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DIFFRACTION OF SOUND FROM SEMI-INFINITE PERFORATED AND SEMI-INFINITE COATED DUCT WITH RING SOURCE

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Abstract. This study delves into the analysis of acoustic waves emanating from a ring source in an infinite cylindrical duct. The duct is equipped with an acoustically absorbing lining on its outer surface when z is less than l, and is perforated when z is greater than l. The inclusion of acoustically absorbing lining results in a substantial increase in the complexity of the equations compared to the scenario without such lining. Through rigorous efforts, these intricate equation systems are numerically solved, and graphs are generated across various parameter values. Furthermore, by adjusting the parameter values, a physical resemblance is established with an existing study in the literature, showcasing impeccable alignment in the results.

1. Introduction

In recent years, the problem of diffraction or radiation of sound waves has been a significant subject analyzed by researchers. Duct and pipe structures have a widespread application in industrial systems such as exhaust systems, ventilation systems, aircraft jets, and modern turbofan engines to control unwanted and potentially harmful noise. Therefore, there is a need to explore more precise mathematical models to address these complex engineering issues.

The Wiener–Hopf method [1], widely recognized for its convenience in analyzing such applications, is included in numerous studies within the literature [2–8]. First, Levine and Schwinger employed the Wiener–Hopf method to investigate the radiation of sound through a semi-infinite rigid duct [9].

There are some techniques used to reduce unwanted noise in duct and pipe modeling. Some of these methods include covering the entire or part of the duct

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with acoustically absorbing lining [10–14] using expansion or contraction chambers [15, 16] and using perforated structures [17, 18].

This study is approached by taking inspiration from the previous study [19]. Unlike the previous one, an acoustically absorbing lining is utilized in this study. This change, although not creating a significant difference at first glance, can be said to transform the existing problem into a quite complex and practical one, especially with the use of acoustically absorbing lining. Significantly, this adjustment introduces a nuanced challenge, particularly evident in terms of mathematical analysis, where it yields a greater number of unknown functions and a binary system of equations. The primary objective of this research is to augment the existing perforated structure by introducing an acoustically absorbing lining. It is well-established that such linings contribute to a measurable reduction in sound pressure levels, typically by a few decibels. Consequently, the simultaneous utilization of both a perforated structure and an acoustically absorbing lining is posited as the key to achieving optimal efficiency in acoustic control.

This study consists of the following sections. In Section 2, the formulation of the problem and the presentation of boundary-continuity conditions are provided. Section 3 addresses the derivation and solution of the Wiener–Hopf equation. Far field analysis is performed in Section 4. In Section 5, numerical results are presented using graphics generated for various parameter values. Finally, Section 6 summarizes the findings obtained from solving the problem.

2. Problem Setting

The problem's geometric configuration is illustrated in Figure 1. Here, the duct walls are modeled as infinitely thin, occupying the region $\{r = a, z \in (-\infty, \infty)\}$, illuminated by a ring source [20] situated at $\{r = b, z = -c, c > 0\}$. For $z < l$, the exterior surface of the cylinder assumes an acoustically absorbing lining characterized by Z, while the interior surface is assumed to be rigid. The duct for $z > l$ is considered perforated. Due to the symmetry of both the problem's geometry and the ring source, the total field remains independent of azimuth ϕ throughout the circular cylindrical coordinate system (r, ϕ, z).

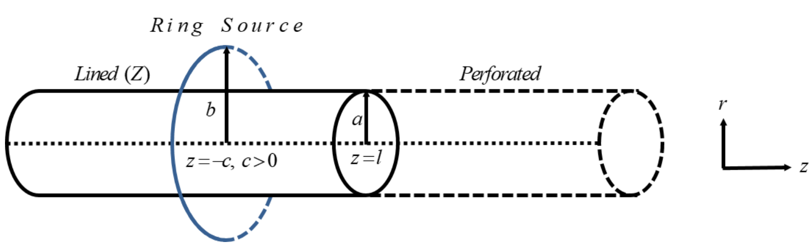

FIGURE 1. The geometry of the problem.

The scalar potential $\psi(r,z)$ can be ascertained by establishing a relationship between velocity v and acoustic pressure p, as expressed by the following equations. $\overrightarrow{v} = \operatorname{grad}\psi$ and $p = \rho_0 \partial\psi/\partial t$, where ρ_0 represents the density of the undisturbed medium, and t denotes time. In the course of this study, we assume a harmonic time dependence of the form $e^{-i\omega t}$, where ω stands for the angular frequency.

For analytical simplicity, the total field $\psi_t(r,z)$ can be represented as:

$$\psi_t\left(r,z\right) = \begin{cases} \psi_1\left(r,z\right), & r > b \\ \psi_2\left(r,z\right), & a < r < b \\ \psi_3\left(r,z\right), & r < a \end{cases}, \tag{1}$$

where $\psi_j(r,z)$, $j = 1,2,3$ appearing in (1) are an unknown functions that satisfy the wave equation

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} + k^2\right]\psi_j(r,z) = 0, \quad j = 1,2,3. \tag{2}$$

Here, $k = \omega/c_0$ represents the wavenumber, where ω is the angular frequency, and c_0 is the speed of sound.

From the geometry of the problem, one can write the following boundary conditions and continuity relations. The outer surface of the duct is lined with acoustically absorbing lining for $z < l$, this means

$$\left(\frac{ik}{Z} - \frac{\partial}{\partial r}\right)\psi_2(a,z) = 0, \quad z < l, \tag{3}$$

where Z denotes the acoustic impedance of the lined wall. The inner surface of the duct is rigid for $z < l$, one obtains

$$\frac{\partial}{\partial r}\psi_3(a,z) = 0, \quad z < l. \tag{4}$$

For the perforated duct for $z > l$, the following conditions can be written

$$\begin{aligned} \frac{\partial}{\partial r}\psi_2(a,z) &= \frac{\partial}{\partial r}\psi_3(a,z), \quad z > l \\ \psi_2(a,z) &= \psi_3(a,z) + i\frac{\zeta_p}{k}\frac{\partial}{\partial r}\psi_3(a,z), \quad z > l \end{aligned}, \tag{5}$$

where ζ_p is the specific impedance [21], which characterizes the acoustic properties of the perforated duct.

$$\zeta_p = [0.006 - ik(t_w + 0.75d_h)]/\sigma,$$

with t_w representing the screen thickness, d_h denoting the perforate hole diameter, and σ indicating the porosity.

The last following conditions can be obtained from the ring source

$$\begin{aligned}\frac{\partial}{\partial r}\psi_1(b,z)-\frac{\partial}{\partial r}\psi_2(b,z)=\delta(z+c),\quad z\in(-\infty,\infty)\\ \psi_1(b,z)-\psi_2(b,z)=0,\quad z\in(-\infty,\infty)\end{aligned},\tag{6}$$

where δ represents the Dirac delta function.

By taking Fourier transform of (2) we obtain the following integral representations

$$\begin{aligned}\psi_1(r,z)&=\frac{k}{2\pi}\int_{\mathcal{L}}A(\alpha)H_0^{(1)}(\lambda kr)e^{-i\alpha kz}d\alpha\\ \psi_2(r,z)&=\frac{k}{2\pi}\int_{\mathcal{L}}[B(\alpha)J_0(\lambda kr)+C(\alpha)Y_0(\lambda kr)]e^{-i\alpha kz}d\alpha,\\ \psi_3(r,z)&=\frac{k}{2\pi}\int_{\mathcal{L}}D(\alpha)J_0(\lambda kr)e^{-i\alpha kz}d\alpha\end{aligned}\tag{7}$$

In the context where $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, and $D(\alpha)$ are the spectral coefficients, determined by solving the equations (3)-(6), the integration contour $\mathcal{L}$ is chosen to be a suitable inverse Fourier transform along or near the real axis in the complex α-plane. The functions J_0 and Y_0 correspond to the Bessel and Neumann functions of order zero, respectively. Additionally, $H_0^{(1)}=J_0+iY_0$ represents the Hankel function of the first type [22], while λ stands for the square root function, defined as:

$$\lambda(\alpha)=\sqrt{1-\alpha^2},\quad \lambda(0)=1.$$

3. Wiener–Hopf Procedure

In this section, the Wiener–Hopf equation will be derived and the soltuion obtained.

3.1. **Derivation of the Wiener–Hopf Equations.**

Enforcing the boundary conditions at $r=a$ and performing Fourier transforms of (3) and (4) yields:

$$k[B(\alpha)J(Z,\alpha)+C(\alpha)Y(Z,\alpha)]=e^{i\alpha kl}\Phi_1^+(\alpha),\tag{8}$$

$$-\lambda kD(\alpha)J_1(\lambda ka)=e^{i\alpha kl}\Phi_2^+(\alpha),\tag{9}$$

where

$$\begin{aligned}J(Z,\alpha)&=iJ_0(\lambda ka)/Z+\lambda J_1(\lambda ka),\\ Y(Z,\alpha)&=iY_0(\lambda ka)/Z+\lambda Y_1(\lambda ka).\end{aligned}$$

Similarly, from equation (5), we get

$$-\lambda kD(\alpha)J_1(\lambda ka)+\lambda kB(\alpha)J_1(\lambda ka)+\lambda kC(\alpha)Y_1(\lambda ka)=e^{i\alpha kl}\Phi_1^-(\alpha),\tag{10}$$

$$D(\alpha)\left[J_0(\lambda ka)-i\lambda\zeta_pJ_1(\lambda ka)\right]-B(\alpha)J_0(\lambda ka)-C(\alpha)Y_0(\lambda ka)=e^{i\alpha kl}\Phi_2^-(\alpha).\tag{11}$$

In the upper half-plane, $\Phi^+_{1,2}$, and in the lower half-plane, $\Phi^-_{1,2}$ are analytical functions, as described in [19], and their definitions are as follows:

$$\Phi_1^+(\alpha) = \int_l^\infty \left[\frac{ik}{Z}\psi_2(a,z) - \frac{\partial}{\partial r}\psi_2(a,z)\right] e^{i\alpha k(z-l)}dz$$
$$\Phi_2^+(\alpha) = \int_l^\infty \frac{\partial}{\partial r}\psi_3(a,z)e^{i\alpha k(z-l)}dz$$
$$\Phi_1^-(\alpha) = \int_{-\infty}^l \left[\frac{\partial}{\partial r}\psi_3(a,z) - \frac{\partial}{\partial r}\psi_2(a,z)\right] e^{i\alpha k(z-l)}dz$$
$$\Phi_2^-(\alpha) = \int_{-\infty}^l \left[\psi_3(a,z) - \psi_2(a,z)\right] e^{i\alpha k(z-l)}dz$$

Finally, from the ring source on $r = b$, we obtain

$$\lambda k A(\alpha) H_1^{(1)}(\lambda kb) = \lambda k B(\alpha) J_1(\lambda kb) + \lambda k C(\alpha) Y_1(\lambda kb) - e^{-i\alpha kc}, \tag{12}$$

$$A(\alpha) H_0^1(\lambda kb) = B(\alpha) J_0(\lambda kb) + C(\alpha) Y_0(\lambda kb), \tag{13}$$

where $H_0^{(1)} = J_1 + Y_1$. By using the above two equation (12) and (13), one gets

$$\begin{aligned} B(\alpha) &= A(\alpha) + e^{-i\alpha kc}\frac{\pi b}{2}Y_0(\lambda kb) \\ C(\alpha) &= iA(\alpha) - e^{-i\alpha kc}\frac{\pi b}{2}J_0(\lambda kb) \end{aligned}. \tag{14}$$

The coefficients $A(\alpha)$, $B(\alpha)$, and $C(\alpha)$ are interrelated through the equation (14), whereas the coefficient $D(\alpha)$ can be readily derived from the equation (9).

$$D(\alpha) = -\frac{e^{i\alpha kl}\Phi_2^+(\alpha)}{\lambda k J_1(\lambda ka)}. \tag{15}$$

By using (8) and (14), $A(\alpha)$ can be obatined as follows

$$A(\alpha) = \frac{e^{i\alpha kl}\Phi_1^+(\alpha)}{kH(Z,\alpha)} - \frac{e^{-i\alpha kc}\pi b}{2H(Z,\alpha)}[Y_0(\lambda kb)J(Z,\alpha) - J_0(\lambda kb)Y(Z,\alpha)], \tag{16}$$

where

$$H(Z,\alpha) = iH_0^{(1)}(\lambda ka)/Z + \lambda H_1^{(1)}(\lambda ka).$$

By substituting these coefficients ($A(\alpha)$, $B(\alpha)$, $C(\alpha)$ and $D(\alpha)$) into (5), we obtain the following equations

$$\Phi_2^+(\alpha) + \frac{\lambda H_1^{1)}(\lambda ka)}{H(Z,\alpha)}\Phi_1^+(\alpha) + \frac{ib}{aZ}e^{-i\alpha k(c+l)}\frac{H_0^{(1)}(\lambda kb)}{H(Z,\alpha)} = \Phi_1^-(\alpha), \tag{17}$$

$$-\frac{J_0(\lambda ka) - i\lambda\zeta_p J_1(\lambda ka)}{\lambda k J_1(\lambda ka)}\Phi_2^+(\alpha) - \frac{H_0^{(1)}(\lambda ka)}{kH(Z,\alpha)}\Phi_1^+(\alpha) + \frac{b}{ka}e^{-i\alpha k(c+l)}\frac{H_0^{(1)}(\lambda kb)}{H(Z,\alpha)} = \Phi_2^-(\alpha). \tag{18}$$

$H_0^{(1)}(\lambda ka)/H(Z,\alpha)$ can be eliminated from equations (17) and (18), we get

$$\Phi_1^+(\alpha) + L(\alpha)\Phi_2^+(\alpha) = \left[\Phi_1^-(\alpha) - \frac{ik}{Z}\Phi_2^-(\alpha)\right], \tag{19}$$

where

$$L(\alpha) = \frac{J(Z,\zeta_p,\alpha)}{\lambda J_1(\lambda ka)}, \tag{20}$$

and

$$J(Z,\zeta_p,\alpha) = J(Z,\alpha) + \zeta_p J_1(\lambda ka)/Z.$$

Then eliminating $\Phi_2^+(\alpha)$ from equation (18) and (19), we get

$$\Phi_1^+(\alpha)M(\alpha) + \frac{b}{a}\frac{H_0^{(1)}(\lambda kb)}{H(Z,\alpha)}e^{-i\alpha k(c+l)} = \left[\Phi_1^-(\alpha) - \frac{ik}{Z}\Phi_2^-(\alpha)\right]\frac{J_0(\lambda ka) - i\lambda\zeta_p J_1(\lambda ka)}{J(Z,\zeta_p,\alpha)} + k\Phi_2^-(\alpha), \tag{21}$$

where

$$M(\alpha) = \frac{J_0(\lambda ka) - i\lambda\zeta_p J_1(\lambda ka)}{J(Z,\zeta_p,\alpha)} - \frac{H_0^{(1)}(\lambda ka)}{H(Z,\alpha)}. \tag{22}$$

(19) and (21) are two coupled Wiener–Hopf equations and $L(\alpha)$, $M(\alpha)$ are kernel functions given in (20) and (22), respectively, to be factorized.

3.2. **Solution of the Wiener–Hopf Equations.** Examine the first Wiener–Hopf equation in (19) and reorganize it using (20) in the subsequent format:

$$\frac{\lambda J_1(\lambda ka)}{J(Z,\zeta_p,\alpha)}\Phi_1^+(\alpha)L_+(\alpha) + \Phi_2^+(\alpha)L_+(\alpha) = \left[\Phi_1^-(\alpha) - \frac{ik}{Z}\Phi_2^-(\alpha)\right]L_-(\alpha). \tag{23}$$

Similarly, by using equation (21) and (22), we get

$$\Phi_1^+(\alpha)M_+(\alpha) + \frac{b}{a}\frac{H_0^{(1)}(\lambda kb)}{H(Z,\alpha)}e^{-i\alpha k(c+l)}M_-(\alpha) = \left[\Phi_1^-(\alpha) - \frac{ik}{Z}\Phi_2^-(\alpha)\right]\frac{J_0(\lambda ka) - i\lambda\zeta_p J_1(\lambda ka)}{J(Z,\zeta_p,\alpha)}M_-(\alpha) + k\Phi_2^-(\alpha)M_-(\alpha). \tag{24}$$

Here, $L_+(\alpha)$, $M_+(\alpha)$, $L_-(\alpha)$, and $M_-(\alpha)$ represent analytic functions devoid of zeros in the upper and lower half-planes, respectively. The factorization of $L(\alpha)$ and $M(\alpha)$ is provided as follows [19]:

$$L(\alpha) = \frac{L_+(\alpha)}{L_-(\alpha)}, \quad M(\alpha) = \frac{M_+(\alpha)}{M_-(\alpha)}.$$

In the upper half-plane, the left-hand side of (23) is analytic, except for the poles originating from the zeros of $J(Z,\zeta_p,\alpha)$ situated in the same half-plane, specifically at $\alpha = \alpha_m^-$ with

$$iJ_0\left(\sqrt{1-\left(\alpha_m^-\right)^2}ka\right)/Z+\sqrt{1-\left(\alpha_m^-\right)^2+\zeta_p/Z}J_1\left(\sqrt{1-\left(\alpha_m^-\right)^2}ka\right)=0. \quad (25)$$

By subtracting the infinite system of poles from both sides of (23), we obtain:

$$\frac{\lambda J_1(\lambda ka)}{J(Z,\zeta_p,\alpha)}\Phi_1^+(\alpha)L_+(\alpha)-\sum_{m=1}^{\infty}\frac{c_m^+}{\alpha-\alpha_m^-}+\Phi_2^+(\alpha)L_+(\alpha)$$
$$=\left[\Phi_1^-(\alpha)-\frac{ik}{Z}\Phi_2^-(\alpha)\right]L_-(\alpha)-\sum_{m=1}^{\infty}\frac{c_m^+}{\alpha-\alpha_m^-}, \quad (26)$$

where

$$c_m^+=\Phi_1^+(\alpha_m^-)L_+(\alpha_m^-)\lim_{\alpha\to\alpha_m^-}\frac{\lambda J_1(\lambda ka)}{\frac{d}{d\alpha}J(Z,\zeta_p,\alpha)}. \quad (27)$$

By applying the analytical continuation principle along with Liouville's theorem to (26), one obtains:

$$\left[\Phi_1^-(\alpha)-\frac{ik}{Z}\Phi_2^-(\alpha)\right]=\frac{1}{L_-(\alpha)}\sum_{m=1}^{\infty}\frac{c_m^+}{\alpha-\alpha_m^-}. \quad (28)$$

By applying similar procedure to (24), we get the followings

$$\Phi_1^+(\alpha)M_+(\alpha)=I_+(\alpha)+\sum_{m=1}^{\infty}\frac{c_m^-}{\alpha-\alpha_m^+}, \quad (29)$$

where

$$I(u)=-\frac{b}{a}\frac{H_0^{(1)}(\lambda kb)}{H(Z,\alpha)}e^{-i\alpha k(c+l)}M_-(\alpha)=I_+(\alpha)+I_-(\alpha), \quad (30)$$

and

$$c_m^-=\left[\Phi_1^-(\alpha_m^+)-\frac{ik}{Z}\Phi_2^-(\alpha_m^+)\right]M_-(\alpha_m^+)\lim_{\alpha\to\alpha_m^+}\frac{J_0(\lambda ka)-i\lambda\zeta_pJ_1(\lambda ka)}{\frac{d}{d\alpha}J(Z,\zeta_p,\alpha)}. \quad (31)$$

Through the decomposition of (30), we can derive analytical functions in both the upper and lower half planes, given by $I(\alpha)=I_+(\alpha)+I_-(\alpha)$.

$$I_+(u)=-\frac{1}{2\pi i}\frac{b}{a}\int_{\mathcal{L}^+}\frac{H_0^{(1)}(\lambda kb)M_-(\tau)}{H(Z,\tau)(\tau-\alpha)}e^{-i\tau k(c+l)}d\tau. \quad (32)$$

3.3. **Determining the Coefficient c_m^+ and c_m^-.** Wiener–Hopf solutions (28) and (29) involve unknown coefficients c_m^+ and c_m^- that need to be determined. By substituting $\alpha = \alpha_n^+$ into (28) and utilizing the relation in (31), we get:

$$\frac{1}{M_-(\alpha_n^+)\lim_{\alpha\to\alpha_n^+}\frac{J_0(\lambda ka)-i\lambda\zeta_p J_1(\lambda ka)}{\frac{d}{d\alpha}J(Z,\zeta_p,\alpha)}}c_n^- = \frac{1}{L_-(\alpha_n^+)}\sum_{m=1}^{\infty}\frac{c_m^+}{\alpha_n^+-\alpha_m^-}, \quad n=1,2,\dots \tag{33}$$

Similarly, by substituting $\alpha = \alpha_n^-$ into (29) and utilizing the relation in (27), we obtain:

$$\frac{M_+(\alpha_n^-)}{L_+(\alpha_n^-)\lim_{\alpha\to\alpha_n^-}\frac{\lambda J_1(\lambda ka)}{\frac{d}{d\alpha}J(Z,\zeta_p,\alpha)}}c_n^+ = I_+(\alpha_n^-) + \sum_{m=1}^{\infty}\frac{c_m^-}{\alpha_n^- - \alpha_m^+}, \quad n=1,2,\dots \tag{34}$$

To find the unknown coefficients c_m^+ and c_m^-, numerical solutions will be obtained for these coupled systems of algebraic equations. Given the rapid convergence of the infinite series, truncation can be efficiently performed. Therefore, all numerical results will be obtained by truncating the infinite series and the infinite systems of linear algebraic equations after the first N terms. As shown in Figure 2, it was observed that the amplitude of the diffracted field becomes insensitive to the increase in the truncation number after reaching $N = 15$ [23,24].

4. Far Field

The expression for the total field in the region $r > b$ can be derived from (7).

$$\psi_1(r,z) = \frac{k}{2\pi}\int_{\mathcal{L}} A(\alpha)H_0^{(1)}(\lambda kr)e^{-i\alpha kz}d\alpha. \tag{35}$$

By using (16) and (29), the total field can be formulated in the following manner:

$$\psi_1(r,z) = \psi_d(r,z) + \psi_i(r,z) + \psi_r(r,z), \tag{36}$$

where

$$\psi_d(r,z) = \frac{k}{2\pi}\int_{\mathcal{L}}\frac{\Phi_1^+(\alpha)}{kH(Z,\alpha)}H_0^{(1)}(\lambda kr)e^{-i\alpha kz}d\alpha, \tag{37}$$

$$\psi_i(r,z)+\psi_r(r,z) = -\frac{kb}{4i}\int_{\mathcal{L}}\frac{Y_0(\lambda kb)J(Z,\alpha)-J_0(\lambda kb)Y(Z,\alpha)}{H(Z,\alpha)}H_0^{(1)}(\lambda kr)e^{-i\alpha k(z+c)}d\alpha. \tag{38}$$

Substituting the following asymptotic expressions, valid for $kr \gg 1$, for $H_0^{(1)}(\lambda kr)$ and $J_0(\lambda kb)$ in place of them

$$H_0^{(1)}(\lambda kr) \sim \sqrt{\frac{2}{\pi\lambda kr}}e^{i(\lambda kr-\pi/4)},$$

and employing the saddle point technique [25], we obtain:

$$\psi_1(r,z) = \psi_d(R_1,\theta_1) + \psi_i(R_2,\theta_2) + \psi_r(R_2,\theta_2), \tag{39}$$

where

$$\psi_d(R_1,\theta_1) = -\frac{i}{\pi}\frac{\Phi_1^+(-\cos\theta_1)}{H(Z,-\cos\theta_1)}\frac{e^{ikR_1}}{kR_1}, \tag{40}$$

$$\psi_i(R_2,\theta_2) + \psi_r(R_2,\theta_2) = \frac{ikb}{2}\frac{Y_0(kb\sin\theta_2)J(Z,-\cos\theta_2) - J_0(kb\sin\theta_2)Y(Z,-\cos\theta_2)}{H(Z,-\cos\theta_2)}\frac{e^{ikR_2}}{kR_2}. \tag{41}$$

Φ_1^+ is given in (29). Here, R_1, θ_1 and R_2, θ_2 represent the spherical coordinates defined as follows:

$$r = R_1\sin\theta_1, \quad z - l = R_1\cos\theta_1,$$
$$r = R_2\sin\theta_2, \quad z + c = R_2\cos\theta_2.$$

5. Numerical Results

In this section, graphs are produced for different parameter values by taking advantage of the properties of geometry. Some parameter values are taken as unchanged and given in Table 1. The values taken for different parameter values are selected from some studies in the literature [19,21]. Figures are produced using the Sound Pressure Level (SPL) formula defined below.

$$\mathrm{SPL} = 20\log_{10}\left|\frac{p}{2\cdot 10^{-5}}\right|.$$

Table 1. The values of the parameter.

Truncation number	N	15
Density of Un. Med.	ρ_0	1.255 kg/m^3
Speed of sound	c_0	340 m/s
Screen thickness	t_w	0.00081 m
Hole diameter	d_h	0.0249 m
Ring source axis	c	0.050 m
Lining length	l	0.010 m
Far radius	R	46 m

In Figure 2, the sound pressure level is plotted according to the increase in truncation number (N) values. As can be seen, after a certain value of N, the change in sound pressure level is insignificant. Therefore, other graphs are produced for $N = 15$.

The change of sound pressure level according to both duct radius and ring source radius are plotted in Figure 3 and Figure 4, respectively. As expected, the sound

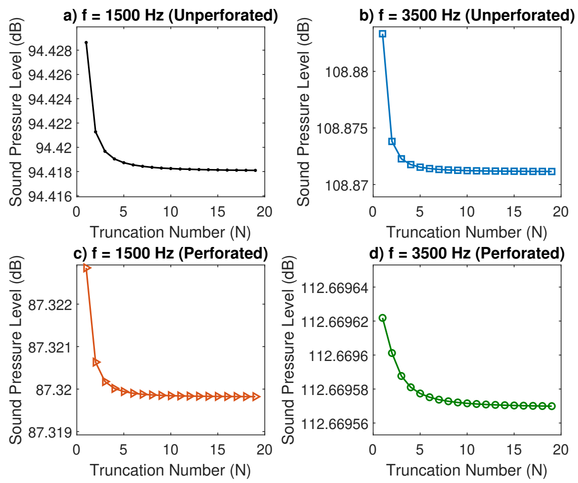

FIGURE 2. Sound Pressure Level (SPL) against the truncation number (N) for $a = 0.010$ m, $b = 0.075$ m, $Z = 1 - 2i$ and $\sigma = 0.057$.

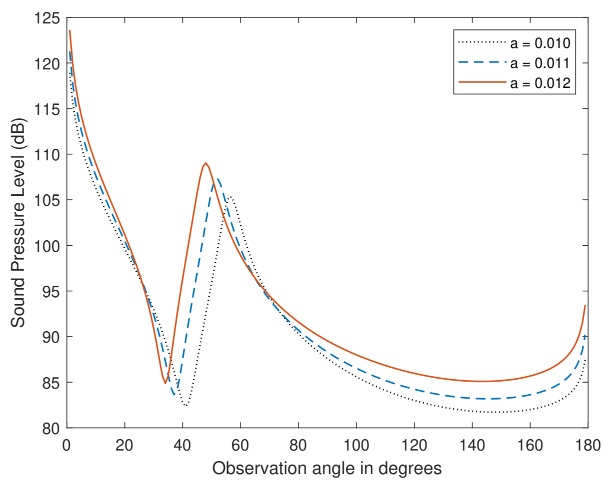

FIGURE 3. The impact of varied duct radius values (a) on Sound Pressure Level (SPL) at different observation angles, with fixed parameters: $f = 1500$ Hz, $b = 0.075$ m, $Z = 1 - 2i$, and $\sigma = 0.057$.

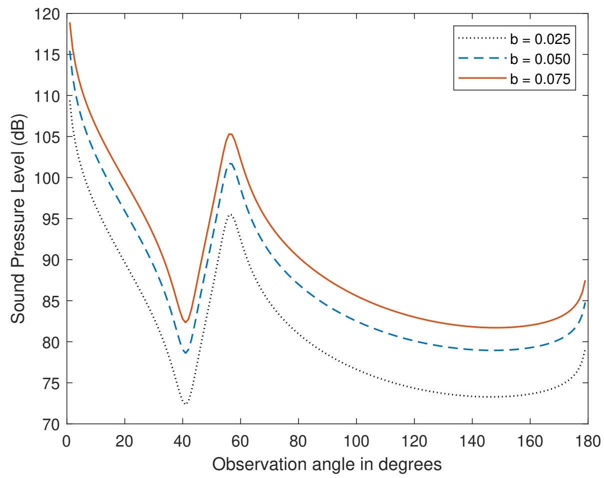

FIGURE 4. The impact of varied ring source radius values (b) on Sound Pressure Level (SPL) at different observation angles, with fixed parameters: $f = 1500$ Hz, $a = 0.010$ m, $Z = 1 - 2i$, and $\sigma = 0.057$.

pressure level increases with increasing the values of duct radius and ring source radius. On the other hand, the opposite situation exists for increasing values of c and l. As in the previous study [19], a decrease in the sound pressure level is observed both as the ring source moves away and as the duct extension increases.

The effect of acoustically absorbing lining is examined in Figure 5 and Figure6. In Figure 5, the real part is kept constant for the complex impedance value, and the variation of sound pressure level is plotted with respect to the imaginary part. In Figure 6, the imaginary part is kept constant, and the variation of sound pressure level is plotted with respect to the real part. As seen, a decrease in sound pressure level can be achieved by selecting appropriate $\mathfrak{Re}Z$ and $\mathfrak{Im}Z$ values.

Figure 7 depicts the variation in sound pressure level for different porosity rates. As expected, the sound pressure level increases with an increase in porosity.

Consistency with the previous study [19] is examined in the last two graphs. These comparison figures indicate a meticulous handling of complex problems that particularly emerged in this study. In Figure 8, the impact of the lining is eliminated to establish physical similarity with the previous study. For this purpose, the impedance value is taken to infinity to obtain a graph for the rigid surface. It should be noted that the sound pressure level graphs obtained by taking the impedance value to infinity are identical to the previous ones. A similar situation holds true for Figure 9 as well. In this graph, normalized values of ka, kb, kc, and kl are employed for consistency with the previous study. To achieve physical similarity,

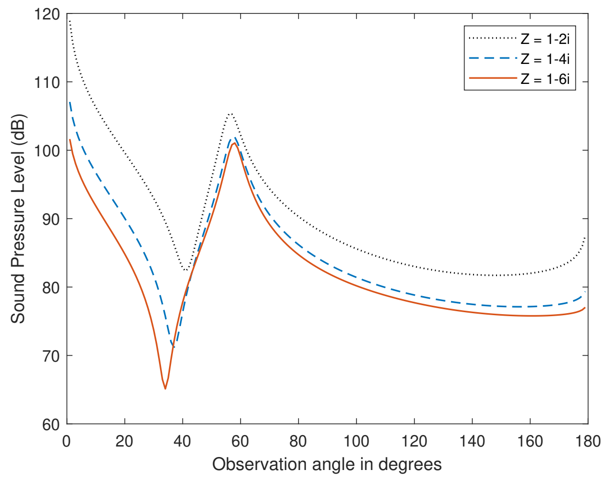

FIGURE 5. The impact of varied absorbing lining (reactance) values ($\mathfrak{Im} Z$) on Sound Pressure Level (SPL) at different observation angles, with fixed parameters: $f = 1500$ Hz, $a = 0.010$ m, $b = 0.075$ m, and $\sigma = 0.057$.

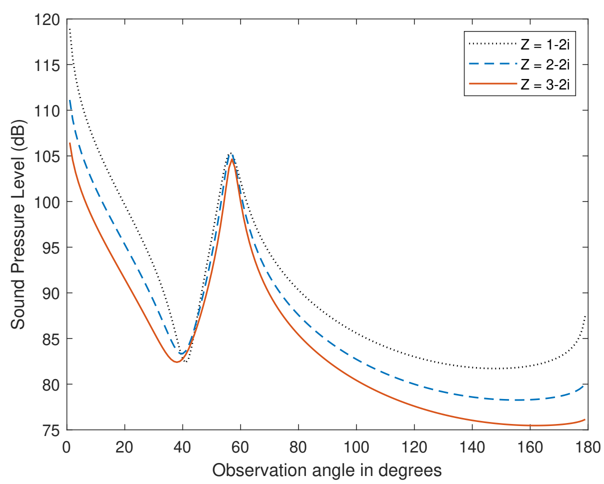

FIGURE 6. The impact of varied absorbing lining (resistance) values ($\mathfrak{Re} Z$) on Sound Pressure Level (SPL) at different observation angles, with fixed parameters: $f = 1500$ Hz, $a = 0.010$ m, $b = 0.075$ m, and $\sigma = 0.057$.

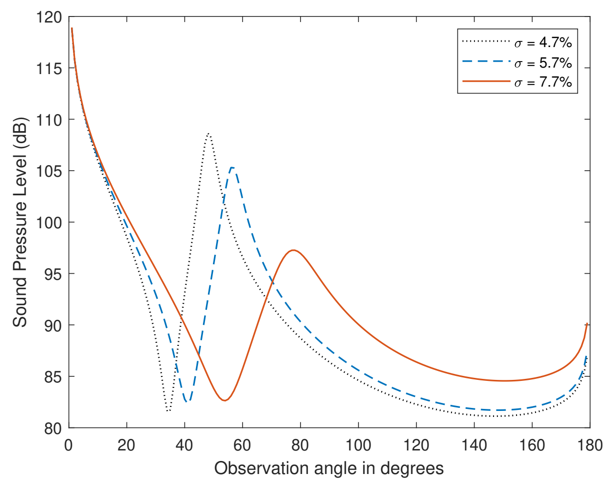

FIGURE 7. The impact of varied porosity values (σ) on Sound Pressure Level (SPL) at different observation angles, with fixed parameters: $f = 1500$ Hz, $a = 0.010$ m, $b = 0.075$ m and $Z = 1-2i$.

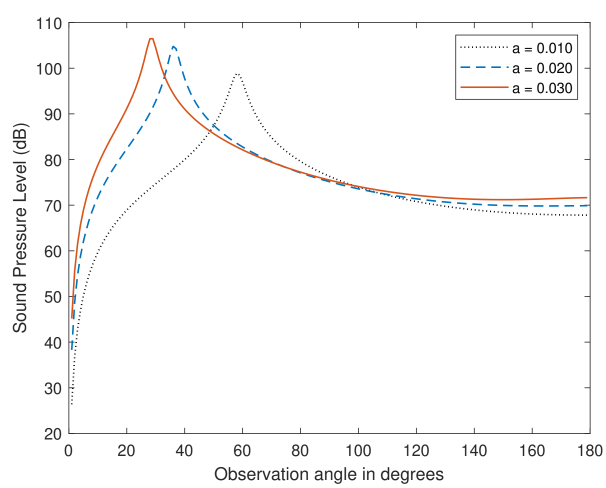

FIGURE 8. At parameters $f = 1500$ Hz, $a = 0.010$ m, $b = 0.075$ m, $Z \to \infty$, and $\sigma = 0.057$, comparison of the Sound Pressure Level (SPL) with different duct radius values (a) with the study of [19].

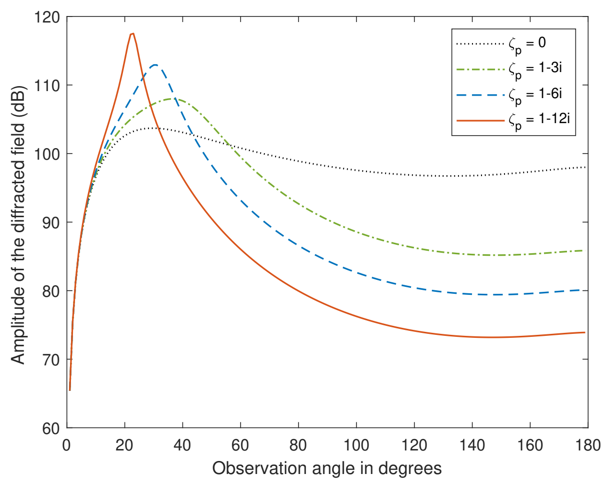

FIGURE 9. At parameters $ka = 1$, $kb = 10$, $kc = 6$, $kl = 10$ and $Z \to \infty$, comparison of the Sound Pressure Level (SPL) with open perforated duct with the study of [19].

the impedance value is taken to infinity. The obtained result, as in the previous graph, demonstrates very good alignment.

6. Conclusions

In this study, the diffraction of sound waves emanating from a ring source in an infinite duct with an acoustically lined outer surface for $z < l$ and a perforated surface for $z > l$ has been investigated using the Wiener Hopf technique. Due to the symmetry of both the problem's geometry and the ring source, the problem has been modeled in two dimensions. By solving the Wiener–Hopf equation, a solution is obtained. Graphs are presented for some specific values of problem parameters, such as the radius of the duct and the ring source, the effect of acoustically absorbing lining, and the perforated duct, to better understand their impact on the sound pressure level. An increase in the values of the duct radius (a) and the ring source radius (b) is observed to result in an increase in the sound pressure level. A similar trend is valid for both real and complex values of acoustically absorbing lining (Z). The effect of the perforated duct on the sound pressure level is also significant, with a decrease in the sound pressure level observed as the porosity of the perforated duct decreases. Finally, when compared to the study of [19] for $Z \to \infty$, it is observed that the conformity is excellent.

This study can also be used to consider the case where mean flow is present both inside and outside the duct. It is worth noting that when mean flow is included, the current problem will be much more complicated to solve analytically and will require careful analysis.

Declaration of Competing Interests The author has no competing interests to declare.

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ON THE JACOBSTHAL NUMBERS WHICH ARE THE PRODUCT OF TWO MODIFIED PELL NUMBERS

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Abstract. This paper presents an analytic study of determining all the possible solutions of the Diophantine equations such that $q_k = J_m J_n$ and $J_k = q_m q_n$. These give intersections of the Modified Pell and Jacobsthal numbers too for the case where $m = 1$ or $n = 1$.

1. Introduction

It is well-known that the Pell, Modified Pell, and Jacobsthal numbers are defined by the recurrence relations

$$P_0 = 0,\ P_1 = 1 \text{ and } P_{n+1} = 2P_n + P_{n-1} \text{ for all } n \geqslant 2, \tag{1}$$

$$q_0 = 1,\ q_1 = 1 \text{ and } q_{n+1} = 2q_n + q_{n-1} \text{ for all } n \geqslant 2, \tag{2}$$

and

$$J_0 = 0,\ J_1 = 1 \text{ and } J_{n+1} = J_n + 2J_{n-1} \text{ for all } n \geqslant 2, \tag{3}$$

respectively. These integer sequences have very interesting characteristics. For this reason, a heavy interest has been devoted to investigation of the subject by a great number of researchers. Here, it is proposed that two fundamental books given by Vajda [1] and Koshy [2] are investigated for a piece of wide information.

As shown from Equations (1)-(3), all the desired terms of the related sequence can be computed recursively by using the respective recurrence relation. Also, as a second way, we can employ the following equations that are called Binet's formulas:

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta},\ q_n = \frac{\gamma^n + \delta^n}{\gamma + \delta},\ \text{and } J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{4}$$

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where γ and δ are the positive and negative roots of $x^2 - 2x - 1 = 0$, and α and β are the positive and negative roots of $x^2 - x - 2 = 0$.

Up to the present, many articles have been governed related to the identities and applications of the Modified Pell and Jacobsthal sequences. Let us briefly mention some of the relevant research. In [3], Horadam gave the definition of the Modified Pell numbers, including some elementary identities, and showed that $Q_n = 2q_n$, where Q_n is the nth Pell-Lucas numbers. In [4], the author defined the Jacobsthal numbers and presented their characteristic identities. In [5], Daşdemir developed an interesting matrix technique to find relationships between the Pell, Pell-Lucas, and Modified Pell numbers. In [6] and [7], Daşdemir brought rich elementary context related to the Jacobsthal and Jacobsthal-Lucas numbers to the available literature by using some matrix identities. In [8], Arslan and Köken presented the Jacobsthal and Jacobsthal-Lucas numbers with rational subscripts based on the idea of computing square roots of the matrices of order 2×2. In [9], Catarino and Campos introduced the Gaussian Modified Pell numbers, including Binet's formula, the generating function, and some sum formula. In [10], Radicic computed determinants, eigenvalues, and the values and boundaries of certain norms for a k-circulant matrix involving the Pell Numbers. In [11], Daşdemir expanded the usual Mersene, Jacobsthal, and Jacobsthal-Lucas numbers to the ones with negative indexes. In [12], Soykan and Göcen presented the definition, Binet formula, and generating functions of the generalized hyperbolic Pell numbers over the bi-dimensional Clifford algebra. In [13], Uygun defined the bi-periodic Jacobsthal and bi-periodic Jacobsthal-Lucas numbers and discovered some features between them.

The above brief literature survey shows that many researchers have genuinely interested in investigating the elementary identities and properties of the Modified Pell numbers and the Jacobsthal numbers with structural configurations and this trend is growing day by day. Motivated by these developments, in this paper, we consider the Diophantine equations

$$J_k = q_m q_n \tag{5}$$

and

$$q_k = J_m J_n \tag{6}$$

for any positive integer k, m, and n under $m \leq n$. The fundamental outputs of the paper are to determine the m, n, and k numbers that satisfy Equations (5) and (6).

2. Auxiliary Descriptions

The following will be used extensively in the rest of the paper

Definition 1. *Let η be an algebraic number of degree d with minimal primitive polynomial over*

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} \left(x - \eta^{(i)}\right),$$

where a_0 is positive and $\eta^{(i)}$ is the conjugate of η. Then,

$$h(\eta) = \frac{1}{d}\left(\log|a_0| + \sum\nolimits_{i=1}^{d} \log\left(\max\left\{\left|\eta^{(i)}\right|, 1\right\}\right)\right), \tag{7}$$

is called the logarithmic height of η.

It should be noted that this function satisfies the following properties:

$$h(\alpha \mp \beta) \leqslant h(\alpha) + h(\beta) + \log 2,\ h\left(\alpha\beta^{\mp 1}\right) \leqslant h(\alpha) + h(\beta), \text{ and } h(\alpha^s) = sh(\alpha).$$

Theorem 1 (Matveev [14]). *Let $\eta_1, \eta_2, \ldots, \eta_s$ be real algebraic numbers and let $b_1, b_2, \ldots, b_s$ be nonzero rational integers. Let $d_{\mathbb{K}}$ be the degree of the number field $\mathbb{Q}(\eta_1, \eta_2, \ldots, \eta_s)$ over $\mathbb{Q}$ and let A_j be the positive real number defined by*

$$A_j \geqslant h'(\eta_j) = \max\left\{d_{\mathbb{K}} h(\eta_j), \left|\log(\eta_j)\right|, 0.16\right\} \text{ for } j = 1, 2, \ldots, l.$$

Put

$$\Lambda = \eta_1\eta_2 \ldots \eta_l - 1 \text{ and } D = \max\{|b_1|, \ldots, |b_l|\}.$$

If $\Lambda \neq 0$, then

$$\log(|\Lambda|) > -1.4 \times 30^{l+3} \times l^{4.5} \times d_{\mathbb{K}}^2 \times (1 + \log(d_{\mathbb{L}}))(1 + \log(D)) A_1 A_2 \ldots A_l.$$

Lemma 1 (Dujella and Pethö [16]). *Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and let A, B, τ be positive rational numbers with $A > 0$ and $B > 1$. Let $\varepsilon = \|\mu q\| - M\|\tau q\|$, where $\|\cdot\|$ is the distance from the nearest integer. If $\varepsilon > 0$, then there is no integer solution (m, n, k) of inequality*

$$0 < m\tau - n + \mu < AB^{-k}$$

with

$$m \leqslant M \text{ and } k \geqslant \frac{\log(Aq/\varepsilon)}{\log B}.$$

Lemma 2. *Let k be a positive integer and let x, y, and z be positive real numbers. Further, let $\sqrt{x}$ and $\sqrt{z}$ be irrational numbers. Then, $\sqrt{x}(y + \sqrt{z})^k$ is an irrational number.*

Proof. Introduce $A := \sqrt{x}(y + \sqrt{z})^k$. From Binomial expansion, we can write

$$\begin{aligned} A &= \sqrt{x}\sum_{i=0}^{k} \binom{k}{i} y^{k-i}(\sqrt{z})^i \\ &= \sqrt{x}\left[\binom{k}{0} y^k + \binom{k}{1} y^{k-1}\sqrt{z} + \binom{k}{2} y^{k-2}(\sqrt{z})^2 + \ldots + \binom{k}{k}(\sqrt{z})^k\right]. \end{aligned}$$

If k is even, then we have

$$A = \sqrt{x}(B + C) = B\sqrt{x} + C\sqrt{x},$$

where

$$B := y^k + \binom{k}{2} y^{k-2}\left(\sqrt{z}\right)^2 + \ldots + \binom{k}{k-2} y^2\left(\sqrt{z}\right)^{k-2} + \left(\sqrt{z}\right)^k,$$

$$C := \binom{k}{1} y^{k-1}\left(\sqrt{z}\right)^1 + \ldots + \binom{k}{k-1} y\left(\sqrt{z}\right)^{k-1}.$$

Here, $B \in \mathbb{Q}$, $C \in \mathbb{R} - \mathbb{Q}$, and $B, C > 0$. $C\sqrt{x}$ can be rational or irrational depending on xz. However, $B\sqrt{x} \in \mathbb{R} - \mathbb{Q}$ due to $B \in \mathbb{Q}$. As a result, $A \in \mathbb{R} - \mathbb{Q}$. When k is odd, a similar evaluation can be done. This completes the proof. □

3. Main Results

In this section, we present the fundamental outcomes of the paper.

Theorem 2. *Let k, m, and n be any positive integers $m \leqslant n$. Then, all the solutions to Equation (5) are*

$$(k, m, n) \in \{(1,1,1), (2,1,1), (3,1,2), (6,2,3)\} \tag{8}$$

and the ones of Equation (6)

$$(k, m, n) \in \{(1,1,1), (1,1,2), (1,2,2), (2,1,3), (2,2,3)\}. \tag{9}$$

Proof. For validation, we apply a proof strategy of two steps. To this aim, appropriate boundaries will be computed separately for Equations (5) and (6). First, let us consider Equation (5) by considering the equations

$$\alpha^{n-2} \leqslant J_n \leqslant \alpha^{n-1} \tag{10}$$

and

$$\gamma^{n-1} \leqslant q_n \leqslant \gamma^n. \tag{11}$$

These can be proved easily by applying the induction method n. Then, we can write

$$\alpha^{k-2} \leqslant J_k = q_n.q_m \leqslant \gamma^{m+n} \text{ and } \alpha^{k-1} \geqslant J_k = q_n.q_m \geqslant \gamma^{m+n-2}.$$

and

$$1 + \frac{\log\gamma}{\log\alpha}(m+n-2) \leqslant k \leqslant 2 + \frac{\log\gamma}{\log\alpha}(m+n),$$

concluding $k < 4n$. Further, using the Binet's formulas of the Jacobsthal and Modified Pell numbers yields

$$\begin{aligned}\left|\frac{\alpha^k}{3} - \frac{\gamma^{n+m}}{4}\right| &= \left|\frac{\beta^k}{3} + \frac{\gamma^n\delta^m + \gamma^m\delta^n + \delta^{n+m}}{4}\right| < \left|\frac{\beta^k}{3} + \frac{3\gamma^{n-m}}{4}\right| \\ &< \frac{3}{2}\max\left\{|\beta|^k, \gamma^{n-m}\right\} = \frac{3\gamma^{n-m}}{2}\end{aligned}$$

or equally

$$\left|\frac{4}{3}\alpha^k\gamma^{-n-m} - 1\right| < \frac{6}{\gamma^{2m}}. \tag{12}$$

Considering the Matveev's theroem, we consider the following case:

$$\Lambda_1 = \frac{4}{3}\alpha^k\gamma^{-n-m} - 1, l = 3, \eta_1 = \frac{1}{3}, \eta_2 = \alpha, \eta_3 = \gamma, d_1 = 1, d_2 = k+2, d_3 = -n-m.$$

Here, it is easy to verify $\Lambda_1 \neq 0$. If the reverse were true anyway, $\frac{4}{3}\alpha^k = \gamma^{n+m}$ would have to be. But, while $\frac{4}{3}\alpha^k \in \mathbb{Q}$, $\gamma^{n+m} \notin \mathbb{Q}$. In this case, the assertion is true. If choosing $\eta_1, \eta_2, \eta_3 \in \mathbb{L} := \mathbb{Q}\left(\sqrt{2}\right)$, $d_L = 2$. This means that

$$h(\eta_1) = \log 3,\ h(\eta_2) = \log\alpha,\ h(\eta_3) = \frac{\log\gamma}{2},\ A_1 = 2\log 3,\ A_2 = 2\log\alpha,\ A_3 = \log\gamma$$

and $D = 4n$. As a result, we have

$$\log|\Lambda_1| > -2.61 \times 10^{12}\left(1 + \log 4n\right). \tag{13}$$

As compared Equation (12) to Equation (13), we finally get

$$m\log\gamma + \log 3 < 1.4 \times 10^{12}\left(1 + \log 4n\right). \tag{14}$$

Doing some mathematical arrangements by using the Binet's formulas in Equations (4), we compute

$$\left|\frac{2}{3q_m}\alpha^k\gamma^{-n} - 1\right| < \frac{2}{\gamma^n}. \tag{15}$$

Accordingly, from Matveev's theorem, the following equations can be obtained:

$$\Lambda_2 = \frac{2}{3q_m}\alpha^k\gamma^{-n} - 1,\ l = 3, \eta_1 = 3q_m,\ \eta_2 = \alpha,\ \eta_3 = \gamma,\ d_1 = -1,\ d_2 = k+1,$$

and, $d_3 = -n$.

$$h(\eta_2) = \log\alpha,\ h(\eta_3) = \frac{1}{2}\log\gamma\, A_2 = 2\log\alpha, \text{ and } A_3 = \log\gamma.$$

It should be noted that η_1 is also a root of the polynomial $2x^2 - 9{P_m}^2$. Then,

$$h(\eta_1) = \frac{1}{2}\left(\log 1 + \log|3q_m| + \log|-3q_m|\right) = \log q_m + \log 3 \leqslant m\log\gamma + \log 3.$$

From Equation (15), we have

$$A_1 = 2.8 \times 10^{12}\left(1 + \log 4n\right) > 2h(\eta_1).$$

Letting $D = 4n$. In this case, by Lemma 2, we can find

$$\log|\Lambda_2| > -3.32 \times 10^{24}(1 + \log 4n)^2. \tag{16}$$

and from Equation (15),

$$\log|\Lambda_2| < \log 2 - n\log\gamma. \tag{17}$$

Solving Equations (16) and (17) together, we get

$$n < 1.72 \times 10^{28} \text{ and } k < 4n. \tag{18}$$

A similar method can be applied to Equation (6). Here, to reduce the size of the current paper, we neglect an explicit proof. But, for Equation (6), we get the following boundaries:

$$n < 9 \times 10^{28} \text{ and } k < 3n. \tag{19}$$

Accordingly, from both Equations (18) and (19), our widest solution range is as follows.

$$k < 4n \text{ and } n < 9 \times 10^{28}. \tag{20}$$

Everything is ok but since our last range is not economical, investigating a solution is very difficult. Therefore, we will take an additional approach, taking into account four different situations.

Case I: For the case where $m \geqslant 2$ in Equation (12), we define

$$\Gamma_1 := k \log \alpha - (n+m) \log \gamma + \log \frac{4}{3},$$

or in another form,

$$\left| e^{\Gamma_1} - 1 \right| = |\Lambda_1| < \frac{6}{\gamma^{2m}} < \frac{1}{5}, \tag{21}$$

which means that $|\Gamma_1| < \frac{1}{4}$. By the way, $|x| < \frac{1}{4}$, $|x| < \frac{3}{2} |e^x - 1|$ holds for $x \in \mathbb{R}$ without the loss of generality. For the case where $x = \Gamma_1$, we obtain

$$|\Gamma_1| < \frac{9}{\gamma^{2m}}. \tag{22}$$

Due to $\Lambda_1 \neq 0$, $\Gamma_1 \neq 0$ too. As a result, $\Gamma_1 < 0$ or $\Gamma_1 > 0$. When $\Gamma_1 > 0$,

$$0 < k \left(\frac{\log \alpha}{\log \gamma} \right) - (n+m) + \left(\frac{\log(4/3)}{\log \gamma} \right) < \frac{9}{(\log \gamma)\, \gamma^{2m}} < \frac{11}{\gamma^{2m}}.$$

According to Dujella and Pethö's lemma for $M = 3.6 \times 10^{29}$, we get

$$\tau = \frac{\log \alpha}{\log \gamma}, \ \mu = \frac{\log(4/3)}{\log \gamma}, \ A = 11, \text{ and } B = \gamma^2. \tag{23}$$

$\tau = [a_0, a_1, ...]$ is turned on to the continued fraction as follows:

$$[a_0, ..., a_{60}] = \frac{p_{60}}{q_{60}} = \frac{633284722967420948224436714420 3}{805255281332277030875937803968 5}$$

such that $6M < 2.2 \times 10^{30} < q$. As a result, $\varepsilon = \|\mu q\| - M \|\tau q\| > 0.41$. This means that $m \leqslant 42$. Also, for the case where $\Gamma_1 < 0$, a similar result can be found.

Case II: In Equation (15) for $n > 1$ under the same assumptions, we find that $n \leqslant 89$.

Case III: In Equation (6) for the case $m \geqslant 4$, we conclude that $m \leqslant 109$.

Case IV: Similarly, In Equation (6) for the case where $n > 1$, we can obtain that $n \leqslant 117$.

According to all the above results, we obtain the widest range such as $n \leqslant 117$ and $k < 469$. If checking the possible cases by using a PC algorithm composed of in Mathematica, we see the intersection set such that $\{1, 3\}$. This exhausts the proof.

□

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CONFORMAL η-RICCI-YAMABE SOLITONS ON SUBMANIFOLDS OF AN $(\mathcal{LCS})_n$-MANIFOLD ADMITTING A QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract. This paper presents some results for conformal η-Ricci-Yamabe solitons (CERYS) on invariant and anti-invariant submanifolds of a $(\mathcal{LCS})_n$-manifold admitting a quarter-symmetric metric connection (QSMC). In addition, we developed the characterization of CERYS on $\mathcal{M}$-projectively flat, $\mathcal{Q}$-flat, and concircularly flat anti-invariant submanifolds of a $(\mathcal{LCS})_n$-manifold with respect to the aforementioned connection. Finally, we construct an extensive example that appoints some of our inferences.

1. Background and Motivations

Conformal Ricci flow is defined in a Riemannian n-manifold $(\mathbb{V}, g)$ as a generalisation of classical Ricci flow by [6]

$$\frac{\partial g}{\partial t} = -2(\mathcal{R}ic + \frac{g}{n}) - pg, \quad \tau(g) = -1,$$

where p is called the conformal pressure, g is the Riemannian metric; τ and $\mathcal{R}ic$ denote the scalar curvature and the Ricci tensor of $\mathbb{V}$, respectively.

A conformal Ricci soliton on $(\mathbb{V}, g)$ is defined as follows [2]:

$$\mathfrak{L}_{\mathcal{F}_1} g + 2\mathcal{R}ic = [\frac{1}{n}(pn + 2) - 2\mu]g,$$

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where $\mu \in \Re$ ($\Re$ is the set of real numbers) and $\mathfrak{L}_{\mathcal{F}_1}$ denotes the Lie-derivative operator along a smooth vector field $\mathcal{F}_1$

A Ricci-Yamabe flow of type (κ, l), which is a scalar combination of Ricci and Yamabe flows, is defined as follows [7]:

$$\frac{\partial}{\partial t}g(t) = 2\kappa\mathcal{R}ic(g(t)) - l\tau(t)g(t), \quad g(0) = g_0,$$

for some scalars κ and l.
A Riemannian manifold is said to have a Ricci-Yamabe solitons of type (κ, l) (briefly, RYS) if [4,29]

$$\mathfrak{L}_{\mathcal{F}_1}g + 2\kappa\mathcal{R}ic + (2\mu - l\tau)g = 0,$$

where $l, \kappa, \mu \in \Re$.
In [30], Zhang et al. studied conformal Ricci-Yamabe soliton (briefly, CRYS), which is defined on $(\mathbb{V}, g)$ by

$$\mathfrak{L}_{\mathcal{F}_1}g + 2\kappa\mathcal{R}ic + [2\mu - l\tau - \frac{1}{n}(pn + 2)]g = 0.$$

In this follow-up, the conformal η-Ricci-Yamabe soliton (briefly, CERYS) on $(\mathbb{V}, g)$ is defined by [28]

$$\mathfrak{L}_{\mathcal{F}_1}g + 2\kappa\mathcal{R}ic + [2\mu - l\tau - \frac{1}{n}(pn + 2)]g + 2\nu\,\eta \otimes \eta = 0, \tag{1}$$

where $l, \kappa, \mu, \nu \in \Re$. If $\mathcal{F}_1=grad(f)$, then the Equation (1) is called a gradient conformal η-Ricci-Yamabe soliton (briefly, GCERYS) and given by

$$\nabla^2 f + \kappa\mathcal{R}ic + [\mu - \frac{l\tau}{2} - \frac{1}{2}(p + \frac{2}{n})]g + \nu\,\eta \otimes \eta = 0,$$

where $\nabla^2 f$ is said to be the Hessian of f. A CRYS (or GCRYS) is said to be shrinking, steady or expanding if $\mu < 0$, $= 0$ or > 0, respectively. A CERYS (or GCERYS) reduces to (i) CERS if $\kappa = 1,\ l = 0$, (ii) CEYS if $\kappa = 0$, $l = 1$, and (iii) conformal η-Einstein soliton (briefly, CEES) if $\kappa = 1$, $l = -1$.

Shaikh [22] introduced the concept of n-dimensional Lorentzian concircular structure manifold (briefly, $(\mathcal{LCS})_n$-manifold) and demonstrated its existence with several examples [24], which generalises the concept of $\mathcal{LP}$-Sasakian manifolds introduced in [13,14]. We refer to the works [1,10,23] for more extensive studies. Mantica and Molinari [18] recently demonstrated that a $(\mathcal{LCS})_n$-manifold $(n > 3)$ is equal to the GRW spacetime. The authors also examined the applicability of $(\mathcal{LCS})_n$-manifolds in general theory of relativity and cosmology in [3]. Thus the geometry of submanifolds has grown in popularity in modern analysis due to its importance in practical mathematics and theoretical physics.

A linear connection $\bar{\nabla}$ on $(\mathbb{V}, g)$ is said to be a quarter-symmetric connection (briefly, QSC) [8] if its torsion tensor $\bar{\mathcal{T}}$ has the form

$$\bar{\mathcal{T}}(\mathcal{F}_1, \mathcal{F}_2) = \bar{\nabla}_{\mathcal{F}_1}\mathcal{F}_2 - \bar{\nabla}_{\mathcal{F}_2}\mathcal{F}_1 - [\mathcal{F}_1, \mathcal{F}_2] = \mathcal{A}(\mathcal{F}_2)\psi^*(\mathcal{F}_1) - \mathcal{A}(\mathcal{F}_1)\psi^*(\mathcal{F}_2), \tag{2}$$

where $\mathcal{A}$ is a 1-form and ψ^* is a $(1,1)$ type tensor field. If a quarter-symmetric linear connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_{\mathcal{F}_1} g)(\mathcal{F}_2, \mathcal{F}_3) = 0,$$

for all $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in \chi(\mathbb{V})$, then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection (briefly, QSMC). If a contact metric manifold admits a QSC, then we take $\mathcal{A}$=η and ψ^*=ϕ and hence (2) takes the form $\bar{\mathcal{T}}(\mathcal{F}_1, \mathcal{F}_2) = \eta(\mathcal{F}_2)\phi(\mathcal{F}_1) - \eta(\mathcal{F}_1)\phi(\mathcal{F}_2)$.

The relation between the Levi-Civita connection ∇ and a QSMC $\bar{\nabla}$ on a contact metric manifold is given by

$$\bar{\nabla}_{\mathcal{F}_1} \mathcal{F}_2 = \nabla_{\mathcal{F}_1} \mathcal{F}_2 - \eta(\mathcal{F}_1)\phi(\mathcal{F}_2).$$

Recently, the QSMC have been studied by many authors such as [9, 12, 19, 31] and many others.

2. Preliminaries

Let $\widetilde{\mathbb{V}}$ be an n-dimensional Lorentzian manifold admitting a unit time-like concircular vector field ζ. Then there is

$$g(\zeta, \zeta) = -1.$$

Since ζ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(\mathcal{F}_1, \zeta) = \eta(\mathcal{F}_1)$$

satisfies [25]

$$\begin{aligned} (\widetilde{\nabla}_{\mathcal{F}_1} \eta)\mathcal{F}_2 &= \alpha[g(\mathcal{F}_1, \mathcal{F}_2) + \eta(\mathcal{F}_1)\eta(\mathcal{F}_2)],\ \alpha \neq 0, \\ \widetilde{\nabla}_{\mathcal{F}_1} \zeta &= \alpha[\mathcal{F}_1 + \eta(\mathcal{F}_1)\zeta],\ \alpha \neq 0, \end{aligned} \tag{3}$$

for $\mathcal{F}_1, \mathcal{F}_2 \in \chi(\widetilde{\mathbb{V}})$, where $\widetilde{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function that satisfies

$$\widetilde{\nabla}_{\mathcal{F}_1} \alpha = (\mathcal{F}_1 \alpha) = d\alpha(\mathcal{F}_1) = \rho\eta(\mathcal{F}_1),$$

ρ being a certain scalar function given by ρ=-$(\zeta\alpha)$. Let us have a look

$$\phi\mathcal{F}_1 = \frac{1}{\alpha}\widetilde{\nabla}_{\mathcal{F}_1} \zeta, \tag{4}$$

then utilizing (3) and (4) we acquire

$$\phi\mathcal{F}_1 = \mathcal{F}_1 + \eta(\mathcal{F}_1)\zeta,$$
$$g(\phi\mathcal{F}_1, \mathcal{F}_2) = g(\mathcal{F}_1, \phi\mathcal{F}_2).$$

Thus the Lorentzian manifold $\widetilde{\mathbb{V}}$ admits the unit time-like concircular vector field ζ, its associated 1-form η and a (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(\mathcal{LCS})_n$-manifold) [17, 22]. Especially, if we take α=1, then we can obtain the $\mathcal{LP}$-Sasakian structure of Matsumoto [13].
In an $(\mathcal{LCS})_n$-manifold, we have [22]:

$$\eta(\zeta) = -1,\ \ \phi \circ \zeta = 0,\ \ \eta(\phi\mathcal{F}_1) = 0,\ \ g(\phi\mathcal{F}_1, \phi\mathcal{F}_2) = g(\mathcal{F}_1, \mathcal{F}_2) + \eta(\mathcal{F}_1)\eta(\mathcal{F}_2),$$

$$\phi^2\mathcal{F}_1 = \mathcal{F}_1 + \eta(\mathcal{F}_1)\zeta,$$
$$\eta(\widetilde{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3) = (\alpha^2-\rho)[g(\mathcal{F}_2,\mathcal{F}_3)\eta(\mathcal{F}_1) - g(\mathcal{F}_1,\mathcal{F}_3)\eta(\mathcal{F}_2)],$$
$$\widetilde{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\zeta = (\alpha^2-\rho)[\eta(\mathcal{F}_2)\mathcal{F}_1 - \eta(\mathcal{F}_1)\mathcal{F}_2],$$
$$\widetilde{\mathcal{R}}ic(\mathcal{F}_1,\zeta) = (n-1)(\alpha^2-\rho)\eta(\mathcal{F}_1),$$
$$\widetilde{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 = \phi\widetilde{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 + (\alpha^2-\rho)[g(\mathcal{F}_2,\mathcal{F}_3)\eta(\mathcal{F}_1) - g(\mathcal{F}_1,\mathcal{F}_3)\eta(\mathcal{F}_2)]\zeta,$$
$$(\widetilde{\nabla}_{\mathcal{F}_1}\phi)\mathcal{F}_2) = \alpha[g(\mathcal{F}_1,\mathcal{F}_2)\zeta + 2\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)\zeta + \eta(\mathcal{F}_2)\mathcal{F}_1],$$

for all $\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3 \in \chi(\widetilde{\mathbb{V}})$.

Let $\mathbb{N}$ be an m-dimensional ($m < n$) submanifold of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with induced metric g. Also, let ∇ be the induced connection on the tangent bundle $T\mathbb{N}$ and $\nabla^\perp$ be the induced connection on the normal bundle $T^\perp\mathbb{N}$ of $\mathbb{N}$, respectively. Then the Gauss and Weingarten formulae are respectively given by

$$\widetilde{\nabla}_{\mathcal{F}_1}\mathcal{F}_2 = \nabla_{\mathcal{F}_1}\mathcal{F}_2 + \hbar(\mathcal{F}_1,\mathcal{F}_2), \tag{5}$$

and

$$\widetilde{\nabla}_{\mathcal{F}_1}\mathcal{F}_3 = -\mathcal{A}_{\mathcal{F}_3}\mathcal{F}_1 + \nabla^\perp_{\mathcal{F}_1}\mathcal{F}_3,$$

for all $\mathcal{F}_1,\mathcal{F}_2 \in \chi(\mathbb{N})$ and $\mathcal{F}_3 \in \chi^\perp(\mathbb{N})$, where $\hbar$ and $\mathcal{A}_{\mathcal{F}_3}$ are second fundamental form and the shape operator (corresponding to the normal vector field $\mathcal{F}_3$), respectively for the immersion of $\mathbb{N}$ into $\widetilde{\mathbb{V}}$. The second fundamental form $\hbar$ and the shape operator $\mathcal{A}_{\mathcal{F}_3}$ are related by [26]

$$g(\hbar(\mathcal{F}_1,\mathcal{F}_2),\mathcal{F}_3) = g(\mathcal{A}_{\mathcal{F}_3}\mathcal{F}_1,\mathcal{F}_2),$$

for all $\mathcal{F}_1,\mathcal{F}_2 \in \chi(\mathbb{N})$ and $\mathcal{F}_3 \in \chi^\perp(\mathbb{N})$. We note that $\hbar(\mathcal{F}_1,\mathcal{F}_2)$ is bilinear and since $\nabla_{f\mathcal{F}_1}\mathcal{F}_2 = f\nabla_{\mathcal{F}_1}\mathcal{F}_2$ for any smooth function f on a manifold, then we have

$$\hbar(f\mathcal{F}_1,\mathcal{F}_2) = f\hbar(\mathcal{F}_1,\mathcal{F}_2).$$

A submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is said to be totally umbilical if

$$\hbar(\mathcal{F}_1,\mathcal{F}_2) = g(\mathcal{F}_1,\mathcal{F}_2)\mathcal{H}, \tag{6}$$

where $\mathcal{F}_1,\mathcal{F}_2 \in T\mathbb{N}$ and the mean curvature vector $\mathcal{H}$ on $\mathbb{N}$ is given by $\mathcal{H} = \frac{1}{m}\sum_{i=1}^{m}\hbar(\upsilon_i,\upsilon_i)$, where $\{\upsilon_1,\upsilon_2,....,\upsilon_m\}$ is a local orthonormal frame of vector fields on $\mathbb{N}$. Moreover, if $\hbar(\mathcal{F}_1,\mathcal{F}_2)=0$ for all $\mathcal{F}_1,\mathcal{F}_2 \in T\mathbb{N}$, then $\mathbb{N}$ is said to be totally geodesic and if $\mathcal{H}=0$ then $\mathbb{N}$ is called minimal in $\widetilde{\mathbb{V}}$.

A submanifold $\mathbb{N}$ of $\widetilde{\mathbb{V}}$ is said to be invariant if the structure vector field ζ is tangent to $\mathbb{N}$ at every point of $\mathbb{N}$ and $\phi\mathcal{F}_1$ is tangent to $\mathbb{N}$ for every vector field $\mathcal{F}_1$ tangent to $\mathbb{N}$ at every point of $\mathbb{N}$, i.e., $\phi(T\mathbb{N}) \subset T\mathbb{N}$ at every point of $\mathbb{N}$. Whereas, $\mathbb{N}$ is said to be anti-invariant if for any $\mathcal{F}_1$ tangent to $\mathbb{N}$, $\phi\mathcal{F}_1$ is normal to $\mathbb{N}$, i.e., $\phi(T\mathbb{N}) \subset T^\perp\mathbb{N}$ at every point of $\mathbb{N}$, where $T^\perp\mathbb{N}$ is the normal bundle of $\mathbb{N}$.

Now we recall the following results:

Lemma 1. *[11] On an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with a QSMC $\bar{\widetilde{\nabla}}$, we have*

(i) $\bar{\widetilde{\nabla}}_{\mathcal{F}_1}\mathcal{F}_2 = \widetilde{\nabla}_{\mathcal{F}_1}\mathcal{F}_2 + \eta(\mathcal{F}_2)\phi\mathcal{F}_1 - g(\phi\mathcal{F}_1, \mathcal{F}_2)\zeta$,

$$\begin{aligned}(ii)\quad \bar{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 &= \widetilde{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 + (2\alpha-1)[g(\phi\mathcal{F}_1,\mathcal{F}_3)\phi\mathcal{F}_2 - g(\phi\mathcal{F}_2,\mathcal{F}_3)\phi\mathcal{F}_1] \\ &+ \alpha[\eta(\mathcal{F}_2)\mathcal{F}_1 - \eta(\mathcal{F}_1)\mathcal{F}_2]\eta(\mathcal{F}_3) + \alpha[g(\mathcal{F}_2,\mathcal{F}_3)\eta(\mathcal{F}_1) - g(\mathcal{F}_1,\mathcal{F}_3)]\zeta,\end{aligned}$$

$$\begin{aligned}(iii)\quad \bar{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) &= \widetilde{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) + (\alpha-1)g(\mathcal{F}_2,\mathcal{F}_3) + (n\alpha-1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3) \\ &-(2\alpha-1)\varepsilon g(\phi\mathcal{F}_2,\mathcal{F}_3),\end{aligned}$$

where $\bar{\mathcal{R}}$, $\bar{\mathcal{R}}ic$ are the curvature and the Ricci tensors of $\widetilde{\mathbb{V}}$ with respect to $\bar{\widetilde{\nabla}}$ and $\varepsilon = \mathrm{trace}\phi$.

3. Cerys on Submanifolds of $(\mathcal{LCS})_n$-Manifolds

Let $(g, \zeta, \mu, \kappa, l)$ be a CERYS on submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$. Then in view of (1) we obtain

$$\begin{aligned}\mathfrak{L}_\zeta g(\mathcal{F}_2,\mathcal{F}_3) &= -2\kappa\mathcal{R}ic(\mathcal{F}_2,\mathcal{F}_3) - [2\mu - l\tau - \frac{1}{n}(pn+2)]g(\mathcal{F}_2,\mathcal{F}_3) \\ &-2\nu\eta(\mathcal{F}_2)\eta(\mathcal{F}_3).\end{aligned} \tag{7}$$

With the help of (4) and (5) one can get

$$\alpha\phi\mathcal{F}_1 = \widetilde{\nabla}_{\mathcal{F}_1}\zeta = \nabla_{\mathcal{F}_1}\zeta + \hbar(\mathcal{F}_1,\zeta). \tag{8}$$

If $\mathbb{N}$ is invariant in $\widetilde{\mathbb{V}}$, then $\phi\mathcal{F}_1, \zeta \in T\mathbb{N}$. So from (8) we yields

$$(i)\ \ \alpha\phi\mathcal{F}_1 = \nabla_{\mathcal{F}_1}\zeta, \ \ (ii)\ \ \hbar(\mathcal{F}_1,\zeta) = 0. \tag{9}$$

Using (9)(i) in (7), we obtain

$$\mathcal{R}ic(\mathcal{F}_2,\mathcal{F}_3) = -\frac{1}{\kappa}[\mu + \alpha - \frac{l\tau}{2} - \frac{1}{2n}(pn+2)]g(\mathcal{F}_2,\mathcal{F}_3) - \frac{(\nu+\alpha)}{\kappa}\eta(\mathcal{F}_2)\eta(\mathcal{F}_3), \tag{10}$$

where $\mathfrak{L}_\zeta g(\mathcal{F}_2,\mathcal{F}_3) = 2\alpha[g(\mathcal{F}_2,\mathcal{F}_3) + \eta(\mathcal{F}_2)\eta(\mathcal{F}_3)]$.

Also, with the help of (9)(ii), we get from (6) that $\eta(\mathcal{E})\mathcal{H} = 0 \implies \mathcal{H} = 0$. So, we obtain the result:

Theorem 1. *If $(g, \zeta, \mu, \nu, \kappa, l)$ be a CERYS on an invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$, then $\mathbb{N}$ is an η-Einstein manifold and also minimal in $\widetilde{\mathbb{V}}$.*

Also, we have

$$\mathcal{R}(\mathcal{F}_2,\mathcal{F}_3)\zeta = \nabla_{\mathcal{F}_2}\nabla_{\mathcal{F}_3}\zeta - \nabla_{\mathcal{F}_3}\nabla_{\mathcal{F}_2}\zeta - \nabla_{[\mathcal{F}_2,\mathcal{F}_3]}\zeta = (\alpha^2-\rho)[\eta(\mathcal{F}_3)\mathcal{F}_2 - \eta(\mathcal{F}_2)\mathcal{F}_3],$$

which by using (9)(i), we lead to

$$\mathcal{R}ic(\mathcal{F}_2,\zeta) = (m-1)(\alpha^2-\rho)\eta(\mathcal{F}_2), \text{ for all } \mathcal{F}_2. \tag{11}$$

By fixing $\mathcal{F}_3$=ζ in (10) and using (11), we get

$$\mu = \nu - \kappa(m-1)(\alpha^2 - \rho) + \frac{l\tau}{2} + \frac{1}{2}(p + \frac{2}{n}).$$

As consequence, we can make the following claim:

Theorem 2. *If $(g, \zeta, \mu, \nu, \kappa, l)$ be a CERYS on an invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$, then the CERYS reduces to*
(i) CERS if $\mu = \nu - (m-1)(\alpha^2 - \rho) + \frac{1}{2}(p + \frac{2}{n})$,
(ii) CEYS if $\mu = \nu + \frac{\tau}{2} + \frac{1}{2}(p + \frac{2}{n})$,
(iii) CEES if $\mu = \nu - (m-1)(\alpha^2 - \rho) - \frac{\tau}{2} + \frac{1}{2}(p + \frac{2}{n})$.

Corollary 1. *An η-Yamabe soliton on an invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ of type $(0, 1)$, is contracting, stable or increasing accordingly as $\tau < -2\nu$, $\tau = -2\nu$, or $\tau > -2\nu$, respectively.*

Corollary 2. *An η-Ricci soliton on an invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifolds $\widetilde{\mathbb{V}}$ of type $(1, 0)$, is contracting, stable or increasing accordingly as $\nu < (m-1)(\alpha^2 - \rho)$, $\nu = (m-1)(\alpha^2 - \rho)$ or $\nu > (m-1)(\alpha^2 - \rho)$, provided $\alpha^2 \neq \rho$.*

Corollary 3. *An η-Einstein soliton on an invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifolds $\widetilde{\mathbb{V}}$ of type $(1, -1)$, is contracting, stable or increasing accordingly as $\tau > 2[\nu - (m-1)(\alpha^2 - \rho)]$, $\tau = 2[\nu - (m-1)(\alpha^2 - \rho)]$ or $\tau < 2[\nu - (m-1)(\alpha^2 - \rho)]$, provided $\alpha^2 \neq \rho$.*

In particular, if $\mathbb{N}$ is an anti-invariant submanifold on $\widetilde{\mathbb{V}}$. Then for any $\mathcal{F}_1 \in T\mathbb{N}$ and $\phi\mathcal{F}_1 \in T^{\perp}\mathbb{N}$, we get from (8) that $\nabla_{\mathcal{F}_1}\zeta$=0, $\hbar(\mathcal{F}_1, \zeta)$=$\alpha\phi\mathcal{F}_1$. Thus, $\mathfrak{L}_\zeta g(\mathcal{F}_1, \mathcal{F}_2)$=0, that is, ζ is a Killing vector field (briefly, KVF) and in this case from (7), we have

$$\mathcal{R}ic(\mathcal{F}_2, \mathcal{F}_3) = -\frac{1}{\kappa}[\mu - \frac{l\tau}{2} - \frac{1}{2}(p + \frac{2}{n})]g(\mathcal{F}_2, \mathcal{F}_3) - \frac{\nu}{\kappa}\eta(\mathcal{F}_2)\eta(\mathcal{F}_3). \tag{12}$$

This results in the following outcomes:

Theorem 3. *If $(g, \zeta, \mu, \nu, \kappa, l)$ be a CERYS on an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifolds $\widetilde{\mathbb{V}}$, then $\mathbb{N}$ is an η-Einstein and ζ is a KVF.*

Again, for an anti-invariant submanifold $\mathbb{N}$ of $\widetilde{\mathbb{V}}$, we have $\mathcal{R}(\mathcal{F}_2, \mathcal{F}_3)\zeta$=0 and hence $\mathcal{R}ic(\mathcal{F}_2, \zeta)$=0. Also, from (12) we obtain $\mathcal{R}ic(\mathcal{F}_2, \zeta) = -\frac{1}{\kappa}[\mu - \frac{l\tau}{2} - \frac{1}{2}(p + \frac{2}{n}) - \nu]\eta(\mathcal{F}_1)$. So, we get $\mu = \frac{l\tau}{2} + \frac{1}{2}(p + \frac{2}{n}) + \nu$. Thus, we have finalized the result:

Corollary 4. *A CERYS of type (κ, l) on an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is contracting, stable or increasing accordingly as $\tau < \frac{-1}{l}[2\nu + (p + \frac{2}{n})]$, $\tau = \frac{-1}{l}[2\nu + (p + \frac{2}{n})]$ or $\tau > \frac{-1}{l}[2\nu + (p + \frac{2}{n})]$.*

4. CERYS ON SUBMANIFOLDS OF $(\mathcal{LCS})_n$-MANIFOLDS ADMITTING $\bar{\tilde{\bar{\nabla}}}$

Assume that $(g, \zeta, \mu, \nu, \kappa, l)$ be a CERYS on a submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ in view of QSMC $\bar{\tilde{\bar{\nabla}}}$. Then from (1) we obtain

$$\begin{aligned}\bar{\mathfrak{L}}_{\mathcal{F}_1} g(\mathcal{F}_2,\mathcal{F}_3) &= -2\kappa\bar{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) - [2\mu - l\bar{\tau} - \frac{1}{n}(pn+2)]g(\mathcal{F}_2,\mathcal{F}_3) \\ &\quad -2\nu\eta(\mathcal{F}_2)\eta(\mathcal{F}_3) = 0.\end{aligned} \tag{13}$$

In view of QSMC $\bar{\nabla}$, the second fundamental form $\bar{\hbar}$ on $\mathbb{N}$ is given by

$$\bar{\tilde{\bar{\nabla}}}_{\mathcal{F}_1}\mathcal{F}_2 = \bar{\nabla}_{\mathcal{F}_1}\mathcal{F}_2 + \bar{\hbar}(\mathcal{F}_1,\mathcal{F}_2). \tag{14}$$

Using Lemma 2.1(i) and (5) in (14), we lead to

$$\bar{\nabla}_{\mathcal{F}_1}\mathcal{F}_2 + \bar{\hbar}(\mathcal{F}_1,\mathcal{F}_2) = \nabla_{\mathcal{F}_1}\mathcal{F}_2 + \hbar(\mathcal{F}_1,\mathcal{F}_2) + \eta(\mathcal{F}_2)\phi\mathcal{F}_1 - g(\phi\mathcal{F}_1,\mathcal{F}_2)\xi. \tag{15}$$

We suppose that $\mathbb{N}$ is invariant in $\widetilde{\mathbb{V}}$, then $\phi\mathcal{F}_1, \xi \in T\mathbb{N}$. Thus from (15) we have

$$\bar{\nabla}_{\mathcal{F}_1}\mathcal{F}_2 = \nabla_{\mathcal{F}_1}\mathcal{F}_2 + \eta(\mathcal{F}_2)\phi\mathcal{F}_1 - g(\phi\mathcal{F}_1,\mathcal{F}_2)\zeta, \tag{16}$$

which means $\mathbb{N}$ admits QSME $\bar{\tilde{\bar{\nabla}}}$. Also, in view of (9)(i), it follows that $\bar{\nabla}_{\mathcal{F}_1}\zeta = (\alpha - 1)\phi\mathcal{F}_1$ and hence

$$\bar{\mathfrak{L}}_{\mathcal{F}_1} g(\mathcal{F}_2,\mathcal{F}_3) = 2(\alpha-1)[g(\mathcal{F}_2,\mathcal{F}_3) + \eta(\mathcal{F}_2)\eta(\mathcal{F}_3)]. \tag{17}$$

Let $\bar{\mathcal{R}}$ be the curvature tensor of submanifold $\mathbb{N}$ with respect to the QSMC $\bar{\tilde{\bar{\nabla}}}$. Then we get

$$\begin{aligned}\bar{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2),\mathcal{F}_3 &= \widetilde{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 + (2\alpha-1)[g(\phi\mathcal{F}_1,\mathcal{F}_3)\phi\mathcal{F}_2 - g(\phi\mathcal{F}_2,\mathcal{F}_3)\phi\mathcal{F}_1)] \\ &\quad +\alpha[\eta(\mathcal{F}_2)\mathcal{F}_1 - \eta(\mathcal{F}_1)\mathcal{F}_2]\eta(\mathcal{F}_3) \\ &\quad +\alpha[g(\mathcal{F}_2,\mathcal{F}_3)\eta(\mathcal{F}_1) - g(\mathcal{F}_1,\mathcal{F}_3)\eta(\mathcal{F}_2)]\zeta,\end{aligned} \tag{18}$$

where $\bar{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 = \bar{\tilde{\bar{\nabla}}}_{\mathcal{F}_1}\bar{\tilde{\bar{\nabla}}}_{\mathcal{F}_2}\mathcal{F}_3 - \bar{\tilde{\bar{\nabla}}}_{\mathcal{F}_2}\bar{\tilde{\bar{\nabla}}}_{\mathcal{F}_1}\mathcal{F}_3 - \bar{\tilde{\bar{\nabla}}}_{[\mathcal{F}_1,\mathcal{F}_2]}\mathcal{F}_3$.
On contracting (18), we obtain

$$\begin{aligned}\bar{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) &= \widetilde{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) + [\alpha(1-2\varepsilon)+\varepsilon]g(\mathcal{F}_2,\mathcal{F}_3) \\ &\quad +[\alpha(m-2\varepsilon)+\varepsilon-1]\eta(\mathcal{F}_2)\eta(\mathcal{F}_3).\end{aligned} \tag{19}$$

In view of (17) and (19), equation (13) reduces to

$$\begin{aligned}\widetilde{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) &= -\frac{1}{\kappa}\big[\mu - \frac{l\bar{\tau}}{2} - \frac{1}{2n}(pn+2) + (\alpha-1) + \kappa\{\alpha(1-2\varepsilon)+\varepsilon\}\big]g(\mathcal{F}_2,\mathcal{F}_3) \\ &\quad -\big[\kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\} + \alpha - 1 + \nu\big]\eta(\mathcal{F}_2)\eta(\mathcal{F}_3).\end{aligned}$$

Thus, we state:

Theorem 4. *Let $(g, \zeta, \mu, \nu, \kappa, l)$ be a CERYS on an invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to QSMC $\bar{\tilde{\bar{\nabla}}}$. If $\bar{\nabla}$ be the induced connection on $\mathbb{N}$ from the connection $\bar{\tilde{\bar{\nabla}}}$, then $\mathbb{N}$ is an η-Einstein manifold.*

Next, if $\mathbb{N}$ is anti-invariant submanifold on $\widetilde{\mathbb{V}}$ as per $\bar{\bar{\nabla}}$, then from (15), we get $\bar{\nabla}_{\mathcal{F}_1}\zeta$=0 and hence we find $\bar{\mathfrak{L}}_\zeta g(\mathcal{F}_2, \mathcal{F}_3)$=0. So from (13) we leads to the outcome:

Theorem 5. *Let $(g, \zeta, \mu, \nu, \kappa, l)$ be a CERYS on an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ admits QSMC $\bar{\bar{\nabla}}$. Then $\mathbb{N}$ is η-Einstein with respect to induced Riemannian connection.*

Corollary 5. *There does not exist a CEYS on an invariant $(or, anti-invariant)$ submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to the QSMC $\bar{\bar{\nabla}}$.*

5. Cerys on $\mathcal{M}$-Projectively Flat Anti-Invariant Submanifolds Admitting $\bar{\bar{\nabla}}$

The $\mathcal{M}$-projective curvature tensor $\mathcal{M}^\flat$ of rank three on $(\mathbb{N}^n, g)$ is given by [5,20]

$$\begin{aligned}\mathcal{M}^\flat(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 &= \mathcal{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 - \frac{1}{2(n-1)}[\mathcal{R}ic(\mathcal{F}_2, \mathcal{F}_3)\mathcal{F}_1 - \mathcal{R}ic(\mathcal{F}_1, \mathcal{F}_3)\mathcal{F}_2] \\ &\quad -\frac{1}{2(n-1)}[g(\mathcal{F}_2, \mathcal{F}_3)\mathcal{Q}\mathcal{F}_1 - g(\mathcal{F}_1, \mathcal{F}_3)\mathcal{Q}\mathcal{F}_2] \qquad (20)\end{aligned}$$

for all smooth vectors fields $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in \chi(\mathbb{N})$, where $\mathcal{Q}$ is the Ricci operator.

We suppose that, $\mathbb{N}$ is $\mathcal{M}$-projectively flat with respect to QSMC $\bar{\bar{\nabla}}$, i.e., $\mathcal{M}^\flat(\mathcal{E}, \mathcal{F})\mathcal{G} = 0$, then from (20) we have

$$\begin{aligned}\bar{\mathcal{R}}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 &= \frac{1}{2(n-1)}[\bar{\mathcal{R}}ic(\mathcal{F}_2, \mathcal{F}_3)\mathcal{F}_1 - \bar{\mathcal{R}}ic(\mathcal{F}_1, \mathcal{F}_3)\mathcal{F}_2] \\ &\quad +\frac{1}{2(n-1)}[g(\mathcal{F}_2, \mathcal{F}_3)\bar{\mathcal{Q}}\mathcal{F}_1 - g(\mathcal{F}_1, \mathcal{F}_3)\bar{\mathcal{Q}}\mathcal{F}_2],\end{aligned}$$

which implies that

$$\bar{\mathcal{R}}ic(\mathcal{F}_2, \mathcal{F}_3) = \frac{\bar{\tau}}{n}g(\mathcal{F}_2, \mathcal{F}_3). \qquad (21)$$

With the help of (21) and Lemma 2.1 (iii), we obtain

$$\begin{aligned}\widetilde{\mathcal{R}}ic(\mathcal{F}_2, \mathcal{F}_3) &= [\frac{\bar{\tau}}{n} + \varepsilon(2\alpha - 1) + (1-\alpha)]g(\mathcal{F}_2, \mathcal{F}_3) \\ &+ [\varepsilon(2\alpha - 1) - (n\alpha - 1)]\eta(\mathcal{F}_2)\eta(\mathcal{F}_3). \qquad (22)\end{aligned}$$

Putting $\mathcal{F}_3$=ζ in (22) and then multiplying both sides by 2κ, we get

$$2\kappa\widetilde{\mathcal{R}}ic(\mathcal{F}_2, \zeta) = [\frac{2\kappa\bar{\tau}}{n} + 2\kappa\alpha(n-1)]\eta(\mathcal{F}_2). \qquad (23)$$

Next, let $(g, \zeta, \mu, \nu, \kappa, l)$ be a CERYS on $\mathbb{N}$ and $\mathbb{N}$ is anti-invariant, then from (1), we lead to

$$2\kappa\widetilde{\mathcal{R}}ic(\mathcal{F}_2, \mathcal{F}_3) = -[2\mu - l\tau - \frac{1}{n}(pn+2)]g(\mathcal{F}_2, \mathcal{F}_3) - 2\nu\eta(\mathcal{F}_2)\eta(\mathcal{F}_3). \qquad (24)$$

Again setting $\mathcal{F}_3=\zeta$ in (24), we have

$$2\kappa\widetilde{\mathcal{R}}ic(\mathcal{F}_2,\zeta) = [-2\mu + l\tau + \frac{1}{n}(pn+2) + 2\nu]\eta(\mathcal{F}_2). \tag{25}$$

Equating (23) and (25), we get

$$\mu = -\frac{\kappa\bar{\tau}}{n} - \kappa\alpha(n-1) + \frac{l\tau}{2} + \frac{1}{2n}(pn+2) + \nu. \tag{26}$$

We assert the outcome:

Theorem 6. *If an anti-invariant submanifold* $\mathbb{N}$ *of an* $(\mathcal{LCS})_n$*-manifold* $\widetilde{\mathbb{V}}$ *is* $\mathcal{M}$*-projectively flat with respect to QSMC* $\bar{\bar{\widetilde{\nabla}}}$*, then the CERYS of type* (κ, l) *on* $\mathbb{N}$ *is contracting, stable or increasing accordingly as*

$$-\frac{\kappa\bar{\tau}}{n} - \kappa\alpha(n-1) + \frac{l\tau}{2} + \frac{1}{2n}(pn+2) + \nu \lesseqgtr 0.$$

It is clear, from (26) that, if $\kappa = 0$, then $\mu = \frac{l\tau}{2} + \frac{1}{2n}(pn+2) + \nu$ and if $l = 0$, then $\mu = -\frac{\kappa\bar{\tau}}{2} - \kappa\alpha(n-1) + \frac{1}{2n}(np+2) + \nu$. Thus, we state:

Corollary 6. *If an anti-invariant submanifold* $\mathbb{N}$ *of an* $(\mathcal{LCS})_n$*-manifold* $\widetilde{\mathbb{V}}$ *is* $\mathcal{M}$*-projectively flat with respect to QSMC* $\bar{\bar{\widetilde{\nabla}}}$*, then the CEYS of type* $(0,1)$ *on* $\mathbb{N}$ *is contracting, stable or increasing accordingly as* $\tau < -\frac{1}{n}[n(p+2\nu)+2]$*,* $\tau = -\frac{1}{n}[n(p+2\nu)+2]$*, or* $\tau > -\frac{1}{n}[n(p+2\nu)+2]$*, respectively.*

Corollary 7. *If an anti-invariant submanifold* $\mathbb{N}$ *of an* $(\mathcal{LCS})_n$*-manifold* $\widetilde{\mathbb{V}}$ *is* $\mathcal{M}$*-projective flat with respect to QSMC* $\bar{\bar{\widetilde{\nabla}}}$*, then the CERS of type* $(1,0)$ *on* $\mathbb{N}$ *is contracting, stable or increasing accordingly as*

$$-\frac{\bar{\tau}}{2} - \alpha(n-1) + \frac{1}{2n}(np+2) + \nu \lesseqgtr 0.$$

Again taking $\mathcal{F}_2=\mathcal{F}_3=\upsilon_i$, $i\,(1 \le i \le n)$ in (1) and using (21), we have

$$\bar{\mathfrak{L}}_{\mathcal{F}_1} g(\upsilon_i,\upsilon_i) + \left\{\frac{2\kappa\bar{\tau}}{n} + 2\mu - l\tau - \frac{1}{n}(pn+2)\right\} g(\upsilon_i,\upsilon_i) + 2\nu\eta(\upsilon_i)\eta(\upsilon_i) = 0,$$

which leads to

$$div(\mathcal{F}_1) + \left\{\kappa\bar{\tau} + n\mu - \frac{ln\tau}{2} - \frac{1}{2}(pn+2)\right\} - \nu = 0. \tag{27}$$

If $\mathcal{F}_1$ is solenoidal, then $div(\mathcal{F}_1)$=0 and hence (27) reduces to

$$\mu = (\frac{p}{2} + \frac{1}{n}) + \frac{l\tau}{2} - \frac{\kappa\bar{\tau}}{2} + \frac{\nu}{n}.$$

Again, if $\mathcal{F}_1=grad(f)$, then the equation (27) becomes

$$\nabla^2 f = -\kappa\bar{\tau} - n\mu + \frac{ln\tau}{2} + \frac{1}{2}(pn+2) + \nu. \tag{28}$$

As a result, we may state:

Theorem 7. *Let the metric g of an $\mathcal{M}$-projectively flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to QSMC $\bar{\bar{\nabla}}$ be a CERYS of type (κ, l), where $\mathcal{F}_1$=grad(f) then (28) holds.*

Corollary 8. *Let the metric g of an $\mathcal{M}$-projectively flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to QSMC $\bar{\bar{\nabla}}$ be a CERYS of type (κ, l). Then the vector field $\mathcal{F}_1$ is solenoidal iff*

$$\mu = \frac{1}{2}(p + \frac{2}{n}) + \frac{l\tau}{2} - \frac{\kappa\bar{\tau}}{n} + \frac{\nu}{n}.$$

6. Cerys on Pseudo-Projectively Flat Anti-Invariant Submanifolds Admitting $\bar{\bar{\nabla}}$

The pseudo-projective curvature tensor $\widetilde{\mathcal{P}}$ of rank three on $(\mathbb{N}^n, g)$ is given by [21]

$$\begin{aligned} \widetilde{\mathcal{P}}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 &= \sigma\mathcal{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 + \varsigma[\mathcal{R}ic(\mathcal{F}_2, \mathcal{F}_3)\mathcal{F}_1 - \mathcal{R}ic(\mathcal{F}_1, \mathcal{F}_3)\mathcal{F}_2] \\ &\quad + \varrho\tau[g(\mathcal{F}_2, \mathcal{F}_3)\mathcal{F}_1 - g(\mathcal{F}_1, \mathcal{F}_3)\mathcal{F}_2], \end{aligned} \tag{29}$$

for all smooth vectors fields $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in \chi(\mathbb{N})$, where σ, ς, ϱ are non-zero constants related by $\varrho = -\frac{1}{n}(\frac{\sigma}{n-1} + \varsigma)$.

Let $(\mathbb{N}^n, g)$ is pseudo-projectively flat with respect to QSMC $\bar{\bar{\nabla}}$, then from (29), we yields

$$\begin{aligned} \sigma\bar{\mathcal{R}}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 &= -\varsigma[\bar{\mathcal{R}}ic(\mathcal{F}_2, \mathcal{F}_3)\mathcal{F}_1 - \bar{\mathcal{R}}ic(\mathcal{F}_1, \mathcal{F}_3)\mathcal{F}_2] \\ &- \varrho\bar{\tau}[g(\mathcal{F}_2, \mathcal{F}_3)\mathcal{F}_1 - g(\mathcal{F}_1, \mathcal{F}_3)\mathcal{F}_2], \end{aligned}$$

which is equivalent to

$$[\sigma + \varsigma(n-1)]\bar{\mathcal{R}}ic(\mathcal{F}_2, \mathcal{F}_3) = -\varrho\bar{\tau}(n-1)g(\mathcal{F}_2, \mathcal{F}_3). \tag{30}$$

Using (30) in Lemma 2.1-(iii), we obtain

$$\begin{aligned} \widetilde{\mathcal{R}}ic(\mathcal{F}_2, \mathcal{F}_3) &= \Big[\frac{-\varrho\bar{\tau}(n-1)}{\{\sigma + \varsigma(n-1)\}} + \varepsilon(2\alpha - 1) - (\alpha - 1)\Big]g(\mathcal{F}_2, \mathcal{F}_3) \\ &\quad - [(n\alpha - 1) - \varepsilon(2\alpha - 1)]\eta(\mathcal{F}_2)\eta(\mathcal{F}_3). \end{aligned} \tag{31}$$

By fixing $\mathcal{G} = \xi$ in (31) and then multiplying both sides by 2κ, we have

$$2\kappa\widetilde{\mathcal{R}}ic(\mathcal{F}_2, \zeta) = [\frac{-2\kappa\varrho\bar{\tau}(n-1)}{\{\sigma + \varsigma(n-1)\}} + 2\alpha\kappa(n-1)]\eta(\mathcal{F}_2). \tag{32}$$

In view of (25) and (32), we get

$$\mu = \frac{\kappa\varrho\bar{\tau}(n-1)}{\{\sigma - \varsigma(1-n)\}} + \frac{l\tau}{2} + (\frac{p}{2} + \frac{1}{n}) + \alpha\kappa(1-n) + \nu.$$

Accordingly, as the Section 5, we claim:

Theorem 8. *If an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is pseudo-projectively flat with respect to QSMC $\bar{\bar{\nabla}}$, then the CERYS of type (κ, l) on $\mathbb{N}$ is contracting, stable or increasing accordingly as*

$$\frac{\kappa\varrho\bar{\tau}(n-1)}{\{\sigma-\varsigma(1-n)\}}+\alpha\kappa(1-n)+\frac{l\tau}{2}+(\frac{p}{2}+\frac{1}{n})+\nu \lesseqqgtr 0.$$

Corollary 9. *If an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is pseudo-projectively flat admits QSMC $\bar{\bar{\nabla}}$, then the CEYS of type $(0,1)$ on $\mathbb{N}$ is contracting, stable or increasing accordingly as $\tau<-[(p+\frac{2}{n})+2\nu]$, $\tau=-[(p+\frac{2}{n})+2\nu]$ or $\tau>-[(p+\frac{2}{n})+2\nu]$.*

Corollary 10. *If an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is pseudo-projectively flat admits QSMC $\bar{\bar{\nabla}}$, then the CERYS of type $(1,0)$ on $\mathbb{N}$ is contracting, stable or increasing accordingly as*

$$\frac{\varrho\bar{\tau}(n-1)}{\{\sigma-\varsigma(1-n)\}}+\alpha(1-n)+(\frac{p}{2}+\frac{1}{n})+\nu \lesseqqgtr 0.$$

Next, we replace $\mathcal{F}_2$=$\mathcal{F}_3$=υ_i $i(1\leq i\leq n)$ in (1) we have

$$\begin{aligned}\bar{\mathfrak{L}}_{\mathcal{F}_1}g(\upsilon_i,\upsilon_i) &= \left\{\frac{2\kappa\varrho\bar{\tau}(n-1)}{\sigma+\varsigma(n-1)}+2\kappa\{\alpha(1-2\varepsilon)+\varepsilon\}-\{2\mu-l\tau-\frac{1}{n}(pn+2)\}\right\}g(\upsilon_i,\upsilon_i)\\ &- [2\nu-2\kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}]\eta(\upsilon_i)\eta(\upsilon_i),\end{aligned}$$

which implies that

$$\begin{aligned}div(\mathcal{F}_1) &= \left\{\frac{n\kappa\varrho\bar{\tau}(n-1)}{\sigma+\varsigma(n-1)}+n\kappa\{\alpha(1-2\varepsilon)+\varepsilon\}-\{n\mu-\frac{nl\tau}{2}-\frac{1}{2}(pn+2)\}\right\}\\ &- [\nu-\kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}]. \end{aligned} \tag{33}$$

If $\mathcal{F}_1$ is solenoidal, then $div(\mathcal{F}_1)$=0 and hence equation (33) reduces to

$$\begin{aligned}\mu &= \left[\frac{\kappa\varrho\bar{\tau}(n-1)}{\sigma+\varsigma(n-1)}+\frac{l\tau}{2}+\frac{1}{2n}(pn+2)+\kappa\{\alpha(1-2\varepsilon)+\varepsilon\}\right]\\ &- \frac{1}{n}[\nu-\kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}].\end{aligned} \tag{34}$$

Again, if $\mathcal{F}_1$=$grad(f)$, then the equation (33) becomes

$$\begin{aligned}\nabla^2 f &= \frac{n\kappa\varrho\bar{\tau}(n-1)}{\sigma-\varsigma(n-1)}+n\kappa\{\alpha(1-2\varepsilon)+\varepsilon\}-n\mu+\frac{nl\tau}{2}+\frac{1}{2}(pn+2)\\ &- [\nu-\kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}].\end{aligned} \tag{35}$$

Thus, we assert:

Theorem 9. *Let the metric g of a pseudo-projectively flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to QSMC $\bar{\bar{\nabla}}$ be a CERYS of type (κ, l), where $\mathcal{F}_1$=$grad(f)$, then (35) holds.*

Corollary 11. *Let the metric g of a pseudo-projectively flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to QSMC $\bar{\bar{\nabla}}$ be a CERYS of type (κ, l), then the vector field $\mathcal{F}_1$ is solenoidal iff the relation (34) holds.*

7. Cerys on $\mathcal{Q}$ Flat Anti-Invariant Submanifolds Admitting $\bar{\bar{\nabla}}$

A curvature tensor of type $(1,3)$ on $(\mathbb{N}^n, g)(n > 2)$ is denoted by $\mathcal{Z}$ and defined by

$$\mathcal{Z}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 = \mathcal{R}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 - \frac{\psi}{n-1}[g(\mathcal{F}_2,\mathcal{F}_3)\mathcal{F}_1 - g(\mathcal{F}_1,\mathcal{F}_3)\mathcal{F}_2], \tag{36}$$

where ψ can be any scalar function. This type of tensor $\mathcal{Z}$ is known as a $\mathcal{Q}$-curvature tensor [15, 16]. If $\psi = \frac{\tau}{n}$, then the $\mathcal{Q}$ curvature tensor is reduced to the concircular curvature tensor.
Let the submanifold $\mathbb{N}$ be $\mathcal{Q}$-flat with respect to $\bar{\bar{\nabla}}$, i.e., $\bar{\mathcal{Z}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 = 0$. Then from (36), we have

$$\bar{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 = \frac{\psi}{n-1}[g(\mathcal{F}_2,\mathcal{F}_3)\mathcal{F}_1 - g(\mathcal{F}_1,\mathcal{F}_3)\mathcal{F}_2],$$

which implies that

$$\bar{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) = \psi g(\mathcal{F}_2,\mathcal{F}_3). \tag{37}$$

With the help of (9) and Lemma 2.1-(iii), we obtain

$$\begin{aligned} \widetilde{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) &= [\psi + \varepsilon(2\alpha - 1) + (1-\alpha)]g(\mathcal{F}_2,\mathcal{F}_3) \\ &\quad -[n\alpha - 1 + \varepsilon(1-2\alpha)]\eta(\mathcal{F}_2)\eta(\mathcal{F}_3). \end{aligned} \tag{38}$$

After taking $\mathcal{F}_3 = \zeta$ in (38) and then multiplying both sides by 2κ we lead to

$$2\kappa\widetilde{\mathcal{R}}ic(\mathcal{F}_2,\zeta) = 2\kappa[\psi + \alpha(n-1)]\eta(\mathcal{F}_2). \tag{39}$$

Equating (25) and (39), we find

$$\mu = \frac{1}{2}(p + \frac{2}{n}) + \frac{l\tau}{2} - \kappa[\psi + \alpha(n-1)] + \nu. \tag{40}$$

Thus, likewise section 6 we bring the outcome:

Theorem 10. *If an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is $\mathcal{Q}$-flat with respect to QSMC $\bar{\bar{\nabla}}$, then the CERYS of type (κ, l) on $\mathbb{N}$ is contracting, stable or increasing accordingly as*

$$\frac{1}{2}(p + \frac{2}{n}) + \frac{l\tau}{2} - \kappa[\psi + \alpha(n-1)] + \nu \lesseqqgtr 0.$$

As a result of the aforementioned theorem, we have the following result:

Corollary 12. *If an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is concircularly flat with respect to QSMC $\bar{\bar{\nabla}}$, then the CERYS of type (κ, l) on $\mathbb{N}$ is contracting, stable or increasing accordingly as*

$$\tau \lesseqqgtr \frac{1}{(nl-2\kappa)}[2\kappa\alpha n(n-1) - (np+2) - 2n\nu].$$

Also, from (40), if $\kappa = 0$, $l = 1$, then $\mu = \frac{\tau}{2} + \frac{1}{2}(p + \frac{2}{n}) + \nu$, and if $l = 0$, $\kappa = 1$,then $\mu = \frac{1}{2}(p + \frac{2}{n}) - [\psi - \alpha(1-n)] + \nu$. Thus, we state the results:

Corollary 13. *If an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is concircularly flat with respect to QSMC $\bar{\bar{\nabla}}$, then the CEYS of type $(0, 1)$ on $\mathbb{N}$ is contracting, stable or increasing accordingly as $\tau < -[(p + \frac{2}{n}) + 2\nu]$, $\tau = -[(p + \frac{2}{n}) + 2\nu]$ or $\tau > -[(p + \frac{2}{n}) + 2\nu]$, respectively.*

Corollary 14. *If an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is concircularly flat with respect to QSMC $\bar{\bar{\nabla}}$, then the CERS of type $(1, 0)$ on $\mathbb{N}$ is contracting, stable or increasing accordingly as*

$$(\frac{p}{2} + \frac{1}{n}) - \kappa[\psi - \alpha(1-n)] + \nu \lesseqqgtr 0.$$

Finally, using (37) in (1) and replacing $\mathcal{F}_2 = \mathcal{F}_3 = \upsilon_i, i (1 \le i \le n)$, we get

$$\begin{aligned} \bar{\mathfrak{L}}_{\mathcal{F}_1} g(\upsilon_i, \upsilon_i) &= -\left\{2\mu - l\tau - \frac{1}{n}(pn+2) + 2\kappa\psi - 2\kappa\{\alpha(1-2\varepsilon) + \varepsilon\}\right\} g(\upsilon_i, \upsilon_i) \\ &\quad -[2\nu - 2\kappa\{\alpha(m-2\varepsilon) + \varepsilon - 1\}]\eta(\upsilon_i)\eta(\upsilon_i), \end{aligned}$$

it leads to the conclusion that

$$\begin{aligned} div(\mathcal{F}_1) &= -[n\mu - \frac{nl\tau}{2} - \frac{1}{2}(pn+2) + n\kappa\psi - n\kappa\{\alpha(1-2\varepsilon) + \varepsilon\}] \\ &\quad -[\nu - \kappa\{\alpha(m-2\varepsilon) + \varepsilon - 1\}]. \end{aligned} \tag{41}$$

If $\mathcal{F}_1$ is solenoidal, then $div(\mathcal{F}_1) = 0$ and hence (41) reduces to

$$\mu = \frac{l\tau}{2} + \frac{1}{2n}(pn+2) - \psi\kappa + \kappa\{\alpha(1-2\varepsilon) + \varepsilon\} - \frac{1}{n}[\nu - \kappa\{\alpha(m-2\varepsilon) + \varepsilon - 1\}]. \tag{42}$$

Again, if $\mathcal{F}_1 = grad(f)$, then the equation (41) becomes

$$\begin{aligned} \nabla^2 f &= [-n\mu + \frac{nl\tau}{2} + \frac{1}{2}(pn+2) - n\kappa\psi + n\kappa\{\alpha(1-2\varepsilon) + \varepsilon\}] \\ &\quad -[\nu - \kappa\{\alpha(m-2\varepsilon) + \varepsilon - 1\}]. \end{aligned} \tag{43}$$

Theorem 11. *If the metric g of a $\mathcal{Q}$-flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to QSMC $\bar{\bar{\nabla}}$ be a CERYS of type (κ, l), where $\mathcal{F}_1 = grad(f)$, then (43) holds.*

Corollary 15. *Let the metric g of a $\mathcal{Q}$-flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to QSMC $\bar{\bar{\nabla}}$ be a CERYS of type (κ, l). Then the vector field $\mathcal{F}_1$ is solenoidal iff the relation (42) holds.*

8. Harmonic Aspect of CERYS on Anti-Invariant Submanifolds Admitting $\tilde{\tilde{\nabla}}$

Taking a look at a function $f:\mathbb{N} \to \Re$. We say that f harmonic if $\nabla^2 f=0$, where ∇^2 is the Lalplacian operator on $\mathbb{N}$ [27]. Since, $\zeta=grad(f)$. Then, utilizing Theorems 7, 9, and 11, we convey the following outcomes:

Theorem 12. *If the metric g of an $\mathcal{M}$-projectively flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\tilde{\mathbb{V}}$ admits a CERYS of type (κ, l) with respect to QSMC $\tilde{\tilde{\nabla}}$ and $\mathcal{F}_1=grad(f)$. If f is a harmonic function on $\mathbb{N}$, then the soliton is increasing, stable, or contracting*
(i) $\tau > \frac{2}{nl}[\kappa\bar{\tau} - \frac{1}{2}(pn+2) - \nu]$,
(ii) $\tau > \frac{2}{nl}[\kappa\bar{\tau} - \frac{1}{2}(pn+2) - \nu]$, *or*
(iii) $\tau > \frac{2}{nl}[\kappa\bar{\tau} - \frac{1}{2}(pn+2) - \nu]$, *respectively.*

Proof. With the help of (28), We may just accomplish the needed results. □

Theorem 13. *If the metric g of a pseudo-projectively flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\tilde{\mathbb{V}}$ admits a CERYS of type (κ, l) with respect to QSMC $\tilde{\tilde{\nabla}}$ and $\mathcal{F}_1=grad(f)$. If f is a harmonic on $\mathbb{N}$, then the soliton is growing, stable, or collapsing*
(i) $\tau > \frac{-1}{l}\left[\frac{2\kappa\varrho\bar{\tau}(n-1)}{(\sigma+\varsigma(n-1))} + 2\kappa\{\alpha(1-2\varepsilon)+\varepsilon\} + (p+\frac{2}{n}) - \frac{2}{n}[\nu - \kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}\right]$,
(ii) $\tau = \frac{-1}{l}\left[\frac{2\kappa\varrho\bar{\tau}(n-1)}{(\sigma+\varsigma(n-1))} + 2\kappa\{\alpha(1-2\varepsilon)+\varepsilon\} + (p+\frac{2}{n}) - \frac{2}{n}[\nu - \kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}\right]$,
or
(iii) $\tau < \frac{-1}{l}\left[\frac{2\kappa\varrho\bar{\tau}(n-1)}{(\sigma+\varsigma(n-1))} + 2\kappa\{\alpha(1-2\varepsilon)+\varepsilon\} + (p+\frac{2}{n}) - \frac{2}{n}[\nu - \kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}\right]$,
respectively.

Proof. We arrive at our conclusions using the equation (35). □

Theorem 14. *If the metric g of a $\mathcal{Q}$-flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\tilde{\mathbb{V}}$ admits a CERYS of type (κ, l) with respect to QSMC $\tilde{\tilde{\nabla}}$ and $\mathcal{F}_1=grad(f)$. If f is a harmonic on $\mathbb{N}$, then the soliton is growing, stable, or collapsing*
(i) $\tau > -\frac{2}{l}[\frac{1}{2}(p+\frac{2}{n}) - \kappa\psi + \kappa\{\alpha(1-2\varepsilon)+\varepsilon\} - \frac{1}{n}[\nu - \kappa\{\alpha(m-2\varepsilon)-1\}]]$,
(ii) $\tau = -\frac{2}{l}[\frac{1}{2}(p+\frac{2}{n}) - \kappa\psi + \kappa\{\alpha(1-2\varepsilon)+\varepsilon\} - \frac{1}{n}[\nu - \kappa\{\alpha(m-2\varepsilon)-1\}]]$,
(iii) $\tau < -\frac{2}{l}[\frac{1}{2}(p+\frac{2}{n}) - \kappa\psi + \kappa\{\alpha(1-2\varepsilon)+\varepsilon\} - \frac{1}{n}[\nu - \kappa\{\alpha(m-2\varepsilon)-1\}]]$, *respectively.*

Proof. By virtue of equation (43) we may simply obtain the desired outcome. □

9. Example

We define $\tilde{\mathbb{V}}^5=\{(r,s,t,u,v) \in \Re^5 : u \neq 0\}$, where $\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5\}$ being standard coordinates of linearly independent vector fields of $\tilde{\mathbb{V}}^5$ given by

$$\upsilon_1 = e^u \frac{\partial}{\partial r} + e^u s \frac{\partial}{\partial t},\ \upsilon_2 = \frac{\partial}{\partial s},\ \upsilon_3 = \frac{\partial}{\partial t} = \zeta,\ \upsilon_4 = \frac{\partial}{\partial u} + e^u v \frac{\partial}{\partial t},\ \upsilon_5 = \frac{\partial}{\partial v}.$$

Also, the metric g of $\widetilde{\mathbb{V}}^5$ has the following relations

$$g(\upsilon_1,\upsilon_1)=g(\upsilon_2,\upsilon_2)=g(\upsilon_3,\upsilon_3)=g(\upsilon_4,\upsilon_4)=g(\upsilon_5,\upsilon_5)=1,\ \ ,g(\upsilon_3,\upsilon_3)=-1.$$

Let the 1-form η is given by $\eta(\mathcal{F}_1)$=$g(\mathcal{F}_1,\upsilon_3)$, $\forall\ \mathcal{F}_1\in\widetilde{\mathbb{V}}^5$ and the $(1,1)$-tensor field ϕ of $\widetilde{\mathbb{V}}^5$ as follows

$$\phi\upsilon_1=\upsilon_2,\ \phi\upsilon_2=\upsilon_1,\ \phi\upsilon_3=0,\ \phi\upsilon_4=\upsilon_5,\ \phi\upsilon_5=\upsilon_4.$$

Utilizing the linearity qualities of ϕ and g dictates how they interact.

$$\phi^2\upsilon_i=\upsilon_i+\eta(\upsilon_i)\zeta,\ \ \eta(\upsilon_3)=-1,$$

hold for i=$1,2,3,4,5$ and ζ=υ_3. Also, for ζ=υ_3, $\widetilde{\mathbb{V}}^5$ satisfies $g(\upsilon_i,\upsilon_3)$=$\eta(\upsilon_i)$, $g(\phi\upsilon_i,\upsilon_j)$=$g(\upsilon_i,\phi\upsilon_j)$ and $g(\phi\upsilon_i,\phi\upsilon_j)$=$g(\upsilon_i,\upsilon_j)$+$\eta(\upsilon_i)\eta(\upsilon_j)$, where $i,j=1,2,3,4,5$. Now, we can compute

$$[\upsilon_i,\upsilon_j]=\begin{cases}-e^u\upsilon_3, & \text{if } i=1,\ j=2,\\ -e^u\upsilon_1, & \text{if } i=1,\ j=4,\\ -e^u\upsilon_3, & \text{if } i=4,\ j=5,\\ 0, & \text{otherwise.}\end{cases}$$

We may use Koszul's formula for getting

$$\widetilde{\nabla}_{\upsilon_1}\upsilon_1=0,\ \ \widetilde{\nabla}_{\upsilon_1}\upsilon_2=\frac{e^u}{2}\upsilon_3,\ \ \widetilde{\nabla}_{\upsilon_1}\upsilon_3=-\frac{e^u}{2}\upsilon_2,\ \ \widetilde{\nabla}_{\upsilon_1}\upsilon_4=0,\ \ \widetilde{\nabla}_{\upsilon_1}\upsilon_5=0,$$
$$\widetilde{\nabla}_{\upsilon_2}\upsilon_1=-\frac{e^u}{2}\upsilon_3,\ \ \widetilde{\nabla}_{\upsilon_2}\upsilon_2=0,\ \ \widetilde{\nabla}_{\upsilon_2}\upsilon_3=-\frac{e^u}{2}\upsilon_1\ \ \widetilde{\nabla}_{\upsilon_2}\upsilon_4=0,\ \ \widetilde{\nabla}_{\upsilon_2}\upsilon_5=0,$$
$$\widetilde{\nabla}_{\upsilon_3}\upsilon_1=-\frac{e^u}{2}\upsilon_2,\ \ \widetilde{\nabla}_{\upsilon_3}\upsilon_2=-\frac{e^u}{2}\upsilon_1,\ \ \widetilde{\nabla}_{\upsilon_3}\upsilon_3=0,\ \ \widetilde{\nabla}_{\upsilon_3}\upsilon_4=-\frac{e^u}{2}\upsilon_5,\ \ \widetilde{\nabla}_{\upsilon_3}\upsilon_5=-\frac{e^u}{2}\upsilon_4,$$
$$\widetilde{\nabla}_{\upsilon_4}\upsilon_1=0,\ \ \widetilde{\nabla}_{\upsilon_4}\upsilon_2=0,\ \ \widetilde{\nabla}_{\upsilon_4}\upsilon_3=-\frac{e^u}{2}\upsilon_5,\ \ \widetilde{\nabla}_{\upsilon_4}\upsilon_4=0,\ \ \widetilde{\nabla}_{\upsilon_4}\upsilon_5=-\frac{e^u}{2}\upsilon_3,$$
$$\widetilde{\nabla}_{\upsilon_5}\upsilon_1=0,\ \ \widetilde{\nabla}_{\upsilon_5}\upsilon_2=0,\ \ \widetilde{\nabla}_{\upsilon_5}\upsilon_3=-\frac{e^u}{2}\upsilon_4,\ \ \widetilde{\nabla}_{\upsilon_5}\upsilon_4=-\frac{e^u}{2}\upsilon_3,\ \ \widetilde{\nabla}_{\upsilon_5}\upsilon_5=0.$$

Thus for υ_3=ζ and α=-$\frac{e^u}{2}$ we verified that $\widetilde{\nabla}_{\mathcal{F}_1}\zeta$=$\alpha\phi\mathcal{F}_1$ for all $\mathcal{F}_1\in\mathcal{T}\widetilde{\mathbb{V}}^5$, where $\mathcal{F}_1$=$\mathcal{F}_1\upsilon_1+\mathcal{F}_2\upsilon_2+\mathcal{F}_3\upsilon_3+\mathcal{F}_4\upsilon_4+\mathcal{F}_5\upsilon_5$. So, the manifold $\widetilde{\mathbb{V}}^5$ equipped with the structure (ϕ,ζ,η,g) is an $(\mathcal{LCS})_5$-manifold with α=-$\frac{e^u}{2}$ and $\varrho^\star$=-$\mathcal{F}_4\alpha$.

Let $\widetilde{\pi}:\mathbb{N}\to\widetilde{\mathbb{V}}$ and given by $\widetilde{\pi}(r,s,t)$=$(r,s,u,0,0)$. Then we define $\mathbb{N}$=$\{(r,s,u)\in\Re^3:u\neq0\}$, where (r,s,u) are the standard coordinates in $\Re^3$. Let $\{\upsilon_1,\upsilon_2,\upsilon_3\}$ on $\mathbb{N}$ given by

$$\upsilon_1=e^u\frac{\partial}{\partial r}+e^us\frac{\partial}{\partial u},\ \upsilon_2=\frac{\partial}{\partial s},\ \upsilon_3=\frac{\partial}{\partial u}.$$
$$g(\upsilon_1,\upsilon_1)=g(\upsilon_2,\upsilon_2)=1,\ \ g(\upsilon_3,\upsilon_3)=-1.$$

Also, the $(1,1)$-tensor field ϕ of $\mathbb{N}^3$ is given by

$$\phi\upsilon_1=\upsilon_2,\ \phi\upsilon_2=\upsilon_1,\ \phi\upsilon_3=0.$$

Utilizing the linearity qualities of ϕ and g dictates how they interact

$$\phi^2 \upsilon_i = \upsilon_i + \eta(\upsilon_i)\zeta, \ \eta(\zeta) = -1,$$

for i=$1,2,3$ and ζ=υ_3. Again, for ζ=υ_3, $\mathbb{N}^3$ satisfies

$$g(\phi\upsilon_i, \phi\upsilon_j) = g(\upsilon_i, \upsilon_j) + \eta(\upsilon_i)\eta(\upsilon_j),$$

where i, j=$1,2,3$. Next, one can easily obtain

$$[\upsilon_1, \upsilon_2] = -e^u\upsilon_3, \quad [e_1, \upsilon_3] = -e^u\upsilon_1, \quad [\upsilon_2, \upsilon_3] = 0.$$

We acquire assuming Koszul's formula

$$\nabla_{\upsilon_1}\upsilon_1 = 0, \quad \nabla_{\upsilon_1}\upsilon_2 = \frac{e^u}{2}\upsilon_3, \quad \nabla_{\upsilon_1}\upsilon_3 = -\frac{e^u}{2}\upsilon_2, \quad \nabla_{\upsilon_2}e_1 = -\frac{e^u}{2}\upsilon_3, \quad \nabla_{\upsilon_2}\upsilon_2 = 0,$$

$$\nabla_{\upsilon_2}\upsilon_3 = -\frac{e^u}{2}\upsilon_1, \quad \nabla_{\upsilon_3}\upsilon_1 = -\frac{e^u}{2}\upsilon_2, \quad \nabla_{\upsilon_3}\upsilon_2 = -\frac{e^u}{2}\upsilon_1, \quad \nabla_{\upsilon_3}\upsilon_3 = 0.$$

Thus the data (ϕ, ζ, η, g) is an $(\mathcal{LCS})_3$-structure on $\mathbb{N}$. Consequently, if $\mathbb{N}^3$ equipped with the structure (ϕ, ζ, η, g) is $(\mathcal{LCS})_3$ manifold with α=-$\frac{e^u}{2}$ and $\varrho^\star$=-$\mathcal{F}_3\alpha$. We define the tangent space $\mathcal{T}\mathbb{N}$ of $\mathbb{N}^3$ as follows

$$\mathcal{T}\mathbb{N} = \mathcal{D} \oplus \mathcal{D}^\perp \oplus < \zeta >,$$

where $\mathcal{D}$=$< \upsilon_1 >$, $\mathcal{D}^\perp$=$< \upsilon_2 >$. Since $\phi\upsilon_1$=$\upsilon_2 \in \mathcal{D}^\perp$, for $\upsilon_1 \in \mathcal{D}$ and $\phi\upsilon_2$=$\upsilon_1 \in \mathcal{D}$, for $\upsilon_2 \in \mathcal{D}^\perp$. Then, $\mathbb{N}^3$ is an invariant submanifold of $\widetilde{\mathbb{V}}^5$. Also, from (5) we have $\hbar(\upsilon_i, \upsilon_j)$=$\widetilde{\nabla}_{\upsilon_i}\upsilon_j - \nabla_{\upsilon_i}\upsilon_j$. Using the values of $\widetilde{\nabla}_{\upsilon_i}\upsilon_j$ and $\nabla_{\upsilon_i}\upsilon_j$, we notice that $\hbar(\upsilon_i, \upsilon_j)$=0, $\forall\, i,\ j = 1,2,3$. i.e., $\mathbb{N}^3$ is totally geodesic. So, Theorem 1 is verified. Now, using (16) we get the QSMC $\bar{\bar{\nabla}}$ on $\mathbb{N}$ as follows

$$\bar{\bar{\nabla}}_{\upsilon_1}\upsilon_3 = -\left\{\frac{e^u+2}{2}\right\}\upsilon_2, \quad \bar{\bar{\nabla}}_{\upsilon_1}\upsilon_1 = 0, \quad \bar{\bar{\nabla}}_{\upsilon_1}\upsilon_2 = \left\{\frac{e^u-2}{2}\right\}\upsilon_3,$$

$$\bar{\bar{\nabla}}_{\upsilon_2}\upsilon_3 = -\left\{\frac{e^u+2}{2}\right\}\upsilon_1, \quad \bar{\bar{\nabla}}_{\upsilon_3}\upsilon_2 = -\frac{e^u}{2}\upsilon_1, \quad \bar{\bar{\nabla}}_{\upsilon_2}\upsilon_1 = -\left\{\frac{e^u+2}{2}\right\}\upsilon_3,$$

$$\bar{\bar{\nabla}}_{\upsilon_3}\upsilon_3 = 0, \quad \bar{\bar{\nabla}}_{\upsilon_2}\upsilon_2 = 0, \quad \bar{\bar{\nabla}}_{\upsilon_3}\upsilon_1 = 0.$$

By using the preceding relations, one can get $\bar{\mathcal{R}}$.

$$\bar{\mathcal{R}}(\upsilon_1, \upsilon_2)\upsilon_1 = \frac{(e^u+2)^2}{4}\upsilon_2, \ \bar{\mathcal{R}}(\upsilon_1, \upsilon_2)\upsilon_2 = -\frac{(3e^{2u}-4)}{4}\upsilon_1, \ \bar{\mathcal{R}}(\upsilon_2, \upsilon_3)\upsilon_2 = \frac{e^u(e^u+2)}{4}\upsilon_3.$$

Also, the $\bar{\mathcal{R}}ic$ and $\bar{\tau}$ have the value

$$\bar{\mathcal{R}}ic(\upsilon_1, \upsilon_1) = -\frac{(3e^{2u}-4)}{4}, \ \bar{\mathcal{R}}ic(\upsilon_2, \upsilon_2) = 0, \ \bar{\mathcal{R}}ic(\upsilon_3, \upsilon_3) = \frac{e^u(e^u+2)}{4},$$

$$\bar{\tau} = -[(e^{2u}-1) + \frac{e^u}{2}].$$

Since, $\mathbb{N}$ in invariant on $\widetilde{\mathbb{V}}$. Therefore, from the equations (1) and (17) we obtain

$$2\kappa\bar{\mathcal{R}}ic(\upsilon_i,\upsilon_i)+[2(\alpha-1)+2\mu-l\bar{\tau}-\frac{1}{n}(pn+2)]g(\upsilon_i,\upsilon_i) \tag{44}$$
$$+2[\alpha-1+\nu]\eta(\upsilon_i)\eta(\upsilon_i)=0,$$

for all $i\in\{1,2,3\}$. From the equation (44), we can easily calculate

$$\mu = \frac{1}{6}[(3p+2)-(3l-2\kappa)\bar{\tau}+2\nu-4(\alpha-1)]. \tag{45}$$

$$\nu = -\frac{1}{6}(3p+2)-\frac{\kappa e^u(e^u+2)}{4}+\mu-\frac{l\bar{\tau}}{2}. \tag{46}$$

With help of equations (45), (46) and the value of $\bar{\tau}$, we obtain

$$\mu = \frac{(3p+2)}{6}-\frac{l(2e^{2u}-2+e^u)}{4}+\frac{\kappa(3e^{2u}-4)}{8}-\alpha+1.$$

Thus the data $(g,\mathcal{F}_1,\mu,\nu,\kappa,l)$ is a CERYS of type (κ,l) with respect to QSMC $\bar{\widetilde{\nabla}}$ on $(\mathbb{N}^3,g)$. Now, we conclude that:

Case(a):
For $\kappa=1$ and $l=0$, $(\mathbb{N}^3,g)$ also admits the CERS, which is
(i) expanding if $p>-\frac{3}{4}e^{2u}+2\alpha-\frac{5}{3}$,
(ii) steady if $p=-\frac{3}{4}e^{2u}+2\alpha-\frac{5}{3}$,
(iii) shrinking if $p<-\frac{3}{4}e^{2u}+2\alpha-\frac{5}{3}$.

Case(b):
For $\kappa=0$ and $l=1$, then $(\mathbb{N}^3,g)$ admits the CEYS, which is
(i) expanding if $p>e^u(e^u+\frac{1}{2})+2\alpha-\frac{11}{3}$,
(ii) steady if $p=e^u(e^u+\frac{1}{2})+2\alpha-\frac{11}{3}$,
(iii) shrinking if $p<e^u(e^u+\frac{1}{2})+2\alpha-\frac{11}{3}$.

Case(c):
For $\kappa=1$ and $l=-1$, $(\mathbb{N}^3,g)$ admits the CEES, which is
(i) expanding if $p>-\frac{e^u}{4}(7e^u+2)-\frac{2}{3}+2\alpha$,
(ii)steady if $p=-\frac{e^u}{4}(7e^u+2)-\frac{2}{3}+2\alpha$,
(iii) shrinking if $p<-\frac{e^u}{4}(7e^u+2)-\frac{2}{3}+2\alpha$.

10. Conclusion

The investigation of a CERYS on Riemannian (or pseudo-Riemannian) manifolds is crucial in differential geometry, relativity theory and physics. RY flow is the most visible representative of modern physics. In addition to differential geometry, the CERYS is a new idea that works with geometric and physical applications. We characterized the submanifolds of a $(\mathcal{LCS})_n$-manifold that admits the CERYS with a QSMC in our study.

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ON THE MELLIN-GAUSS-WEIERSTRASS OPERATORS IN THE MELLIN-LEBESGUE SPACES

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Abstract. In this paper, we present the modulus of smoothness of a function $f \in X_c^p$, which the Mellin-Lebesgue space, and later we state some properties of it. In this way, the rate of convergence is gained. Moreover, we elucidate some pointwise convergence results for the Mellin-Gauss-Weierstrass operators. Especially, we acquire the pointwise convergence of them at any Lebesgue point of a function f.

1. Introduction

Mellin analysis is famous in approximation theory and Mellin operators are broadly investigated in this field (see [13], [18] for a comprehensive theory and, for other approximation results, [7], [12]). The reputation of Mellin operators is both mathematically and due to their applications in different fields. For instance, they are relevant to various problems of Signal Processing: actually, Mellin analysis is quite helpful in situations, where the samples to reconstruct a signal are exponentially spaced rather than equally spaced as in the classical Shannon Sampling Theorem (see, e.g., [14]).

The singular integrals of Mellin convolution type were first-time presented by Kolbe and Nessel [17] in 1972. They play a remarkable role in the Mellin analysis, likewise the traditional convolution operators in the Fourier analysis. These convolution integrals are utilized to explain the attitude of solutions of certain boundary value problems in the wedge-shaped regions. Butzer and Jansche [13] broadly analyzed them, relating to the L_p convergence. The pointwise convergence of linear singular integrals of the Fejer-type in the periodic case or in the line group is was broadly investigated in the classical book by P.L. Butzer and R.J. Nessel [15] in

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1971, where specially an almost everywhere convergence is gained by using the concept of the Lebesgue point of a function $f \in L_p$, $1 \leq p \leq +\infty$.

In [18], the approximation theory by Mellin convolution operators is evolved using a more direct and inherent way, totally unconnected from the Fourier theory, bottomed on a 'logarithmic'version of Taylor formula, Mellin derivatives, and the concepts of 'logarithmic' uniform continuity and 'logarithmic'moment of kernel function, which gives a different and powerful approach.

From the early 2000s until today, Mellin convolution operators have been worked intensively, particularly by Bardaro and Mantellini, and quite significant guidances have been accomplished to this field. In [4] and [5], the authors asserted a convenient linear composition of Mellin type operators to accelerate convergence. In another view to gain better order of approximation, Bardaro and Mantellini [8] took into account linear compositions of Mellin type operators using the iterated kernels instead of the basic kernels. Same authors, in [5], improved the pointwise approximation theory for Mellin convolution operators including Mellin-Gauss-Weierstrass operators, acting on functions defined on the multiplicative group $\mathbb{R}^+$.

Bardaro and Mantellini [7] considered Mellin convolution operators of type

$$(T_w f)(s) = \int_0^\infty K_w(t) f(ts) \frac{dt}{t}, \qquad s \in \mathbb{R}^+$$

where f pertains to domain of the operator T_w and $K_w : (0,\infty) \to \mathbb{R}$ is a set of the kernels, which provides the condition $\int_0^\infty K_w(t) \frac{dt}{t} = 1$. Check against the usual classical convolution, the translation operator is changed by a dilation operator, and let $\mathbb{R}^+$ be the multiplicative topological group granted with the Haar measure $\mu = \frac{dt}{t}$ becoming the Lebesgue measure. We will indicate by $L_p(\mu, \mathbb{R}^+) = L_p(\mu)$, $1 \leq p \leq +\infty$, the Lebesgue spaces according to the measure μ and we will demonstrate by $\|f\|_p$ the matching norm of a function $f \in L_p(\mu)$.

Moreover, in recent important papers, the authors have been working on the Mellin-Lebesgue spaces. For example, in [10], the authors study convergence theorems to a function f of its generalized exponential sampling series in the weighted Lebesgue spaces. In [2], some results on exponential sampling operators in the weighted Lebesgue spaces have been performed recently. In the very recent papers, in [6] and [11], Bardaro et al. examine the boundedness properties and the convergence features of certain semi-discrete exponential-type sampling operators in the weighted Lebesgue spaces, respectively.

Additionally, many studies have been carried out for similar operators on the subject. For instance, in [3], q analogue of the Stancu-Beta operators is introduced, and direct results in terms of the modulus of continuity and the weighted approximation theorem are expressed. In [16], Gupta et al. deal with the semi-exponential type Gauss-Weierstrass operators and they estimate some direct results using suitable modulus of continuity, weighted approximation, quantitative asymptotic formula and pointwise convergence. In the last year, in [1], a new modulus of continuity for locally integrable function spaces is presented and the obtained results are applied to the Gauss-Weierstrass operators.

The rest of the paper is organised as follows. In the next part, elementary informations related to the subject are reminded. After that, the definition of modulus of smoothness of a function $f \in X_c^p$ and its some properties are given. In this way, the rate of convergence is gained. Other than these, the definition of Lebesgue point of a function $f \in X_c^p$ is expressed. Later, we state pointwise convergence of the linear Mellin-Gauss-Weierstrass operators.

2. Basic Notations

Let us represent by $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ the sets of positive integers, positive real numbers and nonnegative real numbers, respectively. By $\mathbb{C}$, we symbolize the set of complex numbers. Throughly the paper, $C(\mathbb{R}^+)$ settles for the space of all continuous and bounded functions defined on $\mathbb{R}^+$and by $C_{comp}(\mathbb{R}^+)$ the subspace of $C(\mathbb{R}^+)$ including all functions with compact support in $\mathbb{R}^+$. Moreover, $C_{comp}^{\infty}(\mathbb{R}^+)$ denotes the subspace of $C_{comp}(\mathbb{R}^+)$ including all test functions, i.e., the functions of compact support which are infinitely differentiable.

For $1 \leq p \leq \infty$, we represent by $L^p(\mathbb{R}^+)$ the ordinary Lebesgue space comprising all Lebesgue measurable function such that

$$\|f\|_p := \left\{ \int_0^{\infty} |f(x)|^p \, dx \right\}^{1/p} < \infty \quad (1 \leq p < \infty)$$

and

$$\|f\|_{\infty} := \underset{x \in \mathbb{R}^+}{ess\sup} |f(x)| < \infty.$$

We should point out that $C(\mathbb{R}^+) \subset L^{\infty}(\mathbb{R}^+)$ and the norm of two spaces is the same.

Let's assume that $c \in \mathbb{R}$ is constant. For $1 \leq p < \infty$, we symbolize by X_c^p the weighted Lebesgue space, so called Mellin-Lebesgue space, which represent the natural Mellin counterpart of the classical Lebesgue spaces, defined by

$$X_c^p := \left\{ f : \mathbb{R}^+ \to \mathbb{C} : f(\cdot)(\cdot)^{c-1/p} \in L^p\left(\mathbb{R}^+\right) \right\}$$

and equipped with the norm

$$\begin{aligned} \|f\|_{X_c^p} \quad : \quad &= \left\{ \int_0^{\infty} |f(x)|^p \, x^{cp-1} dx \right\}^{1/p} \\ &= \quad \left\{ \int_0^{\infty} |f(x)|^p \, x^{cp} \frac{dx}{x} \right\}^{1/p} < \infty \quad . \end{aligned}$$

In case $p = 1$, we will simply write $X_c^1 \equiv X_c$. In an equal form, X_c^p is the space of all functions f such that $f(\cdot)(\cdot)^c \in L_{\mu}^p(\mathbb{R}^+)$, where $L_{\mu}^p(\mathbb{R}^+)$ represents the Lebesgue space in connection with the invariant measure $\mu(A) = \int_A \frac{dt}{t}$ for any measurable set $A \subset \mathbb{R}^+$. For details, see [13].

We consider the linear Mellin-Gauss-Weierstrass operators defined in [13, Page 342 Definition 8] as follows

$$(\mathcal{T}_w f)(s) = \frac{w}{\sqrt{4\pi}} \int_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} f(st) \frac{dt}{t}, \qquad s \in (0,\infty).$$

It is easy to see that

$$\frac{w}{\sqrt{4\pi}} \int_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} \frac{dt}{t} = 1. \tag{1}$$

3. Pointwise Convergence and Quantitative Estimate

This part is seperated to state pointwise convergence of $(\mathcal{T}_w)$ and the rate of convergence through modulus of smoothness which will also be defined.

To acquire convergence theorems for the operators $\mathcal{T}_w$, we need the following density result (see [10]). We accept the following impression: for a subspace $H \subset X_c^p$, we represent by $cls_{X_c^p}(H)$ the closure of H in connection with the norm-topology of X_c^p.

Theorem 1. *[10] For every $p \geq 1$ and $c \in \mathbb{R}$, we have*

$$cls_{X_c^p}\left(C_{comp}^\infty\left(\mathbb{R}^+\right)\right) = X_c^p.$$

Firstly, we begin with the following lemma.

Lemma 1. *If $f \in X_c^p$, then we get*

$$\|\mathcal{T}_w f\|_{X_c^p} \leq e^{c^2/w^2} \|f\|_{X_c^p}.$$

Proof. We can write

$$\begin{aligned}
\|\mathcal{T}_w f\|_{X_c^p} &= \left\{\int_0^\infty |(\mathcal{T}_w f)(s)|^p s^{cp} \frac{ds}{s}\right\}^{1/p} \\
&= \frac{w}{\sqrt{4\pi}} \left\{\int_0^\infty \left|\int_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} f(st) \frac{dt}{t}\right|^p s^{cp} \frac{ds}{s}\right\}^{1/p} \\
&\leq \frac{w}{\sqrt{4\pi}} \int_0^\infty \left\{\int_0^\infty |f(st))|^p s^{cp} \frac{ds}{s}\right\}^{1/p} e^{-\left(\frac{w}{2}\log t\right)^2} \frac{dt}{t} \\
&= \frac{w}{\sqrt{4\pi}} \|f\|_{X_c^p} \int_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} t^{-c} \frac{dt}{t} \\
&= e^{c^2/w^2} \|f\|_{X_c^p}.
\end{aligned}$$

□

Definition 1. *We present the first modulus of smoothness of a function $f \in X_c^p$ with*

$$\omega_{X_c^p}(f;\delta) = \sup_{|\ln t|<\delta} \|f(t\cdot) - f(\cdot)\|_{X_c^p}, \quad \delta > 0.$$

The modulus has the following properties:

Theorem 2. *If $f \in X_c^p$, we have*

$$\lim_{\delta \to 0} \omega_{X_c^p}(f;\delta) = 0. \tag{2}$$

Proof. Let be $|\ln t| < \delta$. Assuming first that $c > 0$, since $f \in X_c^p$, for every $\varepsilon > 0$ there exists $A > 1$ such that for any $\delta > 1$

$$I_1 := \left(\int_0^{e^{-A}} |f(s)|^p s^{cp} \frac{ds}{s} \right)^{1/p} < \frac{\varepsilon}{4e^{c\delta}} \quad \text{and} \quad I_2 := \left(\int_{e^A}^{\infty} |f(s)|^p s^{cp} \frac{ds}{s} \right)^{1/p} < \frac{\varepsilon}{4e^{c\delta}}. \tag{3}$$

From (3), we have

$$\begin{aligned} \left(\int_{s \notin (e^{-A}, e^A)} |f(s)|^p s^{cp} \frac{ds}{s} \right)^{1/p} &= \left(\left(\int_0^{e^{-A}} + \int_{e^A}^{\infty} \right) |f(s)|^p s^{cp} \frac{ds}{s} \right)^{1/p} \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

It is obvious that for any $\delta > 1$

$$\left(\int_{s \notin [e^{-A-\delta}, e^{A+\delta}]} |f(s)|^p s^{cp} \frac{ds}{s} \right)^{1/p} < \frac{\varepsilon}{2}. \tag{4}$$

Then, using the change of variable $ts = u$, with $|\ln t| < \delta$ $(\delta > 1)$, and from (3), we obtain

$$\begin{aligned} \left(\int_0^{e^{-\delta-A}} |f(ts)|^p s^{cp} \frac{ds}{s} \right)^{1/p} &< t^{-c} \left(\int_0^{te^{-\delta-A}} |f(u)|^p u^{cp} \frac{du}{u} \right)^{1/p} \\ &< e^{c\delta} \left(\int_0^{e^{-2\delta-A}} |f(u)|^p u^{cp} \frac{du}{u} \right)^{1/p} < \frac{\varepsilon}{4} \end{aligned} \tag{5}$$

and

$$\begin{aligned} \left(\int_{e^{\delta+A}}^{\infty} |f(ts)|^p s^{cp} \frac{ds}{s} \right)^{1/p} &< t^{-c} \left(\int_{te^{\delta+A}}^{\infty} |f(u)|^p u^{cp} \frac{du}{u} \right)^{1/p} \\ &< e^{c\delta} \left(\int_{e^{2\delta+A}}^{\infty} |f(u)|^p u^{cp} \frac{du}{u} \right)^{1/p} < \frac{\varepsilon}{4}. \end{aligned} \tag{6}$$

From (4), (5) and (6), we obtain

$$\sup_{|\ln t|<\delta}\left(\int_0^{e^{-\delta-A}} |f(ts)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p} + \sup_{|\ln t|<\delta}\left(\int_{e^{\delta+A}}^{\infty} |f(ts)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p} < \varepsilon.$$

In this case, we can write the inequality

$$\omega_{X_c^p}(f;\delta) \le \varepsilon + \sup_{|\ln t|<\delta}\left(\int_{e^{-\delta-A}}^{e^{\delta+A}} |f(ts)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p}.$$

For every $f \in X_c^p$, using Theorem 1, there is $g \in C_{comp}(\mathbb{R}^+)$ such that

$$\left(\int_{e^{-2\delta-A}}^{e^{2\delta+A}} |g(s)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p} < \frac{\varepsilon}{e^{c\delta}}. \tag{7}$$

Using the Minkowsky inequality and the logarithmic continuity of smoothness of the function g in the closed interval, we attain

$$\begin{aligned}\left(\int_{e^{-\delta-A}}^{e^{\delta+A}} |f(ts)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p} \le\ & \left(\int_{e^{-\delta-A}}^{e^{\delta+A}} |f(ts)-g(ts)|^p s^{cp}\frac{ds}{s}\right)^{1/p}\\ &+\left(\int_{e^{-\delta-A}}^{e^{\delta+A}} |g(ts)-g(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p}\\ &+\left(\int_{e^{-\delta-A}}^{e^{\delta+A}} |g(s)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p}.\end{aligned}$$

According to (7), we get

$$\left(\int_{e^{-\delta-A}}^{e^{\delta+A}} |f(ts)-g(ts)|^p s^{cp}\frac{ds}{s}\right)^{1/p} < \varepsilon \quad\text{and}\quad \left(\int_{e^{-\delta-A}}^{e^{\delta+A}} |g(s)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p} < \varepsilon.$$

As g is a continuous function, for $|\ln t| < \delta$, we can take

$$|g(ts)-g(s)| < \frac{\varepsilon}{(2(A+\delta))^{1/p}\, e^{(A+\delta)c}}.$$

Hence, we have

$$\sup_{|\ln t|<\delta}\left(\int\limits_{e^{-\delta-A}}^{e^{\delta+A}}|f(ts)-f(s)|^{p}\,s^{cp}\frac{ds}{s}\right)^{1/p}\leq 3\varepsilon$$

and this theorem proves.

A similar result is obtained when $c<0$. Thus, the desired result emerges again in a similar way. □

Theorem 3. *If $f\in X_c^p$ and $n\in\mathbb{N}$, then we get*

$$\omega_{X_c^p}(f;n\delta)\leq n\omega_{X_c^p}(f;\delta).$$

Proof. With the aid of the definition of $\omega_{X_c^p}$, we obtain

$$\begin{aligned}\omega_{X_c^p}(f;n\delta) &= \sup_{|\ln t|<n\delta}\|f(t\cdot)-f(\cdot)\|_{X_c^p}\\ &= \sup_{|\ln t|<\delta}\left\{\int_0^\infty|f(t^n s)-f(s)|^p\,s^{cp}\frac{ds}{s}\right\}^{1/p}\\ &= \sup_{|\ln t|<\delta}\left\{\int_0^\infty\left|\sum_{k=1}^{n}f\left(t^k s\right)-f\left(t^{k-1}s\right)\right|^p s^{cp}\frac{ds}{s}\right\}^{1/p}\\ &\leq \sum_{k=1}^{n}\sup_{|\ln t|<\delta}\left\{\int_0^\infty\left|f\left(t^k s\right)-f\left(t^{k-1}s\right)\right|^p s^{cp}\frac{ds}{s}\right\}^{1/p}\\ &= n\omega_{X_c^p}(f;\delta).\end{aligned}$$

□

Corollary 1. *If $f\in X_c^p$ and $\lambda\in\mathbb{R}$, then we get*

$$\omega_{X_c^p}(f;\lambda\delta)\leq(1+\lambda)\,\omega_{X_c^p}(f;\delta).$$

Now, we give the following:

Definition 2. *We will call that a point $s\in\mathbb{R}^+$ is a Lebesgue point of a function $f\in X_c^p$ $(c\neq 0)$ if*

$$\lim_{z\to 1}\left|\frac{1}{\log z}\int_1^z|f(su)-f(s)|^p\,u^{cp}\frac{du}{u}\right|^{1/p}=0.$$

This is equivalent to

$$\lim_{z\to 1^-}\left(\frac{1}{-\log z}\int_z^1|f(su)-f(u)|^p\,u^{cp}\frac{du}{u}\right)^{1/p}+\lim_{z\to 1^+}\left(\frac{1}{\log z}\int_1^z|f(su)-f(u)|^p\,u^{cp}\frac{du}{u}\right)^{1/p}=0.$$

You can refer to [9] for the situation in X_0^1 space.

The main conclusion of this part is on pointwise convergence as following:

Theorem 4. *If $f \in X_c^p$, then we get*

$$\lim_{w\to\infty} \left(\mathcal{T}_w f\right)(s) = f(s)$$

for any Lebesgue point $s \in \mathbb{R}^+$.

Proof. Using the property (1), we can obtain that

$$\left|\left(\mathcal{T}_w f\right)(s) - f(s)\right| \leq \frac{w}{\sqrt{4\pi}} \int_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} \left|f(st) - f(s)\right| \frac{dt}{t}.$$

Using Hölder's inequality, we attain

$$\begin{aligned} \left|\left(\mathcal{T}_w f\right)(s) - f(s)\right|^p &\leq \left(\int_0^\infty \frac{w}{\sqrt{4\pi}} e^{-\left(\frac{w}{2}\log t\right)^2} \left|f(st) - f(s)\right|^p t^{cp} \frac{dt}{t}\right) \\ &\times \left(\int_0^\infty \frac{w}{\sqrt{4\pi}} e^{-\left(\frac{w}{2}\log t\right)^2} t^{-cq} \frac{dt}{t}\right)^{\frac{p}{q}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Consider the integral

$$\int_0^\infty \frac{w}{\sqrt{4\pi}} e^{-\left(\frac{w}{2}\log t\right)^2} \left|f(st) - f(s)\right|^p t^{cp} \frac{dt}{t}.$$

Let $\delta > 1$ be fixed and let us consider $H_\delta = \left(\delta^{-1}, \delta\right)$. Then

$$\begin{aligned} &\frac{w}{\sqrt{4\pi}} \int_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} \left|f(st) - f(s)\right|^p t^{cp} \frac{dt}{t} \\ &= \frac{w}{\sqrt{4\pi}} \int_{1/\delta}^\delta e^{-\left(\frac{w}{2}\log t\right)^2} \left|f(st) - f(s)\right|^p t^{cp} \frac{dt}{t} \\ &+ \frac{w}{\sqrt{4\pi}} \int_{\mathbb{R}^+ \setminus H_\delta} e^{-\left(\frac{w}{2}\log t\right)^2} \left|f(st) - f(s)\right|^p t^{cp} \frac{dt}{t} \\ &= I_1 + I_2. \end{aligned}$$

Firtsly, we take into account I_1.

$$\begin{aligned} I_1 &= \frac{w}{\sqrt{4\pi}} \int_{1/\delta}^1 e^{-\left(\frac{w}{2}\log t\right)^2} \left|f(st) - f(s)\right|^p t^{cp} \frac{dt}{t} \\ &+ \frac{w}{\sqrt{4\pi}} \int_1^\delta e^{-\left(\frac{w}{2}\log t\right)^2} \left|f(st) - f(s)\right|^p t^{cp} \frac{dt}{t} \\ &= I_1^1 + I_1^2. \end{aligned}$$

Let us define

$$F^-(z) := \int_z^1 \left|f(su) - f(u)\right|^p u^{cp} \frac{du}{u}$$

for every $z \in \left(\delta^{-1}, 1\right)$. Let $\varepsilon > 0$ be fixed. Since $s \in \mathbb{R}^+$ is a Lebesgue point of f, we can choose $\delta > 1$ such that

$$F^-(z) \leq -\varepsilon \log z.$$

Then, we have

$$\begin{aligned} I_1^1 &= \frac{w}{\sqrt{4\pi}} \int_{1/\delta}^{1} e^{-\left(\frac{w}{2}\log t\right)^2} \left|f(st)-f(s)\right|^p t^{cp} \frac{dt}{t} \\ &= -\frac{w}{\sqrt{4\pi}} \int_{1/\delta}^{1} e^{-\left(\frac{w}{2}\log z\right)^2} dF^-(z) \\ &= \frac{w}{\sqrt{4\pi}} e^{-\left(\frac{w}{2}\log 1/\delta\right)^2} F^-(1/\delta) + \int_{1/\delta}^{1} F^-(z)\, d\left(\frac{w}{\sqrt{4\pi}} e^{-\left(\frac{w}{2}\log z\right)^2}\right) \\ &\leq \varepsilon \frac{w}{\sqrt{4\pi}} e^{-\left(\frac{w}{2}\log 1/\delta\right)^2} \log\delta - \varepsilon \int_{1/\delta}^{1} \log z d\left(\frac{w}{\sqrt{4\pi}} e^{-\left(\frac{w}{2}\log z\right)^2}\right) \\ &\leq \varepsilon \frac{w}{\sqrt{4\pi}} e^{-\left(\frac{w}{2}\log 1/\delta\right)^2} \log\delta - \varepsilon \left[\log\delta \frac{w}{\sqrt{4\pi}} e^{-\left(\frac{w}{2}\log 1/\delta\right)^2} - \frac{w}{\sqrt{4\pi}} \int_{1/\delta}^{1} e^{-\left(\frac{w}{2}\log z\right)^2} \frac{dz}{z}\right] \\ &\leq \varepsilon. \end{aligned}$$

Next for I_1^2, utilizing the similar ways and paying attention to the function

$$F^+(z) := \int_1^z \left|f(su)-f(u)\right|^p u^{cp} \frac{du}{u},$$

we obtain analogous estimate. Thus, we achieve $I_1 \to 0$ for $\omega \to \infty$.

Now, we handle

$$\begin{aligned} I_2 &= \frac{w}{\sqrt{4\pi}} \int_{\mathbb{R}^+ \setminus H_\delta} e^{-\left(\frac{w}{2}\log t\right)^2} \left|f(st)-f(s)\right|^p t^{cp} \frac{dt}{t} \\ &\leq 2^p \frac{w}{\sqrt{4\pi}} \int_0^{1/\delta} e^{-\left(\frac{w}{2}\log t\right)^2} \left|f(st)\right|^p t^{cp} \frac{dt}{t} + 2^p \left|f(s)\right|^p \frac{w}{\sqrt{4\pi}} \int_0^{1/\delta} e^{-\left(\frac{w}{2}\log t\right)^2} t^{cp} \frac{dt}{t} \\ &\quad + 2^p \frac{w}{\sqrt{4\pi}} \int_\delta^\infty e^{-\left(\frac{w}{2}\log t\right)^2} \left|f(st)\right|^p t^{cp} \frac{dt}{t} + 2^p \left|f(s)\right|^p \frac{w}{\sqrt{4\pi}} \int_\delta^\infty e^{-\left(\frac{w}{2}\log t\right)^2} t^{cp} \frac{dt}{t}. \end{aligned}$$

As $\frac{w}{\sqrt{4\pi}} e^{-\left(\frac{w}{2}\log t\right)^2}$ is increasing in $(0,1)$, we can write

$$\frac{w}{\sqrt{4\pi}} \int_0^{1/\delta} e^{-\left(\frac{w}{2}\log t\right)^2} \left|f(st)\right|^p t^{cp} \frac{dt}{t} \leq \frac{w}{\sqrt{4\pi}} e^{-\left(\frac{w}{2}\log 1/\delta\right)^2} s^{-cp} \left\|f\right\|_{X_c^p}.$$

Similarly,

$$\frac{w}{\sqrt{4\pi}} \int_0^{1/\delta} e^{-\left(\frac{w}{2}\log t\right)^2} t^{cp} \frac{dt}{t}$$

tends to zero for $w \to \infty$. On the other hand, we obtain

$$\frac{w}{\sqrt{4\pi}} \int_\delta^\infty e^{-\left(\frac{w}{2}\log t\right)^2} t^{cp} \frac{dt}{t} = \frac{1}{\sqrt{4\pi}} \int_{\delta^w}^\infty e^{-\left(\frac{1}{2}\log t\right)^2} t^{\frac{cp}{w}} \frac{dt}{t},$$

which tends to zero for $w \to \infty$. The last term can be estimated similarly. □

Corollary 2. *If $f \in X_c^p$, then we get*

$$\lim_{w \to \infty} \left(\mathcal{T}_w f\right)(s) = f(s)$$

almost everywhere in $\mathbb{R}^+$.

Theorem 5. *If $f \in X_c^p$, then we get*

$$\left\|\mathcal{T}_w f - f\right\|_{X_c^p} \leq \left(1 + \frac{2}{\sqrt{\pi}}\right) \omega_{X_c^p}\left(f; w^{-1}\right).$$

Proof. Since the property (1), we have

$$\left(\mathcal{T}_w f\right)(s) - f(s) = \frac{w}{\sqrt{4\pi}} \int_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} \left(f(ts)) - f(s)\right) \frac{dt}{t}.$$

Then, we deduce

$$\begin{aligned}
\left\|\mathcal{T}_w f - f\right\|_{X_c^p} &= \frac{w}{\sqrt{4\pi}} \left\{\int_0^\infty \left|\int_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} \left(f(ts)) - f(s)\right) \frac{dt}{t}\right|^p s^{cp} \frac{ds}{s}\right\}^{1/p} \\
&\leq \frac{w}{\sqrt{4\pi}} \int_0^\infty \left\{\int_0^\infty \left|\left(f(ts)) - f(s)\right)\right|^p s^{cp} \frac{ds}{s}\right\}^{1/p} e^{-\left(\frac{w}{2}\log t\right)^2} \frac{dt}{t} \\
&\leq \frac{w}{\sqrt{4\pi}} \int_0^\infty \omega_{X_c^p}\left(f; \left|\ln t\right|\right) e^{-\left(\frac{w}{2}\log t\right)^2} \frac{dt}{t} \\
&\leq \frac{w}{\sqrt{4\pi}} \omega_{X_c^p}(f; \delta) \int_0^\infty \left(1 + \frac{1}{\delta}\left|\ln t\right|\right) e^{-\left(\frac{w}{2}\log t\right)^2} \frac{dt}{t} \\
&= \omega_{X_c^p}(f; \delta) \left(1 + \frac{1}{\delta} \frac{2}{w\sqrt{\pi}}\right).
\end{aligned}$$

Choosing $\delta = w^{-1}$, we obtain desired result. □

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BETA GENERATED SLASH DISTRIBUTION: DERIVATION, PROPERTIES AND APPLICATION TO LIFETIME DATA

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Abstract. In this paper, we introduce a new distribution called beta generated slash distribution by applying the slash construction idea to the existing beta distribution of first kind. The statistical properties of the distribution such as moments, skewness, kurtosis, median, moment generating function, mean deviations, Lorenz and Bonferroni curves, order statistics, Mills ratio, hazard rate functions have been discussed. The location-scale form of the beta generated slash distribution is also established. The hazard rate function is seen to assume different shapes depending upon the values of the parameters. The method of maximum likelihood is used to estimate the unknown parameters of beta generated slash distribution and a simulation study is conducted to check the performance of these estimates. Finally, the proposed distribution is applied to a real-life data set on failure times and the goodness-of-fit of the fitted distribution is compared with four other competing distributions to show its flexibility and advantage particularly in modeling heavy tailed data sets.

1. Introduction

The beta distribution is a continuous type of probability distribution. This distribution represents a family of probabilities and is a versatile way to represent outcomes for percentages or population. The basic beta distribution is called the beta distribution of first kind and is used in a range of disciplines including rule of succession, Bayesian statistics and task duration modelling. The probability density function of beta distribution of first kind is:

$$f(x,a,b) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}, 0 \leq x \leq 1 \tag{1}$$

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The shape of the distribution is controlled by the two shape parameters a and b. Beta distribution is more useful than the normal distribution if we need to model a behaviour that is obviously bounded. In (1), $\beta(a, b)$ is referred to as the beta function and with its help, incomplete beta function ratio, incomplete beta function, the members of several beta generalised distributions have been introduced. For example, Beta Normal distribution by Eugene et al. [1], Beta Gumbel distribution by Nadarajah and Kotz [2], Beta Exponential distribution by Nadarajah and Kotz [3], Beta Exponentiated Weibull distribution by Cordeiro et al. [4] and Beta - Dagum distribution by Domma and Condino [5].

The slash distribution is defined by Rogers and Tukey [6] as the ratio of standard normal random variable to the uniform random variable following the stochastic representation

$$Y = \frac{X}{U^{\frac{1}{q}}} \tag{2}$$

where $X \sim N(0,1)$ and $U \sim U(0,1)$. $q > 0$ is the shape parameter which controls the kurtosis of the distribution. The fundamental studies on slash distribution have focused on its properties and application to model heavy-tailed data. A modified version of slash distribution has been proposed by Reyes et al. [7] who considered the distribution of U in (2) as exponential distribution with parameter 2. Reyes et al. [8] have introduced generalised modified slash distribution by considering U in (2) to be distributed as two - parameter gamma distribution. The authors have showed that this generalised modified- slash distribution performs better than the existing slash distribution and modified-slash distribution in modelling heavy-tailed data. The logit slash distribution [9] is a new extension of slash distribution having support in (0,1). This distribution offers flexible forms depending on the values of the shape parameter q, thus making it useful for bounded heavy-tailed data. The slash distribution is particularly useful when models with heavy tails are neccesary to fit a real data set. This simple concept has launched a remarkable creativity among the reseachers. In the last decade, slash distribution for many popular parent distributions have been extensively explored. For example, the slashed versions for the epsilon half-normal has been established by Gui et al. [10] where a slash distribution is naturally defined with the help of an extension of half normal distribution, the extended slash distribution of sum of two independent logistic variables [11], the modified slash Birnbaum−Saunders distribution by Reyes et al. [12] where an extension of Birnbaum−Saunders has been introduced on the basis of modified slash distribution approach proposed by [7]. An extension of Akash distribution has been introduced by Gomez et al. [13] by using slash construction approach to make the kurtosis of the Akash distribution more flexible. Extensive works on multivariate slash distributions have also been carried out by several authors. For instance, the multivariate skew - slash distribution by Wang and Genton [14] where they discussed the multivariate skew version of the distribution and studied its properties and inferences and used it to fit some skewed data sets. The multivariate asymmetric slash Laplace distribution has been established by Punthumparambath [15]. An alternative to multivariate skew - slash distribution has been introduced by Arslan [16]. Genc established the generalisation of slash distribution by using the scale mixture of exponential power distribution [17]. A family of skew-slash distributions generated by normal and Cauchy kernels was established by Punthumparambath [18] [19]. The general properties of the canonical form of slash distribution have been studied by Rogers and Tukey [6] and Mosteller and Tukey [20]. The maximum likelihood estimators of the location and scale parameters of the standard slash distribution have been studied by Kafadar [21]. Both the discrete and continous structure of the uniform slash and α-slash distributions have been established by Jones and Higuchi [22]. A new family of modified slash distribution along with their applications has been studied by Reyes et al. [23] where type II modified slash distribution is introduced by considering the distribution of U in (2) to be Birnbaum−Saunders distribution.

An extensive review of the existing works on slash distribution revealed that the slash distribution is particulary useful when models with heavy tails are necessary to fit a real data set. In presence of extreme values, the heavy-tailed models are required to perform better modelling. Skewed models provide better prospect in modelling heavy - tailed data and slash distribution is one type of skewed distribution. The usual regression model which finds application across diverse fields of biology, sociology, economics, psychology, epidemiology, marketing etc may not conform to the normal probability law all the time. In such cases the error structures should be handled from the perspective of asymmetry or skewness. Also, slash distribution offers flexibility in modelling extreme events as it is associated with augmenting the kurtosis of the underlying data, thereby accomodating the outliers. Slash distribution has been more popular in robust statistical analysis. Slash distribution remains robust where traditional distribution may fail to adequately capture the tails of the data.

Heavy - tailed lifetime data often arise in real life which requires a flexible heavy-tailed probability model for describing its behaviour. One may also need to look for a probability model which is able to account for the outliers in lifetime data. A slash distribution being a flexible heavy-tailed model is equipped to handle such type of data. Further, most of the existing works focuses on establishing the slash version of random variables having support in the range $(-\infty, \infty)$ and $(0, \infty)$. However till now, not much work on developing the slash distribution for finitely bounded random variable has been carried out. This motivated us to carry out our work on constructing the slashed version of a finite bounded r.v. which is particulary applicable to lifetime data. In particular, the beta random variable of first kind has been considered for this research work.

Here, we introduce an extension of beta distribution through the slash construction idea and the proposed distribution has been named as the beta generated slash (BGSl) distribution. The newly proposed distribution is expected to be useful in modelling data with higher level of kurtosis, providing a more precise representation of extreme outcomes.

We shall say that Y follows the BGSl distribution with parameters a, b and q or $Y \sim BGSl(a,b,q)$ if it can be stochastically expressed as

$$Y = \frac{X}{U^{\frac{1}{q}}}$$

where $X \sim beta(a,b)$ and $U \sim U(0,1)$ and are distributed independently of each other.

The rest of the paper is organised as follows. Section 2 introduces the density function of the proposed distribution. Expressions for pdf, cdf, various descriptive statistics are derived and and behaviour of the curve of the proposed distribution for varying values of the parameters graphically are shown in Section 3. The maximum likelihood estimation of the parameters of the distribution are dealt with in Section 4. In Section 5, some stochastic simulations are performed to illustrate the behaviour of the parameters of the proposed distribution. In Section 6, the proposed model is applied to data set on failure times to exhibit the potential of the distribution in modeling real-life data sets. Finally, the conclusions of this paper are given in Section 7 .

2. Definition and Derivation of the BGSl Distribution

Theorem 1. *Let $Y \sim BGSl(a,b,q)$. Then the pdf of Y is given by:*

$$f(y;a,b,q) = \begin{cases} \frac{q}{\beta(a,b)y^{q+1}}\beta(y;a+q,b), & 0 \le y < 1 \\ \\ \frac{q}{\beta(a,b)y^{q+1}}\beta(a+q,b), & 1 \le y < \infty \end{cases} \tag{3}$$

where a,b are the scale parameters, q is the shape parameter and $\beta(y;a+q,b)$ is the incomplete beta function which is given by:

$$\beta(y;a,b) = \int_0^y u^{a-1}(1-u)^{b-1}du$$

Proof. Let us consider X to be distributed as $Beta(a,b)$. Then the pdf of X is given by

$$f(x;a,b) = \frac{x^{a-1}(1-x)^{b-1}}{\beta(a,b)}, \qquad 0 \le x \le 1$$

Let us now consider the following stochastic representation:

$$Y = \frac{X}{U^{\frac{1}{q}}}$$

where $U \sim U(0,1)$

Suppose

$$W = U \implies X = YW^{\frac{1}{q}}$$

Then the jacobian of the transformation is:

$$J = \begin{vmatrix} \frac{\partial X}{\partial Y} & \frac{\partial X}{\partial W} \\ \frac{\partial U}{\partial Y} & \frac{\partial U}{\partial W} \end{vmatrix} = \begin{vmatrix} w^{\frac{1}{q}} & \frac{yw^{\frac{1}{q}-1}}{q} \\ 0 & 1 \end{vmatrix} = w^{\frac{1}{q}}$$

$\therefore$

$$\begin{aligned} f_Y(y,w) &= f_{x,u}(yw^{\frac{1}{q}}, w)|J| \\ &= \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} w^{\frac{1}{q}} \\ &= \frac{1}{\beta(a,b)} y^{a-1} w^{\frac{a}{q}} (1-yw^{\frac{1}{q}})^{b-1} \end{aligned}$$

When 0< X <1 $\Longrightarrow$ 0< $yw^{\frac{1}{q}}$ <1 $\Longrightarrow$ 0< $y < \frac{1}{w^{\frac{1}{q}}}$
When 0< $U < 1 \implies$ 0< $W < 1$
$\therefore$ The required joint pdf is

$$f(y,w) = \begin{cases} \frac{1}{\beta(a,b)} y^{a-1} w^{\frac{a}{q}} (1-yw^{\frac{1}{q}})^{b-1}, 0 < y < \frac{1}{w^{\frac{1}{q}}}, 0 < w < 1 \\ 0, \qquad otherwise \end{cases} \tag{4}$$

Hence, the marginal distribution function of Y is given by:

$$f(y,w) = \begin{cases} f_1(y), & 0 \le y < 1 \\ f_2(y), & 1 \le y < \infty \end{cases} \tag{5}$$

where

$$\begin{aligned} f_1(y) &= \frac{y^{a-1}}{\beta(a,b)} \int_0^1 w^{\frac{a}{q}} (1-yw^{\frac{1}{q}})^{b-1} dw \\ &= \frac{q}{\beta(a,b) y^{q+1}} \beta(y; a+q, b) \end{aligned} \tag{6}$$

$\beta(y; a+b, q)$ being the incomplete beta function and

$$\begin{aligned} f_2(y) &= \frac{y^{a-1}}{\beta(a,b)} \int_0^{\frac{1}{y^q}} w^{\frac{a}{q}} (1-yw^{\frac{1}{q}})^{b-1} dw \\ &= \frac{q}{\beta(a,b) y^{q+1}} \beta(a+q, b) \end{aligned} \tag{7}$$

□

The pdf of BGSl distribution for different values of parameters, is plotted in Figure 1. From the figure it is seen that the kurtosis of the distribution increases with an increase in the value of q.

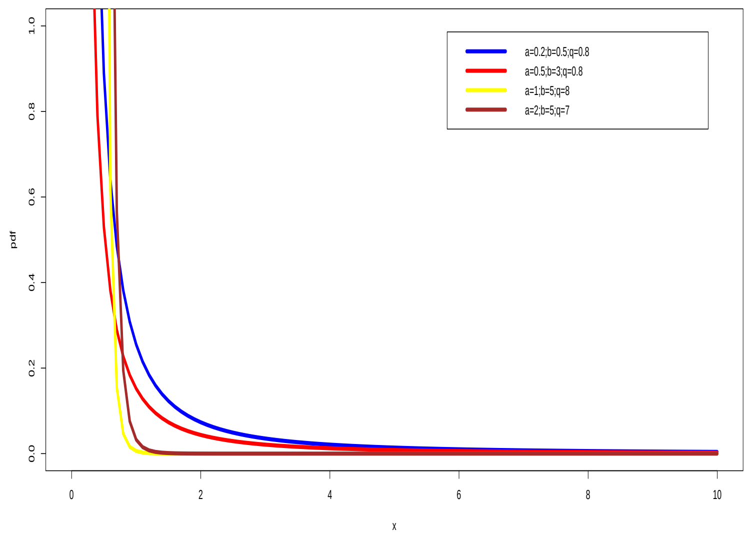

FIGURE 1. Probability density function plots of the BGSl distribution for different values of a,b and q

Again the cdf of Y is given by :

$$F(y) = \begin{cases} F_1(y), & 0 \leq y < 1 \\ F_2(y), & 1 \leq y < \infty \end{cases} \tag{8}$$

where

$$\begin{aligned} F_1(y) &= P(Y \leq y) \\ &= \int_0^y \frac{q}{\beta(a,b)t^{q+1}} \beta(t; a+q, b) dt \\ &= \frac{q}{\beta(a,b)} \int_0^y \beta(t; a+q, b) t^{-(q+1)} dt \\ &= \frac{\beta(y; a, b)}{\beta(a,b)} - y^{-q} \frac{\beta(y; a+q, b)}{\beta(a,b)} \end{aligned} \tag{9}$$

$$\begin{aligned} F_2(y) &= P(Y \leq y) \\ &= \int_0^1 f_1(t) dt + \int_1^y f_2(t) dt \\ &= 1 - \frac{\beta(a+q,b)}{\beta(a,b)} + \frac{\beta(a+q,b)}{\beta(a,b)} (1 - y^{-q}) \end{aligned} \tag{10}$$

The cdf plot for BGSl distribution is shown in Figure 2.

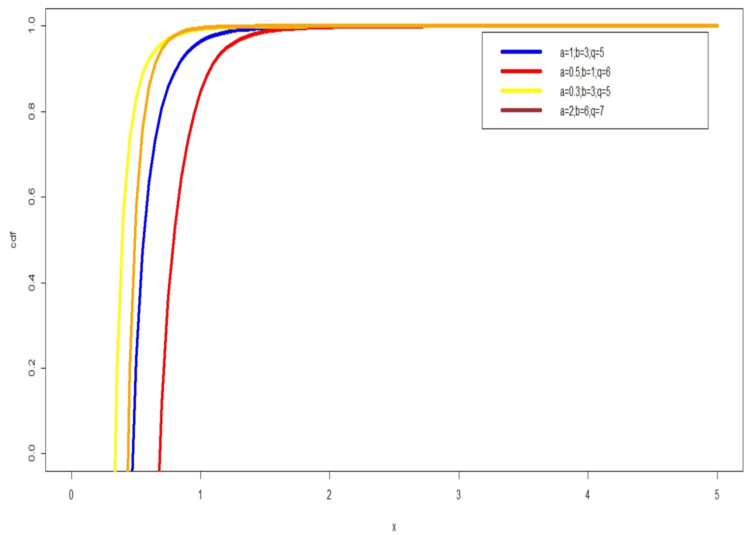

FIGURE 2. Cumulative distribution function plots of the BGSl distribution for different values of a,b and q

2.1. **Location Scale form of BGSl(a,b,q).** Another form of beta generated slash distribution is the location - scale form. By applying the well known location - scale transformation and considering the general form of BGSl distribution, we get the location - scale transformed BGSl variate as

$$T = \mu + \sigma \frac{X}{U^{\frac{1}{q}}} \tag{11}$$

where $X \sim$ Beta(a,b) and $U \sim U(0,1)$ are independent, $q > 0$, $0 < \mu < \infty$ and $\sigma > 0$. μ and σ are the location and scale parameters respectively. The location-scale form of BGSl distribution has the following pdf:

$$f(t;a,b,q) = \begin{cases} \frac{q\sigma^q (t-\mu)^{-(q+1)}}{\beta(a,b)} \beta(\frac{t-\mu}{\sigma}; a+q, b), & \mu < T < \mu + \sigma \\ \\ \frac{q\sigma^q (t-\mu)^{-(q+1)}}{\beta(a,b)} \beta(a+q, b), & \mu + \sigma \leq T < \infty \end{cases} \tag{12}$$

It is denoted by $T \sim BGSlLS(a,b,q,\mu,\sigma)$.

2.1.1. *Special cases of BGSlLS(a, b, q, μ,σ):*

- If $\mu = 0$, $\sigma = 1$, then BGSlLS(a,b,q,μ,σ) reduces to BGSl(a,b,q).
- If $q\to\infty$ then BGSlLS(a,b,q,μ,σ) tends to $\beta(a, b, \mu, \sigma)$ which is the location scale form of beta distribution.

3. Properties of BGSL(a,b,q)

3.1. **Moments and Other Descriptive Measures.** If $Y \sim BGSl(a, b, q)$, then the r^{th} raw moment of Y is given by:

$$\begin{aligned} \mu_r^{/} &= E(Y^r) \\ &= \int_0^\infty y^r f(y)dy \\ &= \int_0^1 y^r f_1(y)dy + \int_1^\infty y^r f_2(y)dy \end{aligned}$$

where r *=1, 2, 3……* and $q>$ *0.*

In particular,

$$\begin{aligned} \mu_1^{/} &= \frac{a}{(a+b)}\frac{q}{(q-1)}, \quad q>1 \\ \mu_2^{/} &= \frac{a(a+1)}{(a+b)(a+b+1)}\frac{q}{(q-2)}, \quad q>2 \\ \mu_3^{/} &= \frac{a(a+1)(a+2)}{(a+b)(a+b+1)(a+b+2)}\frac{q}{(q-3)}, \quad q>3 \\ \mu_4^{/} &= \frac{a(a+1)(a+2)(a+3)}{(a+b)(a+b+1)(a+b+2)(a+b+3)}\frac{q}{(q-4)}, \quad q>4 \end{aligned}$$

The measures of skewness and kurtosis denoted by γ_1 and γ_2 respectively, are defined as:

$$\begin{aligned} \gamma_1 &= \frac{\mu_3^{/} - 3\mu_3^{/}\mu_1^{/} + 2\mu_1^{/3}}{(\mu_2^{/} - \mu_1^{/2})^{\frac{3}{2}}}, \quad q>3 \\ \gamma_2 &= \frac{\mu_4^{/} - 4\mu_3^{/}\mu_1^{/} + 6\mu_2^{/}\mu_1^{/2} - 3\mu_1^{/4}}{(\mu_2^{/} - \mu_1^{/2})^2}, \quad q>4 \end{aligned} \tag{13}$$

In Table 1, the skewness and kurtosis values for some selected values of a,b and q are displayed. From Table 1, it is observed that skewness decreases and kurtosis increases with an increase in q. When b is fixed for some values, skewness and kurtosis increase for $a < 0.5$ but kurtosis decreases slowly as $a > 0.5$.

TABLE 1. Skewness and Kurtosis measurements of BGSl(a,b,q) distribution for different values of a, b and q

a	q	b	Skewness	Kurtosis
0.25	5	0.5	1.286	69.346
		1	1.339	109.617
		2	1.894	130.506
	6	0.5	1.017	58.723
		1	1.707	64.635
		2	2.442	97.201
	10	0.5	0.765	53.397
		1	1.465	56.188
		2	2.185	81.890
0.5	5	0.5	0.860	128.556
		1	1.339	109.617
		2	1.894	140.506
	6	0.5	0.488	130.7417
		1	1.025	139.622
		2	1.603	160.215
	10	0.5	0.126	141.748
		1	0.736	139.866
		2	1.336	157.930
1	5	0.5	0.9102	348.533
		1	1.069	235.526
		2	1.411	210.333
	6	0.5	0.285	352.952
		1	0.610	218.938
		2	1.037	183.11
	10	0.5	0.384	402.169
		1	0.158	223.903
		2	0.684	174.225

3.2. **Median.** The median (M) of a probability distribution is the value which divides the total area under the probability curve into two equal halves. Since the area under the probability curve of BGSl distribution is different in the range $[0, 1)$ and $[1, \infty)$, so the median of the proposed distribution can appear in either one of the two ranges - $[0, 1)$ or $[1, \infty)$. To find the median, the following steps are used:

(1) Compute F(1)=$\int_0^1 f_1(y)dy$.

(2) If F(1) ≥ 0.5 then the median will lie in $[0, 1)$ and M is obtained by solving the following equation:

$$\int_0^M f_1(y)dy = 0.5$$

$$\implies \frac{\beta(M; a, b)}{\beta(a, b)} - \frac{M^{-q}\beta(M; a+q, b)}{\beta(a, b)} = 0.5$$

(3) If F(1) < 0.5 then the median will lie in $[1, \infty)$ and M is obtained by solving the following equation:

$$\int_0^1 f_1(y)dy + \int_1^M f_2(y)dy = 0.5$$

$$\implies 1 - \frac{\beta(a+q, b)}{\beta(a, b)} + \frac{\beta(a+q, b)}{\beta(a, b)}\left[1 - M^{-q}\right] = 0.5 \tag{14}$$

The median values for different set of parameters are given in Table 2:

TABLE 2. Median values for diffrent set of parameters

Parameters	Median
(0.9,0.3,2)	0.75938
(1,1.5,2)	0.56557
(2,0.3,0.5)	4.23087
(0.9,0.3,0.5)	9.96179

3.3. **Moment Generating Function.** For a random variable Y with pdf *f(y)*, the moment generating function is given by:

$$M_Y(t) = E(e^{ty})$$

Hence the moment generating function of BGSL distribution is given by:

$$\begin{aligned}
M_Y(t) &= E\left(e^{ty}\right) \\
&= \int_0^1 e^{ty} f_1(y)dy + \int_1^\infty e^{ty} f_2(y)dy \\
&= \int_0^1 e^{ty} \frac{q\beta(y; a+q, b)}{\beta(a, b)y^{q+1}}dy + \int_1^\infty e^{ty} \frac{q\beta(a+q, b)}{\beta(a, b)y^{q+1}}dy \\
&= \frac{q}{\beta(a, b)} \int_0^1 e^{ty}y^{-(q+1)}\beta(y; a+q, b)dy + \frac{q\beta(a+q, b)}{\beta(a, b)} \int_0^1 e^{ty}y^{-(q+1)}dy \\
&= \frac{q}{\beta(a, b)} \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(ty)^k}{k!}\right) y^{-(q+1)}\beta(y; a+q, b)dy +
\end{aligned}$$

$$\frac{q\beta(a+q,b)}{\beta(a,b)}\int_1^\infty\left(\sum_{k=0}^{\infty}\frac{(ty)^k}{k!}\right)y^{-(q+1)}dy$$
$$=\frac{q}{\beta(a,b)}\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^1 y^{k-q-1}\beta(y;a+q,b)dy+\frac{q\beta(a+q,b)}{\beta(a,b)}\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_1^\infty y^{k-q-1}dy$$
$$=\frac{q}{\beta(a,b)}\sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{1}{k-q}\left\{\beta(a+q,b)-\beta(a+k,b)\right\}+\frac{q\beta(a+q,b)}{\beta(a,b)}\sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{1}{q-k}$$
$$=\frac{q}{\beta(a,b)}\frac{t^0}{0!}\frac{1}{(-q)}\left\{\beta(a+q,b)-\beta(a,b)\right\}+\frac{q}{\beta(a,b)}\sum_{k=1}^{\infty}\frac{t^k}{k!}\frac{1}{k-q}$$
$$\left\{\beta(a+q,b)-\beta(a+k,b)\right\}+\frac{q\beta(a+q,b)}{q\beta(a,b)}+\sum_{k=1}^{\infty}\frac{t^k}{k!(q-k)}\left\{\frac{q\beta(a+q,b)}{\beta(a,b)}\right\}$$
$$=1-\frac{q}{\beta(a,b)}\sum_{k=1}^{\infty}\frac{t^k}{k!(k-q)}\beta(a+k,b) \tag{15}$$

3.4. **Additive Property.** In this section, the additive property of BGSl(a,b,q) is discusssed through the following theorem.

Theorem 2. *$BGSl(a,b,q)$ does not satisfy the additive property i.e., if $X\sim BGSl(a_1, b_1, q_1)$ and $Y\sim BGSl(a_2,b_2,q_2)$, then $(X+Y)$ does not follow the BGSl distribution.*

Proof. The m.g.f. of $BGSl(a,b,q)$ is given by:

$$\begin{aligned}M_Y(t)&=E\left(e^{ty}\right)\\&=\int_0^1 e^{ty}f_1(y)dy+\int_1^\infty e^{ty}f_2(y)dy\\&=1-\frac{q}{\beta(a,b)}\sum_{k=1}^{\infty}\frac{t^k}{k!(k-q)}\beta(a+k,b)\end{aligned} \tag{16}$$

Let $Z=X+Y$ where $X\sim$ BGSl$(a_1,\ b_1,q_1)$ and $Y\sim$ BGSl(a_2,b_2,q_2) and are independently distributed of each other. Then the m.g.f. of Z is

$$\begin{aligned}M_Z(t)&=M_{X+Y}(t)\\&=M_X(t)M_Y(t)\\&=\left(1-\frac{q_1}{\beta(a_1,b_1)}\sum_{k=1}^{\infty}\frac{t^k}{k!(k-q_1)}\beta(a_1+k,b_1)\right)\\&\times\left(1-\frac{q_2}{\beta(a_2,b_2)}\sum_{k=1}^{\infty}\frac{t^k}{k!(k-q_2)}\beta(a_2+k,b_2)\right)\end{aligned} \tag{17}$$

which is not the m.g.f. of BGSl distribution.
Thus, $X+Y$ does not follow BGSl distribution or in other words, the BGSl distribution does not satisfy the additive property. □

3.5. **Mean Deviation About Mean.** The mean deviation about mean of a population measure the amount of scatter in a population to some extent. For a random variable Y with pdf *f(y)*, cdf *F(Y)*, mean $\mu = E(Y)$, the mean deviation about mean is defined by:

$$\begin{aligned}
\delta_1(y) &= \int_0^\infty |y-\mu| f(y)dy \\
&= \int_0^\infty (\mu - y) f(y)dy + \int_\mu^\infty (y-\mu) f(y)dy \\
&= \mu F(\mu) - \int_0^\infty y f(y)dy - \mu\left[1 - F(\mu)\right] + \int_\mu^\infty y f(y)dy \\
&= 2\mu F(\mu) - 2\mu + 2\int_\mu^\infty y f(y)dy \\
&= 2\mu F(\mu) - 2\int_0^\mu y f(y)dy \qquad (18)
\end{aligned}$$

Hence the mean deviation for BGSl distribution is given by:

$$\begin{aligned}
\delta_1(y) = I_{[0,1)}(y)&\Bigg[\frac{2aq}{(a+b)(q-1)}\left\{\frac{\beta(\mu;a,b)}{\beta(a,b)} - \mu^{-q}\beta\left(\mu;a+q,b\right)\right\} \\
&- \left\{\frac{1}{q-1}\beta\left(\mu,a+1,b\right) - \frac{\mu^{1-q}}{q-1}\beta(\mu;a+q,b)\right\}\Bigg] \\
+(1-I_{[0,1)})(y)&\left[\frac{2q\beta(a+q,b)}{\beta(a,b)(1-q)}\left\{(1-\mu^{-q}) - \frac{\beta(a+q,b)}{\beta(a,b)}\right\}\right] \qquad (19)
\end{aligned}$$

where

$$I_{[0,1)}(y) = \begin{cases} 1, & if \quad 0 \le y < 1 \\ 0, & if \quad 1 \le y < \infty \end{cases}$$

3.6. **Mills Ratio.** The Mills Ratio is the ratio of complementary cumulative distribution function to the probability density function. Mills ratio can be used in regression analysis to take account of a possible selection bias. Mills Ratio for BGSl distribution is :

$$\begin{aligned}
M(y) &= \frac{1-F_1(y)}{f_1(y)} + (1 - I_{[0,1)}(y))\frac{1-F_2(y)}{f_2(y)} \\
&= I_{[0,1)}(y)\left[\frac{1 - \frac{\beta(y;a,b)}{\beta(a,b)} + y^{-q}\beta\left(y;a+q,b\right)}{\frac{q\beta(y;a+q,b)}{\beta(a,b)y^{q+1}}}\right]
\end{aligned}$$

$$+ (1 - I_{[0,1)})(y)\left[\frac{1 - \frac{\beta(a+q,b)}{\beta(a,b)}(1-y^{-q})}{\frac{q}{\beta(a,b)y^{q+1}}\beta\left(a+q,b\right)}\right]$$

$$= I_{[0,1)}(y)\left[\frac{y^{q+1}\beta\left(a,b\right) - y^{q+1}\beta\left(y;a,b\right) - y\beta(y;a+q,b)}{q\beta(y;a+q,b)}\right]$$

$$+ (1 - I_{[0,1)})(y)\left[\frac{y^{q+1}\left\{\beta(a,b) - \beta(a+q,b)(1-y^{-q})\right\}}{q\beta(a+q,b)}\right] \tag{20}$$

3.7. **Order Statistics.** Consider a random sample $y_1,\ y_2, \ldots\ldots, y_n$ of size n drawn from BGSl(a,b,q). Further, let $y_{(1)} < y_{(2)} < \ldots\ldots\ldots < y_{(n)}$ denote the order statistics corresponding to this sample. Then the probability density function of the k^{th} order statistic is given by

$$f_{(k)}(y) = \frac{n!}{(k-1)!(n-k)!}\left[F(y)\right]^{k-1}\left[1 - F(y)\right]^{n-k} f(y)$$

Hence the density of k^{th} order statistic for BGSl(a,b,q) is

$$f_{(k)}(y) = I_{[0,1)}(y)\left[\frac{n!}{(k-1)!(n-k)!}\left\{\frac{\beta(y;a,b)}{\beta(a,b)} - y^{-q}\frac{\beta(y;a+q,b)}{\beta(a,b)}\right\}^{k-1}\right.$$

$$\left.\left\{1 - \frac{\beta(y;a,b)}{\beta(a,b)} + y^{-q}\frac{\beta(y;a+q,b)}{\beta(a,b)}\right\}^{n-k}\frac{q}{\beta(a,b)y^{q+1}}\beta(y;a+q,b)\right]$$

$$+ (1 - I_{[0,1)}(y))\left[\left\{\frac{\beta(a+q,b)}{\beta(a,b)}(1-y^{-q})\right\}^{k-1}\left\{1 - \frac{\beta(a+q,b)}{\beta(a,b)}(1-y^{-q})\right\}^{n-k}\right.$$

$$\left.\frac{q}{\beta(a,b)y^{q+1}}\beta(a+q,b)\right] \tag{21}$$

The p.d.f of the smallest order statistic $y_{(1)}$ is

$$f_{(1)}(y) = I_{[0,1)}(y)\left[n\left\{1 - \frac{\beta(y;a,b)}{\beta(a,b)} + y^{-q}\frac{\beta(y;a+q,b)}{\beta(a,b)}\right\}^{n-1}\right.$$

$$\left.\frac{q}{\beta(a,b)y^{q+1}}\beta(y;a+q,b)\right] + (1 - I_{[0,1)}(y))$$

$$\times\left[\left\{1 - \frac{\beta(a+q,b)}{\beta(a,b)}(1-y^{-q})\right\}^{n-1}\frac{q}{\beta(a,b)y^{q+1}}\beta(a+q,b)\right] \tag{22}$$

The pdf of the largest order statistic $y_{(n)}$ is

$$f_{(n)}(y) = I_{[0,1)}(y)\left[n\left\{\frac{\beta(y;a,b)}{\beta(a,b)} - y^{-q}\frac{\beta(y;a+q,b)}{\beta(a,b)}\right\}^{n-1}\frac{q}{\beta(a,b)y^{q+1}}\beta(y;a+q,b)\right]$$
$$+(1-I_{[0,1)}(y))\left[n\left\{\frac{\beta(a+q,b)}{\beta(a,b)}(1-y^{-q})\right\}^{n-1}\frac{q}{\beta(a,b)y^{q+1}}\beta(a+q,b)\right] \tag{23}$$

3.8. **Lorenz and Bonferroni Curve.** The Bonferroni and Lorenz Curve are the most used tools in income inequality measurement. These two curves are widely used in the field of reliability, demography, medicine and insurance. The Bonferroni and Lorenz curves are defined as:

$$L(F(y)) = I_{[0,1)}(y)\left[\frac{1}{\mu}\int_0^y tf_1(t)dt\right] + (1-I_{[0,1)}(y))\left[\frac{1}{\mu}\int_0^y tf_2(t)dt\right] \tag{24}$$

$$\begin{aligned} B(F(y)) &= I_{[0,1)}(y)\left[\frac{1}{\mu F_1(y)}\int_0^y tf_1(t)dt\right] + (1-I_{[0,1)}(y))\left[\frac{1}{\mu F_2(y)}\int_0^y tf_2(t)dt\right] \\ &= I_{[0,1)}(y)\left[\frac{L(F_1(y))}{F_1(y)}\right] + (1-I_{[0,1)}(y))\left[\frac{L(F_2(y))}{F_2(y)}\right] \end{aligned} \tag{25}$$

After simplifications,

$$\begin{aligned} L(F(y)) = I_{[0,1)}(y)&\left[\frac{\left[\beta(y;a+1,b) - t^{1-q}\beta(y;a+q,b)\right](a+b)}{a\beta(a,b)}\right] \\ &-(1-I_{[0,1)}(y))\left[\frac{\beta(a+q,b)(a+b)y^{1-q}}{a\beta(a,b)}\right] \end{aligned} \tag{26}$$

$$\begin{aligned} B(F(y)) &= I_{[0,1)}(y)\frac{\left[\beta(y;a+1,b) - t^{1-q}\beta(y;a+q,b)\right](a+b)}{a\beta(a,b)} \\ &\times \left[\frac{\beta(y;a,b)}{\beta(a,b)} - y^{-q}\beta(y;a+q,b)\right] \\ &+ (1-I_{[0,1)}(y))\left[\frac{\beta(a+q,b)(a+b)y^{1-q}}{a\left(\beta(a,b) - \beta(a+q,b) + \beta(a+q,b)(1-y^{-q})\right)}\right] \end{aligned} \tag{27}$$

3.9. **Hazard Rate Function.** The hazard rate function is a very important tool in understanding about the failure mechanism of a lifetime distribution. Hazard rate function can be used to postulate life distributions in the presence of several competing risk factors. It measures the instantaneous rate at which a system or component is likely to fail, given that it has survived up to a certain time. The hazard rate function of BGSl(a,b,q) is obtained by using the following formula:

$$h(y) = I_{[0,1)}(y)\frac{f_1(y)}{1-F_1(y)} + (1-I_{(0,1)})\frac{f_2(y)}{1-F_2(y)}$$

$$= I_{[0,1)}(y)\left[\frac{q\beta(y,a+q,b)}{y^{q+1}\left\{\beta(a,b)-\beta(y,a,b)+y^{-q}\beta(y,a+q,b)\right\}}\right]$$
$$+ (1-I_{[0,1)})(y)\left[\frac{q\beta(y,a+q,b)}{y^{q+1}\left\{1-(1-y^{-q}\beta(a+q,b))\right\}}\right] \quad (28)$$

The HRF plots of the BGSl distribution for various values of the parameters are shown in Figure 3. For all the combinations of a,b and q, the initial hazard is high which decreases consistently. This shows the flexibility of the hazard rate function of the beta generated slash distribution, thereby indicating that various real-life situations can be suitably modeled using this distribution. For example, for a patient who has undergone a surgery, the hazard (probability of death because of post-surgical complications in this case) is high for a specific period post the procedure. The hazard keeps on decreasing with time and after a certain period of time,i.e. after full recovery, the hazard drops to approximately 0 and remains constant thereafter.

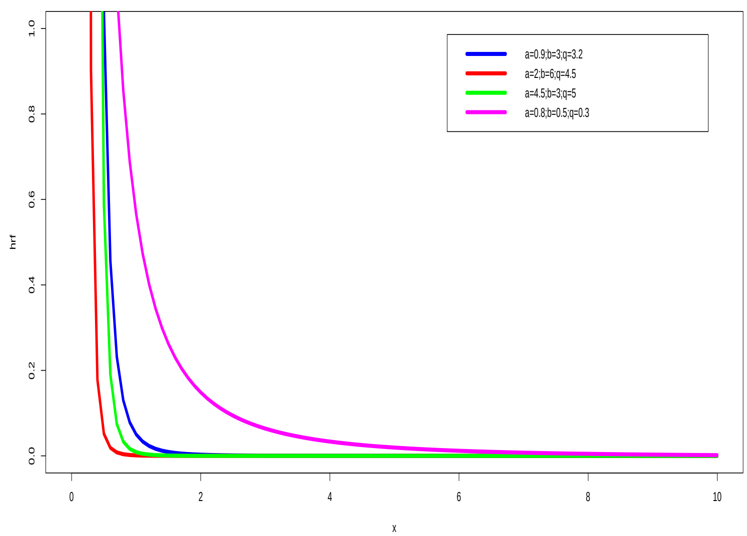

FIGURE 3. HRF plots of the BGSl distribution for different values of a,b and q

We shall now discuss the marginal probability of the variate obtained via the conditional distribution of the location-scale form of beta distribution given a Uniform$(0,1)$ variate, which is presented in Theorem 3.

Theorem 3. *If* $Z|U \sim \beta(a,b,0,\ \sigma\ u^{-\frac{1}{q}})$ *where* $\beta(a,b,0,\ \sigma\ u^{-\frac{1}{q}})$ *is location -scale form of beta distribution and* $U \sim U(0,1)$ *then* $Z \sim BGSl(a,b,q,0,\sigma\ u^{-\frac{1}{q}})$*.*

Proof.

$$\begin{aligned} P(Z|U=u) &= f(z|u) \\ &= \frac{1}{\sigma u^{\frac{1}{q}}} \left(\frac{z}{\sigma}\right)^{a-1} \left(1-\frac{z}{\sigma}\right)^{b-1} \end{aligned}$$

$\therefore$

$$\begin{aligned} f_z &= \int_0^1 f(z|u) f(u) du \\ &= \int_0^1 \frac{u^{\frac{1}{q}}}{\sigma\beta(a,b)} \left(u^{\frac{1}{q}} \frac{z}{\sigma}\right)^{a-1} \left(1-\frac{z}{\sigma}\right)^{b-1} du \\ &= \frac{q\sigma^q}{z^{q+1}\beta(a,b)} \beta(\frac{z}{\sigma}, a+q, b) \end{aligned}$$

□

4. Estimation

In this section, we discuss the maximum likelihood method of estimation for the unknown model parameters of BGSl(a,b,q). Let $y_1,y_2,\ldots\ldots,y_n$ be a random sample of size n from BGSl(a,b,q). Then the log - likelihood function is obtained as:

$$\begin{aligned} L(a,b,q,\boldsymbol{y}) &= \prod_{i=1}^{n} f(y_i,a,b,q) \\ &= L_1(a,b,q,\boldsymbol{y}) * L_2(a,b,q,\boldsymbol{y}) \end{aligned}$$

where

$$\begin{aligned} L_1(a,b,q,\boldsymbol{y}) &= \textstyle\prod_{i=1}^{n} f_1^{I_{(0,1)}(y_i)} \\ &= f_1^{\sum_{i=1}^{n} I_{(0,1)}(y_i)} \end{aligned}$$

$$\begin{aligned} \log L_1(a,b,q,\boldsymbol{y}) = \sum_{i=1}^{n} I_{[0,1)}(y_i) \Big[&\log q + \log\beta(y_i,a+q,b) \\ &- \log\beta(a,b) - (q+1)\log y_i \Big] \end{aligned} \tag{29}$$

Again,

$$\begin{aligned} L_2(a,b,q,\boldsymbol{y}) &= \prod_{i=1}^{n} f_2^{1-I_{(0,1)}(y_i)} \\ &= {f_2}^{\left(n-\sum_{i=1}^{n} I_(0,1)(y_i)\right)} \end{aligned}$$

$$\log L_2(a,b,q,\boldsymbol{y}) = \sum_{i=1}^{n} \left(n - I_{[0,1)}(y_i)\right) \Big[\log q + \log \beta(a+q,b) - \log \beta(a,b) - (q+1)\log y_i\Big] \tag{30}$$

$$\log L(a,b,q,\boldsymbol{y}) = \sum_{i=1}^{n} I_{[0,1)}(y_i)\Big[\log q + \log \beta(y_i,a+q,b) - \log\beta(a,b) - (q+1)\log y_i\Big] + \sum_{i=1}^{n} \left(n - I_{[0,1)}(y_i)\right) \Big[\log q + \log \beta(a+q,b) - \log\beta(a,b) - (q+1)\log y_i\Big] \tag{31}$$

The maximum likelihood estimates (MLE) of the parameters are computed by solving the maximum likelihood equations, which are given by

$$\frac{\partial \log L}{\partial a} \implies \sum_{i=1}^{n} I_{[0,1)}(y_i)\left[\frac{1}{\beta(y_i,a+q,b)}\frac{d}{da}\beta(y_i,a+q,b) - \{\psi_0(a) - \psi_0(a+b)\}\right] + \left(n - I_{[0,1)}(y_i)\right)\left[\frac{1}{\beta(a+q,b)}\frac{d}{da}\beta(a+q,b) - \{\psi_0(a) - \psi_0(a+b)\}\right] = 0 \tag{32}$$

$$\frac{\partial \log L}{\partial b} \implies \sum_{i=1}^{n} I_{[0,1)}(y_i)\left[\frac{1}{\beta(y_i,a+q,b)}\frac{d}{db}\beta(y_i,a+q,b) - \{\psi_0(b) - \psi_0(a+b)\}\right] + \left(n - I_{[0,1)}(y_i)\right)\left[\frac{1}{\beta(a+q,b)}\frac{d}{db}\beta(a+q,b) - \{\psi_0(b) - \psi_0(a+b)\}\right] = 0 \tag{33}$$

$$\frac{\partial \log L}{\partial q} \implies \sum_{i=1}^{n} I_{[0,1)}(y_i)\left[\frac{1}{q} + \frac{1}{\beta(y_i,a+q,b)}\frac{d}{dq}\beta(y_i,a+q,b) - log y_i\right] + \left(n - I_{[0,1)}(y_i)\right)\left[\frac{1}{q} + \frac{1}{\beta(a+q,b)}\frac{d}{dq}\beta(a+q,b) - log y_i\right] = 0 \tag{34}$$

The above maximum likelihood Equations 32- 34 are not in closed form and so, they are difficult to be solved analytically. Hence, we shall use a suitable numerical technique to solve the above equations for a, b and q.

Here all the calculations have been carried out using the R software version 3.6.3. The maxLik package is used to obtain the maximum likelihood estimates of the parameters, the rootSolve package is used to generate random variables from BGSl(a,b,q) and zipfR package is used to evaluate the incomplete beta function.

5. Simulation

In this section, generation of random numbers from BGSl(a,b,q) is discussed. For different values of the parameters a,b and q, we generate random samples of size 50, 100, 200 and 500 from BGSl(a,b,q) . Then the MLEs of the parameters are obtained for each of the generated samples. Finally, the average values of bias and mean squared error (MSE) of these estimates are calculated by using the Monte Carlo approximation technique, taking $N = 1{,}000$ replicates. The algorithm used in this simulation study is shown below:

(1) Simulate $X \sim Beta(a, b)$
(2) Simulate $U \sim U(0, 1)$
(3) Compute $Y = \frac{X}{U^{\frac{1}{q}}}$

Y thus generated is a random number from the BGSl(a,b,q). To calculate the average bias and MSE of the likelihood estimates, we use the formulae as shown below :

Let the true value of the parameter a be a^* and estimate be $\hat{a}$. Then the bias and mean square error (MSE) of $\hat{a}$ in estimating a is given by:

$$Bias(\hat{a}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{a}_i - a*)$$

$$MSE(\hat{a}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{a}_i - a*)^2$$

where N is the number of replications and $\hat{a}_i$ is the MLE of $\hat{a}$ obtained in the i^{th} replicate. Similarly, the bias and MSE of b and q are calculated. It is well known that an estimate is consistent if the bias and MSE decrease (approaches to zero) with an increase in the sample size. Table 3 shows the results of the simulation studies.

From Table 3, it has been found that the parameters are well estimated and the bias and MSE of all the estimators approaches towards zero with an increase in the sample size. Hence, the estimates of the parameters can be believed to be consistent.

6. Application

To show the flexibility of the proposed distribution over some existing distributions in modeling heavy - tailed data we apply these distributions to a real life data set. The dataset is taken from Proschan [24] which describes the times among airconditioning equipment consecutive failures in a Boeing 720 airplane. The data set comprises of the observations:
74, 57, 48, 29, 502, 12, 70, 21, 29, 386, 59, 27, 153, 26, 326.
The histogram of the data set exhibits a right skewed behavior, which may be aptly

TABLE 3. Average bias and RMSE of BGSl(a,b,q) distribution.

parameters	n	$\hat{a}$			$\hat{b}$			$\hat{q}$		
		Mean	Bias($\hat{a}$)	RMSE($\hat{a}$)	Mean	Bias($\hat{b}$)	RMSE($\hat{b}$)	Mean	Bias($\hat{q}$)	RMSE($\hat{q}$)
	30	0.32641	0.02643	0.01384	0.59527	0.03287	0.08526	0.13275	0.00457	0.00042
a=0.3	50	0.31664	0.01658	0.00763	0.54859	0.02548	0.06785	0.10034	0.00331	0.00030
b=0.5	100	0.30603	0.00609	0.00368	0.53248	0.00825	0.05964	0.00957	0.00275	0.00018
q=0.9	200	0.30304	0.00327	0.00181	0.53008	0.00803	0.04327	00932	0.00187	0.00004
	500	0.30238	0.00235	0.00072	0.51635	0.00629	0.03219	0.00854	0.00104	0.00001
	30	0.17883	0.27816	0.07949	0.05267	0.05034	00034	1.91939	0.80460	0.00896
a=1.2	50	0.59274	0.09273	0.02966	0.53127	0.04520	0.00028	1.87617	0.78347	0.00725
b=0.5	100	0.52979	0.02979	0.01298	0.52958	0.04278	0.00023	1.86524	0.76219	0.00658
q=0.8	200	0.52421	0.02421	0.00664	0.51653	0.40595	0.00002	1.85928	0.75727	0.00568
	500	0.50996	0.00996	0.00252	0.50216	0.03863	0.00001	1.84872	0.73527	0.00504
	30	0.87878	0.07873	0.03837	1.23645	0.49152	2.96958	1.99152	0.87210	1.28645
a=0.8	50	0.83583	0.03583	0.02111	0.96879	0.00630	1.12792	1.94481	0.73594	1.00257
b= 1.3	100	0.82729	0.02729	0.01039	0.00435	0.00952	1.71888	0.70267	0.36485	0.99854
q=1.8	200	0.80816	0.00816	0.00498	0.94876	0.00380	0.00634	1.69458	0.65468	0.97380
	500	0.80684	0.00684	0.00203	0.92135	0.00303	0.00439	1.54896	0.58642	0.93276

modelled by the proposed distribution. Since beta generated slash distribution is an extended distribution having support on positive real line, we compare its fit to the considered data sets with some other extended distributions, namely Modified Slash Lindley (MSL) distribution, Generalised beta distribution of first kind (GB1), Generalised Exponential distribution(GE) and Generalised Gamma(GG) distribution distributed on the range $(0,\infty)$. The various values of log-likelihood, AIC and BIC statistic for BGSl and its competing distributions are shown in Table 4.

From the Table 4, it is seen that the BGSl distribution has maximum likelihood and minimum AIC, BIC statistics. Hence the BGSl distribution performs better than the other competing distributions. Furthermore, Figure 4 and Figure 5 show the histogram of the data set along with the fitted densities and the empirical cdf versus fitted cdfs respectively for the times among airconditioning equipment consecutive failures in a Boeing 720 airplane. These figures confirm the best fit of BGSL(a, b, q) as compared to the other competing distributions.

7. Conclusion

This paper introduces a new distribution called the beta generated slash distribution having three parameters, which is obtained from the beta distribution by applying slash construction idea. The various distributional aspects such as moments, skewness, kurtosis, median, moment generating function, mean deviation, mills ratio, order statistics, Lorenz and Bonferroni curves are studied. The method of maximum likelihood is used to estimate the parameters and a simulation study is performed to study the finite sample behaviour of the ML estimates. The MLE's

TABLE 4. Estimated parameters and discrimination criteria of *BGSl(a,b,q)* distribution, *Modified Slash Lindley (MSL) distribution*, *Generalised beta distribution of first kind (GB1)*, *Generalised Exponential distribution(GE)* and *Generalised Gamma(GG) distribution* fitted to the data on failure times

Distribution	MLE	log-likelihood	AIC	BIC
BGSl(a,b,q)	$\hat{a}$=0.01976			
	$\hat{b}$=7.9395	-16.9600	39.92008	37.4483
	$\hat{q}$=4.05135			
MSL	$\hat{}\lambda$=0.05035			
	$\hat{q}$=1.196	-26.0660	56.1321	54.4843
GB1	$\hat{a}$=0.4980			
	$\hat{b}$=0.0271	-74.2625	156.5254	153.2293
	$\hat{p}$=0.5655			
	$\hat{q}$=0.0513			
GE	$\hat{a}$=0.4980153			
	$\hat{\lambda}$=0.02710	-76.7627	157.5254	155.8773
GG	$\hat{a}$=97.9396			
	$\hat{d}$=0.6999	-84.4641	174.9282	172.4565
	$\hat{p}$=0.5655			

are found to be consistent and precise in estimating the true value of the parameters. To show the application of the proposed distribution, it is applied to a dataset consisting of failure times and its fit is compared with that of Modified Slash Lindley (MSL) distribution, Generalised beta distribution of 1st kind (GB1), Generalised Gamma distribution (GG)and Generalised Exponential (GE) distribution using log - likelihood measure, Akaike information criterion(AIC) and Bayesian information criterion(BIC). It is observed that the BGSl distribution is a better fit to the data as compared to the others. Thus it can be concluded that the proposed distribution is more flexible and has advantage in modeling right skewed heavy - tailed datasets occuring in $[0, \infty)$ or any subset of it.

Author Contribution Statements The authors have made equal contributions in this work.

Declaration of Competing Interests The authors declares that they have no conflict of interest about the publication of this paper.

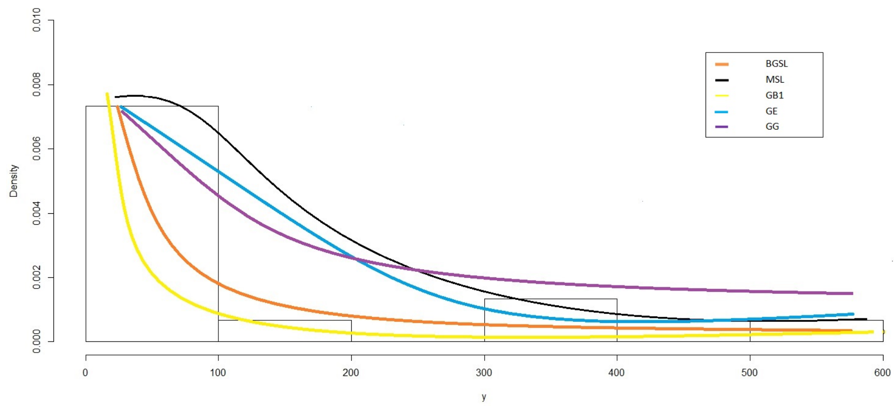

FIGURE 4. Histogram of and fitted densities to the data on air-conditioning equipment consecutive failure times in a Boeing 720 airplane

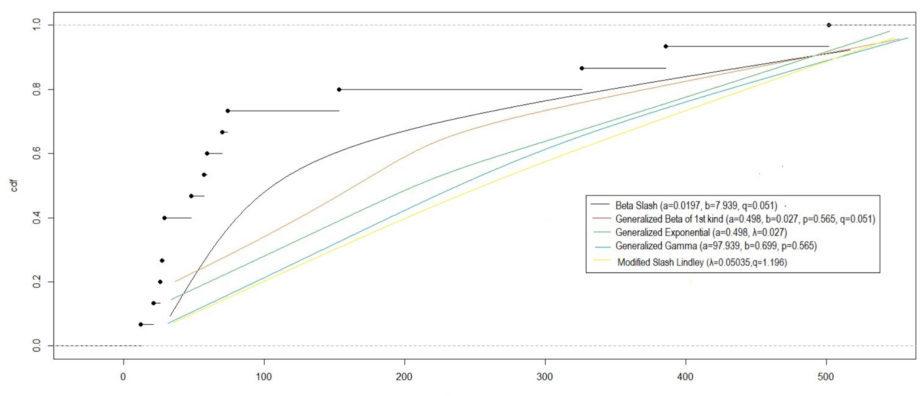

FIGURE 5. CDF plot of the observed data and fitted distributions

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ON A CLASS OF BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER RELATED TO FABER POLYNOMIALS AND $q-$SĂLĂGEAN OPERATOR

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Abstract. In this paper, we define a new class of bi-univalent functions of complex order $\sum_q^n(\tau, \zeta; \phi)$ which is defined by subordination in the open unit disc $\mathbb{D}$. By using $\mathcal{D}_q^n F(\varsigma)$ operator. Furthermore, using the Faber polynomial expansions, we get upper bounds for the coefficients of function belonging to this class.

1. Introduction

Let $\mathcal{A}$ be the class of functions

$$F(\varsigma) = \varsigma + \sum_{\rho=2}^{\infty} a_\rho \varsigma^\rho, \tag{1}$$

defined in $\mathbb{D} = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$ normalized by the conditions $F(0) = F^{'}(0) - 1 = 0$ for every $\varsigma \in \mathbb{D}$ and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$. For every $F \in \mathcal{S}$ there exists an inverse function F^{-1} which is defined in some neighborhood of the origin, and satisfying the conditions

$$F^{-1}(F(\varsigma)) = \varsigma, \quad (\varsigma \in \mathbb{D}),$$

and

$$F(F^{-1}(\omega)) = \omega, \quad (|\omega| < r_0(F);\ r_0(F) \geq \frac{1}{4}),$$

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where

$$\begin{aligned} g(\omega) &= F^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 + -(5a_2^3 - 5a_2a_3 + a_4)\omega^4 + ... \\ &= \omega + \sum_{\rho=2}^{\infty} A_\rho \omega^\rho. \end{aligned} \tag{2}$$

If both F and F^{-1} are univalent in $\mathbb{D}$, then $F \in \mathcal{A}$ is called bi-univalent in $\mathbb{D}$ and the class of these functions is denoted by σ. For more study this class (see [5,7,13, 24,26,27]).

In [17] Faber introduced a polynomial which bears his name and is very important role in geometric function theory.

By using the expansion of this polynomial for $F \in \mathcal{S}$, the coefficients of its inverse $g = F^{-1}$ may be expressed, (see [3] and [4]) as

$$g(\omega) = F^{-1}(\omega) = \omega + \sum_{\rho=2}^{\infty} \frac{1}{\rho} \chi_{\rho-1}^{-\rho}(a_2, a_3, ..., a_\rho)\omega^\rho, \tag{3}$$

where

$$\begin{aligned} \chi_{\rho-1}^{-\rho} &= \frac{(-\rho)!}{(-2\rho+1)!(\rho-1)!}a_2^{\rho-1} + \frac{(-\rho)!}{(2(-\rho+1))!(\rho-3)!}a_2^{\rho-3}a_3 \\ &+ \frac{(-\rho)!}{(-2\rho+3)!(\rho-4)!}a_2^{\rho-4}a_4 + \frac{(-\rho)!}{(2(-\rho+2))!(\rho-5)!}a_2^{\rho-5} \\ &\times \left[a_5 + (-\rho+2)a_3^2\right] + \frac{(-\rho)!}{(-2\rho+5)!(\rho-6)!}a_2^{\rho-6} \times \left[a_6 + (-2\rho+5)a_3a_4\right] \\ &+ \sum_{j\geq 7}^{\infty} a_2^{\rho-j}V_j, \end{aligned}$$

such that V_j with $7 \leq j \leq \rho$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_\rho$, (see [4]). The first three terms of $\chi_{\rho-1}^{-\rho}$ are

$$\chi_1^{-2} = -2a_2, \; \chi_2^{-3} = 3(2a_2^2 - a_3), \; \chi_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4).$$

In general, for any $p \in \mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$, an expansion of χ_ρ^p is (see for details [3,33] or [4])

$$\chi_\rho^p = pa_{\rho+1} + \frac{p(p-1)}{2}D_\rho^2 + \frac{p!}{(p-3)!3!}D_\rho^3 + ... + \frac{p!}{(p-\rho)!\rho!}D_\rho^\rho,$$

where $D_\rho^p = D_\rho^p(a_2, a_3, ...)$ and by [22] (see for details [2,14,16,20,23,31–33,35])

$$D_\rho^m(a_2, a_3, ..., a_{\rho+1}) = \sum_{\rho=0}^{\infty} \frac{m!(a_2)^{\mu_1}...(a_{\rho+1})^{\mu_\rho}}{\mu_1!...\mu_\rho!}, \tag{4}$$

where the sum is taken $\forall \; \mu_1, ..., \mu_\rho \in \mathbb{N} = \{1, 2, ...\}$ satisfying

$$\begin{cases} \mu_1 + \mu_2 + ... + \mu_\rho = m, \\ \mu_1 + 2\mu_2 + ... + k\mu_\rho = \rho. \end{cases}$$

Note that $D^{\rho}_{\rho}(a_2, a_3, ..., a_{\rho+1}) = a_2^{\rho}$.

In the rest of this paper, assume that ϕ is an analytic function with positive real part in $\mathbb{D}$, satisfying $\phi(0) = 1,\ \phi^{'}(0) > 0$ and $\phi(\mathbb{D})$ is symmetric w. r. to the real axis and has the expansion

$$\phi(\varsigma) = 1 + \psi_1\varsigma + \psi_2\varsigma^2 + \psi_3\varsigma^3 + ... \qquad (\psi_1 > 0).$$

Let $u(\varsigma)$ and $v(\omega)$ are analytic in $\mathbb{D}$ with $u(0) = v(0) = 0,\ |u(\varsigma)| < 1,\ |v(\omega)| < 1$, and

$$u(\varsigma) = \varsigma(p_1 + \sum_{\rho=2}^{\infty} p_\rho\varsigma^{\rho-1}) \ and \ v(\omega) = \omega(q_1 + \sum_{\rho=2}^{\infty} q_\rho\omega^{\rho-1}) \quad (\varsigma, \omega \in \mathbb{D}). \tag{5}$$

Then

$$|p_1| \le 1,\ |p_\rho| \le 1 - |p_1|^2,\ |q_1| \le 1,\ |q_\rho| \le 1 - |q_1|^2,\ (\rho \ge 2), \tag{6}$$

see ([28]).

The Jackson [21] $q-$derivative, $0 < q < 1$, was defined by (see also [6], [8,9,11], [18], [30]):

$$\nabla_q F(\varsigma) = \begin{cases} \frac{F(\varsigma)-F(q\varsigma)}{(1-q)\varsigma} & ,\varsigma\neq 0 \\ F^{'}(0) & ,\varsigma=0 \end{cases},$$

that is

$$\nabla_q F(\varsigma) = 1 + \sum_{\rho=2}^{\infty}[\rho]_q a_\rho \varsigma^{\rho-1}, \tag{7}$$

where

$$[j]_q = \frac{1-q^j}{1-q},\ [0]_q = 0. \tag{8}$$

As $q \to 1^-$, $[j]_q = j$ and $\nabla_q F(\varsigma) = F^{'}(\varsigma)$.

Now [19,34] defined $q-$ Sălăgean operator by

$$\mathcal{D}_q^0 F(\varsigma) = F(\varsigma)$$

$$\mathcal{D}_q^1 F(\varsigma) = \varsigma\nabla_q(F(\varsigma)) = \varsigma + \sum_{\rho=2}^{\infty}[\rho]_q a_\rho\varsigma^\rho,$$

$$\mathcal{D}_q^2 F(\varsigma) = \varsigma\nabla_q(\mathcal{D}_q F(\varsigma)),$$

and

$$\begin{aligned} \mathcal{D}_q^n F(\varsigma) &= \varsigma\nabla_q(\mathcal{D}_q^{n-1}F(\varsigma)) \\ &= \varsigma + \sum_{\rho=2}^{\infty}[\rho]_q^n a_\rho\varsigma^\rho, \quad n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}. \end{aligned} \tag{9}$$

Note that: Putting $q \to 1^-$in (9) we have the Sălăgean operator $\mathcal{D}^n$ ([29]).

Definition 1. *([12, 25]) For F and g, analytic in $\mathbb{D}$, F is subordinate to g in $\mathbb{D}$ written $F \prec g$, if $\exists\ \Omega(\varsigma)$, analytic in $\mathbb{D}$, with $\Omega(0) = 0$ and $|\Omega(\varsigma)| < 1 \quad (\varsigma \in \mathbb{D})$, such that $F(\varsigma) = g(\Omega(\varsigma)) \quad (\varsigma \in \mathbb{D})$.*

Definition 2. *For $\tau \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $0 \le \zeta \le 1$, $0 < q < 1$, $n \in \mathbb{N}_0$ and $F \in \sigma$, $F \in \sum_q^n(\tau, \zeta; \phi)$ if for all $\varsigma, \omega \in \mathbb{D}$:*

$$1 + \frac{1}{\tau}\left[\nabla_q(\mathcal{D}_q^n F(\varsigma)) + \zeta\varsigma\nabla_q(\nabla_q\mathcal{D}_q^n F(\varsigma)) - 1\right] \prec \phi(\varsigma), \tag{10}$$

and

$$1 + \frac{1}{\tau}\left[\nabla_q(\mathcal{D}_q^n g(\omega)) + \zeta\omega\nabla_q(\nabla_q\mathcal{D}_q^n g(\omega)) - 1\right] \prec \phi(\omega), \tag{11}$$

where $g(\omega) = F^{-1}(\omega)$.

Note that:
(i) $\sum_q^0(\tau, \zeta; \phi) = \sum_q(\tau, \zeta; \phi)$;
(ii) $\sum_q^0(1, \zeta; \phi) = \sum_q(\zeta; \phi)$;
(iii) $\sum_q^n((1-\alpha)e^{-i\theta}\cos\theta, \zeta; \phi) = \sum_q^n(\zeta, \alpha, \theta; \phi)$ $(0 \le \alpha < 1, |\theta| < \frac{\pi}{2})$, where

$$= \begin{cases} F \in \sigma : \frac{e^{i\theta}\left[\nabla_q(\mathcal{D}_q^n F(\varsigma)) + \zeta\varsigma\nabla_q(\nabla_q\mathcal{D}_q^n F(\varsigma))\right] - (\alpha\cos\theta + i\sin\theta)}{(1-\alpha)\cos\theta} \prec \phi(\varsigma) \\ g \in \sigma : \frac{e^{i\theta}\left[\nabla_q(\mathcal{D}_q^n g(\omega)) + \zeta\omega\nabla_q(\nabla_q\mathcal{D}_q^n g(\omega))\right] - (\alpha\cos\theta + i\sin\theta)}{(1-\alpha)\cos\theta} \prec \phi(\omega) \end{cases};$$

(iv) $\lim_{q\to 1^-} \sum_q^0(\tau, \zeta; \phi) = \sum(\tau, \zeta; \phi)$ (see [15]);
(v) $\lim_{q\to 1^-} \sum_q^0(1, \zeta; \phi) = \sum(\zeta; \phi)$ (see [1]);
(vi) $\lim_{q\to 1^-} \sum_q^0((1-\alpha)e^{-i\theta}\cos\theta, \zeta; \frac{1+\varsigma}{1-\varsigma}) = \sum(\zeta, \alpha, \theta; \frac{1+\varsigma}{1-\varsigma})$ $(0 \le \alpha < 1, |\theta| < \frac{\pi}{2})$, where

$$= \begin{cases} F \in \sigma : \frac{e^{i\theta}\left[F'(\varsigma) + \zeta\varsigma F''(\varsigma)\right] - (\alpha\cos\theta + i\sin\theta)}{(1-\alpha)\cos\theta} \prec \frac{1+\varsigma}{1-\varsigma} \\ g \in \sigma : \frac{e^{i\theta}\left[g'(\omega) + \zeta\omega g''(\omega)\right] - (\alpha\cos\theta + i\sin\theta)}{(1-\alpha)\cos\theta} \prec \frac{1+\omega}{1-\omega} \end{cases}.$$

2. Main Results

We assume that $\tau \in \mathbb{C}^*$, $0 < q < 1$, $0 \le \zeta \le 1$, $n \in \mathbb{N}_0$ and $F(\varsigma) \in \sigma$.
In this section we obtain some inequalities for the function class $\sum_q^n(\tau, \zeta; \phi)$.

Theorem 1. *Let $F \in \sum_q^n(\tau, \zeta; \phi)$. If $a_\varepsilon = 0$ for $2 \le \varepsilon \le \rho - 1$, then*

$$|a_\rho| \le \frac{\psi_1 |\tau|}{(1 + \zeta[\rho-1]_q)[\rho]_q^{n+1}} \qquad (\rho \ge 3), \tag{12}$$

Proof. For functions $\mathcal{D}_q^n F(\varsigma)$ given by (9) and $g = F^{-1}$, we have

$$\begin{aligned} & 1 + \frac{1}{\tau}\left[\nabla_q(\mathcal{D}_q^n F(\varsigma)) + \zeta\varsigma\nabla_q(\nabla_q\mathcal{D}_q^n F(\varsigma)) - 1\right] \\ = \; & 1 + \frac{1}{\tau}\sum_{\rho=2}^{\infty}(1 + \zeta[\rho-1]_q)[\rho]_q^{n+1} a_\rho \varsigma^{\rho-1}, \end{aligned} \tag{13}$$

$$1 + \frac{1}{\tau}\left[\nabla_q(\mathcal{D}_q^n g(\omega)) + \zeta\omega\nabla_q(\nabla_q\mathcal{D}_q^n g(\omega)) - 1\right]$$

$$= \quad 1+\frac{1}{\tau}\sum_{\rho=2}^{\infty}\left(1+\zeta[\rho-1]_q\right)[\rho]_q^{n+1}A_\rho\omega^{\rho-1}. \tag{14}$$

Using (3), we have

$$\begin{aligned} & 1+\frac{1}{\tau}\left[\nabla_q(\mathcal{D}_q^n g(\omega))+\zeta\omega\nabla_q(\nabla_q\mathcal{D}_q^n g(\omega))-1\right] \\ = \quad & 1+\frac{1}{\tau}\sum_{\rho=2}^{\infty}\left(1+\zeta[\rho-1]_q\right)[\rho]_q^{n+1}\frac{1}{\rho}\chi_{\rho-1}^{-\rho}(a_2,a_3,...,a_\rho)\omega^{\rho-1}. \end{aligned} \tag{15}$$

Considering (10) and (11), there are two Schwarz functions $u,\ v:\mathbb{D}\to\mathbb{D}$ with $u(0)=v(0)=0$, which are given by (5), so that

$$1+\frac{1}{\tau}\left[\nabla_q(\mathcal{D}_q^n F(\varsigma))+\zeta\varsigma\nabla_q(\nabla_q\mathcal{D}_q^n F(\varsigma))-1\right]=\phi(u(\varsigma)), \tag{16}$$

$$1+\frac{1}{\tau}\left[\nabla_q(\mathcal{D}_q^n g(\omega))+\zeta\omega\nabla_q(\nabla_q\mathcal{D}_q^n g(\omega))-1\right]=\phi(v(\omega)). \tag{17}$$

Also, by (4) we get

$$\begin{aligned} \phi(u(\varsigma)) \quad &= \quad 1+\psi_1p_1\varsigma+(\psi_1p_2+\psi_2p_1^2)\varsigma^2+... \\ &= \quad 1+\sum_{\rho=1}^{\infty}\sum_{\varepsilon=1}^{\rho}\psi_\varepsilon D_\rho^\varepsilon(p_1,p_2,...,p_\rho)\varsigma^\rho \quad (\varsigma\in\mathbb{D}), \end{aligned} \tag{18}$$

and

$$\begin{aligned} \phi(v(\omega)) \quad &= \quad 1+\psi_1q_1\omega+(\psi_1q_2+\psi_2q_1^2)\omega^2+... \\ &= \quad 1+\sum_{\rho=1}^{\infty}\sum_{\varepsilon=1}^{\rho}\psi_\varepsilon D_\rho^\varepsilon(q_1,q_2,...,q_\rho)\omega^\rho \quad (\omega\in\mathbb{D}). \end{aligned} \tag{19}$$

Comparing the coefficients of (13) and (16) with (18), we get

$$\frac{1}{\tau}\left(1+\zeta[\rho-1]_q\right)[\rho]_q^{n+1}a_\rho=\sum_{\varepsilon=1}^{\rho-1}\psi_\varepsilon D_{\rho-1}^\varepsilon(p_1,p_2,...,p_{\rho-1}) \qquad (\rho\geq 2). \tag{20}$$

Similarly, from (15) and (17) with (19), we get

$$\frac{1}{\tau}\left(1+\zeta[\rho-1]_q\right)[\rho]_q^{n+1}\frac{1}{\rho}\chi_{\rho-1}^{-\rho}(a_2,a_3,...,a_\rho)=\sum_{\varepsilon=1}^{\rho-1}\psi_\varepsilon D_{\rho-1}^\varepsilon(q_1,q_2,...,q_{\rho-1}) \quad (\rho\geq 2). \tag{21}$$

Now, from $a_\varepsilon=0$ for $2\leq\varepsilon\leq\rho-1$, we have $A_\rho=-a_\rho$ and the equalities (20) and (21) yield

$$\begin{aligned} \left(1+\zeta[\rho-1]_q\right)[\rho]_q^{n+1}a_\rho \quad &= \quad \tau\psi_1p_{\rho-1}, \\ -\left(1+\zeta[\rho-1]_q\right)[\rho]_q^{n+1}a_\rho \quad &= \quad \tau\psi_1q_{\rho-1}. \end{aligned} \tag{22}$$

Taking the modulus of each of the two equations in (22) and using (6), we obtain (12). □

Corollary 1. *For* $\phi(\varsigma) = \left(\frac{1+\varsigma}{1-\varsigma}\right)^{\alpha}$ $(0 < \alpha \leq 1)$, *let* $F \in \sum_q^n(\tau, \zeta; \phi)$, *then*

$$|a_\rho| \leq \frac{2\alpha\,|\tau|}{(1+\zeta[\rho-1]_q)\,[\rho]_q^{n+1}} \qquad (\rho \geq 3). \tag{23}$$

Corollary 2. *For* $\phi(\varsigma) = \frac{1+(1-2\beta)\varsigma}{1-\varsigma}$ $(0 \leq \beta < 1)$, *let* $F \in \sum_q^n(\tau, \zeta; \phi)$, *then*

$$|a_\rho| \leq \frac{2\,|\tau|\,(1-\beta)}{(1+\zeta[\rho-1]_q)\,[\rho]_q^{n+1}} \qquad (\rho \geq 3). \tag{24}$$

Remark 1. *For* $\tau = 1,\ n = 0,\ q \to 1^-$ *Corollary 2, reduces to results for [31, Theorem 1], for all* $0 \leq \zeta \leq 1$.

Theorem 2. *Let* $F \in \sum_q^n(\tau, \zeta; \phi)$. *Then*

$$|a_2| \leq \frac{\psi_1\sqrt{\psi_1}\,|\tau|}{\sqrt{\psi_1[2]_q^{2n+2}\,(1+\zeta)^2 + \left|\tau[3]_q^{n+1}\,(1+\zeta[2]_q)\,\psi_1^2 - [2]_q^{2n+2}\,(1+\zeta)^2\,\psi_2\right|}}, \tag{25}$$

$$|a_3| \leq \min\left\{\mathcal{K}(\zeta), \mathcal{L}(\zeta)\right\}, \tag{26}$$

where

$$\mathcal{L}(\zeta) = \begin{cases} \dfrac{\frac{\psi_1|\tau|}{[3]_q^{n+1}(1+\zeta[2]_q)} \times \left([3]_q^{n+1}(1+\zeta[2]_q)|\tau|\psi_1^2 + \left|[3]_q^{n+1}(1+\zeta[2]_q)\tau\psi_1^2 - [2]_q^{2n+2}(1+\zeta)^2\psi_2\right|\right)}{[2]_q^{2n+2}(1+\zeta)^2\psi_1 + \left|[3]_q^{n+1}(1+\zeta[2]_q)\tau\psi_1^2 - [2]_q^{2n+2}(1+\zeta)^2\psi_2\right|} & , \psi_1 \geq \frac{[2]_q^{2n+2}(1+\zeta)^2}{[3]_q^{n+1}(1+\zeta[2]_q)|\tau|} \\ \dfrac{\psi_1|\tau|}{[3]_q^{n+1}(1+\zeta[2]_q)}, & 0 \leq \psi_1 \leq \frac{[2]_q^{2n+2}(1+\zeta)^2}{[3]_q^{n+1}(1+\zeta[2]_q)|\tau|} \end{cases} \tag{27}$$

and

$$\mathcal{K}(\zeta) = \begin{cases} \frac{|\psi_2||\tau|}{[3]_q^{n+1}(1+\zeta[2]_q)} & , |\psi_2| > \psi_1 \\ \frac{\psi_1|\tau|}{[3]_q^{n+1}(1+\zeta[2]_q)} & , |\psi_2| \leq \psi_1 \end{cases}. \tag{28}$$

Proof. If we set $\rho = 2$ and $\rho = 3$ in (20) and (21), respectively, we have

$$\frac{1}{\tau}[2]_q^{n+1}\,(1+\zeta)\,a_2 = \psi_1 p_1, \tag{29}$$

$$\frac{1}{\tau}[3]_q^{n+1}\,(1+\zeta[2]_q)\,a_3 = \psi_1 p_2 + \psi_2 p_1^2, \tag{30}$$

$$-\frac{1}{\tau}[2]_q^{n+1}\,(1+\zeta)\,a_2 = \psi_1 q_1, \tag{31}$$

and

$$\frac{1}{\tau}[3]_q^{n+1}\,(1+\zeta[2]_q)\,(2a_2^2 - a_3) = \psi_1 q_2 + \psi_2 q_1^2. \tag{32}$$

From (29) and (31), we obtain

$$p_1 = -q_1. \tag{33}$$

Adding (30) and (32), and using (33), we have

$$\frac{2}{\tau}[3]_q^{n+1}\left(1+\zeta[2]_q\right)a_2^2-2p_1^2\psi_2=\psi_1(p_2+q_2). \tag{34}$$

From (29), we get

$$\left[2\tau[3]_q^{n+1}\psi_1^2\left(1+\zeta[2]_q\right)-2[2]_q^{2n+2}\left(1+\zeta\right)^2\psi_2\right]a_2^2=\tau^2\psi_1^3(p_2+q_2). \tag{35}$$

By (6), (29) and (33), we obtain

$$\begin{aligned} &\left|2\tau[3]_q^{n+1}\psi_1^2\left(1+\zeta[2]_q\right)-2[2]_q^{2n+2}\left(1+\zeta\right)^2\psi_2\right|\left|a_2\right|^2 \\ \le\ & \left|\tau\right|^2\psi_1^3(|p_2|+|q_2|) \\ \le\ & 2\left|\tau\right|^2\psi_1^3(1-\left|p_1\right|^2) \\ =\ & 2\left|\tau\right|^2\psi_1^3-2[2]_q^{2n+2}\left(1+\zeta\right)^2\psi_1\left|a_2\right|^2. \end{aligned} \tag{36}$$

Consequently

$$\left|a_2\right|^2\le\frac{\left|\tau\right|^2\psi_1^3}{[2]_q^{2n+2}\left(1+\zeta\right)^2\psi_1+\left|\tau[3]_q^{n+1}\psi_1^2\left(1+\zeta[2]_q\right)-[2]_q^{2n+2}\left(1+\zeta\right)^2\psi_2\right|}.$$

So we obtain the bound on $|a_2|$ in (25).

Next, in order to find the bound on the coefficient $|a_3|$, by subtracting (32) from (30), and using (33), we get

$$\frac{-2}{\tau}[3]_q^{n+1}\left(1+\zeta[2]_q\right)a_2^2+\frac{2}{\tau}[3]_q^{n+1}\left(1+\zeta[2]_q\right)a_3=\psi_1(p_2-q_2). \tag{37}$$

Using (6), we have

$$\begin{aligned} 2[3]_q^{n+1}\left(1+\zeta[2]_q\right)\left|a_3\right| &\le 2[3]_q^{n+1}\left(1+\zeta[2]_q\right)\left|a_2\right|^2+\left|\tau\right|\psi_1(|p_2|+|q_2|) \\ &\le 2[3]_q^{n+1}\left(1+\zeta[2]_q\right)\left|a_2\right|^2+2\left|\tau\right|\psi_1(1-\left|p_1\right|^2). \end{aligned} \tag{38}$$

From (29), we get

$$\begin{aligned} [3]_q^{n+1}\left(1+\zeta[2]_q\right)\left|\tau\right|\psi_1\left|a_3\right| &\le \left|\tau\right|^2\psi_1^2 \\ &+ \left[\left|\tau\right|[3]_q^{n+1}\psi_1\left(1+\zeta[2]_q\right)-[2]_q^{2n+2}\left(1+\zeta\right)^2\right]\left|a_2\right|^2. \end{aligned} \tag{39}$$

On the other hand from (30), we have

$$[3]_q^{n+1}\left(1+\zeta[2]_q\right)\left|a_3\right|\le\left|\tau\right|\left[\psi_1(1-\left|p_1\right|^2)+\left|\psi_2\right|\left|p_1\right|^2\right].$$

Consequently,

$$\left|a_3\right|\le\begin{cases}\frac{|\psi_2||\tau|}{[3]_q^{n+1}(1+\zeta[2]_q)} &, |\psi_2|>\psi_1\\ \frac{\psi_1|\tau|}{[3]_q^{n+1}(1+\zeta[2]_q)} &, |\psi_2|\le\psi_1\end{cases}. \tag{40}$$

Hence, from (39) and (40), we obtain (26). □

By letting $\tau = 1,\ n = 0$, we have:

Corollary 3. *Let* $F \in \sum_q^0(1, \zeta; \phi)$. *Then*

$$|a_3| \leq \min\{\mathcal{K}(\zeta), \mathcal{L}(\zeta)\}, \tag{41}$$

where

$$\mathcal{L}(\zeta) = \begin{cases} \frac{\psi_1}{[3]_q(1+\zeta[2]_q)} \times \frac{[3]_q(1+\zeta[2]_q)\psi_1^2 + \left|[3]_q(1+\zeta[2]_q)\psi_1^2 - [2]_q^2(1+\zeta)^2\psi_2\right|}{[2]_q^2(1+\zeta)^2\psi_1 + \left|[3]_q(1+\zeta[2]_q)\psi_1^2 - [2]_q^2(1+\zeta)^2\psi_2\right|} & , \psi_1 \geq \frac{[2]_q^2(1+\zeta)^2}{[3]_q(1+\zeta[2]_q)} \\ \frac{\psi_1}{[3]_q(1+\zeta[2]_q)} & , 0 \leq \psi_1 \leq \frac{[2]_q^2(1+\zeta)^2}{[3]_q(1+\zeta[2]_q)} \end{cases}, \tag{42}$$

and

$$\mathcal{K}(\zeta) = \begin{cases} \frac{|\psi_2|}{[3]_q(1+\zeta[2]_q)} & , |\psi_2| > \psi_1 \\ \frac{\psi_1}{[3]_q(1+\zeta[2]_q)} & , |\psi_2| \leq \psi_1 \end{cases}. \tag{43}$$

3. Future Work

The authors suggest to find upper bounds for the coefficients of function class $\sum_{\lambda,q}^m(\tau, \zeta; \phi)$ for all $\vartheta, \omega \in \mathbb{D}$:

$$1 + \frac{1}{\tau}\left[\nabla_q(\mathcal{D}_{\lambda,q}^m F(\varsigma)) + \zeta\varsigma\nabla_q(\nabla_q\mathcal{D}_{\lambda,q}^m F(\varsigma)) - 1\right] \prec \phi(\varsigma), \tag{44}$$

and

$$1 + \frac{1}{\tau}\left[\nabla_q(\mathcal{D}_{\lambda,q}^m g(\omega)) + \zeta\omega\nabla_q(\nabla_q\mathcal{D}_{\lambda,q}^m g(\omega)) - 1\right] \prec \phi(\omega), \tag{45}$$

where

$$\nabla_{\lambda,q}^m(F(\varsigma)) = \varsigma + \sum_{k=2}^{\infty}[1 + \lambda([k]_q - 1)]^m a_k\varsigma^k,\ \lambda \geq 0,\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{46}$$

is the $q-$ Al-Oboudi operator is defined by Aouf et al. [10].

4. Conclusions

Throughout the paper, we defined a new subclass of bi-univalent functions of complex order $\sum_q^n(\tau, \zeta; \phi)$ by using $\mathcal{D}_q^n F(\varsigma)$ operator. Furthermore, using the Faber polynomial expansions, we find the initial coefficient bounds for this function class.

Author Contribution Statements A.O. MOSTAFA, S. MOHAMED: Conceptualization, methodology, resources, review, editing and supervision; All Authors: validation, formal analysis, investigation; Z. NSAR: data acuration, writing, original draft preparation.

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A NEW APPROACH TO CURVE COUPLES WITH BISHOP FRAME

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Abstract. This paper presents a detailed study of a new generation of the Bishop frame with components including three orthogonal unit vectors, which are tangent vector, normal vector and binormal vector. It is a frame field described on a curve in Euclidean space, which is an alternative to the Frenet frame. It is useful for curves for which the second derivative is not available. Moreover, the conditions which the Bishop frame of one curve coincides with the Bishop frame of another curve are defined. It would be valuable to replicate similar approaches in the Bishop frame of one curve coincides with the Bishop frame of another curve.

1. Introduction

In 1975, Bishop [1] defined a frame which is called the Bishop frame as an alternative to Frenet frame. This frame is useful for the curves that the second derivative is not available. The Bishop frame consists of three orthogonal unit vectors. These vectors are tangent vector, normal vector and binormal vector. Unlike the Frenet frame, the Bishop frame does not require the second derivative of the curve to be defined. The curvature functions κ and τ give important information about a curve in the curve theory. For instance, for any curve, if κ and τ are zero, this curve is geodesic. So, with the help of their curvatures, we get information about the shape and properties of a curve. There are different methods for characterization of a curve [2–4]. The position vector of a curve is very important in these methods. The orthogonal coordinate system $\{T, N, B\}$ spanned by tangent (T), basic normal (N) and binormal (B) vectors at each point of a unit-speed curve. The coordinate system $\{T, N, B\}$ is called the Frenet frame. This frame is spanned by osculating plane, rectifying plane and normal plane. The osculating plane is spanned by

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$\{T, N\}$, the rectifying plane is spanned by $\{T, B\}$ and the normal plane is spanned by $\{N, B\}$, many researchers have done investigation that is related to classical differential geometry topics in the Euclidean space, Minkowski space, dual space, and a new version of Bishop frame and its related applications were given [5–12].

2. Preliminaries

In this section, a brief summary of the concept of Bishop frame is given to provide a background to understand the main idea and the results of this study. The Bishop frame consists of three orthonormal vectors that evolve along a curve, off eringanatural way to describe the curve's orientation in space. Unlike the Frenet-Serret frame, which relies on curvature and torsion defi ned for curves with non-zero curvature and non-vanishing speed, the Bishop frame's adaptability makes it particularly suited for analyzing curves with varying geometric behaviors. Our study develops this adaptability to investigate the cases associated with the tangential, normal, or binormal planes of two distinct curves.

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ is an arbitrary curve [13–15]. If α is unit speed curve, then $\langle \alpha'(s), \alpha'(s)\rangle = 1$, where s is arc length parameter and

$$\langle \gamma, v\rangle = \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3$$

is the standard scalar product of $\mathbb{E}^3$. Furthermore, $\|\gamma\| = \sqrt{\langle \gamma, \gamma\rangle}$ is the norm of γ. Bishop frame along the unit speed curve α is denoted by the tangent T, the principal normal N_1 and the binormal N_2 vector fields. Bishop derivative formulas are written as follows:

$$T' = k_1 N_1 + k_2 N_2,\ N_1' = -k_1 T,\ N_2' = -k_2 T$$

where we shall call the set T, N_1, N_2 as Bishop trihedra and k_1 and k_2 (k_2 is not equal to zero) as Bishop curvatures. For all $t \in I$, if a curve γ is considered spherical, then the curve $\gamma(t)$ lies on a sphere of radius $\mathbb{R}$, centered at the origin in $\mathbb{E}^3$. This can be mathematically expressed as $\|\gamma\| = \mathbb{R}$. The normal curves in $\mathbb{E}^3$ are spherical curves [16]. The Bishop frame, an alternative to the traditional Frenet frame, offers a more flexible representation of curves in $\mathbb{E}^3$ by eliminating the torsion component, thus providing a simpler and often more intuitive understanding of curve geometry. This research leverages the Bishop frame to examine the conditions under which two distinct curves in $\mathbb{E}^3$ can share a common plane in their respective Bishop frames. We denote the tangent vector of a curve at a point by T, and the principal normal vector by N_1. The notation $\bar{T} \in Sp\{T, N_1\}$ signifies that the modified tangent vector $\bar{T}$ of one curve lies within the plane spanned by the tangent and principal normal vectors of other curve. This study not only contributes a new perspective to the analysis of curves within Euclidean space but also underscores the utility and versatility of the Bishop frame in uncovering complex geometric relationships. Through a careful and thorough classification of the possible cases where curve pairs share a Bishop frame plane, combined with the derivation of several interesting results, we enhance the existing knowledge base in differential geometry and open the door to future investigations in this rich and captivating field.

3. On Characterizations of the Curve Couples and Bishop Frame

Let α and $\bar{\alpha}$ are any curves in 3-dimensional Euclidean space where they are described on the same open interval $I \subset \mathbb{R}$. The frames $\{T, N_1, N_2\}$ and $\{\bar{T}, \bar{N}_1, \bar{N}_2\}$ are Bishop frames of α and $\bar{\alpha}$, respectively. Let's denote the are length parameters, first Bishop curvatures and second Bishop curvatures for α and $\bar{\alpha}$ with s, k_1, k_2 and $\bar{s}, \bar{k}_1, \bar{k}_2$ respectively. $\{T, N_1\}, \{N_1, N_2\}, \{T, N_2\}$ planes are the osculating plane, the normal plane and the rectifying plane according to Bishop frame, respectively. These planes are donated, respectively, by OB, AB and CA. Let $\bar{\alpha}$ be a unit speed curve with first and second curvatures $\bar{k}_1, \bar{k}_2$ and Bishop vectors $\bar{T}, \bar{N}_1, \bar{N}_2$. Similarly, $\{\bar{T}, \bar{N}_1\}, \{\bar{N}_1, \bar{N}_2\}, \{\bar{T}, \bar{N}_2\}$ planes are the osculating plane, the normal plane and the rectifying plane according to Bishop frame. These planes are denoted with $\overline{OB}, \overline{AB}$ and $\overline{CA}$, respectively. If we take $r = \frac{d\bar{s}}{ds}$, Bishop formulas are

$$T' = k_1 N_1 + k_2 N_2,\ N_1' = -k_1 T,\ N_2' = -k_2 T \tag{1}$$

$$T' = \bar{k}_1 \bar{N}_1 + \bar{k}_2 \bar{N}_2,\ \bar{N}_1' = -\bar{k}_1 \bar{T},\ \bar{N}_2' = -\bar{k}_2 T \tag{2}$$

where $(')$ denotes $\frac{d}{ds}$. Classify whether any Bishop frame plane of a curve is the any Bishop frame plane of another curve or not and we will give the following cases and their related theorems, here a_1 and a_2 are the non-zero functions of the parameter s and the position vectors of $\bar{\alpha}$ and α are $\bar{\gamma}$ and γ.

Table 1. The cases of curve couples

Cases	Bishop plane for α	Bishop plane for $\bar{\alpha}$	Conditions
1	$\mathrm{sp}\{\mathrm{T}, \mathrm{N}_1\} = \mathrm{OB}$	$\mathrm{sp}\{\bar{T}, \bar{N}_1\} = \overline{OB}$	$\mathrm{OB} = \overline{OB}$
2	$\mathrm{sp}\{\mathrm{T}, \mathrm{N}_1\} = \mathrm{OB}$	$\mathrm{sp}\{\bar{N}_1, \bar{N}_2\} = \overline{AB}$	$\mathrm{OB} = \overline{AB}$
3	$\mathrm{sp}\{\mathrm{T}, \mathrm{N}_1\} = \mathrm{OB}$	$\mathrm{sp}\{\bar{T}, \bar{N}_2\} = \overline{CA}$	$\mathrm{OB} = \overline{CA}$
4	$\mathrm{sp}\{\mathrm{N}_1, \mathrm{N}_2\} = \mathrm{AB}$	$\mathrm{sp}\{\bar{T}, \bar{N}_1\} = \overline{OB}$	$\mathrm{AB} = \overline{OB}$
5	$\mathrm{sp}\{\mathrm{N}_1, \mathrm{N}_2\} = \mathrm{AB}$	$\mathrm{sp}\{\bar{N}_1, \bar{N}_2\} = \overline{AB}$	$\mathrm{AB} = \overline{AB}$
6	$\mathrm{sp}\{\mathrm{N}_1, \mathrm{N}_2\} = \mathrm{AB}$	$\mathrm{sp}\{\bar{T}, \bar{N}_2\} = \overline{CA}$	$\mathrm{AB} = \overline{CA}$
7	$\mathrm{sp}\{\mathrm{T}, \mathrm{N}_2\} = \mathrm{CA}$	$\mathrm{sp}\{\bar{T}, \bar{N}_1\} = \overline{OB}$	$\mathrm{CA} = \overline{OB}$
8	$\mathrm{sp}\{\mathrm{T}, \mathrm{N}_2\} = \mathrm{CA}$	$\mathrm{sp}\{\bar{N}_1, \bar{N}_2\} = \overline{AB}$	$\mathrm{CA} = \overline{AB}$
9	$\mathrm{sp}\{\mathrm{T}, \mathrm{N}_2\} = \mathrm{CA}$	$\mathrm{sp}\{\bar{T}, \bar{N}_2\} = \overline{CA}$	$\mathrm{CA} = \overline{CA}$

Now we investigate these possible cases step by step (see TABLE 1):

Case 1. *Assume that $OB = \overline{OB}$. The osculating plane according to Bishop frame plane of a curve α is the osculating plane according to Bishop frame plane of curve $\bar{\alpha}$.*

Theorem 1. *There isn't a curve pair* $(\alpha, \bar{\alpha})$ *in* $\mathbb{E}^3$ *such that the osculating plane according to Bishop frame of* α *is same osculating plane according to Bishop frame of* $\bar{\alpha}$.

Proof. Assume that the osculating plane according to Bishop frame of a curve α is the osculating plane according to Bishop frame of a other curve $\bar{\alpha}$. Then, the relation can be given that

$$\bar{\gamma} = \gamma + a_1 T + a_2 N_1, \ a_1 \neq 0, \ a_2 \neq 0 \tag{3}$$

where $\bar{\gamma}$ and γ are position vectors of the curves $\bar{\alpha}$ and α respectively, a_1 and a_2 are the non-zero functions of the parameter s. If we take the derivative of (3) with respect to s and by using Bishop formulas in (1) and (2), we have

$$\bar{T} = (1 + a_1' - a_2 k_1) \frac{1}{r} T + (a_1 k_1 + a_2') \frac{1}{r} N_1 + a_1 k_2 \frac{1}{r} N_2 \tag{4}$$

If we rewrite in (4) for constant $\lambda = (1 + a_1' - a_2 k_1) \frac{1}{r}$ and $\mu = (a_1 k_1 + a_2') \frac{1}{r}$ then we obtain

$$\bar{T} = \lambda T + \mu N_1 = (1 + a_1' - a_2 k_1) \frac{1}{r} T + (a_1 k_1 + a_2') \frac{1}{r} N_1 + a_1 k_2 \frac{1}{r} N_2 \tag{5}$$

where $\bar{T} \in Sp\{T, N_1\}$. If we multiply (5) by N_2, we find

$$a_1 k_2 \frac{1}{r} = 0.$$

Then, a_1 or k_2 must be zero. This situation contradicts our assumption. □

Case 2. *Suppose that* $OB = \overline{AB}$. *In the osculating plane according to Bishop frame plane of a curve is the normal plane according to Bishop frame plane of other curve.*

Theorem 2. *Let* α *be any unit speed curve in* $\mathbb{E}^3$ *with non-zero first Bishop curvature* k_1*, second Bishop curvature* k_2 *and Bishop vectors* T, N_1, N_2*. If the osculating plane according to Bishop frame of the curve* α *is the normal plane according to Bishop frame of the space curve* $\bar{\alpha}$*, then* $\bar{\alpha}$ *is written as*

$$\bar{\alpha} = \alpha + \frac{d\bar{s}}{ds} \frac{1}{k_2} T + \frac{1}{k_1} + \frac{1}{k_1 k_2^2} \left(\frac{d^2 \bar{s}}{ds^2} k_2 - \frac{d\bar{s}}{ds} k_2' \right) N_1.$$

Proof. Assume that the osculating plane according to Bishop frame of a curve α is the normal plane according to Bishop frame of a another curve $\bar{\alpha}$. Then, the relation can be given that

$$\bar{\gamma} = \gamma + a_1 T + a_2 N_1, \ a_1 \neq 0, \ a_2 \neq 0. \tag{6}$$

Taking the derivative of (6) with respect to s and apply (1) and (2). We have

$$\bar{T} = (1 + a_1' - a_2 k_1) \frac{1}{r} T + (a_1 k_1 + a_2') \frac{1}{r} N_1 + a_1 k_2 \frac{1}{r} N_2. \tag{7}$$

Since $N_2^{\perp} = Sp\{T, N_1\} = Sp\{\bar{N}_1, \bar{N}_2\} = \bar{T}^{\perp}$, N_2 and $\bar{T}$ are parallel. If we multiply 7 by N_2 and T, respectively, then we obtain

$$a_1 = \frac{r}{k_2}$$
$$a_2 = \frac{1}{k_1} + \frac{1}{k_1 k_2^2}\left(\frac{d^2\bar{s}}{ds^2}k_2 - r{k_2}'\right).$$

□

Case 3. *Assume that $OB = \overline{CA}$. The osculating plane according to Bishop frame plane of a curve is the rectifying plane according to Bishop frame plane of other curve.*

Theorem 3. *There isn't a curve pair $(\alpha, \bar{\alpha})$ in $\mathbb{E}^3$ such that the osculating plane according to Bishop frame of α is the rectifying plane according to Bishop frame of $\bar{\alpha}$.*

Proof. Let's assume that, the osculating plane according to Bishop frame of a curve α be the rectifying plane according to Bishop frame of the curve $\bar{\alpha}$. Then, the relation can be given that

$$\bar{\gamma} = \gamma + a_1 T + a_2 N_1,\ a_1 \neq 0,\ a_2 \neq 0.$$

Taking the derivative of the last equality with respect to s and apply (1) and (2) then, we get

$$\bar{T} = (1 + a_1' - a_2 k_1)\frac{1}{r}T + (a_1 k_1 + a_2')\frac{1}{r}N_1 + a_1 k_2 \frac{1}{r}N_2.$$

For some constant λ and μ, since $\bar{T} \in Sp\{T, N_1\}$, we can write

$$\bar{T} = \lambda T + \mu N_1 = (1 + a_1' - a_2 k_1)\frac{1}{r}T + (a_1 k_1 + a_2')\frac{1}{r}N_1 + a_1 k_2 \frac{1}{r}N_2 \quad (8)$$

if we multiply (8) by N_2, we obtain

$$a_1 k_2 \frac{1}{r} = 0$$

where a_1 or k_2 must be zero. This situation contradicts our assumption. □

Case 4. *Assume that $AB = \overline{OB}$. We examine the case that the normal plane according to Bishop frame plane of a curve is the osculating plane according to Bishop frame plane of other curve.*

Theorem 4. *Let α be any unit speed curve with non-zero first Bishop curvature k_1, second Bishop curvature k_2 and Bishop vectors T, N_1, N_2. If the normal plane according to Bishop frame of the curve α is the osculating plane according to Bishop frame of a other space curve $\bar{\alpha}$, then α is a spherical curve.*

Proof. Let's assume that, the normal plane according to Bishop frame of a curve α is the osculating plane according to Bishop frame of a other curve $\bar{\alpha}$. Then, the relation can be given that

$$\bar{\gamma} = \gamma + a_1 N_1 + a_2 N_2, \; a_1 \neq 0, \; a_2 \neq 0. \tag{9}$$

Let's take the derivative of (9) with respect to s and applying (1) and (2). Then, we get

$$\bar{T} = (1 - a_1 k_1 - a_2 k_2) \frac{1}{r} T + a_1' \frac{1}{r} N_1 + a_2' \frac{1}{r} N_2.$$

For some constant λ and μ, if we take $\bar{T} = \lambda_1 N_1 + \mu_1 N_2$ and $\bar{T} \in \mathrm{Sp}\{N_1, N_2\}$, we can write

$$\lambda_1 N_1 + \mu_1 N_2 = (1 - a_1 k_1 - a_2 k_2) \frac{1}{r} T + a_1' \frac{1}{r} N_1 + a_2' \frac{1}{r} N_2. \tag{10}$$

Now, if we multiply (10) by T, we have

$$a_1 k_1 + a_2 k_2 = 1.$$

Here the curve γ is a spherical curve. □

Case 5. *Assume that $AB = \overline{AB}$. The normal plane according to Bishop frame plane of a curve is the normal plane according to Bishop frame plane of other curve.*

Theorem 5. *Let's take any α and $\bar{\alpha}$ curves with non-zero first Bishop curvature k_1, non-zero second Bishop curvature k_2 in $\mathbb{E}^3$ and Bishop vectors T, N_1, N_2. If the normal plane according to Bishop frame of the curve α is the normal plane according to Bishop frame of a another curve $\bar{\alpha}$, then α is a spherical curve where $r \neq 1$.*

Proof. Let's assume that, the normal plane according to Bishop frame of a curve α be the normal plane according to Bishop frame of the curve $\bar{\alpha}$. Then, the relation can be given that

$$\bar{\gamma} = \gamma + a_1 N_1 + a_2 N_2, \; a \neq 0, \; b \neq 0 \tag{11}$$

Taking the derivative of (11) with respect to s and applying the Bishop formulas given in (1) and (2), we obtain

$$\bar{T} = (1 - a_1 k_1 - a_2 k_2) \frac{1}{r} T + a_1' \frac{1}{r} N_1 + b' \frac{1}{r} N_2. \tag{12}$$

T and $\bar{T}$ are parallel and $T^{\perp} = Sp\{N_1, N_2\} = Sp\{\bar{N}_1, \bar{N}_2\} = \bar{T}^{\perp}$. So, we write

$$< \bar{T}, T >= 1$$

and multiplying (12) by T, we have

$$a_1 k_1 + a_2 k_2 = 1 - r,$$

then, we can write

$$\frac{a_1}{c} k_1 + \frac{a_2}{c} k_2 = 1$$

where $1-r=c$ and $r\neq 1$. Then, the curve α is a spherical curve. □

Case 6. *Suppose that $AB=\overline{CA}$. Now, we give the normal plane according to Bishop frame plane of a curve is the rectifying plane according to Bishop frame plane of other curve.*

Theorem 6. *Let α be unit speed curve in $\mathbb{E}^3$ with non-zero first and second Bishop curvatures k_1,k_2, and Bishop vectors T,N_1,N_2. If the normal plane according to Bishop frame of α is the rectifying plane according to Bishop frame of the curve $\bar{\alpha}$, then α is a spherical curve.*

Proof. Let's assume that, the normal plane according to Bishop frame of a curve α be the rectifying plane according to Bishop frame of another curve $\bar{\alpha}$. Then, the relation can be given that

$$\bar{\gamma}=\gamma+a_1N_1+a_2N_2,\ a_1\neq 0,\ a_2\neq 0.$$

Let's take the derivative of the last equality with respect to s and apply in (1) and (2). Then, we get

$$\bar{T}=(1-a_1k_1-a_2k_2)\frac{1}{r}T+a_1'\frac{1}{r}N_1+a_2'\frac{1}{r}N_2. \tag{13}$$

For some constants λ and μ, we write $\bar{T}=\lambda N_1+\mu N_2$ and $\bar{T}\in \mathrm{Sp}\{N_1,N_2\}$. From (13), we get

$$\lambda N_1+\mu N_2=(1-a_1k_1-a_2k_2)\frac{1}{r}T+a_1'\frac{1}{r}N_1+a_2'\frac{1}{r}N_2. \tag{14}$$

By multiplying (14) by T, we have

$$a_1k_1+a_2k_2=1.$$

□

Case 7. *Let us consider $CA=\overline{OB}$. The rectifying plane according to Bishop frame of a curve α is the osculating plane according to Bishop frame of other curve $\bar{\alpha}$. Then, the relation can be given that*

$$\bar{\gamma}=\gamma+a_1T+a_2N_1,\ a_1\neq 0,\ a_2\neq 0. \tag{15}$$

By taken the derivative of (15) with respect to s and apply in (1) and (2). We will have

$$\bar{T}=\left[(1+a_1'-k_1a_2)\,T+(a_1k_1+a_2')\,N_1+a_1k_2N_2\right]\frac{1}{r} \tag{16}$$

where $\bar{T}=\lambda T+\mu N_2$ and for some constant λ and μ, since $\bar{T}\in \mathrm{Sp}\{T,N_2\}$. From (16), we obtain

$$\lambda T+\mu N_2=\left[(1+a_1'-k_1a_2)\,T+(a_1k_1+a_2')\,N_1+a_1k_2N_2\right]\frac{1}{r}. \tag{17}$$

Then multiplying (17) by N_1, we get

$$a_1k_1+a_2'=0.$$

Case 8. *Assume that* $CA = \overline{AB}$*. The rectifying plane according to Bishop frame plane of a curve is the normal plane according to Bishop frame plane of other curve.*

Theorem 7. *Let* α *be unit speed curve in* $\mathbb{E}^3$ *with non-zero first and second Bishop curvatures* k_1, k_2 *and Bishop vectors* T, N_1, N_2*. If the rectifying plane according to Bishop frame of* α *is the normal plane according to Bishop frame of other curve* $\overline{\alpha}$*, then we can write*

$$\overline{\alpha} = \alpha + \frac{1}{k_1}T + [\frac{1}{k_2} + \frac{1}{k_2}(\frac{1}{k_1})']N_2.$$

Proof. Assume that, the rectifying plane according to Bishop frame of a curve α be the normal plane according to Bishop frame of the curve $\overline{\alpha}$. Then, the relation can be given as

$$\overline{\gamma} = \gamma + a_1T + a_2N_2, \ a_1 \neq 0, \ a_2 \neq 0 \tag{18}$$

Taking the derivative of (18) with respect to s and applying the (1) and (2), then, we have

$$\overline{T} = \ [(1 + a_1' - a_2k_2)T + a_1k_1N_1 + (a_1k_2 + a_2')N_2] \ \frac{1}{r}. \tag{19}$$

N_1 and $\overline{T}$ are parallel. Since $N^{\perp} = Sp\{T, \ N_2\} = Sp\{\overline{N}_1, \overline{N}_2\} = \overline{T}^{\perp}$, multiplying (19) by N_1, we get

$$a_1 = \frac{1}{k_1}.$$

If we Multiply (19) by T, we get

$$1 + a_1' - a_2k_2 = 0,$$

$$b = \frac{1}{k_2} + \frac{1}{k_2}(\frac{1}{k_1})'.$$

□

Case 9. *Assume that* $CA = \overline{CA}$*. The rectifying plane according to Bishop frame plane of a curve is the rectifying plane according to Bishop frame plane of other curve.*

Theorem 8. *There isn't a curve pair* $(\alpha, \overline{\alpha})$ *in* $\mathbb{E}^3$ *such that the rectifying plane according to Bishop frame of* α *is the rectifying plane according to Bishop frame of* $\overline{\alpha}$*.*

Proof. Assume that the rectifying plane according to Bishop frame of a curve α be the rectifying plane according to Bishop frame of other curve $\overline{\alpha}$. Then, the relation can be given that

$$\overline{\gamma} = \gamma + a_1T + a_2N_2, \ a_1 \neq 0, \ a_2 \neq 0. \tag{20}$$

Taking the derivative of (20) with respect to s and using (1) and (2), then, we write

$$\overline{T} = \ [(1 + a' - bk_2)T + ak_1N_1 + (ak_2 + b')N_2] \ \frac{1}{r}. \tag{21}$$

For some constant λ and μ, we can write $\overline{T} = \lambda T + \mu N_2$. Since $\overline{T} \in Sp\{T,\ N_2\}$, by using (21), we obtain

$$\lambda T + \mu N_2 = [(1 + a_1' - a_2 k_2)T + a_1 k_1 N_1 + (a_1 k_2 + a_2')N_2]\ \frac{1}{r}. \tag{22}$$

If we multiply (22) by N_1, we obtain $a_1 k_1 \frac{1}{r} = 0$ and $a_1 = 0$ or $k_1 = 0$. □

4. Conclusion

Bishop frame is considered as an alternative to Frenet frame in 3-dimensional Euclidean space. This idea is a useful method for curves where the second derivative is not available. Here, the conditions for Bishop frame plane to be the same as Bishop frame plane of another curve for a curve are investigated in 3 dimensional Euclidean space. Thus, nine possible cases are examined. The results found extend previous studies. The Bishop frame provides an alternative framework for studying curves, and the results of this paper demonstrate the usefulness of the Bishop frame in this context.

Overall, this paper provides new insights into the geometry of curves in Euclidean space, and highlights the importance of alternative frames such as the Bishop frame in studying curves.

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BIOMEDICAL MODELLING THROUGH PATH ANALYSIS APPROACH

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Abstract. Since blood disease markers are one of the most prevalent health problems in this era, the aim of this study is to forecast pathological subjects from a population through biomedical variables of individuals using the currently produced path analysis (PA) model. In terms of the dataset, 539 subjects were used to implement this study. A mathematical approach based on the PA has been used to create a reliable biomedical model in this research that investigates if there exists a relation between the various anemia types and the biomedical variables through observational variables (the blood variables, age, and sex) and anemia types. Other linear approaches were taken into consideration for comparison, in terms of R^2 value of the model, which has a value of 0.699. The findings reveal that the model has great predictive potential. It is believed that the developed model, which includes observational variables, will help healthcare providers predictively plan appropriate treatment programs for their patients.

1. Introduction

Medical models are frequently used in many healthcare processes and also as a tool for analyzing pathological features. Modelling has become a substantial tool in studies of prediction because of its ease of interpretation for pathological data; therefore, these studies commonly use mathematical modeling to portray the interrelationship among the multiple variables in a mathematical equation. According

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to the research [1–5], how effectively any aim can be achieved depends on knowledge about the problem and how well the modeling is done. Any disease state has different effects for a single disease, which may be used to evaluate the circumstances shown in hospitals. As a result, several endogenous variables determine the majority of outcomes in real-life problems.

The purpose of this study is to use biomedical variables and anemia types to predict pathological people in a population and know the relationship between blood variables. The literature [6, 7] indicates that there has been an increase in anemia among various segments of the population of the community; thus, it is crucial to predict the anemia types. Therefore, our aim is to develop a new biomedical model to study the effect of the blood variables, sex, and age together and their effect on the anemia types.

Our model differs from the other models contained in the literature [8–10] and has been successfully used to predict several anemia types using a wide range of blood variables, sex, and age.

The previous studies [8, 9, 11–13] produced relatively less accurate results by using a relatively small or limited number of blood variables to predict anemia types or a small number of anemia types.

The PA is a generalization of multiple regression that allows one to estimate the strength and sign of directional relationships for complicated causal schemes with multiple dependent variables. A major advantage of the PA is that, in addition to studying the direct effects, indirect effects through intervening variables can be studied, as can estimating the values of coefficients in the model, adding non-recurring paths, reporting additional appropriate indicators, and estimating residuals and their potential relationships with each other. Also, it is seen when there are two or more dependent variables. Therefore, a great deal of research has been done in a variety of fields using the PA found in the literature. Examples of these include the COVID-19 emergency [14], stability analysis and an epidemic model [15], odds of elevated systolic blood pressure [16], blood-associated parasites [17], predicting blood donation behavior among donors [18], the relationship between iron deficiency anemia and blood cadmium and vitamin D [19], the relationship between blood pressure and the structures of the health promotion model [20], and assessing the healthcare empowerment model among HIV-positive individuals [21]. In this study, a realistic model has been developed that allows the best relationship between blood variables to be found through the PA by using many input variables. Because there are more input variables in the created model, it is therefore more realistic in the field of biomedicine. Therefore, the primary goal of the current study is to identify the different forms of anemia utilizing a large number of practical observational characteristics. As a result, it is believed that this study significantly advances the understanding of anemia types

2. Materials and Methods

2.1. **Collection of the study samples.** Here the samples for people and for each subject readings of blood variables are [22,23] Hemoglobin (HB), Red Blood Cells (RBC), White Blood Cell (WBC), Mean Corpuscular Volume (MCV), Mean Corpuscular Hemoglobin (MCH), Hematocrit (HCT), Mean Corpuscular Hemoglobin Concentration (MCHC), Platelets (PLT) and sex and age. The following explanation applies to the blood variables. Within the RBC, the HB is a transportable protein that is made of iron atoms. The worthless nucleus of the RBCs, which are concave cells, contains the HB. The computed result obtained from the HB measurement and a few red cells is the MCH. The immune system cells known as WBCs are responsible for defending the body against infectious diseases. The volume of RBCs in total blood volume divided by the percentage is the HCT. The estimated amount of HB in a particular volume of RBC is known as the MCHC. The PLT is a haphazard, disc-shaped component of blood that promotes blood coagulation. The MCV calculates the average size of the red blood cells in a sample.

Table 1 displays the many forms of anemia and the biological variables used to collect the data. Table 3 additionally lists the many forms of anemia.

Table 1. Some study samples of the data

HB	RBC	MCH	WBC	MCV	HCT	MCHC	PLT	Sex	Age	Anemia type
11	4.16	26.5	7.7	88	36.6	30.1	180	2	10	0
16.3	6.07	26.9	8.16	80.9	49.1	33.2	349	1	23	0
14.1	4.5	31.2	5.5	90	40.3	34.9	198	1	56	0
11	4.86	22.6	13.6	78	37.7	29.2	482	2	6	1
5.79	9.3	16	14.6	60	34.8	26.6	411	1	16	1
4.65	10.2	21.9	6.7	77	35.8	28.5	409	2	48	1
4.1	1.13	36.2	1.6	114	12.8	31.8	6	2	9	2
10.4	3.7	28	9.2	91	33.7	30.7	607	1	16	2
1.46	4.4	30.4	59.8	108	15.8	28.2	330	2	29	2
7.4	3.1	23.9	5.37	70.6	21.9	33.8	233	1	6	3
10.6	4.78	22.1	16.4	76	36.2	29.2	351	2	27	3
5.4	1.92	28	21.2	94	18	29.9	107	1	54	3
8.3	2.58	31.9	12.4	103	26.7	30.9	458	1	11	4
2.73	8.9	32.6	21.78	94.9	25.9	34.4	546	2	20	4
2.66	8.2	30.8	15.7	90.6	24.1	34	437	1	29	4
11.5	4.63	24.9	7	78	35.9	32	180	1	7	5
7.3	2.06	35.4	15.6	117	24.2	30.2	478	2	16	5
4.41	11	24.9	8.67	73.5	32.4	34	280	1	37	5

The purpose of the paper is to predict diseased people from a community using various biomedical data. In order to determine whether a subject was healthy or infected, data were collected through observations of blood variables, and 539 patients provided from the work of Sari and Ahmad [2,24], we ran investigative analysis using SPSS, AMOS. Some related calculated variables are shown and the anemia types and the number of individuals (see Tables 2, 3)

Table 2. Descriptive statistics about the values of the sample

Parameters	HB	RBC	MCH	WBC	MCV	HCT	MCHC	PLT	Sex	Age
Average	9.59	5.10	26.67	13.18	82.60	33.18	32.08	363.26	1.54	21.48
Min.	1.46	0.96	11.7	1.6	38.6	7.7	22.6	2	1	6
Max.	18.2	11.9	77	146.1	117	51.7	60.5	1892	2	56
Standard deviation	4.34	2.26	4.56	16.23	8.71	9.01	3.15	210.13	0.499	10.65
Sig.	0.000	0.001	0.054	0.549	0.988	0.565	0.643	0.009	0.000	0.022

Table 3. The anemia types and their population

Parameters	Type0	Type1	Type2	Type3	Type4	Type5
Definition	Healthy	Iron Deficiency Anemia	Deficiency Vitamin B12	Thalassemia	Sickle Cell	Spherocytosis
No. of subjects	211	83	9	217	10	9

2.2. **Path Analysis.** The directed dependencies between a group of variables are described by the PA, which was created as a technique for examining the direct and indirect impacts of variables. Furthermore, the PA is viewed as a type of multiple regression that focuses on causation; its goal is to clarify the plausibility of the causal models that researchers develop using their theoretical understanding and knowledge, rather than to identify causes.

In path models, the independent and dependent variables are shown as boxes or rectangles. Single-headed arrows are emanating from the exogenous; a double-headed arrow signifies that the variables are only correlated. 'Endogenous' variables are those that are dependent variables. There is one or more single-headed arrows pointing at endogenous variables [25].

While determining the amount and direction of the relation between two or more variables by correlation analysis, the mathematical structure of the relation is determined by regression analysis. For this reason, correlation and regression analysis are often used to examine causal relationships. However, in some cases, these methods are often insufficient to reveal the relationship between variables. In multivariate regression analysis, this situation becomes somewhat complicated. Entering independent variables with a high correlation between them can distort the importance of the model, or make transactions in the model meaningless, or either make the effect on the model more or less necessary or make it act against the outcome of the correlation. Because of this, it is critical to assess the structural relationship between quantitative variables and illustrate the connections between independent variables by analyzing the pathway.

3. Results and Discussion

According to the literature, various strategies of many approaches are used to analyze blood variables [8, 9, 12, 26, 27]. Many researchers [28–34] have taken into account the multiple regression analysis while addressing various anemia problems at various levels. Also, the PA has been studied of blood disease in the literature [16, 17, 19, 20]. Although they investigated a relationship for the prediction of various types of anemia, they only used a few blood factors and were aware of the relationship between them. As a result, the current study focuses on the relationship between a number of observational variables and different forms of anemia. However, some of the observed factors have a greater effect than others.

The correlation analysis was carried out to determine the link between blood variables. Pearson correlation analysis was used to determine the relationship between blood variables. The connections between the blood variables were as follows: HB and RBC (-0.307), HB and MCH (0.420), HB and WBC (-0.276), HB and MCV (0.195), HB and HCT (0.807), HB and MCHC (0.504), HB and PLT (-0.335), HB and sex (-0.2), HB and age (0.258), and other blood variables. Pearson correlation analysis was performed to examine whether our data is suitable for path analysis or to understand the relationship between variables (see Table 4 and Figure 1).

TABLE 4. Correlations relationship between the blood variables

Independent variables	HB	RBC	MCH	WBC	MCV	HCT	MCHC	PLT	Sex	Age
HB	1.000	-0.307***	0.420***	-0.276***	0.195***	0.807***	0.504***	-0.335***	-0.200***	0.258***
RBC	-0.307***	1.000	-0.131**	0.016	-0.050	0.268***	-0.178***	0.015	0.202***	0.323***
MCH	0.420***	-0.131**	1.000	-0.096*	0.570***	0.266***	0.564***	-0.211***	-0.126**	0.197***
WBC	-0.276***	0.016	-0.096*	1.000	0.050	-0.249***	-0.183***	0.574***	0.006	-0.077
MCV	0.195***	-0.050	0.570***	0.050	1.000	0.254***	0.231***	-0.113**	-0.034	0.149***
HCT	0.807***	0.268***	0.266***	-0.249***	0.254***	1.000	0.310***	-0.322***	-0.051	0.434***
MCHC	0.504***	-0.178***	0.564***	-0.183***	0.231***	0.310***	1.000	-0.221***	-0.234***	0.265***
PLT	-0.335***	0.015	-0.211***	0.574***	-0.113**	-0.322***	-0.221***	1.000	-0.013	-0.113**
sex	-0.200***	0.202***	-0.126**	0.006	-0.034	-0.051	-0.234***	-0.013	1.000	-0.049
age	0.258***	0.323***	0.197***	-0.077	0.149***	0.434***	0.265***	-0.113**	-0.049	1.000

$*P < 0.05, **p < 0.01$

Some blood variables had a beneficial effect on anemia types, whilst others had a negative effect. That is, the value of the anemia will rise as the value of the variable rises, and fall as the value of the variable declines. As a result, the standardized coefficient assesses the relative impact of each blood characteristic, sex, and age on different anemia types. It is thus given by $Standard\ Estimate_j = B_j * SD(X_j)/SD(Y)$.

In comparison to the other variables, the HB absolute value of the coefficient, which is (-0.663), has the largest correlation with the disease categories, i.e., when HB goes up by 1 standard deviation, anemia types go down by 0.663 standard deviations, also, the other blood variables. The Standard Estimate value for the HB, means that the dependent variable will vary by the Standard Estimate coefficient value for every change in the HB (see Table 5).

The regression weight for HB in the prediction of anemia types is significantly different from zero at the 0.001 level. Also, RBC and sex are significantly different from zero at the 0.001 level, PLT at the 0.01 level and age at the 0.05 level. As for MCH, WBC, MCV, HCT, and MCHC, the regression weight in the prediction of anemia types is not significantly different from zero at the 0.05 level (see Table 5). As for the regression weight estimate, when HB goes up by 1, anemia types go down by 0.224. In addition, when RBC, MCH, HCT, sex, and age go up by 1, anemia types go down by (0.224), (0.029), (0.016), (0.311), and (0.009), respectively. Also,

Table 5. Regression weights of the blood variables and Anemia by using the PA

	Estimate	Standard Estimate	S.E.	C.R.	P
Diseases < − − − HB	-0.224	-0.663	0.062	-3.615	***
Diseases < − − − RBC	-0.224	-0.345	0.065	-3.424	***
Diseases < − − − MCH	-0.029	-0.090	0.015	-1.949	0.051
Diseases < − − − WBC	0.001	0.016	0.003	0.554	0.580
Diseases < − − − MCV	0.000	-0.001	0.007	-0.015	0.988
Diseases < − − − HCT	-0.016	-0.100	0.028	-0.581	0.561
Diseases < − − − MCHC	0.007	0.016	0.016	0.469	0.639
Diseases < − − − PLT	0.001	0.080	0.000	2.662	0.008
Diseases < − − − sex	-0.311	-0.106	0.073	-4.231	***
Diseases < − − − age	-0.009	-0.065	0.004	-2.325	0.020

when WBC, MCV, MCHC, and PLT go up by 1, anemia types go up by (0.001), (0.000), (0.007), and (0.001), respectively.

The P values were applied to measure the partial effect of the observations of blood variables, sex, and age on the various anemia types compare with $p < 0.05$. The biomedical variables have been seen to affect the various anemia types but in varying rates (see Table 5).

The critical ratio (C.R.) is equal to the parameter estimate divided by the parameter's standard error estimate. This statistic has a conventional normal distribution under the null hypothesis that the parameter has a population value of zero if the necessary distributional assumptions are made. The critical ratio was used to calculate the partial effect of age, sex, and variables on the various anemia types. These biomedical variables have been demonstrated to have differing degrees of effect on the various anemia types (see Table 5).

As for the standard error, the regression weight estimate, -0.224, -0.224, -0.029, 0.001, 0.000, -0.016, 0.007, 0.001, -0.311, and -0.009 have a standard error of about 0.062, 0.065, 0.015, 0.003, 0.007, 0.028, 0.016, 0.000, 0.073, and 0.004, respectively (see Table 5). This means that each blood variable has a different effect than the other on anemia types.

Variance is a statistical measure of how much a set of observations differ from each other, it measures how far a data set is spread out. The variance of HB is estimated to be 18.831, has a standard error of about 1.148. In addition, the variance of RBC, MCH, WBC, MCV, HCT, MCHC, PLT, sex, and age are estimated and have a standard error (see Table 6). The results have been shown that the variance estimate for blood variables is significantly different from zero at the 0.001 level.

In the outcome of the current PA, the biomedical model has been found to be highly effective, on the prediction of the various anemia types, it is 0.699 of the model. Which clarify 69.90% of the change in the relationship of the biomedical model between all the observational variables and the various anemia types. In other words, the error variance of anemia types is approximately 30.1% (see Table 7). As a result, it is concluded that the model including the blood variables, sex,

Table 6. Variances

	Estimate	S.E.	C.R.	P
HB	18.831	1.148	16.401	***
RBC	5.113	0.312	16.401	***
MCH	20.777	1.267	16.401	***
WBC	263.233	16.050	16.401	***
MCV	75.850	4.625	16.401	***
HCT	81.030	4.941	16.401	***
MCHC	9.966	0.608	16.401	***
PLT	44075.983	2687.360	16.401	***
sex	0.248	0.015	16.401	***
age	113.296	6.908	16.401	***
e	0.645	0.039	16.401	***

and age is significant ($p < 0.000$). By correlating the independent variables among themselves, the model achieved similar results in this study.

In this study, the concept was used principle of whether the approach provides an acceptable prediction or not. In comparison to the other approaches [2,24,34], the findings show that the PA has a good fit for the initial dataset (see Table 7). Therefore, the current study offers a reliable model for predicting the various anemia types.

Table 7. Comparison of the PA results with the other approaches

Approaches	R^2
PA	0.699
Particle Swarm Optimization [24]	0.699
Linear Regression Analysis [2,34]	0.699
Linear Deep Learning Methods (LSTM) [2]	0.695

LSTM: Long Short-Term Memory

To know how significant the model, results were investigated of the root mean square residual (RMR) is the square root of the average squared amount by which the sample variances and covariances differ from their estimates obtained for the model. Therefore, the smaller the RMR is indicating the better fit. Thus, the result shows that, according to RMR=0.001 (see Table 8). However, value for the normed fit index (NFI) should range between 0 and 1, NFI values close to 1 indicate a perfect fit.

Incremental Fit Index (IFI): adjusts the NFI for sample size and degrees of freedom. IFI values close to 1 indicate a good fit. The comparative fit index (CFI) analyzes the model fit by examining the discrepancy between the data and the hypothesized model. CFI values range from 0 to 1, CFI values close to 1 indicate a very good fit. Thus, the results show that each from NFI, IFI, and CFI is equal to 1.000. Therefore, the model is considered a suitable fit (see Table 8).

Table 8. Model fit

CMIN	RMR	NFI	IFI	CFI
0.000	0.001	1.000	1.000	1.000

Since blood variables and anemia types are related to each other, blood variables are displayed double-headed arrows between boxes. As the blood variables affect anemia types, therefore, the model has consisted of single-headed arrows when creating the model. The coefficients obtained from the PA are displayed in the model (see Figure 1).

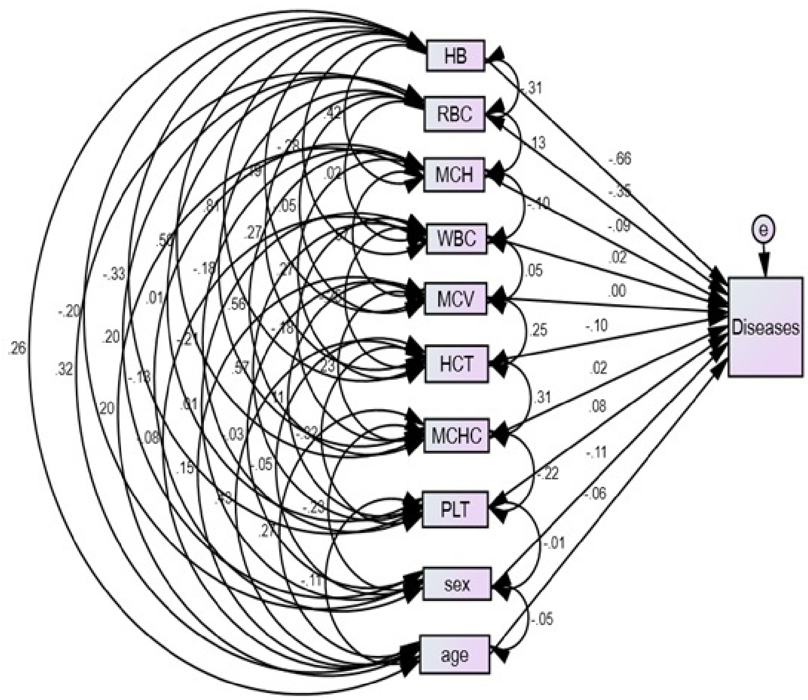

Figure 1. Diagram of the PA for prediction of anemia

4. Conclusions and Future Research

This study has predicted the anemia types through biomedical information (the blood variables, age, and sex of individuals). A mathematical method based on the PA has been implemented for the first time because it is a form of multiple regression centering on causality and sheds light on the tenability of the causal models a researcher formulates based on knowledge and theoretical considerations. So, it was achieved to develop a biomedical model that investigates whether there is a relationship between the various anemia types and the blood variables and finds the best relationship between biomedical variables through the PA in predicting the anemia types. The outcomes showed that the present biomedical model is highly promising and capable of making predictions. In analyzing the present anemia problem, the PA approach has been discovered to be significant compared to other linear methods. It has been concluded that the biomedical model is predicted to be beneficial for the diagnosis of the various anemia types and the provision of effective treatment plans for their patients. This model could be improved in further studies by taking into account various statistical techniques and apply the model to other data and compare it.

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SECOND-ORDER HANKEL DETERMINANT FOR A SUBCLASS OF ANALYTIC FUNCTIONS SATISFYING SUBORDINATION CONDITION CONNECTED WITH MODIFIED q-OPOOLA DERIVATIVE OPERATOR

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Abstract. This paper introduces a new subclass of analytic functions employing the operator that was recently defined by the authors. The coefficients estimate $|a_s|$ $(s = 2, 3)$ of the Taylor-Maclaurin series in this new class, as well as the Fekete-Szegö functional problems, have been derived. Furthermore, we obtained the sharp upper bound for the functional $|\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2|$ for functions belonging to this new subclass.

1. Introduction

By $\mathcal{A}$, we express the functions class f of the form

$$f(\mathbf{z}) = \mathbf{z} + \sum_{s=2}^{\infty} \mathfrak{a}_s \mathbf{z}^s, \tag{1}$$

which are considered analytic with respect to the symmetric open unit disk $\mathbb{U} = \{\mathbf{z} \in \mathbb{C} : |\mathbf{z}| < 1\}$, with the normalization conditions given by $f(0) = f'(0) - 1 = 0$. Furthermore, we denote $\mathcal{S}$ as the subclass of $\mathcal{A}$, which are univalent in $\mathbb{U}$.

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Let $\mathfrak{P}$ be the family of all analytic functions p having positive real parts, given by

$$p(\mathbf{z}) = 1 + \sum_{s=1}^{\infty} \mathfrak{d}_s \mathbf{z}^s \quad , (\Re e\{\mathfrak{d}_s\} > 0, \, \mathbf{z} \in \mathbb{U}) \, . \tag{2}$$

A number of subclasses with respect to normalized analytic functions is studied as part of Geometric Function Theory.

The concept of quantum calculus, also known as q-calculus, has played a significant role in the advancement of Geometric Function Theory (GFT) and its extensive application in diverse fields, including mathematical science and quantum physics. For analyzing a variety of subclasses, q-calculus technique are essential. In Geometric Function Theory, the fundamental q-hypergeometric functions were initially applied by Srivastava and Owa (1989), who also provided a clear foundation for employing calculus inside this theory.

Additionally, using q-calculus theory, it is possible to express univalent function theory. More recently, the use of a fractional q-derivative operator has been observed in the creation of numerous families of analytic functions (for example, in Alsoboh and Darus [8], Elhaddad and Darus [10,11], Mahmood and Darus [23]). For instance, Purohit and Raina [28] investigated the usage of q-fractional operators with respect to defining several analytic function classes for $\mathbb{U}$ as an open unit disc. Meanwhile, Mohammed and Darus [23] assessed properties q-analogue operator with respect to approximation and geometry concerning specific families of analytic function within the compact disc. A rather comprehensive analysis of applied q-analysis in the theory of operators can be discovered in Aral et al. [9] and Exton [12], also see ([7], [14], [15], [16], [17], [29], [31], [34]) for further studies.
The k^{th} Hankel determinant was explored by Noonan and Thomas [24] in 1976, which is expressed as

$$H_k(s) = \begin{vmatrix} \mathfrak{a}_s & \mathfrak{a}_{s+1} & \mathfrak{a}_{s+2} & \cdots & \mathfrak{a}_{s+k+1} \\ \mathfrak{a}_{s+1} & \mathfrak{a}_{s+2} & \mathfrak{a}_{s+3} & \cdots & \mathfrak{a}_{s+k+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathfrak{a}_{s+k-1} & \mathfrak{a}_{s+k} & \mathfrak{a}_{s+k+1} & \cdots & \mathfrak{a}_{s+2k-2} \end{vmatrix}, \qquad (s, k \in \mathbb{N}).$$

This determinant has garnered significant attention from several researchers. The rate of growth of $H_k(s)$ as s tends to ∞ was determined by Noor [25] with bounded boundary. For $k = 2$ and $s = 1$, we have $H_2(1) = |\mathfrak{a}_3 - \mathfrak{a}_2^2|$, which is well-known by Fekete–Szegö functional, and this may be generalized to $|\mathfrak{a}_3 - \mu\mathfrak{a}_2^2|$ for $(\mu \in \mathbb{C})$ (see, for example, [8,10]). For $k = 2$ and $s = 2$, we obtain the second Hankel determinant $H_2(2) = |\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2|$.

Determining the upper bounds for $H_2(2)$ attracts the attention of many authors who have determined several families of analytic functions. In 1967, Pommerenke [27] estimated the sharp upper bounds for the class $\mathcal{A}$. Some recent applications are studied by Abubaker and Darus [1], Ullah et al. [33], and Elhaddad and Darus [11].

Several authors have investigated it before, which may be referred to ([2,19,22,32,34,35,37]).

More recently, Alatawi and Darus [5,6] provided the new q-derivative operator $\mathsf{D}_q^n(\mu,\beta,\eta,t)f(\mathbf{z}):\mathcal{A}\to\mathcal{A}$, which is a modified Opoola operator as follows:

$$\mathsf{D}_q^n(\mu,\beta,\eta,t)f(\mathbf{z})=\mathbf{z}+\sum_{s=2}^{\infty}\Omega_s^n(\eta,\beta,\mu,t)\mathfrak{a}_s\mathbf{z}^s, \tag{3}$$

where

$$\Omega_s^n(\eta,\beta,\mu,t)=\Big[\eta+\Big([s]_q+\beta-\mu-\eta\Big)t\Big]^n,$$

where $n\in\mathbb{N}_0$, $t\geq 0$ and $1\leq\mu+\eta\leq\beta$.

Remark 1. *Some special operators are listed here:*

(1) *When $q\to 1^-$ and $\eta=1$, then $\mathsf{D}_q^n(\mu,\beta,\eta,t)f(\mathbf{z})$ becomes the Opoola differential operator [26].*
(2) *When $q\to 1^-$, $t=1$ and $\mu=\beta$, then $\mathsf{D}_q^n(\mu,\beta,\eta,t)f(\mathbf{z})$ becomes the Sãlãgean differential operator [30].*
(3) *When $t=1$ and $\mu=\beta$, then $\mathsf{D}_q^n(\mu,\beta,\eta,t)f(\mathbf{z})$ becomes the q- Sãlãgean differential operator [13].*
(4) *When $q\to 1^-$, $\mu=\beta$ and $\eta=1$, then $\mathsf{D}_q^n(\mu,\beta,\eta,t)f(\mathbf{z})$ becomes the Al-Oboudi differential operator [4].*

Definition 1. *Let f be given by (1). Hence, $f\in\mathcal{L}_{q,b}^n(\mu,\beta,\eta,t)$ if it complies with the inequality condition given below*

$$\mathfrak{Re}\Big\{\partial_q\mathsf{D}_q^n(\mu,\beta,\eta,t)f(\mathbf{z})\Big\}>0,\quad(\mathbf{z}\in\mathbb{U}). \tag{4}$$

If $q\to 1-$ and $n=0$, then the subclass $\mathcal{L}_{q,b}^n(\mu,\beta,\eta,t)$ is reduced to the class of positive real parts, denoted by $\mathcal{R}$, which was created by MacGregor [21] then studied by Janteng et al. [18].

To demonstrate our main findings, we require the lemmas as expressed below:

Lemma 1. *[20] Let $p\in\mathfrak{P}$ as is in (2), then $|\mathfrak{d}_2-\nu\mathfrak{d}_1^2|\leq 2\max\{1,|2\nu-1|\}$ and the sharpness result of the functions given by*

$$p(\mathbf{z})=\frac{1+\mathbf{z}^2}{1-\mathbf{z}^2},\quad p(\mathbf{z})=\frac{1+\mathbf{z}}{1-\mathbf{z}}.$$

.

Lemma 2. *[27] Suppose $p\in\mathfrak{P}$ given by (2), therefore $|\mathfrak{d}_m|\leq 2$ for all $m\geq 1$.*

Lemma 3. *[19] Let $p\in\mathfrak{P}$ as in (2), then*

$$2\mathfrak{d}_2=\mathfrak{d}_1^2+x(4-\mathfrak{d}_1^2),\quad |x|<1 \tag{5}$$

and

$$4\mathfrak{d}_3=\mathfrak{d}_1^3+2x(4-\mathfrak{d}_1^2)\mathfrak{d}_1-\mathfrak{d}_1(4-\mathfrak{d}_1^2)x^2+2(4-\mathfrak{d}_1^2)(1-|x|^2)\mathbf{z},\quad |\mathbf{z}|<1. \tag{6}$$

In this current work, we determine the sharp upper limits with respect to $H_2(2)$ for the class of analytic functions $\mathcal{L}^n_{q,b}(\mu,\beta,\eta,t)$ as follows.

2. Main Results

In our first theorem, motivated by the result of Zaprawa [36], we determine the coefficients estimate $|a_s|$ $(s = 2,3)$ of the Taylor-Maclaurin series in this new class, as well as the Fekete-Szegö functional problems for functions in $\mathcal{L}^n_{q,b}(\mu,\beta,\eta,t)$.

Theorem 1. *If $f \in \mathcal{L}^n_{q,b}(\mu,\beta,\eta,t)$, then*

$$\left|\mathfrak{a}_2\right| \leq \frac{2}{[2]_q \left(\eta + \left([2]_q + \beta - \mu - \eta\right)t\right)^n},$$

$$\left|\mathfrak{a}_3\right| \leq \frac{2}{[3]_q \left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n},$$

and

$$|a_3 - \aleph a_2^2| \leq \frac{2}{[3]_q \left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n} \max\left\{1; \frac{2\aleph[3]_q \left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n}{[2]_q^2 \left(\eta + \left([2]_q + \beta - \mu - \eta\right)t\right)^{2n}} - 1\right\}.$$

The best possible result is achieved by Köebe function.

Proof. Since $f \in \mathcal{L}^n_{q,b}(\mu,\beta,\eta,t)$. From (3), we have

$$1 + \sum_{s=2}^{\infty} [s]_q \left[\eta + \Big([s]_q + \beta - \mu - \eta\Big)t\right]^n \mathfrak{a}_s \mathbf{z}^{s-1} = 1 + \sum_{s=1}^{\infty} \mathfrak{d}_s \mathbf{z}^s. \tag{7}$$

By equating the coefficients on both sides of (7) yields

$$\mathfrak{a}_2 = \frac{\mathfrak{d}_1}{[2]_q \left(\eta + \left([2]_q + \beta - \mu - \eta\right)t\right)^n}, \tag{8}$$

$$\mathfrak{a}_3 = \frac{\mathfrak{d}_2}{[3]_q \left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n}, \tag{9}$$

$$\mathfrak{a}_4 = \frac{\mathfrak{d}_3}{[4]_q \left(\eta + \left([4]_q + \beta - \mu - \eta\right)t\right)^n}. \tag{10}$$

From (8), (9) and using Lemma 2, yields

$$\left|\mathfrak{a}_2\right| \leq \frac{2}{[2]_q \left(\eta + \left([2]_q + \beta - \mu - \eta\right)t\right)^n},$$

and

$$\left|\mathfrak{a}_3\right| \leq \frac{2}{[3]_q \left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n}.$$

Now,

$$a_3 - \aleph a_2^2 = \frac{\mathfrak{d}_2}{[3]_q\left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n} - \frac{\aleph \mathfrak{d}_1^2}{[2]_q^2\left(\eta + \left([2]_q + \beta - \mu - \eta\right)t\right)^{2n}}$$
$$= \frac{1}{[3]_q\left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n}\left(\mathfrak{d}_2 - \frac{\aleph[3]_q\left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n}{[2]_q^2\left(\eta + \left([2]_q + \beta - \mu - \eta\right)t\right)^{2n}}\mathfrak{d}_1^2\right).$$

Using Lemma 1, we have $|\mathfrak{d}_2 - \nu\mathfrak{d}_1^2| \leq 2\max\{1, |2\nu - 1|\}$

$$|a_3 - \aleph a_2^2| \leq \frac{2}{[3]_q\left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n}\max\left\{1; \frac{2\aleph[3]_q\left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n}{[2]_q^2\left(\eta + \left([2]_q + \beta - \mu - \eta\right)t\right)^{2n}} - 1\right\}.$$

Using the techniques employed by Abubaker and Darus [1], Libera and Zlotkiewicz [19], and Janteng et al. [18], we prove the theorem given below.

Theorem 2. *If $f \in \mathcal{L}_{q,b}^n(\mu, \beta, \eta, t)$, then*

$$\left|\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2\right| \leq \frac{4}{[3]_q^2\left(\Omega_3^n(\eta, \beta, \mu, t)\right)^2}.$$

The best possible result is achieved by Köebe function.

Proof. Since $f \in \mathcal{L}_{q,b}^n(\mu, \beta, \eta, t)$. from (8), (9) and (10), we observe the following

$$\left|\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2\right| = \left|\frac{\mathfrak{d}_1\mathfrak{d}_3}{[2]_q[4]_q\left(\eta + \left([2]_q + \beta - \mu - \eta\right)t\right)^n\left(\eta + \left([4]_q + \beta - \mu - \eta\right)t\right)^n} - \frac{\mathfrak{d}_2^2}{[3]_q^2\left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^{2n}}\right|.$$

Since the function $p(\mathbf{z}) \in \mathfrak{P}$, we assume without loss of generality that $\mathfrak{a}_1 > 0$, and for the sake of notation's accessibility, we let $\mathfrak{a}_1 = \mathbf{z}$, $(0 \leq \mathbf{z} \leq 2)$. By substituting the values of $\mathfrak{a}_2$ and $\mathfrak{a}_3$ from the system of equations (9), we have

$$\left|\mathfrak{a}_2\mathfrak{a}_4-\mathfrak{a}_3^2\right| = \frac{1}{4}\left|\frac{\left(\mathbf{z}^4+2x(4-\mathbf{z}^2)\mathbf{z}^2-\mathbf{z}^2(4-\mathbf{z}^2)x^2+2\mathbf{z}(4-\mathbf{z}^2)(1-|x|^2)\right)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}\right.$$
$$\left.-\frac{\mathbf{z}^4+2\mathbf{z}^2(4-\mathbf{z}^2)x+x^2(4-\mathbf{z}^2)^2}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right|$$
$$=\frac{1}{4}\left|\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)\mathbf{z}^4\right.$$
$$+\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)2x(4-\mathbf{z}^2)\mathbf{z}^2$$
$$-\left(\frac{\mathbf{z}^2}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{4-\mathbf{z}^2}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)x^2(4-\mathbf{z}^2)$$
$$\left.+\frac{2\mathbf{z}(4-\mathbf{z}^2)(1-|x|^2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}\right|.$$

Employing the triangle inequality, $|\mathbf{z}|\leq 1$ and replacing $|x|$ by ν, we obtain

$$\left|a_2a_4-a_3^2\right| = \left|\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)\mathbf{z}^4\right.$$
$$+\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)2\nu(4-\mathbf{z}^2)\mathbf{z}^2$$
$$-\left(\frac{\mathbf{z}^2}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{4-\mathbf{z}^2}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)\nu^2(4-\mathbf{z}^2)$$
$$\left.+\frac{2\mathbf{z}(4-\mathbf{z}^2)(1-\nu^2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}\right|.$$

$$\left|\mathfrak{a}_2\mathfrak{a}_4-\mathfrak{a}_3^2\right| = \frac{1}{4}\left\{\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)\mathbf{z}^4\right.$$
$$+\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)2\nu(4-\mathbf{z}^2)\mathbf{z}^2$$
$$-\left(\frac{\mathbf{z}(\mathbf{z}-2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{4-\mathbf{z}^2}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)\nu^2(4-\mathbf{z}^2)$$
$$\left.+\frac{2\mathbf{z}(4-\mathbf{z}^2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}\right\}=\mathcal{H}(\nu,\mathbf{z}),\tag{11}$$

where $\mathbf{z} \in [0,2]$ and $|x| = \nu \leq 1$.

Subsequently, we maximize the function $\mathfrak{H}(\nu, \mathbf{z})$ on the closed square $[0,1]\times[0,2]$. We now partially differentiate $\mathfrak{H}(\nu, \mathbf{z})$ given in (11) with respect to ν, which yields

$$\frac{\partial \mathfrak{H}_q(\nu,\mathbf{z})}{\partial \nu} = \frac{1}{2}\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)} - \frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)(4-\mathbf{z}^2)\mathbf{z}^2 - \frac{1}{2}\left(\frac{\mathbf{z}(\mathbf{z}-2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)} - \frac{4-\mathbf{z}^2}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)\nu(4-\mathbf{z}^2),$$

implying that $\mathfrak{H}(\nu, \mathbf{z})$ increases with respect to $\mathbf{z}$. This suggests that $\mathfrak{H}(\nu, \mathbf{z})$ may not possess a maximum value in the closed square $[0,1]\times[0,2]$. Apart from that, by fixing $\mathbf{z} \in [0,2]$ we obtain

$$\max_{\nu\in[0,1]} \mathfrak{H}(\nu,\mathbf{z}) = \mathfrak{H}(1,\mathbf{z}) = \mathfrak{K}(\mathbf{z}).$$

$$\begin{aligned}\mathfrak{K}(\mathbf{z}) = \frac{1}{4}\Bigg\{&\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)} - \frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)\mathbf{z}^4\\ &+\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)} - \frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)2(4-\mathbf{z}^2)\mathbf{z}^2\\ &-\left(\frac{\mathbf{z}(\mathbf{z}-2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)} - \frac{4-\mathbf{z}^2}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)(4-\mathbf{z}^2)\\ &+\frac{2\mathbf{z}(4-\mathbf{z}^2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}\Bigg\}.\end{aligned}$$

Then

$$\mathfrak{K}'(\mathbf{z}) = \frac{2\mathbf{z}(4-\mathbf{z}^2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)} - \frac{2\mathbf{z}(4-\mathbf{z}^2)}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}.$$

It is now clear that $\mathfrak{K}'(\mathbf{z}) < 0$ for $0 < \mathbf{z} < 2$ and $\mathfrak{K}(\mathbf{z})$ possess real critical points at $\mathbf{z} = 0$, implying the upper bound with respect to (11) corresponding to $\mathbf{z} = 0$ and $\nu = 1$. Here,

$$\left|\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2\right| \leq \frac{4}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}.$$

Setting $n = 0$ and $q \to 1-$, we obtain the following results.

Corollary 1. *[18] If $f \in \mathcal{R}$, then*

$$\left|\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2\right| \leq \frac{4}{9}.$$

3. Conclusions

The q-calculus gained great importance among many researchers due to its numerous various applications in geometric function theory, especially in analytic function theory. This article primarily aims to estimates of the Taylor-Maclaurin coefficients $|a_s|$ $(s = 2, 3)$ for functions in this new class, as well as solve the Fekete-Szegö functional problems. Additionally, we aim to derive the second Hankel determinants for functions within the new subclass $\mathcal{L}^n_{q,b}(\mu, \beta, \eta, t)$ of analytic functions in the open unit disk $\mathbb{U}$. This subclass is attained by using a differential Operator Involving q-Opoola Operator. Using the results obtained in this article, we can generalize and enhance some recently published articles.

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ON IDEAL BOUNDED SEQUENCES

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Abstract. In this paper, we study the notion of ideal bounded sequences, related to a given ideal, generalizing an earlier concept known as statistical boundedness of a sequence. We proceed to prove some results connecting ideal boundedness of a sequence to that of its subsequences. For this purpose, we use Lebesgue measure and Baire category to measure size.

1. Introduction

The convergence of sequences has undergone numerous generalizations, one of the first and most important being the concept of statistical convergence, introduced by Fast (1951), [8]. Later on, other types of summability including almost convergence, uniform statistical convergence and more generally ideal convergence of sequences were researched by many authors in different directions.

In classical and recent works the relationships between a given sequence and its subsequences regarding different kinds of summability have been studied using measure or category as gauges of size. It is well known that every $x \in (0,1]$ has a binary expansion $x = \sum_{n=1}^{\infty} 2^{-n} d_n(x)$ such that $d_n(x) = 1$ for infinitely many positive integers n, that is unique. Then for any $x \in (0,1]$ and any sequence $s = (s_n)$ we can construct a subsequence (sx) of s in such a way that: $(sx)_i = s_{n_i}$, where $n_1 < n_2 < ... < n_i < ...$ is the set of $n \in \mathbb{N}$ for which $d_n(x) = 1$.

Using this one-to-one correspondence, the sets of all almost convergent, statistically convergent, uniformly statistically convergent, ideal convergent subsequences of a sequence s have been studied in detail in several papers (see [3, 4, 12–15, 17–19, 21–23]).

The concept of statistical boundedness of a sequence first appeared in the work of Fridy and Orhan (1997), [10]. Theorems researching statistical boundedness

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and its relation to statistical convergence were proved by Tripathy (1997) [20], and Bhardwaj and Gupta (2014), [5]. Recently the authors, Miller-Van Wieren (2022), [16] studied statistical boundedness of a sequence and its relation to statistical boundedness of its subsequences using Lebesgue measure and category. In this paper we wish to generalize this concept to ideal boundedness, and to obtain results connecting ideal boundedness of a sequence to that of its subsequences, again with regards to measure and category.

We will first introduce some necessary notation. A family $I \subseteq P(\mathbb{N})$ of subsets of $\mathbb{N}$ is said to be an ideal on $\mathbb{N}$ if I is closed under subsets and finite unions, i.e. for each $A, B \in I$ we have $A \cup B \in I$ and for each $A \in I$ and $B \subset A$, we have $B \in I$. An ideal I is said to be proper if it does not contain $\mathbb{N}$. We say a proper ideal is admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$. Clearly any admissible ideal contains all finite subsets of $\mathbb{N}$. Throughout the paper, we will assume that the ideal I is admissible.

A sequence of real numbers s is said to be I-convergent to L if for every $\varepsilon > 0$ the set $K_\varepsilon = \{n \in \mathbb{N} : |s_n - L| > \varepsilon\}$ belongs to I, and we write $I - \lim s = L$ (see Kostyrko, Šalát and Wilczyński, 2000/01 , Balaz and Šalát , 2006) [2, 11]. It is easy to see that if $I = I_d = \{A \subset \mathbb{N} : d(A) = 0\}$, then I_d-convergence is statistical convergence where $d(A)$ denotes the asymptotic density of A [8], and if $I = I_u = \{A \subset \mathbb{N} : u(A) = 0\}$, then I_u-convergence coincides with uniform statistical convergence where $u(A)$ denotes the uniform density of A (Yurdakadim and Miller-Van Wieren 2016, Yurdakadim and Miller-Van Wieren 2017) [21, 22]. Ideals on $\mathbb{N}$ can be observed as subsets of the Polish space $\{0,1\}^{\mathbb{N}}$. Therefore ideals can have the Baire property or can be Borel, analytic, coanalytic etc. (Farah, 2000) [7]. From now on, we will refer to sets of first Baire category as meager, and to sets whose complement is of first category as comeager.

Next we state a well known lemma that can be found in several sources, recently in (M. Balcerzak, S. Glab, A. Wachowicz , 2016) [3].

Lemma 1. *Suppose I is an ideal on $\mathbb{N}$. The following conditions are equivalent:*

- *I has the Baire property;*
- *I is meager;*
- *There exists a sequence $n_1 < n_2 < \ldots < n_k < \ldots$ of integers in $\mathbb{N}$ such that no member of I contains infinitely many intervals $[n_k, n_{k+1})$ in $\mathbb{N}$.*

It is simple to verify that I_d and I_u have the Baire property. Additionally any analytic or coanalytic ideal has the Baire property.

2. Main Results

First, we recall the definition of a statistically bounded sequence of reals.

Definition 1. *A sequence of reals $s = (s_n)$ is said to be statistically bounded if there exists $L > 0$ such that $d(\{n : |s_n| \geq L\}) = 0$.*

Statistical boundedness of sequences was studied by Tripathy (1997) [20], Bhardwaj and Gupta (2014) [5], Aytar and Pehlivan (2006) [1] and by the authors (Miller-Van Wieren, 2022) [16].

Now we present a generalization of Definition 1, for a given ideal I introduced by Demirci (2001) [6].

Definition 2. *A sequence of reals $s = (s_n)$ is said to be I-bounded if there exists $L > 0$ such that $\{n : |s_n| \geq L\} \in I$.*

Given a sequence $s = (s_n)$ and $n_1 < n_2 < ... < n_k < ...$ we say that $s = (s_{n_k})$ is I-dense in s if $\mathbb{N} \setminus \{n_k : k \in \mathbb{N}\} \in I$. It is clear that $s = (s_{n_k})$ is I-bounded if and only if it has an I-dense subsequence that is bounded.

We will study the relationship of sequences and their subsequences regarding their I-boundedness, using Lebesgue measure as gauge of size.

In (Miller-Van Wieren, 2022) [16], we have shown the following theorem.

Theorem 1. *Suppose s is a sequence of reals. Then s is statistically bounded if and only if the set $\{x \in (0,1] : (sx)$ is statistically bounded$\}$ has Lebesgue measure 1. Additionally, s is not statistically bounded if and only if the set $\{x \in (0,1] : (sx)$ is statistically bounded$\}$ has Lebesgue measure 0.*

Now we direct our attention to sequences and their subsequences with regard to their I-boundedness. The discussion in the theorems that follow is related to some results obtained in [17, 18].

Theorem 2. *Suppose s is a I-bounded sequence, I is an analytic or coanalytic ideal. Then the set $\{x \in (0,1] : (sx)$ is I-bounded$\}$ has Lebesgue measure 0 or 1. Both cases of measure 0 and 1 can occur.*

Proof. Let us first prove that $\{x \in (0,1] : (sx)$ is I-bounded$\}$ is measurable. We have

$$\{x \in (0,1] : (sx) \text{ is } I\text{-bounded}\} = \bigcup_{M \in \mathbb{N}} \{x : \{i : |\,(sx)_i\,| \geq M\} \in I\}.$$

We define the characteristic function

$$\chi_M : (0,1] \to \{0,1\}^{\mathbb{N}}$$

by setting $(\chi_M(x))_i = \begin{cases} 1 & , \quad |\,(sx)_i\,| \geq M \\ 0 & , \quad \textit{otherwise} \end{cases}$

for $M \in \mathbb{N}$. We will verify that χ_M is continuous. For this purpose it is sufficient to check that the i-th component of χ_M, $(\chi_M)_i$ is continuous on $(0,1]$. We will check that the set $(\chi_M)_i^{-1}(\{1\})$ is open. Suppose that $x \in (\chi_M)_i^{-1}(\{1\})$ is arbitrarily fixed. Easily if $y \in (0,1]$ is such that $(sx)_j = (sy)_j$ for $1 \leq j \leq i$, then $y \in (\chi_M)_i^{-1}(\{1\})$. We can conclude that there exists a $k \geq i$ such that: if $y \in (0,1]$ satisfies $x_j = y_j$ for $1 \leq j \leq k$ (where x_j, y_j are the j-th coordinates of x, y respectively as $0-1$ sequences), then $y \in (\chi_M)_i^{-1}(\{1\})$. We obtain that $(\chi_M)_i^{-1}(\{1\})$ is open. In the same manner, we conclude that $(\chi_M)_i^{-1}(\{0\})$ is open.

Since I is analytic or coanalytic, we conclude that $\chi_M^{-1}(I)$ is analytic or coanalytic and hence measurable. Therefore
$\{x : \{i : |(sx)_i| \geq M\} \in I\} = \chi_M^{-1}(I)$ is measurable for $M \in \mathbb{N}$ and consequently $\{x \in (0,1] : (sx) \text{ is } I\text{-bounded}\}$ is measurable. Clearly $\{x \in (0,1] : (sx) \text{ is } I\text{-bounded}\}$ is a tail set. Since we proved it is measurable we conclude that $X = \{x \in (0,1] : (sx) \text{ is } I\text{-bounded}\}$ must have Lebesgue measure 0 or 1.

To see that both values can occur observe the following. If the sequence s is bounded (and consequently I-bounded), then for every $x \in (0,1]$, (sx) is bounded and consequently I-bounded, therefore $m(X) = 1$. Additionaly we can remark that in the case when $I = I_d$, the authors proved in [16] that the set X is of measure 1 for any I-bounded sequence s. Now we construct an example in which $m(X) = 0$ occurs.

In [13], Miller and Orhan (2001) constructed a sequence t of 0's and 1's,

$$t = 01001001...00010001....$$

that we made use of in (Yurdakadim and Miller-Van Wieren, 2016) [21] showing that t uniformly statistically converges to 0, $u(\{n : t_n = 1\}) = 0$, and $X^* = \{x \in (0,1] : \{n : (tx)_n = 1\} \text{is not in } I_u\}$ has measure 1.

Now we will construct a sequence s that is I_u-bounded but $m(X) = 0$. We define $s = (s_n)$ as follows:

$$s_n = \begin{cases} 0 & , \quad t_n = 0 \\ n & , \quad t_n = 1 \end{cases} \quad \text{for } n \in \mathbb{N}.$$

Now from this definition it follows that $u(\{n : s_n \neq 0\}) = 0$, so s is I_u-bounded. Suppose $x \in X^*$. From the definitions of s and X^* we conclude that there exists a subset of $\mathbb{N}$, $\{n_k : k \in \mathbb{N}\}$ not in I_u such that $(sx)_{n_k} \to \infty$ and therefore sx is not I_u-bounded. Since $m(X^*) = 1$, it follows that $m(X) = 0$. This completes the proof.

□

Now we will observe the case when I is an analytic or coanalytic ideal with property (G). We will use some notation from (M. Balcerzak, S. Glab, A. Wachowicz , 2016) [3].

We will denote by T the set of all 0-1 sequences that have an infinite number of ones. A mapping $f : \mathbb{N} \to \mathbb{N}$ is said to be bi-I-invariant if $E \in I$ if and only if $f[E] \in I$ whenever $E \subset \mathbb{N}$. Given a sequence $x \in T$ we can denote $\{n_1 < n_2 < ... < n_i < ...\} = \{k \in \mathbb{N} : x_k = 1\}$. Define $f_x : \mathbb{N} \longrightarrow \mathbb{N}$ by $f_x(k) = n_k$ and define $T_I = \{x \in T : f_x \text{is bi-}I\text{-invariant}\}$.

An ideal I is said to have property (G) if $\mu(T_I) = 1$. For instance, it is easy to check that I_d has property (G) while I_u does not. Now we have an analog of Theorem 1 for ideals with property (G).

Theorem 3. *Suppose s is a sequence, I is an analytic or coanalytic ideal with property (G). Then s is I-bounded if and only if the set $X = \{x \in (0,1] :$*

(sx) is I-bounded$\}$ has Lebesgue measure 1. Additionally, s is not I-bounded if and only if the set $\{x \in (0,1] : (sx)$ is I-bounded$\}$ has Lebesgue measure 0.

Proof. Suppose s is I-bounded. Suppose $M > 0$ is fixed so that $\{n : |s_n| \geq M\} \in I$. Let $x \in T_I$ be arbitrarily fixed (using the earlier mentioned definition of T_I). Then $(sx) = (s_{n_i})_i$ where $n_1 < n_2 < ... < n_i < ....$ Then,

$\{n_i : |s_{n_i}| \geq M\} \subseteq \{n : |s_n| \geq M\}$ and consequently from above $\{n_i : |s_{n_i}| \geq M\} \in I$. Now since $x \in T_I$, we have $\{n_i : |s_{n_i}| \geq M\} \in I \to f_x^{-1}(\{n_i : |s_{n_i}| \geq M\}) \in I \to \{i : |s_{n_i}| \geq M\} \in I$. Hence (sx) is I-bounded. We conclude that $T_I \subseteq X$. Since $m(T_I) = 1$, $m(X) = 1$.

Conversely suppose that $m(X) = 1$.

Let $T = X \cap (1 - X) \cap T_I \cap (1 - T_I)$ where $1 - X = \{x : 1 - x \in X\}$ and $1 - T_I$ is defined analogously. Then $m(T) = 1$ and $x \in T \to 1 - x \in T$. Suppose $x \in T$ is fixed. We will denote by $\{n_i\}$ the set of indices corresponding to x and by $\{n_j\}$ the set of indices corresponding to $1 - x$. Trivially $\{n_i\} \cap \{n_j\} = \varnothing$, $\{n_i\} \cup \{n_j\} = \mathbb{N}$. Then there exists $M > 0$ for which $\{i : |s_{n_i}| \geq M\} \in I$ and $\left\{j : |s_{n_j}| \geq M\right\} \in I$. From the above, $f_x(\{i : |s_{n_i}| \geq M\}) \in I$ and $f_{1-x}(\left\{j : |s_{n_j}| \geq M\right\}) \in I$. Therefore $\{n_i : |s_{n_i}| \geq M\} \in I$ and $\left\{n_j : |s_{n_j}| \geq M\right\} \in I$ and consequently

$$\{n : |s_n| \geq M\} = \{n_i : |s_{n_i}| \geq M\} \bigcup \left\{n_j : |s_{n_j}| \geq M\right\} \in I.$$

Therefore s is I-bounded. This completes the proof of the first statement.

To prove the second statement observe that in the proof of Theorem 2, we have shown that X is a measurable tail set with measure 0 or 1. Therefore the second statement follows immediately from the first one. The proof is complete.

□

Next we observe the relationship of the subsequences of a given sequence regarding I-boundedness, using Baire category as a gauge of size. In (Miller-Van Wieren, 2022) [16] we showed the following theorem .

Theorem 4. *Suppose $s = (s_n)$ is an unbounded sequence of reals, and let $X = \{x \in (0,1] : (sx)$ is statistically bounded$\}$. Then X is meager.*

We focus on I-boundedness with the assumption that I is an ideal with the Baire property. If s is a bounded sequence of reals, then all of its subsequences are likewise bounded, and hence I-bounded as well. If that is not the case we can show the following theorem.

Theorem 5. *Suppose $s = (s_n)$ is an unbounded sequence of reals, I an ideal with the Baire property and $X = \{x \in (0,1] : (sx)$ is I–bounded$\}$. Then X is meager.*

Proof. Since s is unbounded, it has ∞ or $-\infty$ as a limit point. Let us assume that ∞ is a limit point of s (the case of $-\infty$ is analogous) . Now since I has the Baire property, we can find a sequence $n_1 < n_2 < ... < n_k < ...$ of integers such that no member of I contains infinitely many intervals $[n_k, n_{k+1})$.

For arbitrary $m, j \in \mathbb{N}$, let

$$K_{m,j} = \{x \in (0,1] : there\ exists\ k \in \mathbb{N}, n_k > m : |(sx)_i| > j\ for\ i \in [n_k, n_{k+1})\}. \tag{1}$$

Let $m, j \in \mathbb{N}$ be arbitrarily fixed. We proceed to prove that $K_{m,j}$ is comeager.

Fix an arbitrary finite sequence of $0's$ and $1's$ denoted by $\overline{x} = (x_1, x_2, ..., x_d)$. It suffices to prove that we can find a finite extension x^* of $\overline{x}$ such that any $x \in (0,1]$ starting with x^* belongs to $K_{m,j}$. Suppose that $\overline{x}$ has t $1's$ where $t \geq m$ (we can assume this without loss of generality). Let $k = \min\{i : n_i > t\}$. We first extend $\overline{x}$ to a sequence $(x_1, x_2, ..., x_g)$, $g \geq d$ that has exactly $n_k - 1$ $1's$. Since ∞ is a limit point of s we can find $i_{n_k} < i_{n_k+1} < ... < i_{n_{k+1}-1}$ greater than g such that the terms of s corresponding to those indices are greater than j. Now define the following extension of $\overline{x}$

$$x^* = (x_1, x_2, ..., x_g, ..., x_{i_{n_k}}, ..., x_{i_{n_k+1}}, ...x_{i_{n_{k+1}-1}})$$

where for $i > g$: $x_i = 1\ for\ i \in \{i_{n_k}, i_{n_k+1}, ..., i_{n_{k+1}-1}\}$ and $x_i = 0,\ otherwise$. It is clear that any $x \in (0,1]$ that extends x^* belongs to $K_{m,j}$. We conclude $K_{m,j}$ is comeager. Consequently $K = \cap_m \cap_j K_{m,j}$ is also comeager. Now if $x \in K$, for every j the set $\{n : |(sx)_n| > j\}$ contains infinitely many $[n_k, n_{k+1})$. Consequently for $x \in K$, sx cannot be I-bounded, since if we assumed otherwise, there would exist j for which $\{n : |(sx)_n| > j\} \in I$, a contradiction. Since K is comeager, it follows that X is meager. □

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TWO PARAMETER RIDGE ESTIMATOR FOR THE BELL REGRESSION MODEL

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Abstract. One solution to the multicollinearity problem in the Bell regression model, which is utilized for over-dispersion issues, is biased estimators. In recent years, some biased estimators have been proposed in the Bell regression model that can be used in modelling correlated count data. In this article, Bell two-parameter ridge estimator (BTPRE) is proposed. This two-parameter estimator has some advantages over the previously proposed estimators. More efficient results are obtained than the Maximum Likelihood estimator (MLE) and Bell Ridge estimator (BRE) in the case of multicollinearity by using BT-PRE. Monte Carlo simulation study and real data results are obtained to show that the proposed estimator is better. Estimators have been compared according to the Mean Squared Error (MSE) criterion. BTPRE is superior to other estimators.

1. Introduction

In count data modelling, the key distribution is the Poisson distribution because of its simplicity. It has only one parameter, the location parameter, to be estimated. However, the main drawback of the Poisson distribution is that the mean and variance of the Poisson distribution are equal, which is called equidispersion. But, in many real datasets, this assumption does not hold since the variance is greater than the mean of the data. This situation is called an overdispersion problem. When the variability of the data is greater than the mean, an overdispersion problem arises. The most popular overdispersed model is the Negative Binomial regression model (NBRM). NBRM is a mixture model which obtains a mixture of Poisson and Gamma distributions.

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The advantage of Poisson regression model (PRM) over NBRM is that it has one parameter. In response, the advantage of NBRM is that it can be used to model overdispersed data. As an alternative to this model, the Bell regression model (BRM), which has a single parameter, has been proposed by Castellars *et al*. for modelling overdispersed count data [8]. BRM has the advantages of both PRM and NBRM; it has been widely preferred recently. As compared with the NBRM, BRM is more flexible than the NBRM.

One of the general assumptions of regression analysis is that the independent variables are not collinear. But often, in real-life datasets, the independent variables are correlated. This problem is called multicollinearity. If the assumptions are met, a maximum likelihood estimator (MLE) efficiently estimates the parameter. Highly correlated independent variables affect the performance of MLE. In the case of multicollinearity problems, the variance of MLE increases, and the confidence intervals widen. There are many studies on biased estimators to solve this problem. The variance of MLE, which is an unbiased estimator, is very high in the case of multicollinearity problems. In this case, alternative estimators with a bias value and a smaller variance than the variance of the MLE can be used. Thus, the MSE of the biased estimators is smaller than that of the MLE.

One of the most widely used biased estimators is the Ridge estimator (RE) proposed by Hoerl and Kennard [9]. This estimator depends on the k biased parameter. As with many biased estimators, RE was first proposed in a linear regression model (LRM). There are many studies on RE in the literature regarding both its definitions in different regression models and the estimation of the biased parameter. The logistic ridge estimator was defined by Schaefer *et al*., the Gamma ridge estimator was defined by Algamal, and the inverse Gaussian ridge estimator was defined by Algamal and its performances were examined [2, 3, 20]. Regarding modeling of counting data, RE studies were carried out by Månsson and Shukur, Månsoon and Amin *et al*. for PRM, NBRM and BRM, respectively [4, 12, 13].There are alternative estimators to the RE in the literature. Many of these estimators have also been identified in modeling count data [1, 17, 18].

In the ordinary least squares estimator (OLSE), there is an orthogonality between the residuals and the dependent variable. The orthogonality of this estimator is not available in the RE. In the RE, the aim is to reduce the variance, and model fit is not considered. A two-parameter ridge estimator (TPRE) was proposed by Lipovetsky and Conklin [10, 11] as a generalized version of the ridge estimator to increase the regression fit. The TPRE consists of k and q parameters. With the added parameter q, orthogonality between the dependent variable and residuals is provided. In addition, more efficient estimates are obtained from MLE and RE estimators. The TPRE for the linear model was compared with the OLSE and RE by Toker and Kaçıranlar according to the matrix MSE criterion [21]. Asar and Genç proposed TPRE for the logistic regression model [5]. Then, TPRE was defined for the inverse Gaussian regression model by Bulut and Işılar [7].

In this study, we propose the BTPRE for the Bell regression model used in modelling the count data. For this purpose, BRM and BRE are given in Section 2. BTPRE has been defined. In Section 3, the Monte Carlo Simulation study and actual data results are given to examine the performance of the proposed estimator. In the Section 4, the results of the studies are examined.

2. Methodology

A discrete random variable Y is said to be Bell distribution with the parameter $\theta > 0$, $Y \sim Bell(\theta)$, if its probability mass function (pmf) is given as

$$P(Y = y) = \frac{\theta^y e^{-e^\theta + 1} B_y}{y!},\ y = 0, 1, 2, \ldots \tag{1}$$

where $B_y = \frac{1}{n}\sum_{q=0}^{\infty} \frac{q^y}{q!}$ is called the Bell number [6]. Since the Bell distribution is a member of the exponential family, the Bell regression model can be written as a special case of the generalized linear models (GLM's), which are widespread to model the mean of the response variable. Using the reparametrization given by Castellares *et al*. [8], the pmf can be rewritten as follows:

$$P(Y = y) = exp\Big\{1 - exp\big\{W_0(\mu)\big\}\Big\} \frac{W_0(\mu)^y B_y}{y!},\ y = 0, 1, 2, \ldots \tag{2}$$

where $\theta = W_0(\mu)$ and $W_0(.)$ is the Lambert function. The mean and variance can be written using this parametrization as follows

$$E(y) = \mu, \tag{3}$$
$$Var(y) = \mu[1 + W_0(\mu)]. \tag{4}$$

The BRM is a good alternative to NBRM to model count data with overdispersion. The response variable distributed as $y_i \sim Bell(W_0(\mu_i))$ where $\mu_i = exp\{x_i^T\beta\}exp\{exp\{x_i^T\beta\}\}$ for $i = 1, 2, \ldots, n$. Using the Eq. (2), the log-likelihood function is given as follows

$$\begin{aligned}\ell(\mu_i; y_i) &= n - \sum_{i=1}^{n} exp\big\{W_0(\mu_i)\big\} + \sum_{i=1}^{n} y_i log\big(W_0(\mu_i)\big) + \sum_{i=1}^{n} log\big(B_{y_i}\big) - \sum_{i=1}^{n} log\big(y_i!\big) \\ &\propto \sum_{i=1}^{n} y_i log\big(exp\{x_i^T\beta\}exp\{exp\{c_i^T\beta\}\}\big) - exp\big\{exp\{x_i^T\beta\}exp\{exp\{x_i^T\beta\}\}\big\} \end{aligned} \tag{5}$$

Taking the derivative of the log-likelihood function concerning β parameter, we can obtain the following score function

$$S(\beta) = \frac{d\ell(\mu_i; y_i)}{d\beta} = \sum_{i=1}^{n} \left[x_i \Big(1 + exp\big\{x_i^T\beta\big\}\Big)(y_i - \mu_i) \right] \tag{6}$$

The most commonly used estimation method in the GLM's is the maximum likelihood estimation (MLE) method. To obtain the MLE of the BRM, we have to solve Eq. (6). Since the Eq. (6) is a non-linear according to the β, we can use the method of scoring:

$$\beta^{(m+1)} = \beta^{(m)} + I^{-1}\beta^{(m)} S\Big(\beta^{(r)}\Big) \tag{7}$$

where $S\big(\beta^{(m)}\big)$ is the score function evaluated at $\beta^{(m)}$, and

$$I^{-1}\big(\beta^{(m)}\big) = E\left[\frac{d^2\ell(\mu_i; y_i)}{d\beta d\beta^T}\right] = X^T W\big(\beta^{(m)}\big)X,$$

where $W\big(\beta^{(m)}\big) = diag\left\{\frac{\mu_i(\beta^{(m)})}{1+exp\left\{x_i^T\beta^{(m)}\right\}}\right\}$ evaluated at $\beta^{(m)}$. The final step of the Eq. (7) can also be written as

$$\widehat{\beta}_{MLE} = (X^T\widehat{W}X)^{-1}X'\widehat{W}\widehat{z}, \tag{8}$$

where $\widehat{z} = log(\widehat{\mu}) + W^{-\frac{1}{2}}V^{-\frac{1}{2}}(y-\mu)$, and $V = Var(y)$. The covariance matrix of the MLE can be computed as

$$Cov\big(\widehat{\beta}_{MLE}\big) = \big(X^T\widehat{W}X\big)^{-1}, \tag{9}$$

which equals the inverse of the Hessian matrix. The matrix mean square error (MMSE) and scaler mean square error (SMSE) of the MLE are given by

$$MMSE\big(\widehat{\beta}_{MLE}\big) = D^{-1}, \tag{10}$$

$$SMSE\big(\widehat{\beta}_{MLE}\big) = \sum_{j=1}^{l}\frac{1}{\lambda_j}, \tag{11}$$

where $D = X^T\widehat{W}X$, λ_j are the eigenvalues of D matrix and l is a total number of parameter.

When the multicollinearity exits, the MLE inflates. So, Amin *et al*. [4] proposed the Ridge estimator for the BRM to handle the multicollinearity problem as given in the following subsection.

2.1. **Ridge Estimator in the BRM.** Amin *et al*. [4] introduced the Bell Ridge estimator (BRE) to cope with the multicollinearity problem's adverse effects. BRE is given as follows

$$\widehat{\alpha}_k = D_k^{-1}D\alpha \tag{12}$$

where $D_k = (X^T\widehat{W}X + kI_l)$ and $k > 0$ is a biasing parameter. $\alpha = Z^T\beta_{MLE}$ where Z is a eigenvector of D. The MMSE and SMSE of the BRE are given as

$$MMSE\Big(\widehat{\alpha}_k\Big) = D_k D D_k + k^2 D_k^{-1} \alpha\alpha^T D_k^{-1}, \quad (13)$$

$$SMSE\Big(\widehat{\alpha}_k\Big) = \sum_{j=1}^{l} \frac{\lambda_j}{(\lambda_j + k)^2} + k^2 \sum_{j=1}^{l} \frac{\alpha_j^2}{(\lambda_j + k)^2}. \quad (14)$$

In this study, the biased parameter has estimated as follows

$$\widehat{k} = \frac{l}{\widehat{\alpha}^T \widehat{\alpha}}. \quad (15)$$

2.2. **Two Parameter Ridge Estimator in BRM.** Lipovetsky and Conklin [11] has proposed an objective function for the TPRE as follows

$$S^2 = ||Y - X\beta||^2 + q_1||\beta||^2 + q_2||X^T Y - \beta||^2 + q_3||Y^T(Y - X\beta)||^2. \quad (16)$$

The generalization of the TPRE in the BRM obtained from the objective function given in Eq. (16) is given below

$$\widehat{\alpha}_{qk} = q D_k^{-1} D \widehat{\alpha} \quad (17)$$

where $k > 0$ and $q > 0$. This estimator is the Bell two-parameter Ridge estimator (BTPRE) in which BRE and MLE are special cases of it. For example, if $q = 1$ is taken in Eq. (17), we can obtain $\widehat{\alpha}_k$. If we takes $q = 1$ and $k = 0$, $\widehat{\alpha}_{MLE}$ can be obtained. The coefficient of determination for the BTPRE is given in Eq. (18).

$$R^2 = 2qr^T D_k^{-1} r - q^2 r^T D_k^{-1} D D_k^{-1} r \quad (18)$$

where $r = X^T \widehat{W} \widehat{z}$. In order to maximize the model fit, optimal q is as follow

$$q = \frac{r^T D_k^{-1} r}{r^T D_k^{-1} D D_k^{-1} r}. \quad (19)$$

MMSE and MSE are computed as

$$MMSE(\widehat{\alpha}_{qk}) = q^2 D_k^{-1} D D_k^{-1} + (q D_k^{-1} D - I)\alpha\alpha^T (q D_k^{-1} D - I), \quad (20)$$

$$MSE(\widehat{\alpha}_{qk}) = q^2 \sum_{j=1}^{l} \frac{\lambda_j}{(\lambda_j + k)^2} + \sum_{j=1}^{l} \frac{\alpha_j^2 (q\lambda_j - \lambda_j - k)^2}{(\lambda_j + k)^2}. \quad (21)$$

In the literature related to biased estimators, there are different estimation equations for the parameters of the estimators. In order to minimize the MSE in the BTPRE, the derivatives of Eq. (12) for k and q, respectively, were calculated. The optimal parameter estimates obtained by equating the equations to zero are given below.

$$k = \frac{\sum_{j=1}^{l} q\lambda_j + (q-1)\lambda_j^2 \alpha_j^2}{\sum_{j=1}^{l} \lambda_j \alpha_j^2} \quad (22)$$

$$q = \frac{\sum_{j=1}^{l} \frac{\lambda_j \alpha_j^2}{\lambda_j + k}}{\sum_{j=1}^{l} \frac{\lambda_j + \lambda_j \alpha_j^2}{(\lambda_j + k)^2}} \tag{23}$$

Two methods were used for the proposed BTPRE in this study. First, the parameters for the BTPRE symbolized as $\widehat{\alpha}_{qk_1}$ were calculated by following the steps below.

Step 1. The initial value is determined so that $\widehat{k}^0 > \frac{1}{\widehat{\alpha}^T \widehat{\alpha}}$.
Step 2. Calculate $\widehat{q}$ using Eq. (23) with $\widehat{k}^0$ given in Step 1.
Step 3. It is calculated as $\widehat{k} = \frac{1}{l} \sum_{j=1}^{l} \frac{\widehat{q}\lambda_j + (\widehat{q}-1)\lambda_j^2 \widehat{\alpha}_j^2}{\lambda_j \widehat{\alpha}_j^2}$.

Secondly, the TPRE calculated with the following steps is given as $\widehat{\alpha}_{qk_2}$.

Step 1. Calculate the initial value as $q^0 > \sum_{j=1}^{l} \frac{\lambda_j \widehat{\alpha}_j^2}{1 + \lambda_j \widehat{\alpha}_j^2}$.
Step 2. Eq. (22) using q^0 yields k^0.
Step 3. $\widehat{q}$ is calculated from Eq. (19).
Step 4. Using Eq. (23), $\widehat{q}$ is updated.

Theorem 1. *Let $k > 0$, BTPRE is superior to MLE if $k > \lambda_j(q-1)$ where $j = 1, ..., l$.*

Proof. The difference between MSE's of the MLE and BTPRE is obtained by

$$\begin{aligned} \boldsymbol{\delta} &= MSE(\widehat{\alpha}) - MSE(\widehat{\alpha}_{qk}) \\ &= \sum_{j=1}^{l} \frac{1}{\lambda_j} - q^2 \sum_{j=1}^{l} \frac{\lambda_j}{(\lambda_j + k)^2} - \sum_{j=1}^{l} \frac{(q\lambda_j - \lambda_j - k)^2 \alpha_j^2}{(\lambda_j + k)^2}. \end{aligned} \tag{24}$$

The difference between MSE's is pozitif definite, if $\frac{1}{\lambda_j} - \frac{\lambda_j}{(\lambda_j+k)^2}$ is pozitif. The fact that $\boldsymbol{\delta}$ is a p.d. iff $k > \lambda_j(q-1)$. The proof is finished. □

Theorem 2. *Let $k > 0$, $MSE(\widehat{\alpha}_k) - MSE(\widehat{\alpha}_{qk}) > 0$, if only $q > 1$.*

Proof. The difference between MSE's of the BRE and BTPRE is obtained by

$$\begin{aligned} \boldsymbol{\delta} &= MSE(\widehat{\alpha}_k) - MSE(\widehat{\alpha}_{qk}) \\ &= \sum_{j=1}^{l} \frac{\lambda_j}{(\lambda_j + k)^2} + k^2 \sum_{j=1}^{l} \frac{\alpha_j^2}{(\lambda_j + k)^2} - q^2 \sum_{j=1}^{l} \frac{\lambda_j}{(\lambda_j + k)^2} - \sum_{j=1}^{l} \frac{(q\lambda_j - \lambda_j - k)^2 \alpha_j^2}{(\lambda_j + k)^2} \\ &= (1 - q^2) \sum_{j=1}^{l} \frac{\lambda_j}{(\lambda_j + k)^2} + \sum_{j=1}^{l} \frac{[k^2 - (q\lambda_j - \lambda_j - k)^2]\alpha_j^2}{(\lambda_j + k)^2}. \end{aligned} \tag{25}$$

For the δ to be positive, the difference between variances must be positive. If only $q < 1$ then $(1 - q^2) > 0$. The proof is completed. □

3. Simulation Study and Real Data Example

The performances of the estimators are compared according to the MSE criterion in the simulation and a real data example.

3.1. **Simulation Study.** The $X_{n\times p}$ independent variable matrix formed in the studies on biased estimators was created by McDonald and Galarneau [14] using the equation given in Eq. (26).

$$x_{ij} = (1-\rho^2)^{1/2} z_{ij} + \rho z_{ip}, \tag{26}$$

where ρ is the correlation coefficient. The z_{ij} are pseudo random numbers. $y_{n\times 1}$ is generated as

$$y_i \sim Bell(W_0(exp\{\beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip}\})), \tag{27}$$

where $\beta_{p\times 1}$ was selected using the method given in [16]. In the simulation study, the sample size is chosen as $n = 50, 100, 150, 200, 250,$ and 300 , and the correlation coefficient is $\rho = 0.90, 0.95,$ and 0.99, and finally, the number of independent variables is taken as $p = 3, 5, 7$.

This study was done in R program [19] with 2000 repetitions. The results obtained by calculating the performances of the estimators with the MSE equation given in Eq. (28) are given in Table (1)-(3).

$$MSE(\widehat{\beta}) = \frac{1}{2000}\sum_{r=1}^{2000}(\widehat{\beta}_r - \beta)'(\widehat{\beta}_r - \beta). \tag{28}$$

When the Tables (1)-(3) are examined, the MSE of all estimators is decreasing as the sample size increases. As the correlation coefficient increases, the MSE values of all estimators increase in all scenarios. Similarly, increasing the number of independent variables negatively affects the performance of the estimators. The BTPRE has the smallest MSE value for each sample size and correlation coefficient in all designs. The result is that the proposed estimator is superior to MLE and BRE. In addition, two different parameter selection methods were used in the study. It is seen from the MSE values that the method mentioned as $BTPRE_2$ is better than $BTPRE_1$. It is seen that the smallest MSE value belongs to $BTPRE_2$ for all cases.

3.2. **Real Data Example.** In this subsection, an application study is given to support the simulation study. Mine fracture dataset provided by Myers *et al.*, consisting of n=44 observations, was used as the real dataset [15]. Dependent variable comprises the number of injuries in coal mines in the Appalachian region. Models used in modelling the dependent variable are PRM, NBRM and BRM. Akaike Information Information (AIC) value has been used to select the best model from the Poisson, Negative Binomial and Bell distributions. The results of the AIC are given in the Table (4). According to the results from the Table (4), the

Table 1. MSE values for p=3

n	ρ	$\widehat{\beta}_{MLE}$	$\widehat{\beta}_{k}$	$\widehat{\beta}_{qk_1}$	$\widehat{\beta}_{qk_2}$
	0.90	6.73214	5.87381	2.68612	1.68484
50	0.95	9.28130	7.71959	4.12875	2.35018
	0.99	33.36669	24.60985	17.43750	7.44138
	0.90	5.42670	4.97601	2.19685	1.37202
100	0.95	6.61325	5.79349	2.87992	1.72569
	0.99	16.08049	12.06670	7.10425	3.62146
	0.90	5.08468	4.78100	2.12913	1.29036
150	0.95	5.83406	5.27756	2.54194	1.50144
	0.99	12.69978	9.89156	6.30262	3.24474
	0.90	4.93791	4.71320	2.10739	1.25240
200	0.95	5.35580	4.92607	2.41291	1.39102
	0.99	10.10888	8.12280	4.82236	2.56908
	0.90	4.86601	4.68486	2.02834	1.19642
250	0.95	5.21363	4.87818	2.38204	1.35741
	0.99	9.04588	7.37682	4.44209	2.37815
	0.90	4.75503	4.60505	2.01804	1.19031
300	0.95	4.89462	4.62580	2.27716	1.31450
	0.99	7.99842	6.65926	3.86502	2.15689

appropriate model is chosen as the BRM since the Bell distribution has the smallest AIC value. Independent variables used in the dataset are as follows

X1: inner burden thickness in feet,
X2: percent extraction of the lower previously pricked mined seam,
X3: the lower seam height
X4: the time that the mine

The number of conditions used to determine whether the multicollinearity occurred in the data set is 296.5585. The correlation chart showing the correlation between the independent variables is given in Figure 1.

Because of existing multicollinearity, we calculate the MLE, RE and BTPRE coefficients for the data set. Then, the estimated coefficients, the standard errors and the square root of MSE values are given in Table (5).

When the Table (5) is examined, it is seen that BTPRE has the smallest MSE. The real data results show that the performance of the proposed BTPRE is superior to the MLE and RE, like the simulation studies. In addition, the method used to estimate k and q parameters in $TPRE_2$ is more effective than that of $TPRE_1$.

TABLE 2. MSE values for p=5

n	ρ	$\widehat{\beta}_{MLE}$	$\widehat{\beta}_{k}$	$\widehat{\beta}_{qk_1}$	$\widehat{\beta}_{qk_2}$
	0.90	9.82742	8.67058	2.91611	1.74571
50	0.95	14.70160	12.30335	5.20670	2.66722
	0.99	58.62601	43.50621	31.47012	10.27462
	0.90	7.71904	7.18999	2.27929	1.36482
100	0.95	9.53292	8.44894	3.16134	1.75353
	0.99	30.45618	23.44846	13.20561	5.28299
	0.90	7.00468	6.66178	1.99375	1.21469
150	0.95	8.02216	7.27704	2.60181	1.51511
	0.99	21.49925	16.84574	8.92770	3.77031
	0.90	6.65705	6.41170	1.96885	1.18348
200	0.95	7.47940	6.92626	2.35127	1.33888
	0.99	17.04084	13.75045	6.44487	3.01546
	0.90	6.51391	6.31932	1.88913	1.15233
250	0.95	6.94140	6.51257	2.27448	1.31927
	0.99	14.35469	11.78576	5.82564	2.69328
	0.90	6.43463	6.27732	1.87978	1.12367
300	0.95	6.87342	6.33572	2.00512	1.29277
	0.99	12.73392	10.52779	4.84047	2.41665

4. Conclusion

PRM and NBRM have been generally used in the modelling of count data. BRM has been widely preferred as an alternative to these models in recent years. BRM may be more suitable for modelling overdispersed count data. As seen in the real data set discussed in the study, the Bell distribution is more convenient than the alternative distributions. Considering this situation, alternative biased estimators are proposed for the Bell regression model to handle the multicollinearity problem.

In this article, we propose BTPRE as an alternative to these estimators. It is concluded from the simulation study and a real data example that the performance of the proposed estimator is superior to MLE and BRE.

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TABLE 3. MSE values for p=7

n	$\boldsymbol{\rho}$	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{MLE}}$	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{k}}$	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{qk_1}}$	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{qk_2}}$
	0.90	14.06553	12.77779	5.05469	2.72267
50	0.95	19.69486	16.76399	6.42533	3.13107
	0.99	82.85781	63.98084	55.16817	15.71594
	0.90	13.21807	12.63627	4.93636	2.41681
100	0.95	15.44038	14.15015	5.53295	2.74144
	0.99	43.16660	34.19774	19.14917	7.47775
	0.90	11.84296	11.46179	4.80369	2.39698
150	0.95	14.01502	13.16318	5.12539	2.67410
	0.99	31.62615	26.13034	13.52178	5.87188
	0.90	11.57151	11.30324	4.51272	2.10850
200	0.95	12.76125	12.16552	4.86460	2.19485
	0.99	25.18756	21.20189	10.95476	5.17243
	0.90	10.40783	10.19874	3.40672	2.03554
250	0.95	11.78194	11.30935	4.13962	2.10319
	0.99	21.31977	18.07933	7.76840	3.92102
	0.90	8.88708	8.71718	2.14829	1.34627
250	0.95	11.21841	10.82050	4.07510	2.07894
	0.99	18.49267	15.92869	7.09717	3.11654

TABLE 4. AIC values of dependent variable

	POISSON	**NEGATIVE BINOMIAL**	**BELL**
AIC	173.2554	172.3399	169.4784

TABLE 5. Results of the Mine fracture dataset

	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{0}}$	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{1}}$	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{2}}$	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{3}}$	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{4}}$	**MSE**
$\widehat{\boldsymbol{\beta}}$	0.00293	-0.01126	0.01819	-0.02384	-4.00837	1.38936
	(1.38907)	(0.00106)	(0.01684)	(0.00679)	(0.02179)	
$\widehat{\boldsymbol{\beta}}_{\boldsymbol{k}}$	0.00294	-0.01126	0.01819	-0.02384	-3.57858	1.24044
	(1.31249)	(0.00106)	(0.01599)	(0.00676)	(0.02178)	
$\widehat{\boldsymbol{\beta}}_{\boldsymbol{qk_1}}$	0.00322	-0.00545	0.00117	-0.00021	-0.00001	0.00147
	(4.00788)	(0.00247)	(0.06173)	(0.00514)	(0.02970)	
$\widehat{\boldsymbol{\beta}}_{\boldsymbol{qk_2}}$	-0.00294	-0.00140	0.00020	-0.00003	-0.00001	0.00043
	(4.00784)	(0.00359)	(0.06547)	(0.00367)	(0.02969)	

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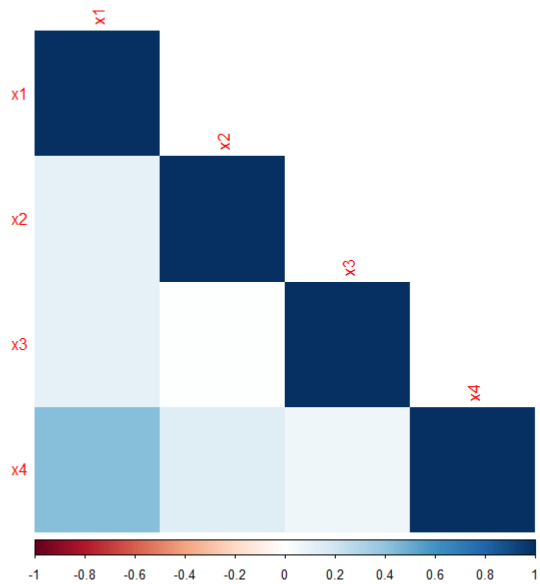

Figure 1. Correlation chart between independent variables

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SOME RESULTS ON $\mathcal{I}_2$-DEFERRED STATISTICALLY CONVERGENT DOUBLE SEQUENCES IN FUZZY NORMED SPACES

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Abstract. The primary objective of this study is to introduce the concepts of $\mathcal{I}_2$-deferred Cesàro summability and $\mathcal{I}_2$-deferred statistical convergence for double sequences in fuzzy normed spaces (FNS). Furthermore, the aim is to explore the connections between these concepts and subsequently establish several theorems pertaining to the notion of $\mathcal{I}_2$-deferred statistical convergence in FNS for double sequences. We further define $\mathcal{I}_2$-deferred statistical limit points and $\mathcal{I}_2$-deferred statistical cluster points of a sequence within FNS and explore the relationships among these concepts.

1. Introduction

The concept of statistical convergence, initially introduced in Zygmund's monograph [40], was later revisited by Fast [11] and independently reexamined for both real and complex sequences by Schoenberg [32]. Mursaleen and Edely [26] extended this investigation to double sequences. Additionally, Fridy [13] explored statistical limit points and cluster points in the context of real number sequences.

Kostyrko et al. [20] introduced the concept of ideal convergence, which encompasses various convergence notions such as usual convergence and statistical convergence. Das et al. [7] extended this concept to double sequences in a metric space, termed $\mathcal{I}$-convergence. In a subsequent work, Savaş and Das [30] further

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expanded the idea of ideal convergence introduced by Kostyrko et al. [20], exploring its application to $\mathcal{I}$-statistical convergence and investigating its fundamental properties.

Zadeh [39] pioneered the theory of fuzzy sets, laying its foundation. Matloka [24] explored the convergence of sequences of fuzzy numbers, while Nanda [27] demonstrated that the set of convergent sequences of fuzzy numbers forms a complete metric space. Nuray and Savaş [28] extended the notion of convergence to statistically Cauchy and statistical convergence sequences of fuzzy numbers. Kumar et al. [21, 22] delved into $\mathcal{I}$-convergence, $\mathcal{I}$-limit points, and $\mathcal{I}$-cluster points for sequences of fuzzy numbers. Tripathy et al. [36] further investigated $\mathcal{I}$-statistically limit points and $\mathcal{I}$-statistically cluster points for sequences of fuzzy numbers. In [19], researchers extended existing theories on the convergence of fuzzy number sequences to $\mathcal{I}_2$-statistical convergence, broadened the notions of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points to double sequences, and investigated fundamental features and relationships between sets of $\mathcal{I}_2$-statistical cluster points and $\mathcal{I}_2$-statistical limit points of double sequences of fuzzy numbers. Katsaras [17] initially introduced the concept of fuzzy norm while examining fuzzy topological vector spaces. In 1992, Felbin [12] extended this concept to a fuzzy norm on linear spaces, drawing from the idea of fuzzy numbers initially proposed by Kaleva and Seikkala in the context of fuzzy metric treatment. Further research, including studies in [6, 38], investigated diverse topological characteristics of these FNS, while works such as [2, 3] explored various types of FNS.

Agnew [1] introduced the concept of deferred Cesàro mean for real (or complex) sequences, followed by Küçükaslan and Yilmaztürk's [23] presentation of deferred statistical convergence for single sequences. Subsequently, Şengül et al. [34] introduced deferred $\mathcal{I}$-convergence. Dağadur and Sezgek [4, 5, 35] investigated deferred Cesàro summability and deferred statistical convergence for double sequences. Statistical convergence and deferred statistical convergence have been explored in various studies, as referenced in [8, 9, 15, 16, 18, 25, 29, 31, 37].

In this study, we adhere to the approach delineated in Felbin's work. Within the realm of FNS analysis, the convergence of sequences of fuzzy numbers plays a pivotal role in defining standard convergence within these spaces. This paper seeks to utilize the concept of generalized statistical convergence of fuzzy number sequences via ideal to explore a more extensive form of convergence, particularly $\mathcal{I}_2$-deferred statistical convergence within an FNS. The goal is to establish fundamental principles and key insights in this domain.

This paper is dedicated to introducing a novel form of convergence for sequences of fuzzy numbers within FNS. In Section 2, we provide some preliminary definitions and theorems concerning fuzzy number sequences, FNS, and deferred statistical convergence. In Section 3, we intend to define the concepts of $\mathcal{I}_2$-deferred Cesàro summability and $\mathcal{I}_2$-deferred statistical convergence for double sequences within FNS. In Section 4, our goal is to investigate the interconnections between these concepts and subsequently establish several theorems regarding the notion of $\mathcal{I}_2$-deferred statistical convergence in FNS for double sequences. Additionally, we define $\mathcal{I}_2$-deferred statistical limit points and $\mathcal{I}_2$-deferred statistical cluster points of sequences within FNS, and delve into the relationships among these concepts.

2. Definitions and Preliminaries

In this section, we commence by revisiting some fundamental definitions related to fuzzy numbers, fuzzy number sequences (FNS), and deferred convergence.

Definition 1. *([33]) Suppose $\mu : \mathbb{R} \to [0,1]$ represents a fuzzy subset of $\mathbb{R}$. For any $\alpha \in [0,1]$, the α-level set of μ, symbolized as μ_α, is described as the set of real numbers $\mathbb{R}$, where the measure μ is at least α. When $0 < \alpha \leqslant 1$, the notation $[\mu]_\alpha$ refers to the collection of points t in $\mathbb{R}$ where μ evaluates to at α. In the case where $\alpha = 0$, $[\mu]_\alpha$ indicates the closure of the set of points t in $\mathbb{R}$ where μ evaluates to strictly greater than 0.*

Definition 2. *([33]) A fuzzy set denoted by μ defined on the real numbers $\mathbb{R}$ is termed a fuzzy number subject to the specified conditions:*
(i) μ is normal, signifying the existence of a specific point t_0 in $\mathbb{R}$ where μ reaches its maximum membership grade of 1.
(ii) μ is fuzzy convex, implying that for any pair of real numbers t_1 and t_2, and any λ in the interval $[0,1]$, $\mu(\lambda t_1 + (1-\lambda)t_2)$ is greater than or equal to the minimum of $\mu(t_1)$ and $\mu(t_1)$.
(iii) μ is upper semi-continuous.
(iv) The set $[\mu]_0$, comprising all t in $\mathbb{R}$ where $\mu(t)$ is greater than 0, is compact.

A real number r can be represented as a fuzzy number $\tilde{r}$ defined by, if t equals r, then $\tilde{r}(t)$ equals 1, if t is not equal to r, then $\tilde{r}(t)$ equals 0. It can be demonstrated that μ qualifies as a fuzzy number if and only if each α-level set $[\mu]_\alpha$ forms a non-empty, bounded, and closed interval. We denote $[\mu]_\alpha = [\mu_\alpha^-, \mu_\alpha^+]$.

Definition 3. *([33]) Let's consider $L(\mathbb{R})$, the collection of all fuzzy numbers. If a fuzzy number μ is a member of $L(\mathbb{R})$ and its membership grade $\mu(t)$ is zero for $t < 0$, it is termed a non-negative fuzzy number.*

By $L^*(\mathbb{R})$, we denote the set of all non-negative fuzzy numbers. We can express that $\mu \in L^*(\mathbb{R})$ iff $\mu_\alpha^- \geqslant 0$ for each $\alpha \in [0,1]$. Clearly, $\widetilde{0} \in L^*(\mathbb{R})$.

A partial order denoted by $\preccurlyeq$ on $L(\mathbb{R})$ is defined as follows:

$$\mu \preccurlyeq \nu \text{ iff } \mu_\alpha^- \leqslant \nu_\alpha^- \quad \text{and} \quad \mu_\alpha^+ \leqslant \nu_\alpha^+ \text{for all } \alpha \in [0,1].$$

The strict inequality denoted by $\prec$ on $L(\mathbb{R})$ is established as $\mu \prec \nu($ or $\nu \succ \mu)$ iff $\mu_\alpha^- < \nu_\alpha^-$and $\mu_\alpha^+ < \nu_\alpha^+$for all $\alpha \in [0,1]$.

Definition 4. *([33]) We define the operations of addition $(\oplus)$, multiplication $(\otimes)$, and scalar multiplication on the set $L(\mathbb{R})$ as follows:*

(i) The convolution of two functions μ and ν, denoted as $(\mu \oplus \nu)(t)$, is defined for any t in the real numbers $\mathbb{R}$ as the supremum of the minimum values obtained by shifting and overlapping the functions μ and ν.
(ii) The product convolution of two functions μ and ν, denoted as $(\mu \otimes \nu)(t)$, is defined for any t in the real numbers $\mathbb{R}$ as the supremum of the minimum values obtained by scaling and overlapping the functions μ and ν.
(iii) Scalar multiplication of a function μ by a scalar k is defined for any t in the real numbers $\mathbb{R}$ as μ evaluated at t/k, where k is a real number not equal to zero. Additionally, when $k = 0$, the result is the zero function $\widetilde{0}(t)$.

Theorem 1. *([33]) Let $\mu, \nu \in L(\mathbb{R})$ and $\alpha \in [0,1]$. Then we have*

$$(i)\ [\mu \oplus \nu]_\alpha = [\mu_\alpha^- + \nu_\alpha^-, \mu_\alpha^+ + \nu_\alpha^+],$$

$$(ii)\ [\mu \otimes \nu]_\alpha = [\mu_\alpha^- \nu_\alpha^-, \mu_\alpha^+ \nu_\alpha^+]\,(\mu, \nu \in L^*(\mathbb{R}))$$

$$(iii)\ [k\mu]_\alpha = k[\mu]_\alpha = \begin{cases} [k\mu_\alpha^-, k\mu_\alpha^+] & \text{if } k \geqslant 0 \\ [k\mu_\alpha^+, k\mu_\alpha^-] & \text{if } k < 0 \end{cases}$$

Theorem 2. *([14]) Let μ be a fuzzy number in $L(\mathbb{R})$, with α-level sets denoted by $[\mu]_\alpha = [\mu_\alpha^-, \mu_\alpha^+]$. The theorem establishes the following:*
(i) μ_α^- is a bounded, left-continuous, non-decreasing function on $(0,1]$,
(ii) μ_α^+ is a bounded, right-continuous, non-increasing function on $(0,1]$,
(iii) at $\alpha = 0$, both μ_0^- and μ_0^+ are continuous,
(iv) μ_1^- is less than or equal to μ_1^+.

On the other hand, given functions $p(\alpha)$ and $q(\alpha)$ satisfying conditions (i)-(iv), there exists a unique fuzzy number $\mu \in L(\mathbb{R})$ such that $[\mu]_\alpha = [p(\alpha), q(\alpha)]$ for all $\alpha \in [0,1]$.

Definition 5. *([33]) Considering μ and ν as elements of the space $L(\mathbb{R})$, we define the discrepancy between two measures μ and ν, denoted as $\mathcal{F}(\mu, \nu)$, as the supremum over all possible values of α in the interval $[0,1]$ of the maximum absolute differences between the lower and upper variations of μ and ν. This function, $\mathcal{F}$, is known as the supremum metric on the set $L(\mathbb{R})$. If (μ_u) is a sequence in $L(\mathbb{R})$ and μ is an element of $L(\mathbb{R})$, we say that the sequence (μ_u) converges to μ in the metric $\mathcal{F}$, indicated as $\mu_u \xrightarrow{\mathcal{F}} \mu$ or $(\mathcal{F}) - \lim_{u\to\infty} \mu_u = \mu$, if the limit as u approaches infinity of the supremum metric $\mathcal{F}(\mu_u, \mu)$ is equal to zero.*

Definition 6. *([12]) Consider a vector space X over $\mathbb{R}$, equipped with a mapping $\|\cdot\| : X \to L^*(\mathbb{R})$, and let symmetric, non-decreasing mappings $L, R : [0,1]\times[0,1] \to [0,1]$ be given, satisfying $L(0,0) = 0$ and $R(1,1) = 1$. We denote this quadruple as $(X, \|\cdot\|, L, R)$, termed an FNS, where $\|\cdot\|$ is referred to as a fuzzy norm, if it adheres to the following conditions:*
(i) The norm of x equals zero iff x is the zero vector θ.

(ii) For any vector x in X and scalar r, the norm of the scalar multiple rx is equal to the absolute value of r multiplied by the norm of x.
(iii) For any vectors x and y in X:
(a) The norm of their sum $x+y$ is greater than or equal to the minimum of their norms.
(b) The norm of their sum $x+y$ is less than or equal to the maximum of their norms.

Additionally, functions $L(x,y)$ and $R(x,y)$ are defined as the minimum and maximum of x and y, respectively, when x and y are within the interval $[0,1]$. The FNS is denoted as $(X, |\cdot|)$ or simply X when L and R adhere to these definitions.

Lemma 1. *([12]) In a FNS, the norm of the sum of two vectors is less than or equal to the sum of their individual norms as defined in Definition 6 (iii) (a) (with L being the minimum function) is equivalent to the inequality $\|x+y\|_\alpha^- \leqslant \|x\|_\alpha^- + \|y\|_\alpha^-$, holding for all $\alpha \in (0,1]$ and $x, y \in X$.*

Lemma 2. *([12]) The triangle inequality specified in Definition 6 (iii)(b) (with $R = \max$) is equivalent to the inequality $|x+y|_\alpha^+ \leqslant |x|_\alpha^+ + |y|_\alpha^+$ for all $\alpha \in (0,1]$ and $x, y \in X$.*

Remark 1. *By referring to Theorem 1 (iii) and Lemma 1, we can infer that the condition described in Definition 6 (iii)(a) (with $L = \min$) implies that*

$$\lim_{\alpha\to 0} \|x+y\|_\alpha^- \leqslant \lim_{\alpha\to 0} \|x\|_\alpha^- + \lim_{\alpha\to 0} \|y\|_\alpha^-$$

that is, $\|x+y\|_0^- \leqslant \|x\|_0^- + \|y\|_0^-$. Similarly, according to Definition 6 (iii)(b) (with $R = \max$), it follows that the non-negative part of the sum of two elements, denoted by $\|x+y\|_0^+$, is bounded above by the sum of their respective non-negative parts, $\|x\|_0^+ + \|y\|_0^+$. Consequently, in a FNS $(X, \|\cdot\|)$, the triangle inequality specified in Definition 6 (iii) suggests that the norm of the sum of two elements, denoted by $\|x+y\|$, is less than or equal to the composition of the norms of x and y, denoted by $\|x\| \oplus \|y\|$.

According to Definition 6, we have $x = \theta$ iff $\|x\| = \widetilde{0}$, iff $\|x\|_\alpha^- = \|x\|_\alpha^+ = 0$ for all $\alpha \in [0,1]$. Furthermore, we have $\|x\|_0^- > 0$ whenever $x \neq \theta$. Now if $r = 0$, then $[\|rx\|]_\alpha = [\|\theta\|]_\alpha = [0,0] = [|r|\|x\|]_\alpha$ for all $\alpha \in [0,1]$ and $x \in X$. For $r \neq 0$, we have $[\|rx\|]_\alpha = [|r|\|x\|]_\alpha$ for each $\alpha \in [0,1]$, i.e., $\|rx\|_\alpha^- = |r|\|x\|_\alpha^-$and $\|rx\|_\alpha^+ = |r|\|x\|_\alpha^+$for each $\alpha \in [0,1]$. Thus, we can say that $\|\cdot\|_\alpha^-$and $\|\cdot\|_\alpha^+$are norms on X in the usual sense in view of Definition 6, with the choice of $L = \min$ and $R = \max$, where $\alpha \in [0,1]$.

Example 1. *([33]) Let $(X, \|\cdot\|_C)$ be an ordinary normed linear space. Then a fuzzy norm $\|\cdot\|$ on X can be obtained as*

$$\|x\|(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant \zeta\|x\|_C \text{ or } t \geqslant \eta\|x\|_C \\ \frac{t}{(1-\zeta)\|x\|_C} - \frac{\zeta}{1-\zeta} & \text{if } \zeta\|x\|_C \leqslant t \leqslant \|x\|_C \\ \frac{-t}{(\eta-1)\|x\|_C} + \frac{b}{\eta-1} & \text{if } \|x\|_C \leqslant t \leqslant \eta\|x\|_C \end{cases}$$

in the given context, $\|x\|_C$ denotes the standard norm of x (excluding the zero vector), where $0 < \zeta < 1$ and $1 < \eta < \infty$. For the zero vector $x = \theta$, we define $\|x\| = \widetilde{0}$. Consequently, $(X, \|\cdot\|)$ constitutes a FNS. The specific fuzzy norm discussed here is referred to as the triangular fuzzy norm.

Example 2. *Let $(\mathbb{R}, \|\cdot\|_{\mathbb{R}})$ is a normed linear space. Then the fuzzy norm $\|\cdot\|$ on $\mathbb{R}$ can be obtained as*

$$\|x\|(s) = \begin{cases} \frac{s-|x|}{s+|x|} & s > |x| \\ 0, & s \leq |x| \end{cases}$$

and $(\mathbb{R}, \|x\|)$ is a FNS.

Definition 7. *([10]) Let $(X, \|\cdot\|)$ be an FNS. A sequence (x_r) in X converges to $x \in X$ with respect to the fuzzy norm on X, denoted by $x_r \xrightarrow{FN} x$, if $(\mathcal{F}) - \lim_{r\to\infty} \|x_r - x\| = \widetilde{0}$, where for every $\varepsilon > 0$, there exists an $N(\varepsilon) \in \mathbb{N}$ such that $\mathcal{F}\left(\|x_r - x\|, \widetilde{0}\right) < \varepsilon$ for all $r \geqslant N$. This means that for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that $\sup_{\alpha\in[0,1]} \|x_r - x\|_\alpha^+ = \|x_r - x\|_0^+ < \varepsilon$.*

Definition 8. *([28]) A sequence (x_r) of fuzzy numbers is considered to be statistically convergent to the fuzzy number x, denoted as $st\text{-}\lim x_r = x$, if for each $\varepsilon > 0$, there exists a positive integer N such that,*

$$\delta\left(\{r \in \mathbb{N} : \mathcal{F}(x_r, x) \geqslant \varepsilon\}\right) = 0.$$

Definition 9. *([1]) Let K be a subset of the positive integers $\mathbb{N}$, and let $K_{d,c}(n)$ denote the set of integers in the interval $[d(n)+1, c(n)]$ that belong to K, where $d = (d(n))$ and $c = (c(n))$ are sequences of non-negative integers satisfying the conditions:*

$$d(n) < c(n) \quad \text{for all} \quad n \in \mathbb{N} \text{ and } \quad \lim_{n\to\infty} c(n) = \infty.$$

The deferred density of K is denoted and defined by

$$\delta_{d,c}(K) = \lim_{n\to\infty} \frac{1}{c(n) - d(n)} \left|K_{d,c}(n)\right|.$$

Definition 10. *([23]) Consider a sequence (x_r) of real numbers. We say that (x_r) is deferred statistically convergent to $l \in \mathbb{R}$ if, for every $\varepsilon > 0$, the following holds:*

$$\lim_{n\to\infty} \frac{1}{c(n) - d(n)} \left|\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : |x_r - l| \geq \varepsilon\}\right| = 0$$

where $d=(d(n))$ and $c=(c(n))$ are sequences of non-negative integers satisfying the conditions specified in Equation (1).

For a double sequence $w=(w_{uv})$, the deferred Cesàro mean $D_{\rho,\phi}$ is defined by

$$(D_{\rho,\phi}w)_{\alpha\beta}=\frac{1}{\rho_\alpha\phi_\beta}\sum_{\substack{u=t_\alpha+1\\ v=k_\beta+1}}^{v_\alpha,l_\beta} w_{uv}=\frac{1}{\rho_\alpha\phi_\beta}\sum_{\substack{u=t_\alpha+1\\ v=k_\beta+1}}^{v_\alpha,l_\beta} w_{uv}$$

where $(t_\alpha),(v_\alpha),(k_\beta),(l_\beta)$ are non-negative integer sequences satisfying following conditions:

$$\begin{aligned} &t_\alpha<v_\alpha,\quad \lim_{\alpha\to\infty} v_\alpha=\infty;\quad k_\beta<l_\beta,\quad \lim_{\beta\to\infty} l_\beta=\infty \\ &v_\alpha-t_\alpha=\rho_\alpha;\quad l_\beta-k_\beta=\phi_\beta. \end{aligned} \tag{1}$$

Note here that the method $D_{\rho,\phi}$ is openly regular for any selection of the sequences $(t_\alpha),(v_\alpha),(k_\beta),(l_\beta)$.

All through this study, except where otherwise stated, $(t_\alpha),(v_\alpha),(k_\beta),(l_\beta)$ are conceived non-negative integer sequences satisfying 1.

A double sequence (w_{uv}) is strongly deferred Cesàro summable to w provided that

$$\lim_{\alpha,\beta\to\infty}\frac{1}{\rho_\alpha\phi_\beta}\sum_{\substack{u=t_\alpha+1\\ v=k_\beta+1}}^{v_\alpha,l_\beta}\|w_{uv}-w\|_0^+=0,$$

A double sequence (w_{uv}) is considered bounded with respect to the fuzzy norm X, if there exists $\mathcal{U}>0$ such that $\|w_{uv}-w\|_0^+\le\mathcal{U}$ for all $(u,v)\in\mathbb{N}^2$. Additionally, L_∞^2 denotes the set of all bounded double sequences.

By double lacunary sequence, we mean that a double sequence $\theta_2=\{(p_\alpha,q_\beta)\}$ of two increasing integer sequences (p_α) and (q_β) such that

$$p_0=0, h_\alpha=p_\alpha-p_{\alpha-1}\to\infty \text{ and } q_0=0, \bar{h}_\beta=q_\beta-q_{\beta-1}\to\infty \text{ as } \alpha,\beta\to\infty.$$

3. New Concepts

In this section, we present the notions of deferred statistical convergence, $\mathcal{I}_2$-deferred Cesàro summability, and $\mathcal{I}_2$-deferred statistical convergence for double sequences in the context of FNS. We establish essential properties concerning these concepts and delve into defining $\mathcal{I}_2$-deferred statistical limit points as well as $\mathcal{I}_2$-deferred statistical cluster points for double sequences in FNS. Our inquiry centers on elucidating the interconnections among these introduced concepts, presenting pivotal findings that enrich the comprehension of $\mathcal{I}_2$-statistical convergence within FNS.

Throughout the article, we will consider $(X,\|.\|)$ as a FNS.

Definition 11. *A sequence (w_{uv}) in X is considered to be deferred statistically convergent to $w \in X$ regarding the fuzzy norm on X, where $(t_\alpha), (v_\alpha), (k_\beta), (l_\beta)$ are sequences of non-negative integers satisfying the conditions specified in Equation 1, then we write $w_{uv} \overset{DSt_2(FN)}{\rightarrow} w$ or $DSt_2(FN) - \lim w_{uv} = w$, provided that $DSt_2(FN) - \lim \|w_{uv} - w\| = \widetilde{0}$; i.e., for each $\lambda > 0$, we have*

$$\delta_2\left(\left\{(u,v) : t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta,\ \mathcal{F}\left(\|w_{uv} - w\|, \widetilde{0}\right) \geqslant \lambda\right\}\right) = 0,$$

or equivalently,

$$\lim_{\alpha,\beta\to\infty} \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta,\ \mathcal{F}\left(\|w_{uv} - w\|, \widetilde{0}\right) \geq \lambda\right\}\right| = 0.$$

This implies that for each $\lambda > 0$, the set

$$K(\lambda) = \left\{(u,v) : t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta,\ \|w_{uv} - w\|_0^+ \geqslant \lambda\right\}$$

has a natural density of zero. That is, for each $\lambda > 0$, $\|w_{uv} - w\|_0^+ < \lambda$ for a.a. u, v (all most all u, v). The element w belongs to the set X serves as the deferred statistical limit of the double sequence (w_{uv}).
A concise and insightful interpretation of the mentioned definition is as follows:

$$w_{uv} \overset{DSt_2(FN)}{\rightarrow} w \text{ iff } DSt_2(FN) - \lim \|w_{uv} - w\|_\alpha^+ = 0.$$

Noting that $DSt_2(FN) - \lim \|w_{uv} - w\|_\alpha^+ = 0$ implies that

$$DSt_2(FN) - \lim \|w_{uv} - w\|_\alpha^- = DSt_2(FN) - \lim \|w_{uv} - w\|_\alpha^+ = 0$$

for each $\alpha \in [0,1]$ since

$$0 \leqslant \|w_{uv} - w\|_\alpha^- \leqslant \|w_{uv} - w\|_\alpha^+ \leqslant \|w_{uv} - w\|_0^+$$

holds for every $u, v \in \mathbb{N}$ and for each $\alpha \in [0,1]$.

Example 3. *Let $(\mathbb{R}, \|\cdot\|_{\mathbb{R}})$ be an FNS. Then, a fuzzy norm $\|\cdot\|$ on $\mathbb{R}$ is define in Example 2 and $(t_\alpha), (v_\alpha), (k_\beta), (l_\beta)$ are sequences of non-negative integers satisfying the conditions specified in Equation 1. Define the sequence $w = (w_{uv})$ as*

$$w_{uv} := \begin{cases} u^2v^2; & [\sqrt{v_\alpha}] - 1 < u \leq [\sqrt{v_\alpha}] \\ & [\sqrt{l_\beta}] - 1 < v \leq [\sqrt{l_\beta}] \quad \alpha, \beta = 1, 2, 3, \ldots \\ 0; & \text{otherwise.} \end{cases}$$

where $0 < t_\alpha \leq [\sqrt{v_\alpha}] - 1$, $0 < k_\beta \leq [\sqrt{l_\beta}] - 1$ and (v_α), (l_β) are monotonic increasing sequences. Then, $w_{uv} \overset{DSt_2(FN)}{\rightarrow} 0$.
Justification: For every $0 < \lambda < 1$, $s > \|w\|$ we have

$$K(\lambda) = \left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta : \|w_{uv} - 0\|_0^+ \geqslant \lambda\right\}.$$

This implies that,

$$K(\lambda) = \left\{(u,v) : t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta : \tfrac{s-\|w_{uv}\|}{s+\|w_{uv}\|} \geqslant \lambda\right\}$$

$$= \left\{(u,v) : t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta : \|w_{uv}\| \leqslant \tfrac{s(1-\lambda)}{1+\lambda}\right\}$$

for suitable value of s *and* λ*, we get* $\{(u,v) : \|w_{uv}\| \geqslant 0\}$*. Hence*

$$K(\lambda) = \left\{(u,v) : t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta,\ \|w_{uv} - 0\|_0^+ \ge \lambda\right\}$$

$$= \left\{(u,v) : t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta,\ w_{uv} = u^2v^2\right\}$$

$$= \{(1,1), (4,4), (9,9), (16,16), \ldots\} \in \mathcal{I}_2.$$

As a result, $w_{uv} \overset{DSt_2(FN)}{\to} 0$.

Definition 12. *The double sequence* (w_{uv}) *is called to be* $\mathcal{I}_2$*-deferred Cesàro summable to* $w \in X$ *regarding the fuzzy norm on* X*, if for each* $\lambda > 0$

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \left\|\frac{1}{\rho_\alpha \phi_\beta} \sum_{\substack{u=t_\alpha+1 \\ v=k_\beta+1}}^{v_\alpha, l_\beta} w_{uv} - w\right\|_0^+ \ge \lambda\right\} \in \mathcal{I}_2,$$

and this condition is denoted in the format $w_{uv} \overset{DC_1(\mathcal{I}_2)(FN)}{\to} w$ *or* $DC_1(\mathcal{I}_2)(FN) - \lim w_{uv} = w$.

Definition 13. *The double sequence* (w_{uv}) *is said to be strongly* $\mathcal{I}_2$*-deferred Cesàro summable to* $w \in X$ *regarding the fuzzy norm on* X*, if for each* $\lambda > 0$

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta} \sum_{\substack{u=t_\alpha+1 \\ v=k_\beta+1}}^{v_\alpha, l_\beta} \|w_{uv} - w\|_0^+ \ge \lambda\right\} \in \mathcal{I}_2$$

and this case is denoted in $w_{uv} \overset{DC_1[\mathcal{I}_2](FN)}{\to} w$ *or* $DC_1[\mathcal{I}_2] - \lim w_{uv} = w$ *format.*

The notation $DC_1[\mathcal{I}_2](FN)$ represents the collection of all double sequences that exhibit strongly $\mathcal{I}_2$-deferred Cesàro summability with respect to the fuzzy norm X.

Remark 2. $DC_1(\mathcal{I}_2)(FN)$ *and* $DC_1[\mathcal{I}_2](FN)$*-summability concepts;*
(i) For $t_\alpha = 0$, $v_\alpha = \alpha$ *and* $k_\beta = 0$, $l_\beta = \beta$*, match with* $\mathcal{I}_2$*-Cesàro and strongly* $\mathcal{I}_2$*-Cesàro summability concepts regarding the fuzzy norm on* X*, respectively.*
(ii) For $t_\alpha = p_{\alpha-1}$, $v_\alpha = p_\alpha$ *and* $k_\beta = q_{\beta-1}$, $l_\beta = q_\beta$ $\{(p_\alpha, q_\beta)\}$ *states double lacunary sequence), match with* $\mathcal{I}_2$*-lacunary and strongly* $\mathcal{I}_2$*-lacunary convergence concepts regarding the fuzzy norm on* X*, respectively.*
(iii) For the ideal $\mathcal{I}_2^f$ *(the ideal of density zero sets of* $\mathbb{N}^2$*), match with deferred*

Cesàro and strongly deferred Cesàro summability concepts regarding the fuzzy norm on X, respectively.

Definition 14. *The double sequence (w_{uv}) is considered to be $\mathcal{I}_2$-deferred statistical convergent to $w \in X$ regarding the fuzzy norm on X, if for every $\lambda, \mu > 0$, the set*

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta,\ \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right| \geq \mu\right\} \in \mathcal{I}_2.$$

This scenario is denoted as $w_{uv} \overset{DS(\mathcal{I}_2)(FN)}{\rightarrow} w$ or $DS(\mathcal{I}_2)(FN) - \lim w_{uv} = w$. The collection of all double sequences of sets that are $\mathcal{I}_2$-deferred statistically convergent with respect to the fuzzy norm X is represented by $DS(\mathcal{I}_2)(FN)$.

Remark 3. *The concepts outlined match with different forms of convergence within the framework of $DS(\mathcal{I}_2)(FN)$;*
(i) When $t_\alpha = 0, v_\alpha = \alpha$ and $k_\beta = 0, l_\beta = \beta$, it corresponds to the $\mathcal{I}_2$-statistical convergence concept with respect to the fuzzy norm X.
(ii) When $t_\alpha = p_{\alpha-1}, v_\alpha = p_\alpha$ and $k_\beta = q_{\beta-1}, l_\beta = q_\beta$ (where $\{(p_\alpha, q_\beta)\}$ denotes a double lacunary sequence), it aligns with the $\mathcal{I}_2$-lacunary statistical convergence concept with respect to the fuzzy norm X.
(iii) When considering the ideal $\mathcal{I}_2^f$ (the ideal of density zero sets of $\mathbb{N}$), it corresponds to the deferred statistical convergence concept with respect to the fuzzy norm X.

Definition 15. *Let (w_{uv}) be a sequence in $(X, \|\cdot\|)$ with $(t_\alpha), (v_\alpha), (k_\beta), (l_\beta)$ being sequences of non-negative integers satisfying the conditions specified in Equation 1. We say that the sequence (w_{uv}) in X is $\mathcal{I}_2$-deferred statistically Cauchy with respect to the fuzzy norm on X if, for every $\lambda, \mu > 0$, there exist natural numbers $N = N(\lambda)$, $M = M(\lambda)$ such that*

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv} - w_{NM}\|_0^+ \geq \lambda\right\}\right| \geq \mu\right\} \in \mathcal{I}_2.$$

Theorem 3. *Let (w_{uv}) be a sequence in $(X, \|\cdot\|)$ with $(t_\alpha), (v_\alpha), (k_\beta), (l_\beta)$ are sequences of non-negative integers satisfying the conditions specified in Equation 1. Then, every $\mathcal{I}_2$-deferred statistically convergent sequence is also a $\mathcal{I}_2$-deferred statistically Cauchy sequence.*

Proof. Let $DS(\mathcal{I}_2)(FN) - \lim w_{uv} = w$ and $\lambda, \mu > 0$. Then, we have

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \frac{\lambda}{2}\right\}\right| \geq \frac{\mu}{2}\right\} \in \mathcal{I}_2.$$

Choose N, $M \in \mathbb{N}$ such that

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv} - w_{NM}\|_0^+ \geq \frac{\lambda}{2}\right\}\right| \geq \frac{\mu}{2}\right\} \in \mathcal{I}_2.$$

Now $\|\cdot\|_0^+$ being a norm in the usual sense, we get

$$\left\{(\alpha,\beta)\in\mathbb{N}^2:\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta,\|w_{uv}-w_{NM}\|_0^+\ge\lambda\right\}\right|\ge\mu\right\}$$

$$=\left\{(\alpha,\beta)\in\mathbb{N}^2:\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta,\|(w_{uv}-w)+(w-w_{NM})\|_0^+\ge\lambda\right\}\right|\ge\mu\right\}$$

$$\subseteq\left\{(\alpha,\beta)\in\mathbb{N}^2:\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta,\|w_{uv}-w\|_0^+\ge\frac{\lambda}{2}\right\}\right|\ge\frac{\mu}{2}\right\}$$

$$\cup\left\{(\alpha,\beta)\in\mathbb{N}^2:\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta,\|w_{uv}-w_{NM}\|_0^+\ge\frac{\lambda}{2}\right\}\right|\ge\frac{\mu}{2}\right\}\in\mathcal{I}_2.$$

This indicates that the double sequence (w_{uv}) is $\mathcal{I}_2$-deferred statistically Cauchy. □

4. Main Results

In this section, we initially explore the connections between $DC_1[\mathcal{I}_2](FN)$-summability and $DS(\mathcal{I}_2)(FN)$-convergence concepts.

Theorem 4. *Let (w_{uv}), (t_{uv}) be sequences of real numbers, then*
(i) If $DS(\mathcal{I}_2)(FN)-\lim w_{uv}=w_0$ and $DS(\mathcal{I}_2)(FN)-\lim t_{uv}=t_0$, then
$DS(\mathcal{I}_2)(FN)-\lim(w_{uv}+t_{uv})=w_0+t_0$,
(ii)If $DS(\mathcal{I}_2)(FN)-\lim w_{uv}=w_0$ and $q\in\mathbb{C}$, then
$DS(\mathcal{I}_2)(FN)-\lim(qw_{uv})=qw_0$,
(iii) If $DS(\mathcal{I}_2)(FN)-\lim w_{uv}=w_0$ and $DS(\mathcal{I}_2)(FN)-\lim t_{uv}=t_0$, and there are positive numbers u and v such that $\|w_{uv}\|\le u$ and $\|t_0\|\le v$ for any u,v, then
$DS(\mathcal{I}_2)(FN)-\lim(w_{uv}t_{uv})=w_0t_0$.

Proof. (i) Assume that $DS(\mathcal{I}_2)(FN)-\lim w_{uv}=w_0$ and $DS(\mathcal{I}_2)(FN)-\lim t_{uv}=t_0$. Since $\|\cdot\|_0^+$ is a norm in the usual sense, we get

$$\|(w_{uv}+t_{uv})-(w_0+t_0)\|_0^+\leqslant\|w_{uv}-w_0\|_0^++\|t_{uv}-t_0\|_0^+\tag{2}$$

for all $u,v\in\mathbb{N}$. Now let us write

$$K(\lambda)=\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta:\|(w_{uv}+t_{uv})-(w_0+t_0)\|_0^+\geqslant\lambda\right\},$$

$$K_1(\lambda)=\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta:\|w_{uv}-w_0\|_0^+\geqslant\frac{\lambda}{2}\right\}$$

$$K_2(\lambda)=\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta:\|t_{uv}-t_0\|_0^+\geqslant\frac{\lambda}{2}\right\}.$$

Therefore, based on Equation 2, it follows that $K(\lambda)\subseteq K_1(\lambda)\cup K_2(\lambda)$. Given our assumption that $K_1(\lambda),K_2(\lambda)\in\mathcal{I}_2$. We conclude that $K(\lambda)\in\mathcal{I}_2$, thereby completing the proof.

(ii) Let, $DS(\mathcal{I}_2)(FN) - \lim w_{uv} = w_0$, then $q \in \mathbb{R} - \{0\}$ for every $\lambda > 0$, we have

$$\left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta : \|w_{uv} - w_0\|_0^+ \geqslant \frac{\lambda}{|q|}\right\} \in \mathcal{I}_2,$$

$$\Longrightarrow \left\{(u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta : \|qw_{uv} - qw_0\|_0^+ \geqslant \lambda\right\} \in \mathcal{I}_2.$$

So $DS(\mathcal{I}_2)(FN) - \lim (qw_{uv}) = qw_0$, $(q \in \mathbb{R})$.
(iii) Assume $\lambda, \mu > 0$ and $u, v > 0$ then

$$K = \left\{(\alpha,\beta) \in \mathbb{N}^2 : \tfrac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv} - w_0\|_0^+ \geq \lambda\right\}\right| < \tfrac{\mu}{2v}\right\} \in \mathcal{F}(\mathcal{I}_2)$$

and

$$L = \left\{(\alpha,\beta) \in \mathbb{N}^2 : \tfrac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|t_{uv} - t_0\|_0^+ \geq \lambda\right\}\right| < \tfrac{\mu}{2u}\right\} \in \mathcal{F}(\mathcal{I}_2).$$

Since $K \cap L \in \mathcal{F}(\mathcal{I}_2)$ and $\emptyset \notin \mathcal{F}(\mathcal{I}_2)$ this means $K \cap L \neq \emptyset$. So, for all $(u,v) \in K \cap L$ we have

$$\begin{aligned}
\|w_{uv}t_{uv} - w_0 t_0\|_0^+ &= \|w_{uv}t_{uv} - w_{uv}t_0 + w_{uv}t_0 - w_0 t_0\|_0^+ \\
&\leq \|w_{uv}t_{uv} - w_{uv}t_0\|_0^+ + \|w_{uv}t_0 - w_0 t_0\|_0^+ \\
&\leq u\|t_{uv} - t_0\|_0^+ + v\|w_{uv} - w_0\|_0^+ < u\tfrac{\mu}{2u} + v\tfrac{\mu}{2v} = \mu,
\end{aligned}$$

i.e.,

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv}t_{uv} - w_0 t_0\|_0^+ \geq \lambda\right\}\right| < \mu\right\} \in \mathcal{F}(\mathcal{I}_2).$$

Hence $DS(\mathcal{I}_2)(FN) - \lim (w_{uv}t_{uv}) = w_0 t_0$. □

Theorem 5. *$DS(\mathcal{I}_2)(FN) \cap L_\infty^2$ is a closed subset of L_∞^2.*

Proof. Suppose that $(w^j)_{j\in\mathbb{N}} = (w_{uv}^j) \subseteq DS(\mathcal{I}_2)(FN) \cap L_\infty^2$ is convergent sequence and that it converges to $w \in L_\infty^2$. We need to prove that $w \in DS(\mathcal{I}_2)(FN) \cap L_\infty^2$. Assume that $w^j \to L_j(DS(\mathcal{I}_2)(FN))$ for $\forall j \in \mathbb{N}$. Take a positive strictly decreasing sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ where $\lambda_j = \frac{\lambda}{2^j}$ for a given $\lambda > 0$. It is evident that the sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ converges to 0. Let's select positive integer j such that $\|w - w^j\|_\infty < \frac{\lambda_j}{4}$. Let $0 < \mu < 1$. Then

$$\begin{aligned}
A = \Big\{(\alpha,\beta) \in \mathbb{N}^2 : \tfrac{1}{\rho_\alpha \phi_\beta} \big|\{t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \\
\|w_{uv}^j - L_j\|_0^+ \geq \tfrac{\lambda_j}{4}\Big\}\Big| < \tfrac{\mu}{3}\Big\} \in \mathcal{F}(\mathcal{I}_2),
\end{aligned}$$

and

$$\begin{aligned}
B = \Big\{(\alpha,\beta) \in \mathbb{N}^2 : \tfrac{1}{\rho_\alpha \phi_\beta} \big|\{t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \\
\|w_{uv}^{j+1} - L_{j+1}\|_0^+ \geq \tfrac{\lambda_{j+1}}{4}\Big\}\Big| < \tfrac{\mu}{3}\Big\} \in \mathcal{F}(\mathcal{I}_2).
\end{aligned}$$

Since $A \cap B \in \mathcal{F}(\mathcal{I}_2)$ and $\phi \notin \mathcal{F}(\mathcal{I}_2)$, we can choose $(\alpha,\beta) \in A \cap B$. Then

$$\frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv}^j - L_j\|_0^+ \geq \frac{\lambda_j}{4}\right\}\right| < \frac{\mu}{3},$$

and

$$\frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \left\|w_{uv}^{j+1} - L_{j+1}\right\|_0^+ \ge \frac{\lambda_{j+1}}{4}\right\}\right| < \frac{\mu}{3}$$

and so

$$\tfrac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \left\|w_{uv}^{j} - L_{j}\right\|_0^+ \ge \tfrac{\lambda_{j}}{4} \vee \left\|w_{uv}^{j+1} - L_{j+1}\right\|_0^+ \ge \tfrac{\lambda_{j+1}}{4}\right\}\right| < \mu < 1.$$

Hence, there exist $t_\alpha < u \le v_\alpha$, $k_\beta < v \le l_\beta$ for which $\left\|w_{uv}^{j} - L_{j}\right\|_0^+ \ge \frac{\lambda_j}{4}$ and $\left\|w_{uv}^{j+1} - L_{j+1}\right\|_0^+ \ge \frac{\lambda_{j+1}}{4}$. Then, we can write

$$\begin{aligned}
\|L_j - L_{j+1}\|_0^+ &\le \left\|L_j - w_{uv}^j\right\|_0^+ + \left\|w_{uv}^j - w_{uv}^{j+1}\right\|_0^+ + \left\|w_{uv}^{j+1} - L_{j+1}\right\|_0^+ \\
&\le \left\|w_{uv}^j - L_j\right\|_0^+ + \left\|w_{uv}^{j+1} - L_{j+1}\right\|_0^+ + \left\|w - w^j\right\|_\infty + \left\|w - w^{j+1}\right\|_\infty \\
&\le \tfrac{\lambda_j}{4} + \tfrac{\lambda_{j+1}}{4} + \tfrac{\lambda_j}{4} + \tfrac{\lambda_{j+1}}{4} \le \lambda_j.
\end{aligned}$$

This implies that $\{L_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and so there is a real number L such that $L_j \to L$, as $j \to \infty$. We need to prove that $w \to L\,(DS\,(\mathcal{I}_2)\,(FN))$. For any $\lambda > 0$, choose $j \in \mathbb{N}$ such that $\lambda_j < \frac{\lambda}{4}$, $\left\|w - w^j\right\|_\infty < \frac{\lambda}{4}$, $\|L_j - L\|_0^+ < \frac{\lambda}{4}$. Then

$$\begin{aligned}
&\tfrac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \|w_{uv} - L\|_0^+ \ge \lambda\right\}\right| \\
&\le \tfrac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \left\|w_{uv}^j - L_j\right\|_0^+ + \left\|w_{uv} - w_{uv}^j\right\|_\infty + \|L_j - L\|_0^+ \ge \lambda\right\}\right| \\
&\le \tfrac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \left\|w_{uv}^j - L_j\right\|_0^+ + \tfrac{\lambda}{4} + \tfrac{\lambda}{4} \ge \lambda\right\}\right| \\
&\le \tfrac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \left\|w_{uv}^j - L_j\right\|_0^+ \ge \tfrac{\lambda}{2}\right\}\right|.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left\{(\alpha,\beta) \in \mathbb{N}^2 : \tfrac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \|w_{uv} - L\|_0^+ \ge \lambda\right\}\right| < \mu\right\} \\
&\supseteq \left\{(\alpha,\beta) \in \mathbb{N}^2 : \tfrac{1}{\rho_\alpha \phi_\beta}\left|\{t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta,\right.\right. \\
&\left.\left.\left\|w_{uv}^j - L_j\right\|_0^+ \ge \tfrac{\lambda}{2}\right\}\right| < \mu\right\} \in \mathcal{F}\,(\mathcal{I}_2)\,.
\end{aligned}$$

So

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \|w_{uv} - L\|_0^+ \ge \lambda\right\}\right| < \mu\right\} \in \mathcal{F}\,(\mathcal{I}_2)\,,$$

and hence

$$\left\{(\alpha,\beta)\in\mathbb{N}^2:\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta,\|w_{uv}-L\|_0^+\ge\lambda\right\}\right|\ge\mu\right\}\in\mathcal{I}_2.$$

This implies that $w\to L\left(DS\left(\mathcal{I}_2\right)(FN)\right)$, thereby completing the proof of the theorem. □

Theorem 6. *If a double sequence (w_{uv}) is strongly $\mathcal{I}_2$-deferred Cesàro summable to $w\in X$, then this sequence is $\mathcal{I}_2$-deferred statistical convergent to $w\in X$. Also, the inclusion $DC_1\left[\mathcal{I}_2\right](FN)\subseteq DS\left(\mathcal{I}_2\right)(FN)$ is strict.*

Proof. Suppose that $w_{uv}\overset{DC_1[\mathcal{I}_2](FN)}{\to}w$. For each $\lambda>0$, we can express

$$\sum_{\substack{u=t_\alpha+1\\v=k_\beta+1}}^{v_\alpha,l_\beta}\|w_{uv}-w\|_0^+\ge\sum_{\substack{u=t_\alpha+1\\v=k_\beta+1\\\|w_{uv}-w\|_0^+\ge\lambda}}^{v_\alpha,l_\beta}\|w_{uv}-w\|_0^+$$

$$\ge\lambda\left|\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta,\|w_{uv}-w\|_0^+\ge\lambda\right\}\right|,$$

and therefore, we have

$$\frac{1}{\lambda}\frac{1}{\rho_\alpha\phi_\beta}\sum_{\substack{u=t_\alpha+1\\v=k_\beta+1}}^{v_\alpha,l_\beta}\|w_{uv}-w\|_0^+\ge\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\le v_\alpha,k_\beta<v\le l_\beta,\|w_{uv}-w\|_0^+\ge\lambda\right\}\right|.$$

For every $\mu>0$ we obtain

$$\left\{(\alpha,\beta)\in\mathbb{N}^2:\tfrac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\le v_\alpha,\ k_\beta<v\le l_\beta,\|w_{uv}-w\|_0^+\ge\lambda\right\}\right|\ge\mu\right\}$$

$$\subseteq\left\{(\alpha,\beta)\in\mathbb{N}^2:\tfrac{1}{\rho_\alpha\phi_\beta}\sum_{\substack{u=t_\alpha+1\\v=k_\beta+1}}^{v_\alpha,l_\beta}\|w_{uv}-w\|_0^+\ge\lambda\mu\right\}\in\mathcal{I}_2.$$

Thus, we get $w_{uv}\overset{DS(\mathcal{I}_2)(FN)}{\to}w$.

To show the strictness of the inclusion, choose $v_\alpha=\alpha$, $t_\alpha=0$ and $l_\beta=\beta$, $k_\beta=0$ define a sequence (w_{uv}) by

$$w_{uv}=\begin{cases}\sqrt{pq}, & u=p^2,\ v=q^2\\ 0, & u\ne p^2,\ v\ne q^2.\end{cases}$$

Then, for every $\lambda>0$, we have

$$\frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v):t_\alpha<u\le v_\alpha,\ k_\beta<v\le l_\beta,\|w_{uv}-0\|_0^+\ge\lambda\right\}\right|\le\frac{\sqrt{pq}}{pq}$$

and for any $\mu > 0$ we get

$$\left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v) : t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \|w_{uv}-0\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\}$$

$$\subseteq \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{\sqrt{pq}}{pq} \ge \mu\right\}.$$

As the set on the right-hand side is finite and thus falls within $\mathcal{I}_2$, it implies that $DS\left(\mathcal{I}_2\right)(FN) - \lim w_{uv} = 0$. On the other hand

$$\frac{1}{\rho_\alpha\phi_\beta}\sum_{\substack{u=t_\alpha+1\\ v=k_\beta+1}}^{v_\alpha,l_\beta}\|w_{uv}-0\|_0^+ = \frac{[\sqrt{pq}][\sqrt{pq}]}{pq} \to 1.$$

Then

$$\left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho_\alpha\phi_\beta}\sum_{\substack{u=t_\alpha+1\\ v=k_\beta+1}}^{v_\alpha,l_\beta}\|w_{uv}-0\|_0^+ \ge 1\right\}$$

$$= \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{\sqrt{pq}}{pq} \ge 1\right\} = \{(m,n),(m+1,n+1),(m+2,n+2),\ldots\}$$

for some $m,n \in \mathbb{N}$ which belongs to $\mathcal{F}(\mathcal{I}_2)$, since $\mathcal{I}_2$ is admissible. So $w_{uv} \nrightarrow 0\left(DC_1\left[\mathcal{I}_2\right](FN)\right)$. □

Corollary 1. *If* $w_{uv} \overset{\mathcal{I}_2(FN)}{\to} w$, *then* $w_{uv} \overset{DS(\mathcal{I}_2)(FN)}{\to} w$.

The converse of Theorem 6 is not generally valid. To illustrate this point, we can consider the following example by choosing $\mathcal{I}_2 = \mathcal{I}_2^f$ (the ideal of density zero sets of $\mathbb{N}$).

Example 4. *Consider* $X = \mathbb{R}^2$ *and let* (w_{uv}) *denote a double sequence defined as follows:*

$$w_{uv} := \begin{cases} u^2v^2; & v_\alpha - \left[\left|\sqrt{\rho_\alpha}\right|\right] < u \le v_\alpha, \\ & l_\beta - \left[\left|\sqrt{\phi_\beta}\right|\right] < v \le l_\beta, \quad (u,v)\in\mathbb{N}^2, \\ 0; & otherwise. \end{cases}$$

This sequence is unbounded. Additionally, it is $\mathcal{I}_2$*-deferred statistical convergent to* $w = 0$ *with respect to the fuzzy norm* X*, but it is not strongly* $\mathcal{I}_2$*-deferred Cesàro summable with respect to the fuzzy norm* X*.*

Theorem 7. *If a double sequence* $(w_{uv}) \in L_\infty^2$ *is* $\mathcal{I}_2$*-deferred statistical convergent to* $w \in X$ *with respect to the fuzzy norm* X*, then this sequence is also strongly* $\mathcal{I}_2$*-deferred Cesàro summable to the same limit.*

Proof. Suppose that $(w_{uv}) \in L^2_\infty$ and $w_{uv} \overset{DS(\mathcal{I}_2)(FN)}{\rightarrow} w$. Let $(w_{uv}) \in L^2_\infty$. Therefore, there exists $\mathcal{U} > 0$ such that $\|w_{uv} - w\|_0^+ \leq \mathcal{U}$ for all $(u,v) \in \mathbb{N}^2$. For each $\lambda > 0$, we have

$$\frac{1}{\rho_\alpha \phi_\beta} \sum_{\substack{u=t_\alpha+1 \\ v=k_\beta+1}}^{v_\alpha, l_\beta} \|w_{uv} - w\|_0^+$$
$$= \frac{1}{\rho_\alpha \phi_\beta} \sum_{\substack{u=t_\alpha+1 \\ v=k_\beta+1 \\ \|w_{uv}-w\|_0^+ \geq \lambda}}^{v_\alpha, l_\beta} \|w_{uv} - w\|_0^+ + \frac{1}{\rho_\alpha \phi_\beta} \sum_{\substack{u=t_\alpha+1 \\ v=k_\beta+1 \\ \|w_{uv}-w\|_0^+ < \lambda}}^{v_\alpha, l_\beta} \|w_{uv} - w\|_0^+$$
$$\leq \frac{\mathcal{U}}{\rho_\alpha \phi_\beta} \left| \left\{ (u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda \right\} \right| + \lambda.$$

So, for all $\mu > 0$ we get

$$\left\{ (\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta} \sum_{\substack{u=t_\alpha+1 \\ v=k_\beta+1}}^{v_\alpha, l_\beta} \|w_{uv} - w\|_0^+ \geq \mu \right\}$$
$$\subseteq \left\{ (\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta} \left| \left\{ (u,v) : t_\alpha < u \leq v_\alpha, k_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda \right\} \right| \geq \frac{\mu}{\mathcal{U}} \right\} \in \mathcal{I}_2.$$

As a result, we get $w_{uv} \overset{DC_1[\mathcal{I}_2](FN)}{\rightarrow} w$. □

By Theorem 6 and Theorem 7, we obtain the following corollary.

Corollary 2. $L^2_\infty \cap DC_1[\mathcal{I}_2](FN) = L^2_\infty \cap DS(\mathcal{I}_2)(FN)$.

Theorem 8. *Let* $\left(\frac{t_\alpha}{\rho_\alpha}\right)$ *and* $\left(\frac{k_\beta}{\phi_\beta}\right)$ *be bounded, then*

$$w_{uv} \overset{S(\mathcal{I}_2)(FN)}{\rightarrow} w \Rightarrow w_{uv} \overset{DS(\mathcal{I}_2)(FN)}{\rightarrow} w.$$

Proof. To begin, given that $\left(\frac{t_\alpha}{\rho_\alpha}\right)$ is bounded, there exists exists a positive value $\sigma > 0$ such that $\frac{t_\alpha}{\rho_\alpha} < \sigma$ for all $\alpha \in \mathbb{N}$. Therefore, we express this as:

$$\frac{t_\alpha}{\rho_\alpha} < \sigma \Rightarrow \frac{\rho_\alpha}{v_\alpha} > \frac{1}{1+\sigma}.$$

Likewise, for each $\kappa \in \mathbb{N}$, we can derive the following inequalities

$$\frac{k_\beta}{\phi_\beta} < \kappa \Rightarrow \frac{\phi_\beta}{l_\beta} > \frac{1}{1+\kappa}.$$

Assume that $w_{uv} \overset{S(\mathcal{I}_2)(FN)}{\rightarrow} w$. For each $\lambda > 0$, we have

$$\begin{aligned}&\left\{(u,v): t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\\ &\subseteq \left\{(u,v): u \le v_\alpha, v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\},\end{aligned}$$

and so

$$\begin{aligned}&\tfrac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v): t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right|\\ &\le \tfrac{v_\alpha l_\beta}{\rho_\alpha\phi_\beta}\tfrac{1}{v_\alpha l_\beta}\left|\left\{(u,v): u \le v_\alpha, v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right|.\end{aligned}$$

Thus, for any positive value $\mu > 0$ we obtain

$$\begin{aligned}&\left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v): t_\alpha < u \le v_\alpha, k_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\}\\ &\subseteq \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{v_\alpha l_\beta}\left|\left\{(u,v): u \le v_\alpha, v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \tfrac{\mu}{(1+\sigma)(1+\kappa)}\right\}.\end{aligned}$$

As a result, we get $w_{uv} \overset{DS(\mathcal{I}_2)(FN)}{\rightarrow} w$. □

We will examine the following theorems under the given constraints:

$$t_\alpha \le t'_\alpha < v'_\alpha \le v_\alpha \quad \text{and} \quad k_\beta \le k'_\beta < l'_\beta \le l_\beta$$

for all $\alpha, \beta \in \mathbb{N}$, where each of these represents sequences of non-negative integers.

Theorem 9. *If* $\left(\frac{\rho_\alpha\phi_\beta}{\rho'_\alpha\phi'_\beta}\right)$ *is bounded, then*

$$DS(\mathcal{I}_2)(FN)_{[\rho,\phi]} \subseteq DS(\mathcal{I}_2)(FN)_{[\rho',\phi']}.$$

Proof. To begin, given that $\left(\frac{\rho_\alpha\phi_\beta}{\rho'_\alpha\phi'_\beta}\right)$ is bounded, there exists an $\varpi > 0$ such that $\frac{\rho_\alpha\phi_\beta}{\rho'_\alpha\phi'_\beta} < \varpi$ for all $\alpha, \beta \in \mathbb{N}$. Assuming $(w_{uv}) \in DS(\mathcal{I}_2)(FN)_{[\rho,\phi]}$ and $w_{uv} \overset{DS(\mathcal{I}_2)(FN)_{[\rho,\phi]}}{\rightarrow} w$. For any $\lambda > 0$ since

$$\begin{aligned}&\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ k'_\beta < v \le l'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\\ &\subseteq \left\{(u,v): t_\alpha < u \le v_\alpha,\ k_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\},\end{aligned}$$

we can express

$$\begin{aligned}&\tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ k'_\beta < v \le l'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right|\\ &\le \tfrac{\rho_\alpha\phi_\beta}{\rho'_\alpha\phi'_\beta}\left(\tfrac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v): t_\alpha < u \le v_\alpha,\ k_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right|\right)\end{aligned}$$

So, for each $\mu > 0$ we obtain

$$\begin{aligned}&\left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ k'_\beta < v \le l'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\}\\ &\subseteq \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v): t_\alpha < u \le v_\alpha,\ k_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \tfrac{\mu}{\varpi}\right\}.\end{aligned}$$

Thus, we get $w_{uv} \overset{DS(\mathcal{I}_2)(FN)_{[\rho',\phi']}}{\rightarrow} w$. As a result,

$$DS\left(\mathcal{I}_2\right)\left(FN\right)_{[\rho,\phi]} \subseteq DS\left(\mathcal{I}_2\right)\left(FN\right)_{[\rho',\phi']}.$$

□

Theorem 10. *If the sets* $\{u : t_\alpha < u \leq t'_\alpha\}, \{u : v'_\alpha < u \leq v_\alpha\}, \left\{v : k_\beta < v \leq k'_\beta\right\}$, $\left\{v : l'_\beta < v \leq l_\beta\right\}$ *are finite for every* $\alpha, \beta \in \mathbb{N}$, *then*

$$DS\left(\mathcal{I}_2\right)\left(FN\right)_{[\rho',\phi']} \subseteq DS\left(\mathcal{I}_2\right)\left(FN\right)_{[\rho,\phi]}.$$

Proof. Let $(w_{uv}) \in DS\left(\mathcal{I}_2\right)\left(FN\right)_{[\rho',\phi']}$ and $w_{uv} \overset{DS(\mathcal{I}_2)(FN)_{[\rho',\phi']}}{\rightarrow} w$. Then, for all $\lambda, \mu > 0$ we have

$$\left\{(\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v) : t'_\alpha < u \leq v'_\alpha,\ k'_\beta < v \leq l'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right| \geq \mu\right\} \in \mathcal{I}_2.$$

Additionally, for each $\lambda > 0$, since

$$\begin{aligned}
&\left\{(u,v) : t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&= \left\{(u,v) : t_\alpha < u \leq t'_\alpha,\ k_\beta < v \leq k'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : t_\alpha < u \leq t'_\alpha,\ k'_\beta < v \leq l'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : t_\alpha < u \leq t'_\alpha,\ l'_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : t'_\alpha < u \leq v'_\alpha,\ k_\beta < v \leq k'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : t'_\alpha < u \leq v'_\alpha,\ k'_\beta < v \leq l'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : t'_\alpha < u \leq v'_\alpha,\ l'_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : v'_\alpha < u \leq v_\alpha,\ k_\beta < v \leq k'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : v'_\alpha < u \leq v_\alpha,\ k'_\beta < v \leq l'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\} \\
&\cup \left\{(u,v) : v'_\alpha < u \leq v_\alpha,\ l'_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\},
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{1}{\rho_\alpha \phi_\beta}\left|\left\{(u,v) : t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right| \\
&\leq \frac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v) : t_\alpha < u \leq t'_\alpha,\ k_\beta < v \leq k'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right| \\
&+ \frac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v) : t_\alpha < u \leq t'_\alpha,\ k'_\beta < v \leq l'_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right| \\
&+ \frac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v) : t_\alpha < u \leq t'_\alpha,\ l'_\beta < v \leq l_\beta, \|w_{uv} - w\|_0^+ \geq \lambda\right\}\right|
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ k_\beta < v \le k'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \\
&+ \frac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ k'_\beta < v \le l'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \\
&+ \frac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ l'_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \\
&+ \frac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v): v'_\alpha < u \le v_\alpha,\ k_\beta < v \le k'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \\
&+ \frac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v): v'_\alpha < u \le v_\alpha,\ k'_\beta < v \le l'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \\
&+ \frac{1}{\rho'_\alpha \phi'_\beta}\left|\left\{(u,v): v'_\alpha < u \le v_\alpha,\ l'_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right|,
\end{aligned}
$$

and hence, for all $\mu > 0$

$$
\begin{aligned}
&\left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v): t_\alpha < u \le v_\alpha,\ k_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\subseteq \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t_\alpha < u \le t'_\alpha,\ k_\beta < v \le k'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t_\alpha < u \le t'_\alpha,\ k'_\beta < v \le l'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t_\alpha < u \le t'_\alpha,\ l'_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ k_\beta < v \le k'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ k'_\beta < v \le l'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): t'_\alpha < u \le v'_\alpha,\ l'_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): v'_\alpha < u \le v_\alpha,\ k_\beta < v \le k'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): v'_\alpha < u \le v_\alpha,\ k'_\beta < v \le l'_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \\
&\cup \left\{(\alpha,\beta)\in\mathbb{N}^2 : \tfrac{1}{\rho'_\alpha\phi'_\beta}\left|\left\{(u,v): v'_\alpha < u \le v_\alpha,\ l'_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\}.
\end{aligned}
$$

If the sets $\{u : t_\alpha < u \le t'_\alpha\}$, $\{u : v'_\alpha < u \le v_\alpha\}$, $\left\{v : k_\beta < v \le k'_\beta\right\}$, $\left\{v : l'_\beta < v \le l_\beta\right\}$ are all finite for every $\alpha, \beta \in \mathbb{N}$ within the given expression, based on the assumption, we conclude

$$
\left\{(\alpha,\beta)\in\mathbb{N}^2 : \frac{1}{\rho_\alpha\phi_\beta}\left|\left\{(u,v): t_\alpha < u \le v_\alpha,\ k_\beta < v \le l_\beta, \|w_{uv}-w\|_0^+ \ge \lambda\right\}\right| \ge \mu\right\} \in \mathcal{I}_2,
$$

This indicates that $w_{uv} \overset{DS(\mathcal{I}_2)(FN)_{[\rho,\phi]}}{\rightarrow} w$ and $(w_{uv}) \in DS(\mathcal{I}_2)(FN)_{[\rho,\phi]}$. Hence, $DS(\mathcal{I}_2)(FN)_{[\rho',\phi']} \subseteq DS(\mathcal{I}_2)(FN)_{[\rho,\phi]}$. □

Based on Theorem 6, Theorem 7, Theorem 9 and Theorem 10, we derive the following corollary.

Corollary 3. *(i) Assuming that* $\left(\frac{\rho_\alpha \phi_\beta}{\rho'_\alpha \phi'_\beta}\right)$ *is bounded. If a double sequence* (w_{uv}) *is* $DC_1\left[\mathcal{I}_2\right](FN)_{[\rho,\phi]}$*-summable to* $w \in X$ *with respect to the fuzzy norm* X*, then this sequence is also* $DS\left(\mathcal{I}_2\right)(FN)_{[\rho',\phi']}$*-convergent to* $w \in X$.
(ii) Let the sets $\{u : t_\alpha < u \leq t'_\alpha\}$, $\{u : v'_\alpha < u \leq v_\alpha\}$, $\left\{v : k_\beta < v \leq k'_\beta\right\}$, $\left\{v : l'_\beta < v \leq l_\beta\right\}$ *are finite for all* $\alpha, \beta \in \mathbb{N}$*. If a double sequence* $(w_{uv}) \in L^2_\infty$ *is* $DS\left(\mathcal{I}_2\right)(FN)_{[\rho',\phi']}$*-convergent to* $w \in X$ *with respect to the fuzzy norm* X*, then this sequence is* $DC_1\left[\mathcal{I}_2\right](FN)_{[\rho,\phi]}$*-summable to* $w \in X$.

Now, we will examine the notions of $\mathcal{I}_2$-deferred statistical limit points and $\mathcal{I}_2$-deferred statistical cluster points of a double sequence of fuzzy numbers, expanding upon the concepts previously discussed regarding a sequence of fuzzy numbers. Additionally, attention will be directed towards significant fundamental characteristics pertaining to the set of all $\mathcal{I}_2$-deferred statistical cluster points and the set of all $\mathcal{I}_2$-deferred statistical limit points of a double sequence of fuzzy numbers, and an exploration of the relationship between them will be conducted.

Definition 16. *An element* $w_0 \in X$ *is termed an* $\mathcal{I}_2$*-deferred statistical limit point of double sequence* (w_{uv}) *with respect to the fuzzy norm* X *if, for each* $\lambda > 0$ *there exists a set*

$$U = \{(u_1, v_1) < (u_2, v_2) < ... < (u_r, v_s) < ...\} \subset \mathbb{N}^2$$

such that $U \notin \mathcal{I}_2$ *and* $DSt_2(FN) - \lim w_{u_r, v_s} = w_0$.

The notation $\mathcal{I}_2^{FN} - S(\Lambda_w)$ *represents the set comprising all* $\mathcal{I}_2$*-deferred statistical limit point of a double sequence* (w_{uv}).

Theorem 11. *If* $DS\left(\mathcal{I}_2\right)(FN) - \lim w_{uv} = w_0$*, then* $\mathcal{I}_2^{FN} - S(\Lambda_w) = \{w_0\}$.

Proof. Given that $DS\left(\mathcal{I}_2\right)(FN) - \lim w_{uv} = w_0$, for each $\lambda, \mu > 0$, the set

$$T = \left\{(\alpha, \beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w_0\|_0^+ \geq \lambda\right\}\right| \geq \mu\right\} \in \mathcal{I}_2,$$

where $\mathcal{I}_2$ is an admissible ideal.

Let's assume that $\mathcal{I}_2^{FN} - S(\Lambda_w)$ includes q_0 distinct from w_0, that is, $q_0 \in \mathcal{I}_2^{FN} - S(\Lambda_w)$. Therefore, there exists a $U \subset \mathbb{N}^2$ such that $U \notin \mathcal{I}_2$ and $DSt_2(FN) - \lim w_{u_r, v_s} = q_0$.

Let

$$P = \left\{(\alpha, \beta) \in M : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - q_0\|_0^+ \geq \lambda\right\}\right| \geq \mu\right\}.$$

So, P is a finite set, implying that $P \in \mathcal{I}_2$. So

$$P^c = \left\{(\alpha, \beta) \in M : \frac{1}{\rho_\alpha \phi_\beta}\left|\left\{t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - q_0\|_0^+ \geq \lambda\right\}\right| < \mu\right\} \in \mathcal{F}\left(\mathcal{I}_2\right).$$

Let T_1 be defined as follows:

$$T_1 = \left\{ (\alpha,\beta) \in M : \frac{1}{\rho_\alpha \phi_\beta} \left| \left\{ t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w_0\|_0^+ \geq \lambda \right\} \right| \geq \mu \right\}.$$

So $T_1 \subset T \in \mathcal{I}_2$, i.e., $T_1^c \in \mathcal{F}(\mathcal{I}_2)$. Therefore, $T_1^c \cap P^c \neq \emptyset$, since $T_1^c \cap P^c \in \mathcal{F}(\mathcal{I}_2)$.

Suppose $(i,j) \in K_1^c \cap P^c$ and let $\lambda := \frac{\|w_0 - q_0\|_0^+}{3} > 0$. Then

$$\tfrac{1}{\rho_i \phi_j} \left| \left\{ t_i < u \leq v_i,\ k_j < v \leq l_j, \|w_{uv} - w_0\|_0^+ \geq \lambda \right\} \right| < \mu \text{ and}$$

$$\tfrac{1}{\rho_i \phi_j} \left| \left\{ t_i < u \leq v_i,\ k_j < v \leq l_j, \|w_{uv} - q_0\|_0^+ \geq \lambda \right\} \right| < \mu,$$

which means, for the maximum $t_i < u \leq v_i$, $k_j < v \leq l_j$ we have $\|w_{uv} - w_0\|_0^+ < \lambda$ and $\|w_{uv} - q_0\|_0^+ < \lambda$ for a very small $\mu > 0$. Therefore, we need to obtain

$$\begin{aligned} &\left\{ t_i < u \leq v_i,\ k_j < v \leq l_j, \|w_{uv} - w_0\|_0^+ < \lambda \right\} \\ \cap &\left\{ t_i < u \leq v_i,\ k_j < v \leq l_j, \|w_{uv} - q_0\|_0^+ < \lambda \right\} \neq \emptyset, \end{aligned}$$

which leads to a contradiction, as the neighborhoods of w_0 and q_0 are disjoint. Thus, $\mathcal{I}_2^{FN} - S(\Lambda_w) = \{w_0\}$. □

Definition 17. *An element w_0 is considered as $\mathcal{I}_2$-deferred statistical cluster point of a double sequence $w = (w_{uv})$ if, for each $\lambda, \mu > 0$, the set*

$$\left\{ (\alpha,\beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta} \left| \left\{ t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w_0\|_0^+ \geq \lambda \right\} \right| < \mu \right\} \notin \mathcal{I}_2.$$

$\mathcal{I}_2^{FN} - S(\Gamma_w)$ represents the set of all $\mathcal{I}_2$-deferred statistical cluster point of a double sequence (w_{uv}).

Theorem 12. *For any double sequence (w_{uv}),*

$$\mathcal{I}_2^{FN} - S(\Lambda_w) \subseteq \mathcal{I}_2^{FN} - S(\Gamma_w).$$

Proof. Let $w_0 \in \mathcal{I}_2^{FN} - S(\Lambda_w)$. In that case, there is a set

$$U = \{(u_1, v_1) < (u_2, v_2) < ... < (u_r, v_s) < ...\} \subset \mathbb{N}^2$$

such that $U \notin \mathcal{I}_2$ and $DSt_2(FN) - \lim w_{u_r,v_s} = w_0$. So, we have

$$\lim_{\alpha,\beta\to\infty} \frac{1}{\rho_\alpha \phi_\beta} \left| \left\{ t_\alpha < u_r \leq v_\alpha, k_\beta < v_s \leq l_\beta, \|w_{u_r,v_s} - w_0\|_0^+ \geq \lambda \right\} \right| = 0.$$

Take $\mu > 0$, so there is $n_0 \in \mathbb{N}$ such that for $m, n > n_0$ we obtain

$$\frac{1}{\rho_m \phi_n} \left| \left\{ t_m < u_r \leq v_m, k_n < v_s \leq l_n, \|w_{u_r,v_s} - w_0\|_0^+ \geq \lambda \right\} \right| < \mu.$$

Let

$$K = \left\{ (m,n) \in \mathbb{N}^2 : \frac{1}{\rho_m \phi_n} \left| \left\{ t_m < u_r \leq v_m, k_n < v_s \leq l_n, \|w_{u_r,v_s} - w_0\|_0^+ \geq \lambda \right\} \right| < \mu \right\}.$$

In addition, we get

$$K \supset U \setminus \{(u_1, v_1), (u_2, v_2), ..., (u_{n_0}, v_{n_0})\}.$$

Given that $\mathcal{I}_2$ is an admissible ideal and $U \notin \mathcal{I}_2$, therefore $K \notin \mathcal{I}_2$. Consequently, according to the definition of an $\mathcal{I}_2$-deferred statistical cluster point $w_0 \in \mathcal{I}_2^{FN} - S(\Gamma_w)$. This concludes the proof. □

Theorem 13. *If $w = (w_{uv})$ and $q = (q_{uv})$ are two double sequences such that*

$$\left\{(u, v) \in \mathbb{N}^2 : w_{uv} \neq q_{uv}\right\} \in \mathcal{I}_2,$$

then

(*i*) $\mathcal{I}_2^{FN} - S(\Lambda_w) = \mathcal{I}_2^{FN} - S(\Lambda_q)$.

(*ii*) $\mathcal{I}_2^{FN} - S(\Gamma_w) = \mathcal{I}_2^{FN} - S(\Gamma_q)$.

Proof. (*i*) Let $w_0 \in \mathcal{I}_2^{FN} - S(\Lambda_w)$. As per the definition, there exists a set $U \subseteq \mathbb{N}^2$, arranged as

$$U = \{(u_1, v_1) < (u_2, v_2) < ... < (u_r, v_s) < ...\} \subset \mathbb{N}^2$$

such that $U \notin \mathcal{I}_2$ and $DSt_2(FN) - \lim w_{u_r, v_s} = w_0$. Since

$$\{(u, v) \in U : w_{uv} \neq q_{uv}\} \subseteq \left\{(u, v) \in \mathbb{N}^2 : w_{uv} \neq q_{uv}\right\} \in \mathcal{I}_2,$$

$$U' = \{(u, v) \in U : w_{uv} = q_{uv}\} \notin \mathcal{I}_2 \text{ and } U' \subseteq U.$$

Thus, the fact that $DSt_2(FN) - \lim q_{u'_r, v'_s} = w_0$ implies that $w_0 \in \mathcal{I}_2^{FN} - S(\Lambda_q)$, and consequently

$$\mathcal{I}_2^{FN} - S(\Lambda_w) \subseteq \mathcal{I}_2^{FN} - S(\Lambda_q).$$

By symmetry,

$$\mathcal{I}_2^{FN} - S(\Lambda_q) \subseteq \mathcal{I}_2^{FN} - S(\Lambda_w).$$

Hence, we obtain

$$\mathcal{I}_2^{FN} - S(\Lambda_w) = \mathcal{I}_2^{FN} - S(\Lambda_q).$$

(*ii*) Let $w_0 \in \mathcal{I}_2^{FN} - S(\Gamma_w)$. So, according to the definition for each $\lambda, \mu > 0$, we have

$$K = \left\{(\alpha, \beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta} \left| \left\{t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|w_{uv} - w_0\|_0^+ \geq \lambda\right\} \right| < \mu \right\} \notin \mathcal{I}_2.$$

Let

$$T = \left\{(\alpha, \beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta} \left| \left\{t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|q_{uv} - w_0\|_0^+ \geq \lambda\right\} \right| < \mu \right\}.$$

We have to prove that $T \notin \mathcal{I}_2$. Suppose that $T \in \mathcal{I}_2$, So

$$T^c = \left\{(\alpha, \beta) \in \mathbb{N}^2 : \frac{1}{\rho_\alpha \phi_\beta} \left| \left\{t_\alpha < u \leq v_\alpha,\ k_\beta < v \leq l_\beta, \|q_{uv} - w_0\|_0^+ \geq \lambda\right\} \right| \geq \mu \right\} \in \mathcal{F}(\mathcal{I}_2).$$

According to the hypothesis,

$$P = \left\{(u, v) \in \mathbb{N}^2 : w_{uv} = q_{uv}\right\} \in \mathcal{F}(\mathcal{I}_2).$$

Hence, $T^c \cap P \in \mathcal{F}(\mathcal{I}_2)$. Furthermore, it's evident that $T^c \cap P \subseteq K^c \in \mathcal{F}(\mathcal{I}_2)$, implying $K \in \mathcal{I}_2$, which contradicts the initial assumption. Therefore, $T \notin \mathcal{I}_2$ and thus the desired result is achieved. □

5. Conclusion

In conclusion, this study has advanced the understanding of convergence in FNS by introducing the novel concepts of $\mathcal{I}_2$-deferred Cesàro summability and $\mathcal{I}_2$-deferred statistical convergence for double sequences. Through rigorous investigation, we have uncovered significant connections between these concepts and have established several theorems elucidating the notion of $\mathcal{I}_2$-deferred statistical convergence in FNS for double sequences. Moreover, we have defined and explored the properties of $\mathcal{I}_2$-deferred statistical limit points and $\mathcal{I}_2$-deferred statistical cluster points within the context of FNS, providing valuable insights into their relationships. These findings not only contribute to the theoretical framework of convergence in FNS but also pave the way for future research directions and applications in related fields.

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SPECTRAL PROPERTIES OF A FUNCTIONAL BINOMIAL MATRIX

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Abstract. In this article, we consider the definition of the Fibonacci polynomial sequence with the second-order linear recurrence relation, where coefficients and initial conditions depend on the variable t. And then, we introduce the functional binomial matrix depending on the coefficients of the second-order linear recurrence relation. In the following, we study the spectral properties of the functional binomial matrix using the Fibonacci polynomial sequence and we obtain a diagonal decomposition for it using the Vandermunde matrix. Finally, by applying some linear algebra tools we obtain a number of combinatorial identities involving the Fibonacci polynomial sequence.

1. Introduction

The Fibonacci sequence and the Lucas sequence are among the most well-known second-order linear recurrence sequences that are of particular importance in number theory and combinatorics (see [17]):

$$F_n = F_{n-1} + F_{n-2}, \qquad F_0 = 0, \quad F_1 = 1, \tag{1}$$

$$L_n = L_{n-1} + L_{n-2}, \qquad L_0 = 2, \quad L_1 = 1. \tag{2}$$

Usually, second-order linear recurrence relations are generalized with two ideas, first by preserving the recurrence relation and second by preserving the initial conditions. The most prominent examples of Fibonacci-Like sequences are given as follows:

- The Jacobsthal sequence [11] is defined by the recurrence relation

$$J_n = J_{n-1} + J_{n-2} \quad (n \geq 2), \qquad J_0 = 1, \quad J_1 = 1. \tag{3}$$

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- The Jacobsthal-Lucas sequence [11] is defined by the recurrence relation

$$j_n = j_{n-1} + 2j_{n-2} \quad (n \geq 2), \qquad j_0 = 2, \quad j_1 = 1. \tag{4}$$

- Singh et al. [26] defined Fibonacci-Like sequence

$$S_n = S_{n-1} + S_{n-2} \quad (n \geq 2), \qquad S_0 = 2, \quad S_1 = 2. \tag{5}$$

- Horadam [11], Kalman [14], Stanimirović [27] and Gupta [10] generalized the Fibonacci sequence by considering a new initial condition and a new recurrence relation:

$$F_n = AF_{n-1} + BF_{n-2} \quad (n \geq 2), \qquad F_0 = a, \quad F_1 = b, \tag{6}$$

where A, B, a and b are positive integers.

A natural way to generalize the Fibonacci sequence is to use the Fibonacci polynomials. For over a century, both Fibonacci and Lucas polynomials have appeared in the literature in the study of several subjects such as algebra, geometry, combinatorics, approximation theory, statistics, and number theory [23]. Fibonacci polynomials were studied in 1883 by Catalan and Jacobsthal [8, 13]. Many works dealt with different properties of these polynomials and their applications. Fibonacci polynomials appear in different frameworks. Fibonacci polynomials are special cases of Chebyshev polynomials and have been studied on a more advanced level by many mathematicians. Large classes of Fibonacci-Like polynomials can be defined with the help of recurrence relations and the properties of the resulting Fibonacci numbers can be studied [17].

The most prominent examples of Fibonacci polynomials sequences are given as follows:

- The polynomials $F_n(t)$ studied by Catalan [8] are defined by the recurrence relation:

$$F_n(t) = tF_{n-1}(t) + F_{n-2}(t) \quad (n \geq 2), \qquad F_0(t) = 1, \quad F_1(t) = t. \tag{7}$$

- The Fibonacci polynomials studied by Jacobsthal [13] were defined by

$$J_n(t) = J_{n-1}(t) + tJ_{n-2}(t) \quad (n \geq 2), \qquad J_0(t) = 1, \quad J_1(t) = 1. \tag{8}$$

- The Pell polynomials [12] are defined by

$$P_n(t) = 2tP_{n-1}(t) + P_{n-2}(t) \quad (n \geq 2), \qquad P_0(t) = 0, \quad P_1(t) = 1. \tag{9}$$

- The Lucas polynomials [5] are defined by

$$L_n(t) = tL_{n-1}(t) + L_{n-2}(t) \quad (n \geq 2), \qquad L_0(t) = 2, \quad L_1(t) = t. \tag{10}$$

Many authors have studied Fibonacci polynomials with different ideas [4, 19, 21, 22, 26]. But recently Kaygisiz and Sahin [15] have presented new generalizations of Lucas numbers with matrix representation using generalized Lucas polynomials. Also, Lee and Asci [18] have defined a new generalization of Fibonacci polynomial called (A, B)-Fibonacci polynomial with the help of Pascal matrix. They obtain combinatorial identities and, using Riordan's method, obtain Pascal matrix factorizations

including (A, B)-Fibonacci polynomials. In this paper, we present generalization of the Fibonacci and Lucas polynomials by changing the initial terms and the recurrence relation.
In [7], Carlits (for $a = b = 1$) and in [1], Akkuse studied the $(n+1) \times (n+1)$ matrix $\mathcal{B}_n = \left[a^{i+j-n} b^{n-j} \binom{i}{n-j}\right]_{0 \le i,j \le n}$, and derived many interesting results on spectral and powers of these matrices. In this paper, introducing a generalized functional matrix $\mathcal{B}_n[x(t), y(t)]$ which call it the generalized functional binomial matrix of two variables $x(t)$ and $y(t)$ (both variables are dependent on t), we find the eigenvalues, eigenvectors and characteristic polynomial of it. We also obtain a decomposition for the matrix $\mathcal{B}_n[x(t), y(t)]$ and some identities for the polynomials Fibonacci sequence.

Definition 1. *The functional binomial matrix of two variables of order* $(n+1) \times (n+1)$ *is defined by*

$$\mathcal{B}_n[x(t), y(t)] = \left[x(t)^{i+j-n} y(t)^{n-j} \binom{i}{n-j}\right]_{0 \le i,j \le n}. \tag{11}$$

Example 1. *The functional binomial matrix of two variables of order* 4×4 *is as follows*

$$\mathcal{B}_3[x(t), y(t)] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & y(t) & x(t) \\ 0 & y(t)^2 & 2x(t)y(t) & x(t)^2 \\ y(t)^3 & 3y(t)^2 x(t) & 3y(t)x(t)^2 & x(t)^3 \end{pmatrix}.$$

In the following lemma, we can easily obtain a decomposition for the functional binomial matrix of two variables, considering $\mathcal{B}_n[x(t), 1] = \mathcal{B}_n[x(t)]$.

Lemma 1.

$$\mathcal{B}_n[x(t), y(t)] = \mathcal{B}_n[x(t)] \operatorname{diag}\bigl(y(t)^n, \cdots, y(t), 1\bigr).$$

For finding $\mathcal{B}_n^{-1}[x(t), y(t)]$, it is enough to find $\mathcal{B}_n^{-1}[x(t)]$. Now, consider the matrix $\tilde{I} = [\delta_{i,n-j}]_{0 \le i,j \le n}$, where $\delta_{i,n-j}$ is the Kronecker delta. It is easy to see that $\mathcal{B}_n[x(t)] = \mathcal{P}_n[x(t)]\tilde{I}_{n+1}$, where $\mathcal{P}_n[x(t)] = \left[\binom{i}{j} x(t)^{i-j}\right]_{0 \le i,j \le n}$ is the Pascal matrix with one variable, has the following properties (see [2, 3, 6, 16]):

(1) $\mathcal{P}_n[x(t)]\mathcal{P}_n[y(t)] = \mathcal{P}_n[x(t) + y(t)]$,
(2) $\mathcal{P}_n[x(t)]\mathcal{P}_n[-x(t)] = \mathcal{P}_n[0] = I_{n+1}$ namely $\mathcal{P}_n^{-1}[x(t)] = \mathcal{P}_n[-x(t)]$.

Therefore

$$\mathcal{B}_n^{-1}[x(t)] = \tilde{I}\mathcal{P}_n[-x(t)] = \left[\binom{n-i}{j}(-x(t))^{n-i-j}\right]_{0 \le i,j \le n}.$$

According to above topics, we present the inverse of the functional binomial matrix of two variables as follows $\mathcal{B}_n^{-1}\bigl[x(t), y(t)\bigr] = \left[(-x(t))^{n-i-j}(y(t))^{i-n}\binom{n-i}{j}\right]_{0 \le i,j \le n}$.

Example 2.

$$\mathcal{B}_4^{-1}[x(t),y(t)] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & y(t) & -x(t) \\ 0 & y(t)^2 & -2x(t)y(t) & x(t)^2 \\ y(t)^3 & -3y(t)^2x(t) & 3y(t)x(t)^2 & -x(t)^3 \end{pmatrix}.$$

2. The generalized Fibonacci polynomial and the functional binomial matrix

According to relations (1)-(5), a natural and general definition (6) can be presented, where coefficients and initial conditions are positive integers. Now, with the same idea and according to the recurrence relations (8)-(10), the following general definition can be presented, where the coefficients of the recursive and initial relation are considered as polynomials with integer coefficients.

Definition 2. *Let $A(t), B(t), a(t)$ and $b(t)$ be polynomials with integer coefficients. The generalized Fibonacci polynomials $\{F_n(a(t), b(t); A(t), B(t))\}_{n\geq 0}$ (we shall often drop the argument $(a(t), b(t); A(t), B(t))$ and simply write $\{F_n(t)\}_{n\geq 0}$ are defined by the recurrence relation*

$$F_n(t) = A(t)F_{n-1}(t) + B(t)F_{n-2}(t) \quad (n \geq 2), \tag{12}$$

$$F_0(t) = a(t), \quad F_1(t) = b(t). \tag{13}$$

For easy notation, we shall sometimes write A, B, a, b for $A(t), B(t), a(t)$ and $b(t)$. We display some special cases of the sequence $\{F_n(t)\}_{n\geq 0}$, in Table 1.

Table 1. Some special cases of $\{F_n(t)\}_{n\geq 0}$

Polynomial Type	$F_n(a, b; A, B)$	$A(t)$	$B(t)$	$a(t)$	$b(t)$
generalized Fibonacci	$F_n(t)$	t	1	1	t
generalized Lucas	$L_n(t)$	t	1	2	t
generalized Pell	$P_n(t)$	$2t$	1	0	1
Jacobsthal	$J_n(t)$	1	t	1	1
1st kind Chebyshev	$T_n(t)$	$2t$	-1	1	t
2nd kind Chebyshev	$U_n(t)$	$2t$	-1	1	$2t$
3th kind Chebyshev	$V_n(t)$	$2t$	-1	1	$2t-1$
4th kind Chebyshev	$W_n(t)$	$2t$	-1	1	$2t+1$

Theorem 1. *The non-degenerated second-order recurrent sequence $F_n(t)$, defined in (12), satisfies the following generalization of the Binet formula*

$$F_n(t) = \left(\frac{b - a\beta}{\alpha - \beta}\right)\alpha^n + \left(\frac{a\alpha - b}{\alpha - \beta}\right)\beta^n \quad (n \geq 0), \tag{14}$$

where α and β are the roots of the characteristic equation $\lambda^2 - A\lambda - B = 0$.

Corollary 1. *For $a = 0$ and $b \neq 0$, we have*

$$F_n(t) = \frac{b\big(\alpha^n - \beta^n\big)}{\alpha - \beta} \quad (n \geq 0), \tag{15}$$

and for $a \neq 0$ and $b = 0$, we have

$$F_n(t) = \frac{-a\alpha\beta\big(\alpha^{n-1} - \beta^{n-1}\big)}{\alpha - \beta} \quad (n \geq 0), \tag{16}$$

and also for $b = ka$ where k is a non-zero fixed number, we have

$$F_n(t) = \frac{a\big[\alpha^n - \beta^n - k\alpha\beta\big(\alpha^{n-1} - \beta^{n-1}\big)\big]}{\alpha - \beta} \quad (n \geq 0). \tag{17}$$

Corollary 2. *For $n \geq 1$ and $k \geq 0$, we have*

$$F_{k(n+1)}(t) = \mathcal{A}_k F_{kn}(t) - (-B)^k F_{k(n-1)}(t),$$

where $\mathcal{A}_k$ satisfy $\mathcal{A}_{k+1} = A\mathcal{A}_k + B\mathcal{A}_{k-1}$ with the boundary conditions $\mathcal{A}_0 = 2$ and $\mathcal{A}_1 = A$.

Proof. By the Binet formula (14) and since $\mathcal{A}_k = \alpha^k + \beta^k$ and $\alpha\beta = -B$, we have

$$\begin{aligned}
\mathcal{A}_k F_{kn}(t) - (-B)^k F_{k(n-1)}(t) = \\
&= \big(\alpha^k + \beta^k\big)\left[\left(\frac{b - a\beta}{\alpha - \beta}\right)\alpha^{kn} + \left(\frac{a\alpha - b}{\alpha - \beta}\right)\beta^{kn}\right] \\
&\quad - \big(\alpha\beta\big)^k\left[\left(\frac{b - a\beta}{\alpha - \beta}\right)\alpha^{k(n-1)} + \left(\frac{a\alpha - b}{\alpha - \beta}\right)\beta^{k(n-1)}\right] \\
&= \left(\frac{b - a\beta}{\alpha - \beta}\right)\alpha^{k(n+1)} + \left(\frac{a\alpha - b}{\alpha - \beta}\right)\beta^{k(n+1)} \\
&= F_{k(n+1)}(t).
\end{aligned}$$

□

The following theorem is the main result of this paper which gives the relation of the characteristic polynomial of the generalized binomial matrix of two variables $\mathcal{B}_n[A, B]$ with the generalized Fibonacci sequence $\big\{F_n(t)\big\}_{n\geq 0}$.

Theorem 2. *If $\Big(F_\ell(t)^{n-i}(t)F_{\ell+1}^i(t)\Big)_{0\leq i\leq n}$ be a column vector of $(n+1)$-dimension, then*

$$\mathcal{B}_n[A, B]\Big(F_\ell(t)^{n-i}F_{\ell+1}^i(t)\Big)_{0\leq i\leq n} = \Big(F_{\ell+1}^{n-i}(t)F_{\ell+2}^i(t)\Big)_{0\leq i\leq n}. \tag{18}$$

Proof. Let $\mathcal{B}_n[A,B]\Big(F_\ell^{n-i}(t)F_{\ell+1}^i(t)\Big)_{0\le i\le n} = [a_i]$, we have

$$\begin{aligned} a_i &= \sum_{k=0}^{n}\binom{i}{n-k}A^{i+k-n}B^{n-k}F_\ell^{n-k}(t)F_{\ell+1}^k(t) \\ &= F_{\ell+1}^{n-i}(t)\sum_{k=n-i}^{n}\binom{i}{n-k}\big(BF_\ell(t)\big)^{n-k}\big(AF_{\ell+1}(t)\big)^{i+k-n}, \end{aligned}$$

which substituting $r = k-n+i$, we obtain

$$\begin{aligned} a_i &= F_{\ell+1}^{n-i}(t)\sum_{r=0}^{i}\binom{i}{r}\Big(BF_\ell(t)\Big)^{i-r}\Big(AF_{\ell+1}(t)\Big)^{r} \\ &= F_{\ell+1}^{n-i}(t)\Big(BF_\ell(t)+AF_{\ell+1}(t)\Big)^{i} \\ &= F_{\ell+1}^{n-i}(t)F_{\ell+2}^{i}(t). \end{aligned}$$

□

Example 3.

$$\begin{aligned} \mathcal{B}_3[A,B]\Big(F_\ell^{3-i}(t)F_{\ell+1}^i(t)\Big)_{0\le i\le 3} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & B & A \\ 0 & B^2 & 2AB & A^2 \\ B^3 & 3B^2A & 3BA^2 & A^3 \end{pmatrix}\begin{pmatrix} F_\ell^3(t) \\ F_\ell^2(t)F_{\ell+1}(t) \\ F_\ell(t)F_{\ell+1}^2(t) \\ F_{\ell+1}^3(t) \end{pmatrix} \\ &= \begin{pmatrix} F_{\ell+1}^3(t) \\ F_{\ell+1}^2(t)F_{\ell+2}(t) \\ F_{\ell+1}(t)F_{\ell+2}^2(t) \\ F_{\ell+2}^3(t) \end{pmatrix}. \end{aligned}$$

Corollary 3.

$$F_{\ell+1}^{n-i}(t)F_{\ell+2}^{i}(t) = \sum_{i_1,\cdots,i_\ell}\binom{i}{n-i_1}\binom{i_1}{n-i_2}\cdots\binom{i_{\ell+1}}{n-i_\ell}A^{i+i_\ell-n\ell+2\sum_{r=1}^{\ell-1}i_r}B^{n\ell-\sum_{r=1}^{\ell}i_r}a^{n-i_\ell}b^{i_\ell}.$$

Proof. By induction on ℓ and using (18), we have

$$\mathcal{B}_n^\ell[A,B]\Big(a^{n-i}b^i\Big)_{0\le i\le n} = \Big(F_{\ell+1}^{n-i}(t)F_{\ell+2}^{i}(t)\Big)_{0\le i\le n}.$$

Now, if we consider the i-th rows, we get

$$F_{\ell+1}^{n-i}(t)F_{\ell+2}^{i}(t) = \Big(\mathcal{B}_n^\ell[A,B]\big(a^{n-s}b^s\big)_{s=0}^{n}\Big)_i = \sum_{i_1,\cdots,i_\ell}a_{i,i_1}\cdots a_{i_{\ell-1},i_\ell}a^{n-i_\ell}b^{i_\ell}$$

$$= \sum_{i_1,\cdots,i_\ell} \binom{i}{n-i_1}\binom{i_1}{n-i_2}\cdots\binom{i_{\ell-1}}{n-i_\ell} A^{i+i_\ell-n\ell+2\sum_{r=1}^{\ell-1} i_r} B^{n\ell-\sum_{r=1}^{\ell} i_r} a^{n-i_\ell} b^{i_\ell}.$$

□

The matrix $\left[F_j^{n-i}(t)F_{j+1}^i(t)\right]_{0\le i,j\le n}$ is invertible.

Proof. If we divide the j-th column by $F_{j+1}^n(t)$, we obtain the Vandermonde matrix $\left[\left(\frac{F_j(t)}{F_{j+1}(t)}\right)^{n-i}\right]$ which has nonzero determinant. □

Theorem 3. *For the sequence* $\{F_n(t)\}_{n\ge 0}$ *and* $k\ge 2$, *we have*

$$\begin{aligned}\big(xF_k(t)+BF_{k-1}(t)\big)^r&\big(xF_{k+1}(t)+BF_k(t)\big)^{n-r}\\ &= \sum_{r_0,r_1,\cdots,r_k}\binom{r}{r_0}\binom{n-r}{r_1}\cdots\binom{n-r_{k-1}}{r_k} A^{(k-1)n-r_0-r_1-2\sum_{\ell=2}^{k} r_\ell - r_{k+1}}\\ &\quad\times B^{\sum_{\ell=0}^{k} r_\ell} a^{r_0} b^{r-r_0+r_1}\big(Ab+Ba\big)^{n-r_1-r} x^{n-r_k}. \qquad (19)\end{aligned}$$

Proof. Using the binomial expansion, we have

$$\begin{aligned}&\big(xF_1(t)+BF_0(t)\big)^r\big(xF_2(t)+BF_1(t)\big)^{n-r}\\ &\quad= \sum_{r_0,r_1}\binom{r}{r_0}\binom{n-r}{r_1} B^{r_0+r_1} a^{r_0} b^{r-r_0+r_1}\big(Ab+Ba\big)^{n-r_1-r} x^{n-r_0-r_1}. \qquad (20)\end{aligned}$$

For all integers $k\ge 2$, we prove equality (19) by induction. For $k=2$, in (20), we replace x by $A+Bx^{-1}$ and multiply the result by x^n, and the conclusion is obtained. Assuming that (19) holds for the value k, we replace x by $A+Bx^{-1}$ and multiply the result by x^n. The left side of the formula is as follows

$$\begin{aligned}&\big(AF_k(t)x+BF_{k-1}(t)x+BF_k(t)\big)^r\big(AF_{k+1}(t)x+BF_k(t)x+BF_{k+1}(t)\big)^{n-r}\\ &\qquad\qquad= \big(F_{k+1}(t)x+BF_k(t)\big)^r\big(F_{k+2}(t)x+BF_{k+1}(t)\big)^{n-r},\end{aligned}$$

the right side of the formula is as follows

$$\begin{aligned}&\sum_{r_0,r_1,\cdots,r_{k+1}}\binom{r}{r_0}\binom{n-r}{r_1}\cdots\binom{n-r_{k-1}}{r_k}\binom{n-r_k}{r_{k+1}}\\ &\quad\times A^{kn-r_0-r_1-2\sum_{\ell=2}^{k} r_\ell - r_{k+1}} B^{\sum_{\ell=0}^{k+1} r_\ell} a^{r_0} b^{r-r_0+r_1}\big(Ab+Ba\big)^{n-r_1-r} x^{n-r_{k+1}}.\end{aligned}$$

This evidently completes the proof of (19). □

Corollary 4. *For* $k\ge 2$, *we have*

$$\begin{aligned}\big(xF_k(t)+BF_{k-1}(t)\big)^r&\big(xF_{k+1}(t)+BF_k(t)\big)^{n-r} \qquad (21)\\ &= \sum_{r_1,\cdots,r_k}\binom{n-r}{r_1}\cdots\binom{n-r_{k-1}}{r_k} A^{(k+1)n-2\sum_{\ell=1}^{k-1} r_\ell - r} B^{\sum_{\ell=1}^{k} r_\ell} x^{n-r_k},\end{aligned}$$

where $F_0(t) = 0$ *and* $F_1(t) = A$.

Lemma 2. *For all* $k \geq 1$, *we have*

$$\operatorname{tr}\left(\mathcal{B}_n^k[A,B]\right) = \frac{F_{k(n+1)}(t)}{F_k(t)}, \tag{22}$$

where $F_0(t) = 0$ *and* $F_1(t) = A$.

Proof. We multiply (21) by x^r and sum over r. This gives

$$\sum_{r=0}^{n} x^r \left(xF_k(t) + BF_{k-1}(t)\right)^r \left(xF_{k+1}(t) + BF_k(t)\right)^{n-r} \tag{23}$$

$$= \sum_{r,r_1,\cdots,r_k} \binom{n-r}{r_1} \cdots \binom{n-r_{k-1}}{r_k} A^{(k+1)n-2\sum_{\ell=1}^{k-1} r_\ell - r} B^{\sum_{\ell=1}^{k} r_\ell} x^{n-r_k+r}.$$

The coefficient of x^n on the right of (23) is $\operatorname{tr}\left(\mathcal{B}_n^k[A,B]\right)$ and the coefficient of x^n on the left of (23) is

$$\sum_{r+s+u=n} \binom{r}{s}\binom{n-r}{u} \left(BF_{k-1}(t)\right)^{r-s} \left(F_k(t)\right)^s \left(BF_k(t)\right)^{n-r-u} \left(F_{k+1}(t)\right)^u$$

$$= \sum_{r+s\leq n} B^r \binom{r}{s}\binom{n-r}{s} F_{k-1}^{r-s}(t) F_k^{2s}(t) F_{k+1}^{n-r-s}(t) = c_n^k.$$

Then

$$\sum_{n=0}^{\infty} c_n^k x^n = \sum_{r,s=0}^{\infty} \binom{r}{s} B^r F_{k-1}^{r-s}(t) F_k^{2s}(t) x^{r+s} \sum_{n=r+s} \binom{n-r}{s} \left(F_{k+1}(t)x\right)^{n-r-s}$$

$$= \sum_{r,s=0}^{\infty} B^r \binom{r}{s} F_{k-1}^{r-s}(t) F_k^{2s}(t) x^{r+s} \left(1 - F_{k+1}(t)x\right)^{-s-1}$$

$$= \sum_{s=0}^{\infty} B^s F_k^{2s}(t) \left(1 - F_{k+1}(t)x\right)^{-s-1} \sum_{r\geq 0} \binom{r}{s} \left(F_{k-1}(t)x\right)^{r+s}$$

$$= \sum_{s=0}^{\infty} B^s F_k^{2s}(t) \left(1 - F_{k+1}(t)x\right)^{-s-1} \left(1 - F_{k-1}(t)x\right)^{-s-1}$$

$$= \frac{1}{\left(1 - F_{k+1}(t)x\right)\left(1 - BF_{k-1}(t)x\right)} \times \frac{1}{1 - \frac{BF_k^2(t)x^2}{\left(1-F_{k+1}(t)x\right)\left(1-BF_{k-1}(t)x\right)}}$$

$$= \frac{1}{\left(1 - F_{k+1}(t)x\right)\left(1 - BF_{k-1}(t)x\right) - BF_k^2(t)x^2}.$$

Here by the Binet formula (15), we have

$$\sum_{n=0}^{\infty} c_n^k x^n = \frac{1}{1 - \left(\alpha^k + \beta^k\right)x + (\alpha\beta)^k x^2}$$

$$= \frac{1}{\left(1-\alpha^k x\right)\left(1-\beta^k x\right)}$$
$$= \frac{1}{\alpha^k-\beta^k}\left(\frac{\alpha^k}{1-\alpha^k x}-\frac{\beta^k}{1-\beta^k x}\right).$$

It follows that

$$c_n^k = \frac{\alpha^{k(n+1)}-\beta^{k(n+1)}}{\alpha^k-\beta^k} = \frac{F_{k(n+1)}(t)}{F_k(t)}.$$

□

Theorem 4. *The eigenvalues of* $\mathcal{B}_n[A,B]$ *are*

$$\alpha^n, \alpha^{n-1}\beta, \cdots, \alpha\beta^{n-1}, \beta^n,$$

and the characteristic polynomial of $\mathcal{B}_n[A,B]$ *is*

$$\chi_n(\tau) = \prod_{i=0}^{n}\left(\tau-\alpha^i\beta^{n-i}\right).$$

Proof. Let $\chi_{n+1}(\tau) = \det\left(\tau I_{n+1} - \mathcal{B}_n[A,B]\right)$ and $\lambda_0, \lambda_1, \cdots, \lambda_n$ denote the eigenvalues of $\mathcal{B}_n[A,B]$. Then by Lemma 2,

$$\begin{aligned}
\frac{\chi'_{n+1}(\tau)}{\chi_{n+1}(\tau)} &= \sum_{k=0}^{n}\frac{1}{\tau-\lambda_\ell} = \sum_{k=0}^{\infty}\tau^{-k-1}\sum_{j=0}^{k}\lambda_j^k \\
&= \sum_{k=0}^{\infty}\tau^{-k-1}\mathrm{tr}\left(\mathcal{B}_n^\ell[A,B]\right) = \sum_{k=0}^{\infty}\tau^{-k-1}\frac{F_{k(n+1)}(t)}{F_k(t)} \\
&= \sum_{k=0}^{\infty}\tau^{-k-1}\frac{\alpha^{k(n+1)}-\beta^{k(n+1)}}{\alpha^k-\beta^k} = \sum_{k=0}^{\infty}\tau^{-k-1}\sum_{j=0}^{n}\alpha^{jk}\beta^{(n-j)k} \\
&= \sum_{j=0}^{n}\frac{1}{\tau-\alpha^j\beta^{n-j}}.
\end{aligned}$$

It follows that

$$\chi_{n+1}(\tau) = \prod_{j=0}^{n}\left(\tau-\alpha^j\beta^{n-j}\right).$$

□

Theorem 5. *For* $a=0$ *and* $b \neq 0$*, we have*

$$\chi_{n+1}(\tau) = \sum_{\ell=0}^{n+1}(-1)^{\frac{\ell(\ell+1)}{2}}B^{\frac{\ell(\ell-1)}{2}}\begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)}\tau^{n+1-\ell},$$

where $\begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)}$ *is defined as*

$$\begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)} = \begin{cases} 1, & \ell = 0, n+1; \\ \frac{F_{n+1}(t)F_n(t)\cdots F_{n-\ell+2}(t)}{F_1(t)F_2(t)\cdots F_\ell(t)}, & 0 < \ell < n+1. \end{cases}$$

Proof. We use the following identity (see [20])

$$\prod_{j=0}^{n}(1-q^j\tau) = \sum_{\ell=0}^{n+1}(-1)^\ell q^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_q \tau^\ell,$$

where $\begin{bmatrix} n+1 \\ \ell \end{bmatrix}_q$ is the q-binomial coefficient (Gaussian binomial), and is defined by

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)},$$

where m and r are non-negative integers. If $r > m$, this evaluates to 0 and for $r = 0, m$, the value is 1.
Replacing q in the above equation by $\frac{\beta}{\alpha}$ and using the Binet formula (15), we have

$$\begin{bmatrix} n+1 \\ \ell \end{bmatrix}_q = \alpha^{\ell^2-(n+1)\ell} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)}.$$

Therefore

$$\prod_{j=0}^{n}(1-\alpha^{-j}\beta^j\tau) = \sum_{\ell=0}^{n+1}(-1)^\ell \alpha^{\frac{\ell(\ell-1)}{2}} \beta^{\frac{\ell(\ell-1)}{2}-n\ell} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)} \tau^\ell.$$

Substituting τ by $\alpha^n\tau^{-1}$ and using $\alpha\beta = -B$, we get

$$\prod_{j=0}^{n}(\tau-\alpha^{n-j}\beta^j) = \sum_{\ell=0}^{n+1}(-1)^{\frac{\ell(\ell+1)}{2}} B^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)} \tau^{n+1-\ell},$$

which is the desired result. □

Example 4. *The characteristic polynomials of* $\chi_{n+1}(\tau)$ *for* $n = 0, 1, 2$ *are*

$$\chi_1(\tau) = \tau - 1$$
$$\chi_2(\tau) = \tau^2 - A\tau - B$$
$$\chi_3(\tau) = \tau^3 - (B + A^2)\tau^2 - (A^2B + B^2)\tau + B^3.$$

3. Diagonalization of the Functional Binomial Matrix

The results of this section are for a specific case of the recurrence relation (12) with (13) for $a(t) = 0$, $b(t) = 1$ and coefficients $A(t)$ and $B(t)$ which are arbitrary functions of t.
Let $n \geq 1$ and $\mathcal{C}_n[A, B]$ be the companion matrix of the characteristic polynomial $\chi_n(\tau)$, where

$\mathcal{C}_n[A,B] = \big(c_{i,j}(A,B)\big),$

$$\begin{cases} c_{i,i+1}(A,B) = 1, & i = 0,1,\cdots,n-1; \\ c_{n,n-j}(A,B) = -(-1)^{\frac{(j+1)(j+2)}{2}} B^{\frac{j(j+1)}{2}} \left[{n+1 \atop j+1}\right]_{F_n(t)}, & j = 0,1,\cdots,n-1; \\ c_{i,j}(A,B) = 0, & \text{otherwise.} \end{cases}$$

$\mathcal{R}_n[A,B] = \big(r_{i,j}(A,B)\big)$ and $\mathcal{M}_n[A,B] = \big(m_{i,j}(A,B)\big),$

$$\begin{cases} r_{0,j}(A,B) = r_{1,j}(A,B) = \delta_{n,j}, \\ r_{i,j}(A,B) = \binom{n}{j}\big(BF_{i-1}(t)\big)^{n-j}\big(F_i(t)\big)^j, \qquad i = 2,\cdots,n,\ j = 0,1,\cdots,n, \end{cases}$$

$$\begin{cases} m_{0,j}(A,B) = \delta_{n,j}, \\ m_{i,j}(A,B) = \binom{n}{j}\big(BF_i(t)\big)^{n-j}\big(F_{i+1}(t)\big)^j, \qquad i = 1,\cdots,n,\ j = 0,1,\cdots,n. \end{cases}$$

Lemma 3. *For every positive integer k, we have*

$$\Big(\mathcal{B}_n^k[A,B]\Big)_{nj} = \binom{n}{j}\big(BF_k(t)\big)^{n-j}\big(F_{k+1}(t)\big)^j, \qquad j = 0,1,\cdots,n.$$

Proof. Let n be a fixed natural number. We will prove the assertion by induction on k. The above equality is valid for $k = 0$. Now assume the results is valid for $k > 0$. Then, since $\mathcal{B}_n^{k+1}[A,B] = \mathcal{B}_n^k[A,B]\mathcal{B}_n[A,B]$, we have

$$\begin{aligned} \big(\mathcal{B}_n^{k+1}[A,B]\big)_{nj} &= \sum_{i=0}^{n}\big(\mathcal{B}_n^k[A,B]\big)_{ni}\big(\mathcal{B}_n[A,B]\big)_{ij} \\ &= \sum_{i=0}^{n}\binom{n}{i}\big(BF_k(t)\big)^{n-i}\big(F_{k+1}(t)\big)^i\binom{i}{n-j}A^{i+j-n}B^{n-j} \\ &= \big(BF_{k+1}(t)\big)^{n-j}\sum_{i=0}^{n}\binom{n}{n-j}\binom{j}{i+j-n}\big(AF_{k+1}(t)\big)^{i-n+j}\big(BF_k(t)\big)^{n-i} \\ &= \binom{n}{j}\big(BF_{k+1}(t)\big)^{n-j}\sum_{i=0}^{n}\binom{j}{i+j-n}\big(AF_{k+1}(t)\big)^{i+j-n}\big(BF_k(t)\big)^{n-i} \\ &= \binom{n}{j}\big(BF_{k+1}(t)\big)^{n-j}\sum_{m=0}^{j}\binom{j}{m}\big(AF_{k+1}(t)\big)^{m}\big(BF_k(t)\big)^{j-m} \\ &= \binom{n}{j}\big(BF_{k+1}(t)\big)^{n-j}\big(AF_{k+1}(t) + BF_k(t)\big)^{j} \\ &= \binom{n}{j}\big(BF_{k+1}(t)\big)^{n-j}\big(F_{k+2}(t)\big)^{j}. \end{aligned}$$

□

Theorem 6. *Let $F_0(t) = 0$ and $F_1(t) = 1$. Then*

$$\sum_{\ell=0}^{n+1}(-1)^{\frac{\ell(\ell+1)}{2}} B^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)} \big(F_{n-\ell+1}(t)\big)^{n-j} \big(F_{n-\ell+2}(t)\big)^{j} = 0.$$

Proof. The characteristic polynomials of $\mathcal{B}_n[A,B]$ is

$$\sum_{\ell=0}^{n+1}(-1)^{\frac{\ell(\ell+1)}{2}} B^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)} \tau^{n+1-\ell} = 0.$$

Now by the Cayley-Hamilton Theorem [24], we get

$$\sum_{k=0}^{n+1}(-1)^{\frac{\ell(\ell+1)}{2}} B^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)} \mathcal{B}_n^{n-\ell+l}[A,B] = 0, \tag{24}$$

where 0 denotes the $(n+1)\times(n+1)$ zero matrix. So by Lemma 3 and substituting this result into (24), we obtain

$$\sum_{\ell=0}^{n+1}(-1)^{\frac{\ell(\ell+1)}{2}} B^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)} \Big(\mathcal{B}_n^{n-\ell+1}[A,B]\Big)_{nj} = 0.$$

Therefore

$$\sum_{\ell=0}^{n+1}(-1)^{\frac{\ell(\ell+1)}{2}} B^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)} \big(F_{n-\ell+1}(t)\big)^{n-j} \big(F_{n-\ell+2}(t)\big)^{j} = 0.$$

□

Theorem 7. *Let $a(t) = 0$ and $b(t) = 1$. For all n, we have*

$$\mathcal{M}_n[A,B] = \mathcal{C}_n[A,B]\mathcal{R}_n[A,B] = \mathcal{R}_n[A,B]\mathcal{B}_n[A,B],$$

and so

$$\mathcal{B}_n[A,B] = \mathcal{R}_n^{-1}[A,B]\mathcal{C}_n[A,B]\mathcal{R}_n[A,B].$$

Proof. At first, we prove $\mathcal{M}_n[A,B] = \mathcal{C}_n[A,B]\mathcal{R}_n[A,B]$. In fact, multiplying the first n rows of $\mathcal{C}_n[A,B]$ by $\mathcal{R}_n[A,B]$, clearly we get the first n rows of $\mathcal{M}_n[A,B]$. For the last row, for each $0 \le j \le n$, we have

$$\begin{aligned}
&\big(\mathcal{C}_n[A,B]\mathcal{R}_n[A,B]\big)_{nj} = \\
&\quad= \sum_{k=0}^{n}\big(\mathcal{C}_n[A,B]\big)_{n,n-k}\big(\mathcal{R}_n[A,B]\big)_{n-k,j} \\
&\quad= \sum_{k=0}^{n} -(-1)^{\frac{(k+1)(k+2)}{2}} B^{\frac{k(k+1)}{2}} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{F_n(t)} \binom{n}{j} \big(BF_{n-k-1}(t)\big)^{n-j} \big(F_{n-k}(t)\big)^{j} \\
&\quad= \binom{n}{j} B^{n-j} \sum_{\ell=1}^{n+1} -(-1)^{\frac{\ell(\ell+1)}{2}} B^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)} \big(F_{n-\ell}(t)\big)^{n-j} \big(F_{n-\ell+1}(t)\big)^{j}
\end{aligned}$$

$$= \binom{n}{j} B^{n-j} \Bigg(\big(F_n(t)\big)^{n-j} \big(F_{n+1}(t)\big)^j + \sum_{\ell=0}^{n+1} -(-1)^{\frac{\ell(\ell+1)}{2}} B^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)} \big(F_{n-\ell}(t)\big)^{n-j} \big(F_{n-\ell+1}(t)\big)^j \Bigg)$$
$$= \binom{n}{j} \big(BF_n(t)\big)^{n-j} \big(F_{n+1}(t)\big)^j,$$

which is clearly true by Theorem 6. This proves,

$$\mathcal{M}_n[A,B] = \mathcal{C}_n[A,B]\mathcal{R}_n[A,B].$$

Since for each i, j with $0 \le i \le j \le n$, we have

$$\begin{aligned}
\big(\mathcal{R}_n[A,B]\mathcal{B}_n[A,B]\big)_{ij} &= \sum_{k=0}^{n} \big(\mathcal{R}_n[A,B]\big)_{ik} \big(\mathcal{B}_n[A,B]\big)_{kj} \\
&= \sum_{k=0}^{n} \binom{n}{k} \big(BF_{i-1}(t)\big)^{n-k} \big(F_i(t)\big)^k A^{k+j-n} B^{n-j} \binom{k}{n-j} \\
&= \binom{n}{j} \sum_{k=0}^{n} \binom{j}{n-k} A^{k+j-n} B^{2n-j-k} \big(F_{i-1}(t)\big)^{n-k} \big(F_i(t)\big)^k \\
&= \binom{n}{j} \big(BF_i(t)\big)^{n-j} \sum_{\ell=0}^{j} \binom{j}{\ell} \big(BF_{i-1}(t)\big)^{\ell} \big(AF_i(t)\big)^{n-\ell} \\
&= \binom{n}{j} \big(BF_i(t)\big)^{n-j} \big(BF_{i-1}(t) + AF_i(t)\big)^j \\
&= \binom{n}{j} \big(BF_i(t)\big)^{n-j} \big(F_{i+1}(t)\big)^j \\
&= \big(\mathcal{M}_n[A,B]\big)_{ij},
\end{aligned}$$

we get $\mathcal{M}_n[A,B] = \mathcal{R}_n[A,B]\mathcal{B}_n[A,B]$. □

Example 5.

$$\mathcal{M}_3[A,B] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ B^3 & 3B^2A & 3BA^2 & A^3 \\ A^3B^3 & 3B^2A^2(B+A^2) & 3BA(B+A^2)^2 & (B+A^2)^3 \\ B^3(B+A^2)^3 & 3B^2A(B+A^2)^2(2B+A^2) & 3BA^2(B+A^2)(2B+A^2)^2 & A^3(2B+A^2)^3 \end{pmatrix},$$

$$\mathcal{C}_3[A,B] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -B^6 & -B^3A(2B+A^2) & B(B+A^2)(2B+A^2) & (2B+A^2)A \end{pmatrix},$$

$$\mathcal{R}_3[A,B] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ B^3 & 3B^2A & 3BA^2 & A^3 \\ A^3B^3 & 3B^2A^2(B+A^2) & 3BA(B+A^2)^2 & (B+A^2)^3 \end{pmatrix},$$

and so

$$\mathcal{M}_3[A,B] = \mathcal{C}_3[A,B]\mathcal{R}_3[A,B].$$

Also,

$$\mathcal{R}_3[A,B] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ B^3 & 3B^2A & 3BA^2 & A^3 \\ A^3B^3 & 3B^2A^2(B+A^2) & 3BA(B+A^2)^2 & (B+A^2)^3 \end{pmatrix},$$

$$\mathcal{B}_3[A,B] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & B & 1 \\ 0 & B^2 & 2BA & A^2 \\ B^3 & 3B^2A & 3BA^2 & A^3 \end{pmatrix},$$

and therefore $\mathcal{M}_3[A,B] = \mathcal{R}_3[A,B]\mathcal{B}_3[A,B]$.

Let $\mathcal{V}_n$ be the Vandermonde matrix which is defined by

$$\mathcal{V}_n = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \alpha^n & \alpha^{n-1}\beta & \cdots & \alpha\beta^{n-1} & \beta^n \\ \alpha^{2n} & (\alpha^{n-1}\beta)^2 & \cdots & (\alpha\beta^{n-1})^2 & \beta^{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha^{n^2} & (\alpha^{n-1}\beta)^n & \cdots & (\alpha\beta^{n-1})^n & \beta^{n^2} \end{pmatrix}.$$

By the relation between the component matrix and the Vandermonde matrix, we can obtain Theorems 8 and 9. For this purpose, we need the following lemma.

Lemma 4 ([24], P. 4)**.** *If M be the following matrix*

$$M = \begin{pmatrix} 0 & m_1 & 0 & \cdots & 0 \\ 0 & 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m_{n-1} \\ p_1 & p_2 & p_3 & \cdots & p_n \end{pmatrix},$$

then its eigenvalues are the roots of $p_1 + p_2\lambda + \cdots + p_n\lambda^{n-1} = \lambda^n$ and $v_1 = \left(\alpha, \alpha\lambda, , \alpha\lambda^2, \cdots, \alpha\lambda^{n-1}\right)^T$ is an eigenvector for the root λ.

Theorem 8. *Let $a(t) = 0$ and $b(t) = 1$. Eigenvectors of the matrix $\mathcal{C}_n[A, B]$ are $\mathcal{V}_n$, and also eigenvectors of the matrix $\mathcal{B}_n[A, B]$ are $E_n[A, B] = \mathcal{R}_n^{-1}[A, B]\mathcal{V}_n$.*

Proof. According to Lemma 4, columns of $\mathcal{V}_n$ are eigenvectors of $\mathcal{C}_n[A, B]$. □

Theorem 9. *For $a(t) = 0$ and $b(t) = 1$, we have*

$$\left(\mathcal{R}_n^{-1}[A, B]\mathcal{V}_n\right)^{-1}\mathcal{B}_n[A, B]\left(\mathcal{R}_n^{-1}[A, B]\mathcal{V}_n\right) = \operatorname{diag}\left(\alpha^n, \alpha^{n-1}\beta, \cdots, \alpha\beta^{n-1}, \beta^n\right).$$

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AN EXTENDED FRAMEWORK FOR BIHYPERBOLIC GENERALIZED TRIBONACCI NUMBERS

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Abstract. The aim of this article is to identify and analyze a new type special number system which is called bihyperbolic generalized Tribonacci numbers ($\mathcal{BGTN}$ for short). For this purpose, we give both classical and several new properties such as; recurrence relation, Binet formula, generating function, exponential generating function, summation formulae, matrix formula, and special determinant equations of $\mathcal{BGTN}$. Also, the system of $\mathcal{BGTN}$ is quite a big family and includes several type special cases with respect to initial values and r, s, t values, we give the subfamilies and special cases of it. In addition to these, we construct some numerical algorithms including recurrence relation and special two types determinant equations related to calculating the terms of this new type special number system. Then, we examine several properties by taking two special cases and including some illustrative numerical examples.

1. Introduction

Numbers and number systems are well-established fundamental and important topics in not only mathematics but also other disciplines with varied applications and benefits. In spite of their long history, numbers systems are still an interesting and important area to work for lots of researchers since there are several applications in different and several areas such as; differential geometry, engineering, robotics, graph theory, etc. There exist several types of number systems in the existing literature. A hyperbolic (perplex, split-complex) number is a number of the form $z = x + yj$ where $x, y \in \mathbb{R}$, $j^2 = 1, j \neq \pm 1$, $j \notin \mathbb{R}$ [39, 43, 61]. Also, a bihyperbolic

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number (canonical hyperbolic quaternion [10], hyperbolic four complex numbers [35]) is written as a linear combination of a pair of hyperbolic numbers. There exists a relationship between the bihyperbolic numbers and 4–dimensional pseudo-Euclidean spaces. Bihyperbolic numbers are denoted by $\mathcal{H}$ and are defined as [4,10,35,37]:

$$\mathcal{H} := \{\zeta = \rho_0 + \rho_1 j_1 + \rho_2 j_2 + \rho_3 j_3 : \rho_0, \rho_1, \rho_2, \rho_3 \in \mathbb{R},\ j_1, j_2, j_3 \notin \mathbb{R}\},$$

where j_1, j_2, j_3 satisfy the multiplication rules:

$$j_1^2 = j_2^2 = j_3^2 = 1, \quad j_1 j_2 = j_2 j_1 = j_3, \quad j_1 j_3 = j_3 j_1 = j_2, \quad j_2 j_3 = j_3 j_2 = j_1. \tag{1}$$

On the other hand, several studies have been done and are ongoing on the special recurrence sequences which can have different orders. For example, Fibonacci and Lucas sequences [18,33] can be given as examples related to second-order recurrence sequences. The most general form of the second-order recurrence sequences is called as Horadam [26]. In this study, we deal with the generalization of third-order recurrence sequences which is called generalized Tribonacci sequence (or numbers). Generalized Tribonacci sequence $\{T_n(T_0, T_1, T_2; r, s, t)\}_{n\geq 0}$ (for short: $\{T_n\}_{n\geq 0}$) given by the following recurrence relation:

$$T_n = rT_{n-1} + sT_{n-2} + tT_{n-3}, \quad n \geq 3 \tag{2}$$

with the initial conditions $T_0 = a, T_1 = b, T_2 = c$ are arbitrary integers and r, s, t are real numbers [11]. Generalization of special third-order numbers was studied in [1,12,13,15,16,19,20,36,40–42,44–54,59,60,62].

Furthermore, the framework of bringing together the quaternions and special recurrence sequences is a popular and interesting concept for researchers. Real quaternions were investigated by W. R. Hamilton as an expansion of the complex numbers [23,24] (see also Section "Conclusion"). There exist several studies with respect to combining several different types quaternions such as; split [17], generalized [29,30,34,38], etc. Additionally, special recurrence sequences were examined considering quaternions, for instance, Fibonacci and Lucas real quaternions [20,22,25,27], Fibonacci and Lucas generalized quaternions [2,20], Narayana (or Fibonacci-Narayana) generalized quaternions [20]. Besides, the researchers started to examine the combining the third-order recurrence sequences and several types quaternions, such as; Padovan and Perrin quaternions [21,28,58], generalized Tribonacci real quaternions [11]. Also, generalized bicomplex Tribonacci quaternions were introduced in [32].

In the same manner, a great deal of researchers started to investigate the bihyperbolic numbers with several special recurrence sequences. Studies on bihyperbolic numbers, and bringing together the bihyperbolic numbers and some special recurrence numbers have been gathered speed in the existing literature. Bród et al. studied the generalization of bihyperbolic Pell numbers in [5]. Also, Bród et al. examined the one-parameter and two-parameter generalizations of the bihyperbolic

Jacobsthal numbers in [6, 7], respectively. Then, bihyperbolic numbers of the Fibonacci type and idempotent representation of them were investigated in [8]. In [9], some combinatorial properties of bihyperbolic numbers of the Fibonacci type are investigated. Azak examined some new identities related to bihyperbolic Fibonacci and Lucas numbers in [3]. Further, Fibonacci and Lucas bihypernomials [55] and certain bihypernomials with respect to Pell and Pell-Lucas numbers [56] examined.

In this study, we investigate a new type of number system which is called as bihyperbolic generalized Tribonacci numbers ($\mathcal{BGTN}$) and give some special cases with respect to the initial and r, s, t values. Then, we obtain the recurrence relation, Binet formula, generating function, exponential generating function, summation formulae, several new special properties, matrix formula, and special determinant equations related to these new types special numbers. Moreover, we establish some numerical algorithms including recurrence relation and special two types determinant equations related to calculating the terms of $\mathcal{BGTN}$. As a final part, we review the overall conclusions and give several contributions for future studies.

2. Basic Concepts

In this section, we give some background about bihyperbolic numbers and generalized Tribonacci numbers.

The addition and multiplication operations are commutative and associative on $\mathcal{H}$. $(\mathcal{H}, +, .)$ is a commutative ring [4]. Besides, a bihyperbolic number $\zeta = \rho_0 + \rho_1 j_1 + \rho_2 j_2 + \rho_3 j_3 \in \mathcal{H}$ has three conjugations, as follows:

$$\begin{cases} \overline{\zeta}^{j_1} = \rho_0 + \rho_1 j_1 - \rho_2 j_2 - \rho_3 j_3, \\ \overline{\zeta}^{j_2} = \rho_0 - \rho_1 j_1 + \rho_2 j_2 - \rho_3 j_3, \\ \overline{\zeta}^{j_3} = \rho_0 - \rho_1 j_1 - \rho_2 j_2 + \rho_3 j_3, \end{cases}$$

which are called as the principal conjugations of ζ [10].

Additionally, the characteristic equation of generalized Tribonacci numbers given in Eq. (2) is $x^3 - rx^2 - sx - t = 0$. The roots of this equation are given as follows:

$$x_1 = \frac{r}{3} + \alpha + \beta, \quad x_2 = \frac{r}{3} + \varepsilon\alpha + \varepsilon^2\beta, \quad x_3 = \frac{r}{3} + \varepsilon^2\alpha + \varepsilon\beta, \tag{3}$$

where

$$\begin{cases} \alpha = \sqrt[3]{\dfrac{r^3}{27} + \dfrac{rs}{6} + \dfrac{t}{2} + \sqrt{\mu}}, \\ \beta = \sqrt[3]{\dfrac{r^3}{27} + \dfrac{rs}{6} + \dfrac{t}{2} - \sqrt{\mu}}, \\ \varepsilon = \dfrac{-1 + i\sqrt{3}}{2}, \\ \mu = \dfrac{r^3 t}{27} - \dfrac{r^2 s^2}{108} + \dfrac{rst}{6} - \dfrac{s^3}{27} + \dfrac{t^2}{4}, \end{cases}$$

and

$$x_1 + x_2 + x_3 = r, \quad x_1x_2 + x_1x_3 + x_2x_3 = -s, \quad x_1x_2x_3 = t.$$

Providing $\mu > 0$, Eq. (2) has one real and two non-real solutions, the latter being conjugate complex. The following equation is called as Binet formula for generalized Tribonacci numbers [11]:

$$T_n = \frac{\widetilde{P}x_1^n}{(x_1 - x_2)(x_1 - x_3)} + \frac{\widetilde{R}x_2^n}{(x_2 - x_1)(x_2 - x_3)} + \frac{\widetilde{S}x_3^n}{(x_3 - x_1)(x_3 - x_2)}, \tag{4}$$

where

$$\begin{cases} \widetilde{P} = c - (x_2 + x_3)b + x_2x_3a, \\ \widetilde{R} = c - (x_1 + x_3)b + x_1x_3a, \\ \widetilde{S} = c - (x_1 + x_2)b + x_1x_2a. \end{cases} \tag{5}$$

Besides, the quite beneficial and functional method to generate T_n is applying S-matrix which is determined in [41,59] and is a generalization of the R-matrix. The S-matrix is determined as follows (see also [31,60]):

$$S = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In Table 1, some special subfamilies (9 pieces) of generalized Tribonacci numbers are given with respect to r, s, t values. Additionally, Table 2 includes several members of the family of generalized Tribonacci numbers (38 pieces) regarding both initial values and r, s, t values [1,12,13,15,16,19,20,36,40–42,44–54,59,60,62].

Table 1. A brief classification for generalized Tribonacci numbers

Name	$\{T_n\} = \{T_n(T_0, T_1, T_2; r, s, t)\}$	**Recurrence Relation**
G. Tribonacci (usual)	$\{\mathscr{A}_n\} = \{T_n(T_0, T_1, T_2; 1, 1, 1)\}$	$\mathscr{A}_n = \mathscr{A}_{n-1} + \mathscr{A}_{n-2} + \mathscr{A}_{n-3}$
G. Padovan	$\{\mathscr{G}_n\} = \{T_n(T_0, T_1, T_2; 0, 1, 1)\}$	$\mathscr{G}_n = \mathscr{G}_{n-2} + \mathscr{G}_{n-3}$
G. Pell-Padovan	$\{\mathscr{M}_n\} = \{T_n(T_0, T_1, T_2; 0, 2, 1)\}$	$\mathscr{M}_n = 2\mathscr{M}_{n-2} + \mathscr{M}_{n-3}$
G. T. Pell	$\{\mathscr{S}_n\} = \{T_n(T_0, T_1, T_2; 2, 1, 1)\}$	$\mathscr{S}_n = 2\mathscr{S}_{n-1} + \mathscr{S}_{n-2} + \mathscr{S}_{n-3}$
G. T. Jacobsthal	$\{\mathscr{X}_n\} = \{T_n(T_0, T_1, T_2; 1, 1, 2)\}$	$\mathscr{X}_n = \mathscr{X}_{n-1} + \mathscr{X}_{n-2} + 2\mathscr{X}_{n-3}$
G. Jacobsthal-Padovan	$\{\boldsymbol{\chi}_n\} = \{T_n(T_0, T_1, T_2; 0, 1, 2)\}$	$\boldsymbol{\chi}_n = \boldsymbol{\chi}_{n-2} + 2\boldsymbol{\chi}_{n-3}$
G. Narayana	$\{\boldsymbol{\vartheta}_n\} = \{T_n(T_0, T_1, T_2; 1, 0, 1)\}$	$\boldsymbol{\vartheta}_n = \boldsymbol{\vartheta}_{n-1} + \boldsymbol{\vartheta}_{n-3}$
G. 3-primes	$\{\boldsymbol{\kappa}_n\} = \{T_n(T_0, T_1, T_2; 2, 3, 5)\}$	$\boldsymbol{\kappa}_n = 2\boldsymbol{\kappa}_{n-1} + 3\boldsymbol{\kappa}_{n-1} + 5\boldsymbol{\kappa}_{n-3}$
G. Reverse 3-primes	$\{\boldsymbol{\nabla}_n\} = \{T_n(T_0, T_1, T_2; 5, 3, 2)\}$	$\boldsymbol{\nabla}_n = 5\boldsymbol{\nabla}_{n-1} + 3\boldsymbol{\nabla}_{n-1} + 2\boldsymbol{\nabla}_{n-3}$

*G.: Generalized, T.: Third Order

Table 2. Some special cases of generalized Tribonacci numbers

Name	$\{T_n\} = \{T_n(T_0, T_1, T_2; r, s, t)\}$	**Recurrence Relation**
Tribonacci	$\{A_n\} = \{T_n(0,1,1;1,1,1)\}$	$A_n = A_{n-1} + A_{n-2} + A_{n-3}$
Tribonacci-Lucas	$\{B_n\} = \{T_n(3,1,3;1,1,1)\}$	$B_n = B_{n-1} + B_{n-2} + B_{n-3}$
Tribonacci-Perrin	$\{C_n\} = \{T_n(3,0,2;1,1,1)\}$	$C_n = C_{n-1} + C_{n-2} + C_{n-3}$
M. Tribonacci	$\{D_n\} = \{T_n(1,1,1;1,1,1)\}$	$D_n = D_{n-1} + D_{n-2} + D_{n-3}$
M. Tribonacci-Lucas	$\{E_n\} = \{T_n(4,4,10;1,1,1)\}$	$E_n = E_{n-1} + E_{n-2} + E_{n-3}$
A. Tribonacci-Lucas	$\{F_n\} = \{T_n(4,2,0;1,1,1)\}$	$F_n = F_{n-1} + F_{n-2} + F_{n-3}$
Padovan (Cordonnier)	$\{G_n\} = \{T_n(1,1,1;0,1,1)\}$	$G_n = G_{n-2} + G_{n-3}$
Perrin	$\{H_n\} = \{T_n(3,0,2;0,1,1)\}$	$H_n = H_{n-2} + H_{n-3}$
Van der Laan	$\{I_n\} = \{T_n(1,0,1;0,1,1)\}$	$I_n = I_{n-2} + I_{n-3}$
Padovan-Perrin	$\{J_n\} = \{T_n(0,0,1;0,1,1)\}$	$J_n = J_{n-2} + J_{n-3}$
M. Padovan	$\{K_n\} = \{T_n(3,1,3;0,1,1)\}$	$K_n = K_{n-2} + K_{n-3}$
A. Padovan	$\{L_n\} = \{T_n(0,1,0;0,1,1)\}$	$L_n = L_{n-2} + L_{n-3}$
Pell-Padovan	$\{M_n\} = \{T_n(1,1,1;0,2,1)\}$	$M_n = 2M_{n-2} + M_{n-3}$
Pell-Perrin	$\{N_n\} = \{T_n(3,0,2;0,2,1)\}$	$N_n = 2N_{n-2} + N_{n-3}$
T. Fibonacci-Pell	$\{O_n\} = \{T_n(1,0,2;0,2,1)\}$	$O_n = 2O_{n-2} + O_{n-3}$
T. Lucas-Pell	$\{P_n\} = \{T_n(3,0,4;0,2,1)\}$	$P_n = 2P_{n-2} + P_{n-3}$
A. Pell-Padovan	$\{R_n\} = \{T_n(0,1,0;0,2,1)\}$	$R_n = 2R_{n-2} + R_{n-3}$
T. Pell	$\{S_n\} = \{T_n(0,1,2;2,1,1)\}$	$S_n = 2S_{n-1} + S_{n-2} + S_{n-3}$
T. Pell-Lucas	$\{U_n\} = \{T_n(3,2,6;2,1,1)\}$	$U_n = 2U_{n-1} + U_{n-2} + U_{n-3}$
T. modified Pell	$\{V_n\} = \{T_n(0,1,1;2,1,1)\}$	$V_n = 2V_{n-1} + V_{n-2} + V_{n-3}$
T. Pell-Perrin	$\{W_n\} = \{T_n(3,0,2;2,1,1)\}$	$W_n = 2W_{n-1} + W_{n-2} + W_{n-3}$
T. Jacobsthal	$\{X_n\} = \{T_n(0,1,1;1,1,2)\}$	$X_n = X_{n-1} + X_{n-2} + 2X_{n-3}$
T. Jacobsthal-Lucas	$\{Y_n\} = \{T_n(2,1,5;1,1,2)\}$	$Y_n = Y_{n-1} + Y_{n-2} + 2Y_{n-3}$
M. T. Jacobsthal	$\{Z_n\} = \{T_n(3,1,3;1,1,2)\}$	$Z_n = Z_{n-1} + Z_{n-2} + 2Z_{n-3}$
T. Jacobsthal-Perrin	$\{\Gamma_n\} = \{T_n(3,0,2;1,1,2)\}$	$\Gamma_n = \Gamma_{n-1} + \Gamma_{n-2} + 2\Gamma_{n-3}$
Jacobsthal-Padovan	$\{\chi_n\} = \{T_n(1,1,1;0,1,2)\}$	$\chi_n = \chi_{n-2} + 2\chi_{n-3}$
Jacobsthal-Perrin	$\{\Delta_n\} = \{T_n(3,0,2;0,1,2)\}$	$\Delta_n = \Delta_{n-2} + 2\Delta_{n-3}$
A. Jacobsthal-Padovan	$\{\omega_n\} = \{T_n(0,1,0;0,1,2)\}$	$\omega_n = \omega_{n-2} + 2\omega_{n-3}$
M. Jacobsthal-Padovan	$\{\Omega_n\} = \{T_n(3,1,3;0,1,2)\}$	$\Omega_n = \Omega_{n-2} + 2\Omega_{n-3}$
Narayana	$\{\vartheta_n\} = \{T_n(0,1,1;1,0,1)\}$	$\vartheta_n = \vartheta_{n-1} + \vartheta_{n-3}$
Narayana-Lucas	$\{\tau_n\} = \{T_n(3,1,1;1,0,1)\}$	$\tau_n = \tau_{n-1} + \tau_{n-3}$
Narayana-Perrin	$\{\sigma_n\} = \{T_n(3,0,2;1,0,1)\}$	$\sigma_n = \sigma_{n-1} + \sigma_{n-3}$
3-primes	$\{\kappa_n\} = \{T_n(0,1,2;2,3,5)\}$	$\kappa_n = 2\kappa_{n-1} + 3\kappa_{n-2} + 5\kappa_{n-3}$
Lucas 3-primes	$\{\theta_n\} = \{T_n(3,2,10;2,3,5)\}$	$\theta_n = 2\theta_{n-1} + 3\theta_{n-2} + 5\theta_{n-3}$
M. 3-primes	$\{\gamma_n\} = \{T_n(0,1,1;2,3,5)\}$	$\gamma_n = 2\gamma_{n-1} + 3\gamma_{n-2} + 5\gamma_{n-3}$
Reverse 3-primes	$\{\nabla_n\} = \{T_n(0,1,5;5,3,2)\}$	$\nabla_n = 5\nabla_{n-1} + 3\nabla_{n-2} + 2\nabla_{n-3}$
Reverse Lucas 3-primes	$\{\Lambda_n\} = \{T_n(3,5,31;5,3,2)\}$	$\Lambda_n = 5\Lambda_{n-1} + 3\Lambda_{n-2} + 2\Lambda_{n-3}$
Reverse M. 3-primes	$\{\phi_n\} = \{T_n(0,1,4;5,3,2)\}$	$\phi_n = 5\phi_{n-1} + 3\phi_{n-2} + 2\phi_{n-3}$

*M.: Modified, A.: Adjusted, T.: Third order

3. The Bihyperbolic Generalized Tribonacci Numbers

In this section, we introduce bihyperbolic generalized Tribonacci numbers ($\mathcal{BGTN}$) by taking into account several special cases with respect to r, s, t values, and initial values. Besides, we scrutinize not only classical several properties but also some new and interesting equations. Then, we support these new results with some numerical algorithms. Finally, we examine two special cases of $\mathcal{BGTN}$.

Definition 1. *The nth $\mathcal{BGTN}$ is defined as:*

$$\mathcal{T}_n = T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3, \quad n \geq 0 \tag{6}$$

with the initial values

$$\begin{cases} \mathcal{T}_0 = a + bj_1 + cj_2 + (rc + sb + ta)j_3, \\ \mathcal{T}_1 = b + cj_1 + (rc + sb + ta)j_2 + \left(\left(r^2 + s\right)c + (rs + t)\,b + rta\right)j_3, \\ \mathcal{T}_2 = c + (rc + sb + ta)\,j_1 + \left(\left(r^2 + s\right)c + (rs + t)\,b + rta\right)j_2 \\ \qquad + \left(\left(r^3 + 2rs + t\right)c + \left(r^2 s + s^2 + rt\right)b + \left(r^2 t + st\right)a\right)j_3, \end{cases}$$

where the rules of j_1, j_2, j_3 are given in Eq. (1) and T_n is the nth generalized Tribonacci number given in Eq. (2).

In the following Definition 2, we give some basic algebraic properties such as; equality, summation, subtraction, multiplication with a constant (a constant is a real number), multiplication of any two $\mathcal{BGTN}$, and also three types principal conjugations of $\mathcal{BGTN}$.

Definition 2 (Algebraic Properties)**.** *Let $\mathcal{T}_n$ and $\mathcal{T}_m$ be the nth and mth $\mathcal{BGTN}$, respectively. Then, the followings are defined:*

- ***Equality:***

$$\mathcal{T}_n = \mathcal{T}_m \Leftrightarrow \quad T_n = T_m, \quad T_{n+1} = T_{m+1}, \quad T_{n+2} = T_{m+2}, \quad T_{n+3} = T_{m+3},$$

- ***Addition/Subtraction:***

$$\mathcal{T}_n \pm \mathcal{T}_m = T_n \pm T_m + (T_{n+1} \pm T_{m+1})j_1 + (T_{n+2} \pm T_{m+2})j_2 + (T_{n+3} \pm T_{m+3})j_3,$$

- ***Multiplication by a scalar:***

$$\upsilon\mathcal{T}_n = \upsilon T_n + \upsilon T_{n+1}j_1 + \upsilon T_{n+2}j_2 + \upsilon T_{n+3}j_3, \quad \upsilon \in \mathbb{R},$$

- ***Multiplication:***

$$\begin{aligned} \mathcal{T}_n\mathcal{T}_m = & T_nT_m + T_{n+1}T_{m+1} + T_{n+2}T_{m+2} + T_{n+3}T_{m+3} \\ & + (T_nT_{m+1} + T_{n+1}T_m + T_{n+2}T_{m+3} + T_{n+3}T_{m+2})j_1 \\ & + (T_nT_{m+2} + T_{n+1}T_{m+3} + T_{n+2}T_m + T_{n+3}T_{m+1})j_2 \\ & + (T_nT_{m+3} + T_{n+1}T_{m+2} + T_{n+2}T_{m+1} + T_{n+3}T_m)j_3, \end{aligned}$$

by using the rules in Eq. (1) for multiplication.

• ***Principal Conjugates:*** *Also, the following three types principal conjugations of* $\mathcal{T}_n$ *are defined by:*

$$\begin{cases} \overline{\mathcal{T}}_n^{\,j_1} = T_n + T_{n+1}j_1 - T_{n+2}j_2 - T_{n+3}j_3, \\ \overline{\mathcal{T}}_n^{\,j_2} = T_n - T_{n+1}j_1 + T_{n+2}j_2 - T_{n+3}j_3, \\ \overline{\mathcal{T}}_n^{\,j_3} = T_n - T_{n+1}j_1 - T_{n+2}j_2 + T_{n+3}j_3. \end{cases} \tag{7}$$

Now, let us give the recurrence relation of $\mathcal{BGTN}$.

Theorem 1 (Recurrence Relation)**.** *Let* $\mathcal{T}_n$ *be the nth* $\mathcal{BGTN}$*. Then, the following recurrence relation is satisfied:*

$$\mathcal{T}_n = r\mathcal{T}_{n-1} + s\mathcal{T}_{n-2} + t\mathcal{T}_{n-3}, \quad n \geq 3. \tag{8}$$

Proof. Using Eqs. (2) and (6), we complete the proof:

$$\begin{aligned} r\mathcal{T}_{n-1} + s\mathcal{T}_{n-2} + t\mathcal{T}_{n-3} &= r(T_{n-1} + T_n j_1 + T_{n+1} j_2 + T_{n+2} j_3) \\ &\quad + s(T_{n-2} + T_{n-1} j_1 + T_n j_2 + T_{n+1} j_3) \\ &\quad + t(T_{n-3} + T_{n-2} j_1 + T_{n-1} j_2 + T_n j_3) \\ &= T_n + T_{n+1} j_1 + T_{n+2} j_2 + T_{n+3} j_3 \\ &= \mathcal{T}_n. \end{aligned}$$

□

In the following, we construct a numerical algorithm (Algorithm 1) in order to calculate the *nth* term of $\mathcal{BGTN}$ based on the recurrence relation given in Eq. (8).

Algorithm 1 A numerical algorithm for finding *nth* term of $\mathcal{BGTN}$

1: Begin
2: Input $\mathcal{T}_0, \mathcal{T}_1$ and $\mathcal{T}_2$
3: Compose $\mathcal{T}_n$ with respect to Eq. (8) for every $n \geq 3$
4: Count up $\mathcal{T}_n$
5: Output $\mathcal{T}_n = T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3$
6: Complete

With the same logic of Table 1 and Table 2 in Section "Basic Concepts", we can also obtain the same classifications and give special cases of $\mathcal{BGTN}$ in the following Table 3 and Table 4. The members of the $\mathcal{BGTN}$ which are written in Table 3 can be also classified and expressed in detail linked to Table 2 regarding recurrence relations and the initial values. For the sake of brevity, the small parts of them are written in Table 4 and Table 5 for readers to examine. The other members can be easily observed and examined, as well. The first three initial values for special cases written in Table 4 are given in Table 5.

TABLE 3. A brief classification for special cases of $\mathcal{BGTN}$

Name	Definition	Recurrence Relation
B. G. Tribonacci (usual)	$\widehat{\mathscr{A}_n} = \mathscr{A}_n + \mathscr{A}_{n+1}j_1 + \mathscr{A}_{n+2}j_2 + \mathscr{A}_{n+3}j_3$	$\widehat{\mathscr{A}_n} = \widehat{\mathscr{A}_{n-1}} + \widehat{\mathscr{A}_{n-2}} + \widehat{\mathscr{A}_{n-3}}$
B. G. Padovan	$\widehat{\mathscr{G}_n} = \mathscr{G}_n + \mathscr{G}_{n+1}j_1 + \mathscr{G}_{n+2}j_2 + \mathscr{G}_{n+3}j_3$	$\widehat{\mathscr{G}_n} = \widehat{\mathscr{G}_{n-2}} + \widehat{\mathscr{G}_{n-3}}$
B. G. Pell-Padovan	$\widehat{\mathscr{M}_n} = \mathscr{M}_n + \mathscr{M}_{n+1}j_1 + \mathscr{M}_{n+2}j_2 + \mathscr{M}_{n+3}j_3$	$\widehat{\mathscr{M}_n} = 2\widehat{\mathscr{M}_{n-2}} + \widehat{\mathscr{M}_{n-3}}$
B. G. T. Pell	$\widehat{\mathscr{S}_n} = \mathscr{S}_n + \mathscr{S}_{n+1}j_1 + \mathscr{S}_{n+2}j_2 + \mathscr{S}_{n+3}j_3$	$\widehat{\mathscr{S}_n} = 2\widehat{\mathscr{S}_{n-1}} + \widehat{\mathscr{S}_{n-2}} + \widehat{\mathscr{S}_{n-3}}$
B. G. T. Jacobsthal	$\widehat{\mathscr{X}_n} = \mathscr{X}_n + \mathscr{X}_{n+1}j_1 + \mathscr{X}_{n+2}j_2 + \mathscr{X}_{n+3}j_3$	$\widehat{\mathscr{X}_n} = \widehat{\mathscr{X}_{n-1}} + \widehat{\mathscr{X}_{n-2}} + 2\widehat{\mathscr{X}_{n-3}}$
B. G. Jacobsthal-Padovan	$\widehat{\boldsymbol{\chi}}_n = \boldsymbol{\chi}_n + \boldsymbol{\chi}_{n+1}j_1 + \boldsymbol{\chi}_{n+2}j_2 + \boldsymbol{\chi}_{n+3}j_3$	$\widehat{\boldsymbol{\chi}}_n = \widehat{\boldsymbol{\chi}}_{n-2} + 2\widehat{\boldsymbol{\chi}}_{n-3}$
B. G. Narayana	$\widehat{\boldsymbol{\vartheta}}_n = \boldsymbol{\vartheta}_n + \boldsymbol{\vartheta}_{n+1}j_1 + \boldsymbol{\vartheta}_{n+2}j_2 + \boldsymbol{\vartheta}_{n+3}j_3$	$\widehat{\boldsymbol{\vartheta}}_n = \widehat{\boldsymbol{\vartheta}}_{n-1} + \widehat{\boldsymbol{\vartheta}}_{n-3}$
B. G. 3-primes	$\widehat{\boldsymbol{\kappa}}_n = \boldsymbol{\kappa}_n + \boldsymbol{\kappa}_{n+1}j_1 + \boldsymbol{\kappa}_{n+2}j_2 + \boldsymbol{\kappa}_{n+3}j_3$	$\widehat{\boldsymbol{\kappa}}_n = 2\widehat{\boldsymbol{\kappa}}_{n-1} + 3\widehat{\boldsymbol{\kappa}}_{n-2} + 5\widehat{\boldsymbol{\kappa}}_{n-3}$
B. G. Reverse 3-primes	$\widehat{\boldsymbol{\nabla}}_n = \boldsymbol{\nabla}_n + \boldsymbol{\nabla}_{n+1}j_1 + \boldsymbol{\nabla}_{n+2}j_2 + \boldsymbol{\nabla}_{n+3}j_3$	$\widehat{\boldsymbol{\nabla}}_n = 5\widehat{\boldsymbol{\nabla}}_{n-1} + 3\widehat{\boldsymbol{\nabla}}_{n-2} + 2\widehat{\boldsymbol{\nabla}}_{n-3}$

*B.: Bihyperbolic, G.: Generalized, T.: Third Order

TABLE 4. Some special cases of $\mathcal{BGTN}$

Name	Definition	Recurrence Relation
B. Tribonacci-Lucas	$\mathcal{B}_n = B_n + B_{n+1}j_1 + B_{n+2}j_2 + B_{n+3}j_3$	$\mathcal{B}_n = \mathcal{B}_{n-1} + \mathcal{B}_{n-2} + \mathcal{B}_{n-3}$
B. Perrin	$\mathcal{H}_n = H_n + H_{n+1}j_1 + H_{n+2}j_2 + H_{n+3}j_3$	$\mathcal{H}_n = \mathcal{H}_{n-2} + \mathcal{H}_{n-3}$
B. Pell-Padovan	$\mathcal{M}_n = M_n + M_{n+1}j_1 + M_{n+2}j_2 + M_{n+3}j_3$	$\mathcal{M}_n = 2\mathcal{M}_{n-2} + \mathcal{M}_{n-3}$
B. T. Pell	$\mathcal{S}_n = S_n + S_{n+1}j_1 + S_{n+2}j_2 + S_{n+3}j_3$	$\mathcal{S}_n = 2\mathcal{S}_{n-1} + \mathcal{S}_{n-2} + \mathcal{S}_{n-3}$
B. T. Jacobsthal	$\mathcal{X}_n = X_n + X_{n+1}j_1 + X_{n+2}j_2 + X_{n+3}j_3$	$\mathcal{X}_n = \mathcal{X}_{n-1} + \mathcal{X}_{n-2} + 2\mathcal{X}_{n-3}$
B. Jacobsthal-Padovan	$\widetilde{\chi}_n = \chi_n + \chi_{n+1}j_1 + \chi_{n+2}j_2 + \chi_{n+3}j_3$	$\widetilde{\chi}_n = \widetilde{\chi}_{n-2} + 2\widetilde{\chi}_{n-3}$
B. Narayana	$\widetilde{\vartheta}_n = \vartheta_n + \vartheta_{n+1}j_1 + \vartheta_{n+2}j_2 + \vartheta_{n+3}j_3$	$\widetilde{\vartheta}_n = \widetilde{\vartheta}_{n-1} + \widetilde{\vartheta}_{n-3}$
B. 3-primes	$\widetilde{\kappa}_n = \kappa_n + \kappa_{n+1}j_1 + \kappa_{n+2}j_2 + \kappa_{n+3}j_3$	$\widetilde{\kappa}_n = 2\widetilde{\kappa}_{n-1} + 3\widetilde{\kappa}_{n-2} + 5\widetilde{\kappa}_{n-3}$
B. reverse 3-primes	$\widetilde{\nabla}_n = \nabla_n + \nabla_{n+1}j_1 + \nabla_{n+2}j_2 + \nabla_{n+3}j_3$	$\widetilde{\nabla}_n = 5\widetilde{\nabla}_{n-1} + 3\widetilde{\nabla}_{n-2} + 2\widetilde{\nabla}_{n-3}$

*B.: Bihyperbolic, T.: Third Order

Table 5. Initial values of special cases

For	$n=0$	$n=1$	$n=2$
$\mathcal{B}_n$	$3+j_1+3j_2+7j_3$	$1+3j_1+7j_2+11j_3$	$3+7j_1+11j_2+21j_3$
$\mathcal{H}_n$	$3+2j_2+3j_3$	$2j_1+3j_2+2j_3$	$2+3j_1+2j_2+5j_3$
$\mathcal{M}_n$	$1+j_1+j_2+3j_3$	$1+j_1+3j_2+3j_3$	$1+3j_1+3j_2+7j_3$
$\mathcal{S}_n$	$j_1+2j_2+5j_3$	$1+2j_1+5j_2+13j_3$	$2+5j_1+13j_2+33j_3$
$\mathcal{X}_n$	$j_1+j_2+2j_3$	$1+j_1+2j_2+5j_3$	$1+2j_1+5j_2+9j_3$
$\widetilde{\chi}_n$	$1+j_1+j_2+3j_3$	$1+j_1+3j_2+3j_3$	$1+3j_1+3j_2+5j_3$
$\widetilde{\vartheta}_n$	$j_1+j_2+j_3$	$1+j_1+j_2+2j_3$	$1+j_1+2j_2+3j_3$
$\widetilde{\kappa}_n$	$j_1+2j_2+7j_3$	$1+2j_1+7j_2+25j_3$	$2+7j_1+25j_2+81j_3$
$\widetilde{\nabla}_n$	$j_1+5j_2+28j_3$	$1+5j_1+28j_2+157j_3$	$5+28j_1+157j_2+879j_3$

Theorem 2. $\forall n \in \mathbb{N}$, *the Binet formula for the* $\mathcal{BGTN}$ *is as follows:*

$$\mathcal{T}_n = \frac{\widetilde{P}x_1^n\widetilde{x}_1}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^n\widetilde{x}_2}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^n\widetilde{x}_3}{(x_3-x_1)(x_3-x_2)},$$

where

$$\begin{cases} \widetilde{x}_1 = 1+x_1j_1+x_1^2j_2+x_1^3j_3, \\ \widetilde{x}_2 = 1+x_2j_1+x_2^2j_2+x_2^3j_3, \\ \widetilde{x}_3 = 1+x_3j_1+x_3^2j_2+x_3^3j_3. \end{cases} \tag{9}$$

Here $\widetilde{P}, \widetilde{R}, \widetilde{S}$ *are given in Eq.* (5) *and* x_1, x_2, x_3 *are given in Eq.* (3).

Proof. Using Eqs. (4) and (6), we manage to prove:

$$\begin{aligned} \mathcal{T}_n = \; & \frac{\widetilde{P}x_1^n}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^n}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^n}{(x_3-x_1)(x_3-x_2)} \\ & + \left(\frac{\widetilde{P}x_1^{n+1}}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^{n+1}}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^{n+1}}{(x_3-x_1)(x_3-x_2)}\right) j_1 \\ & + \left(\frac{\widetilde{P}x_1^{n+2}}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^{n+2}}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^{n+2}}{(x_3-x_1)(x_3-x_2)}\right) j_2 \\ & + \left(\frac{\widetilde{P}x_1^{n+3}}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^{n+3}}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^{n+3}}{(x_3-x_1)(x_3-x_2)}\right) j_3 . \end{aligned}$$

Finally, we reach $\mathcal{T}_n = \frac{\widetilde{P}x_1^n\widetilde{x}_1}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^n\widetilde{x}_2}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^n\widetilde{x}_3}{(x_3-x_1)(x_3-x_2)}$. □

Theorem 3. *The generating function of* $\mathcal{BGTN}$ *is as follows:*

$$\sum_{n=0}^{\infty} \mathcal{T}_n x^n = \frac{\mathcal{T}_0 + (\mathcal{T}_1 - r\mathcal{T}_0)\,x + (\mathcal{T}_2 - r\mathcal{T}_1 - s\mathcal{T}_0)\,x^2}{1 - rx - sx^2 - tx^3}. \tag{10}$$

Proof. Let the following function

$$G(x) = \sum_{n=0}^{\infty} \mathcal{T}_n x^n = \mathcal{T}_0 + \mathcal{T}_1 x + \mathcal{T}_2 x^2 + \ldots + \mathcal{T}_n x^n + \ldots$$

be generating function of $\mathcal{T}_n$. Then, if both sides of this equation are multiplied by rx, sx^2, tx^3, the followings are obtained:

$$\begin{aligned} rxG(x) &= r\mathcal{T}_0 x + r\mathcal{T}_1 x^2 + r\mathcal{T}_2 x^3 + \ldots + r\mathcal{T}_n x^{n+1} + \ldots \\ sx^2 G(x) &= s\mathcal{T}_0 x^2 + s\mathcal{T}_1 x^3 + s\mathcal{T}_2 x^4 + \ldots + s\mathcal{T}_n x^{n+2} + \ldots \\ tx^3 G(x) &= t\mathcal{T}_0 x^3 + t\mathcal{T}_1 x^4 + t\mathcal{T}_2 x^5 + \ldots + t\mathcal{T}_n x^{n+3} + \ldots \end{aligned}$$

Then, by using Eq. (8), we get:

$$(1 - rx - sx^2 - tx^3)G(x) = \mathcal{T}_0 + (\mathcal{T}_1 - r\mathcal{T}_0)x + (\mathcal{T}_2 - r\mathcal{T}_1 - s\mathcal{T}_0)x^2.$$

Consequently, we obtain Eq. (10). □

Theorem 4. *The exponential generating function of $\mathcal{BGTN}$ is as follows:*

$$\sum_{n=0}^{\infty} \mathcal{T}_n \frac{y^n}{n!} = \frac{\widetilde{P}\widetilde{x}_1 e^{x_1 y}}{(x_1 - x_2)(x_1 - x_3)} + \frac{\widetilde{R}\widetilde{x}_2 e^{x_2 y}}{(x_2 - x_1)(x_2 - x_3)} + \frac{\widetilde{S}\widetilde{x}_3 e^{x_3 y}}{(x_3 - x_1)(x_3 - x_2)}$$

(see $\widetilde{x}_1, \widetilde{x}_2$ and $\widetilde{x}_3$ in Eq. (9)).

Proof. By using Eq. (2), we get:

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathcal{T}_n \frac{y^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{\widetilde{P} x_1^n \widetilde{x}_1}{(x_1 - x_2)(x_1 - x_3)} + \frac{\widetilde{R} x_2^n \widetilde{x}_2}{(x_2 - x_1)(x_2 - x_3)} + \frac{\widetilde{S} x_3^n \widetilde{x}_3}{(x_3 - x_1)(x_3 - x_2)} \right) \frac{y^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\widetilde{P} x_1^n \widetilde{x}_1}{(x_1 - x_2)(x_1 - x_3)} \frac{y^n}{n!} + \sum_{n=0}^{\infty} \frac{\widetilde{R} x_2^n \widetilde{x}_2}{(x_2 - x_1)(x_2 - x_3)} \frac{y^n}{n!} \\ &\quad + \sum_{n=0}^{\infty} \frac{\widetilde{S} x_3^n \widetilde{x}_3}{(x_3 - x_1)(x_3 - x_2)} \frac{y^n}{n!} \\ &= \frac{\widetilde{P}\widetilde{x}_1}{(x_1 - x_2)(x_1 - x_3)} \sum_{n=0}^{\infty} \frac{(x_1 y)^n}{n!} + \frac{\widetilde{R}\widetilde{x}_2}{(x_2 - x_1)(x_2 - x_3)} \sum_{n=0}^{\infty} \frac{(x_2 y)^n}{n!} \\ &\quad + \frac{\widetilde{S}\widetilde{x}_3}{(x_3 - x_1)(x_3 - x_2)} \sum_{n=0}^{\infty} \frac{(x_3 y)^n}{n!} \\ &= \frac{\widetilde{P}\widetilde{x}_1 e^{x_1 y}}{(x_1 - x_2)(x_1 - x_3)} + \frac{\widetilde{R}\widetilde{x}_2 e^{x_2 y}}{(x_2 - x_1)(x_2 - x_3)} + \frac{\widetilde{S}\widetilde{x}_3 e^{x_3 y}}{(x_3 - x_1)(x_3 - x_2)}. \end{aligned}$$

The proof is completed. □

Thanks to the study [52], we can get the summation formulae for $\mathcal{BGTN}$ in the following theorem. The proof is omitted due to the fact that it can be completed with mathematical induction, easily.

Theorem 5. *$\forall m \in \mathbb{N}$, the following summation formulae for $\mathcal{BGTN}$ are satisfied:*

$$\text{(i)} \quad \sum_{n=0}^{m} \mathfrak{T}_n = \frac{\begin{array}{l}\mathfrak{T}_{m+3} + (1-r)\mathfrak{T}_{m+2} + (1-r-s)\mathfrak{T}_{m+1} - \mathfrak{T}_2 + (r-1)\mathfrak{T}_1 \\ +(r+s-1)\mathfrak{T}_0\end{array}}{r+s+t-1},$$

$$\text{(ii)} \quad \sum_{n=0}^{m} \mathfrak{T}_{2n} = \frac{\begin{array}{l}(1-s)\mathfrak{T}_{2m+2} + (t+rs)\mathfrak{T}_{2m+1} + (t^2+rt)\mathfrak{T}_{2m} + (s-1)\mathfrak{T}_2 \\ +(-t-rs)\mathfrak{T}_1 + (r^2-s^2+rt+2s-1)\mathfrak{T}_0\end{array}}{(r+s+t-1)(r-s+t+1)},$$

$$\text{(iii)} \quad \sum_{n=0}^{m} \mathfrak{T}_{2n+1} = \frac{\begin{array}{l}(r+t)\mathfrak{T}_{2m+2} + (s-s^2+t^2+rt)\mathfrak{T}_{2m+1} + (t-st)\mathfrak{T}_{2m} \\ +(-r-t)\mathfrak{T}_2 + (-1+s+r^2+rt)\mathfrak{T}_1 + (-t+st)\mathfrak{T}_0\end{array}}{(r-s+t+1)(r+s+t-1)},$$

where denominators are not equal to zero.

Particular Case 1. *If $s = 1$, we can get the following summation formulae for special cases of part (ii) and (iii) of the previous Theorem 5:*

$$\text{(i)} \quad \sum_{n=0}^{m} \mathfrak{T}_{2n} = \frac{\mathfrak{T}_{2m+1} + t\mathfrak{T}_{2m} - \mathfrak{T}_1 + r\mathfrak{T}_0}{r+t},$$

$$\text{(ii)} \quad \sum_{n=0}^{m} \mathfrak{T}_{2n+1} = \frac{\mathfrak{T}_{2m+2} + t\mathfrak{T}_{2m+1} - \mathfrak{T}_2 + r\mathfrak{T}_1}{r+t},$$

where denominators are not equal to zero.

Thanks to the study [11], we get the following Theorem 6:

Theorem 6. *$\forall m \in \mathbb{N}$, the following summation property holds for $\mathcal{BGTN}$:*

$$\sum_{n=0}^{m} \mathfrak{T}_n = \frac{\mathfrak{T}_{m+2} + (1-r)\mathfrak{T}_{m+1} + t\mathfrak{T}_m + \eta}{\delta},$$

where

$$\begin{cases} \delta = r+s+t-1, \\ \lambda = (r+s-1)a + (r-1)b - c, \\ \eta = \lambda + (\lambda - \delta a)j_1 + (\lambda - \delta(a+b))j_2 + (\lambda - \delta(a+b+c))j_3. \end{cases}$$

Proof. Using Eq. (6) and utilizing the Lemma 2.3 on page 6 in the study [11], then we can complete the proof:

$$\begin{aligned}
\sum_{n=0}^{m} \mathfrak{T}_n &= \sum_{n=0}^{m} \left(T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3\right) \\
&= \sum_{n=0}^{m} T_n + \sum_{n=0}^{m} T_{n+1}j_1 + \sum_{n=0}^{m} T_{n+2}j_2 + \sum_{n=0}^{m} T_{n+3}j_3 \\
&= \frac{1}{\delta} \begin{bmatrix} T_{m+2} + (1-r)T_{m+1} + tT_m + \lambda \\ + \left(T_{m+3} + (1-r)T_{m+2} + tT_{m+1} + \lambda - \delta a\right) j_1 \\ + \left(T_{m+4} + (1-r)T_{m+3} + tT_{m+2} + \lambda - \delta\left(a+b\right)\right) j_2 \\ + \left(T_{m+5} + (1-r)T_{m+4} + tT_{m+3} + \lambda - \delta\left(a+b+c\right)\right) j_3 \end{bmatrix} \\
&= \frac{\mathfrak{T}_{m+2} + (1-r)\mathfrak{T}_{m+1} + t\mathfrak{T}_m + \eta}{\delta}.
\end{aligned}$$

We get the desired result. □

Theorem 7. $\forall n \in \mathbb{N}$, *the following properties are satisfied:*

(i) $\mathfrak{T}_n + \overline{\mathfrak{T}}_n^{\,j_1} = 2(T_n + T_{n+1}j_1)$,
(ii) $\mathfrak{T}_n + \overline{\mathfrak{T}}_n^{\,j_2} = 2(T_n + T_{n+2}j_2)$,
(iii) $\mathfrak{T}_n + \overline{\mathfrak{T}}_n^{\,j_3} = 2(T_n + T_{n+3}j_2)$.

Proof. (i) Using Eqs. (6) and (7), the proof is completed as:

$$\begin{aligned}
\mathfrak{T}_n + \overline{\mathfrak{T}}_n^{\,j_1} =& T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3 + T_n + T_{n+1}j_1 - T_{n+2}j_2 - T_{n+3}j_3 \\
=& 2(T_n + T_{n+1}j_1).
\end{aligned}$$

By the same way, the other parts can be obtained. □

Theorem 8. $\forall n \in \mathbb{N}$, *the following property holds:*

$$\mathfrak{T}_n - \mathfrak{T}_{n+1}j_1 - \mathfrak{T}_{n+2}j_2 - \mathfrak{T}_{n+3}j_3 = T_n - T_{n+2} - T_{n+4} + T_{n+6} - 2\mathfrak{T}_{n+3}j_3.$$

Proof. Using Eqs. (6) and (1), we have:

$$\begin{aligned}
\mathfrak{T}_n - \mathfrak{T}_{n+1}j_1 - \mathfrak{T}_{n+2}j_2 - \mathfrak{T}_{n+3}j_3 =& T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3 \\
&- (T_{n+1} + T_{n+2}j_1 + T_{n+3}j_2 + T_{n+4}j_3)j_1 \\
&- (T_{n+2} + T_{n+3}j_1 + T_{n+4}j_2 + T_{n+5}j_3)j_2 \\
&- (T_{n+3} + T_{n+4}j_1 + T_{n+5}j_2 + T_{n+6}j_3)j_3 \\
=& T_n - T_{n+2} - T_{n+4} + T_{n+6} - 2\mathfrak{T}_{n+3}j_3.
\end{aligned}$$

Hence, this proof is completed. □

Theorem 9. $\forall n \in \mathbb{Z}^+$, *the following is obtained:*

$$\begin{pmatrix} \mathfrak{T}_{n+2} \\ \mathfrak{T}_{n+1} \\ \mathfrak{T}_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathfrak{T}_2 \\ \mathfrak{T}_1 \\ \mathfrak{T}_0 \end{pmatrix}.$$

Proof. The proof can be conducted by mathematical induction, therefore we omit it. □

By inspiring the study [32], we present the following determinant equation for $\mathcal{BGTN}$ which enables a different way to find the nth term.

Theorem 10. $\forall n \in \mathbb{N}$, *the following equation holds:*

$$\mathfrak{T}_n = \begin{vmatrix} \mathfrak{T}_0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mathfrak{T}_1 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \mathfrak{T}_2 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & t & s & r & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & r & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & s & r \end{vmatrix}_{(n+1)\times(n+1)} . \tag{11}$$

Proof. It can be proved by using Eq. (8) and Theorem 5 on page 5 in [32]. □

In the following, we construct a numerical algorithm (Algorithm 2) with respect to the determinant equation given by Theorem 10.

Algorithm 2 A numerical algorithm for finding nth term of $\mathcal{BGTN}$

1: Begin
2: Input $\mathfrak{T}_0, \mathfrak{T}_1$ and $\mathfrak{T}_2$
3: Form $\mathfrak{T}_n$ with respect to Eq. (11)
4: Compute $\mathfrak{T}_n$
5: Output $\mathfrak{T}_n = T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3$
6: Complete

Also, thanks to the study [16] and [14], we get the other method which can be examined in Theorem 11, in order to calculate the nth terms of $\mathcal{BGTN}$.

Theorem 11. *$\forall n \in \mathbb{N}$, the following equation is satisfied:*

$$\mathcal{T}_n = \begin{vmatrix} \mathcal{T}_0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ r\mathcal{T}_0 - \mathcal{T}_1 & r & \frac{1}{\mathcal{T}_0} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & r\mathcal{T}_1 - \mathcal{T}_2 & r & t & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathcal{T}_0 & -\frac{s}{t} & r & t & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{1}{t} & -\frac{s}{t} & r & t & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & r & t \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -\frac{s}{t} & r \end{vmatrix}_{(n+1)\times(n+1)} \tag{12}$$

where $\mathcal{T}_0 \overline{\mathcal{T}}_0^{j_1} \overline{\mathcal{T}}_0^{j_2} \overline{\mathcal{T}}_0^{j_3} \neq 0$ and $t \neq 0$.

Proof. For the sake of brevity, we also skip this proof. □

Now, let us give a numerical algorithm in the following (Algorithm 3) related to the Theorem 11.

Algorithm 3 A numerical algorithm for finding *nth* term of $\mathcal{BGTN}$

1: Begin
2: Input $\mathcal{T}_0, \mathcal{T}_1$ and $\mathcal{T}_2$
3: Form $\mathcal{T}_n$ according to Eq. (12)
4: Compute $\mathcal{T}_n$
5: Output $\mathcal{T}_n = T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3$
6: Complete

According to the Theorem 1-Theorem 11, we can get the following two corollaries consisting of several features for bihyperbolic Tribonacci numbers and bihyperbolic Padovan numbers, respectively. With the same logic, these concepts are also valid for the other $\mathcal{BGTN}$ which are not need to be written here for the sake of brevity (see subfamilies in Table 3 and a small part of them in Table 4).

Corollary 1. *Let consider the nth bihyperbolic Tribonacci number $\mathcal{A}_n$ with the initial values*

$$\begin{cases} \mathcal{A}_0 = j_1 + j_2 + 2j_3, \\ \mathcal{A}_1 = 1 + j_1 + 2j_2 + 4j_3, \\ \mathcal{A}_2 = 1 + 2j_1 + 4j_2 + 7j_3. \end{cases}$$

Then the followings hold:

(i) *The recurrence relation for* $\mathcal{A}_n$ *is as:*

$$\mathcal{A}_n = \mathcal{A}_{n-1} + \mathcal{A}_{n-2} + \mathcal{A}_{n-3}, \quad n \geq 3.$$

(ii) *The Binet formula of* $\mathcal{A}_n$ *is as:*

$$\mathcal{A}_n = \frac{x_1^{n+1}\widetilde{x}_1}{(x_1 - x_2)(x_1 - x_3)} + \frac{x_2^{n+1}\widetilde{x}_2}{(x_2 - x_1)(x_2 - x_3)} + \frac{x_3^{n+1}\widetilde{x}_3}{(x_3 - x_1)(x_3 - x_2)} \cdot$$

(iii) *The generating function of* $\mathcal{A}_n$ *is as:*

$$\sum_{n=0}^{\infty} \mathcal{A}_n x^n = \frac{\mathcal{A}_0 + (\mathcal{A}_1 - \mathcal{A}_0)x + (\mathcal{A}_2 - \mathcal{A}_1 - \mathcal{A}_0)x^2}{1 - x - x^2 - x^3} \cdot$$

(iv) *The exponential generating function of* $\mathcal{A}_n$ *is as:*

$$\sum_{n=0}^{\infty} \mathcal{A}_n \frac{y^n}{n!} = \frac{x_1\widetilde{x}_1 e^{x_1 y}}{(x_1 - x_2)(x_1 - x_3)} + \frac{x_2\widetilde{x}_2 e^{x_2 y}}{(x_2 - x_1)(x_2 - x_3)} + \frac{x_3\widetilde{x}_3 e^{x_3 y}}{(x_3 - x_1)(x_3 - x_2)} \cdot$$

(v) $\forall m \in \mathbb{N}$, *the summation formulae for* $\mathcal{A}_n$ *are satisfied:*

- $\sum\limits_{n=0}^{m} \mathcal{A}_n = \frac{1}{2}(\mathcal{A}_{m+3} - \mathcal{A}_{m+1} - \mathcal{A}_2 + \mathcal{A}_0)$,
- $\sum\limits_{n=0}^{m} \mathcal{A}_{2n} = \frac{1}{2}(\mathcal{A}_{2m+1} + \mathcal{A}_{2m} - \mathcal{A}_1 + \mathcal{A}_0)$,
- $\sum\limits_{n=0}^{m} \mathcal{A}_{2n+1} = \frac{1}{2}(\mathcal{A}_{2m+2} + \mathcal{A}_{2m+1} - \mathcal{A}_2 + \mathcal{A}_1)$.

(vi) $\forall m \in \mathbb{N}$, *the following summation property holds for* $\mathcal{A}_n$*:*

$$\sum_{n=0}^{m} \mathcal{A}_n = \frac{\mathcal{A}_{m+2} + \mathcal{A}_m + (-1 - j_1 - 3j_2 - 5j_3)}{2} \cdot$$

(vii) *The following properties are derived:*

- $\mathcal{A}_n + \overline{\mathcal{A}}_n^{\,j_1} = 2(A_n + A_{n+1}j_1)$,
- $\mathcal{A}_n + \overline{\mathcal{A}}_n^{\,j_2} = 2(A_n + A_{n+2}j_2)$,
- $\mathcal{A}_n + \overline{\mathcal{A}}_n^{\,j_3} = 2(A_n + A_{n+3}j_2)$.

(viii) *The following property for* $\mathcal{A}_n$ *is supplied as:*

$$\mathcal{A}_n - \mathcal{A}_{n+1}j_1 - \mathcal{A}_{n+2}j_2 - \mathcal{A}_{n+3}j_3 = A_n - A_{n+2} - A_{n+4} + A_{n+6} - 2\mathcal{A}_{n+3}j_3.$$

(ix) *The following property for* $\mathcal{A}_n$ *is maintained as:*

$$\begin{pmatrix} \mathcal{A}_{n+2} \\ \mathcal{A}_{n+1} \\ \mathcal{A}_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathcal{A}_2 \\ \mathcal{A}_1 \\ \mathcal{A}_0 \end{pmatrix}.$$

(x) *The following equation for $\mathcal{A}_n$ holds as:*

$$\mathcal{A}_n = \begin{vmatrix} \mathcal{A}_0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{A}_1 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{A}_2 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 \end{vmatrix}_{(n+1)\times(n+1)}.$$

(xi) *Since the value of $\mathcal{A}_0\overline{\mathcal{A}_0}^{j_1}\overline{\mathcal{A}_0}^{j_2}\overline{\mathcal{A}_0}^{j_3}$ is zero, we cannot construct the method with respect to the determinant equation for the bihyperbolic Tribonacci numbers given in Eq. (12) written in the Theorem 11.*

Now, let us present an example with respect to the method given in part (x) of Corollary 1. Consider $n = 7$ and let us calculate the $7th$ term of the $\mathcal{BGTN}$:

$$\begin{vmatrix} \mathcal{A}_0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{A}_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{A}_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{vmatrix}_{8\times 8} = 24 + 44j_1 + 81j_2 + 149j_3 = \mathcal{A}_7.$$

Corollary 2. *Let consider the nth bihyperbolic Padovan number $\mathcal{G}_n$ with the initial values*

$$\begin{cases} \mathcal{G}_0 = 1 + j_1 + j_2 + 2j_3, \\ \mathcal{G}_1 = 1 + j_1 + 2j_2 + 2j_3, \\ \mathcal{G}_2 = 1 + 2j_1 + 2j_2 + 3j_3. \end{cases}$$

Then, the followings hold:

(i) *The recurrence relation for $\mathcal{G}_n$ is as:*

$$\mathcal{G}_n = \mathcal{G}_{n-2} + \mathcal{G}_{n-3}, \quad n \geq 3.$$

(ii) *The Binet formula of $\mathcal{G}_n$ is as:*

$$\mathcal{G}_n = \frac{(x_2-1)(x_3-1)x_1^n\widetilde{x}_1}{(x_1-x_2)(x_1-x_3)} + \frac{(x_1-1)(x_3-1)x_2^n\widetilde{x}_2}{(x_2-x_1)(x_2-x_3)} + \frac{(x_1-1)(x_2-1)x_3^n\widetilde{x}_3}{(x_3-x_1)(x_3-x_2)}.$$

(iii) *The generating function of $\mathcal{G}_n$ is as:*

$$\sum_{n=0}^{\infty} \mathcal{G}_n x^n = \frac{\mathcal{G}_0 + \mathcal{G}_1 x + (\mathcal{G}_2 - \mathcal{G}_0)x^2}{1 - x^2 - x^3}.$$

(iv) *The exponential generating function of $\mathcal{G}_n$ is as:*

$$\sum_{n=0}^{\infty} \mathcal{G}_n \frac{y^n}{n!} = \frac{(x_2-1)(x_3-1)\widetilde{x}_1 e^{x_1 y}}{(x_1-x_2)(x_1-x_3)} + \frac{(x_1-1)(x_3-1)\widetilde{x}_2 e^{x_2 y}}{(x_2-x_1)(x_2-x_3)} + \frac{(x_1-1)(x_2-1)\widetilde{x}_3 e^{x_3 y}}{(x_3-x_1)(x_3-x_2)}.$$

(v) $\forall m \in \mathbb{N}$, *the summation formulae for $\mathcal{G}_n$ are satisfied:*

- $\sum\limits_{n=0}^{m} \mathcal{G}_n = \mathcal{G}_{m+3} + \mathcal{G}_{m+2} - \mathcal{G}_2 - \mathcal{G}_1,$
- $\sum\limits_{n=0}^{m} \mathcal{G}_{2n} = \mathcal{G}_{2m+1} + \mathcal{G}_{2m} - \mathcal{G}_1,$
- $\sum\limits_{n=0}^{m} \mathcal{G}_{2n+1} = \mathcal{G}_{2m+2} + \mathcal{G}_{2m+1} - \mathcal{G}_2.$

(vi) $\forall m \in \mathbb{N}$, *the following summation property holds for $\mathcal{G}_n$:*

$$\sum_{n=0}^{m} \mathcal{G}_n = \mathcal{G}_{m+2} + \mathcal{G}_{m+1} + \mathcal{G}_m + (-2 - 3j_1 - 4j_2 - 5j_3).$$

(vii) *The following properties for $\mathcal{G}_n$ are derived:*

- $\mathcal{G}_n + \overline{\mathcal{G}}_n^{\,j_1} = 2(G_n + G_{n+1}j_1),$
- $\mathcal{G}_n + \overline{\mathcal{G}}_n^{\,j_2} = 2(G_n + G_{n+2}j_2),$
- $\mathcal{G}_n + \overline{\mathcal{G}}_n^{\,j_3} = 2(G_n + G_{n+3}j_2).$

(viii) *The following property for $\mathcal{G}_n$ is supplied:*

$$\mathcal{G}_n - \mathcal{G}_{n+1}j_1 - \mathcal{G}_{n+2}j_2 - \mathcal{G}_{n+3}j_3 = G_n - G_{n+2} - G_{n+4} + G_{n+6} - 2\mathcal{G}_{n+3}j_3.$$

(ix) *The following property for $\mathcal{G}_n$ is maintained as:*

$$\begin{pmatrix} \mathcal{G}_{n+2} \\ \mathcal{G}_{n+1} \\ \mathcal{G}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathcal{G}_2 \\ \mathcal{G}_1 \\ \mathcal{G}_0 \end{pmatrix}.$$

(x) *The following equality for $\mathcal{G}_n$ holds:*

$$\mathcal{G}_n = \begin{vmatrix} \mathcal{G}_0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{G}_1 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{G}_2 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 0 \end{vmatrix}_{(n+1)\times(n+1)}.$$

(xi) *The following equation for $\mathcal{G}_n$ is satisfied:*

$$\mathcal{G}_n = \begin{vmatrix} \mathcal{G}_0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -\mathcal{G}_1 & 0 & \dfrac{1}{\mathcal{G}_0} & 0 & \dots & 0 & 0 \\ 0 & -\mathcal{G}_2 & 0 & 1 & \dots & 0 & 0 \\ 0 & \mathcal{G}_0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{vmatrix}_{(n+1)\times(n+1)},$$

where $\mathcal{G}_0\,\overline{\mathcal{G}_0}^{j_1}\overline{\mathcal{G}_0}^{j_2}\overline{\mathcal{G}_0}^{j_3} = 5 \neq 0$.

Let us give an example with respect to the method given in part (xi) of Corollary 2. Consider $n = 3$, and let us calculate the $3th$ term of the $\mathcal{BGTN}$:

$$\begin{vmatrix} \mathcal{G}_0 & 1 & 0 & 0 \\ -\mathcal{G}_1 & 0 & \dfrac{1}{\mathcal{G}_0} & 0 \\ 0 & -\mathcal{G}_2 & 0 & 1 \\ 0 & \mathcal{G}_0 & -1 & 0 \end{vmatrix}_{4\times 4} = 2 + 2j_1 + 3j_2 + 4j_3 = \mathcal{G}_3.$$

4. Conclusions

In this present study, we introduce the $\mathcal{BGTN}$ by examining several well-known relations and identities. By putting this theory into literature, we have an extended framework for third-order linear recurrence sequences with bihyperbolic number components.

For future works, let us make a brief introduction associated with the topic: quaternions with $\mathcal{BGTN}$ components. Quaternions were defined by W. R. Hamilton [23, 24], and the algebra of quaternions is associative, non-commutative, and 4-dimensional Clifford algebra. Quaternions have huge significance in lots of areas such as; pure/applied mathematics, motion geometry, differential geometry, graph theory, differential equations, computer animation, robotics, and so on. A quaternion is represented by $q = q_0 + q_1 i + q_2 j + q_3 k$ where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and i, j, k are quaternionic units which satisfy:

$$i^2 = -1,\ j^2 = -1,\ k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j. \quad (13)$$

Hence, the nth quaternion with $\mathcal{BGTN}$ components can be defined as:

$$\mathbb{T}_n = \mathcal{T}_n + \mathcal{T}_{n+1} i + \mathcal{T}_{n+2} j + \mathcal{T}_{n+3} k, \quad n \geq 0$$

with the initial conditions $\mathbb{T}_0, \mathbb{T}_1$ and $\mathbb{T}_2$ considering Eq. (13). As an illustration

$$
\begin{aligned}
\mathbb{T}_0 =& a + bj_1 + cj_2 + (rc + sb + ta)j_3 \\
&+ \left\{ \begin{array}{l} b + cj_1 + (rc + sb + ta)\, j_2 + \left[\left(r^2 + s\right) c \right. \\ \left. + (rs + t)\, b + rta\right] j_3 \end{array} \right\} i \\
&+ \left\{ \begin{array}{l} c + (rc + sb + ta)\, j_1 \\ + \left[\left(r^2 + s\right) c + (rs + t)\, b + rta\right] j_2 \\ + \left[\begin{array}{l} \left(r^3 + 2rs + t\right) c + \left(r^2 s + s^2 + rt\right) b \\ + \left(r^2 t + st\right) a \end{array} \right] j_3 \end{array} \right\} j \\
&+ \left\{ \begin{array}{l} rc + sb + ta + \left[\left(r^2 + s\right) c + (rs + t)\, b + rta\right] j_1 \\ + \left[\begin{array}{l} \left(r^3 + 2rs + t\right) c + \left(r^2 s + s^2 + rt\right) b \\ + \left(r^2 t + st\right) a \end{array} \right] j_2 \\ + \left[\begin{array}{l} \left(r^3 t + 2str + t^2\right) a \\ + \left(r^3 s + r^2 t + 2s^2 r + 2st\right) b \\ + \left(r^4 + 3r^2 s + s^2 + 2tr\right) c \end{array} \right] j_3 \end{array} \right\} k.
\end{aligned}
$$

Additionally, the recurrence relation $\mathbb{T}_n = r\mathbb{T}_{n-1} + s\mathbb{T}_{n-2} + t\mathbb{T}_{n-3},\ n \geq 3$ holds for $\mathbb{T}_n$. So, quaternions with several members of $\mathcal{BGTN}$ components can be easily understood by taking into account Table 3 and Table 4.

As an another aspect, the type of quaternion can also be changed in line with this objective, for instance generalized quaternion case. Additionally, with the guidance of the study [57], combining 3-parameter generalized quaternions (as a special generalization of 2-parameter generalized quaternions) with Tribonacci numbers and bihyperbolic number are our another forthcoming goals. We intend to examine these topics exhaustively in future works.

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APPROXIMATION PROPERTIES OF THE UNIVARIATE AND BIVARIATE BERNSTEIN-STANCU OPERATORS OF MAX-PRODUCT KIND

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Abstract. This paper presents the nonlinear maximum product type of univariate and bivariate Bernstein–Stancu operators and uses new definitions to investigate the approximation properties. The order of approximation obtained with the nonlinear maximum product type of operator sequences would be better than the degree of approximation of the known linear operator sequences.

1. Introduction

In 1969, Stancu [3] introduced the Bernstein-Stancu polynomials as follows

$$B_n^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n}{k}x^k(1-x)^k, \tag{1}$$

where $n \in \mathbb{N}$, $f \in C[0,1]$, which is the space of all real valued continuous functions defined on $[0,1]$, real numbers α and β are fixed, indicating that $0 \leq \alpha \leq \beta$. The classical Bernstein polynomials are obtained in the condition $\alpha = \beta = 0$.

The approximation of a continuous function by a series of linear positive operators is the main topic of Korovkin-type approximation theory (see [1], [2], [18]- [21]). Recently, Bede et al., [4] have introduced nonlinear positive operators in place of linear positive operators. Bede et al., introduced the max-product version of families of linear approximation operators (see [7], [8]), which generated a new field of

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approximation theory. Up to now researchers have investigated and explored many aspects of max product operators (see [13]- [17]). It is concluded that they have an even better order of approximation for certain subclasses of functions and the same order as in the case of positive linear operators.

Lemma 1. *([5]) Let $I \subset \mathbb{R}$ be a bounded or unbounded interval, $CB_+(I)$ be the space of real valued continuous and bounded functions defined on I, and $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the next requirements:*

(i) (Monotonicity) If $f, g \in CB_+(I)$ provide $f \leq g$ then $L_n(f) \leq L_n(g)$ for all $n \in \mathbb{N}$;

(ii) (Sublinearity) For all $f, g \in CB_+(I)$ $L_n(f+g) \leq L_n(f) + L_n(g)$.

In [11], the function of two real variables function f be given over the unit square $s : [0,1][0,1]$ then the bivariate Bernstein polynomial of degree (n,m), corresponding to the function f, is defined by means of the formula

$$B_{n,m}(f)(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} p_{n,i}(x) p_{m,j}(y) f(i/n, j/m).$$

The square interval of bivariate Bernstein polynomials connected to a function of $f(x,y)$ given by

$$\begin{aligned} B_{n,m}(f)(x,y) &= \sum_{i=0}^{n} \sum_{j=0}^{m} p_{n,i}(x) p_{m,j}(y) f(i/n, j/m) \\ &= \frac{\sum_{i=0}^{n} \sum_{j=0}^{m} p_{n,i}(x) p_{m,j}(y) f(i/n, j/m)}{\sum_{i=0}^{n} \sum_{j=0}^{m} p_{n,i}(x) p_{m,j}(y)}, \end{aligned}$$

where $f : [0,1] \times [0,1] \to \mathbb{R}$, $(x,y) \in [0,1]^2$, $n, m \in \mathbb{N}$ (see Hildebrandt–Schoenberg [9], Butzer [10]). Also, in [6] max-product Bernstein operators of two variables defined by

$$B_{n,m}^{(M)}(f)(x,y) = \frac{\bigvee_{i=0}^{n} \bigvee_{j=0}^{m} p_{n,i}(x) p_{m,j}(y) f(i/n, j/m)}{\bigvee_{i=0}^{n} \bigvee_{j=0}^{m} p_{n,i}(x) p_{m,j}(y)}, \ (x,y) \in [0,1]^2, n, m \in \mathbb{N}.$$

In this work, we study the max-product Bernstein-Stancu operators. Firstly, we give the definition of the bivariate Bernstein-Stancu max product operators and investigate approximation properties of these operators. Then, the bivariate Bernstein-Stancu max product operators will be defined and approximation properties for bivariate Bernstein-Stancu max product operators will be investigated.

2. Construction of the Bernstein–Stancu Operators of Max-Product Kind and the Approximation Properties

We describe the max-product type nonlinear Bernstein-Stancu operators as follows

$$P_z^{(M)}(\kappa;x)=\frac{\bigvee_{m=0}^{z}p_{z,m}(x)\kappa\left(\frac{m+\rho}{z+\theta}\right)}{\bigvee_{m=0}^{z}p_{z,m}(x)},z\in\mathbb{N} \tag{2}$$

with $p_{z,m}(x)=\binom{z}{m}x^m(1-x)^{z-m}$ where $\kappa:[0,1]\to\mathbb{R}^+,\ x\in[0,1]\ ;\ \rho,\theta\in\mathbb{R}^+$ $0\le\rho\le\theta$ and

$$lim_{z\to\infty}\frac{1}{z+\theta}=0.$$

Here, $P_z^{(M)}(\kappa;x)$ is positive and continuous on $[0,1]$ for the continuous function κ (see [12]). We also know that for $\forall z\in\mathbb{N}$, $P_z^{(M)}(\kappa;0)-\kappa(0)=0$ and the following definitions will be given for the arbitrary $0\le x\le 1$.

Now, the approximation rate will be calculated with the help of the modulus of continuity for maximum product type Bernstein-Stancu operators. To prove the main results, we need the following notations and auxiliary results. For each $m,\varpi\in\{0,1,...,z\}$ and $x\in\left[\frac{\varpi+\rho}{z+\theta+1}\frac{\varpi+\rho+1}{z+\theta+1}\right]$, $z\in\mathbb{N}$ and $0\le\rho\le\theta$, let us indicate

$$N_{m,z,\varpi}(x)=\frac{p_{z,m}(x)|\frac{m+\rho}{z+\theta}-x|}{p_{z,\varpi}(x)},\ n_{m,z,\varpi}(x)=\frac{p_{z,m}(x)}{p_{z,\varpi}(x)}.$$

Let $m,\varpi\in\{0,1,...,z\}$, $x\in\left[\frac{\varpi+\rho}{z+\theta+1},\frac{\varpi+\rho+1}{z+\theta+1}\right]$, $z\in\mathbb{N}$ and for $\rho,\theta\in\mathbb{R}^+$, we have $0\le\rho\le\theta$. Hence, it is obvious that

i. If $\varpi+1\le m$ then $N_{m,z,\varpi}(x)=\frac{p_{z,m}(x)(\frac{m+\rho}{z+\theta}-x)}{p_{z,\varpi}(x)}$,

ii. if $m\le\varpi-1$ then $N_{m,z,\varpi}(x)=\frac{p_{z,m}(x)(x-\frac{m+\rho}{z+\theta})}{p_{z,\varpi}(x)}$.

Additionally, for each $m,\varpi\in\{0,1,...,z\}$, $x\in\left[\frac{\varpi+\rho}{z+\theta+1},\frac{\varpi+\rho+1}{z+\theta+1}\right]$, $z\in\mathbb{N}$ and for $\rho,\theta\in\mathbb{R}^+$, we get $0\le\rho\le\theta$. Let us indicate

i. If $\varpi+2\le m$, then $\overline{N}_{m,z,\varpi}(x)=\frac{p_{z,m}(x)(\frac{m+\rho}{z+\theta+1}-x)}{p_{z,\varpi}(x)}$,

ii. if $m\le\varpi-2$ then $\underline{N}_{m,z,\varpi}(x)=\frac{p_{z,m}(x)(x-\frac{m+\rho}{z+\theta+1})}{p_{z,\varpi}(x)}$.

Lemma 2. *Let* $x\in\left[\frac{\varpi+\rho}{z+\theta+1},\frac{\varpi+\rho+1}{z+\theta+1}\right]$ *and for all* $m,\varpi\in\{0,1,...,z\}$,

i. *If* $\varpi+2\le m$, *then* $\overline{N}_{m,z,\varpi}(x)\le N_{m,z,\varpi}(x)\le 3\overline{N}_{m,z,\varpi}(x)$.

ii. *If* $m\le\varpi-2$, *then* $N_{m,z,\varpi}(x)\le\underline{N}_{m,z,\varpi}(x)\le 6N_{m,z,\varpi}(x)$.

Proof. (i) This inequality $\overline{N}_{m,z,\varpi}(x) \leq N_{m,z,\varpi}(x)$ is immediate. Besides,

$$\begin{aligned}
\frac{N_{m,z,\varpi}(x)}{\overline{N}_{m,z,\varpi}(x)} &= \frac{\frac{m+\rho}{z+\theta} - x}{\frac{m+\rho}{z+\theta+1} - x} \leq \frac{\frac{m+\rho}{z+\theta} - \frac{\varpi+\rho}{z+\theta+1}}{\frac{m+\rho}{z+\theta+1} - \frac{\varpi+\rho+1}{z+\theta+1}} \\
&\leq \frac{mz + m\theta + m + \rho - z\varpi - \theta\varpi}{(z+\theta)(m-\varpi-1)} \qquad (3)\\
&\leq \frac{m-\varpi}{m-\varpi-1}.\frac{\rho}{(z+\theta)(m-\varpi-1)} \leq 3
\end{aligned}$$

which proves (i).

(ii) The inequality $N_{m,z,\varpi}(x) \leq \underline{N}_{m,z,\varpi}(x)$ is immediate. Also

$$\begin{aligned}
\frac{\underline{N}_{m,z,\varpi}(x)}{N_{m,z,\varpi}(x)} &= \frac{x - \frac{m+\rho}{z+\theta+1}}{x - \frac{m+\rho}{z+\theta}} \leq \frac{\frac{\varpi+\rho+1}{z+\theta+1} - \frac{m+\rho}{z+\theta+1}}{\frac{\varpi+\rho}{z+\theta+1} - \frac{m+\rho}{z+\theta}} \\
&= \frac{(z+\theta+1)(\varpi+\rho+1-m-\rho)}{(z+\theta)(\varpi+\rho-m-\rho) - m - \rho} \\
&\leq \frac{(z+\theta+1)(\varpi+1-m)}{(z+\theta)(\varpi+\rho-m-\rho-1)} \\
&= \frac{z+\theta+1}{z+\theta}.\frac{\varpi+1-m}{\varpi-m-1} \leq 2\frac{\varpi+1-m}{\varpi-m-1} \\
&= 2\left(1 + \frac{1}{\varpi-m-1}\right) \leq 6
\end{aligned}$$

Therefore, we get the following inequality

$$N_{m,z,\varpi}(x) \leq \underline{N}_{m,z,\varpi}(x) \leq 6N_{m,z,\varpi}(x),$$

and it is the proof of the lemma. □

Lemma 3. *For all* $m, \varpi \in \{0, 1, ..., z\}$ *and* $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$, *we have*

$$n_{m,z,\varpi}(x) \leq 1$$

Proof. We have two states 1)$\varpi \leq m$, 2)$m \leq \varpi$

1)It is obvious that the function $g(x) = \frac{1-x}{x}$ is nonincreasing on $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$ it means that

$$\frac{n_{m,z,\varpi}(x)}{n_{m+1,z,\varpi}(x)} = \frac{m+1}{z-m}.\frac{1-x}{x} \geq \frac{m+1}{z-m}.\frac{1-\frac{\varpi+\rho+1}{z+\theta}}{\frac{\varpi+\rho+1}{z+\theta}} \geq 1,$$

which implies that

$$... \leq n_{m+2,z,\varpi}(x) \leq n_{m+1,z,\varpi}(x) \leq n_{m,z,\varpi}(x).$$

2)For $m \leq \varpi$, we obtain

$$\frac{n_{m,z,\varpi}(x)}{n_{m-1,z,\varpi}(x)} = \frac{z-m+1}{m}.\frac{x}{1-x} \geq \frac{z-m+1}{m}.\frac{\frac{\varpi+\rho}{z+\theta}}{1-\frac{\varpi+\rho}{z+\theta}} \geq 1,$$

which immediately implies

$$n_{0,z,\varpi}(x) \leq ... \leq n_{m-2,z,\varpi}(x) \leq n_{m-1,z,\varpi}(x) \leq n_{m,z,\varpi}(x)$$

Since for $\forall m, \varphi \in \{1, 2, ..., z\}$, $n_{m,z,\varphi}(x) = 1$ the conclusion of the lemma is obvious. □

Lemma 4. *Let* $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$, $z \in \mathbb{N}$ *ve* $\rho, \theta \in \mathbb{R}^+$ $0 \leq \rho \leq \theta$ *olmak üzere,*

i) *If* $m \in \{\varpi+2, \varpi+3, ..., z-1\}$ *is such that* $\varpi \leq (m+\rho) - \sqrt{m+\rho+1}$, *then* $\overline{N}_{m+1,z,\varpi}(x) \leq \overline{N}_{m,z,\varpi}(x)$.

ii) *If* $m \in \{1, 2, ..., \varpi-2\}$ *is such that* $(m+\rho)+\sqrt{m+\rho} \leq \varpi$, *then* $\underline{N}_{m-1,z,\varpi}(x) \leq \underline{N}_{m,z,\varpi}(x)$.

Proof. (i) We observe that

$$\frac{\overline{N}_{m,z,\varpi}(x)}{\overline{N}_{m+1,z,\varpi}(x)} = \frac{m+\rho+1}{z+\theta-m-\rho}.\frac{1-x}{x}.\frac{\frac{m+\rho}{z+\theta+1}-x}{\frac{m+\rho+1}{z+\theta+1}-x}.$$

Since the function $\psi(x) = \frac{1-x}{x}.\frac{\frac{m+\rho}{z+\theta+1}-x}{\frac{m+\rho+1}{z+\theta+1}-x}$ is nonincreasing on $[0, 1]$, it means that

$$\psi(x) \geq \psi(\frac{\varpi+\rho+1}{z+\theta+1}) = \frac{z+\theta-\varpi-\rho}{\varpi+\rho+1}\frac{m+\rho-\varpi-\rho-1}{m+\rho-\varpi-\rho} = \frac{z+\theta-\varpi-\rho}{\varpi+\rho+1}\frac{m-\varpi-1}{m-\varpi},$$

for all $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1},\right]$. Then since the contidion $\varpi \leq (m+\rho) - \sqrt{m+\rho+1}$ implies

$$\begin{aligned}(m+\rho+1)(m+\rho-\varpi-\rho-1) \geq& (\varpi+\rho+1)(m+\rho-\varpi-\rho),\\ (m+\rho+1)(m-\varpi-1) \geq& (\varpi+\rho+1)(m-\varpi),\end{aligned}$$

we obtain

$$\frac{\overline{N}_{m,z,\varpi}(x)}{\overline{N}_{m+1,z,\varpi}(x)} \geq \frac{m+\rho+1}{z+\theta-m-\rho}.\frac{z+\theta-\varpi-\rho}{\varpi+\rho+1}.\frac{m-\varpi-1}{m-\varpi} \geq 1.$$

(ii) We observe that

$$\frac{\underline{N}_{m,z,\varpi}(x)}{\underline{N}_{m+1,z,\varpi}(x)} = \frac{z+\theta-m-\rho+1}{m+\rho}.\frac{x}{1-x}.\frac{x-\frac{m+\rho}{z+\theta+1}}{x-\frac{m+\rho-1}{z+\theta+1}}.$$

Since the function $g(x) = \frac{1-x}{x}\frac{x-\frac{m+\rho}{z+\theta+1}}{x-\frac{m+\rho-1}{z+\theta+1}}$ is nondecreasing on $[0, 1]$ it means that $g(x) \geq g(\frac{\varpi+\rho}{z+\theta+1}) = \frac{\varpi+\rho}{z+\theta+1-\varpi-\rho}\frac{\varpi-m}{\varpi-m+1}$, for all $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$. Then,since the condition $(m+\rho)+\sqrt{m+\rho} \leq \varpi+\rho$ implies $(\varpi+\rho)(\varpi-m) \geq (m-\rho)(\varpi-m-1)$, we obtain

$$\frac{\underline{N}_{m,z,\varpi}(x)}{\underline{N}_{m+1,z,\varpi}(x)} \geq \frac{z+\theta-m-\rho+1}{m+\rho}.\frac{\varpi+\rho}{z+\theta+1-\varpi-\rho}.\frac{\varpi-m}{\varpi-m+1} \geq 1,$$

which proves the lemma. □

Lemma 5. *We have*

$$\bigvee_{m=0}^{z} p_{z,m}(x) = p_{z,\varpi}(x),\ \forall x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right],\ \varpi \in \{0,1,...,z\}$$

which $p_{z,m}(x) = C_z^m x^m (1-x)^{z-m}$.

Proof. Firstly, we demonstrate that for fixed $z \in \mathbb{N}$ and $0 \leq m+\rho \leq m+\rho+1 \leq z$, we have

$$0 \leq p_{z,m+1}(x) \leq p_{z,m}(x) \text{ if and only if } x \in \left[0, \frac{\varpi+\rho+1}{z+\theta+1}\right].$$

Actually, the inequality one reduces to

$$0 \leq \binom{z}{m+1} x^{m+1}(1-x)^{z-m-1} \leq \binom{z}{m} x^m (1-x)^z - m,$$

which, after being simplified, is equivalent to

$$0 \leq x\left[\binom{z}{m+1} + \binom{z}{m}\right] \leq \binom{z}{m}.$$

Regarding the equality $\binom{z}{m+1} + \binom{z}{m} = \binom{z+1}{m+1}$, The inequality mentioned above straight away equals $0 \leq x \leq \frac{m+1}{z+1}$. Using $m = 0, ..., z$ in the inequality that was just demonstrated above, we obtain

$$p_{z,\rho+1}(x) \leq p_{z,\rho}(x),\ \text{if and only if } x \in \left[0, \frac{\rho+1}{z+\theta+1}\right],$$

$$p_{z,\rho+2}(x) \leq p_{z,\rho+1}(x),\ \text{if and only if } x \in \left[0, \frac{\rho+2}{z+\theta+1}\right],$$

$$p_{z,\rho+3}(x) \leq p_{z,\rho+2}(x),\ \text{if and only if } x \in \left[0, \frac{\rho+3}{z+\theta+1}\right],$$

so on,

$$p_{z,m+1}(x) \leq p_{z,m}(x),\ \text{if and only if } x \in \left[0, \frac{m+\rho+1}{z+\theta+1}\right],$$

so on,

$$p_{z,z-2}(x) \le p_{z,z-3}(x), \text{ if and only if } x \in \left[0, \frac{z+\theta-2}{z+\theta+1}\right],$$

$$p_{z,z-1}(x) \le p_{z,z-2}(x), \text{ if and only if } x \in \left[0, \frac{z+\theta-1}{z+\theta+1}\right],$$

$$p_{z,z}(x) \le p_{z,z-1}(x), \text{ if and only if } x \in \left[0, \frac{z+\theta}{z+\theta+1}\right].$$

From all these inequalities, we easily get:

If $x \in \left[0, \frac{\rho+1}{z+\theta+1}\right]$, then $p_{z,m+\rho}(x) \le p_{z,\rho}(x)$, for all $m = 0, ..., z$,

If $x \in \left[\frac{\rho+1}{z+\theta+1}, \frac{\rho+2}{z+\theta+1}\right]$, then $p_{z,m+\rho}(x) \le p_{z,\rho+1}(x)$, for all $m = 0, ..., z$,

If $x \in \left[\frac{\rho+2}{z+\theta+1}, \frac{\rho+3}{z+\theta+1}\right]$, then $p_{z,m+\rho}(x) \le p_{z,\rho+2}(x)$, for all $m = 0, ..., z$,

and so on finally

$$\text{if } x \in \left[\frac{\rho+z}{z+\theta+1}, 1\right] \text{ then } p_{z,m}(x) \le p_{z,z}(x), \text{ for all } m = 0, ..., z,$$

that proves the lemma. □

Theorem 1. *Let* $\kappa : [0,1] \to [0,1]$, κ, *be a continuous function on* $[0,1]$. *Then, we obtain*

$$|P_z^{(M)}(\kappa)(x) - \kappa(x)| \le 12\omega_1\left(\kappa, \frac{\sqrt{x+\rho}}{\sqrt{z+\theta+1}}\right), x \in [0,1], \forall n \in \mathbb{N}.$$

Here, $\omega_1(\kappa, \delta) = sup(|\kappa(x) - \kappa(t)| : x, t \in [0,1], |x-t| \le \delta)$.

Proof. Checking that the max-product Bernstein-Stancu operators satisfy the requirements in Lemma 1 is easy.

$$|P_z^{(M)}(\kappa)(x) - \kappa(x)| \le \left(1 + \frac{1}{\delta_z} P_z^{(M)}(\varrho_x)(x)\right)\omega_1(\kappa; \delta_z), \tag{4}$$

which $\varrho_x(t) = |t - x|$. So, it is enough to estimate

$$E_z(x) = P_z^{(M)}(\varrho)(x) = \frac{\bigvee_{m=0}^{z} p_{z,m}(x)\kappa\left|\frac{m+\rho}{z+\theta+1} - x\right|}{\bigvee_{m=0}^{z} p_{z,m}(x)}.$$

Let $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$ where $\varpi \in \{0,1,...,z\}$ is fixed, arbitrary. By Lemma 5 we easily obtain

$$E_z(x) = \max_{m=0,1,...,z}\{N_{m,z,\varpi}(x)\}, x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right].$$

Now, we can assume $\varpi = 0,1,...,z$, since simple calculation for $\varpi = 0$ shows that in this case we get $E_z(x) \leq \frac{1}{z+\theta+1}$, for all $x \in \left[0, \frac{1}{z+\theta+1}\right]$. Consequently, an upper estimate for each $N_{m,z,\varpi}(x)$ must still be obtained, when $m = 0,1,...$ and $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$. Actually, we will demonstrate that

$$N_{m,z,\varpi}(x) \leq 6\frac{\sqrt{x+\rho}}{z+\theta+1}, \tag{5}$$

which immediately implies that

$$E_z(x) \leq 6\frac{\sqrt{x+\rho}}{z+\theta+1},\ x \in \left[0, \frac{1}{z+\theta+1}\right],$$

and taking $\delta_z = 6\frac{\sqrt{x+\rho}}{z+\theta+1}$ in (4) we immediately get the estimate in the statement. To demonstrate (5), we consider the subsequent circumstances: i)$m \in \{\varpi-1, \varpi, \varpi+1\}$; ii)$m \geq \varpi+2$ and iii)$m \leq \varpi-2$

Case i). If $m = \varpi$, then $N_{\varpi,z,\varpi}(x) = \left|\frac{\varpi+\rho}{z+\theta} - x\right|$. Since $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$ it means that $N_{\varpi,z,\varpi}(x) \leq \frac{1}{z+\theta+1}$.

If $m = \varpi+1$, then $N_{\varpi+1,z,\varpi}(x) = n_{\varpi+1,z,\varpi}(x)\left(\frac{\varpi+\rho+1}{z+\theta} - x\right)$. Since by Lemma 3 we have $n_{\varpi+1,z,\varpi}(x) \leq 1$, we obtain $N_{\varpi+1,z,\varpi}(x) \leq \frac{\varpi+\rho+1}{z+\theta} - x \leq \frac{\varpi+\rho+1}{z+\theta} - \frac{\varpi+\rho}{z+\theta+1} \leq \frac{3}{z+\theta+1}$.

If $m = \varpi-1$, then $N_{\varpi-1,z,\varpi}(x) = n_{\varpi-1,z,\varpi}(x)\left(x - \frac{\varpi+\rho-1}{z+\theta}\right) \leq \frac{\varpi+\rho+1}{z+\theta+1} - \frac{\varpi+\rho-1}{z+\theta} \leq \frac{2}{z+\theta+1}$

Case ii). Subcase a) Suppose first that $m+\sqrt{m+1} \leq \varpi$, we get

$$\begin{aligned}\overline{N}_{m,z,\varpi}(x) &= n_{m,z,\varpi}(x)\left(\frac{m+\rho}{z+\theta+1} - x\right) \leq \frac{m+\rho}{z+\theta+1} - x\\ &\leq \frac{m+\rho}{z+\theta+1} - \frac{\varpi+\rho}{z+\theta+1} \leq \frac{m+\rho}{z+\theta+1} - \frac{m+\sqrt{m+1}}{z+\theta+1}\\ &= \frac{\sqrt{m+1}}{z+\theta+1} \leq \frac{1}{\sqrt{z+\theta+1}}.\end{aligned}$$

Subcase b) Assume that $m-\sqrt{m+1} \geq \varpi$. Since the function $\vartheta(x) = x-\sqrt{x+1}$ is nondecreasing on $x \in [0,1]$ it follows that there exists $\overline{m} \in \{0,...,z\}$,of maximum value, such that $\overline{m}-\sqrt{\overline{m}+1} < \varpi$. Then for $m_1 = \overline{m}+1$ we obtain $m_1-\sqrt{m_1+1} \geq$

ϖ and

$$\begin{aligned}\overline{N}_{\overline{m},z,\varpi}(x) &= n_{\overline{m},z,\varpi}(x)\left(\frac{\overline{m}+\rho+1}{z+\theta+1}-x\right) \leq \frac{\overline{m}+\rho+1}{z+\theta+1}-x \\ &\leq \frac{\overline{m}+\rho+1}{z+\theta+1}-\frac{\varpi+\rho}{z+\theta+1} \leq \frac{\overline{m}+\rho+1}{z+\theta+1}-\frac{\overline{m}-\sqrt{\overline{m}+1}}{z+\theta+1} \\ &= \frac{\sqrt{\overline{m+1}}+1}{z+\theta+1} \leq \frac{2}{\sqrt{z+\theta+1}}.\end{aligned}$$

Also, we have $m_1 \geq \varpi + 2$. Indeed, this is a consequence of the fact that ϑ is nondecreasing on the interval $x \in [0,1]$ and because it is easy to see that $\vartheta(\varpi+1) \leq \varpi$. By Lemma 4 it follows that $\overline{N}_{\overline{m}+1,z,\varpi}(x) \geq \overline{N}_{\overline{m}+2,z,\varpi}(x) \geq ... \geq \overline{N}_{z,z,\varpi}$. We thus obtain $\overline{N}_{\overline{m},z,\varpi}(x) = \frac{2}{\sqrt{z+\theta+1}}$ for any $m \in \{\overline{m}+1, \overline{m}+2, ..., z\}$. Therefore, in both subcases, we get $N_{m,z,\varpi}(x) \leq \frac{6}{\sqrt{z+\theta+1}}$.

Case iii). Subcase a) Assume first that $m + \sqrt{m} \geq \varpi$. Then we get

$$\begin{aligned}\underline{N}_{m,z,\varpi}(x) &= n_{m,z,\varpi}(x)\left(x - \frac{m+\rho}{z+\theta+1}\right) \\ &\leq x - \frac{m+\rho}{z+\theta+1} \leq \frac{\varpi+\rho+1}{z+\theta+1} - \frac{m+\rho}{z+\theta+1} \\ &\leq \frac{m+\sqrt{m}+\rho+1}{z+\theta+1} - \frac{m+\rho}{z+\theta+1} \\ &= \frac{\sqrt{m}+\rho+1}{z+\theta+1} \leq \frac{2}{\sqrt{z+\theta+1}}.\end{aligned}$$

Subcase b) Assume now that $m + \sqrt{m} \leq \varpi$. Let $\underline{m} \in \{0, ..., z\}$ be the minimum value such that $\underline{m} - \sqrt{\underline{m}} < \varpi$. Then $m_2 = \underline{m} - 1$ satisfies $m_2 - \sqrt{m_2} \geq \varpi$ and

$$\begin{aligned}\underline{N}_{\underline{m},z,\varpi}(x) &= n_{\underline{m},z,\varpi}(x)\left(x - \frac{\underline{m}+\rho-1}{z+\theta+1}\right) \\ &\leq x - \frac{\underline{m}+\rho+1}{z+\theta+1} \leq \frac{\varpi+\rho+1}{z+\theta+1} - \frac{\underline{m}+\rho-1}{z+\theta+1} \\ &\leq \frac{\underline{m}+\sqrt{\underline{m}+1}}{z+\theta+1} - \frac{\underline{m}+\rho-1}{z+\theta+1} = \frac{\sqrt{\underline{m}}+2+\rho}{z+\theta+1} \leq \frac{3}{\sqrt{z+\theta+1}}.\end{aligned}$$

Additionally, since in this case we have $\varpi \geq 2$ it is immediate that $m_2 \geq \varpi - 2$. By Lemma 4, it follows that $\underline{N}_{\underline{m}-1,z,\varpi}(x) \geq \underline{N}_{\underline{m}-2,z,\varpi}(x) \geq ... \geq \underline{N}_{0,z,\varpi}$. We obtain $\underline{N}_{m,z,\varpi}(x) \leq \frac{3}{\sqrt{z+\theta+1}}$ for $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$ and for any $m \leq \varpi - 2$. In both subcases, we get $N_{m,z,\varpi}(x) \leq \frac{3}{\sqrt{z+\theta+1}}$.

As a result, by collecting all the predictions in the above cases and subcases, we easily obtain the relation (5) that completes the proof.

□

3. Nonlinear Max-Product Type Bivariate Bernstein-Stancu Operators

We introduce nonlinear bivariate Bernstein-Stancu operators of max-product type in this section.

Let us $\kappa : [0,1] \times [0,1] \to \mathbb{R}^+$ be a continuous function and for $k = 1, 2$, $\rho_k, \theta_k \in \mathbb{R}_+$, with $0 \le \rho_k \le \theta_k$. Then the nonlinear maximum product type of bivariate Bernstein-Stancu operators is defined as follows:

$$P^{(M)}_{z,h,\varpi_k,\iota_k}(\kappa : x, y) = \frac{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y) \kappa\left(\frac{m+\rho_1}{z+\theta_1}, \frac{j+\rho_2}{h+\theta_2}\right)}{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)}, \tag{6}$$

with

$$p_{z,m}(x) = C_z^m x^m (1-x)^{z-m} \text{ and } p_{h,j}(y) = C_h^j y^j (1-y)^{h-j}, \tag{7}$$

for all $x, y \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, $\varpi, \iota \in \mathbb{N}$.

The subsequent sections provide an error estimate the nonlinear maximum product type of bivariate Bernstein-Stancu operators in terms of modulus of continuity, along with some features of the $P^{(M)}_{z,h,\varpi_k,\iota_k}$ operators.

We require the following notations and auxiliary results for the main result proofs. Now, some definitions for the x and y variables and lemmas will be given.

For each $m = \{0, 1, ..., z\}$, $j = \{0, 1, ..., h\}$ and $x, y \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, $\varpi, \iota \in \mathbb{N}$, let us denote

$$N_{m,z,\varpi}(x) = \frac{p_{z,m}(x)\left|\frac{m+\rho_1}{z+\theta_1} - x\right|}{p_{z,\varpi}(x)}, \qquad n_{m,z,\varpi}(x) = \frac{p_{z,m}(x)}{p_{z,\varpi}(x)}$$

$$N_{j,h,\iota}(y) = \frac{p_{h,j}(y)\left|\frac{j+\rho_2}{h+\theta_2} - y\right|}{p_{h,\iota}(y)}, \qquad n_{j,h,\iota}(y) = \frac{p_{h,j}(y)}{p_{h,\iota}(y)}.$$

Let $x, y \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, $\varpi, \iota \in \mathbb{N}$, $m = \{0, 1, ..., z\}$, $j = \{0, 1, ..., h\}$, and for $k = 1, 2$, $\rho_k, \theta_k \in \mathbb{R}_+$, with $0 \le \rho_k \le \theta_k$. Hence, it is obvious that

i) If $\varpi + 1 \le m$, then $N_{m,z,\varpi}(x) = \frac{p_{z,m}(x)(\frac{m+\rho_1}{z+\theta_1} - x)}{p_{z,\varpi}(x)}$,

ii) If $m \le \varpi - 1$, then $N_{m,z,\varpi}(x) = \frac{p_{z,m}(x)(x - \frac{m+\rho_1}{z+\theta_1})}{p_{z,\varpi}(x)}$,

iii) If $\iota + 1 \le j$, then $N_{j,h,\iota}(y) = \frac{p_{h,j}(y)(\frac{j+\rho_2}{h+\theta_2} - y)}{p_{h,\iota}(y)}$,

iv) If $j \le \iota - 1$, then $N_{j,h,\iota}(y) = \frac{p_{h,j}(y)(y - \frac{j+\rho_2}{h+\theta_2})}{p_{h,\iota}(y)}$.

Additionally, let $x, y \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, $\varpi, \iota \in \mathbb{N}$, $m = \{0, 1, ..., z\}$, $j = \{0, 1, ..., h\}$, and for $k = 1, 2$, $\rho_k, \theta_k \in \mathbb{R}_+$, with $0 \le \rho_k \le \theta_k$. Then, we can denote

$$\underline{N}_{m,z,\varpi}(x) = \frac{p_{z,m}(x)(x - \frac{m+\rho_1}{z+\theta_1+1})}{p_{z,\varpi}(x)}, \text{ for } m \le \varpi - 2$$
$$\underline{N}_{j,h,\iota}(y) = \frac{p_{h,j}(y)(y - \frac{j+\rho_2}{h+\theta_2+1})}{p_{h,\iota}(y)}, \text{ for } j \le \iota - 2$$

Lemma 6. *Let $x, y \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, $\varpi, \iota \in \mathbb{N}$, and for $k = 1, 2$, $\rho_k, \theta_k \in \mathbb{R}_+$, with $0 \le \rho_k \le \theta_k$.*

i) *For all $m, \varpi = \{0, 1, ..., z\}$ and $\varpi+2 \le m$, we have $\overline{N}_{m,z,\varpi}(x) \le N_{m,z,\varpi}(x) \le 3\overline{N}_{m,z,\varpi}(x)$.*
ii) *For all $j, \iota = \{0, 1, ..., h\}$ and $\iota + 2 \le j$, we have $\overline{N}_{j,h,\iota}(y) \le N_{j,h,\iota}(y) \le 3\overline{N}_{j,h,\iota}(y)$.*
iii) *For all $m, \varpi = \{0, 1, ..., z\}$ and $m \le \varpi-2$, we have $N_{m,z,\varpi}(x) \le \underline{N}_{m,z,\varpi}(x) \le 6N_{m,z,\varpi}(x)$*
iv) *For all $j, \iota = \{0, 1, ..., h\}$ and $\iota + 2 \le j$, we have $j \le \iota - 2$, $N_{j,h,\iota}(y) \le \underline{N}_{j,h,\iota}(y) \le 6N_{j,h,\iota}(y)$*

The proof is in a similar way to the univariate given in Lemma 2.

Lemma 7. *For all $m, \varpi = \{0, 1, ..., z\}$, $j, \iota = \{0, 1, ..., h\}$ and $x, y \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, we have*

$$n_{m,z,\varpi}(x) \le 1 \text{ and } n_{j,h,\iota}(y) \le 1. \tag{8}$$

The proof is in a similar way to the univariate given in Lemma 3.

Lemma 8. *Let $p_{z,m}(x)$ and $p_{h,j}(y)$ defined as given in (7). Then we have*

$$\bigvee_{m=0}^{z} p_{z,m}(x). \bigvee_{j=0}^{h} p_{h,j}(y) = p_{z,\varpi}(x).p_{h,\iota}(y),$$

for all $x, y \in [0, 1]^2 ve$ $(x, y) \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, $\varpi = 0, ...z$, $\iota = 0, ...h$.

Proof. Since we have $\bigvee_{m=0}^{z} p_{z,m}(x) > 0$, $\bigvee_{j=0}^{h} p_{h,j}(y) > 0$ for all $x, y \in [0, 1]$ Firstly, we claim that for $\varpi = 0, ...z, \iota = 0, ...h$, we have

$$0 \le p_{z,m+1}(x) \le p_{z,m}(x),$$
$$0 \le p_{h,j+1}(y) \le p_{h,j}(y),$$

if and only if $x \in \left[0, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right]$ and $y \in \left[0, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$. Writing the following inequality is simple:

$$0 \leq \binom{z}{\varpi+1} x^{\varpi+1}(1-x)^{z-\varpi-1} \leq \binom{z}{\varpi} x^{\varpi}(1-x)^{z-\varpi},$$
$$0 \leq \binom{h}{\iota+1} y^{\iota+1}(1-y)^{h-\iota-1} \leq \binom{h}{\iota} y^{\iota}(1-y)^{h-\iota}$$

and after simplifcation,

$$0 \leq x\left[\binom{z}{\varpi+1} + \binom{z}{\varpi}\right] \leq \binom{z}{\varpi},$$
$$0 \leq y\left[\binom{z}{\iota+1} + \binom{h}{\iota}\right] \leq \binom{h}{\iota}.$$

From the equality $\binom{z}{\varpi+1} + \binom{z}{\varpi} = \binom{z+1}{\varpi+1}$, we get $0 \leq x \leq \frac{\varpi+\rho_1+1}{z+\theta 1+1}$, $0 \leq y \leq \frac{\iota+\rho_2+1}{h+\theta_2+1}$. Also, denoting

$$A_{m,z,\varpi}(x) = \frac{p_{z,m}(x)}{p_{z,\varpi}(x)} = \frac{\binom{z}{m}}{\binom{z}{\varpi}} \left(\frac{x}{1-x}\right)^{m-\varpi}, \tag{9}$$

$$A_{j,h,\iota}(y) = \frac{p_{h,m}(y)}{p_{h,\iota}(y)} = \frac{\binom{h}{j}}{\binom{h}{\iota}} \left(\frac{y}{1-y}\right)^{j-\iota}, \tag{10}$$

so, we can write $A_{m,z,\varpi,j,h,\iota}(x,y) = A_{m,z,\varpi}(x).A_{j,h,\iota}(y)$. Therefore, we can use the following formula to prove the approximation results

$$P^{(M)}_{z,h,\varpi_k,\iota_k}(\kappa : x, y) = \bigvee_{m=0}^{z} \bigvee_{j=0}^{h} p_{h,j}(y) A_{m,z,\varpi,j,h,\iota}(x,y) \kappa\left(\frac{m+\rho_1}{z+\theta_1}, \frac{j+\rho_2}{h+\theta_2}\right), \tag{11}$$

for all $(x,y) \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, $\varpi = 0, ...z$, $\iota = 0, ...h$.

It easily follows that we can write

$$P^{(M)}_{z,h,\varpi_k,\iota_k}(\kappa : x, y) = P^{(M)}_{z,\varpi_k,x}\left[P^{(M)(\kappa)}_{h,\iota_k,y}\right](x,y)$$

where, if $F = F(x,y)$ then the notations $P^{(M)}_{z,x}(F)$ means that the univariate max-product Bernstein operator $P^{(M)}_{z}(F)$ is applied to F considered as function of x while $P^{(M)}_{z,y}(F)$ means that the univariate max-product Bernstein operator $P^{(M)}_{z}(F)$ is applied to F considered as function of y. In other words, the bivariate max-product Bernstein operators are tensor products of the univariate max- product Bernstein operators.

□

Definition 1. *Suppose that $\kappa : I : [0,1] \times [0,1] \to \mathbb{R}^+$ olmak üzere,*

(i) *Let for all $y, x, x+\phi \in [0,1], \phi > 0$, $\kappa(x+\phi, y) - \kappa(x,y) \geq 0 (resp., \leq 0)$. Then, the function κ is increasing with respect to x on I (resp., decreasing).*

(ii) *Let for all* $x, y, y+\Phi \in [0,1], \Phi > 0,\ \kappa(x, y+\Phi) - \kappa(x,y) \geq 0 (resp., \leq 0)$. *Ten, the function* κ *is increasing with respect to* y *on* I *(resp., decreasing).*

(iii) *Let for all* $x, x+\phi, y, y+\Phi \in [0,1], \phi, \Phi > 0\ \Delta_2\kappa(x,y) = \kappa(x+phi, y+\Phi) - \kappa(x, y+\Phi) - \kappa(x+\phi, y) + \kappa(x,y) \geq 0 (resp., \leq 0)$ *Ten, the function* κ *is upper bidimensional monotone on* I *(resp., lower bidimensional monotone). (Bede, Coroianu ve Gal, 2016).*

Theorem 2. *Let* $\kappa : [0,1] \times [0,1] \to \mathbb{R}_+$ *be a continuous function. We have*

$$|P_{z,h}^{(M)}(\kappa)(x,y) - \kappa(x,y)| \leq 18\omega_1 \left(\kappa; \frac{\sqrt{x+\rho_1}}{\sqrt{z+\theta_1+1}}, \frac{\sqrt{y+\rho_2}}{\sqrt{h+\theta_2+1}}\right),$$

for all $x, y \in [0,1]$ *and* $z, h \in \mathbb{N}$*. Here*

$\omega_1(\kappa;\gamma,\delta) = \sup\left\{|\ \kappa)(x,y) - \kappa)(z,t)\ |; x,y,z,t[0,1]\times[0,1], |\ x-z\ |\leq\gamma, |\ y-t\ |\leq\delta\right\}.$

Proof. Taking into account the inequality valid for the positive numbers A_k, B_k, $k \in \{0,1,...,s\}$,

$$|max_{k\in\{0,1,\ldots,s\}}\{A_k\} - max_{k\in\{0,1,\ldots,s\}}\{B_k\}| \leq max_{k\in\{0,1,\ldots,s\}}\{|A_k - B_k|\}$$

we obtain

$$\begin{aligned}
&|P_{z,h}^{(M)}(\kappa)(x,y) - \kappa(x,y)| \\
&= \left| \frac{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)\kappa\left(\frac{m+\rho_1}{z+\theta_1}, \frac{j+\rho_2}{h+\theta_2}\right)}{\bigvee_{m=0}^{z} p_{z,m}(x)\bigvee_{j=0}^{h} p_{h,j}(y)} - \frac{\bigvee_{m=0}^{z} p_{z,m}(x)\bigvee_{j=0}^{h} p_{h,j}(y)\kappa(x,y)}{\bigvee_{m=0}^{z} p_{z,m}(x)\bigvee_{j=0}^{h} p_{h,j}(y)} \right| \\
&\leq \frac{\bigvee_{m=0}^{z} p_{z,m}(x)\bigvee_{j=0}^{h} p_{h,j}(y)\left|\kappa\left(\frac{m+\rho_1}{z+\theta_1}, \frac{j+\rho_2}{h+\theta_2}\right) - \kappa(x,y)\right|}{\bigvee_{m=0}^{z} p_{z,m}(x)\bigvee_{j=0}^{h} p_{h,j}(y)} \\
&\leq \frac{\bigvee_{m=0}^{z} p_{z,m}(x)\bigvee_{j=0}^{h} p_{h,j}(y)\omega_1\left(\kappa; |m+\rho_1/z+\theta_1-x|, |j+\rho_2/h+\theta_2-y)\,|\right)}{\bigvee_{m=0}^{z} p_{z,m}(x)\bigvee_{j=0}^{h} p_{h,j}(y)} \\
&= \frac{\bigvee_{m=0}^{z} p_{z,m}(x)\bigvee_{j=0}^{h} p_{h,j}(y)\omega_1\left(\kappa; \delta\frac{|m+\rho_1/z+\theta_1-x|}{\delta}, \nu\frac{|j+\rho_2/h+\theta_2-y|)}{\nu}\right)}{\bigvee_{m=0}^{z} p_{z,m}(x)\bigvee_{j=0}^{h} p_{h,j}(y)}
\end{aligned}$$

$$= \frac{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)\omega_1 \left(1 + \frac{|m+\rho_1/z+\theta_1-x|}{\delta} + \frac{|j+\rho_2/h+\theta_2-y|)}{\nu}\right)\omega_1(\kappa,\delta,\nu)}{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)}$$

$$= \omega_1(\kappa,\delta,\nu)\left(1 + \frac{1}{\delta}\frac{\bigvee_{m=0}^{z} p_{z,m}(x)\left(\frac{m+\rho_1}{z+\theta_1-x}\right)}{\bigvee_{m=0}^{z} p_{z,m}(x)} + + \frac{1}{\nu}\frac{\bigvee_{j=0}^{h} p_{h,j}(y)\left(\frac{m+\rho_2}{h+\theta_2-y}\right)}{\bigvee_{j=0}^{h} p_{h,j}(y)}\right)$$

Burada, $\delta = \frac{6\sqrt{x+\rho_1}}{\sqrt{z+\theta_1+1}}$ ve $\nu = \frac{6\sqrt{y+\rho_2}}{\sqrt{h+\theta_2+1}}$

$$|P_{z,h}^{(M)}(\kappa)(x,y) - \kappa(x,y)| \leq 3\omega_1\left(\kappa; \frac{6\sqrt{x+\rho_1}}{\sqrt{z+\theta_1+1}}, \frac{6\sqrt{y+\rho_2}}{\sqrt{h+\theta_2+1}}\right)$$
$$\leq 18\omega_1\left(\kappa; \frac{\sqrt{x+\rho_1}}{\sqrt{z+\theta_1+1}}, \frac{\sqrt{y+\rho_2}}{\sqrt{h+\theta_2+1}}\right)$$

□

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SOME VARIABLE EXPONENT BOUNDEDNESS AND COMMUTATORS ESTIMATES FOR FRACTIONAL ROUGH HARDY OPERATORS ON CENTRAL MORREY SPACE

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Abstract. In this article, we study the boundedness of the fractional Rough Hardy operator and its adjoint operators on the central Morrey space with a variable exponent. We also establish the same boundedness for their commutators when the symbol functions are on the λ-central BMO space with a variable exponent.

1. Introduction

The Hardy operator is a key operator in mathematical analysis and has been extensively used in recent times. In 1920, Hardy [1] defined an operator for a locally integrable $f \in \mathbb{R}^n$.

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x > 0. \tag{1}$$

He also established a sharp inequality for it. Later, Faris (as seen in [2]) formulated the n-dimensional form of (1). Grafakos and Christ [3] determined the exact norm value on the Lebesgue space for the n-dimensional Hardy operator. Additionally, the Hardy integral inequality has garnered significant attention. Alternate proofs, variants, applications, and generalizations of this inequality were explored in various articles. Some of these inequalities are discussed in [3,4]. Furthermore, in [5], the

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authors introduced the n-dimensional fractional Hardy operator as follows:

$$H_\beta f(z) = \frac{1}{|z|^{n-\beta}} \int_{|t|\le|z|} f(t)dt, \quad H_\beta^* f(z) = \int_{|t|>|z|} \frac{f(t)}{|t|^{n-\beta}}dt, \quad z \in \mathbb{R}^n \setminus \{0\}, \tag{2}$$

where $|z| = \sqrt{\sum_{i=1}^n z_i^2}$. Moreover, commutators of these operators are defined as follows:

$$[b,H]f = bHf - H(bf), \qquad [b,H^*]f = bH^*f - H^*(bf), \tag{3}$$

where b is a locally integrable function. Fu, Liu, and Wang [5] established boundedness for the commutator of the n-dimensional fractional Hardy operator. Firstly, Ren and Tao [6] provided the definition of the n-dimensional rough Hardy operator and its adjoint operator as follows:

$$H_{\Omega,\beta} f(z) = \frac{1}{|z|^{n-\beta}} \int_{|t|\le|z|} \Omega(z-t) f(t)dt,$$
$$H_{\Omega,\beta}^* f(z) = \int_{|t|>|z|} \Omega(z-t) \frac{f(t)}{|t|^{n-\beta}}dt,$$

where $\Omega \in L^s(S^{n-1})$, $1 < s \le \infty$, and is homogeneous of degree zero. Commutators of rough hardy operators are defined as:

$$H_{\Omega,\beta}^b f(z) = \frac{1}{|z|^{n-\beta}} \int_{|t|\le|z|} (b(z)-b(t))\Omega(z-t) f(t)dt,$$
$$H_{\Omega,\beta}^{*,b} f(z) = \int_{|t|>|z|} (b(z)-b(t))\Omega(z-t) \frac{f(t)}{|t|^{n-\beta}}dt,$$

which were used by Wei, Zhen, and Wang [7] to develop estimates for the commutator on the Herz space.

It's important to highlight that the function space featuring varying exponents plays a pivotal role in both harmonic analysis and applied mathematics. Orlicz [8] initiated the theory of variable exponent Lebesgue space for the first time. Musieliak Orlicz spaces are defined in [9]. Sobolev and Lebesgue spaces with integrability exponents have been thoroughly examined, as seen in [10–12] and the references therein. Following that, work on variable Lebesgue spaces began, along with the exploration of the boundedness of numerous operators, including the maximum operator on Lebesgue spaces $L^{p(\cdot)}$ [13]. At the same time, the central bounded mean oscillation space, λ-central Morrey space, and similar function spaces offer compelling practical uses through the exploration of operator estimates in tandem with singular integral operators, as detailed in [14, 26]. The analysis of Morrey space can be traced back to Morrey's [15] work on the regularity of solutions of partial differential equations. In [14], the authors defined λ-central Morrey space and central bounded mean oscillation (BMO) space, which are generalized based on bounded central mean oscillation. λ-central Morrey space and central BMO space have impressive applications in analyzing the boundedness of many operators; see, for example, [16,17]. Furthermore, with substantial applications in image processing [18], electrorheological fluid [19], and partial differential equations [20], variable exponent functions have garnered significant attention. Following Kováčik's [21] seminal work, such theories have made significant advances. For the first time, the idea of non-homogeneous variable exponent central Morrey spaces was formulated by Mizuta [22]. In the recent past, Wang et al. defined variable exponent central BMO and established the boundedness of some operators in [23], which was later extended by Zunwei Fu [24] to variable exponent λ-central Morrey space and central BMO space.

In [25,27], the authors obtained results for the boundedness of several integral operators on function spaces with variable exponents. Additionally, some authors proved results for the boundedness of multilinear integral operators and their commutators as well, as seen in [28,29].

Motivated by [24,30,31], we are going to examine the boundedness of fractional Rough Hardy operators, as well as the boundedness of commutators, on the variable exponent λ-central Morrey space.

Let's elucidate the structure of this paper. In Section 2, we will revisit certain definitions, lemmas, and propositions in the context of variable exponent Lebesgue space. In the third section of this article, we consider the boundedness of the fractional Rough Hardy operator and its adjoint operator on the central Morrey space with a variable exponent, respectively. In Section 4, we consider the boundedness of commutators of the fractional Rough Hardy operator and its adjoint operator on the λ-central BMO space with a variable exponent, respectively..

Additionally, $|B|$ and χ_B represent the Lebesgue measure and the characteristic function of a measurable set $B \subset \mathbb{R}^n$, respectively. When we write $g \approx h$, we are indicating the existence of constants $c_1, c_2 > 0$ such that $c_1 g \leq h \leq c_2 g$. Here, $B_k = B(0, 2^k) = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and the characteristics function $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$ (see [35]).

2. Function spaces along variable exponent

First of all, we will provide some basic definitions and notations concerning Lebesgue spaces over variable exponents. Let's consider an open set $E \subseteq \mathbb{R}^n$, and let $q(\cdot) : E \to [1, \infty)$ be a measurable function. We denote the conjugate exponent as $q'(\cdot)$, which is defined as

$$\frac{1}{q'(\cdot)} + \frac{1}{q(\cdot)} = 1$$

The set $P(E)$ consists of all pairs of exponents $(q(\cdot), q'(\cdot))$ that satisfy

$$1 < q^- = ess \inf\{q(x) : x \in E\}$$

$$q^+ = ess \sup\{q(x) : x \in E\} < \infty.$$

We use $L^{q(\cdot)}$ to represent the space of all measurable functions f such that for some $\zeta > 0$,

$$\int_E \left(\frac{|f(x)|}{\zeta}\right)^{q(x)} dx < \infty,$$

This space is a Banach function space equipped with the Luxemburg norm:

$$||f||_{L^{q(\cdot)}(E)} = \inf\left\{\zeta > 0 : \int_E \left(\frac{|f(x)|}{\zeta}\right)^{q(x)} dx \leq 1\right\}.$$

We define $L^{q(\cdot)}_{\text{loc}}(\delta)$ as the set of functions f belonging to $L^{q(\cdot)}(E)$ for any compact subset $E \subset \delta$:

$$L^{q(\cdot)}_{loc}(\delta) = \left\{f : f \in L^{q(\cdot)}(E) \ for\ every\ compact\ subset\ E \subset \delta\right\}.$$

Here, M denotes the Hardy-Littlewood maximal operator acting on a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and is defined by

$$Mf = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f| dy$$

where $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$ is the ball centered at x with radius r.

$\mathfrak{B}$ is a set containing $q(\cdot) \in \mathbb{R}^n$ that satisfy the condition that M is bounded on $L^{q(\cdot)}$. Now, we express a few properties of variable exponents associated with the class $\mathfrak{B}(E)$. Neugebauer, Cruz Uribe, and Fiorenza [12], as well as Nakvinda [32], established the inequalities presented in the proposition below.

Proposition 1. *[32] Let E be an open set, and let $q(\cdot) \in P(E)$ satisfy the requirements given below:*

$$|q(y) - q(x)| \leq \frac{-C}{ln(|y - x|)},\ \frac{1}{2} \geq |y - x|, \tag{4}$$

$$|q(y) - q(x)| \leq \frac{C}{ln(|x| + e)},\ |x| \leq |y|, \tag{5}$$

then $q(\cdot) \in \mathfrak{B}(E)$, where C stands for a positive constant independent of y and x.

Lemma 1. *[21] (Generalized Hölder inequality) Let $q(\cdot)$, $q_1(\cdot)$, and $q_2(\cdot)$ be in $P(E)$.*

- *If $h \in L^{q(\cdot)}$ and $f \in L^{q'(\cdot)}$, then*

$$\int_E |h(x)f(x)|dx \leq r_q \|h\|_{L^{q(\cdot)}} \|f\|_{L^{q'(\cdot)}},$$

  where $r_q = 1 + \frac{1}{q^-} - \frac{1}{q^+}$.
- *If $h \in L^{q_1(\cdot)}(E)$ and $f \in L^{q_2(\cdot)}(E)$, and $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$, then*

$$\|hf|_{L^{q(\cdot)}} \leq r_{q,q_1} \|h\|_{L^{q_1(\cdot)}} \|f\|_{L^{q_2(\cdot)}},$$

  where $r_{q,q_1} = \left(1 + \frac{1}{(q_1)^-} - \frac{1}{(q_1)^+}\right)^{1/q^-}$.

Lemma 2. *[34] Assuming that $q(\cdot) \in \mathfrak{B}$ for all measurable subsets I of S, and $I \subset \mathbb{R}^n$, there exists a constant $0 < \delta < 1$ and a constant C such that*

$$\frac{\|\chi_I\|_{L^{q(\cdot)}}}{\|\chi_S\|_{L^{q(\cdot)}}} \leq C \left(\frac{|I|}{|S|}\right)^{\delta}.$$

$$\frac{\|\chi_S\|_{L^{q(\cdot)}}}{\|\chi_I\|_{L^{q(\cdot)}}} \leq C \frac{|S|}{|I|}.$$

Remark 1. *Suppose that $q(\cdot) \in P(\mathbb{R}^n)$ and satisfies conditions (4) and (5) in Proposition 1. Then so does $q^{'}(\cdot)$. Generally, we can see that both $q(\cdot)$ and $q^{'}(\cdot)$ belong to $\mathfrak{B}(\mathbb{R}^n)$ based on Proposition 1. Therefore, by virtue of Lemma 2, we can consider a constant $\delta_1 \in (0, \frac{1}{(q_2)_+})$ such that*

$$\frac{\|\chi_I\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|I|}{|S|}\right)^{\delta_1} \tag{6}$$

which holds for all balls S in $\mathbb{R}^n$ and for $I \subset S$. If $q_1(\cdot) \in P$, using Lemma 2, we can take constant $\delta_3 \in (0, \frac{1}{(q_1^{'})+})$ such that

$$\frac{\|\chi_I\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|I|}{|S|}\right)^{\delta_3}. \tag{7}$$

Lemma 3. *[34] Let $q(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$. The following inequality holds for all balls $B \subset \mathbb{R}^n$ and a positive constant C:*

$$C^{-1} \leq \frac{1}{|B|} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Definition 1. *[35] Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define*

$$\|b\|_{BMO} = \sup_B \frac{1}{B} \int_B |b(x) - \underset{B}{Avg}\, b| dx,$$

where the function b is considered to have bounded mean oscillation if $\|b\|_{BMO} < \infty$.

Lemma 4. *[36] Assuming that $q(\cdot) \in P(\mathbb{R}^n)$, for $b \in BMO$, and for $j, i \in \mathbb{Z}$ with $j > i$, we have the following inequalities:*

$$C^{-1}\|b\|_{BMO} \leq \sup_{B:Ball} \frac{1}{\|\chi_B\|_{L^{q(\cdot)}}} \|(b - b_B)\chi_B\|_{L^{q(\cdot)}} \leq C\|b\|_{BMO} \tag{8}$$

$$\|(b - b_{B_i})\chi_{B_j}\|_{L^{q(\cdot)}} \leq C(j - i)\|b\|_{BMO}\|\chi_{B_j}\|_{L^{q(\cdot)}} \tag{9}$$

Definition 2. *[24] Let $\lambda \in \mathbb{R}$ and $q(\cdot) \in P(\mathbb{R}^n)$. Then the central Morrey space for the variable exponent $\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)$ is defined as*

$$\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n) = \left\{ f \in L^{q(\cdot)}_{Loc}(\mathbb{R}^n) : \|f\|_{\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|f\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}$$

Definition 3. *[24] Let $\lambda < \frac{1}{n}$ and $q(\cdot) \in P(\mathbb{R}^n)$. The variable exponent λ-central BMO space $CBMO^{q(\cdot),\lambda}$ is defined as*

$$CBMO^{q(\cdot),\lambda} = \left\{ f \in L^{q(\cdot)}_{Loc}(\mathbb{R}^n) : \|f\|_{CBMO^{q(\cdot),\lambda}(\mathbb{R}^n)} < \infty \right\}$$

where

$$\|f\|_{CBMO^{q(\cdot),\lambda}} = \sup_{R>0} \frac{\|(f - f_{B(0,R)})\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}$$

By using the boundedness results of the integral operator I_β, we will demonstrate the boundedness of the fractional rough Hardy operator:

$$I_\beta(f)(t) = \int_{\mathbb{R}^n} \frac{f(z)}{|t-z|^{n-\beta}} dz.$$

Proposition 2. *[37] Let $q_1(\cdot) \in P(\mathbb{R}^n)$, $0 < \beta < \frac{n}{(q_1)_+}$, and define $q_2(\cdot)$ as*

$$\frac{1}{q_2(\cdot)} = \frac{1}{q_1(\cdot)} - \frac{\beta}{n}.$$

Then,

$$\|I_\beta f\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.$$

Lemma 5. *[30]Assuming that β, $q_1(\cdot)$, and $q_2(\cdot)$ are defined similarly to proposition 2, we have*

$$\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \le C2^{-j\beta}\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.$$

3. Boundedness Of Fractional Rough Hardy Operators

Theorem 1. *Assume that $\Omega \in L^s(S^{n-1})$, where $\frac{n}{n-1} < s$. Let $q_1(\cdot), p(\cdot), q_2(\cdot) \in P(\mathbb{R}^n)$ satisfy the inequalities (4) and (5) in proposition 1. Define the variable exponent $p(\cdot)$ by*

$$\frac{1}{q_1(\cdot)} = \frac{1}{p(\cdot)} + \frac{\beta}{n}.$$

Let λ_1 satisfy the following condition:

When $\frac{1}{q_2(\cdot)} = \frac{1}{q_1'(\cdot)} - \frac{1}{s}$, there exist $\lambda_1 > -\frac{\beta}{n}$ and $\lambda = \lambda_1 + \frac{\beta}{n}$. If $\delta_3 - \frac{1}{s} + \lambda + \delta_1 > 0$, then the fractional rough Hardy operator is bounded from $\dot{B}^{q_1(\cdot),\lambda_1}$ to $\dot{B}^{p(\cdot),\lambda}$, and the following inequality holds:

$$||H_{\beta,\Omega}f||_{\dot{B}^{p(\cdot),\lambda}} \le C||f||_{\dot{B}^{q_1(\cdot),\lambda_1}}.$$

Proof.

$$|H_{\beta,\Omega}f(x)\cdot\chi_k(x)|\le\frac{1}{|x|^{n-\beta}}\int_{B_k}|f(t)||\Omega(x-t)|dt\cdot\chi_k(x)$$
$$\le C2^{-k(n-\beta)}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\chi_k(x).$$

Using $\frac{1}{q_2(\cdot)}+\frac{1}{s}=\frac{1}{q_1'(\cdot)}$

$$|H_{\beta,\Omega}f(x)\cdot\chi_k(x)|\le C2^{-k(n-\beta)}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^{s}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_2(\cdot)}}\cdot\chi_k(x).$$

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C2^{-k(n-\beta)}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^{s}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_2(\cdot)}}\|\chi_k\|_{L^{p(\cdot)}}.$$

Hence we have

$$\|\chi_k\|_{L^{q_2(\cdot)}}\approx|B_k|^{\frac{1}{q_2(\cdot)}}\approx|B_k|^{\frac{1}{q_1'(\cdot)}-\frac{1}{s}}\approx|B_k|^{-\frac{1}{s}}\|\chi_k\|_{L^{q_1'(\cdot)}}$$

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C2^{-k(n-\beta)}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^{s}(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|_{L^{p(\cdot)}}. \tag{10}$$

Based on Proposition 2, we have

$$I_\beta(\chi_{B_k})(x)\ge C2^{k\beta}\chi_{B_k}(x)$$
$$\chi_{B_k}(x)\le C2^{-k\beta}I_\beta(\chi_{B_k})(x)$$

$$\begin{aligned}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}&\le C2^{-k\beta}\|I_\beta\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\&\le C2^{-k\beta}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\\&\le C2^{k(n-\beta)}\|\chi_{B_k}\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{11}$$

Using inequality (11) in (10), we obtain

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^{s}(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|^{-1}_{L^{q_1'(\cdot)}}. \tag{12}$$

Using condition (7), we get

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\sum_{j=-\infty}^{k}2^{(j-k)n\delta_3}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^{s}(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}. \tag{13}$$

For $t \in C_j$, $x \in C_k$, and $j \leq k$, we have $0 \leq |x-t| \leq |x| + 2^j \leq 2.2^k$, and

$$\int_{C_j} |\Omega(x-t)|^s dt \leq \int_0^{2^{k+1}} \int_{s^{n-1}} |\Omega(x^{'})|^s d\sigma(x^{'}) r^{n-1} dr \leq C 2^{kn}$$

$$\begin{aligned}\|H_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq C \sum_{j=-\infty}^{k} 2^{(j-k)(n\delta_3 - \frac{n}{s})} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\ &\leq C \sum_{j=-\infty}^{k} 2^{(j-k)(n\delta_3 - \frac{n}{s})} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)} |B_j|^{\lambda_1} ||\chi_j||_{L^{q_1(\cdot)}}.\end{aligned} \tag{14}$$

$$\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \approx |B|^{\frac{1}{q_1(\cdot)}} \approx |B|^{\frac{1}{p(\cdot)} + \frac{\beta}{n}} \approx |B|^{\frac{\beta}{n}} \|\chi_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\begin{aligned}\|H_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq C \sum_{j=-\infty}^{k} 2^{(j-k)(n\delta_3 - \frac{n}{s})} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)} |B_j|^{\lambda_1 + \frac{\beta}{n}} ||\chi_j||_{L^{p(\cdot)}} \\ \leq C \sum_{j=-\infty}^{k} 2^{(j-k)(n\delta_3 - \frac{n}{s})} &\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)} \frac{|B_j|^{\lambda}}{|B_k|^{\lambda}} |B_k|^{\lambda} \frac{||\chi_j||_{L^{p(\cdot)}}}{||\chi_k||_{L^{p(\cdot)}}} ||\chi_k||_{L^{p(\cdot)}}.\end{aligned} \tag{15}$$

Using inequality (6), we have

$$\|H_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \sum_{j=-\infty}^{k} 2^{n(j-k)(\delta_3 - \frac{1}{s} + \lambda + \delta_1)} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)} |B_k|^{\lambda} ||\chi_k||_{L^{p(\cdot)}}. \tag{16}$$

Since $\delta_3 - \frac{1}{s} + \lambda + \delta_1 > 0$, we obtain

$$\|H_{\beta,\Omega} f\|_{\dot{B}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leq C \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)}. \tag{17}$$

Theorem 2. *Let $p(\cdot), q_1(\cdot)$, $q_2(\cdot)$ and β be defined the same as in Theorem 1, and $\Omega \in L^s(S^{n-1})$. If $\lambda = \lambda_1 + \frac{\beta}{n}$ and $\lambda < \frac{1}{s} - \frac{\beta}{n} - 1$, then*

$$\|H^*_{\beta,\Omega} f\|_{\dot{B}^{p(\cdot),\lambda}} \leq C \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}.$$

Proof.

$$\begin{aligned}|H^*_{\beta,\Omega} f(x) \cdot \chi_k| &\leq \int_{\mathbb{R}^n \setminus B_k} |f(t)\Omega(x-t)||t|^{\beta-n} dt \cdot \chi_k(x) \\ &\leq C \sum_{j=k+1}^{\infty} 2^{j(\beta-n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \cdot \chi_k(x).\end{aligned}$$

By using $\frac{1}{q_2(\cdot)} + \frac{1}{s} = \frac{1}{q_1'(\cdot)}$

$$\begin{aligned}&\|H^*_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\leq C \sum_{j=k+1}^{\infty} 2^{j(\beta-n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\|_{L^s} \|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Using inequality (11), we have

$$\begin{aligned}&\|H^*_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\leq C \sum_{j=k+1}^{\infty} 2^{(j-k)(\beta-n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\|_{L^s} \|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|\chi_k\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

As we know, we obtain

$$\|\chi_j\|_{L^{q_2(\cdot)}} \approx |B_j|^{\frac{1}{q_2(\cdot)}} \approx |B_j|^{\frac{1}{q_1'(\cdot)} - \frac{1}{s}} \approx |B_j|^{-\frac{1}{s}} \|\chi_j\|_{L^{q_1'(\cdot)}}$$

$$\begin{aligned}&\|H^*_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\leq C \sum_{j=k+1}^{\infty} 2^{(j-k)(\beta-n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\|_{L^s} |B_j|^{-\frac{1}{s}} \|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|\chi_k\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Now by using condition (7),

$$\begin{aligned}&\|H^*_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\leq C \sum_{j=k+1}^{\infty} 2^{(j-k)(\beta-n+n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\|_{L^s} |B_j|^{-\frac{1}{s}}.\end{aligned}$$

For further calculations following Theorem 1, we get

$$\begin{aligned}&\|H^*_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\leq C \sum_{j=k+1}^{\infty} 2^{(j-k)(\beta-\frac{n}{s})} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Hence we have

$$\|H^*_{\beta,\Omega} f \cdot \chi_k\|_{\dot{B}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leq C \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)} \sum_{j=k+1}^{\infty} 2^{(j-k)(\beta-\frac{n}{s}+n\lambda+n)}.$$

By utilizing $\lambda < \frac{1}{S} - \frac{\beta}{n} - 1$, we obtain the desired result:

$$\|H^*_{\beta,\Omega} f\|_{\dot{B}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leq C \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

4. Boundedness Commutators Of Fractional Rough Hardy Operators

Theorem 3. *Let $0<\beta<n$, $\Omega\in L^s(S^{n-1})$, and $\frac{n}{n-1}<s$. Suppose that $q_1(\cdot),p(\cdot),q(\cdot)\in P(\mathbb{R}^n)$ satisfy conditions (4) and (5) in Proposition 1, and let the variable exponent $q_2(\cdot)$ be define by*

$$\frac{1}{q_1(\cdot)}=\frac{1}{q_2(\cdot)}-\frac{1}{q(\cdot)}+\frac{\beta}{n}.$$

Let λ_1 satisfy the following condition:

When $\frac{1}{q(\cdot)}=\frac{1}{q_1'(\cdot)}-\frac{1}{s}$, there exists $\lambda_1>-\lambda-\frac{\beta}{n}$ such that $\lambda_2=\lambda_1+\lambda+\frac{\beta}{n}$. If $b\in\|b\|_{CBMO^{q(\cdot),\lambda}}$ and $\delta_3-\frac{1}{s}+\lambda_2+\delta_1>0$, then the following inequality holds:

$$||[b,H_{\beta,\Omega}]f||_{\dot{B}^{q_2(\cdot),\lambda_2}}\le C\|b\|_{CBMO^{q(\cdot),\lambda}}||f||_{\dot{B}^{q_1(\cdot),\lambda_1}}.$$

Proof.

$$\begin{aligned}|[b,H_{\beta,\Omega}]f(x)\cdot\chi_B(x)|&\le\frac{1}{|x|^{n-\beta}}\int_{B(0,|x|)}|f(t)(b(x)-b(t))\Omega(x-t)|dt\cdot\chi_B(x)\\&\le\frac{1}{|x|^{n-\beta}}\int_{B(0,|x|)}|f(t)(b(x)-b_B)\Omega(x-t)|dt\cdot\chi_B(x)\\&+\frac{1}{|x|^{n-\beta}}\int_{B(0,|x|)}|f(t)(b(t)-b_B)\Omega(x-t)|dt\cdot\chi_B(x)\\&=A_1+A_2.\end{aligned}$$

First, we estimate A_1. Let $\frac{1}{p(x)}=\frac{1}{q_1(x)}-\frac{\beta}{n}$. This implies $\frac{1}{q_2(x)}=\frac{1}{p(x)}+\frac{1}{q(x)}$

$$A_1=|(b(x)-b_B)\chi_B(x)||H_{\beta,\Omega}f(x)|,$$

$$\|A_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}=\|(b(x)-b_B)\chi_B(x)H_{\beta,\Omega}f\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.$$

Let $\frac{1}{p(\cdot)}=\frac{1}{q_2(\cdot)}-\frac{1}{q(\cdot)}$, and use Hölder inequality ($\frac{1}{q_2(\cdot)}=\frac{1}{p(\cdot)}+\frac{1}{q(\cdot)}$)

$$\begin{aligned}\|A_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}&\le C\|H_{\beta,\Omega}f\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|(b(x)-b_B)\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&=C\|H_{\beta,\Omega}f\|_{\dot{B}^{\mu,p(\cdot)}}|B|^{\mu}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|b\|_{CBMO^{q(\cdot),\lambda}}|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)},\end{aligned}$$

Given that $\mu=\lambda_1+\frac{\beta}{n}$, and using the result of Theorem 1

$$\|A_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|b\|_{CBMO^{q(\cdot),\lambda}}|B|^{\lambda_2}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

Next,

$$\begin{aligned}A_2 &= \sum_{k=-\infty}^{0} \frac{1}{|x|^{n-\beta}} \sum_{j=-\infty}^{k} \int_{2^jB\backslash 2^{j-1}B} |f(t)(b(t)-b_B)\Omega(x-t)|dt \cdot \chi_{2^kB\backslash 2^{k-1}B}(x)\\ &\le \sum_{k=-\infty}^{0} \frac{1}{|x|^{n-\beta}} \sum_{j=-\infty}^{k} \int_{2^jB\backslash 2^{j-1}B} |f(t)(b(t)-b_{2^jB})\Omega(x-t)|dt \cdot \chi_{2^kB\backslash 2^{k-1}B}(x)\\ &+ \sum_{k=-\infty}^{0} \frac{1}{|x|^{n-\beta}} \sum_{j=-\infty}^{k} \int_{2^jB\backslash 2^{j-1}B} |f(t)(b_B-b_{2^jB})\Omega(x-t)|dt \cdot \chi_{2^kB\backslash 2^{k-1}B}(x)\\ &= A_{21} + A_{22}\end{aligned}$$

$$A_{21} = \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} \int_{2^jB\backslash 2^{j-1}B} |f(t)(b(t)-b_{2^jB})\Omega(x-t)|dt \cdot \chi_{2^kB\backslash 2^{k-1}B}(x)$$

given that $\frac{1}{q(\cdot)} + \frac{1}{q_1(\cdot)} + \frac{1}{s} = 1$, we can now use Holder's inequality

$$\begin{aligned}A_{21} &\le C \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta}{n}-1} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=-\infty}^{k} \|(b(t)-b_{2^jB})\chi_{2^jB}\|_{L^{q(\cdot)}}\\ &\times \|f\chi_{2^jB}\|_{L^{q_1(\cdot)}} \|\Omega(x-t)\chi_{2^jB}\|_{L^s}\\ &= C \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta}{n}-1} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=-\infty}^{k} \|b\|_{CBMO^{q(\cdot),\lambda}} |2^jB|^{\lambda} \|\chi_{2^jB}\|_{L^{q(\cdot)}}\\ &\times \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |2^jB|^{\lambda_1} \|\chi_{2^jB}\|_{L^{q_1(\cdot)}} \|\Omega(x-t)\chi_{2^jB}\|_{L^s}\\ &= C \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta}{n}-1} \chi_{2^kB\backslash 2^{k-1}B}(x) \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\\ &\times \sum_{j=-\infty}^{k} |2^jB|^{\lambda+\lambda_1} |2^jB|^{\frac{1}{s}+\frac{1}{q_1(\cdot)}+\frac{1}{q(\cdot)}}\\ &= C \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta}{n}-1} \chi_{2^kB\backslash 2^{k-1}B}(x) \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\\ &\times \sum_{j=-\infty}^{k} |2^j|^{\lambda+\lambda_1+1} |B|^{\lambda+\lambda_1+1}\\ &= C \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^k|^{\frac{\beta}{n}+\lambda+\lambda_1} \chi_{2^kB\backslash 2^{k-1}B}(x) |B|^{\lambda+\lambda_1+\frac{\beta}{n}}\end{aligned}$$

$$\begin{aligned}
\|A_{21}\|_{L^{q_2(\cdot)}} &\le C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^k|^{\frac{\beta}{n}+\lambda+\lambda_1} \|\chi_{2^kB}\|_{L^{q_2(\cdot)}} |B|^{\lambda+\lambda_1+\frac{\beta}{n}} \\
&= C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^k|^{\frac{\beta}{n}+\lambda+\lambda_1} |2^kB|^{\frac{1}{q_2(\cdot)}} |B|^{\lambda+\lambda_1+\frac{\beta}{n}} \\
&= C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |B|^{\frac{1}{q_2(\cdot)}} |B|^{\lambda_2} \sum_{k=-\infty}^{0} |2|^{k(\lambda_2+\frac{1}{q_2(\cdot)})}
\end{aligned}$$

$$\|A_{21}\|_{L^{q_2(\cdot)}} \le C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \|\chi_B\|_{L^{q_2(\cdot)}} |B|^{\lambda_2}$$

$$A_{22} = \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} \int_{2^jB\setminus 2^{j-1}B} |(b_B - b_{2^jB}) f(t)\Omega(x-t)| dt \cdot \chi_{2^kB\setminus 2^{k-1}B}(x)$$

$$\begin{aligned}
|(b_B - b_{2^jB})| &= \sum_{i=j}^{-1} |(b_{2^{i+1}B} - b_{2^iB})| \\
&= \sum_{i=j}^{-1} \frac{1}{|2^iB|} \int_{2^iB} |b(t) - b_{2^{i+1}B}| dy \\
&\le C \sum_{i=j}^{-1} \frac{1}{|2^iB|} \|(b - b_{2^{i+1}b})\chi_{2^{i+1}B}\|_{L^{q(\cdot)}} \|\chi_{2^{i+1}B}\|_{L^{q'(\cdot)}}
\end{aligned}$$

By virtue of Lemma 3, we have

$$\begin{aligned}
|(b_B - b_{2^jB})| &\le C \sum_{i=j}^{-1} \frac{1}{|2^iB|} \|(b - b_{2^{i+1}b})\chi_{2^{i+1}B}\|_{L^{q(\cdot)}} \frac{|2^{i+1}B|}{\|\chi_{2^{i+1}B}\|_{L^{q(\cdot)}}} \\
&\le C \sum_{i=j}^{-1} \|b\|_{CBMO^{q(\cdot),\lambda}} |2^{i+1}B|^{\lambda} \\
&\le C\|b\|_{CBMO^{q(\cdot),\lambda}} \sum_{i=j}^{-1} |2^{i+1}B|^{\lambda} \\
&\le C\|b\|_{CBMO^{q(\cdot),\lambda}} |2^{j+1}B|^{\lambda} |j| \qquad (18)
\end{aligned}$$

$$A_{22} \leq C\sum_{k=-\infty}^{0}\chi_{2^kB\setminus 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1}\sum_{j=-\infty}^{k}\|b\|_{CBMO^{q(\cdot),\lambda}}|2^{j+1}B|^{\lambda}|j|$$
$$\times\|f\chi_{2^jB}\|_{L^{q_1(\cdot)}}\|\Omega(x-t)\chi_{2^jB}\|_{L^s}\|\chi_{2^jB}\|_{L^{q(\cdot)}}$$
$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\sum_{k=-\infty}^{0}\chi_{2^kB\setminus 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1}\sum_{j=-\infty}^{k}|2^jB|^{\lambda+\lambda_1}|j|\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}$$
$$\times\|\chi_{2^jB}\|_{L^{q_1(\cdot)}}\|\Omega(x-t)\chi_{2^jB}\|_{L^s}\|\chi_{2^jB}\|_{L^{q(\cdot)}}$$
$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\sum_{k=-\infty}^{0}\chi_{2^kB\setminus 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1}$$
$$\times\sum_{j=-\infty}^{k}|2^jB|^{\lambda+\lambda_1}|j|\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}|2^jB|^{\frac{1}{q_1(\cdot)}+\frac{1}{s}+\frac{1}{q(\cdot)}}$$
$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}\chi_{2^kB\setminus 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1}\sum_{j=-\infty}^{k}|2^jB|^{\lambda+\lambda_1+1}|j|$$
$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}\chi_{2^kB\setminus 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1}|2^kB|^{\lambda+\lambda_1+1}|k|$$
$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}|k||2^kB|^{\lambda+\lambda_1+\frac{\beta}{n}}\chi_{2^kB\setminus 2^{k-1}B}(x)$$

$$\|A_{22}\|_{L^{q_2(\cdot)}} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}|k||2^kB|^{\lambda+\lambda_1+\frac{\beta}{n}}\|\chi_{2^kB}\|_{L^{q_2(\cdot)}}$$
$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}|k||2^kB|^{\lambda+\lambda_1+\frac{\beta}{n}}|2^kB|^{\frac{1}{q_2(\cdot)}}$$
$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}|k||2^k|^{\lambda_2+\frac{1}{q_2(\cdot)}}|B|^{\lambda_2+\frac{1}{q_2(\cdot)}}$$
$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}|B|^{\lambda_2}\|\chi_B\|_{L^{q_2(\cdot)}}$$

Combine all results of A_1, A_2, A_{21}, and A_{22}, we obtain the required result

$$\|[b,H_{\beta,\Omega}]f\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}|B|^{\lambda_2}\|\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}$$
$$\|[b,H_{\beta,\Omega}]f\|_{\dot{B}^{q_2(\cdot),\lambda_2}} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}$$

Theorem 4. *Let $p(\cdot), q_1(\cdot)$, $q_2(\cdot)$, and β be defined as in Theorem 2 , and let $\Omega \in L^s(S^{n-1})$. If $b \in \|b\|_{CBMO^{q(\cdot),\lambda}}$, $\lambda_2 = \lambda+\lambda_1+\frac{\beta}{n}$, and $\beta < n(1-\delta_3-\delta_1-\lambda_2+\frac{1}{s})$, then*

$$\|[b,H^*_{\beta,\Omega}]f\|_{\dot{B}^{q_2(\cdot),\lambda_2}} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}$$

Proof.

$$\begin{aligned}|[b,H^*_{\beta,\Omega}]f(x)\cdot\chi_B(x)| &\le \int_{B(0,|x|)^c}\frac{|f(t)(b(x)-b(t))\Omega(x-t)|}{|t|^{n-\beta}}dt\cdot\chi_B(x)\\ &\le \int_{B(0,|x|)^c}\frac{|f(t)(b(x)-b_B)\Omega(x-t)|}{|t|^{n-\beta}}dt\cdot\chi_B(x)\\ &+\int_{B(0,|x|)^c}\frac{|f(t)(b(t)-b_B)\Omega(x-t)|}{|t|^{n-\beta}}dt\cdot\chi_B(x)\\ &= D_1+D_2.\end{aligned}$$

$$D_1=|(b(x)-b_B)\chi_B(x)||H^*_{\beta,\Omega}f(x)|,$$
$$\|D_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}=\|(b(x)-b_B)\chi_B(x)H^*_{\beta,\Omega}f(x)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.$$

By using Hölder inequality $(\frac{1}{q_2(\cdot)}=\frac{1}{p(\cdot)}+\frac{1}{q(\cdot)})$

$$\begin{aligned}\|D_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} &\le C\|(b(x)-b_B)\chi_B(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|H^*_{\beta,\Omega}f(x)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &= C\|b\|_{CBMO^{q(\cdot),\lambda}}|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}|B|^{\mu}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|H^*_{\beta,\Omega}f\|_{\dot{B}^{\mu,p(\cdot)}},\end{aligned}$$

Given that $\mu=\lambda_1+\frac{\beta}{n}$, and using the result of Theorem 2

$$\|D_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\le C\|b\|_{CBMO^{q(\cdot),\lambda}}|B|^{\lambda_2}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}},$$

Next,

$$D_2=\int_{B(0,|x|)^c}\frac{|f(t)(b(t)-b_B)\Omega(x-t)|}{|t|^{n-\beta}}dt\cdot\chi_B(x).$$

$$\begin{aligned}D_2 &= \sum_{k=-\infty}^{0}\int_{2^jB\backslash 2^{j-1}B}\frac{|f(t)(b(t)-b_B)\Omega(x-t)|}{|t|^{n-\beta}}dt\cdot\chi_{2^kB\backslash 2^{k-1}B}(x)\\ &\le \sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{j=k+1}^{\infty}|2^jB|^{\frac{\beta}{n}-1}\int_{2^jB\backslash 2^{j-1}B}|f(t)(b(t)-b_{2^jB})\Omega(x-t)|dt\\ &+\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{j=k+1}^{\infty}|2^jB|^{\frac{\beta}{n}-1}\int_{2^jB\backslash 2^{j-1}B}|f(t)(b_B-b_{2^jB})\Omega(x-t)|dt\\ &= D_{21}+D_{22}\end{aligned}$$

$$D_{21}=\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{j=k+1}^{\infty}|2^jB|^{\frac{\beta}{n}-1}\int_{2^jB\backslash 2^{j-1}B}|f(t)(b(t)-b_{2^jB})\Omega(x-t)|dt$$

Using Holder's inequality $(\frac{1}{q(\cdot)}+\frac{1}{q_1(\cdot)}+\frac{1}{s}=1)$.

$$
\begin{aligned}
D_{21} \leq C & \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \\
& \times \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} \|(b(t)-b_{2^jB})\chi_{2^jB}\|_{L^{q(\cdot)}} \|f\chi_{2^jB}\|_{L^{q_1(\cdot)}} \|\Omega(x-t)\chi_{2^jB}\|_{L^s} \\
= C & \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \\
& \times \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} \|b\|_{CBMO^{q(\cdot),\lambda}} |2^jB|^{\lambda} \|\chi_{2^jB}\|_{L^{q(\cdot)}} \\
& \qquad \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |2^jB|^{\lambda_1} \|\chi_{2^jB}\|_{L^{q_1(\cdot)}} \|\Omega(x-t)\chi_{2^jB}\|_{L^s} \\
= C & \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \\
& \times \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} |2^jB|^{\lambda+\lambda_1} |2^jB|^{\frac{1}{s}+\frac{1}{q_1(\cdot)}+\frac{1}{q(\cdot)}} \\
= C & \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{j=k+1}^{\infty} |2^jB|^{\lambda_2} \\
= C & \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^{(k+1)}B|^{\lambda_2} \chi_{2^kB\backslash 2^{k-1}B}(x)
\end{aligned}
$$

$$
\begin{aligned}
\|D_{21}\|_{L^{q_2(\cdot)}} & \leq C \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^{(k+1)}B|^{\lambda_2} \|\chi_{2^kB}\|_{L^{q_2(\cdot)}} \\
& \leq C \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^{(k+1)}B|^{\lambda_2} |2^kB|^{\frac{1}{q_2(\cdot)}} \\
& \leq C \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |B|^{\lambda_2+\frac{1}{q_2(\cdot)}} \sum_{k=-\infty}^{0} |2^{(k+1)}|^{\lambda_2+\frac{1}{q_2(\cdot)}} \\
& \leq C \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |B|^{\lambda_2+\frac{1}{q_2(\cdot)}}
\end{aligned}
$$

$$
D_{22} = \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} \int_{2^jB\backslash 2^{j-1}B} |(b_B - b_{2^jB})f(t)|dt
$$

Here we use inequality (18)

$$\begin{aligned}
D_{22} &\leq C\sum_{k=-\infty}^{0}\chi_{2^kB\setminus 2^{k-1}B}(x)\sum_{j=k+1}^{\infty}|2^jB|^{\frac{\beta}{n}-1}\|b\|_{CBMO^{q(\cdot),\lambda}}|2^{j+1}B|^{\lambda}|j|\|f\chi_{2^jB}\|_{L^{q_1(\cdot)}}\\
&\quad\times\|\Omega(x-t)\chi_{2^jB}\|_{L^s}\|\chi_{2^jB}\|_{L^{q(\cdot)}}\\
&\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\sum_{k=-\infty}^{0}\chi_{2^kB\setminus 2^{k-1}B}(x)\sum_{j=k+1}^{\infty}|2^jB|^{\frac{\beta}{n}-1}|2^jB|^{\lambda+\lambda_1}|j|\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\\
&\quad\times\|\Omega(x-t)\chi_{2^jB}\|_{L^s}\|\chi_{2^jB}\|_{L^{q(\cdot)}}\|\chi_{2^jB}\|_{L^{q_1(\cdot)}}\\
&\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}\chi_{2^kB\setminus 2^{k-1}B}(x)\sum_{j=k+1}^{\infty}|2^jB|^{\lambda+\lambda_1+\frac{\beta}{n}}|j|\\
&\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}\chi_{2^kB\setminus 2^{k-1}B}(x)|2^{k+1}B|^{\lambda_2}|k+1|\\
&\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}|k+1||2^{k+1}B|^{\lambda_2}\chi_{2^kB\setminus 2^{k-1}B}(x)
\end{aligned}$$

$$\begin{aligned}
\|D_{22}\|_{L^{q_2(\cdot)}} &\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}|k+1||2^{k+1}B|^{\lambda_2}\|\chi_{2^kB}\|_{L^{q_2(\cdot)}}\\
&\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}|k+1||2^{k+1}B|^{\lambda_2}|2^kB|^{\frac{1}{q_2(\cdot)}}\\
&\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}|k+1||2^{k+1}|^{\lambda_2+\frac{1}{q_2(\cdot)}}|B|^{\lambda_2+\frac{1}{q_2(\cdot)}}\\
&\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}|B|^{\lambda_2}\|\chi_B\|_{L^{q_2(\cdot)}}
\end{aligned}$$

Combining all the results from D_1, D_2, D_{21}, and D_{22}, we obtain the required results:

$$\|[b,H^*_{\beta,\Omega}]f\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}|B|^{\lambda_2}\|\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}$$

$$\|[b,H^*_{\beta,\Omega}]f\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}.$$

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ON THE EIGENSTRUCTURE OF THE q-STANCU OPERATOR

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Abstract. The main goal of this research is to find the eigenvalues and the corresponding eigenfunctions of the q-Stancu operator, $L_{n,s,q}$, introduced by L. Yun and R. Wang. In this work, an explicit representation for moments of all orders has been derived. Further, it has been proved that $L_{n,s,q}$ possesses $n-s+1$ linearly independent eigenfunctions whose explicit expression and the corresponding eigenvalues are derived. In addition, for special choices of parameters, several eigenfunctions are depicted.

1. Introduction

The discovery of the Bernstein polynomials by S. N. Bernstein in 1912 [2] paved the way for a vast number of studies in the approximation theory. Due to their elegant structure and remarkable properties, these polynomials have formed the basis for research not only in mathematics but also in many other fields such as physics, statistics, engineering (see [6, 8, 16]). The extensive research on the Bernstein operators has enabled the development of various generalizations and modified forms.

In 1981, Stancu proposed a generalization of the Bernstein operator, representing an extension based on the non-negative integer parameter s, of the classical Bernstein operator as follows:

Definition 1. *[17] Let n and s be integers such that $0 \leqslant s < n/2$. Then, for any function $f \in C[0,1]$, the Stancu operator is defined by*

$$L_{n,s}(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k,s}(x), \tag{1}$$

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where

$$b_{n,k,s}(x)=\begin{cases}(1-x)p_{n-s,k}(x), & 0\leqslant k<s,\\ (1-x)p_{n-s,k}(x)+xp_{n-s,k-s}(x), & s\leqslant k\leqslant n-s,\\ xp_{n-s,k-s}(x), & n-s<k\leqslant n.\end{cases}$$

Here, $p_{n,k}(x)$ *are the Bernstein basis polynomials given by*

$$p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k},\quad k=0,1,\dots,n.$$

Observe that, for $s=0,1$, (1) reveals the classical Bernstein operator.

In [17], Stancu examined the remainder term of the approximation formulas for operator $L_{n,s}$ and established its asymptotic estimate using Voronovskaja-type formula. He also estimated the order of approximation for operator (1) in terms of the modulus of continuity of a function f and its derivative f'. Moreover, he found the eigenvalues of this Bernstein-type operator and proved that the sequence of the eigenvalues is monotonically decreasing. In 2008, L. Yun and X. Xiang delved into the monotonicity-preserving and convexity-preserving properties of the aforementioned operator. They provided a proof regarding the operator's monotonicity for convex functions and gave the theorem about simultaneous approximation [19]. Recently, the Kantorovich extension of Stancu operator was proposed and investigated in [3].

Another way to extend the operator is to obtain a modified version of the classical operator by employing q-calculus. The first steps of this generalization were taken by Lupaş [12] and Phillips [15], who introduced q-generalizations of the Bernstein operator. Owing to their works, the idea of generalizing operator using q-calculus has been extended to many operators and this idea is still fruitful, see for example, [7, 9, 14].

In 2011, L. Yun and R. Wang [20] introduced a q-generalization of the Stancu operator, known as q-Stancu operator. There, they studied shape-preserving and approximation properties of this generalization. A year later, X. Xiang [18] obtained more results pertinent to the q-Stancu operator.

For the convenience of the reader, some notations and definitions related to q-calculus are provided, see [1, Chapter 10], and afterward, the definition of the q-Stancu operator will be given.

Let $q>0$. For any non-negative integer n, the q-integer $[n]_q$ is defined by

$$[0]_q:=0,\quad [n]_q:=1+q+\dots+q^{n-1},\quad n=1,2,\dots \tag{2}$$

The expressions below are q-variants of factorials and binomial coefficients known as q-factorials and q-binomial coefficients, respectively,

$$[0]_q!:=1,\quad [n]_q!:=[1]_q[2]_q\cdots[n]_q,\quad n=1,2,\dots,$$

and

$$\begin{bmatrix}n\\k\end{bmatrix}_q:=\frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Also, for each $x \in \mathbb{C}$, the q-analogue of the Pochhammer symbol is defined by

$$(x;q)_0 := 1, \quad (x;q)_n := \prod_{j=0}^{n-1}(1-xq^j).$$

Definition 2. *[20] Let n and s be integers such that $0 \leqslant s < n/2$. Then, for $0 < q < 1$ and $f \in C[0,1]$, the q-Stancu operator, $L_{n,s,q} : C[0,1] \to C[0,1]$, is given by*

$$L_{n,s,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) b_{n,k,s}(q;x),$$

where

$$b_{n,k,s}(q;x) = \begin{cases} (1-q^{n-k-s}x)p_{n-s,k}(q;x), & 0 \leqslant k < s, \\ (1-q^{n-k-s}x)p_{n-s,k}(q;x) + q^{n-k}xp_{n-s,k-s}(q;x), & s \leqslant k \leqslant n-s, \\ q^{n-k}xp_{n-s,k-s}(q;x), & n-s < k \leqslant n, \end{cases}$$

and

$$p_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (x;q)_{n-k}, \quad k = 0,1,\dots,n. \tag{3}$$

The polynomials (3) are known as q-Bernstein basis polynomials.

Along with changing index as $k - s = i$ in the sum and then denoting again the summation index by k, it becomes evident that the operator can be represented for $n = 1, 2, \dots,$ as follows:

$$L_{n,s,q}(f;x) = \sum_{k=0}^{n-s} \left\{ (1-q^{n-k-s}x) f\left(\frac{[k]_q}{[n]_q}\right) + q^{n-k-s}xf\left(\frac{[k+s]_q}{[n]_q}\right) \right\} p_{n-s,k}(q;x). \tag{4}$$

See [20, formula (1.2)].

Note that, q-Stancu operator reduces to the classical Stancu operator, as introduced in [17], when q is set to 1. Additionally, in the cases where $s = 0, 1$, the operators $L_{n,s,q}$ coincide with the q-Bernstein operators defined by Phillips [15]. Furthermore, this operator possesses some properties of the q-Bernstein polynomials. In the case $0 < q < 1$, q-Stancu operator is a positive linear operator, while in the case $q > 1$, it is not. This operator enjoys the end-point interpolation property, that is, $L_{n,s,q}(f;0) = f(0)$ and $L_{n,s,q}(f;1) = f(1)$ for all $q > 0$. Due to $L_{n,s,q}(1;x) = 1$ and $L_{n,s,q}(t;x) = x$, the q-Stancu operator leaves the linear functions invariant.

The eigenvalues and eigenvectors of linear operators are important issues in the applications of linear algebra to the theory of algorithms, the theory of Markov chains and computer science. The spectral theory of linear operators is also used extensively in other disciplines, like quantum mechanics and the field theory, see, e.g., [11] and [21]. Even though quantum systems are generally described in L^2 spaces of infinite dimensions, the quantum perturbation theory routinely uses their finite-dimensional approximations, see, e.g., [11, Chapter 5]. Apart from that, eigenvalues and eigenvectors are used in the theory of parametric excitation of oscillating systems, see [10, Section 27].

The present paper is devoted to examining the eigenvalues and the eigenfunctions of the q-Stancu operators $L_{n,s,q}$. The structure of this paper is as follows: In Section 1, some preliminary results that will be used through the paper and the explicit formula for the moments of all orders for the q-Stancu operator are provided. Hitherto, only the first three moments have been calculated. Section 2 focuses on the eigenvalues and the corresponding eigenfunctions of the q-Stancu operator. It is demonstrated that while $\xi = 1$ is a double eigenvalue, the others are simple. In the last section, the eigenvectors are graphically illustrated for selected values of parameters.

2. Moments of q-Stancu Operators

The calculation of the moments of linear positive operators plays a significant role when studying their approximation properties. Regarding the q-generalization of the Stancu operator $L_{n,s,q}$, only the first three moments, $L_{n,s,q}(e_i; x)$, where $e_i = t^i$, $i = 0, 1, 2$ have been found so far, see [20, Proposition 2]. In this section, explicit formulae for all the moments of the q-Stancu operators will be presented through moments of the q-Bernstein operator. To begin with, let us provide the essential details regarding the q-Bernstein operator.

The explicit form of the moments of $B_{n,q}$, mentioned in [5, formula (2.4)], is provided below:

$$B_{n,q}(e_k; x) = \sum_{i=0}^{k} \frac{S_q(k,i)}{[n]_q^{k-i}} \lambda_{i,q}^{(n)} x^i, \tag{5}$$

where $S_q(i,j)$ is defined as the q-Stirling numbers of the second kind [5, formula (2.5)] as follows,

$$S_q(i,j) = \frac{1}{[j]_q! \, q^{j(j-1)/2}} \sum_{r=0}^{j} (-1)^r q^{r(r-1)/2} \begin{bmatrix} j \\ r \end{bmatrix}_q [j-r]_q^i,$$

with $S_q(0,0) = 1$, $S_q(i,0) = 0$ for $i > 0$, $S_q(i,j) = 0$ for $j > i$.

Here are the eigenvalues $\lambda_{m,q}^{(n)}$ of the q-Bernstein operator [13]: for $m = 2, 3, \dots, n$,

$$\lambda_{0,q}^{(n)} = \lambda_{1,q}^{(n)} = 1, \quad \lambda_{m,q}^{(n)} = \left(1 - \frac{1}{[n]_q}\right)\left(1 - \frac{[2]_q}{[n]_q}\right) \cdots \left(1 - \frac{[m-1]_q}{[n]_q}\right).$$

Theorem 1. *For $m = 1, 2 \dots$, there holds*

$$L_{n,s,q}(e_m; x) = \sum_{r=1}^{m} a_{n,s,q}(r,m) x^r, \tag{6}$$

where

$$a_{n,s,q}(r,m) = \frac{[n-s]_q^r}{[n]_q^m} S_q(m,r)\lambda_{r,q}^{(n-s)} + \sum_{j=1}^{m-r+1}\sum_{i=0}^{j-1} A(j,i,r-1) + \sum_{j=m-r+2}^{m}\sum_{i=r-1-m+j}^{j-1} A(j,i,r-1) \quad (7)$$

and

$$A(j,i,r) = \binom{m}{j}\binom{j-1}{i} q^{n-s}(-1)^i(1-q)^i \frac{[n-s]_q^r[s]_q^j}{[n]_q^m} S_q(m-j+i,r)\lambda_{r,q}^{(n-s)}. \quad (8)$$

Proof. From the definition (4), one has

$$\begin{aligned} L_{n,s,q}(e_m;x) &= \sum_{k=0}^{n-s}\left\{(1-q^{n-k-s}x)\left(\frac{[k]_q}{[n]_q}\right)^m + q^{n-k-s}x\left(\frac{[k+s]_q}{[n]_q}\right)^m\right\}p_{n-s,k}(q;x) \\ &= \sum_{k=0}^{n-s}\left(\frac{[k]_q}{[n]_q}\right)^m p_{n-s,k}(q;x) + \frac{x}{[n]_q^m}\sum_{k=0}^{n-s} q^{n-k-s}\left([k+s]_q^m - [k]_q^m\right)p_{n-s,k}(q;x) \\ &= \left(\frac{[n-s]_q}{[n]_q}\right)^m B_{n-s,q}(e_m;x) + \frac{x}{[n]_q^m}\sum_{k=0}^{n-s} q^{n-k-s}\left([k+s]_q^m - [k]_q^m\right)p_{n-s,k}(q;x). \end{aligned}$$

Using the relation $[k+s]_q = [k]_q + q^k[s]_q$ and the binomial expansion formula in the second sum, we get

$$\begin{aligned} &L_{n,s,q}(e_m;x) \\ &= \left(\frac{[n-s]_q}{[n]_q}\right)^m B_{n-s,q}(e_m;x) + \frac{x}{[n]_q^m}\sum_{k=0}^{n-s}\sum_{j=1}^{m}\binom{m}{j}[k]_q^{m-j}q^{n-k-s}\left(q^k[s]_q\right)^j p_{n-s,k}(q;x) \\ &= \left(\frac{[n-s]_q}{[n]_q}\right)^m B_{n-s,q}(e_m;x) + \frac{q^{n-s}x}{[n]_q^m}\sum_{j=1}^{m}\binom{m}{j}[s]_q^j\sum_{k=0}^{n-s}[k]_q^{m-j}q^{k(j-1)}p_{n-s,k}(q;x). \end{aligned}$$

Applying $q^k = 1-(1-q)[k]_q$, one can write

$$\begin{aligned} L_{n,s,q}(e_m;x) &= \left(\frac{[n-s]_q}{[n]_q}\right)^m B_{n-s,q}(e_m;x) \\ &\quad + \frac{q^{n-s}x}{[n]_q^m}\sum_{j=1}^{m}\binom{m}{j}[s]_q^j\sum_{k=0}^{n-s}[k]_q^{m-j}(1-(1-q)[k]_q)^{j-1}p_{n-s,k}(q;x). \end{aligned}$$

With the use of the binomial theorem,

$$\begin{aligned} L_{n,s,q}(e_m;x) &= \left(\frac{[n-s]_q}{[n]_q}\right)^m B_{n-s,q}(e_m;x) \\ &\quad + \frac{q^{n-s}x}{[n]_q^m}\sum_{j=1}^{m}\binom{m}{j}[s]_q^j\sum_{i=0}^{j-1}\binom{j-1}{i}(-1)^i(1-q)^i\sum_{k=0}^{n-s}[k]_q^{m-j+i}p_{n-s,k}(q;x) \end{aligned}$$

$$= \left(\frac{[n-s]_q}{[n]_q}\right)^m B_{n-s,q}(e_m;x) + \frac{q^{n-s}x}{[n]_q^m}\sum_{j=1}^{m}\binom{m}{j}[s]_q^j$$
$$\times \sum_{i=0}^{j-1}\binom{j-1}{i}(-1)^i(1-q)^i[n-s]_q^{m-j+i}B_{n-s,q}(e_{m-j+i};x).$$

Employing (5), one obtains

$$L_{n,s,q}(e_m;x) = \frac{1}{[n]_q^m}\sum_{r=0}^{m}[n-s]_q^r S_q(m,r)\lambda_{r,q}^{(n-s)}x^r$$
$$+\sum_{j=1}^{m}\sum_{i=0}^{j-1}\sum_{r=0}^{m-j+i}\binom{m}{j}\binom{j-1}{i}q^{n-s}(-1)^i(1-q)^i\frac{[n-s]_q^r[s]_q^j}{[n]_q^m}S_q(m-j+i,r)\lambda_{r,q}^{(n-s)}x^{r+1}.$$

Changing the order of triple sums leads to:

$$\sum_{j=1}^{m}\sum_{i=0}^{j-1}\sum_{r=0}^{m-j+i}A(j,i,r) = \sum_{r=0}^{m-1}\sum_{j=1}^{m-r}\sum_{i=0}^{j-1}A(j,i,r) + \sum_{r=1}^{m-1}\sum_{j=m-r+1}^{m}\sum_{i=r-m+j}^{j-1}A(j,i,r),$$

which allows us to write

$$L_{n,s,q}(e_m;x) = \frac{1}{[n]_q^m}\sum_{r=0}^{m}[n-s]_q^r S_q(m,r)\lambda_{r,q}^{(n-s)}x^r$$
$$+\sum_{r=0}^{m-1}\sum_{j=1}^{m-r}\sum_{i=0}^{j-1}A(j,i,r)x^{r+1} + \sum_{r=1}^{m-1}\sum_{j=m-r+1}^{m}\sum_{i=r-m+j}^{j-1}A(j,i,r)x^{r+1},$$

where $A(j,i,r)$ is given by (8). The first sum can be started from $r=1$ due to the equality $S_q(i,0)=0$ for $i>0$. In the last triple sum, an empty sum is obtained for $r=0$, so it can be started from zero. Additionally, if we make the shift of index $r\mapsto r-1$ in the second and third sums, we arrive at:

$$L_{n,s,q}(e_m;x) = \frac{1}{[n]_q^m}\sum_{r=1}^{m}[n-s]_q^r S_q(m,r)\lambda_{r,q}^{(n-s)}x^r$$
$$+\sum_{r=1}^{m}\sum_{j=1}^{m-r+1}\sum_{i=0}^{j-1}A(j,i,r-1)x^r + \sum_{r=1}^{m}\sum_{j=m-r+2}^{m}\sum_{i=r-1-m+j}^{j-1}A(j,i,r-1)x^r$$
$$=: \sum_{r=1}^{m}a_{n,s,q}(r,m)x^r,$$

where the coefficients $a_{n,s,q}(r,m)$ are as in (7). This completes the proof. □

Remark 1. *It should be noted that the expression for $m=1,2$ in (6) recovers the same result as the ones in [20, Proposition 2].*

3. Spectrum of the q-Stancu Operator

In this section, we will investigate the spectral properties of the q-Stancu operator, including its eigenvalues and associated eigenvectors. In the next theorem, we will prove that, similar to the q-Bernstein operators, the subsequent eigenvalues, excluding the first two, will be found as simple eigenvalues.

Theorem 2. *For all $0 < q < 1$, the q-Stancu operator owns $n-s+1$ eigenvalues $\xi_{m,q}^{(n,s)}$ expressed as*

$$\xi_{0,q}^{(n,s)} = \xi_{1,q}^{(n,s)} = 1,$$

$$\xi_{m,q}^{(n,s)} = \frac{[n-s]_q^{m-1}}{[n]_q^m}\left([n-s]_q - [m-1]_q + q^{n-s}[ms]_q\right)\lambda_{m-1,q}^{(n-s)}, \quad m = 2,3,\ldots,n-s.$$

Moreover, they obey the following order:

$$1 = \xi_{0,q}^{(n,s)} = \xi_{1,q}^{(n,s)} > \xi_{2,q}^{(n,s)} > \xi_{3,q}^{(n,s)} > \ldots > \xi_{n-s,q}^{(n,s)} > 0.$$

Proof. The polynomial $L_{n,s,q}(e_m;x)$ can be written as

$$L_{n,s,q}(e_m;x) = \xi_{m,q}^{(n,s)} x^m + P_{m-1}(x), \tag{9}$$

where

$$\xi_{m,q}^{(n,s)} = a_{n,s,q}(m,m) = \frac{[n-s]_q^{m-1}}{[n]_q^m}\left([n-s]_q - [m-1]_q + q^{n-s}[ms]_q\right)\lambda_{m-1,q}^{(n-s)},$$

and P_{m-1} is a polynomial of degree at most $m-1$.

By (9), the matrix representation of $L_{n,s,q}$ in the standard basis $\{1, x, x^2, \ldots, x^n\}$ is an upper triangular matrix, whose diagonal entries are $\{\xi_{m,q}^{(n,s)}\}$. Therefore, the numbers $\{\xi_{m,q}^{(n,s)}\}$, $m = 0, \ldots, n-s$ are the eigenvalues of $L_{n,s,q}$.

Next, let us demonstrate that the sequence $\{\xi_{m,q}^{(n,s)}\}_{m\geqslant 1}$ is monotonically decreasing. Obviously,

$$\begin{aligned}
\frac{\xi_{m+1,q}^{(n,s)}}{\xi_{m,q}^{(n,s)}} &= \frac{\frac{[n-s]_q^m}{[n]_q^{m+1}}\left([n-s]_q - [m]_q + q^{n-s}[(m+1)s]_q\right)\lambda_{m,q}^{(n-s)}}{\frac{[n-s]_q^{m-1}}{[n]_q^m}\left([n-s]_q - [m-1]_q + q^{n-s}[ms]_q\right)\lambda_{m-1,q}^{(n-s)}} \\
&= \frac{[n-s]_q}{[n]_q}\left(1 - \frac{[m-1]_q}{[n-s]_q}\right)\frac{[n-s]_q - [m]_q + q^{n-s}[(m+1)s]_q}{[n-s]_q - [m-1]_q + q^{n-s}[ms]_q} \\
&= \frac{[n-s]_q - [m-1]_q}{[n]_q} \cdot \frac{[n-s]_q - [m]_q + q^{n-s}[(m+1)s]_q}{[n-s]_q - [m-1]_q + q^{n-s}[ms]_q} \\
&= \frac{[n-s]_q - [m-1]_q}{[n]_q} \cdot \frac{[n+ms]_q - [m]_q}{[n+ms-s]_q - [m-1]_q} \\
&= \frac{q^{m-1} - q^{n-s}}{1-q^n} \cdot \frac{q^m - q^{n+ms}}{q^{m-1} - q^{n+ms-s}}.
\end{aligned}$$

In order to prove that $\xi_{m,q}^{(n,s)}$ is monotonically decreasing, one needs to show

$$\frac{\xi_{m+1,q}^{(n,s)}}{\xi_{m,q}^{(n,s)}} = \frac{q^{m-1}-q^{n-s}}{1-q^n} \cdot \frac{q^m - q^{n+ms}}{q^{m-1}-q^{n+ms-s}} < 1,$$

which means that

$$\begin{aligned}(1-q^n)&(q^{m-1}-q^{n+ms-s}) - (q^{m-1}-q^{n-s})(q^m-q^{n+ms})\\ &= q^{m-1} - q^{n+ms-s} - q^{n+m-1} - q^{2m-1} + q^{n+ms+m-1} + q^{n+m-s} > 0.\end{aligned}$$

Dividing both sides by q^{m-1}, the latter inequality takes the form

$$1 - q^{n+ms-s-m+1} - q^n - q^m + q^{n+ms} + q^{n-s+1} > 0.$$

Adding and subtracting q^{n+m} on the left side of the inequality and making some simplifications, one gets

$$\begin{aligned}&1 - q^{n+ms-s-m+1} - q^n - q^m + q^{n+ms} + q^{n-s+1} + q^{n+m} - q^{n+m} > 0\\ \Leftrightarrow&(1-q^n) - q^m(1-q^n) + q^{n+m}(-q^{ms-s-2m+1} + q^{ms-m} + q^{-m-s+1} - 1) > 0\\ \Leftrightarrow&(1-q^n)(1-q^m) + q^{n+m}(q^{ms-m}(1-q^{-m-s+1}) - (1-q^{-m-s+1}) > 0\\ \Leftrightarrow&(1-q^n)(1-q^m) + q^{n+m}(q^{ms-m}-1)(1-q^{-m-s+1}) > 0\\ \Leftrightarrow&(1-q^n)(1-q^m) + q^{n-s+1}(1-q^{ms-m})(1-q^{m+s-1}) > 0,\end{aligned}$$

which yields that, for $s \geqslant 1$ and $m \geqslant 1$, the sequence $\{\xi_{m,q}^{(n,s)}\}_{m\geqslant 1}$ is decreasing, implying that the numbers $\xi_{m,q}^{(n,s)}$, $m = 1, \ldots, n-s$ are distinct. □

Remark 2. *It is worth mentioning that* $\{\xi_{m,1}^{(n,s)}\}_{m=0}^{n-s}$ *are the eigenvalues of the classical Stancu operator found in [17, Theorem 5.1]. Additionally, when* $s = 0$ *or* $s = 1$*, we obtain the eigenvalues of the q-Bernstein operator defined by Phillips [15], and, accordingly, when q equals 1, we recover the eigenvalues of the classical Bernstein operator given in [4].*

Theorem 3. *For* $n \in \mathbb{N}$ *and* $m = 0, 1, \ldots, n-s$*, the monic polynomials* $\varphi_m^{(n,s)}(q;x)$*, which are the eigenfunctions of* $L_{n,s,q}(f;x)$ *associated with the eigenvalues* $\xi_{m,q}^{(n,s)}$*, are given by*

$$\varphi_m^{(n,s)}(q;x) = \sum_{u=0}^{m} d_{n,s,q}(u,m)x^u,$$

where $d_{n,s,q}(m,m) = 1$ *and* $\varphi_0^{(n,s)}(q;x) = 1$, $\varphi_1^{(n,s)}(q;x) = x$*, while for* $m > 1$ *and* $v = 1, 2, \ldots, m$*,*

$$d_{n,s,q}(m-v,m) = \frac{1}{\xi_{m,q}^{(n,s)} - \xi_{m-v,q}^{(n,s)}} \sum_{u=0}^{v-1} d_{n,s,q}(m-u,m)a_{n,s,q}(m-v,m-u).$$

Proof. Consider the monic eigenfunctions of $L_{n,s,q}(f;x)$:

$$\varphi_m^{(n,s)}(q;x) = \sum_{u=0}^{m} d_{n,s,q}(u,m)x^u, \quad d_{n,s,q}(m,m) := 1, \tag{10}$$

corresponding to the eigenvalue $\xi_{m,q}^{(n,s)}$. Then,

$$L_{n,s,q}(\varphi_m^{(n,s)}(q;x);x) = \xi_{m,q}^{(n,s)}\varphi_m^{(n,s)}(q;x). \tag{11}$$

Taking expression (10) into account, (11) can be written as

$$\begin{aligned}\xi_{m,q}^{(n,s)}\sum_{v=0}^{m} d_{n,s,q}(v,m)x^v &= \sum_{u=0}^{m} d_{n,s,q}(u,m)L_{n,s,q}(t^u;x)\\ &= \sum_{u=0}^{m} d_{n,s,q}(u,m)\sum_{v=1}^{u} a_{n,s,q}(v,u)x^v = \sum_{v=1}^{m}\sum_{u=v}^{m} d_{n,s,q}(u,m)a_{n,s,q}(v,u)x^v.\end{aligned}$$

Comparing the coefficient of x^s in both sides results in

$$\xi_{m,q}^{(n,s)} d_{n,s,q}(v,m) = \sum_{u=v}^{m} d_{n,s,q}(u,m)a_{n,s,q}(v,u).$$

Substituting v with $m-v$ and u with $m-u$ leads to

$$\xi_{m,q}^{(n,s)} d_{n,s,q}(m-v,m) = \sum_{u=0}^{v} d_{n,s,q}(m-u,m)a_{n,s,q}(m-v,m-u),$$

resulting

$$d_{n,s,q}(m-v,m) = \frac{1}{\xi_{m,q}^{(n,s)} - \xi_{m-v,q}^{(n,s)}}\sum_{u=0}^{v-1} d_{n,s,q}(m-u,m)a_{n,s,q}(m-v,m-u),$$

which completes the proof. □

As an application of this theorem, the following result on the convergence of the iterates can be stated.

Corollary 1. *Let $0<q<1$, $f \in C[0,1]$ and $L_{n,s,q}^m$ stand for the m-th iterate of $L_{n,s,q}$, which is defined by $L_{n,s,q}^1(f;x) = L_{n,s,q}(f;x)$,*

$$L_{n,s,q}^m(f;x) = L_{n,s,q}\left(L_{n,s,q}^{m-1}(f;x)\right), \quad m = 2,3,\dots$$

Then, for fixed n and s,

$$\lim_{m\to\infty} L_{n,s,q}^m(f;x) = f(0)(1-x) + f(1)x$$

and the convergence is uniform on $[0,1]$.

4. Numerical Examples

In this part, we will present the visual representation of the eigenfunctions $\varphi_m^{(n,s)}(q;x)$ for some specific parameter values. Figure 1 illustrates the eigenfunctions $\varphi_m^{(9,3)}(q;x)$ for $m = 0, 1, \dots, 6$ normalized to establish a uniform norm 1. Figure 2 shows how the eigenfunctions $\varphi_3^{(n,4)}(q;x)$ behave as the parameter n varies, whereas Figure 3 displays the eigenfunctions $\varphi_5^{(15,s)}(q;x)$ for different values of s. In Figure 4, while keeping all parameters fixed except for q, the eigenfunctions $\varphi_3^{(10,4)}(q;x)$ are demonstrated with respect to the varying values of q.

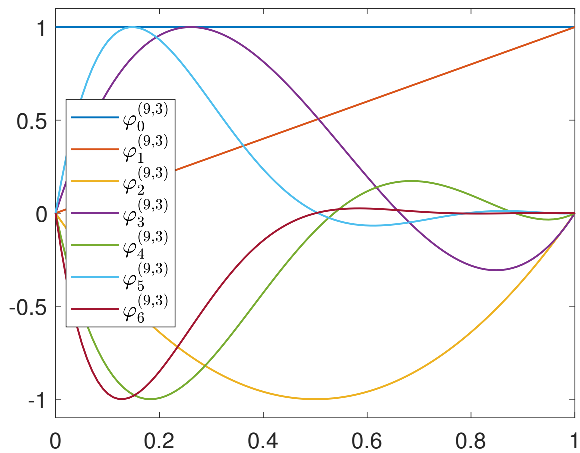

Figure 1. The normalized eigenfunctions of $L_{9,3,q}$ for $q = 0.5$.

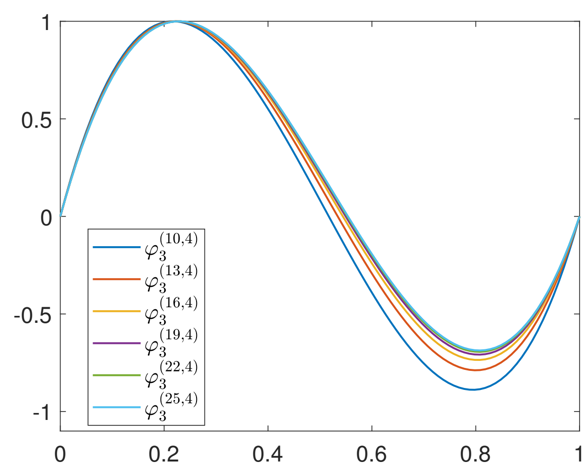

FIGURE 2. The eigenfunctions $\varphi_3^{(n,4)}(q;x)$ for different values of n and $q = 0.8$.

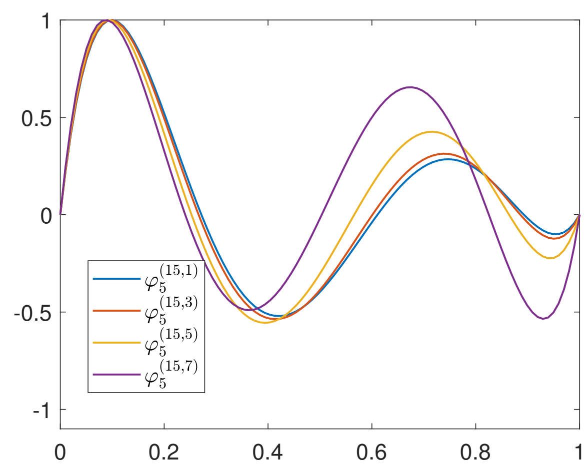

FIGURE 3. The eigenfunctions $\varphi_5^{(15,s)}(q;x)$ for different values of s and $q = 0.8$.

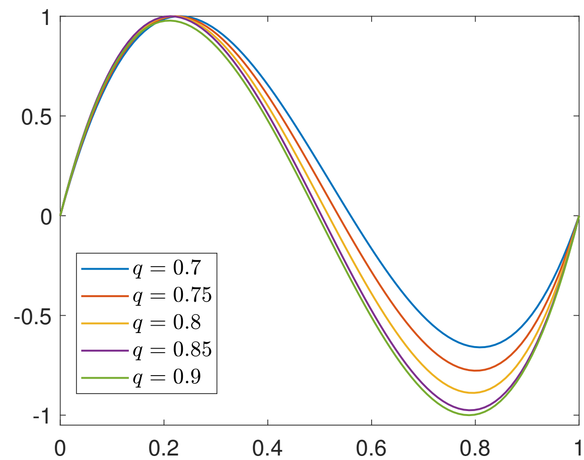

FIGURE 4. The eigenfunctions $\varphi_3^{(10,4)}(q; x)$ for different values of q.

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MATHEMATICAL ANALYSIS AND NUMERICAL SIMULATIONS FOR A NONLINEAR KLEIN GORDON EQUATION IN AN EXTERIOR DOMAIN

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Abstract. In this study, the finite propogation speed properties investigated for a two dimensional exterior problem defined by nonlinear Klein-Gordon equation. Under some assumptions on the initial data and the nonlinearity, the solution is shown to have a finite propogation speed. Furthermore, it is demonstrated that the problem has a unique solution, and accurate numerical solutions have been produced by the use of the dual reciprocity boundary element approach with linear radial basis functions.

1. Introduction

Nonlinear wave equations are used to describe various physical problems, including free surface problems, fields generated at the speed of light, large amplitude problems. The nonlinearity may originate from the material constitutive relations, from the large amplitude of the motion, or from the presence of a free boundary [1,2]. In most cases, nonlinear exterior wave problems are difficult to analyze theoretically and computationally, since there is an added nonlinearity, the problem is time dependent, the domain is unbounded and periodic waves are not possible.

It is known that solutions of the nonhomogenous linear wave equations have quite different behaviour from solutions of parabolic equations since the energy of a pure wave equation is constant and the initial data are transported with finite velocity [3].

For the linear wave equation as well as for the general class of linear hyperbolic equations in [4] it has been shown that any disturbance originated outside the light

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cone with a fixed vertex $(\mathbf{x_0}, \mathbf{t_0})$ has no effect on the solution within the cone and consequently has finite propogation speed. The theoretical and computational analyses of nonlinear wave problems are usually complicated, since there is an added nonlinearity, the problem is time dependent, the domain is unbounded and periodic waves are not possible. The trial equation method has been used to find exact solutions of a nonlinear Klein Gordon equation [5]. An energy decay estimate has been derived in [6] and asymptotic behaviour of the energy for periodic solutions has been studied for a particular semi-linear wave equation,namely damped Klein Gordon equation in [7]. The longtime behaviour of a nonlinear exterior wave problem has been studied in [8]. The existence of a global solution has been studied for exponential type nonlinearity and for a Cauchy problem with small data in [9] and [10], respectively. On the other hand, local energy decay properties have been studied for the dissipative exterior Klein-Gordon equation [11]

It is known that the theoretical solutions are not easy to obtain when the equation is nonlinear and particularly if the problem domain is unbounded. The radial basis functions [12], Taylor matrix method [13], finite difference method(FDM) [14, 15] have been used for the solution of the problems defined by nonlinear Klein Gordon equations. One can find differential quadrature solution for the 2-D IBVP in [16] and artificial boundary method has been applied for the initial value problem (IVP) defined by a coupled nonlinear Klein Gordon equations [17].

Most numerical methods such as the finite element method(FEM), FDM and the differential quadrature method (DQM) have some difficulties for unbounded regions, since they need to discretize the domain itself.

A FEM based method, Dirichlet to Neumann FEM (DNFEM) is a general method for the solution of problems in unbounded domains. DNFEM method constructs an equaivalent problem by introducing an artificial boundary and a map is derived between the original domain and the and the artificial boundary. A detailed review on the method can be found in [18]. Later, another alternative numerical method dual reciprocity boundary element method (DRBEM), which has the advantage of discretizing only the boundary of the region, has been applied to the same problem in [19]. However for the unbounded domains one should be careful with the selection of the approximating radial basis functions (RBF), unless the problem is not guaranteed that the solution vanishes far away from the time-space cylinder.

In this paper, an IBVP which has significant applications in quantum physics and defined by a Klein Gordon equation (Section 2)) has been considered. In Sections 3 and 4 the IBVP has been shown to have finite propogation speed under some assumptions on the initial data and the solution has also shown to be unique. Unlike [19], one has the advantage of having freedom in the selection of the approximating RBF, since it is guaranteed by the mathematical analysis. For the numerical solution procedure (Section 5), DRBEM is used with linear RBF. The numerical results have been seen to be consistent with the behaviour of the solution and a well agreement with previously given reference solution [20] has been obtained in terms of absolute maximum error.

2. The Problem Definition

In this paper, the finite propogation properties, the uniqueness and the numerical solution of the IBVP

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = \phi(u) \quad \text{in} \quad \Omega \times (0,\infty) \tag{1}$$

$$u = g_1 \quad \text{on} \quad \Gamma_{g_1}, \quad \tfrac{\partial u}{\partial n} = g_2 \quad \text{on} \quad \Gamma_{g_2} \tag{2}$$

$$u(x,y,0) = u_0(x,y), \qquad \frac{\partial u}{\partial t}(x,y,0) = v_0(x,y) \tag{3}$$

are considered. In (1), c is a constant and $\phi\left(u\right)$ is a given function of the unknown u. In Equations (1)-(3), the infinite exterior problem domain Ω has an inner boundary $\Gamma = \Gamma_{g_1} \cup \Gamma_{g_2}$, n is inward unit normal, $g_1,\ g_2,\ u_0,\ v_0$ are given functions, u_0 and v_0 have compact support.

3. Domain of Dependence

In this section the domain of dependence of solutions to the nonlinear Klein-Gordon equation is examined. In order to prove finite propogation speed a curved 'cone-like' region C is found as in [4]. To this end, the boundary of C is estimated as a level set $\{\mathfrak{p} = 0\}$ where $\mathfrak{p}$ solves the Hamilton-Jacobi equation

$$\mathfrak{p}_t - c^2\left(\mathfrak{p}_x^2 + \mathfrak{p}_y^2\right)^{1/2} = 0 \qquad \text{in} \qquad \Omega \times (0,\infty)\,. \tag{4}$$

Separating the variables one can write

$$\mathfrak{p}\left(x,y,t\right) = \mathfrak{q}\left(x,y\right) + t - t_0 \qquad \left((x,y)\in\Omega, 0\le t\le t_0\right) \tag{5}$$

where $\mathfrak{q}$ solves

$$\begin{cases} c^2\left(\left(\mathfrak{q}_x\right)^2 + \left(\mathfrak{q}_y\right)^2\right) = 1 \quad \mathfrak{q}>0 \quad \text{in} \quad \Omega - \{x_0\} \\ \mathfrak{q}\left(x_0,y_0\right) = 0. \end{cases} \tag{6}$$

for a fixed $(x_0,y_0)\in\Omega$, $t_0>0$. Therefore it is assumed that $\mathfrak{q}$ is a smooth solution of (6) on $\Omega - \{(x_0,y_0)\}$. Now one can define C as,

$$C := \{(x,y,t)\,|\mathfrak{p}\left(x,y,t\right)<0\} = \{(x,y,t)\,|\mathfrak{q}\left(x,y\right)<t_0-t\} \subset \Omega\times(0,\infty)\,.$$

with the cross section of C_t of C for each $t>0$,

$$C_t := \{x|\mathfrak{q}\left(x,y\right)<t_0-t\}\,. \tag{7}$$

Moreover, the cone C is taken far enough such that both boundaries do not touch each other, i.e., $\bar{C}\cap\bar{\Omega^i}\times[0,T]=\emptyset$ where Ω^i is the interior domain bounded by Γ.

Theorem 1 (Finite Propogation Speed)**.** *Assume u is a smooth solution of Equation (1) with $\phi(u) = -mu^n$, $m > 0$, n is a positive odd integer. If $u \equiv u_t \equiv 0$ on C_0, then $u \equiv 0$ within the cone C.*

Proof. Defining the energy

$$e(t) := \frac{1}{2}\left\{\int_{C_t}\left(u_t^2 + c^2\left|\nabla u\right|^2\right)dx\right\} - \int_{C_t}\Phi\left(u\right)dx \tag{8}$$

where $\phi(u) = \frac{\partial \Phi}{\partial u}$, $\dot{e}\left(t\right)$ can be computed by making use of the Coarea formula, [4]

$$\begin{aligned}\dot{e}\left(t\right) &= \underbrace{\left[\int_{C_t}\left(u_t u_{tt} + c^2 \nabla u . \nabla u_t\right)dx\right]}_{A} \\ &- \underbrace{\left[\frac{1}{2}\int_{\partial C_t}\left(u_t^2 + c^2\left|\nabla u\right|^2\right)\frac{1}{\left|\nabla q\right|}dS + \int_{C_t}\left(\frac{d}{dt}\phi\left(u\right)\right)dx\right]}_{B} - \underbrace{\left[\int_{\partial C_t}\frac{\phi\left(u\right)}{\left|\nabla q\right|}dS\right]}_{C}\end{aligned} \tag{9}$$

Integration by parts in A yields,

$$A = \int_{C_t} u_t\left(u_{tt} - c^2\nabla^2 u\right)dx + c^2\int_{\partial C_t} u_t\left(\nabla u . \nu\right)dS \tag{10}$$

where $\nu = \frac{\nabla q}{\left|\nabla q\right|}$ is the outer normal to ∂C_t. Using (6), Cauchy-Schwarz and Cauchy inequalities and the fact that u is a solution of (1) one gets,

$$\left|A\right| \leq \int_{C_t} u_t\phi\left(u\right)dx + c^2\int_{\partial C_t}\left|u_t\right|\left|\nabla u\right|\frac{1}{\left|\nabla q\right|}dS \leq \int_{C_t}\frac{d}{dt}\Phi\left(u\right)dx + B \tag{11}$$

and thus

$$\dot{e}\left(t\right) \leq \int_{\partial C_t}\frac{\Phi\left(u\right)}{\left|\nabla q\right|}dS \leq 0 \tag{12}$$

since $\Phi(u) = -m\frac{u^{n+1}}{n+1}$ $(m, n > 0, n$ is an odd integer) Thus $e\left(t\right)$ is a nonincreasing function of t and hence,

$$e(t) \leq e(0) = 0 \quad \forall \;\; 0 \leq t \leq t_0.$$

On the other hand, by its definition (Equation (8)) $e(t)$ is nonnegative and therefore $u_t, \nabla u, u \equiv 0$ within the cone C.

□

4. Uniqueness of the Solution

In this section the aim is to show the uniqueness of the solution for (1-3) for some particular choice of the nonlinear funciton ϕ in equation 1.

Theorem 2 (Uniqueness)**.** *There exists at most one function $u \in L^2(\Omega)$ solving the initial and boundary value problem (1) -(3) with $\phi(u) = -mu^n, \quad m > 0, \quad n$ is a positive odd integer.*

Proof. To show uniqueness u and $\tilde{u}$ are assumed to be two different solutions of (1-3). If one considers the L^2 inner product of the functions $u_t - \tilde{u}_t$ then obtains $f(u) - f(\tilde{u})$

$$\begin{aligned}
\langle u_t - \tilde{u}_t, \phi(u) - \phi(\tilde{u})\rangle &= \int_\Omega (u_t - \tilde{u}_t)(\phi(u) - \phi(\tilde{u}))\, dx \\
&= \int_\Omega \left(\tfrac{1}{2}\tfrac{d}{dt}\left(u_t^2 - 2u_t.\tilde{u}_t + \tilde{u}_t^2\right)\right) \\
&+ \int_\Omega \left(\tfrac{1}{2}\tfrac{d}{dt}\left[(\nabla u)^2 - 2(\nabla u.\nabla\tilde{u}) + (\nabla\tilde{u})^2\right]\right) dx \\
&+ \int_\Gamma \left(\tfrac{\partial u}{\nabla\nu} - \tfrac{\partial\tilde{u}}{\nabla\nu}\right)(\tilde{u}_t - u_t)\, dS
\end{aligned} \tag{13}$$

In (13) integration by parts and the fact that u and $\tilde{u}$ are both solutions of the equation (1) are made use of. Also by the boundary conditions the last integral in (13) vanishes, since both solutions satisfy the same boundary condition. Therefore one finally has,

$$2(u_t - \tilde{u}_t, \phi(u) - \phi(\tilde{u})) = \tfrac{d}{dt}\|u_t - \tilde{u}_t\|^2_{L^2(\Omega)} + \tfrac{d}{dt}\|\nabla u - \nabla\tilde{u}\|^2_{L^2(\Omega).}$$

Using Cauchy's inequality with ϵ one gets,

$$\tfrac{d}{dt}\|u_t - \tilde{u}_t\|^2_{L^2(\Omega)} + \tfrac{d}{dt}\|\nabla u - \nabla\tilde{u}\|^2_{L^2(\Omega)} \leq \epsilon\|u_t - \tilde{u}_t\|^2_{L^2(\Omega)} + \tfrac{1}{\epsilon}\|\phi(u) - \phi(\tilde{u})\|^2_{L^2(\Omega).}$$

If one selects $\epsilon > 0$ sufficiently small, uses Poincare inequality and Lipschitz continuity of the polynomial $\phi(u)$ then these lead to

$$\frac{d}{dt}\|u - \tilde{u}\|_{L^2(\Omega)} \leq C\|u - \tilde{u}\|_{L^2(\Omega)}$$

Finally Gronwall's inequality gives

$$\|u - \tilde{u}\|_{L^2(\Omega)} \leq C\|u(0) - \tilde{u}(0)\|_{L^2(\Omega)}$$

which results with $u = \tilde{u}$.

□

5. Numerical Solution

In this section, the IBVP defined by (1)-(3) is solved approximately by using a combination of DRBEM and FDM is used as in [19].The DRBEM has the benefit of discretizing only the region's boundary, and it has been used to solve a variety of issues in a wide range of scientific fields, including fluid dynamics and medicine e.g., [22,23]. However here the nonlinearity function is chosen as given in Theorem 1, so that by the theory given in Section 3 , one is free for the selection of approximating RBF.

To see that, before the application of DRBEM, consider the family of cones

$$\begin{aligned} C_{i,T}(x_i) \quad = \quad & \left\{(x,t) \in \mathbb{R}^2 \times [0,T] \,|\, |x - x_i| \, \tfrac{1}{c} \leq t, \right. \\ & \left. 0 \leq t \leq T,\, x_i \in (supp\,(u_0) \cup supp\,(v_0))\right\}. \end{aligned}$$

By using these cones one can define the following domain

$$B := \cup C_{i,T}(x_i)$$

where, $B^C : \mathbb{R}^2 \times [0,T]/B$ and naturally the cones in Section 3 are included in B^C. By Theorem 1 in Section 3 it is obvious that for $t \in [0,T]$ all x with $(x,t) \in B^C$, u vanishes, i.e. $u\,(x,t) = 0$, since $x \notin supp\,(u_0) \cup supp\,(v_0)$.

To apply DRBEM, Equation(1) is multiplied by the fundamental solution of the Laplace equation (u^*) and integrated over Ω [21], i.e.,

$$\int_\Omega (c^2\nabla^2 u)u^* d\Omega = \int_\Omega \left(\frac{\partial^2 u}{\partial t^2} - \phi(u)\right) u^* d\Omega \tag{14}$$

Then, if for the left hand side of Equation 14 integration by parts is applied, one obtains

$$\begin{aligned} \int_\Omega \left(c^2\nabla^2 u\right) u^* d\Omega \quad &= \quad \int_{\Omega \cap B^T} \left(c^2\nabla^2 u\right) u^* d\Omega \\ &= \quad c^2 (d_i u_i + \int_\Gamma (\tfrac{\partial u^*}{\partial n} u - u^* \tfrac{\partial u}{\partial n}) d\Gamma) \end{aligned} \tag{15}$$

where $B^T := \{x \in \Omega |\, (x,T) \in B\}$, i denotes the source (fixed) point, and $d_i = \int_\Omega \Delta(x_i, y_i, x, y) dR$. Here the only integral coming from the boundary is coming from the boundary is the obstacle, namely Γ, because of the fact that the solution u vanishes in the region $\bar{B}^C$ identically.

The choice of approximating functions is not restricted, since for $u \in C^2\,(\Omega \times (0,T))$ the integrand of the right hand side,i.e. , $\left(\frac{\partial^2 u}{\partial t^2} - \phi(u)\right)$ vanishes outside B by Theorem 1 in Section 3 and the right hand side of (14) can be approximated using linear RBF $f = 1 + r$ with r being the distance function and considered as the modulus of a vector r_{kj} where for each boundary point k on the obstacle, j represents each of the other boundary and internal nodes. The approximation would be as follows:

$$\left(\frac{\partial^2 u}{\partial t^2} - \phi(u)\right) \approx \sum_{j=1}^{N+L} \alpha_j(t) f_j \tag{16}$$

with N and L being the number of discretization points on the inner boundary and inside the exterior domian, respectively. Choosing the RBF f_j s to be related to the other distance functions $\hat{u}_j(x, y)$ through the relation $\nabla^2 \hat{u}_j = f_j$ with the condition $\sum_{j=1}^{N+L} \alpha_j(t)\, \hat{u}_j \equiv 0$ in B_C^T one obtains,

$$\begin{aligned} \int_\Omega (\tfrac{\partial^2 u}{\partial t^2} - \phi(u)) u^* d\Omega &= \int_{B^T \cap \Omega} (\tfrac{\partial^2 u}{\partial t^2} - \phi(u)) u^* d\Omega \\ &= \int_{B^T \cap \Omega} \nabla^2 u^* \textstyle\sum_{j=1}^{N+L} \alpha_j(t) \hat{u}_j d\Omega \\ &+ \int_\Gamma \left(\tfrac{\partial}{\partial n} u^*\right) \textstyle\sum_{j=1}^{N+L} \alpha_j(t) \hat{u}_j d\Gamma \\ &- \int_\Gamma u^* \tfrac{\partial}{\partial n} \left(\textstyle\sum_{j=1}^{N+L} \alpha_j(t) \hat{u}_j\right) d\Gamma \end{aligned} \tag{17}$$

Here there is no boundary integral coming from the boundary $B_T \cap \Omega$, since $\sum_{j=1}^{N+L} \alpha_j(t)\, \hat{u}_j \equiv 0$ on the boundary of $B^T \cap \Omega$ by the assumption.

Combining (15) and (17) yields,

$$\begin{aligned} & c^2 \left(c_i u_i + \int_\Gamma \left(\tfrac{\partial u^*}{\partial n} u - u^* q\right) d\Gamma\right) = \\ & \textstyle\sum_{j=1}^{N+L} \alpha_j(t) \left[c_i \hat{u}_{ij} + \int_\Gamma \left(\tfrac{\partial u^*}{\partial n} \hat{u}_j - u^* \hat{q}_j\right) d\Gamma\right] \quad . \end{aligned} \tag{18}$$

where $\hat{q}_j = \frac{\partial \hat{u}_j}{\partial n}$.

Finally, rewriting Equation (18) in matrix vector form, one gets a system of ordinary differential equations of size $(N + L) \times (N + L)$ as;

$$c^2 (\mathbf{Hu} - \mathbf{Gq}) \quad = \quad \mathbf{S} (\ddot{\mathbf{u}} - \boldsymbol{\phi}) \tag{19}$$

where $\mathbf{S} = \left(\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{Q}}\right) \mathbf{F}^{-1}$ with G and H being the matrices with entries containing the fundamental solution of Laplace equation and its normal derivative, respectively. Furthermore, each coloumn of the matrices F , $\hat{U}$ and $\hat{Q}$ consists of the vectors of approximating functions f_j, particular solutions $\hat{u}_j$ and $\hat{q}_j$, respectively. Note that, $\ddot{\mathbf{u}}$ is the $(N+L)\times 1$ vector containing second order time derivative at discretization points.

In order to approximate the solution at the discretization points at a time T, the time interval $[0, T]$ is divided into K with an equal time step size of Δt. The second order time derivative in (19) is discretized by using the central difference scheme having $O(\Delta t^2)$, i.e.,

$$\ddot{\mathbf{u}} = \frac{1}{\Delta t^2}\left(\mathbf{u}^{k+1} - 2\mathbf{u}^k + \mathbf{u}^{k-1}\right) \quad \text{for} \quad k = 0, 1, 2, \cdots N \tag{20}$$

with k denoting the time level. In order to obtain the DRBEM solution for the first time level, the initial conditions (3) are made use of. To this end, the first order derivative in (3) is discretized by using the $O(\Delta t)$ backward difference scheme which gives $\mathbf{u}^{-1}$ at the discretization points as,

$$\mathbf{u}^{-1} = \mathbf{u}^0 - \Delta t \mathbf{v}^0 = \mathbf{u}_0 - \Delta t \mathbf{v}_0 \tag{21}$$

where $\mathbf{u}_0$ and $\mathbf{v}_0$ contains the values of the initial conditions in (3) at the discretization points.

In order to overcome the stability problems a relaxation procedure is applied for the unknown u as

$$\mathbf{u} = (1-\beta)\mathbf{u}^k + \beta \mathbf{u}^{k+1} \tag{22}$$

positioning the values of $\mathbf{u}$ between the time levels k and $k+1$. Together with the relaxation procedure (22) the unknown $\mathbf{u}$ is obtained at the discretization points by solving the system of linear algebraic equations

$$\mathbf{A_1 u^1} = \mathbf{A_2 u^0} + \mathbf{\Delta t^2} \boldsymbol{\phi}\left(\mathbf{u^0}\right) + \mathbf{A_3 q^1} + \mathbf{\Delta t v^0} \tag{23}$$

$$\mathbf{B_1 u^{k+1}} = \mathbf{B_2 u^k} + \mathbf{\Delta t^2} \boldsymbol{\phi}\left(\mathbf{u^k}\right) + \mathbf{A_3 q^{k+1}} - \mathbf{u^{k-1}} \tag{24}$$

at the first and $(k+1)$-st time levels, respectively. Note that the matrices in equations (23) and (24) are given as

$$\begin{gathered} \mathbf{A_1} = \left((1-\beta)\,\mathbf{I} + \Delta t^2 \beta \bar{\mathbf{H}}\right), \\ \mathbf{A_2} = \left((1-\beta)\, I - \Delta t^2 \,(1-\beta)\, \bar{\mathbf{H}}\right), \quad \mathbf{A_3} = \Delta t^2 \bar{\mathbf{G}} \\ \mathbf{B_1} = (1-2\beta)\, I + \Delta t^2 \beta \bar{\mathbf{H}}, \\ \mathbf{B_2} = 2\,(1-\beta)\,\mathbf{I} - \Delta t^2\,(1-\beta)\,\bar{\mathbf{H}} \end{gathered}$$

with

$$\bar{\mathbf{H}} = -c^2 \mathbf{S}^{-1} \mathbf{H}, \quad \bar{\mathbf{G}} = -c^2 \mathbf{S}^{-1} \mathbf{G}.$$

Example: Computations have been carried out for the initial and boundary value problem (1)-(3) with $\phi(u) = -mu^3$ and $c = 200$. The region is taken as the exterior of a circle with radius $a = 0.25$. The initial conditions u_0 and v_0 are taken to be 0 and the solution is assumed to be 1 on the interior boundary. A reference solution [20] is used to compare the numerical results with the technique described here. In the calculations, $N = 120$ constant boundary elements and $L = 100$ interior points are used, and the time step is taken as 0.01. In order to compare the results with the reference solution (denoted by u_{ref}) in [20], the interior points are taken along a portion (between $a = 0.25$ and $R = 0.5$) of a straight line radiating from the origin of the circle.

In Tables 1 and 2, both of the solutions, obtained here(u_{DRBEM}) and the reference solution [20] (u_{ref}), are presented, for different times at the point $R = 0.5$ with $\phi(u) = -mu^3$ for $m = 0.0$, $m = 10000$, respectively. In the third row of the table the reference and DRBEM solutions are compared in terms of absolute relative error τ which is calculated by

$$\tau = \left| \frac{u_{ref} - u_{DRBEM}}{u_{ref}} \right|$$

One can observe that the DRBEM solution with linear RBF are accurate almost with 4 significant digits for both linear and nonlinear cases.

In the Figures 1 and 2, the reference solution and the DRBEM solution are illustrated at different times for the linear and nonlinear ($m = 10000$) cases; respectively. One can see that DRBEM solution agrees well with the reference solution.

TABLE 1. $R = 0.5, m = 0$

t	0.02	0.05	0.08	0.1
u_{ref}	0.9441	0.8174	0.8673	0.8598
u_{DRBEM}	0.9448	0.8176	0.8677	0.8593
τ	7×10^{-4}	2×10^{-4}	4×10^{-4}	5×10^{-4}

TABLE 2. $R = 0.5, m = 10000$

t	0.02	0.05	0.08	0.1
u_{ref}	0.9164	0.7509	0.7867	0.7736
u_{DRBEM}	0.9165	0.7511	0.7861	0.7742
τ	1×10^{-4}	2×10^{-4}	6×10^{-4}	6×10^{-4}

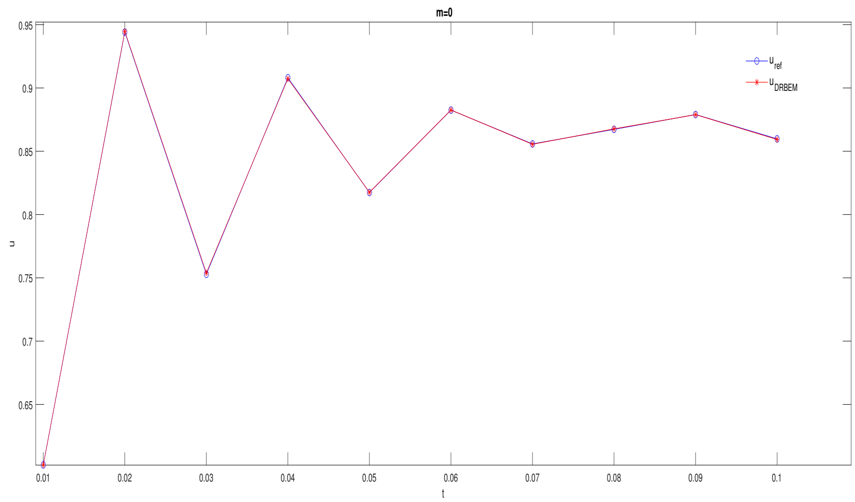

FIGURE 1. Reference and DRBEM Solutions for $m = 0$ at different times

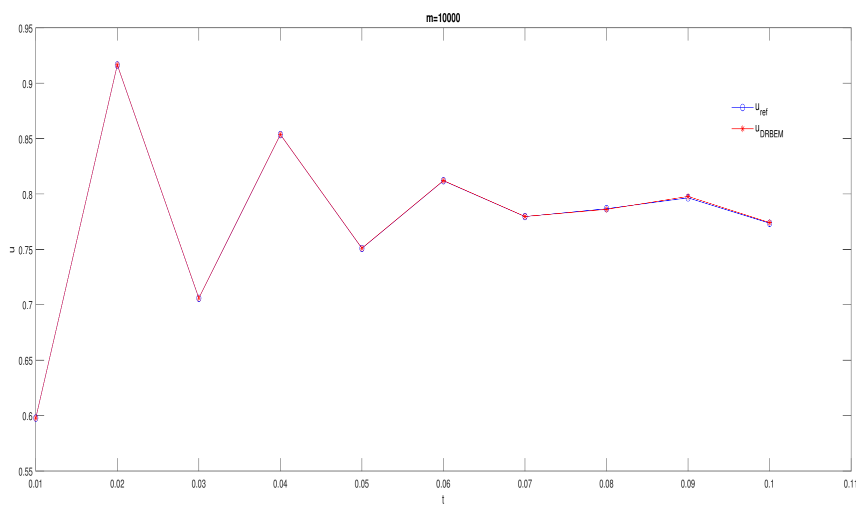

FIGURE 2. Reference and DRBEM Solutions for $m = 10000$ at different times

6. Conclusion

In this paper, finite propogation speed properties are shown for a nonlinear two-dimensional exterior Klein Gordon problem. The solution of the problem is shown to be unique. For the numerical solution of the corresponding problem DRBEM is used and the nonhomogenity is approximated with the help of linear RBF which is known to have some difficulties when the domain is an exterior one. However, the theory in Section 3 shows that under certain conditions on the nonlinearity and initial conditions the solution is vanishing far away from the time-space cyclinder and thus the approximation by the RBF is taken only within the finite region of integration. The numerical results show good agreement with a previously obtained reference solution in terms of absolute relative error.

Declaration of Competing Interests This work does not have any conflict of interest.

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NOTES ON THE GEOMETRY OF COTANGENT BUNDLE AND UNIT COTANGENT SPHERE BUNDLE

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Abstract. Let $(\mathtt{N}, \mathfrak{g})$ be a Riemannian manifold, by using the musical isomorphisms ♪ and ♮ induced by $\mathfrak{g}$, we built a bridge between the geometry of the tangent bundle $\mathtt{TN}$ (resp. the unit tangent sphere bundle $\mathtt{T}_1\mathtt{N}$) equipped with the Sasaki metric $\mathfrak{g}_S$ (resp. the induced Sasaki metric $\bar{\mathfrak{g}}_S$) and that of the cotangent bundle $\mathtt{T}^*\mathtt{N}$ (resp. the unit cotangent sphere bundle $\mathtt{T}^*_1\mathtt{N}$) endowed with the Sasaki metric $\mathfrak{g}_{\widetilde{S}}$ (resp. the induced Sasaki metric $\tilde{\mathfrak{g}}_{\widetilde{S}}$). Moreover, we prove that $\mathtt{T}^*_1\mathtt{N}$ carries a contact metric structure and study some of its properties.

1. Introduction

The geometry of tangent bundles of differentiable manifolds is of particular interest in different areas of mathematics and physics. The research in this domain began in 1958, with a very fundamental paper by Sasaki [16]. He constructed a Riemannian metric $\mathfrak{g}_S$ (called the Sasaki metric) on the tangent bundle $\mathtt{TN}$ of a Riemannian manifold $(\mathtt{N}, \mathfrak{g})$, which depends on the metric $\mathfrak{g}$. Since then, the geometry of $(\mathtt{TN}, \mathfrak{g}_S)$ or the (unit) tangent sphere bundle $\mathtt{T}_1\mathtt{N}$ endowed with the induced Sasaki metric $\bar{\mathfrak{g}}_S$ has acquired extensive literature; see, for instance, [4,5,9,11,12] and the survey [18].

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On the other hand, the geometry of cotangent bundles of differentiable manifolds developed in parallel with that of tangent bundles, as can be seen in [18]. In classical mechanics, the cotangent bundle can be viewed as the phase space. Akbulut, Özdemir and Salimov [1] defined the Sasaki metric analogue $\mathfrak{g}_{\widetilde{S}}$ on the cotangent bundle $\mathtt{T}^*\mathtt{N}$ of $(\mathtt{N}, \mathfrak{g})$ and studied some of its properties. Afterwards, Salimov and Agca [15] gave some of its curvature properties. In [10], the authors proved that the musical isomorphisms ♪ and ♮ induced by Riemannian metric $\mathfrak{g}$ are isometries between $(\mathtt{TN}, \mathfrak{g}_S)$ and $(\mathtt{T}^*\mathtt{N}, \mathfrak{g}_{\widetilde{S}})$.

In this paper, we shed more light on the geometry of cotangent bundle (resp. unit cotangent sphere bundle). Firstly, by using ♪ and ♮ we studied the relationship between the geometry of $(\mathtt{T}^*\mathtt{N}, \mathfrak{g}_{\widetilde{S}})$ and that of $(\mathtt{TN}, \mathfrak{g}_S)$ and vice versa, and this improved the results of [15]. Secondly, after we defined the unit cotangent sphere bundle $\mathtt{T}_1^*\mathtt{N}$ of $(\mathtt{N}, \mathfrak{g})$ and endowed it by the induced Sasaki metric $\tilde{\mathfrak{g}}_{\widetilde{S}}$, we showed that ♪ and ♮ induced by Riemannian metric $\mathfrak{g}$ are isometries between $(\mathtt{T}_1\mathtt{N}, \bar{\mathfrak{g}}_S)$ and $(\mathtt{T}_1^*\mathtt{N}, \tilde{\mathfrak{g}}_{\widetilde{S}})$, which allowed us to deduce the geometric properties of $(\mathtt{T}_1^*\mathtt{N}, \tilde{\mathfrak{g}}_{\widetilde{S}})$ from those of $(\mathtt{T}_1\mathtt{N}, \bar{\mathfrak{g}}_S)$. Finally, like the unit tangent sphere bundle of $(\mathtt{N}, \mathfrak{g})$, we showed that the unit cotangent sphere bundle of $(\mathtt{N}, \mathfrak{g})$ carries a contact metric structure and studied some of its properties.

2. Some Results on the Geometry of Cotangent Bundle

First, we start with a brief review on the geometry of tangent and cotangent bundles following [7, 11, 18]. Let $(\mathtt{N}, \mathfrak{g})$ be an n-dimensional Riemannian manifold, ∇ its Levi-Civita connection and $(\mathtt{TN}, \pi, \mathtt{N})$ (resp. $(\mathtt{T}^*\mathtt{N}, \widetilde{\pi}, \mathtt{N})$) be its tangent bundle (resp. its cotangent bundle). The tangent space $\mathtt{T}_\mathfrak{p}\mathtt{TN}$ (resp. $\mathtt{T}_\mathfrak{q}\mathtt{T}^*\mathtt{N}$) at a point $\mathfrak{p} = (p, \mathtt{v})$ in $\mathtt{TN}$ (resp. at a point $\mathfrak{q} = (p, \mathfrak{v})$ in $\mathtt{T}^*\mathtt{N}$) splits into the direct sum of the vertical subspace $\mathcal{V}_\mathfrak{p} = \ker(d\pi\ |_\mathfrak{p})$ (resp. $\widetilde{\mathcal{V}}_\mathfrak{q} = \ker(d\widetilde{\pi}\ |_\mathfrak{q})$) and the horizontal subspace $\mathcal{H}_\mathfrak{p}$ (resp. $\widetilde{\mathcal{H}}_\mathfrak{q}$), with respect to ∇: $\mathtt{T}_\mathfrak{p}\mathtt{TN} = \mathcal{H}_\mathfrak{p} \oplus \mathcal{V}_\mathfrak{p}$ (resp. $\mathtt{T}_\mathfrak{q}\mathtt{T}^*\mathtt{N} = \widetilde{\mathcal{H}}_\mathfrak{q} \oplus \widetilde{\mathcal{V}}_\mathfrak{q}$).

The Sasaki metric $\mathfrak{g}_S$ on $\mathtt{TN}$ is defined for any $\Upsilon_1, \Upsilon_2 \in \Gamma(\mathtt{TN})$ by

$$\mathfrak{g}_S(\Upsilon_1^h, \Upsilon_2^h) = \mathfrak{g}_S(\Upsilon_1^v, \Upsilon_2^v) = \mathfrak{g}(\Upsilon_1, \Upsilon_2) \circ \pi,\ \mathfrak{g}_S(\Upsilon_1^v, \Upsilon_2^h) = 0,$$

where Υ_1^h and Υ_1^v are the horizontal and the vertical lifts of Υ_1 respectively. It is well known from [7] that $(\mathtt{TN}, \mathfrak{g}_S, J)$ is an almost Hermitian manifold, where the structure J is defined by

$$\begin{cases} J(\Upsilon_1^h) = \Upsilon_1^v\ , \\ J(\Upsilon_1^v) = -\Upsilon_1^h\ , \end{cases} \tag{1}$$

for any $\Upsilon_1 \in \Gamma(\mathtt{TN})$. Furthermore, J defines an almost Kählerian structure. It is a Kählerian manifold if and only if $(\mathtt{N}, \mathfrak{g})$ is flat.

On the other hand, the Sasaki metric $\mathfrak{g}_{\widetilde{S}}$ on $\mathtt{T}^*\mathtt{N}$ is defined for any $\Upsilon_1, \Upsilon_2 \in \Gamma(\mathtt{TN})$ and any $\omega, \theta \in \Gamma(\mathtt{T}^*\mathtt{N})$ by:

$$\mathfrak{g}_{\widetilde{S}}(\omega^{\widetilde{v}}, \theta^{\widetilde{v}}) = \mathfrak{g}^{-1}(\omega, \theta) \circ \widetilde{\pi},\ \mathfrak{g}_{\widetilde{S}}(\Upsilon_1^{\widetilde{h}}, \Upsilon_2^{\widetilde{h}}) = \mathfrak{g}(\Upsilon_1, \Upsilon_2) \circ \widetilde{\pi},\ \mathfrak{g}_{\widetilde{S}}(\omega^{\widetilde{v}}, \Upsilon_2^{\widetilde{h}}) = 0,$$

where $\Upsilon_1^{\widetilde{h}}$ and $\omega^{\widetilde{v}}$ are the horizontal lift of Υ_1 and the vertical lift of ω respectively. Here, $\mathfrak{g}^{-1}(\omega,\theta)=\mathfrak{g}(\natural\omega,\natural\theta)$, in which $\natural:\omega\mapsto\natural(\omega)$, such that $\natural(\omega)$ is the vector field on $\mathtt{N}$ defined by $\mathfrak{g}(\natural(\omega),\Upsilon_2)=\omega(\Upsilon_2)$. Note that the musical isomorphisms $\natural$ and $♪:\Upsilon_1\mapsto ♪(\Upsilon_1)=\mathfrak{g}(\Upsilon_1,.)$, define a bundle isomorphism between the tangent and the cotangent bundle of $\mathtt{N}$; moreover, $♪$ and $\natural$ are isometries between $(\mathtt{TN},\mathfrak{g}_S)$ and $(\mathtt{T^*N},\mathfrak{g}_{\widetilde{S}})$ [10]. Further we have

$$♪_*(\Upsilon_1^v)=(♪\Upsilon_1)^{\widetilde{v}},\ ♪_*(\Upsilon_1^h)=\Upsilon_1^{\widetilde{h}}, \tag{2}$$

and

$$\natural_*(\omega^{\widetilde{v}})=(\natural\omega)^v,\ \natural_*(\Upsilon_1^{\widetilde{h}})=\Upsilon_1^h. \tag{3}$$

Taking account of Eq. (2) and Eq. (3) and Lemma 2 in [7], the Lie brackets of vertical and horizontal lifts to $\mathtt{T^*N}$ are given as follows:

Lemma 1.

$$\begin{aligned}\left[\omega^{\widetilde{v}},\theta^{\widetilde{v}}\right]_\zeta&=0,\\ \left[\Upsilon_1^{\widetilde{h}},\omega^{\widetilde{v}}\right]_\zeta&=(\nabla_{\Upsilon_1}\omega)_\zeta^{\widetilde{v}},\\ \left[\Upsilon_1^{\widetilde{h}},\Upsilon_2^{\widetilde{h}}\right]_\zeta&=[\Upsilon_1,\Upsilon_2]_\zeta^{\widetilde{h}}-(R(\Upsilon_1,\Upsilon_2)\varpi)_\zeta^{\widetilde{v}},\end{aligned}$$

for any $\Upsilon_1,\Upsilon_2\in\Gamma(\mathtt{TN})$ *and any* $\omega,\theta\in\Gamma(\mathtt{T^*N})$*, where* $\zeta\in\mathtt{T^*N}$*,* ϖ *is a 1-form on* $\mathtt{N}$ *such that* $\varpi_{\widetilde{\pi}(\zeta)}=\zeta$ *and* $R(\Upsilon_1,\Upsilon_2)=[\nabla_{\Upsilon_1},\nabla_{\Upsilon_2}]-\nabla_{[\Upsilon_1,\Upsilon_2]}$ *is the curvature tensor of* $\mathtt{N}$*.*

From Eq. (2) and Eq. (3) and the Levi-Civita connection $\overline{\nabla}$ of $\mathfrak{g}_S$ given by the formulas (8)-(11) in [11], we obtain the following

Lemma 2. *The Levi-Civita connection* $\widetilde{\nabla}$ *of* $\mathfrak{g}_{\widetilde{S}}$ *is described completely by*

$$\begin{aligned}(\widetilde{\nabla}_{\Upsilon_1^{\widetilde{h}}}\Upsilon_2^{\widetilde{h}})_\zeta&=(\nabla_{\Upsilon_1}\Upsilon_2)_\zeta^{\widetilde{h}}-\frac{1}{2}(R(\Upsilon_1,\Upsilon_2)\varpi)_\zeta^{\widetilde{v}},\\ (\widetilde{\nabla}_{\Upsilon_1^{\widetilde{h}}}\theta^{\widetilde{v}})_\zeta&=(\nabla_{\Upsilon_1}\theta)_\zeta^{\widetilde{v}}+\frac{1}{2}(R(\natural(\varpi),\natural(\theta))\Upsilon_1)_\zeta^{\widetilde{h}},\\ (\widetilde{\nabla}_{\omega^{\widetilde{v}}}\Upsilon_2^{\widetilde{h}})_\zeta&=\frac{1}{2}(R(\natural(\varpi),\natural(\omega))\Upsilon_2)_\zeta^{\widetilde{h}},\\ (\widetilde{\nabla}_{\omega^{\widetilde{v}}}\theta^{\widetilde{v}})_\zeta&=0,\end{aligned} \tag{4}$$

for any $\Upsilon_1,\Upsilon_2\in\Gamma(\mathtt{TN})$ *and any* $\omega,\theta\in\Gamma(\mathtt{T^*N})$*, where* $\zeta\in\mathtt{T^*N}$ *and* ϖ *is a 1-form on* $\mathtt{N}$ *such that* $\varpi_{\widetilde{\pi}(\zeta)}=\zeta$*.*

Proposition 1. *Let* $\widetilde{R}$ *be the curvature tensor of* $(\mathtt{T^*N},\mathfrak{g}_{\widetilde{S}})$*. Then the following formulae hold*

$$\left\{\widetilde{R}(\Upsilon_1^{\widetilde{h}},\Upsilon_2^{\widetilde{h}})\Upsilon_3^{\widetilde{h}}\right\}_\zeta=\left\{R(\Upsilon_1,\Upsilon_2)\Upsilon_3+\frac{1}{4}R(\natural\varpi,R(\Upsilon_3,\Upsilon_2)\natural\varpi)\Upsilon_1\right.$$

$$+\frac{1}{4}R(\natural\varpi, R(\Upsilon_1,\Upsilon_3)\natural\varpi)\Upsilon_2+\frac{1}{2}R(\natural\varpi, R(\Upsilon_1,\Upsilon_2)\natural\varpi)\Upsilon_3\Big\}^{\widetilde{h}}_{\zeta}$$

$$+\frac{1}{2}\Big\{(\nabla_{\Upsilon_3}R)(\Upsilon_1,\Upsilon_2)\varpi\Big\}^{\widetilde{v}}_{\zeta},$$

$$\Big\{\widetilde{R}(\Upsilon_1^{\widetilde{h}},\Upsilon_2^{\widetilde{h}})\omega^{\widetilde{v}}\Big\}_{\zeta}=\Big\{R(\Upsilon_1,\Upsilon_2)\omega+\frac{1}{4}R(R(\natural\varpi,\natural\omega)\Upsilon_2,\Upsilon_1)\varpi$$
$$-\frac{1}{4}R(R(\natural\varpi,\natural\omega)\Upsilon_1,\Upsilon_2)\varpi\Big\}^{\widetilde{v}}_{\zeta}+\frac{1}{2}\Big\{(\nabla_{\Upsilon_1}R)(\natural\varpi,\natural\omega)\Upsilon_2$$
$$-(\nabla_{\Upsilon_2}R)(\natural\varpi,\natural\omega)\Upsilon_1\Big\}^{\widetilde{h}}_{\zeta},$$

$$\Big\{\widetilde{R}(\Upsilon_1^{\widetilde{h}},\omega^{\widetilde{v}})\Upsilon_3^{\widetilde{h}}\Big\}_{\zeta}=\Big\{\frac{1}{4}R(R(\natural\varpi,\natural\omega)\Upsilon_3,\Upsilon_1)\varpi+\frac{1}{2}R(\Upsilon_1,\Upsilon_3)\omega\Big\}^{\widetilde{v}}_{\zeta}$$
$$+\frac{1}{2}\Big\{(\nabla_{\Upsilon_1}R)(\natural\varpi,\natural\omega)\Upsilon_3\Big\}^{\widetilde{h}}_{\zeta},$$

$$\Big\{\widetilde{R}(\Upsilon_1^{\widetilde{h}},\omega^{\widetilde{v}})\theta^{\widetilde{v}}\Big\}_{\zeta}=-\Big\{\frac{1}{2}R(\natural\omega,\natural\theta)\Upsilon_1+\frac{1}{4}R(\natural\varpi,\natural\omega)R(\natural\varpi,\natural\theta)\Upsilon_1\Big\}^{\widetilde{h}}_{\zeta},$$

$$\Big\{\widetilde{R}(\omega^{\widetilde{v}},\theta^{\widetilde{v}})\Upsilon_3^{\widetilde{h}}\Big\}_{\zeta}=\Big\{R(\natural\omega,\natural\theta)\Upsilon_3+\frac{1}{4}R(\natural\varpi,\natural\omega)R(\natural\varpi,\natural\theta)\Upsilon_3$$
$$-\frac{1}{4}R(\natural\varpi,\natural\theta)R(\natural\varpi,\natural\omega)\Upsilon_3\Big\}^{\widetilde{h}}_{\zeta},$$

$$\Big\{\widetilde{R}(\omega^{\widetilde{v}},\theta^{\widetilde{v}})\mu^{\widetilde{v}}\Big\}_{\zeta}=0,$$

for any $\Upsilon_1,\Upsilon_2,\Upsilon_3\in\Gamma(\mathtt{TN})$ *and any* $\omega,\theta,\mu\in\Gamma(\mathtt{T^*N})$*, where* $\zeta\in\mathtt{T^*N}$ *and* ϖ *is a 1-form on* $\mathtt{N}$ *such that* $\varpi_{\widetilde{\pi}(\zeta)}=\zeta$*.*

Proof. Using Eq. (2) and Eq. (3), we obtain

$$♪_*\left(\Big\{\bar{R}(\natural_*\widetilde{\Upsilon}_1,\natural_*\widetilde{\Upsilon}_2)\natural_*\widetilde{\Upsilon}_3\Big\}_w\right)=\Big\{\widetilde{R}(\widetilde{\Upsilon}1,\widetilde{\Upsilon}_2)\widetilde{\Upsilon}_3\Big\}_{\zeta},\tag{5}$$

such that $♪(w)=\zeta$ and for any vector fields $\overline{\Upsilon}_1,\overline{\Upsilon}_2,\overline{\Upsilon}_3$ on TN and any vector fields $\widetilde{\Upsilon}_1,\widetilde{\Upsilon}_2,\widetilde{\Upsilon}_3$ on $\mathtt{T^*N}$. Thus, by Eq. (5) and Theorem 1 in [11], we find the required formulae. □

3. Unit Cotangent Sphere Bundle

The unit tangent sphere bundle $\mathtt{T}_1\mathtt{N}$ of a Riemannian manifold $(\mathtt{N}, \mathfrak{g})$ consists of all unit tangent vectors to $\mathtt{N}$. As a hypersurface of $\mathtt{TN}$ it is given by

$$\mathtt{T}_1\mathtt{N} = \{\mathfrak{p} = (p, \mathtt{v}) \in \mathtt{TN} \,|\, \mathfrak{g}_p(\mathtt{v}, \mathtt{v}) = 1\}.$$

The vector field $N_{\mathfrak{p}} = \mathtt{v}^v$ is a unit normal of $\mathtt{T}_1\mathtt{N}$. In contrast with the horizontal lift of a vector field, the vertical lift is not in general tangent to $\mathtt{T}_1\mathtt{N}$ [3]; for this reason, it was defined the tangential lift of $\Upsilon_1 \in \mathtt{T}_p\mathtt{N}$ to $\mathfrak{p} \in \mathtt{T}_1\mathtt{N}$ as following [3]

$$\Upsilon^t_{1\,\mathfrak{p}} = \Upsilon^v_{1\,\mathfrak{p}} - \mathfrak{g}(\Upsilon_1, \mathtt{v})N_{\mathfrak{p}} = (\Upsilon_1 - \mathfrak{g}(\Upsilon_1, \mathtt{v})\mathtt{v})^v_{\mathfrak{p}}.$$

Clearly, the tangent space $\mathtt{T}_{\mathfrak{p}}\mathtt{T}_1\mathtt{N}$ is spanned by vectors of the form Υ^h_1 and Υ^t_1, where there is $\Upsilon_1 \in \mathtt{T}_p\mathtt{N}$. To simplify notation, we will use $\overline{\Upsilon}_1$ for $\Upsilon_1 - \mathfrak{g}(\Upsilon_1, \mathtt{v})\mathtt{v}$, then $\Upsilon^t_1 = \overline{\Upsilon}^v_1$. The Riemannian metric $\bar{\mathfrak{g}}_S$ on the hypersurface $\mathtt{T}_1\mathtt{N}$ induced by $\mathfrak{g}_S$ on $\mathtt{TN}$ is uniquely determined by the formulae

$$\begin{aligned} \bar{\mathfrak{g}}_S(\Upsilon^h_1, \Upsilon^h_2) &= \mathfrak{g}_S(\Upsilon^h_1, \Upsilon^h_2), \\ \bar{\mathfrak{g}}_S(\Upsilon^t_1, \Upsilon^h_2) &= 0, \\ \bar{\mathfrak{g}}_S(\Upsilon^t_1, \Upsilon^t_2) &= \mathfrak{g}_S(\Upsilon^v_1, \Upsilon^v_2) - \mathfrak{g}_S(\Upsilon^v_1, N)\mathfrak{g}_S(\Upsilon^v_2, N). \end{aligned}$$

Now by analogy with unit tangent sphere bundle, we define the unit cotangent sphere bundle $\mathtt{T}^*_1\mathtt{N}$, as the set of all unit tangent covectors to $\mathtt{N}$. As a hypersurface of $\mathtt{T}^*\mathtt{N}$ it is defined by

$$\mathtt{T}^*_1\mathtt{N} = \{\mathfrak{q} = (p, \mathfrak{v}) \in \mathtt{T}^*\mathtt{N} \,|\, \mathfrak{g}^{-1}_p(\mathfrak{v}, \mathfrak{v}) = 1\}.$$

The vector field $\widetilde{N}_{\mathfrak{q}} = \mathfrak{v}^{\widetilde{v}}$ is a unit normal of $\mathtt{T}^*_1\mathtt{N}$. The horizontal lift of a vector field is tangent to $\mathtt{T}^*_1\mathtt{N}$, but in general the vertical lift is not tangent. Thus, for $\omega \in \mathtt{T}^*_p\mathtt{N}$ we define the tangential lift of ω to $\mathfrak{q} \in \mathtt{T}^*_1\mathtt{N}$ by

$$\omega^{\widetilde{t}}_{\mathfrak{q}} = \omega^{\widetilde{v}}_{\mathfrak{q}} - \mathfrak{g}^{-1}(\omega, \mathfrak{v})\widetilde{N}_{\mathfrak{q}} = (\omega - \mathfrak{g}^{-1}(\omega, \mathfrak{v})\mathfrak{v})^{\widetilde{v}}_{\mathfrak{q}}.$$

The tangent space $\mathtt{T}_{\mathfrak{q}}\mathtt{T}^*_1\mathtt{N}$ is spanned by vectors of the form Υ^h_1 and $\omega^{\widetilde{t}}$. For the sake of notation clarity, we will use $\overline{\omega}$ as a shorthand for $\omega - \mathfrak{g}^{-1}(\omega, \mathfrak{v})\mathfrak{v}$, then $\omega^{\widetilde{t}} = \overline{\omega}^{\widetilde{v}}$. The Riemannian metric $\tilde{\mathfrak{g}}_{\widetilde{S}}$ on the hypersurface $\mathtt{T}^*_1\mathtt{N}$ induced by $\mathfrak{g}_{\widetilde{S}}$ on $\mathtt{T}^*\mathtt{N}$ is uniquely determined by the formulae

$$\begin{aligned} \tilde{\mathfrak{g}}_{\widetilde{S}}(\Upsilon^{\widetilde{h}}_1, \Upsilon^{\widetilde{h}}_2) &= \mathfrak{g}_{\widetilde{S}}(\Upsilon^{\widetilde{h}}_1, \Upsilon^{\widetilde{h}}_2), \\ \tilde{\mathfrak{g}}_{\widetilde{S}}(\omega^{\widetilde{t}}, \Upsilon^{\widetilde{h}}_2) &= 0, \\ \tilde{\mathfrak{g}}_{\widetilde{S}}(\omega^{\widetilde{t}}, \theta^{\widetilde{t}}) &= \mathfrak{g}_{\widetilde{S}}(\omega^{\widetilde{v}}, \theta^{\widetilde{v}}) - \mathfrak{g}_{\widetilde{S}}(\omega^{\widetilde{v}}, \widetilde{N})\mathfrak{g}_{\widetilde{S}}(\theta^{\widetilde{v}}, \widetilde{N}). \end{aligned} \tag{6}$$

We have the following

Theorem 1. *Let be an n-dimensional Riemannian manifold* $(\mathtt{N}, \mathfrak{g})$ *with Riemannian metric* $\mathfrak{g}$. *Subsequently, the musical isomorphisms generated by the metric* $\mathfrak{g}$ *represent isometric mappings between* $(\mathtt{T}_1\mathtt{N}, \bar{\mathfrak{g}}_S)$ *and* $(\mathtt{T}^*_1\mathtt{N}, \tilde{\mathfrak{g}}_{\widetilde{S}})$.

Proof. From Eq. (2) and Eq. (3), we find

$$♪_*(\Upsilon_1^t) = (♪\Upsilon_1)^{\widetilde{t}}, \tag{7}$$

$$\natural_*(\omega^{\widetilde{t}}) = (\natural\omega)^t. \tag{8}$$

Thus

$$\begin{aligned} ♪^*(\tilde{\mathfrak{g}}_{\widetilde{S}})(\Upsilon_1^t, \Upsilon_2^t) &= \tilde{\mathfrak{g}}_{\widetilde{S}}(♪_*\Upsilon_1^t, ♪_*\Upsilon_2^t) \\ &= \tilde{\mathfrak{g}}_{\widetilde{S}}((♪\Upsilon_1)^{\widetilde{t}}, (♪\Upsilon_2)^{\widetilde{t}}) \\ &= \bar{\mathfrak{g}}_S(\Upsilon_1^t, \Upsilon_2^t), \end{aligned} \tag{9}$$

$$♪^*(\tilde{\mathfrak{g}}_{\widetilde{S}})(\Upsilon_1^t, \Upsilon_2^h) = 0 = \bar{\mathfrak{g}}_S(\Upsilon_1^t, \Upsilon_2^h), \tag{10}$$

and

$$\begin{aligned} ♪^*(\tilde{\mathfrak{g}}_{\widetilde{S}})(\Upsilon_1^h, \Upsilon_2^h) &= \tilde{\mathfrak{g}}_{\widetilde{S}}(♪_*\Upsilon_1^h, ♪_*\Upsilon_2^h) \\ &= \tilde{\mathfrak{g}}_{\widetilde{S}}(\Upsilon_1^{\widetilde{h}}, \Upsilon_2^{\widetilde{h}}) \\ &= \bar{\mathfrak{g}}_S(\Upsilon_1^h, \Upsilon_2^h), \end{aligned} \tag{11}$$

then from Eq. (9)-(11), we find that $♪ : (\mathtt{T}_1\mathtt{N}, \bar{\mathfrak{g}}_S) \to (\mathtt{T}_1^*\mathtt{N}, \tilde{\mathfrak{g}}_{\widetilde{S}})$ is an isometry. In a similar way, we can also prove that $\natural : (\mathtt{T}_1^*\mathtt{N}, \tilde{\mathfrak{g}}_{\widetilde{S}}) \to (\mathtt{T}_1\mathtt{N}, \bar{\mathfrak{g}}_S)$ is an isometry. □

By virtue of Eq. (2) and Eq. (7) and the formulae (3.2)-(3.3) in [3], the Lie brackets of vector fields on $\mathtt{T}_1^*\mathtt{N}$ involving tangential lifts are given as follows:

$$\left[\Upsilon_1^{\widetilde{h}}, \omega^{\widetilde{t}}\right]_\nu = (\nabla_{\Upsilon_1}\omega)_\nu^{\widetilde{t}}\ , \tag{12}$$

$$\left[\omega^{\widetilde{t}}, \theta^{\widetilde{t}}\right]_\nu = \mathfrak{g}^{-1}(\omega, \vartheta)\theta_\nu^{\widetilde{t}} - \mathfrak{g}^{-1}(\theta, \vartheta)\omega_\nu^{\widetilde{t}}\ ,$$

for any $\Upsilon_1 \in \Gamma(\mathtt{TN})$ and any $\omega, \theta \in \Gamma(\mathtt{T}^*\mathtt{N})$, here $\nu = (x, q) \in \mathtt{T}_1^*\mathtt{N}$ and ϑ is a 1-form on $\mathtt{N}$ such that $\vartheta_{\widehat{\pi}(\nu)} = \nu$ where $\widehat{\pi} : \mathtt{T}_1^*\mathtt{N} \longrightarrow \mathtt{N}$ is the natural projection. Using Eq. (2), Eq. (3), Eq. (7), Eq. (8) and the Levi-Civita connection $\overline{\nabla}^S$ of $\bar{\mathfrak{g}}_S$ given by Proposition 3.1 in [3], we obtain the following:

Proposition 2. *The Levi-Civita connection* $\widetilde{\nabla}^{\widetilde{S}}$ *of Riemannian metric* $\tilde{\mathfrak{g}}_{\widetilde{S}}$ *is entirely described by*

$$\begin{aligned} (\widetilde{\nabla}^{\widetilde{S}}_{\Upsilon_1^{\tilde{h}}}\Upsilon_2^{\widetilde{h}})_\nu &= (\nabla_{\Upsilon_1}\Upsilon_2)_\nu^{\widetilde{h}} - \frac{1}{2}(R(\Upsilon_1, \Upsilon_2)\vartheta)_\nu^{\widetilde{t}}, \\ (\widetilde{\nabla}^{\widetilde{S}}_{\Upsilon_1^{\tilde{h}}}\theta^{\widetilde{t}})_\nu &= (\nabla_{\Upsilon_1}\theta)_\nu^{\widetilde{t}} + \frac{1}{2}(R(\natural\vartheta, \natural\theta)\Upsilon_1)_\nu^{\widetilde{h}}, \\ (\widetilde{\nabla}^{\widetilde{S}}_{\omega^{\tilde{t}}}\Upsilon_2^{\widetilde{h}})_\nu &= \frac{1}{2}(R(\natural\vartheta, \natural\omega)\Upsilon_2)_\nu^{\widetilde{h}}, \\ (\widetilde{\nabla}^{\widetilde{S}}_{\omega^{\tilde{t}}}\theta^{\widetilde{t}})_\nu &= -\mathfrak{g}^{-1}(\theta, \vartheta)\omega_\nu^{\widetilde{t}}, \end{aligned} \tag{13}$$

for any $\Upsilon_1, \Upsilon_2 \in \Gamma(\mathtt{TN})$ *and any* $\omega, \theta \in \Gamma(\mathtt{T}^*\mathtt{N})$, *here* $\nu = (p, \mathfrak{v}) \in \mathtt{T}_1^*\mathtt{N}$ *and* ϑ *is a 1-form on* $\mathtt{N}$ *such that* $\vartheta_{\widehat{\pi}(\nu)} = \nu$.

Proposition 3. *The curvature tensor* $\widetilde{R}^{\widetilde{S}}$ *of* $(\mathtt{T}_1^*\mathtt{N}, \tilde{\mathfrak{g}}_{\widetilde{S}})$ *is entirely described by*

$$\left\{\widetilde{R}^{\widetilde{S}}(\Upsilon_1^{\widetilde{h}}, \Upsilon_2^{\widetilde{h}})\Upsilon_3^{\widetilde{h}}\right\}_\nu = \left\{R(\Upsilon_1, \Upsilon_2)\Upsilon_3 + \frac{1}{4}R(\natural\vartheta, R(\Upsilon_3, \Upsilon_2)\natural\vartheta)\Upsilon_1 + \frac{1}{4}R(\natural\vartheta, R(\Upsilon_1, \Upsilon_3)\natural\vartheta)\Upsilon_2 + \frac{1}{2}R(\natural\vartheta, R(\Upsilon_1, \Upsilon_2)\natural\vartheta)\Upsilon_3\right\}_\nu^{\widetilde{h}} + \frac{1}{2}\left\{(\nabla_{\Upsilon_3}R)(\Upsilon_1, \Upsilon_2)\vartheta\right\}_\nu^{\widetilde{t}},$$

$$\left\{\widetilde{R}^{\widetilde{S}}(\Upsilon_1^{\widetilde{h}}, \Upsilon_2^{\widetilde{h}})\omega^{\widetilde{t}}\right\}_\nu = \left\{R(\Upsilon_1, \Upsilon_2)(\omega - \mathfrak{g}^{-1}(\omega, \vartheta)\vartheta) + \frac{1}{4}R(R(\natural\vartheta, \natural\omega)\Upsilon_2, \Upsilon_1)\vartheta - \frac{1}{4}R(R(\natural\vartheta, \natural\omega)\Upsilon_1, \Upsilon_2)\vartheta\right\}_\nu^{\widetilde{t}} + \frac{1}{2}\left\{(\nabla_{\Upsilon_1}R)(\natural\vartheta, \natural\omega)\Upsilon_2 - (\nabla_{\Upsilon_2}R)(\natural\vartheta, \natural\omega)\Upsilon_1\right\}_\nu^{\widetilde{h}},$$

$$\left\{\widetilde{R}^{\widetilde{S}}(\Upsilon_1^{\widetilde{h}}, \omega^{\widetilde{t}})\Upsilon_3^{\widetilde{h}}\right\}_\nu = \left\{\frac{1}{2}R(\Upsilon_1, \Upsilon_3)(\omega - \mathfrak{g}^{-1}(\omega, \vartheta)\vartheta) + \frac{1}{4}R(R(\natural\vartheta, \natural\omega)\Upsilon_3, \Upsilon_1)\vartheta\right\}_\nu^{\widetilde{v}} + \frac{1}{2}\left\{(\nabla_{\Upsilon_1}R)(\natural\vartheta, \natural\omega)\Upsilon_3\right\}_\nu^{\widetilde{h}},$$

$$\left\{\widetilde{R}^{\widetilde{S}}(\Upsilon_1^{\widetilde{h}}, \omega^{\widetilde{t}})\theta^{\widetilde{t}}\right\}_\nu = -\left\{\frac{1}{2}R(\natural\omega - \mathfrak{g}^{-1}(\omega, \vartheta)\natural\vartheta, \natural\theta - \mathfrak{g}^{-1}(\theta, \vartheta)\natural\vartheta)\Upsilon_1 + \frac{1}{4}R(\natural\vartheta, \natural\omega)R(\natural\vartheta, \natural\theta)\Upsilon_1\right\}_\nu^{\widetilde{h}},$$

$$\left\{\widetilde{R}^{\widetilde{S}}(\omega^{\widetilde{t}}, \theta^{\widetilde{t}})\Upsilon_3^{\widetilde{h}}\right\}_\nu = \left\{R(\natural\omega - \mathfrak{g}^{-1}(\omega, \vartheta)\natural\vartheta, \natural\theta - \mathfrak{g}^{-1}(\theta, \vartheta)\natural\vartheta)\Upsilon_3 + \frac{1}{4}[R(\natural\vartheta, \natural\omega), R(\natural\vartheta, \natural\theta)]\Upsilon_3\right\}_\nu^{\widetilde{h}},$$

$$\left\{\widetilde{R}^{\widetilde{S}}(\omega^{\widetilde{t}}, \theta^{\widetilde{t}})\mu^{\widetilde{t}}\right\}_\nu = -\tilde{\mathfrak{g}}_{\widetilde{S}}(\omega^{\widetilde{t}}, \mu^{\widetilde{t}})\theta_\nu^{\widetilde{t}} + \tilde{\mathfrak{g}}_{\widetilde{S}}(\mu^{\widetilde{t}}, \theta^{\widetilde{t}})\omega_\nu^{\widetilde{t}},$$

for any $\Upsilon_1, \Upsilon_2, \Upsilon_3 \in \Gamma(\mathtt{TN})$ *and any* $\omega, \theta, \mu \in \Gamma(\mathtt{T}^*\mathtt{N})$*, here* $\nu = (p, \mathfrak{v}) \in \mathtt{T}_1^*\mathtt{N}$ *and* ϑ *is a 1-form on* $\mathtt{N}$ *such that* $\vartheta_{\widehat{\pi}(\nu)} = \nu$.

Proof. Let $\bar{R}^S$ be the curvature tensor of $(\mathtt{T}_1\mathtt{N}, \bar{\mathfrak{g}}_S)$. Using Eq. (2) and Eq. (7), we obtain

$$♪_*\left(\left\{\bar{R}^S(\natural_*\widetilde{\Upsilon}_1, \natural_*\widetilde{\Upsilon}_2)\natural_*\widetilde{\Upsilon}_3\right\}_v\right) = \left\{\widetilde{R}^{\widetilde{S}}(\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2)\widetilde{\Upsilon}_3\right\}_\nu, \tag{14}$$

such that $♪(v) = \nu$ and for any vector fields $\overline{\Upsilon}_1, \overline{\Upsilon}_2, \overline{\Upsilon}_3$ on $\mathtt{T}_1\mathtt{N}$ and any vector fields $\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2, \widetilde{\Upsilon}_3$ on $\mathtt{T}_1^*\mathtt{N}$. Thus, the required formulae follow from Eq. (14) and Proposition 3.2 in [3]. □

Theorem 2. *The pair* $(\mathtt{T}_1^*\mathtt{N}, \tilde{\mathfrak{g}}_{\widetilde{S}})$ *is locally symmetric if and only if the base manifold* $(\mathtt{N}, \mathfrak{g})$ *is flat or* $\mathtt{N}$ *is a* 2*-dimensional manifold with a constant curvature* 1*.*

Proof. It's clear that

$$♪_*\left[(\overline{\nabla}^S_{\natural_*\widetilde{W}}\bar{R}^S)(\natural_*\widetilde{\Upsilon}_1, \natural_*\widetilde{\Upsilon}_2)\natural_*\widetilde{\Upsilon}_3\right] = (\widetilde{\nabla}^{\widetilde{S}}_{\widetilde{W}}\widetilde{R}^S)(\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2)\widetilde{\Upsilon}_3,$$

and

$$\natural_*\left[(\widetilde{\nabla}^{\widetilde{S}}_{♪_*\overline{W}}\widetilde{R}^S)(♪_*\overline{\Upsilon}_1, ♪_*\overline{\Upsilon}_2)♪_*\overline{\Upsilon}_3\right] = (\overline{\nabla}^S_{\overline{W}}\bar{R}^S)(\overline{\Upsilon}_1, \overline{\Upsilon}_2)\overline{\Upsilon}_3,$$

for any vector fields $\overline{\Upsilon}_1, \overline{\Upsilon}_2, \overline{\Upsilon}_3$ on $\mathtt{T}_1\mathtt{N}$ and any vector fields $\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2, \widetilde{\Upsilon}_3$ on $\mathtt{T}_1^*\mathtt{N}$. Therefore $(\mathtt{T}_1^*\mathtt{N}, \tilde{\mathfrak{g}}_{\widetilde{S}})$ is locally symmetric if and only if $(\mathtt{T}_1\mathtt{N}, \bar{\mathfrak{g}}_S)$ is locally symmetric, combining this fact with the main result in [2] we deduce the required assertion. □

3.1. **An Almost Contact Structure on $\mathtt{T}_1^*\mathtt{N}$.** We first recall some notions on almost contact structure, for more details we refer to [2]. Let $\mathtt{N}^{2n+1}$ be an odd-dimensional smooth manifold, we say that $\mathtt{N}^{2n+1}$ has an almost contact structure if the relations

$$\mu(\varsigma) = 1 \quad \text{and} \quad \mathcal{F}^2\Upsilon_1 = -\Upsilon_1 + \mu(\Upsilon_1)\varsigma$$

hold on $\mathtt{N}^{2n+1}$, where ς is a vector field, μ is a 1-form, and $\mathcal{F}$ is a (1,1)-tensor field on $\mathtt{N}^{2n+1}$.

Then there exists a compatible Riemannian metric $\mathfrak{g}$

$$\mathfrak{g}(\mathcal{F}\Upsilon_1, \mathcal{F}\Upsilon_2) = \mathfrak{g}(\Upsilon_1, \Upsilon_2) - \mu(\Upsilon_1)\mu(\Upsilon_2)$$

for all vector fields Υ_1 and Υ_2 on $\mathtt{N}$. We call $(\mu, \varsigma, \mathcal{F}, \mathfrak{g})$ an almost contact metric manifold, ς being known as its characteristic vector field. For an almost contact metric manifold $\mathtt{N}$, its fundamental 2-form Φ is defined by $\Phi(\Upsilon_1, \Upsilon_2) = \mathfrak{g}(\mathcal{F}\Upsilon_1, \Upsilon_2)$. If

$$\Phi = d\mu,$$

$\mathtt{N}$ is called a contact metric manifold. A contact metric manifold for which ς is a Killing vector field (resp. harmonic vector field) is called a K-contact manifold (resp. H-contact manifold). Recall that a unit vector field Υ_1 on $\mathtt{N}$ is harmonic if and only if $\bar{\Delta}\Upsilon_1$ is parallel to Υ_1, where $\bar{\Delta}\Upsilon_1$ is the rough Laplacian of Υ_1 (see [8]). In [14] Perrone showed that a contact metric manifold is H-contact if and only if the characteristic vector field ς is an eigenvector of the Ricci operator.

A contact metric structure is called Sasakian structure if it is normal. Recall that an almost contact structure $(\mu, \varsigma, \mathcal{F}, \mathfrak{g})$ is said to be normal if

$$N(\Upsilon_1, \Upsilon_2) = [\mathcal{F}, \mathcal{F}](\Upsilon_1, \Upsilon_2) + 2d\mu(\Upsilon_1, \Upsilon_2)\varsigma = 0,$$

for all $\Upsilon_1, \Upsilon_2 \in \Gamma(\mathtt{TN})$, here $N(\Upsilon_1, \Upsilon_2)$ is $(1,2)$-tensor field and $[\mathcal{F}, \mathcal{F}]$ is the Nijenhuis torsion of $\mathcal{F}$,

$$[\mathcal{F}, \mathcal{F}](\Upsilon_1, \Upsilon_2) = \mathcal{F}^2[\Upsilon_1, \Upsilon_2] + [\mathcal{F}\Upsilon_1, \mathcal{F}\Upsilon_2] - \mathcal{F}[\mathcal{F}\Upsilon_1, \Upsilon_2] - \mathcal{F}[\Upsilon_1, \mathcal{F}\Upsilon_2].$$

A powerful characterization for Sasakian manifolds is the following: An almost contact metric manifold $(\mu, \varsigma, \mathcal{F}, \mathfrak{g})$ is Sasakian if and only if

$$(\nabla_{\Upsilon_1}\mathcal{F})\Upsilon_2 = \mathfrak{g}(\Upsilon_1, \Upsilon_2)\varsigma - \mu(\Upsilon_2)\Upsilon_1; \quad \Upsilon_1, \Upsilon_2 \in \Gamma(\mathtt{TN}),$$

where ∇ is the Levi-Civita connection of $(\mathtt{N}, \mathfrak{g})$.

Next, it's well known from [17] that the unit tangent sphere bundle $\mathtt{T}_1\mathtt{N}$ has a standard contact metric structure $(\varsigma', \mu', \mathcal{F}', \bar{\mathfrak{g}}'_S) = (2\varsigma, \frac{1}{2}\mu, \mathcal{F}, \frac{1}{4}\bar{\mathfrak{g}}_S)$, where ς, μ and $\mathcal{F}$ are given by

$$\varsigma = -JN = \mathtt{v}^i \left(\frac{\partial}{\partial x^i}\right)^h,$$

$$\mu(\Upsilon_1^t) = 0, \quad \mu(\Upsilon_1^h) = \mathfrak{g}(\Upsilon_2, \mathtt{v}), \tag{15}$$

$$\mathcal{F}(\Upsilon_1^t) = -\Upsilon_1^h + \mathfrak{g}(\Upsilon_1, \mathtt{v})\varsigma, \quad \mathcal{F}(\Upsilon_1^h) = \Upsilon_1^t, \tag{16}$$

here $(p, \mathtt{v}) \in \mathtt{TN}$ and $\Upsilon_1 \in \Gamma(\mathtt{TN})$. Note that ς is the geodesic flow.

3.1.1. *An almost Kählerian structure on* $\mathtt{T}^*\mathtt{N}$. Let $(\mathtt{N}, \mathfrak{g})$ be a Riemannian manifold of dimension n and $(\mathtt{T}^*\mathtt{N}, \mathfrak{g}_{\widetilde{S}})$ its cotangent bundle endowed with the Sasaki metric. On $\mathtt{T}^*\mathtt{N}$ we define the structure $\widetilde{J}$ by

$$\begin{cases} \widetilde{J}(\Upsilon_1^{\widetilde{h}}) = (\flat\Upsilon_1)^{\widetilde{v}}, \\ \widetilde{J}(\omega^{\widetilde{v}}) = -(\natural\omega)^{\widetilde{h}}, \end{cases} \tag{17}$$

for any $\Upsilon_1 \in \Gamma(\mathtt{TN})$ and $\omega \in \Gamma(\mathtt{T^*N})$. It is clear that $(\mathtt{T}^*\mathtt{N}, \widetilde{J})$ is an almost complex manifold. Moreover, since

$$\mathfrak{g}_{\widetilde{S}}(\widetilde{J}(\Upsilon_1^h), \omega^v) = \omega(\Upsilon_1) = -\mathfrak{g}_{\widetilde{S}}(\Upsilon_1^h, \widetilde{J}(\omega^v)),$$

$$\mathfrak{g}_{\widetilde{S}}(\widetilde{J}(\omega^v), \theta^v) = 0 = -\mathfrak{g}_{\widetilde{S}}(\omega^v, \widetilde{J}(\theta^v)),$$

and

$$\mathfrak{g}_{\widetilde{S}}(\widetilde{J}(\Upsilon_1^h), \Upsilon_2^h) = 0 = -\mathfrak{g}_{\widetilde{S}}(\Upsilon_1^h, \widetilde{J}(\Upsilon_2^h)),$$

for any $\Upsilon_1, \Upsilon_2 \in \Gamma(\mathtt{TN})$ and any covectors ω and θ on $\mathtt{N}$, then $(\mathtt{T}^*\mathtt{N}, \mathfrak{g}_{\widetilde{S}}, \widetilde{J})$ is an almost hermitian manifold. Furthermore, the 2-form $\Omega_{\widetilde{S}}$ defined by:

$$\Omega_{\widetilde{S}} = \mathfrak{g}_{\widetilde{S}}(\widetilde{J}\cdot, \cdot)$$

is closed. In fact, we know

$$d\Omega_{\widetilde{S}} = 0 \Leftrightarrow \mathfrak{g}_{\widetilde{S}}((\widetilde{\nabla}_{\widetilde{\Upsilon}_1}\widetilde{J})\widetilde{\Upsilon}_2, \widetilde{\Upsilon}_3) + \mathfrak{g}_{\widetilde{S}}((\widetilde{\nabla}_{\widetilde{\Upsilon}_2}\widetilde{J})\widetilde{\Upsilon}_3, \widetilde{\Upsilon}_1) + \mathfrak{g}_{\widetilde{S}}((\widetilde{\nabla}_{\widetilde{\Upsilon}_3}\widetilde{J})\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2) = 0,$$

for any vector fields $\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2, \widetilde{\Upsilon}_3$ on $\mathtt{T}^*\mathtt{N}$. Using the algebraic Bianchi identity, Eq. (4) and Eq. (17), we get $d\Omega_{\widetilde{S}} = 0$. Hence, we may state the following:

Theorem 3. *Let be an n-dimensional Riemannian manifold* $(\mathtt{N}, \mathfrak{g})$ *with Riemannian metric* $\mathfrak{g}$ *be. Then* $(\mathtt{T}^*\mathtt{N}, \mathfrak{g}_{\widetilde{S}}, \widetilde{J})$ *is an almost Kählerian manifold.*

Theorem 4. *Let be an n-dimensional Riemannian manifold* $(\mathtt{N}, \mathfrak{g})$ *with Riemannian metric* $\mathfrak{g}$ *be. The musical isomorphisms* ♪ *and* ♮ *are holomorphic maps between* $(\mathtt{TN}, \mathfrak{g}_S, J_S)$ *and* $(\mathtt{T}^*\mathtt{N}, \mathfrak{g}_{\widetilde{S}}, \widetilde{J})$*. Moreover,* $(\mathtt{T}^*\mathtt{N}, \mathfrak{g}_{\widetilde{S}}, \widetilde{J})$ *is a Kählerian manifold if and only if* $(\mathtt{N}, \mathfrak{g})$ *is flat.*

Proof. From Eq. (1), Eq. (2), Eq. (3) and Eq. (17) we obtain $♪_* J = \widetilde{J} ♪_*$ and $\natural_* \widetilde{J} = J \natural_*$, it follows that ♪ and ♮ are holomorphic maps. Thus, by a direct computations we get

$$♪_*(\overline{\nabla}_{\natural_* \widetilde{\Upsilon}_1} J)\natural_* \widetilde{\Upsilon}_2 = (\widetilde{\nabla}_{\widetilde{\Upsilon}_1} \widetilde{J})\widetilde{\Upsilon}_2,$$

and

$$\natural_*(\widetilde{\nabla}_{♪_* \overline{\Upsilon}_1} \widetilde{J})♪_* \overline{\Upsilon}_2 = (\overline{\nabla}_{\overline{\Upsilon}_1} J)\overline{\Upsilon}_2,$$

for any vector fields $\overline{\Upsilon}_1, \overline{\Upsilon}_2$ on $\mathtt{TN}$ and any vector fields $\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2$ on $\mathtt{T}^*\mathtt{N}$, then $(\mathtt{T}^*\mathtt{N}, \mathfrak{g}_{\widetilde{S}}, \widetilde{J})$ is a Kählerian manifold if and only if $(\mathtt{TN}, \mathfrak{g}_S, J)$ is, or equivalently $(\mathtt{N}, \mathfrak{g})$ is flat. □

3.1.2. *An almost contact structure on* $\mathtt{T}_1^*\mathtt{N}$. With the help of the almost complex structure $\widetilde{J}$, we can define a unit vector field $\widetilde{\varsigma}$, a 1-form $\widetilde{\mu}$ and a $(1,1)$-tensor field $\widetilde{\mathcal{F}}$ on $\mathtt{T}^*\mathtt{N}$, as given below:

$$\widetilde{\varsigma} = -\widetilde{J}\widetilde{N}, \quad \widetilde{\mathcal{F}} = \widetilde{J} - \widetilde{\mu} \otimes \widetilde{N}.$$

Explicitly $\widetilde{\varsigma}$, $\widetilde{\mu}$ and $\widetilde{\mathcal{F}}$ are given by

$$\widetilde{\varsigma} = \mathfrak{v}^i (\frac{\partial}{\partial x^i})^{\widetilde{h}}, \tag{18}$$

$$\widetilde{\mu}(\omega^{\widetilde{t}}) = 0, \quad \widetilde{\mu}(\Upsilon_1^{\widetilde{h}}) = \mathfrak{g}^{-1}(♪\Upsilon_1, \mathfrak{v}), \tag{19}$$

$$\widetilde{\mathcal{F}}(\omega^{\widetilde{t}}) = -(\natural\omega)^{\widetilde{h}} + \mathfrak{g}^{-1}(\omega, \mathfrak{v})\widetilde{\varsigma}, \quad \widetilde{\mathcal{F}}(\Upsilon_1^{\widetilde{h}}) = (♪\Upsilon_1)^{\widetilde{t}}. \tag{20}$$

Note that $\widetilde{\varsigma}$ is the cogeodesic flow.

Proposition 4. $(\mathtt{T}_1^*\mathtt{N}, \widetilde{\varsigma}', \widetilde{\mu}', \widetilde{\mathcal{F}}', \widetilde{\mathfrak{g}}'_{\widetilde{S}})$ *is an almost contact metric manifold, where we have* $(\widetilde{\varsigma}', \widetilde{\mu}', \widetilde{\mathcal{F}}', \widetilde{\mathfrak{g}}'_{\widetilde{S}}) = (2\widetilde{\varsigma}, \frac{1}{2}\widetilde{\mu}, \widetilde{\mathcal{F}}, \frac{1}{4}\widetilde{\mathfrak{g}}_{\widetilde{S}})$.

Proof. By definition, we shall show that $(\widetilde{\varsigma}', \widetilde{\mu}', \widetilde{\mathcal{F}}', \widetilde{\mathfrak{g}}'_{\widetilde{S}})$ satisfies

$$\widetilde{\mu}'(\widetilde{\varsigma}') = 1, \quad \widetilde{\mathcal{F}}'^2 = -I + \widetilde{\mu}' \otimes \widetilde{\varsigma}' \quad \text{and} \quad \widetilde{\mathfrak{g}}'_{\widetilde{S}}(\widetilde{\mathcal{F}}'\widetilde{\Upsilon}_1, \widetilde{\mathcal{F}}'\widetilde{\Upsilon}_2) = \widetilde{\mathfrak{g}}'_{\widetilde{S}}(\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2) - \widetilde{\mu}'(\widetilde{\Upsilon}_1)\widetilde{\mu}'(\widetilde{\Upsilon}_2)$$

for all vector fields $\widetilde{\Upsilon}_1$ and $\widetilde{\Upsilon}_2$ on $\mathtt{T}_1^*\mathtt{N}$. From Eq. (18)-(20), we yield

$$\widetilde{\mu}'(\widetilde{\varsigma}') = 1, \quad \widetilde{\mathcal{F}}'(\widetilde{\varsigma}') = 0,$$

$$\begin{aligned}\widetilde{\mathcal{F}}'^2(\omega^{\widetilde{t}}) &= -\widetilde{\mathcal{F}}'((\natural\omega)^{\widetilde{h}}) \\ &= -\omega^{\widetilde{t}} \end{aligned} \tag{21}$$

and

$$\begin{aligned}\widetilde{\mathcal{F}}'^2(\Upsilon_1^{\widetilde{h}}) &= \widetilde{\mathcal{F}}'((♪\Upsilon_1)^{\widetilde{t}}) \\ &= -\Upsilon_1^{\widetilde{h}} + \mathfrak{g}^{-1}(♪\Upsilon_1, \mathfrak{v})\widetilde{\varsigma} \\ &= -\Upsilon_1^{\widetilde{h}} + \widetilde{\mu}'(\Upsilon_1^{\widetilde{h}})\widetilde{\varsigma}'. \end{aligned} \tag{22}$$

By Eq. (21) and Eq. (22), we see that $\widetilde{\phi}'^2 = -I + \widetilde{\mu}' \otimes \widetilde{\varsigma}'$. By virtue of Eq. (3) and Eq. (20), it follows that

$$\begin{aligned}\tilde{\mathfrak{g}}'_{\widetilde{S}}(\widetilde{\phi}'(\omega^{\widetilde{t}}), \widetilde{\phi}'(\theta^{\widetilde{t}})) &= \frac{1}{4}\tilde{\mathfrak{g}}_{\widetilde{S}}(\widetilde{\phi}'(\omega^{\widetilde{t}}), \widetilde{\phi}'(\theta^{\widetilde{t}})) \\ &= \frac{1}{4}(\mathfrak{g}^{-1}(\omega,\theta) - \mathfrak{g}^{-1}(\omega,\mathfrak{v})\mathfrak{g}^{-1}(\theta,\mathfrak{v})) \\ &= \frac{1}{4}(\tilde{\mathfrak{g}}_{\widetilde{S}}(\omega^{\widetilde{t}}, \theta^{\widetilde{t}}) - \widetilde{\mu}(\omega^{\widetilde{t}})\widetilde{\mu}(\theta^{\widetilde{t}})) \\ &= \tilde{\mathfrak{g}}'_{\widetilde{S}}(\omega^{\widetilde{t}}, \theta^{\widetilde{t}}) - \widetilde{\mu}'(\omega^{\widetilde{t}})\widetilde{\mu}'(\theta^{\widetilde{t}}), \end{aligned} \tag{23}$$

and

$$\begin{aligned}\tilde{\mathfrak{g}}'_{\widetilde{S}}(\widetilde{\phi}'(\Upsilon_1^{\widetilde{h}}), \widetilde{\phi}'(\Upsilon_2^{\widetilde{h}})) &= \frac{1}{4}\tilde{\mathfrak{g}}_{\widetilde{S}}(\widetilde{\phi}'(\Upsilon_1^{\widetilde{h}}), \widetilde{\phi}'(\Upsilon_2^{\widetilde{h}})) \\ &= \frac{1}{4}\tilde{\mathfrak{g}}_{\widetilde{S}}((♪\Upsilon_1)^{\widetilde{t}}, (♪\Upsilon_2)^{\widetilde{t}}) \\ &= \frac{1}{4}(\mathfrak{g}^{-1}(♪\Upsilon_1, ♪\Upsilon_2) - \mathfrak{g}^{-1}(♪\Upsilon_1,\mathfrak{v})\mathfrak{g}^{-1}(♪\Upsilon_2,\mathfrak{v})) \\ &= \tilde{\mathfrak{g}}'_{\widetilde{S}}(\Upsilon_1^{\widetilde{h}}, \Upsilon_2^{\widetilde{h}}) - \widetilde{\mu}'(\Upsilon_1^{\widetilde{h}})\widetilde{\mu}'(\Upsilon_2^{\widetilde{h}}). \end{aligned} \tag{24}$$

From Eq. (23) and Eq. (24), we see that

$$\tilde{\mathfrak{g}}'_{\widetilde{S}}(\widetilde{\mathcal{F}}'(\widetilde{\Upsilon}_1), \widetilde{\mathcal{F}}'(\widetilde{\Upsilon}_2)) = \tilde{\mathfrak{g}}'_{\widetilde{S}}(\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2) - \widetilde{\mu}'(\widetilde{\Upsilon}_1)\widetilde{\mu}'(\widetilde{\Upsilon}_2),$$

for all vector fields $\widetilde{\Upsilon}_1$ and $\widetilde{\Upsilon}_2$ on $\mathtt{T}_1^*\mathtt{N}$. Therefore $(\mathtt{T}_1^*\mathtt{N}, \widetilde{\varsigma}', \widetilde{\mu}', \widetilde{\mathcal{F}}', \tilde{\mathfrak{g}}'_{\widetilde{S}})$ is an almost contact metric manifold. □

Proposition 5. $(\mathtt{T}_1^*\mathtt{N}, \widetilde{\varsigma}', \widetilde{\mu}', \widetilde{\mathcal{F}}', \tilde{\mathfrak{g}}'_{\widetilde{S}})$ *is a contact metric manifold, where we have* $(\widetilde{\varsigma}', \widetilde{\mu}', \widetilde{\mathcal{F}}', \tilde{\mathfrak{g}}'_{\widetilde{S}}) = (2\widetilde{\varsigma}, \frac{1}{2}\widetilde{\mu}, \widetilde{\mathcal{F}}, \frac{1}{4}\tilde{\mathfrak{g}}_{\widetilde{S}})$.

Proof. By using Eq. (20), we yield

$$\begin{aligned}\tilde{\mathfrak{g}}'_{\widetilde{S}}(\omega^{\widetilde{t}}, \widetilde{\mathcal{F}}'(\Upsilon_1^{\widetilde{h}})) &= \tilde{\mathfrak{g}}'_{\widetilde{S}}(\omega^{\widetilde{t}}, (♪\Upsilon_1)^{\widetilde{t}}) \\ &= \frac{1}{4}\{\mathfrak{g}^{-1}(\omega, ♪\Upsilon_1) - \mathfrak{g}^{-1}(\omega,\mathfrak{v})\mathfrak{g}^{-1}(♪\Upsilon_1,\mathfrak{v})\}. \end{aligned}$$

On the other side, from the definition of the vertical lift to $\mathtt{T}^*\mathtt{N}$ we get

$$\omega^{\widetilde{v}}(\mathfrak{g}^{-1}(♪\Upsilon_1,\mathfrak{v}))=\mathfrak{g}^{-1}(♪\Upsilon_1,\omega),$$

and

$$q^{\widetilde{v}}(\mathfrak{g}^{-1}(♪\Upsilon_1,\mathfrak{v}))=\mathfrak{g}^{-1}(♪\Upsilon_1,\mathfrak{v}).$$

Thus, we obtain

$$\omega^{\widetilde{t}}(\mathfrak{g}^{-1}(♪\Upsilon_1,\mathfrak{v}))=\mathfrak{g}^{-1}(\omega,♪\Upsilon_1)-\mathfrak{g}^{-1}(\omega,\mathfrak{v})\mathfrak{g}^{-1}(♪\Upsilon_1,\mathfrak{v}), \tag{25}$$

it follows from Eq. (12), Eq. (19) and Eq. (25) that

$$\begin{aligned} d\widetilde{\mu}'(\omega^{\widetilde{t}},\Upsilon_1^{\widetilde{h}}) &= \frac{1}{2}\{\omega^{\widetilde{t}}\widetilde{\mu}'(\Upsilon_1^{\widetilde{h}})-\Upsilon_1^{\widetilde{h}}\widetilde{\mu}'(\omega^{\widetilde{t}})-\widetilde{\mu}'([\omega^{\widetilde{t}},\Upsilon_1^{\widetilde{h}}])\}\\ &= \frac{1}{4}\{\omega^{\widetilde{t}}\widetilde{\mu}(\Upsilon_1^{\widetilde{h}})\}\\ &= \frac{1}{4}\{\mathfrak{g}^{-1}(\omega,♪\Upsilon_1)-\mathfrak{g}^{-1}(\omega,\mathfrak{v})\mathfrak{g}^{-1}(♪\Upsilon_1,\mathfrak{v})\}. \end{aligned}$$

Then we get the contact metric structure $(\widetilde{\varsigma}',\widetilde{\mu}',\widetilde{\phi}',\tilde{\mathfrak{g}}'_{\widetilde{S}})$ on $\mathtt{T}_1^*\mathtt{N}$. □

Theorem 5. *The contact metric structure on* $\mathtt{T}_1^*\mathtt{N}$ *is* K*-contact if and only if the contact metric structure on* $\mathtt{T}_1\mathtt{N}$ *is.*

Proof. As ♪ and ♮ are isometries between $(\mathtt{T}_1\mathtt{N},\bar{\mathfrak{g}}_S)$ and $(\mathtt{T}_1^*\mathtt{N},\tilde{\mathfrak{g}}_{\widetilde{S}})$, we deduce that

$$♪^*(L_{\widetilde{\varsigma}'}\tilde{\mathfrak{g}}_{\widetilde{S}})(\overline{\Upsilon}_1,\overline{\Upsilon}_2)=(L_{♮_*\widetilde{\varsigma}'}♪^*\tilde{\mathfrak{g}}_{\widetilde{S}})(\overline{\Upsilon}_1,\overline{\Upsilon}_2)=(L_{\varsigma'}\bar{\mathfrak{g}}_S)(\overline{\Upsilon}_1,\overline{\Upsilon}_2), \tag{26}$$

and

$$♮^*(L_{\varsigma'}\bar{\mathfrak{g}}_S)(\widetilde{\Upsilon}_1,\widetilde{\Upsilon}_2)==(L_{♪_*\varsigma'}♮^*\bar{\mathfrak{g}}_S)(\widetilde{\Upsilon}_1,\widetilde{\Upsilon}_2)=(L_{\widetilde{\varsigma}'}\tilde{\mathfrak{g}}_{\widetilde{S}})(\widetilde{\Upsilon}_1,\widetilde{\Upsilon}_2), \tag{27}$$

for any vector fields $\overline{\Upsilon}_1,\overline{\Upsilon}_2$ on $\mathtt{T}_1\mathtt{N}$ and any vector fields $\widetilde{\Upsilon}_1,\widetilde{\Upsilon}_2$ on $\mathtt{T}_1^*\mathtt{N}$. Then, from Eq. (26) and Eq. (27) we get our assertion. □

Theorem 6. *The contact metric structure on* $\mathtt{T}_1^*\mathtt{N}$ *is Sasakian if and only if the contact metric structure on* $\mathtt{T}_1\mathtt{N}$ *is.*

Proof. Let $(\varsigma',\mu',\mathcal{F}',\bar{\mathfrak{g}}'_S)$ (resp. $(\widetilde{\varsigma}',\widetilde{\mu}',\widetilde{\mathcal{F}}',\tilde{\mathfrak{g}}'_{\widetilde{S}})$ be the standard contact metric structure of $\mathtt{T}_1\mathtt{N}$ (resp. $\mathtt{T}_1^*\mathtt{N}$). From Eq. (2), Eq. (3), Eq. (7), Eq. (8), Eq. (15), Eq. (16), Eq. (19) and Eq. (20) we have

$$\begin{aligned} ♪_*\mathcal{F}' &= \widetilde{\mathcal{F}}'♪_*,\\ ♮_*\widetilde{\mathcal{F}}' &= \mathcal{F}'♮_*,\\ ♪^*\widetilde{\mu}' &= \mu',\\ ♮^*\mu' &= \widetilde{\mu}'. \end{aligned}$$

Hence ♪ is $(\mathcal{F}',\widetilde{\mathcal{F}}')$-holomorphic map and ♮ is $(\widetilde{\mathcal{F}}',\mathcal{F}')$-holomorphic map. Therefore, we get

$$♪_*((\overline{\nabla}^S_{\overline{\Upsilon}_1}\mathcal{F}')\overline{\Upsilon}_2)=(\widetilde{\nabla}^{\widetilde{S}}_{♪_*\overline{\Upsilon}_1}\widetilde{\mathcal{F}}')♪_*\overline{\Upsilon}_2,$$

and

$$\natural_*(\widetilde{\nabla}^{\widetilde{S}}_{\widetilde{\Upsilon}_1}\widetilde{\mathcal{F}}')\widetilde{\Upsilon}_2 = (\overline{\nabla}^{S}_{\natural_*\widetilde{\Upsilon}_1}\mathcal{F}')\natural_*\widetilde{\Upsilon}_2.$$

Thus, we obtain

$$\begin{aligned}♪_*((\overline{\nabla}^{S}_{\natural_*\widetilde{\Upsilon}_1}\mathcal{F}')\natural_*\widetilde{\Upsilon}_2 - \bar{\mathfrak{g}}'_S(\natural_*\widetilde{\Upsilon}_1,\natural_*\widetilde{\Upsilon}_2)\varsigma' + \mu'(\natural_*\widetilde{\Upsilon}_2)\natural_*\widetilde{\Upsilon}_1) = (&\widetilde{\nabla}^{\widetilde{S}}_{\widetilde{\Upsilon}_1}\widetilde{\mathcal{F}}')\widetilde{\Upsilon}_2\\ &- \tilde{\mathfrak{g}}'_{\widetilde{S}}(\widetilde{\Upsilon}_1,\widetilde{\Upsilon}_2)\widetilde{\varsigma}'\\ &+ \widetilde{\mu}'(\widetilde{\Upsilon}_2)\widetilde{\Upsilon}_1,\end{aligned}$$

and

$$\begin{aligned}\natural_*((\widetilde{\nabla}^{\widetilde{S}}_{♪_*\overline{\Upsilon}_1}\widetilde{\mathcal{F}}')♪_*\overline{\Upsilon}_2 - \tilde{\mathfrak{g}}'_{\widetilde{S}}(♪_*\overline{\Upsilon}_1,♪_*\overline{\Upsilon}_2)\varsigma' + \widetilde{\mu}'(♪_*\overline{\Upsilon}_2)♪_*\overline{\Upsilon}_1) = (&\overline{\nabla}^{S}_{\overline{\Upsilon}_1}\mathcal{F}')\overline{\Upsilon}_2\\ &- \bar{\mathfrak{g}}'_S(\overline{\Upsilon}_1,\overline{\Upsilon}_2)\varsigma'\\ &+ \mu'(\overline{\Upsilon}_2)\overline{\Upsilon}_1,\end{aligned}$$

then, the contact metric structure on $\mathtt{T}_1^*\mathtt{N}$ is Sasakian if and only if the contact metric structure on $\mathtt{T}_1\mathtt{N}$ is. □

Theorem 7. *The contact metric structure present on $\mathtt{T}_1^*\mathtt{N}$ is categorized as K-contact if and only if the Riemannian manifold $(\mathtt{N},\mathfrak{g})$ possesses a constant curvature of 1. In such instances, the structure established on $\mathtt{T}_1^*\mathtt{N}$ is denoted as Sasakian.*

Proof. Combining Theorems 5 and 6 with Theorem 8 in [17], we get our assertion. □

Finally, recall that a Riemannian manifold $(\mathtt{N},\mathfrak{g})$ of dimension n is said to be 2-stein if there exist two functions α_1, $\alpha_2 : \mathtt{N} \longrightarrow \mathbb{R}$ such that for every $p \in \mathtt{N}$ and every vector Υ_1 tangent to $\mathtt{N}$ at p we have

$$\mathrm{Tr}(R_{\Upsilon_1}) = \alpha_1(p)\left\|\Upsilon_1\right\|^2, \qquad \mathrm{Tr}(R^2_{\Upsilon_1}) = \alpha_2(p)\left\|\Upsilon_1\right\|^4,$$

where R_{Υ_1} is the Jacobi operator [6].

Theorem 8. *The contact metric structure on $\mathtt{T}_1^*\mathtt{N}$ is H-contact if and only if $(\mathtt{N},\mathfrak{g})$ is 2-stein.*

Proof. It is obvious that the Ricci operators $\widetilde{Q}(\widetilde{\varsigma})$ on $\mathtt{T}_1\mathtt{N}$ and $\overline{Q}(\varsigma)$ on $\mathtt{T}_1^*\mathtt{N}$ are related by:

$$♪_*\overline{Q}(\varsigma) = \widetilde{Q}(\widetilde{\varsigma}),$$

and

$$\natural_*\widetilde{Q}(\widetilde{\varsigma}) = \overline{Q}(\varsigma).$$

Thus, it follows from the main Theorem in [13] that the contact metric structure on $\mathtt{T}_1^*\mathtt{N}$ is H-contact if and only if $(\mathtt{N},\mathfrak{g})$ is 2-stein. □

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NORM RETRIEVAL IN DYNAMICAL SAMPLING FORM

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Abstract. In this article, we study the construction of norm retrievable frames that have a dynamical sampling structure. For a closed subspace W of $\mathbb{R}^n$, we show that when the collection of subspaces $\{A^\ell W\}_{i\in I}$ is norm retrievable in $\mathbb{R}^n$ for a unitary or Jordan operator A, then there always exists a collection of norm retrievable frame vectors that have a dynamical sampling structure in $\mathbb{R}^n$.

1. Introduction

Given a signal $x \in \mathbb{H}$ in a separable Hilbert space with a given orthonormal basis $\{e_i\}_{i\in I}$ in $\mathbb{H}$, Parseval's identity allows us to reconstruct the signal x from the measurements$\{\langle x, e_i\rangle\}_{i\in I}$. The set of coefficients $\{\langle x, e_i\rangle\}_{i\in I}$ is unique. We are unable to recreate the signal x from the remaining data if a measurement is missing or damaged. We can see the need for a set of vectors that allows for some loss resilience while also having a reconstruction property similar to Parseval's identity. A frame $\{x_i\}_{i\in I}$ for $\mathbb{H}$ allows for redundancy while preserving a structure so that reconstruction is possible. Now, the set of measurements $\{\langle x, x_i\rangle\}_{i\in I}$ is not necessarily unique.

We can reconstruct the signal x from the measurements $\{\langle x, x_i\rangle\}$ using the frame vectors $\{x_i\}_{i\in I}$ in $\mathbb{H}$. But let's say that the measurements' phase was lost or was impossible to determine. These restrictions may apply in a setting like a tomography or crystallography. We are unable to create the exact signal x when we just have the phaseless measurements $\{|\langle x, x_i\rangle|\}$. The idea of phase retrieval for Hilbert space frames was first proposed by Casazza, Balan, and Edidin [11] in 2006 to extract the phase of a signal from a redundant linear system using the intensity measurements $\{|\langle x, x_i\rangle|\}$. They showed that we require a minimum $2n-1$ vectors to have phase retrieval in $\mathbb{R}^n$. Phase retrieval is a stronger condition than being a

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frame. A set of vectors does not meet the requirements for phase retrieval if it is not a frame. Norm retrieval is a different condition that is less strong than phase retrieval. The notion of norm retrieval is described in [10], a collection of vectors performs norm retrieval if two vectors in the Hilbert space have the same intensity measurements, then they have the same norm. The phase retrieval conditions are relaxed by the norm retrieval property. There exist norm retrievable sets that are not phase retrievable, but every phase retrievable set is also a norm retrievable set. Fewer vectors are needed for norm retrieval compared to phase retrieval. For instance, orthonormal bases are not phase retrievable but they are norm retrievable sets.

When $\Omega \subseteq \{1, 2, ..., n\}$ are the coarse sample points in $\mathbb{H}$, the measurements $\{\langle x, e_i\rangle : i \in \Omega\}$ have insufficient information in general to recover the original signal x. Given an operator A on $\mathbb{H}$, suppose the signal $x \in \mathbb{H}$ evolves through the operator A over time to become $A^\ell x$ at time ℓ. Now, we can have extra information $\{A^\ell x(i) : i \in \Omega\}$ about the signal x. In [6], Aldroubi and his collaborators recently showed that x can be recovered from the measurements of $\{\langle A^\ell x, e_i\rangle : \ell = 0, 1, \dots, L;\ i \in \Omega\}$ if and only if the time-space samples is a set of frame vectors.

In this article, We will look at how these two most recent advancements in frame theory cross. We will attempt to demonstrate when norm retrieval is feasible under the unitary and the Jordan operators using samples obtained from the dynamical sampling structure. We consider the norm retrieval problem in the dynamical sampling setting in the finite-dimensional real Hilbert space $\mathbb{R}^n$.

2. Preliminaries

In this section, we provide some of the terminology and findings in frame theory, phase retrieval, norm retrieval, and dynamical sampling that are essential to understanding the conclusions we reach.

2.1. Frames.

Definition 1. *[23] A set of vectors $\{x_i\}_{i\in I}$ is said to be a **frame** in a Hilbert space $\mathbb{H}$ if there exist constants A and B with $0 < A \le B < \infty$ such that*

$$A||x||^2 \le \sum_{i\in I} |\langle x, x_i\rangle|^2 \le B||x||^2, \ \textit{for all} \quad x \in \mathbb{H}. \tag{1}$$

A and B are called upper and lower frame bounds of the frame $\{x_i\}_{i\in I}$, respectively. If $A = B$, then $\{x_i\}_{i\in I}$ is called a **tight frame**. The set of vectors $\{x_i\}_{i\in I}$ is called a **Parseval frame** if $A = B = 1$.

Let $\{e_i\}_{i\in I}$ be the standart orthonormal basis in $\ell^2(I)$. Given the set $\{x_i\}_{i\in I}$ in $\mathbb{H}$, the operator $\Phi : \mathbb{H} \to \ell^2(I)$, which is generated from the set $\{x_i\}_{i\in I}$,

$$\Phi(x) = \sum_{i\in I} \langle x, x_i\rangle e_i \quad \text{for all} \quad x \in \mathbb{H} \tag{2}$$

is called the **analysis operator** associated with the set $\{x_i\}_{i\in I}$.

The **synthesis operator** is the adjoint $\Phi^* : \ell^2(I) \to \mathbb{H}$ of the analysis operator Φ and is defined by

$$\Phi^* : \ell^2(I) \to \mathbb{H}, \quad \Phi^*((c_i)_{i\in I}) = \sum_{i\in I} c_i x_i. \tag{3}$$

The operator $S = \Phi^*\Phi : \mathbb{H} \to \mathbb{H}$ is called the **frame operator** of the frame $\{x_i\}_{i\in I}$ and is defined by

$$S(x) = \Phi^*\Phi(x) = \sum_{i\in I} \langle x, x_i\rangle x_i. \tag{4}$$

The operator S is a bounded, self adjoint, positive, and invertible operator that satisfies the operator inequality $AI \leq S \leq BI$ where A and B signify the upper and lower frame limits and I denotes the identity operator on $\mathbb{H}$. A frame is complete if it meets the lowest frame criteria. On the other hand, the upper frame condition requires a well-defined analysis operator.

Definition 2. *[25] Given a frame $\{x_i\}_{i\in I}$ in $\mathbb{H}$, if there are scalars $\{c_i\}_{i\in I}$ such that $\{c_i x_i\}_{i\in I}$ is a Parseval frame, then the frame $\{x_i\}_{i\in I}$ for a Hilbert space $\mathbb{H}$ is said to be scalable. If there is a value of $\delta > 0$, such that $c_i > \delta$ for all $i \in I$, then the set $\{x_i\}_{i\in I}$ is known as a strictly scalable frame.*

2.2. **Phase Retrieval and Norm Retrieval.** For the given set $\{x_i\}_{i\in I}$ in $\mathbb{H}$, the reconstruction of x up to a constant phase from the absolute value of the inner product of the coefficients measurements $\{\langle x, x_i\rangle\}_{i\in I}$ is called phase retrieval which defined by Balan, Casazza, and Edidin in [11].

Applications, where measurements of a signal can only identify by amplitude rather than the phase, are included in speech recognition [30], optics applications like X-ray crystallography [20, 29], quantum state tomography [28], and electron microscopy [27, 31]. Phase retrieval problem has been extensively studied in [10–13, 15–18, 24].

Definition 3. *[11] A collection of vectors $\{x_i\}_{i=1}^M$ in $\mathbb{R}^n$ is called **phase retrieval** if for all $x, y \in \mathbb{R}^n$ which satisfies $|\langle x, x_i\rangle| = |\langle y, x_i\rangle|$ for all $i = 1, .., M$, then $x = cy$ where $c = \pm 1$ in $\mathbb{R}^n$.*

Definition 4. *[10] A collection of vectors $\{x_i\}_{i=1}^M$ in $\mathbb{R}^n$ is called **norm retrieval** if for all $x, y \in \mathbb{R}^n$ which satisfies $|\langle x, x_i\rangle| = |\langle y, x_i\rangle|$ for all $i = 1, .., M$, then $||x|| = ||y||$.*

Lemma 1. *[17] In $\mathbb{R}^n$, if the number of n vectors $\{x_i\}_{i=1}^n$ do norm retrieval, they have to be orthogonal to each other.*

There is also the idea of phase and norm retrieval by projections, which is agree with our earlier definitions when the projections are one-dimensional.

Definition 5. *[10] Let $\{W_i\}_{i=1}^M$ be a collection of subspaces in $\mathbb{R}^n$ and define $\{P_i\}_{i=1}^M$ to be the orthogonal projections onto each of these subspaces. We say that $\{W_i\}_{i=1}^M$ (or $\{P_i\}_{i=1}^M$) yields* ***phase retrieval*** *if for $x, y \in \mathbb{R}^n$ satisfying $||P_i x|| = ||P_i y||$ for all $i = 1, ..., M$, then $x = cy$ for some scalar c with $c = \pm 1$.*

Definition 6. *[10] Let $\{W_i\}_{i=1}^M$ be a collection of subspaces in $\mathbb{R}^n$ and define $\{P_i\}_{i=1}^M$ to be the orthogonal projections onto each of these subspaces. We say that $\{W_i\}_{i=1}^M$ (or $\{P_i\}_{i=1}^M$) yields* ***norm retrieval*** *if for $x, y \in \mathbb{R}^n$ satisfying $||P_i x|| = ||P_i y||$ for all $i = 1, ..., M$, then $||x|| = ||y||$.*

Definition 7. *[11] A frame $\{x_i\}_{i=1}^M$ in $\mathbb{R}^n$ satisfies the* ***complement property*** *if for any index set $I \subset \{1, \dots M\}$, either span$\{x_i\}_{i\in I} = \mathbb{R}^n$ or span$\{x_i\}_{i\in I^c} = \mathbb{R}^n$.*

Theorem 1. *[11] A frame $\{x_i\}_{i=1}^M$ in $\mathbb{R}^n$ yields phase retrieval if and only if it has the complement property.*

2.3. **Dynamical Sampling.** Given a bounded operator A, a vector $b \in \mathbb{H}$ and $\ell \in \mathbb{N}$, we can get a collection of vectors $\{b, Ab, A^2 b, ...A^\ell b\}$ by applying the operator A to the vector b. The dynamical sampling problem which defined by Aldroubi, Davis, and Krishtal in [7] is looking for the conditions on the set of vectors $\{b_i \in \mathbb{H} : i \in \Omega, |\Omega| < dim(\mathbb{H})\}$, the operator A and $\ell_i \in \mathbb{N}$ such that the collection of vectors

$$\{b_i, Ab_i, ..., A^{\ell_i} b_i\}_{\{i\in\Omega, \ell_i\in\mathbb{N}\}}$$

is a frame in $\mathbb{H}$. In 2012, Aldroubi and his collaborators created a mathematical system for a dynamical sampling structure with results appearing in [5, 6]. The dynamical sampling problem gets the attention of other researchers and has been recently studied by [1–4, 8, 9, 14, 22, 26].

Let A be a matrix that can be written as $A^* = B^{-1} D B$ where D is a diagonal and B is an invertible matrix. Let $\{\lambda_j\}_{j\in J}$ be distinct eigenvectors of D and $\{P_j\}_{j\in J}$ denote the orthogonal projections in $\mathbb{H}$ onto the eigenspaces $\{E_j\}_{j\in J}$ of D associated to the eigenvalues $\{\lambda_j\}_{j\in J}$. Then we have the following result.

Theorem 2. *[6, Thm: 2.2] Let $\Omega \subseteq \{1, 2, \dots, n\}$ and $\{b_i : i \in \Omega\}$ be vectors in $\mathbb{R}^n$. Let D be a diagonal matrix and r_i be the degree of the D-annihilator of b_i. Then $\{D^j b_i : i \in \Omega; j = 0, 1, ..., l_i; l_i = r_i - 1\}$ is a frame of $\mathbb{R}^n$ if and only if $\{P_j(b_i) : i \in \Omega\}$ is a frame of E_j for all $j \in J$.*

The authors of [6] extended the Theorem 2 to non-diagonalizable operators as follows.

Theorem 3. *[6, Thm 2.6] Let J be a matrix in Jordan form as in 9. Let $\Omega \subseteq \{1, 2, \dots, n\}$ and $\{b_i : i \in \Omega\}$ be vectors in $\mathbb{R}^n$, r_i be the degree of the J-annihilator of the vector b_i and $l_i = r_i - 1$. Then the following propositions are equivalent.*

1. *The set of vectors $\{J^j b_i : i \in \Omega,\ j = 0, 1, ..., l_i,\}$ is a frame for $\mathbb{R}^n$.*
2. *For every $s = 1, .., n$, $\{P_s(b_i) : i \in \Omega\}$ is a frame for W_s.*

3. Results

We first start with creating a standard dynamical sampling system in $\mathbb{R}^n$ using a bounded linear operator A. Assume that the vector $b \in \mathbb{R}^n$ evolves through the operator A to become the vector $A^\ell b$ at time $\ell \in \mathbb{N}$. Let $\Omega \subseteq \{1, 2, ..., n\}$ be the sample points and define $A^\ell W = span\{A^\ell b_i \in \mathbb{R}^n; i \in \Omega\}$.

In [14], we show the construction of norm retrievable sets $\{A^\ell b_i\}_{\{\ell=0,1,...M, i\in\Omega\}}$ that arise from a dynamical sampling system in a finite-dimensional real Hilbert space $\mathbb{R}^n$. In this paper, we show the relations between the norm retrievable set of $\{A^\ell b_i\}_{\{\ell=0,1,...M,i\in\Omega\}}$ in $\mathbb{R}^n$ and the set of projections $\{P_\ell\}_{\{\ell=0,1,...M\}}$ under the unitary and Jordan operator, where P_ℓ is the orthogonal projection onto the subspace $A^\ell W$.

First, we show that the collection of vectors $\{A^\ell b_i\}_{\{\ell=0,1,...M,i\in\Omega\}}$ is norm retrievable in $\mathbb{R}^n$ if the identity operator on $\mathbb{R}^n$ is in the spanning set of the rank one projection of the vectors $\{b_i \in \mathbb{R}^n : i \in \Omega, |\Omega| < n\}$ as shown in the following Theorem 4.

Theorem 4. *Let A be a bounded linear operator on $\mathbb{R}^n$ and $\{b_i \in \mathbb{R}^n : i \in \Omega, |\Omega| < n\}$ be a collection of vectors in $\mathbb{R}^n$. The collection of vectors $\{A^\ell b_i\}_{\{0\le\ell\le M,i\in\Omega\}}$ accomplishes norm retrieval condition in $\mathbb{R}^n$ for some $M \in \mathbb{N}$ if there exists a solution $\{C_{\ell,i}\}_{\{0\le\ell\le M,i\in\Omega\}}$ to the following system of linear equations*

$$\sum_{\ell,i} C_{\ell,i}|\langle e_j, A^\ell b_i\rangle|^2 = 1 \tag{5}$$

$$\sum_{\ell,i} C_{\ell,i}\langle e_j, A^\ell b_i\rangle\langle e_k, A^\ell b_i\rangle = 0 \tag{6}$$

for all $j, k = 1, 2, ..n$ with $j \neq k$.

Proof. Assume that given the operator A on $\mathbb{R}^n$ and the collection of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega, |\Omega| < n\}$, the measurements, $|\langle x, A^\ell b_i\rangle| = |\langle y, A^\ell b_i\rangle| \quad \forall\, 0 \le \ell \le M, i \in \Omega$ for fixed $x, y \in \mathbb{R}^n$, are known. Then we have

$$\langle x - y, A^\ell b_i\rangle = 0 \quad \text{or} \quad \langle x + y, A^\ell b_i\rangle = 0 \quad \forall \ell, i$$

and

$$\langle x - y, \langle x + y, A^\ell b_i\rangle A^\ell b_i\rangle = \langle x - y, A^\ell b_i(A^\ell b_i)^*(x + y)\rangle = 0 \quad \forall \ell, i.$$

Given any scalar value $C_{\ell,i}$, we have $C_{\ell,i}\langle x - y, A^\ell b_i(A^\ell b_i)^*(x + y)\rangle = 0 \quad \forall \ell, i.$

If $I \in \text{span}\{A^\ell b_i(A^\ell b_i)^*\}_{\{0\le\ell\le M,i\in\Omega\}}$, then $\langle x - y, x + y\rangle = 0$ and $||x|| = ||y||$.

Now, we show that $I \in \text{span}\{A^\ell b_i(A^\ell b_i)^*\}_{\{0\le\ell\le M,i\in\Omega\}}$ if and only if the equations (5) and (6) have a solution. Let $\{e_j\}_{j=1}^n$ be the standard orthonormal bases in $\mathbb{R}^n$. Then, we can express any vector $A^\ell b_i \in \mathbb{R}^n$ as

$$A^\ell b_i = \begin{bmatrix} \langle e_1, A^\ell b_i \rangle \\ \langle e_2, A^\ell b_i \rangle \\ \vdots \\ \langle e_n, A^\ell b_i \rangle \end{bmatrix}.$$

Hence, we have

$$A^\ell b_i (A^\ell b_i)^* = \begin{bmatrix} |\langle e_1, A^\ell b_i \rangle|^2 & \langle e_1, A^\ell b_i \rangle \langle e_2, A^\ell b_i \rangle & \cdots \langle e_1, A^\ell b_i \rangle \langle e_n, A^\ell b_i \rangle \\ \langle e_2, A^\ell b_i \rangle \langle e_1, A^\ell b_i \rangle & |\langle e_2, A^\ell b_i \rangle|^2 & \cdots \langle e_2, A^\ell b_i \rangle \langle e_n, A^k b_i \rangle \\ \vdots & \vdots & \vdots \\ \langle e_n, A^\ell b_i \rangle \langle e_1, A^\ell b_i \rangle & \langle e_n, A^\ell b_i \rangle \langle e_2, A^{\ell_i} b_i \rangle & \cdots |\langle e_n, A^\ell b_i \rangle|^2 \end{bmatrix}.$$

The linear equation systems in (5) and (6) have a solution if and only if the identity operator $I \in \text{span}\{A^\ell b_i (A^\ell b_i)^*\}_{\{\ell=0,1,\dots M,\, i\in\Omega\}}$. If so, we also have the collection of vectors $\{A^\ell b_i\}_{\{\ell=0,1,\dots M,\, i\in\Omega\}}$ which does norm retrieval in $\mathbb{R}^n$ as demonstrated in the following example. □

Example 1. *Let*

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then the set

$$F = \{b, Ab, A^2b, A^3b\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

contains an orthogonal basis. Hence, it does norm retrieval. Since the number of vectors is less than 5, it does not do phase retrieval in $\mathbb{R}^3$. Note that the span of the rank one operators generated by the vectors $\{b, Ab, A^2b, A^3b\}$ contains the identity operator.

$$bb^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Ab(Ab)^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^2b(A^2b)^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad A^3b(A^3b)^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and

$$I = bb^* + \frac{1}{2} A^2b(A^2b)^* + \frac{1}{2} A^3b(A^3b)^*.$$

When A is an $n \times n$ diagonal operator, the authors in [3, Thm.3] showed that the set of vectors $\{A^\ell b_i\}_{0\leq \ell\leq M, i\in\Omega\}}$ is a scalable frame if and only if there exists a positive solution $\{C_{\ell,i}\}_{0\leq \ell\leq M, i\in\Omega\}}$ to the system of equations in (5) and (6). Theorem 4 illustrates that if the solution $\{C_{\ell,i}\}_{0\leq \ell\leq M, i\in\Omega\}}$ to the system of linear equations in (5) and (6) is not a positive solution, there exist norm retrievable frames $\{A^\ell b_i\}_{0\leq \ell\leq M, i\in\Omega\}}$ which are not scalable frames.

Theorem 4 does not give the conditions on the operator A, the set of sample points $\{b_i \in \mathbb{R}^n : i \in \Omega,\ |\Omega| < n\}$ and the time increments ℓ but we show later sections how it works to obtain dynamical sampling frame which does norm retrieval.

Theorem 5. *[10] Given a collection of vectors $\{x_i\}_{i=1}^M$ in a Hilbert space $\mathbb{H}^n$. The following statements are equivalent to each other.*
(1) *The set of vectors $\{x_i\}_{i=1}^M$ is phase retrievable in $\mathbb{H}^n$*
(2) *The set of vectors $\{Ax_i\}_{i=1}^M$ is phase retrievable for all invertible operator A on $\mathbb{H}^n$*
(3) *The set of vectors $\{Ax_i\}_{i=1}^M$ is norm retrievable for all invertible operator A on $\mathbb{H}^n$.*

Given the collection of vectors $\{b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ in $\mathbb{R}^n$. Let $W = span\{b_i \in \mathbb{R}^n; i \in \Omega\}$. For every $\ell \in \mathbb{N}$, the subspaces, which are generated by iteration of W under the operator A, can be defined as

$$A^\ell W = span\{A^\ell b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\} \subset \mathbb{R}^n.$$

Let $\{P_\ell\}$ be the orthogonal projections from $\mathbb{R}^n$ onto $A^\ell W$ for each $\ell \in \mathbb{N}$. Theorem 5 states that if the collection of vectors $\{b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ is phase retrievable in W, then the collection of vectors $\{A^\ell b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ is phase retrievable in $A^\ell W$ for every $\ell \in \mathbb{N}$ when A is an invertible operator on $\mathbb{R}^n$. Assume there exists an $M \in \mathbb{N}$ such that $\mathbb{R}^n = span\{A^\ell b_i\}_{\{i\in\Omega,\, \ell=0,1,\ldots M\}}$. The collection of vectors $\{A^\ell b_i\}_{\{i\in\Omega\}}$ satisfies phase retrieval in $A^\ell W$ for every $\ell = 0, 1, \ldots M$ but it does not imply that $\{A^\ell b_i\}_{\{i\in\Omega\, \ell=0,1,\ldots M\}}$ is phase retrievable in $\mathbb{R}^n$.

Example 2. *Define* $W = span\{e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, e_1 + e_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}\}.$

Let A be an invertible operator on $\mathbb{R}^3$ such that $Ae_1 = e_2$ and $Ae_2 = e_3$. The iteration of the subspace W under A can be shown as

$$AW = span\{e_2, e_3, e_2 + e_3\}.$$

The collection of vectors in $\{e_1, e_2, e_1+e_2\}$ and $\{e_2, e_3, e_2+e_3\}$ is phase retrievable in W and AW, respectively. On the other hand, the collection of vectors $\{e_1, e_2, e_3, e_1+e_2, e_2+e_3\}$ is not phase retrievable because when we get the partion $\{e_1, e_2, e_1+e_2\}$ and $\{e_3, e_2+e_3\}$ of the set $\{e_1, e_2, e_3, e_1+e_2, e_2+e_3\}$, neither of these sets spans $\mathbb{R}^3$. This implies that the collection of vectors $\{e_1, e_2, e_3, e_1+e_2, e_2+e_3\}$ does not satisfy the complementary property (Definition 7) and fails the phase retrieval condition in R^3.

Theorem 6. *Let the set of vectors $\{b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ is phase retrievable in $W \subset \mathbb{R}^n$ and A is an invertible operator on $\mathbb{R}^n$. The collection of vectors $\{A^\ell b_i\}_{\{i\in\Omega\, \ell=0,1,\ldots M\}}$ is norm retrievable in $\mathbb{R}^n$ if the set of projections $\{P_\ell\}_{\ell=0}^M$ is norm retrievable in $\mathbb{R}^n$ for some $M \in \mathbb{N}$, where P_ℓ is the orthogonal projection onto the subspace $A^\ell W = span\{\{A^\ell b_i\}_{i\in\Omega}\}$.*

Proof. For $x, y \in \mathbb{R}^n$, assume $|\langle x, A^\ell b_i\rangle| = |\langle y, A^\ell b_i\rangle|$ for all $i \in \Omega$, $\ell = 0, 1, \ldots M$. Let P_ℓ be the orthogonal projection onto the subspace $A^\ell W$ for each ℓ. Thus,

$$P_\ell A^\ell b_i = A^\ell b_i \quad \text{and} \quad |\langle P_\ell x, P_\ell A^\ell b_i\rangle| = |\langle P_\ell y, P_\ell A^\ell b_i\rangle|, \text{ for all } i \in \Omega.$$

According to Theorem 5, the set of vectors $\{A^\ell b_i\}_{i\in\Omega}$ is phase retrievable (and consequently norm retrievable) in $A^\ell W$ for all ℓ since A is an invertible operator and the collection of vectors $\{b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ performs phase retrieval in W. This states that $||P_\ell x|| = ||P_\ell y||$ for all $\ell = 0, 1, \ldots M$. By our supposition, $\{P_\ell\}_{\ell=0}^M$ is norm retrievable in $\mathbb{R}^n$ and we have $||x|| = ||y||$.

□

3.1. **Iteration of Subspaces Under the Unitary and Jordan Operator.** We can do norm retrieval more smoothly if our dynamical sampling operator is unitary.

Given the index set $\Omega \subset \{1, 2, \ldots, n\}$ and the orthonormal bases $\{e_i\}_{i=1}^n$ of R^n. Suppose U is a unitary operator on R^n. Let $W = span\{e_i : i \in \Omega, |\Omega| < n\}$ and $U^\ell W = span\{U^\ell e_i : i \in \Omega, |\Omega| < n\}$ for any $\ell \in \mathbb{N}$. Given any $\ell \in \mathbb{N}$, since U is a unitary operator, it preserves the inner product. Thus, we have $\langle U^\ell e_i, U^\ell e_k\rangle = \langle e_i, e_k\rangle = 0$ for all $i \neq k$. Which says that $\{U^\ell e_i\}_{i\in\Omega}$ is an orthonormal basis for $U^\ell W$ for each ℓ.

Lemma 2. *Suppose $W = span\{e_i : i \in \Omega, |\Omega| \leq n\}$ and $U^\ell W = span\{U^\ell e_i : i \in \Omega, |\Omega| < n\}$ for $\ell \geq 0$, where U is a unitary operator on $\mathbb{R}^n$ and $\{e_i\}_{i=1}^n$ is an orthonormal bases of R^n. Let P_ℓ be the orthogonal projection onto $U^\ell W$ for each ℓ. If the collection of projections $\{P_\ell\}_{\ell=0}^M$ is norm retrievable in $\mathbb{R}^n$ for some $M \in \mathbb{N}$, then the collection of vectors $\{U^\ell e_i\}_{\{i\in\Omega,\ell=0,1,..M\}}$ is norm retrievable in $\mathbb{R}^n$.*

Proof. For $x, y \in \mathbb{R}^n$, assume that $|\langle x, U^\ell e_i\rangle| = |\langle y, U^\ell e_i\rangle|$ for any $i \in \Omega$ and $\ell = 0, 1, ..M$. Since $U^\ell e_i \in U^\ell W$ for any $\ell = 0, 1, ..M$, we see that $P_\ell U^\ell e_i = U^\ell e_i$ and

$$\begin{aligned} |\langle x, U^\ell e_i\rangle| = |\langle y, U^\ell e_i\rangle| &\Longrightarrow |\langle x, P_\ell U^\ell e_i\rangle| = |\langle y, P_\ell U^\ell e_i\rangle| \\ &\Longrightarrow |\langle P_\ell x, U^\ell e_i\rangle| = |\langle P_\ell y, U^\ell e_i\rangle|. \end{aligned}$$

For each fixed ℓ, since P_ℓ is a projection on $U^\ell W$ and $\{U^\ell e_i\}_{i\in\Omega}$ is an orthonormal basis in $U^\ell W$, we have

$$||P_\ell x|| = \sum_{i\in\Omega} |< P_\ell x, U^\ell e_i >|^2 = \sum_{i\in\Omega} |< P_\ell y, U^\ell e_i >|^2 = ||P_\ell y|| \qquad (7)$$

By assumption, since the collection of projections $\{P_\ell\}_{\ell=0}^M$ is norm retrievable in $\mathbb{R}^n$, we have $||x|| = ||y||$.

□

In Lemma 2, the collection of vectors $\{U^\ell e_i : i \in \Omega, |\Omega| < n\}$ in $U^\ell W$ is orthonormal because U is a unitary operator. Obtaining orthonormal bases as sample sets is a strong condition but we can reduce this presumption as the following lemma.

Corollary 1. *Let U be a unitary operator on $\mathbb{R}^n$. For a collection of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega, |\Omega| < n\}$ in $\mathbb{R}^n$, define $W = span\{b_i \in \mathbb{R}^n : i \in \Omega, |\Omega| < n\}$ and$U^\ell W = span\{U^\ell b_i : i \in \Omega, |\Omega| <\}$ for $\ell \in \mathbb{N}$. Let P_ℓ be the orthogonal projection onto $U^\ell W$ for $\ell \in \mathbb{N}$. If the collection of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ is norm retrievable in W and the set of projections $\{P_\ell\}_{\{\ell=0,1,...M\}}$ is norm retrievable in $\mathbb{R}^n$ for some $M \in \mathbb{N}$, then the collection of vectors $\{U^\ell b_i\}_{\{i\in\Omega,\ell=0,1,...M\}}$ is norm retrievable in $\mathbb{R}^n$.*

Proof. For $x, y \in \mathbb{R}^n$, assume that $|\langle x, U^j b_i\rangle| = |\langle y, U^j b_i\rangle| \quad \forall i \in \Omega, \ell = 0, 1, ..M$. For each fixed ℓ, since the unitary operators preserve norm retrieval condition, the collection of vectors $\{U^\ell b_i : i \in \Omega, |\Omega| < n\}$ is norm retrievable in $U^\ell W$. This says that for any given $x, y \in \mathbb{R}^n$ and $\ell \in \mathbb{N}$, $|\langle x, U^j b_i\rangle| = |\langle y, U^j b_i\rangle|$, $\forall i \in \Omega$ implies that $||P_\ell x|| = ||P_\ell y||$. Since we assumed that the set of projections $\{P_\ell\}_{\{\ell=0,1,...M\}}$ is norm retrievable in $\mathbb{R}^n$, we have $||x|| = ||y||$.

□

In Corollary 1 and Lemma 2, we assumed that the set of projections $\{P_\ell\}_{\{\ell=0,1,...M\}}$ is norm retrievable in $\mathbb{R}^n$ for some $M \in \mathbb{N}$. In general, we do not know whether such an $M \in \mathbb{N}$ exists or not. Now that we have a condition, we can guarantee that the projection set $\{P_\ell\}_{\{\ell=0,1,...M\}}$ performs norm retrieval on $\mathbb{R}^n$. We need the definition of fusion frames defined in [19].

Definition 8. *[19] Let I be an index set and $\{v_i\}_{i\in I}$ be a family of weights. That is $v_i > 0$ for all $i \in I$. Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of a Hilbert space $\mathbb{H}$ and P_{Wi} is the orthogonal projection onto the subspace Wi for each $i \in I$. Then $\{(W_i, v_i)\}_{i\in I}$ is a **fusion frame** for $\mathbb{H}$, if there exists constants $0 < A \leq B < \infty$ such that*

$$A||x||^2 \leq \sum_{i\in I} {v_i}^2 ||P_{W_i}(x)||^2 \leq B||x||^2, \ \textit{for all} \quad x \in \mathbb{H}. \tag{8}$$

*A and B are called the fusion frame bounds. The family (W_i, v_i) is called a **Parseval fusion frame** if $A = B = 1$ and a **tight fusion frame** if $A = B$.*

Theorem 7. *Let U be a unitary operator on $\mathbb{R}^n$ and $\{b_i \in \mathbb{R}^n : i \in \Omega \quad |\Omega| < n\}$ be a set of orthonormal vectors in $\mathbb{R}^n$. The set of vectors $\{U^\ell b_i : i \in \Omega, \quad \ell = 0, 1, ...M\}$ is a tight frame in $\mathbb{R}^n$ if and only if the set of orthogonal projections $\{P_\ell\}_{\{\ell=0,1,...M\}}$ is a tight fusion frame with weights $v_\ell = 1$ for all ℓ, where P_ℓ is the orthogonal projection onto $U^\ell W$ for each ℓ.*

Proof. ($\Longrightarrow$) Suppose the set of vectors $\{U^\ell b_i : i \in \Omega,\, \ell = 0, 1, ...M\}$ does tight frame in $\mathbb{R}^n$ with frame bound $A > 0$. Then, given any $x \in \mathbb{R}^n$, we can write

$$||x||^2 = \frac{1}{A} \sum_{i\in\Omega,\ell=0,1,\ldots,M} |\langle x, U^\ell b_i\rangle|^2.$$

Since $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ is a set of orthonormal vectors in $\mathbb{R}^n$ and U is a unitary operator on $\mathbb{R}^n$, $\{U^\ell b_i : i \in \Omega\}$ is also orthonormal set of vectors in $U^\ell W$ for each ℓ. Hence, the orthogonal projection P_ℓ onto the subspace $U^\ell W = span\{U^\ell\ b_i : i \in \Omega\}$ can be written as

$$P_\ell(x) = \sum_{i\in\Omega} \langle x, U^\ell b_i\rangle U^\ell b_i.$$

Thus,

$$||x||^2 = \frac{1}{A} \sum_{i\in\Omega,\ell=0,1,\ldots,M} |\langle x, U^\ell b_i\rangle|^2 = \frac{1}{A} \sum_{\ell=0,1,\ldots,M} ||P_\ell(x)||^2$$

and the set of orthogonal projections $\{P_\ell\}_{\{\ell=0,1,\ldots,M\}}$ is a A -tight fusion frame with weights $v_\ell = 1$.

($\Longleftarrow$) These follow from the definition of a tight fusion frame with weights $v_\ell = 1$ for all ℓ.

□

If $\{b_i \in \mathbb{R}^n : i \in \Omega \quad |\Omega| < n\}$ is a set of vectors that are orthogonal but not orthonormal in $\mathbb{R}^n$, then the set $\{U^\ell b_i : i \in \Omega, \quad \ell = 0, 1, ...M\}$ is not necessarily a tight frame in $\mathbb{R}^n$ anymore. In this case, we have the following corollary that follows from Theorem 5, Lemma 1 and Corollary 1.

Corollary 2. *Let U be a unitary operator on $\mathbb{R}^n$ and $\{b_i \in \mathbb{R}^n : i \in \Omega \quad |\Omega| < n\}$ consists of orthogonal vectors in $\mathbb{R}^n$. The set of vectors $\{U^\ell b_i : i \in \Omega,\, \ell = 0, 1, ...M\}$ is norm retrievable in $\mathbb{R}^n$ if $x \in span\{P_\ell(x)\}_{\ell=0}^M$, for any $x \in \mathbb{R}^n$.*

Now, we are interested in the linear operator A on $\mathbb{R}^n$ that has all real eigenvalues and is unitarily similar to the Jordan form. We want to construct subspaces $A^\ell W$ in $\mathbb{R}^n$ which are not necessarily orthogonal to each other and show the relations between the norm retrievable set of vectors $\{A^\ell b_i\}_{\{\ell=0,1,\ldots M, i\in\Omega\}}$ in $\mathbb{R}^n$ and the set of projections $\{P_\ell\}_{\{\ell=0,1,\ldots M\}}$, where P_ℓ is the orthogonal projection onto the subset $A^\ell W$. To set up the following construction, we apply the notation from [6].

Suppose $J \in M_n(\mathbb{R})$ is a Jordan matrix that has all real eigenvalues, then we have

$$J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_s \end{pmatrix}. \tag{9}$$

For each $j = 1, 2, ..s$, $J_j = \lambda_j I_j + N_j$ where I_j is an $r_j \times r_j$ identity matrix and N_j is a $r_j \times r_j$ nilpotent block-matrix of the form

$$N_j = \begin{pmatrix} N_{j_1} & 0 & \cdots & 0 \\ 0 & N_{j_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{j_i} \end{pmatrix}. \tag{10}$$

Each N_{ji} is a $r_j^i \times r_j^i$ cyclic nilpotent matrix of the form

$$N_{ji} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{11}$$

with $r_j^1 \geq r_j^2 \geq ... \geq r_j^i$ and $r_j^1 + r_j^2 + ... + r_j^i = r_j$. The matrix J has distinct eigenvalues λ_j, $j = 1, 2, ..s$ and $r_1 + r_2 + ... + r_s = n$.

Before we state our theorem related to the Jordan form, we would like to give an illustrative example to interested readers.

Example 3. *Let* $J = \lambda I + N \in \mathbb{R}^{4\times 4}$ *and assume that*

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

where

$$N_i = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

for $i = 1, 2$. *Then, we have the subspaces*

$$\begin{aligned} W &= span\{e_1, e_3\} \\ JW &= span\{\lambda e_1 + e_2, \lambda e_3 + e_4\} \\ J^2W &= span\{\lambda^2 e_1 + 2\lambda e_2, \lambda^2 e_3 + 2\lambda e_4\} \end{aligned}$$

Let P_ℓ *be the orthogonal projection onto the subspace* $J^\ell W$ *for each* $\ell = 0, 1, 2$. *For fixed* ℓ,

$$||J^\ell e_1||^2 = \lambda^{2\ell} + \ell^2\lambda^{2(\ell-1)} = ||J^\ell e_3||^2.$$

Let $c_\ell = \lambda^{2\ell} + \ell^2\lambda^{2(\ell-1)}$ for $\ell = 0, 1, 2$, then the orthogonal projection P_ℓ onto the subspace $J^\ell W$ can be written as

$$P_\ell(x) = \frac{1}{c_\ell}\sum_{i=1,3}\langle x, J^\ell e_i\rangle J^\ell e_i \quad and \quad ||P_\ell(x)||^2 = \frac{1}{c_\ell}\sum_{i=1,3}|\langle x, J^\ell e_i\rangle|^2.$$

If $\lambda = 0$, then $P_0+P_1 = I$ and the set of vectors $\{J^\ell e_i\}_{i=1,3,\ell=0,1,2} = \{e_1, e_2, e_3, e_4\}$ is an orthonormal bases and it does norm retrieval in $\mathbb{R}^n$. Assume $\lambda \neq 0$.

$$For\ any\ \ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^n, c_0P_0(x) = \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{bmatrix}$$

$$c_1P_1(x) = \begin{bmatrix} \lambda^2x_1 + \lambda x_2 \\ \lambda x_1 + x_2 \\ \lambda^2x_3 + \lambda x_4 \\ \lambda x_3 + x_4 \end{bmatrix} \ and\ \ c_2P_2(x) = \begin{bmatrix} \lambda^4x_1 + 2\lambda^3x_2 \\ 2\lambda^3x_1 + 4\lambda^2x_2 \\ \lambda^4x_3 + 2\lambda^3x_4 \\ 2\lambda^3x_3 + 4\lambda^2x_4 \end{bmatrix}.$$

This states that $\frac{\lambda^4+1}{2\lambda^2}c_0P_0 - c_1P_1 + \frac{1}{\lambda^2}c_2P_2 = I$ and the set of projections $\{P_\ell\}_{\ell=0,1,2}$ does norm retrieval in $\mathbb{R}^n$ since the coefficients $\{c_\ell\}_{\ell=0,1,2}$ are independent from choice of x. This implies that the set of vectors $\{J^\ell e_i\}_{i=1,3,\ell=0,1,2}$ does norm retrieval in $\mathbb{R}^n$.

Theorem 8. *Let $J \in M_n(\mathbb{R})$ be a Jordan matrix in the form of Equation (9) that has all real eigenvalues and $W_j = span\{e_{k_{ji}} : j = 1, 2, ..., s\}$, where s is the number of distinct eigenvalues in J and $e_{k_{ji}}$ is the standard orthonormal bases vector of $\mathbb{R}^n$ corresponding to the first row of the cyclic nilpotent matrix N_{ji} in (11). Let $P_{\ell j}$ be the orthogonal projections onto the subsets $J^\ell W_j$. Suppose the order r_j^i of N_{ji} is the same for all i, j. Then the collection of vectors $\{J^\ell e_{k_{ji}}\}_{\{j=1,2,...,s,1\leq i\leq k(j),\ell=0,1,...r_j^i\}}$ is norm retrievable in $\mathbb{R}^n$, where $k(j)$ is the number of cyclic nilpotent matrices N_{ji} in N_j if the set of projections is norm retrievable $\mathbb{R}^n$.*

Proof. By choice of $e_{k_{ji}}$ as a standard orthonormal basis vector corresponding to the first row of N_{ji}, the set of vectors $\{J^\ell e_{k_{ji}}\}_{\{j=1,2,...,s,1\leq i\leq k(j)\}}$ forms an orthogonal bases in $J^\ell W_j$ for each ℓ. As shown in Example 3, for fixed ℓ, the norm of the vectors $J^\ell e_{k_{ij}}$ is the same for all i, j. Suppose $||J^\ell e_{k_{ij}}|| = c_{\ell j}$ for some $c_{\ell j} \in \mathbb{R}$. Since the set of vectors $\{J^\ell e_{k_{ji}}\}_{\{j=1,2,...,s,1\leq i\leq k(j)\}}$ forms an orthogonal basis in $J^\ell W_j$ for each ℓ, the set of vectors $\{\frac{1}{c_{\ell j}}J^\ell e_{k_{ji}}\}_{\{j=1,2,...,s,1\leq i\leq k(j)\}}$ forms an orthonormal bases in $J^\ell W_j$ for each ℓ. For fixed ℓ, the orthogonal projection $P_{\ell j}$ onto $J^l W_j$ can be defined by

$$P_{\ell j}(x) = \sum_{i,j}\langle x, \frac{1}{c_{\ell j}}J^\ell e_{k_{ji}}\rangle \frac{1}{c_{\ell i}}J^\ell e_{k_{ji}}.$$

This implies $\{J^l e_{k_{ij}}\}$ does norm retrieval in $\mathbb{R}^n$ if and only if $I = \sum_\ell c_{\ell,i} P^i_\ell$. Since the constants $c_{\ell j}$ is same for fixed ℓ, for any $x \in \mathbb{R}^n$, we have

$$||P_{\ell j}(x)||^2 = \frac{1}{c^2_{\ell j}} \sum_{i,j} |\langle x, J^\ell e_{k_{ji}} \rangle|^2.$$

To show that the set of vectors $\{J^\ell e_{k_{ji}}\}_{\{j=1,2,\ldots,s,1\leq i\leq k(j),\ell=0,1,\ldots r^i_j\}}$ is norm retrievable in $\mathbb{R}^n$, assume $|\langle x, J^\ell e_{k_{ji}}\rangle| = |\langle y, J^\ell e_{k_{ji}}\rangle|$ for all ℓ, j, i for any given $x, y \in \mathbb{R}^n$. Since the constants $c_{\ell j}$ are independent of the choice of x and y, we have

$$||P_{\ell j}(x)||^2 = \frac{1}{c^2_{\ell j}} \sum_{i,j} |\langle x, J^\ell e_{k_{ji}} \rangle|^2 = \frac{1}{c^2_{\ell j}} \sum_{i,j} |\langle y, J^\ell e_{k_{ji}} \rangle|^2 = ||P_{\ell j}(y)||^2.$$

We assumed that the set of orthogonal projections $\{P^\ell e_{k_{ji}}\}_{\{j=1,2,\ldots,s,\ell=0,1,\ldots r^i_j\}}$ is norm retrievable in $\mathbb{R}^n$. This implies that $||x|| = ||y||$ and the collection of vectors $\{J^\ell e_{k_{ji}}\}_{\{j=1,2,\ldots,s,1\leq i\leq k(j),\ell=0,1,\ldots r^i_j\}}$ is norm retrievable in $\mathbb{R}^n$.

□

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