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RESEARCH ARTICLE

Trace formula for finite groups and the Macdonald correspondence for $\mathsf{GL}_n(\mathbb{F}_q)$*

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*A part of this work was presented at the Selçuk University Mathematics Colloquium in April 27, 2022 and completed while visiting the Abdus Salam ICTP in June 2024.

ABSTRACT

Let G be a finite group. The trace formula for G, which is the trivial case of the Arthur trace formula, is well known with many applications. In this note, we further consider a subgroup Γ of G and a representation $\rho : \Gamma \to \mathsf{GL}(V_\rho)$ of Γ on a finite dimensional $\mathbb{C}$-vector space V_ρ, and compute the trace $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ of the operator $\mathsf{Ind}_\Gamma^G \rho(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ for any function $f : G \to \mathbb{C}$ in two different ways. The expressions for $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ denoted by $J(\rho, f)$ and $I(\rho, f)$ are the spectral side and the geometric side of the trace formula for $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$, respectively. The identity $J(\rho, f) = \mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) = I(\rho, f)$ is a generalization of the trace formula for the finite group G. This theory is then applied to the "automorphic side" of the Macdonald correspondence for $\mathsf{GL}_n(\mathbb{F}_q)$; namely, to the "automorphic side" of the local 0-dimensional Langlands correspondence for $\mathsf{GL}(n)$, where new identities are obtained for the ϵ-factors of representations of $\mathsf{GL}_n(\mathbb{F}_q)$.

Mathematics Subject Classification (2020): 20C15, 11F72

Keywords: Trace formula, finite group, Macdonald correspondence

1. INTRODUCTION

This short note which is the revised version of our colloquium talk notes that we delivered at Selçuk University, Konya in 2022, concerns the trivial case of the Arthur trace formula, namely the trace formula for finite groups. Let G be a finite group. The trace formula for G is an identity involving characters of the finite group G, and although simple, this formula is of central importance with deep applications in the theory of linear representations of G over $\mathbb{C}$. For instance, using the trace formula for G, it is well known that major theorems like the Frobenius reciprocity law (Example 6.1) and the Plancherel formula (Example 6.2) for G follow directly.

In this note, we further consider a subgroup Γ of G and a representation

$$\rho : \Gamma \to \mathsf{GL}(V_\rho)$$

of Γ on a finite dimensional vector space V_ρ over $\mathbb{C}$, and for any function $f : G \to \mathbb{C}$, we compute the trace $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ of the operator

$$\mathsf{Ind}_\Gamma^G \rho(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$$

which is defined by

$$(\mathsf{Ind}_\Gamma^G(\rho)(f))(\varphi) = \sum_{g \in G} f(g)[\mathsf{Ind}_\Gamma^G(\rho)(g)](\varphi), \ {}^\forall \varphi \in \mathsf{Ind}_\Gamma^G(V_\rho)$$

in two different ways. The first expression for $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ denoted by $J(\rho, f)$ involves multiplicities of irreducible representations appearing in $\mathsf{Ind}_\Gamma^G(\rho)$ and called the spectral side of the trace formula for $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$. The second expression $I(\rho, f)$ for $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$, which is called the geometric side of the trace formula for $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$, is

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constructed by the conjugacy classes of Γ and G. The identity (Theorem 3.1 and Theorem 5.1)

$$J(\rho, f) = \mathrm{Tr}(\mathsf{Ind}_{\Gamma}^{G}(\rho)(f)) = I(\rho, f)$$

that we derive and call the trace formula for G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ in this note, is a generalization of the well known trace formula for the finite group G which corresponds to the case $\Gamma = \langle 1 \rangle$ and $\rho = \mathbb{1}_{\langle 1 \rangle} : \langle 1 \rangle \to \mathrm{GL}(\mathbb{C})$. In the remainder of this work, that is in Section 7, we apply this theory to the "automorphic side" of the Macdonald correspondence for $\mathsf{GL}_n(\mathbb{F}_q)$; namely, to the "automorphic side" of the local 0-dimensional Langlands correspondence for $\mathsf{GL}(n)$, where new identities are obtained for the ϵ-factors of representations of $\mathsf{GL}_n(\mathbb{F}_q)$.

The main references that we follow closely in this work are Terras Terras (1999) and Yang Yang (2006). However, we will deal with the trace formula for finite groups in full generality. The trace formula stated and proved in this note should be considered "folklore", videlicet, well-known to researchers in the area, and seems only treated recently in the M.Sc. thesis of Chasek Chasek (2023) and in a note of Lee Lee (2022). Therefore, the only contribution of this note is the last section on the Macdonald correspondence for $\mathsf{GL}_n(\mathbb{F}_q)$, where the main references that we follow are Macdonald (1980); Piatetski-Shapiro (1983); Silberger and Zink (2008); Ye and Zelingher (2021)

2. THE REPRESENTATION $\mathsf{IND}_{\Gamma}^{G}(\rho)$ OF G ON $\mathsf{IND}_{\Gamma}^{G}(V_\rho)$ OVER $\mathbb{C}$ INDUCED FROM $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ UP TO G

To fix the very basic notation, let G be a finite group, Γ a fixed subgroup of G of index $(G : \Gamma) = \iota$, and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ a representation of Γ on a d-dimensional vector space V_ρ over $\mathbb{C}$ whose character is denoted by $\chi_\rho : \Gamma \to \mathbb{C}$ as usual. Set $\chi_\rho(1) = d_\rho$ the dimension $\dim_{\mathbb{C}}(V_\rho)$ of V_ρ called the degree of $\rho : \Gamma \to \mathrm{GL}(V_\rho)$). Introduce further the map

$$\widetilde{\rho} : G \to \left\langle \mathrm{GL}(V_\rho) \cup \{\mathbf{0} : V_\rho \to \{0_{V_\rho}\}\}\right\rangle$$

by defining

$$\widetilde{\rho}(x) = \begin{cases} \rho(x), & x \in \Gamma; \\ \mathbf{0}, & x \in G - \Gamma, \end{cases} \quad {}^{\forall}x \in G.$$

Recall that Piatetski-Shapiro (1983); Serre (1972), the $\mathbb{C}[G]$-module $\mathrm{Ind}_{\Gamma}^{G}(V_\rho)$ induced from the $\mathbb{C}[\Gamma]$-module V_ρ is defined by

$$\mathrm{Ind}_{\Gamma}^{G}(V_\rho) := \{\varphi : G \to V_\rho \mid \varphi(\gamma x) = \rho(\gamma)(\varphi(x)), {}^{\forall}x \in G, {}^{\forall}\gamma \in \Gamma\},$$

which defines a representation

$$\mathsf{Ind}_{\Gamma}^{G}(\rho) : G \to \mathrm{GL}(\mathsf{Ind}_{\Gamma}^{G}(V_\rho))$$

of the finite group G on the induced $\mathbb{C}$-linear space $\mathsf{Ind}_{\Gamma}^{G}(V_\rho)$ by

$$[\mathsf{Ind}_{\Gamma}^{G}(\rho)(g)](\varphi)(x) = \varphi(xg), \ {}^{\forall}g \in G, {}^{\forall}\varphi \in \mathsf{Ind}_{\Gamma}^{G}V_\rho, {}^{\forall}x \in G,$$

called the representation induced from $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ up to G.

Recall that $\mathsf{Ind}_{\Gamma}^{G}(V_\rho)$ is an inner product space under the inner product

$$\langle \bullet \mid \bullet \rangle : \mathsf{Ind}_{\Gamma}^{G}(V_\rho) \times \mathsf{Ind}_{\Gamma}^{G}(V_\rho) \to \mathbb{C}$$

on $\mathsf{Ind}_{\Gamma}^{G}(V_\rho)$ defined by

$$\langle \varphi_1 \mid \varphi_2 \rangle := \sum_{i=1}^{\iota} {}^{\mathsf{H}}[\varphi_2(g_i)]_{\mathcal{B}_{V_\rho}} [\varphi_1(g_i)]_{\mathcal{B}_{V_\rho}}, \ {}^{\forall}\varphi_1, \varphi_2 \in \mathsf{Ind}_{\Gamma}^{G}(V_\rho),$$

which neither depends on a choice of a complete set of representatives $\mathcal{R}_{\Gamma\backslash G} = \{g_1, \cdots, g_\iota\} \subset G$ of the coset space $\Gamma\backslash G$ nor a choice of an ordered basis $\mathcal{B}_{V_\rho} = \{v_1, \cdots, v_d\}$ of V_ρ over $\mathbb{C}$ (in particular, nor a choice of an ordered orthonormal basis $\mathcal{N}_{V_\rho} = \{e_1, \cdots, e_d\}$ of V_ρ over $\mathbb{C}$), and

$$\mathcal{N}_{\mathsf{Ind}_{\Gamma}^{G}(V_\rho)} := \{\varphi_{ij}\}_{\substack{i=1,\cdots,\iota \\ j=1,\cdots,d}}$$

is an orthonormal basis of $\mathsf{Ind}_{\Gamma}^{G}(V_\rho)$ over $\mathbb{C}$, where for $i = 1, \cdots, \iota$ and $j = 1, \cdots, d$

$$\varphi_{ij} : G \to V_\rho$$

is defined by

$$\varphi_{ij}(x) = \widetilde{\rho}(xg_i^{-1})e_j, \ {}^{\forall}x \in G.$$

The $(G:\Gamma)\dim(V_\rho) = \iota d$-dimensional representation $\mathsf{Ind}_\Gamma^G(\rho)$ of G on $\mathsf{Ind}_\Gamma^G(V_\rho)$ over $\mathbb{C}$ decomposes into the direct sum

$$\mathsf{Ind}_\Gamma^G(\rho) = \bigoplus_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathsf{Ind}_\Gamma^G(\rho))\pi_o \tag{1}$$

of non-isomorphic irreducible representations π_o of G over $\mathbb{C}$ with $m(\pi_o, \mathsf{Ind}_\Gamma^G(\rho)) \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\Pi(G)$ denotes the set of all isomorphism classes $[\pi]$ of irreducible representations π of G over $\mathbb{C}$.

3. THE TRACE FORMULA FOR G WITH RESPECT TO THE SUBGROUP Γ AND $\rho : \Gamma \to \mathrm{GL}(V_\rho)$

Let $\mathsf{r} : G \to \mathrm{GL}(V_\mathsf{r})$ be a representation of the finite group G on a d-dimensional vector space V_r over $\mathbb{C}$. For any function $f : G \to \mathbb{C}$, there exists a $\mathbb{C}$-linear operator

$$\mathsf{r}(f) : V_\mathsf{r} \to V_\mathsf{r}$$

on V_r defined by

$$(\mathsf{r}(f))(v) = \sum_{g\in G} f(g)\mathsf{r}(g)(v), \ {}^\forall v \in V_\mathsf{r}.$$

We can compute the trace $\mathrm{Tr}(\mathsf{r}(f))$ of the operator $\mathsf{r}(f) : V_\mathsf{r} \to V_\mathsf{r}$ on V_r.

In this note, in particular, we are interested in computing the trace $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ of the linear operator $\mathsf{Ind}_\Gamma^G(\rho)(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ on $\mathsf{Ind}_\Gamma^G(V_\rho)$. Observe that, the operator

$$\mathsf{Ind}_\Gamma^G(\rho)(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$$

on the $\mathbb{C}$-linear space $\mathsf{Ind}_\Gamma^G(V_\rho)$ given by

$$(\mathsf{Ind}_\Gamma^G(\rho)(f))(\varphi) = \sum_{g\in G} f(g)[\mathsf{Ind}_\Gamma^G(\rho)(g)](\varphi), \ {}^\forall \varphi \in \mathsf{Ind}_\Gamma^G(V_\rho)$$

is defined explicitly by

$$(\mathsf{Ind}_\Gamma^G(\rho)(f)\varphi)(x) = \sum_{g\in G} f(g)[\mathsf{Ind}_\Gamma^G(\rho)(g)](\varphi)(x) = \sum_{g\in G} f(g)\varphi(xg), \ {}^\forall \varphi \in \mathsf{Ind}_\Gamma^G(V_\rho), \ {}^\forall x \in G.$$

Therefore changing the G-variables $xg \rightsquigarrow y$ and partitioning G as $G = \bigsqcup_{i=1}^{\iota} \Gamma g_i$, the equalities

$$\begin{aligned}(\mathsf{Ind}_\Gamma^G(\rho)(f)\varphi)(x) &= \sum_{y\in G} f(x^{-1}y)\varphi(y) \\ &= \sum_{i=1}^{\iota}\sum_{\gamma\in\Gamma} f(x^{-1}\gamma g_i)\varphi(\gamma g_i) \qquad {}^\forall \varphi \in \mathsf{Ind}_\Gamma^G(V_\rho), \ {}^\forall x \in G, \\ &= \sum_{i=1}^{\iota}\sum_{\gamma\in\Gamma} f(x^{-1}\gamma g_i)\rho(\gamma)(\varphi(g_i))\end{aligned} \tag{2}$$

follow immediately. Define now an $\mathsf{End}_\mathbb{C}(V_\rho)$-valued function

$$K_f : G \times G \to \mathsf{End}_\mathbb{C}(V_\rho)$$

on $G \times G$ by

$$K_f(x, y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\rho(\gamma), \ {}^\forall x, y \in G.$$

The operator $\mathsf{Ind}_\Gamma^G(\rho)(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ on $\mathsf{Ind}_\Gamma^G(V_\rho)$ is then an "integral operator" on $\mathsf{Ind}_\Gamma^G(V_\rho)$ with "kernel" $K_f : G \times G \to \mathsf{End}_\mathbb{C}(V_\rho)$, given by

$$(\mathsf{Ind}_\Gamma^G(\rho)(f)\varphi)(x) = \sum_{i=1}^{\iota} K_f(x, g_i)(\varphi(g_i)), \qquad {}^\forall \varphi \in \mathsf{Ind}_\Gamma^G(V_\rho), \ {}^\forall x \in G. \tag{3}$$

The trace formula for G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ is an identity that computes the trace $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ of the operator $\mathsf{Ind}_\Gamma^G(\rho)(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ on $\mathsf{Ind}_\Gamma^G(V_\rho)$ in two different ways:

$$\text{"Spectral side"} = \mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) = \text{"Geometric side"}.$$

More precisely, the trace formula for the finite group G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ states the following:

Theorem 3.1. (Trace formula for finite groups – Version 1) *For any function $f : G \to \mathbb{C}$, the trace $\mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f))$ of the operator*

$$\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$$

on the $\mathbb{C}$-linear space $\mathrm{Ind}_\Gamma^G(V_\rho)$ satisfies the identity

$$J(\rho, f) = \sum_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho))\mathrm{Tr}(\pi_o(f)) = \mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f)) = \sum_{\gamma \in \{\Gamma\}} \chi_\rho(\gamma)\frac{|G_\gamma|}{|\Gamma_\gamma|} \sum_{t \in G_\gamma \backslash G} f(t^{-1}\gamma t) = I(\rho, f), \tag{4}$$

where

- $\{\Gamma\}$ = *a set consisting of all representatives for the conjugacy classes in* Γ*;*
- $\Gamma_\gamma = \{\delta \in \Gamma \mid \delta^{-1}\gamma\delta = \gamma\}$ *for* $\gamma \in \{\Gamma\}$*;*
- $G_\gamma = \{g \in G \mid g^{-1}\gamma g = \gamma\}$ *for* $\gamma \in \{\Gamma\}$*.*

Here, $J(\rho, f)$ and $I(\rho, f)$ are called the spectral side and the geometric side of the trace formula for the finite group G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$, respectively.

In the next section we sketch a proof of this theorem.

4. PROOF OF THE TRACE FORMULA FOR G WITH RESPECT TO THE SUBGROUP Γ AND $\rho : \Gamma \to \mathrm{GL}(V_\rho)$

To establish the spectral side $J(\rho, f)$ of the trace formula for G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$, observe that the irreducible decomposition

$$\mathrm{Ind}_\Gamma^G(\rho) = \bigoplus_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho))\pi_o$$

of the representation $\mathrm{Ind}_\Gamma^G(\rho)$ of the finite group G on the vector space $\mathrm{Ind}_\Gamma^G(V_\rho)$ over $\mathbb{C}$ induced from $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ up to G, stated in (1) yields, for any function $f : G \to \mathbb{C}$, the decomposition

$$\mathrm{Ind}_\Gamma^G(\rho)(f) = \bigoplus_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho))\pi_o(f)$$

of the operator $\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$ on $\mathrm{Ind}_\Gamma^G(V_\rho)$. Therefore, the trace $\mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f))$ of $\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$ is given by

$$\mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f)) = \mathrm{Tr}\left(\bigoplus_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho))\pi_o(f)\right) = \sum_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho))\mathrm{Tr}(\pi_o(f)) = J(\rho, f),$$

which is the spectral side of the formula stated in (4).

For the geometric side $I(\rho, f)$ of the trace formula for G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$, recall that, by (3), the operator

$$\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$$

on the $\mathbb{C}$-linear space $\mathrm{Ind}_\Gamma^G(V_\rho)$ has an expression of the form

$$(\mathrm{Ind}_\Gamma^G(\rho)(f)\varphi)(x) = \sum_{i=1}^{\iota} K_f(x, g_i)(\varphi(g_i)), \qquad {}^\forall\varphi \in \mathrm{Ind}_\Gamma^G(V_\rho),\ {}^\forall x \in G.$$

On the other hand, the matrix representation $[\mathrm{Ind}_\Gamma^G(\rho)(f)]_{\mathcal{N}_{\mathrm{Ind}_\Gamma^G(V_\rho)}} \in \mathbb{C}^{\iota d \times \iota d}$ of the operator $\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$ on $\mathrm{Ind}_\Gamma^G(V_\rho)$ with respect to the "lexicographically" ordered orthonormal basis $\mathcal{N}_{\mathrm{Ind}_\Gamma^G(V_\rho)} = \{\varphi_{ij}\}_{\substack{i=1,\cdots,\iota \\ j=1,\cdots,d}}$ of $\mathrm{Ind}_\Gamma^G(V_\rho)$ is given by

$$[\mathrm{Ind}_\Gamma^G(\rho)(f)]_{\mathcal{N}_{\mathrm{Ind}_\Gamma^G(V_\rho)}} = \begin{bmatrix} \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{11}) \mid \varphi_{11} \right\rangle & \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{12}) \mid \varphi_{11} \right\rangle & \cdots & \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{\iota d}) \mid \varphi_{11} \right\rangle \\ \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{11}) \mid \varphi_{12} \right\rangle & \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{12}) \mid \varphi_{12} \right\rangle & \cdots & \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{\iota d}) \mid \varphi_{12} \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{11}) \mid \varphi_{\iota d} \right\rangle & \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{12}) \mid \varphi_{\iota d} \right\rangle & \cdots & \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{\iota d}) \mid \varphi_{\iota d} \right\rangle \end{bmatrix}.$$

Therefore, the trace $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ of the operator $\mathsf{Ind}_\Gamma^G(\rho)(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ on $\mathsf{Ind}_\Gamma^G(V_\rho)$ is given by

$$\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) = \sum_{(s,t)=(1,1)}^{(\iota,d)} \left\langle \mathsf{Ind}_\Gamma^G(\rho)(f)(\varphi_{st}) \mid \varphi_{st} \right\rangle,$$

which is in explicit form given by

$$\begin{aligned}
\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) &= \sum_{(s,t)=(1,1)}^{(\iota,d)} \left\langle \mathsf{Ind}_\Gamma^G(\rho)(f)(\varphi_{st}) \mid \varphi_{st} \right\rangle \\
&= \sum_{(s,t)=(1,1)}^{(\iota,d)} \sum_{i_o=1}^{\iota} {}^{\mathsf{H}}[\varphi_{st}(g_{i_o})]_{\mathcal{N}_{V_\rho}} [(\mathsf{Ind}_\Gamma^G(\rho)(f)(\varphi_{st}))(g_{i_o})]_{\mathcal{N}_{V_\rho}} \\
&= \sum_{(s,t)=(1,1)}^{(\iota,d)} \sum_{i_o=1}^{\iota} {}^{\mathsf{H}}[\varphi_{st}(g_{i_o})]_{\mathcal{N}_{V_\rho}} \left[\sum_{i=1}^{\iota} K_f(g_{i_o}, g_i)(\varphi_{st}(g_i)) \right]_{\mathcal{N}_{V_\rho}} \\
&= \sum_{(s,t)=(1,1)}^{(\iota,d)} \sum_{i_o=1}^{\iota} {}^{\mathsf{H}}[\varphi_{st}(g_{i_o})]_{\mathcal{N}_{V_\rho}} \left[\sum_{i=1}^{\iota} \sum_{\gamma\in\Gamma} f(g_{i_o}^{-1}\gamma g_i)\rho(\gamma)(\varphi_{st}(g_i)) \right]_{\mathcal{N}_{V_\rho}} \\
&= \sum_{(s,t)=(1,1)}^{(\iota,d)} \sum_{i_o=1}^{\iota} \sum_{j_o=1}^{d} \overline{\left\langle \varphi_{st}(g_{i_o}) \mid e_{j_o} \right\rangle} \sum_{i=1}^{\iota} \sum_{\gamma\in\Gamma} f(g_{i_o}^{-1}\gamma g_i) \left\langle \rho(\gamma)(\varphi_{st}(g_i)) \mid e_{j_o} \right\rangle.
\end{aligned}$$

Note that, for $s, i_o = 1, \cdots, \iota$; $t, j_o = 1, \cdots, d$

$$\varphi_{st}(g_{i_o}) = \widetilde{\rho}(g_{i_o}g_s^{-1})e_t = \begin{cases} \rho(g_{i_o}g_s^{-1})e_t = e_t, & \Gamma g_{i_o} = \Gamma g_s \Leftrightarrow i_o = s; \\ 0_{V_\rho}, & \Gamma g_{i_o} \neq \Gamma g_s \Leftrightarrow i_o \neq s \end{cases}$$

and

$$\left\langle \varphi_{st}(g_{i_o}) \mid e_{j_o} \right\rangle = \begin{cases} 1, & i_o = s \text{ and } j_o = t; \\ 0, & i_o = s \text{ and } j_o \neq t; \\ 0, & i_o \neq s. \end{cases}$$

Also, for $\gamma \in \Gamma$; $s, i, i_o = 1, \cdots, \iota$; $t, j_o = 1, \cdots, d$,

$$\rho(\gamma)(\varphi_{st}(g_i)) = \varphi_{st}(\gamma g_i) = \widetilde{\rho}(\gamma g_i g_s^{-1})e_t = \begin{cases} \rho(\gamma g_i g_s^{-1})e_t = \rho(\gamma)e_t, & \Gamma\gamma g_i = \Gamma g_i = \Gamma g_s \Leftrightarrow i = s; \\ 0_{V_\rho}, & \Gamma\gamma g_i = \Gamma g_i \neq \Gamma g_s \Leftrightarrow i \neq s \end{cases}$$

and

$$f(g_{i_o}^{-1}\gamma g_i) \left\langle \rho(\gamma)(\varphi_{st}(g_i)) \mid e_{j_o} \right\rangle = \begin{cases} f(g_{i_o}^{-1}\gamma g_i) \left\langle \rho(\gamma)e_t \mid e_{j_o} \right\rangle, & \Gamma\gamma g_i = \Gamma g_i = \Gamma g_s \Leftrightarrow i = s; \\ 0, & \Gamma\gamma g_i = \Gamma g_i \neq \Gamma g_s \Leftrightarrow i \neq s. \end{cases}$$

Therefore, for a fixed $s = 1, \cdots, \iota$ and a fixed $t = 1, \cdots, d$,

$$\begin{aligned}
\left\langle \mathsf{Ind}_\Gamma^G(\rho)(f)(\varphi_{st}) \mid \varphi_{st} \right\rangle &= \sum_{i_o=1}^{\iota} \sum_{j_o=1}^{d} \overline{\left\langle \varphi_{st}(g_{i_o}) \mid e_{j_o} \right\rangle} \sum_{i=1}^{\iota} \sum_{\gamma\in\Gamma} f(g_{i_o}^{-1}\gamma g_i) \left\langle \rho(\gamma)(\varphi_{st}(g_i)) \mid e_{j_o} \right\rangle \\
&= \sum_{i=1}^{\iota} \sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_i) \left\langle \rho(\gamma)(\varphi_{st}(g_i)) \mid e_t \right\rangle \\
&= \sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_s) \left\langle \rho(\gamma)(\varphi_{st}(g_s)) \mid e_t \right\rangle \\
&= \sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_s) \left\langle \rho(\gamma)e_t \mid e_t \right\rangle,
\end{aligned}$$

proving that

$$\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) = \sum_{(s,t)=(1,1)}^{(\iota,d)} \left\langle \mathsf{Ind}_\Gamma^G(\rho)(f)(\varphi_{st}) \mid \varphi_{st} \right\rangle = \sum_{\substack{1\le s\le \iota \\ 1\le t\le d}} \sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_s) \left\langle \rho(\gamma)e_t \mid e_t \right\rangle. \tag{5}$$

Introduce:

- $\{\Gamma\}$ = a set consisting of all representatives γ for the conjugacy classes C_γ^Γ in Γ;

– $\Gamma_\gamma = \{\delta \in \Gamma \mid \delta^{-1}\gamma\delta = \gamma\}$ for $\gamma \in \{\Gamma\}$;
– $G_\gamma = \{g \in G \mid g^{-1}\gamma g = \gamma\}$ for $\gamma \in \{\Gamma\}$.

Then the subgroup Γ of G decomposes as

$$\Gamma = \bigsqcup_{\gamma\in\{\Gamma\}} C_\gamma = \bigsqcup_{\gamma\in\{\Gamma\}} \{\delta^{-1}\gamma\delta \mid \delta \in \mathcal{R}_{\Gamma_\gamma\backslash\Gamma}\},$$

where $\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}$ denotes any fixed complete set of representatives of $\Gamma_\gamma\backslash\Gamma$. Therefore, for $s = 1, \cdots, \iota$ and $t = 1, \cdots, d$, the following identities hold

$$\begin{aligned}\sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_s)\,\langle\rho(\gamma)e_t \mid e_t\rangle &= \sum_{\gamma\in\{\Gamma\}}\sum_{\omega\in C_\gamma} f(g_s^{-1}\omega g_s)\,\langle\rho(\omega)e_t \mid e_t\rangle \\ &= \sum_{\gamma\in\{\Gamma\}}\sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}} f(g_s^{-1}\delta^{-1}\gamma\delta g_s)\,\left\langle\rho(\delta^{-1}\gamma\delta)e_t \mid e_t\right\rangle.\end{aligned} \tag{6}$$

Now, substituting eq. (6) into eq. (5),

$$\begin{aligned}\mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f)) &= \sum_{\substack{1\le s\le\iota\\1\le t\le d}}\sum_{\gamma\in\{\Gamma\}}\sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}} f(g_s^{-1}\delta^{-1}\gamma\delta g_s)\,\left\langle\rho(\delta^{-1}\gamma\delta)e_t \mid e_t\right\rangle \\ &= \sum_{\gamma\in\{\Gamma\}}\sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}}\sum_{\substack{1\le s\le\iota\\1\le t\le d}} f(g_s^{-1}\delta^{-1}\gamma\delta g_s)\,\left\langle\rho(\delta^{-1}\gamma\delta)e_t \mid e_t\right\rangle \\ &= \sum_{\gamma\in\{\Gamma\}}\sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}}\left(\sum_{1\le s\le\iota} f(g_s^{-1}\delta^{-1}\gamma\delta g_s)\right)\left(\sum_{1\le t\le d}\left\langle\rho(\delta^{-1}\gamma\delta)e_t \mid e_t\right\rangle\right),\end{aligned}$$

where for $\gamma \in \{\Gamma\}$ and $\delta \in \mathcal{R}_{\Gamma_\gamma\backslash\Gamma}$,

$$\sum_{1\le t\le d}\left\langle\rho(\delta^{-1}\gamma\delta)e_t \mid e_t\right\rangle = \mathrm{Tr}\rho(\delta^{-1}\gamma\delta) = \mathrm{Tr}\rho(\gamma) = \chi_\rho(\gamma).$$

Therefore,

$$\mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f)) = \sum_{\gamma\in\{\Gamma\}}\chi_\rho(\gamma)\sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}}\sum_{1\le s\le\iota} f(g_s^{-1}\delta^{-1}\gamma\delta g_s).$$

Observe that, for $\gamma \in \{\Gamma\}$, the group G has the following subgroups as seen in the Hasse diagram:

$$\begin{tikzcd}[column sep=small] & G \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma \arrow[dr, dash] & & G_\gamma \arrow[dl, dash] \\ & \Gamma_\gamma & \end{tikzcd}$$

Each chain of subgroups

$$\Gamma_\gamma \le \Gamma \le G$$

and

$$\Gamma_\gamma \le G_\gamma \le G$$

of G produces different partitions of G into cosets modulo Γ_γ, as will be discussed in the next two observations.

Observation 4.1. For $\gamma \in \{\Gamma\}$, the group G partitions into cosets modulo Γ_γ as

$$G = \bigsqcup_{s=1}^{\iota}\bigsqcup_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}} \Gamma_\gamma\delta g_s,$$

where $\mathcal{R}_{\Gamma_\gamma\backslash\Gamma} \subset \Gamma$ is any fixed complete set of representatives of the coset space $\Gamma_\gamma\backslash\Gamma$.

Proof. We have already fixed a complete set of representatives $\mathcal{R}_{\Gamma\backslash G} = \{g_1, \cdots, g_\iota\} \subset G$ of the coset space $\Gamma\backslash G$. So, there is a decomposition of the group G into cosets modulo Γ as $G = \bigsqcup_{s=1}^{\iota}\Gamma g_s$.

There is also the decomposition of the subgroup Γ of G into cosets modulo Γ_γ as

$$\Gamma = \bigsqcup_{\delta \in \mathcal{R}_{\Gamma_\gamma \backslash \Gamma}} \Gamma_\gamma \delta.$$

Therefore the group G decomposes into cosets modulo Γ_γ as:

$$G = \Gamma g_1 \sqcup \cdots \sqcup \Gamma g_\iota = (\bigsqcup_{\delta \in \mathcal{R}_{\Gamma_\gamma \backslash \Gamma}} \Gamma_\gamma \delta) g_1 \sqcup \cdots \sqcup (\bigsqcup_{\delta \in \mathcal{R}_{\Gamma_\gamma \backslash \Gamma}} \Gamma_\gamma \delta) g_\iota = \bigsqcup_{s=1}^{\iota} \bigsqcup_{\delta \in \mathcal{R}_{\Gamma_\gamma \backslash \Gamma}} \Gamma_\gamma \delta g_s.$$

Observation 4.2. For $\gamma \in \{\Gamma\}$, there is another partitioning of the group G into cosets modulo Γ_γ given by

$$G = \bigsqcup_{t \in \mathcal{R}_{G_\gamma \backslash G}} \bigsqcup_{y \in \mathcal{R}_{\Gamma_\gamma \backslash G_\gamma}} \Gamma_\gamma y t,$$

where $\mathcal{R}_{G_\gamma \backslash G} \subset G$ and $\mathcal{R}_{\Gamma_\gamma \backslash G_\gamma} \subset G_\gamma$ are any fixed complete set of representatives of the coset spaces $G_\gamma \backslash G$ and $\Gamma_\gamma \backslash G_\gamma$, respectively.

Proof. We have already fixed a complete system of representatives $\mathcal{R}_{G_\gamma \backslash G} \subset G$ of the coset space $G_\gamma \backslash G$. So, there is a decomposition of the group G into cosets modulo G_γ as $G = \bigsqcup_{t \in \mathcal{R}_{G_\gamma \backslash G}} G_\gamma t$.

There is also the decomposition of the subgroup G_γ of G into cosets modulo Γ_γ as

$$G_\gamma = \bigsqcup_{y \in \mathcal{R}_{\Gamma_\gamma \backslash G_\gamma}} \Gamma_\gamma y.$$

Therefore the group G decomposes into cosets modulo Γ_γ as:

$$G = \bigsqcup_{t \in \mathcal{R}_{G_\gamma \backslash G}} G_\gamma t = \bigsqcup_{t \in \mathcal{R}_{G_\gamma \backslash G}} \left(\bigsqcup_{y \in \mathcal{R}_{\Gamma_\gamma \backslash G_\gamma}} \Gamma_\gamma y \right) t = \bigsqcup_{t \in \mathcal{R}_{G_\gamma \backslash G}} \bigsqcup_{y \in \mathcal{R}_{\Gamma_\gamma \backslash G_\gamma}} \Gamma_\gamma y t.$$

So by Observation 4.1,

$$\begin{aligned}
\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) &= \sum_{\gamma \in \{\Gamma\}} \chi_\rho(\gamma) \sum_{\delta \in \mathcal{R}_{\Gamma_\gamma \backslash \Gamma}} \sum_{1 \le s \le \iota} f(g_s^{-1} \delta^{-1} \gamma \delta g_s) \\
&= \sum_{\gamma \in \{\Gamma\}} \chi_\rho(\gamma) \sum_{x \in \mathcal{R}_{\Gamma_\gamma \backslash G}} f(x^{-1} \gamma x),
\end{aligned}$$

and by Observation 4.2,

$$\begin{aligned}
\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) &= \sum_{\gamma \in \{\Gamma\}} \chi_\rho(\gamma) \sum_{x \in \mathcal{R}_{\Gamma_\gamma \backslash G}} f(x^{-1} \gamma x) \\
&= \sum_{\gamma \in \{\Gamma\}} \chi_\rho(\gamma) \sum_{t \in \mathcal{R}_{G_\gamma \backslash G}} \sum_{y \in \mathcal{R}_{\Gamma_\gamma \backslash G_\gamma}} f(t^{-1} y^{-1} \gamma y t) \\
&= \sum_{\gamma \in \{\Gamma\}} \chi_\rho(\gamma) \frac{|G_\gamma|}{|\Gamma_\gamma|} \sum_{t \in \mathcal{R}_{G_\gamma \backslash G}} f(t^{-1} \gamma t) = I(\rho, f),
\end{aligned}$$

which is the geometric side of the formula stated in (4), completing the proof of Theorem 3.1.

5. TRACE FORMULA FOR G IN "ARTHUR FORM"

Finite groups are (locally) compact groups under the discrete topology. By compactness of G, there is a *unique* (left and right invariant) Haar measure $d\mu_G^{\mathrm{Haar}}$ on G, which in this case is nothing but the counting measure $d\mu_G^{\mathrm{Count}}$ on G. The same holds true for the subgroup Γ of G as well as the subgroups Γ_γ and G_γ of Γ and G, respectively, for $\gamma \in \{\Gamma\}$. So the invariant measures on the coset spaces $\Gamma_\gamma \backslash G_\gamma$ and $G_\gamma \backslash G$, which are defined by the products $d\mu_{G_\gamma}^{\mathrm{Haar}} = d\mu_{\Gamma_\gamma}^{\mathrm{Haar}} \times d\mu_{\Gamma_\gamma \backslash G_\gamma}^{\mathrm{Haar}}$ and $d\mu_G^{\mathrm{Haar}} = d\mu_{G_\gamma}^{\mathrm{Haar}} \times d\mu_{G_\gamma \backslash G}^{\mathrm{Haar}}$ for $\gamma \in \{\Gamma\}$ are all uniquely defined and are the counting measures on $\Gamma_\gamma \backslash G_\gamma$ and $G_\gamma \backslash G$, respectively.

Therefore, in this setting, for $\gamma \in \{\Gamma\}$,

$$a_\Gamma^G(\gamma) := \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) = \int_{\Gamma_\gamma \backslash G_\gamma} d\mu_{G_\gamma}^{\mathrm{Haar}} = \int_{\Gamma_\gamma \backslash G_\gamma} d\mu_{G_\gamma}^{\mathrm{Count}} = \sum_{y \in \mathcal{R}_{\Gamma_\gamma \backslash G_\gamma}} 1 = \frac{|G_\gamma|}{|\Gamma_\gamma|}.$$

Define the orbital integral $\mathrm{O}(\gamma, f)$ of $f : G \to \mathbb{C}$ over the conjugacy class C_γ^G of $\gamma \in \{\Gamma\}$ in G by

$$\mathrm{O}(\gamma, f) := \int_{G_\gamma\backslash G} f(t^{-1}\gamma t)\frac{d\mu_G^{\mathrm{Haar}}}{d\mu_{G_\gamma}^{\mathrm{Haar}}} = \int_{G_\gamma\backslash G} f(t^{-1}\gamma t)\frac{d\mu_G^{\mathrm{Count}}}{d\mu_{G_\gamma}^{\mathrm{Count}}} = \sum_{t\in\mathcal{R}_{G_\gamma\backslash G}} f(t^{-1}\gamma t).$$

So, the trace formula for G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ stated in (4) becomes:

Theorem 5.1. (Trace formula for finite groups – Arthur form) *For any function $f : G \to \mathbb{C}$, the trace $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ of the operator*

$$\mathsf{Ind}_\Gamma^G(\rho)(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$$

on the $\mathbb{C}$-linear space $\mathsf{Ind}_\Gamma^G(V_\rho)$ satisfies the identity

$$J(\rho, f) = \sum_{\pi_o\in[\pi]\in\Pi(G)} m(\pi_o, \mathsf{Ind}_\Gamma^G(\rho))\mathrm{Tr}(\pi_o(f)) = \mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) = \sum_{\gamma\in\{\Gamma\}} \chi_\rho(\gamma)a_\Gamma^G(\gamma)\mathrm{O}(\gamma, f) = I(\rho, f), \tag{7}$$

where

- *$\{\Gamma\}$ = a set consisting of all representatives for the conjugacy classes in Γ;*
- *$\Gamma_\gamma = \{\delta \in \Gamma \mid \delta^{-1}\gamma\delta = \gamma\}$ for $\gamma \in \{\Gamma\}$;*
- *$G_\gamma = \{g \in G \mid g^{-1}\gamma g = \gamma\}$ for $\gamma \in \{\Gamma\}$.*

Here, $J(\rho, f)$ and $I(\rho, f)$ are the spectral side and the geometric side of the trace formula for the finite group G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$, respectively.

Theorem 5.1 is exactly the trace formula given in Arthur (Arthur 2005, eq (1.3)).

6. TEST FUNCTIONS $F : G \to \mathbb{C}$

Clearly, the operator $\mathsf{Ind}_\Gamma^G(\rho)(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ on the $\mathbb{C}$-linear space $\mathsf{Ind}_\Gamma^G(V_\rho)$ depends on the choice of $f : G \to \mathbb{C}$. Therefore the trace formula for the finite group G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$, namely, the spectral side $J(\rho, f)$ and the geometric side $I(\rho, f)$ of the formula corresponding to the trace $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ of $\mathsf{Ind}_\Gamma^G(\rho)(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ depend on the function $f : G \to \mathbb{C}$ as well. The function $f : G \to \mathbb{C}$ is called a "test function" of the trace formula for G (with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$), and choosing $f : G \to \mathbb{C}$ carefully, the trace identities for the operator $\mathsf{Ind}_\Gamma^G(\rho)(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ on $\mathsf{Ind}_\Gamma^G(V_\rho)$ yield extremely deep results.

The first example is the Frobenius reciprocity law for the finite group G.

Example 6.1. (Frobenius reciprocity law for G) For any $\sigma_o \in [\sigma] \in \Pi(G)$,

$$\left\langle \mathsf{Ind}_\Gamma^G(\rho), \sigma_o \right\rangle_G = \left\langle \rho, \mathsf{Res}_\Gamma^G(\sigma_o) \right\rangle_\Gamma.$$

Proof. For $\sigma_o \in [\sigma] \in \Pi(G)$, define a test function $f_{\sigma_o} : G \to \mathbb{C}$ by $f_{\sigma_o}(g) = \overline{\chi}_{\sigma_o}(g) = \overline{\chi_{\sigma_o}(g)}$ for $g \in G$. The spectral side $J(\rho, f_{\sigma_o})$ of the trace formula for the operator $\mathsf{Ind}_\Gamma^G(\rho)(f_{\sigma_o}) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ on $\mathsf{Ind}_\Gamma^G(V_\rho)$ is then given by

$$\begin{aligned} J(\rho, f_{\sigma_o}) &= \sum_{\pi_o\in[\pi]\in\Pi(G)} m(\pi_o, \mathsf{Ind}_\Gamma^G(\rho))\mathrm{Tr}(\pi_o(f_{\sigma_o})) \\ &= \sum_{\pi_o\in[\pi]\in\Pi(G)} m(\pi_o, \mathsf{Ind}_\Gamma^G(\rho))\mathrm{Tr}(\sum_{g\in G} f_{\sigma_o}(g)\pi_o(g)) \\ &= \sum_{\pi_o\in[\pi]\in\Pi(G)} m(\pi_o, \mathsf{Ind}_\Gamma^G(\rho)) \sum_{g\in G} f_{\sigma_o}(g)\mathrm{Tr}(\pi_o(g)). \end{aligned}$$

Now, by the orthogonality of irreducible characters of G, for $\pi_o \in [\pi] \in \Pi(G)$,

$$\sum_{g\in G} f_{\sigma_o}(g)\mathrm{Tr}(\pi_o(g)) = \sum_{g\in G} \overline{\chi}_{\sigma_o}(g)\chi_{\pi_o}(g) = \begin{cases} |G|, & \pi_o = \sigma_o, \\ 0, & [\pi_o] \cap [\sigma_o] = \varnothing. \end{cases}$$

Therefore, the spectral side $J(\rho, f_{\sigma_o})$ of the trace formula for the operator $\mathsf{Ind}_\Gamma^G(\rho)(f_{\sigma_o}) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ on $\mathsf{Ind}_\Gamma^G(V_\rho)$ is

$$J(\rho, f_{\sigma_o}) = \sum_{\pi_o\in[\pi]\in\Pi(G)} m(\pi_o, \mathsf{Ind}_\Gamma^G(\rho)) \sum_{g\in G} f_{\sigma_o}(g)\mathrm{Tr}(\pi_o(g)) = |G|m(\sigma_o, \mathsf{Ind}_\Gamma^G(\rho)) = |G| \left\langle \mathsf{Ind}_\Gamma^G(\rho), \sigma_o \right\rangle_G,$$

that is,

$$\frac{1}{|G|}J(\rho, f_{\sigma_o}) = \left\langle \mathsf{Ind}_\Gamma^G(\rho), \sigma_o \right\rangle_G.$$

Next, computing the "$|G|^{-1}$ multiple" of the geometric side $I(\rho, f_{\sigma_o})$ of the trace formula for the operator $\mathsf{Ind}_\Gamma^G(\rho)(f_{\sigma_o}) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ on $\mathsf{Ind}_\Gamma^G(V_\rho)$,

$$\begin{aligned}
\frac{1}{|G|}I(\rho, f_{\sigma_o}) &= \frac{1}{|G|}\sum_{\gamma\in\{\Gamma\}} \chi_\rho(\gamma) a_\Gamma^G(\gamma) \mathrm{O}(\gamma, f_{\sigma_o}) \\
&= \frac{1}{|G|}\sum_{\gamma\in\{\Gamma\}} \chi_\rho(\gamma)\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \sum_{t\in\mathcal{R}_{G_\gamma\backslash G}} f_{\sigma_o}(t^{-1}\gamma t) \\
&= \frac{1}{|G|}\sum_{\gamma\in\{\Gamma\}} \chi_\rho(\gamma)\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\mathrm{vol}(G_\gamma\backslash G)\overline{\chi}_{\sigma_o}(\gamma) \\
&= \sum_{\gamma\in\{\Gamma\}} \frac{\chi_\rho(\gamma)\overline{\chi}_{\sigma_o}(\gamma)}{|\Gamma_\gamma|} \\
&= \frac{1}{|\Gamma|}\sum_{\gamma\in\{\Gamma\}} \frac{|\Gamma|}{|\Gamma_\gamma|}\chi_\rho(\gamma)\overline{\chi}_{\sigma_o}(\gamma).
\end{aligned}$$

Observe that, for $\gamma \in \{\Gamma\}$, the coset space $\Gamma_\gamma\backslash\Gamma$ is in bijective correspondence with the conjugacy class C_γ^Γ of $\gamma \in \{\Gamma\}$ in Γ under the bijection $\Gamma_\gamma\backslash\Gamma \to C_\gamma^\Gamma$ defined by $\Gamma_\gamma\delta \mapsto x^{-1}\gamma\delta$ for all $\delta \in \Gamma$. Therefore, $\frac{|\Gamma|}{|\Gamma_\gamma|} = |C_\gamma^\Gamma|$, and the following identities

$$\frac{1}{|G|}I(\rho, f_{\sigma_o}) = \frac{1}{|\Gamma|}\sum_{\gamma\in\{\Gamma\}}\frac{|\Gamma|}{|\Gamma_\gamma|}\chi_\rho(\gamma)\overline{\chi}_{\sigma_o}(\gamma) = \frac{1}{|\Gamma|}\sum_{\gamma\in\{\Gamma\}}|C_\gamma^\Gamma|\chi_\rho(\gamma)\overline{\chi}_{\sigma_o}(\gamma) = \frac{1}{|\Gamma|}\sum_{\delta\in\Gamma}\chi_\rho(\delta)\overline{\chi}_{\sigma_o}(\delta) = \left\langle \rho, \mathsf{Res}_\Gamma^G(\sigma_o)\right\rangle_H$$

follow at once, completing the proof of the Frobenius reciprocity law.

For the second example, define an inner product

$$(\bullet \mid \bullet) : \mathbb{C}[G] \times \mathbb{C}[G] \to \mathbb{C}$$

on the group algebra $\mathbb{C}[G]$ of G over $\mathbb{C}$ by

$$(f \mid h) := |G|\sum_{g\in G} f(g^{-1})h(g), \quad {}^\forall f, h \in \mathbb{C}[G].$$

The Fourier transform

$$\mathcal{F}_\mathsf{r} : \mathbb{C}[G] \to \mathsf{End}_\mathbb{C}(V_\mathsf{r})$$

on G coupled to a representation $\mathsf{r} : G \to \mathsf{GL}(V_\mathsf{r})$ of G on a d-dimensional vector space V_r over $\mathbb{C}$ is defined by

$$\mathcal{F}_\mathsf{r} : f \mapsto \mathcal{F}_\mathsf{r} f =: \widehat{f}(\mathsf{r}) \overset{def}{=} \sum_{g\in G} f(g)\mathsf{r}(g) = \mathsf{r}(f), \quad {}^\forall f \in \mathbb{C}[G].$$

It is well known and easy to derive that

$$\widehat{f * h}(\mathsf{r}) = \widehat{f}(\mathsf{r}) \circ \widehat{h}(\mathsf{r}), \quad {}^\forall f, h \in \mathbb{C}[G]. \tag{8}$$

Here, for $f, h \in \mathbb{C}[G]$, their convolution product $f * h \in \mathbb{C}[G]$ is defined by $(f * h)(x) = \sum_{g\in G} f(xg^{-1})h(g)$ for all $x \in G$. So $(f * h)(1) = \sum_{g\in G} f(g^{-1})h(g)$.

Now, having set the stage, we can now state and prove the Plancherel formula for G in the following example.

Example 6.2. (Plancherel formula for G) For any two functions $f, h : G \to \mathbb{C}$,

$$(f \mid h) = \sum_{\pi_o\in[\pi]\in\Pi(G)} \mathrm{d}_{\pi_o}\mathrm{Tr}(\widehat{f}(\pi_o) \circ \widehat{h}(\pi_o)).$$

Proof. For any two functions $f, h : G \to \mathbb{C}$ and for any $\pi_o \in [\pi] \in \Pi(G)$, by (8) the identity $\widehat{f * h}(\pi_o) = \widehat{f}(\pi_o) \circ \widehat{h}(\pi_o)$ holds true. Therefore, the right-hand side of the Plancherel formula becomes

$$\sum_{\pi_o\in[\pi]\in\Pi(G)} \mathrm{d}_{\pi_o}\mathrm{Tr}(\widehat{f}(\pi_o) \circ \widehat{h}(\pi_o)) = \sum_{\pi_o\in[\pi]\in\Pi(G)} \mathrm{d}_{\pi_o}\mathrm{Tr}(\widehat{f * h}(\pi_o)) = \sum_{\pi_o\in[\pi]\in\Pi(G)} \mathrm{d}_{\pi_o}\mathrm{Tr}(\pi_o(f * h)).$$

Now, we consider the standard case of the trace formula for the finite group G; that is, the trace formula for G with respect to the subgroup $\Gamma = \langle 1 \rangle$ and $\rho = \mathbb{1}_\Gamma : \Gamma \longrightarrow \mathrm{GL}(\mathbb{C})$. Then, by (7), the spectral side $J(\mathbb{1}_{\langle 1 \rangle}, f * h)$ of the trace formula for G with respect to the subgroup $\langle 1 \rangle$ and $\mathbb{1}_{\langle 1 \rangle} : \langle 1 \rangle \longrightarrow \mathrm{GL}(\mathbb{C})$ reads as

$$J(\mathbb{1}_{\langle 1 \rangle}, f * h) = \sum_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathsf{Ind}_{\langle 1 \rangle}^G(\mathbb{1}_{\langle 1 \rangle}))\mathrm{Tr}(\pi_o(f * h)) = \sum_{\pi_o \in [\pi] \in \Pi(G)} \mathrm{d}_{\pi_o}\mathrm{Tr}(\pi_o(f * h)),$$

and the geometric side of $I(\mathbb{1}_{\langle 1 \rangle}, f * h)$ of the trace formula for G with respect to the subgroup $\langle 1 \rangle$ and $\mathbb{1}_{\langle 1 \rangle} : \langle 1 \rangle \longrightarrow \mathrm{GL}(\mathbb{C})$ reduces to

$$I(\mathbb{1}_{\langle 1 \rangle}, f * h) = \sum_{\gamma \in \{\langle 1 \rangle\}} \chi_{\mathbb{1}_{\langle 1 \rangle}}(\gamma) a_{\langle 1 \rangle}^G(\gamma) \mathrm{O}(\gamma, f * h) = |G|(f * h)(1) = |G| \sum_{g \in G} f(g^{-1})h(g) = (f \mid h),$$

completing the proof of the Plancherel formula for G.

Our third example needs some preliminaries. Hence deserves discussion in a new section.

7. THE "AUTOMORPHIC SIDE" OF THE MACDONALD CORRESPONDENCE FOR $\mathrm{GL}_N(\mathbb{F}_Q)$

Macdonald proved Macdonald (1980) an analogue of the local Langlands correspondence for $\mathsf{GL}_n(F)$ where F is a local field with finite residue class field $\kappa_F = \mathbb{F}_q$ for the finite group $\mathsf{GL}_n(\mathbb{F}_q)$; namely, following the reformulation of Vogan Vogan (2020), there exists a bijective correspondence

$$\mathcal{M}_n(\mathbb{F}_q) : \mathcal{X}_n(\mathsf{WM}_{\mathbb{F}_q}) \rightleftarrows \Pi(\mathsf{GL}_n(\mathbb{F}_q)) \tag{9}$$

between the set $\mathcal{X}_n(\mathsf{WM}_{\mathbb{F}_q})$ of "isomorphism classes of complex n-dimensional admissible, that is Frob_q equivariant and semisimple representations of the absolute Weil-Macdonald group $\mathsf{WM}_{\mathbb{F}_q}$ of $\mathbb{F}_q$" and the set $\Pi(\mathsf{GL}_n(\mathbb{F}_q))$ of "isomorphism classes of irreducible representations of $\mathsf{GL}_n(\mathbb{F}_q)$ over $\mathbb{C}$" (but we can also assume over $\overline{\mathbb{Q}}_\ell, \Omega_\ell, ...$) where $q = p^f$ with p a prime number and $0 < f \in \mathbb{Z}$. Here, $\mathsf{WM}_{\mathbb{F}_q}$ is defined by $\mathsf{WM}_{\mathbb{F}_q} = (\varprojlim_m \mathbb{F}_{q^m}^\times) \ltimes \mathbb{C}^+$ where the inverse limit $\varprojlim_m \mathbb{F}_{q^m}^\times$ is with respect to the connecting maps given by the norm maps $\mathbb{F}_{q^m}^\times \xleftarrow{\mathrm{N}_{md/m}} \mathbb{F}_{q^{md}}^\times$ for all $0 < m, d \in \mathbb{Z}$. The action of Frob_q on $\mathsf{WM}_{\mathbb{F}_q}$ is defined by the q-th power map and the multiplication by q on the components of $\mathsf{WM}_{\mathbb{F}_q}$, and the action of $\varprojlim_m \mathbb{F}_{q^m}^\times$ on the additive group $\mathbb{C}^+$ is defined via the canonical isomorphism $\varprojlim_m \mathbb{F}_{q^m}^\times \approx I_F/P_F$ where F is a local field with $\kappa_F = \mathbb{F}_q$, and I_F, P_F are the inertia and the wild inertia subgroups of the Weil group W_F of F together with the homomorphism $|.|_F \circ \mathsf{Art}_F : W_F \xrightarrow{\mathsf{Art}_F} F^\times \xrightarrow{|.|_F} \mathbb{R}_{>0}^\times$, where $\mathsf{Art}_F : W_F \longrightarrow F^\times$ is the local Artin reciprocity law of F. This correspondence satisfies the "naturality" properties; that is, matching of corresponding local ϵ-factors and corresponding local ζ- and L-factors, and corresponding conductors. The bijection (9) is called the Macdonald correspondence for $\mathrm{GL}_n(\mathbb{F}_q)$, which needs further discussion and postponed to a future study

In this note we are interested in $\Pi(\mathsf{GL}_n(\mathbb{F}_q))$, namely the "automorphic side" of the Macdonald correspondence (9) for $\mathrm{GL}_n(\mathbb{F}_q)$, and the main references that we follow closely are Carter (1993); Green (1955) and Macdonald (1980); Ye and Zelingher (2021).

7.1. Parabolic induction and cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$

To describe the "automorphic side" of the Macdonald correspondence (9) for $\mathrm{GL}_n(\mathbb{F}_q)$ precisely, let us fix :

- $\mathcal{J} = \{j_1, \cdots, j_s\}$ an ordered partition of n; that is, $0 < j_1, \cdots, j_s \in \mathbb{Z}$ such that the ordered sum $j_1 + \cdots + j_s = n$.
- $\mathsf{P}_\mathcal{J}$ the standard parabolic subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ with respect to the partition $\mathcal{J}$ defined by

$$\mathsf{P}_\mathcal{J} := \begin{bmatrix} \mathsf{GL}_{j_1}(\mathbb{F}_q) & \times & \times & \times & \times \\ & \mathsf{GL}_{j_2}(\mathbb{F}_q) & \times & \times & \times \\ & & \ddots & \times & \times \\ & 0 & & \ddots & \times \\ & & & & \mathsf{GL}_{j_s}(\mathbb{F}_q) \end{bmatrix}_{n \times n} ;$$

– $\mathsf{M}_{\mathcal{J}}$ the Levi factor of $\mathsf{P}_{\mathcal{J}}$ by

$$\mathsf{M}_{\mathcal{J}} := \begin{bmatrix} \mathsf{GL}_{j_1}(\mathbb{F}_q) & & & & \\ & \mathsf{GL}_{j_2}(\mathbb{F}_q) & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \mathsf{GL}_{j_s}(\mathbb{F}_q) \end{bmatrix}_{n\times n};$$

– $\mathsf{N}_{\mathcal{J}}$ the unipotent radical of $\mathsf{P}_{\mathcal{J}}$ by

$$\mathsf{N}_{\mathcal{J}} := \begin{bmatrix} 1_{j_1} & \times & \times & \times & \times \\ & 1_{j_2} & \times & \times & \times \\ & & \ddots & \times & \times \\ & 0 & & \ddots & \times \\ & & & & 1_{j_s} \end{bmatrix}_{n\times n}.$$

Note that, $\mathsf{M}_{\mathcal{J}}$ acts on $\mathsf{N}_{\mathcal{J}}$ by conjugation, and

$$\mathsf{P}_{\mathcal{J}} = \mathsf{M}_{\mathcal{J}} \ltimes \mathsf{N}_{\mathcal{J}}.$$

For $1 \leq i \leq s$, let $\pi_i : \mathsf{GL}_{j_i}(\mathbb{F}_q) \longrightarrow \mathsf{GL}(V_{\pi_i})$ be a representation of $\mathsf{GL}_{j_i}(\mathbb{F}_q)$ on a d_i-dimensional vector space V_{π_i} over $\mathbb{C}$. Then, the natural isomorphism $\mathsf{M}_{\mathcal{J}} \xrightarrow{\sim} \prod_{1\leq i\leq s} \mathsf{GL}_{j_i}(\mathbb{F}_q)$ yields a representation $\mathsf{M}_{\mathcal{J}} \xrightarrow{\sim} \prod_{1\leq i\leq s} \mathsf{GL}_{j_i}(\mathbb{F}_q) \xrightarrow{\otimes_{1\leq i\leq s}\pi_i} \mathsf{GL}(\bigotimes_{1\leq i\leq s} V_{\pi_i})$ of the Levi factor $\mathsf{M}_{\mathcal{J}}$ of $\mathsf{P}_{\mathcal{J}}$ on the $\mathbb{C}$-linear space $\bigotimes_{1\leq i\leq s} V_{\pi_i}$, which in return defines via inflation to $\mathsf{P}_{\mathcal{J}}$ a representation $\widetilde{\otimes}_{1\leq i\leq s}\pi_i : \mathsf{P}_{\mathcal{J}} \longrightarrow \mathsf{GL}(\bigotimes_{1\leq i\leq s} V_{\pi_i})$ of $\mathsf{P}_{\mathcal{J}}$ on $\bigotimes_{1\leq i\leq s} V_{\pi_i}$ by letting the matrices in $\mathsf{N}_{\mathcal{J}}$ act as the identity on $\bigotimes_{1\leq i\leq s} V_{\pi_i}$. Consider the representation

$$\pi_1 \circ \cdots \circ \pi_s := \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\widetilde{\otimes}_{1\leq i\leq s}\pi_i) : \mathrm{GL}_n(\mathbb{F}_q) \longrightarrow \mathsf{GL}(\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\bigotimes_{1\leq i\leq s} V_{\pi_i}))$$

of $\mathrm{GL}_n(\mathbb{F}_q)$ on $\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\bigotimes_{1\leq i\leq s} V_{\pi_i})$ constructed by this parabolic induction process. For $1 < n \in \mathbb{Z}$, an irreducible representation $\pi : \mathsf{GL}_n(\mathbb{F}_q) \longrightarrow \mathsf{GL}(V_\pi)$ of $\mathsf{GL}_n(\mathbb{F}_q)$ on a d-dimensional vector space V_π over $\mathbb{C}$ is called cuspidal, if it does not occur in any representation of $\mathsf{GL}_n(\mathbb{F}_q)$ of the form $\pi_1 \circ \pi_2$, where $\pi_1 : \mathsf{GL}_{j_1}(\mathbb{F}_q) \longrightarrow \mathsf{GL}(V_{\pi_1})$ and $\pi_2 : \mathsf{GL}_{j_2}(\mathbb{F}_q) \longrightarrow \mathsf{GL}(V_{\pi_2})$ are both irreducible and $0 < j_1, j_2 \in \mathbb{Z}$ such that $j_1 + j_2 = n$. For $n = 1$, all irreducible representations of $\mathsf{GL}_1(\mathbb{F}_q) = \mathbb{F}_q^\times$ over $\mathbb{C}$ are by definition cuspidal. The "Philosophy of Cusp Forms" of Harish-Chandra (for details Bump (2013)) states that the cuspidal representations of $\mathsf{GL}_n(\mathbb{F}_q)$ are the basic building blocks of all irreducible representations of $\mathsf{GL}_n(\mathbb{F}_q)$ in the sense that for $\pi_o \in [\pi] \in \Pi(\mathsf{GL}_n(\mathbb{F}_q))$, there exists an ordered partition $\mathcal{J} = \{j_1, \cdots, j_s\}$ of n and cuspidal representations $\pi_1, \cdots, \pi_s$ of $\mathsf{GL}_{j_1}(\mathbb{F}_q), \cdots, \mathsf{GL}_{j_s}(\mathbb{F}_q)$, respecively, such that π_o is an irreducible constituent of $\pi_1 \circ \cdots \circ \pi_s$. This completes the description of the set $\Pi(\mathsf{GL}_n(\mathbb{F}_q))$ of isomorphism classes of irreducible representations of $\mathsf{GL}_n(\mathbb{F}_q)$ over $\mathbb{C}$ via the "Philosophy of Cusp Forms".

7.2. ζ-functions (integrals) of $\mathsf{GL}_n(\mathbb{F}_q)$

For a representation $\pi : \mathrm{GL}_n(\mathbb{F}_q) \longrightarrow \mathsf{GL}(V_\pi)$ of $\mathrm{GL}_n(\mathbb{F}_q)$ on a d-dimensional vector space V_π over $\mathbb{C}$, denoting the set of all $n \times n$ matrices over $\mathbb{F}_q$ by $\mathsf{M}_n(\mathbb{F}_q)$, Macdonald attached a function

$$\mathsf{Z}(\bullet, \pi) : \mathbb{C}[\mathsf{M}_n(\mathbb{F}_q)] \longrightarrow \mathsf{End}_{\mathbb{C}}(V_\pi)$$

defined by

$$\mathsf{Z}(\Phi, \pi) := \pi(\Phi) = \sum_{g\in\mathsf{GL}_n(\mathbb{F}_q)} \Phi(g)\pi(g), \quad {}^\forall\Phi \in \mathbb{C}[\mathsf{M}_n(\mathbb{F}_q)],$$

called the ζ-function of the representation $\pi : \mathrm{GL}_n(\mathbb{F}_q) \longrightarrow \mathsf{GL}(V_\pi)$ of $\mathrm{GL}_n(\mathbb{F}_q)$ on V_π over $\mathbb{C}$ (Macdonald 1980, Section 2).

Now, the trace formula for $\mathrm{GL}_n(\mathbb{F}_q)$ with respect to the parabolic subgroup $\mathsf{P}_{\mathcal{J}}$ of $\mathrm{GL}_n(\mathbb{F}_q)$ corresponding to an ordered partition $\mathcal{J}$ of n and $\rho : \mathsf{P}_{\mathcal{J}} \longrightarrow \mathrm{GL}(V_\rho)$ computes the trace $\mathrm{Tr}(\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho)))$ of the operator $\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho)) : \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho) \longrightarrow \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho)$ on the $\mathbb{C}$-linear space $\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho)$. More precisely, we have the following theorem, which is essentially a reformulation of Theorem 5.1 in this setting.

Theorem 7.1. *For any function $\Phi : \mathsf{M}_n(\mathbb{F}_q) \to \mathbb{C}$, the trace $\mathrm{Tr}(\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho)))$ of the operator*

$$\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho)) : \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho) \to \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho)$$

on the $\mathbb{C}$-linear space $\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho)$ satisfies the identity

$$\sum_{\pi_o \in [\pi] \in \Pi(\mathsf{GL}_n(\mathbb{F}_q))} m(\pi_o, \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho))\mathrm{Tr}(\mathsf{Z}(\Phi, \pi_o)) = \mathrm{Tr}(\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho))) = \sum_{\gamma \in \{\mathsf{P}_{\mathcal{J}}\}} \chi_\rho(\gamma) a_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma)\mathsf{O}(\gamma, \Phi),$$

where

- $\{\mathsf{P}_{\mathcal{J}}\}$ = *a set consisting of all representatives for the conjugacy classes in* $\mathsf{P}_{\mathcal{J}}$;
- $\mathsf{P}_{\mathcal{J}_\gamma} = \{\delta \in \mathsf{P}_{\mathcal{J}} \mid \delta^{-1}\gamma\delta = \gamma\}$ *for* $\gamma \in \{\mathsf{P}_{\mathcal{J}}\}$;
- $\mathrm{GL}_n(\mathbb{F}_q)_\gamma = \{g \in \mathrm{GL}_n(\mathbb{F}_q) \mid g^{-1}\gamma g = \gamma\}$ *for* $\gamma \in \{\mathsf{P}_{\mathcal{J}}\}$.

Furthermore, for $\gamma \in \{\mathsf{P}_{\mathcal{J}}\}$,

$$a_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma) = \frac{|\mathrm{GL}_n(\mathbb{F}_q)_\gamma|}{|\mathsf{P}_{\mathcal{J}_\gamma}|}$$

and the orbital integral $\mathsf{O}(\gamma, \Phi)$ of $\Phi : \mathsf{M}_n(\mathbb{F}_q) \to \mathbb{C}$ over the conjugacy class $C_\gamma^{\mathrm{GL}_n(\mathbb{F}_q)}$ of γ in $\mathrm{GL}_n(\mathbb{F}_q)$ is given by

$$\mathsf{O}(\gamma, \Phi) = \sum_{t \in \mathcal{R}_{\mathrm{GL}_n(\mathbb{F}_q)_\gamma \backslash \mathrm{GL}_n(\mathbb{F}_q)}} \Phi(t^{-1}\gamma t).$$

There is an important special case of Theorem 7.1. Let $\mathcal{J}_1 = \{1, \cdots, 1\}$ be the ordered partition of n given by $\overbrace{1 + \cdots + 1}^{n-\text{copies}} = n$. Then $\mathsf{N}_{\mathcal{J}_1}$ becomes the subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ consisting of the upper traingular unipotent matrices in $\mathrm{GL}_n(\mathbb{F}_q)$. Now, define a 1-dimensional representation

$$\theta_\psi : \mathsf{N}_{\mathcal{J}_1} \to \mathbb{C}^\times$$

of $\mathsf{N}_{\mathcal{J}_1}$ over $\mathbb{C}$ by

$$\theta_\psi : [x_{ij}]_{\substack{1\le i\le n\\ 1\le j\le n}} \mapsto \psi(x_{12} + x_{23} + \cdots + x_{(n-1)n}), \quad {}^\forall [x_{ij}]_{\substack{1\le i\le n\\ 1\le j\le n}} \in \mathsf{N}_{\mathcal{J}_1}, \tag{10}$$

where $\psi : \mathbb{F}_q^+ \to \mathbb{C}^\times$ is a non-trivial additive character of $\mathbb{F}_q$. Set $\mathbb{C} = V_{\theta_\psi}$. The representation

$$\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi) : \mathsf{GL}_n(\mathbb{F}_q) \to \mathsf{GL}(\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi}))$$

of $\mathrm{GL}_n(\mathbb{F}_q)$ on the $\mathbb{C}$-linear space $\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi})$ is multiplicity free; that is, it has multiplicity one property, which states:

- If $\pi_o \in [\pi] \in \Pi(\mathsf{GL}_n(\mathbb{F}_q))$ then $m(\pi_o, \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi)) \le 1$.

Now, $\pi_o \in [\pi] \in \Pi(\mathsf{GL}_n(\mathbb{F}_q))$ is said to have a Whittaker model if there exists a non-trivial additive character $\psi : \mathbb{F}_q^+ \to \mathbb{C}^\times$ of $\mathbb{F}_q$ such that $m(\pi_o, \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi)) = 1$. Moreover,

- If π_o is a cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$, then π_o has a Whittaker model.

The trace formula for $\mathrm{GL}_n(\mathbb{F}_q)$ with respect to the subgroup $\mathsf{N}_{\mathcal{J}_1}$ of upper triangular unipotent matrices in $\mathrm{GL}_n(\mathbb{F}_q)$ and $\theta_\psi : \mathsf{N}_{\mathcal{J}_1} \to \mathbb{C}^\times$ given by (10) computes the trace $\mathrm{Tr}(\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi)))$ of the operator $\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi)) : \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi}) \to \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi})$ on the $\mathbb{C}$-linear space $\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi})$, which has a simpler form thanks to the multiplicity one property of the representation $\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi) : \mathsf{GL}_n(\mathbb{F}_q) \to \mathsf{GL}(\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi}))$ of $\mathrm{GL}_n(\mathbb{F}_q)$ on the $\mathbb{C}$-linear space $\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi})$, which follows as a corollary of Theorem 7.1.

Corollary 7.2. For any function $\Phi : \mathsf{M}_n(\mathbb{F}_q) \to \mathbb{C}$, the trace $\mathrm{Tr}(\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi)))$ of the operator

$$\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi)) : \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi}) \to \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi})$$

on the $\mathbb{C}$-linear space $\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi})$ satisfies the identity

$$\underbrace{\sum_{\pi_o\in[\pi]\in\Pi(\mathsf{GL}_n(\mathbb{F}_q))} m(\pi_o, \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi))\mathrm{Tr}(\mathsf{Z}(\Phi,\pi_o))}_{\substack{\sum\limits_{\substack{\pi_o\in[\pi]\in\Pi(\mathsf{GL}_n(\mathbb{F}_q))\\ m(\pi_o,\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi))=1}} \mathrm{Tr}(\mathsf{Z}(\Phi,\pi_o))}} = \mathrm{Tr}(\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi))) = \underbrace{\sum_{\gamma\in\{\mathsf{N}_{\mathcal{J}_1}\}} \chi_{\theta_\psi}(\gamma) a_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma)\mathrm{O}(\gamma,\Phi)}_{\sum\limits_{\gamma\in\{\mathsf{N}_{\mathcal{J}_1}\}} \theta_\psi(\gamma) a_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma)\mathrm{O}(\gamma,\Phi)},$$

where

- $\{\mathsf{N}_{\mathcal{J}_1}\}$ = a set consisting of all representatives for the conjugacy classes in $\mathsf{N}_{\mathcal{J}_1}$;
- $\mathsf{N}_{\mathcal{J}_1\gamma} = \{\delta \in \mathsf{N}_{\mathcal{J}_1} \mid \delta^{-1}\gamma\delta = \gamma\}$ for $\gamma \in \{\mathsf{N}_{\mathcal{J}_1}\}$;
- $\mathrm{GL}_n(\mathbb{F}_q)_\gamma = \{g \in \mathrm{GL}_n(\mathbb{F}_q) \mid g^{-1}\gamma g = \gamma\}$ for $\gamma \in \{\mathsf{N}_{\mathcal{J}_1}\}$.

Furthermore, for $\gamma \in \{\mathsf{N}_{\mathcal{J}_1}\}$,

$$a_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma) = \frac{|\mathsf{GL}_n(\mathbb{F}_q)_\gamma|}{|\mathsf{N}_{\mathcal{J}_1\gamma}|}$$

and the orbital integral $\mathrm{O}(\gamma,\Phi)$ of $\Phi : \mathsf{M}_n(\mathbb{F}_q) \to \mathbb{C}$ over the conjugacy class $C_\gamma^{\mathrm{GL}_n(\mathbb{F}_q)}$ of γ in $\mathrm{GL}_n(\mathbb{F}_q)$ is given by

$$\mathrm{O}(\gamma,\Phi) = \sum_{t\in\mathcal{R}_{\mathrm{GL}_n(\mathbb{F}_q)_\gamma\backslash\mathrm{GL}_n(\mathbb{F}_q)}} \Phi(t^{-1}\gamma t).$$

7.3. ϵ-factors of representations of $\mathrm{GL}_n(\mathbb{F}_q)$

Let $\psi : \mathbb{F}_q^+ \to \mathbb{C}^\times$ be a non-trivial additive character of $\mathbb{F}_q$. For $x \in \mathsf{M}_n(\mathbb{F}_q)$, let ${}_x\mu : \mathsf{M}_n(\mathbb{F}_q) \to \mathsf{M}_n(\mathbb{F}_q)$ be the additive homomorphism defined by ${}_x\mu : y \mapsto xy$ for all $y \in \mathsf{M}_n(\mathbb{F}_q)$. For $x \in \mathsf{M}_n(\mathbb{F}_q)$, we consider the Fourier transform

$$\mathcal{F}_{\mathsf{r}_x} : \mathbb{C}[\mathsf{M}_n(\mathbb{F}_q)] \to \mathsf{End}_{\mathbb{C}}(V_{\mathsf{r}_x})$$

on $\mathsf{M}_n(\mathbb{F}_q)$ coupled to the representation $\mathsf{r}_x : \mathsf{M}_n(\mathbb{F}_q) \xrightarrow{{}_x\mu} \mathsf{M}_n(\mathbb{F}_q) \xrightarrow{\mathsf{Tr}} \mathbb{F}_q \xrightarrow{\psi} \mathbb{C}^\times$ of the additive group $\mathsf{M}_n(\mathbb{F}_q)$ on the 1-dimensional vector space $V_{\mathsf{r}_x} = \mathbb{C}$ over $\mathbb{C}$ defined by

$$\mathcal{F}_{\mathsf{r}_x} : \Phi \mapsto \mathcal{F}_{\mathsf{r}_x}\Phi =: \widehat{\Phi}(\mathsf{r}_x) =: \widehat{\Phi}(x) \overset{\text{def}}{=} |\mathsf{M}_n(\mathbb{F}_q)|^{-\frac{1}{2}} \sum_{g\in\mathsf{M}_n(\mathbb{F}_q)} \Phi(g)\mathsf{r}_x(g) = \mathsf{r}_x(\Phi), \quad {}^\forall\Phi \in \mathbb{C}[\mathsf{M}_n(\mathbb{F}_q)].$$

Let $\pi : \mathrm{GL}_n(\mathbb{F}_q) \to \mathsf{GL}(V_\pi)$ be a representation of $\mathrm{GL}_n(\mathbb{F}_q)$ on a d-dimensional vector space V_π over $\mathbb{C}$. For each $\Phi \in \mathbb{C}[\mathsf{M}_n(\mathbb{F}_q)]$, by Macdonald (Macdonald 1980, eq. (2.3)),

$$\mathsf{Z}(\Phi,\pi) = \sum_{g\in\mathsf{GL}_n(\mathbb{F}_q)} \Phi(g)\pi(g) = \sum_{x\in\mathsf{M}_n(\mathbb{F}_q)} \widehat{\Phi}(-x)W(\pi,\psi;x),$$

where

$$W(\pi,\psi;x) = |\mathsf{M}_n(\mathbb{F}_q)|^{-\frac{1}{2}} \sum_{h\in\mathsf{GL}_n(\mathbb{F}_q)} \psi(\mathsf{Tr}(hx))\pi(h), \quad {}^\forall x \in \mathsf{M}_n(\mathbb{F}_q). \tag{11}$$

Now choosing $x = 1$ for $x \in \mathsf{M}_n(\mathbb{F}_q)$, Macdonald proved (Macdonald 1980, eq. (2.4)) that

$$W(\pi,\psi;1)\pi(g) = \pi(g)W(\pi,\psi;1), \quad {}^\forall g \in \mathsf{GL}_n(\mathbb{F}_q).$$

Therefore, there exists a constant $(\pi,\psi) \in \mathbb{C}$ such that

$$W(\pi,\psi;1) = (\pi,\psi)\pi(1).$$

The epsilon factor $\epsilon(\pi,\psi)$ of the representation $\pi : \mathrm{GL}_n(\mathbb{F}_q) \to \mathsf{GL}(V_\pi)$ of $\mathrm{GL}_n(\mathbb{F}_q)$ on the d-dimensional vector space V_π over $\mathbb{C}$ with respect to the choice of a non-trivial additive character $\psi : \mathbb{F}_q^+ \to \mathbb{C}^\times$ is defined by

$$\epsilon(\pi,\psi) := (\check{\pi},\psi),$$

where $\check{\pi} : \mathsf{GL}_n(\mathbb{F}_q) \to \mathsf{GL}(V_{\check{\pi}})$ denotes the contragradient of $\pi : \mathsf{GL}_n(\mathbb{F}_q) \to \mathsf{GL}(V_\pi)$; that is, the representation of $\mathsf{GL}_n(\mathbb{F}_q)$ on the dual space $V_{\check{\pi}}$ of V_π defined by $\check{\pi}(g) = {}^t\pi(g^{-1})$.

Thus, it follows from (11) that

$$W(\check{\pi},\psi;1) = |\mathsf{M}_n(\mathbb{F}_q)|^{-\frac{1}{2}} \sum_{h\in\mathsf{GL}_n(\mathbb{F}_q)} \psi(\mathsf{Tr}(h))\check{\pi}(h) = \epsilon(\pi,\psi)\check{\pi}(1).$$

Therefore,

$$\mathrm{Tr}(W(\breve{\pi},\psi;1)) = |\mathsf{M}_n(\mathbb{F}_q)|^{-\frac{1}{2}} \sum_{h\in\mathsf{GL}_n(\mathbb{F}_q)} \psi(\mathrm{Tr}(h))\mathrm{Tr}(\breve{\pi}(h)) = \epsilon(\pi,\psi)\mathrm{Tr}(\breve{\pi}(1)),$$

proving that

$$\epsilon(\pi,\psi) = \frac{|\mathsf{M}_n(\mathbb{F}_q)|^{-\frac{1}{2}}}{\dim(\pi)} \sum_{h\in\mathsf{GL}_n(\mathbb{F}_q)} \psi(\mathrm{Tr}(h))\chi_{\breve{\pi}}(h), \tag{12}$$

as $\mathrm{Tr}(\breve{\pi}(1)) = \dim(\pi)$.

Moreover, Ye and Zelingher studied the effect of the linear algebraic operations $\boxplus$ and $\boxtimes$ on $\Pi(\mathsf{GL}_*(\mathbb{F}_q)) = \bigsqcup_{0\leq n\in\mathbb{Z}} \Pi(\mathsf{GL}_n(\mathbb{F}_q))$ to ϵ-factors in Ye and Zelingher (2021). More precisely, for $\pi_1 \in \Pi(\mathsf{GL}_{n_1}(\mathbb{F}_q))$ and $\pi_2 \in \Pi(\mathsf{GL}_{n_2}(\mathbb{F}_q))$, there exist $\pi_1 \boxplus \pi_2 \in \Pi(\mathsf{GL}_{n_1+n_2}(\mathbb{F}_q))$ and $\pi_1 \boxtimes \pi_2 \in \Pi(\mathsf{GL}_{n_1n_2}(\mathbb{F}_q))$, whose ϵ-factor, and ζ- and L-factors are known instead of $\pi_1 \boxplus \pi_2$ and $\pi_1 \boxtimes \pi_2$ themselves, and it is proved Ye and Zelingher (2021) by Ye and Zelingher that

$$\epsilon(\pi_1 \boxplus \pi_2,\psi) = \epsilon(\pi_1,\psi)\epsilon(\pi_2,\psi). \tag{13}$$

Now, combining (13) with the identity (12), for $\pi_1 \in \Pi(\mathsf{GL}_{n_1}(\mathbb{F}_q))$ and $\pi_2 \in \Pi(\mathsf{GL}_{n_2}(\mathbb{F}_q))$, we have

$$\begin{aligned}\epsilon(\pi_1 \boxplus \pi_2,\psi) &= \frac{|\mathsf{M}_{n_1}(\mathbb{F}_q)|^{-\frac{1}{2}}}{\dim(\pi_1)} \sum_{h'\in\mathsf{GL}_{n_1}(\mathbb{F}_q)} \psi(\mathrm{Tr}(h'))\chi_{\breve{\pi}_1}(h') \frac{|\mathsf{M}_{n_2}(\mathbb{F}_q)|^{-\frac{1}{2}}}{\dim(\pi_2)} \sum_{h''\in\mathsf{GL}_{n_2}(\mathbb{F}_q)} \psi(\mathrm{Tr}(h''))\chi_{\breve{\pi}_2}(h'') \\ &= \frac{|\mathsf{M}_{n_1}(\mathbb{F}_q)\times\mathsf{M}_{n_1}(\mathbb{F}_q)|^{-\frac{1}{2}}}{\dim(\pi_1)\dim(\pi_2)} \sum_{\substack{h'\in\mathsf{GL}_{n_1}(\mathbb{F}_q)\\ h''\in\mathsf{GL}_{n_2}(\mathbb{F}_q)}} \psi\left(\mathrm{Tr}\begin{bmatrix} h' & 0\\ 0 & h''\end{bmatrix}\right)\chi_{\breve{\pi}_1\oplus\breve{\pi}_2}\left(\begin{bmatrix} h' & 0\\ 0 & h''\end{bmatrix}\right).\end{aligned} \tag{14}$$

In case the representation $\pi : \mathrm{GL}_n(\mathbb{F}_q) \to \mathsf{GL}(V_\pi)$ has no 1-component; that is, π is not a constituent of $\rho_{n-1}\circ(1)$, where ρ_{n-1} is the regular representation of $\mathsf{GL}_{n-1}(\mathbb{F}_q)$ over $\mathbb{C}$ and (1) is the trivial representation $\mathsf{GL}_1(\mathbb{F}_q) \to \mathbb{C}^\times$, Macdonald further proved (Macdonald 1980, Proposition 2.7) that

$${}^t\mathsf{Z}(\widehat{\Phi},\breve{\pi}) = \epsilon(\pi,\psi)\mathsf{Z}(\Phi,\pi), \tag{15}$$

where ${}^t\mathsf{Z}(\widehat{\Phi},\breve{\pi})$ is the transpose of $\mathsf{Z}(\widehat{\Phi},\breve{\pi})$. Applying Theorem 5.1, the trace formula for finite groups in Arthur form, to the identity (15), the following identities follow immediately.

Theorem 7.3. *Let $\rho : \mathsf{P}_{\mathcal{J}} \to \mathrm{GL}(V_\rho)$ be a representation of the standard parabolic subgroup $\mathsf{P}_{\mathcal{J}}$ of $\mathrm{GL}_n(\mathbb{F}_q)$ with respect to the partition $\mathcal{J}$ on the vector space V_ρ over $\mathbb{C}$. Assume that the representation*

$$\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho) : \mathrm{GL}_n(\mathbb{F}_q) \to \mathsf{GL}(\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho))$$

of $\mathrm{GL}_n(\mathbb{F}_q)$ on the induced $\mathbb{C}$-linear space $\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho)$ has no 1-component. The epsilon factor $\epsilon(\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho),\psi)$ of the representation $\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho) : \mathrm{GL}_n(\mathbb{F}_q) \to \mathsf{GL}(\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho))$ with respect to the choice of a non-trivial additive character $\psi : \mathbb{F}_q^+ \to \mathbb{C}^\times$ has then the following description:

$$\frac{\sum_{\pi_o\in[\pi]\in\Pi(\mathsf{GL}_n(\mathbb{F}_q))} m(\pi_o,\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\breve{\rho}))\,\overbrace{\mathrm{Tr}({}^t\mathsf{Z}(\widehat{\Phi},\pi_o))}^{\mathrm{Tr}(\mathsf{Z}(\widehat{\Phi},\pi_o))}}{\sum_{\pi_o\in[\pi]\in\Pi(\mathsf{GL}_n(\mathbb{F}_q))} m(\pi_o,\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho))\mathrm{Tr}(\mathsf{Z}(\Phi,\pi_o))} = \epsilon(\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho),\psi) = \frac{\sum_{\gamma\in\{\mathsf{P}_{\mathcal{J}}\}} \chi_{\breve{\rho}}(\gamma)a_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma)\mathrm{O}(\gamma,\widehat{\Phi})}{\sum_{\gamma\in\{\mathsf{P}_{\mathcal{J}}\}} \chi_{\rho}(\gamma)a_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma)\mathrm{O}(\gamma,\Phi)},$$

where

- $\{\mathsf{P}_{\mathcal{J}}\}$ *= a set consisting of all representatives for the conjugacy classes in* $\mathsf{P}_{\mathcal{J}}$*;*
- $\mathsf{P}_{\mathcal{J}_\gamma} = \{\delta\in\mathsf{P}_{\mathcal{J}} \mid \delta^{-1}\gamma\delta = \gamma\}$ *for* $\gamma\in\{\mathsf{P}_{\mathcal{J}}\}$*;*
- $\mathrm{GL}_n(\mathbb{F}_q)_\gamma = \{g\in\mathrm{GL}_n(\mathbb{F}_q) \mid g^{-1}\gamma g = \gamma\}$ *for* $\gamma\in\{\mathsf{P}_{\mathcal{J}}\}$.

Furthermore, for $\gamma\in\{\mathsf{P}_{\mathcal{J}}\}$,

$$a_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma) = \frac{|\mathrm{GL}_n(\mathbb{F}_q)_\gamma|}{|\mathsf{P}_{\mathcal{J}_\gamma}|},$$

and the orbital integral $\mathrm{O}(\gamma,\Phi)$ of $\Phi : \mathsf{M}_n(\mathbb{F}_q) \to \mathbb{C}$ over the conjugacy class $C_\gamma^{\mathrm{GL}_n(\mathbb{F}_q)}$ of γ in $\mathrm{GL}_n(\mathbb{F}_q)$ is given by

$$\mathrm{O}(\gamma,\Phi) = \sum_{t\in\mathcal{R}_{\mathrm{GL}_n(\mathbb{F}_q)_\gamma\backslash\mathrm{GL}_n(\mathbb{F}_q)}} \Phi(t^{-1}\gamma t)$$

and the orbital integral $\mathrm{O}(\gamma, \widehat{\Phi})$ *of* $\widehat{\Phi} : \mathsf{M}_n(\mathbb{F}_q) \to \mathbb{C}$ *over the conjugacy class* $C_\gamma^{\mathrm{GL}_n(\mathbb{F}_q)}$ *of* γ *in* $\mathrm{GL}_n(\mathbb{F}_q)$ *is given by*

$$\mathrm{O}(\gamma, \widehat{\Phi}) = \sum_{t \in \mathcal{R}_{\mathrm{GL}_n(\mathbb{F}_q)_\gamma \backslash \mathrm{GL}_n(\mathbb{F}_q)}} \widehat{\Phi}(t^{-1}\gamma t)$$

The identities given by Theorem 7.3 do not seem to appear in the literature and may have applications in the ϵ-factor analysis of representations of $\mathrm{GL}_n(\mathbb{F}_q)$ over $\mathbb{C}$.

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RESEARCH ARTICLE

Backward Shift Operators on Bergman-Besov Spaces as Bergman Projections

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ABSTRACT

We express backward shift operators on all Bergman-Besov spaces in terms of Bergman projections in one and several variables including the Banach function spaces and the special Hilbert spaces such as Drury-Arveson and Dirichlet spaces. These operators are adjoints of the shift operators and their definitions for the case $p = 1$ and proper Besov spaces require the use of nontrivial imbeddings of the spaces into Lebesgue classes. Our results indicate that the backward shifts are compositions of imbeddings into Lebesgue classes followed by multiplication operators by the conjugates of the coordinate variables followed by Bergman projections on appropriate spaces. We apply our results to the wandering subspace property of invariant subspaces of the shift operators on certain of our Hilbert spaces.

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1. INTRODUCTION

Shift operators and their adjoints the backward shift operators have a central position in operator theory. Forward shift operators on holomorphic function spaces have simple representations as operators of multiplication by the coordinate variables. This is true also for the adjoint of the shift operator $f(z) \mapsto zf(z)$ on the Hardy space H^2 on the unit disc. This operator is the backward shift operator with the explicit formula

$$f(z) \mapsto \frac{f(z) - f(0)}{z} \qquad (z \in \mathbb{D},\ f \in H^2). \tag{1}$$

Backward shift operators on other holomorphic function spaces such as the weighted Bergman spaces can be written in terms of the Taylor series of the functions in the spaces, but simple explicit expressions in the spirit of (1) have been lacking until recently.

In Gu and Luo (2024), for weighted Bergman Hilbert spaces A^2_n on the unit disc with nonnegative integer weight parameter n, explicit expressions akin to (1) have been obtained. In the same paper, another formula on the same spaces have been obtained using the Bergman projections again with integer parameters.

It turns out that, by judicious use of dual representations, it is possible to extend the Bergman projection formulas considerably. We obtain expressions for the backward shift operators on weighted Bergman and Bergman-Besov spaces B^p_q on the unit disc and the ball using Bergman projections. The spaces on which our formulas work include weighted Bergman spaces with non-integer weight parameter $q > -1$, Besov spaces which correspond to parameter values $q \leq -1$, Banach Bergman-Besov with parameters $1 \leq p < \infty$, the same spaces of holomorphic functions of several complex variables on the unit ball of $\mathbb{C}^N$, and in particular the Drury-Arveson and Dirichlet spaces.

To place our results in context, we introduce some notation. Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^N$ with respect to the norm $|z| = \sqrt{\langle z, z\rangle}$ induced by the usual Hermitian inner product $\langle w, z\rangle = w_1\bar{z}_1 + \cdots + w_N\bar{z}_N$, which is the unit disc $\mathbb{D}$ for $N = 1$. Let $H(\mathbb{B})$ denote the space of all holomorphic functions on $\mathbb{B}$, respectively.

We let ν be the Lebesgue measure on $\mathbb{B}$ normalized so that $\nu(\mathbb{B}) = 1$. For $q \in \mathbb{R}$, we also define on $\mathbb{B}$ the measures

$$d\nu_q(z) := (1 - |z|^2)^q \, d\nu(z).$$

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These measures are finite for $q > -1$ and σ-finite otherwise. For $0 < p < \infty$, we denote the Lebesgue classes with respect to ν_q by L^p_q, writing also $L^p = L^p_0$.

The standard *weighted Bergman spaces* on $\mathbb{B}$ are $A^p_q = L^p_q \cap H(\mathbb{B})$ for $q > -1$ normed by $\|f\|_{A^p_q} := \|f\|_{L^p_q}$. Equivalently, the Bergman space A^p_q is imbedded isometrically in L^p_q by the *inclusion map* i. We again write $A^p = A^p_0$ for the unweighted Bergman spaces.

Bergman spaces are generalized to two-parameter Besov spaces B^p_q for $q \leq -1$. We defer the precise definition of Besov spaces to a later section; see Definition 2.1. It suffices now to note that the Besov space B^p_q is imbedded isometrically in the Lebesgue space L^p_q with the same parameters p, q by a map that is a combination of a derivative of the function and the product of a power of $1 - |z|^2$, where the order of the derivative and the power are the same. It is also true that an $f \in H(\mathbb{B})$ belongs to B^p_q whenever sufficiently high-order derivatives of f lie in a Bergman space.

We use the notation B^p_q for the full collection of Bergman-Besov spaces for $q \in \mathbb{R}$. For all $q \in \mathbb{R}$ and $1 \leq p < \infty$, Bergman-Besov projections P_s exist from the Lebesgue class L^p_q onto the Bergman-Besov space B^p_q for s satisfying a certain well-known inequality; see Theorem 3.1.

The *shift operator* S on a holomorphic function space on the unit disc $\mathbb{D}$ is simply the operator of multiplication by the coordinate variable z; so $S(f)(z) = zf(z)$. When the space on which S acts matters, we attach the parameters of the space to S and write $S^p_q : B^p_q \to B^p_q$.

For function spaces on the unit ball $\mathbb{B}$, we have N coordinate variables $z_1, \ldots, z_N$ and hence N shift operators; so $S_j(f)(z) = z_j f(z)$ for $j = 1, \ldots, N$. But we can also indicate the spaces on which the shifts act and write $(S^p_q)_j : B^p_q \to B^p_q$.

The adjoints $(S^p_q)^*$ and $(S^p_q)^*_j$ of the shift operators on B^p_q act on the dual spaces $(B^p_q)^*$ and are called the *backward shift operators*. For $1 < p < \infty$, we have $(B^p_q)^* = B^{p'}_q$, where p' is the exponent conjugate to p, that is, $p' = p/(p-1)$. For $p = 1$, the corresponding dual spaces are the weighted Bloch spaces $\mathcal{B}^\infty_\alpha$, whose definitions are also deferred to a later section; see Definition 2.2. For the dual spaces, see Theorem 3.3.

The main purpose of this paper is to establish expressions for the backward shift operators on A^p_q and B^p_q in terms of Bergman projections. We use the fact that the adjoint of a shift operator on a complex Lebesgue space is merely the operator of multiplication by the *conjugate* of the coordinate variable used in the shift operator. Our formulas have the following form: We take a function in B^p_q and imbed it in the associated Lebesgue space L^p_q as explained above, then we multiply the imbedded function by the conjugate of the corresponding coordinate variable, and lastly we project back to the Bergman or Besov space by a suitable Bergman projection.

Our main results in this direction are Theorems 4.2, 4.4, 6.2, and 6.4. They and their proofs occupy Sections 4 and 6. In Sections 2 and 3 following, we summarize all relevant information on the spaces we work on and the projections used in the theorems. In Section 5, we show that the invariant subspaces of the backward shifts in certain Besov spaces on the disc have the wandering subspace property.

2. PRELIMINARIES

In this section, we give the remaining notation and the necessary details on the function spaces. We use multi-index notation in which $\alpha = (\alpha_1, \ldots, \alpha_N)$ is an N-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \cdots \alpha_N!$, $0^0 = 1$, and $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$ for $z \in \mathbb{C}^n$. A star $(\ )^*$ indicates adjoints for operators and duals for spaces. We show an integral inner product on a function space X by $[\,\cdot, \cdot\,]_X$.

The *Pochhammer symbol* $(a)_b$ is defined by

$$(a)_b := \frac{\Gamma(a+b)}{\Gamma(a)}$$

when a and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function Γ. This is a shifted rising factorial since $(a)_k = a(a+1)\cdots(a+k-1)$ for positive integer k. In particular, $(1)_k = k!$ and $(a)_0 = 1$. A very useful identity is

$$(a)_{n+m} = (a)_n (a+n)_m \tag{2}$$

for $n, m \in \mathbb{N}$. Stirling formula gives

$$\frac{\Gamma(c+a)}{\Gamma(c+b)} \sim c^{a-b}, \qquad \frac{(a)_c}{(b)_c} \sim c^{a-b}, \qquad \frac{(c)_a}{(c)_b} \sim c^{a-b} \qquad (\operatorname{Re} c \to \infty), \tag{3}$$

where $A \sim B$ means that $|A/B|$ is bounded above and below by two strictly positive constants, that is, $A = O(B)$ and $B = O(A)$ for all A, B of interest. For $A = O(B)$, we also write $A \lesssim B$.

For $q \in \mathbb{R}$ and $w, z \in \mathbb{B}$, the *Bergman-Besov kernels* are

$$K_q(z,w) := \begin{cases} \dfrac{1}{(1-\langle z,w\rangle)^{1+N+q}} = \displaystyle\sum_{k=0}^{\infty} \frac{(1+N+q)_k}{k!} \langle z,w\rangle^k, & q > -(1+N), \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) = \displaystyle\sum_{k=0}^{\infty} \frac{k!\ \langle z,w\rangle^k}{(1-(N+q))_k}, & q \le -(1+N), \end{cases}$$

where ${}_2F_1 \in H(\mathbb{D})$ is the Gauss hypergeometric function. They are the reproducing kernels of Hilbert Bergman-Besov spaces. Notice that

$$K_{-(1+N)}(z,w) = \frac{1}{\langle z,w\rangle} \log \frac{1}{1-\langle z,w\rangle}.$$

These kernels are positive definite and sesquiholomorphic, and hence give rise to reproducing kernel Hilbert spaces of holomorphic functions which are the Bergman-Besov Hilbert spaces B^2_q. In particular, for $q > -1$, the B^2_q are weighted Bergman spaces A^2_q, B^2_{-1} is the Hardy space H^2, B^2_{-N} is the Drury-Arveson space, and $B^2_{-(1+N)}$ is the Dirichlet space. When $N = 1$, the Hardy and the Drury-Arveson spaces coincide.

To define the Bergman-Besov spaces for $p \neq 2$, we proceed as follows. Let the coefficient of $\langle z,w\rangle^k$ in the series expansion of $K_q(z,w)$ be $c_k(q)$ for any $q \in \mathbb{R}$. Note that $c_0(q) = 1$, $c_k(q) > 0$ for any k, and by (3),

$$c_k(q) \sim k^{N+q} \qquad (k \to \infty), \tag{4}$$

for every q. This explains the choice of the parameters of the hypergeometric function in K_q.

Let $f \in H(\mathbb{B})$ be given on $\mathbb{B}$ by its convergent *homogeneous expansion* $f = \sum_{k=0}^{\infty} f_k$ in which f_k is a homogeneous polynomial in $z_1, \dots, z_N$ of degree k. For $N = 1$, f_k is simply the kth term in the Taylor series of $f \in H(\mathbb{D})$. For any $s, t \in \mathbb{R}$, we define the *radial fractional differential operator* D^t_s on $H(\mathbb{B})$ by

$$D^t_s f := \sum_{k=0}^{\infty} d_k(s,t) f_k := \sum_{k=0}^{\infty} \frac{c_k(s+t)}{c_k(s)} f_k. \tag{5}$$

Note that $d_0(s,t) = 1$ so that $D^t_s(1) = 1$, $d_k(s,t) > 0$ for any k, and

$$d_k(s,t) \sim k^t \qquad (k \to \infty),$$

for any s, t by (4). So D^t_s is a continuous operator on $H(\mathbb{B})$ and is of order t. In particular, $D^t_s z^\gamma = d_{|\gamma|}(s,t) z^\gamma$ for any multi-index γ. More importantly,

$$D^0_s = I, \qquad D^u_{s+t} D^t_s = D^{t+u}_s, \qquad \text{and} \qquad (D^t_s)^{-1} = D^{-t}_{s+t}$$

for any s, t, u, where the inverse is two-sided. Thus any D^t_s maps $H(\mathbb{B})$ onto itself. The coefficients $d_k(s,t)$ are chosen in such a way that

$$D^t_q K_q(z,w) = K_{q+t}(z,w) \tag{6}$$

for any $q, t \in \mathbb{R}$, where differentiation is performed in the holomorphic variable z.

Consider now the linear transformation I^t_s defined for $f \in H(\mathbb{B})$ by

$$I^t_s f(z) := (1-|z|^2)^t\, D^t_s f(z).$$

When $t = 0$, s is irrelevant and I^0_s is just the inclusion i.

Definition 2.1. For $q \in \mathbb{R}$ and $0 < p < \infty$, we define the *Bergman-Besov space* B^p_q to consist of all $f \in H(\mathbb{B})$ for which $I^t_s f$ belongs to L^p_q for some s, t satisfying

$$q + pt > -1. \tag{7}$$

It is well-known that under (7), Definition 2.1 is independent of s, t and the norms $\|f\|_{B^p_q} := \|I^t_s f\|_{L^p_q}$ are all equivalent. For $p = 2$, these norms are also equivalent to the norms obtained from the reproducing kernels. Explicitly,

$$\|f\|^p_{B^p_q} = \int_{\mathbb{B}} |D^t_s f(z)|^p (1-|z|^2)^{q+pt}\, d\nu(z) \qquad (q + pt > -1). \tag{8}$$

When $q > -1$, we can take $t = 0$ in (7) and obtain the weighted Bergman spaces $A^p_q = B^p_q$. When $q \le -1$, we call the spaces B^p_q *proper* Besov spaces. For $0 < p < 1$, what we call norms are actually quasinorms.

The Lebesgue class of all essentially bounded functions on $\mathbb{B}$ with respect to any ν_q is the same, so we denote them all by $\mathcal{L}^\infty$.

For $\alpha \in \mathbb{R}$, we also define the weighted Lebesgue spaces $\mathcal{L}^\infty_\alpha$ to consist of all measurable φ defined on $\mathbb{B}$ for which $(1-|z|^2)^\alpha \varphi(z)$ belongs to $\mathcal{L}^\infty$ normed by

$$\|\varphi\|_{\mathcal{L}^\infty_\alpha} := \operatorname*{ess\,sup}_{z\in\mathbb{B}} (1-|z|^2)^\alpha \, |\varphi(z)|.$$

Definition 2.2. For $\alpha \in \mathbb{R}$, we define the *Bloch-Lipschitz space* $\mathcal{B}^\infty_\alpha$ to consist of all $f \in H(\mathbb{B})$ for which $I^t_s f$ belongs to $\mathcal{L}^\infty_\alpha$ for some s, t satisfying

$$\alpha + t > 0. \tag{9}$$

It is well-known that under (9), Definition 2.2 is independent of s, t and the norms $\|f\|_{\mathcal{B}^\infty_\alpha} := \|I^t_s f\|_{\mathcal{L}^\infty_\alpha}$ are all equivalent. Explicitly,

$$\|f\|_{\mathcal{B}^\infty_\alpha} = \sup_{z\in\mathbb{B}} |D^t_s f(z)|(1-|z|^2)^{\alpha+t} \qquad (\alpha+t>0). \tag{10}$$

If $\alpha > 0$, we can take $t = 0$ in (9) and obtain the weighted Bloch spaces. When $\alpha < 0$, these spaces are the holomorphic Lipschitz spaces $\Lambda_{-\alpha} = \mathcal{B}^\infty_\alpha$. The *usual Bloch space* $\mathcal{B}^\infty_0 = \mathcal{B}^\infty$ corresponds to $\alpha = 0$. There is no mention of the little Bloch space in this paper.

Remark 2.3. Definitions 2.1 and 2.2 imply that I^t_s imbeds B^p_q isometrically into L^p_q if and only if (7) holds, and I^t_s imbeds $\mathcal{B}^\infty_\alpha$ isometrically into $\mathcal{L}^\infty_\alpha$ if and only if (9) holds. By (8) and (10), $f \in B^p_q$ if and only if $D^t_s f \in A^p$ for $q + pt = 0$, and $f \in \mathcal{B}^\infty_\alpha$ if and only if $D^t_s f \in \mathcal{B}^\infty$ for $\alpha + t = 0$.

The reproducing property in the B^2_q is this: Given a $q \in \mathbb{R}$, there are $t, s_1, s_2 \in \mathbb{R}$ such that for any $f \in B^2_q$ and $z \in \mathbb{B}$, we have

$$f(z) = [f(\cdot), K_q(z,\cdot)]_{B^2_q} = C_q \int_{\mathbb{B}} D^t_{s_1} f(w) D^t_{s_2} K_q(z,w)(1-|w|^2)^{q+2t}\, d\nu(w),$$

where $[\cdot,\cdot]$ are the inner products associated to the norms in (8) and the C_q are normalizing constants. For Bergman spaces, $q > -1$ and naturally $t = 0$.

Proposition 2.4. *(Kaptanoğlu and Üreyen 2008, Proposition 3.1) (Kaptanoğlu and Tülü 2011, Proposition 2.1) For any $p > 0$ and $q, \alpha, s, t \in \mathbb{R}$, the maps $D^t_s : B^p_q \to B^p_{q+pt}$ and $D^t_s : \mathcal{B}^\infty_\alpha \to \mathcal{B}^\infty_{\alpha+t}$ are Banach space isomorphisms. They are also isometries when the parameters of the imbeddings I in the norms of the spaces are chosen as s, u for the domain and as $s+t, u-t$ for the target space.*

3. BERGMAN-BESOV PROJECTIONS

Bergman-Besov projections are the linear transformations P_s defined for $s \in \mathbb{R}$ and suitable φ by

$$P_s\varphi(z) = \int_{\mathbb{B}} \varphi(w) K_s(z,w)\, d\nu_s(w) \qquad (z \in \mathbb{B}).$$

Theorem 3.1. *(Kaptanoğlu 2005, Theorem 1.2) (Kaptanoğlu and Tülü 2011, Theorem 1.3) For $1 \le p < \infty$, the map $P_s : L^p_q \to B^p_q$ is bounded if and only if*

$$q + 1 < p(s+1). \tag{11}$$

Given an s satisfying (11), if t satisfies (7), then

$$P_s I^t_s f = \frac{N!}{(1+s+t)_N} f \tag{12}$$

holds for $f \in B^p_q$. Further, $P_s : \mathcal{L}^\infty_\alpha \to \mathcal{B}^\infty_\alpha$ is bounded if and only if

$$\alpha < s + 1. \tag{13}$$

Given an s satisfying (13), if t satisfies (9), then (12) holds for $f \in \mathcal{B}^\infty_\alpha$.

Remark 3.2. Note that $1 + s + t > 0$ if either (7) and (11), or (9) and (13) hold, thus in all cases considered in Theorem 3.1.

The *dual* of a Banach (or Hilbert) space is the space of all bounded linear functionals on the space. For $1 < p < \infty$, the dual of L^p_q is $L^{p'}_q$ under the pairing $[\cdot,\cdot]_q$, where

$$[\varphi, \psi]_q := \int_{\mathbb{B}} \varphi \overline{\psi}\, d\nu_q. \tag{14}$$

The dual of any L^1_q can be realized as any one of $\mathcal{L}^\infty_\alpha$ under the pairing $[\cdot,\cdot]_{q+\alpha}$.

Theorem 3.3. *(Kaptanoğlu 2005, Remark 7.3) (Kaptanoğlu and Tülü 2011, Theorem 6.2) For $1 < p < \infty$, the dual space of B^p_q can be identified with $B^{p'}_q$ under each of the pairings*

$$[f, g]_{q,s,t} := \int_{\mathbb{B}} I^t_s f \, \overline{I^{-q+s}_{q+t} g} \, d\nu_q \tag{15}$$

for s, t satisfying (11) and (7), that is, for every bounded linear functional Φ on B^p_q, there is a unique $g \in B^{p'}_q$ such that $\Phi f = [f, g]_{q,s,t}$ for $f \in B^p_q$. The dual space of any B^1_q can be identified with any $\mathcal{B}^\infty_\alpha$ under each of the pairings $[f, g]_{q+\alpha,s,t}$ for s, t satisfying (11) and (7) with $p = 1$, that is, for every bounded linear functional Φ on B^1_q, there is a unique $g \in \mathcal{B}^\infty_\alpha$ such that $\Phi f = [f, g]_{q+\alpha,s,t}$ for $f \in B^1_q$.

Equation (15) takes simpler forms for Bergman spaces for which $q > -1$ and we can take $t = 0$. For $1 < p < \infty$, we can also take $s = q$, but for $p = 1$, we must have $s > q$. Then the pairings are

$$\int_{\mathbb{B}} f \, \overline{g} \, d\nu_q \qquad \text{and} \qquad \int_{\mathbb{B}} f \, \overline{I^{-q-\alpha+s}_{q+\alpha} g} \, d\nu_{q+\alpha} \tag{16}$$

for $1 < p < \infty$ and $p = 1$, respectively.

So we have two pairings (14) and (15) with one and three parameters to use without and with an I^t_s, respectively.

Most useful is the Banach space adjoint of $I^t_s : B^p_q \to L^p_q$ for $1 \le p < \infty$ under the conditions (7) and (11). We use Theorem 3.3. For $1 < p < \infty$, the adjoint is the operator $(I^t_s)^* : L^{p'}_q \to B^{p'}_q$ such that $[I^t_s f, \psi]_q = [f, (I^t_s)^* \psi]_{q,s,t}$ for $f \in B^p_q$ and $\psi \in L^{p'}_q$, where s satisfies (11). For $p = 1$, it is the operator $(I^t_s)^* : \mathcal{L}^\infty_\alpha \to \mathcal{B}^\infty_\alpha$ such that $[I^t_s f, \psi]_{q+\alpha} = [f, (I^t_s)^* \psi]_{q+\alpha,s,t}$ for $f \in B^1_q$ and $\psi \in \mathcal{L}^\infty_\alpha$.

Theorem 3.4. *Let $q \in \mathbb{R}$. For $1 < p < \infty$, we have $(I^t_s)^* = \dfrac{(1+s+t)_N}{N!} P_{q+t}$, and for $p = 1$, we have $(I^t_s)^* = \dfrac{(1+s+t)_N}{N!} P_{q+\alpha+t}$ for any $\alpha \in \mathbb{R}$, where s, t satisfy (11) and (7). Explicitly, for $f \in B^p_q$ and $\psi \in (L^p_q)^*$,*

$$\int_{\mathbb{B}} I^t_s f \, \overline{\psi} \, d\nu_{q+\alpha} = \frac{(1+s+t)_N}{N!} \int_{\mathbb{B}} I^t_s f \, \overline{I^{-q-\alpha+s}_{q+\alpha+t} P_{q+\alpha+t} \psi} \, d\nu_{q+\alpha},$$

where $\alpha = 0$ for $1 < p < \infty$ and $\alpha \in \mathbb{R}$ is arbitrary for $p = 1$.

This theorem says that the composition of an I-type operator following a P-type operator can be removed under certain integrals.

Proof. We give the proof only for $1 < p < \infty$; the proof for $p = 1$ follows the same lines and is omitted.

Let $f \in B^p_q$, $\psi \in L^{p'}_q$, and put $F = D^t_s f$. By Proposition 2.4, $F \in B^p_{q+pt}$; but since $q + pt > -1$, actually $F \in A^p_{q+pt}$, a Bergman space which can be described without using any derivative. We have $q + pt + 1 < p(s + t + 1)$, so by Theorem 3.1, P_{s+t} maps A^p_{q+pt} onto itself and $P_{s+t} F = \dfrac{N!}{(1+s+t)_N} F$. We compute by first writing the integrals explicitly, next differentiating under the integral sign using (6), then interchanging the order of integration by the Fubini theorem, and finally using the information about F just stated. We obtain

$$\begin{aligned}
[f, P_{q+t}\psi]_{q,s,t} &= \int_{\mathbb{B}} (1-|z|^2)^t D^t_s f(z) \, \overline{(1-|z|^2)^{-q+s}} \\
&\quad \overline{\cdot D^{-q+s}_{q+t} \int_{\mathbb{B}} \psi(w) K_{q+t}(z,w) (1-|w|^2)^{q+t} \, d\nu(w)} \, (1-|z|^2)^q \, d\nu(z) \\
&= \int_{\mathbb{B}} (1-|z|^2)^{s+t} F(z) \int_{\mathbb{B}} \overline{\psi(w) K_{s+t}(z,w)} \, (1-|w|^2)^{q+t} \, d\nu(w) \, d\nu(z) \\
&= \int_{\mathbb{B}} (1-|w|^2)^{q+t} \, \overline{\psi(w)} \int_{\mathbb{B}} F(z) K_{s+t}(w,z) (1-|z|^2)^{s+t} \, d\nu(z) \, d\nu(w) \\
&= \int_{\mathbb{B}} (1-|w|^2)^{q+t} \, \overline{\psi(w)} \, P_{s+t} F(w) \, d\nu(w) \\
&= \int_{\mathbb{B}} (1-|w|^2)^t \frac{N!}{(1+s+t)_N} F(w) \, \overline{\psi(w)} \, d\nu_q(w) \\
&= \int_{\mathbb{B}} \frac{N!}{(1+s+t)_N} I^t_s f(w) \, \overline{\psi(w)} \, d\nu_q(w) = \left[\frac{N!}{(1+s+t)_N} I^t_s f, \, \psi \right]_q .
\end{aligned}$$

The proof is complete.

Theorem 3.4 takes simpler forms for Bergman spaces for which $q > -1$ and we can take $t = 0$. For $1 < p < \infty$, we can further take $s = q$ and then $i^* = \dfrac{(1+q)_N}{N!} P_q$, where i is the inclusion from A^p_q into L^p_q. For $p = 1$, we must have $q < s$, and then

$i^* = \dfrac{(1+s)_N}{N!} P_q$ upon taking $\alpha = 0$. The simpler explicit forms for $f \in A^p_q$ and $\psi \in (L^p_q)^*$ for general α are

$$\int_{\mathbb{B}} f\,\overline{\psi}\,d\nu_q = \frac{(1+q)_N}{N!} \int_{\mathbb{B}} f\,\overline{P_q \psi}\,d\nu_q \qquad (1 < p < \infty), \tag{17}$$

$$\int_{\mathbb{B}} f\,\overline{\psi}\,d\nu_{q+\alpha} = \frac{(1+s)_N}{N!} \int_{\mathbb{B}} f\,\overline{I^{-q-\alpha+s}_{q+\alpha} P_{q+\alpha} \psi}\,d\nu_{q+\alpha} \qquad (p = 1). \tag{18}$$

We need the following results when we compute the backward shifts explicitly on specific Bergman-Besov spaces. First, using multi-index notation,

$$\langle z, w\rangle^k = \sum_{|\gamma|=k} \frac{k!}{\gamma!} z^\gamma \overline{w}^\gamma. \tag{19}$$

Lemma 3.5. *Let α, β be two multi-indices, $|\alpha| = n$, and $s > -1$. Then*

$$\int_{\mathbb{B}} z^\alpha \overline{z}^\beta (1-|z|^2)^s\,d\nu(z) = \begin{cases} 0, & \text{if } \beta \neq \alpha, \\ \dfrac{N!\,\alpha!}{(1+s)_{N+n}}, & \text{if } \beta = \alpha. \end{cases}$$

Proof. This is (Alpay and Kaptanoğlu 2001, Lemma 1).

Lemma 3.6. *Let α, β be two multi-indices, $n = |\alpha|$, $m = |\beta|$, and $r, s \in \mathbb{R}$ with $r + s > -1$. Let $J = P_s\big(z^\alpha \overline{z}^\beta (1-|z|^2)^r\big)$. Then*

$$J = \begin{cases} \dfrac{N!\,(1+N+s)_{n-m}}{(1+r+s)_{N+n}} \, \dfrac{\alpha!}{(\alpha-\beta)!} \, z^{\alpha-\beta}, & \text{if } s > -(1+N), \\ \dfrac{N!\,((n-m)!)^2}{(1-(N+s))_{n-m}\,(1+r+s)_{N+n}} \, \dfrac{\alpha!}{(\alpha-\beta)!} \, z^{\alpha-\beta}, & \text{if } s \leq -(1+N), \end{cases}$$

for $\alpha \geq \beta$, and $J = 0$ otherwise, where $\alpha \geq \beta$ means $\alpha_j \geq \beta_j$ for all $j = 1, \ldots, N$.

Proof. In what follows, by Lemma 3.5, the only value of γ that gives a nonzero integral is $\gamma = \alpha - \beta \geq 0$, which also explains why the integral is 0 for $\alpha < \beta$.

For $s > -(1+N)$, by the way Bergman-Besov projections and kernels are defined, (19), and Lemma 3.5,

$$\begin{aligned} J &= \int_{\mathbb{B}} w^\alpha \overline{w}^\beta (1-|w|^2)^{r+s} \sum_{k=0}^{\infty} \frac{(1+N+s)_k}{k!} \sum_{|\gamma|=k} \frac{k!}{\gamma!} z^\gamma \overline{w}^\gamma \,d\nu(w) \\ &= \frac{(1+N+s)_{n-m}}{(\alpha-\beta)!} z^{\alpha-\beta} \int_{\mathbb{B}} |w^\alpha|^2 (1-|w|^2)^{r+s}\,d\nu(w) \\ &= \frac{(1+N+s)_{n-m}}{(\alpha-\beta)!} z^{\alpha-\beta} \frac{N!\,\alpha!}{(1+r+s)_{N+n}}. \end{aligned}$$

For $s \leq -(1+N)$, similarly,

$$\begin{aligned} J &= \int_{\mathbb{B}} w^\alpha \overline{w}^\beta (1-|w|^2)^{r+s} \sum_{k=0}^{\infty} \frac{k!}{(1-(N+s))_k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} z^\gamma \overline{w}^\gamma \,d\nu(w) \\ &= \frac{((n-m)!)^2}{(1-(N+s))_{n-m}} \frac{1}{(\alpha-\beta)!} z^{\alpha-\beta} \int_{\mathbb{B}} |w^\alpha|^2 (1-|w|^2)^{r+s}\,d\nu(w) \\ &= \frac{((n-m)!)^2}{(1-(N+s))_{n-m}} \frac{1}{(\alpha-\beta)!} z^{\alpha-\beta} \frac{N!\,\alpha!}{(1+r+s)_{N+n}}. \end{aligned}$$

We use Lemma 3.6 only for $m = 1$.

4. BACKWARD SHIFT OPERATORS ON SPACES ON UNIT DISC

The spaces we work on have infinite families of equivalent norms. The pairings under which the dual spaces are realized depend strongly on the particular norms used. Likewise, the adjoint operators take different forms depending on the pairings. For this reason, for each type of space, we define the adjoints anew.

Throughout this section, $N = 1$. When (7) and (11) both hold, always $s + t > -1$.

Definition 4.1. For $q > -1$, let $S^p_q : A^p_q \to A^p_q$ be the shift operator acting on a Bergman space. If $1 < p < \infty$, we define its adjoint $(S^p_q)^* : A^{p'}_q \to A^{p'}_q$ by $\big[S^p_q f,\, g\big]_q = \big[f,\, (S^p_q)^* g\big]_q$, where $f \in A^p_q$ and $g \in A^{p'}_q$. If $p = 1$, we define its adjoint $(S^1_q)^* : \mathcal{B}^\infty_\alpha \to \mathcal{B}^\infty_\alpha$ by $\big[S^1_q f,\, g\big]_{q+\alpha} = \big[f,\, (S^1_q)^* g\big]_{q+\alpha,s,0}$, where $f \in A^1_q$ and $g \in \mathcal{B}^\infty_\alpha$.

Theorem 4.2. *The adjoint of a Bergman shift operator is*

$$(S_q^p)^* g(z) = (1+s)P_{q+\alpha}\big(\overline{z}g(z)\big) = (1+s)\int_{\mathbb{D}} \frac{\overline{w}\, g(w)}{(1-\overline{w}z)^{2+q+\alpha}}\, d\nu_{q+\alpha}(w),$$

where $g \in A_q^{p'}$, $\alpha = 0$, *and* $s = q$ *for* $1 < p < \infty$, *and* $g \in \mathscr{B}_\alpha^\infty$, $\alpha \in \mathbb{R}$, *and* s *satisfies* (11) *for* $p = 1$.

Proof. Let $1 < p < \infty$ and $g \in A_q^{p'}$ first. Definition 4.1 and (17) give

$$\begin{aligned}\big[S_q^p f, g\big]_q &= \int_{\mathbb{D}} z f(z)\, \overline{g(z)}\, d\nu_q(z) = \int_{\mathbb{D}} f(z)\, \overline{\overline{z}g(z)}\, d\nu_q(z)\\ &= (1+q)\int_{\mathbb{D}} f(z)\, \overline{P_q\big(\overline{z}g(z)\big)}\, d\nu_q(z) = \big[f, (1+q)P_q\big(\overline{z}g(z)\big)\big]_q.\end{aligned}$$

Then

$$(S_q^p)^* g(z) = (1+q)P_q\big(\overline{z}g(z)\big) \qquad (1 < p < \infty).$$

Let $p = 1$ and $g \in \mathscr{B}_\alpha^\infty$ next. Definition 4.1 and (18) give

$$\begin{aligned}\big[S_q^1 f, g\big]_{q+\alpha} &= \int_{\mathbb{D}} z f(z)\, \overline{g(z)}\, d\nu_{q+\alpha}(z) = \int_{\mathbb{D}} f(z)\, \overline{\overline{z}g(z)}\, d\nu_{q+\alpha}(z)\\ &= (1+s)\int_{\mathbb{D}} f(z)\, \overline{I_{q+\alpha}^{-q-\alpha+s} P_{q+\alpha}\big(\overline{z}g(z)\big)}\, d\nu_{q+\alpha}(z)\\ &= \big[f, (1+s)P_{q+\alpha}\big(\overline{z}g(z)\big)\big]_{q+\alpha,s,0}.\end{aligned}$$

Thus $(S_q^1)^* g(z) = (1+s)P_{q+\alpha}\big(\overline{z}g(z)\big)$.

We keep α when $p = 1$ for flexibility. But if we choose $\alpha = 0$, then the only difference between the two cases in Theorem 4.2 is whether the coefficient is $1+q$ or $1+s$ with $s > q$.

Definition 4.3. Let $q \le -1$ and t, s satisfy (7) and (11). Also let $s > -2$ for convenience. Let $S_q^p : B_q^p \to B_q^p$ be the shift operator acting on a proper Besov space. If $1 < p < \infty$, we define its adjoint $(S_q^p)^* : B_q^{p'} \to B_q^{p'}$ by the identity $\big[S_q^p f, g\big]_{q,s,t} = \big[f, (S_q^p)^* g\big]_{q,s+1,t}$, where $f \in B_q^p$ and $g \in B_q^{p'}$. If $p = 1$, we define its adjoint $(S_q^1)^* : \mathscr{B}_\alpha^\infty \to \mathscr{B}_\alpha^\infty$ by $\big[S_q^1 f, g\big]_{q+\alpha,s,t} = \big[f, (S_q^1)^* g\big]_{q+\alpha,s+1,t}$, where $f \in B_q^1$ and $g \in \mathscr{B}_\alpha^\infty$.

Theorem 4.4. *The adjoint of a proper Besov shift operator is*

$$(S_q^p)^* g(z) = \frac{(2+s+t)^2}{2+s} P_{q+\alpha+t}\big(\overline{z} I_{q+\alpha+t}^{-q-\alpha+s} g(z)\big),$$

where $g \in B_q^{p'}$ *and* $\alpha = 0$ *for* $1 < p < \infty$, $g \in \mathscr{B}_\alpha^\infty$ *and* $\alpha \in \mathbb{R}$ *for* $p = 1$, t, s *satisfy* (7) *and* (11), *and* $s > -2$ *for convenience.*

The explicit integral expression for $P_{q+\alpha+t}$ depends on whether $q + \alpha + t > -2$ or $q + \alpha + t \le -2$.

Proof. Let $f \in B_q^p$ be given by its Taylor series $f(z) = \sum_{k=0}^\infty f_k z^k$. Since $s > -2$ and $s + t > -1 > -2$, by (5) we have

$$D_s^t(z f(z)) = \sum_{k=0}^\infty \frac{(2+s+t)_{k+1}}{(2+s)_{k+1}} f_k z^{k+1} = \frac{2+s+t}{2+s}\, z \sum_{k=0}^\infty \frac{(2+s+1+t)_k}{(2+s+1)_k} f_k z^k = \frac{2+s+t}{2+s}\, z D_{s+1}^t f(z),$$

and hence $I_s^t(z f(z)) = \dfrac{2+s+t}{2+s}\, z I_{s+1}^t f(z)$.

Let $1 < p < \infty$ and $g \in B_q^{p'}$ first. Definition 4.3, the previous calculation, and Theorem 3.4 give

$$\begin{aligned}\big[S_q^p f, g\big]_{q,s,t} &= \int_{\mathbb{D}} I_s^t(z f(z))\, \overline{I_{q+t}^{-q+s} g(z)}\, d\nu_q(z)\\ &= \int_{\mathbb{D}} I_{s+1}^t f(z)\, \overline{\frac{2+s+t}{2+s}\, \overline{z} I_{q+t}^{-q+s} g(z)}\, d\nu_q(z)\\ &= (2+s+t)\int_{\mathbb{D}} I_{s+1}^t f(z)\, \overline{\frac{2+s+t}{2+s} I_{q+t}^{-q+s+1} P_{q+t}\big(\overline{z} I_{q+t}^{-q+s} g(z)\big)}\, d\nu_q(z)\\ &= \left[f(z), \frac{(2+s+t)^2}{2+s} P_{q+t}\big(\overline{z} I_{q+t}^{-q+s} g(z)\big)\right]_{q,s+1,t}.\end{aligned}$$

The desired formula is obtained.

Let $p = 1$ and $g \in \mathcal{B}_\alpha^\infty$ next. Similar to the previous case, Definition 4.3, the above calculation, and Theorem 3.4 give

$$\begin{aligned}
\left[S_q^p f,\, g\right]_{q+\alpha,s,t} &= \int_{\mathbb{D}} I_s^t(zf(z))\,\overline{I_{q+\alpha+t}^{-q-\alpha+s}\, g(z)}\, d\nu_{q+\alpha}(z) \\
&= \int_{\mathbb{D}} I_{s+1}^t f(z)\,\overline{\frac{2+s+t}{2+s}\,\overline{z} I_{q+\alpha+t}^{-q-\alpha+s}\, g(z)}\, d\nu_{q+\alpha}(z) \\
&= \int_{\mathbb{D}} I_{s+1}^t f(z)\,\overline{\frac{(2+s+t)^2}{2+s}\, I_{q+\alpha+t}^{-q-\alpha+s+1} P_{q+\alpha+t}\big(\overline{z} I_{q+\alpha+t}^{-q-\alpha+s}\, g(z)\big)}\, d\nu_{q+\alpha}(z) \\
&= \left[f(z),\, \frac{(2+s+t)^2}{2+s} P_{q+\alpha+t}\big(\overline{z} I_{q+\alpha+t}^{-q-\alpha+s}\, g(z)\big)\right]_{q+\alpha,s+1,t}.
\end{aligned}$$

The desired formula follows.

We check some well-known Hilbert spaces to see the differences between our formulas and the more commonly known ones. Such differences are bound to happen since we base our formulas on integral inner products while many formulas in the literature are based on the norms derived from reproducing kernels. Since all the spaces involved consist of holomorphic functions on $\mathbb{D}$, it is enough to check the results on $g(z) = z^n$ for $n = 0, 1, 2, \ldots$.

Remark 4.5. Let $q > -1$ and consider first the Bergman Hilbert spaces A_q^2. Here there must be no difference in the literature among the backward shift operators since the reproducing kernels of Bergman spaces are derived from standard integral norms. Theorem 4.2 and Lemma 3.6 give

$$(S_q^2)^*(z^n) = (1+q)P_q(z^n\overline{z}) = (1+q)\frac{(2+q)_{n-1}}{(1+q)_{1+n}}\, nz^{n-1} = \frac{n}{1+q+n}\, z^{n-1},$$

which agrees with (Kaptanoğlu 2014, (26)), as expected.

Remark 4.6. For the proper Besov spaces B_q^2 with $q \le -1$, there are two possibilities, $q+t > -2$ or $q+t \le -2$. In the first possibility ($q+t > -2$), we have $D_{q+t}^{-q+s}(z^n) = \dfrac{(2+s+t)_n}{(2+q+t)_n}\, z^n$ by (5). Then Theorem 4.4 and Lemma 3.6 give

$$\begin{aligned}
(S_q^2)^*(z^n) &= \frac{(2+s+t)^2}{2+s}\,\frac{(2+s+t)_n}{(2+q+t)_n}\, P_{q+t}\big(\overline{z}(1-|z|^2)^{-q+s} z^n\big) \\
&= \frac{(2+s+t)^2}{2+s}\,\frac{(2+s+t)_n}{(2+q+t)_n}\,\frac{(2+q+t)_{n-1}}{(1+s+t)_{1+n}}\, nz^{n-1} \\
&= \frac{(2+s+t)^2}{2+s}\,\frac{1}{1+s+t}\,\frac{n}{1+q+t+n}\, z^{n-1}.
\end{aligned}$$

Let $s \to \infty$ since it can be as large as we wish; then essentially

$$(S_q^2)^*(z^n) = \frac{n}{1+q+t+n}\, z^{n-1}. \tag{20}$$

For $q = -1$, we have $B_{-1}^2 = H^2$ for which essentially $(S_{-1}^2)^*(z^n) = \dfrac{n}{t+n}\, z^{n-1}$. For H^2, any small $t > 0$ works. If we further let $t \to 0^+$, we obtain $(S_{-1}^2)^*(z^n) \to z^{n-1}$, which is what (1) says.

Remark 4.7. In the second possibility ($q+t \le -2$) when $q \le -1$, by (5) again, we have $D_{q+t}^{-q+s}(z^n) = \dfrac{(2+s+t)_n}{n!}\,\dfrac{(-(q+t))_n}{n!}\, z^n$. Then Theorem 4.4 and Lemma 3.6 give

$$\begin{aligned}
(S_q^2)^*(z^n) &= \frac{(2+s+t)^2}{2+s}\,\frac{(2+s+t)_n}{n!}\,\frac{(-(q+t))_n}{n!}\, P_{q+t}\big(\overline{z}(1-|z|^2)^{-q+s} z^n\big) \\
&= \frac{(2+s+t)^2}{2+s}\,\frac{(2+s+t)_n}{n!}\,\frac{(-(q+t))_n}{n!}\,\frac{(n-1)!}{(-(q+t))_{n-1}}\,\frac{n!}{(1+s+t)_{1+n}}\, z^{n-1} \\
&= \frac{(2+s+t)^2}{2+s}\,\frac{1}{1+s+t}\,\frac{-(1+q+t)+n}{n}\, z^{n-1}.
\end{aligned}$$

As $s \to \infty$ again, essentially

$$(S_q^2)^*(z^n) = \frac{-(1+q+t)+n}{n}\, z^{n-1}. \tag{21}$$

For $q = -2$, B_{-2}^2 is the Dirichlet space for which $(S_{-2}^2)^*(z^n) = \dfrac{1-t+n}{n}\, z^{n-1}$ essentially. For this space, we must have $t > 1/2$. In spite of this, if we further let $t \to 0^+$, we obtain $(S^2_{-2})^*(z^n) \to \dfrac{1+n}{n}\, z^{n-1}$, which is what (Kaptanoğlu 2014, (26)) says, contrary to intuition.

5. WANDERING SUBSPACE PROPERTY

In this section, we identify some shift operators acting on Bergman-Besov Hilbert spaces B^2_q whose invariant subspaces have the wandering subspace property.

Let T be a left-invertible operator on a Hilbert space H and let $E \subset H$ be a closed T-invariant subspace of H. We say E has the *wandering subspace property* if E is the smallest closed T-invariant subspace including $E \ominus TE$, where $\ominus$ indicates orthogonal complement, that is, if $E = \bigvee_{n=0}^{\infty} T^n(E \ominus TE)$, where $\bigvee$ indicates closed linear span.

In (Richter 1988, Theorem 1 and Corollary), it is shown that the invariant subspaces of the shift operator on the Besov Hilbert spaces B^2_q with $-2 \le q \le -1$, that is, on those spaces between the Hardy Hilbert and Dirichlet spaces, have the wandering subspace property. In (Aleman et al. 1996, Theorem 3.5), it is shown that the invariant subspaces of the shift operator on the unweighted Bergman Hilbert space has the wandering subspace property. It should be noted that all results of this form are norm (or inner product) dependent since the adjoint depends on it, except perhaps those on Bergman spaces in whose norms there is universal agreement. In fact, in Gallardo-Gutiérrez et al. (2020) it is shown that by renorming, one can force the wandering subspace property.

In (Shimorin 2001, Theorem 4.1), a very practical sufficient condition is given for the wandering subspace property in which $A \le B$ means $B - A$ is a positive operator.

Theorem 5.1. (Shimorin (2001)) *If S is the shift operator on a space of holomorphic functions on $\mathbb{D}$ and $SS^* + (S^*S)^{-1} \le 2I$, then the invariant subspaces of S have the wandering subspace property.*

Checking the hypothesis of this theorem is especially easy since both $S^2_q(S^2_q)^*$ and $(S^2_q)^*S^2_q$ are diagonal operators on the orthogonal basis $\{1, z, z^2 \dots\}$ for all B^2_q. We also see that renorming does have an effect.

Theorem 5.2. *For $-1 < q \le 0$, the shift operator S^2_q on the Bergman space A^2_q has the wandering subspace property.*

Proof. By Remark 4.5, we have $S^2_q(S^2_q)^*(z^n) = \dfrac{n}{1+q+n}\, z^n$ for $n \ge 1$, $S^2_q(S^2_q)^*(1) = 0$, $(S^2_q)^*S^2_q(z^n) = \dfrac{1+n}{2+q+n}\, z^n$, and $((S^2_q)^*S^2_q)^{-1}(z^n) = \dfrac{2+q+n}{1+n}\, z^n$. Applying Theorem 5.1, $S^2_q(S^2_q)^* + (S^2_q)^*S^2_q \le 2I$ if and only if

$$\frac{2+q}{1} \le 2 \qquad \text{and} \qquad \frac{n}{1+q+n} + \frac{2+q+n}{1+n} \le 2.$$

The first inequality gives $q \le 0$ and the second $q \ge -1$.

Theorem 5.3. *For $q \le -1$, the shift operator S^2_q on the Besov space B^2_q using the adjoints from (Kaptanoğlu 2014, (26)) has the wandering subspace property if $q = -1$, that is, for the Hardy space H^2 with the usual norm and adjoint.*

Proof. For $-2 < q \le -1$, $(S^2_q)^*(z^n) = \dfrac{n}{1+q+n}\, z^{n-1}$, which is identical to the adjoints in Theorem 5.2. So we have only $q = -1$ from the proof of that theorem.

For $q \le -2$, $(S^2_q)^*(z^n) = \dfrac{-1-q+n}{n}\, z^{n-1}$. Then $S^2_q(S^2_q)^*(z^n) = \dfrac{-1-q+n}{n}\, z^n$ and $((S^2_q)^*S^2_q)^{-1}(z^n) = \dfrac{1+n}{-q+n}\, z^n$. Hence $S^2_q(S^2_q)^* + (S^2_q)^*S^2_q \le 2I$ if and only if

$$\frac{1}{-q} \le 2 \qquad \text{and} \qquad \frac{-1-q+n}{n} + \frac{1+n}{-q+n} \le 2.$$

The first inequality gives $q \le -1/2$ and the second $q \ge -1$. Thus there is no $q \le -2$ with desired properties.

Theorem 5.4. *For $q \le -1$, the shift operator S^2_q on the Besov space B^2_q using the adjoints in Theorem 4.4 has the wandering subspace property if t is chosen to obtain $-1 \le q + t \le 0$.*

Note that it is compulsory to have $q + 2t > -1$ by (7).

Proof. To make the formulas amenable to computation, we use the adjoints in Remarks 4.6 and 4.7 in their limiting form as $s \to \infty$.

For $q + t > -2$, by using (20) we have $(S^2_q)^*(z^n) = \dfrac{n}{1+q+t+n}\, z^{n-1}$. Then also $S^2_q(S^2_q)^*(z^n) = \dfrac{n}{1+q+t+n}\, z^n$ and

$((S_q^2)^* S_q^2)^{-1}(z^n) = \dfrac{2+q+t+n}{1+n}\, z^n$. Hence $S_q^2 (S_q^2)^* + (S_q^2)^* S_q^2 \le 2I$ if and only if

$$\frac{2+q+t}{1} \le 2 \qquad \text{and} \qquad \frac{n}{1+q+t+n} + \frac{2+q+t+n}{1+n} \le 2.$$

The first inequality gives $q+t \le 0$ and the second $q+t \ge -1$. Thus for $q \le -1$, if we choose t with $-1 \le q+t \le 0$, then S_q^2 on B_q^2 has the wandering subspace property.

For $q+t \le -2$, by using (21) we have $(S_q^2)^*(z^n) = \dfrac{-1-q-t+n}{n}\, z^{n-1}$. Then also $S_q^2(S_q^2)^*(z^n) = \dfrac{-1-q-t+n}{n}\, z^n$ and $((S_q^2)^* S_q^2)^{-1}(z^n) = \dfrac{1+n}{-q-t+n}\, z^n$. Hence $S_q^2 (S_q^2)^* + (S_q^2)^* S_q^2 \le 2I$ if and only if

$$\frac{1}{-q-t} \le 2 \qquad \text{and} \qquad \frac{-1-q-t+n}{n} + \frac{1+n}{-q-t+n} \le 2.$$

The first inequality gives $q+t \ge 1/2$ and the second $q+t \ge -1$, which contradict $q+t \le -2$. So no q and t can be found with the desired properties in this case.

Some other shift operators are checked in Gu and Luo (2024).

6. BACKWARD SHIFT OPERATORS ON SPACES ON UNIT BALL

Shift operators S_j, $j = 1, \dots, N$, on holomorphic function spaces on the unit ball $\mathbb{B}$ in $\mathbb{C}^N$ are investigated from many perspectives in Kaptanoğlu (2014). Here we concentrate only on their adjoints represented as Bergman-Besov projections. For readability, we refrain from attaching the parameters q, p of the spaces to the shift operators since they are clear from the context. We also take $\alpha = 0$ when $p = 1$ again for simplicity.

Definition 6.1. For $q > -1$, let $S_j : A_q^p \to A_q^p$ be a shift operators acting on a Bergman space, $j = 1, \dots, N$. If $1 < p < \infty$, we define its adjoint $S_j^* : A_q^{p'} \to A_q^{p'}$ by $[S_j f, g]_q = [f, S_j^* g]_q$, where $f \in A_q^p$ and $g \in A_q^{p'}$. If $p = 1$, we define its adjoint $S_j^* : \mathcal{B}^\infty \to \mathcal{B}^\infty$ by $[S_j f, g]_q = [f, S_j^* g]_{q,s,0}$, where $f \in A_q^1$ and $g \in \mathcal{B}^\infty$.

Theorem 6.2. *For $j = 1, \dots, N$, the adjoint of the Bergman shift operator S_j is*

$$S_j^* g(z) = \frac{(1+s)_N}{N!}\, P_q\big(\overline{z}_j g(z)\big) = \frac{(1+s)_N}{N!} \int_{\mathbb{B}} \frac{\overline{w}_j\, g(w)}{(1-\langle z, w\rangle)^{1+N+q}}\, d\nu_q(w),$$

where $g \in A_q^{p'}$ and $s = q$ for $1 < p < \infty$, and $g \in \mathcal{B}^\infty$ and $s > q$ for $p = 1$.

Proof. The proof is very similar to that of Theorem 4.2 and we omit it. The only thing that requires attention is that now we work in $\mathbb{C}^N$ with $N > 1$. The same are true for the proof of Theorem 6.4 below.

Definition 6.3. Let $q \le -1$ and t, s satisfy (7) and (11). Also let $s > -(1+N)$ for convenience. Let $S_j : B_q^p \to B_q^p$ be a shift operator acting on a proper Besov space, $j = 1, \dots, N$. If $1 < p < \infty$, we define its adjoint $S_j^* : B_q^{p'} \to B_q^{p'}$ by the identity $[S_j f, g]_{q,s,t} = [f, S_j^* g]_{q,s+1,t}$, where $f \in B_q^p$ and $g \in B_q^{p'}$. If $p = 1$, we define its adjoint $S_j^* : \mathcal{B}^\infty \to \mathcal{B}^\infty$ by the same identity, where $f \in B_q^1$ and $g \in \mathcal{B}^\infty$.

Theorem 6.4. *For $j = 1, \dots, N$, the adjoint of the proper Besov shift operator S_j is*

$$S_j^* g(z) = \frac{1+N+s+t}{1+N+s}\, \frac{(2+s+t)_N}{N!}\, P_{q+t}\big(\overline{z}_j I_{q+t}^{-q+s} g(z)\big),$$

where $g \in B_q^{p'}$ for $1 < p < \infty$, $g \in \mathcal{B}^\infty$ for $p = 1$, t, s satisfy (7) and (11), and $s > -(1+N)$ for convenience.

The explicit integral expression for P_{q+t} depends on whether $q+t > -(1+N)$ or $q+t \le -(1+N)$.

Let's evaluate the formulas in Theorems 6.2 and 6.4 on a monomial for all values of q and see their actual effects on certain standard reproducing kernel Hilbert spaces. In the Remarks below $g(z) = z^\alpha$, $n = |\alpha|$, $\beta = e_j$, $e_j = (0, \dots, 0, 1, 0 \dots, 0)$ with 1 in jth position, $j = 1, \dots, N$, and $|\beta| = m = 1$. Note that $\dfrac{\alpha!}{(\alpha - e_j)!} = \alpha_j$.

Remark 6.5. Let $q > -1$ and consider the Bergman Hilbert spaces A_q^2 on $\mathbb{B}$. Theorem 6.2, Lemma 3.6, and (2) give

$$S_j(z^\alpha) = \frac{(1+q)_N}{N!}\, P_q(\overline{z}_j z^\alpha) = \frac{(1+q)_N}{N!}\, \frac{N!\,(1+N+q)_{n-1}}{(1+q)_{N+n}}\, \frac{\alpha!}{(\alpha-e_j)!}\, z^{\alpha - e_j} = \frac{\alpha_j}{N+q+n}\, z^{\alpha-e_j},$$

which agrees with (Kaptanoğlu 2014, (26)).

Remark 6.6. For the proper Besov spaces B^2_q with $q \le -1$, when $q+t > -2$, we have $D^{-q+s}_{q+t}(z^\alpha) = \dfrac{(1+N+s+t)_n}{(1+N+q+t)_n} z^\alpha$ by (5). Theorem 6.4, Lemma 3.6, and (2) give

$$\begin{aligned}(S^2_q)^*(z^\alpha) &= \frac{1+N+s+t}{1+N+s} \frac{(2+s+t)_N}{N!} \frac{(1+N+s+t)_n}{(1+N+q+t)_n} P_{q+t}\big(\bar z_j (1-|z|^2)^{-q+s} z^n\big)\\ &= \frac{1+N+s+t}{1+N+s} \frac{(2+s+t)_N}{N!} \frac{(1+N+s+t)_n}{(1+N+q+t)_n} \frac{N!\,(1+N+q+t)_{n-1}}{(1+s+t)_{N+n}} \alpha_j z^{\alpha-e_j}\\ &= \frac{(1+N+s+t)^2}{(1+N+s)(1+s+t)} \frac{\alpha_j}{N+q+t+n} z^{\alpha-e_j}.\end{aligned}$$

Let $s \to \infty$ as before since it can be as large as we wish; then essentially

$$(S^2_q)^*(z^n) = \frac{\alpha_j}{N+q+t+n} z^{\alpha-e_j}.$$

The cases $q=-1$ and $q=-N$ pertain to the Hardy space H^2 and the Drury-Arveson space. We must have $q+2t>0$ for this formula to make sense by Definition 6.3. But again contrary to intuition, if we let $t \to 0+$, we obtain the adjoint formulas in (Kaptanoğlu 2014, (26)) that are derived from the reproducing kernel norms.

Remark 6.7. For the proper Besov spaces B^2_q with $q \le -1$, when $q+t \le -2$, we have

$$D^{-q+s}_{q+t}(z^\alpha) = \frac{(1+N+s+t)_n}{n!} \frac{(1-(N+q+t))_n}{n!} z^\alpha$$

by (5). Theorem 6.4, Lemma 3.6, and (2) give

$$\begin{aligned}(S^2_q)^*(z^\alpha) &= \frac{1+N+s+t}{1+N+s} \frac{(2+s+t)_N}{N!} \frac{(1+N+s+t)_n}{n!} \frac{(1-(N+q+t))_n}{n!} P_{q+t}\big(\bar z_j (1-|z|^2)^{-q+s} z^n\big)\\ &= \frac{1+N+s+t}{1+N+s} \frac{(2+s+t)_N}{N!} \frac{(1+N+s+t)_n}{n!} \frac{(1-(N+q+t))_n}{n!} \frac{N!\,(n-1)!\,(n-1)!}{(1-(N+q+t))_{n-1}\,(1+s+t)_{N+n}} \alpha_j z^{\alpha-e_j}\\ &= \frac{(1+N+s+t)^2}{(1+N+s)(1+s+t)} \frac{-(N+q+t)+n}{n^2} \alpha_j z^{\alpha-e_j}.\end{aligned}$$

Let $s \to \infty$ again; then essentially

$$(S^2_q)^*(z^n) = \frac{-(N+q+t)+n}{n^2} \alpha_j z^{\alpha-e_j}.$$

The case $q=-(1+N)$ pertains to the Dirichlet space. We must have $q+2t>0$ for this formula to make sense by Definition 6.3. But again contrary to intuition, if we let $t \to 0+$, we obtain the adjoint formulas in (Kaptanoğlu 2014, (26)) that are derived from the reproducing kernel norms.

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RESEARCH ARTICLE

The nX-complementary generations of the group $PSL(3,5)$

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ABSTRACT
Let G be a finite non-abelian simple group and nX be a non-trivial conjugacy class of elements of order n in G. We say that G is *nX-complementary generated*, if for every $x \in G$, there exists an element $y \in nX$ such that $G = \langle x, y \rangle$. In this paper we study the nX-complementary generations for all the non-trivial conjugacy classes of the projective special linear group $PSL(3,5)$. We approach this kind of generation using the structure constant method. GAP [The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.9.3;* 2018. (http://www.gap-system.org)] is used frequently in our computations.

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Keywords: Conjugacy classes, generation, nX-complementary generation, structure constant

1. INTRODUCTION

The problem of generation of finite groups has great interest and has many applications to groups and their representations. The classification of finite simple groups is involved heavily and plays a pivotal role in most general results on the generation of finite groups. The study of generating sets in finite groups has a rich history, with numerous applications. We are interested in three kinds of generations of a finite simple group G, namely the (p,q,r)-generation, the nX-complementary generation and the ranks of the non-trivial conjugacy classes of G.

Definition 1.1. A finite group G is called *(l,m,n)-generated*, if it is a quotient group of the triangle group $T(l,m,n) = \left\langle x, y, z | x^l = y^m = z^n = xyz = 1 \right\rangle$.

In our work we generally restrict ourselves to the cases when l, m and n are primes and we use the notation (p,q,r)-generation rather than (l,m,n)-generation.

Definition 1.2. Let G be a finite simple group and nX be a non-trivial conjugacy class of elements of G. We define the *rank* of nX in G, denoted by $rank(G : nX)$, to be the minimum number of elements in nX that generate G.

Definition 1.3. A finite group G is said to be *nX-complementary generated*, if for every $x \in G$, there exists an element $y \in nX$ such that $G = \langle x, y \rangle$. The element y is called *complementary*.

In Woldar (1994), Woldar proved that every sporadic simple group G is pX-complementary generated where p is the largest prime dividing the order of G. In Ganief (1997), Ganief conjectured that every finite simple group is nX-complementary generated for some conjugacy class nX. In an attempt to further the theory on nX-complementary generation, he posed the problem:
Problem 1: Given a finite non-abelian simple group G, find all conjugacy classes nX of G such that G is nX-complementary generated.

In Ganief (1997) Ganief gave a complete answer to Problem 1 for the following sporadic simple group: the Janko groups J_1, J_2, J_3, J_4; the Higman-Sims group HS; the McLaughlin group McL; the Conway groups Co_3, Co_2 and the Fischer group Fi_{22}. The previous results have been published by Ganief and Moori in a series of papers Ganief and Moori (1997, 1998), Ganief and Moori (1998, 1999). Ashrafi in Ashrafi (2003, 2004) did the same for the sporadic groups He, Th, HN and $O'N$. Also Darafsheh, Ashrafi and Moghani established in Ashrafi et. al (2006); Darafsheh et. al. (2003, 2004), the nX-complementary generations for the sporadic groups Fi_{23}, Ru, Ly and Co_1.

With nX being a non-trivial conjugacy class of the group $G = PSL(3,5)$ as in the Atlas ATLAS (2024), the main result on the nX-complementary generation of the projective special linear group $PSL(3,5)$ can be summarized in the following theorem.

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Theorem 1.4. *The group $G = PSL(3,5)$ is nX-complementary generated for all the conjugacy classes of G except when $nX \in \{1A, 2A, 4A, 4B, 5A\}$.*

The proof of Theorem 1.4 will be done through sequence of propositions and corollaries that will be established in Section 3.

The (p,q,r)-generations and the ranks of the conjugacy classes of the group $PSL(3,5)$ have been determined by the authors in Basheer and Seretlo (2019). We mainly used the structure constant method. In this paper we study the nX-complementary generation for the non-trivial conjugacy classes of the group $PSL(3,5)$. For the notation, description of the structure constant method and known results, we follow precisely Basheer and Seretlo (2019); Basheer et. al. (2019); Basheer and Seretlo (2021, 2020); Basheer et. al. (2024).

2. THE PROJECTIVE SPECIAL LINEAR GROUP $PSL(3,5)$

The projective special linear group $PSL(3,5)$ is a simple group of order $372000 = 2^5 \times 3 \times 5^3 \times 31$. By the Atlas ATLAS (2024), the group $PSL(3,5)$ has exactly 30 conjugacy classes of its elements, of which 14 of these classes have elements of prime orders. These are the classes $2A$, $3A$, $5A$, $5B$, $31A$, $31B$, $31C$, $31D$, $31E$, $31F$, $31G$, $31H$, $31I$ and $31J$. Also $PSL(3,5)$ has 5 conjugacy classes of maximal subgroups, where representatives of these classes of maximal subgroups can be taken as follows:

Table 1. The maximal subgroups of $G = PSL(3,5)$

	$\lvert H_i \rvert$	$[G : H_i]$
$H_1 = 5^2{:}GL(5,2)$	12000	31
$H_2 = 5^2{:}GL(5,2)$	12000	31
$H_3 = S_5$	120	3100
$H_4 = 4^2{:}S_3$	96	3875
$H_5 = 31{:}3$	93	4000

Throughout this paper and unless otherwise stated, by G we always mean the projective special linear group $PSL(3,5)$.

3. THE NX-COMPLEMENTARY GENERATIONS OF THE $PSL(3,5)$

The following lemma is very crucial in the determination of which conjugacy classes nX of G are nX-complementary generated.

Lemma 3.1. *A group G is nX-complementary generated if and only if for every conjugacy class pY of G, p prime, there exists a conjugacy class $t_{pY}Z$, depending on pY, such that the group G is $(pY, nX, t_{pY}Z)$-generated. Moreover; if G is a finite simple group, then G is not a $2X$-complementary generated, for any conjugacy class of involutions.*

Proof. See Lemma 2.3.8 of Ganief Ganief (1997).

Remark 3.2. Lemma 3.1 gives us a necessary and sufficient condition to determine whether a group G is nX-complementary generated or not. To check that the group G is nX-complementary generated, we need to check that G is $(pY, nX, t_{pY}Z)$-generated group for all the classes pY, where p is a prime number divides the order of G.

Remark 3.3. Recall that two involutions generate a dihedral group. Thus if G is a finite simple group and $2X$ is an involution class of G, then G is not a $2X$-complementary generated. To see this, suppose that G is a $2X$-complementary generated group. Then by Lemma 3.1, it follows that G is $(2X, 2X, t_{2X}Z)$-generated, which means that G is a dihedral group, contradicting the fact that G is a finite simple group. Therefore in the investigation on the classes of G whether they are nX-complementary generated or not we start by those classes where elements are of order at least 3.

The following result also helps a lot in determining whether G is an nX-complementary generated with the information that G is complementary generated by some conjugacy class sY and a power map of nX gives sY.

Lemma 3.4. *If G is sY-complementary generated and $(rX)^n = sY$, then G is rX-complementary generated.*

Proof. Let rX and sY be non-trivial conjugacy classes of G such that $(rX)^n = sY$ for some positive integer n. Now assume that G is not rX-complementary generated group. It follows that there exits an element x of prime order such that $\langle x, y \rangle < G$ for all $y \in rX$. Since $x, y^n \in \langle x, y \rangle$, it follows that $\langle x, y^n \rangle \leq \langle x, y \rangle < G$, for all $y^n \in sY$. We conclude that if G is not rX-complementary generated, then it is also not sY-complementary generated. The contrapositive of this gives the result.

It is clear that the group $G = PSL(3,5)$ is neither $1A$- nor $2A$-complementary generated. In the next proposition we show that G is not $3A$-complementary generated.

Proposition 3.5. *The group G is $3A$-complementary generated.*

Proof. By Propositions 3.1, 3.7, 3.11 and 3.12 of Basheer and Seretlo (2019), the group G is $(2A,3A,31X)$-, $(3A,3A,5B)$-, $(3A,5Y,31X)$- and $(3A,31X,31X)$-generated, for $Y \in \{A,B\}$ and $X \in \{A,B,C,D,E,F,G,H,I,J\}$. We know that if $G = \langle x,y \rangle$, then it is also $\langle y,x \rangle$. Therefore G is $(5Y,3A,31X)$- and $(31X,3A,31X)$-generated. The result follows by Lemma 3.1.

Proposition 3.6. *The group G is neither $4A$- nor $4B$-complementary generated.*

Proof. Here we show that G is not $(2A,4X,nZ)$-generated for $X \in \{A,B\}$ and all the conjugacy classes of G. The direct computations with GAP show that $\Delta_G(2A,4X,nZ) = 0$ for $X \in \{A,B\}$ and all the classes nZ of G except for the cases $(2A,4X,4Y)$, $(2A,4X,4C)$, $(2A,4X,20Y)$, $(2A,4A,24Z)$ and $(2A,4B,24Y)$ for $X,Y \in \{A,B\}$, $X \neq Y$ and $Z \in \{C,D\}$. Here we have

$$\begin{aligned}
\Delta_G(2A,4X,4Y) &= 49 < 480 = |C_G(g)|, g \in 4Y, Y \in \{A,B\}, \\
\Delta_G(2A,4X,4C) &= 9 < 16 = |C_G(g)|, g \in 4C, \\
\Delta_G(2A,4X,8Y) &= 6 < 24 = |C_G(g)|, g \in 8Y, Y \in \{A,B\}, \\
\Delta_G(2A,4X,20Y) &= 4 < 20 = |C_G(g)|, g \in 20Y, Y \in \{A,B\}, \\
\Delta_G(2A,4A,24Z) &= 6 < 24 = |C_G(g)|, g \in 24Z, Z \in \{C,D\}, \\
\Delta_G(2A,4B,24Y) &= 6 < 24 = |C_G(g)|, g \in 24Y, Y \in \{A,B\}.
\end{aligned}$$

It follows by Lemma 2.7 of Basheer and Seretlo (2020) that G is not generated by any of the previous 3-tuples. Therefore G is not generated by $(2A,4X,nZ)$, $X \in \{A,B\}$ and all the classes nZ of G. This completes the proof that G is not $4X$-complementary generated for $X \in \{A,B\}$.

Proposition 3.7. *The group G is $4C$-complementary generated.*

Proof. Here we do some computations regarding the triples $(pY,4C,31A)$ for all the primes p dividing the order of G. We firstly prove that G is $(2A,4C,31A)$-generated group. The direct computations with GAP show that $\Delta_G(2A,4C,31A) = 31$. Now from Table 1, we can see that only $H_5 = 31{:}3$ is the maximal subgroup of G that contains elements of order 31. However it is clear that H_5 neither contains elements of order 2 nor of order 4. Thus in the computations of $\Delta^*_G(2A,4C,31A)$ there is no contribution from any maximal subgroup of G. It follows that $\Delta^*_G(2A,4C,31A) = \Delta_G(2A,4C,31A) = 31$ and thus G is $(2A,4C,31A)$-generated group. Next we show that G is $(3A,4C,31A)$-generated group. The direct computations with GAP gives $\Delta_G(3A,4C,31A) = 961$. Again since H_5 is the only maximal subgroup of G with elements of order 31, however it does not contain an element of order 4. Therefore $\Delta^*_G(3A,4C,31A) = \Delta_G(3A,4C,31A) = 961$ and consequently G is $(3A,4C,31A)$-generated group. Similar arguments show that G is $(5X,4C,31A)$-generated group for $X \in \{A,B\}$ ($\Delta^*_G(5A,4C,31A) = \Delta_G(5A,4C,31A) = 31$, while $\Delta^*_G(5B,4C,31A) = \Delta_G(5B,4C,31A) = 930$). Finally using similar arguments we obtain for $X \in \{A,B,C,D,E,F,G,H,I,J\}$ that $\Delta^*_G(31X,4C,31A) = \Delta_G(31X,4C,31A) = 806$, establishing the generation of G by the triple $(31X,4C,31A)$ for X in the previous set. Now by applying Lemma 3.1 we deduce that G is $4C$-complementary generated group.

Proposition 3.8. *The group G is not $5A$-complementary generated.*

Proof. To show that G is not $5A$-complementary generated, we prove that G is not $(2A,5A,nZ)$-generated for all the conjugacy classes nZ of G. The direct computations with GAP reveal that $\Delta_G(2A,5A,nZ) = 0$ for all the classes nZ except when $nZ \in \{2A,10A,12A,12B,20A,20B\}$. Clearly the group G cannot be $(2A,5A,2A)$-generated as this violets the condition $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$ for the group to be (l,m,n)-generated. Now we have the following:

$$\begin{aligned}
\Delta_G(2A,5A,10A) &= 9 < 20 = |C_G(g)|, g \in 10A, \\
\Delta_G(2A,5A,12X) &= 6 < 24 = |C_G(g)|, g \in 12X, X \in \{A,B\}, \\
\Delta_G(2A,5A,20X) &= 5 < 20 = |C_G(g)|, g \in 20X, X \in \{A,B\}.
\end{aligned}$$

It follows by Lemma 2.7 of Basheer and Seretlo (2020) that G is not generated by any of the previous 3-tuples. Therefore G is not generated by $(2A,5A,nZ)$ and all the classes nZ of G. Hence G is not $5A$-complementary generated.

Proposition 3.9. *The group G is $5B$-complementary generated.*

Proof. By Propositions 3.4, 3.11, 3.15 and 3.17 of Basheer and Seretlo (2019), the group G is $(2A,5B,31X)$-, $(3A,5B,31X)$-, $(5Z,5B,31Y)$- and $(5B,31X,31Y)$-generated, for $Z \in \{A,B\}$ and $X,Y \in \{A,B,C,D,E,F,G,H,I,J\}$. We know that if $G = \langle x,y \rangle$, then it is also $\langle y,x \rangle$. Therefore G is $(31X,5B,31Y)$-generated for X and Y in the previous set. The result follows by Lemma 3.1.

Proposition 3.10. *The group G is $6A$-complementary generated.*

Proof. The proof is similar to the one of Proposition 3.7. We show that G is $(pY, 6A, 31A)$-generated group for all $pY \in \{2A, 3A, 5A, 5B, 31X\}$, $X \in \{A, B, C, D, E, F, G, H, I, J\}$. From Table 1, the only maximal subgroup that has elements of order 31 is $H_5 = 31{:}3$. However it is clear that H_5 does not contain an element of order 6. Thus there is no contribution from any maximal subgroup of G in the computations of $\Delta^*_G(pY, 6A, 31A)$. Using GAP we obtain that

$$\begin{aligned}
\Delta^*_G(2A, 6A, 31A) &= \Delta_G(2A, 6A, 31A) = 31,\\
\Delta^*_G(3A, 6A, 31A) &= \Delta_G(3A, 6A, 31A) = 651,\\
\Delta^*_G(5A, 6A, 31A) &= \Delta_G(5A, 6A, 31A) = 31,\\
\Delta^*_G(5B, 6A, 31A) &= \Delta_G(5B, 6A, 31A) = 620,\\
\Delta^*_G(31X, 6A, 31A) &= \Delta_G(31X, 6A, 31A) = 496,
\end{aligned}$$

for all $X \in \{A, B, C, D, E, F, G, H, I, J\}$. It follows that G is $(pY, 6A, 31A)$-generated group for all $pY \in \{2A, 3A, 5A, 5B, 31X\}$ and $X \in \{A, B, C, D, E, F, G, H, I, J\}$. By Lemma 3.1 we deduce that G is $6A$-complementary generated.

Proposition 3.11. *The group G is $12A$- and $12B$-complementary generated.*

Proof. Since G has only one class of elements of order 6, it becomes obvious that the power maps $(12A)^2 = (12B)^2 = 6A$. By Proposition 3.10 we have G is $6A$-complementary generated. It follows by Lemma 3.4 that G is also $12A$- and $12B$-complementary generated group.

Proposition 3.12. *The group G is $24X$-complementary generated group for $X \in \{A, B, C, D\}$.*

Proof. The power maps of the classes of elements of order 24 as follows: $(24A)^2 = 12A$, $(24B)^2 = 12A$, $(24C)^2 = 12B$ and $(24D)^2 = 12B$. By Corollary 3.11, we know that G is $12A$- and $12B$-complementary generated. It follows by Lemma 3.4 that G is $24X$-complementary generated group for $X \in \{A, B, C, D\}$.

Proposition 3.13. *The group G is $8A$- and $8B$-complementary generated.*

Proof. The proof is similar to the one of Propositions 3.7 and 3.10. We show that G is $(pY, 8X, 31A)$-generated group for all $pY \in \{2A, 3A, 5A, 5B, 31Z\}$, $Z \in \{A, B, C, D, E, F, G, H, I, J\}$ and $X \in \{A, B\}$. From Table 1, the only maximal subgroup that has elements of order 31 is $H_5 = 31{:}3$. However it is clear that H_5 does not contain an element of order 8. Thus there is no contribution from any maximal subgroup of G in the computations of $\Delta^*_G(pY, 8X, 31A)$. Using GAP we obtain that

$$\begin{aligned}
\Delta^*_G(2A, 8X, 31A) &= \Delta_G(2A, 8X, 31A) = 31,\\
\Delta^*_G(3A, 8X, 31A) &= \Delta_G(3A, 8X, 31A) = 651,\\
\Delta^*_G(5A, 8X, 31A) &= \Delta_G(5A, 8X, 31A) = 31,\\
\Delta^*_G(5B, 8X, 31A) &= \Delta_G(5B, 8X, 31A) = 620,\\
\Delta^*_G(31Z, 8X, 31A) &= \Delta_G(31Z, 8X, 31A) = 496,
\end{aligned}$$

for all $Z \in \{A, B, C, D, E, F, G, H, I, J\}$ and $X \in \{A, B\}$. It follows that G is $(pY, 8X, 31A)$-generated group for all $pY \in \{2A, 3A, 5A, 5B, 31Z\}$ and $Z \in \{A, B, C, D, E, F, G, H, I, J\}$ and $X \in \{A, B\}$. By Lemma 3.1 we deduce that G is $8X$-complementary generated group for $X \in \{A, B\}$.

Proposition 3.14. *The group G is $10A$-complementary generated.*

Proof. The proof is similar to the previous ones handling the cases $4C$, $6A$, $8A$ and $8B$. We show that G is $(pY, 10A, 31A)$-generated group for all $pY \in \{2A, 3A, 5A, 5B, 31X\}$ and $X \in \{A, B, C, D, E, F, G, H, I, J\}$. From Table 1, the only maximal subgroup that has elements of order 31 is $H_5 = 31{:}3$. However it is clear that H_5 does not contain an element of order 10. Thus there is no contribution from any maximal subgroup of G in the computations of $\Delta^*_G(pY, 10A, 31A)$. Using GAP we obtain that

$$\begin{aligned}
\Delta^*_G(2A, 10A, 31A) &= \Delta_G(2A, 10A, 31A) = 31,\\
\Delta^*_G(3A, 10A, 31A) &= \Delta_G(3A, 10A, 31A) = 775,\\
\Delta^*_G(5A, 10A, 31A) &= \Delta_G(5A, 10A, 31A) = 31,\\
\Delta^*_G(5B, 10A, 31A) &= \Delta_G(5B, 10A, 31A) = 774,\\
\Delta^*_G(31X, 10A, 31A) &= \Delta_G(31X, 10A, 31A) = 620,
\end{aligned}$$

for all $X \in \{A, B, C, D, E, F, G, H, I, J\}$. It follows that G is $(pY, 6A, 31A)$-generated group for all $pY \in \{2A, 3A, 5A, 5B, 31X\}$ and $X \in \{A, B, C, D, E, F, G, H, I, J\}$. By Lemma 3.1 we deduce that G is $6A$-complementary generated.

Proposition 3.15. *The group G is $20A$-complementary generated.*

Proof. Since G has only one class of elements of order 10, it becomes obvious that the power map $(20A)^2 = 10A$. By Proposition 3.14 we have G is $10A$-complementary generated. It follows by Lemma 3.4 that G is also $20A$-complementary generated group.

Proposition 3.16. *The group G is $31X$-complementary generated for $X \in \{A, B, C, D, E, F, G, H, I, J\}$.*

Proof. By Propositions 3.5, 3.12, 3.17 and 3.18 of Basheer and Seretlo (2019), the group G is $(2A, 31X, 31Y)$-, $(3A, 31X, 31Y)$-, $(5B, 31X, 31Y)$- and $(31X, 31Y, 31Z)$-generated, for $X, Y \in \{A, B, C, D, E, F, G, H, I, J\}$. Also for $X, Y \in \{A, B, C, D, E, F, G, H, I, J\}$ and $X \neq Y$, the group G is $(5A, 31X, 31Y)$-generated. Therefore G is $(pY, 31X, nZ)$-generated for all the conjugacy classes pY of G, where p is a prime divides the order of G. The result follows by Lemma 3.1.

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RESEARCH ARTICLE

Generalizations of third-order recurrence relation

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ABSTRACT

This paper presents a generalization of the sequence defined by the third-order recurrence relation $V_n(a_j, p_j) = \sum_{j=1}^{3} p_j V_{n-j},\ n \geq 4,$, $p_3 \neq 0$ with initial terms $V_j = a_j$, where a_j and p_j $j = 1, 2, 3$, are any non-zero real numbers. The generating function and Binet's formula are derived for this generalized tribonacci sequence. Classical second-order generalized Fibonacci sequences and other existing sequences based on second-order recurrence relations are implicitly included in this analysis. These derived sequences are discussed as special cases of the generalization. A pictorial representation is provided, illustrating the growth and variation of tribonacci numbers for different initial terms a_j and coefficients p_j. Additionally, the tribonacci constant is examined and visually represented. It is observed that the constant is influenced solely by the coefficients p_j of the recurrence relation and is unaffected by the initial terms a_j.

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Keywords: Generalized Tribonacci sequence, generating function, Binet's formula, Tribonacci constant.

1. INTRODUCTION

Tribonacci sequences are the generalization of the classical Fibonacci sequence, defined by a recurrence relation involving the sum of the three preceding terms, where each term is the sum of the three preceding terms. The enigmatic tribonacci number sequences with its captivating properties, has piqued the curiosity of mathematicians and researchers, opening doors to a world of intriguing applications as it has attracted attention in various branches of physical sciences and its applications. Sequence terms in a recursive relations are generated sequentially, the process of calculating any specific term is computationally intensive, as it necessitates the calculation of all its predecessors. Alternatively, using the index form of a generating function or Binet's formula provides efficient methods for directly computing any term of a recursive sequence. Although extensive research has been conducted on second-order Fibonacci sequences and their generalizations, the exploration of third-order recurrence relations, particularly in the context of third-order Fibonacci-like sequences, has received comparatively less attention. A generalized tribonacci sequence, $\{V_n\}$, is result of the recurrence relations with coefficients p_j and arbitrary first three initial terms a_j. The concept of tribonacci sequence mentioned and studied, first time by Feinberg M. Feinberg (1963), then number of generalizations of the Fibonacci sequence have been considered and examined by many authors W. R. Spickerman (1982); T. Komatsu (2018); R. Frontczak (2018); A. G. Shannon (1972); A. C. F. Bueno (2015); T. Komatsu and R. Li (2017); T. Koshy (2001); P. Y. Lin (1988); S. Pethe (1988); Y. Soykan (2019); C. C. Yalavigi (1972). F. T. Howard (2001) extended and generalize the main result obtained by F. T. Howard (1999) for tribonacci sequences. Generalization of Tribonacci sequences for quaternions studied by G. Cerda-Morales (2017). In the literature. Generalized Tribonacci sequence has also been considered and studied by A. G. Shannon and A. F. Horadam (1972); M. E. Waddill and L. Sacks (1967); T. Komatsu and R. Li (2017) and Y. Soykan, I. et al. (2020); A. Scott, T. et al. (1997). This research aims to address by considering and exploring the properties, patterns, and potential applications of generalized third-order Fibonacci sequences. In this article, generalized third-order recurrence relations with variable coefficients p_j and initial terms a_j are taken to derive the generalized form of generating function and the Binet's formula. Classical second-order generalized Fibonacci sequences and other existing sequences based on second-order recurrence relations are implicitly included in this analysis. These derived sequences are discussed as special cases of the generalization. A pictorial representation is provided, illustrating the growth and variation of tribonacci numbers for different initial terms a_j and coefficients p_j. Additionally, the

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Tribonacci constant is examined and visually represented. It is observed that the constant is influenced solely by the coefficients p_j of the recurrence relation and is unaffected by the initial terms a_j.

Definition 1.1. We define the Generalized Tribonacci sequence $\{V_n\}$ by the following linear recurrence relation:

$$V_n(a_1, a_2, a_3, p_1, p_2, p_3) = p_1 V_{n-1} + p_2 V_{n-2} + p_3 V_{n-3}, \; n \geq 4, \tag{1}$$

with the initial conditions,$a_j = V_j$, a_j, p_j, $j = 1, 2, 3$ are any non–zero real numbers.

The expression for $\{V_n\}$ in (1) is holds true T. Koshy (2001) for every integer $n \geq 4$.

Terms of the Generalized Tribonacci Sequence The first few terms in the generalized form of the sequence defined in (1) are:

$$\{V_n\} = \begin{Bmatrix} a_1, a_2, a_3, p_1 a_3 + p_2 a_2 + p_3 a_1, \left(p_1^2 + p_2\right) a_3 + (p_1 p_2 + p_3)\, a_2 + p_1 p_3 a_1, \\ \left(p_1^3 + p_3 + 2p_1 p_2\right) a_3 + \left(p_1^2 p_2 + p_2^2 + p_1 p_3\right) a_2 + \left(p_1^2 p_3 + p_2 p_3\right) a_1 + \cdots \end{Bmatrix}.$$

Tribonacci Sequences pictorial representations A few values Y. Soykan, I. et al. (2020) of Tribonacci sequences represented in the following figure.

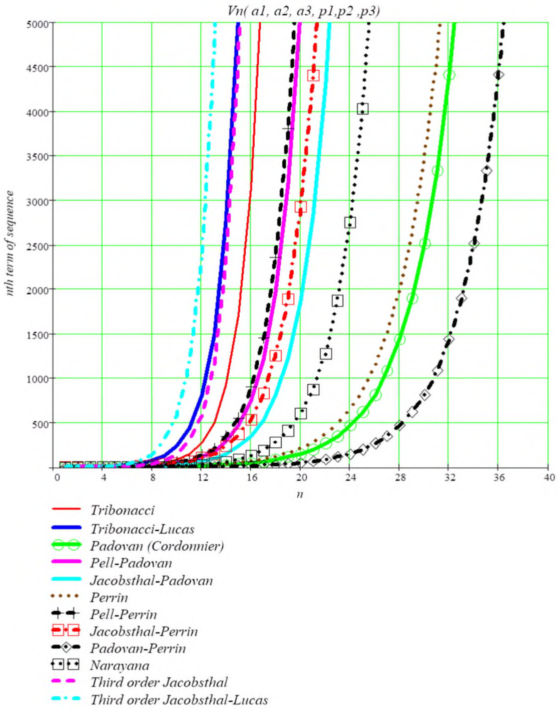

Figure 1. Tribonacci sequences progression and comparison

Special Cases

Remark 1.2. With initial conditions $V_0 = 0, V_1 = 1,\ V_2 = 1$, and $p_1 = p_2 = p_3 = 1$, recurrence relation (1) is known as the generalized Lucas tribonacci sequence and is denoted by T_n in F. T. Howard (1999). The first few terms of the sequence deduced from the above generalization:

$$\{V_n\}_{n\geq 0} = \{T_n\} = \{0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, \cdots\}.$$

This tribonacci number sequence is A000073 on the OEIS, N. J. A. Sloane (1973).

Remark 1.3. If we substitute the initial conditions $V_0 = 3, V_1 = 1,\ V_2 = 3$, and $p_1 = p_2 = p_3 = 1$ in (1), it reduces to K_n sequence which is explained in **?**. The first few terms of the sequence K_n are:

$$\{V_n\}_{n\geq 0} = \{K_n\} = \{3, 1, 3, 7, 11, 21, 39, 71, 131, 241, 443, 815, 1499, 2757, 5071, 9327, \cdots\}.$$

This tribonacci number sequence is A001644 on the OEIS, N. J. A. Sloane (1973).

2. GENERALIZED GENERATING FUNCTIONS

A generating function W. Watkins (1987) is a representation of a sequence as the coefficients of a power series in mathematics. By analyzing the generating function, we can derive various properties of the generalized Tribonacci sequence, such as closed-form expressions, asymptotic behavior, and generating function identities.

Theorem 2.1. (Generalized Generating Functions) *The generalized generating function of the sequence defined in (1) is*

$$V(x) = \frac{f(x)}{1 - p_1 x - p_2 x^2 - p_3 x^3}, \tag{2}$$

where $f(x) = V_0 + (V_1 - p_1 V_0)\, x + (V_2 - p_1 V_1 - p_2 V_0)\, x^2$.

Proof. If $V(x)$ is the generating function of $V_n = p_1 V_{n-1} + p_2 V_{n-2} + p_3 V_{n-3}$, then we have

$$V(x) = \sum_{n=0}^{\infty} V_n x^n = V_0 + V_1 x + V_2 x^2 + V_3 x^3 + \cdots \tag{3}$$

Multiplying $V(x)$ by $p_1 x$, $p_2 x^2$ and $p_3 x^3$, we have

$$\begin{aligned} p_1 x V(x) &= p_1 V_0 x + p_1 V_1 x^2 + p_1 V_2 x^3 + p_1 V_3 x^4 + \cdots \\ p_2 x^2 V(x) &= p_2 V_0 x^2 + p_2 V_1 x^3 + p_2 V_2 x^4 + p_2 V_3 x^5 + \cdots . \\ p_3 x^3 V(x) &= p_3 V_0 x^3 + p_3 V_1 x^4 + p_3 V_2 x^5 + p_3 V_3 x^6 + \cdots \end{aligned} \tag{4}$$

Subtracting (3)- (4) and rearranging the above equations, we have

$$V(x)\left[1 - p_1 x - p_2 x^2 - p_3 x^3\right] = f(x).$$

Solving for $V(x)$, we obtain

$$V(x) = \frac{f(x)}{1 - p_1 x - p_2 x^2 - p_3 x^3}, \tag{5}$$

where $f(x) = V_0 + (V_1 - p_1 V_0)\, x + (V_2 - p_1 V_1 - p_2 V_0)\, x^2$ is a polynomial.
Hence $V(x)$ is the generating function of the sequence $\{V_n\}$.

Remark 2.2. If we substitute $V_0 = 3, V_1 = 1,\ V_2 = 3$, and $p_1 = p_2 = p_3 = 1$ in the result obtained in (5), it reduces to the generating function

$$V(x) = \frac{3 - 2x - x^2}{1 - x - x^2 - x^3} = T(x),$$

which is the same as result, which is explained in M. Elia (2001); M. Catalani (2002).

Remark 2.3. If we substitute $V_0 = 0, V_1 = 1,\ V_2 = 1$, and $p_1 = p_2 = p_3 = 1$ in result of (5), it reduces to the generating function

$$V(x) = \frac{x}{1 - x - x^2 - x^3} = K(x),$$

which is the same as result, which is explained in M. Elia (2001); M. Catalani (2002).

2.1. Even and odd terms Generating Functions of the Generalized Tribonacci Sequence

Theorem 2.4. *[Even and odd terms Generating Functions] The generating functions of even $V_{2n}(x)$ and odd $V_{2n+1}(x)$ terms of the Generalized Tribonacci Sequence (1) are:*

$$V_{even}(x) = \frac{V_0 - \left[(2p_2 + p_1^2)V_0 - V_2\right] x + \left[(p_2^2 - p_1 p_3)V_0 + (p_1 p_2 + p_3)V_1 - p_2 V_2\right] x^2}{1 - (p_1^2 + 2p_2)x - (2p_1 p_3 - p_2^2)x^2 - p_3^2 x^3},$$

and

$$V_{odd}(x) = \frac{V_1 + \left[V_0 p_3 - (p_1^2 + p_2)V_1 + p_1 V_2\right] x + \left[p_3 V_2 - p_1 p_3 V_1 - p_2 p_3 V_0\right] x^2}{1 - (p_1^2 + 2p_2)x - (2p_1 p_3 - p_2^2)x^2 - p_3^2 x^3}.$$

Proof. From the definition of the even $V_{2n}(x) = \frac{V_n(\sqrt{x}) + V_n(-\sqrt{x})}{2}$ and odd $V_{2n+1}(x) = \frac{V_n(\sqrt{x}) - V_n(-\sqrt{x})}{2\sqrt{x}}$ functions and employing the Generalized generating function of the Tribonacci sequence (1) obtained in the Theorem (2.1) we have

$$V_n(x)=\frac{f(x)}{1-p_1x-p_2x^2-p_3x^3},$$

where $f(x)=V_0+(V_1-p_1V_0)\,x+(V_2-p_1V_1-p_2V_0)\,x^2$.
On simplification we obtained the Generalized Generating function of even and odd terms of Tribonacci sequence

$$V_{even}(x)=\frac{V_0-\left[(2p_2+p_1^2)V_0-V_2\right]x+\left[(p_2^2-p_1p_3)V_0+(p_1p_2+p_3)V_1-p_2V_2\right]x^2}{1-(p_1^2+2p_2)x-(2p_1p_3-p_2^2)x^2-p_3^2x^3}, \tag{6}$$

and

$$V_{odd}(x)=\frac{V_1+\left[V_0p_3-(p_1^2+p_2)V_1+p_1V_2\right]x+\left[p_3V_2-p_1p_3V_1-p_2p_3V_0\right]x^2}{1-(p_1^2+2p_2)x-(2p_1p_3-p_2^2)x^2-p_3^2x^3}. \tag{7}$$

2.2. Special cases of Even and odd terms Generating Functions

Remark 2.5. With initial conditions $V_0=0, V_1=1,\ V_2=1$, and $p_1=p_2=p_3=1$, the even and odd terms Generating Functions of the generalized Lucas sequence T_n T. Koshy (2001)) are deduced from the (6) and (7) generalized even and odd terms generating functions as:

$$V_{even}(x)=T_{even}(x)=\frac{x+x^2}{1-3x-x^2-x^3},$$

and

$$V_{odd}(x)=T_{odd}(x)=\frac{1-x}{1-3x-x^2-x^3}.$$

Similarly with initial conditions $V_0=3, V_1=1,\ V_2=3$, and $p_1=p_2=p_3=1$ in (1), the even and odd terms Generating Functions of the generalized Lucas sequence K_n. T. Koshy (2001)) are deduced from the (6) and (7) generalized even and odd terms Generating Functions are:

$$V_{2n}(x)=K_{even}(x)=\frac{3-6x-x^2}{1-3x-x^2-x^3},$$

and

$$V_{odd}(x)=K_{odd}(x)=\frac{3+4x-x^2}{1-3x-x^2-x^3}.$$

These even and odd terms of the Generating Functions of T_n and K_n are same as obtained by T. Komatsu (2018).

Theorem 2.6. (Generalized Binet's formula for Tribonacci sequence) *Generalized form of the Binet's formula for the sequence defined in (1) is*

$$V_n(x)=\sum_{j=1}^{3}\left[\frac{A_1\alpha_j^2+A_2\alpha_j+A_3}{\prod\limits_{\substack{1\le i\le 3\\ i\ne j}}(\alpha_j-\alpha_i)}\right]\alpha_j^n=\sum_{j=1}^{3}\frac{\left(\alpha_j^2A_1+\alpha_jA_2+A_3\right)p_3\alpha_j^{n+1}}{\alpha_j^3p_3-\alpha_jp_1+2}.$$

Proof. Since $V(x)$ is the generating function of the sequence $\{V_n\}$

$$V(x)=\frac{f(x)}{1-p_1x-p_2x^2-p_3x^3},$$

where $f(x)=V_0+(V_1-p_1V_0)\,x+(V_2-p_1V_1-p_2V_0)\,x^2$.
Consider the partial fraction decomposition of the right-hand side of the generating function , we have

$$V(x)=\frac{A_1+A_2x+A_3x^2}{1-p_1x-p_2x^2-p_3x^3}=\frac{A_1+A_2x+A_3x^2}{(1-\alpha_1x)(1-\alpha_2x)(1-\alpha_3x)},$$

where $A_1=V_0, A_2=V_1-p_1V_0, A_3=V_2-p_1V_1-p_2V_0$ and $\alpha_i,\ i=1,2,3$, are roots of the equation $1-p_1x-p_2x^2-p_3x^3=0$.

On simplification we have

$$V_n(x)=\sum_{j=1}^{3}\left[\frac{A_1\alpha_j^2+A_2\alpha_j+A_{3k}}{\prod\limits_{\substack{1\le i\le 3\\ i\ne j}}(\alpha_j-\alpha_i)}\right]\alpha_j^n=\sum_{j=1}^{3}\frac{\left(\alpha_j^2A_1+\alpha_jA_2+A_3\right)p_3\alpha_j^{n+1}}{p_3\alpha_j^3-\alpha_jp_1+2}. \tag{8}$$

The above relation is the Generalized Binet's formula for Tribonacci sequence

2.3. Special cases: Generalized Binet's formula for Tribonacci sequence

Remark 2.7. Generalized form of the Binet's formula (8) for the generalized sequence(1) can also be written as

$$\begin{aligned}V_n&=\left[\begin{array}{l}\left(\dfrac{\alpha_3^{n+2}-\alpha_2^{n+2}}{\alpha_3-\alpha_2}\right)\dfrac{\alpha_1}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_1)}+\left(\dfrac{\alpha_1^{n+2}-\alpha_3^{n+2}}{\alpha_1-\alpha_3}\right)\dfrac{\alpha_2}{(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)}\\ +\left(\dfrac{\alpha_2^{n+2}-\alpha_1^{n+2}}{\alpha_2-\alpha_1}\right)\dfrac{\alpha_3}{(\alpha_3-\alpha_2)(\alpha_1-\alpha_3)}\end{array}\right].\\ &=\sum_{j=0}^{n}\left(\sum_{k=0}^{n-j}\alpha_1^j\alpha k\alpha_3^{n-j-k}\right)\end{aligned}$$

Remark 2.8. If we put $V_3=0,p_3=0$, in equation (1) then tribonacci sequences becomes the generalized classical Fibonacci sequence, and the Binet's formula (8) in this case reduces to

$$\begin{aligned}V_n(x)&=\frac{A_1+A_2x}{1-p_1x-p_2x^2}\\ V_n(x)&=\frac{(\alpha_1A_1+A_2)\left(\alpha_1^n\right)-(\alpha_2A_1+A_2)\left(\alpha_2^n\right)}{\alpha_1-\alpha_2}\\ V_n(x)&=\frac{A_1\left(\alpha_1^{n+1}-\alpha_2^{n+1}\right)+A_2\left(\alpha_1^n-\alpha_2^n\right)}{\alpha_1-\alpha_2}=A_1\left(\frac{\alpha_1^{n+1}-\alpha_2^{n+1}}{\alpha_1-\alpha_2}\right)+A_2\left(\frac{\alpha_1^n-\alpha_2^n}{\alpha_1-\alpha_2}\right)\end{aligned},$$

where $A_1=V_0$,$A_2=(V_1-p_1V_0)$ and $\alpha_i,\ i=1,2$ are roots of the equation $1-p_1x-p_2x^2=0$.

Remark 2.9. If we take $V_0=0,V_1=1,\ V_2=1$, and $p_1=p_2=p_3=1$, in the expression (8) this reduces to

$$V_n(x)=\frac{\alpha_1^{n+1}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}+\frac{\alpha_2^{n+1}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}+\frac{\alpha_3^{n+1}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}=T_n$$

. which is same, as obtained by R. Frontczak (2018).

When$V_0=3,V_1=1,\ V_2=3$, and $p_1=p_2=p_3=1$, in the expression (8) this reduces to

$$V(x)=\alpha_1^n+\alpha_2^n+\alpha_3^n=K_n$$

where $\alpha_i,\ i=1,2,3$ are roots of the equation $1-x-x^2-x^3=0$. This is in agreement with W. R. Spickerman (1982); R. Frontczak (2018). .

Theorem 2.10. *If*

$$V_n=\begin{cases}a_i \text{ if } 1\le n\le 3\\ p_1V_{n-1}+p_2V_{n-2}+\cdots+p_3V_{n-3} \text{ if } n>3\end{cases}$$

then for $n\ge 4$, we have

$$V_n=2p_1V_{n-1}+\left(p_2-{p_1}^2\right)V_{n-2}+(p_3-p_1p_2)V_{n-3}-p_1p_3V_{n-4}.$$

Proof. Rewrite the recurrence relation (1) as

$$\begin{aligned}V_n&=p_1V_{n-1}+p_2V_{n-2}+p_3V_{n-3}+0\\ &=p_1V_{n-1}+p_2V_{n-2}+p_3V_{n-3}+\left(p_1V_{n-1}-{p_1}^2V_{n-2}-p_1p_2V_{n-3}-p_1p_3V_{n-4}\right)\\ &=2p_1V_{n-1}+\left(p_2-{p_1}^2\right)V_{n-2}+(p_3-p_1p_2)V_{n-3}-p_1p_3V_{n-4}\end{aligned}$$

$$\left[\begin{array}{l}\because\ V_n = p_1V_{n-1} + p_2V_{n-2} + p_3V_{n-3} \text{ by multipling } p_1 \text{ and replacing } n \text{ by } (n-1)\text{, we have} \\ p_1V_{n-1} - {p_1}^2V_{n-2} - p_1p_2V_{n-3} - p_1p_3V_{n-4}\end{array}\right]$$

$$V_n = 2p_1V_{n-1} + \left(p_2 - {p_1}^2\right)V_{n-2} + (p_3 - p_1p_2)\,V_{n-3} - p_1p_3V_{n-4}. \tag{9}$$

Remark 2.11. On substituting $p_1 = p_2 = p_3 = 1$,in the result of above Theorem (2.10) we have

$$V_n = 2V_{n-1} + (0)\,V_{n-2} + (0)\,V_{n-3} - V_{n-4}$$

, this implies that

$$V_n = 2V_{n-1} - V_{n-4}$$

. which is in agreement with F. T. Howard and C. Cooper (1970); M. E. Waddill and L. Sacks (1967).

Theorem 2.12. *If* $V_n = p_1V_{n-1} + p_2V_{n-2} + p_3V_{n-3}$*,* $n > 3$*,* $f(x) = x^3 - p_1x^2 - p_2x - p_3 = 0$*,, then*

$$\lim_{n\to\infty} \frac{V_{n+1}(a_1, a_2, a_3, p_1, p_2, p_3)}{V_n(a_1, a_2, a_3, p_1, p_2, p_3)}$$
$$= \begin{cases} \alpha,\ \textit{real root of } f(x) = 0,\ p_i > 0,\ \textit{others roots are complex,} \\ \alpha\ \textit{(largest root), if all roots of } f(x) = 0\ \textit{are real,} \\ p_i > 0,\ \textit{others roots are complex,} \\ 1.839,\ \textit{if } p_j = 1 \textit{ and } a_j\ (j = 1, 2, 3)\ \textit{are any real numbers,} \\ 1.618,\ \textit{if } p_3 = 0,\ a_3 = 0,\ \textit{and } p_j,\ a_j,\ (j = 1, 2)\ \textit{are any real numbers.} \end{cases}$$

Remark 2.13. Graphical representation of the theorem (2.12)for the polynomials
$f(x) = x^3 - x^2 - x = 0$, $f(x) = x^3 - x^2 - x - 1 = 0$, and $f(x) = x^3 - x^2 - x - 3 = 0$.

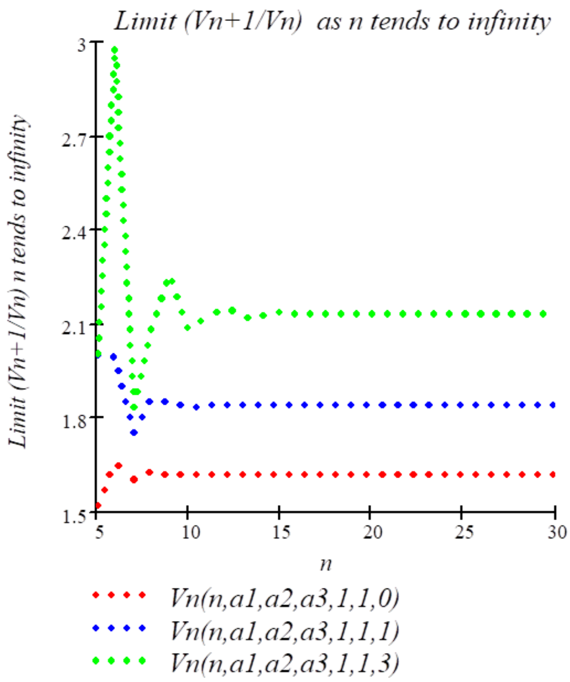

Figure 2. Tribonacci sequences progression and comparison

3. IDENTITIES

Theorem 3.1. *If $n \geq m$, then on employing the result of theorem (2.6)*

$$V_nV_{n+m} = \left(\sum_{j=1}^{3}\left[\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\le i\le 3\\ i\neq j}}(\alpha_j-\alpha_i)}\right]\right)\alpha_j^n.\left(\sum_{j=1}^{3}\left[\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\le i\le 3\\ i\neq j}}(\alpha_j-\alpha_i)}\right]\right)\alpha_j^{n+m}.$$

Proof. Using

$$V_n(x) = \sum_{j=1}^{3}\left[\frac{A_1\alpha_j^2 + A_2\alpha_j + A_{3k}}{\prod\limits_{\substack{1\le i\le 3\\ i\neq j}}(\alpha_j-\alpha_i)}\right]\alpha_j^n.$$

On simplifying the RHS, we obtain

$$V_nV_{n+m} = V_{2n+m} + V_nV_m - \sum_{j=1}^{3}\left(\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\le i\le 3\\ i\neq j}}(\alpha_j-\alpha_i)}\right)^2\alpha_j^{2n+m}.$$

3.1. Special cases:Identities

If we replace n by $n-1$ and taking $m=1$, in (3.1) then we obtain

$$V_{n-1}V_n = V_{2n-1} + V_{n-1}V_1 - \sum_{j=1}^{3}\left(\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\le i\le 3\\ i\neq j}}(\alpha_j-\alpha_i)}\right)^2\alpha_j^{2n-1}.$$

If we take $m = n$ then we get

$$V_nV_{2n} = V_{3n} + V_nV_n - \sum_{j=1}^{3}\left(\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\le i\le 3\\ i\neq j}}(\alpha_j-\alpha_i)}\right)^2\alpha_j^{3n}.$$

If we take $m = 0$ in (3.1) then we get

$$V_n{}^2 = V_{2n} + V_nV_0 - \sum_{j=1}^{3}\left(\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\le i\le 3\\ i\neq j}}(\alpha_j-\alpha_i)}\right)^2\alpha_j^{2n}$$

$$V_n{}^3 = V_n{}^2V_n = V_{2n}V_n + V_n{}^2V_0 - V_n\sum_{j=1}^{3}\left(\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\le i\le 3\\ i\neq j}}(\alpha_j-\alpha_i)}\right)^2\alpha_j^{2n}V_n$$

In general from (3.1), we have

$$V_nV_{nm} = V_{mn+n} + V_nV_{nm-n} - \sum_{j=1}^{3}\left(\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\le i\le 3\\ i\neq j}}(\alpha_j-\alpha_i)}\right)^2\alpha_j^{nm-n}.$$

4. DISCUSSION AND CONCLUSION

This study investigates a generalized third-order recurrence relation. After defining the initial terms in general form, we present a graphical representation in Figure 1 to illustrate the progression of Tribonacci numbers for various cases considered by previous authors. Figure 2 depicts the ratio of consecutive terms as the number of terms approaches infinity. We observe that the Tribonacci constant is solely influenced by the coefficients p_j of the recurrence relation and is unaffected by the terms a_j.We derive the generating function and Binet formula in their general forms. By applying these results, we show that many existing results from previous studies emerge as special cases. Future research could delve deeper into this generalized third-order sequence, extending the analysis to explore additional properties and applications. Employing alternative approaches, such as matrix methods, combinatorial arguments, or number theory, may lead to the discovery of new identities and theorems.

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RESEARCH ARTICLE

A New Liu-Ratio Estimator For Linear Regression Models

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ABSTRACT
In statistical modeling, regression analysis is a set of statistical processes for estimating the relationships between a dependent variable and one or more independent variables. Although there are various methods for estimating parameters, the most popular is the Ordinary Least Squares (OLS) method. However, in the presence of multicollinearity and outliers, the OLS estimator may give inaccurate values and also misleading inference results. There are many modified biased robust estimators for the simultaneous occurrence of outliers and multicollinearity in the data. In this paper, a new estimator called the Liu-Ratio Estimator (LRE), which can be used as an alternative to the Least Squares Ratio (LSR) estimator and the Ridge Ratio estimator (RRE), is proposed to mitigate the effect of y-direction outliers and multicollinearity in the data. The performance of the proposed estimator is examined in two Monte Carlo simulation studies in the presence of multicollinearity and y-direction outliers. According to the simulation results, LRE is a strong alternative to LSR and RRE in the presence of multicollinearity and y-direction outliers in the data.

Mathematics Subject Classification (2020): 62J05,62J07

Keywords: Least Squares Ratio Estimator, Liu Estimator, Multicollinearity, Ridge Estimator.

1. INTRODUCTION

Regression analysis is a statistical technique for investigating and modeling the relationship between variables. Applications for regression models are numerous and occur in almost every field, including engineering, the physical and chemical sciences, economics, management, life and biological sciences, and social sciences. The classical linear regression model assumes a relation of the form:

$$y_i = \beta_0 + \sum_{j=1}^{p} x_{ij}\beta_j + \varepsilon_i, \quad i = 1, 2, ..., n \tag{1}$$

where n is the number of observations, x_{ij} $j = 1, 2, ..., p$ are the independent variables for observation i, y_i the observed response variable, the ε_i is the error term for the observation i and β_j are the coefficients to be estimated, representing the relationship between each independent variable and the dependent variable.

The most popular way of estimating β is to minimize the Ordinary Least Squares (OLS) criterion. Unfortunately, the well-known problem of multicollinearity in regression analysis due to high correlation between independent variables affects the OLS estimator. As a result of multicollinearity between explanatory variables, the variance of OLS becomes so large that estimates become unstable (Montgomery et al. 2001). Many biased estimators have been proposed for the multicollinearity problem, but the Ridge Estimator (RE) proposed by Hoerl and Kennard (1970) and the Liu Estimator (LE) proposed by Liu (1993) are some of the most widely used estimators.

In addition, there are many situations where the distribution of errors is nonnormal. In the case of nonnormal distributions, particularly heavy-tailed distributions, the OLS estimator no longer has the desirable properties. These heavy-tailed distributions tend to generate outliers, which may have an improper effect on the OLS estimates (Montgomery et al. 2001). Numerous robust estimating techniques, including the M-estimator, the least squares median estimator, the least truncated sum of squares estimator, the S-estimator, and the MM-estimator, have been presented to generate parameter estimates in the presence of outliers (Rousseeuw and Leroy 1987), (Maronna et al. 2006). However, while robust estimators are robust techniques for obtaining parameter estimates that are not affected by outliers, some unstable estimates may still be obtained due to the presence of multicollinearity between variables. Therefore, to mitigate the effects of both outliers and multicollinearity to some extent is to use biased-robust estimators.

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For example, various modifications of RE and LE, which are used for the multicollinearity problem, are widely used to address both outliers and multicollinearity (Silvapulle 1991), (Arslan and Billor 2000), (Maronna 2011), (Kan et al. 2013), (Jadhav and Kashid 2016), (Ertaş et al. 2017), (Filzmoser and Kurnaz 2018).

Recently, Akbilgic and Akinci (2009) proposed the Least Squares Ratio (LSR) as an alternative for OLS in order to estimate the beta parameter vector in the presence of y-direction outliers. On the other hand, Jadhav and Kashid (2018) developed an estimator called the Ridge Ratio Estimator (RRE) as an alternative to RE and LSR in the presence of outliers and multicollinearity in the data. Therefore, one of the objectives of this paper is to propose a new estimator as an alternative to LSR and RRE to overcome the simultaneous occurrence of outliers and multicollinearity in the data, based on the fact that LE is always an alternative to RE as known from the multicollinearity problem. Another objective is to investigate the performance of the proposed estimator with respect to LSR and RRE through extensive simulation studies.

The organization of the paper is as follows: The main ideas underlying the proposed estimator are highlighted in Section 2. In Section 3, two separate Monte Carlo simulation studies are conducted to evaluate the performance of the proposed estimator with respect to LSR and RRE. In Section 4, the performance of the proposed estimator is evaluated against that of other estimators on artificial data. Finally, the conclusions of the study are presented in Section 5.

2. A NEW ROBUST LIU RATIO ESTIMATOR

For the regression model given by (1), OLS minimizes the sum of squares of the distances between the observed value y_i and the fitted value $\hat{y}_i$ where $i = 1, 2, \dots, n$. As an alternative to OLS, LSR method starts with the same goal $y_i = \hat{y}_i$, or $y_i - \hat{y}_i = 0$, $i = 1, 2, \dots, n$ as in OLS. Note that the OLS approach satisfies this aim by finding the regression parameters minimizing the sum of $(y_i - \hat{y}_i)^2$. However, LSR proceeds by dividing through by y_i and so $\frac{\hat{y}_i}{y_i} = 1$ is obtained under an assumption of $y_i \neq 0$ where $i = 1, 2, \dots, n$ (Akbilgic and Akinci 2009). Hence, it is obvious that, equations $\frac{\hat{y}_i}{y_i} - 1 = 0$ and thus $\frac{y_i - \hat{y}_i}{y_i} = 0$ where $i = 1, 2, \dots, n$ are obtained by basic mathematical operations. As a result, the LSR estimator is obtained by minimizing the objective function as follows:

$$\min_{\beta} \sum_{i=1}^{n} \left(\frac{y_i - \hat{y}_i}{y_i} \right)^2 \quad \text{or} \quad \min_{\beta} \sum_{i=1}^{n} \left(1 - \hat{\beta}_j \frac{x_{ij}}{y_i} \right)^2 \tag{2}$$

where $\hat{y}_i = \beta_0 + \sum_{j=1}^{p} \hat{\beta}_j x_{ij}$, $i = 1, 2, \dots, n$. Taking the partial derivatives of (2) with respect to the β components and setting them equal to zero, Akbilgic and Akinci (2009) defined the LSR estimator as follows:

$$\hat{\beta}_{LSR} = \left(\left(\frac{X}{Y} \right)' \left(\frac{X}{Y} \right) \right)^{-1} \left(\frac{X}{Y^2} \right)' Y \tag{3}$$

where X/Y matrix is obtained by dividing the values x_{ij} by y_i, and X/Y^2 is computed by dividing the values x_{ij} by y_i^2 where $j = 1, 2, \dots, p$.

On the other hand, Jadhav and Kashid (2018) developed an estimator called RRE as an alternative to RE and LSR. Note that RRE using RE and LSR estimator is proposed to tackle the problem of outliers and multicollinearity. For the parameters β in Equation (1), the RRE is defined as:

$$\hat{\beta}_{RRE} = \left(\left(\frac{X}{Y} \right)' \left(\frac{X}{Y} \right) + kI \right)^{-1} \left(\frac{X}{Y^2} \right)' Y, \quad k > 0, \tag{4}$$

where k is a biasing parameter.

Let us state that the LSR and RRE given by (3) and (4) are obtained by minimization of the objective function given below:

$$S(\beta) = (1 - \underline{X}\beta)' (1 - \underline{X}\beta) + k\beta'\beta \tag{5}$$

where 1 is the $n \times 1$ dimensional matrix of 1s, $\underline{X}$ is obtained by dividing the values x_{ij} by y_i for $j = 1, \dots, p$ and the parameter $k \geq 0$ controls the amount of shrinkage. Note that minimization of the objective function given by (5) with respect to the parameter vector β yields the LSR estimator given by (3) when $k = 0$ and the RRE given by (4) when $k > 0$.

As an alternative to the objective function (5), which yields the LSR and RRE given by (3) and (4), consider the following penalized objective function:

$$S(\beta) = (1 - \underline{X}\beta)' (1 - \underline{X}\beta) + (d\hat{\beta}_{LSR} - \beta)' (d\hat{\beta}_{LSR} - \beta), \qquad 0 < d < 1 \tag{6}$$

where $\hat{\beta}_{LSR}$ is the LSR estimator given in (3) and $\underline{X}$ is obtained by dividing the values x_{ij} by y_i for $j = 1, \dots, p$. When $S(\beta)$ in (6) is differentiated with respect to β, the following equation is obtained:

$$\left. \frac{\partial S}{\partial \beta} \right|_{\hat{\beta}} = -2\underline{X}' + 2\underline{X}'\underline{X}\beta - 2d\hat{\beta}_{LSR} + 2\beta = 0. \tag{7}$$

Solving the system given in (7) with respect to β defines the Liu Ratio Estimator (LRE) as follows:

$$\hat{\beta}_{LRE} = \left(\underline{X}'\underline{X} + I\right)^{-1}\left(\underline{X}' + d\hat{\beta}_{LSR}\right), \qquad 0 < d < 1, \tag{8}$$

where d is a biasing parameter. If the estimator (8) is restated in the structure of (3) or (4), LRE is obtained as follows:

$$\hat{\beta}_{LRE} = \left(\left(\frac{X}{Y}\right)'\left(\frac{X}{Y}\right) + I\right)^{-1}\left(\left(\frac{X}{Y^2}\right)' Y + d\hat{\beta}_{LSR}\right), \qquad 0 < d < 1 \tag{9}$$

where X/Y matrix is obtained by dividing the values x_{ij} by y_i, and X/Y^2 is computed by dividing the values x_{ij} by y_i^2 where $j = 1, 2, ..., p$.

3. THE MONTE CARLO SIMULATION STUDIES

In this section, the performance of LRE is compared with other existing estimators, OLS, RE, LE, LSR and RRE using two different Monte Carlo simulation designs. In the first design, we investigated the effects of sample size (n), the degree of the collinearity (ρ), the number of the explanatory variables (p) and the variance (σ^2) on the performances of the considered estimators. In the second simulation design, we examined LSR, RRE and LRE performances for each of n, p, ρ and σ^2 values at certain values of k and d. For both simulation designs, we generate the explanatory variables by the following McDonald and Galarneau (1975) as

$$x_{ij} = \left(1-\rho^2\right)^{1/2} u_{ij} + \rho u_{ip+1}, \quad i = 1, 2, .., n, \quad j = 1, 2, ..., p \tag{10}$$

where u_{ij} are independent standard normal pseudo-random numbers. ρ is specified so that the correlation between any two variables is given by ρ^2. These variables are standardized such that $X'X$ is a correlation matrix. Investigations are conducted on three distinct sets of correlations that correspond to $\rho = 0.8, 0.9$ and 0.95. The response variable is generated by

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \ldots + \beta_p x_{pi} + \varepsilon_i, \quad i = 1, 2, \ldots, n \tag{11}$$

where $\varepsilon_i \sim N\left(0, \sigma^2\right)$ and β_0 is equal to zero. The values of σ^2 are 1, 5, and 10 for various comparisons of the error term. For each set of explanatory variables, the parameter vector β is chosen as the normalized eigenvector corresponding to the largest eigenvalue of $X'X$ so that $\beta'\beta = 1$. The sample sizes n are 50, 100 and 200. The number of explanatory variables is chosen as $p = 4$, 8, and 12.

We examine the effects of y-direction outliers on the estimators by considering three different cases such as no outlier, one outlier and two outliers. When there is no outlier, dependent variables are taken into consideration as in Equation (11). In the case of one outlier, the n observation is changed as $y(n) = 500$. For two outlier case, $y(1) = 500$ and $y(n) = 500$ altered observations are used.

In order to estimate the biasing parameters in the simulation, based on the studies of Kibria (2003) and Qasim et al. (2020), the biasing parameters for RE, LE, RRE, and LRE are taken as follows:

RE: $\hat{k}_{RE} = \dfrac{\hat{\sigma}^2_{OLS}}{\left(\prod_{j=1}^{p+1}\hat{\beta}^2_{OLS(j)}\right)^{\frac{1}{p+1}}}$ where $\hat{\sigma}^2_{OLS} = \dfrac{\sum_{i=1}^{n}\left(y_i - \hat{y}_{OLS(i)}\right)^2}{n-p-1}$

LE: $\hat{d}_{LE} = \max\left(0, \min\left(\dfrac{\hat{\beta}^2_{OLS(j)} - \hat{\sigma}^2_{OLS}}{\max\left(\frac{\hat{\sigma}^2_{OLS}}{\lambda_j}\right) + \max\left(\hat{\beta}^2_{OLS(j)}\right)}\right)\right)$ where λ_j is the jth eigenvalues of $X'X$, $j = 1, 2, ..., p+1$.

RRE: $\hat{k}_{RRE} = \dfrac{\hat{\sigma}^2_{LSR}}{\left(\prod_{j=1}^{p+1}\hat{\beta}^2_{LSR(j)}\right)^{\frac{1}{p+1}}}$ where $\hat{\sigma}^2_{LSR} = \dfrac{\sum_{i=1}^{n}\left(y_i - \hat{y}_{LSR(i)}\right)^2}{n-p-1}$

LRE: $\hat{d}_{LRE} = \max\left(0, \min\left(\dfrac{\hat{\beta}^2_{LSR(j)} - \hat{\sigma}^2_{LSR}}{\max\left(\frac{\hat{\sigma}^2_{LSR}}{\lambda_j}\right) + \max\left(\hat{\beta}^2_{LSR(j)}\right)}\right)\right)$ where λ_j is the jth eigenvalues of $\underline{X}'\underline{X}$, $j = 1, 2, ..., p+1$.

As a measure of performance we use the estimated Mean Squared Error (MSE) between the estimated parameters in the l-th repetition, $\hat{\beta}^{(l)}$, and the true parameters β:

$$\text{MSE} = \frac{1}{m}\sum_{l=1}^{m}\frac{1}{p}\left\|\beta - \hat{\beta}^{(l)}\right\|^2 \tag{12}$$

where p is the number of explanatory variables. The simulation experiment is replicated $m = 2000$ times by generating new pseudo-random numbers. The R programming language was used to carry out the calculations. The results are given in Tables 1-3 where the lowest estimated MSE values in each row are indicated by bold.

In all 81 scenarios in Tables 1-3, the LSR, RRE and LRE outperformed other estimators according to criterion (12). With the

Table 1. The estimated MSE values of the considered estimators for the model when $p = 2$

			No outlier						One outlier						Two outliers					
σ^2	n	ρ	**OLS**	**RE**	**LE**	**LSR**	**RRE**	**LRE**	**OLS**	**RE**	**LE**	**LSR**	**RRE**	**LRE**	**OLS**	**RE**	**LE**	**LSR**	**RRE**	**LRE**
1	50	0.8	3.874	1.731	**0.836**	1.104	1.008	1.043	66639.985	33649.654	10840.251	1.115	**1.011**	1.045	47482.474	13934.638	6376.756	1.119	**1.012**	1.048
5	50	0.8	18.756	7.000	3.602	1.796	1.243	**1.229**	66755.55	33738.42	10868.439	1.893	1.266	**1.242**	47585.798	13989.384	6389.664	1.906	1.27	**1.245**
10	50	0.8	39.375	14.953	8.504	2.547	1.466	**1.324**	66525.692	33542.163	10818.309	2.753	1.53	**1.345**	47403.547	13909.209	6374.627	2.781	1.534	**1.340**
1	50	0.9	8.586	3.144	1.367	1.33	**1.072**	1.102	10400.479	275.513	271.375	1.340	**1.076**	1.104	335131.132	160914.664	89564.179	1.392	**1.088**	1.114
5	50	0.9	42.251	14.49	7.867	3.02	1.605	**1.443**	10412.218	279.069	271.616	3.058	1.611	**1.444**	335134.314	160891.027	89610.788	3.364	1.699	**1.499**
10	50	0.9	81.964	27.041	14.999	4.822	2.097	**1.629**	10387.775	283.39	273.565	4.875	2.116	**1.637**	334751.036	160565.563	89372.943	5.368	2.259	**1.711**
1	50	0.95	14.332	4.986	2.313	1.565	**1.159**	1.178	110682.31	22309.457	16422.522	1.595	**1.167**	1.181	25303.971	310.034	239.406	1.619	**1.175**	1.188
5	50	0.95	72.727	25.501	15.425	4.03	1.88	**1.614**	110530.006	22269.075	16397.836	4.228	1.94	**1.646**	25272.394	313.56	245.276	4.375	1.981	**1.663**
10	50	0.95	142.487	46.331	28.765	6.875	2.787	**2.124**	110708.454	22374.722	16479.652	7.279	2.878	**2.203**	25391.9	321.936	253.227	7.340	2.865	**2.200**
1	100	0.8	3.34	1.471	**0.720**	1.036	0.996	1.02	10198.92	2144.602	984.846	1.038	**0.996**	1.021	480.365	3.225	85.917	1.042	**0.998**	1.024
5	100	0.8	17.474	6.768	3.864	1.396	**1.121**	1.169	10203.427	2151.653	999.082	1.404	**1.123**	1.171	496.307	3.431	87.961	1.421	**1.127**	1.175
10	100	0.8	34.438	12.756	7.631	1.771	1.233	**1.225**	10251.047	2176.151	1018.585	1.795	1.237	**1.228**	505.923	3.628	89.693	1.818	1.243	**1.233**
1	100	0.9	4.788	1.916	**0.851**	1.062	1.003	1.035	6251.16	362.017	264.557	1.063	**1.003**	1.036	1603.233	83.091	659.89	1.065	**1.004**	1.037
5	100	0.9	23.743	8.541	4.500	1.432	**1.109**	1.147	6287.381	372.356	270.937	1.442	**1.113**	1.151	1625.594	85.726	662.281	1.449	**1.114**	1.153
10	100	0.9	49.678	17.979	9.756	2.015	1.312	**1.281**	6314.094	384.904	279.665	2.021	1.31	**1.28**	1644.439	87.799	661.543	2.045	1.319	**1.286**
1	100	0.95	13.699	4.709	2.265	1.262	**1.058**	1.104	16134.448	1079.382	504.879	1.264	**1.059**	1.104	3786.59	659.29	1599.82	1.272	**1.062**	1.107
5	100	0.95	71.713	23.493	14.264	2.583	1.463	**1.378**	16206.988	1099.17	516.107	2.598	1.462	**1.378**	3838.046	669.735	1598.714	2.615	1.470	**1.382**
10	100	0.95	129.272	41.231	24.767	3.997	1.865	**1.656**	16270.935	1124.121	529.387	4.054	1.879	**1.663**	3910.352	688.784	1606.804	4.083	1.886	**1.662**
1	200	0.8	3.846	1.639	**0.774**	1.01	0.993	1.006	7926.647	1957.891	1343.945	1.010	**0.993**	1.006	5383.217	563.529	526.902	1.010	**0.993**	1.006
5	200	0.8	18.474	6.78	3.755	1.203	**1.062**	1.114	7965.217	1981.025	1363.837	1.206	**1.064**	1.115	5415.279	572.762	529.15	1.207	**1.063**	1.116
10	200	0.8	39.446	14.739	8.691	1.406	**1.125**	1.176	7985.94	1998.333	1381.326	1.420	**1.130**	1.18	5435.076	581.752	530.902	1.419	**1.129**	1.178
1	200	0.9	5.757	2.24	**0.964**	1.04	1.003	1.029	1723.361	483.201	632.332	1.040	**1.003**	1.029	14523.593	4669.841	2203.649	1.040	**1.002**	1.029
5	200	0.9	29.047	10.274	5.329	1.276	**1.072**	1.124	1741.125	488.224	633.732	1.279	**1.073**	1.125	14504.736	4667.649	2204.356	1.285	**1.076**	1.127
10	200	0.9	56.529	19.381	10.749	1.645	**1.185**	1.219	1771.988	495.737	634.393	1.651	**1.188**	1.221	14577.619	4691.152	2204.976	1.652	**1.189**	1.223
1	200	0.95	10.419	3.527	1.636	1.096	**1.018**	1.058	40589.394	15480.793	8489.903	1.099	**1.02**	1.06	38599.933	8068.549	3190.09	1.099	**1.020**	1.059
5	200	0.95	54.636	17.793	10.267	1.58	**1.154**	1.197	40580.392	15466.929	8503.168	1.595	**1.157**	1.199	38592.549	8082.718	3204.612	1.604	**1.164**	1.202
10	200	0.95	113.452	38.663	23.827	2.237	1.359	**1.337**	40655.728	15509.057	8562.385	2.299	1.377	**1.346**	38665.172	8132.565	3237.893	2.297	1.376	**1.344**

Table 2. The estimated MSE values of the considered estimators for the model when $p = 4$

			No outlier						One outlier						Two outliers					
σ^2	n	ρ	**OLS**	**RE**	**LE**	**LSR**	**RRE**	**LRE**	**OLS**	**RE**	**LE**	**LSR**	**RRE**	**LRE**	**OLS**	**RE**	**LE**	**LSR**	**RRE**	**LRE**
1	50	0.8	8.284	2.658	**0.908**	1.471	1.064	1.21	44899.659	19414.94	12989.991	1.490	**1.069**	1.217	69474.34	18211.508	10069.709	1.505	**1.073**	1.220
5	50	0.8	42.778	12.939	4.247	3.907	1.693	**1.481**	44931.556	19423.137	12921.074	4.051	1.73	**1.492**	69486.601	18214.168	10070.666	4.154	1.763	**1.497**
10	50	0.8	85.032	25.567	8.414	6.154	2.173	**1.501**	44959.982	19429.209	12797.552	6.405	2.223	**1.509**	69527.885	18228.755	10074.035	6.501	2.242	**1.509**
1	50	0.9	22.801	6.951	**0.703**	2.657	1.366	1.393	271966.879	123807.12	8147.935	2.821	1.407	**1.406**	751440.542	418378.421	105601.218	2.949	1.444	**1.410**
5	50	0.9	110.602	32.634	3.429	8.52	2.783	**1.515**	272216.47	123954.815	8198.149	9.110	2.921	**1.513**	751857.779	418678.157	105852.306	9.589	3.018	**1.514**
10	50	0.9	223.867	65.53	6.668	16.46	4.793	**1.542**	272119.315	123804.205	8228.282	17.662	5.085	**1.530**	751521.878	418260.815	105688.226	18.564	5.238	**1.533**
1	50	0.95	33.631	10.103	**0.761**	3.376	1.532	1.436	259525.725	63099.131	1318.697	3.548	1.576	**1.435**	378257.454	62328.384	3624.173	3.587	1.582	**1.432**
5	50	0.95	163.61	47.601	3.455	13.223	4.054	**1.497**	259308.276	62984.916	1318.486	14.122	4.298	**1.489**	378000.4	62252.271	3634.782	14.358	4.322	**1.492**
10	50	0.95	319.828	90.852	6.899	22.63	6.208	**1.539**	259489.553	63068.59	1320.502	23.951	6.522	**1.534**	378210.213	62347.954	3646.722	24.527	6.674	**1.538**
1	100	0.8	8.016	2.682	**0.900**	1.211	1.023	1.138	14856.18	1785.804	1109.349	1.218	**1.024**	1.140	49141.245	9956.498	3139.145	1.224	**1.025**	1.143
5	100	0.8	40.411	12.628	4.317	2.308	**1.285**	1.416	14857.174	1790.82	1110.635	2.325	**1.287**	1.417	49114.256	9945.318	3137.823	2.349	**1.291**	1.420
10	100	0.8	80.741	25.239	8.529	3.739	1.686	**1.532**	14892.954	1805.585	1114.098	3.773	1.692	**1.533**	49165.728	9973.805	3144.359	3.811	1.697	**1.534**
1	100	0.9	14.385	4.531	**0.782**	1.458	1.090	1.238	49290.511	15515.865	2919.152	1.473	**1.093**	1.243	71044.994	16209.444	4777.114	1.496	**1.097**	1.250
5	100	0.9	69.687	20.61	3.67	3.355	1.536	**1.488**	49375.88	15551.268	2924.204	3.422	1.555	**1.491**	70991.087	16197.107	4780.66	3.531	1.578	**1.491**
10	100	0.9	140.746	41.436	7.265	5.571	2.050	**1.520**	49387.911	15562.731	2928.893	5.654	2.06	**1.519**	71333.569	16331.478	4792.289	5.898	2.127	**1.520**
1	100	0.95	27.765	8.287	**0.729**	1.942	1.199	1.364	97667.155	29058.228	1401.182	1.973	**1.207**	1.369	381563.018	188415.657	4475.914	2.011	**1.217**	1.375
5	100	0.95	135.335	38.539	3.433	5.633	2.085	**1.529**	97701.742	29054.883	1407.93	5.726	2.096	**1.529**	381755.21	188479.119	4477.368	5.862	2.124	**1.531**
10	100	0.95	279.99	82.078	6.700	10.147	3.199	**1.550**	97951.323	29170.418	1427.14	10.369	3.244	**1.553**	381791.619	188419.736	4479.598	10.69	3.322	**1.561**
1	200	0.8	8.202	2.668	**0.895**	1.118	1.018	1.092	1707.04	88.718	257.37	1.118	**1.018**	1.093	24028.221	7941.71	3735.021	1.120	**1.018**	1.094
5	200	0.8	42.141	13.247	4.313	1.728	**1.167**	1.336	1740.713	91.726	260.112	1.731	**1.167**	1.336	24039.073	7941.875	3734.765	1.743	**1.170**	1.338
10	200	0.8	84.549	26.201	8.524	2.425	**1.336**	1.461	1780.684	95.502	263.682	2.433	**1.339**	1.461	24058.517	7946.63	3736.432	2.447	**1.342**	1.460
1	200	0.9	14.696	4.537	**0.738**	1.217	1.035	1.151	22948.371	5544.276	956.247	1.219	**1.035**	1.152	44633.513	9553.022	1201.213	1.220	**1.034**	1.152
5	200	0.9	73.322	21.798	3.692	2.211	**1.276**	1.419	22950.016	5543.911	981.318	2.225	**1.278**	1.421	44615.021	9546.238	1201.764	2.261	**1.296**	1.423
10	200	0.9	147.872	43.559	7.195	3.704	1.660	**1.525**	23038.957	5582.191	1008.822	3.721	1.664	**1.526**	44693.284	9583.597	1205.384	3.728	1.663	**1.526**
1	200	0.95	31.562	9.17	**0.702**	1.517	1.110	1.276	35295.806	5531.456	638.058	1.521	**1.110**	1.276	124111.006	33503.082	1870.807	1.523	**1.110**	1.277
5	200	0.95	162.176	47.719	3.637	3.817	1.681	**1.528**	35410.954	5580.152	638.615	3.841	1.682	**1.528**	124218.044	33546.479	1863.449	3.868	1.689	**1.529**
10	200	0.95	314.547	89.211	6.875	6.052	2.125	**1.531**	35640.676	5671.806	643.211	6.078	2.129	**1.530**	124527.453	33700.271	1854.619	6.125	2.136	**1.528**

Table 3. The estimated MSE values of the considered estimators for the model when $p = 8$

			No outlier						**One outlier**						**Two outliers**					
σ^2	n	ρ	**OLS**	**RE**	**LE**	**LSR**	**RRE**	**LRE**	**OLS**	**RE**	**LE**	**LSR**	**RRE**	**LRE**	**OLS**	**RE**	**LE**	**LSR**	**RRE**	**LRE**
1	50	0.8	21.089	5.793	1.509	3.755	**1.507**	1.835	139147.844	39737.4	9468.393	3.926	**1.539**	1.848	388397.223	139159.787	24145.655	4.155	**1.592**	1.864
5	50	0.8	106.443	28.954	7.558	14.335	3.739	**2.097**	139300.65	39785.071	9474.007	15.044	3.876	**2.093**	388623.289	139237.387	24149.224	15.918	4.042	**2.089**
10	50	0.8	208.961	55.83	15.185	27.613	6.508	**2.052**	139270.452	39762.681	9479.12	29.249	6.814	**2.040**	388525.391	139172.257	24155.227	30.819	7.109	**2.027**
1	50	0.9	35.800	9.638	**1.161**	6.039	2.017	2.013	242807.428	54992.036	4742.136	6.431	2.103	**2.021**	497556.099	121583.337	13511.857	6.555	2.126	**2.026**
5	50	0.9	183.061	49.255	5.776	23.738	5.628	**2.025**	242939.822	55058.251	4749.194	25.179	5.919	**2.012**	497559.784	121571.713	13513.03	26.199	6.149	**2.022**
10	50	0.9	360.535	95.785	11.446	46.145	10.312	**1.931**	242981.461	55032.333	4748.732	49.155	10.892	**1.920**	497814.968	121686.567	13524.012	50.782	11.185	**1.925**
1	50	0.95	102.571	27.969	**0.664**	15.941	4.263	2.131	479567.128	101864.307	2984.616	16.385	4.331	**2.119**	1142542.861	252811.176	5952.728	16.856	4.397	**2.111**
5	50	0.95	506.765	134.981	3.114	66.258	14.882	**1.861**	479603.915	101823.237	2985.231	69.592	15.601	**1.845**	1142701.658	252830.074	5953.948	72.775	16.400	**1.832**
10	50	0.95	1016.342	272.293	6.258	127.465	27.21	**1.704**	479642.683	101827.648	2988.019	132.843	28.202	**1.689**	1142611.687	252799.378	5957.66	138.349	29.28	**1.682**
1	100	0.8	21.952	6.325	1.459	2.345	**1.243**	1.665	35509.399	6622.481	3294.372	2.375	**1.248**	1.674	89907.325	19893.191	7589.355	2.413	**1.255**	1.687
5	100	0.8	112.031	31.04	7.279	7.852	2.422	**2.156**	35577.205	6640.059	3298.547	7.977	2.453	**2.160**	89975.62	19911.924	7593.955	8.195	2.505	**2.163**
10	100	0.8	218.515	59.876	14.193	14.36	3.682	**2.148**	35740.879	6695.676	3311.712	14.624	3.743	**2.146**	90108.298	19967.624	7608.083	15.140	3.869	**2.150**
1	100	0.9	41.298	11.274	**1.090**	3.62	1.504	1.911	56569.198	7152.557	1800.411	3.655	**1.509**	1.914	128857.612	20803.63	4113.459	3.706	**1.520**	1.921
5	100	0.9	203.455	54.211	5.328	13.419	3.55	**2.167**	56732.487	7193.956	1804.164	13.66	3.612	**2.167**	129042.961	20856.726	4118.128	13.878	3.645	**2.165**
10	100	0.9	400.159	104.894	10.413	26.425	6.337	**2.060**	56883.601	7234.105	1809.091	26.822	6.413	**2.055**	129052.864	20854.859	4119.762	27.291	6.512	**2.053**
1	100	0.95	76.929	20.342	**0.703**	6.198	2.035	2.106	126025.455	13035.134	727.896	6.276	**2.049**	2.107	190278.589	17611.911	1537.059	6.358	**2.064**	2.111
5	100	0.95	387.956	103.51	3.498	25.797	6.187	**2.092**	126070.222	13050.755	730.481	26.149	6.243	**2.084**	190356.293	17629.49	1537.948	26.723	6.389	**2.082**
10	100	0.95	799.35	213.639	7.16	49.689	10.981	**1.92**	126764.214	13228.797	743.915	50.439	11.154	**1.918**	191033.781	17772.868	1542.641	51.127	11.354	**1.916**
1	200	0.8	21.895	6.134	1.463	1.654	**1.111**	1.444	10012.152	843.329	870.751	1.656	**1.111**	1.445	28849.262	3309.752	2082.147	1.661	**1.112**	1.447
5	200	0.8	108.547	29.233	7.085	4.42	**1.683**	2.031	10084.569	857.11	875.245	4.440	**1.688**	2.033	28875.314	3316.304	2084.089	4.459	**1.690**	2.035
10	200	0.8	220.731	60.855	14.441	7.742	2.390	**2.203**	10186.622	877.742	881.913	7.773	2.394	**2.203**	29050.204	3361.629	2095.271	7.812	2.398	**2.201**
1	200	0.9	35.310	9.526	**1.098**	2.132	1.215	1.638	16948.57	1290.932	625.896	2.139	**1.217**	1.640	127540.396	38822.079	3355.094	2.150	**1.218**	1.643
5	200	0.9	180.901	48.881	5.537	6.55	**2.130**	2.148	17079.771	1316.061	630.099	6.584	**2.141**	2.149	127685.158	38866.985	3359.624	6.691	2.170	**2.153**
10	200	0.9	364.599	98.811	11.02	12.313	3.367	**2.225**	17254.997	1348.93	635.024	12.421	3.390	**2.226**	127877.033	38929.171	3365.18	12.616	3.420	**2.225**
1	200	0.95	80.118	21.452	**0.692**	3.538	1.510	1.952	95780.669	20585.028	676.354	3.572	**1.519**	1.958	220670.252	57282.291	1760.579	3.599	**1.524**	1.961
5	200	0.95	398.392	106.157	3.443	13.267	3.517	**2.191**	96094.469	20689.105	679.066	13.345	3.526	**2.188**	220860.583	57326.758	1762.045	13.542	3.557	**2.188**
10	200	0.95	806.223	215.746	6.852	25.588	6.081	**2.094**	96396.472	20776.776	681.414	25.801	6.122	**2.092**	221089.695	57358.344	1764.309	26.000	6.158	**2.090**

inclusion of the y-direction outliers, the performance of the commonly used OLS, RE and LE is quite poor. On the other hand, LSR, RRE and LRE exhibited different behaviors in different scenarios. The following observations can be obtained from Tables 1-3:

1. When the number of outliers is gradually increased along with the number of variables in the model by keeping ρ, n, and σ^2 constant, an increase in the estimated MSE values of all estimators is observed.
2. When n, p and σ^2 are held constant, we observe that the estimated MSE values of LSR generally increase as the correlation between variables is increased, while the estimated MSE values of RRE and LRE remain almost constant. On the other hand, when the correlation between the variables and the outliers in the data are increased, the estimated MSE values of the LSR and RRE estimators increase. On the other hand, for $p = 8$, the estimated MSE value of LRE decreases when the model variance is large.
3. When n, p and ρ are kept constant and the variance σ^2 is increased, the MSE values of the LSR, RRE and LRE estimators generally increase. When the model variance increases with the number of outliers, the estimated MSE values of the LSR and RRE increase. On the contrary, for $p = 8$, the estimated MSE values of LRE decrease at high correlation and small sample size.
4. When p, ρ and σ^2 are kept constant and the number of observations in the model is increased, a decrease is observed in the estimated MSE values of all estimators. When the number of outliers and the number of variables in the model are increased, a decrease is observed in the estimated MSE values of the LSR and RRE. On the other hand, the estimated MSE values of the LRE for $p = 8$ show an increase at high correlation and large variance values.

As a result, we can conclude that the estimated MSE values for LSR, RRE and LRE for variables such as n, p, ρ , and σ^2 with the change in the number of outliers are considerably lower than OLS, RE and LE.

In the second simulation scheme, we investigate the performance of LSR, RRE and LRE in the presence of y-direction outliers for each $n,\ p,\ \rho$, and σ^2. The purpose of this simulation is to investigate the performance of LSR, RRE and LRE with respect to MSE values given in (12) with various values of the biasing parameter k and d and the presence of outliers in the y-direction. The biasing parameters k and d are not estimated in the second simulation scheme. Only the MSE values obtained by increasing k and d values in the range $[0,\ 1]$ by 0.1 are compared. We only consider the cases $\rho = 0.8,\ 0.95$, $n = 50,\ \ 200$, and $p = 2,\ 8$, and $\sigma^2 = 1,\ \ 10$. Depending on these n, ρ, p, and σ^2 values, the explanatory variables are generated according to equation (10). Similar to the previous simulation scheme, we examine the effects of outliers in the y-direction on the estimators considering three different cases: no outliers, one outlier and two outliers. For every values of k and d, the simulation is run 2000 times. The results are collectively presented graphically in Figures 1-6.

Figures 1-6 clearly show the effects of varying the biasing parameter k and d between 0 and 1 on the estimated MSE values of the estimators. According to the figures, we can obtain the following results depending on each$\left(n, \rho, p, \sigma^2\right)$.

1) The LSR estimator showed an increase in the estimated MSE values in the presence of none, one and two outliers in the y-direction, but generally showed a stable behavior.

2) Although the MSE values estimated for RRE decreased with increasing values of the biasing parameter k, it did not affect the MSE values estimated from the outliers in the y-direction.

3) Although the MSE values estimated for LRE increased with increasing values of the biasing parameter d, it did not affect the MSE values estimated from the outliers in the y-direction.

As a result, no dramatic change is observed in the MSE values estimated by comparing LSR, RRE and LSR among themselves as OLS, RE and LE. On the other hand, for large values of the biasing parameter k, RRE and for small values of the biasing parameter d, LRE stand out due to their performance.

4. AN EMPIRICAL APPLICATION

In this section, we created an experimental dataset to study the performance of LSR, RRE and LRE. To do this, we created a dataset using Equation (10) with $n = 100$,$p = 4$ and $\rho = 0.95$. We used set.seed(4) in the R Program. Using equation (11) to create the response variable with $\sigma^2 = 5$. Modified observations $y\,(1) = 500$ and $y\,(n) = 500$ were used to create two outliers. In this case, the eigenvalues of the $X'X$ matrix were calculated as 100.000, 3.738, 0.106, 0.092, and 0.064. The condition number is approximately 39.410, therefore the matrix X is moderate ill-conditioned. The eigenvalues of the $\underline{X}'\underline{X}$ matrix were calculated as 164.869, 5.437, 0.122, 0.077, and 0.060. The condition number is approximately 52.593, therefore the matrix $\underline{X}$ is moderate ill-conditioned. The numerical results are summarized in Table 4.

From Table 4, it can be observed that the estimated MSE values of LSR, RRE, and LRE give smaller values compared to OLS, RE, and LE. As a result, RRE and LRE outperform LSR in the presence of multicollinearity and y-direction outliers. It also seems that LRE can be a strong alternative to RRE.

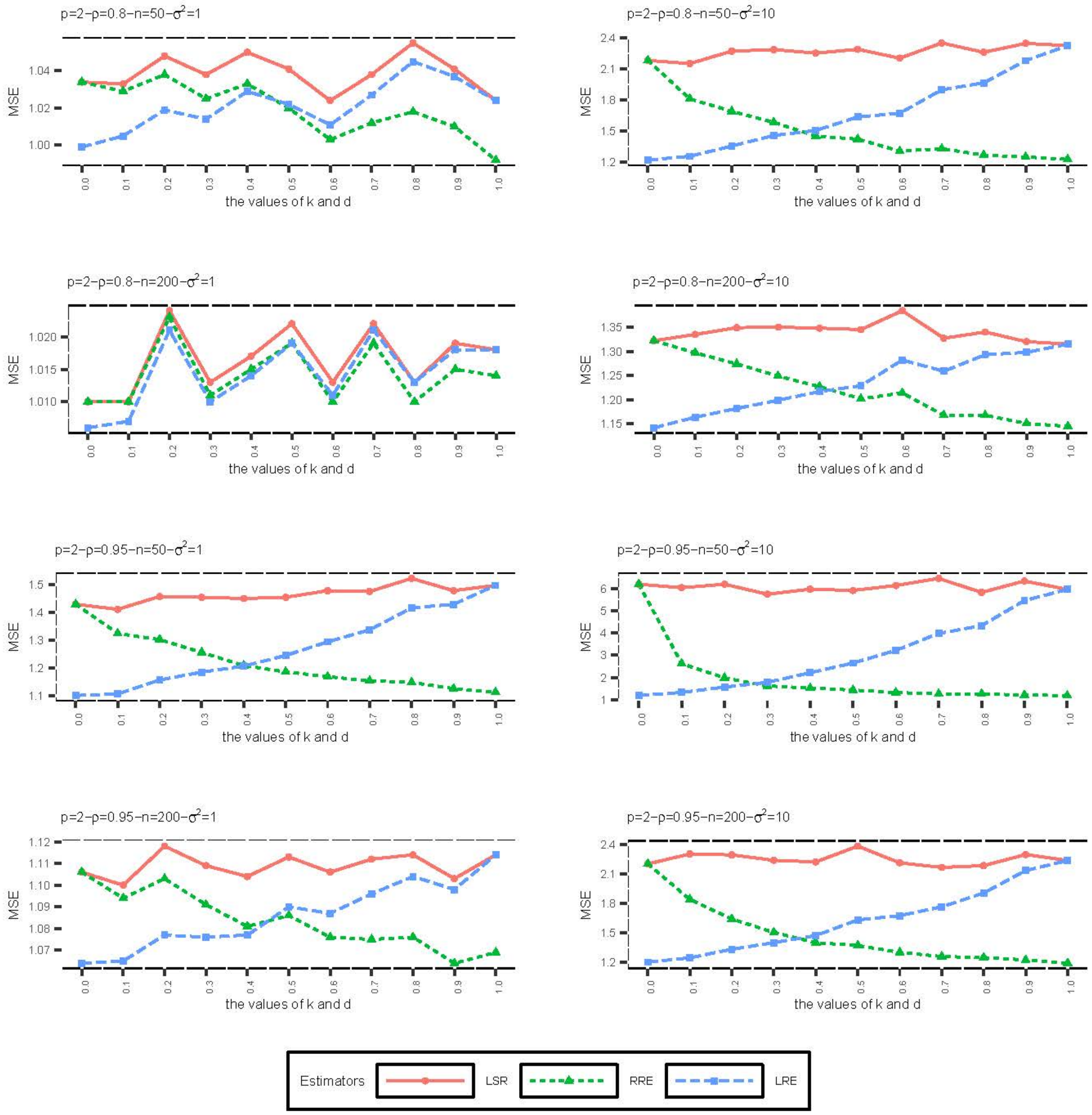

Figure 1. The estimated MSE values of LSR, RRE and LRE as a function k and d where $p = 2$ with no outlier

Table 4. The estimated parameter values and the estimated MSE values of the estimators

	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$MSE\left(\hat{\beta}\right)$
$\hat{\beta}_{OLS}$	11.0549	59.8429	455.3388	26.2779	-451.0035	83022.074
$\hat{\beta}_{RE}\left(\hat{k}_{RE} = 0.7087\right)$	10.9771	14.8855	67.4696	22.2895	-30.0934	1255.141
$\hat{\beta}_{LE}\left(\hat{d}_{LE} = 0\right)$	10.9454	14.598	53.2606	20.1941	-18.1441	777.143
$\hat{\beta}_{LSR}$	0.1297	6.8485	-4.443	-5.1036	1.8292	19.359
$\hat{\beta}_{RRE}\left(\hat{k}_{RRE} = 0.2234\right)$	0.1239	1.2467	-1.202	-1.4109	0.6017	1.224
$\hat{\beta}_{LRE}\left(\hat{d}_{LRE} = 0\right)$	0.1253	0.2183	-0.439	-0.5042	0.0808	0.253

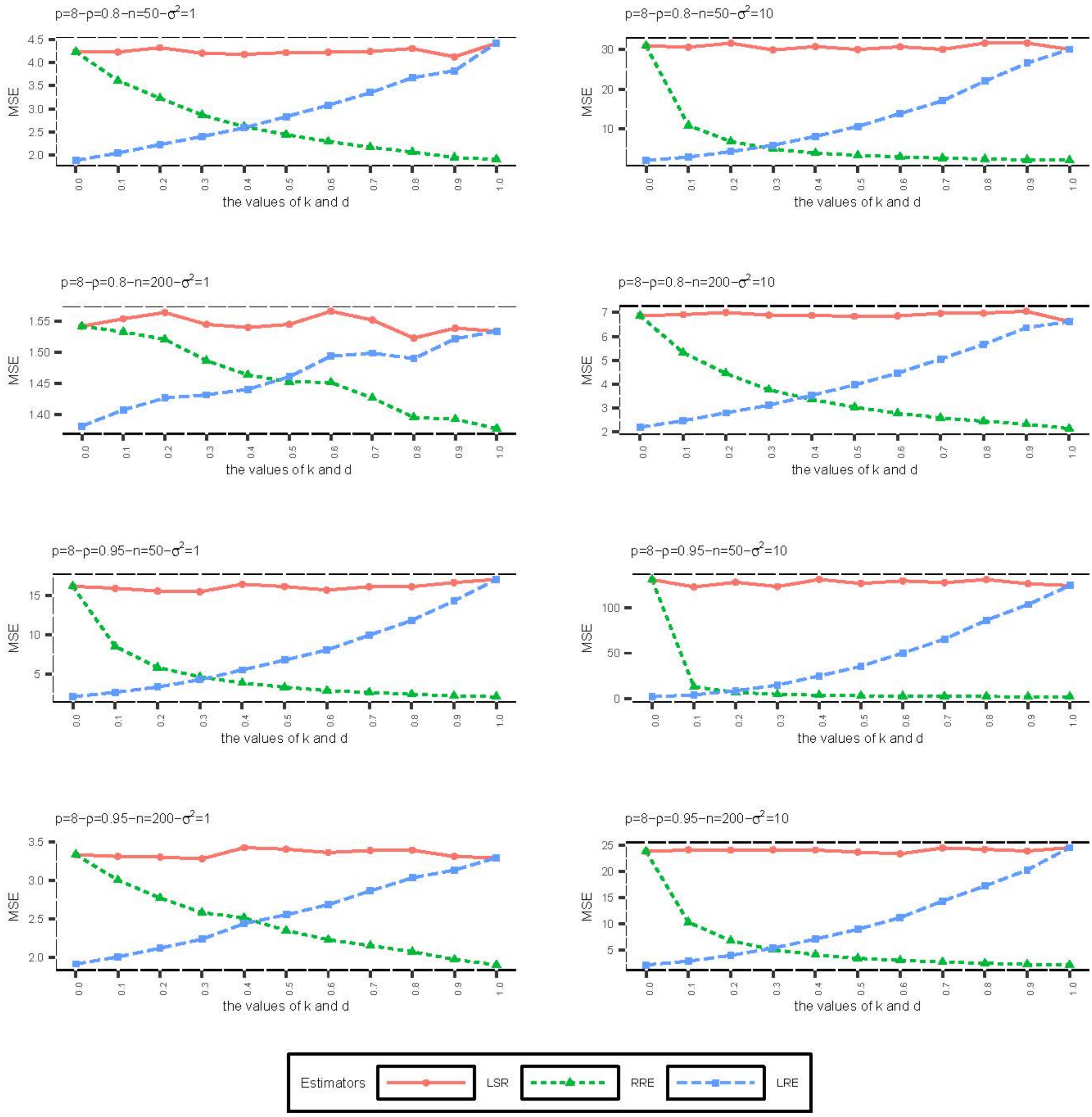

Figure 2. The estimated MSE values of LSR, RRE and LRE as a function k and d where $p = 8$ with no outlier

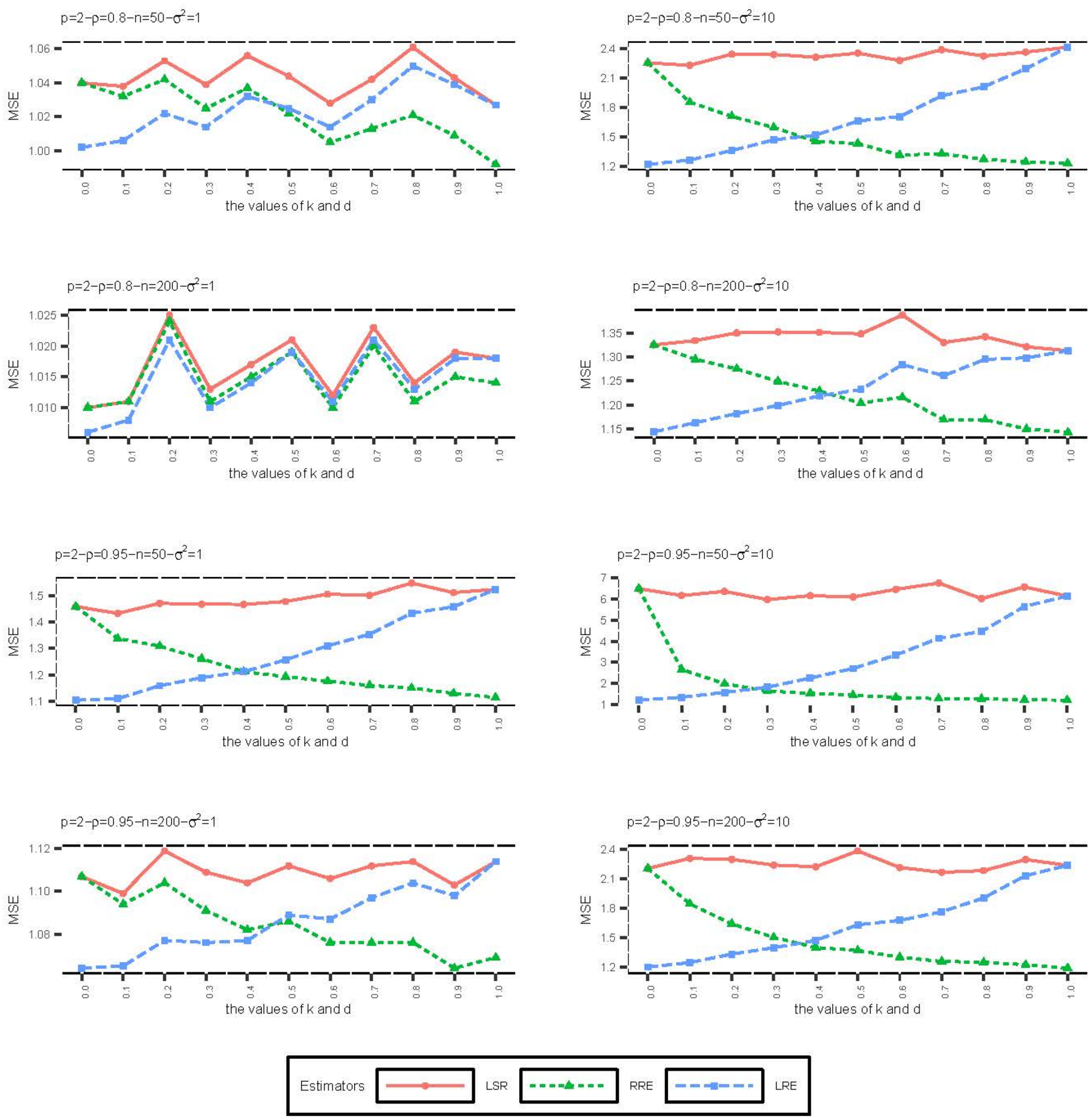

Figure 3. The estimated MSE values of LSR, RRE and LRE as a function k and d where $p = 2$ with one outlier

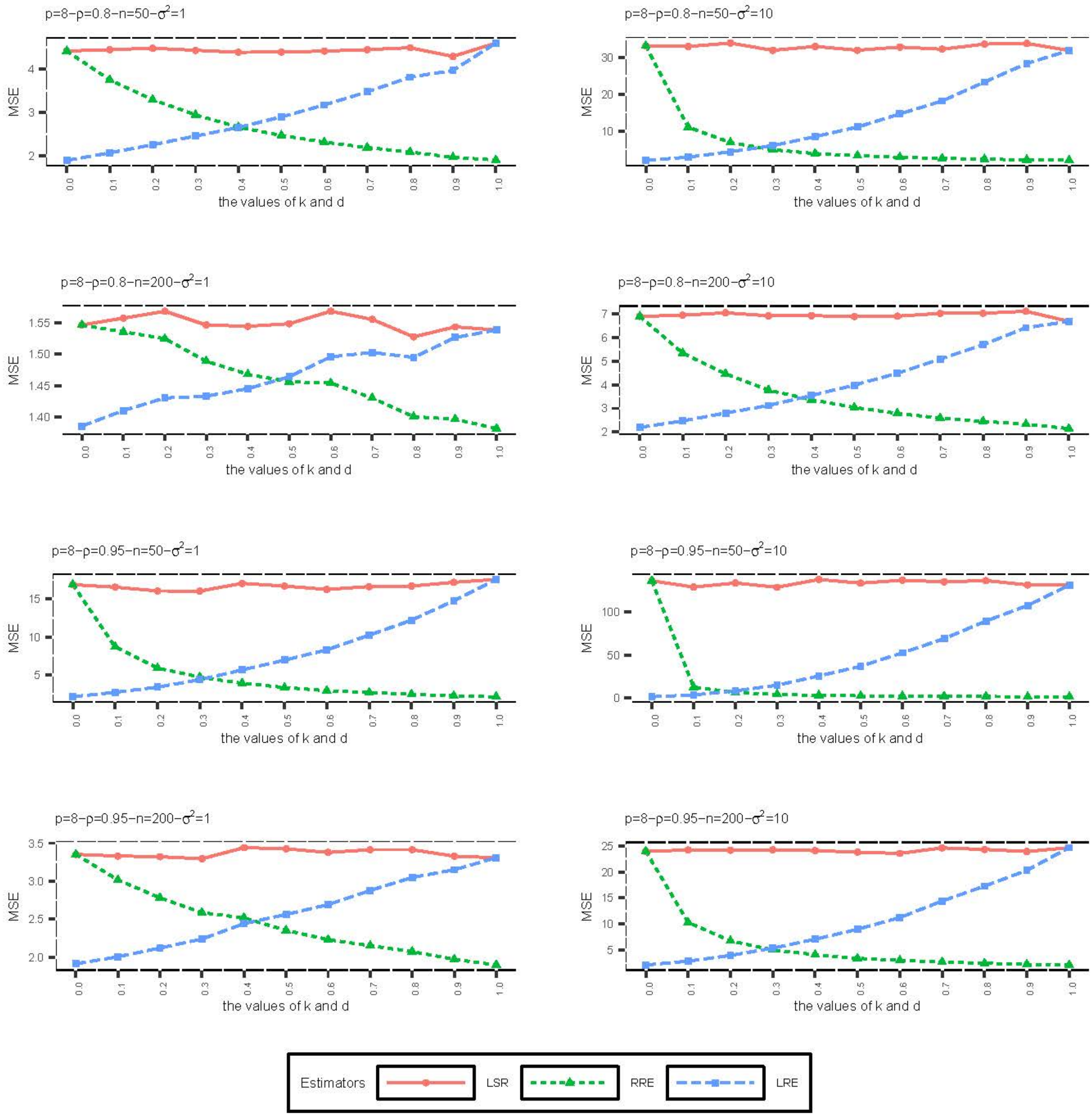

Figure 4. The estimated MSE values of LSR, RRE and LRE as a function k and d where $p = 8$ with one outlier

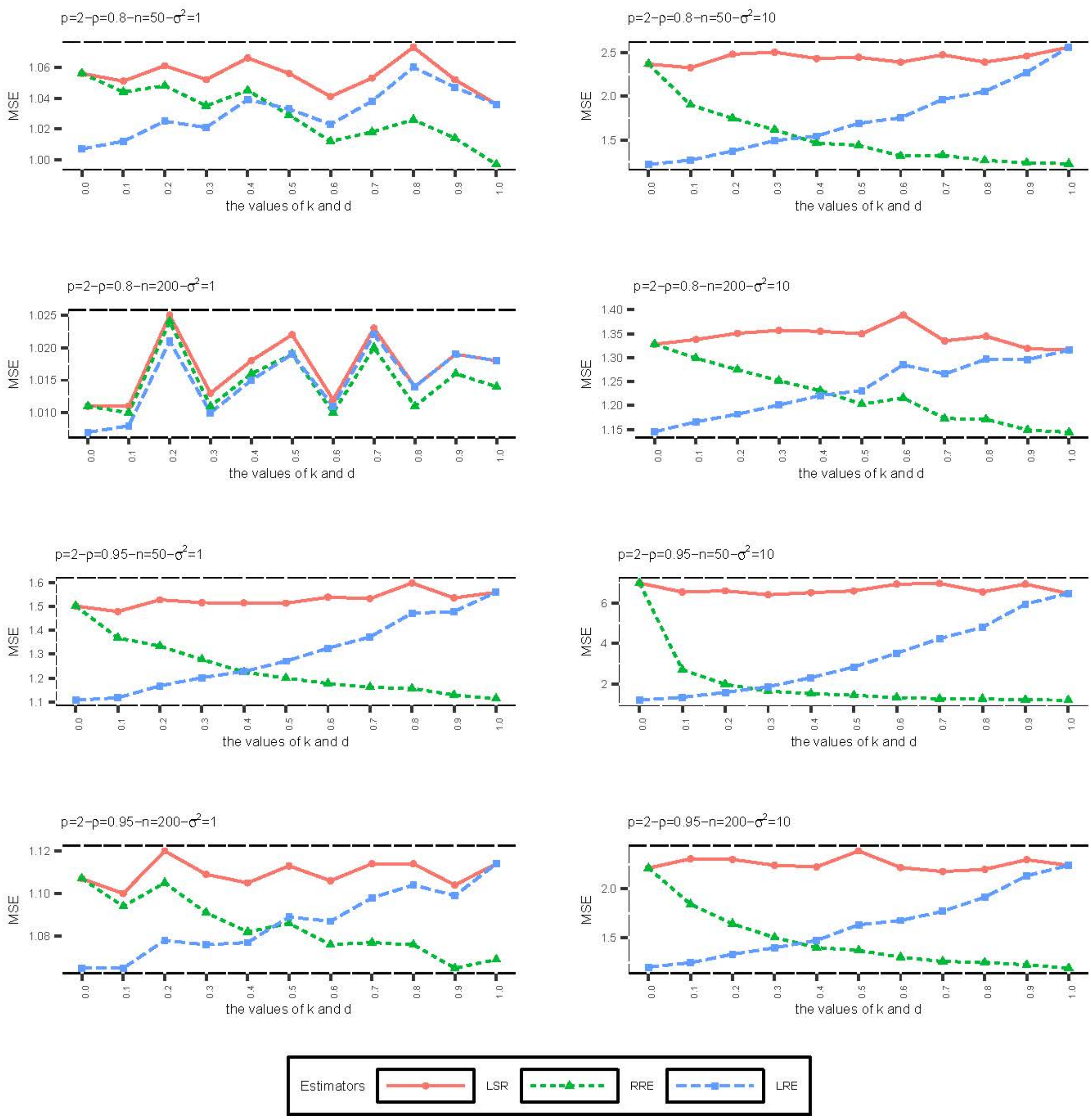

Figure 5. The estimated MSE values of LSR, RRE and LRE as a function k and d where $p = 2$ with two outliers

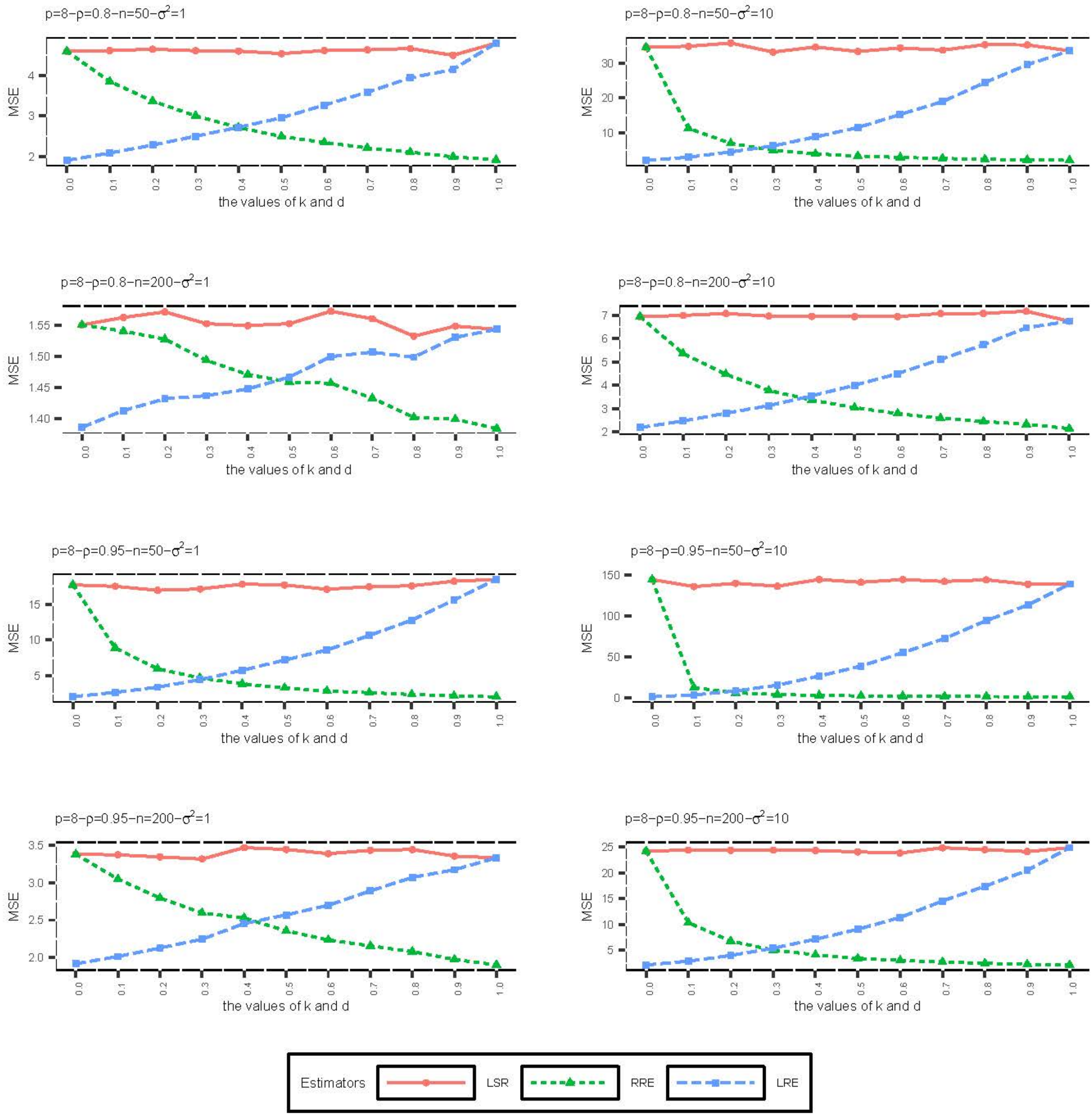

Figure 6. The estimated MSE values of LSR, RRE and LRE as a function k and d where $p = 8$ with two outliers

5. CONCLUSION

In this article, we proposed a new estimator named the LRE as an alternative to LSR and RRE in the presence of multicollinearity and y-direction outliers. Two separate Monte Carlo simulation study are conducted to examine the performance of LRE. In the first simulation study, we compared the considered estimators together with the estimates of the biasing parameters k and d. When the y-direction outliers are taken into account, the performance of OLS, RE and LE is considerably poor, while the performance of LSR, RRE and LRE is more stable. In the second simulation study, the performance of LSR, RRE and LRE are analyzed by choosing k and d values as fixed and equally spaced. According to the simulation results, LRE performs better for small values of d and RRE performs better for large values of k. According to the simulation results and the analysis of synthetic data, we recommend LRE as an alternative to RRE in the presence of y-direction outliers and multicollinearity between variables.

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