ &= C_p^{i,j}\left(\sum_{n=-\infty}^{pl+i} \overline{\hat{\phi}(n)} z^{pl+i-n}\right) \\ &= C_p^{i,j}\left(\sum_{n=0}^{\infty} \overline{\hat{\phi}(pl+i-n)} z^n\right). \end{aligned} \tag{2.5}$$

Similarly,

$$T_\phi C_p^{i,j} z^{pl+j} = \sum_{n=0}^{\infty} \hat{\phi}(n-pl-i) z^n \tag{2.6}$$

and

$$C_p^{i,j} T_\phi^* z^{pl+j} = C_p^{i,j} \left(\sum_{n=0}^{\infty} \overline{\hat{\phi}(pl+j-n)} z^n \right). \tag{2.7}$$

Suppose $i \neq 0$ and k is any integer such that $0 \leq k \leq i-1$. We claim that $\hat{\phi}(n) = 0$ for $|n| = pl + i - k$. Comparing the coefficients of z^k in the right most expressions of (2.4) and (2.5) as well as those of z^k in (2.6) and (2.7), we have

$$\hat{\phi}(k-pl-j) = \hat{\phi}(pl+i-k) \text{ and } \hat{\phi}(k-pl-i) = \hat{\phi}(pl+j-k), \tag{2.8}$$

respectively. Since $i < j$, we also have

$$pl+i-k \leq pl+j-1-k \leq pl+j. \tag{2.9}$$

When $k = 0$, it follows from (2.8) and the fact $\hat{\phi}(-pl-j) = 0 = \hat{\phi}(pl+j)$ that $\hat{\phi}(n) = 0$ for $|n| = pl + i$. Assume there exists an integer k' such that $0 \leq k' < i-1$ and $\hat{\phi}(n) = 0$ if $|n| = pl+i-k', \ldots, pl+i$. By taking $k = k'+1$ and $k = k'$ in (2.8) and (2.9) respectively, it follows from the induction assumption and (2.3) that $\hat{\phi}(pl+i-k'-1) = 0 = \hat{\phi}(k'+1-pl-i)$. Therefore,

$$\hat{\phi}(n) = 0 \text{ for } |n| = pl+1, \ldots, pl+j \tag{2.10}$$

(if $i = 0$, then (2.10) is also true in light of (2.3)).

It remains to show that $\hat{\phi}(n) = 0$ for $|n| = pl + k$, where k is any integer with $j+1 \leq k \leq p-1$. Since p does not divide $2pl + k$, we have $2pl + i + k \neq pl' + i$ for all $l' \in \mathbb{N}$. Moreover, $2pl + i + k \neq pl' + j$ for every $l' \in \mathbb{N}$. Otherwise, $i - j + k = p(l' - 2l)$. This equality is absurd, because $i+1 \leq i-j+k \leq p-1+i-j \leq p-2 < p$. Comparing the coefficients of $z^{2pl+i+k}$ in the right most expressions of (2.4) and (2.5) as well as those of $z^{2pl+i+k}$ in (2.6) and (2.7) gives

$$\hat{\phi}(pl+i-j+k) = \hat{\phi}(-pl-k) \text{ and } \hat{\phi}(pl+k) = \hat{\phi}(-pl-i+j-k), \tag{2.11}$$

respectively. Furthermore,

$$pl+i+1 \leq pl+i-j+k \leq pl+k-1. \tag{2.12}$$

When $k = j+1$, it follows from (2.11) and the fact $\hat{\phi}(pl+i+1) = 0 = \hat{\phi}(-pl-i-1)$ that $\hat{\phi}(n) = 0$ for $|n| = pl + j + 1$. Assume there is an integer k' for which $j+1 \leq k' < p-1$ and $\hat{\phi}(n) = 0$ for $|n| = pl+j+1, \ldots, pl+k'$. Taking $k = k'+1$ in (2.11) and (2.12), it follows from the induction assumption, (2.3), (2.11) and (2.12) that $\hat{\phi}(n) = 0$ for $|n| = pl + k' + 1$.

Hence $\hat{\phi}(pl+r) = 0$ for all integers l and $r = 1, \ldots, p-1$. Comparing the coefficients of z^j in the right most expressions of (2.4) and (2.5), we have

$$\hat{\phi}(-pl) = \hat{\phi}(pl)$$

for all non-negative integers l. The proof of the theorem is now complete. □

Chattopadhyaya et al. introduced another special case of C_σ in [1], namely the operator $C_n : H^2 \to H^2$ defined by

$$C_n \left(\sum_{l=0}^{\infty} \sum_{r=0}^{n-1} a_{nl+r} z^{nl+r} \right) = \sum_{l=0}^{\infty} \sum_{r=0}^{n-1} \overline{a_{nl+n-r-1}} z^{nl+r},$$

where $\sum_{l=0}^{\infty}\sum_{r=0}^{n-1} a_{nl+r}z^{nl+r} \in H^2$ and n is any fixed positive integer. They also obtained the following characterizations for T_ϕ to be C_n-symmetric.

Theorem 2.2. [1, Theorem 3.1] *Let $\phi(z) = \sum_{k=-\infty}^{\infty} \hat{\phi}(k)z^k \in L^\infty(\mathbb{T})$. Then the operator T_ϕ is C_n-symmetric if and only if*

$$\hat{\phi}(nl) = \hat{\phi}(-nl)$$

and

$$\hat{\phi}(nl + r) = 0$$

for every integer l and $r = 1, 2, \ldots, n-1$.

The method adopted in proving Theorem 2.1 furnishes an alternative proof to the necessity part of Theorem 2.2: Assume T_ϕ is C_n-symmetric. Since $C_n z^{nl} = z^{nl+n-1}$, we have

$$\langle P(\overline{\phi}z^{nl}), C_n z^m\rangle_{H^2} = \overline{\hat{\phi}(m - nl - n + 1)}$$

for all non-negative integers l and m. The fact that $\{C_n z^m\}_{m=0}^{\infty}$ is an orthonormal basis for H^2 implies

$$\left\|P(\overline{\phi}z^{nl})\right\|^2 = \sum_{m=0}^{\infty} |\hat{\phi}(m - nl - n + 1)|^2 = \sum_{k=-nl-n+1}^{\infty} |\hat{\phi}(k)|^2.$$

Moreover,

$$\left\|P(\overline{\phi}z^{nl})\right\|^2 = \sum_{k=-\infty}^{nl} |\hat{\phi}(k)|^2.$$

Thus,

$$\sum_{k=-nl-n+1}^{\infty} |\hat{\phi}(k)|^2 = \sum_{k=-\infty}^{nl} |\hat{\phi}(k)|^2. \tag{2.13}$$

Considering $P(\overline{\phi}z^{nl+n-1})$ likewise, we obtain

$$\sum_{k=-nl}^{\infty} |\hat{\phi}(k)|^2 = \sum_{k=-\infty}^{nl+n-1} |\hat{\phi}(k)|^2. \tag{2.14}$$

Now, it follows from (2.13) and (2.14) that

$$\sum_{k=-nl-n+1}^{-nl-1} |\hat{\phi}(k)|^2 = -\sum_{k=nl+1}^{nl+n-1} |\hat{\phi}(k)|^2,$$

i.e., $\hat{\phi}(k) = 0$ for $|k| = nl + 1, \ldots, nl + n - 1$. Furthermore,

$$\begin{aligned} T_\phi C_n z^{nl+n-1} = T_\phi z^{nl} &= P\left(\sum_{k=-\infty}^{\infty} \hat{\phi}(k)z^{nl+k}\right) \\ &= \sum_{k=-nl}^{\infty} \hat{\phi}(k)z^{nl+k} = \sum_{k=0}^{\infty} \hat{\phi}(k - nl)z^k \end{aligned} \tag{2.15}$$

and

$$
\begin{aligned}
C_n T_\phi^* z^{nl+n-1} &= C_n P\left(\sum_{k=-\infty}^{\infty} \overline{\hat{\phi}(k)} z^{nl+n-1-k}\right) \\
&= C_n \left(\sum_{k=-\infty}^{nl+n-1} \overline{\hat{\phi}(k)} z^{nl+n-1-k}\right) \\
&= C_n \left(\sum_{k=0}^{\infty} \overline{\hat{\phi}(nl+n-1-k)} z^{k}\right). \qquad (2.16)
\end{aligned}
$$

Upon comparing the constant terms in the right most expressions of (2.15) and (2.16), we have

$$\hat{\phi}(-nl) = \hat{\phi}(nl).$$

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CONSTRUCTIVE MATHEMATICAL ANALYSIS
7 (2024), No. 3, pp. 134-149
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

Solutions for nonhomogeneous degenerate quasilinear anisotropic problems

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ABSTRACT. In this article, we consider a class of nonlinear elliptic problems, where anisotropic leading differential operator incorporates the unbounded coefficients and the nonlinear term is a convection term. We prove the solvability of degenerate Dirichlet problem with convection, i.e. the existence of at least one bounded weak solution via the theory of pseudomonotone operators, Nemytskii-type operator and a priori estimate in the degenerate anisotropic Sobolev spaces.

Keywords: Degenerate anisotropic $\vec{p}$-Laplacian, degenerate anisotropic Sobolev spaces, unbounded coefficient, bounded solution, truncation, pseudomonotone operator.

2020 Mathematics Subject Classification: 35P30, 35J20, 47J30, 35J70, 35J92.

1. INTRODUCTION

Anisotropic partial differential equations have various applications in the mathematical modelling of physical and mechanical processes. In particular, they are used in models for the dynamics of fluids in anisotropic media when the conductivities of the media are distinct in different directions, or in biology as a model for the propagation of epidemic diseases in nonhomogeneous clusters. The interest in anisotropic problems has deeply increased recently, because many difficulties arise in passing from the isotropic setting to the anisotropic one. For example some fundamental tools available for the isotropic problem (such as the strong maximum principle) cannot be extended to the anisotropic problem (see [1–5, 7–9, 13, 19–21, 23–28] and the references therein).

One of the most interesting problem in a bounded domain $\Omega \subset \mathbb{R}^N$ is the isotropic case of the degenerate quasilinear Dirichlet elliptic equations with convection

$$-div(a(x)|\nabla u|^{p-2}\nabla u) = f(x, u, \nabla u). \tag{1.1}$$

Motreanu and Tornatore [17] developed a sub-supersolution approach to prove the existence of nontrivial, nonnegative and bounded solutions for (1.1). In the anisotropic setting, they analyzed the problem (see [19]),

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(G_i(u)|\frac{\partial u}{\partial x_i}|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = F(x, u, \nabla u), \tag{1.2}$$

Received: 24.06.2024; Accepted: 09.09.2024; Published Online: 11.09.2024
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DOI: 10.33205/cma.1504337

where the coefficients in the principal part are unbounded from above, and obtained the existence of solutions in a weak sense for degenerate anisotropic quasilinear Dirichlet problem (1.2).

In the present work, we extend the results above to a more general case. We consider the problem

$$(1.3)\qquad \begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\nu_i(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = f(x,u,\nabla u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $N \geq 3$ with a Lipschitz boundary $\partial\Omega$, p_i are given real numbers ($1 < p_i < \infty,\ i = 1, 2, \dots, N$) and, $f : \Omega \times \mathbb{R} \times \mathbb{R}^N$ is a Carathéodory function. The function f depends on the solution and its gradient (usually called convection term) satisfies hypotheses (H_1) and (H_2) (see Section 2). Notice that the problem (1.3) includes the differential operator which is anisotropic with measurable coefficients $\nu_i(x,t)$ ($i = 1, 2, \dots, N$) that can be written in the form $\nu_i(x,t) = a_i(x)g_i(|t|)$ with functions a_i and g_i that will be defined in Section 2.

The novelty of the paper is the new extension of problems (1.1) and (1.2) to a degenerate one in the anisotropic setting. The extended problem (1.3) is degenerate because the weight functions are decomposed in two parts. The first part, $a_i(x)$ can approach zero or be unbounded, the second part $g_i(t)$ can be unbounded from above. Thus, we need to consider the degenerate anisotropic Sobolev space $W_0^{1,\vec{p}}(\vec{a},\Omega)$ (see Section 2) as a suitable function space. By using the theory of pseudomonotone operators, as well as Nemytskii-type operator, and considering an appropriate truncation and a priori estimate in the anisotropic Sobolev spaces, we prove the existence of at least a bounded weak solution for (1.3) as well as the existence of a uniform bound for the solution set in the anisotropic setting. Our existence result for problem (1.3) is formulated as follows:

Theorem 1.1. *Assume that the weight functions $\nu_i : \Omega \times \mathbb{R} \to \mathbb{R}$ have the structure in (2.4) with positive functions $a_i \in L^1_{loc}(\Omega)$ and continuous functions $g_i : [0,+\infty) \to [\alpha_i,+\infty)$ with $\alpha_i > 0$ for $i = 1, 2, \dots, N$ satisfying the condition (H_1). Assume also that the Caratheodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the conditions (H_2) and (H_3). Then, problem (1.3) possesses at least a bounded weak solution $u \in W_0^{1,\vec{p}}(\vec{a},\Omega) \cap L^\infty(\Omega)$ in the sense of Definition 3.2.*

The rest of the paper is organized as follows. In Section 2, we state the main hypotheses and the structure of the problem (1.3) and we review some facts about the degenerate anisotropic Sobolev spaces which will be used in the sequel. In Section 3, we study the estimate of the solution set of problem (1.3) in $W_0^{1,\vec{p}}(\vec{a},\Omega)$. In Section 4, we prove the solvability of the auxiliary problem (4.26) obtained, which is used as an appropriate truncation, via the theory of pseudomonotone operators and we prove that the problem (1.3) possesses at least a bounded weak solution $u \in W_0^{1,\vec{p}}(\vec{a},\Omega)$ in the sense of Definition 3.2.

2. Preliminaries

In this section, first, we state the main hypotheses and the structure of the problem (1.3) in Sec. 2.1. Then we recall some facts about the suitable function space $W_0^{1,\vec{p}}(\vec{a},\Omega)$ which is necessary for studying the problem (1.3) in Sec. 2.2.

2.1. **Structure of the problem.** The structure that we admit for the weights ν_i entering problem (1.3) is of the form

$$\nu_i(x,t) := a_i(x)g_i(|t|) \text{ for a.e. } x \in \Omega \text{ and for all } t \in \mathbb{R}, \tag{2.4}$$

with positive functions $a_i \in L^1_{loc}(\Omega)$ and positive continuous functions $g_i : [0,+\infty[\to [\alpha_i,+\infty[$, with $\alpha_i > 0$ for $i = 1,2,\dots,N$. Moreover, for the functions a_i we assume the following hypothesis

(H_1) $a_i^{-s_i} \in L^1(\Omega)$ for some $s_i \in \left(\max\{\frac{N}{p_i}, \frac{1}{p_i-1}\}, +\infty\right)$ for $i = 1,2,\dots,N$.

We point out that the problem (1.3) is degenerate because the weight functions are decomposed in two parts, the first part $a_i(x)$ can approach zero or be unbounded, the second part $g_i(t)$ can be unbounded from above. We set $\vec{p} := (p_1,p_2,\dots,p_N)$, $\vec{a} = (a_1,a_2,\dots,a_N)$ and $p_{s_i} := (p_s)_i = \frac{p_i s_i}{s_i+1}$ for $i = 1,2,\dots,N$, where the real numbers s_i are given by hypothesis (H_1) and consider the vector $\vec{p}_s = (p_{s_1},\dots,p_{s_N})$. We say $\vec{q} \le \vec{p}$ iff $q_i \le p_i$ for all $i = 1,2,\dots,N$, notice that the definition of $\vec{p}_s$ implies $\vec{p}_s \le \vec{p}$. Using (H_1), we have

$$p_{s_i} > 1, \quad i = 1,2,\dots,N$$

and we assume

$$\sum_{i=1}^{N} \frac{1}{p_{s_i}} > 1, \tag{2.5}$$

and we set the critical exponent

$$p_s^* := \frac{N}{\sum_{i=1}^{N} \frac{1}{p_{s_i}} - 1}. \tag{2.6}$$

We introduce

$$p^+ := \max\{p_1,\dots,p_N\} \text{ and } p^- := \min\{p_1,\dots,p_N\}, \tag{2.7}$$

and assume that

$$p^+ < p_s^*. \tag{2.8}$$

For the nonlinear term $f : \Omega \times \mathbb{R} \times \mathbb{R}^N$, we assume the following hypotheses

(H_2) there exist the constants $b_1 \ge 0$, $b_2 \ge 0$, $b_3 \ge 0$ and $q \in (p^+, p_s^*)$ such that

$$|f(x,t,\xi)| \le b_1 + b_2|t|^{q-1} + b_3\left(\sum_{i=1}^{N} a_i|\xi_i|^{p_i}\right)^{\frac{1}{q'}},$$

(H_3) there exist the constants $c_1 \ge 0$, $c_2 \ge 0$ with $c_1 + c_2\eta^{p^-}N^{p^--1} < \alpha_i$ for all $i = 1,2,\dots,N$ and a function $\varrho \in L^1(\Omega)$ such that

$$f(x,t,\xi)t \le c_1 \sum_{i=1}^{N} a_i(x)|\xi_i|^{p_i} + c_2|t|^{p^-} + \varrho(x)$$

for all $x \in \Omega$, $t \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, where η is given by (2.11).

2.2. **Function space.** In this section, we define the degenerate anisotropic Sobolev spaces (see [14, 15, 21, 22, 26, 27] and references therein). Set

$$\vec{p} := (p_1, p_2, \dots, p_N)$$

with $1 < p_1, p_2, \dots, p_N < \infty$ and $\sum_{i=1}^{N} \frac{1}{p_i} > 1$. We introduce p^+, p^- and p^* as in (2.7) and (2.6), respectively. We recall the anisotropic Sobolev space

$$W^{1,\vec{p}}(\Omega) := \left\{ u \in W^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega),\ i = 1, 2, \dots, N \right\}$$

with the norm $\|u\|_{W^{1,\vec{p}}(\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega)}$. The space $W_0^{1,\vec{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to this norm.

We recall the following theorem [12, Theorem 1].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. If*

$$p_i > 1, \text{ for all } i = 1, 2, \dots, N, \quad \sum_{i=1}^{N} \frac{1}{p_i} > 1,$$

then for all $r \in [1, p_\infty]$ where, $p_\infty = \max\{p^*,\ p^+\}$, there is a continuous embedding $W_0^{1,\vec{p}}(\Omega) \subset L^r(\Omega)$. For $r < p_\infty$, the embedding is compact.

The degenerate Banach space with weight $a \in L^1_{loc}(\Omega)$ which satisfies the condition $a^{-s} \in L^1(\Omega)$ for some $s \in (\frac{N}{p_i}, +\infty) \cap [\frac{1}{p_i - 1}, +\infty)$ is

$$L^{p_i}(a, \Omega) := \left\{ u : \Omega \to \mathbb{R} :\ u \text{ is measurable and } \int_\Omega a(x)|u(x)|^{p_i} dx < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^{p_i}(a,\Omega)} = \left(\int_\Omega a(x)|u(x)|^{p_i} dx \right)^{\frac{1}{p_i}}.$$

The degenerate weighted Sobolev space is defined by

$$W^{1,p_i}(a, \Omega) := \left\{ u \in L^{p_i}(\Omega) \ :\ \int_\Omega a(x)|u(x)|^{p_i} dx < \infty \right\}$$

and endowed with the norm

$$\|u\|_{W^{1,p_i}(a,\Omega)} = \|u\|_{L^{p_i}(\Omega)} + \||\nabla u|\|_{L^{p_i}(a,\Omega)}.$$

The space $W_0^{1,p_i}(a, \Omega)$ is the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p_i}(a,\Omega)}$. Furthermore,

$$\|u\|_{W_0^{1,p}(a,\Omega)} := \left(\int_\Omega a(x)|\nabla u(x)|^p dx \right)^{\frac{1}{p}} \tag{2.9}$$

for all $u \in W_0^{1,p}(a, \Omega)$, is an equivalent norm on $W_0^{1,p}(a, \Omega)$ for which $W_0^{1,p}(a, \Omega)$ becomes a uniformly convex Banach space.

Now, we recall the next Proposition from [16, Proposition 1] which establishes the continuous embedding of degenerate Sobolev space $W^{1,p}(a, \Omega)$ into the classical Sobolev space $W^{1,p_s}(\Omega)$.

Proposition 2.1. *Let be $p > 1$ and $a \in L^1_{loc}(\Omega)$ which satisfies the condition $a^{-s} \in L^1(\Omega)$ for some $s \in (\frac{N}{p}, +\infty) \cap [\frac{1}{p-1}, +\infty)$. Then, there are continuous embeddings*

$$W^{1,p}(a,\Omega) \hookrightarrow W^{1,p_s}(\Omega) \hookrightarrow L^p(\Omega),$$

where $p_s = \frac{ps}{s+1}$. In addition, the embedding $W^{1,p_s}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.

Let be $\vec{p} = (p_1, p_2, \ldots, p_N)$ and $\vec{a} = (a_1, a_2, \ldots, a_N)$ such that condition (H_1) holds, the degenerate anisotropic Sobolev space is given by

$$W^{1,\vec{p}}(\vec{a},\Omega) = \left\{ u \in W^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(a_i,\Omega) \text{ for } i = 1,\ldots,N \right\}$$

with the norm $\|u\|_{W^{1,\vec{p}}(\vec{a},\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^N \|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(a_i,\Omega)}$. The anisotropic Sobolev space $W_0^{1,\vec{p}}(\vec{a},\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to this norm. $W_0^{1,\vec{p}}(\vec{a},\Omega)$ with the following norm

$$\|u\| = \|u\|_{W_0^{1,\vec{p}}(\vec{a},\Omega)} := \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)}$$

is a separable and reflexive Banach space [11, 15].

Finally, by Proposition 2.1, we can prove the following proposition.

Proposition 2.2. *Assume that (H_1), (2.5), (2.6) and (2.8) hold. There are continuous embeddings*

$$W_0^{1,\vec{p}}(\vec{a},\Omega) \hookrightarrow W_0^{1,\vec{p}_s}(\Omega) \hookrightarrow L^r(\Omega) \tag{2.10}$$

for all $1 \le r \le {p_s}^*$ In addition, the embedding $W^{1,\vec{p}}(\vec{a},\Omega) \hookrightarrow L^r(\Omega)$ is compact for $r < {p_s}^*$. Furthermore $W_0^{1,\vec{p}}(\vec{a},\Omega)$ is a uniformly convex Banach space.

Proof. In order to prove the first inclusion in (2.10), let $u \in W_0^{1,\vec{p}}(\vec{a},\Omega)$. Using Hölder's inequality and condition (H_1) (note $p_{s_i} < p_i$), we infer that

$$\begin{aligned}
\int_\Omega |\frac{\partial u}{\partial x_i}|^{p_{s_i}} dx &= \int_\Omega \left(a_i(x)^{\frac{p_{s_i}}{p_i}} |\frac{\partial u}{\partial x_i}|^{p_{s_i}} \right) a_i(x)^{-\frac{p_{s_i}}{p_i}} dx \\
&\le \left(\int_\Omega a_i(x) |\frac{\partial u}{\partial x_i}|^{p_i} dx \right)^{\frac{p_{s_i}}{p_i}} \left(\int_\Omega a_i(x)^{-\frac{p_{s_i}}{p_i - p_{s_i}}} dx \right)^{\frac{p_i - p_{s_i}}{p_i}} \\
&\le \|a_i^{-s_i}\|_{L^1(\Omega)}^{\frac{1}{s_i+1}} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)}^{p_{s_i}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|u\|_{W_0^{1,\vec{p}_s}(\Omega)} &= \sum_{i=1}^N \left(\int_\Omega |\frac{\partial u}{\partial x_i}|^{p_{s_i}} dx \right)^{\frac{1}{p_{s_i}}} \le \sum_{i=1}^N \|a_i^{-s_i}\|_{L^1(\Omega)}^{\frac{1}{p_i s_i}} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)} \\
&\le \Upsilon \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)} = \Upsilon \|u\|
\end{aligned}$$

where $\Upsilon = \max\{\|a_i^{-s_i}\|_{L^1(\Omega)}^{\frac{1}{p_i s_i}} : i = 1, 2, \ldots, N\}$. The continuous inclusion $W_0^{1,\vec{p}}(\vec{a},\Omega) \hookrightarrow W_0^{1,\vec{p}_s}(\Omega)$ is proven.

Also we know by Theorem 2.2 and using (2.8) that the embedding $W_0^{1,\vec{p}_s}(\Omega) \hookrightarrow L^r(\Omega)$ for $1 \le r < p_s^*$ is compact, then the compactness of the second inclusion in (2.10) follows.

It remains to show that $W_0^{1,\vec{p}}(\vec{a},\Omega)$ is a uniformly convex Banach space. It suffices to have $a_i^{-\frac{1}{p_i-1}} \in L^1(\Omega)$ for all $i = 1, 2, \ldots, N$(see [10, Theorem 1.3]). From hypothesis (H_1), it is known that $a_i^{-s_i} \in L^1(\Omega)$ with $s_i \geq \frac{1}{p_i-1}$, for all $i = 1, 2, \ldots, N$, which results in

$$\begin{aligned} \int_\Omega a_i(x)^{-\frac{1}{p_i-1}} dx &= \int_{\{a_i(x)<1\}} a_i(x)^{-\frac{1}{p_i-1}} + \int_{\{a_i(x)\geq 1\}} a_i(x)^{-\frac{1}{p_i-1}} \\ &\leq \int_\Omega a_i(x)^{-s_i} + |\Omega| < \infty, \end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Thus completing the proof. □

Taking into account Proposition 2.2, definition of $\vec{p}_s$, (2.7) and (2.8), there exists a positive constant η such that

$$\|u\|_{L^{p^-}(\Omega)} \leq \eta \|u\|, \quad \text{for all } u \in W_0^{1,\vec{p}}(\vec{a},\Omega). \tag{2.11}$$

The degenerate anisotropic $\vec{p}$-Laplacian operator with the weights $a_i \in L^1_{loc}(\Omega)$, $i = 1, 2, \ldots, N$ is defined by the map

$$-\Delta_{\vec{p},\vec{a}}(\cdot) = -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(a_i(x)\left|\frac{\partial(\cdot)}{\partial x_i}\right|^{p_i-2}\frac{\partial(\cdot)}{\partial x_i}\right) : W_0^{1,\vec{p}}(\vec{a},\Omega) \to W_0^{1,\vec{p}}(\vec{a},\Omega)^*.$$

This means that

$$\langle -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right), v\rangle = \sum_{i=1}^N \int_\Omega a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx$$

for all $u,\ v \in W_0^{1,\vec{p}}(\vec{a},\Omega)$.
The definition makes sense as can be seen through Hölder's inequality

$$\begin{aligned} &\left|\sum_{i=1}^N \int_\Omega a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx\right| \\ \leq &\sum_{i=1}^N \int_\Omega \left(a_i(x)^{\frac{p_i-1}{p_i}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-1}\right)\left(a_i(x)^{\frac{1}{p_i}}\left|\frac{\partial v}{\partial x_i}\right|\right) dx \\ \leq &\sum_{i=1}^N \left(\int_\Omega a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\right)^{\frac{p_i-1}{p_i}}\left(\int_\Omega a_i(x)\left|\frac{\partial v}{\partial x_i}\right|^{p_i} dx\right)^{\frac{1}{p_i}} < \infty. \end{aligned}$$

Remark 2.1. *The ordinary definition of p-Laplacian is recovered when $p_i = p$ and $a_i(x) = 1$ in Ω, for $i = 1, 2, \ldots, N$.*

Before ending this section we recall the definition of pseudomonotone map.

Definition 2.1. *The map $A : X \to X^*$ is called pseudomonotone if for each sequence $\{u_n\} \subset X$ satisfying $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0$, it holds*

$$\langle A(v), u - v\rangle \leq \liminf_{n\to\infty}\langle A(u_n), u_n - v\rangle \text{ for all } v \in X.$$

The main theorem for pseudomonotone operators reads as follows (see, e.g., [6, Theorem 2.99]).

Theorem 2.3. *Let X be a reflexive Banach space. If the mapping $A : X \to X^*$ is pseudomonotone, bounded and coercive, then it is surjective.*

3. Bounded solutions

We start with the estimate of the solution set of problem (1.3) in $W_0^{1,\vec{p}}(\vec{a},\Omega)$. But before that, we recall the definition of a weak solution for problem (1.3).

Definition 3.2. *The function $u \in W_0^{1,\vec{p}}(\vec{a},\Omega)$ is called a weak solution to problem (1.3) if $f(x,u,\nabla u)v$ and $\nu_i(x,u)|\frac{\partial u}{\partial x_i}|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}$ for $i=1,2,\dots,N$ are integrable on Ω and*

$$\sum_{i=1}^{N}\int_{\Omega}\nu_i(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx=\int_{\Omega}f(x,u,\nabla u)v dx \tag{3.12}$$

for all $v \in W_0^{1,\vec{p}}(\vec{a},\Omega)$.

Lemma 3.1. *Under assumptions (H_1) and (H_3), the set of solutions to problem (1.3) is bounded in $W_0^{1,\vec{p}}(\vec{a},\Omega)$ with a bound depending on g_i only through its lower bound α_i, for $i=1,2,\dots,N$.*

Proof. Set $v=u\in W_0^{1,\vec{p}}(\vec{a},\Omega)$ in (3.12), we get

$$\sum_{i=1}^{N}\int_{\Omega}\nu_i(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx=\int_{\Omega}f(x,u,\nabla u)u dx.$$

Hypothesis (H_3), $p_i>1$ for $i=1,2,\dots,N$ in conjunction with (2.4) and Proposition 2.2 ensures that

$$\begin{aligned}
&\sum_{i=1}^{N}\alpha_i\left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)}\\
&\leq\sum_{i=1}^{N}\int_{\Omega}\nu_i(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx=\int_{\Omega}f(x,u,\nabla u)u dx\\
&\leq c_1\sum_{i=1}^{N}\left(\int_{\Omega}a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\right)+c_2\left(\int_{\Omega}|u|^{p^-}dx\right)+\int_{\Omega}\varrho(x)dx\\
&\leq c_1\sum_{i=1}^{N}\left(\int_{\Omega}a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\right)+c_2\left\|u\right\|^{p^-}_{L^{p^-}(\Omega)}+\|\varrho\|_{L^1(\Omega)}.\\
&\leq c_1\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)}+c_2\eta^{p^-}\|u\|^{p^-}+\|\varrho\|_{L^1(\Omega)}\\
&\leq c_1\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)}+c_2\eta^{p^-}\left(\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(a_i,\Omega)}\right)^{p^-}+\|\varrho\|_{L^1(\Omega)}\\
&\leq c_1\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)}+c_2\eta^{p^-}N^{p^--1}\left(N+\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)}\right)+\|\varrho\|_{L^1(\Omega)}.
\end{aligned}$$

Thus

$$\sum_{i=1}^{N}(\alpha_i-c_1-c_2\eta^{p^-}N^{p^--1})\left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)}\leq\|\varrho\|_{L^1(\Omega)}+c_2\eta^{p^-}N^{p^-}.$$

But $\alpha_i-c_1-c_2\eta^{p^-}N^{p^--1}>0$ for all $i=1,2,\dots,N$ and $\varrho\in L^1(\Omega)$ hence the proof is complete. ∎

Theorem 3.4. *Assume that conditions (H_1), (H_2) and (H_3) are fulfilled. Then there exists a constant $C > 0$ such that for each weak solution $u \in W_0^{1,\vec{p}}(\vec{a}, \Omega)$ to problem (1.3) it holds the uniform estimate $\|u\|_{L^\infty(\Omega)} \leq C$. The constant C depends on g_i, for $i = 1, 2, \ldots, n$, only through its lower bound α_i $(i = 1, 2, \ldots, N)$.*

Proof. Let $u \in W_0^{1,\vec{p}}(\vec{a}, \Omega)$ be a weak solution to problem (1.3). We can write $u = u^+ - u^-$, where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. We have to show that u^+ and u^- are both uniformly bounded by a constant independent of u. We only provide the proof for u^+ because in the case of u^- one can argue similarly.

Our first goal is to prove that

$$u^+ \in L^r(\Omega) \text{ for all } r \in [1, +\infty). \tag{3.13}$$

To this end we insert in (3.12) the test function $v = u^+ u_h^{kp_j} \in W_0^{1,\vec{p}}(\vec{a}, \Omega)$, where $u_h := \min\{u^+, h\}$ with arbitrary constants $h > 0$, $k > 0$ and $1 \leq j \leq N$, thus obtaining

$$\sum_{i=1}^{N} \int_\Omega \nu_i(x, u) \left|\frac{\partial u}{\partial x_i}\right|^{p_i - 2} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_i}(u^+ u_h^{kp_j}) dx = \int_\Omega f(x, u, \nabla u) u^+ u_h^{kp_j} dx. \tag{3.14}$$

By means of (2.4), the left-hand side of (3.14) can be estimated from below as

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega \nu_i(x, u) \left|\frac{\partial u}{\partial x_i}\right|^{p_i - 2} \frac{\partial u}{\partial x_i} \frac{\partial u^+ u_h^{kp_j}}{\partial x_i} dx \\
=&\sum_{i=1}^{N} \int_\Omega a_i(x) g_i(|u|) \left|\frac{\partial u}{\partial x_i}\right|^{p_i - 2} \frac{\partial u}{\partial x_i} \left(u_h^{kp_j} \frac{\partial u^+}{\partial x_i} + kp_j u^+ u_h^{kp_j - 1} \frac{\partial u_h}{\partial x_i} \right) dx \\
\geq&\sum_{i=1}^{N} \alpha_i (kp_j + 1) \int_\Omega a_i(x) u_h^{kp_j} \left|\frac{\partial u^+}{\partial x_i}\right|^{p_i} dx > \sum_{i=1}^{N} \alpha_i \int_\Omega a_i(x) u_h^{kp_j} \left|\frac{\partial u^+}{\partial x_i}\right|^{p_i} dx.
\end{aligned} \tag{3.15}$$

On the other hand the right hand side of (3.14) by (H_2) implies

$$\begin{aligned}
&\int_\Omega f(x, u, \nabla u) u^+ u_h^{kp_j} dx \\
\leq& b_3 \int_\Omega \left(\sum_{i=1}^{N} a_i(x) \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \right)^{\frac{1}{q'}} u^+ u_h^{kp_j} dx + b_2 \int_\Omega |u|^{q-1} u^+ u_h^{kp_j} dx + b_1 \int_\Omega u^+ u_h^{kp_j} dx \\
\leq& b_4 \sum_{i=1}^{N} \left(\int_\Omega a_i(x)^{\frac{1}{q'}} \left|\frac{\partial u}{\partial x_i}\right|^{\frac{p_i}{q'}} u^+ u_h^{kp_j} dx \right) + b_2 \int_\Omega (u^+)^q u_h^{kp_j} dx + b_1 \int_\Omega u^+ u_h^{kp_j} dx
\end{aligned} \tag{3.16}$$

with a constant $b_4 > 0$. We observe that through Young's inequality, for any $\varepsilon > 0$ and a constant $c(\varepsilon) > 0$ we get

$$\begin{aligned}
&b_4 \sum_{i=1}^{N} \left(\int_\Omega a_i(x)^{\frac{1}{q'}} \left|\frac{\partial u}{\partial x_i}\right|^{\frac{p_i}{q'}} u^+ u_h^{kp_j} dx \right) \\
=&\varepsilon \sum_{i=1}^{N} \int_\Omega a_i(x) \left|\frac{\partial u^+}{\partial x_i}\right|^{p_i} u_h^{kp_j} dx + C(\varepsilon) \sum_{i=1}^{N} \int_\Omega (u^+)^q u_h^{kp_j} dx,
\end{aligned} \tag{3.17}$$

taking into account of previous relation and since

$$\int_\Omega u^+ u_h^{kp_j} dx \leq \int_\Omega (u^+)^q u_h^{kp_j} dx + |\Omega|$$

from (3.15) we obtain

$$\begin{aligned} &\int_\Omega f(x,u,\nabla u)u^+u_h^{kp_j}\,dx \\ &\leq \varepsilon \sum_{i=1}^N \int_\Omega a_i(x)\left|\frac{\partial u^+}{\partial x_i}\right|^{p_i} u_h^{kp_j}\,dx + b_5\left(\sum_{i=1}^N \int_\Omega (u^+)^q u_h^{kp_j}\,dx+1\right) \end{aligned} \tag{3.18}$$

with a constant $b_5 > 0$. Then, we derive by (3.15) and (3.18)

$$\sum_{i=1}^N(\alpha_i-\varepsilon)\int_\Omega a_i(x)\left|\frac{\partial u^+}{\partial x_i}\right|^{p_i} u_h^{kp_j}\,dx \leq b_6\left(\int_\Omega (u^+)^q u_h^{kp_j}\,dx+1\right), \tag{3.19}$$

with a constant $b_6 > 0$ and for every $j = 1, 2, \ldots, N$.

We observe that

$$\left|\frac{\partial}{\partial x_i}(u^+u_h^k)\right| \leq (k+1)u_h^k\left|\frac{\partial u^+}{\partial x_i}\right|.$$

Taking into account this relation, from (3.19) if $\varepsilon > 0$ is sufficiently small, we obtain for each $j = 1, 2, \ldots, N$

$$\left\|\frac{\partial(u_h^k u^+)}{\partial x_j}\right\|_{L^{p_j}(a_j,\Omega)} \leq (k+1)b_6^{\frac{1}{p_j}}\left(\int_\Omega (u^+)^q u_h^{kp_j}\,dx+1\right)^{\frac{1}{p_j}}. \tag{3.20}$$

By hypothesis (H_2), we can find $r \in (p_j, q)$ satisfying

$$\frac{(q-p_j)r}{r-p_j} \leq p_s^*, \quad j = 1, 2, \ldots, N. \tag{3.21}$$

We can estimate the left-hand side of (3.19), using Hölder's inequality getting for any $k > 0$

$$\begin{aligned} \int_\Omega u_h^{kp_j}\left(u^+\right)^q dx &= \int_\Omega \left(u_h^k u^+\right)^{p_j}\left(u^+\right)^{q-p_j} dx \\ &\leq \left(\int_\Omega \left(u^+\right)^{\frac{(q-p_j)r}{r-p_j}} dx\right)^{\frac{r-p_j}{r}}\left(\int_\Omega \left(u_h^k u^+\right)^r dx\right)^{\frac{p_j}{r}} \\ &\leq M\|u_h^k u^+\|_{L^r(\Omega)}^{p_j}, \end{aligned}$$

where $M > 0$ is a constant that does not depend on the solution u of (1.3). The independence of M with respect to the solution u is a consequence of Lemma 3.1 and the continuous embedding $W_0^{1,\vec{p}}(\vec{a},\Omega) \hookrightarrow L^{\frac{(q-p_j)r}{r-p_j}}(\Omega)$ that follows from Proposition 2.2 and (3.21), morover constant M depends on g_i, for $i = 1, 2, \ldots, N$, only for its lower bound α_i.

Inserting the previous inequality into (3.20), we obtain

$$\left\|\frac{\partial(u_h^k u^+)}{\partial x_j}\right\|_{L^{p_j}(a_j,\Omega)} \leq (k+1)b_6^{\frac{1}{p_j}}\left(M\|u_h^k u^+\|_{L^r(\Omega)}^{p_j}+1\right)^{\frac{1}{p_j}}.$$

Summing on j from 1 to N, we have

$$\|u_h^k u^+\| \leq (k+1)b_7 N\left(\|u_h^k u^+\|_{L^r(\Omega)}+1\right)$$

with a constant $b_7 > 0$.

From the continuous embedding $W_0^{1,\vec{p}}(\vec{a},\Omega) \hookrightarrow L^{p_s^*}(\Omega)$ and using Fatou's lemma, we get

$$\|u^+\|_{L^{(k+1)p_s^*}(\Omega)} \leq b_8^{\frac{1}{k+1}}(k+1)^{\frac{1}{k+1}}N^{\frac{1}{k+1}}\left(\|u^+\|_{L^{(k+1)r}(\Omega)}+1\right) \tag{3.22}$$

with a constant $b_8 > 0$.

Without loss of generality, we may suppose that $\|u^+\|_{L^{(k+1)r}(\Omega)} > 1$ except for finitely many k (otherwise conclusion readily follows), moreover, since the sequence $(k+1)^{\frac{1}{\sqrt{k+1}}}$ is bounded, (3.22) gives rise to a constant $b > 0$ such that accordingly, (3.22) amounts to saying that

$$\|u^+\|_{L^{(k+1)p_s^*}(\Omega)} \leq b^{\frac{1}{\sqrt{k+1}}} \|u^+\|_{L^{(k+1)r}(\Omega)} \tag{3.23}$$

with a constant $b > 0$ independent of k and of the solution u, and for which the dependence on g_i, for $i = 1, 2, \ldots, N$, reduces to the dependence on α_i. At this point, we can implement the Moser iteration with $(k_n+1)r = (k_{n-1}+1)p_s^*$ posing $(k_1+1)r = p_s^*$ if $\|u^+\|_{L^{(k+1)r}(\Omega)} > 1$ for all k and $(k_1+1)r = (k_0+1)p_s^*$ if $\|u^+\|_{L^{(k_0+1)r}(\Omega)} \leq 1$ and $\|u^+\|_{L^{(k+1)r}(\Omega)} > 1$ for all $k > k_0$. Then (3.23) renders

$$\|u^+\|_{L^{(k_n+1)p_s^*}(\Omega)} \leq b^{\sum_{1\leq i\leq n} \frac{1}{\sqrt{k_i+1}}} \|u^+\|_{L^{p_s^*}(\Omega)}, \text{ for all } n \geq 1. \tag{3.24}$$

Letting $n \to \infty$ in (3.24) since the series converges and $k_n \to +\infty$ as $n \to \infty$ the uniform boundedness of the solution set of (1.3) is achieved then there exists a positive constant C such that $\|u\|_{L^\infty(\Omega)} \leq C$. A careful reading of the proof shows that the dependence of the uniform bound C on g_i, for $i = 1, 2, \ldots, N$, arises just through the lower bound α_i of g_i for $i = 1, 2, \ldots, N$. This completes the proof. □

4. Truncated weight and associated operator

For any number $R > 0$ we consider the following truncation of the weights $\nu_i(x, u)$, for $i = 1, 2, \ldots, N$ in problem (1.3):

$$\nu_{iR}(x, t) = a_i(x) g_{iR}(|t|), \text{ for all } (x, t) \in \Omega \times \mathbb{R},$$

where

$$g_{iR}(t) = \begin{cases} g_i(t) & \text{if } \quad t \in [0, R], \\ g(R) & \text{if } \quad t > R. \end{cases} \tag{4.25}$$

Corresponding to the truncation in (4.25), we state the auxiliary problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\nu_{iR}(x, u) \left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \right) = f(x, u, \nabla u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.26}$$

Our approach to study problem (4.26) is based on the theory of pseudomonotone operators. In this respect, we introduce the mapping, corresponding to an $R > 0$

$$A_R : W_0^{1,\vec{p}}(\vec{a}, \Omega) \to W_0^{1,\vec{p}}(\vec{a}, \Omega)^*$$

as

$$A_R = \mathcal{A} - \mathcal{N}, \tag{4.27}$$

with the degenerate anisotropic operator associated to the truncated weights $\nu_{iR}(x, t)$ ($i = 1, 2, \ldots, N$), $\mathcal{A} : W_0^{1,\vec{p}}(\vec{a}, \Omega) \to W_0^{1,\vec{p}}(\vec{a}, \Omega)^\star$ defined by

$$\langle \mathcal{A}(u), v \rangle = \sum_{i=1}^{N} \int_\Omega \nu_{iR}(x, u) |\frac{\partial u}{\partial x_i}|^{p_i - 2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx \text{ for all } u, v \in W_0^{1,\vec{p}}(\vec{a}, \Omega) \tag{4.28}$$

and a Nemytskii-type operator $\mathcal{N} : W_0^{1,\vec{p}}(\vec{a}, \Omega) \to L^{\overline{p}'}(\Omega)$ defined by

$$\langle \mathcal{N}(u), v \rangle = \int_\Omega f(x, u(x), \nabla u(x)) v(x) dx \text{ for all } u, v \in W_0^{1,\vec{p}}(\vec{a}, \Omega). \tag{4.29}$$

Remark 4.2. *One has that $u \in W_0^{1,\vec{p}}(\vec{a},\Omega)$ is a (weak) solution to problem* (4.26) *if and only if it solves the equation $A_R(u)=0$ with A_R given in* (4.27).

The next propositions focus on the properties of the operators $\mathcal{A}$ and $\mathcal{N}$.

Proposition 4.3. *Given $R>0$, let $\mathcal{A}: W_0^{1,\vec{p}}(\vec{a},\Omega) \to W_0^{1,\vec{p}}(\vec{a},\Omega)^*$ be as in* (4.28). *Then, $\mathcal{A}$ is well defined, bounded and continuous, moreover it has the S_+-property, that is, any sequence $\{u_n\} \subset W_0^{1,\vec{p}}(\vec{a},\Omega)$ with $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)$ and*

$$\limsup_{n\to+\infty}\langle \mathcal{A}(u_n), u_n-u\rangle \le 0 \tag{4.30}$$

satisfies $u_n \to u$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)$.

Proof. By (2.4), (4.25), the continuity of g_i, for $i=1,2,\dots,N$ and Hölder's inequality, we get

$$\begin{aligned}
|\langle \mathcal{A}(u),v\rangle| &\le \sum_{i=1}^N \int_\Omega a_i(x) g_{iR}(u)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-1}\left|\frac{\partial v}{\partial x_i}\right| dx \\
&\le \sum_{i=1}^N \max_{t\in[0,R]} g_i(t)\int_\Omega a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-1}\left|\frac{\partial v}{\partial x_i}\right| dx \\
&\le G\sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(a_i,\Omega)}^{p_i-1}\left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}(a_i,\Omega)}
\end{aligned}$$

for all $u,v \in W_0^{1,\vec{p}}(\vec{a},\Omega)$, where $G=\max_{1\le i\le N}\left(\max_{t\in[0,R]} g_i(t)\right)$. The operator $\mathcal{A}$ in (4.28) is thus well defined and bounded.

Assume that $u_n \to u$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)$ we can prove that $\mathcal{A}(u_n)\to\mathcal{A}(u)$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)^*$. We observe that

$$\begin{aligned}
&\|\mathcal{A}(u_n)-\mathcal{A}(u)\|_{W_0^{1,\vec{p}}(\vec{a},\Omega)^*} \\
\le& G\sum_{i=1}^N\left(\int_\Omega a_i(x)\left|\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2}\frac{\partial u_n}{\partial x_i}-\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right|^{\frac{p_i}{p_i-1}} dx\right)^{\frac{p_i-1}{p_i}} \\
&+\sum_{i=1}^N\left(\int_\Omega a_i(x)\,|g_{iR}(|u_n|)-g_{iR}(|u|)|^{\frac{p_i}{p_i-1}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx\right)^{\frac{p_i-1}{p_i}} \\
=& G\sum_{i=1}^N\left\|\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2}\frac{\partial u_n}{\partial x_i}-\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right\|_{L^{\frac{p_i-1}{p_i}}(a_i,\Omega)} \\
&+\sum_{i=1}^N\left\|(g_{iR}(|u_n|)-g_{iR}(|u|))^{\frac{1}{p_i-1}}\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(a_i,\Omega)}^{p_i-1}.
\end{aligned}$$

Applying Lebesgue's Dominated Convergence Theorem on the basis of the continuity of g_i ($i=1,2,\dots,N$) and the strong convergence $u_n\to u$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)$ we have

$$\begin{aligned}
&\left\|\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2}\frac{\partial u_n}{\partial x_i}-\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right\|_{L^{\frac{p_i}{p_i-1}}(a_i,\Omega)} \to 0 \text{ and} \\
&\left\||g_{iR}(|u_n|)-g_{iR}(|u|)|^{\frac{1}{p_i-1}}\left|\frac{\partial u}{\partial x_i}\right|\right\|_{L^{p_i}(a_i,\Omega)} \to 0
\end{aligned}$$

as $n \to \infty$ which establishes the desired conclusion.

Let $\{u_n\} \in W_0^{1,\vec{p}}(\vec{a}, \Omega)$ be a sequence which satisfies (4.30). By (4.28) and Hölder's inequality, we have

$$\begin{aligned} &\langle \mathcal{A}(u_n) - \mathcal{A}(u), u_n - u \rangle \\ &= \sum_{i=1}^{N} \int_\Omega a_i(x) g_{iR}(|u_n|) \left(\left| \frac{\partial u_n}{\partial x_i} \right|^{p_i-2} \frac{\partial u_n}{\partial x_i} - \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) dx \\ &+ \sum_{i=1}^{N} \int_\Omega a_i(x) \left(g_{iR}(|u_n|) - g_{iR}(|u|) \right) \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial (u_n - u)}{\partial x_i} dx \\ &\geq \alpha^- \sum_{i=1}^{N} \left[(\|u_n\|^{p_i-1}_{W_0^{1,p_i}(a_i,\Omega)} - \|u\|^{p_i-1}_{W_0^{1,p_i}(a_i,\Omega)})(\|u_n\|_{W_0^{1,p_i}(a_i,\Omega)} - \|u\|_{W_0^{1,p_i}(a_i,\Omega)}) \right] \\ &+ \sum_{i=1}^{N} \int_\Omega a_i(x) \left(g_{iR}(|u_n|) - g_{iR}(|u|) \right) \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial (u_n - u)}{\partial x_i} dx, \end{aligned} \tag{4.31}$$

with $\alpha^- = \min\{\alpha_1, \ldots, \alpha_N\}$. The assumptions $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\vec{a}, \Omega)$ and (4.30) imply

$$\limsup_{n\to\infty} \langle \mathcal{A}_R(u_n) - \mathcal{A}_R(u), u_n - u \rangle \leq 0. \tag{4.32}$$

We also have

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x) \left(g_{iR}(|u_n|) - g_{iR}(|u|) \right) \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial (u_n - u)}{\partial x_i} dx = 0. \tag{4.33}$$

Indeed, through Hölder's inequality and the boundedness of $\{u_n\}$ in $W_0^{1,\vec{p}}(\vec{a}, \Omega)$ (note that $u_n \rightharpoonup u$), there is a constant $\tilde{\alpha} > 0$ such that

$$\begin{aligned} &\left| \sum_{i=1}^{N} \int_\Omega a_i(x) \left(g_{iR}(|u_n|) - g_{iR}(|u|) \right) \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial (u_n - u)}{\partial x_i} dx \right| \\ \leq &\tilde{\alpha} \sum_{i=1}^{N} \left(\int_\Omega a_i(x) \left| g_{iR}(|u_n|) - g_{iR}(|u|) \right|^{\frac{p_i-1}{p_i}} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \right)^{\frac{p_i-1}{p_i}}. \end{aligned}$$

Then (4.33) is achieved by applying Lebesgue's Dominated Convergence Theorem on the basis of the continuity of g_i ($i = 1, 2, \ldots, N$). Combining (4.31), (4.32) and (4.33) we have

$$\lim_{n\to\infty} \left(\left\| \frac{\partial u_n}{\partial x_i} \right\|^{p_i-1}_{L^{p_i}(a_i,\Omega)} - \left\| \frac{\partial u}{\partial x_i} \right\|^{p_i-1}_{L^{p_i}(a_i,\Omega)} \right) \left(\left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)} - \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)} \right) = 0.$$

Then

$$\lim_{n\to\infty} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)} = \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)}, \quad i = 1, 2, \ldots, N.$$

Since the space $W_0^{1,\vec{p}}(\vec{a}, \Omega)$ is uniformly convex (see Proposition 2.2), we obtain $u_n \to u$ in $W_0^{1,\vec{p}}(\vec{a}, \Omega)$, this proves the S_+-property of the operator $\mathcal{A}$. □

Proposition 4.4. *Assume that hypotheses (H_1)-(H_3) hold. Then the map $\mathcal{N} : W_0^{1,\vec{p}}(\vec{a}, \Omega) \to L^{q'}(\Omega)$ in (4.29) is well defined, continuous and bounded.*

Proof. Assumption (H_2) yields

$$\int_\Omega |f(x,u,\nabla u)|^{q'}dx \leq \theta\left(\int_\Omega \sum_{i=1}^N \left(a_i(x)|\frac{\partial u}{\partial x_i}|^{p_i}\right)dx + \int_\Omega |u|^q dx + 1\right)$$
$$\leq \theta\left(\sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(a_i,\Omega)}^{p_i} + \|u\|_{L^q(\Omega)}^q + 1\right)$$

for all $u \in W_0^{1,\vec{p}}(\vec{a},\Omega)$ with a constant θ. Hence $\mathcal{N}(u) \in L^{q'}(\Omega)$ whenever $u \in W_0^{1,\vec{p}}(\vec{a},\Omega)$, Thus the previous estimate shows that

$$\mathcal{N} : W_0^{1,\vec{p}}(\vec{a},\Omega) \to L^{q'}(\Omega)$$

is well defined and bounded. Therefore the mapping $\mathcal{N}$ is continuous. Let $u_n \to u$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)$, so $\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}$ in $L^{p_i}(a_i,\Omega)$ $(i = 1,2,\ldots,N$ and $u_n \to u$ in $L^q(\Omega)$. Hypothesis (H_2) and Krasnoseleskii's theorem on Nemitskii operator assure that $f(x,u_n,\nabla u_n) \to f(x,u,\nabla u)$ in $L^{q'}(\Omega)$ whence $\mathcal{N}$ is a continuous operator. □

Now we are able to prove the solvability of auxiliary problem (4.26).

Theorem 4.5. *Assume that the weights $\nu_i : \Omega \times \mathbb{R} \to \mathbb{R}$ has the structure in (2.4) with a positive functions $a_i \in L^1_{loc}(\Omega)$ satisfying the condition (H_1) and continuous functions $g_{iR} : [0,+\infty) \to [\alpha_i,+\infty)$ with $\alpha_i > 0$ and $i = 1,2,\ldots,N$. If $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying the conditions (H_2) and (H_3), then problem (4.26) has a weak solution $u_R \in W_0^{1,\vec{p}}(\vec{a},\Omega)$ for every $R > 0$.*

Proof. We are going to apply Theorem 2.3 to the operator A_R in (4.27) with any fixed $R > 0$. By Propositions 4.3 and 4.4 it is known that the mapping A_R is bounded. Let us show that A_R is a pseudomonotone operator. To this end, let $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)$ and

$$\limsup_{n\to\infty}\langle A_R(u_n), u_n - u\rangle \leq 0. \tag{4.34}$$

There holds

$$\limsup_{n\to\infty}\langle \mathcal{N}(u_n), u_n - u\rangle = 0 \tag{4.35}$$

as can be noticed from Proposition 4.3 since

$$|\langle \mathcal{N}(u_n), u_n - u\rangle| \leq \|\mathcal{N}(u_n)\|_{L^{q'}(\Omega)}\|u_n - u\|_{L^q(\Omega)}$$

and $u_n \to u$ in $L^q(\Omega)$ (refer to the compact embedding of $W_0^{1,\vec{p}}(\vec{a},\Omega)$ into $L^q(\Omega)$ and that $\mathcal{N}(u_n)$ is bounded in $L^{q'}(\Omega)$).

On the basis of (4.27) and (4.35), we note that (4.34) reduces to (4.30). We are thus enabled to apply Proposition 4.3 obtaining the strong convergence $u_n \to u$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)$. In view of the continuity of the maps $\mathcal{A} : W_0^{1,\vec{p}}(\vec{a},\Omega) \to W_0^{1,\vec{p}}(\vec{a},\Omega)^*$ and $\mathcal{N} : W_0^{1,\vec{p}}(\vec{a},\Omega) \to L^{q'}(\Omega)$ for which we address to Proposition 4.3 and Proposition 4.4, we infer that $A_R(u_n) \to A_R(u)$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)^*$ and $\langle A_R(u_n), u_n\rangle \to \langle A_R(u), u\rangle$, thus A_R is pseudomonotone.

We turn our attention to show that the operator A_R in (4.27) is coercive which reads as

$$\lim_{\|u\|\to\infty}\frac{\langle A_R(u), u\rangle}{\|u\|} = +\infty. \tag{4.36}$$

The proof is carried out by making use of hypothesis (H_3) that implies for $\|u\| > 1$ we have

$$\begin{aligned}
&\langle A_R(u), u\rangle \\
&= \sum_{i=1}^{N} \int_\Omega \nu_{iR}(x,u)|\frac{\partial u}{\partial x_i}|^{p_i} dx - \int_\Omega f(x,u,\nabla u) u dx \\
&\geq \sum_{i=1}^{N} \alpha_i \int_\Omega a_i(x)|\frac{\partial u}{\partial x_i}|^{p_i} dx - c_1 \sum_{i=1}^{N} \int_\Omega a_i(x)|\frac{\partial u}{\partial x_i}|^{p_i} dx - c_2 \int_\Omega |u|^{p^-} dx - \int_\Omega \varrho(x) dx \\
&= \sum_{i=1}^{N} \alpha_i \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)}^{p_i} - c_1 \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)}^{p_i} - c_2 \|u\|_{L^{p^-}(\Omega)}^{p^-} - \|\varrho\|_{L^1(\Omega)} \\
&\geq \sum_{i=1}^{N} (\alpha_i - c_1 - c_2 \eta^{p^-} N^{p^- -1}) \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)}^{p_i} - c_2 \eta^{p^-} N^{p^-} - \|\varrho\|_{L^1(\Omega)}.
\end{aligned}$$

Since $\alpha_i - c_1 - c_2\eta^{p^-} N^{p^- -1} > 0$ for all $i = 1, 2, \ldots, N$ and $p^- > 1$, we infer that (4.36) holds true. Therefore it is allowed to apply Theorem 2.3, which provides a solution $u_R \in W_0^{1,\vec{p}}(\vec{a}, \Omega)$ for the operator equation $A_R(u_R) = 0$. Invoking Remark 4.2, u_R represents a weak solution to equation (4.26). The proof is complete. □

Now we can state the proof of the main result, i.e. the proof of Theorem 1.1.

Proof. Theorem 3.4 ensures that the entire set of solutions of problem (1.3) is uniformly bounded, that is, there exists a constant $C > 0$ such that $\|u\|_{L^\infty(\Omega)} \leq C$ for all weak solutions $u \in W_0^{1,\vec{p}}(\vec{a}, \Omega)$ to problem (1.3). The truncated problem (4.26) satisfies exactly the same hypotheses, and with the same coefficients, as the original problem (1.3) with g_{iR} in place of g_i. It is essential to note that the inequality $\alpha_i - c_1 - c_2\eta^{p^-} N^{p^- -1} > 0$ for all $i = 1, 2, \ldots, N$, assumed in hypothesis (H_3) is independent of $R > 0$. Consequently, Theorem 3.4 applies to the truncated problem (4.26) involving the truncation g_{iR} and produces the same uniform bound $C > 0$ for the solution set of (4.26) with any $R > 0$. Actually, the statements of Theorem 3.4 and Lemma 3.1 show that the uniform bound $C > 0$ for the solution set depends on the function g_i only through the lower bound α_i of g_i, which is the same for each truncation g_{iR}. In particular, we have that the solution $u_R \in W_0^{1,\vec{p}}(\vec{a}, \Omega)$ to problem (4.26) provided by Theorem 4.5 satisfies the estimate $\|u_R\|_{L^\infty(\Omega)} \leq C$ whenever $R > 0$.

Now choose $R \geq C$. Then the estimate $\|u_R\|_{L^\infty(\Omega)} \leq C$ and (4.25) imply

$$g_{iR}(|u_R(x)|) = g_i(|u_R(x)|) \text{ for all } x \in \Omega \text{ and } i = 1, 2, \ldots, N,$$

hence due to (2.4),

$$\nu_{iR}(x, u_R(x)) = \nu_i(x, u_R(x)) \text{ for all } x \in \Omega \text{ and } i = 1, 2, \ldots, N.$$

It follows that the solution $u_R \in W_0^{1,\vec{p}}(\vec{a}, \Omega)$ to the auxiliary problem (4.26) is a bounded weak solution to the original problem (1.3), which completes the proof of Theorem 1.1. □

Remark 4.3. *We end this section by observing that, in the case of variable exponents, that is when $p_j = p_j(x)$, there are many applications to electrorheological fluids, thermorheological fluids, elastic materials, and image restoration. Thus (1.3) can be studied in the case of variable exponents, for the future studies.*

5. Conclusion

We study the nonlinear elliptic problem (1.3) characterized by an anisotropic leading differential operator that includes unbounded coefficients, with the nonlinear component being a convection term. This problem is a new extension of problems (1.1) and (1.2) to a degenerate one in the anisotropic setting. Our result shows the solvability of the degenerate Dirichlet problem associated with convection, demonstrating the existence of at least one bounded weak solution. This is achieved through the application of the theory of pseudomonotone operators, the Nemytskii-type operator, and a priori estimates within the framework of degenerate anisotropic Sobolev spaces. To the best of our knowledge, our result represents the first result in this literature.

Acknowledgments. Author E. Tornatore is member of GNAMPA. This research is partially supported by FFR2024-Tornatore and PRIN 2022-Progetti di Ricerca di rilevante Interesse Nazionale-Nonlinear differential problems with applications to real phenomena (Project N. 2022ZXZTN2)

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