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Spectral Characteristics of the Sturm-Liouville Problem with Spectral Parameter-Dependent Boundary Conditions

Aynur Çöl[1]

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Abstract — We consider the Sturm-Liouville problem on the half line ($0 \leq x < \infty$), where the boundary conditions contain polynomials of the spectral parameter. We define the scattering function and present the spectrum of the boundary value problem. The continuity of the scattering function is discussed. In a special case, the Levinson-type formula is introduced, demonstrating that the increment of the scattering function's logarithm is related to the number of eigenvalues.

Keywords *Levinson-type formula, scattering function, spectral parameter dependent boundary condition*

Mathematics Subject Classification (2020) 34L25, 81U40

1. Introduction

Consider the boundary value problem

$$\ell v := -v'' + \varphi(x)v = \lambda^2 v,\ 0 \leq x < \infty \tag{1.1}$$

$$(\beta_3 v(0) - \alpha_3 v'(0))i\lambda^3 + (\beta_2 v(0) - \alpha_2 v'(0))\lambda^2 - (\beta_1 v(0) - \alpha_1 v'(0))i\lambda - \beta_0 v(0) + \alpha_0 v'(0) = 0 \tag{1.2}$$

known as a Sturm Liouville problem, where λ is a spectral parameter, the potential function $\varphi\left(x\right)$ is real valued such that

$$\int\limits_0^\infty (1+x)\left|\varphi\left(x\right)\right| dx < \infty \tag{1.3}$$

and for $\alpha_i,\ \beta_i \in \mathbb{R}$, $i = \overline{0,3}$, $\alpha_3 \neq 0$, and $\beta_3 \neq 0$,

$$(-1)^k \delta_{ik} \leq 0,\ k \in \{1,2\};\ \delta_{ik} = 0,\ k = 3 \ \text{ where } \ \delta_{ik} = \alpha_{i+k}\beta_i - \alpha_i \beta_{i+k} \tag{1.4}$$

An important part of scattering theory is the study of boundary value problems involving the spectral parameter. Sturm-Liouville problems with spectral parameter-dependent boundary conditions arise in studies of heat conduction problems and vibrating string problems. Cohen introduced a method to solve an initial-boundary value problem arising in the diffusion and heat flow theory [1]. Various examples of spectral problems that occur in mechanical engineering and contain an eigenparameter in the boundary conditions were presented in [2]. Moreover, problems with boundary conditions concerning spectral parameters were investigated in finite intervals [3–10] and on the half line in [11–14].

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Levinson's theorem provides a relation between the number of bound states of a quantum mechanical system and the phase shift of that system [15–17]. It is a fundamental tool in quantum mechanics and scattering theory, as it is responsible for solving the inverse scattering problem [18]. In work [19], the Levinson formula was obtained for Sturm Liouville operator, which is not only a necessary condition but also sufficient for the given collection $\{S(\lambda);\ \lambda_j;\ m_j\ (j=\overline{1,n})\}$ to be scattering data of the reconstructed equation. The Levinson-type formulas for boundary conditions containing a spectral parameter were studied in [20–22].

The paper aims to analyze the spectral characteristics of the Sturm-Liouville problem with a nonlinear spectral parameter in the boundary condition. In progress, we provide the scattering function and the spectrum of the boundary value problem (1.1) and (1.2), and present the relation between the number of eigenvalues and the argument's variation of the scattering function. This relation is referred to as the Levinson-type formula.

The remaining paper is structured as follows: Section 2 presents the scattering function and the spectrum for (1.1) and (1.2). Section 3 investigates the scattering function's continuity. Finally, section 4 derives the Levinson-type formula.

2. The Scattering Function $S(\lambda)$ and the Discrete Spectrum

Let (1.3) hold. Then, as known in [19], there exists a unique solution $e(\lambda,x)$ of (1.1) which holds the asymptotic behavior $\lim\limits_{x\to+\infty} e^{-i\lambda x}e(\lambda,x)=1$, for $\Im\geq 0$, and can be expressed as

$$e(\lambda,x)=e^{i\lambda x}+\int_x^\infty K(x,t)\,e^{i\lambda t}dt \tag{2.1}$$

called Jost solution. The function $e(\lambda,x)$ is analytic with respect to λ in the upper-half plane ($\Im>0$) and continuous on the real line. Moreover, the kernel function $K(x,t)$ is related as follows:

$$K(x,x)=\frac{1}{2}\int_x^\infty \varphi(\zeta)\,d\zeta$$

Let $\psi(\lambda,x)$ represent the solution of (1.1) with the conditions:

$$\psi(\lambda,0)=\alpha_0+i\alpha_1\lambda-\alpha_2\lambda^2-i\alpha_3\lambda^3 \quad \text{and} \quad \psi'(\lambda,0)=\beta_0+i\beta_1\lambda-\beta_2\lambda^2-i\beta_3\lambda^3$$

It is obvious that the solution $\psi(\lambda,x)$ holds (1.2).

Let $W[y;z]:=y'z-yz'$ denote the Wronskian. For any solutions $e(\lambda,x)$ and $e(-\lambda,x)$ of (1.1), the Wronskian $W[e(\lambda,x);e(-\lambda,x)]$ is independent of x and is equal to $2i\lambda$. Therefore, for all $\lambda\in\mathbb{R}\backslash\{0\}$, $e(\lambda,x)$ and $e(-\lambda,x)$ constitute a fundamental set of solutions of (1.1), and any solution $\psi(\lambda,x)$ of (1.1) can be expressed as

$$\psi(\lambda,x)=e(\lambda,x)\,\gamma_1(\lambda)+e(-\lambda,x)\,\gamma_2(\lambda) \tag{2.2}$$

By evaluating the following Wronskians of $e(\lambda,x)$ and $\psi(\lambda,x)$,

$$W[e(\lambda,x),\psi(\lambda,x)]=\gamma_2(\lambda)2i\lambda=\psi(\lambda,0)e'(\lambda,0)-\psi'(\lambda,0)e(\lambda,0)$$

and

$$W[e(-\lambda,x),\psi(\lambda,x)]=-\gamma_1(\lambda)2i\lambda=\psi(\lambda,0)e'(-\lambda,0)-\psi'(\lambda,0)e(-\lambda,0)$$

we find $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$ and substitute in (2.2). Let $\Theta(\lambda)$ and $\Theta_1(\lambda)$ be functions such that

$$\Theta(\lambda)=\psi(\lambda,0)e'(\lambda,0)-\psi'(\lambda,0)e(\lambda,0) \tag{2.3}$$

and

$$\Theta_1(\lambda) = \psi(\lambda,0)e'(-\lambda,0) - \psi'(\lambda,0)e(-\lambda,0) \tag{2.4}$$

Therefore, we obtain the solution of (1.1) with (1.2) such that

$$\psi(\lambda,x) = (2i\lambda)^{-1}\left[-\Theta_1(\lambda)e(\lambda,x) + \Theta(\lambda)e(-\lambda,x)\right] \tag{2.5}$$

Define the function

$$S(\lambda) = \Theta_1(\lambda)[\Theta(\lambda)]^{-1} \tag{2.6}$$

called the scattering function of (1.1) and (1.2).

We state some properties of $S(\lambda)$. Show that $\Theta(\lambda) \neq 0$, for all $\lambda \in \mathbb{R}\backslash\{0\}$. Assuming the contrary, then there exists a $\lambda_0 \in \mathbb{R}$, $\lambda_0 \neq 0$, such that

$$(\alpha_0 + i\alpha_1\lambda_0 - \alpha_2\lambda_0^2 - i\alpha_3\lambda_0^3)e'(\lambda_0,0) = (\beta_0 + i\beta_1\lambda_0 - \beta_2\lambda_0^2 - i\beta_3\lambda_0^3)e(\lambda_0,0)$$

Besides,

$$\begin{aligned} 2i\lambda_0 &= W[e(\lambda_0,0),\overline{e(\lambda_0,0)}] \\ &= e'(\lambda_0,0)\overline{e(\lambda_0,0)} - e(\lambda_0,0)\overline{e'(\lambda_0,0)} \\ &= |e(\lambda_0,0)|^2\, 2i\Im\left(\tfrac{\beta_0+i\beta_1\lambda_0-\beta_2\lambda_0^2-i\beta_3\lambda_0^3}{\alpha_0+i\alpha_1\lambda_0-\alpha_2\lambda_0^2-i\alpha_3\lambda_0^3}\right) \end{aligned}$$

From the result,

$$\frac{|e(\lambda_0,0)|^2\left[\alpha_1\beta_0 - \alpha_0\beta_1 + (\alpha_2\beta_1 - \alpha_1\beta_2)|\lambda_0|^2 + (\alpha_3\beta_2 - \alpha_2\beta_3)|\lambda_0|^4\right]}{\left|\alpha_0 + i\alpha_1\lambda_0 - \alpha_2\lambda_0^2 - i\alpha_3\lambda_0^3\right|^2} = -1$$

This is a contradiction since the left hand is positive, which proves the claim.

Therefore, firstly, $S(\lambda)$ is defined on $(-\infty,0)$ and $(0,\infty)$, and secondly, it is continuous in these intervals, which can be observed from the definition of $\Theta(\lambda)$. In section 3, the continuity of $S(\lambda)$ at $\lambda = 0$ is investigated. Next, $\Theta(\lambda)$ is analytic function of λ since $e(\lambda,0)$ and $e'(\lambda,0)$ are analytic in the upper half plane.

From the definition of $S(\lambda)$, we derive that the function $-1 - S(\lambda)$ belongs to the space $L_2(-\infty,\infty)$. Using (2.1) and substituting related expressions into the $\Theta(\lambda)$,

$$\begin{aligned} \Theta(\lambda) &= (\alpha_0 + i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3)\left\{i\lambda - K(0,0) + \int_0^\infty K_x(0,t)e^{i\lambda t}dt\right\} \\ &\quad -(\beta_0 + i\beta_1\lambda - \beta_2\lambda^2 - i\beta_3\lambda^3)\left\{1 + \int_0^\infty K(0,t)e^{i\lambda t}dt\right\} \\ &= (i\lambda)^4\left[\alpha_3 + O\left(\tfrac{1}{\lambda}\right)\right] \end{aligned} \tag{2.7}$$

as $|\lambda| \to \infty$. Similarly,

$$\begin{aligned} \Theta_1(\lambda) &= (\alpha_0 + i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3)\left\{-i\lambda - K(0,0) + \int_0^\infty K_x(0,t)e^{-i\lambda t}dt\right\} \\ &\quad -(\beta_0 + i\beta_1\lambda - \beta_2\lambda^2 - i\beta_3\lambda^3)\left\{1 + \int_0^\infty K(0,t)e^{-i\lambda t}dt\right\} \\ &= (i\lambda)^4\left[-\alpha_3 + O\left(\tfrac{1}{\lambda}\right)\right] \end{aligned} \tag{2.8}$$

Then, the following result is obtained:

$$-1-S(\lambda)=O\left(\frac{1}{\lambda}\right), \quad |\lambda|\to\infty \tag{2.9}$$

Therefore, $-1-S(\lambda)\in L_2(-\infty,\infty)$.

Lemma 2.1. For all $\lambda\in\mathbb{R}\backslash\{0\}$,

$$S(\lambda)=\overline{S(-\lambda)}, \quad |S(\lambda)|<1$$

Proof. Since $q(x)$ is real, it follows that $\overline{e(\lambda,0)}=e(-\lambda,0)$. For $\lambda\in\mathbb{R}\backslash\{0\}$, $\overline{\psi(\lambda,0)}=\psi(-\lambda,0)$ and $\overline{\psi'(\lambda,0)}=\psi'(-\lambda,0)$, it follows from (2.3) and (2.4) that $\overline{\Theta(\lambda)}=\Theta(-\lambda)$ and $\overline{\Theta_1(\lambda)}=\Theta_1(-\lambda)$, which shows $\overline{S(\lambda)}=S(-\lambda)$, for all $\lambda\in\mathbb{R}\backslash\{0\}$.

To show $|S(\lambda)|<1$, the following equality is obtained:

$$\begin{aligned}|S(\lambda)|^2 &= S(\lambda)\cdot\overline{S(\lambda)}\\ &= \frac{|\psi(\lambda,0)|^2\cdot|e'(\lambda,0)|^2+|\psi'(\lambda,0)|^2\cdot|e(\lambda,0)|^2-2\Re\left(\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e(\lambda,0)\cdot\overline{e'(\lambda,0)}\right)}{|\psi(\lambda,0)|^2\cdot|e'(\lambda,0)|^2+|\psi'(\lambda,0)|^2\cdot|e(\lambda,0)|^2-2\Re\left(\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e'(\lambda,0)\cdot\overline{e(\lambda,0)}\right)}\end{aligned}$$

for all $\lambda\in\mathbb{R}\backslash\{0\}$. Using (1.4),

$$\begin{aligned}&\left[\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}-\psi'(\lambda,0)\cdot\overline{\psi(\lambda,0)}\right]\cdot\left[e'(\lambda,0)\,\overline{e(\lambda,0)}-\overline{e'(\lambda,0)}e(\lambda,0)\right]\\ &\quad=2i\Im\left(\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\right)\cdot W[e(\lambda,0),\overline{e(\lambda,0)}]\\ &\quad=-4\lambda^2\left[\alpha_1\beta_0-\alpha_0\beta_1+(\alpha_2\beta_1-\alpha_1\beta_2)\lambda^2+(\alpha_3\beta_2-\alpha_2\beta_3)\lambda^4\right]\\ &\quad<0\end{aligned}$$

which yields

$$\begin{aligned}&-\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e(\lambda,0)\cdot\overline{e'(\lambda,0)}-\overline{\psi(\lambda,0)}\cdot\psi'(\lambda,0)\cdot e'(\lambda,0)\cdot\overline{e(\lambda,0)}\\ &<-\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e'(\lambda,0)\cdot\overline{e(\lambda,0)}-\overline{\psi(\lambda,0)}\cdot\psi'(\lambda,0)\cdot\overline{e'(\lambda,0)}\cdot e(\lambda,0)\end{aligned}$$

i.e.,

$$-2\Re\left(\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e(\lambda,0)\cdot\overline{e'(\lambda,0)}\right)<-2\Re\left(\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e'(\lambda,0)\cdot\overline{e(\lambda,0)}\right)$$

and then

$$\begin{aligned}&|\psi(\lambda,0)|^2\cdot|e'(\lambda,0)|^2+|\psi'(\lambda,0)|^2\cdot|e(\lambda,0)|^2-2\Re\left(\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e(\lambda,0)\cdot\overline{e'(\lambda,0)}\right)\\ &<|\psi(\lambda,0)|^2\cdot|e'(\lambda,0)|^2+|\psi'(\lambda,0)|^2\cdot|e(\lambda,0)|^2-2\Re\left(\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e'(\lambda,0)\cdot\overline{e(\lambda,0)}\right)\end{aligned}$$

which shows $|S(\lambda)|^2<1$, that is, $|S(\lambda)|<1$, for all $\lambda\in\mathbb{R}\backslash\{0\}$. Thus, the lemma is proved. □

We proceed to research the spectrum of the boundary value problem (1.1) and (1.2). Therefore, we investigate the scattering function in more detail. It is a meromorphic function in the upper half plane $\Im\lambda>0$, with poles at the zeros of the function $\Theta(\lambda)$.

Lemma 2.2. The function $\Theta(\lambda)$ has only finitely many zeros in the upper half plane $\Im>0$. The zeros of $\Theta(\lambda)$ are simple and pure imaginary.

Proof. If we assume that $\rho(x)\equiv 1$ in [11], the proof of the lemma can be obtained similarly. □

Let $i\lambda_j$ such that $\lambda_j>0$, for all $j=\overline{1,n}$, be the zeros of the function $\Theta(\lambda)$, called the singular values of (1.1) and (1.2). Thus, the numbers m_j, for all $j=\overline{1,n}$, are defined by

$$m_j^{-2} \equiv \int_0^\infty |e(i\lambda_j, x)|^2 dx + \frac{|e(i\lambda_j, 0)|^2 \left[\frac{1}{2}\sum_{k=0}^{2}(\alpha_{1+k}\beta_k - \alpha_k\beta_{1+k})\lambda_j^{2k-1} + \sum_{k=0}^{1}(\alpha_k\beta_{2+k} - \alpha_{2+k}\beta_k)\lambda_j^{2k}\right]}{|\alpha_0 - \alpha_1\lambda_j + \alpha_2\lambda_j^2 - \alpha_3\lambda_j^3|^2}$$

and called the normalized numbers for (1.1) and (1.2). As a result, we can give the following definition.

Definition 2.3. The collection of quantities $\left\{S(\lambda);\, i\lambda_j;\, m_j\ \ (j = \overline{1,n})\right\}$ is called the scattering data of the boundary value problem (1.1) and (1.2).

Based on the scattering data, form an integral equation for the kernel $K(x, y)$.

Theorem 2.4. For every fixed $x \geq 0$, the kernel $K(x, y)$ of the solution (2.1) satisfies the integral equation, called the main equation

$$K(x,y) + F(x+y) + \int_x^\infty K(x,t)F(t+y)dt = 0, \quad y > x \tag{2.10}$$

where

$$F(x) = \sum_{j=1}^{n} m_j^2 e^{-\lambda_j x} + \frac{1}{2\pi}\int_{-\infty}^{\infty}(-1 - S(\lambda))e^{i\lambda x}d\lambda$$

Proof. The proof is obtained similarly for the case $\rho(x) \equiv 1$ in [11]. □

3. The Scattering Function's Continuity

This section presents the scattering function's continuity.

Theorem 3.1. For all $\lambda \in \mathbb{R}$, the function $S(\lambda)$ is continuous.

Proof. From section 2, $\Theta(\lambda) \neq 0$, for all λ in the intervals $(-\infty, 0)$ and $(0, \infty)$, and $S(\lambda)$ is defined on $(-\infty, 0)$ and $(0, \infty)$ and continuous in these intervals. From the form (2.7) of $\Theta(\lambda)$, if $\Theta(0) \neq 0$, then $S(\lambda)$ is continuous at zero and $S(0) = 1$. It remains to investigate the case:

$$\Theta(0) = \alpha_0\left\{-K(0,0) + \int_0^\infty K_x(0,t)dt\right\} - \beta_0\left\{1 + \int_0^\infty K(0,t)dt\right\} = 0$$

Moreover, if we substitute $x = 0$ into (2.10), then

$$K(0,y) + F(y) + \int_0^\infty K(0,t)F(t+y)dt = 0 \tag{3.1}$$

Integrating (3.1) according to y from z to ∞, letting $t + y = \xi$, and applying the integration by parts,

$$\left\{1 + \int_0^\infty K(0,y)dy\right\}\int_z^\infty F(y)dy + \int_z^\infty K(0,y)dy - \int_0^\infty F(t+z)\left\{\int_t^\infty K(0,\xi)d\xi\right\}dt = 0 \tag{3.2}$$

We now apply the same procedure to the derivation of the main equation concerning x for obtaining

$$\left\{-K(0,0) + \int_0^\infty K_x(0,y)dy\right\}\int_z^\infty F(y)dy - F(z) + \int_z^\infty K_x(0,y)dy - \int_0^\infty F(t+z)\left\{\int_t^\infty K_x(0,\xi)d\xi\right\}dt = 0 \tag{3.3}$$

Multiplying (3.3) by $(\alpha_0 + i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3)$ and (3.2) by $(\beta_0 + i\beta_1\lambda - \beta_2\lambda^2 - i\beta_3\lambda^3)$ and subtracting the latter from the former,

$$0 = \Bigg[(\alpha_0 + i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3) \left\{ -K(0,0) + \int\limits_0^\infty K_x(0,y)dy \right\}$$

$$- \; (\beta_0 + i\beta_1\lambda - \beta_2\lambda^2 - i\beta_3\lambda^3) \left\{ 1 + \int\limits_0^\infty K(0,y)dy \right\} \Bigg] \int\limits_z^\infty F(y)dy$$

$$+ (\alpha_0 + i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3) \int\limits_z^\infty K_x(0,y)dy$$

$$- (\beta_0 + i\beta_1\lambda - \beta_2\lambda^2 - i\beta_3\lambda^3) \int\limits_z^\infty K(0,y)dy - (\alpha_0 + i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3)F(z)$$

$$- \int\limits_0^\infty \left\{ \int\limits_t^\infty \left[(\alpha_0 + i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3)K_x(0,\xi) - (\beta_0 + i\beta_1\lambda - \beta_2\lambda^2 - i\beta_3\lambda^3)K(0,\xi) \right] d\xi \right\} F(t+z)dt$$

Letting $\lambda \to 0$,

$$\alpha_0 F(z) = \left[\alpha_0 \left\{ -K(0,0) + \int\limits_0^\infty K_x(0,y)dy \right\} - \beta_0 \left\{ 1 + \int\limits_0^\infty K(0,y)dy \right\} \right] \int\limits_z^\infty F(y)dy$$

$$+ \int\limits_z^\infty \left[\alpha_0 K_x(0,y)dy - \beta_0 K(0,y) \right] dy$$

$$- \int\limits_0^\infty \left\{ \int\limits_t^\infty \left[\alpha_0 K_x(0,\xi) - \beta_0 K(0,\xi) \right] d\xi \right\} F(t+z)dt$$

Define the functions $G(z)$ and $H(z)$ as follows:

$$G(z) := \int\limits_z^\infty \left[\alpha_0 K_x(0,y)dy - \beta_0 K(0,y) \right] dy$$

and

$$H(z) := \alpha_0 F(z)$$

Hence, the integral equation is as follows:

$$G(z) - \int\limits_0^\infty F(t+z)G(t)dt = H(z)$$

$G(z)$ is a bounded solution of the equation

$$G(z) - \int\limits_0^\infty F(t+z)G(t)dt = 0, \quad 0 \leq z < \infty$$

and every bounded solution of this equation is summable on the half line $[0,\infty)$. It means that $G(z) \in L_1(0,\infty)$ (see p. 211 [19]). Thus, for

$$\widehat{K_1}(\lambda) = \alpha_0 + i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3 - \beta_1 - i\beta_2\lambda + \beta_3\lambda^2$$

$$-(\alpha_1 + i\alpha_2\lambda - \alpha_3\lambda^2)K(0,0) + (\alpha_1 + i\alpha_2\lambda - \alpha_3\lambda^2) \int\limits_0^\infty K_x(0,t)e^{i\lambda t}dt$$

$$-(\beta_1 + i\beta_2\lambda - \beta_3\lambda^2) \int\limits_0^\infty K(0,t)e^{i\lambda t}dt + \int\limits_0^\infty G(t)e^{i\lambda t}dt$$

$$
\begin{aligned}
\Theta(\lambda) = {} & \alpha_0 \left(i\lambda - K(0,0) + \int_0^\infty K_x(0,t)e^{i\lambda t}dt \right) - \beta_0 \left(1 + \int_0^\infty K(0,t)e^{i\lambda t}dt \right) \\
& +(i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3) \left(i\lambda - K(0,0) + \int_0^\infty K_x(0,t)e^{i\lambda t}dt \right) \\
& -(i\beta_1\lambda - \beta_2\lambda^2 - i\beta_3\lambda^3) \left(1 + \int_0^\infty K(0,t)e^{i\lambda t}dt \right) \\
= {} & \alpha_0 \left\{ i\lambda - K(0,0) + \int_0^\infty K_x(0,t)dt + i\lambda \int_0^\infty \left(\int_t^\infty K_x(0,y)dy \right) e^{i\lambda t}dt \right\} \\
& -\beta_0 \left\{ 1 + \int_0^\infty K(0,t)dt + i\lambda \int_0^\infty \left(\int_t^\infty K(0,y)dy \right) e^{i\lambda t}dt \right\} \\
& +i\lambda \left[i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3 - \beta_1 - i\beta_2\lambda + \beta_3\lambda^2 - (\alpha_1 + i\alpha_2\lambda - \alpha_3\lambda^2)K(0,0) \right] \\
& +i\lambda \left[(\alpha_1 + i\alpha_2\lambda - \alpha_3\lambda^2) \int_0^\infty K_x(0,t)e^{i\lambda t}dt - (\beta_1 + i\beta_2\lambda - \beta_3\lambda^2) \int_0^\infty K(0,t)e^{i\lambda t}dt \right] \\
= {} & \alpha_0 \left\{ -K(0,0) + \int_0^\infty K_x(0,t)dt \right\} - \beta_0 \left\{ 1 + \int_0^\infty K(0,t)dt \right\} \\
& +i\lambda \int_0^\infty \left\{ \int_t^\infty \left[\alpha_0 K_x(0,y) - \beta_0 K(0,y) \right] dy \right\} e^{i\lambda t}dt \\
& +i\lambda \left[i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3 - \beta_1 - i\beta_2\lambda + \beta_3\lambda^2 - (\alpha_1 + i\alpha_2\lambda - \alpha_3\lambda^2)K(0,0) \right] \\
& +i\lambda \left[(\alpha_1 + i\alpha_2\lambda - \alpha_3\lambda^2) \int_0^\infty K_x(0,t)e^{i\lambda t}dt - (\beta_1 + i\beta_2\lambda - \beta_3\lambda^2) \int_0^\infty K(0,t)e^{i\lambda t}dt \right] \\
= {} & i\lambda \widehat{K_1}(\lambda)
\end{aligned} \tag{3.4}
$$

In a similar manner, from (2.8),

$$\Theta_1(\lambda) = -i\lambda \widehat{K_2}(\lambda) \tag{3.5}$$

where

$$
\begin{aligned}
\widehat{K_2}(\lambda) = {} & \alpha_0 + i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3 + \beta_1 + i\beta_2\lambda - \beta_3\lambda^2 \\
& +(\alpha_1 + i\alpha_2\lambda - \alpha_3\lambda^2)K(0,0) - (\alpha_1 + i\alpha_2\lambda - \alpha_3\lambda^2) \int_0^\infty K_x(0,t)e^{-i\lambda t}dt \\
& +(\beta_1 + i\beta_2\lambda - \beta_3\lambda^2) \int_0^\infty K(0,t)e^{-i\lambda t}dt + \int_0^\infty G(t)e^{-i\lambda t}dt
\end{aligned}
$$

According to (2.6), (3.4), and (3.5),

$$S(\lambda) = -\frac{\widehat{K_2}(\lambda)}{\widehat{K_1}(\lambda)}$$

Taking into account the identity (2.5),

$$2\psi(\lambda, x) = \widehat{K_1}(\lambda) \left\{ e(-\lambda, x) - S(\lambda)e(\lambda, x) \right\}$$

from which it follows that $\widehat{K_1}(0) \neq 0$, otherwise it would be $\psi(0,x) = 0$ and it contradicts $\psi(0,0) \neq 0$. This shows that $S(\lambda)$ is continuous at $\lambda = 0$ and completes the proof. □

Consequently, from these results and section 2, $S(\lambda)$ is defined over $(-\infty, \infty)$ and continuous in this interval. Moreover, in the case $\alpha_1 = \beta_1 = 0$,

$$S(0) = \begin{cases} 1, & \Theta(\lambda) \neq 0 \\ -1, & \Theta(\lambda) = 0 \end{cases} \tag{3.6}$$

4. The Levinson-Type Formula

This section describes the Levinson-type formula for the considered boundary value problem.

Theorem 4.1. The following formula holds:

$$n - t(\Theta) = \frac{\mu(+\infty) - \mu(+0)}{\pi} \tag{4.1}$$

where n is the number of the zeros of the function $\Theta(\lambda)$ in the upper half plane,

$$\mu(\lambda) = \arg \Theta(\lambda), \text{ and } t(\Theta) = \begin{cases} 2, & \Theta(0) \neq 0 \\ \frac{3}{2}, & \Theta(0) = 0 \end{cases} \tag{4.2}$$

Proof. To achieve formula (4.1), the function $\Theta(\lambda)$ is analyzed using the argument principle. We now assume that

$$\Gamma_{R,\epsilon} = C_R^+ \cup C_R^- \cup [-R, -\epsilon] \cup [\epsilon, R]$$

for sufficiently large $R > 0$ and sufficiently small ϵ, where C_R^+ is a circle oriented counterclockwise and centered at the origin with radius R, and C_ϵ^- is a circle oriented clockwise and centered at the origin with radius ϵ.

Define the function $\arg \Theta(\lambda) = \mu(\lambda)$. Then, the function $\Theta(\lambda)$ is analytic in the upper half plane and continuous along the real axis. Hence, the increment of $\mu(\lambda)$ equals the number of zeros of $\Theta(\lambda)$ multiplied by 2π as λ runs over the real axis from $-\infty$ to ∞, bypassing the point $\lambda = 0$ along semicircle of sufficiently small radius ϵ in the upper half-plane.

As $R \to \infty$,

$$\{\mu(-\epsilon) - \mu(-\infty)\} + \{\mu(+\epsilon) - \mu(-\epsilon)\} + \{\mu(+\infty) - \mu(+\epsilon)\} + 4\pi = 2\pi n$$

because

$$\Theta(\lambda) = (i\lambda)^4 \left[\alpha_3 + O(\frac{1}{\lambda})\right], \quad |\lambda| \to \infty$$

for $\Im \geq 0$. If $\Theta(0) \neq 0$, then

$$\lim_{\epsilon \to 0} \{\mu(+\epsilon) - \mu(-\epsilon)\} = 0$$

However, if $\Theta(0) = 0$, then $\Theta(\lambda) = i\lambda \widehat{K_1}(\lambda), \widehat{K_1}(0) \neq 0$ by (3.4). Hence,

$$\lim_{\epsilon \to 0} \{\mu(+\epsilon) - \mu(-\epsilon)\} = -\pi$$

When $\epsilon \to 0$,

$$2\{\mu(\infty) - \mu(0)\} + \begin{cases} 0, & \Theta(0) \neq 0 \\ -\pi, & \Theta(0) = 0 \end{cases} + 4\pi = 2\pi n$$

since $\lim_{\epsilon \to 0} \{\mu(-\epsilon) - \mu(-\infty)\} = \lim_{\epsilon \to 0} \{\mu(\infty) - \mu(\epsilon)\}$. Thus,

$$n - t(\Theta) = \frac{\mu(+\infty) - \mu(0)}{\pi}$$

where $t(\Theta)$ is defined by the formula (4.2), which proves the theorem. □

Proposition 4.2. For $\alpha_1 = \beta_1 = 0$, the increase in the logarithm of the scattering function is associated with the number of eigenvalues of the problem (1.1) and (1.2) by the following equality

$$n-2=\frac{\ln S(+0)-\ln S(\infty)}{2\pi i}-\frac{1-S(0)}{4} \tag{4.3}$$

PROOF. According to (2.9) and (3.6), $|S(0)| = |S(\infty)| = 1$, and hence $\ln S(+0) = -2i\mu(0)$ and $\ln S(\infty) = -2i\mu(\infty)$. Considering these results in (4.1), (4.3) holds. □

Definition 4.3. (4.3) is called *the Levinson-type formula* for (1.1) and (1.2).

5. Conclusion

Levinson's theorem is a valuable tool for understanding quantum scattering phenomena. In this work, we have provided the scattering function and the spectrum for (1.1) and (1.2). The scattering function's continuity has been studied. The formula connecting the number of eigenvalues of (1.1) and (1.2) to the argument's variation of the function $\Theta(\lambda)$ over the interval $(-\infty, \infty)$ has been introduced. In a special case, we have derived the Levinson-type formula.

The study described in the text focuses on conducting spectral analysis of a second-order differential operator with nonlinear dependence on spectral parameters in boundary conditions. In future research, this methodology can be extended to various boundary value problems, and the boundary value problem (1.1) and (1.2) can be generalized for boundary conditions involving higher order polynomials of the spectral parameter.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Unveiling the Dynamics of Nonlinear Landau-Ginzburg-Higgs (LGH) Equation: Wave Structures through Multiple Auxiliary Equation Methods

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Abstract — This comprehensive investigation delves deeply into the intricate dynamics governed by the nonlinear Landau-Ginzburg-Higgs equation. It uncovers a diversity of semi-analytical solutions by leveraging three auxiliary equation methods within the traveling wave framework. This article effectively utilizes the improved Kudryashov, Kudryashov's R, and Sardar's subequation methods. The methods discussed are advantageous because they are easy to implement and suitable for use with the Mathematica package program. Each method yields a distinct set of solutions, scrutinized across all cases. We elucidate the complex wave structures through 3D, 2D, and contour graphical representations, providing profound insights into their underlying characteristics. Furthermore, we scrutinize the influence of parameter variations on these wave structures, thereby offering a comprehensive understanding of their dynamic behavior.

Keywords *Landau-Ginzburg-Higgs equation, improved Kudryashov method, Kudryashov's R method, Sardar's subequation method*

Mathematics Subject Classification (2020) 35C07, 35C11

1. Introduction

It has long been known that nonlinear structures are used to model many natural phenomena and basic science fields such as physics, chemistry, and biology. It is very important to obtain the solutions of these models in various engineering fields. Making sense of scientific phenomena and solving the obtained structures with today's knowledge and technology has been the goal of researchers for decades. For this purpose, they have worked on developing different perspectives by applying various analytical and numerical solution methods such as the (G'/G) expansion method [1, 2], Bernoulli (G'/G) expansion method [3], sub-equation method [4, 5], sine-Gordon expansion method (SGEM) [6], rational sine-Gordon expansion method (rSGEM) [7], exponential function method [8], modified exponential function method [9, 10], exponential rational function method [11, 12], unified method [13, 14], Kudryashov methods [15–19], Khater methods [20], natural decomposition method [21], variational approximation methods, Hirota direct method [22]. Besides, most of the put forward semi-analytical methods are based on the same starting point, it is seen that even small changes in the method steps affect the structure of the solution functions. Considering that a small change causes big consequences, called the butterfly effect in today's age, small changes in the solution structures will allow the scientific phenomenon discussed to be interpreted differently.

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This study discusses the Landau-Ginzburg-Higgs equation (LGHE), created to understand and describe phase transitions, superconductivity condensed matter physics, and the behavior of certain types of fields in high-energy physics. Solutions of this equation can represent the distribution of the superconducting order parameter within the material and provide information on properties, such as the penetration depth of magnetic fields and the critical temperature of the superconducting transition. For this reason, LGHE has attracted the attention of many researchers, and some soliton structures have been obtained by applying various methods. Our aim in doing this study is to add new ones to the solution structures of LGHE and to show the suitability of the methods discussed by comparing them to this equation. In Section 2, we provide the mathematical algorithms of the improved Kudryashov method (IKM), Kudryashov's R method (KRM), and Sardar's subequation method (SSM) to figure out the solitary wave solitons. In section 3, we apply the proposed methods to the nonlinear LGHE. In the last section, we provide the concluding remarks on the obtained solutions.

2. Preliminaries

This section provides the basic steps of IKM, KRM, and SSM.

2.1. Improved Kudryashov Method (IKM)

First, we handle the general expression of a nonlinear partial differential equation in the form [17]:

$$\kappa\left(u, u_x, u_y, u_t, u_{xx}, u_{xy}, ...\right) = 0 \tag{2.1}$$

where κ is polynomial function in u and its partial derivatives are included. Then, the given nonlinear partial differential equation (NPDE) (2.1) can be converted into ordinary differential equation (ODE) by traveling wave transformation as follows:

$$u(x, y, t) = u(\psi), \quad \psi = \mu(x + y - ct) \tag{2.2}$$

where c is an arbitrary constant. After applying the above transformation and the chain rule, we obtain the following equality:

$$K(u, u', u'', ...) = 0 \tag{2.3}$$

Specific items of the method can be applied after that reduction. In this step, according to the proposed method, we assume (2.3) has a solution in the form:

$$u(\psi) = \sum_{s=-1}^{N-1} a_{s+1}\chi^{s+1}(\psi) \tag{2.4}$$

where

$$\chi(\psi) = \pm(1 + exp(2\psi))^{-1/2}$$

and $\chi(\psi)$ satisfies the following ODE:

$$\chi_\psi^2 = (\chi^2(\chi^4 - 2\chi^2 + 1))^{1/2}$$

After taking this auxiliary differential equation and considering the solution (2.4), we can write the first, second, and third-order derivatives of (2.4) as follows:

$$u_\psi = \sum_{s=0}^{N-1} a_{s+1}(s + 1)(\chi^{s+3}(\psi) - \chi^{s+1}(\psi)) \tag{2.5}$$

$$u_{\psi\psi} = \sum_{s=0}^{N-1} a_{s+1}\left[(s + 1)(s + 3)\chi^{s+5}(\psi) - 2(s + 1)(s + 2)\chi^{s+3}(\psi) + (s + 1)^2\chi^{s+1}(\psi)\right] \tag{2.6}$$

and

$$u_{\psi\psi\psi} = \sum_{s=0}^{N-1} a_{s+1}\left[(s+1)(s+3)(s+5)\chi^{s+7}(\psi) - 3(s+1)(s+3)^2\chi^{s+5}(\psi) + \left(3(s+1)^2(s+3)+4(s+1)\right)\chi^{s+3}(\psi) - (s+1)^3\chi^{s+1}(\psi)\right] \tag{2.7}$$

If necessary, other higher-order derivatives can also be written. Substituting (2.4) and (2.5)-(2.7) into (2.3) then, equating degrees of highest order linear term $(u^{(p)}(\chi))^r$ and highest degree nonlinear term $u^l(\chi)u^{(s)}(\chi)$ as required by the principle of homogeneous balance, we can define the pole order N clearly. After this implementation, we obtain an algebraic system. By using the computer package program, we can solve the algebraic system according to degrees of the χ, then the coefficients of the polynomial (2.4) and parameters of (2.2) can be obtained. Finally, substituting the coefficients, parameters, and the traveling wave transformation into the obtained polynomial, the solutions of (2.1) are obtained.

2.2. Kudryashov's R Method (KRM)

The major items of the method proposed above are indicated as follows [19]: In the first item, we handle the general impression of nonlinear partial differential equations in the form:

$$\kappa\left(u, u_x, u_y, u_t, u_{xx}, u_{xy}, ...\right) = 0 \tag{2.8}$$

where κ is a polynomial function in u and its assorted order partial derivatives and nonlinear terms are included. Secondly, we assume that the subsequent traveling wave transformation is done to reduce (2.8) to an ordinary differential equation:

$$u(x,y,t) = u(\psi), \quad \psi = \mu(x+y-ct) \tag{2.9}$$

where μ and c are arbitrary constants. After applying the above transformation and the chain rule, we get the following equality:

$$K(u, u', u'', ...) = 0 \tag{2.10}$$

In this step, according to the proposed method, we assume (2.10) has a solution in the form:

$$u(\psi) = \sum_{s=-1}^{N-1} a_{s+1}\chi^{s+1}(\psi) \tag{2.11}$$

where

$$\chi(\psi) = \frac{4\alpha}{4\alpha^2 e^{\psi} + \gamma e^{-\psi}}$$

such that $\gamma = 4\alpha\beta$ and $u(\psi)$ adopts the given ordinary differential equation:

$$\chi_\psi^2 = \chi^2(1-\gamma\chi^2)$$

After taking this auxiliary differential equation and considering the solution (2.11), we can write the first, second, and third-order derivatives of (2.11) as follows:

$$u_\psi = \sum_{s=0}^{N-1} a_{s+1}\chi^s(\psi)\chi_\psi(\psi)$$

$$u_{\psi\psi} = \sum_{s=0}^{N-1} a_{s+1}\left[(s+1)^2\chi^{s+1}(\psi) - (s+1)^2\gamma\chi^{s+3}(\psi) + (s+1)\gamma\chi^{s+4}(\psi)\right]$$

and

$$u_{\psi\psi\psi} = \sum_{s=0}^{N-1} a_{s+1}\left[(s+1)^3\chi^s(\psi) - \gamma(s+1)^2(s+3)\chi^{s+2}(\psi) - \gamma(s+1)(s+3)\chi^{s+2}(\psi)\right]\chi_\psi(\psi)$$

By equating the degrees of highest order linear term $(u^{(p)}(\chi))^r$ and highest degree nonlinear term $u^l(\chi)u^{(s)}(\chi)$ as required by the principle of homogeneous balance, we can define the pole order N clearly. After this implementation, we obtain an algebraic system. By using the computer package program, we can solve the algebraic system according to degrees of the χ, then the coefficients of the polynomial (2.11) and parameters of (2.9) can be obtained. Finally, substituting the coefficients, parameters, and the traveling wave transformation into the obtained polynomial, the solutions of (2.8) are obtained.

2.3. Subequation Method in Sardar's Sense (SSM)

The major items of the method proposed above are indicated as follows [24]: In the first item, we handle the general impression of nonlinear partial differential equations in the form:

$$\kappa\left(u, u_x, u_y, u_t, u_{xx}, u_{xy}, ...\right) = 0 \tag{2.12}$$

where κ is a polynomial function in u and its assorted order partial derivatives and nonlinear terms are included. Secondly, we assume that the subsequent traveling wave transformation is done to reduce (2.12) to an ordinary differential equation:

$$u(x,y,t) = u(\psi), \quad \psi = \mu(x+y-ct) \tag{2.13}$$

where μ and c are arbitrary constants. After applying the above transformation and the chain rule, we get the following equality:

$$K(u, u', u'', ...) = 0 \tag{2.14}$$

In this step, according to the proposed method, we assume (2.14) has a solution in the form:

$$u(\psi) = \sum_{s=-1}^{N-1} a_{s+1}\chi^{s+1}(\psi) \tag{2.15}$$

where $\chi(\psi)$ is the solution of the following differential equation

$$\chi_\psi^2 = \eta + \gamma\chi^2(\psi) + \chi^4(\psi) \tag{2.16}$$

and the solutions of (2.16) has four cases of solutions:

Case 1: If $\gamma > 0$ and $\eta = 0$, then

$$\chi_1(\psi) = \pm\sqrt{-\alpha\beta\gamma}\,\text{sech}_{\alpha\beta}\left(\sqrt{\gamma}\psi\right)$$

and

$$\chi_2(\psi) = \pm\sqrt{\alpha\beta\gamma}\,\text{csch}_{\alpha\beta}\left(\sqrt{\gamma}\psi\right)$$

where $\text{sech}_{\alpha\beta}(\psi) = \frac{2}{\alpha e^{\psi}+\beta e^{-\psi}}$ and $\text{csch}_{\alpha\beta}(\psi) = \frac{2}{\alpha e^{\psi}-\beta e^{-\psi}}$.

Case 2: If $\gamma < 0$ and $\eta = 0$, then

$$\chi_3(\psi) = \pm\sqrt{-\alpha\beta\gamma}\,\sec_{\alpha\beta}\left(\sqrt{-\gamma}\psi\right)$$

and

$$\chi_4(\psi) = \pm\sqrt{-\alpha\beta\gamma}\,\csc_{\alpha\beta}\left(\sqrt{-\gamma}\psi\right)$$

where $\sec_{\alpha\beta}(\psi) = \frac{2}{\alpha e^{i\psi}+\beta e^{-i\psi}}$ and $\csc_{\alpha\beta}(\psi) = \frac{2}{\alpha e^{i\psi}-\beta e^{-i\psi}}$.

Case 3: If $\gamma < 0$ and $\eta = \frac{\gamma^2}{4\beta}$, then

$$\chi_5(\psi) = \pm\sqrt{-\frac{\gamma}{2}}\tanh_{\alpha\beta}\left(\sqrt{-\frac{\gamma}{2}}\psi\right)$$

$$\chi_6(\psi) = \pm\sqrt{-\frac{\gamma}{2}}\coth_{\alpha\beta}\left(\sqrt{-\frac{\gamma}{2}}\psi\right)$$

$$\chi_7(\psi) = \pm\sqrt{-\frac{\gamma}{2}}\left(\tanh_{\alpha\beta}\left(\sqrt{-2\gamma}\psi\right) \pm i\sqrt{\alpha\beta}\operatorname{sech}_{\alpha\beta}\left(\sqrt{-2\gamma}\psi\right)\right)$$

$$\chi_8(\psi) = \pm\sqrt{-\frac{\gamma}{2}}\left(\coth_{\alpha\beta}\left(\sqrt{-2\gamma}\psi\right) \pm \sqrt{\alpha\beta}\operatorname{csch}_{\alpha\beta}\left(\sqrt{-2\gamma}\psi\right)\right)$$

and

$$\chi_9(\psi) = \pm\sqrt{-\frac{\gamma}{8}}\left(\tanh_{\alpha\beta}\left(\sqrt{-\frac{\gamma}{8}}\psi\right) + \coth_{\alpha\beta}\left(\sqrt{-\frac{\gamma}{8}}\psi\right)\right)$$

where $\tanh_{\alpha\beta}(\psi) = \frac{\alpha e^{\psi} - \beta e^{-\psi}}{\alpha e^{\psi} + \beta e^{\psi}}$ and $\coth_{\alpha\beta}(\psi) = \frac{\alpha e^{\psi} + \beta e^{-\psi}}{\alpha e^{\psi} - \beta e^{\psi}}$.

Case 4: If $\gamma > 0$ and $\eta = \frac{\gamma^2}{4}$, then

$$\chi_{10}(\psi) = \pm\sqrt{\frac{\gamma}{2}}\tan_{\alpha\beta}\left(\sqrt{\frac{\gamma}{2}}\psi\right)$$

$$\chi_{11}(\psi) = \pm\sqrt{\frac{\gamma}{2}}\cot_{\alpha\beta}\left(\sqrt{\frac{\gamma}{2}}\psi\right)$$

$$\chi_{12}(\psi) = \pm\sqrt{\frac{\gamma}{2}}\left(\tan_{\alpha\beta}\left(\sqrt{2\gamma}\psi\right) \pm \sqrt{\alpha\beta}\sec_{\alpha\beta}\left(\sqrt{2\gamma}\psi\right)\right)$$

$$\chi_{13}(\psi) = \pm\sqrt{\frac{\gamma}{2}}\left(\cot_{\alpha\beta}\left(\sqrt{2\gamma}\psi\right) \pm \sqrt{\alpha\beta}\csc_{\alpha\beta}\left(\sqrt{2\gamma}\psi\right)\right)$$

and

$$\chi_{14}(\psi) = \pm\sqrt{\frac{\gamma}{8}}\left(\tan_{\alpha\beta}\left(\sqrt{\frac{\gamma}{8}}\psi\right) + \cot_{\alpha\beta}\left(\sqrt{\frac{\gamma}{8}}\psi\right)\right)$$

where $\tanh_{\alpha\beta}(\psi) = \frac{\alpha e^{\psi} - \beta e^{-\psi}}{\alpha e^{\psi} + \beta e^{\psi}}$ and $\coth_{\alpha\beta}(\psi) = \frac{\alpha e^{\psi} + \beta e^{-\psi}}{\alpha e^{\psi} - \beta e^{\psi}}$. After taking this auxiliary differential equation and considering the solution (2.15), we can write the first, second, and third-order derivatives of (2.15) as follows:

$$u_{\psi} = \sum_{s=0}^{N-1}(s+1)a_{s+1}\chi^{s}(\psi)\chi_{\psi}(\psi)$$

$$u_{\psi\psi} = \sum_{s=0}^{N-1} a_{s+1}\left[s(s+1)\eta\chi^{s-1}(\psi) + (s+1)(s\gamma+1)\chi^{s+1}(\psi) + (s+1)(s+2)\chi^{s+3}(\psi)\right]$$

and

$$u_{\psi\psi\psi} = \sum_{s=0}^{N-1} a_{s+1}\left[(s^2-1)s\eta\chi^{s-2}(\psi) + (s+1)^2(s\gamma+1)\chi^{s}(\psi) + (s+1)(s+2)(s+3)\chi^{s+2}(\psi)\right]\chi_{\psi}(\psi)$$

By equating the degrees of highest order linear term $(u^{(p)}(\chi))^r$ and highest degree nonlinear term $u^l(\chi)u^{(s)}(\chi)$ as required by the principle of homogeneous balance, we can define the pole order N clearly. After this implementation, we obtain an algebraic system. By using the computer package program, we can solve the algebraic system according to degrees of the χ, then the coefficients of the polynomial (2.15) and parameters of (2.13) can be obtained. Finally, substituting the coefficients, parameters, and the traveling wave transformation into the obtained polynomial, the solutions of (2.12) are obtained.

3. Solutions of LGHE

The Landau-Ginzburg-Higgs equation (LGHE) is a physics equation that arises in the fields of condensed matter physics and high energy physics, especially in the study of phase transitions, superconductivity, cosmology, optics and the behavior of certain field theories and it can be represented as follows [23]:

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} - g^2 U + h^2 U^3 = 0. \tag{3.1}$$

Here, $\frac{\partial^2 U}{\partial t^2}$ represents the second partial derivative of a field U with respect to time t, which describes the time evolution of the field. Moreover, $\frac{\partial^2 U}{\partial x^2}$ represents the second partial derivative of the field U with respect to space x, which describes how the field varies in space. Besides, g is a constant that determines the strength of the linear term, and h is another constant that determines the strength of the quadratic nonlinear term. Using the following transformation

$$U(x,t) = u(\psi), \quad \psi = \mu x - ct$$

we can reduce (3.1) into the following ODE:

$$(c^2 - \mu^2)\frac{d^2 u}{d\psi^2} - g^2 u + h^2 u^2 = 0 \tag{3.2}$$

After this reduction, we can use analytical approaches as follows:

3.1. IKM Sense

According to the IKM, we can think (3.2) has polynomial solution as:

$$u(\psi) = \sum_{s=-1}^{N-1} a_{s+1}\chi^{s+1}(\psi)$$

where

$$\chi(\psi) = \pm(1 + \exp(2\psi))^{-1/2}$$

and $u(\psi)$ adopts the given ordinary differential equation:

$$\chi_\psi^2 = (\chi^2(\chi^4 - 2\chi^2 + 1))^{1/2}$$

Then, according to the previous adoption and using the homogeneous balance principle, we obtain pole order $N = 2$ and thus

$$u(\psi) = a_0 + a_1\chi(\psi) + a_2\chi^2(\psi) \tag{3.3}$$

After determining the quadratic polynomial,

$$u''(\psi) = a_1\chi(\psi) + 4a_2\chi^2(\psi) - 4a_1\chi^3(\psi) - 12a_2\chi^4(\psi) + 3a_1\chi^5(\psi) + 8a_2\chi^6(\psi) \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.1), we obtain four cases solutions of LGHE.

Case 1:

$$a_0 = -\frac{g}{h^{3/2}}, \qquad a_1 = 0, \qquad a_2 = \frac{2g}{h^{3/2}}, \qquad c = -\frac{2\mu^2 - g^2}{2}$$

$$U_1 = -\frac{g}{h^{3/2}}\tanh\left(\mu x - \sqrt{\frac{2\mu^2 - g^2}{2}}t\right) \tag{3.5}$$

Case 2:

$$a_0 = -\frac{g}{h^{3/2}}, \qquad a_1 = 0, \qquad a_2 = \frac{2g}{h^{3/2}}, \qquad c = \frac{2\mu^2 - g^2}{2}$$

$$U_2 = -\frac{g}{h^{3/2}} \tanh\left(\mu x + \sqrt{\frac{2\mu^2 - g^2}{2}} t\right)$$

Case 3:

$$a_0 = \frac{g}{h^{3/2}}, \qquad a_1 = 0, \qquad a_2 = -\frac{2g}{h^{3/2}}, \qquad c = -\frac{2\mu^2 - g^2}{2}$$

$$U_3 = \frac{g}{h^{3/2}} \tanh\left(\mu x - \sqrt{\frac{2\mu^2 - g^2}{2}} t\right)$$

Case 4:

$$a_0 = \frac{g}{h^{3/2}}, \qquad a_1 = 0, \qquad a_2 = -\frac{2g}{h^{3/2}}, \qquad c = -\frac{2\mu^2 - g^2}{2}$$

$$U_4 = \frac{g}{h^{3/2}} \tanh\left(\mu x + \sqrt{\frac{2\mu^2 - g^2}{2}} t\right)$$

3D surface, 2D plots, and contour plot of the kink type solution (3.5) are shown in Figure 1:

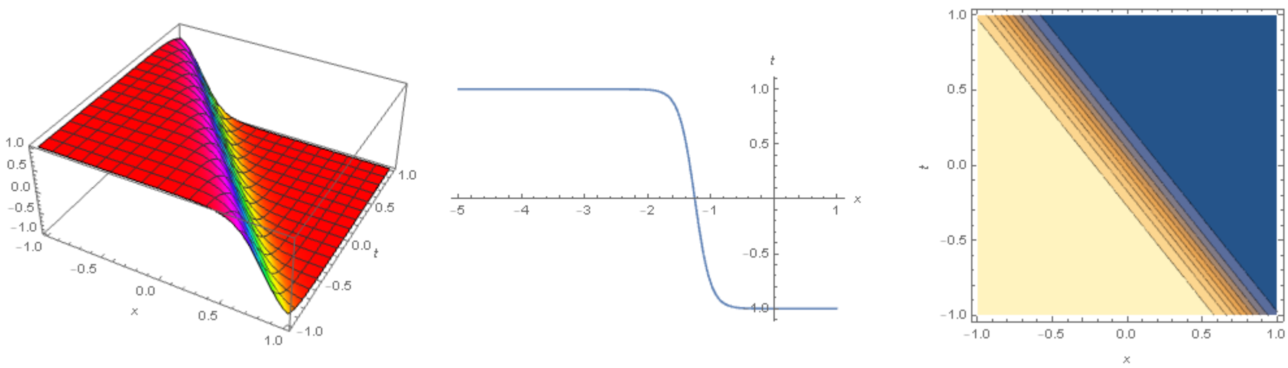

Figure 1. 3D surface, 2D plots, and contour plot of the kink type solution (3.5) for $g = 3\sqrt{2}$, $h = 2$, and $\mu = 5$

3.2. KRM Sense

Supposing the solution of (3.1) in the form:

$$u(\psi) = \sum_{s=-1}^{N-1} a_{s+1} \chi^{s+1}(\psi)$$

where

$$\chi(\psi) = \frac{4\alpha}{4\alpha^2 e^{\psi} + \gamma e^{-\psi}}$$

such that $\gamma = 4\alpha\beta$ and $u(\psi)$ adopts the given ordinary differential equation:

$$\chi_\psi^2 = \chi^2(1 - \gamma\chi^2)$$

Then, according to the previous adoption and using the homogeneous balance principle, we obtain pole order $N = 1$ and thus

$$u(\psi) = a_0 + a_1\chi(\psi). \tag{3.6}$$

After determining the first-degree polynomial,

$$u''(\psi) = a_1\chi(\psi)(1 - 2\gamma\chi^2(\psi)) \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.1), we obtain four cases solutions of LGHE.

Case 1:

$$a_0 = 0, \qquad a_1 = -\frac{g\sqrt{2\gamma}}{h^{3/2}}, \qquad c = \sqrt{\mu^2 + g^2}$$

$$U_1 = -\frac{2g\sqrt{2\alpha\beta}}{h^{3/2}} \operatorname{sech}_{\alpha\beta}\left(\mu x - \sqrt{\mu^2 + g^2 t}\right)$$

Case 2:

$$a_0 = 0, \qquad a_1 = \frac{g\sqrt{2\gamma}}{h^{3/2}}, \qquad c = -\sqrt{\mu^2 + g^2}$$

$$U_2 = \frac{2g\sqrt{2\alpha\beta}}{h^{3/2}} \operatorname{sech}_{\alpha\beta}\left(\mu x + \sqrt{\mu^2 + g^2 t}\right)$$

Case 3:

$$a_0 = 0, \qquad a_1 = -\frac{g\sqrt{2\gamma}}{h^{3/2}}, \qquad c = \sqrt{\mu^2 + g^2}$$

$$U_3 = \frac{2g\sqrt{2\alpha\beta}}{h^{3/2}} \operatorname{sech}_{\alpha\beta}\left(\mu x - \sqrt{\mu^2 + g^2 t}\right)$$

Case 4:

$$a_0 = 0, \qquad a_1 = -\frac{g\sqrt{2\gamma}}{h^{3/2}}, \qquad c = -\sqrt{\mu^2 + g^2}$$

$$U_4 = -\frac{2g\sqrt{2\alpha\beta}}{h^{3/2}} \operatorname{sech}_{\alpha\beta}\left(\mu x + \sqrt{\mu^2 + g^2 t}\right) \tag{3.8}$$

3D surface, 2D plots, and contour plot of the bell shaped bright soliton solution (3.8) are shown in Figure 2:

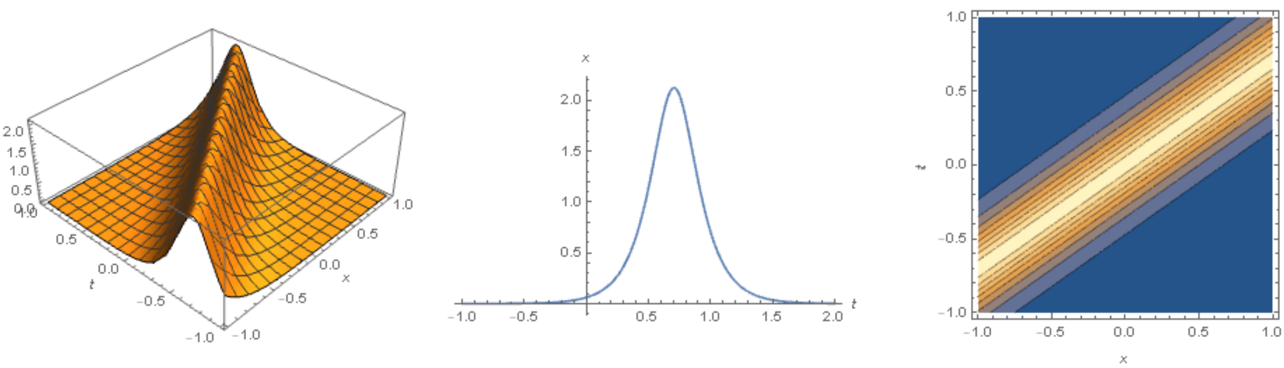

Figure 2. 3D surface, 2D plots, and contour plot of the bell shaped bright soliton solution (3.8) for $g = 3\sqrt{2}$, $h = 2$, $\mu = 3\sqrt{2}$, α=1, and β=1

3.3. SSM Sense

Supposing the solution of (3.1) in the form:

$$u(\psi) = \sum_{s=-1}^{N-1} a_{s+1}\chi^{s+1}(\psi)$$

where $\chi(\psi)$ is the solution of (2.16). Then, according to the previous adoption and using the homogeneous balance principle, we obtain pole order $N = 1$ and thus

$$u(\psi) = a_0 + a_1\chi(\psi) \tag{3.9}$$

After determining the first-degree polynomial,

$$u''(\psi) = a_1\gamma\chi(\psi) + 2a_1\chi^3(\psi) \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.1), we obtain four cases solutions of LGHE where

$$a_0 = 0, \qquad a_1 = \pm\frac{\sqrt{2(\mu^2 - c^2)}}{h^{3/2}}, \quad \text{and} \quad \gamma = \frac{g^2}{c^2 - \mu^2}$$

Case 1: If $|c| > |\mu|$ and $\eta = 0$, then

$$U_{1,1} = \pm\frac{g\sqrt{2\alpha\beta}}{h^{3/4}}\text{sech}_{\alpha\beta}\left(\frac{g}{\sqrt{c^2 - \mu^2}}\left(\mu x - ct\right)\right)$$

and

$$U_{1,2} = \pm i\frac{g\sqrt{2\alpha\beta}}{h^{3/4}}\text{csch}_{\alpha\beta}\left(\frac{g}{\sqrt{c^2 - \mu^2}}\left(\mu x - ct\right)\right) \tag{3.11}$$

3D surface, 2D plots, and contour plot of the kink type solution (3.11) are shown in Figure 3:

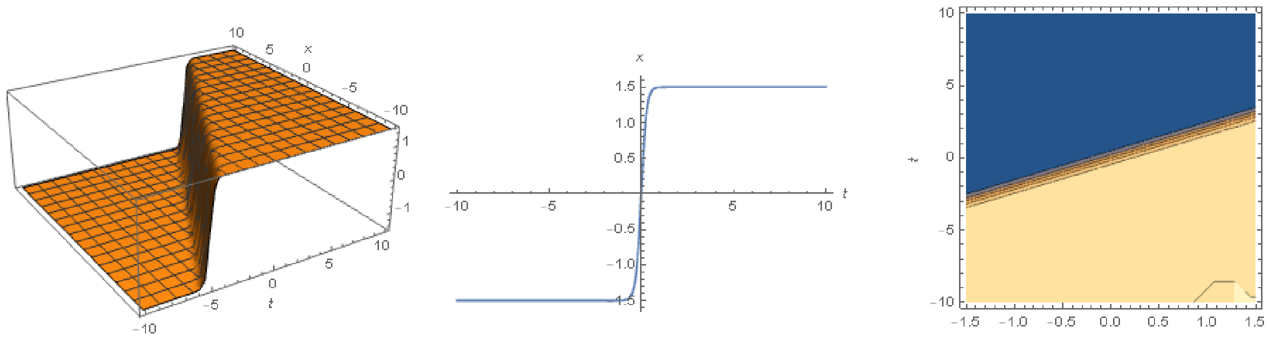

Figure 3. 3D surface, 2D plots, and contour plot of the kink type solution (3.11) for $g = 3\sqrt{2}$, $h = 2$, $\mu = 2$, α=1, β=1, and $c = 1$ and 2D plot for $t = 0$

Case 2: If $|c| < |\mu|$ and $\eta = 0$, then

$$U_{2,1} = \pm\frac{g\sqrt{2\alpha\beta}}{h^{3/4}}\sec_{\alpha\beta}\left(\frac{g}{\sqrt{\mu^2 - c^2}}\left(\mu x - ct\right)\right)$$

and

$$U_{2,2} = \pm\frac{g\sqrt{2\alpha\beta}}{h^{3/4}}\csc_{\alpha\beta}\left(\frac{g}{\sqrt{\mu^2 - c^2}}\left(\mu x - ct\right)\right)$$

Case 3: If $|c| < |\mu|$ and $\eta = \frac{g^4}{4\beta(c^2-\mu^2)^2}$, then

$$U_{3,1} = \pm\frac{g}{h^{3/4}}\tanh_{\alpha\beta}\left(\frac{g}{\sqrt{2(\mu^2 - c^2)}}\left(\mu x - ct\right)\right)$$

$$U_{3,2} = \pm\frac{g}{h^{3/4}}\coth_{\alpha\beta}\left(\frac{g}{\sqrt{2(\mu^2 - c^2)}}\left(\mu x - ct\right)\right) \tag{3.12}$$

$$U_{3,3} = \pm\frac{g}{h^{3/4}}\left[\tanh_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2 - c^2}}\left(\mu x - ct\right)\right) \pm i\sqrt{\alpha\beta}\text{sech}_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2 - c^2}}\left(\mu x - ct\right)\right)\right]$$

$$U_{3,4} = \pm\frac{g}{h^{3/4}}\left[\coth_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2 - c^2}}\left(\mu x - ct\right)\right) \pm \sqrt{\alpha\beta}\text{csch}_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2 - c^2}}\left(\mu x - ct\right)\right)\right]$$

and

$$U_{3,5} = \pm\frac{g}{2h^{3/4}}\left[\tanh_{\alpha\beta}\frac{g}{2\sqrt{2(\mu^2 - c^2)}}\left(\mu x - ct\right) + \coth_{\alpha\beta}\frac{g}{2\sqrt{2(\mu^2 - c^2)}}\left(\mu x - ct\right)\right]$$

3D surface, 2D plots, and contour plot of the multiple singular soliton type solution (3.12) are shown in Figure 4:

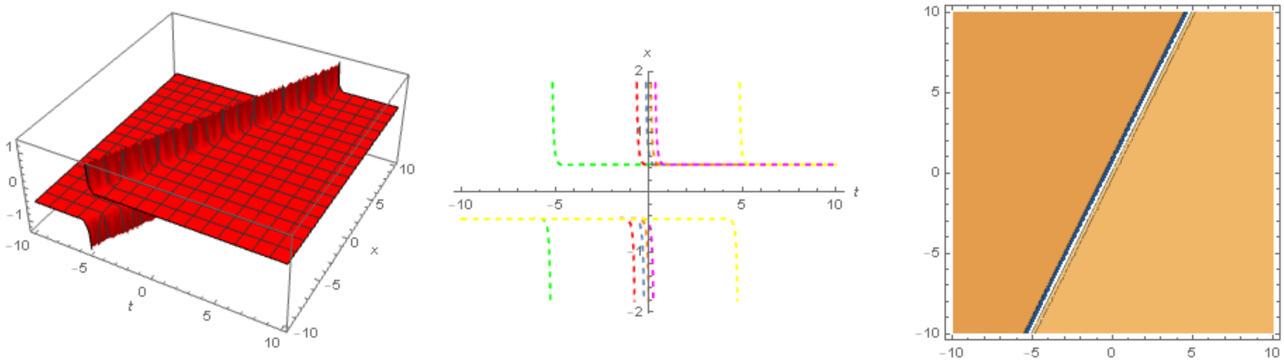

Figure 4. 3D surface, 2D plots, and contour plot of the multiple singular soliton type solution (3.12) for $g = 3\sqrt{2}$, $h = 16$, $\mu = 2$, α=1, β=0.5, and $c = 1$ and 2D plot when $t = -10$, $t = -1$, $t = 0$, $t = 0.5$, $t = 1$, and $t = 10$, respectively

Case 4: If $|c| > |\mu|$ and $\eta = \frac{g^4}{4(c^2-\mu^2)^2}$, then

$$U_{4,1} = \pm i\frac{g}{h^{3/4}}\tan_{\alpha\beta}\left(\frac{g}{\sqrt{2(c^2-\mu^2)}}(\mu x - ct)\right)$$

$$U_{4,2} = \pm i\frac{g}{h^{3/4}}\cot_{\alpha\beta}\left(\frac{g}{\sqrt{2(c^2-\mu^2)}}(\mu x - ct)\right) \tag{3.13}$$

$$U_{4,3} = \pm i\frac{g}{h^{3/4}}\left[\tan_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2-c^2}}(\mu x - ct)\right) \pm \sqrt{\alpha\beta}\sec_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2-c^2}}(\mu x - ct)\right)\right]$$

$$U_{4,4} = \pm i\frac{g}{h^{3/4}}\left[\cot_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2-c^2}}(\mu x - ct)\right) \pm \sqrt{\alpha\beta}\csc_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2-c^2}}(\mu x - ct)\right)\right]$$

and

$$U_{4,5} = \pm i\frac{g}{2\sqrt{2}h^{3/2}}\left[\tan_{\alpha\beta}\frac{g}{2\sqrt{2(\mu^2-c^2)}}(\mu x - ct) + \cot_{\alpha\beta}\frac{g}{2\sqrt{2(\mu^2-c^2)}}(\mu x - ct)\right]$$

3D surface, 2D plots, and contour plot of the singular periodic soliton type solution (3.13) are shown in Figure 5:

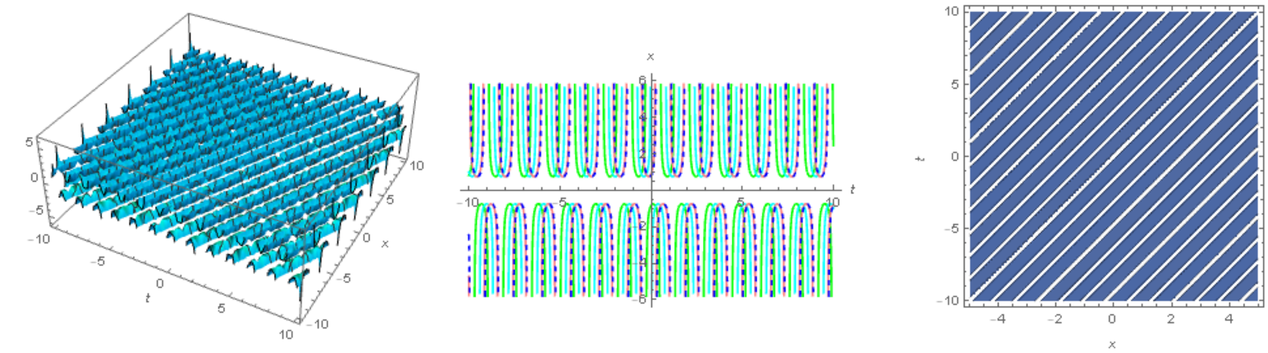

Figure 5. 3D surface, 2D plots, and contour plot of the singular periodic soliton type solution (3.13) for $g = 3\sqrt{2}$, $h = 16$, $\mu = 2$, α=1, β=1, and $c = 1$ and 2D plot when $t = -10$, $t = 0$, $t = 0.5$, and $t = 10$, respectively

4. Conclusion

In conclusion, our investigation of the solitary wave solutions of (3.1) has provided intriguing insights through three distinct approaches. We have identified four distinct solution scenarios using the IKM and the KRM. The IKM approach revealed the emergence of hyperbolic-type solutions, as shown in

Figure 1, featuring non-breaking, smooth traveling wave structures. Moreover, the KRM approach led to the discovery of special hyperbolic-type solutions, exemplified by Figure 2. Furthermore, employing the SSM enabled us to derive several solitary wave solutions, including generalized hyperbolic and trigonometric function solutions. It is worth noting that while these waves progress smoothly, they exhibit a distinct turning point transition within this context. These findings provide valuable contributions to understanding the intricate dynamics governed by (3.1). Given the observed effectiveness of the methods on the double-order model, these methods can also be extended to different nonlinear models. All the methods we have discussed offer the advantage of a systematic algorithmic structure, enabling diverse solution forms through polynomial-type auxiliary equations and supporting the use of computer software packages. However, their sole limitation is that they are only applicable to equations of even order and those with nonlinearity involving the square of the first derivative. Additionally, employing various numerical methods from the literature to obtain different solution structures of the LGH model will be beneficial for comparing the solutions. Furthermore, these methods can be applied to the fractional LGH equation, a more generalized form of the LGH equation discussed in [21]. This allows for easy comparison of the similarities and differences in solution structures.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Bi-f-Harmonic Legendre Curves on (α, β)-Trans-Sasakian Generalized Sasakian Space Forms

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Abstract — In this study, we consider bi-f-harmonic Legendre curves on (α, β)-trans-Sasakian generalized Sasakian space form. We provide the necessary and sufficient conditions for a Legendre curve to be bi-f-harmonic on (α, β)-trans-Sasakian generalized Sasakian space form without any restrictions by a main theorem. Afterward, we investigate these conditions under nine different cases. As a result of these investigations, we obtain the original theorems and corollaries as well as the nonexistence theorems. We perform these investigations according to the ρ_2 and ρ_3 functions from the curvature tensor of the (α, β)-trans-Sasakian generalized Sasakian space form, the curvature and torsion of the bi-f-harmonic Legendre curve, and finally, the positions of the basis vectors relative to each other.

Keywords *Bi-f-harmonic curves, Legendre curves, trans-Sasakian space forms, generalized Sasakian space forms*

Mathematics Subject Classification (2020) 53C25, 53C43

1. Introduction

Let $\mathbb{M}$ and $\mathbb{N}$ be Riemannian manifolds. Then, a map $\dot{\vartheta} : (\mathbb{M}, g) \to (\mathbb{N}, h)$ is called harmonic if it is a critical point of energy functional given by

$$E(\dot{\vartheta}) = \frac{1}{2} \int_{\mathbb{M}} \left| d\dot{\vartheta} \right|^2 v_g$$

Moreover, harmonic maps are defined as solutions of the corresponding Euler-Lagrange equation which is a non-linear elliptic partial differential equation characterized by the vanishing of the tension field $\dot{\tau}(\dot{\vartheta}) = \mathrm{trace} \nabla d\dot{\vartheta}$.

The bienergy functional of a map $\dot{\vartheta}$ is introduced by Eells and Sampson [1] as follows:

$$V_2(\dot{\vartheta}) = \frac{1}{2} \int_{\mathbb{M}} \left| \dot{\tau}(\dot{\vartheta}) \right|^2 v_g$$

Here, if $\dot{\vartheta}$ is a critical point of the bienergy functional, then it is called a biharmonic map. The Euler-Lagrange equation of $V_2(\dot{\vartheta})$ which is characterized by the vanishing of the bitension field is obtained by Jiang [2] as

$$\dot{\tau}_2(\dot{\vartheta}) = -\triangle \dot{\tau}(\dot{\vartheta}) - \mathrm{trace} \mathcal{R}^{\mathbb{N}}(d\dot{\vartheta}, \dot{\tau}(\dot{\vartheta}))d\dot{\vartheta}$$

Here, $\mathcal{R}^{\mathbb{N}}(\mathcal{X}, \mathcal{Y}) = [\nabla_{\mathcal{X}}, \nabla_{\mathcal{Y}}] - \nabla_{[\mathcal{X},\mathcal{Y}]}$ is the curvature operator of $\mathbb{N}$ and $\Delta = -\mathrm{trace}(\nabla^{\dot{\vartheta}} \nabla^{\dot{\vartheta}} - \nabla^{\dot{\vartheta}}_{\nabla})$ is the rough Laplacian on the sections of $\dot{\vartheta}^{-1} T\mathbb{N}$. If $\dot{\tau}_2(\dot{\vartheta}) = 0$, then $\dot{\vartheta}$ is called as a biharmonic map.

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f-harmonic maps are defined as critical points of f-energy functional

$$V_f(\dot\vartheta) = \frac{1}{2}\int_{\mathbb{M}} f\left|d\dot\vartheta\right|^2 v_g$$

for the maps $\dot\vartheta : (\mathbb{M}, g) \to (\mathbb{N}, h)$ where $f \in C^\infty(\mathbb{M}, \mathbb{R})$ [3]. The Euler-Lagrange equation is given by $\dot\tau_f(\dot\vartheta) = f\dot\tau(\dot\vartheta) + d\dot\vartheta(\text{grad}f)$ where $\dot\tau(\dot\vartheta) \equiv \text{trace}\nabla d\dot\vartheta$ is the tension field of $\dot\vartheta$.

The critical points of the f-bienergy functional

$$V_{2,f}\left(\dot\vartheta\right) = \frac{1}{2}\int_{\mathbb{M}} f\left|\dot\tau(\dot\vartheta)\right|^2 v_g$$

for maps $\dot\vartheta : (\mathbb{M}, g) \to (\mathbb{N}, h)$ is called as f-biharmonic maps. The Euler-Lagrange equation provides the f-biharmonic map equation as

$$\dot\tau_{2,\,f}\left(\dot\vartheta\right) \equiv f\,\dot\tau_2(\dot\vartheta) - (\Delta f)\dot\tau(\dot\vartheta) - 2\nabla^{\dot\vartheta}_{gradf}\,\dot\tau(\dot\vartheta)$$

which is called f-bitension field of map $\dot\vartheta$ [4].

Bi-f-harmonic maps are defined as critical points of the bi-f-energy functional

$$V_{f,2}\left(\dot\vartheta\right) = \frac{1}{2}\int_{\mathbb{M}} \left|\dot\tau_f\left(\dot\vartheta\right)\right|^2 v_g$$

for maps $\dot\vartheta : (\mathbb{M}, g) \to (\mathbb{N}, h)$. The Euler-Lagrange equation provides the bi-f-harmonic map equation [5]:

$$\dot\tau_{f,2}\left(\dot\vartheta\right) \equiv f\,J^{\dot\vartheta}(\dot\tau_f\left(\dot\vartheta\right)) - \nabla^{\dot\vartheta}_{\text{grad}f}\,\dot\tau_f\left(\dot\vartheta\right) \tag{1.1}$$

where $J^{\dot\vartheta}$ is the Jacobi operator of the map defined by

$$J^{\dot\vartheta}(\mathcal{X}) = -Tr\left(\nabla^{\dot\vartheta}\nabla^{\dot\vartheta}\mathcal{X} - \nabla^{\dot\vartheta}_{\nabla^{\mathbb{M}}}\mathcal{X} - \mathcal{R}^{\mathbb{N}}(d\dot\vartheta, \mathcal{X})d\dot\vartheta\right)$$

It is obvious that if f is a constant function, then f-biharmonic and bi-f-harmonic maps become biharmonic maps. Bi-f-harmonic and f-biharmonic maps which are not biharmonic are called proper bi-f-harmonic and proper f-biharmonic maps, respectively. For more details about bi-f-harmonic maps, see [4–6].

The notion of generalized Sasakian space forms was introduced by Alegre et al. [7]. Sarkar et.al. [8] studied Legendre curves in 3-dimensional trans-Sasakian manifolds. Then, Fetcu [9] handled biharmonic Legendre curves in Sasakian space forms. Moreover, Güvenç and Özgür [10, 11] investigated some classes of biharmonic Legendre curves in generalized Sasakian space forms and f-biharmonic Legendre curves in Sasakian space forms. In addition, for recent studies, see [12–14].

In this paper, we study bi-f-harmonic Legendre curves in (α,β)-trans-Sasakian generalized Sasakian space forms and provide some characterizations for bi-f-harmonicity of such curves under some special assumptions.

2. Generalized Sasakian Space Forms

In this section, we provide some basic definitions about almost contact metric manifolds and generalized Sasakian space forms in [7, 15].

$\mathbb{M}^{(2n+1)}$ is defined as an almost contact manifold with the almost contact structure $(\dot\vartheta, \varsigma, \dot\eta)$ if a tensor field $\dot\vartheta$ of type $(1, 1)$, a vector field ς, and a $1-$form $\dot\eta$ satisfy the followings

$$\dot\vartheta^2 = -I + \dot\eta \otimes \varsigma \tag{2.1}$$

and

$$\dot\eta(\varsigma) = 1$$

Here, I denotes the identity transformation. As an consequence of the conditions (2.1), $\dot{\vartheta}\varsigma = 0$ and $\dot{\eta} \circ \dot{\vartheta} = 0$.

Let $\mathbb{M}^{(2n+1)}$ be an almost contact manifold with an almost contact structure $(\dot{\vartheta}, \varsigma, \dot{\eta})$. If it admits a Riemannian metric g such that

$$g(\dot{\vartheta}\mathcal{X}, \dot{\vartheta}\mathcal{Y}) = g(\mathcal{X}, \mathcal{Y}) - \dot{\eta}(\mathcal{X})\dot{\eta}(\mathcal{Y}), \quad \mathcal{X}, \mathcal{Y} \in \Gamma(T\mathbb{M}) \tag{2.2}$$

then it becomes an almost contact metric manifold with an almost contact metric structure $(\dot{\vartheta}, \varsigma, \dot{\eta}, g)$. From (2.2),

$$g(\mathcal{X}, \dot{\vartheta}\mathcal{Y}) = -g(\dot{\vartheta}\mathcal{X}, \mathcal{Y})$$

and

$$g(\mathcal{X}, \varsigma) = \dot{\eta}(\mathcal{X})$$

for any $\mathcal{X}, \mathcal{Y} \in T\mathbb{M}$. The fundamental 2-form of $\mathbb{M}$ is defined by

$$\Phi(\mathcal{X}, \mathcal{Y}) = g(\mathcal{X}, \dot{\vartheta}\mathcal{Y})$$

An almost contact metric structure becomes a contact metric structure if

$$g(\mathcal{X}, \dot{\vartheta}\mathcal{Y}) = d\dot{\eta}(\mathcal{X}, \mathcal{Y})$$

for all vector fields $\mathcal{X}, \mathcal{Y} \in \Gamma(T\mathbb{M})$, where

$$d\dot{\eta}(\mathcal{X}, \mathcal{Y}) = \frac{1}{2}\{\mathcal{X}\dot{\eta}(\mathcal{Y}) - \mathcal{Y}\dot{\eta}(\mathcal{X}) - \dot{\eta}([\mathcal{X}, \mathcal{Y}])\}$$

A contact metric manifold with a Killing Reeb vector field ς is called a K-contact manifold. An almost contact metric manifold is called normal if

$$\mathcal{N}_{\dot{\vartheta}}(\mathcal{X}, \mathcal{Y}) + 2d\dot{\eta}(\mathcal{X}, \mathcal{Y})\varsigma = 0$$

where $\mathcal{N}$ is the Nijenhuis torsion tensor of $\dot{\vartheta}$ given by

$$\mathcal{N}_{\dot{\vartheta}}(\mathcal{X}, \mathcal{Y}) = \dot{\vartheta}^2[\mathcal{X}, \mathcal{Y}] + \left[\dot{\vartheta}\mathcal{X}, \dot{\vartheta}\mathcal{Y}\right] - \dot{\vartheta}\left[\dot{\vartheta}\mathcal{X}, \mathcal{Y}\right] - \dot{\vartheta}\left[\mathcal{X}, \dot{\vartheta}\mathcal{Y}\right]$$

for all $\mathcal{X}, \mathcal{Y} \in \Gamma(T\mathbb{M})$. A contact normal metric manifold is said to be a Sasakian manifold. Besides, an almost contact metric manifold is called a Sasakian manifold if and only if

$$(\nabla_{\mathcal{X}}\dot{\vartheta})\mathcal{Y} = g(\mathcal{X}, \mathcal{Y})\varsigma - \dot{\eta}(\mathcal{Y})\mathcal{X}$$

for any $\mathcal{X}, \mathcal{Y} \in \Gamma(T\mathbb{M})$.

An almost contact metric manifold is called a Kenmotsu manifold if and only if $d\dot{\eta} = 0$ and $d\Phi = 2\dot{\eta}\wedge\Phi$, or equivalently

$$(\nabla_{\mathcal{X}}\dot{\vartheta})\mathcal{Y} = -\dot{\eta}(\mathcal{Y})\dot{\vartheta}\mathcal{X} - g(\mathcal{X}, \dot{\vartheta}\mathcal{Y})\varsigma$$

Hence,

$$\nabla_{\mathcal{X}}\varsigma = \mathcal{X} - \dot{\eta}(\mathcal{X})\varsigma$$

Finally, an almost contact metric manifold is called a cosymplectic manifold if and only if $d\dot{\eta} = 0$ and $d\Phi = 0$, or equivalently $\nabla\dot{\vartheta} = 0$ and thus $\nabla\varsigma = 0$.

As a generalization of Kenmotsu and Sasakian manifolds, (α, β)-trans-Sasakian manifolds were introduced by Oubiña [16]. If there exist two functions α and β on an almost contact metric manifold $\mathbb{M}$ satisfying

$$\left(\nabla_{\mathcal{X}}\dot{\vartheta}\right)\mathcal{Y} = \alpha\left(g(\mathcal{X}, \mathcal{Y})\varsigma - \dot{\eta}(\mathcal{Y})\mathcal{X}\right) + \beta\left(g\left(\dot{\vartheta}\mathcal{X}, \mathcal{Y}\right)\varsigma - \dot{\eta}(\mathcal{Y})\dot{\vartheta}\mathcal{X}\right)$$

for any $\mathcal{X}, \mathcal{Y} \in \Gamma(T\mathbb{M})$, then $\mathbb{M}$ is called a trans-Sasakian manifold.

Here,

i. if $\beta = 0$, then $\mathbb{M}$ is called a α-Sasakian manifold,

ii. if $\beta = 0$ and $\alpha = 1$, then $\mathbb{M}$ is called a Sasakian manifold,

iii. if $\alpha = 0$, then $\mathbb{M}$ is called a β-Kenmotsu manifold,

iv. if $\beta = 1$ and $\alpha = 0$, then $\mathbb{M}$ is called a Kenmotsu manifolds, and

v. if $\alpha = \beta = 0$, then $\mathbb{M}$ is a cosymplectic manifold.

For a trans-Sasakian manifold,

$$\nabla_{\mathcal{X}} \varsigma = -\alpha \dot{\vartheta} \mathcal{X} + \beta \left(\mathcal{X} - \dot{\eta}(\mathcal{X}) \varsigma \right)$$

and

$$d\dot{\eta} = \alpha \Phi$$

De and Tripathi [17] showed that on an (α,β)-trans-Sasakian manifold the following relation is hold:

$$\varsigma(\alpha) + 2\alpha\beta = 0$$

It was shown in [18] that an (α,β)-trans-Sasakian manifold with dimension ≥ 5 is either α-Sasakian, β-Kenmotsu or cosymplectic.

A $\dot{\vartheta}$-section of an almost contact metric manifold $(\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ at a point $p \in \mathbb{M}$ is a section $\Pi \subseteq T_p\mathbb{M}$ spanned by a unit vector field $\mathcal{X}_p$ orthogonal to ς_p and $\dot{\vartheta}\mathcal{X}_p$. The $\dot{\vartheta}$-sectional curvature $\mathcal{K}(\mathcal{X} \wedge \dot{\vartheta}\mathcal{X})$ is defined by

$$\mathcal{K}(\mathcal{X} \wedge \dot{\vartheta}\mathcal{X}) = \mathcal{R}(\mathcal{X}, \dot{\vartheta}\mathcal{X}, \dot{\vartheta}\mathcal{X}, \mathcal{X})$$

If $\dot{\vartheta}$-sectional curvature of $\mathbb{M}$ is constant, then it is called a space form.

Moreover, an almost contact metric manifold is called a generalized Sasakian space form [7] if there exist functions ρ_1, ρ_2, and ρ_3 on $\mathbb{M}$ such that

$$\begin{aligned} \mathcal{R}(\mathcal{X},\mathcal{Y})\mathcal{Z} = {} & \rho_1 \left\{ g(\mathcal{Y},\mathcal{Z})\mathcal{X} - g(\mathcal{X},\mathcal{Z})\mathcal{Y} \right\} + \rho_2 \left\{ g(\mathcal{X},\dot{\vartheta}\mathcal{Z})\dot{\vartheta}\mathcal{Y} - g(\mathcal{Y},\dot{\vartheta}\mathcal{Z})\dot{\vartheta}\mathcal{X} + 2g(\mathcal{X},\dot{\vartheta}\mathcal{Y})\dot{\vartheta}\mathcal{Z} \right\} \\ & + \rho_3 \left\{ \dot{\eta}(\mathcal{X})\dot{\eta}(\mathcal{Z})\mathcal{Y} - \dot{\eta}(\mathcal{Y})\dot{\eta}(\mathcal{Z})\mathcal{X} + g(\mathcal{X},\mathcal{Z})\dot{\eta}(\mathcal{Y})\varsigma - g(\mathcal{Y},\mathcal{Z})\dot{\eta}(\mathcal{X})\varsigma \right\} \end{aligned} \tag{2.3}$$

for any vector fields on $\mathbb{M}$, where $\mathcal{R}$ denotes the curvature tensor of $\mathbb{M}$.

For a generalized Sasakian-space-form;

i. if $\rho_1 = \frac{c+3}{4}$ and $\rho_2 = \rho_3 = \frac{c-1}{4}$, then it becomes a Sasakian-space-form,

ii. if $\rho_1 = \frac{c-3}{4}$ and $\rho_2 = \rho_3 = \frac{c+1}{4}$, then it becomes a Kenmotsu-space-form, and

iii. if $\rho_1 = \rho_2 = \rho_3 = \frac{c}{4}$, then it becomes a cosymplectic-space-form

where c is the constant $\dot{\vartheta}$-sectional curvature. The contact distribution of an almost contact metric manifold $(\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is defined by

$$\{\mathcal{X} \in \Gamma(T\mathbb{M}) : \dot{\eta}(\mathcal{X}) = 0\}$$

and an integral curve of the contact distribution is called a Legendre curve [15].

3. Bi-f-Harmonic Curves

Recall the bi-f-harmonic map equation for curves in Riemannian and start with the important proposition for Euler-Lagrange equation of bi-f-harmonic maps [5].

Proposition 3.1. Let $\dot{\vartheta} : (\mathbb{M}, g) \to (\mathbb{N}, h)$ be a smooth map between Riemannian manifolds. Then, $\dot{\vartheta}$ is a bi-f-harmonic map if and only if its bi-f-tension field

$$\dot{\tau}_{f,2}(\dot{\vartheta}) = \Delta_f^2 \dot{\tau}_f \left(\dot{\vartheta}\right) - f \,\mathrm{trace}_g \mathcal{R}^{\mathbb{N}} \left(\dot{\tau}_f \left(\dot{\vartheta}\right), d\dot{\vartheta}\right) d\dot{\vartheta} \tag{3.1}$$

vanishes, where

$$\Delta_f^2 \dot{\tau}_f \left(\dot{\vartheta}\right) = -\mathrm{trace}_g \left(\nabla^{\dot{\vartheta}} f \left(\nabla^{\dot{\vartheta}} \dot{\tau}_f \left(\dot{\vartheta}\right)\right) - f \,\nabla^{\dot{\vartheta}}_{\nabla^{\mathbb{M}}} \dot{\tau}_f \left(\dot{\vartheta}\right)\right) \tag{3.2}$$

and $\dot{\tau}_f \left(\dot{\vartheta}\right)$ is the f-tension field given by (1.1).

By considering a curve, from (3.1) and (3.2), from [6], the following proposition is hold:

Proposition 3.2. Let $\sigma : I \to (\mathbb{N}, h)$ be a curve parameterized by arclenght on a Riemannian manifold $(\mathbb{N}, h)$ and $\sigma' = T$. Then, σ is a bi-f-harmonic curve if and only if

$$(ff'')'T + \left(2\left(f'\right)^2 + 3f''f\right) \nabla_T^{\mathbb{N}} T + 4f'f\,\nabla_T^2 T + f^2 \nabla_T^3 T + f^2 \mathcal{R}^{\mathbb{N}} \left(\nabla_T^{\mathbb{N}} T, T\right) T = 0$$

where $f : I \to \mathbb{R}^+$, I is an interval, $\nabla_T^2 T = \nabla_T^{\mathbb{N}} \nabla_T^{\mathbb{N}} T$, and $\nabla_T^3 T = \nabla_T^{\mathbb{N}} \nabla_T^{\mathbb{N}} \nabla_T^{\mathbb{N}} T$.

Assume that $\sigma : I \to (\mathbb{N}, h)$ is a arclenght parameterized curve in an $n-$dimensional Riemannian manifold $(\mathbb{N}, h)$. If there exist ortonormal vector fields $V_1, V_2, \cdots, V_r$ along σ such that

$$\begin{aligned} \nabla_T V_1 &= k_1 V_2 \\ \nabla_T V_2 &= -k_1 V_1 + k_2 V_3 \\ &\vdots \\ \nabla_T V_r &= -k_{r-1} V_{r-1} \end{aligned} \tag{3.3}$$

then σ is called a Frenet curve of osculating order r, for $1 \leq r \leq n$. Here, $V_1 = \sigma' = T$ is the unit tangent vector field of σ, V_2 is the unit normal vector field of σ with the same direction as $\nabla_T V_1$, and the vectors $V_3, V_4, \cdots, V_r$ are the unit vectors obtained from the Frenet equations for σ, where $k_1 = \|\nabla_T V_1\|$ and $k_2, k_3, \cdots, k_{r-1}$ are real-valued positive functions.

From (3.3),

$$\nabla_T^2 T = \nabla_T^{\mathbb{N}} \nabla_T^{\mathbb{N}} T = -k_1^2 V_1 + k_1' V_2 + k_1 k_2 V_3$$

$$\nabla_T^3 T = \nabla_T^{\mathbb{N}} \nabla_T^{\mathbb{N}} \nabla_T^{\mathbb{N}} T = -3k_1 k_1' V_1 + \left(k_1'' - k_1^3 - k_1 k_2^2\right) V_2 + \left(2k_1' k_2 + k_1 k_2'\right) V_3 + k_1 k_2 k_3 V_4$$

and

$$\mathcal{R}^{\mathbb{N}} \left(\nabla_T^{\mathbb{N}} T, T\right) T = k_1 \mathcal{R}^{\mathbb{N}} \left(V_2, T\right) T$$

Then,

$$\begin{aligned} \dot{\tau}_{f,2}(\sigma) =& \left((ff'')' - 3k_1 k_1' f^2 - 4k_1^2 f f'\right) T + \left(\left(-k_1^3 - k_1 k_2^2 + k_1''\right) f^2 + 4k_1' f f' + 3k_1 f'' f + 2k_1 (f')^2\right) V_2 \\ &+ \left(4k_1 k_2 f f' + f^2 \left(2k_2 k_1' + k_1 k_2'\right)\right) V_3 + \left(k_1 k_2 k_3 f^2\right) V_4 + k_1 f^2 \mathcal{R}^{\mathbb{N}}(V_2, T) T \end{aligned}$$

Theorem 3.3. Let $\sigma : I \to (\mathbb{N}, h)$ be a arclenght parameterized curve on a Riemannian manifold $(\mathbb{N}, h)$. Then, σ is a bi-f-harmonic curve if and only if

$$\begin{aligned} 0 =& \left((ff'')' - 3k_1 k_1' f^2 - 4k_1^2 f f'\right) T + \left(\left(-k_1^3 - k_1 k_2^2 + k_1''\right) f^2 + 4k_1' f f' + 3k_1 f'' f + 2k_1 (f')^2\right) V_2 \\ &+ \left(4k_1 k_2 f f' + f^2 \left(2k_2 k_1' + k_1 k_2'\right)\right) V_3 + \left(k_1 k_2 k_3 f^2\right) V_4 + k_1 f^2 \mathcal{R}^{N}(V_2, T) T \end{aligned} \tag{3.4}$$

4. Bi-f-harmonic Curves in (α,β)-Trans-Sasakian Generalized Sasakian Space Forms

In this section, we first obtain bi-f-harmonic equation of a curve $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ on an (α,β)-trans-Sasakian generalized Sasakian space form. Note that, throughout this paper, we use (α,β)-TSGSSF instead of (α,β)-trans-Sasakian generalized Sasakian space form and cons instead of constant in equations for the sake of simplicity. By using (2.3),

$$\begin{aligned}\mathcal{R}^{\mathbb{M}}(V_2,T)T = &\ \rho_1\left\{g(T,T)V_2 - g(V_2,T)T\right\} + \rho_2\left\{g(V_2,\dot{\vartheta}T)\dot{\vartheta}T - g(T,\dot{\vartheta}T)\dot{\vartheta}V_2 - 2g(T,\dot{\vartheta}V_2)\dot{\vartheta}T\right\}\\ &+\rho_3\left\{\dot{\eta}(V_2)\dot{\eta}(T)T - \dot{\eta}(T)\dot{\eta}(T)V_2 + g(V_2,T)\dot{\eta}(T)\varsigma - g(T,T)\dot{\eta}(V_2)\varsigma\right\}\end{aligned}$$

which implies

$$\mathcal{R}^{\mathbb{M}}(V_2,T)T = \rho_3\dot{\eta}(T)\dot{\eta}(V_2)T + \left(\rho_1 - \rho_3\left(\dot{\eta}(T)\right)^2\right)V_2 - 3\rho_2 g(T,\dot{\vartheta}V_2)\dot{\vartheta}T - \rho_3\dot{\eta}(V_2)\varsigma$$

From (3.4), we get bi-f-tension field of σ.

Theorem 4.1. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF. Then, σ is a bi-f-harmonic curve if and only if

$$\begin{aligned}0 = &\left((ff'')' - 3k_1k_1'f^2 - 4k_1^2 ff' + k_1 f^2\rho_3\dot{\eta}(T)\dot{\eta}(V_2)\right)T\\ &+\left(\left(-k_1^3 - k_1k_2^2 + k_1''\right)f^2 + 4k_1' ff' + 3k_1 f'' f + 2k_1(f')^2 + k_1 f^2\left(\rho_1 - \rho_3\left(\dot{\eta}(T)\right)^2\right)\right)V_2\\ &+\left(4k_1k_2 ff' + f^2\left(2k_2k_1' + k_1k_2'\right)\right)V_3 + \left(k_1k_2k_3 f^2\right)V_4 + 3\rho_2 k_1 f^2 g(\dot{\vartheta}T,V_2)\dot{\vartheta}T - \rho_3 k_1 f^2\dot{\eta}(V_2)\varsigma\end{aligned}$$

For the remaining parts of this study, we consider that $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a Legendre curve in an (α,β)-TSGSSF. If σ is a Legendre curve, then

$$\dot{\eta}(V_2) = -\frac{\beta}{k_1} \tag{4.1}$$

Since σ is a Legendre curve, from (4.1), it is obvious that $V_2 \perp \varsigma$ if and only if $\beta = 0$ [19].

Corollary 4.2. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a Legendre curve parameterized by its arclenght on an (α,β)-TSGSSF. Then, σ is a bi-f-harmonic curve if and only if

$$\begin{aligned}0 = &\left((ff'')' - 3k_1k_1'f^2 - 4k_1^2 ff'\right)T\\ &\left((-k_1^3 - k_1k_2^2 + k_1'')f^2 + 4k_1' ff' + 3k_1 f'' f + 2(f')^2 k_1 + k_1 f^2\rho_1\right)V_2\\ &+f\left(2k_2k_1' f + k_1k_2' f + 4k_1k_2 f'\right)V_3 + \left(k_1k_2k_3 f^2\right)V_4 + 3\rho_2 k_1 f^2 g(\dot{\vartheta}T,V_2)\dot{\vartheta}T + \beta\rho_3 f^2\varsigma\end{aligned} \tag{4.2}$$

Let $m = \min\{r, 4\}$. From (4.2), σ is a bi-f-harmonic Legendre curve if and only if

i. $\rho_2 = 0$ or $\dot{\vartheta}T \perp V_2$ or $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$

ii. $\rho_3 = 0$ or $\varsigma \in \text{span}\{V_2, V_3 \cdots, V_m\}$

iii. $g(\dot{\tau}_{f,2}(\sigma), V_i) = 0$, for all $i \in \{1, 2, \cdots, m\}$

Theorem 4.3. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a Legendre curve parameterized by its arclenght on an (α,β)-TSGSSF. Then, σ is a bi-f-harmonic curve if and only if

i. $\rho_2 = 0$ or $\dot{\vartheta}T \perp V_2$ or $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$

ii. $\rho_3 = 0$ or $\varsigma \in \text{span}\{V_2, V_3, \cdots, V_m\}$

iii. The following equations are satisfied:

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ \left\{ \begin{array}{c} (k_1'' - k_1^3 - k_1k_2^2 + k_1\rho_1)f^2 + 4k_1'f'f + 3k_1f''f + 2(f')^2k_1 \\ +3f^2k_1\rho_2 g(\dot{\vartheta}T, V_2)^2 + f^2\rho_3\beta\dot{\eta}(V_2) \end{array} \right. = 0 \\ 4k_1k_2ff' + f^2\ (2k_2k_1' + k_1k_2') + 3\rho_2k_1f^2g(\dot{\vartheta}T, V_2)g(\dot{\vartheta}T, V_3) + \beta\rho_3f^2\dot{\eta}(V_3) = 0 \\ k_1k_2k_3 + 3\rho_2k_1g(\dot{\vartheta}T, V_2)g(\dot{\vartheta}T, V_4) + \beta\rho_3\dot{\eta}(V_4) = 0 \end{cases} \tag{4.3}$$

CASE I. Let $\rho_2 = \rho_3 = 0$. Then, the manifold $\mathbb{M}$ is a Riemannian space form of constant sectional curvature ρ_2. In this case, $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a proper bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (k_1'' - k_1^3 - k_1k_2^2 + k_1\rho_1)f^2 + 4k_1'f f' + 3k_1f''f + 2k_1(f')^2 = 0 \\ 4k_1k_2f' + 2k_2k_1'f + k_1k_2'f = 0 \\ k_1k_2k_3 = 0 \end{cases} \tag{4.4}$$

Theorem 4.4. There is no any proper bi-f-harmonic Legendre curve of osculating order $r \geq 4$ in an (α, β)-TSGSSF with $\rho_2 = \rho_3 = 0$.

From (4.4), if σ is a geodesic curve, then it is a bi-f-harmonic curve if and only if $ff'' = \text{cons}$.

Theorem 4.5. A geodesic curve in an (α, β)-TSGSSF is bi-f-harmonic if and only if $ff'' = \text{cons}$.

This theorem proves that there are bi-f harmonic curves that are not harmonic. Afterward, we investigate bi-f-harmonicity of $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ considering some special subcases:

CASE I. 1. If $k_1 = \text{cons} \neq 0$ and $k_2 = 0$, then, from (4.4),

$$\begin{cases} (ff'')' - 4k_1^2 ff' = 0, \\ (\rho_1 - k_1^2)f^2 + 2(f')^2 + 3f''f = 0 \end{cases} \tag{4.5}$$

From the second equation of (4.5), $ff'' = \frac{(k_1^2 - \rho_1)f^2 - 2(f')^2}{3}$ which implies

$$10k_1^2 ff' + \rho_1' f^2 + 2\rho_1 ff' + 4f'f'' = 0 \tag{4.6}$$

via the first equation of (4.5).

Theorem 4.6. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α, β)-TSGSSF with $\rho_2 = \rho_3 = 0$, $k_1 = \text{cons} \neq 0$, and $k_2 = 0$. Then, σ is a bi-f-harmonic Legendre curve if and only if f, k_1, and ρ_1 satisfy following differential equation

$$10k_1^2 ff' + \rho_1' f^2 + 2\rho_1 ff' + 4f'f'' = 0$$

Further, if (4.6) is solved by assuming ρ_1 constant, the the following result is obtained.

Theorem 4.7. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an α-Sasakian generalized Sasakian space form dimension ≥ 5 with $\rho_2 = \rho_3 = 0$, $k_1 = \text{cons} \neq 0$, and $k_2 = 0$. Then, σ is a proper bi-f-harmonic Legendre curve if and only if f is a function defined by

$$f(s) = c_1 \cos\left(\sqrt{\frac{5k_1^2 + \rho_1}{2}} s\right) + c_2 \sin\left(\sqrt{\frac{5k_1^2 + \rho_1}{2}} s\right)$$

where $s \in I$ and ρ_1 is a constant.

CASE I. 2. If $k_1 = \text{cons} \neq 0$ and $k_2 = \text{cons} \neq 0$, then (4.4) reduces to

$$\begin{cases} (ff'')' - 4k_1^2 ff' = 0 \\ f^2(-k_1^2 - k_2^2 + \rho_1) + 3f''f + 2(f')^2 = 0 \\ f' = 0 \\ k_3 = 0 \end{cases}$$

which implies

$$\begin{cases} f = \text{cons} \\ k_1^2 + k_2^2 = \rho_1 \\ k_3 = 0 \end{cases}$$

Theorem 4.8. There is no any proper bi-f-harmonic Legendre curve on an (α,β)-TSGSSF with $\rho_2 = \rho_3 = 0$, $k_1 = \text{cons} \neq 0$, and $k_2 = \text{cons} \neq 0$.

CASE I. 3. If $k_1 \neq \text{cons}$ and $k_2 = \text{cons} \neq 0$, then (4.4) reduces to

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 ff' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'')f^2 + 4k_1'ff' + 3k_1f''f + 2k_1(f')^2 + k_1\rho_1 f^2 = 0 \\ 2k_1f' + k_1'f = 0 \\ k_3 = 0 \end{cases}$$

Theorem 4.9. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 = \rho_3 = 0$, $k_1 \neq \text{cons}$, and $k_2 = \text{cons} \neq 0$. Then, σ is a bi-f-harmonic Legendre curve if and only if

$$f = \pm c k_1^{-\frac{1}{2}}$$

for some real constant c, $k_3 = 0$, and the curvature k_1 solves the following second order non-linear differential equations system

$$\begin{cases} 9(k_1')^3 + 4k_1'k_1^4 - 10k_1''k_1'k_1 + 2k_1'''k_1^2 = 0 \\ -3(k_1')^2 + 4k_1^4 + 4k_1^2k_2^2 + 2k_1''k_1 - 4k_1^2\rho_1 = 0 \end{cases}$$

CASE I. 4. If $k_1 \neq \text{cons}$ and $k_2 \neq \text{cons}$, then by using the third equation in (4.4),

$$f = \pm c k_1^{-\frac{1}{2}} k_2^{-\frac{1}{4}}$$

for some real constant c. Besides, from the last equation in (4.4), $k_3 = 0$.

Theorem 4.10. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 = \rho_3 = 0$, $k_1 \neq \text{cons}$, and $k_2 \neq \text{cons}$. Then, σ is a bi-f-harmonic Legendre curve if and only if $f = \pm c k_1^{-\frac{1}{2}} k_2^{-\frac{1}{4}}$, c is a constant, $k_3 = 0$, and k_1 and k_2 satisfy the following second order non-linear differential equation system

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 ff' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'' + k_1\rho_1)f^2 + 4k_1'ff' + 3k_1f''f + 2k_1(f')^2 = 0 \end{cases}$$

Before calculating Case II, we recall the following results [20]:

Proposition 4.11. Let $(\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be an α-Sasakian generalized Sasakian space form. Therefore,

α is independent of the direction of ς and the following equation is valid

$$\rho_1 - \rho_3 = \alpha^2$$

Moreover, if $\mathbb{M}$ is connected, then α is a constant.

Theorem 4.12. Let $(\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a connected α-Sasakian generalized Sasakian space form with dimension $m \geq 5$. Then, ρ_1, ρ_2, and ρ_3 are constant functions related as follows:

i. If $\alpha = 0$, then $\rho_1 = \rho_2 = \rho_3$ and $\mathbb{M}$ is a cosymplectic manifold of constant $\dot{\vartheta}$-sectional curvature

ii. If $\alpha \neq 0$, then $\rho_1 - \alpha^2 = \rho_2 = \rho_3$

CASE II. Let $\rho_2 = 0$, $\rho_3 \neq 0$, and $V_2 \perp \varsigma$. Then, from (4.1), it is obvious that the manifold $\mathbb{M}$ is an α-Sasakian generalized Sasakian space form. By using Proposition 4.11, $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a proper bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'')f^2 + 4k_1' f f' + 3k_1 f'' f + 2k_1(f')^2 + k_1 f^2 (\rho_3 + \alpha^2) = 0 \\ 4k_1k_2 f' + (2k_2 f k_1' + k_1 k_2') f = 0 \\ k_1k_2k_3 = 0. \end{cases} \tag{4.7}$$

Theorem 4.13. There is no any bi-f-harmonic Legendre curve of osculating order $r > 3$ satisfying $\rho_2 = 0$, $\rho_3 \neq 0$, and $V_2 \perp \varsigma$ in an α-Sasakian generalized Sasakian space form.

Theorem 4.14. There is no any bi-f-harmonic Legendre curve satisfying $\rho_2 = 0$, $\rho_3 \neq 0$, and $V_2 \perp \varsigma$ in a connected α-Sasakian generalized Sasakian space form with dimension ≥ 5.

CASE II.1. Let $\rho_2 = 0$, $\rho_3 \neq 0$, $V_2 \perp \varsigma$, and $\alpha \neq 0$.

In this case, we consider bi-f-harmonic Legendre curves satisfying $\rho_2 = 0$, $\rho_3 \neq 0$, and $V_2 \perp \varsigma$ in a connected 3-dimensional α-Sasakian generalized Sasakian space forms. In a 3-dimensional α-Sasakian manifold, a Legendre curve is a Frenet curve of osculating order 3 and its torsion is always α [21]. Then, (4.7) reduces to

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'')f^2 + 4f k_1' f' + 3f'' f k_1 + 2(f')^2 k_1 + k_1 f^2 (\rho_3 + \alpha^2) = 0 \\ 2k_1 f' + f k_1' = 0 \end{cases} \tag{4.8}$$

Theorem 4.15. Let $(\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a 3-dimensional connected α-Sasakian generalized Sasakian space form satisfying $\rho_2 = 0$, $\rho_3 \neq 0$, and $V_2 \perp \varsigma$. Then, $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a bi-f-harmonic Legendre curve if and only if $f = \pm c k_1^{-\frac{1}{2}}$, where c is a constant and k_1 solves the following second order non-linear differential equation system

$$\begin{cases} 9(k_1')^3 + 4k_1'k_1^4 - 10k_1''k_1'k_1 + 2k_1'''k_1^2 = 0 \\ -3(k_1')^2 + 4k_1^4 + 4k_1^2k_2^2 + 2k_1''k_1 - 4k_1^2(\rho_3 + \alpha^2) = 0 \end{cases}$$

If $k_1 = \text{cons} \neq 0$, then f is constant from the third equation in (4.8).

Corollary 4.16. There is no any proper bi-f-harmonic Legendre helix in a 3-dimensional connected α-Sasakian generalized Sasakian space form satisfying $\rho_2 = 0$, $\rho_3 \neq 0$, and $V_2 \perp \varsigma$.

CASE II.2. Let $\rho_2 = 0$, $\rho_3 \neq 0$, $V_2 \perp \varsigma$, and $\alpha = 0$.

Theorem 4.17. Let $(\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a cosymplectic generalized Sasakian space form satisfying $\rho_2 = 0$,

$\rho_3 \neq 0$, and $V_2 \perp \varsigma$. Then, $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a bi-f-harmonic Legendre curve if and only if $\rho_1 = \rho_3$ and the following differential equation system is satisfied

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'')f^2 + 4k_1' f f' + 3k_1 f'' f + 2k_1(f')^2 + k_1 f^2(\rho_3 + \alpha^2) = 0 \\ 4k_1k_2 f' + (2k_2 f k_1' + k_1k_2')f = 0 \\ k_1k_2k_3 = 0 \end{cases}$$

CASE III. Let $\rho_2 = 0$, $\rho_3 \neq 0$, $\dot{\eta}(V_2) \neq 0$, and $\varsigma \in \operatorname{span}\{V_2, V_3, \cdots, V_m\}$. Then, from (4.2), $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a proper bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1 f^2 k_1' - 4k_1^2 f f' = 0 \\ \begin{cases} (-k_1^3 - k_1k_2^2 + k_1'')f^2 + 4f k_1' f' + 3f'' f k_1 + 2(f')^2 k_1 \\ +k_1 f^2 \rho_1 + f^2 \beta \rho_3 \dot{\eta}(V_2) = 0 \end{cases} \\ 2k_2 f k_1' + k_1 f k_2' + 4k_1k_2 f' + \beta\rho_3 f \dot{\eta}(V_3) = 0 \\ k_1k_2k_3 + \beta\rho_3\dot{\eta}(V_4) = 0 \end{cases}$$

Let $m = \min\{r, 4\} = 4$, which implies $r \geq 4$. Then,

$$\varsigma = \cos\theta_1 V_2 + \sin\theta_1 \cos\theta_2 V_3 + \sin\theta_1 \sin\theta_2 V_4$$

which implies

$$\dot{\eta}(V_2) = \cos\theta_1, \quad \dot{\eta}(V_3) = \sin\theta_1\cos\theta_2, \quad \text{and} \quad \dot{\eta}(V_4) = \sin\theta_1 \sin\theta_2$$

Here, $\theta_1 : I \to \mathbb{R}$ denotes the angle function between ς and V_2 and $\theta_2 : I \to \mathbb{R}$ is the angle function between V_3 and the orthogonal projection of ς onto $\operatorname{span}\{V_3, V_4\}$.

Theorem 4.18. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 = 0$, $\rho_3 \neq 0$, $\dot{\eta}(V_2) \neq 0$ and $\varsigma \in \operatorname{span}\{V_2, V_3, \cdots, V_m\}$. Then, σ is a bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1 f^2 k_1' - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'')f^2 + 4f k_1' f' + 3f'' f k_1 + 2(f')^2 k_1 + k_1 f^2 \rho_1 + f^2 \beta\rho_3 \cos\theta_1 = 0 \\ 2k_2 f k_1' + k_1 f k_2' + 4k_1k_2 f' + \beta \sin\theta_1 \cos\theta_2 \rho_3 f = 0 \\ k_1k_2k_3 + \beta \sin\theta_1 \sin\theta_2 \rho_3 = 0 \end{cases} \tag{4.9}$$

provided $r \geq 4$.

As a particular case, if $\beta = 0$, that is, $(\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is an α-Sasakian generalized Sasakian space form, then the following results is obtained:

Corollary 4.19. There is no any bi-f-harmonic Legendre curve of osculating order $r \geq 4$ in an α-Sasakian generalized Sasakian space form, satisfying $\rho_2 = 0$, $\rho_3 \neq 0$, $\dot{\eta}(V_2) \neq 0$, and $\varsigma \in \operatorname{span}\{V_2, V_3, \cdots, V_m\}$.

If ρ_1, ρ_3, and the first three curvatures of σ are constants, then the following result is valid:

Theorem 4.20. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 = 0$, $\rho_3 = \text{cons} \neq 0$, $\dot{\eta}(V_2) \neq 0$ and $\varsigma \in \operatorname{span}\{V_2, V_3, \cdots, V_m\}$. Then, σ is a bi-f-harmonic Legendre curve if and only if f is one of the followings:

$$f(s)=c_1\cos\left(\sqrt{\frac{-5k_1^2+k_2^2-\rho_1+\rho_3(cos\theta_1)^2}{2}}s\right)+c_2\sin\left(\sqrt{\frac{-5k_1^2+k_2^2-\rho_1+\rho_3(cos\theta_1)^2}{2}}s\right) \quad (4.10)$$

$$f(s)=c_3s+c_4 \quad (4.11)$$

and

$$f(s)=c_5e^{-\sqrt{\frac{5k_1^2-k_2^2+\rho_1+\rho_3(cos\theta_1)^2}{2}}s}+c_6e^{\sqrt{\frac{5k_1^2-k_2^2+\rho_1+\rho_3(cos\theta_1)^2}{2}}s} \quad (4.12)$$

provided that

$$5k_1^2-k_2^2+\rho_1+\rho_3(cos\theta_1)^2>0$$

$$5k_1^2-k_2^2+\rho_1+\rho_3(cos\theta_1)^2=0$$

and

$$5k_1^2-k_2^2+\rho_1+\rho_3(cos\theta_1)^2<0$$

respectively, and

$$f(s)=e^{\frac{k_3}{4}\int\cot\theta_2\,ds} \quad (4.13)$$

where $c_1,c_2,\cdots,c_6$, θ_1 and θ_2 are constants.

Proof. By using (4.9),

$$\begin{cases}(ff'')'-4k_1^2ff'=0\\ 3f''f+2(f')^2+f^2\left(-k_1^2-k_2^2+\rho_1-\rho_3\left(cos\theta_1\right)^2\right)=0\\ 4k_1k_2f'+\beta\sin\theta_1\cos\theta_2\rho_3f=0\\ k_1k_2k_3+\beta\sin\theta_1\sin\theta_2\rho_3=0\end{cases} \quad (4.14)$$

From the second equation of (4.14),

$$f''f=\frac{-2(f')^2+\left(k_1^2+k_2^2-\rho_1+\rho_3(cos\theta_1)^2\right)f^2}{3} \quad (4.15)$$

If (4.15) is used in the first equation of (4.14),

$$2f''+\left(5k_1^2-k_2^2+\rho_1-\rho_3(cos\theta_1)^2\right)f=0 \quad (4.16)$$

By solving the differential equation (4.16), the first assertion of the theorem is obtained. Besides,

$$\beta\rho_3\sin\theta_1\left(\cos\theta_2k_3f-4\sin\theta_2f'\right)=0$$

via the last two equations of (4.14) which implies (4.13). □

Let $r=3$. This implies that $\varsigma\in\text{span}\{V_2,V_3\}$ and by choosing $\theta_2=0$, $\varsigma=\cos\theta_1V_2+\sin\theta_1V_3$ where $\theta_1:I\to\mathbb{R}$ denotes the angle function between ς and V_2.

Theorem 4.21. Let $\sigma:I\to(\mathbb{M},\dot{\vartheta},\varsigma,\dot{\eta},g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2=0$, $\rho_3\neq0$, $\dot{\eta}(V_2)\neq0$, and $\varsigma\in\text{span}\{V_2,V_3\}$. Then, σ is a bi-f-harmonic Legendre curve if and only if

$$\begin{cases}(ff'')'-3k_1k_1'f^2-4k_1^2ff'=0\\ (3f''f+2(f')^2)k_1++4k_1'ff'+f^2(-k_1^3-k_1k_2^2+k_1''+k_1\rho_1+\beta\rho_3\cos\theta_1)=0\\ 4k_1k_2f'+f(2k_2fk_1'+k_1k_2'+\beta\sin\theta_1\rho_3)=0\end{cases}$$

provided $r=3$.

If ρ_1, ρ_3, and the first two curvatures of σ are constants, then the following result is obtained:

Corollary 4.22. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 = 0$, $\rho_3 = \text{cons} \neq 0$, $\dot{\eta}(V_2) \neq 0$, and $\varsigma \in \text{span}\{V_2, V_3\}$. Then, σ is a bi-f-harmonic Legendre curve if and only if f is defined by one of the form given in (4.10), (4.11), or (4.12) and

$$f(s) = e^{\frac{\rho_3}{4}\int \sin\theta_1 \cos\theta_1 \, ds}$$

where $s \in I$.

Let $r = 2$. Then, $\varsigma \in \text{span}\{V_2\}$ which implies $\varsigma = \pm V_2$ by taking $\theta_1 \in \{0, \pi\}$ and $\theta_2 = 0$.

Theorem 4.23. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 = 0$, $\rho_3 \neq 0$, $\dot{\eta}(V_2) \neq 0$, and $\varsigma = \pm V_2$. Then, σ is a bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (3f''f + 2(f')^2)k_1 + 4k_1' f f' + f^2(-k_1^3 + k_1'' + k_1\rho_1 + \beta\rho_3\cos\theta_1) = 0 \end{cases}$$

provided $r = 2$.

If ρ_1, ρ_3, and the first curvature of σ are constants, then the following result is obtained:

Corollary 4.24. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 = 0$, $\rho_3 = \text{cons} \neq 0$, $\dot{\eta}(V_2) \neq 0$, and $\varsigma = \pm V_2$. Then, σ is a bi-f-harmonic curve if and only if f is defined by one of the form given in (4.10), (4.11), or (4.12).

CASE IV. Let $\rho_2 \neq 0$, $\rho_3 = 0$, and $V_2 \perp \dot{\vartheta}T$. Then, from (4.3), $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a proper bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'' + k_1\rho_1)f^2 + 4k_1' f f' + 3k_1 f'' f + 2(f')^2 k_1 = 0 \\ 4k_1k_2 f' + (2k_2k_1' + k_1k_2')f = 0 \\ k_1k_2k_3 = 0 \end{cases}$$

Corollary 4.25. There is no any bi-f-harmonic Legendre curve of osculating order $r \geq 4$ in an (α,β)-TSGSSF, satisfying $\rho_2 \neq 0$, $\rho_3 = 0$, and $V_2 \perp \dot{\vartheta}T$.

Note that because the conditions obtained in Cases I and IV are the same, it is not necessary to investigate the subcases for Case IV.

CASE V: Let $\rho_2 \neq 0$, $\rho_3 = 0$, $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$, $g(\dot{\vartheta}T, V_2) \neq 0$, and $m = \min\{r, 4\} = 4$, which implies $r \geq 4$. Then,

$$\dot{\vartheta}T = \cos a_1 V_2 + \sin a_1 \cos a_2 V_3 + \sin a_1 \sin a_2 V_4 \tag{4.17}$$

which implies

$$g(\dot{\vartheta}T, V_2) = \cos a_1$$

$$g(\dot{\vartheta}T, V_3) = \sin a_1 \cos a_2$$

and

$$g(\dot{\vartheta}T, V_4) = \sin a_1 \sin a_2 \tag{4.18}$$

Here, $a_1 : I \to \mathbb{R}$ denotes the angle function between $\dot{\vartheta}T$ and V_2 and $a_2 : I \to \mathbb{R}$ is the angle function between V_3 and the orthogonal projection of $\dot{\vartheta}T$ onto $\text{span}\{V_3, V_4\}$. Thus, the following result is obtained:

Theorem 4.26. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 \neq 0$, $\rho_3 = 0$, $\dot{\vartheta}T \in \operatorname{span}\{V_2, V_3, V_4\}$, and $g(\dot{\vartheta}T, V_2) \neq 0$. Then, σ is a bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'' + k_1\rho_1 + 3k_1\rho_2(\cos a_1)^2)\, f^2 + 4f\, k_1' f' + 3f'' f\, k_1 + 2(f')^2 k_1 = 0 \\ f\ (2k_2 f\, k_1' + k_1 f\, k_2' + 4k_1k_2 f') + 3\rho_2 k_1 f^2 \cos a_1 \cos a_2 \sin a_1 \ = 0 \\ k_1k_2k_3 + 3\rho_2 k_1 \sin a_1 \sin a_2 \cos a_1 = 0 \end{cases}$$

If the first three curvatures are constants, the following result is obtained:

Theorem 4.27. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 \neq 0$, $\rho_3 = 0$, $\dot{\vartheta}T \in \operatorname{span}\{V_2, V_3, \cdots, V_m\}$, and $g(\dot{\vartheta}T, V_2) \neq 0$. Then, σ is a bi-f-harmonic Legendre curve if and only if k_1, k_2, and k_3 satisfy the following differential equations

$$\begin{cases} (ff'')' - 4k_1^2 f f' = 0 \\ (-k_1^2 - k_2^2 + \rho_1 + 3\rho_2(\cos a_1)^2) f^2 + 3f'' f\ + 2(f')^2 = 0 \end{cases}$$

where

$$f(s) = e^{\frac{k_3}{4}\int \cot a_2\, ds}$$

and a_1 and a_2 are constants.

Let $r = 3$. Therefore,

$$\dot{\vartheta}T = \cos a_1 V_2 + \sin a_1 V_3$$

Hence, $g(\dot{\vartheta}T, V_2) = \cos a_1$, $g(\dot{\vartheta}T, V_3) = \sin a_1$, $a_2 = 0$, and $k_3 = 0$.

Theorem 4.28. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 \neq 0$, $\rho_3 = 0$, $\dot{\vartheta}T \in \operatorname{span}\{V_2, V_3\}$, and $g(\dot{\vartheta}T, V_2) \neq 0$. Then, σ is a bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'' + k_1\rho_1 + 3k_1\rho_2(\cos a_1)^2) f^2 + 4f\, k_1' f' + 3f'' f\, k_1 + 2(f')^2 k_1 = 0 \\ f\ (2k_2 f\, k_1' + k_1 f\, k_2' + 4k_1k_2 f') + 3\rho_2 k_1 f^2 \cos a_1 \sin a_1 \ = 0 \end{cases}$$

provided $r = 3$.

Let $r = 2$. Therefore, $\dot{\vartheta}T = \pm V_2$. Hence, $g(\dot{\vartheta}T, V_2) = \pm 1$, $g(\dot{\vartheta}T, V_3) = 0$, $a_1 = a_2 = 0$, and $k_2 = k_3 = 0$.

Theorem 4.29. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 \neq 0$, $\rho_3 = 0$, and $\dot{\vartheta}T = \pm V_2$. Then, σ is a bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 + k_1'' + k_1\rho_1 \pm 3k_1\rho_2) f^2 + 4f\, k_1' f' + 3f'' f\, k_1 + 2(f')^2 k_1 \ = 0 \end{cases}$$

provided $r = 2$.

CASE VI. Let $\rho_2 \neq 0, \rho_3 \neq 0, V_2 \perp \dot{\vartheta}T$, and $V_2 \perp \varsigma$. Then, from (4.3), $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a proper bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (k_1'' - k_1^3 - k_1k_2^2 + k_1\rho_1)f^2 + 4k_1' f f' + 3k_1 f'' f + 2k_1(f')^2 = 0 \\ 4k_1k_2f' + 2k_2k_1'f + k_1k_2'f = 0 \\ k_1k_2k_3 = 0 \end{cases}$$

Corollary 4.30. There is no any bi-f-harmonic Legendre curve of osculating order $r \geq 4$ in an α-Sasakian generalized Sasakian space form, satisfying $\rho_2 \neq 0, \rho_3 \neq 0, V_2 \perp \dot{\vartheta}T$, and $V_2 \perp \varsigma$.

Note that because the conditions obtained in Cases I and VI are the same, it is not necessary to investigate the subcases for Case VI.

CASE VII. Let $\rho_2 \neq 0$, $\rho_3 \neq 0$, $V_2 \perp \dot{\vartheta}T$, $\varsigma \in \text{span}\{V_2, V_3, \cdots, V_m\}$, and $\dot{\eta}(V_2) \neq 0$. Then, from (4.3), $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a proper bi-f-harmonic curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'' + k_1\rho_1 + \beta\rho_3(\cos\theta_1)^2)f^2 + 4f k_1' f' + 3f'' f k_1 + 2(f')^2 k_1 = 0 \\ 4k_1k_2 f f' + f^2(2k_2k_1' + k_1k_2' + \beta\rho_3 \sin\theta_1 \cos\theta_2) = 0 \\ k_1k_2k_3 + \beta\rho_3 k_1 \sin\theta_1 \sin\theta_2 = 0 \end{cases}$$

Corollary 4.31. There is no any bi-f-harmonic curve of osculating order $r \geq 4$ in an α-Sasakian generalized Sasakian space form, satisfying $\rho_2 \neq 0$, $\rho_3 \neq 0$, $V_2 \perp \dot{\vartheta}T$, $\varsigma \in \text{span}\{V_2, V_3, \cdots, V_m\}$, and $\dot{\eta}(V_2) \neq 0$.

Note that because the conditions obtained in Cases III and VII are the same, we omit to investigate the subcases for Case VII.

CASE VIII. Let $\rho_2 \neq 0$, $\rho_3 \neq 0$, $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$, $g(\dot{\vartheta}T, V_2) \neq 0$, and $\varsigma \perp V_2$. Then, from (4.17) and (4.18), the following result is obtained:

Theorem 4.32. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an α-Sasakian generalized Sasakian space form with $\rho_2 \neq 0$, $\rho_3 \neq 0$, $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$, $g(\dot{\vartheta}T, V_2) \neq 0$, and $\varsigma \perp V_2$. Then, σ is a bi-f-harmonic Legendre curve if and only if k_1, k_2, and k_3 satisfy the following differential equations:

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'' + k_1\rho_1 + 3k_1\rho_2(\cos a_1)^2)f^2 + 4f k_1' f' + 3f'' f k_1 + 2(f')^2 k_1 = 0 \\ f\,(2k_2 f k_1' + k_1 f k_2' + 4k_1k_2f') + 3\rho_2 k_1 f^2 \cos a_1 \cos a_2 \sin a_1 + \beta\rho_3\dot{\eta}(V_3) = 0 \\ k_1k_2k_3 + 3\rho_2 k_1 \sin a_1 \sin a_2 \cos a_1 + \beta\rho_3\dot{\eta}(V_4) = 0 \end{cases} \tag{4.19}$$

If $r = 3$, then the first three equations of the (4.19) are satisfied, taking $a_2 = 0$.

If $r = 2$, then the first two equations of the (4.19) are satisfied, taking $a_1 \in \{0, \pi\}$.

CASE IX. Let $\rho_2 \neq 0$, $\rho_3 \neq 0$, $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$, $g(\dot{\vartheta}T, V_2) \neq 0$, $\dot{\eta}(V_2) \neq 0$, and $\varsigma \in \text{span}\{V_2, V_3, \cdots, V_m\}$. Then, the following result is obtained:

Theorem 4.33. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α, β)-trans-Sasakian generalized Sasakian space form with $\rho_2 \neq 0$, $\rho_3 \neq 0$, $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$, $g(\dot{\vartheta}T, V_2) \neq 0$, $\dot{\eta}(V_2) \neq 0$, and $\varsigma \in \text{span}\{V_2, V_3, \cdots, V_m\}$. Then, σ is a bi-f-harmonic curve if and only if k_1, k_2, and k_3 satisfy the following differential equations:

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ \begin{cases} (-k_1^3 - k_1k_2^2 + k_1'' + k_1\rho_1 + 3k_1\rho_2(\cos a_1)^2 + \beta\rho_3\cos\theta_1)f^2 + 4f k_1' f' \\ +(3f''f + 2(f')^2)k_1 = 0 \end{cases} \\ 4k_1k_2 f f' + f^2(2k_2k_1' + k_1k_2' + 3\rho_2k_1\cos a_1\cos a_2\sin a_1 + \beta\rho_3\sin\theta_1\cos\theta_2) = 0 \\ k_1k_2k_3 + 3\rho_2k_1\sin a_1\sin a_2\cos a_1 + \beta\rho_3\sin\theta_1\sin\theta_2 = 0 \end{cases} \tag{4.20}$$

If $r = 3$, then the first three equations of the (4.20) are satisfied, taking $a_2 = 0$ and $\theta_2 = 0$.

If $r = 2$, then the first two equations of the (4.20) are satisfied, taking $\theta_1 \in \{0, \pi\}$ and $a_1 \in \{0, \pi\}$.

5. Conclusion

This study has obtained the necessary and sufficient conditions for a curve to be bi-f-harmonic Legendre in the (α,β)-trans-Sasakian generalized Sasakian space form. While conducting this investigation, the functions from the manifold's curvature tensor, curvature and torsion of the curve, and the relative positions of the basis vectors have been considered. Future studies could focus on different curves, such as Slant, in the (α,β)-trans-Sasakian generalized Sasakian space form. Additionally, research can be conducted on special cases of the (α,β)-trans-Sasakian manifold, including α-Sasakian, Sasakian, β-Kenmotsu, Kenmotsu, and cosymplectic manifold types.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

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Renyi Type Holographic Dark Energy

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Abstract — Dark energy is one of the prominent mysteries of the universe that still awaits a solution. One of the plausible ways to collect data about any formation or understand its information capacity is to investigate the entropy of that formation. In this study, Renyi Holographic Dark Energy (RHDE) matter distribution is analyzed within the framework of General Relativity Theory, considering homogeneous and isotropic Friedmann-Robertson-Walker (FRW) space-time. Hubble parameter and RHDE density were used to obtain exact solutions of Einstein field equations. The analysis of the obtained solutions was performed by drawing evolution graphs for redshift z.

Keywords *Renyi holographic dark energy, FRW universe model, Hubble parameter*

Mathematics Subject Classification (2020) 83C05, 83C15

1. Introduction

Observations suggest that the universe is accelerating in two phases: the very early inflationary phase and the current phase. An exotic component with a large negative pressure, called dark energy (DE), which accounts for about 70% of the universe's energy density, is thought to cause the accelerated expansion [1]. In addition, the second largest component of our universe is dark matter (DM). The origin and nature of dark matter and dark energy are not fully known. Different theoretical models have been constructed to explain and interpret the accelerating universe.

The cosmological constant lambda cold dark matter (ΛCDM) is proposed as the simplest dark energy model. Although ΛCDM is consistent with current observations, it suffers from the problem of coincidence and fine-tuning. [2]. For this reason, we are trying to investigate the origin of dark energy based on the holographic principle [3]. With the introduction of the holographic principle into cosmology, holographic dark energy (HDE) was proposed [4]. Holographic dark energy is based on the use of various horizons as the radius of the universe. It can explain the current acceleration and is supported by many observations [5, 6], making holographic dark energy an interesting model. Recently, dark energy models, such as Tsallis holographic dark energy, Renyi holographic dark energy (RHDE), and Sharma Mitall holographic dark energy, have been proposed to investigate cosmological phenomena using extended entropy formalisms, such as Tsallis [7], Renyi [8], and Sharma Mitall [9]. The energy density of RHDE is as follows:

$$\rho_R = \frac{3c^2}{8\pi L^2}\left(1+\pi\delta L^2\right)^{-1} \tag{1.1}$$

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where δ is the true non-extensibility (NE) parameter measuring the degree of non-extensibility, L is the IR cut-off, and c is a numerical constant [10,11]. If the Hubble horizon is taken as the IR cut-off $\left(L = \frac{1}{H}\right)$, then (1.1) becomes the following equation, and the energy density of the RHDE is obtained as follows [10]:

$$\rho_R = \frac{3c^2H^2}{8\pi\left(\frac{\pi\delta}{H^2}+1\right)} \tag{1.2}$$

There are many studies on RHDE. Some of these studies can be summarized as follows. Prasanthi and Aditya [1] have studied the RHDE model for the Bianchi VI_0 universe in the General Relativity Theory. Bhattacharjee [12] has investigated the dynamics of Tsallis and Renyi holographic dark energy in Friedmann-Robertson-Walker (FRW) space-time using a hybrid scale factor. Dubey et al. [10] have analyzed RHDE model in a flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric using different values of the parameter δ. Dixit et al. [13] have investigated the behavior of RHDE in the flat FRW universe in the framework of $f(R,T)$ gravity using the Granda-Oliveros and Hubble horizons. Sharma and Dubey [14] have obtained a solution using the RHDE deceleration parameter in the flat FRW universe. They have also calculated state finder parameters to understand the geometrical behavior of the model using observational data. Moreover, in 2020, Sharma and Dubey [15] studied the RHDE model for the FRW universe in Brans-Dicke cosmology. Saha et al. [16] have investigated the interacting and non-interacting RHDE models in the Dvali-Gabadadze-Porratidal braneworld framework. In addition, Saha et al. [17] have researched the distribution of matter with Barrow holographic dark energy and viscous fluid in the form of pressure-less dark matter using different scale factors for the flat FRW universe. Liu et al. [18] have analyzed the quintessential dark energy of the Kerr black hole by testing it through observational data using quasi-periodic oscillations. Ranjit et al. [19] have studied models of the universe with interacting Tsallis holographic dark energy in the Chern-Simons alternative gravitational theory. Yılmaz and Güdekli [20] have investigated FLRW universe models with modified Chaplygin gas and cosmological constants, one of the dark energy candidates. Koussour et al. [21] have proposed a new equation of state parameter for dark energy in the $f(Q)$ alternative theory of gravitation. Koussour et al. [22] have obtained solutions for various Hubble parameters in scalar field dark energy models.

The outline of the article is as follows: In section 2, the General Relativity Theory formulation is provided, and the field equations for the RHDE model are obtained using the FRW metric. In section 3, solutions to Einstein field equations are provided. Moreover, in this section, the solutions are analyzed with the help of graphics. Finally, in the section 4, the planned future studies are mentioned.

2. Field Equations

The General Relativity theory attempts to explain the universe's structure on a large scale. The field equations in this theory are expressed as follows:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} \tag{2.1}$$

Here, $8\pi G = c^4 = 1$ can be taken. The general form of the homogeneous and isotropic FRW metric in spherical coordinates (r,θ,ϕ,t) is as follows:

$$ds^2 = dt^2 - A^2\left(\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{2.2}$$

Here, it can take the values $\kappa = -1$, $\kappa = 0$, or $\kappa = 1$. If $\kappa = -1$, then it refers to the open universe model, if $\kappa = 0$, then it refers to the flat universe model, and if $\kappa = 1$, then it refers to the closed universe model. The universe is assumed to be filled with matter and a fluid known as holographic dark energy. The energy-momentum tensor for matter is defined as follows:

$$T^m_{\mu\nu} = \rho_m u_\mu u_\nu$$

where ρ_m shows the density of matter [1]. The energy-momentum tensor ($T^R_{\mu\nu}$) of RHDE is as follows:

$$T^R_{\mu\nu} = (\rho_R + p_R)u_\mu u_\nu - p_R\, g_{\mu\nu}$$

where p_R and ρ_R denote RHDE pressure and energy density, respectively [1]. The total energy-momentum tensor is expressed as follows:

$$\begin{aligned} T_{\mu\nu} &= T^m_{\mu\nu} + T^R_{\mu\nu} \\ &= (\rho_m + \rho_R + p_R)u_\mu u_\nu - p_R g_{\mu\nu} \end{aligned} \tag{2.3}$$

We obtain the Einstein field equations from (2.1)-(2.3) as follows:

$$\frac{2\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{\kappa}{A^2} = -p_R \tag{2.4}$$

and

$$\frac{3\dot{A}^2}{A^2} + \frac{3\kappa}{A^2} = \rho_m + \rho_R \tag{2.5}$$

Here, the dot shows a derivative concerning cosmic time t.

3. Results and Discussions

As can be observed from (2.4) and (2.5), there are two equations with four unknowns as A, ρ_m, ρ_R, and p_R. We need two additional equations to solve this equations system.

i. To obtain a solution, we can first use the Hubble parameter. The Hubble parameter, a cosmological parameter that expresses the universe's expansion rate, defines the universe's expansion rate by a numerical value, also called the Hubble constant (H_0). This parameter helps us understand the universe's expansion rate and obtain some critical information about the universe's past. In this study, the Hubble parameter suggested by Pacif et al. [23] is taken:

$$H = \frac{\dot{a}}{a} = \frac{\beta}{\sqrt{t+\alpha}} \tag{3.1}$$

Here, a denotes scale factor and α and β are real constants.

ii. As the second equation, by estimating the Hubble horizon as an IR cut-off, we can take the energy density of the RHDE as in (1.2).

From (3.1), the metric potential (is also equal to the scale factor a) A is

$$A = c_1 e^{2\sqrt{t+\alpha}\,\beta} \tag{3.2}$$

Here, c_1 is an integral constant. From (1.2) and (3.1), we obtain energy density of RHDE as

$$\rho_R = \frac{3\beta^4 c^2}{8\left(\delta\,(t+\alpha)\,\pi + \beta^2\right)(t+\alpha)\,\pi} \tag{3.3}$$

Furthermore, from (2.4), (2.5), (3.2), and (3.3), the energy density of matter and pressure are obtained as follows:

$$\rho_m = \frac{3e^{-4\sqrt{t+\alpha}\,\beta}\kappa}{{c_1}^2} - \frac{3\beta^2\left(-8\delta\,(t+\alpha)\,\pi^2 - 8\beta^2\pi + c^2\beta^2\right)}{8\left(\delta\,(t+\alpha)\,\pi + \beta^2\right)(t+\alpha)\,\pi}$$

and

$$p_R = -\frac{e^{-4\sqrt{t+\alpha}\,\beta}\kappa}{{c_1}^2} - \frac{3\left(\sqrt{t+\alpha}\,\beta - \frac{1}{3}\right)\beta}{(t+\alpha)^{\frac{3}{2}}} \tag{3.4}$$

Finally, we can determine the equation of state (EoS) parameter by using the formulation $\omega = \frac{p_R}{\rho_R}$ where ω denotes the EoS parameter:

$$\omega = -\frac{8\pi\left(3\beta\sqrt{t+\alpha}-1\right)\left(\delta\left(t+\alpha\right)+\beta^2\right)}{3\,\beta^3c^2\sqrt{t+\alpha}} - \frac{8\pi\kappa\left(\delta\left(t+\alpha\right)\pi+\beta^2\right)\left(t+\alpha\right)}{3{c_1}^2c^2\beta^4e^{4\beta\sqrt{t+\alpha}}}$$

When the solutions are investigated, it is observed that there is a singularity at $t = -\alpha$. However, since the constant α is positive and t can never have a negative value, $t = -\alpha$ has no problems for the solutions. In order to draw and analyze the graphs of the physical variables we obtain in the solutions, we need the values of the constants in the solutions. We can get the values of the constants in the solutions by using some observational values. Table 1 contains the values of the constants we use to draw the graphs. When obtaining the values in Table 1, $t_0 = 13.8$ Gyr was taken.

Table 1. Values of constants

Data Set	α	β	c_1	δ	c
SN Ia [24]	1.0	0.4813	0.0246	1.4	70
SN Ia + H(z) + BAO/CMB [25]	1.6	0.2770	0.1137	1.4	110
SN Ia + BAO +H(z) [26]	1.3	0.3762	0.0537	1.4	90
CC+SN Ia+BAO+R18 [27]	1.6	0.2963	0.0977	1.4	120

Using the values in Table 1, a graph of the variation in metric potential over time was drawn for four different observation values. When Figure 1 is analyzed, it is observed that the metric potential exhibits similar behavior for all four observation values up to a certain point and increases over time. Still, after a certain point, the increase accelerates for SN Ia. The fact that the metric potential increases over time within four different observation values shows that the universe model has an expansion.

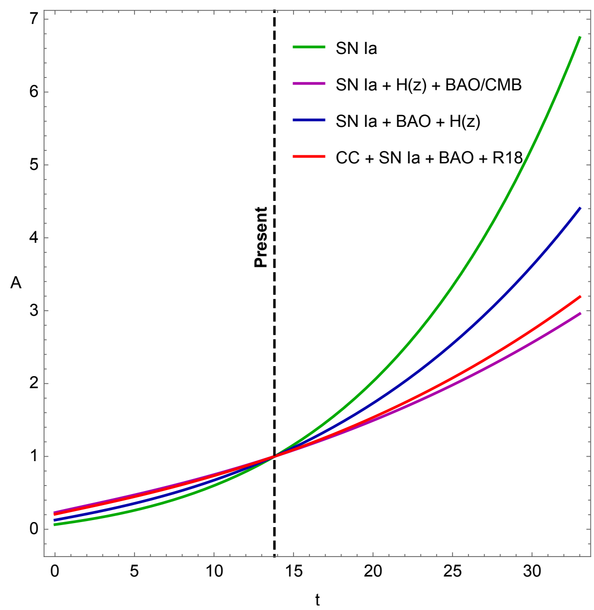

Figure 1. Variation of metric potential versus cosmic time t

The following equation expresses the relationship between redshift and scale factor:

$$1 + z = \frac{a(t_0)}{a(t)} \tag{3.5}$$

where $a(t_0)$ is the present value of the scale factor. In this study, $a(t_0) = 1$ is assumed. From (3.1) and (3.5), a relationship between the redshift variable z and time can be obtained as follows:

$$t = t_0 - \frac{\sqrt{t_0 + \alpha}\,\ln(1+z)}{\beta} + \frac{\ln(1+z)^2}{4\beta^2} \tag{3.6}$$

With the help of (3.3) and (3.6) and using the values in Table 1, the variation graph of the energy density of RHDE according to redshift is presented in Figure 2. The energy density is expected to increase in dark energy models according to z. When Figure 2 is investigated, an increase is observed for all four observation values. However, the increase is faster for the observation values SN Ia + H(z) + BAO/CMB and CC+SN Ia+BAO+R18.

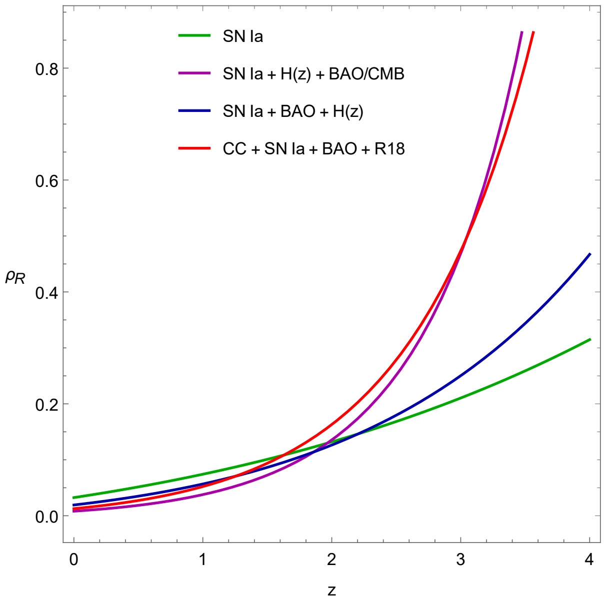

Figure 2. Variation of the energy density of RHDE versus redshift.

Similarly, with the help of (3.4) and (3.6) and using the values in Table 1, the variation graph of the energy density of RHDE according to redshift is provided in Figure 3. In dark energy models, the pressure is expected to decrease concerning z. When Figure 3 is investigated, a decrease is observed for all four observation values. However, it can be observed from Figure 3 that the observation values of SN Ia + H(z) + BAO/CMB and CC + SN Ia + BAO + R18 start to increase after a certain point.

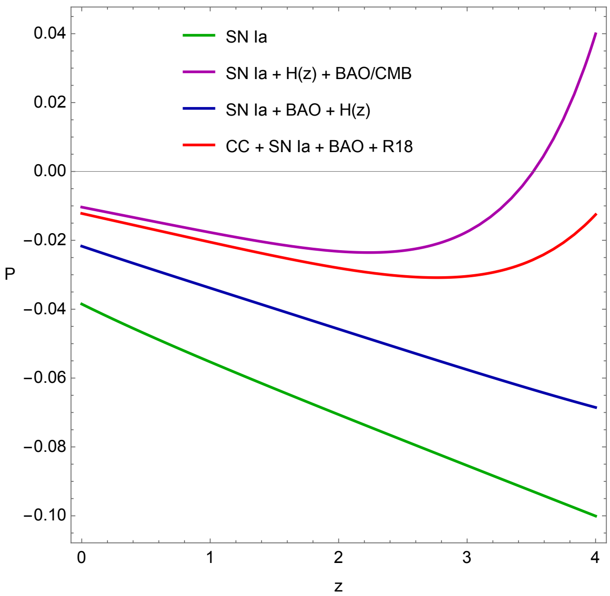

Figure 3. Variation of pressure versus redshift

Figure 4 shows the variation of the equation of state parameter according to redshift. When Figure 4 is investigated, the equation of state parameter behaves similarly in four different observation values.

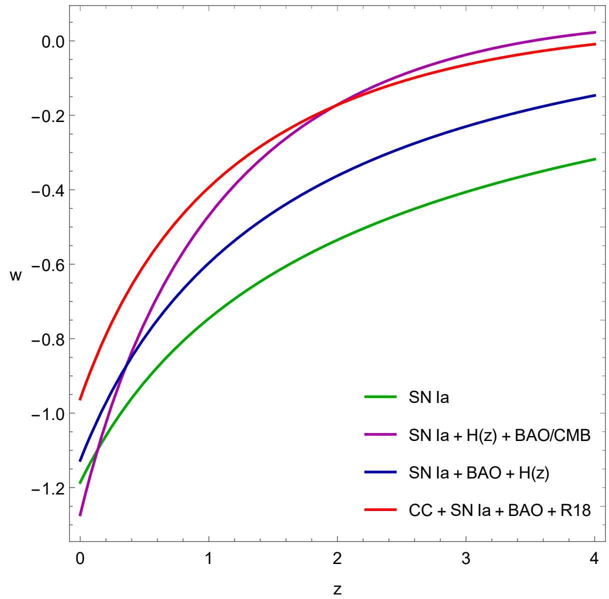

Figure 4. Variation of equation of state versus redshift

4. Conclusion

This article has investigated the behavior of the RHDE matter distribution in the homogeneous and isotropic FRW universe model, which best describes today's universe in the framework of General Relativity Theory. In future studies, it will be worthwhile to investigate universe models with RHDE matter distribution within the framework of Lyra theory, $f(R)$ theory, and $f(Q)$ theory.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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A Study on Zagreb Indices of Vertex-Switching for Special Graph Classes

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Research Article

Abstract — Many graph theorists have studied graph operations due to their applications and the advantages with heavy calculations. In a recent paper, the vertex-switching operation is analyzed, and some vertex-switched graphs are determined for some graph classes. This paper calculates the first Zagreb index and the second Zagreb index of vertex-switched star, complete bipartite, and tadpole graphs. It finally discusses the need for further research.

Keywords *Vertex switching, vertex switched graph, star graph, complete bipartite graph, tadpole graph*

Mathematics Subject Classification (2020) 05C07, 05C09

1. Introduction

A simple graph G consists of a set V of vertices (nodes) and a set E of edges (links) between some vertices so that no loops or multiple edges appear in G. For more information about graphs, see [1]. Many graph operations use vertices and edges for several purposes, such as vertex deletion and addition, edge deletion and addition, and edge contraction. Several such vertex and edge operations' applications can be found in [2–10].

In [11], Seidel has introduced a new type of vertex operation called vertex switching. For some fundamental properties and calculations, see [12–14]. For a graph $G = G(V, E)$ and a subset S of $V = V(G)$, the switching of G by S is the graph $G^S(V, E')$ obtained from G by removing all edges between S and $V - S$ and adding new edges between S and $V - S$ which are not in G. If the set S consists of a single vertex v, then, in particular, G^S, denoted briefly by G^v, is the vertex switching graph of G by v.

Some mathematical formulae called topological graph indices are used in obtaining some required properties of a given graph or a graph class. Some of these indices are useful in molecular chemistry and are alternatively called molecular descriptors. Two of these indices are the first and second Zagreb indices defined by Gutman and Trinajstic [15] as follows:

$$M_1(G) = \sum_{u \in V(G)} du^2 \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} dudv$$

These two topological indices have been studied by many authors [16, 17]. In this study, we calculate

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the first and the second Zagreb indices of vertex switched star, complete bipartite, and tadpole graphs for the special case of $S = \{v\}$ is a single vertex set.

2. First Zagreb Index of Vertex Switched Graphs of Some Graph Classes

This section calculates the first Zagreb index of vertex-switched star, tadpole, and complete bipartite graphs. We start with complete bipartite graphs.

2.1. First Zagreb Index of Vertex Switched Graphs of Complete Bipartite Graphs

Firstly, we study the first Zagreb index of vertex switched complete bipartite graphs. Let two bipartite sets of the vertices in $K_{r,s}$ with $r \leq s$ be $A = \{u_1, u_2, \cdots, u_r\}$ and $B = \{v_1, v_2, \cdots, v_s\}$. By the definition of vertex switching operation, the case $r = 2$ shows a difference with the case $r \geq 3$ as we obtain a star graph in the former case, and in the latter case, a new graph class is obtained. We denote by $K_{r,s(t)}$ the graph obtained by joining t vertices to all of the s vertices in a bipartite graph $K_{r,s}$. For some examples of these graphs, see Figures 1 and 2 and more details, see [18].

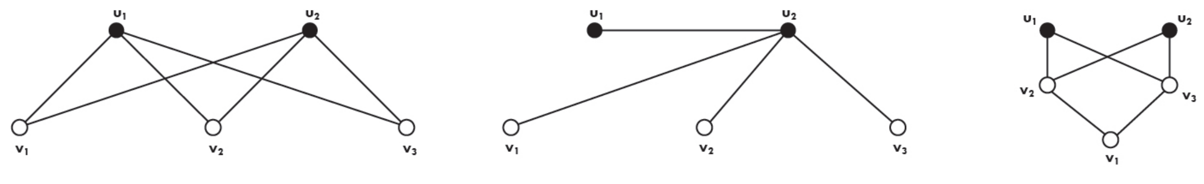

Figure 1. Vertex switching of $K_{2,3}$ where v belongs to different partition sets

We consider the situation where the switched vertex belongs to A:

Theorem 2.1. Let $v \in A$. Then,

$$M_1(K_{2,s}^v) = s^2 + 3s + 2$$

Proof. We need to delete all the edges $vv_1, vv_2, \cdots, vv_s$ incident to v and add a new edge between v and each vertex w in G which are not adjacent to v in G. Thus, v becomes a pendant vertex in $K_{2,s}^v$. There is only one vertex w in G as $r = 2$. Hence, w will be connected to all the other vertices in G when we form $K_{2,s}^v$, and all these vertices are adjacent only to w, causing the graph to be a star graph. The total number of those vertices is $2 + s - 1 = s + 1$. Adding w, we obtain that this star graph will be S_{2+s}. For more details, see [18]. The vertex partition of $K_{2,s}^v$ is provided in Table 1:

Table 1. The vertex partition of $K_{2,s}^v$, for $v \in A$

d_u	# u
1	$s+1$
$s+1$	1

stands for "the number of".

Hence, the first Zagreb index of $K_{2,s}^v$ is

$$M_1(K_{2,s}^v) = M_1(S_{2+s}) = 1^2(s+1) + 1(s+1)^2 = s^2 + 3s + 2$$

□

We consider the situation where the switched vertex belongs to B:

Theorem 2.2. Let $v \in B$. Then,

$$M_1(K_{2,s}^v) = 3s^2 + 3s - 6$$

PROOF. Let $v \in B$. We delete both edges vu_1 and vu_2 and add new $s-1$ edges vv_2, vv_3, $\cdots$, vv_s. Moreover, u_1 and u_2 are adjacent to all the vertices in $B-\{v\}$ which is the graph $K^1_{2,s-1}$ that is the graph obtained by joining v to $s-2$ vertices in $B-\{v\}$ in $K_{2,s-2}$, giving a complete bipartite graph $K_{2,s-1}$. Thus, the vertex partition of $K^v_{2,s}$ is provided in Table 2:

Table 2. The vertex partition of $K^v_{2,s}$, for $v \in B$

d_u	$\#\ u$
$s-1$	3
3	$s-1$

stands for "the number of".

Hence,

$$M_1(K^v_{2,s}) = M_1(K^1_{2,s-1}) = 3(s-1)^2 + 3^2(s-1) = 3s^2 + 3s - 6$$

□

We consider the first Zagreb index of a vertex switched graph $K_{r,s}$ with $3 \leq r \leq s$. For an illustration, see Figure 2, where $r = 3$ and $s = 4$.

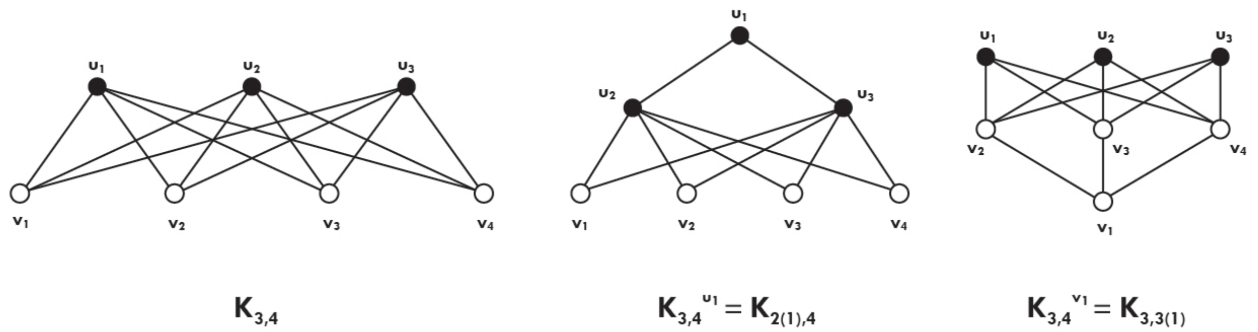

Figure 2. Two possible vertex switchings of the complete bipartite graph $K_{3,4}$

Theorem 2.3. Let $K_{r,s}$ be a complete bipartite graph with $3 \leq r \leq s$. If $v \in A$, then

$$M_1(K^v_{r,s}) = s^2r + r^2s + r^2 - s^2 - r - s$$

and if $v \in B$, then

$$M_1(K^v_{r,s}) = s^2r + r^2s + s^2 - r^2 - r - s$$

PROOF. For $v \in A$, $M_1(K^v_{r,s}) = M_1(K^1_{s,r-1})$. Therefore, the vertex partition of $K^v_{r,s}$ is provided in Table 3:

Table 3. The vertex partition of $K^v_{r,s}$, for $v \in A$

d_u	$\#\ u$
$s+1$	$r-1$
$r-1$	$s+1$

stands for "the number of".

Hence,

$$M_1(K^v_{r,s}) = (s-1)^2(r+1) + (r+1)^2(s-1) = s^2r + r^2s + r^2 - s^2 - r - s$$

For $v \in B$, $M_1(K^v_{r,s}) = M_1(K^1_{r,s-1})$. Therefore, the vertex partition of $K^v_{r,s}$ is provided in Table 4:

Table 4. The vertex partition of $K^v_{r,s}$, for $v \in B$

d_u	# u
$s-1$	$r+1$
$r+1$	$s-1$

stands for "the number of".

Hence,

$$M_1(K^v_{r,s}) = M_1(K^1_{r,s-1}) = (s-1)^2(r+1) + (r+1)^2(s-1) = s^2r + r^2s + s^2 - r^2 - r - s$$

□

2.2. First Zagreb Index of Vertex Switched Graphs of Star Graphs

This section looks for the vertex-switched graphs of the star graphs. As there are two different types of vertices in a star graph, we have the following sub-cases:

Theorem 2.4. Let S_n be a star graph, v_1 be the central vertex, and all the remaining vertices be $v_2, v_3, \cdots, v_n$. Then,

$$M_1(S_n^{v_1}) = 0$$

and

$$M_1(S_n^{v_i}) = 2n^2 - 4n$$

for $i \in \{2, 3, \cdots, n\}$, where $K_{1,1,n-2}$ is the complete tripartite graph.

Proof. First, determine $S_n^{v_1}$. As v_1 is connected to all the other vertices and there are no other edges in S_n, the remaining graph, when we delete all the incident $n-1$ edges to v_1, will have no edges, that is, the resulting graph will be N_n, see the graph in the middle in Figure 3, and hence $M_1(S_n^{v_1}) = 0$.

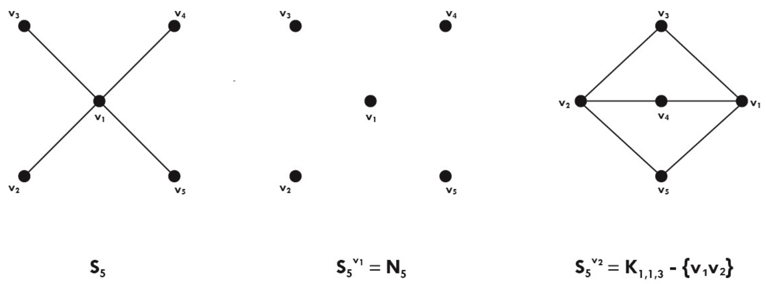

Figure 3. Two possible vertex switchings of a star graph

Secondly, determine $S_n^{v_i}$ where v_i is a vertex different then v_1. When we switch the vertex v_i, all vertices $v_2, v_3, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n$ will be adjacent to only v_1 and v_i, each. This graph is an example of the triangular book graph without a spine; see the graph on the right in Figure 3, for $n = 5$, and Figure 4, for $n > 5$. That is, $S_n^{v_i} = K_{1,1,n-2} - \{v_1v_i\}$.

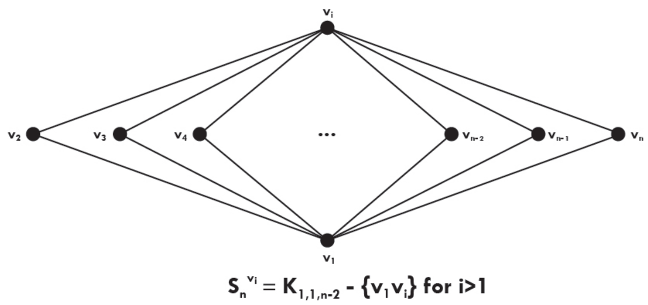

Figure 4. Vertex switching at a pendant vertex of a star graph giving triangular book graph without a spine

Therefore, the vertex partition of $S_n^{v_i}$ is provided in Table 5:

Table 5. Vertex partition of $S_n^{v_i}$, for $v_2, v_3, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n$

d_u	$\#\ u$
2	$n-2$
$n-2$	2

\# stands for "the number of".

Hence,

$$M_1(S_n^{v_i}) = M_1(K_{1,1,n-2} - \{v_1v_i\}) = 2^2(n-2) + (n-2)^2 2 = 2n^2 - 4n$$

□

2.3. First Zagreb Index of Vertex Switched Graphs of Tadpole Graphs

This section calculates the first Zagreb index of vertex-switched tadpole graphs $T_{r,s}$. This case is much more complicated than the previously considered graph classes, as many different types of vertices exist. Let the vertex at which the path and cycle parts of the tadpole graph intersect be v_1, the remaining vertices on the cycle in clockwise order be $v_2, v_3, \cdots, v_r$, and the remaining vertices starting from the neighbor vertex v_{r+1} be $v_{r+1}, v_{r+2}, \cdots, v_{r+s}$ (see Figure 5).

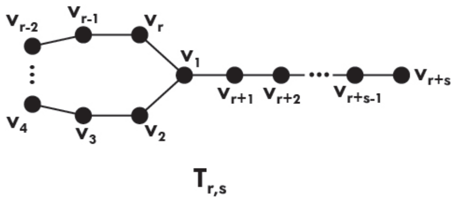

Figure 5. Tadpole graph $T_{r,s}$

Using the symmetry in $T_{r,s}$, we study the first Zagreb index of the vertex switched graphs $T_{r,s}^v$ for v is v_1, v_2, v_3, v_{r+1}, v_{r+2}, v_{r+s-1}, and v_{r+s} as $T_{r,s}^{v_r} = T_{r,s}^{v_2}$, $T_{r,s}^{v_{r-1}} = T_{r,s}^{v_3}$, etc. In Figure 6, we illustrated all the possible vertex-switched tadpole graphs:

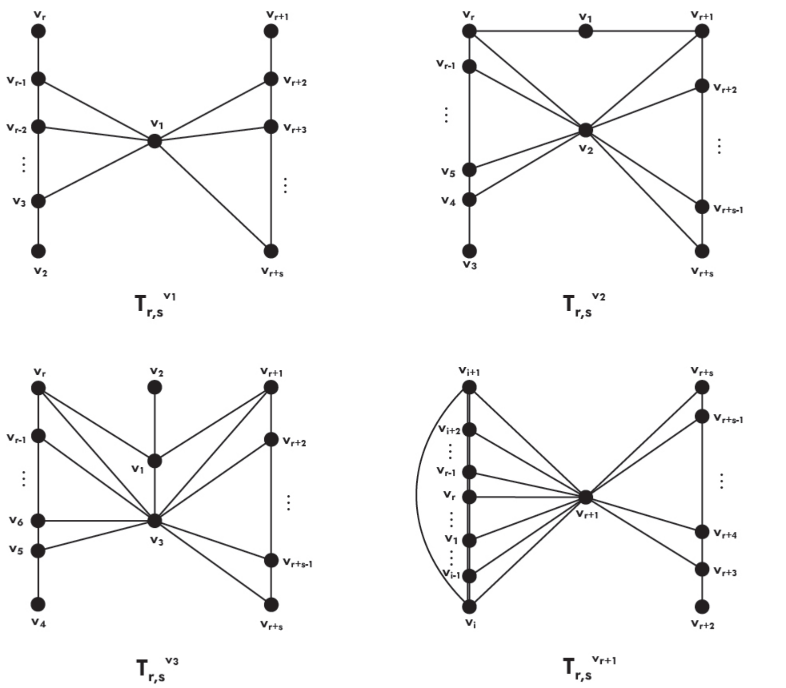

Figure 6. All the possible vertex switched tadpole graphs $T_{r,s}$

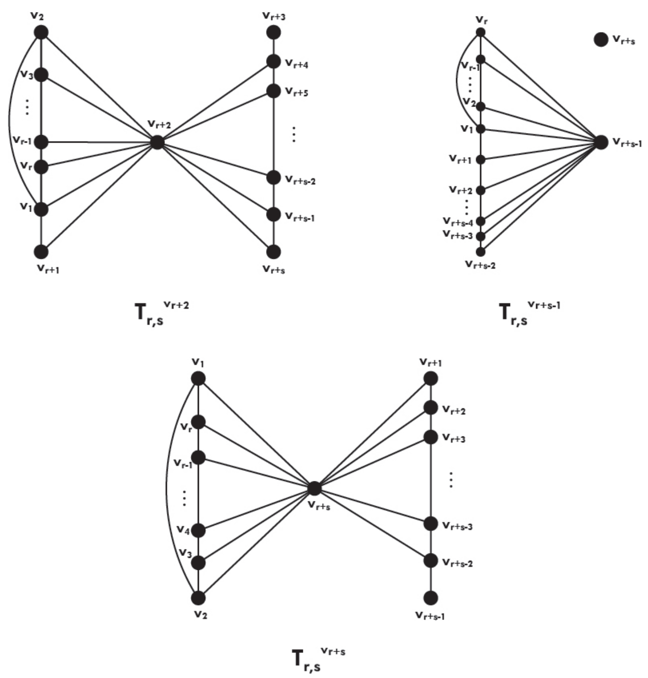

Figure 6. (Continued) All the possible vertex switched tadpole graphs $T_{r,s}$

Theorem 2.5. Let $T_{r,s}$ be a tadpole graph, v_1 be the central vertex, and all the remaining vertices be $v_2, v_3, \cdots, v_n$. Then,

$$M_1(T_{r,s}^{v_1}) = r^2 + s^2 + 2rs + r + s - 22$$

$$M_1(T_{r,s}^{v_2}) = r^2 + s^2 + 2rs + 3r + 3s - 18$$

$$M_1(T_{r,s}^{v_3}) = r^2 + s^2 + 2rs + 3r + 3s - 14$$

$$M_1(T_{r,s}^{v_{r+1}}) = r^2 + s^2 + 2rs + 5r + 5s - 18$$

$$M_1(T_{r,s}^{v_{r+2}}) = r^2 + s^2 + 2rs + 5r + 5s - 20$$

$$M_1(T_{r,s}^{v_{r+s-1}}) = r^2 + s^2 + 2rs + 5r + 5s - 3$$

and

$$M_1(T_{r,s}^{v_{r+s}}) = r^2 + s^2 + 2rs + 5r + 5s - 18$$

PROOF. The vertex partition for $T_{r,s}^{v_1}$ is provided in Table 6:

Table 6. The vertex partition of $T_{r,s}^{v_1}$

d_u	$\#\ u$
1	3
2	1
3	$r+s-5$
$r+s-4$	1

\# stands for "the number of".

Hence, its first Zagreb index is as follows:

$$M_1(T_{r,s}^{v_1}) = 1^2 \cdot 3 + 2^2 \cdot 1 + 3^2 \cdot (r+s-5) + (r+s-4)^2 \cdot 1 = r^2 + s^2 + 2rs + r + s - 22$$

Moreover, the vertex partition for $T_{r,s}^{v_2}$ is provided in Table 7:

Table 7. The vertex partition of $T_{r,s}^{v_2}$

d_u	# u
1	1
2	2
3	$r+s-4$
$r+s-3$	1

stands for "the number of".

Thus,

$$M_1(T_{r,s}^{v_2}) = 1^2 \cdot 1 + 2^2 \cdot 2 + 3^2 \cdot (r+s-4) + (r+s-3)^2 \cdot 1 = r^2 + s^2 + 2rs + 3r + 3s - 18$$

Further, the vertex partition for $T_{r,s}^{v_3}$ is provided in Table 8:

Table 8. The vertex partition of $T_{r,s}^{v_3}$

d_u	# u
1	2
2	1
3	$r+s-5$
4	1
$r+s-3$	1

stands for "the number of".

Thereby,

$$M_1(T_{r,s}^{v_3}) = 1^2 \cdot 2 + 2^2 \cdot 1 + 3^2 \cdot (r+s-5) + 4^2 \cdot 1 + (r+s-3)^2 \cdot 1 = r^2 + s^2 + 2rs + 3r + 3s - 14$$

Besides, the vertex partition for $T_{r,s}^{v_{r+1}}$ is provided in Table 9:

Table 9. The vertex partition of $T_{r,s}^{v_{r+1}}$

d_u	# u
1	1
2	1
3	$r+s-3$
$r+s-2$	1

stands for "the number of".

Therefore,

$$M_1(T_{r,s}^{v_{r+1}}) = 1^2 \cdot 1 + 2^2 \cdot 1 + 3^2 \cdot (r+s-3) + (r+s-2)^2 \cdot 1 = r^2 + s^2 + 2rs + 5r + 5s - 18$$

In addition, the vertex partition for $T_{r,s}^{v_{r+2}}$ is provided in Table 10:

Table 10. The vertex partition of $T_{r,s}^{v_{r+2}}$

d_u	# u
1	1
2	2
3	$r+s-5$
4	1
$r+s-2$	1

stands for "the number of".

Hence,

$$M_1(T_{r,s}^{v_{r+2}}) = 1^2 \cdot 1 + 2^2 \cdot 2 + 3^2 \cdot (r+s-5) + (r+s-2)^2 \cdot 1 + 4^2 \cdot 1 = r^2 + s^2 + 2rs + 5r + 5s - 20$$

Moreover, the vertex partition for $T_{r,s}^{v_{r+s-1}}$ is provided in Table 11:

Table 11. The vertex partition of $T_{r,s}^{v_{r+s-1}}$

d_u	$\# u$
2	1
3	$r+s-3$
4	1
$r+s-2$	1

stands for "the number of".

Thus,

$$M_1(T_{r,s}^{v_{r+s-1}}) = r^2 + s^2 + 2rs + 5r + 5s - 3$$

The vertex partition for $T_{r,s}^{v_{r+s}}$ is provided in Table 12:

Table 12. The vertex partition of $T_{r,s}^{v_{r+s}}$

d_u	$\# u$
1	1
2	1
3	$r+s-3$
$r+s-2$	1

stands for "the number of".

Therefore,

$$M_1(T_{r,s}^{v_{r+s}}) = 1^2 \cdot 1 + 2^2 \cdot 1 + 3^2 \cdot (r+s-3) + (r+s-2)^2 \cdot 1 = r^2 + s^2 + 2rs + 5r + 5s - 18$$

□

3. Second Zagreb index of switched graphs of some graph classes

This section calculates the second Zagreb index of a switched star, tadpole, and complete bipartite graphs. Due to the difference in definitions of the first and second Zagreb indices, we use another method. This method uses edge partition, which depends on determining all the pairs of vertex degrees where these vertex pairs form an edge in the graph.

Theorem 3.1. If $v \in A$, then

$$M_2(K_{2,s}^v) = M_2(S_{2+s}) = (s+1)^2$$

and if $v \in B$, then

$$M_2(K_{2,s}^v) = M_2(K_{2,s-1}^1) = 9(s-1)^2$$

Proof. If $v \in A$, the edge partition of $K_{2,s}^v$ is provided in Table 13:

Table 13. The vertex partition of $K_{2,s}^v$ for $v \in A$

(d_u, d_v)	$\# uv$
$(s+1, 1)$	$s+1$

stands for "the number of".

Then, the second Zagreb index of $K_{2,s}^v$ is as follows:

$$M_2(K_{2,s}^v) = M_2(S_{2+s}) = (s+1)^2$$

If $v \in B$, then the edge partition of $K_{2,s}^v$ is provided in Table 14:

Table 14. The vertex partition of $K_{2,s}^v$ for $v \in B$

(d_u,d_v)	# uv
$(3,s-1)$	$3(s-1)$

stands for "the number of".

Then, the second Zagreb index of $K_{2,s}^v$ is as follows:

$$M_2(K_{2,s}^v) = M_2(K_{2,s-1}^1) = 9(s-1)^2$$

□

Theorem 3.2. Let $K_{r,s}$ be a complete bipartite graph with $3 \leq r \leq s$. If $v \in A$, then

$$M_2(K_{r,s}^v) = M_2(K_{s,r-1}^1) = (s+1)^2(r-1)^2$$

and if $v \in B$, then

$$M_2(K_{r,s}^v) = M_2(K_{r,s-1}^1) = (s-1)^2(r+1)^2$$

Proof. If $v \in A$, then the edge partition of $K_{r,s}^v$ is provided in Table 15:

Table 15. The vertex partition of $K_{r,s}^v$ for $v \in A$

(d_u,d_v)	# uv
$(s+1,r-1)$	$(s+1)(r-1)$

stands for "the number of".

Therefore,

$$M_2(K_{r,s}^v) = M_2(K_{s,r-1}^1) = (s+1)^2(r-1)^2$$

If $v \in B$, then the edge partition of $K_{r,s}^v$ is provided in Table 16:

Table 16. The vertex partition of $K_{r,s}^v$ for $v \in B$

(d_u,d_v)	# uv
$(r+1,s-1)$	$(r+1)(s-1)$

stands for "the number of".

Hence,

$$M_2(K_{r,s}^v) = M_2(K_{s,r-1}^1) = (s-1)^2(r+1)^2$$

□

Theorem 3.3. Let S_n be a star graph, v_1 be the central vertex, and all the remaining vertices be $v_2, v_3, \cdots, v_n$. Then,

$$M_2(S_n^{v_1}) = 0$$

and

$$M_2(S_n^{v_i}) = M_2(K_{1,1,n-2} - \{v_1v_i\}) = 4(n-2)^2$$

for $i \in \{2, 3, \cdots, n\}$, where $K_{1,1,n-2}$ is the tripartite graph.

Proof. Because $M_2(S_n^{v_1}) = N_n$, it is obvious that

$$M_2(S_n^{v_1}) = 0$$

For $i \in \{2, 3, \cdots, n\}$, the edge partition for $S_n^{v_i}$ is provided in Table 17:

Table 17. The edge partition of $S_n^{v_i}$ for $i \in \{2, 3, \cdots, n\}$

(d_u, d_v)	# uv
$(2, n-2)$	$2(n-2)$

stands for "the number of".

Hence,

$$M_2(S_n^{v_i}) = M_2(K_{1,1,n-2} - \{v_1v_i\}) = 4(n-2)^2$$

□

Theorem 3.4. Let $T_{r,s}$ be a tadpole graph, v_1 be the central vertex, and all the remaining vertices be $v_2, v_3, \cdots, v_n$. Then,

$$M_2(T_{r,s}^{v_1}) = 3r^2 + 3s^2 + 6rs - 16r - 16s + 4$$

$$M_2(T_{r,s}^{v_2}) = 3r^2 + 3s^2 + 6rs - 10r - 10s - 3$$

$$M_2(T_{r,s}^{v_3}) = 3r^2 + 3s^2 + 6rs - 9r - 9s + 1$$

$$M_2(T_{r,s}^{v_{r+1}}) = 3r^2 + 3s^2 + 6rs - 4r - 4s - 13$$

$$M_2(T_{r,s}^{v_{r+2}}) = 3r^2 + 3s^2 + 6rs - 4r - 4s - 8$$

$$M_2(T_{r,s}^{v_{r+s-1}}) = 3r^2 + 3s^2 + 6rs - 3r - 3s$$

and

$$M_2(T_{r,s}^{v_{r+s}}) = 3r^2 + 3s^2 + 6rs - 4r - 4s - 13$$

Proof. The edge partition for $T_{r,s}^{v_1}$ is provided in Table 18:

Table 18. The edge partition of $T_{r,s}^{v_1}$

(d_u, d_v)	# uv
$(1,3)$	3
$(2,3)$	1
$(2, r+s-4)$	1
$(3, r+s-4)$	$r+s-5$
$(3,3)$	$r+s-7$

stands for "the number of".

By the definition of the second Zagreb index,

$$M_2(T_{r,s}^{v_1}) = 3r^2 + 3s^2 + 6rs - 16r - 16s + 4$$

The edge partition for $T_{r,s}^{v_2}$ is provided in Table 19:

Table 19. The edge partition of $T_{r,s}^{v_2}$

(d_u, d_v)	# uv
$(1,3)$	1
$(2,3)$	3
$(2, r+s-3)$	1
$(3, r+s-3)$	$r+s-4$
$(3,3)$	$r+s-6$

stands for "the number of".

By the definition of second Zagreb index,

$$M_2(T_{r,s}^{v_2}) = 3r^2 + 3s^2 + 6rs - 10r - 10s - 3$$

The edge partition for $T_{r,s}^{v_3}$ is provided in Table 20:

Table 20. The edge partition of $T_{r,s}^{v_3}$

(d_u,d_v)	$\# \; uv$
$(1,3)$	1
$(1,4)$	1
$(2,3)$	1
$(3,r+s-3)$	$r+s-5$
$(2,r+s-3)$	1
$(4,r+s-3)$	1
$(3,3)$	$r+s-7$

stands for "the number of".

Hence,

$$M_2(T_{r,s}^{v_3}) = 3r^2 + 3s^2 + 6rs - 9r - 9s + 1$$

The edge partition for $T_{r,s}^{v_{r+1}}$ is provided in Table 21:

Table 21. The edge partition of $T_{r,s}^{v_{r+1}}$

(d_u,d_v)	$\# \; uv$
$(1,3)$	1
$(2,3)$	1
$(3,r+s-2)$	$r+s-3$
$(2,r+s-2)$	1
$(3,3)$	$r+s-4$

stands for "the number of".

Therefore,

$$M_2(T_{r,s}^{v_{r+1}}) = 3r^2 + 3s^2 + 6rs - 4r - 4s - 13$$

The edge partition for $T_{r,s}^{v_{r+2}}$ is provided in Table 22:

Table 22. The edge partition of $T_{r,s}^{v_{r+2}}$

(d_u,d_v)	$\# \; uv$
$(1,3)$	1
$(2,3)$	1
$(2,4)$	1
$(3,3)$	$r+s-7$
$(3,r+s-2)$	$r+s-5$
$(4,r+s-2)$	1
$(2,r+s-2)$	2
$(3,4)$	2

stands for "the number of".

Thus,

$$M_2(T_{r,s}^{v_{r+2}}) = 3r^2 + 3s^2 + 6rs - 4r - 4s - 8$$

The edge partition for $T_{r,s}^{v_{r+s-1}}$ is provided in Table 23:

Table 23. The edge partition of $T_{r,s}^{v_{r+s-1}}$

(d_u,d_v)	# uv
$(2,3)$	1
$(3,3)$	$r+s-6$
$(3,r+s-2)$	$r+s-4$
$(2,r+s-2)$	1
$(3,4)$	3
$(4,r+s-2)$	1

stands for "the number of".

Thereby,

$$M_2(T_{r,s}^{v_{r+s-1}}) = 3r^2 + 3s^2 + 6rs - 3r - 3s$$

□

4. Conclusion

Several vertex and edge operations are frequently used in graph theory to obtain several properties of given graph types. A recently introduced operation is vertex switching. Several preliminary results have just been published on this operation. In this paper, we considered two important degree-based topological indices, the first and second Zagreb indices, for the vertex-switched graphs. The other topological indices can be calculated for the vertex-switched graphs of different type graph classes. As there are over 3000 such indices, a large research area is related to this problem. Moreover, graph operations, derived graphs, and other graph parameters can be studied for the vertex-switched graphs. Such studies might have chemical applications, and we can obtain physicochemical properties of many molecular structures, including nanocones, nanomaterials, dendrimers, and chains. This will increase the importance of the studied areas and may cause the occurrence of new study areas. As there are many diverse definitions of Zagreb-type graph indices, one can also study the exponential Zagreb indices defined in [19]. Similarly, an analysis of Zagreb indices over zero divisor graphs can be given in detail using the results in [17].

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Computation of the Golden Matrix Exponential Functions of Special Matrices

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Research Article

Abstract — Computation of the matrix exponential functions is important in solving various scientific and engineering problems due to their active role in solving differential equations. Accurate and effective computation of these functions determines the success of mathematical analysis and practical applications. Therefore, studying and understanding matrix exponential functions is the key to developing mathematical and engineering sciences. In the present paper, we aim to compute the values of the 1st and 2nd type Golden matrix exponential functions for some special matrices. We present the similarities and differences with the value of the well-known matrix exponential function for the same special matrices.

Keywords *Fibonacci sequence, Golden calculus, matrix exponential function*

Mathematics Subject Classification (2020) 15A24, 05A10

1. Introduction

Matrix functions play an important role in both theoretical and applied sciences. For example, they are essential in quantum mechanics, control theory, physics, mathematics, and engineering to solve optimization problems, compute eigenvalues, solve differential equations, and perform transformations that simplify complex systems. Among various matrix functions, matrix exponential and trigonometric functions are particularly noteworthy due to their wide applications and the rich mathematical properties they exhibit. The matrix exponential function is very important in solving linear differential equations and appears prominently in the study of linear dynamical systems. The study and application of these matrix functions are topics of intense research due to their theoretical importance and practical benefits. Researchers constantly explore new methods to compute these functions more efficiently and understand their behavior in different contexts. This ongoing research not only advances our mathematical knowledge but also leads to innovations in various fields of science and engineering. For some of the papers, which include the matrix exponential, trigonometric, and hyperbolic functions, see [1–13].

The matrix exponential function is defined by the Taylor series expansions as

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

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where A is the rth order matrix [9]. The matrix trigonometric and hyperbolic functions are defined by the rules

$$\cos(A) = \frac{e^{iA} + e^{-iA}}{2} \quad \text{and} \quad \sin(A) = \frac{e^{iA} - e^{-iA}}{2i}$$

and

$$\cosh(A) = \frac{e^{A} + e^{-A}}{2} \quad \text{and} \quad \sinh(A) = \frac{e^{A} - e^{-A}}{2}$$

where A is the rth order matrix [8, 14].

Sastre et al. [5] provided an algorithm for computing matrix cosine function with the help of the Taylor series and cosine double angle formula. Some methods were presented by Defez and Jodar [15] to compute matrix exponential, sine, and cosine functions based on Hermite matrix polynomials. Besides, Defez et al. [16] introduced a method to compute hyperbolic matrix functions based on Hermite matrix polynomials.

Moreover, number sequences are indispensable tools in mathematical and scientific research. Investigating these sequences provides an in-depth look at scientific research and enables discoveries. For this reason, number sequences are considered the cornerstones of mathematical thinking and scientific progress. The most popular number sequence is undoubtedly the Fibonacci number sequence. The Fibonacci number sequence is defined by the recurrence relation, for $n \geq 1$,

$$F_{n+1} = F_n + F_{n-1}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$ [17]. The Binet formula for the Fibonacci number sequence is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ which are called the Golden and Silver ratios, respectively [17]. This number sequence finds extensive applications not only in theoretical mathematics but also across diverse scientific disciplines. In computer science, algorithms based on Fibonacci sequences are used for data structures and sorting problems. In economics, Fibonacci retracement is a popular tool in technical analysis to predict market movements. Additionally, the Fibonacci sequence appears in the study of population growth models. The pervasive presence of Fibonacci numbers in theoretical and applied sciences underscores their importance as a fundamental mathematical concept.

The Golden Fibonacci calculus is introduced by Pashaev and Nalci [18], which is an application of the Fibonacci number sequence. The authors defined the Golden derivative operator, Golden binomial expansion, Golden exponential functions, etc. We present some of the principle definitions of the Golden Fibonacci calculus.

The Fibonacci factorial $F_n!$ is defined by

$$F_n! = \prod_{i=1}^{n} F_i = F_n F_{n-1} F_{n-2} \cdots F_2 F_1$$

where F_n is the nth Fibonacci number [18]. The Golden binomial is defined as

$$(x+y)_F^n = \left(x + \alpha^{n-1} y\right)\left(x + \alpha^{n-2}\beta y\right)\cdots\left(x + \alpha\beta^{n-2} y\right)\left(x + \beta^{n-1} y\right)$$

where α and β are the Golden and Silver ratios, respectively [18]. The Golden binomial also holds the equality

$$(x+y)_F^n = \sum_{k=0}^{n} \binom{n}{k}_F (-1)^{\frac{k(k-1)}{2}} x^{n-k} y^k$$

where $\binom{n}{k}_F$ denotes the Fibonacci binomial coefficients which are defined by the rule

$$\binom{n}{k}_F = \frac{F_n!}{F_{(n-k)}!F_k!}$$

with $\binom{n}{0}_F = 1$ [19]. These coefficients are called as Fibonomial coefficients [19]. The 1st and 2nd type Golden exponential functions are defined as [18]

$$e_F^x = \sum_{n=0}^{\infty} \frac{(x)_F^n}{F_n!} \quad \text{and} \quad E_F^x = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{(x)_F^n}{F_n!}$$

Briefly,

$$e_F^x = \sum_{n=0}^{\infty} \frac{x^n}{F_n!} \quad \text{and} \quad E_F^x = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{x^n}{F_n!}$$

Özvatan [19] obtained an estimation for the 1st type Golden exponential base number as

$$3.7041 < e_F < 3.7044$$

Using a similar method, an estimation for the 2nd type Golden exponential base number can be obtained as follows:

$$0.6958 < E_F < 0.6961$$

The Golden Taylor expansions

$$\cos_F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{F_{2n}!} \quad \text{and} \quad \sin_F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{F_{2n+1}!}$$

and

$$\cosh_F(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{F_{2n}!} \quad \text{and} \quad \sinh_F(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{F_{2n+1}!}$$

indicate some Golden trigonometric and Golden hyperbolic functions [18]. These functions also have the following representations [18]:

$$\sin_F(x) = \frac{e_F^{ix} - e_F^{-ix}}{2i} = \frac{E_F^x - E_F^{-x}}{2}$$

$$\cos_F(x) = \frac{e_F^{ix} + e_F^{-ix}}{2} = \frac{E_F^x + E_F^{-x}}{2}$$

$$\sinh_F(x) = \frac{e_F^{x} - e_F^{-x}}{2} = \frac{E_F^{ix} - E_F^{-ix}}{2i}$$

and

$$\cosh_F(x) = \frac{e_F^{x} + e_F^{-x}}{2} = \frac{E_F^{ix} + E_F^{-ix}}{2}$$

By starting from the divisibility problem for the Fibonacci numbers the Fibonacci divisors, and the corresponding hierarchy of Golden derivatives in powers of the Golden ratio are introduced by Pashaev [20]. The author also developed the corresponding quantum calculus. The concepts of the Golden Fibonacci calculus are extended to matrices in [21]. Here, we present some definitions of the Golden Fibonacci matrix calculus.

For the rth order commutable matrices A and B, the Golden binomial is defined as

$$(A+B)_F^n = \left(A + \alpha^{n-1}B\right)\left(A + \alpha^{n-2}\beta B\right)\dots\left(A + \alpha\beta^{n-2}B\right)\left(A + \beta^{n-1}B\right)$$

where α and β represent the Golden and Silver ratios, respectively [21]. The Golden binomial of the rth order commutable matrices A and B also holds the equality [21]:

$$(A+B)_F^n = \sum_{k=0}^{n} \binom{n}{k}_F (-1)^{\frac{k(k-1)}{2}} A^{n-k} B^k$$

The 1st and 2nd type Golden matrix exponential functions have the following Golden Taylor series expansions:

$$e_F^A = \sum_{n=0}^{\infty} \frac{(A)_F^n}{F_n!} \quad \text{and} \quad E_F^A = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{(A)_F^n}{F_n!}$$

where A is the rth order matrix [21]. Briefly, we use the following notations throughout this paper:

$$e_F^A = \sum_{n=0}^{\infty} \frac{A^n}{F_n!} \quad \text{and} \quad E_F^A = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{A^n}{F_n!}$$

There are the following relations for the Golden matrix exponential functions

$$e_{-F}^A = E_F^A \quad \text{and} \quad e_F^A e_{-F}^B = e_F^{A+B}$$

where A and B are rth order commutable matrices [21]. Some Golden matrix trigonometric and hyperbolic functions are defined with the help of the Golden matrix exponential functions as follows:

$$\sin_F(A) = \frac{e_F^{iA} - e_F^{-iA}}{2i} = \frac{E_F^A - E_F^{-A}}{2}$$

$$\cos_F(A) = \frac{e_F^{iA} + e_F^{-iA}}{2} = \frac{E_F^A + E_F^{-A}}{2}$$

$$\sinh_F(A) = \frac{e_F^{A} - e_F^{-A}}{2} = \frac{E_F^{iA} - E_F^{-iA}}{2i}$$

and

$$\cosh_F(A) = \frac{e_F^{A} + e_F^{-A}}{2} = \frac{E_F^{iA} + E_F^{-iA}}{2}$$

where A is the rth order matrix [21].

In the present paper, we conduct a detailed examination of the 1st and 2nd type Golden matrix exponential functions. A thorough understanding of these functions may offer alternative approaches to solving differential equations, which play a significant role in various scientific fields. We explore what these functions represent for certain special matrices. In this process, we compare the findings of [6] and [13] regarding the matrix exponential function when similar matrices are used, with the findings we obtained for the 1st and 2nd type Golden matrix exponential functions. We note that to avoid similarity in proving our results, we provide proofs only for the 1st type Golden matrix exponential function, since the 2nd type can be derived in a similar manner.

2. Main Results

The matrix exponential function holds the equality $e^{0_r} = I_r$, for the rth order zero matrix 0_r, where I_r is the rth order identity matrix. We provide similar equalities for the Golden matrix exponential functions in the first proposition.

Proposition 2.1. For the Golden matrix exponential functions of the rth order zero matrix 0_r,

$$e_F^{0_r} = I_r \quad \text{and} \quad E_F^{0_r} = I_r$$

where I_r is the rth order identity matrix.

Proof. Considering the Golden Taylor series expansion of e_F^A, for $A = 0_r$,

$$e_F^{0_r} = \frac{I_r}{F_0!} + \sum_{n=1}^{\infty} \frac{0_r^n}{F_n!} = I_r$$

□

The matrix exponential function has the property $e^{(A^T)} = (e^A)^T$, where A^T is the transpose matrix of the rth order matrix A. The Golden matrix exponential functions have the following property similar to the matrix exponential function.

Proposition 2.2. Let A^T be the transpose matrix of the rth order matrix A. Then, for the Golden matrix exponential functions,

$$e_F^{(A^T)} = \left(e_F^A\right)^T \quad \text{and} \quad E_F^{(A^T)} = \left(E_F^A\right)^T$$

Proof. Using the Golden Taylor series expansion of the 1st type Golden matrix exponential function and the well known property $\left(A^T\right)^s = (A^s)^T$ of the matrix A, for $s \in \{1, 2, 3, \ldots\}$,

$$\begin{aligned} e_F^{(A^T)} &= \sum_{n=0}^{\infty} \frac{\left(A^T\right)^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \frac{(A^n)^T}{F_n!} \\ &= \left(\sum_{n=0}^{\infty} \frac{A^n}{F_n!}\right)^T \\ &= \left(e_F^A\right)^T \end{aligned}$$

□

For the matrix exponential function, $e^{mI_r} = (e^m)I_r$, where I_r is the rth order identity matrix and $m \in \mathbb{Z}$. The Golden matrix exponential functions have similar properties to the matrix exponential function, as follows:

Proposition 2.3. The Golden matrix exponential functions hold the following equalities

$$e_F^{mI_r} = e_F^m I_r \quad \text{and} \quad E_F^{mI_r} = E_F^m I_r$$

for the rth order identity matrix I_r and $m \in \mathbb{Z}$.

Proof. Considering the Golden Taylor series expansion of the 1st type Golden matrix exponential function and the property $I_r^s = I_r$ of identity matrix I_r, for $s \in \{1, 2, 3, \ldots\}$,

$$\begin{aligned} e_F^{mI_r} &= \sum_{n=0}^{\infty} \frac{(mI_r)^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \frac{m^n}{F_n!} I_r \\ &= e_F^m I_r \end{aligned}$$

□

The matrix exponential function satisfies the equation $e^A e^B = e^{A+B}$, for the rth order commutable matrices A and B. However, this property is not provided for the Golden matrix exponential functions. We investigate the Golden matrix exponential functions in terms of this property in the next proposition.

Proposition 2.4. For the rth order commutable matrices A and B,

$$e_F^A e_F^B \neq e_F^{A+B} \quad \text{and} \quad E_F^A E_F^B \neq E_F^{A+B}$$

PROOF. Consider the matrix $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then,

$$e_F^{A+B} = e_F^{0_r} = I_r$$

where 0_r is the rth order zero matrix. On the other hand, for the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, $e_F^A = \begin{bmatrix} e_F^1 & 0 \\ 0 & e_F^2 \end{bmatrix}$ and $e_F^B = \begin{bmatrix} e_F^{-1} & 0 \\ 0 & e_F^{-2} \end{bmatrix}$. Then,

$$e_F^A e_F^B = \begin{bmatrix} e_F^1 & 0 \\ 0 & e_F^2 \end{bmatrix} \begin{bmatrix} e_F^{-1} & 0 \\ 0 & e_F^{-2} \end{bmatrix}$$

Since $3.7041 < e_F < 3.7044$, it is clear that $e_F^a e_F^{-a} \neq 1$ for an arbitrary number a. Thus, $e_F^A e_F^B \neq I_r$. Therefore,

$$e_F^A e_F^B \neq e_F^{A+B}$$

□

The inverse of the matrix exponential e^A is $(e^A)^{-1} = e^{-A}$, for the rth order matrix A. However, this property is not held by the Golden matrix exponential functions. We provide this property by the next proposition.

Proposition 2.5. Let the matrices $\left(e_F^A\right)^{-1}$ and $\left(E_F^A\right)^{-1}$ be the inverses of e_F^A and E_F^A, respectively, for the rth order matrix A. Then,

$$\left(e_F^A\right)^{-1} \neq e_F^{-A} \quad \text{and} \quad \left(E_F^A\right)^{-1} \neq E_F^{-A}$$

PROOF. If the inverse of e_F^A is equal to e_F^{-A}, the equality $e_F^A e_F^{-A} = I_r$ must be held, where I_r is the rth order identity matrix. It is clear from the example in the proof of Proposition 2.4 that this equality is unsatisfied. □

The matrix exponential function has the property $(e^A)^m = e^{mA}$, where A is the rth order matrix and $m \in \mathbb{Z}$. However, in the following proposition, we state that Golden matrix exponential functions do not have similar properties.

Proposition 2.6. The Golden matrix exponential functions satisfy the following inequalities

$$\left(e_F^A\right)^m \neq e_F^{mA} \quad \text{and} \quad \left(E_F^A\right)^m \neq E_F^{mA}$$

for the rth order matrix A and $m \in \mathbb{Z}$.

The proof is clear from Proposition 2.4.

Let A and B be rth order commutable matrices. Then, the matrix exponential function has the equality $e^A e^B = e^B e^A$. We investigate the same property for the Golden matrix exponential functions in the following proposition.

Proposition 2.7. For the Golden matrix exponential functions,

$$e_F^A e_F^B = e_F^B e_F^A \quad \text{and} \quad E_F^A E_F^B = E_F^B E_F^A$$

where A and B are rth order commutable matrices.

Proof. For the commutable rth order matrices A and B,

$$\begin{aligned} e_F^A e_F^B &= \sum_{n=0}^{\infty} \frac{A^n}{F_n!} \sum_{n=0}^{\infty} \frac{B^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \frac{B^n}{F_n!} \sum_{n=0}^{\infty} \frac{A^n}{F_n!} \\ &= e_F^B e_F^A \end{aligned}$$

□

The matrix exponential function has the property $e^{m\overline{I_r}} = \cosh(m) I_r + \sinh(m) \overline{I_r}$, for the rth order anti-identity matrix $\overline{I_r}$, where I_r is the rth order identity matrix. We investigate the Golden matrix exponential functions in terms of this property in the following proposition.

Proposition 2.8. Let $\overline{I_r} = \operatorname{adiag}[1, 1, \cdots, 1]$ be the rth order anti-identity matrix. Then,

$$e_F^{m\overline{I_r}} = \cosh_F(m) I_r + \sinh_F(m) \overline{I_r} \quad \text{and} \quad E_F^{m\overline{I_r}} = \cos_F(m) I_r + \sin_F(m) \overline{I_r}$$

where I_r is the rth order identity matrix.

Proof. For the rth order anti-identity matrix $\overline{I_r}$, $\overline{I_r}^{2s} = I_r$ and $\overline{I_r}^{2s-1} = \overline{I_r}$, for $s \in \{1, 2, 3, \cdots\}$, where I_r is rth order identity matrix. If we use these equalities in the Golden Taylor series expansion of the 1st type Golden matrix exponential function, then

$$\begin{aligned} e_F^{m\overline{I}} &= \sum_{n=0}^{\infty} \frac{\left(m\overline{I_r}\right)^n}{F_n!} \\ &= \frac{I_r}{F_0!} + \frac{m\overline{I_r}}{F_1!} + \frac{(mI_r)^2}{F_2!} + \frac{\left(m\overline{I_r}\right)^3}{F_3!} + \frac{(mI_r)^4}{F_4!} + \cdots \\ &= \left(\frac{1}{F_0!} + \frac{m^2}{F_2!} + \frac{m^4}{F_4!} + \cdots\right) I_r + \left(\frac{m}{F_1!} + \frac{m^3}{F_3!} + \frac{m^5}{F_5!} + \cdots\right) \overline{I_r} \\ &= \cosh_F(m) I_r + \sinh_F(m) \overline{I_r} \end{aligned}$$

□

The matrix exponential function provide the equality $e^{mD} = \operatorname{diag}\left[e^{md_1}, e^{md_2}, e^{md_3}, \cdots, e^{md_r}\right]$, for the rth order diagonal matrix D. We present the following proposition to indicate that the Golden matrix exponential functions have similar properties.

Proposition 2.9. Let $D = \operatorname{diag}[d_1, d_2, d_3, \cdots, d_r]$ be the rth order diagonal matrix. Then,

$$e_F^{mD} = \operatorname{diag}\left[e_F^{md_1}, e_F^{md_2}, e_F^{md_3}, \cdots, e_F^{md_r}\right]$$

and

$$E_F^{mD} = \operatorname{diag}\left[E_F^{md_1}, E_F^{md_2}, E_F^{md_3}, \cdots, E_F^{md_r}\right]$$

Proof. Considering the Golden Taylor series expansion of the 1st type Golden matrix exponential function $e_F^{mD} = \sum_{n=0}^{\infty} \frac{(mD)^n}{F_n!}$ and the sth power matrices $D^s = \operatorname{diag}[d_1^s, d_2^s, d_3^s, \cdots, d_r^s]$, for $s \in \{1, 2, 3, \ldots\}$, the proof is clear. □

The matrix exponential function holds $e^{m\overline{D}} = \cosh_F(m\xi) I_r + \frac{1}{\xi} \sinh(m\xi) \overline{D}$, for the rth order anti-diagonal matrix $\overline{D} = \operatorname{adiag}[d_1', d_2', d_3', \cdots, d_r']$ and identity matrix I_r, where $d_i' d_{r-i+1}' = \xi^2$ and $i \in \{1, 2, \ldots, r\}$. The Golden matrix exponential functions have similar properties to the matrix exponential functions, as follows:

Proposition 2.10. Let $\overline{D}$ be the rth order anti-diagonal matrix mentioned above. Then, the Golden matrix exponential functions hold the following equalities:

$$e_F^{m\overline{D}} = \cosh_F(m\xi)\, I_r + \frac{1}{\xi}\sinh_F(m\xi)\,\overline{D} \quad \text{and} \quad E_F^{m\overline{D}} = \cos_F(m\xi)\, I_r + \frac{1}{\xi}\sin_F(m\xi)\,\overline{D}$$

where $d'_i d'_{r-i+1} = \xi^2$ and I_r is the rth order identity matrix.

Proof. Let $\overline{D} = \operatorname{adiag}[d'_1, d'_2, d'_3, \cdots, d'_r]$ be the rth order anti-diagonal matrix. Then,

$$\overline{D}^{2s} = \operatorname{adiag}\left[(d'_1 d'_r)^s, (d'_2 d'_{r-1})^s, (d'_3 d'_{r-2})^s, \cdots, (d'_r d'_1)^s\right] = \xi^{2s} I_r$$

and

$$\overline{D}^{2s-1} = \xi^{2s-1}\overline{D}$$

for $s \in \{1, 2, 3, \ldots\}$, where $d'_i d'_{r-i+1} = \xi^2$, $i \in \{1, 2, \cdots, r\}$, and I_r is the rth order identity matrix. If we substitute these equalities in the Golden Taylor series expansion of 1st type Golden matrix exponential function, then

$$\begin{aligned}
e_F^{m\overline{D}} &= \sum_{n=0}^{\infty} \frac{\left(m\overline{D}\right)^n}{F_n!} \\
&= \frac{I_r}{F_0!} + \frac{m\overline{D}}{F_1!} + \frac{\left(m\overline{D}\right)^2}{F_2!} + \frac{\left(m\overline{D}\right)^3}{F_3!} + \frac{\left(m\overline{D}\right)^4}{F_4!} + \cdots \\
&= \left(\frac{I_r}{F_0!} + \frac{\left(m\overline{D}\right)^2}{F_2!} + \frac{\left(m\overline{D}\right)^4}{F_4!} + \cdots\right) + \left(\frac{m\overline{D}}{F_1!} + \frac{\left(m\overline{D}\right)^3}{F_3!} + \frac{\left(m\overline{D}\right)^5}{F_5!} + \cdots\right) \\
&= \left(\frac{1}{F_0!} + \frac{(m\xi)^2}{F_2!} + \frac{(m\xi)^4}{F_4!} + \cdots\right) I_r + \left(\frac{m}{F_1!} + \frac{m^3\xi^2}{F_3!} + \frac{m^5\xi^4}{F_5!} + \cdots\right)\overline{D} \\
&= \cosh_F(m\xi)\, I_r + \frac{1}{\xi}\sinh_F(m\xi)\,\overline{D}
\end{aligned}$$

□

The matrix exponential function holds the equality $e^{mA} = \operatorname{diag}[e^m I_r, e^{-m} I_r]$, for the $2r$th order positive negative identity matrix $A = \begin{bmatrix} I_r & \\ & -I_r \end{bmatrix}$, where I_r is the rth order identity matrix. The Golden matrix exponential functions have properties similar to those of the matrix exponential function, as indicated in the next proposition.

Proposition 2.11. Let A be the $2r$th order positive negative identity matrix mentioned above. Then,

$$e_F^{mA} = \operatorname{diag}\left[e_F^m I_r, e_F^{-m} I_r\right] \quad \text{and} \quad E_F^{mA} = \operatorname{diag}\left[E_F^m I_r, E_F^{-m} I_r\right]$$

where I_r is the rth order identity matrix.

Proof. For the $2r$th order positive negative identity matrix $A = \begin{bmatrix} I_r & \\ & -I_r \end{bmatrix}$,

$$A^{2s} = I_{2r} \quad \text{and} \quad A^{2s+1} = A$$

where I_{2r} is the $2r$th order identity matrix and $s \in \{1, 2, 3, \ldots\}$. Then,

$$
\begin{aligned}
e_F^{mA} &= \sum_{n=0}^{\infty} \frac{(mA)^n}{F_n!} \\
&= \frac{I_{2r}}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\
&= \left(\frac{I_{2r}}{F_0!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^4}{F_4!} + \cdots\right) + \left(\frac{mA}{F_1!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^5}{F_5!} + \cdots\right) \\
&= \left(\frac{1}{F_0!} + \frac{m^2}{F_2!} + \frac{m^4}{F_4!} + \cdots\right) I_{2r} + \left(\frac{m}{F_1!} + \frac{m^3}{F_3!} + \frac{m^5}{F_5!} + \cdots\right) A \\
&= \cosh_F(m)\, I_{2r} + \sinh_F(m)\, A \\
&= \operatorname{diag}\left[e_F^{m} I_r, e_F^{-m} I_r\right]
\end{aligned}
$$

□

The matrix exponential function satisfies $e^{mA} = \cos(m)\, I_{2r} + \sin(m)\, A$, for the $2r$th order positive negative anti-identity matrix $A = \begin{bmatrix} & I_r \\ -I_r & \end{bmatrix}$, where I_r is the rth order identity matrix. The following proposition gives the corresponding equalities for the Golden matrix exponential functions.

Proposition 2.12. Let A be the $2r$th order positive negative anti-identity matrix mentioned above. Then,

$$
e_F^{mA} = \cos_F(m)\, I_{2r} + \sin_F(m)\, A \quad \text{and} \quad E_F^{mA} = \cosh_F(m)\, I_{2r} + \sinh_F(m)\, A
$$

where I_{2r} is the $2r$th order identity matrix.

PROOF. For the $2r$th order positive negative anti-identity matrix A, $A^2 = -I_{2r}$, $A^3 = -A$, $A^4 = I_{2r}$, $A^5 = A$, $A^6 = -I_{2r}$, $A^7 = -A$, $A^8 = I_{2r}$, $A^9 = A$, ..., where I_{2r} is the $2r$th order identity matrix. Considering these equalities and the Golden Taylor series expansion of the Golden matrix trigonometric function,

$$
\begin{aligned}
e_F^{mA} &= \frac{I_{2r}}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\
&= \left(\frac{I_{2r}}{F_0!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^4}{F_4!} + \cdots\right) + \left(\frac{mA}{F_1!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^5}{F_5!} + \cdots\right) \\
&= \left(\frac{1}{F_0!} - \frac{m^2}{F_2!} + \frac{m^4}{F_4!} - \cdots\right) I_{2r} + \left(\frac{m}{F_1!} - \frac{m^3}{F_3!} + \frac{m^5}{F_5!} - \cdots\right) A \\
&= \cos_F(m)\, I_{2r} + \sin_F(m)\, A
\end{aligned}
$$

□

For the rth order positive negative alternating identity matrix $A = \operatorname{diag}\left[1, -1, 1, -1, \cdots, (-1)^{r-1}\right]$, the matrix exponential function is equal to $e^{mA} = \operatorname{diag}\left[e^{m}, e^{-m}, e^{m}, e^{-m}, \cdots, e^{(-1)^{r-1}m}\right]$. In the following proposition, we investigate the Golden matrix exponential functions for this kind of matrix.

Proposition 2.13. For the rth order positive negative alternating identity matrix A mentioned above, the Golden matrix exponential are

$$
e_F^{mA} = \operatorname{diag}\left[e_F^{m}, e_F^{-m}, e_F^{m}, e_F^{-m}, \cdots, e_F^{(-1)^{r-1}m}\right]
$$

and

$$E_F^{mA} = \mathrm{diag}\left[E_F^{m}, E_F^{-m}, E_F^{m}, E_F^{-m}, \cdots, E_F^{(-1)^{r-1}m}\right]$$

Proof. Let A be the rth order positive negative alternating identity matrix. Then, $A^{2s} = I_{2r}$ and $A^{2s+1} = A$, where I_{2r} is the $2r$th order identity matrix and $s \in \{1, 2, 3, \ldots\}$. Thus the proof is similar to the proof of Proposition 2.11.

□

The matrix exponential function holds $e^{mA} = \cos(m) I_r + \sin(m) A$, for the rth order positive negative alternating anti-identity matrix $A = \mathrm{adiag}\left[1, -1, 1, -1, \cdots, (-1)^{r-1}\right]$, where I_r is the rth order identity matrix. The Golden matrix exponential functions have similar properties to the matrix exponential function, as follows:

Proposition 2.14. Let A be the rth order positive negative alternating anti-identity matrix mentioned above. Then,

$$e_F^{mA} = \cos_F(m) I_r + \sin_F(m) A \quad \text{and} \quad E_F^{mA} = \cosh_F(m) I_r + \sinh_F(m) A$$

where I_r is the rth order identity matrix.

Proof. For the rth order positive negative alternating anti-identity matrix A, $A^2 = -I_r$, $A^3 = -A$, $A^4 = I_r$, $A^5 = A$, $A^6 = -I_r$, $A^7 = -A$, $A^8 = I_r$, $A^9 = A$, ..., where I_r is the rth order identity matrix. Using these equalities and the Golden Taylor series expansions of the Golden matrix trigonometric functions,

$$\begin{aligned} e_F^{mA} &= \frac{I_r}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\ &= \left(\frac{I_r}{F_0!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^4}{F_4!} + \cdots\right) + \left(\frac{mA}{F_1!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^5}{F_5!} + \cdots\right) \\ &= \left(\frac{1}{F_0!} - \frac{m^2}{F_2!} + \frac{m^4}{F_4!} - \cdots\right) I_r + \left(\frac{m}{F_1!} - \frac{m^3}{F_3!} + \frac{m^5}{F_5!} - \cdots\right) A \\ &= \cos_F(m) I_r + \sin_F(m) A \end{aligned}$$

□

The matrix exponential function is equal to $e^{mA} = \cosh(m) I_r + \sinh(m) A$, for the rth order square identity matrix A with $A^2 = I_r$, where I_r is the rth order identity matrix. We provide the following proposition to give similar properties of the Golden matrix exponential functions.

Proposition 2.15. For the rth order square identity matrix A mentioned above,

$$e_F^{mA} = \cosh_F(m) I_r + \sinh_F(m) A \quad \text{and} \quad E_F^{mA} = \cos_F(m) I_r + \sin_F(m) A$$

where I_r is the rth order identity matrix.

Proof. The matrix A with $A^2 = I_r$ has the equalities

$$A^{2s} = I_r \quad \text{and} \quad A^{2s-1} = A$$

for $s \in \{1, 2, 3, \ldots\}$, where I_r is the rth order identity matrix. Considering these equalities and the Golden Taylor series expansion of the Golden matrix hyperbolic function,

$$
\begin{aligned}
e_F^{mA} &= \frac{I_r}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\
&= \left(\frac{I_r}{F_0!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^4}{F_4!} + \cdots\right) + \left(\frac{mA}{F_1!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^5}{F_5!} + \cdots\right) \\
&= \left(\frac{1}{F_0!} + \frac{m^2}{F_2!} + \frac{m^4}{F_4!} + \cdots\right) I_r + \left(\frac{m}{F_1!} + \frac{m^3}{F_3!} + \frac{m^5}{F_5!} + \cdots\right) A \\
&= \cosh_F(m)\, I_r + \sinh_F(m)\, A
\end{aligned}
$$

□

The matrix exponential function has the equality $e^{mA} = \cos(m)\, I_r + \sin(m)\, A$, where A is the rth order square anti-identity matrix and I_r is the rth order identity matrix. We investigate the Golden matrix exponential functions in terms of this property in the following proposition.

Proposition 2.16. Let A be the rth order square anti-identity matrix with $A^2 = -I_r$. Then,

$$e_F^{mA} = \cos_F(m)\, I_r + \sin_F(m)\, A \quad \text{and} \quad E_F^{mA} = \cosh_F(m)\, I_r + \sinh_F(m)\, A$$

Proof. For the rth order square anti-identity matrix A with $A^2 = -I_r$, $A^3 = -A$, $A^4 = I_r$, $A^5 = A$, $A^6 = -I_r$, $A^7 = -A$, $A^8 = I_r$, $A^9 = A$, ..., where I_r is the rth order identity matrix. Thus, the proof is similar to the proof of Proposition 2.14. □

The matrix exponential function has the property $e^{mA} = I_r + (e^m - 1)\, A$, for the rth order idempotent matrix A, where I_r is the rth order identity matrix. The Golden matrix exponential functions have similar properties to the matrix exponential function, as follows:

Proposition 2.17. Let A be the rth order idempotent matrix with $A^s = A$, for $s \in \{1, 2, 3, \ldots\}$. Then,

$$e_F^{mA} = I_r + (e_F^m - 1)\, A \quad \text{and} \quad E_F^{mA} = I_r + (E_F^m - 1)\, A$$

where I_r is the rth order identity matrix.

Proof. By the Golden Taylor series expansion of the 1st type Golden matrix exponential function,

$$
\begin{aligned}
e_F^{mA} &= \frac{I_r}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\
&= \frac{I_r}{F_0!} + \left(\frac{m}{F_1!} + \frac{m^2}{F_2!} + \frac{m^3}{F_3!} + \frac{m^4}{F_4!} + \cdots\right) A \\
&= I_r + (e_F^m - 1)\, A
\end{aligned}
$$

□

The matrix exponential function has the equality $e^{mA} = I_r + (1 - e^{-m})\, A$, for the rth order anti-idempotent matrix A. In the following proposition, we research the Golden matrix exponential functions for this kind of matrix.

Proposition 2.18. For the rth order anti-idempotent matrix A with $A^s = (-1)^{s-1} A$, for $s \in \{1, 2, 3, \ldots\}$, the Golden matrix exponential functions are

$$e_F^{mA} = I_r + \left(1 - e_F^{-m}\right) A \quad \text{and} \quad E_F^{mA} = I_r + \left(1 + E_F^{-m}\right) A$$

where I_r is the rth order identity matrix.

PROOF. By the Golden Taylor series expansions of the Golden hyperbolic functions,

$$\begin{aligned} e_F^{mA} &= \frac{I_r}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\ &= \frac{I_r}{F_0!} + \left(-\frac{m^2}{F_2!} - \frac{m^4}{F_4!} - \frac{m^6}{F_6!} - \cdots\right) A + \left(\frac{m}{F_1!} + \frac{m^3}{F_3!} + \frac{m^5}{F_5!} + \cdots\right) A \\ &= I_r + (1 - \cosh_F(m) + \sinh_F(m))\, A \\ &= I_r + \left(1 - e_F^{-m}\right) A \end{aligned}$$

□

Let $A = \begin{bmatrix} I_r & I_r \\ I_r & I_r \end{bmatrix}$ be the $2r$th order block identity matrix, where I_r is the rth order identity matrix. Then, the matrix exponential function has the equality $e^{mA} = I_{2r} + \frac{1}{2}\left(e^{2m} - 1\right) A$, where I_{2r} is the $2r$th order identity matrix. The following proposition indicates the Golden matrix exponential functions have similar equalities to the matrix exponential function for this kind of matrix.

Proposition 2.19. For the $2r$th order block identity matrix A mentioned above,

$$e_F^{mA} = I_{2r} + \frac{1}{2}\left(e_F^{2m} - 1\right) A \quad \text{and} \quad E_F^{mA} = I_{2r} + \frac{1}{2}\left(E_F^{2m} - 1\right) A$$

where I_{2r} is the $2r$th order identity matrix.

PROOF. For the $2r$th order block identity matrix A, $A^s = 2^{s-1}A$, for $s \in \{2, 3, \ldots\}$. Substituting these equalities in the Golden Taylor series expansion of the 1st type Golden matrix exponential function,

$$\begin{aligned} e_F^{mA} &= \frac{I_{2r}}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\ &= \frac{I_{2r}}{F_0!} + \frac{mA}{F_1!} + \frac{(m)^2\, 2A}{F_2!} + \frac{(m)^3\, 2^2 A}{F_3!} + \frac{(m)^4\, 2^3 A}{F_4!} + \cdots \\ &= \frac{I_{2r}}{F_0!} + \frac{1}{2}\left(\frac{2m}{F_1!} + \frac{(2m)^2}{F_2!} + \frac{(2m)^3}{F_3!} + \frac{(2m)^4}{F_4!} + \cdots\right) A \\ &= I_{2r} + \frac{1}{2}\left(e_F^{2m} - 1\right) A \end{aligned}$$

□

Let $A = \begin{bmatrix} -I_r & I_r \\ I_r & -I_r \end{bmatrix}$, where I_r is the rth order identity matrix. Then, the matrix exponential function for A is equal to $e^{mA} = I_{2r} - \frac{1}{2}\left(e^{2m} - 1\right) A$. In the next proposition, we investigate the Golden matrix exponential functions for this kind of matrix.

Proposition 2.20. Let A be the $2r$th order block identity matrix mentioned above. Then,

$$e_F^{mA} = I_{2r} - \frac{1}{2}\left(e_F^{2m} - 1\right) A \quad \text{and} \quad E_F^{mA} = I_{2r} - \frac{1}{2}\left(E_F^{2m} - 1\right) A$$

where I_{2r} is the $2r$th order identity matrix.

PROOF. Using the Golden Taylor series expansion of the 1st type Golden matrix exponential function and considering $A^s = (-2)^{s-1} A$, for $s \in \{2, 3, \ldots\}$,

$$\begin{aligned}
e_F^{mA} &= \frac{I_{2r}}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\
&= \frac{I_{2r}}{F_0!} + \frac{mA}{F_1!} + \frac{(m)^2\, 2A}{F_2!} + \frac{(m)^3\, 2^2 A}{F_3!} + \frac{(m)^4\, 2^3 A}{F_4!} + \cdots \\
&= \frac{I_{2r}}{F_0!} + \frac{1}{2}\left(\frac{2m}{F_1!} - \frac{(2m)^2}{F_2!} + \frac{(2m)^3}{F_3!} - \frac{(2m)^4}{F_4!} + \cdots\right) A \\
&= \frac{I_{2r}}{F_0!} - \frac{1}{2}\left(\frac{(2m)^2}{F_2!} + \frac{(2m)^4}{F_4!} + \frac{(2m)^6}{F_6!} + \cdots\right) A + \frac{1}{2}\left(\frac{2m}{F_1!} + \frac{(2m)^3}{F_3!} + \frac{(2m)^5}{F_5!} + \cdots\right) A \\
&= I_{2r} - \frac{1}{2}\left(\cosh_F(2m) - 1 + \sinh_F(2m)\right) A \\
&= I_{2r} - \frac{1}{2}\left(e_F^{2m} - 1\right) A
\end{aligned}$$

□

For the $2r$th order matrix $A = \begin{bmatrix} I_r & -I_r \\ -I_r & I_r \end{bmatrix}$, where I_r is the rth order identity matrix, the matrix exponential function for A is equal to $e^{mA} = I_{2r} - \frac{1}{2}\left(e^{2m} - 1\right) A$, where I_{2r} is the $2r$th order identity matrix. The Golden matrix exponential functions behave similarly to matrix exponential function, as stated in the proposition below.

Proposition 2.21. Let the matrix A be as mentioned above. Then,

$$e_F^{mA} = I_{2r} - \frac{1}{2}\left(e_F^{2m} - 1\right) A \quad \text{and} \quad E_F^{mA} = I_{2r} - \frac{1}{2}\left(E_F^{2m} - 1\right) A$$

where I_{2r} is the $2r$th order identity matrix.

The proof is similar to the proof of Proposition 2.20, considering $A^s = (2)^{s-1} A$, for $s \in \{2, 3, \ldots\}$.

The matrix exponential function is equal to $e^{mA} = I_r - \frac{1}{r}\left(e^{rm} - 1\right) A$, for the rth order unity matrix A, i.e., all entries of A equal to 1, where I_r is the rth order identity matrix. We investigate the Golden matrix exponential functions for the unity matrix.

Proposition 2.22. Let A be the rth order unity matrix. Then,

$$e_F^{mA} = I_r - \frac{1}{r}\left(e_F^{rm} - 1\right) A \quad \text{and} \quad E_F^{mA} = I_r - \frac{1}{r}\left(E_F^{rm} - 1\right) A$$

where I_r is the rth order identity matrix.

Proof. For the unity matrix A, $A^s = r^{s-1} A$ for $s \in \{2, 3, \ldots\}$. Considering these equalities and the Golden Taylor series expansion of the 1st type Golden matrix exponential function,

$$\begin{aligned}
e_F^{mA} &= \frac{I_r}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots \\
&= \frac{I_r}{F_0!} + \frac{mA}{F_1!} + \frac{(m)^2\, rA}{F_2!} + \frac{(m)^3\, r^2 A}{F_3!} + \frac{(m)^4\, r^3 A}{F_4!} + \cdots \\
&= \frac{I_r}{F_0!} + \frac{1}{r}\left(\frac{rm}{F_1!} + \frac{(rm)^2}{F_2!} + \frac{(rm)^3}{F_3!} + \frac{(rm)^4}{F_4!} + \cdots\right) A \\
&= I_r - \frac{1}{r}\left(e_F^{rm} - 1\right) A
\end{aligned}$$

□

The matrix exponential function is $e^{mA} = \sum_{n=0}^{s-1} \frac{(mA)^n}{n!}$, for the nilponent matrix A with $A^s = 0$, where $s \in \{2, 3, \ldots\}$. The next proposition includes the values of the Golden matrix exponential functions for the nilpotent matrix.

Proposition 2.23. For the rth order nilpotent matrix A with $A^s = 0$, for $s \in \{2, 3, \ldots\}$,

$$e_F^{mA} = \sum_{n=0}^{s-1} \frac{(mA)^n}{F_n!}$$

and

$$E_F^{mA} = \sum_{n=0}^{s-1} (-1)^{\frac{n(n-1)}{2}} \frac{(mA)^n}{F_n!}$$

Proof. Since $A^s = A^{s+1} = A^{s+2} = \cdots = 0_r$, where 0_r is the rth order zero matrix, the result is clear. □

Corollary 2.24. A strictly triangular matrix which is a triangular matrix with zero diagonal entries has similar results to the matrix in Proposition 2.23, because it is also a nilpotent matrix.

The matrix exponential function of the rth order matrix A can be obtained via its similar matrix B, that is, $e^{mA} = Pe^{mB}P^{-1}$, where P is a non-singular matrix. We investigate this property for the Golden matrix exponential functions.

Proposition 2.25. Let the rth order matrices A and B be similar. Then, the Golden matrix exponential functions of A can be calculated by the rules

$$e_F^{mA} = Pe_F^{mB}P^{-1} \quad \text{and} \quad E_F^{mA} = PE_F^{mB}P^{-1}$$

where P is a non singular matrix such that $A = PBP^{-1}$.

Proof. By applying the Golden Taylor series expansion of the 1st type Golden matrix exponential function,

$$\begin{aligned}
e_F^{mA} &= e_F^{mPBP^{-1}} \\
&= \frac{I_r}{F_0!} + \frac{mPBP^{-1}}{F_1!} + \frac{(mPBP^{-1})^2}{F_2!} + \frac{(mPBP^{-1})^3}{F_3!} + \frac{(mPBP^{-1})^4}{F_4!} + \cdots \\
&= I_r + mPBP^{-1} + \frac{m^2PB^2P^{-1}}{F_2!} + \frac{m^3PB^3P^{-1}}{F_3!} + \frac{m^4PB^4P^{-1}}{F_4!} + \cdots \\
&= P\left(I_r + mB + \frac{(mB)^2}{F_2!} + \frac{(mB)^3}{F_3!} + \frac{(mB)^4}{F_4!} + \cdots\right)P^{-1} \\
&= Pe_F^{mB}P^{-1}
\end{aligned}$$

where I_r is the rth order identity matrix. □

For the rth order Jordan matrix A, the matrix exponential function is equal to $e^{mA} = [c_{ij}]_{i,j=1}^r$ with

$$c_{ij} = \begin{cases} \frac{m^k}{k!}e^{m\lambda}, & i = j - k \text{ and } k \in \{0, 1, 2, \cdots, r-1\} \\ 0, & \text{otherwise} \end{cases}$$

Finally, we explore the Golden matrix exponential functions for the Jordan matrix A.

Proposition 2.26. For the rth order Jordan matrix $A = [a_{ij}]_{i,j=1}^{r}$ with $a_{ij} = \begin{cases} \lambda, & i = j \\ 1, & i = j-1 \\ 0, & \text{otherwise} \end{cases}$, the Golden matrix exponential functions are

$$e_F^{mA} = \left[c_{ij}^{*}\right]_{i,j=1}^{r} \quad \text{with} \quad c_{ij}^{*} = \begin{cases} \dfrac{m^k}{F_k!} e_F^{m\lambda}, & i = j-k \\ 0, & \text{otherwise} \end{cases}$$

and

$$E_F^{mA} = \left[c_{ij}^{**}\right]_{i,j=1}^{r} \quad \text{with} \quad c_{ij}^{**} = \begin{cases} \dfrac{m^k}{F_k!} E_F^{m\lambda}, & i = j-k \\ 0, & \text{otherwise} \end{cases}$$

where $k \in \{0, 1, 2, \ldots, r-1\}$.

Proof. The sth power of matrix A is

$$A^s = \begin{bmatrix} \lambda^s & s\lambda^{s-1} & \frac{s(s-1)}{2!}\lambda^{s-2} & \frac{s(s-1)(s-2)}{3!}\lambda^{s-3} & \cdots & \frac{s(s-1)(s-2)\ldots(s-r+1)}{(r-1)!}\lambda^{s-r+1} \\ 0 & \lambda^s & s\lambda^{s-1} & \frac{s(s-1)}{2!}\lambda^{s-2} & \cdots & \vdots \\ 0 & 0 & \lambda^s & s\lambda^{s-1} & \cdots & \frac{s(s-1)(s-2)}{3!}\lambda^{s-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^s & s\lambda^{s-1} \\ 0 & 0 & 0 & \cdots & 0 & \lambda^s \end{bmatrix}$$

and the Golden Taylor series expansion of the 1st type Golden matrix exponential function

$$e_F^{mA} = \frac{I_r}{F_0!} + \frac{mA}{F_1!} + \frac{(mA)^2}{F_2!} + \frac{(mA)^3}{F_3!} + \frac{(mA)^4}{F_4!} + \cdots$$

This completes the proof. □

3. Conclusion

In the present paper, we have computed the values of the 1st and 2nd type Golden matrix exponential functions for some special matrices. We have presented a comparative analysis of these values with the well-known matrix exponential function value for the same special matrices. We believe that a good understanding of these functions will enable the development of new alternative approaches to solving differential equations and optimization problems in various sciences and engineering. For future research, the solution of the linear Golden 1st order autonomous system $D_F^t f = Af$, where D_F is the Golden time derivative, can be written as Golden matrix exponential $f = e_F(At) f_0$, where A is an rth order matrix. Additionally, by selecting different forms for the matrix A, various dynamical systems with Golden evolution can be defined.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Hyper-Dual Leonardo Quaternions

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Abstract — In this paper, hyper-dual Leonardo quaternions are defined and studied. Some basic properties of the hyper-dual Leonardo quaternions, including their relationships with the hyper-dual Fibonacci quaternions and hyper-dual Lucas quaternions, are analyzed. In addition, some formulae and identities, such as the recurrence relations, Binet's formula, generating functions, Vajda's identity, certain sum formulae, and some binomial-sum formulae, are investigated for hyper-dual Leonardo quaternions.

Keywords *Dual numbers, hyper-dual numbers, quaternions, Fibonacci numbers, Leonardo numbers*

Mathematics Subject Classification (2020) 11B39, 11R52

1. Introduction

Dual numbers invented in 1873 by Clifford [1] are an extension of real numbers. Hyper-dual numbers are an extension of dual numbers. Fike and Alonso [2] introduced hyper-dual numbers to demonstrate the advantages of hyper-dual numbers in second-order numerical differentiation. Dual and hyper-dual numbers have become a useful tool in mathematics and engineering. For further information about the applications of dual and hyper-dual numbers, see [3–13]. Quaternions discovered by Hamilton [14] are a 4-dimensional hyper-complex number system. Cohen and Shoham [9] defined hyper-dual quaternions by replacing each real number in a quaternion with the associated hyper-dual number.

Integer sequences are an important field of study in mathematics. The Fibonacci sequence is one of the most well-known examples of special integer sequences. This sequence is widely used in many scientific fields, including mathematics, physics, engineering, and art. Another well-known sequence is the Lucas sequence, closely related to the Fibonacci sequence. Many authors have investigated the Fibonacci and Lucas sequences in [15–17], among others. Another integer sequence studied intensively by researchers in recent years and closely related to the Fibonacci sequence is the Leonardo sequence. Some properties of this sequence have been investigated in [18, 19]. Several authors have investigated the properties of hyper-complex numbers with distinct integer sequences from various perspectives. Some examples of recent studies on quaternions and hyper-dual numbers with the Fibonacci, Lucas, and Leonardo sequences can be found in [20–25].

This paper aims to define the hyper-dual Leonardo quaternions by considering the concepts of hyper-dual numbers, quaternions, and Leonardo numbers and to investigate some of their algebraic and combinatorial properties.

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2. Preliminaries

This section provides some basic notions to provide a background for the next section.

Definition 2.1. [1] Let a and b be arbitrary real numbers. Then, a dual number x has the form

$$x = a + b\varepsilon$$

where ε is the dual unit that satisfies the rules $\varepsilon^2 = 0$ and $\varepsilon \neq 0$.

Definition 2.2. [2] Let x_1 and x_2 be any dual numbers and ε be the dual unit. Then, a hyper-dual number z is represented as follows:

$$z = x_1 + x_2\varepsilon$$

Furthermore, it is easy to see that any hyper-dual number z can be characterized by

$$z = a_1 + a_2\varepsilon_1 + a_3\varepsilon_2 + a_4\varepsilon_1\varepsilon_2$$

where, for all $i \in \{1, 2, 3, 4\}$, a_i is a real number and ε_1 and ε_2 are the dual units that satisfy the rules

$$\varepsilon_1{}^2 = \varepsilon_2{}^2 = (\varepsilon_1\varepsilon_2)^2 = 0, \quad \varepsilon_1 \neq \varepsilon_2, \quad \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1, \quad \varepsilon_1 \neq 0, \quad \varepsilon_2 \neq 0, \quad \text{and} \quad \varepsilon_1\varepsilon_2 \neq 0 \tag{2.1}$$

Let $z_1 = a_1 + a_2\varepsilon_1 + a_3\varepsilon_2 + a_4\varepsilon_1\varepsilon_2$ and $z_2 = b_1 + b_2\varepsilon_1 + b_3\varepsilon_2 + b_4\varepsilon_1\varepsilon_2$ be any two hyper-dual numbers. Then, the addition, scalar multiplication (by a scalar λ), and multiplication of two hyper-dual numbers are defined as follows, respectively:

$$z_1 + z_2 = (a_1 + b_1) + (a_2 + b_2)\varepsilon_1 + (a_3 + b_3)\varepsilon_2 + (a_4 + b_4)\varepsilon_1\varepsilon_2$$

$$\lambda z_1 = \lambda a_1 + \lambda a_2\varepsilon_1 + \lambda a_3\varepsilon_2 + \lambda a_4\varepsilon_1\varepsilon_2$$

and

$$z_1z_2 = (a_1b_1) + (a_1b_2 + a_2b_1)\varepsilon_1 + (a_1b_3 + a_3b_1)\varepsilon_2 + (a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1)\varepsilon_1\varepsilon_2$$

The set of all the hyper-dual numbers forms a 4-dimensional, with the basis $\{1, \varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2\}$, commutative, and associative algebra over the real numbers. For detailed information about hyper-dual numbers, see [2].

Definition 2.3. [14] A quaternion q is of the form

$$q = q_1 + q_2i + q_3j + q_4k$$

where, for all $i \in \{1, 2, 3, 4\}$, q_i is a real number and i, j, and k are the quaternionic units that satisfy the multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = k = -ji, \quad jk = i = -kj, \quad \text{and} \quad ki = j = -ik \tag{2.2}$$

Let $p = p_1 + p_2i + p_3j + p_4k$ and $q = q_1 + q_2i + q_3j + q_4k$ be any two quaternions. Then, the addition, scalar (λ) multiplication, and multiplication of two quaternions are defined as follows, respectively:

$$p + q = (p_1 + q_1) + (p_2 + q_2)i + (p_3 + q_3)j + (p_4 + q_4)k$$

$$\lambda q = \lambda q_1 + \lambda q_2i + \lambda q_3j + \lambda q_4k$$

and

$$\begin{aligned} pq = {} & (p_1q_1 - p_2q_2 - p_3q_3 - p_4q_4) + (p_1q_2 + p_2q_1 + p_3q_4 - p_4q_3)i + (p_1q_3 + p_3q_1 + p_4q_2 - p_2q_4)j \\ & + (p_1q_4 + p_4q_1 + p_2q_3 - p_3q_2)k \end{aligned}$$

The set of all the quaternions forms a 4-dimensional, with the basis $\{1, i, j, k\}$, non-commutative, and associative algebra over the real numbers. For further quaternion information, see [14, 26].

Definition 2.4. [9] A hyper-dual quaternion Q is defined as

$$Q = z_1 + z_2 i + z_3 j + z_4 k$$

where, for all $i \in \{1, 2, 3, 4\}$, z_i is a hyper-dual number and i, j, and k are the quaternionic units defined as in (2.2).

Note that the dual units ε_1 and ε_2 commute with the quaternionic units i, j, and k, e.g., $\varepsilon_1 i = i\varepsilon_1$ [9]. In the rest of this section, we provide some definitions and identities of the sequences of Fibonacci, Lucas, and Leonardo numbers.

Definition 2.5. [15] For $n \geq 2$, the Fibonacci and Lucas numbers are defined by the recurrence relations, respectively:

$$F_n = F_{n-1} + F_{n-2} \quad \text{with} \quad F_0 = 0,\ F_1 = 1$$

and

$$L_n = L_{n-1} + L_{n-2} \quad \text{with} \quad L_0 = 2,\ L_1 = 1$$

Here, F_n and L_n denote the n-th Fibonacci and Lucas numbers, respectively.

Definition 2.6. [18] The Leonardo numbers are defined recursively by

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \quad n \geq 2$$

or

$$Le_n = 2Le_{n-1} - Le_{n-3}, \quad n \geq 3$$

with the initial conditions $Le_0 = Le_1 = 1$ and $Le_2 = 3$. Here, Le_n denotes the n-th Leonardo number.

Moreover, Ömür and Koparal [24] defined the hyper-dual generalized Fibonacci and Lucas numbers. In particular cases of the hyper-dual generalized Fibonacci and Lucas numbers, the hyper-dual Fibonacci and Lucas numbers can be derived as:

Definition 2.7. [24] The hyper-dual Fibonacci and hyper-dual Lucas numbers are defined as follows, respectively:

$$HDF_n = F_n + F_{n+1}\varepsilon_1 + F_{n+2}\varepsilon_2 + F_{n+3}\varepsilon_1\varepsilon_2 \tag{2.3}$$

and

$$HDL_n = L_n + L_{n+1}\varepsilon_1 + L_{n+2}\varepsilon_2 + L_{n+3}\varepsilon_1\varepsilon_2 \tag{2.4}$$

where ε_1 and ε_2 are the dual units defined as in (2.1).

Definition 2.8. [25] The hyper-dual Leonardo numbers are

$$HDLe_n = Le_n + Le_{n+1}\varepsilon_1 + Le_{n+2}\varepsilon_2 + Le_{n+3}\varepsilon_1\varepsilon_2 \tag{2.5}$$

where ε_1 and ε_2 are the dual units in (2.1).

Moreover, the recurrence relation of the hyper-dual Leonardo numbers is provided by

$$HDLe_n = HDLe_{n-1} + HDLe_{n-2} + A, \quad n \geq 2 \tag{2.6}$$

or

$$HDLe_n = 2HDLe_{n-1} - HDLe_{n-3}, \quad n \geq 3 \tag{2.7}$$

Here, $A := 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2$ [25].

Definition 2.9. [20] The Fibonacci and Lucas quaternions are defined as follows, respectively:

$$QF_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$$

and

$$QL_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k$$

where i, j, and k are the quaternionic units in (2.2).

Definition 2.10. [23] The Leonardo quaternions are defined by

$$QLe_n = Le_n + Le_{n+1}i + Le_{n+2}j + Le_{n+3}k \tag{2.8}$$

where i, j, and k are the quaternionic units in (2.2).

Binet's formula for QLe_n is

$$QLe_n = 2\frac{\alpha^{n+1}\hat{\alpha} - \beta^{n+1}\hat{\beta}}{\alpha - \beta} - q_u \tag{2.9}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $\hat{\alpha} := 1 + \alpha i + (1+\alpha)j + (1+2\alpha)k$, $\hat{\beta} := 1 + \beta i + (1+\beta)j + (1+2\beta)k$, and $q_u := 1 + i + j + k$ [23]. Then, the following properties hold [23]:

$$QLe_n = 2QF_{n+1} - q_u \tag{2.10}$$

$$QLe_{n+1} - QLe_n = 2QF_n \tag{2.11}$$

$$QLe_{n+2} = QLe_{n+1} + QLe_n + q_u \tag{2.12}$$

$$\sum_{k=1}^{n} QLe_k = QLe_{n+2} - QLe_2 - nq_u \tag{2.13}$$

$$\sum_{k=1}^{n} QLe_{2k-1} = QLe_{2n} - QLe_0 - nq_u \tag{2.14}$$

and

$$\sum_{k=1}^{n} QLe_{2k} = QLe_{2n+1} - QLe_1 - nq_u \tag{2.15}$$

Here, QF_n is the n-th Fibonacci quaternion and QLe_n is the n-th Leonardo quaternion.

Ait-Amrane et al. [27] defined the hyper-dual Horadam quaternions from two perspectives. In the particular case of the hyper-dual Horadam quaternions, the hyper-dual Fibonacci and Lucas quaternions can be derived as follows:

Definition 2.11. [27] The hyper-dual Fibonacci and Lucas quaternions are defined by

$$QHDF_n = HDF_n + HDF_{n+1}i + HDF_{n+2}j + HDF_{n+3}k$$

and

$$QHDL_n = HDL_n + HDL_{n+1}i + HDL_{n+2}j + HDL_{n+3}k$$

respectively, where HDF_n is the n-th hyper-dual Fibonacci number, HDL_n is the n-th hyper-dual Lucas number, and i, j, and k are the quaternionic units in (2.2).

In addition, the hyper-dual Fibonacci and Lucas quaternions can be defined as:

Definition 2.12. [27] The hyper-dual Fibonacci and Lucas quaternions are defined by

$$QHDF_n = QF_n + QF_{n+1}\varepsilon_1 + QF_{n+2}\varepsilon_2 + QF_{n+3}\varepsilon_1\varepsilon_2$$

and

$$QHDL_n = QL_n + QL_{n+1}\varepsilon_1 + QL_{n+2}\varepsilon_2 + QL_{n+3}\varepsilon_1\varepsilon_2$$

respectively, where ε_1 and ε_2 are the dual units in (2.1).

3. Main Results

This section begins with defining the general term of the hyper-dual Leonardo quaternions.

Definition 3.1. For $n \geq 0$, the n-th hyper-dual Leonardo quaternion is

$$QHDLe_n = HDLe_n + HDLe_{n+1}i + HDLe_{n+2}j + HDLe_{n+3}k \tag{3.1}$$

where $HDLe_n$ is the n-th hyper-dual Leonardo number and i, j, and k are the quaternionic units in (2.2).

Moreover, considering (2.5) and (2.8), we can obtain

$$\begin{aligned} QHDLe_n &= HDLe_n + HDLe_{n+1}i + HDLe_{n+2}j + HDLe_{n+3}k \\ &= (Le_n + Le_{n+1}\varepsilon_1 + Le_{n+2}\varepsilon_2 + Le_{n+3}\varepsilon_1\varepsilon_2) + (Le_{n+1} + Le_{n+2}\varepsilon_1 + Le_{n+3}\varepsilon_2 + Le_{n+4}\varepsilon_1\varepsilon_2)i \\ &\quad +(Le_{n+2} + Le_{n+3}\varepsilon_1 + Le_{n+4}\varepsilon_2 + Le_{n+5}\varepsilon_1\varepsilon_2)j + (Le_{n+3} + Le_{n+4}\varepsilon_1 + Le_{n+5}\varepsilon_2 + Le_{n+6}\varepsilon_1\varepsilon_2)k \\ &= (Le_n + Le_{n+1}i + Le_{n+2}j + Le_{n+3}k) + (Le_{n+1} + Le_{n+2}i + Le_{n+3}j + Le_{n+4}k)\varepsilon_1 \\ &\quad +(Le_{n+2} + Le_{n+3}i + Le_{n+4}j + Le_{n+5}k)\varepsilon_2 + (Le_{n+3} + Le_{n+4}i + Le_{n+5}j + Le_{n+6}k)\varepsilon_1\varepsilon_2 \\ &= QLe_n + QLe_{n+1}\varepsilon_1 + QLe_{n+2}\varepsilon_2 + QLe_{n+3}\varepsilon_1\varepsilon_2 \end{aligned}$$

Therefore, the general term of the hyper-dual Leonardo quaternions can be reidentified in the following.

Definition 3.2. For $n \geq 0$, the n-th hyper-dual Leonardo quaternion is

$$QHDLe_n = QLe_n + QLe_{n+1}\varepsilon_1 + QLe_{n+2}\varepsilon_2 + QLe_{n+3}\varepsilon_1\varepsilon_2 \tag{3.2}$$

where QLe_n is the n-th Leonardo quaternion and ε_1 and ε_2 are the dual units in (2.1).

The first three hyper-dual Leonardo quaternions are as follows:

$$\begin{aligned} QHDLe_0 &= (1 + i + 3j + 5k) + (1 + 3i + 5j + 9k)\varepsilon_1 + (3 + 5i + 9j + 15k)\varepsilon_2 \\ &\quad +(5 + 9i + 15j + 25k)\varepsilon_1\varepsilon_2 \end{aligned}$$

$$\begin{aligned} QHDLe_1 &= (1 + 3i + 5j + 9k) + (3 + 5i + 9j + 15k)\varepsilon_1 + (5 + 9i + 15j + 25k)\varepsilon_2 \\ &\quad +(9 + 15i + 25j + 41k)\varepsilon_1\varepsilon_2 \end{aligned}$$

and

$$\begin{aligned} QHDLe_2 &= (3 + 5i + 9j + 15k) + (5 + 9i + 15j + 25k)\varepsilon_1 + (9 + 15i + 25j + 41k)\varepsilon_2 \\ &\quad +(15 + 25i + 41j + 67k)\varepsilon_1\varepsilon_2 \end{aligned}$$

Throughout this paper, let $A := 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2$, $q_u := 1 + i + j + k$, and $\Delta := Aq_u = q_uA$.

By (2.6) and (3.1), the following recurrence relation of the hyper-dual Leonardo quaternions is obtained:

$$QHDLe_n = QHDLe_{n-1} + QHDLe_{n-2} + \Delta, \quad n \geq 2 \tag{3.3}$$

Moreover, by (2.7) and (3.1), the alternative recurrence relation of the hyper-dual Leonardo quaternions is obtained:

$$QHDLe_n = 2QHDLe_{n-1} - QHDLe_{n-3}, \quad n \geq 3 \tag{3.4}$$

Theorem 3.3. For $n \geq 0$,

i. $QHDLe_n - QHDLe_{n+1}i - QHDLe_{n+2}j - QHDLe_{n+3}k = 3(HDLe_{n+4} + HDLe_{n+2}) + 2A$

ii. $QHDLe_n - QHDLe_{n+1}\varepsilon_1 - QHDLe_{n+2}\varepsilon_2 - QHDLe_{n+3}\varepsilon_1\varepsilon_2 = QLe_n - 2QLe_{n+3}\varepsilon_1\varepsilon_2$

Proof. Let $n \geq 0$.

i. Using (3.1) to the left-hand side (LHS),

$$\begin{aligned} LHS = & HDLe_n + HDLe_{n+1}i + HDLe_{n+2}j + HDLe_{n+3}k \\ & - (HDLe_{n+1} + HDLe_{n+2}i + HDLe_{n+3}j + HDLe_{n+4}k)i \\ & - (HDLe_{n+2} + HDLe_{n+3}i + HDLe_{n+4}j + HDLe_{n+5}k)j \\ & - (HDLe_{n+3} + HDLe_{n+4}i + HDLe_{n+5}j + HDLe_{n+6}k)k \end{aligned}$$

From the multiplication rules of the quaternionic units in (2.2),

$$LHS = HDLe_n + HDLe_{n+2} + HDLe_{n+4} + HDLe_{n+6}$$

Using (2.6),

$$LHS = 3HDLe_{n+4} + 3HDLe_{n+2} + 2A$$

ii. Using (3.2) to the left-hand side (LHS),

$$\begin{aligned} LHS = & QLe_n + QLe_{n+1}\varepsilon_1 + QLe_{n+2}\varepsilon_2 + QLe_{n+3}\varepsilon_1\varepsilon_2 \\ & - (QLe_{n+1} + QLe_{n+2}\varepsilon_1 + QLe_{n+3}\varepsilon_2 + QLe_{n+4}\varepsilon_1\varepsilon_2)\varepsilon_1 \\ & - (QLe_{n+2} + QLe_{n+3}\varepsilon_1 + QLe_{n+4}\varepsilon_2 + QLe_{n+5}\varepsilon_1\varepsilon_2)\varepsilon_2 \\ & - (QLe_{n+3} + QLe_{n+4}\varepsilon_1 + QLe_{n+5}\varepsilon_2 + QLe_{n+6}\varepsilon_1\varepsilon_2)\varepsilon_1\varepsilon_2 \end{aligned}$$

Considering the multiplication rules of the dual units in (2.1),

$$LHS = QLe_n - 2QLe_{n+3}\varepsilon_1\varepsilon_2$$

□

Lemma 3.4. For positive integer n, the followings hold:

i. $HDLe_{n-1} + HDLe_{n+1} = 2HDL_{n+1} - 2A$ [25]

ii. $HDLe_n + HDF_n + HDL_n = 2HDLe_n + A$

where $HDLe_n$, HDF_n, and HDL_n are the n-th hyper-dual Leonardo, hyper-dual Fibonacci, and hyper-dual Lucas numbers, respectively.

Proof. *ii.* From (2.3)-(2.5) and the relation $Le_n + F_n + L_n = 2Le_n + 1$ provided in [19], the proof is clear. □

Theorem 3.5. For $n \geq 0$, the followings hold:

i. $QHDLe_{n-1} + QHDLe_{n+1} = 2QHDL_{n+1} - 2\Delta$

ii. $QHDLe_n + QHDF_n + QHDL_n = 2QHDLe_n + \Delta$

iii. $QHDLe_n = 2QHDF_{n+1} - \Delta$

iv. $QHDLe_{n+1} - QHDLe_n = 2QHDF_n$

where $QHDF_n$ and $QHDL_n$ are the n-th hyper-dual Fibonacci and hyper-dual Lucas quaternions, respectively.

Proof. From (2.10), (2.11), (3.1), and (3.2) and Lemma 3.4, the proofs of *i.*, *ii.*, *iii.*, and *iv.* are obvious. □

Theorem 3.6. For $n \geq 0$, Binet's formula of the hyper-dual Leonardo quaternions is

$$QHDLe_n = 2\left(\frac{\alpha^*\alpha^{n+1} - \beta^*\beta^{n+1}}{\alpha - \beta}\right) - \Delta \tag{3.5}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$,

$$\alpha^* := (1 + \alpha i + (1+\alpha)j + (1+2\alpha)k)(1 + \alpha\varepsilon_1 + (1+\alpha)\varepsilon_2 + (1+2\alpha)\varepsilon_1\varepsilon_2)$$

and

$$\beta^* := (1 + \beta i + (1+\beta)j + (1+2\beta)k)(1 + \beta\varepsilon_1 + (1+\beta)\varepsilon_2 + (1+2\beta)\varepsilon_1\varepsilon_2)$$

Proof. From (2.9) and (3.2) and the equalities $1+\alpha = \alpha^2$, $1+2\alpha = \alpha^3$, $1+\beta = \beta^2$, and $1+2\beta = \beta^3$,

$$\begin{aligned}
QHDLe_n &= QLe_n + QLe_{n+1}\varepsilon_1 + QLe_{n+2}\varepsilon_2 + QLe_{n+3}\varepsilon_1\varepsilon_2 \\
&= \left(2\frac{\alpha^{n+1}\hat{\alpha} - \beta^{n+1}\hat{\beta}}{\alpha - \beta} - q_u\right) + \left(2\frac{\alpha^{n+2}\hat{\alpha} - \beta^{n+2}\hat{\beta}}{\alpha - \beta} - q_u\right)\varepsilon_1 \\
&\quad + \left(2\frac{\alpha^{n+3}\hat{\alpha} - \beta^{n+3}\hat{\beta}}{\alpha - \beta} - q_u\right)\varepsilon_2 + \left(2\frac{\alpha^{n+4}\hat{\alpha} - \beta^{n+4}\hat{\beta}}{\alpha - \beta} - q_u\right)\varepsilon_1\varepsilon_2 \\
&= 2\frac{\alpha^{n+1}\hat{\alpha}}{\alpha - \beta}(1 + \alpha\varepsilon_1 + \alpha^2\varepsilon_2 + \alpha^3\varepsilon_1\varepsilon_2) - 2\frac{\beta^{n+1}\hat{\beta}}{\alpha - \beta}(1 + \beta\varepsilon_1 + \beta^2\varepsilon_2 + \beta^3\varepsilon_1\varepsilon_2) \\
&\quad - q_u(1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2) \\
&= 2\frac{\alpha^*\alpha^{n+1}}{\alpha - \beta} - 2\frac{\beta^*\beta^{n+1}}{\alpha - \beta} - \Delta
\end{aligned}$$

□

Theorem 3.7. The ordinary generating function for the hyper-dual Leonardo quaternions is

$$g(x) = \frac{QHDLe_0 + (QHDLe_1 - 2QHDLe_0)x + (QHDLe_2 - 2QHDLe_1)x^2}{1 - 2x + x^3}$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} QHDLe_n x^n$$

be the ordinary generating function for the hyper-dual Leonardo quaternions. Then, from (3.4),

$$\begin{aligned}
g(x) &= QHDLe_0 + QHDLe_1 x + QHDLe_2 x^2 + \sum_{n=3}^{\infty} QHDLe_n x^n \\
&= QHDLe_0 + QHDLe_1 x + QHDLe_2 x^2 + \sum_{n=3}^{\infty}(2QHDLe_{n-1} - QHDLe_{n-3})x^n \\
&= QHDLe_0 + QHDLe_1 x + QHDLe_2 x^2 + 2x\sum_{n=3}^{\infty} QHDLe_{n-1}x^{n-1} - x^3\sum_{n=3}^{\infty} QHDLe_{n-3}x^{n-3} \\
&= QHDLe_0 + QHDLe_1 x + QHDLe_2 x^2 - 2x(QHDLe_0 + QHDLe_1 x) + 2x\sum_{n=0}^{\infty} QHDLe_n x^n \\
&\quad - x^3\sum_{n=0}^{\infty} QHDLe_n x^n \\
&= QHDLe_0 + (QHDLe_1 - 2QHDLe_0)x + (QHDLe_2 - 2QHDLe_1)x^2 + 2xg(x) - x^3 g(x)
\end{aligned}$$

Hence,

$$g(x)(1-2x+x^3) = QHDLe_0 + (QHDLe_1 - 2QHDLe_0)x + (QHDLe_2 - 2QHDLe_1)x^2$$

□

Theorem 3.8. The exponential generating function for the hyper-dual Leonardo quaternions is

$$eg(x) = \sum_{n=0}^{\infty} QHDLe_n \frac{x^n}{n!} = 2\frac{\alpha^*\alpha}{\alpha-\beta}e^{\alpha x} - 2\frac{\beta^*\beta}{\alpha-\beta}e^{\beta x} - \Delta e^x$$

where α^* and β^* are defined as in Theorem 3.6.

Proof. From (3.5), we obtain

$$\begin{aligned} eg(x) &= \sum_{n=0}^{\infty} QHDLe_n \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(2\left(\frac{\alpha^*\alpha^{n+1} - \beta^*\beta^{n+1}}{\alpha-\beta}\right) - \Delta\right)\frac{x^n}{n!} \\ &= 2\frac{\alpha^*\alpha}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\alpha x)^n}{n!} - 2\frac{\beta^*\beta}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\beta x)^n}{n!} - \Delta\sum_{n=0}^{\infty}\frac{x^n}{n!} \\ &= 2\frac{\alpha^*\alpha}{\alpha-\beta}e^{\alpha x} - 2\frac{\beta^*\beta}{\alpha-\beta}e^{\beta x} - \Delta e^x \end{aligned}$$

□

Corollary 3.9. The Poisson generating function for the hyper-dual Leonardo quaternions is

$$pg(x) = 2\frac{\alpha^*\alpha}{\alpha-\beta}e^{(\alpha-1)x} - 2\frac{\beta^*\beta}{\alpha-\beta}e^{(\beta-1)x} - \Delta$$

Proof. Since $pg(x) = eg(x)e^{-x}$, the proof is straghtforward. □

Theorem 3.10. For $n \geq 1$, the followings hold:

i. $\sum\limits_{k=1}^{n} QHDLe_k = QHDLe_{n+2} - QHDLe_2 - n\Delta$

ii. $\sum\limits_{k=1}^{n} QHDLe_{2k-1} = QHDLe_{2n} - QHDLe_0 - n\Delta$

iii. $\sum\limits_{k=1}^{n} QHDLe_{2k} = QHDLe_{2n+1} - QHDLe_1 - n\Delta$

Proof. *i.* From (2.13) and (3.2),

$$\begin{aligned} \sum_{k=1}^{n} QHDLe_k &= \sum_{k=1}^{n}(QLe_k + QLe_{k+1}\varepsilon_1 + QLe_{k+2}\varepsilon_2 + QLe_{k+3}\varepsilon_1\varepsilon_2) \\ &= \left(\sum_{k=1}^{n} QLe_k\right) + \left(\sum_{k=1}^{n} QLe_{k+1}\right)\varepsilon_1 + \left(\sum_{k=1}^{n} QLe_{k+2}\right)\varepsilon_2 + \left(\sum_{k=1}^{n} QLe_{k+3}\right)\varepsilon_1\varepsilon_2 \\ &= (QLe_{n+2} - QLe_2 - nq_u) + (QLe_{n+2} + QLe_{n+1} - QLe_2 - QLe_1 - nq_u)\,\varepsilon_1 \\ &\quad + (2QLe_{n+2} + QLe_{n+1} - 2QLe_2 - QLe_1 - nq_u)\,\varepsilon_2 \\ &\quad + (QLe_{n+3} + 2QLe_{n+2} + QLe_{n+1} - QLe_3 - 2QLe_2 - QLe_1 - nq_u)\,\varepsilon_1\varepsilon_2 \end{aligned}$$

Then, considering (2.12),

$$\sum_{k=1}^{n} QHDLe_k = (QLe_{n+2} - QLe_2 - nq_u) + (QLe_{n+3} - QLe_3 - nq_u)\,\varepsilon_1 + (QLe_{n+4} - QLe_4 - nq_u)\,\varepsilon_2 + (QLe_{n+5} - QLe_5 - nq_u)\,\varepsilon_1\varepsilon_2$$

Then, it follows that

$$\begin{aligned}\sum_{k=1}^{n} QHDLe_k &= (QLe_{n+2} + QLe_{n+3}\varepsilon_1 + QLe_{n+4}\varepsilon_2 + QLe_{n+5}\varepsilon_1\varepsilon_2) \\ &\quad - (QLe_2 + QLe_3\varepsilon_1 + QLe_4\varepsilon_2 + QLe_5\varepsilon_1\varepsilon_2) - nq_u(1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2) \\ &= QHDLe_{n+2} - QHDLe_2 - n\Delta\end{aligned}$$

This completes the proof of *i.* In a similar manner, *ii.* and *iii.* can be proved by using (2.14) and (2.15). □

Theorem 3.11. For $n \geq 0$, the followings hold:

i. $QHDLe_{2n} = \sum_{k=0}^{n} \binom{n}{k}(QHDLe_k + \Delta) - \Delta$

ii. $QHDLe_{2n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k}(QHDLe_{k-1} + \Delta) - \Delta$

Proof. *i.* From (3.5),

$$\begin{aligned}QHDLe_{2n} &= 2\left(\frac{\alpha^*\alpha^{2n+1} - \beta^*\beta^{2n+1}}{\alpha - \beta}\right) - \Delta \\ &= 2\left(\frac{\alpha^*\alpha(\alpha^2)^n - \beta^*\beta(\beta^2)^n}{\alpha - \beta}\right) - \Delta \\ &= 2\left(\frac{\alpha^*\alpha(1+\alpha)^n - \beta^*\beta(1+\beta)^n}{\alpha - \beta}\right) - \Delta\end{aligned}$$

Since $(1+\alpha)^n = \sum_{k=0}^{n} \binom{n}{k}\alpha^k$ and $(1+\beta)^n = \sum_{k=0}^{n} \binom{n}{k}\beta^k$, then

$$\begin{aligned}QHDLe_{2n} &= 2\left(\frac{\alpha^*\alpha}{\alpha - \beta}\sum_{k=0}^{n}\binom{n}{k}\alpha^k - \frac{\beta^*\beta}{\alpha - \beta}\sum_{k=0}^{n}\binom{n}{k}\beta^k\right) - \Delta \\ &= 2\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\alpha^*\alpha^{k+1} - \beta^*\beta^{k+1}}{\alpha - \beta}\right) - \Delta \\ &= \sum_{k=0}^{n}\binom{n}{k}\left(2\frac{\alpha^*\alpha^{k+1} - \beta^*\beta^{k+1}}{\alpha - \beta} - \Delta\right) + \sum_{k=0}^{n}\binom{n}{k}\Delta - \Delta \\ &= \sum_{k=0}^{n}\binom{n}{k}(QHDLe_k + \Delta) - \Delta\end{aligned}$$

ii. The proof is similar to the proof of *i.* □

Theorem 3.12. (Vajda's Identity) For non-negative integers n, r, and s,

$$\begin{aligned}QHDLe_{n+r}QHDLe_{n+s} - QHDLe_nQHDLe_{n+r+s} &= \frac{4}{\sqrt{5}}(-1)^{n+1}(\beta^*\alpha^*\alpha^s - \alpha^*\beta^*\beta^s)F_r \\ &\quad + \Delta(QHDLe_n + QHDLe_{n+r+s}) \\ &\quad - \Delta(QHDLe_{n+r} + QHDLe_{n+s})\end{aligned}$$

where F_r is the r-th Fibonacci number.

PROOF. Applying (3.5) to the left-hand side (LHS),

$$
\begin{aligned}
LHS &= \left(2\left(\frac{\alpha^*\alpha^{n+r+1}-\beta^*\beta^{n+r+1}}{\alpha-\beta}\right)-\Delta\right)\left(2\left(\frac{\alpha^*\alpha^{n+s+1}-\beta^*\beta^{n+s+1}}{\alpha-\beta}\right)-\Delta\right)\\
&\quad-\left(2\left(\frac{\alpha^*\alpha^{n+1}-\beta^*\beta^{n+1}}{\alpha-\beta}\right)-\Delta\right)\left(2\left(\frac{\alpha^*\alpha^{n+r+s+1}-\beta^*\beta^{n+r+s+1}}{\alpha-\beta}\right)-\Delta\right)\\
&= 4\left(\frac{(\alpha\beta)^{n+1}(\alpha^r-\beta^r)(\beta^*\alpha^*\alpha^s-\alpha^*\beta^*\beta^s)}{(\alpha-\beta)^2}\right)\\
&\quad-\Delta\left(QHDLe_{n+r}+QHDLe_{n+s}-QHDLe_n-QHDLe_{n+r+s}\right)\\
&= \frac{4}{\sqrt{5}}(-1)^{n+1}(\beta^*\alpha^*\alpha^s-\alpha^*\beta^*\beta^s)F_r\\
&\quad+\Delta(QHDLe_n+QHDLe_{n+r+s}-QHDLe_{n+r}-QHDLe_{n+s})
\end{aligned}
$$

Here, $F_r = \frac{\alpha^r-\beta^r}{\alpha-\beta}$ [15]. □

In the particular case of Theorem 3.12, we have the following results:

Corollary 3.13. (Catalan's Identity) For non-negative integers n and s such that $n \geq s$,

$$
\begin{aligned}
QHDLe_{n-s}QHDLe_{n+s}-(QHDLe_n)^2 &= \frac{4}{\sqrt{5}}(-1)^{n+s}(\beta^*\alpha^*\alpha^s-\alpha^*\beta^*\beta^s)F_s\\
&\quad+\Delta(2QHDLe_n-QHDLe_{n-s}-QHDLe_{n+s})
\end{aligned}
$$

PROOF. Taking $r \to -s$ in Theorem 3.12 and considering the relation $F_{-r} = (-1)^{r+1}F_r$ [15], the proof is obvious. □

Corollary 3.14. (Cassini's Identity) For positive integer n,

$$
\begin{aligned}
QHDLe_{n-1}QHDLe_{n+1}-(QHDLe_n)^2 &= \frac{4}{\sqrt{5}}(-1)^{n+1}(\beta^*\alpha^*\alpha-\alpha^*\beta^*\beta)\\
&\quad+\Delta(QHDLe_{n-2}-QHDLe_{n-1})
\end{aligned}
$$

PROOF. Taking $r \to -s$ and $s = 1$ in Theorem 3.12 and using (3.3), the proof is clear. □

Corollary 3.15. (d'Ocagne's Identity) For positive integers n and m,

$$
\begin{aligned}
QHDLe_{n+1}QHDLe_m-QHDLe_nQHDLe_{m+1} &= \frac{4}{\sqrt{5}}(-1)^{n+1}(\beta^*\alpha^*\alpha^{m-n}-\alpha^*\beta^*\beta^{m-n})\\
&\quad+\Delta(QHDLe_{m-1}-QHDLe_{n-1})
\end{aligned}
$$

PROOF. Taking $s \to m-n$ and $r = 1$ in Theorem 3.12 and using (3.3), the proof is clear. □

4. Conclusion

In this study, the hyper-dual Leonardo quaternions have been proposed from two different perspectives. At first, the hyper-dual quaternions have been defined using the hyper-dual Leonardo numbers as coefficients in quaternions. Then, as equivalent to this first definition, the hyper-dual Leonardo quaternions have been defined using the Leonardo quaternions as coefficients in hyper-dual numbers. Some of their properties, such as non-homogeneous and homogeneous recurrence relations, Binet's formula, certain sum formulae, and binomial-sum formulae, have been provided. The ordinary, exponential, and Poisson-generating functions, Vajda's identity, and, in particular cases, Catalan's, Cassini's, and d'Ocagne's identities of the hyper-dual Leonardo quaternions have been presented. For future studies, researchers may define hyper-dual split quaternions provided in [10] with the Leonardo number coefficients.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Chen-Type Inequality for Generic Submanifolds of Quaternionic Space Form and Its Application

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Abstract — In 1993, the theory of Chen invariants started when Chen wrote basic inequalities for submanifolds in space forms. This inequality is called Chen's first inequality. Afterward, many geometers studied many papers dealing with this new inequality. The present paper aims to establish a Chen inequality for quaternionic generic submanifolds in a quaternionic space form and obtain this inequality for real hypersurfaces.

Keywords *Chen-type inequality, generic submanifold, quaternionic space form, real hypersurfaces*

Mathematics Subject Classification (2020) 53B20, 53C40

1. Introduction

One of the most interesting topics in differential geometry is the submanifolds of the almost Hermitian manifold. We note that the Kaehler manifold's submanifolds are determined by its tangent space behavior under the action of a complex structure J. One of the classes of submanifolds of Kaehler manifolds is holomorphic submanifolds and the other is total real submanifolds. In the first case, the tangent bundle of the submanifold is invariant under J where as in the second case, the normal bundle of the submanifold is invariant under J. CR-submanifolds were introduced by Bejancu in [1] as a natural generalization of invariant submanifolds and anti-invariant submanifolds. Chen investigated the first detailed research on this subject in [2]. Moreover, the topology of CR-submanifolds was widely studied [3–7]. The authors defined the quaternion CR-submanifolds in quaternion Kaehler manifolds [8], and were followed by several geometers [9–18]. Generic submanifold was defined as a generalization of the concept of CR-submanifold [19]. These submanifolds are known by relaxing the condition on the complementary distribution of holomorphic distribution. More precisely, if the maximal complex subspaces $D_p = T_pM \cap J(T_pM)$ determine on M a distribution $D : D_p \subseteq T_pM$, the M is called a generic submanifold of $\bar{M}$. Generic submanifolds have been commonly studied [20–26].

The present article is organized as follows: Section 2 recalls basic notions and results of quaternion Kaehler manifolds. New optimal inequalities were introduced in [27] for anti-holomorphic submanifolds in complex space forms. In recent years, this new inequality has been obtained by distinct researchers for different classes of submanifolds in different ambient manifolds [28, 29]. Section 3 establishes new inequality for quaternionic generic submanifolds in a quaternionic space form and gives some results. The last part of this paper obtains this inequality for real hypersurfaces.

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2. Preliminaries

In this section, some basic concepts will be given from [1,30] for the following sections. A Riemannian manifold $(\bar{M}, \tilde{g})$ of dimension $4m$, for $m \geq 1$, is called quaternion Kaehler manifold with 3-dimensional vector bundle σ of local basis of almost Hermitian structures J_1, J_2, and J_3 if the following conditions are satisfied

$$J_1 \circ J_2 = -J_2 \circ J_1 = J_3$$

and

$$\bar{\nabla}_X J_m = \sum_{b=1}^{3} A_{ml}(X) J_l, \quad m \in \{1, 2, 3\}, \quad \forall X \in \left(T\bar{M}\right)$$

where A_{ml} are certain local 1-forms on $\bar{M}$ such that $A_{ml} + A_{lm} = 0$. For a Riemann submanifold $N \subset \bar{M}$ of a Riemannian manifold $\bar{M}$, Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_W Z = \nabla_W Z + h(W, Z)$$

and

$$\bar{\nabla}_W V = -A_V W + \nabla^{\perp}_W V \tag{2.1}$$

for all $W, Z \in TN$ and $V \in TN^{\perp}$, where ∇ and $\bar{\nabla}$ are the Levi-Civita connections of N and $\bar{M}$, respectively. Moreover, h and $\nabla^{\perp}_X$ denote the second fundamental form of N, and the normal connection on the normal bundle, respectively. From (2.1), A_{ξ} the second fundamental tensor and h the second fundamental form are related by

$$\tilde{g}(h(W, Z), \xi) = \tilde{g}(A_{\xi} W, Z)$$

If quaternionic sectional curvature of a quaternionic Kaehler manifold is constant, then it is called a quaternionic space form and denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by

$$\bar{R}(\mathcal{P}, \mathcal{Q})\mathcal{R} = \frac{c}{4}\left\{\tilde{g}(\mathcal{Q}, \mathcal{R})\mathcal{P} - \tilde{g}(\mathcal{P}, \mathcal{R})\mathcal{Q} + \sum_{a=1}^{3} \tilde{g}(\mathcal{R}, J_a\mathcal{Q})J_a\mathcal{P} - \tilde{g}(\mathcal{R}, J_a\mathcal{P})J_a\mathcal{Q} + 2\tilde{g}(\mathcal{P}, J_a\mathcal{Q})J_a\mathcal{R}\right\}$$

for all $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \Gamma(TN)$ [1]. For the second fundamental form h, the covariant derivation $(\nabla_{\mathcal{P}} h)(\mathcal{Q}, \mathcal{R})$ is as follows:

$$(\nabla_{\mathcal{P}} h)(\mathcal{Q}, \mathcal{R}) = \nabla^{\perp}_{\mathcal{P}} h(\mathcal{Q}, \mathcal{R}) - h(\nabla_{\mathcal{P}}\mathcal{Q}, \mathcal{R}) - h(\mathcal{Q}, \nabla_{\mathcal{P}}\mathcal{R})$$

for any $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \Gamma(TN)$. For the submanifold N, the Gauss, Codazzi, and Ricci equations of N are provided as follows, respectively:

$$R(\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{W}) = \bar{R}(\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{W}) + \tilde{g}(h(\mathcal{P}, \mathcal{W}), h(\mathcal{Q}, \mathcal{R})) - \tilde{g}(h(\mathcal{P}, \mathcal{R}), h(\mathcal{Q}, \mathcal{W})) \tag{2.2}$$

$$\left(\bar{R}(\mathcal{P}, \mathcal{Q})\mathcal{R}\right)^{\perp} = (\nabla_{\mathcal{P}} h)(\mathcal{Q})\mathcal{R}) - (\nabla_{\mathcal{Q}} h)(\mathcal{P}, \mathcal{R})$$

and

$$\bar{R}(\mathcal{P}, \mathcal{Q}, \xi, \eta) = R^{\perp}(\mathcal{P}, \mathcal{Q}, \xi, \eta) + \tilde{g}([A_{\xi}, A_{\eta}]\,\mathcal{P}, \mathcal{Q})$$

for all $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{W} \in \Gamma(TN)$ and $\xi, \eta \in \Gamma(TN)^{\perp}$. The H mean curvature vector of a submanifold N is as follows:

$$H = \left(\frac{1}{p}\right) \operatorname{trace} h, \quad p = \dim N$$

Definition 2.1. A submanifold N of a quaternion Kaehler manifold $\bar{M}$ is called a generic submanifold of $\bar{M}$ if there are two subspace D differentiable distribution and $D^{\perp}$ purely real distribution with constant ranks on N such that $D = J_a TN \cap TN$ and $D^{\perp}$ are complementary orthogonal to D.

Thus, from the definition, it is expressed as follows:

$$TN = D \oplus D^{\perp} \quad \text{and} \quad J_a(D) = D$$

For $X \in \Gamma(D^{\perp})$, $a \in \{1, 2, 3\}$,

$$J_a X = T_a X + F_a X$$

where $T_a X \in \Gamma\left(D^{\perp}\right)$ and $F_a X \in \Gamma\left(\vartheta^{\perp}\right)$. Moreover, $\vartheta^{\perp}$ and ϑ are complementary orthogonal to each other. Thus, from the definition, it is expressed as follows:

$$TN^{\perp} = \vartheta \oplus \vartheta^{\perp} \quad \text{and} \quad J_a(\vartheta) = \vartheta$$

For $V \in \Gamma\left(\vartheta^{\perp}\right)$,

$$J_a V = t_a V + f_a V$$

where $t_a V \in \Gamma(TN)$ and $f_a V \in \Gamma\left(\vartheta^{\perp}\right)$ [25].

3. Chen-Type Inequality for Generic Submanifolds of Quaternionic Space Form

Let M be a generic submanifold of a quaternion Kaehler manifold $\bar{M}$ with the differentiable distribution D and the purely real distribution $D^{\perp}$ of M. Consider orthonormal frame $\{e_1, e_2, \cdots, e_{2q+p}\}$ on M in such that $\{e_1, e_2, \cdots, e_{2q}\}$ are in D and $\{e_{2q+1}, e_{2q+2}, \cdots, e_{2q+p}\}$ are in $D^{\perp}$.

Chen [31] investigated new types of Riemannian invariants for submanifolds in space forms, now known as the Chen invariants. Chen [32] defined for CR-submanifold N in a Kaehler manifold $\bar{M}$ with τ the scalar curvature of N, and $\tau(D)$ the scalar curvature of the holomorphic distribution D of N as follows:

$$\delta(D)(p) = \tau(p) - \tau(D_p), \quad p \in N$$

Let $\vec{H}_D$ and $\vec{H}_{D^{\perp}}$ be the two partial mean curvature vectors of M, respectively, i.e.,

$$\vec{H}_D = \frac{1}{2q} \sum_{i=1}^{2q} h(e_i, e_i) \quad \text{and} \quad \vec{H}_{D^{\perp}} = \frac{1}{p} \sum_{r=2q+1}^{2q+p} h(e_r, e_r) \tag{3.1}$$

Theorem 3.1. Let M be a quaternionic generic submanifold of quaternionic space form $\bar{M}$ with minimal codimension, i.e., $\dim \vartheta_x = 0$, for $x \in M$, $\dim D_x = 2q$, $\dim D_x^{\perp} = p$, and $\dim TM^{\perp} = 2m - (2q + p)$, then

$$\delta(D) \leq 2qp\frac{c}{4} + \frac{p(p-1)}{2}\frac{c}{4} + \frac{9c}{4}\|T_a\|^2 + \frac{(2q+p)^2}{2}H^2 + \frac{(p-1)p^2}{p+2}|H_{D^{\perp}}|^2$$

Proof. Let $\{e_1, e_2, \cdots, e_{2q}, e_{2q+1}, \cdots, e_{2q+p}\}$ be orthonormal bases on TM such that $\{e_1, e_2, \cdots, e_{2q}\}$ are in D and $\{e_{2q+1}, e_{2q+2}, \cdots, e_{2q+p}\}$ are in $D^{\perp}$ and let $\{e_{2q+p+1}, e_{2q+p+2}, \cdots, e_{2q+p+2m}\}$ of $TM^{\perp}$. Since

$$\tau = \sum_{1 \leq i < j \leq 2q} K(e_i \wedge e_j) + \sum_{2q+1 \leq r < s \leq 2q+p} K(e_r \wedge e_s) + \sum_{i=1}^{2q} \sum_{r=2q+1}^{2q+p} K(e_i \wedge e_r)$$

and

$$\tau(D) = \sum_{1 \leq i < j \leq 2q} K(e_i \wedge e_j)$$

then

$$\delta(D) = \tau - \tau(D) = \sum_{2q+1 \leq r < s \leq 2q+p} K(e_r \wedge e_s) + \sum_{i=1}^{2q} \sum_{r=2q+1}^{2q+p} K(e_i \wedge e_r)$$

From (2.2),

$$\begin{aligned} K(\mathcal{P}\wedge\mathcal{Q}) &= \bar{K}(\mathcal{P}\wedge\mathcal{Q}) + \tilde{g}(h(\mathcal{P},\mathcal{P}),h(\mathcal{Q},\mathcal{Q})) - \tilde{g}(h(\mathcal{P},\mathcal{Q}),h(\mathcal{P},\mathcal{Q})) \\ &= \frac{c}{4}\left[1+3\textstyle\sum_{a=1}^{3}\tilde{g}(J_a\mathcal{P},\mathcal{Q})^2\right] + \tilde{g}(h(\mathcal{P},\mathcal{P}),h(\mathcal{Q},\mathcal{Q})) - \tilde{g}(h(\mathcal{P},\mathcal{Q}),h(\mathcal{P},\mathcal{Q})) \end{aligned} \tag{3.2}$$

From (3.2), for $\mathcal{P}=e_i$, $\mathcal{Q}=e_r$, $i\in\{1,2,\cdots,2q\}$, and $r\in\{2q+1,2q+2,\cdots,2q+p\}$,

$$K(e_i\wedge e_r) = \frac{c}{4}\left[1+3\sum_{a=1}^{3}\tilde{g}(J_ae_i,e_r)^2\right] + \tilde{g}(h(e_i,e_i),h(e_r,e_r)) - \tilde{g}(h(e_i,e_r),h(e_i,e_r)) \tag{3.3}$$

Since $J_ae_i\in D$ and $e_r\in D^\perp$,

$$\tilde{g}(J_ae_i,e_r)=0$$

By summation in (3.3) over $i\in\{1,2,\cdots,2q\}$ and $r\in\{2q+1,2q+2,\cdots,2q+p\}$,

$$\sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p}K(e_i\wedge e_r) = 2qp\frac{c}{4} + \sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p}\left[\tilde{g}(h(e_i,e_i),h(e_r,e_r)) - \tilde{g}(h(e_i,e_r),h(e_i,e_r))\right]$$

From (3.2), for $\mathcal{P}=e_r$, $\mathcal{Q}=e_s$, and $r,s\in\{2q+1,2q+2,\cdots,2q+p\}$,

$$K(e_r\wedge e_s) = \frac{c}{4}\left[1+3\sum_{a=1}^{3}\tilde{g}(J_ae_r,e_s)^2\right] + \tilde{g}(h(e_r,e_r),h(e_s,e_s)) - \tilde{g}(h(e_r,e_s),h(e_r,e_s))$$

Since $T_ae_r\in D^\perp$ and $e_s\in D^\perp$,

$$\tilde{g}(J_ae_r,e_s)\neq 0$$

By summation in (3.3) over $i\in\{1,2,\cdots,2q\}$ and $r\in\{2q+1,2q+2,\cdots,2q+p\}$,

$$\begin{aligned} \sum_{2q+1\le r<s\le 2q+p}K(e_r\wedge e_s) &= \frac{p(p-1)}{2}\frac{c}{4} + \frac{9c}{4}\|T_a\|^2 \\ &\quad + \sum_{2q+1\le r<s\le 2q+p}\left[\tilde{g}(h(e_r,e_r),h(e_s,e_s)) - \tilde{g}(h(e_r,e_s),h(e_r,e_s))\right] \end{aligned}$$

where

$$\|T_a\|^2 = \sum_{i,j=2q+1}^{2q+p}\tilde{g}(T_ae_r,e_s)^2$$

and

$$\begin{aligned} \delta(D) &= 2qp\frac{c}{4} + \frac{p(p-1)}{2}\frac{c}{4} + \frac{9c}{4}\|T_a\|^2 + \sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p}\tilde{g}(h(e_i,e_i),h(e_r,e_r)) \\ &\quad + \sum_{2q+1\le r<s\le 2q+p}\tilde{g}(h(e_r,e_r),h(e_s,e_s)) - \sum_{2q+1\le r<s\le 2q+p}\|h(e_r,e_s)\|^2 \\ &\quad - \sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p}\|h(e_i,e_r)\|^2 \end{aligned} \tag{3.4}$$

Moreover,

$$\frac{(2q+p)^2}{2}H^2 + p^2|H_{D^\perp}|^2 - \|h_{D^\perp}\|^2 \tag{3.5}$$

where $\|h_{D^\perp}\|^2$ is defined by

$$\|h_{D^\perp}\|^2 = \sum_{2q+1\le r<s\le 2q+p}\|h(e_r,e_s)\|^2 \tag{3.6}$$

Combining (3.4) and (3.5),

$$\delta(D) = 2qp\frac{c}{4} + \frac{p(p-1)}{2}\frac{c}{4} + \frac{9c}{4}\|T_a\|^2 + \frac{(2q+p)^2}{2}H^2 + p^2|H_{D^\perp}|^2 - \|h_{D^\perp}\|^2 - \sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p}\|h(e_i,e_r)\|^2$$

and

$$p^2|H_{D^\perp}|^2 + \frac{(p+2)}{p-1}\left[\frac{(2q+p)^2}{2}H^2 - \sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p}\|h(e_i,e_r)\|^2 - \delta(D)\right] + \frac{c}{4}\left(2qp + \frac{p(p+2)}{2} + 9\|T_a\|^2\right) = \frac{p+2}{p-1}\|h_{D^\perp}\|^2 - \frac{3p^2}{p-1}|H_{D^\perp}|^2$$

From (3.1) and (3.6),

$$\frac{p+2}{p-1}\|h_{D^\perp}\|^2 - \frac{3p^2}{p-1}|H_{D^\perp}|^2 = \frac{p+2}{p-1}\left[\sum_{s=2q+1}^{2q+p}(h_{ss}^r) + \sum_{s\neq t}(h_{st}^r)^2 + \sum_{k=2q+p+1}^{2q+p+2m}\sum_{s,t=2q+1}^{2q+p}(h_{st}^k)^2\right] - \frac{3}{p-1}\left(\sum_{s=2q+1}^{2q+p}h_{ss}^r\right)^2$$

Moreover, since

$$0 \leq \sum_{i\leq j}(a_i - a_j)^2 = (n-1)\sum_{i=1}^{n}a_i^2 - 2\sum_{i\leq j}a_ia_j \tag{3.7}$$

then

$$n^2\|H\|^2 = \left(\sum_{i=1}^{n}a_i\right)^2 = \sum_{i=1}^{n}a_i^2 + 2\sum_{i\leq j}a_ia_j \tag{3.8}$$

Thus, from (3.7) and (3.8),

$$\frac{p+2}{p-1}\|h_{D^\perp}\|^2 - \frac{3p^2}{p-1}|H_{D^\perp}|^2 = \frac{1}{p-1}\left\{(p+2)\sum_{s=2q+1}^{2q+p}(h_{ss}^r)^2 + (p+2)\left[\sum_{s\neq t}(h_{st}^r)^2 + \sum_{k=2q+p+1}^{2q+p+2m}\sum_{s,t=2q+1}^{2q+p}(h_{st}^k)^2\right] - 3\left(\sum_{s=2q+1}^{2q+p}h_{ss}^r\right)^2\right\}$$

$$\begin{aligned}\frac{p+2}{p-1}\|h_{D^\perp}\|^2 - \frac{3p^2}{p-1}|H_{D^\perp}|^2 &= \frac{1}{p-1}\left\{(p-1)\sum_{s=2q+1}^{2q+p}(h_{ss}^r)^2 - 6\sum_{s\leq t}h_{ss}^rh_{tt}^r + (p+2)\left[\sum_{s\neq t}(h_{st}^r)^2 + \sum_{k=2q+p+1}^{2q+p+2m}\sum_{s,t=2q+1}^{2q+p}(h_{st}^k)^2\right]\right\}\\ &= \frac{1}{p-1}\left\{2(1-p)\sum_{s=2q+1}^{2q+p}(h_{ss}^r)^2 + 3\sum_{s\leq t}(h_{ss}^r - h_{tt}^r) + (p+2)\left[\sum_{s\neq t}(h_{st}^r)^2 + \sum_{k=2q+p+1}^{2q+p+2m}\sum_{s,t=2q+1}^{2q+p}(h_{st}^k)^2\right]\right\}\\ &\geq 0\end{aligned}$$

Thus,

$$\|h_{D^\perp}\|^2 \geq \frac{3p^2}{p+2}|H_{D^\perp}|^2$$

and

$$2qp\frac{c}{4} + \frac{p(p-1)}{2}\frac{c}{4} + \frac{9c}{4}\|T_a\|^2 + \frac{(2q+p)^2}{2}H^2 + p^2|H_{D^\perp}|^2 - \delta(D) \geq \frac{3p^2}{p+2}|H_{D^\perp}|^2 + \sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p}\|h(e_i,e_r)\|^2$$

then

$$2qp\frac{c}{4} + \frac{p(p-1)}{2}\frac{c}{4} + \frac{9c}{4}\|T_a\|^2 + \frac{(2q+p)^2}{2}H^2 + p^2|H_{D^\perp}|^2 - \delta(D) \geq \frac{3p^2}{p+2}|H_{D^\perp}|^2$$

Hence,

$$\delta(D) \leq 2qp\frac{c}{4} + \frac{p(p-1)}{2}\frac{c}{4} + \frac{9c}{4}\|T_a\|^2 + \frac{(2q+p)^2}{2}H^2 + \frac{(p-1)p^2}{p+2}|H_{D^\perp}|^2$$

□

Corollary 3.2. If M^n is a quaternionic generic submanifold of the quaternionic Euclidean m space H^m with $c = 0$, then

$$\delta(D) \leq \frac{(2q+p)^2}{2}H^2 + \frac{(p-1)p^2}{p+2}|H_{D^\perp}|^2$$

Corollary 3.3. If M^n is a quaternionic generic submanifold of the quaternionic projective m space $HP^m(4c)$ with $c > 0$, then

$$\delta(D) \leq 2qp + \frac{p(p-1)}{2} + 9\|T_a\|^2 + \frac{(2q+p)^2}{2}H^2 + \frac{(p-1)p^2}{p+2}|H_{D^\perp}|^2$$

Corollary 3.4. If M^n is a quaternionic generic submanifold of the quaternionic hyperbolic m space $HH^m(4c)$ with $c < 0$, then

$$\delta(D) \geq 2qp + \frac{p(p-1)}{2} + 9\|T_a\|^2 + \frac{(2q+p)^2}{2}H^2 + \frac{(p-1)p^2}{p+2}|H_{D^\perp}|^2$$

4. An Inequality for Real Hypersurfaces

Indeed, generic submanifolds with $\vartheta = \{0\}$ and $D^\perp = Sp\{J_aN\}$ are real hypersurface of a quaternion Kaehler manifold, where N is the unit normal vector field of the hypersurface. Therefore, Theorem 3.1 lead to the following

Theorem 4.1. Let M be a real hypersurface of quaternionic space form $\bar{M}(4c)$. Then,

$$\delta(D) \leq \frac{(2q+3)^2}{2}H^2 + 9|H_{D^\perp}|^2 + (6q+3)c$$

PROOF. Let M be a real hypersurface of quaternionic space form $\bar{M}(4c)$. Then, it follows from the definition $\delta(D)$ that

$$\delta(D) = \tau - \tau(D) = \sum_{1\leq a<b\leq 3} K(J_aN \wedge J_bN) + \sum_{i=1}^{2q}\sum_{a=1}^{3} K(e_i \wedge J_aN)$$

For $i \in \{1, 2, \cdots, 2q\}$ and $a \in \{1, 2, 3\}$,

$$\begin{aligned}\delta(D) &= 6qc + 3c + \sum_{i=1}^{2q}\sum_{a=1}^{3} \tilde{g}(h(e_i, e_i), h(J_aN, J_aN)) \\ &\quad + \sum_{1\leq a<b\leq 3} \tilde{g}(h(J_aN, J_aN), h(J_bN, J_bN)) - \sum_{1\leq a<b\leq 3} \|h(J_aN, J_bN)\|^2 \\ &\quad - \sum_{i=1}^{2q}\sum_{a=1}^{3} \|h(e_i, J_aN)\|^2\end{aligned} \tag{4.1}$$

Moreover,

$$\begin{aligned}&\sum_{i=1}^{2q}\sum_{a=1}^{3} \tilde{g}(h(e_i, e_i), h(J_aN, J_aN)) + \sum_{1\leq a<b\leq 3} \tilde{g}(h(J_aN, J_aN), h(J_bN, J_bN)) \\ &\quad - \sum_{1\leq a<b\leq 3} \|h(J_aN, J_bN)\|^2 = \frac{(2q+3)^2}{2}H^2 + 9|H_{D^\perp}|^2 - \|h_{D^\perp}\|^2\end{aligned} \tag{4.2}$$

where $\|h_{D^\perp}\|^2$ is defined by

$$\|h_{D^\perp}\|^2 = \sum_{1\leq a<b\leq 3} \|h(J_aN, J_bN)\|^2$$

Combining (4.1) and (4.2),

$$\begin{aligned}\delta(D) &= 6qc + 3c + \frac{(2q+3)^2}{2}H^2 + 9|H_{D^\perp}|^2 - \|h_{D^\perp}\|^2 - \sum_{i=1}^{2q}\sum_{a=1}^{3}\|h(e_i, J_aN)\|^2 \\ &\leq \frac{(2q+3)^2}{2}H^2 + 9|H_{D^\perp}|^2 + (6q+3)c\end{aligned}$$

□

5. Conclusion

This study investigates an inequality for an intrinsic invariant of Chen-type defined on quaternionic generic submanifolds in a quaternionic space form. Its application obtains this inequality for real hypersurfaces. Although results for certain submanifolds have been obtained in previous studies, their generalized state has not been made. Thus, this study will provide new fields for researchers studying generic submanifolds by working in different space forms such as generalized complex space forms, Sasakian space forms, cosymplectic space forms, and locally conformal Kähler space forms.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Results of Convergence, Stability, and Data Dependency for an Iterative Algorithm

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Research Article

Abstract — In this study, we reconstruct an existing result related to the strong convergence of a recently introduced iterative algorithm by removing certain restrictions on the coefficient sequences. We then obtain some new results on the stability and data dependency of this algorithm. To validate our results, we provide a series of nontrivial complex examples, demonstrating the significance and accuracy of our theoretical contributions.

Keywords *Data dependency, fixed point, iterative algorithm, stability, strong convergence*

Mathematics Subject Classification (2020) 47H09, 47H10

1. Introduction

Fixed point theory provides a powerful tool for solving various problems encountered in fields such as engineering, economics, biology, physics and chemistry [1, 2]. Let X be a non-empty set and S a mapping from X to X. If $Su = u$, for an element u in X, then u is called a fixed point of S. Fixed point theory has been studied on various spaces, including metric spaces, finite dimensional spaces, infinite dimensional Banach spaces, and Hilbert spaces. Various theories have been developed to determine the existence and uniqueness of fixed points of a mapping. However, finding the value of a fixed point is not easy in general. To approximate the fixed point, many effective iterative algorithms have been defined and studied, such as the Mann iterative algorithm [3], Ishikawa iterative algorithm [4], two step Mann iterative algorithm [5], Suantai-Phuengrattana (SP) iterative algorithm [6]. The convergence speed, stability, and data dependency of an iterative algorithm are significant factors in determining the performance of one algorithm compared to another. There are many studies [7–12] in the literature that deal with these factors.

Chauhan et al. [13] introduced a new iterative algorithm inspired by the Karakaya et al. [14], providing better results than the Karakaya iterative algorithm in terms of convergence speed. They named this new algorithm the Surjeet-Naveen-Imdad-Asim (SNIA) iterative algorithm (Naveen et al. iterative algorithm) and proved that the iterative sequence $(\sigma_n)_n$ generated by this algorithm converges strongly to the fixed point of S if the coefficient sequences $(\alpha_n^i)_{n=1}^{\infty}$ are in $(\frac{1}{2}, 1)$, for $i \in \{1, 2, 3, 4, 5\}$, and the mapping S satisfies quasi contraction condition. We denote that the proof of Theorem 2.1 in [13] was done under the assumptions $1 - \alpha_n^2 - \alpha_n^3 \geq 0$ and $1 - \alpha_n^4 - \alpha_n^5 \geq 0$, for all $n \in \mathbb{N}$, the set of all the

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natural numbers. However, under the condition $(\frac{1}{2}, 1)$ on the coefficient sequences, $1-\alpha_n^2-\alpha_n^3<0$ and $1-\alpha_n^4-\alpha_n^5<0$, for all $n \in \mathbb{N}$.

The aim of this paper is to reconstruct the convergence result in Theorem 2.1 in [13], removing the restricting conditions $(\frac{1}{2}, 1)$ on the coefficient sequences and obtain the convergence results for some algorithms. The another aim is to prove the stability and data dependency of the SNIA iterative algorithm generated by quasi-contractive mappings. Nontrivial examples will be presented to confirm the validity and applicability of all obtained theoretical results.

2. Preliminaries

We remind the basic terminology that is connected to our study. Let (X, d) be a metric space and S a mapping from X to X. Osilike [15] considered the mapping S having a fixed point and satisfying the contractive condition:

$$\forall x_1, x_2 \in X,\ d(Sx_1, Sx_2) \leq Ld(x_1, Sx_1) + \delta d(x_1, x_2) \tag{2.1}$$

where $\delta \in [0,1)$ and $L \geq 0$. He obtained stability results for some iterative algorithms generated with the mapping S satisfying (2.1). Imoru and Olatinwo [16] defined a more general conctractive condition than (2.1) as follows:

$$\forall x_1, x_2 \in X,\ d(Sx_1, Sx_2) \leq \varphi(d(x_1, Sx_1)) + \delta d(x_1, x_2) \tag{2.2}$$

where $\delta \in [0,1)$ and $\varphi : R^+ \to R^+$ is monotone increasing such that $\varphi(0) = 0$. They proved some stability results using mappings satisfying (2.2). If $\varphi(x) = Lx$ is taken in (2.2), which $L \geq 0$ is a constant, then the condition (2.2) is reduced to condition (2.1). Thus, (2.2) is more general than (2.1). Bosede and Rhoades [17] made an assumption which makes all generalizations of the form (2.2) meaningless and implied by (2.1). In their assumption, S is a self mapping on a complete metric space that has a fixed point x^* and satisfies the following quasi contractive condition:

$$\forall x \in X,\ d(Sx, x^*) \leq \delta d(x, x^*) \tag{2.3}$$

where $\delta \in [0,1)$. Bosede and Rhoades [17] obtained some stability results using mappings satisfying (2.3). It is clear that, if X is a normed space, then the quasi contractive condition (2.3) turns into

$$\forall x \in X,\ \|Sx - x^*\| \leq \delta \|x - x^*\| \tag{2.4}$$

Throughout this paper, we denote the set of all the fixed points of a mapping S by F_S.

Let C be a nonempty convex subset of a normed space E. Karakaya et al. [14] have described a three-step iterative algorithm that can be used to generate several types of iterative algorithms by choosing specific coefficient sequences as follows:

Karakaya iterative algorithm

Input: Self mapping S on C, initial point s_1, $(\alpha_n^i)_{n=1}^\infty \subset [0,1]$, $i \in \{1,2,3,4,5\}$, such that $(\alpha_n^2+\alpha_n^3)_{n=1}^\infty \subset [0,1], (\alpha_n^4+\alpha_n^5)_{n=1}^\infty \subset [0,1]$, and $N \in \mathbb{N}$.

1: **for** $n \in \{1, 2, \cdots, N\}$ **do**

2: $p_n = (1-\alpha_n^1)\, s_n + \alpha_n^1 S s_n$

$r_n = (1-\alpha_n^2-\alpha_n^3)\, p_n + \alpha_n^2 S p_n + \alpha_n^3 S s_n$

$s_{n+1} = (1-\alpha_n^4-\alpha_n^5)\, r_n + \alpha_n^4 S r_n + \alpha_n^5 S p_n$

3: **end for**

Output: Approximate solution s_N

Some iterative algorithms obtained by special choosing of the coefficient sequences in Karakaya iterative algorithm are given below.

If $\alpha_n^1 = 1$, for all $n \in \mathbb{N}$, and the other coefficient sequences are zero, then Karakaya iterative algorithm turns into Picard iterative algorithm. If all coefficient sequences except for $(\alpha_n^4)_n$ are zero, then it turns into Mann iterative algorithm. If $\alpha_n^5 = \alpha_n^3 = 0$ for all $n \in \mathbb{N}$, then it turns into SP iterative algorithm. If $\alpha_n^5 = \alpha_n^3 = \alpha_n^1 = 0$, for all $n \in \mathbb{N}$, then it turns into two-step Mann iterative algorithm [14].

Let E be a Banach space. SNIA iterative algorithm is defined by Chauhan et al. [13] as follows:

SNIA iterative algorithm

Input: Self mapping S on E, initial point σ_1, $(\alpha_n^i)_{n=1}^{\infty} \subset (\frac{1}{2}, 1)$, $i \in \{1,2,3,4,5\}$, and $N \in \mathbb{N}$.

1: **for** $n \in \{1, 2, \cdots, N\}$ **do**

2: $\varphi_n = S\left[\left(1-\alpha_n^1\right)\sigma_n + \alpha_n^1 S\sigma_n\right]$

$\tau_n = S\left[\left(1-\alpha_n^2-\alpha_n^3\right)\varphi_n + \alpha_n^2 S\varphi_n + \alpha_n^3 S\sigma_n\right]$

$\sigma_{n+1} = S\left[\left(1-\alpha_n^4-\alpha_n^5\right)\tau_n + \alpha_n^4 S\tau_n + \alpha_n^5 S\varphi_n\right]$

3: **end for**

Output: Approximate solution σ_N

Karakaya iterative algorithm is obtained if S is taken as the identity operator in SNIA iterative algorithm. Therefore, SNIA iterative algorithm is more general than Karakaya iterative algorithm [13].

The following definitions and lemmas are important in obtaining the findings stated in this study.

Definition 2.1. [18] Let $(a_n)_n$ be a sequence in a (X,d) metric space. The sequence $(b_n)_n \subset X$ is called the approximate sequence of the sequence $(a_n)_n$ if, for all $m \in \mathbb{N}$, there exists an $\zeta = \zeta(m)$ such that

$$\forall i \geq m, \;\; d(a_i, b_i) \leq \zeta$$

Lemma 2.2. [18] The sequence $(b_n)_n$ is an approximate sequence of the sequence $(a_n)_n$ if and only if there is a decreasing sequence of positive numbers $(c_n)_n$ converging to some $\eta \geq 0$ such that

$$\forall n \geq k \text{ (fixed)}, \;\; d(a_n, b_n) \leq c_n$$

Definition 2.3. [18] Let $S : X \to X$ be a mapping, in which (X,d) is a metric space. Let $a_{n+1} = f(S, a_n)$ be an iterative algorithm such that $(a_n)_n$ converges to the fixed point x^* of S. Let $(b_n)_n \subset X$ be an approximate sequence of $(a_n)_n$ and $\varepsilon_n := d(b_{n+1}, f(S, b_n))$, for all $n \in \mathbb{N}$. The iterative algorithm $a_{n+1} = f(S, a_n)$ is said to be weakly S-stable if

$$\lim_{n\to\infty} \varepsilon_n = 0 \;\Rightarrow\; \lim_{n\to\infty} b_n = x^*$$

Definition 2.4. [18] Let $S, \widetilde{S} : X \to X$ be two mappings, where (X,d) is a metric space. $\widetilde{S}$ is referred to as an approximate mapping for S if there exists a suitable $\varepsilon > 0$ such that $d(Sx, \widetilde{S}x) \leq \varepsilon$, for all $x \in X$.

Lemma 2.5. [19] Let $(p_n)_n$ and $(t_n)_n$ be nonnegative real number sequences and $\theta \in [0,1)$ such that $p_{n+1} \leq \theta p_n + t_n$, for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

3. Main Results

In this section, we reconstruct the strong convergence result in [13] by removing the restriction on the coefficient sequences and provide some convergence results. We then obtain new results related to stability and data dependency for the SNIA iterative algorithm.

The following theorem is a reformulated version of Theorem 2.1 in [13], with the restriction on the coefficient sequences removed.

Theorem 3.1. Let C be a non-empty convex and closed subset of a Banach space E and $S : C \to C$ be a mapping satisfying (2.4) with $F_S \neq \emptyset$. For all $\sigma_1 \in C$, let $(\sigma_n)_n$ be a sequence generated by SNIA iterative algorithm with $(\alpha_n^i)_{n=1}^{\infty} \subset [0,1]$, $i \in \{1,2,3,4,5\}$, such that $(\alpha_n^2 + \alpha_n^3)_{n=1}^{\infty} \subset [0,1]$ and $(\alpha_n^4 + \alpha_n^5)_{n=1}^{\infty} \subset [0,1]$. Then, the sequence $(\sigma_n)_n$ converges strongly to the fixed point of S.

Proof. Assume that x^* is a fixed point of S. It can be observed from (2.4) that x^* is unique fixed point of S. Using (2.4) and $(\alpha_n^1)_n \subset [0,1]$,

$$\|\varphi_n - x^*\| \leq \delta[1 - \alpha_n^1(1-\delta)]\|\sigma_n - x^*\| \tag{3.1}$$

and by (2.4), $1 - \alpha_n^2 - \alpha_n^3 \geq 0$, $\alpha_n^2 \geq 0$, and $\alpha_n^3 \geq 0$, for all $n \in \mathbb{N}$, and $\delta < 1$,

$$\|\tau_n - x^*\| \leq \delta(1 - \alpha_n^2 - \alpha_n^3 + \alpha_n^2\delta)\|\varphi_n - x^*\| + \delta^2\alpha_n^3\|\sigma_n - x^*\| \tag{3.2}$$

If (3.1) is used in (3.2), then the following inequality are valid:

$$\begin{aligned} \|\tau_n - x^*\| &\leq \delta(1 - \alpha_n^2 - \alpha_n^3 + \alpha_n^2\delta)\delta[1 - \alpha_n^1(1-\delta)]\|\sigma_n - x^*\| + \alpha_n^3\delta^2\|\sigma_n - x^*\| \\ &= \delta^2\left[(1 - \alpha_n^2 - \alpha_n^3 + \alpha_n^2\delta)\left(1 - \alpha_n^1(1-\delta)\right) + \alpha_n^3\right]\|\sigma_n - x^*\| \end{aligned} \tag{3.3}$$

Moreover,

$$\begin{aligned} \|\sigma_{n+1} - x^*\| &= \|S\left[\left(1 - \alpha_n^4 - \alpha_n^5\right)\tau_n + \alpha_n^4 S\tau_n + \alpha_n^5 S\varphi_n\right] - x^*\| \\ &\leq \delta(1 - \alpha_n^4 - \alpha_n^5 + \alpha_n^4\delta)\|\tau_n - x^*\| + \delta^2\alpha_n^5\|\varphi_n - x^*\| \end{aligned} \tag{3.4}$$

If (3.1) and (3.3) are used in (3.4), then

$$\begin{aligned} \|\sigma_{n+1} - x^*\| \leq &\{\delta^3(1 - \alpha_n^4 - \alpha_n^5 + \alpha_n^4\delta)\left[(1 - \alpha_n^2 - \alpha_n^3 + \alpha_n^2\delta)\left(1 - \alpha_n^1(1-\delta)\right) + \alpha_n^3\right] \\ &+ \delta^3\alpha_n^5[1 - \alpha_n^1(1-\delta)]\}\|\sigma_n - x^*\| \end{aligned} \tag{3.5}$$

Since $1 - \alpha_n^1(1-\delta) \leq 1$, for all $n \in \mathbb{N}$, by (3.5),

$$\begin{aligned} \|\sigma_{n+1} - x^*\| &\leq \{\delta^3\left(1 - \alpha_n^4 - \alpha_n^5 + \alpha_n^4\delta\right)\left(1 - \alpha_n^2 - \alpha_n^3 + \alpha_n^2\delta + \alpha_n^3\right) + \delta^3\alpha_n^5\}\|\sigma_n - x^*\| \\ &= \delta^3[(1 - \alpha_n^4 - \alpha_n^5 + \alpha_n^4\delta)\left(1 - \alpha_n^2 + \alpha_n^2\delta\right) + \alpha_n^5]\|\sigma_n - x^*\| \end{aligned}$$

and if $1 - \alpha_n^2(1-\delta) \leq 1$, for all $n \in \mathbb{N}$, is used in the last inequality, then

$$\|\sigma_{n+1} - x^*\| \leq \delta^3(1 - \alpha_n^4 - \alpha_n^5 + \alpha_n^4\delta + \alpha_n^5)\|\sigma_n - x^*\| \leq \delta^3\left(1 - \alpha_n^4(1-\delta)\right)\|\sigma_n - x^*\| \tag{3.6}$$

is obtained. Using that $1 - \alpha_n^4(1-\delta) \leq 1$, for all $n \in \mathbb{N}$, in (3.6),

$$\|\sigma_{n+1} - x^*\| \leq \delta^3\|\sigma_n - x^*\|$$

Since $\delta \in [0,1)$, by Lemma 2.5, $\sigma_n \to x^*$ as $n \to \infty$. □

Remark 3.2. Chauhan et al. stated that the main results of Karakaya [14] could be obtained by assuming $S(x) = 0$, for all $x \in C$, in Theorem 2.1 of [13]. However, if $S(x) = 0$ is taken in Theorem 2.1 of [13], then the SNIA iterative algorithm does not denote Karakaya iterative algorithm. Therefore, the main results of Karakaya [14] can not be obtained. We denote that if S is taken as the identity operator in Theorem 2.1 of [13], then the Karakaya iterative algorithm can be obtained but in this case the necessary hypotheses (2.4) on S is not provided. Thus, the main result(s) of Karakaya [14] can not be obtained from Theorem 2.1 in [13].

We observed that if the condition on S in Theorem 3 of [14] is replaced by the quasi contractive condition (2.4), then this theorem is satisfied under the same hypotheses. In the following theorem, we will consider this situation by an extra condition on the sequence $(\alpha_n^4)_n$. It means that if the sequence $(\sigma_n)_n$ in Theorem 3.1 is replaced by the sequence $(s_n)_n$ generated by Karakaya iterative algorithm, then an extra condition is required to the hypotheses in Theorem 3.1. The proof of the theorem will be done by following similar steps in the proof of Theorem 3 in [14].

Theorem 3.3. Let C be a non-empty convex and closed subset of a Banach space E and $S : C \to C$ be a mapping satisfying (2.4) with $F_S \neq \emptyset$. For all $s_1 \in C$, let $(s_n)_n$ be a sequence generated by Karakaya iterative algorithm with $(\alpha_n^i)_{n=1}^\infty \subset [0,1]$, $i \in \{1,2,3,4,5\}$, such that $(\alpha_n^2 + \alpha_n^3)_{n=1}^\infty \subset [0,1]$, $(\alpha_n^4 + \alpha_n^5)_{n=1}^\infty \subset [0,1]$, and $\sum_{n=1}^\infty \alpha_n^4 = \infty$. Then, the sequence $(s_n)_n$ converges strongly to the fixed point of S.

Proof. If similar steps in the proof of Theorem 3 in [14] are followed using (2.4), then the below inequalities are obtained, for all $n \in \mathbb{N}$:

$$\|p_n - x^*\| = \|(1-\alpha_n^1)s_n + \alpha_n^1 S s_n - x^*\| \leq [1 - \alpha_n^1(1-\delta)]\|s_n - x^*\|$$

$$\|r_n - x^*\| \leq [(1 - \alpha_n^2(1-\delta) - \alpha_n^3)(1 - \alpha_n^1(1-\delta)) + \delta\alpha_n^3]\|s_n - x^*\|$$

and

$$\begin{aligned} \|s_{n+1} - x^*\| &\leq \{(1 - \alpha_n^4(1-\delta) - \alpha_n^5)\left[(1 - \alpha_n^2 - \alpha_n^3 + \alpha_n^2\delta)\left(1 - \alpha_n^1(1-\delta)\right) + \delta\alpha_n^3\right] \\ &\Rightarrow +\delta\alpha_n^5(1 - \alpha_n^1(1-\delta))\}\|s_n - x^*\| \\ &\leq \{(1 - \alpha_n^4(1-\delta) - \alpha_n^5) + \delta\alpha_n^5(1 - \alpha_n^1(1-\delta))\}\|s_n - x^*\| \\ &\leq [1 - \alpha_n^4(1-\delta)]\|s_n - x^*\| \end{aligned} \tag{3.7}$$

Thus, by using induction,

$$\|s_{n+1} - x^*\| \leq \|s_1 - x^*\| \prod_{k=1}^{n} \{(1 - \alpha_k^4(1-\delta)\}$$

It is well known that $1 - t \leq e^{-t}$, for all $t \in [0,1]$. Therefore,

$$\|s_{n+1} - x^*\| \leq \|s_1 - x^*\| e^{-(1-\delta)\sum_{i=1}^n \alpha_i^4} \tag{3.8}$$

By using the condition $\sum\limits_{i=1}^{\infty} \alpha_i^4 = \infty$ in (3.8), we obtain $\|s_{n+1} - x^*\| \to 0$ as $n \to \infty$. Thus, the proof is completed. □

Remark 3.4. We observe that the condition $\sum\limits_{n=1}^{\infty} \alpha_n^4 = \infty$ can be replaced by $\sum\limits_{n=1}^{\infty} \alpha_n^5 = \infty$ in Theorem 3.3. In this case, the proof of Theorem 3.3 is followed by rearranging the inequality in (3.7) as follows:

$$\|s_{n+1} - x^*\| \leq [1 - \alpha_k^5(1-\delta)]\|s_n - x^*\|$$

Corollary 3.5. Assume that all the hypotheses in Theorem 3.3 are satisfied. Then, we get the following results, which possibly existing in the literature.

i. Mann iterative algorithm generated by S satisfying quasi contraction condition (2.4) converges strongly to the fixed point of S if taken $\alpha_n^5 = \alpha_n^2 = \alpha_n^3 = \alpha_n^1 = 0$, for all $n \in \mathbb{N}$, in Theorem 3.3.

ii. SP iterative algorithm generated by S satisfying quasi contraction condition (2.4) converges strongly to the fixed point of S if taken $\alpha_n^5 = \alpha_n^3 = 0$, for all $n \in \mathbb{N}$, in Theorem 3.3.

iii. Two-step Mann iterative algorithm generated by S satisfying quasi contraction condition (2.4) converges strongly to the fixed point of S if taken $\alpha_n^5 = \alpha_n^3 = \alpha_n^1 = 0$, for all $n \in \mathbb{N}$, in Theorem 3.3.

3.1. Stability Results

An iterative algorithm that converges to a unique fixed point is stable if the numerical errors that occur in each step have no effect on the convergence of algorithm. In this part, we show the stability of SNIA iterative algorithm for quasi contractive mappings.

Theorem 3.6. Let C be a non-empty convex and closed subset of a Banach space E and $S : C \to C$ be a mapping satisfying (2.4) with $F_S \neq \emptyset$ and $\sigma_1,\ c_1 \in C$. Let $(\sigma_n)_n$ be a sequence generated by SNIA iterative algorithm with $(\alpha_n^i)_{n=1}^\infty \subset [0,1]$, $i \in \{1,2,3,4,5\}$, such that $(\alpha_n^2 + \alpha_n^3)_{n=1}^\infty \subset [0,1]$, $(\alpha_n^4 + \alpha_n^5)_{n=1}^\infty \subset [0,1]$, and $(y_n)_{n=1}^\infty \subset C$ be an approximate sequence of $(\sigma_n)_n$. Define a sequence $(\varepsilon_n)_{n=1}^\infty \subset R^+$ by

$$v_n = S\left[(1-\alpha_n^1)y_n + \alpha_n^1 S y_n\right]$$

$$u_n = S\left[\left(1-\alpha_n^2-\alpha_n^3\right)v_n + \alpha_n^2 S v_n + \alpha_n^3 S y_n\right]$$

$$f(S,y_n) = S\left[\left(1-\alpha_n^4-\alpha_n^5\right)u_n + \alpha_n^4 S u_n + \alpha_n^5 S v_n\right]$$

and

$$\varepsilon_n = \|y_{n+1} - f(S,y_n)\|,\ n \in \mathbb{N}$$

Then, $\lim_{n\to\infty} \varepsilon_n = 0$ implies $\lim_{n\to\infty} y_n = x^*$. In other words, SNIA iterative algorithm is weakly S-stable.

Proof. By Theorem 3.1, the sequence $(\sigma_n)_n$ generated by SNIA iterative algorithm converges the fixed point x^* of S. Assume that $\lim_{n\to\infty} \varepsilon_n = 0$. We will prove that $\lim_{n\to\infty} y_n = x^*$.

$$\begin{aligned} \|y_{n+1} - x^*\| &\leq \|y_{n+1} - S\left[\left(1-\alpha_n^4-\alpha_n^5\right)u_n + \alpha_n^4 S u_n + \alpha_n^5 S v_n\right]\| \\ &\quad + \|S\left[\left(1-\alpha_n^4-\alpha_n^5\right)u_n + \alpha_n^4 S u_n + \alpha_n^5 S v_n\right] - \sigma_{n+1}\| + \|\sigma_{n+1} - x^*\| \\ &= \varepsilon_n + \|S\left[\left(1-\alpha_n^4-\alpha_n^5\right)u_n + \alpha_n^4 S u_n + \alpha_n^5 S v_n\right] - \sigma_{n+1}\| + \|\sigma_{n+1} - x^*\| \end{aligned} \tag{3.9}$$

By (2.4),

$$\|Sx - Sy\| \leq \delta\|x - x^*\| + \delta\|y - x^*\|, \text{ for all } x, y \in C \tag{3.10}$$

If (3.10), (2.4), and the definition of SNIA iterative algorithm are used and operations are continued as in Theorem 3.1, then

$$\begin{aligned} \|S\left[\left(1-\alpha_n^4-\alpha_n^5\right)u_n + \alpha_n^4 S u_n + \alpha_n^5 S v_n\right] - \sigma_{n+1}\| &\leq \left[\delta(1-\alpha_n^4-\alpha_n^5) + \delta^2\alpha_n^4\right]\left[\|u_n - x^*\| + \|\tau_n - x^*\|\right] \\ &\quad + \delta^2\alpha_n^5\left[\|v_n - x^*\| + \|\varphi_n - x^*\|\right] \\ &\leq \delta^2\left[(1-\alpha_n^2(1-\delta)-\alpha_n^3)(1-\alpha_n^1(1-\delta)) + \alpha_n^3\right] \\ &\quad \times\left[\delta\left(1-\alpha_n^4(1-\delta)-\alpha_n^5\right)\right]\left[\|y_n - x^*\| + \|\sigma_n - x^*\|\right] \\ &\quad + \delta^3\alpha_n^5\left(1-\alpha_n^1(1-\delta)\right)\left[\|y_n - x^*\| + \|\sigma_n - x^*\|\right] \end{aligned} \tag{3.11}$$

Using $\delta < 1$ and $1-\alpha_n^1(1-\delta) \leq 1$, $1-\alpha_n^2(1-\delta) \leq 1$, and $1-\alpha_n^4(1-\delta) \leq 1$, for all $n \in \mathbb{N}$, in (3.11),

$$\left\|S\left[\left(1-\alpha_n^4-\alpha_n^5\right)u_n + \alpha_n^4 S u_n + \alpha_n^5 S v_n\right] - \sigma_{n+1}\right\| \leq \delta\left[\|y_n - x^*\| + \|\sigma_n - x^*\|\right] \tag{3.12}$$

If (3.12) is used in (3.9), then it is obtained

$$\|y_{n+1} - x^*\| \leq \delta\|y_n - x^*\| + \varepsilon_n + \delta\|\sigma_n - x^*\| + \|\sigma_{n+1} - x^*\|$$

Let $t_n := \varepsilon_n + \delta\|\sigma_n - x^*\| + \|\sigma_{n+1} - x^*\|$. By hypotheses, $\lim_{n\to\infty} t_n = 0$. Thus, by Lemma 2.5, $\lim_{n\to\infty} y_n = x^*$. This completes the proof. □

3.2. Data Dependency Results

In this part, we give a result regarding the data dependency of SNIA iterative algorithm for mappings satisfying quasi contractive condition (2.4).

Theorem 3.7. Let E, C, and S be as in Theorem 3.1. Let $\widetilde{S}$ be an approximate mapping of S as in Definition 2.4 with a suitable error ε. Let $(\sigma_n)_n$ be the sequence generated by SNIA iterative algorithm and let the sequence $(\widetilde{\sigma}_n)_n$ be as follows:

$$\begin{aligned} &\widetilde{\sigma}_1 \in C \\ &\widetilde{\varphi}_n = \widetilde{S}[(1-\alpha_n^1)\,\widetilde{\sigma}_n + \alpha_n^1 \widetilde{S}\widetilde{\sigma}_n] \\ &\widetilde{\tau}_n = \widetilde{S}[(1-\alpha_n^2-\alpha_n^3)\,\widetilde{\varphi}_n + \alpha_n^2 \widetilde{S}\widetilde{\varphi}_n + \alpha_n^3 \widetilde{S}\widetilde{\sigma}_n] \\ &\widetilde{\sigma}_{n+1} = \widetilde{S}[(1-\alpha_n^4-\alpha_n^5)\,\widetilde{\tau}_n + \alpha_n^4 \widetilde{S}\widetilde{\tau}_n + \alpha_n^5 \widetilde{S}\widetilde{\varphi}_n], \quad n \in \mathbb{N} \end{aligned} \tag{3.13}$$

where $(\alpha_n^i)_{n=1}^{\infty} \subset [0,1]$, for $i \in \{1,2,3,4,5\}$, are sequences satisfying the conditions in Theorem 3.1. If $Sx^* = x^*$ and $\widetilde{S}\widetilde{x}^* = \widetilde{x}^*$ such that $\lim_{n\to\infty} \widetilde{\sigma}_n = \widetilde{x}^*$, then

$$\|x^* - \widetilde{x}^*\| \le \frac{1+\delta}{1-\delta}\,\varepsilon$$

PROOF. By Definition 2.4 and (2.4), the mapping S satisfies the below inequality, for all $x, \widetilde{x} \in C$:

$$\|Sx - \widetilde{S}\widetilde{x}\| \le \|Sx - x^*\| + \|S\widetilde{x} - x^*\| + \varepsilon \le 2\delta\|x - x^*\| + \delta\|x - \widetilde{x}\| + \varepsilon \tag{3.14}$$

By the definition of SNIA iterative algorithm, (3.13), and (3.14),

$$\begin{aligned} \|\sigma_{n+1} - \widetilde{\sigma}_{n+1}\| \le \delta \Big\| \Big[\Big(1-\alpha_n^4-\alpha_n^5\Big)\tau_n + \alpha_n^4 S\tau_n + \alpha_n^5 S\varphi_n\Big] - \Big(1-\alpha_n^4-\alpha_n^5\Big)\widetilde{\tau}_n - \alpha_n^4 \widetilde{S}\widetilde{\tau}_n - \alpha_n^5 \widetilde{S}\widetilde{\varphi}_n] \Big\| \\ + 2\delta \Big\| \Big[\Big(1-\alpha_n^4-\alpha_n^5\Big)\tau_n + \alpha_n^4 S\tau_n + \alpha_n^5 S\varphi_n\Big] - x^* \Big\| + \varepsilon \end{aligned}$$

and, by using (3.14) and (2.4),

$$\begin{aligned} \|\sigma_{n+1} - \widetilde{\sigma}_{n+1}\| &\le \delta\Big(1-\alpha_n^4-\alpha_n^5\Big)\|\tau_n - \widetilde{\tau}_n\| + \delta\alpha_n^4\|S\tau_n - \widetilde{S}\widetilde{\tau}_n\| + +\delta\alpha_n^5\|S\varphi_n - \widetilde{S}\widetilde{\varphi}_n\| \\ &\quad + 2\delta\Big(1-\alpha_n^4-\alpha_n^5\Big)\|\tau_n - x^*\| + 2\delta\alpha_n^4\|S\tau_n - x^*\| + 2\delta\alpha_n^5\|S\varphi_n - x^*\| + \varepsilon \\ &\le \delta\Big(1-\alpha_n^4-\alpha_n^5\Big)\|\tau_n - \widetilde{\tau}_n\| + \delta^2\alpha_n^4\|\tau_n - \widetilde{\tau}_n\| + 2\delta^2\alpha_n^4\|\tau_n - x^*\| + \delta\alpha_n^4\varepsilon \\ &\quad + \delta^2\alpha_n^5\|\varphi_n - \widetilde{\varphi}_n\| + 2\delta^2\alpha_n^5\|\varphi_n - x^*\| + \delta\alpha_n^5\varepsilon + 2\delta\Big(1-\alpha_n^4-\alpha_n^5\Big)\|\tau_n - x^*\| \\ &\quad + 2\delta^2\alpha_n^4\|\tau_n - x^*\| + 2\delta^2\alpha_n^5\|\varphi_n - x^*\| + \varepsilon \end{aligned}$$

By arranging the last inequality,

$$\begin{aligned} \|\sigma_{n+1} - \widetilde{\sigma}_{n+1}\| \le\ &\{\delta\left(1-\alpha_n^4-\alpha_n^5\right) + \delta^2\alpha_n^4\}\|\tau_n - \widetilde{\tau}_n\| + \delta^2\alpha_n^5\|\varphi_n - \widetilde{\varphi}_n\| \\ &+2\delta\{1-\alpha_n^4-\alpha_n^5+2\delta\alpha_n^4\}\|\tau_n - x^*\| + 4\delta^2\alpha_n^5\|\varphi_n - x^*\| + \delta\alpha_n^4\varepsilon + \delta\alpha_n^5\varepsilon + \varepsilon \end{aligned} \tag{3.15}$$

By following similar steps above,

$$\begin{aligned} \|\tau_n - \widetilde{\tau}_n\| \le\ &\{\delta\left(1-\alpha_n^2-\alpha_n^3\right) + \delta^2\alpha_n^2\}\|\varphi_n - \widetilde{\varphi}_n\| + \delta^2\alpha_n^3\|\sigma_n - \widetilde{\sigma}_n\| \\ &+2\delta\{1-\alpha_n^2-\alpha_n^3+2\delta\alpha_n^2\}\|\varphi_n - x^*\| + 4\delta^2\alpha_n^3\|\sigma_n - x^*\| + \delta\alpha_n^2\varepsilon + \delta\alpha_n^3\varepsilon + \varepsilon \end{aligned} \tag{3.16}$$

and

$$\|\varphi_n - \widetilde{\varphi}_n\| \le \delta\Big\{1-\alpha_n^1(1-\delta)\Big\}\|\sigma_n - \widetilde{\sigma}_n\| + 2\delta\{1-\alpha_n^1+2\delta\alpha_n^1\}\|\sigma_n - x^*\| + \delta\alpha_n^1\varepsilon + \varepsilon \tag{3.17}$$

If (3.16) and (3.17) are used in (3.15), we obtain the following inequality:

$$\|\sigma_{n+1} - \widetilde{\sigma}_{n+1}\| \leq A\|\sigma_n - \widetilde{\sigma}_n\| + B\|\sigma_n - x^*\| + C\|\varphi_n - x^*\| + D\|\tau_n - x^*\| + E \tag{3.18}$$

where

$$\begin{aligned} A &:= \delta(1-\alpha_n^1(1-\delta))\left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta(1-\alpha_n^2-\alpha_n^3)+\delta^2\alpha_n^2]+\delta^2\alpha_n^5\right\} \\ &\quad +\delta^2\alpha_n^3\{\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4\} \\ B &:= 2\delta(1-\alpha_n^1+2\delta\alpha_n^1)\left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta(1-\alpha_n^2-\alpha_n^3)+\delta^2\alpha_n^2]+\delta^2\alpha_n^5\right\} \\ &\quad +4\delta^2\alpha_n^3\{\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4\} \\ C &:= 2\delta(1-\alpha_n^2-\alpha_n^3+2\delta\alpha_n^2)[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4]+4\delta^2\alpha_n^5 \\ D &:= 2\delta(1-\alpha_n^4-\alpha_n^5+2\delta\alpha_n^4) \end{aligned}$$

and

$$\begin{aligned} E &:= \left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta(1-\alpha_n^2-\alpha_n^3)+\delta^2\alpha_n^2]+\delta^2\alpha_n^5\right\}(\delta\alpha_n^1\varepsilon+\varepsilon) \\ &\quad +[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta\alpha_n^2\varepsilon+\delta\alpha_n^3\varepsilon+\varepsilon]+\delta\alpha_n^4\varepsilon+\delta\alpha_n^5\varepsilon+\varepsilon \end{aligned}$$

Arrange the number A,

$$\begin{aligned} A &= \left[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4\right]\left\{\left[\delta(1-\alpha_n^2-\alpha_n^3)+\delta^2\alpha_n^2\right]\left[\delta(1-\alpha_n^1(1-\delta))\right]+\delta^2\alpha_n^3\right\} \\ &\quad +\delta^3\alpha_n^5(1-\alpha_n^1(1-\delta)) \end{aligned}$$

Since $\delta \in [0,1)$ and $1-\alpha_n^1(1-\delta) \leq 1$, for all $n \in \mathbb{N}$,

$$\begin{aligned} A &\leq \left[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4\right]\left\{\delta(1-\alpha_n^3)\delta(1-\alpha_n^1(1-\delta))+\delta^2\alpha_n^3\right\}+\delta^3\alpha_n^5 \\ &\leq \left[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4\right]\left\{\delta(1-\alpha_n^3)\delta+\delta^2\alpha_n^3\right\}+\delta^3\alpha_n^5 \\ &\leq \left[\delta(1-\alpha_n^4-\alpha_n^5)+\delta\alpha_n^4\right]\delta^2+\delta^3\alpha_n^5=\delta^3 \end{aligned} \tag{3.19}$$

Since $\delta \in [0,1)$, and $1-\alpha_n^3 \leq 1$ and $1-\alpha_n^5 \leq 1$, for all $n \in \mathbb{N}$,

$$\begin{aligned} B &= 2\delta(1-\alpha_n^1+2\delta\alpha_n^1)\left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta(1-\alpha_n^2-\alpha_n^3)+\delta^2\alpha_n^2]+\delta^2\alpha_n^5\right\} \\ &\quad +4\delta^2\alpha_n^3\{\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4\} \\ &\leq 2\delta(1-\alpha_n^1+2\delta\alpha_n^1)\left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta\alpha_n^4][\delta(1-\alpha_n^2-\alpha_n^3)+\delta\alpha_n^2]+\delta^2\alpha_n^5\right\} \\ &\quad +4\delta^2\alpha_n^3\{\delta(1-\alpha_n^4-\alpha_n^5)+\delta\alpha_n^4\} \\ &= 2\delta(1-\alpha_n^1+2\delta\alpha_n^1)\left\{\delta(1-\alpha_n^5)\delta(1-\alpha_n^3)+\delta^2\alpha_n^5\right\}+4\delta^3\alpha_n^3(1-\alpha_n^5) \\ &\leq 2\delta(1-\alpha_n^1+2\delta\alpha_n^1)\left\{\delta^2+\delta^2\alpha_n^5\right\}+4\delta^3\alpha_n^3 \end{aligned} \tag{3.20}$$

Using $1-\alpha_n^1 \leq 1$, $\alpha_n^5 \leq 1$ and $\alpha_n^3 \leq 1$, for all $n \in \mathbb{N}$ in (3.20),

$$B \leq 2\delta(1-\alpha_n^1+2\delta\alpha_n^1)2\delta^2+4\delta^3 \leq 2\delta(1+2\delta)2\delta^2+4\delta^3=8\delta^3(1+\delta) \tag{3.21}$$

Since $1-\alpha_n^2-\alpha_n^3 \leq 1$, $\alpha_n^2 \leq 1$, $\delta < 1$, and $\alpha_n^5 \leq 1$, $1-\alpha_n^5 \leq 1$, for all $n \in \mathbb{N}$, we get

$$\begin{aligned} C &= 2\delta(1-\alpha_n^2-\alpha_n^3+2\delta\alpha_n^2)[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4]+4\delta^2\alpha_n^5 \\ &\leq 2\delta(1+2\delta)[\delta(1-\alpha_n^4-\alpha_n^5)+\delta\alpha_n^4]+4\delta^2\alpha_n^5 \leq 2\delta(1+2\delta)\delta(1-\alpha_n^5)+4\delta^2 \leq 2\delta^2(3+2\delta) \end{aligned} \tag{3.22}$$

Since $1-\alpha_n^4-\alpha_n^5 \leq 1$ and $\alpha_n^4 \leq 1$, for all $n \in \mathbb{N}$,

$$D = 2\delta(1-\alpha_n^4-\alpha_n^5+2\delta\alpha_n^4) \leq 2\delta(1+2\delta) \tag{3.23}$$

Using $\delta \in [0,1)$ and $\alpha_n^2 + \alpha_n^3 \leq 1$ and $\alpha_n^4 + \alpha_n^5 \leq 1$, for all $n \in \mathbb{N}$,

$$\begin{aligned}
E &= \left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta(1-\alpha_n^2-\alpha_n^3)+\delta^2\alpha_n^2]+\delta^2\alpha_n^5\right\}(\delta\alpha_n^1\varepsilon+\varepsilon) \\
&\quad + [\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta\alpha_n^2\varepsilon+\delta\alpha_n^3\varepsilon+\varepsilon]+\delta\alpha_n^4\varepsilon+\delta\alpha_n^5\varepsilon+\varepsilon \\
&\leq \left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta(1-\alpha_n^3)]+\delta^2\alpha_n^5\right\}(\delta\alpha_n^1\varepsilon+\varepsilon) \\
&\quad + [\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta\varepsilon(\alpha_n^2+\alpha_n^3)+\varepsilon]+\delta\varepsilon(\alpha_n^4+\alpha_n^5)+\varepsilon \\
&\leq \left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta\alpha_n^4][\delta(1-\alpha_n^3)]+\delta^2\alpha_n^5\right\}(\delta\alpha_n^1\varepsilon+\varepsilon)+[\delta(1-\alpha_n^4-\alpha_n^5)+\delta\alpha_n^4][\delta\varepsilon+\varepsilon]+\delta\varepsilon+\varepsilon \\
&= \left\{\delta(1-\alpha_n^5)(1-\alpha_n^3)+\delta\alpha_n^5\right\}\delta\varepsilon(\delta\alpha_n^1+1)+\delta(1-\alpha_n^5)[\delta\varepsilon+\varepsilon]+\delta\varepsilon+\varepsilon
\end{aligned}$$

Since $\delta \in [0,1)$ and $1-\alpha_n^3 \leq 1$, $1-\alpha_n^5 \leq 1$, and $\alpha_n^1 \leq 1$, for all $n \in \mathbb{N}$, we get the following inequality for the number E:

$$\begin{aligned}
E &\leq \left\{\delta(1-\alpha_n^5)+\delta\alpha_n^5\right\}\delta\varepsilon(\delta\alpha_n^1+1)+(\delta\varepsilon+\varepsilon)[\delta(1-\alpha_n^5)+1] \\
&= \delta^2\varepsilon(\delta\alpha_n^1+1)+(\delta\varepsilon+\varepsilon)[\delta(1-\alpha_n^5)+1] \\
&\leq \delta^2\varepsilon(\delta+1)+(\delta\varepsilon+\varepsilon)(\delta+1)=\varepsilon(\delta+1)(\delta^2+\delta+1)
\end{aligned} \tag{3.24}$$

Therefore, using (3.19) and (3.21)-(3.24) in (3.18),

$$\begin{aligned}
\|\sigma_{n+1}-\widetilde{\sigma}_{n+1}\| \leq{}& \delta^3\|\sigma_n-\widetilde{\sigma}_n\|+8\delta^3(1+\delta)\|\sigma_n-x^*\|+2\delta^2(3+2\delta)\|\varphi_n-x^*\| \\
&+2\delta(1+2\delta)\|\tau_n-x^*\|+\varepsilon(\delta+1)(\delta^2+\delta+1)
\end{aligned} \tag{3.25}$$

By (3.1) and (3.3),

$$\|\varphi_n-x^*\| \leq \|\sigma_n-x^*\| \quad \text{and} \quad \|\tau_n-x^*\| \leq \|\sigma_n-x^*\|$$

Besides, under hypotheses, by Theorem 3.1, since $\lim\limits_{n\to\infty}\|\sigma_n-x^*\|=0$,

$$\lim_{n\to\infty}\|\varphi_n-x^*\|=\lim_{n\to\infty}\|\tau_n-x^*\|=0$$

Thus, taking the limit for $n\to\infty$ in (3.25),

$$\|x^*-\widetilde{x}^*\| \leq \frac{\varepsilon(\delta+1)(\delta^2+\delta+1)}{1-\delta^3}=\frac{1+\delta}{1-\delta}\varepsilon$$

□

4. Numerical Examples

In this section, we provide some numerical examples that support our theoretical results.

The first example, built on an infinite dimensional Banach space and satisfying the conditions of Theorem 3.1 and Theorem 3.3, shows that the SNIA iterative algorithm is more effective than Karakaya, SP, and two-step Mann iterative algorithms in terms of convergence.

Example 4.1. Let E be the Banach space $l_1 = \{(x_i)_{i=1}^{\infty} \subset \mathbb{K} : \sum_{i=1}^{\infty}|x_i| < \infty\}$ endowed with norm $\|(x_i)_i\|_1 = \sum\limits_{i=1}^{\infty}|x_i|$ and be defined a sequence $(x_i)_i$ as follows:

$$\forall i \in \mathbb{N}, \quad x_i = (x_n^i)_{n=1}^{\infty}, \quad x_n^i = \begin{cases} 0, n \neq i \\ \dfrac{1}{i}, n = i \end{cases}$$

It is clear that $(x_i)_i$ is a sequence in E. Moreover, $\lim_{i\to\infty} \|x_i - 0\|_1 = 0$. We define the set

$$C := \left\{ \sum_{k=1}^{\infty} \mu_k x_k : (\mu_k)_{k=1}^{\infty} \in B_{l_1} \right\}$$

where B_{l_1} is the closed unit ball of l_1. Since $(x_n)_n$ is a null sequence in E, it is well known in the literature that C is a convex and closed subset in E [20, 21]. Moreover, by Grothendieck's characterization [22], we can say that C is a proper subset of B_E. Using the above definition of the sequence $(x_i)_i$, we get the set C as follows:

$$C = \left\{ \left(\frac{\mu_k}{k}\right)_{k=1}^{\infty} : (\mu_k)_{k=1}^{\infty} \in B_{l_1} \right\}$$

We define a mapping $S : C \to C$ by

$$S\left(\left(\frac{\mu_k}{k}\right)_{k=1}^{\infty}\right) := \left(\frac{k}{4}\left(\frac{\mu_k}{k}\right)^2\right)_{k=1}^{\infty}$$

It can be observed that the mapping S is well defined and S has a unique fixed point $x^* = (0, 0, 0, 0, \cdots)$. We show that there exist a number $\delta \in [0, 1)$ such that $\|Sx - x^*\|_1 \le \delta \|x - x^*\|_1$, for all $x \in C$. If $x \in C$, then there is a $(\mu_k)_{k=1}^{\infty} \in B_{l_1}$ such that $x = \left(\frac{\mu_k}{k}\right)_{k=1}^{\infty}$. Thus,

$$\|Sx - x^*\|_1 = \left\| \left(\frac{1}{4}\frac{\mu_k^2}{k}\right)_k \right\|_1 = \frac{1}{4}\sum_{k=1}^{\infty} \frac{|\mu_k^2|}{k} \le \frac{1}{4}\sum_{k=1}^{\infty} \frac{|\mu_k|}{k} = \frac{1}{4}\|x - x^*\|_1$$

This shows that $\delta = \frac{1}{4}$. That is, S satisfies quasi contractive condition (2.4). However, we denote that for all $x, y \in C$, $\|Sx - Sy\|_1 \nleq \frac{1}{4}\|x - y\|_1$. For example, for $x = (1, 0, 0, 0, \cdots)$ and $y = \left(\frac{1}{2}, 0, 0, 0, \cdots\right)$, $\|Sx - Sy\|_1 \nleq \frac{1}{4}\|x - y\|_1$. Let the initial terms of all mentioned algorithms be $s_0 = \sigma_0 = \left(\frac{1}{n2^n}\right)_n$, $\alpha_n^4 = \alpha_n^2 = \alpha_n^1 = 1 - \frac{1}{n^5 + 1}$, and $\alpha_n^5 = \alpha_n^3 = \frac{1}{2(n^5 + 1)}$, for all $n \in \mathbb{N}$, satisfying $(\alpha_n^4 + \alpha_n^5)_n \subset [0, 1]$ and $(\alpha_n^2 + \alpha_n^3)_n \subset [0, 1]$. Figure 1 manifests that the sequence generated by SNIA iterative algorithm converges the fixed point $x^* = 0$ of S faster than the sequences generated by Karakaya, Mann, SP, and two-step Mann iterative algorithms.

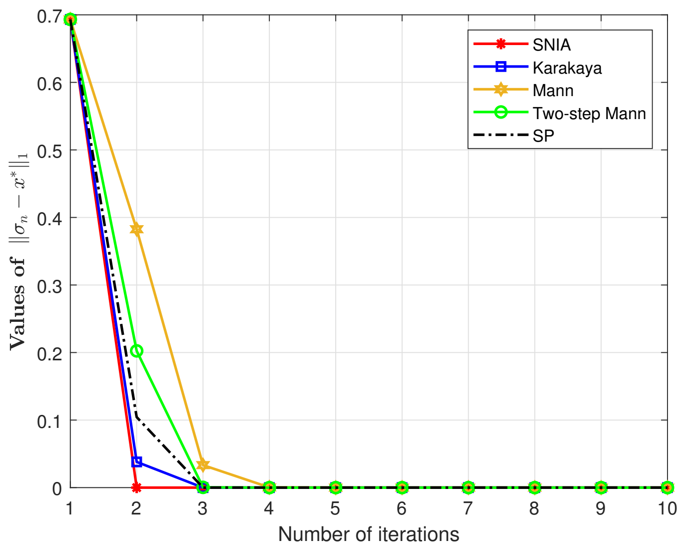


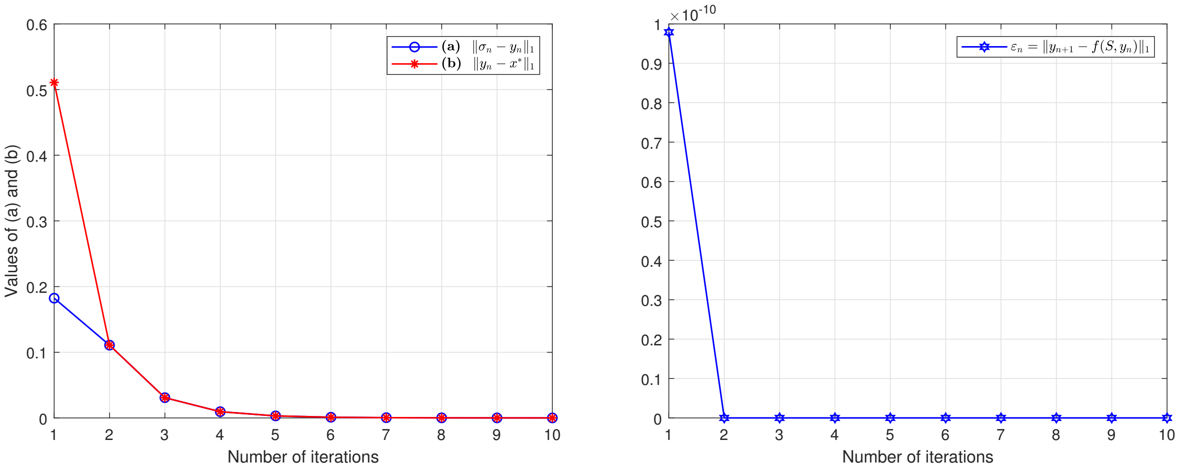
Figure 1. Convergence behaviors of algorithms in Example 4.1

The following example, which supports the accuracy of the result in Theorem 3.6 shows that SNIA iterative algorithm in Example 4.1 is weakly S-stable.

Example 4.2. Let E, C, and S be as in Example 4.1. We define the sequence $(y_n)_n$ in C as follows:

$$\forall n \in \mathbb{N}, \quad y_n = (y_i^n)_{i=1}^{\infty}, \quad y_i^n = \begin{cases} 0, i < n \\ \dfrac{2^i}{i5^i}, i \geq n \end{cases}$$

Figure 2 (a) shows that the $(y_n)_n$ is an approximate sequence of the sequence $(\sigma_n)_n$ generated by SNIA iterative algorithm. Further, Figure 2 (a)-(b) manifests that $\lim_{n\to\infty} \varepsilon_n = 0$ implies $\lim_{n\to\infty} y_n = x^*$. In other words, SNIA iterative algorithm is weakly S-stable.



(a) **(b)**

Figure 2. Graphs showing the convergence states of the sequences $(y_n - \sigma_n)_n$, $(y_n - x^*)_n$, and $(\varepsilon_n)_n$

The following example deals with the data dependency of the sequence $(\sigma_n)_n$ generated by SNIA iterative algorithm in Example 4.1.

Example 4.3. Let E, C, and S be as in Example 4.1. We define a mapping $\widetilde{S} : C \to C$ as in the following:

$$\widetilde{S}\left(\left(\frac{\alpha_k}{k}\right)_k\right) := (\beta_k)_k, \quad \beta_k = \begin{cases} 1/4, & k = 1 \\ \dfrac{\alpha_{k-1}}{k3^{k-1}}, & k \geq 2 \end{cases}$$

where $(\alpha_k)_{k=1}^{\infty} \in B_E$. Then, $\widetilde{S} : C \to C$ is well defined, and for all $x = \left(\frac{\alpha_k}{k}\right)_k \in C$,

$$\begin{aligned} \|Sx - \widetilde{S}x\|_1 &= \frac{1}{4}\left|\alpha_1^2 - 1\right| + \sum_{k=2}^{\infty} \frac{1}{k}\left|\frac{\alpha_k^2}{4} - \frac{\alpha_{k-1}}{3^{k-1}}\right|, \quad ((\alpha_k)_k \in B_{l_1}) \\ &\leq \frac{1}{4} + \frac{1}{8}\sum_{k=2}^{\infty}|\alpha_k^2| + \frac{1}{6}\sum_{k=2}^{\infty}|\alpha_{k-1}|, \quad ((\alpha_k)_k \in B_{l_1}) \\ &\leq \frac{1}{4} + \frac{1}{8} + \frac{1}{6} = 0.5416666 = \varepsilon \end{aligned}$$

Thus, we can consider the mappings S and $\widetilde{S}$ as approximate operators in Definition 2.4. If $\widetilde{S}$ has a fixed point $\widetilde{x}^*$ and the sequence $(\widetilde{\sigma}_n)_n$ generated by (3.13) with the choice of the coefficient sequences satisfying the conditions in Theorem 3.7, converges to $\widetilde{x}^*$, then without knowing and calculating $\widetilde{x}^*$, we can determine an upper bound for $\widetilde{x}^*$ by (3.7) as follows:

$$\|x^* - \widetilde{x}^*\| \leq \frac{1+\delta}{1-\delta}\varepsilon = \frac{1+1/4}{1-1/4}(0.5416666) = 0.902730$$

We get that the fixed point of $\widetilde{S}$ as $\widetilde{x}^* = \left(\frac{1}{3^{\frac{k(k-1)}{2}} 4k}\right)_k$. Figure 3 shows that the sequence $(\widetilde{\sigma}_n)_n$ generated by (3.13) converges to $\widetilde{x}^*$. In addition, $\|x^* - \widetilde{x}^*\| = \frac{1177}{3992} = 0.2948$. That is, (3.7) is satisfied.

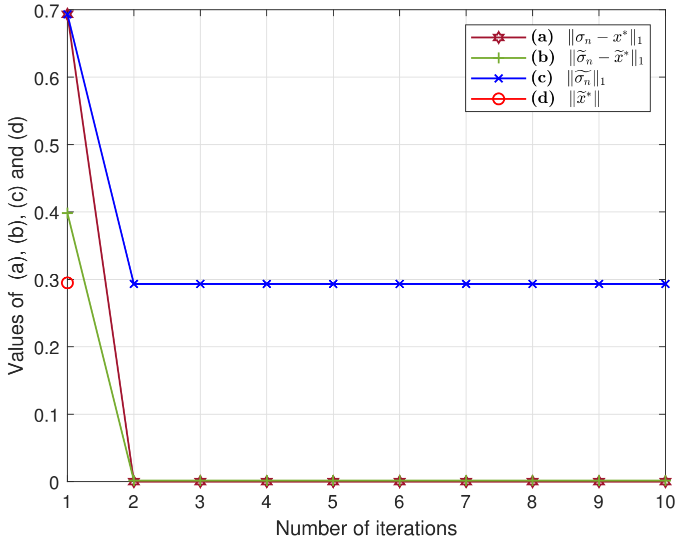


Figure 3. Graphs showing the values of $\|\sigma_n - \widetilde{x}^*\|_1$, $\|\widetilde{\sigma}_n - \widetilde{x}^*\|_1$, $\|\widetilde{\sigma}_n\|_1$, and $\|\widetilde{x}^*\|_1$, for $n \in \{1, 2, ..., 10\}$

5. Conclusion

In this study, the convergence result of the SNIA iterative algorithm introduced by Chauhan et al. [13] has been revised and improved while simultaneously obtaining its weak stability and data dependency. The findings of this study are substantiated by nontrivial examples in an infinite dimensional Banach space, thereby bridging the gap between practice and theory. Based on the graphs presented, it has been observed that the algorithm yields superior results in numerical examples. Furthermore, the algorithm's convergence, which does not necessitate additional conditions (except for convexity) on coefficient sequences, sets it apart from the aforementioned algorithms. Consequently, it can be concluded that the algorithm with the stability and data dependency properties is more effective for quasi-contractive mappings when compared to the algorithms discussed in this study, based on both theoretical and practical outcomes. In future studies, researchers can examine the convergence of the SNIA iterative algorithm for different mapping classes under appropriate conditions. Moreover, they can compare the algorithm's performance speed with existing algorithms in the literature for these mapping classes.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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