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A contribution to complementary soft binary piecewise plus and gamma operations

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Abstract — Molodtsov, in 1999, introduced soft set theory as mathematical a tool to deal with uncertainty. Since then, different kinds of soft set operations have been defined and used in various types. In this paper, it is aimed to contribute to the soft set literature by obtaining the distributions of soft binary piecewise operations over complementary soft binary piecewise plus and gamma operations.

Keywords: *Soft sets, conditional complements, soft set operations*

Subject Classification (2020): 03E20, 03E72

1. Introduction

The existence of some types of uncertainty in the problems of many fields such as economics, environmental and health sciences, engineering prevents us from using classical methods to solve the problems successfully. There are three well-known basic theories that we can consider as a mathematical tool to deal with uncertainties, which are probability theory, fuzzy set theory, and interval mathematics. But since all these theories have their own shortcomings, Molodtsov [1] introduced Soft Set Theory as mathematical tools to overcome these uncertainties. Since then, this theory has been applied to many fields including information systems, decision making, optimization theory, game theory, operations research, measurement theory and so on. In [2,3], first contributions as regards soft set operations were made. After then, Ali et al. [4] introduced and investigated several soft set operations such as restricted and extended soft set operations. in Sezgin and Atagün [5] discussed the basic properties of soft set operations and illustrated the interconnections of soft set operations with each other. They also defined the notion of restricted symmetric difference of soft sets and investigated its properties. Sezgin et al. [6] defined a new soft set operation called extended difference of soft sets and Stojanovic [7] proposed the extended symmetric difference of soft sets and investigated its properties. The two main categories into which the operations of soft set theory fall, according to the research, are restricted soft set operations and extended soft set operations.

Çağman [8] proposed two conditional complements of sets as a new concept of set theory, i.e., inclusive complement and exclusive complement and explored the relationships between them. By the inspiration of this study, Sezgin et al. [9] defined some new binary operations on sets and investigated their basic properties together with their interconnections. Aybek [10] transferred these complements to soft set theory and defined

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some new restricted soft set operations and extended soft set operations. Demirci [11], Sarıalioğlu [12], and Akbulut [13] introduced a new type of extended operation by changing the form of extended soft set operations using the complement at the first and second row of the piecewise function of extended soft set operations and studied the basic properties of them in detail. Moreover, Eren [14] defined a new type of soft difference operations and with the inspiration of this study, Yavuz [15] defined some new soft set operations called soft binary piecewise operations and studied their basic properties. Also, by introducing a new type of soft binary piecewise operation, studies on soft set operations were studied [16-21] by changing the form of soft binary piecewise operation by using the complement at the first row of the soft binary piecewise operations.

Sezgin and Atagün [16] and Sezgin and Aybek [17] defined complementary soft binary piecewise plus and gamma operation, respectively. The algebraic properties of these new operations were investigated. Especially the distributions of these operations over extended soft set operations, complementary extended soft set operations, soft binary piecewise operations, complementary soft binary piecewise operations, and restricted soft set operations were handled. In this study, we aim to contribute to the literature of soft set theory by obtaining the distributions of soft binary piecewise operations over complementary soft binary piecewise plus and gamma operations.

2. Preliminaries

Definition 2.1. [1] Let U be the universal set, E be the parameter set, $P(U)$ be the power set of U and $N \subseteq E$. A pair (K, N) is called a soft set over U where K is a set-valued function such that $K: N \to P(U)$.

The set of all the soft sets over U is designated by $S_E(U)$, and throughout this paper, all the soft sets are the elements of $S_E(U)$. Çağman [8] defined two conditional complements of sets, for the ease of illustration, we show these complements as $+$ and θ, respectively. These complements are defined as following: Let P and C be two subsets of U. C-inclusive complement of P is defined by, $P + C = P' \cup C$ and C-exlusive complement of P is defined by $P\theta C = P' \cap C$. Here, U refers to a universe, P' is the complement of P over U. Sezgin et al. [9] introduced such new three complements as binary operations of sets as following: Let P and C be two subsets of U. Then, $P * C = P' \cup C'$, $P\gamma C = P' \cap C$, $P\lambda C = P \cup C'$ [9]. Aybek [10] conveyed these classical sets to soft sets, and they defined restricted and extended soft set operations and investigated their properties.

As a summary for soft set operations, we can categorize all types of soft set operations as following: Let "∇" be used to represent the set operations (i.e., here ∇ can be $\cap$, $\cup$, $-$, Δ, $+$, θ, $*$, λ, γ), then restricted operations, extended operations, complementary extended operations, soft binary piecewise operations, complementary soft binary piecewise operations are defined in soft set theory as following:

Definition 2.2. [4,6,10] Let (K, P) and (G, C) be soft sets over U. The restricted operation ∇ (restricted intersection, union, difference, symmetric difference, plus, theta, star, gamma, and lambda) of (K, P) and (G, C) is the soft set (Y, S), denoted by $(K, P)\nabla_R(G, C) = (Y, S)$ where $S = P \cap C \neq \emptyset$ and for all $v \in S$, $Y(v) = K(v) \nabla G(v)$.

Definition 2.3. [2,4,6,7,10] Let (K, P) and (G, C) be soft sets over U. The extended operation ∇ (extended union, intersection, difference, symmetric difference, plus, theta, gamma, lambda, and star) of (K, P) and (G, C) is the soft set (Y, S), denoted by $(K, P)\nabla_\varepsilon(G, C) = (Y, S)$ where $S = P \cup C$ and for all $v \in S$,

$$Y(v) = \begin{cases} K(v), & v \in P - C \\ G(v), & v \in C - P \\ K(v)\nabla G(v), & v \in P \cap C \end{cases}$$

Definition 2.4. [11-13] Let (K, P) and (G, C) be soft sets over U. The complementary extended operation ∇ (complementary extended gamma, intersection, star, plus, union, theta, difference, and lambda) of (K, P) and (G, C) is the soft set (Y, S), denoted by $(K, P)\overset{*}{\nabla}_\varepsilon(G, C) = (Y, S)$ where $S = P \cup C$ and for all $v \in S$,

$$Y(v)=\begin{cases}K'(v), & v\in P-C\\ G'(v), & v\in C-P\\ K(v)\nabla G(v), & v\in P\cap C\end{cases}$$

Definition 2.5. [14,15] Let (K,P) and (G,C) be soft sets over U. The soft binary piecewise operation ∇ (soft binary piecewise difference, intersection, union, plus, gamma, theta, lambda, and star) of (K,P) and (G,C) is the soft set (Y,P), denoted by $(K,P)\tilde{\nabla}(G,C)=(Y,P)$ where for all $v\in P$,

$$Y(v)=\begin{cases}K(v), & v\in P-C\\ K(v)\nabla G(v), & v\in P\cap C\end{cases}$$

Definition 2.6. [16-21] Let (K,P) and (G,C) be soft sets over U. The complementary soft binary piecewise operation ∇ (complementary soft binary piecewise star, theta, plus, intersection, union, gamma, lambda, and difference) of (K,P) and (G,C) is the soft set (Y,P), denoted by $(K,P)\overset{*}{\tilde{\nabla}}(G,C)=(Y,P)$ where for all $v\in P$,

$$Y(v)=\begin{cases}K'(v), & v\in P-C\\ K(v)\nabla G(v), & v\in P\cap C\end{cases}$$

Definition 2.7. [16] Let (K,P) and (G,C) be soft sets over U. The complementary soft binary piecewise plus $(+)$ operation of (K,P) and (G,C) is the soft set (Y,P), denoted by $(K,P)\overset{*}{\tilde{+}}(G,C)=(Y,P)$ where for all $v\in P$,

$$Y(v)=\begin{cases}K'(v), & v\in P-C\\ K'(v)\cup G(v), & v\in P\cap C\end{cases}$$

Definition 2.8. [17] Let (K,P) and (G,C) be soft sets over U. The complementary soft binary piecewise gamma (γ) operation of (K,P) and (G,C) is the soft set (Y,P), denoted by $(K,P)\overset{*}{\tilde{\gamma}}(G,C)=(Y,P)$ where for all $v\in P$,

$$Y(v)=\begin{cases}K'(v), & v\in P-C\\ K'(v)\cap G(v), & v\in P\cap C\end{cases}$$

3. Distribution Rules

In this section, distributions of soft binary piecewise operations over complementary soft binary piecewise plus and gamma operation are investigated in detail, and many interesting results are obtained.

Theorem 3.1. Let (K,P), (G,C), and (L,R) be soft sets over U. Then, we have the following distributions of soft binary piecewise operations over complementary soft binary piecewise plus $(+)$ operation:

i. $(K,P)\,\tilde{\cap}\left[(G,C)\overset{*}{\tilde{+}}(L,R)\right]=\left[(K,P)\,\tilde{\setminus}\,(G,C)\right]\tilde{\cap}\,[(L,R)\,\tilde{\cap}\,(K,P)]$ where $P\cap C\cap R=\emptyset$

Proof.

Handle the left-hand side of the equality and let $(G,C)\overset{*}{\tilde{+}}(L,R)=(M,C)$ where for all $I\in C$

$$M(I)=\begin{cases}G'(I), & I\in C-R\\ G'(I)\cup L(I), & I\in C\cap R\end{cases}$$

Let $(K,P)\,\tilde{\cap}\,(M,C)=(N,P)$ where for all $I\in P$

$$N(I)=\begin{cases}K(I), & I\in P-C\\ K(I)\cup M(I), & I\in P\cap C\end{cases}$$

and thus

$$N(I)=\begin{cases} K(I), & I\in P-C \\ K(I)\cup G'(I), & I\in P\cap(C-R)=P\cap C\cap R' \\ K(I)\cap[G'(I)\cup L(I)], & I\in P\cap(C\cap R)=P\cap C\cap R \end{cases} \tag{3.1}$$

Handle the left-hand side of the equality: $\left[(K,P)\,\tilde{\setminus}\,(G,C)\right]\tilde{\cap}\,[(L,R)\,\tilde{\cap}\,(K,P)]$. Let $(K,P)\,\tilde{\setminus}\,(G,C)=(V,P)$ where for all $I\in P$

$$V(I)=\begin{cases} K(I), & I\in P-C \\ K(I)\cap G'(I), & I\in P\cap C \end{cases}$$

Suppose that $(L,R)\,\tilde{\cap}\,(K,P)=(W,R)$ where for all $I\in R$

$$W(I)=\begin{cases} L(I), & I\in R-P \\ L(I)\cap K(I), & I\in R\cap P \end{cases}$$

Let $(V,P)\,\tilde{\cap}\,(W,R)=(T,P)$ where for all $I\in P$

$$T(I)=\begin{cases} V(I), & I\in P-R \\ V(I)\cap W(I), & I\in P\cap R \end{cases}$$

and thus

$$T(I)=\begin{cases} K(I), & I\in(P-C)-R=P\cap C'\cap R' \\ K(I)\cap G'(I), & I\in(P\cap C)-R=P\cap C\cap R' \\ K(I)\cap L(I), & I\in(P-C)\cap(R-P)=\emptyset \\ K(I)\cap[L(I)\cap K(I)], & I\in(P-C)\cap(R\cap P)=P\cap C'\cap R \\ [K(I)\cap G'(I)]\cap L(I), & I\in(P\cap C)\cap(R-P)=\emptyset \\ [K(I)\cap G'(I)]\cap[L(I)\cap K(I)], & I\in(P\cap C)\cap(R\cap P)=P\cap C\cap R \end{cases}$$

Therefore,

$$T(I)=\begin{cases} K(I), & I\in(P-C)-R=P\cap C'\cap R' \\ K(I)\cap G'(I), & I\in(P\cap C)-R=P\cap C\cap R' \\ K(I)\cup L(I), & I\in(P-C)\cap(R-P)=\emptyset \\ K(I), & I\in(P-C)\cap(R\cap P)=P\cap C'\cap R \\ [K(I)\cap G'(I)]\cup L(I), & I\in(P\cap C)\cap(R-P)=\emptyset \\ [K(I)\cap G'(I)]\cup[L(I)\cap K(I)], & I\in(P\cap C)\cap(R\cap P)=P\cap C\cap R \end{cases} \tag{3.2}$$

Handle $I\in P-C$ in the first equation. Since $P-C=P\cap C'$, if $I\in C'$, then $I\in L-C$ or $I\in(C\cup R)'$. Hence, if $I\in P-C$, then $I\in P\cap C'\cap R'$ or $I\in P\cap C'\cap R$. Thus, it can be observed that (3.1)=(3.2). □

ii. $\left[(K,P)\overset{*}{\tilde{\mp}}(G,C)\right]\tilde{\cap}\ (L,R)=\left[(K,P)\overset{*}{\tilde{\gamma}}(L,R)\right]\tilde{\cup}\,[(G,C)\,\tilde{\cap}\ (L,R)]$

Proof.

Handle the left-hand side of the equality and let $(K,P)\overset{*}{\tilde{\mp}}(G,C)=(M,P)$ where for all $I\in P$

$$M(I)=\begin{cases} K'(I), & I\in P-C \\ K'(I)\cup G(I), & I\in P\cap C \end{cases}$$

Let $(M,P)\,\tilde{\cap}\,(L,R)=(N,P)$ where for all $I\in P$

$$N(I)=\begin{cases} M(I), & I\in P-R \\ M(I)\cap L(I), & I\in P\cap R \end{cases}$$

Thus,

$$N(I) = \begin{cases} K'(I), & I \in (P - C) - R = P \cap C' \cap R' \\ K'(I) \cup G(I), & I \in (P \cap C) - R = P \cap C \cap R' \\ K'(I) \cap L(I), & I \in (P - C) \cap R = P \cap C' \cap R \\ [K'(I) \cup G(I)] \cap L(I), & I \in (P \cap C) \cap R = P \cap C \cap R \end{cases} \tag{3.3}$$

Handle the left-hand side of the equality: $\left[(K,P)\overset{*}{\tilde{\gamma}}(L,R)\right] \tilde{\cup} [(G,C) \tilde{\cap} (L,R)]$. Let $(K,P)\overset{*}{\tilde{\gamma}}(L,R) = (V,P)$ where for all $I \in P$

$$V(I) = \begin{cases} K'(I), & I \in P - R \\ K'(I) \cap L(I), & I \in P \cap R \end{cases}$$

Suppose that $(G,C) \tilde{\cap} (L,R) = (W,C)$ where for all $I \in C$

$$W(I) = \begin{cases} G(I), & I \in C - R \\ G(I) \cap L(I), & I \in C \cap R \end{cases}$$

Let $(V,P) \tilde{\cup} (W,R) = (T,P)$ where for all $I \in P$

$$T(I) = \begin{cases} V(I), & I \in P - R \\ V(I) \cup W(I), & I \in P \cap R \end{cases}$$

Thus,

$$T(I) = \begin{cases} K'(I), & I \in (P - R) - C = P \cap C' \cap R' \\ K'(I) \cap L(I), & I \in (P \cap R) - C = P \cap C' \cap R \\ K'(I) \cup G(I), & I \in (P - R) \cap (C - R) = P \cap C \cap R' \\ K'(I) \cup [G(I) \cap L(I)], & I \in (P - R) \cap (C \cap R) = \emptyset \\ [K'(I) \cap L(I)] \cup G(I), & I \in (P \cap R) \cap (C - R) = \emptyset \\ [K'(I) \cap L(I)] \cup [G(I) \cap L(I)], & I \in (P \cap R) \cap (C \cap R) = P \cap C \cap R \end{cases} \tag{3.4}$$

It can be observed that (3.3)=(3.4). □

iii. $(K,P) \tilde{\cup} \left[(G,C) \overset{*}{\tilde{\mp}} (L,R)\right] = \left[(K,P)\tilde{\lambda}(G,C)\right] \tilde{\cup} [(L,R) \tilde{\cup} (K,P)]$ where $P \cap C' \cap R = \emptyset$.

Proof.

Handle the left-hand side of the equality and let $(G,C) \overset{*}{\tilde{\mp}} (L,R) = (M,C)$ where for all $I \in C$

$$M(I) = \begin{cases} G'(I), & I \in C - R \\ G'(I) \cup L(I), & I \in C \cap R \end{cases}$$

Let $(K,P) \tilde{\cup} (M,C) = (N,P)$ where for all $I \in P$

$$N(I) = \begin{cases} K(I), & I \in P - C \\ K(I) \cup M(I), & I \in P \cap C \end{cases}$$

Thus,

$$N(I) = \begin{cases} K(I), & I \in P - C \\ K(I) \cup G'(I), & I \in P \cap (C - R) = P \cap C \cap R' \\ K(I) \cup [G'(I) \cup L(I)], & I \in P \cap (C \cap R) = P \cap C \cap R \end{cases} \tag{3.5}$$

Handle the left-hand side of the equality: $\left[(K,P)\tilde{\lambda}(G,C)\right] \tilde{\cup} [(L,R) \tilde{\cup} (K,P)]$. Let $(K,P)\tilde{\lambda}(G,C) = (V,P)$ where for all $I \in P$

$$V(I) = \begin{cases} K(I), & I \in P - C \\ K(I) \cup G'(I), & I \in P \cap C \end{cases}$$

Suppose that $(L,R) \mathbin{\tilde{\cup}} (K,P) = (W,R)$ where for all $I \in R$

$$W(I) = \begin{cases} L(I), & I \in R - P \\ L(I) \cup K(I), & I \in R \cap P \end{cases}$$

Let $(V,P) \mathbin{\tilde{\cup}} (W,R) = (T,P)$. Then, for all $I \in P$

$$T(I) = \begin{cases} V(I), & I \in P - R \\ V(I) \cup W(I), & I \in P \cap R \end{cases}$$

Thus,

$$T(I) = \begin{cases} K(I), & I \in (P - C) - R = P \cap C' \cap R' \\ K(I) \cup G'(I), & I \in (P \cap C) - R = P \cap C \cap R' \\ K(I) \cup L(I), & I \in (P - C) \cap (R - P) = \emptyset \\ K(I) \cup [L(I) \cup K(I)], & I \in (P - C) \cap (R \cap P) = P \cap C' \cap R \\ [K(I) \cup G'(I)] \cup L(I), & I \in (P \cap C) \cap (R - P) = \emptyset \\ [K(I) \cup G'(I)] \cup [L(I) \cup K(I)], & I \in (P \cap C) \cap (R \cap P) = P \cap C \cap R \end{cases} \tag{3.6}$$

It can be observed that (3.5)=(3.6). □

iv. $\left[(K,P) \overset{*}{\tilde{\mp}} (G,C)\right] \mathbin{\tilde{\cup}} (L,R) = \left[(K,P) \overset{*}{\tilde{\mp}} (L,R)\right] \mathbin{\tilde{\cup}} [(G,C) \mathbin{\tilde{\cup}} (L,R)]$

Proof.

Handle the left-hand side of the equality and let $(K,P) \overset{*}{\tilde{\mp}} (G,C) = (M,P)$ where for all $I \in P$

$$M(I) = \begin{cases} K'(I), & I \in P - C \\ K'(I) \cup G(I), & I \in P \cap C \end{cases}$$

Let $(M,P) \mathbin{\tilde{\cup}} (L,R) = (N,P)$ where for all $I \in P$

$$N(I) = \begin{cases} M(I), & I \in P - R \\ M(I) \cup L(I), & I \in P \cap R \end{cases}$$

Thus,

$$N(I) = \begin{cases} K'(I), & I \in (P - C) - R = P \cap C' \cap R' \\ K'(I) \cup G(I), & I \in (P \cap C) - R = P \cap C \cap R' \\ K'(I) \cup L(I), & I \in (P - C) \cap R = P \cap C' \cap R \\ [K'(I) \cup G(I)] \cup L(I), & I \in (P \cap C) \cap R = P \cap C \cap R \end{cases} \tag{3.7}$$

Handle the left-hand side of the equality: $\left[(K,P) \overset{*}{\tilde{\mp}} (L,R)\right] \mathbin{\tilde{\cup}} [(G,C) \mathbin{\tilde{\cup}} (L,R)]$. Let $(K,P) \overset{*}{\tilde{\mp}} (L,R) = (V,P)$ where for all $I \in P$

$$V(I) = \begin{cases} K'(I), & I \in P - R \\ K'(I) \cup L(I), & I \in P \cap R \end{cases}$$

Suppose that $(G,C) \mathbin{\tilde{\cup}} (L,R) = (W,C)$ where for all $I \in C$

$$W(I) = \begin{cases} G(I), & I \in C - R \\ G(I) \cup L(I), & I \in C \cap R \end{cases}$$

Let $(V,P) \mathbin{\tilde{\cup}} (W,R) = (T,P)$ where for all $I \in P$

$$T(I) = \begin{cases} V(I), & I \in P - R \\ V(I) \cup W(I), & I \in P \cap R \end{cases}$$

and thus

$$T(I)=\begin{cases} K'(I), & I\in(P-R)-C=P\cap C'\cap R' \\ K'(I)\cup L(I), & I\in(P\cap R)-C=P\cap C'\cap R \\ K'(I)\cup G(I), & I\in(P-R)\cap(C-R)=P\cap C\cap R' \\ K'(I)\cup[G(I)\cup L(I)], & I\in(P-R)\cap(C\cap R)=\emptyset \\ [K'(I)\cup L(I)]\cup G(I), & I\in(P\cap R)\cap(C-R)=\emptyset \\ [K'(I)\cup L(I)]\cup[G(I)\cup L(I)], & I\in(P\cap R)\cap(C\cap R)=P\cap C\cap R \end{cases} \tag{3.8}$$

It can be observed that (3.7)=(3.8). □

v. $(K,P)\tilde{\setminus}\left[(G,C)\overset{*}{\tilde{\mp}}(L,R)\right]=[(K,P)\mathbin{\tilde{\cap}}(G,C)]\mathbin{\tilde{\cap}}[(L,R)\tilde{\gamma}(K,P)]$ where $P\cap C'\cap R=\emptyset$

Proof.

Handle the left-hand side of the equality and let $(G,C)\overset{*}{\tilde{\mp}}(L,R)=(M,C)$ where for all $I\in C$

$$M(I)=\begin{cases} G'(I), & I\in C-R \\ G'(I)\cup L(I), & I\in C\cap R \end{cases}$$

Let $(K,P)\tilde{\setminus}(M,C)=(N,P)$ where for all $I\in P$

$$N(I)=\begin{cases} K(I), & I\in P-C \\ K(I)\cap M'(I), & I\in P\cap C \end{cases}$$

Thus,

$$N(I)=\begin{cases} K(I), & I\in P-C \\ K(I)\cap G(I), & I\in P\cap(C-R)=P\cap C\cap R' \\ K(I)\cap[G(I)\cup L'(I)], & I\in P\cap(C\cap R)=P\cap C\cap R \end{cases} \tag{3.9}$$

Handle the left-hand side of the equality: $[(K,P)\mathbin{\tilde{\cap}}(G,C)]\mathbin{\tilde{\cap}}[(L,R)\,\tilde{\gamma}(K,P)]$. Let $(K,P)\mathbin{\tilde{\cap}}(G,C)=(V,P)$ where for all $I\in P$

$$V(I)=\begin{cases} K(I), & I\in P-C \\ K(I)\cap G(I), & I\in P\cap C \end{cases}$$

Suppose that $(L,R)\tilde{\gamma}(K,P)=(W,R)$ where for all $I\in R$

$$W(I)=\begin{cases} L(I), & I\in R-P \\ L'(I)\cap K(I), & I\in R\cap P \end{cases}$$

Let $(V,P)\mathbin{\tilde{\cap}}(W,R)=(T,P)$. Then, for all $I\in P$

$$T(I)=\begin{cases} V(I), & I\in P-R \\ V(I)\cap W(I), & I\in P\cap R \end{cases}$$

Thus,

$$T(I)=\begin{cases} K(I), & I\in(P-C)-R=P\cap C'\cap R' \\ K(I)\cap G(I), & I\in(P\cap C)-R=P\cap C\cap R' \\ K(I)\cap L(I), & I\in(P-C)\cap(R-P)=\emptyset \\ K(I)\cap[L'(I)\cap K(I)], & I\in(P-C)\cap(R\cap P)=P\cap C'\cap R \\ [K(I)\cap G(I)]\cap L(I), & I\in(P\cap C)\cap(R-P)=\emptyset \\ [K(I)\cap G(I)]\cap[L'(I)\cap K(I)], & I\in(P\cap C)\cap(R\cap P)=P\cap C\cap R \end{cases} \tag{3.10}$$

It can be observed that (3.9)=(3.10). □

vi. $\left[(K,P)\ \overset{*}{\tilde{\mp}}\ (G,C)\right]\tilde{\setminus}(L,R) = \left[(K,P)\ \overset{*}{\tilde{\theta}}\ (L,R)\right]\tilde{\cup}\left[(G,C)\tilde{\setminus}(L,R)\right]$

Proof.

Handle the left-hand side of the equality and let $(K,P)\ \overset{*}{\tilde{\mp}}(G,C) = (M,P)$ where for all $I \in P$

$$M(I) = \begin{cases} K'(I), & I \in P - C \\ K'(I) \cup G(I), & I \in P \cap C \end{cases}$$

Let $(M,P)\tilde{\setminus}(L,R) = (N,P)$ where for all $I \in P$

$$N(I) = \begin{cases} M(I), & I \in P - R \\ M(I) \cap L'(I), & I \in P \cap R \end{cases}$$

Thus,

$$N(I) = \begin{cases} K'(I), & I \in (P - C) - R = P \cap C' \cap R' \\ K'(I) \cup G(I), & I \in (P \cap C) - R = P \cap C \cap R' \\ K'(I) \cap L'(I), & I \in (P - C) \cap R = P \cap C' \cap R \\ [K'(I) \cup G(I)] \cap L'(I), & I \in P \cap (C \cap R) = P \cap C \cap R \end{cases} \tag{3.11}$$

Handle the left-hand side of the equality: $\left[(K,P)\overset{*}{\tilde{\theta}}(L,R)\right]\tilde{\cup}\left[(G,C)\tilde{\setminus}(L,R)\right]$. Let $(K,P)\overset{*}{\tilde{\theta}}(L,R) = (V,P)$ where for all $I \in P$

$$V(I) = \begin{cases} K'(I), & I \in P - R \\ K'(I) \cap L'(I), & I \in P \cap R \end{cases}$$

Suppose that $(G,C)\tilde{\setminus}(L,R) = (W,C)$ where for all $I \in C$

$$W(I) = \begin{cases} G(I), & I \in C - R \\ G(I) \cap L'(I), & I \in C \cap R \end{cases}$$

Let $(V,P)\ \tilde{\cup}\ (W,R) = (T,P)$ where for all $I \in P$

$$T(I) = \begin{cases} V(I), & I \in P - R \\ V(I) \cup W(I), & I \in P \cap R \end{cases}$$

Thus,

$$T(I) = \begin{cases} K'(I), & I \in (P - R) - C = P \cap C' \cap R' \\ K'(I) \cap L'(I), & I \in (P \cap R) - C = P \cap C' \cap R \\ K'(I) \cup G(I), & I \in (P - R) \cap (C - R) = P \cap C \cap R' \\ K'(I) \cup [G(I) \cap L'(I)], & I \in (P - R) \cap (C \cap R) = \emptyset \\ [K'(I) \cap L'(I)] \cup G(I), & I \in (P \cap R) \cap (C - R) = \emptyset \\ [K'(I) \cap L'(I)] \cup [G(I) \cap L'(I)], & I \in (P \cap R) \cap (C \cap R) = P \cap C \cap R \end{cases} \tag{3.12}$$

It can be observed that (3.11)=(3.12). ◻

vii. $(K,P)\tilde{\lambda}\left[(G,C)\overset{*}{\tilde{\mp}}(L,R)\right] = [(K,P)\ \tilde{\cup}\ (G,C)]\ \tilde{\cup}\ [(L,R)\tilde{\mp}(K,P)]$ where $P \cap C' \cap R = \emptyset$

Proof.

Handle the left-hand side of the equality and let $(G,C)\overset{*}{\tilde{\mp}}(L,R) = (M,C)$ where for all $I \in C$

$$M(I) = \begin{cases} G'(I), & I \in C - R \\ G'(I) \cup L(I), & I \in C \cap R \end{cases}$$

Let $(K,P)\tilde{\lambda}(M,C)=(N,P)$ where for all $I\in P$

$$N(I)=\begin{cases} K(I), & I\in P-C \\ K(I)\cup M'(I), & I\in P\cap C \end{cases}$$

Thus,

$$N(I)=\begin{cases} K(I), & I\in P-C \\ K(I)\cup G(I), & I\in P\cap(C-R)=P\cap C\cap R' \\ K(I)\cup[G(I)\cup L'(I)], & I\in P\cap(C\cap R)=P\cap C\cap R \end{cases} \tag{3.13}$$

Handle the left-hand side of the equality: $[(K,P)\,\tilde{\cup}\,(G,C)]\,\tilde{\cup}\,[(L,R)\tilde{\mp}(K,P)]$. Let $(K,P)\,\tilde{\cup}\,(G,C)=(V,P)$ where for all $I\in P$

$$V(I)=\begin{cases} K(I), & I\in P-C \\ K(I)\cup G(I), & I\in P\cap C \end{cases}$$

Suppose that $(L,R)\tilde{\mp}(K,P)=(W,R)$ where for all $I\in R$

$$W(I)=\begin{cases} L(I), & I\in R-P \\ L'(I)\cup K(I), & I\in R\cap P \end{cases}$$

Let $(V,P)\,\tilde{\cup}\,(W,R)=(T,P)$. Then, for all $I\in P$

$$T(I)=\begin{cases} V(I), & I\in P-R \\ V(I)\cup W(I), & I\in P\cap R \end{cases}$$

Thus,

$$T(I)=\begin{cases} K(I), & I\in(P-C)-R=P\cap C'\cap R' \\ K(I)\cup G(I), & I\in(P\cap C)-R=P\cap C\cap R' \\ K(I)\cup L(I), & I\in(P-C)\cap(R-P)=\emptyset \\ K(I)\cup[L'(I)\cup K(I)], & I\in(P-C)\cap(R\cap P)=P\cap C'\cap R \\ [K(I)\cup G(I)]\cup L(I), & I\in(P\cap C)\cap(R-P)=\emptyset \\ [K(I)\cup G(I)]\cup[L'(I)\cup K(I)], & I\in(P\cap C)\cap(R\cap P)=P\cap C\cap R \end{cases} \tag{3.14}$$

It can be observed that (3.13)=(3.14). □

viii. $\left[(K,P)\overset{*}{\tilde{\mp}}(G,C)\right]\tilde{\lambda}(L,R)=\left[(K,P)\overset{*}{\tilde{*}}(L,R)\right]\tilde{\cup}\left[(G,C)\tilde{\lambda}(L,R)\right]$

Proof.

Handle the left-hand side of the equality and let $(K,P)\overset{*}{\tilde{\mp}}(G,C)=(M,P)$ where for all $I\in P$

$$M(I)=\begin{cases} K'(I), & I\in P-C \\ K'(I)\cup G(I), & I\in P\cap C \end{cases}$$

Let $(M,P)\tilde{\lambda}(L,R)=(N,P)$ where for all $I\in P$

$$N(I)=\begin{cases} M(I), & I\in P-R \\ M(I)\cup L'(I), & I\in P\cap R \end{cases}$$

Thus,

$$N(I)=\begin{cases} K'(I), & I\in(P-C)-R=P\cap C'\cap R' \\ K'(I)\cup G(I), & I\in(P\cap C)-R=P\cap C\cap R' \\ K'(I)\cup L'(I), & I\in(P-C)\cap R=P\cap C'\cap R \\ [K'(I)\cup G(I)]\cup L'(I), & I\in(P\cap C)\cap R=P\cap C\cap R \end{cases} \tag{3.15}$$

Handle the left-hand side of the equality: $\left[(K,P)\overset{*}{\tilde{*}}(L,R)\right]\tilde{\cup}\left[(G,C)\tilde{\lambda}(L,R)\right]$. Let $(K,P)\overset{*}{\tilde{*}}(L,R)=(V,P)$ where for all $I\in P$

$$V(I)=\begin{cases}K'(I), & I\in P-R\\ K'(I)\cup L'(I), & I\in P\cap R\end{cases}$$

Suppose that $(G,C)\tilde{\lambda}(L,R)=(W,C)$ where for all $I\in C$

$$W(I)=\begin{cases}G(I), & I\in C-R\\ G(I)\cup L'(I), & I\in C\cap R\end{cases}$$

Let $(V,P)\tilde{\cup}(W,R)=(T,P)$ where for all $I\in P$

$$T(I)=\begin{cases}V(I), & I\in P-R\\ V(I)\cup W(I), & I\in P\cap R\end{cases}$$

Thus,

$$T(I)=\begin{cases}K'(I), & I\in(P-R)-C=P\cap C'\cap R'\\ K'(I)\cup L'(I), & I\in(P\cap R)-C=P\cap C'\cap R\\ K'(I)\cup G(I), & I\in(P-R)\cap(C-R)=P\cap C\cap R'\\ K'(I)\cup[G(I)\cup L'(I)], & I\in(P-R)\cap(C\cap R)=\emptyset\\ [K'(I)\cup L'(I)]\cup G(I), & I\in(P\cap R)\cap(C-R)=\emptyset\\ [K'(I)\cup L'(I)]\cup[G(I)\cup L'(I)], & I\in(P\cap R)\cap(C\cap R)=P\cap C\cap R\end{cases}\tag{3.16}$$

It can be observed that (3.15)=(3.16). □

Theorem 3.2. Let (K,P), (G,C), and (L,R) be soft sets over U. Then, we have the following distributions of soft binary piecewise operations over complementary soft binary piecewise gamma (γ) operation:

i. $(K,P)\tilde{\cap}\left[(G,C)\overset{*}{\tilde{\gamma}}(L,R)\right]=\left[(K,P)\tilde{\setminus}(G,C)\right]\tilde{\cup}\left[(L,R)\tilde{\cap}(K,P)\right]$ where $P\cap C\cap R=\emptyset$

Proof.

Let first handle the left-hand side of the equality and let $(G,C)\overset{*}{\tilde{\gamma}}(L,R)=(M,C)$ where for all $I\in C$

$$M(I)=\begin{cases}G'(I), & I\in C-R\\ G'(I)\cap L(I), & I\in C\cap R\end{cases}$$

Let $(K,P)\tilde{\cap}(M,C)=(N,P)$ where for all $I\in P$

$$N(I)=\begin{cases}K(I), & I\in P-C\\ K(I)\cap M(I), & I\in P\cap C\end{cases}$$

Thus,

$$N(I)=\begin{cases}K'(I), & I\in P-C\\ K(I)\cap G'(I), & I\in P\cap(C-R)=P\cap C\cap R'\\ K(I)\cap[G'(I)\cap L(I)], & I\in P\cap(C\cap R)=P\cap C\cap R\end{cases}\tag{3.17}$$

Handle the left-hand side of the equality: $\left[(K,P)\tilde{\setminus}(G,C)\right]\tilde{\cup}\left[(L,R)\tilde{\cap}(K,P)\right]$. Let $(K,P)\tilde{\setminus}(G,C)=(V,P)$ where for all $I\in P$

$$V(I)=\begin{cases}K(I), & I\in P-C\\ K(I)\cap G'(I), & I\in P\cap C\end{cases}$$

Suppose that $(L,R)\tilde{\cap}(K,P)=(W,R)$ where for all $I\in R$

$$W(I) = \begin{cases} L(I), & I \in R - P \\ L(I) \cap K(I), & I \in R \cap P \end{cases}$$

Let $(V, P) \,\tilde{\cup}\, (W, R) = (T, P)$. Then, for all $I \in P$

$$T(I) = \begin{cases} V(I), & I \in P - R \\ V(I) \cup W(I), & I \in P \cap R \end{cases}$$

Thus,

$$T(I) = \begin{cases} K(I), & I \in (P - C) - R = P \cap C' \cap R' \\ K(I) \cap G'(I), & I \in (P \cap C) - R = P \cap C \cap R' \\ K(I) \cup L(I), & I \in (P - C) \cap (R - P) = \emptyset \\ K(I) \cup [L(I) \cap K(I)], & I \in (P - C) \cap (R \cap P) = P \cap C' \cap R \\ [K(I) \cap G'(I)] \cup L(I), & I \in (P \cap C) \cap (R - P) = \emptyset \\ [K(I) \cap G'(I)] \cup [L(I) \cap K(I)], & I \in (P \cap C) \cap (R \cap P) = P \cap C \cap R \end{cases}$$

Therefore,

$$T(I) = \begin{cases} K(I), & I \in (P - C) - R = P \cap C' \cap R' \\ K(I) \cap G'(I), & I \in (P \cap C) - R = P \cap C \cap R' \\ K(I) \cup L(I), & I \in (P - C) \cap (R - P) = \emptyset \\ K(I), & I \in (P - C) \cap (R \cap P) = P \cap C' \cap R \\ [K(I) \cap G'(I)] \cup L(I), & I \in (P \cap C) \cap (R - P) = \emptyset \\ [K(I) \cap G'(I)] \cup [L(I) \cap K(I)], & I \in (P \cap C) \cap (R \cap P) = P \cap C \cap R \end{cases} \tag{3.18}$$

Handle $I \in P - C$ in the first equation. Since $P - C = P \cap C'$, if $I \in C'$, then $I \in R - C$ or $I \in (C \cup R)'$. Hence, if $I \in P - C$, then $I \in P \cap C' \cap R'$ or $I \in P \cap C' \cap R$. Thus, it can be observed that (3.17)=(3.18). □

ii. $\left[(K, P)\overset{*}{\tilde{\gamma}}(G, C)\right] \tilde{\cap}\, (L, R) = \left[(K, P)\overset{*}{\tilde{\gamma}}(L, R)\right] \tilde{\cap}\, [(G, C) \,\tilde{\cap}\, (L, R)]$

Proof.

Handle the left-hand side of the equality and let $(K, P)\overset{*}{\tilde{\gamma}}(G, C) = (M, P)$ where for all $I \in P$

$$M(I) = \begin{cases} K'(I), & I \in P - C \\ K'(I) \cap G(I), & I \in P \cap C \end{cases}$$

Let $(M, P) \,\tilde{\cap}\, (L, R) = (N, P)$ where for all $I \in P$

$$N(I) = \begin{cases} M(I), & I \in P - R \\ M(I) \cap L(I), & I \in P \cap R \end{cases}$$

Thus,

$$N(I) = \begin{cases} K'(I), & I \in (P - C) - R = P \cap C' \cap R' \\ K'(I) \cap G(I), & I \in (P \cap C) - R = P \cap C \cap R' \\ K'(I) \cap L(I), & I \in (P - C) \cap R = P \cap C' \cap R \\ [K'(I) \cap G(I)] \cap L(I), & I \in P \cap (C \cap R) = P \cap C \cap R \end{cases} \tag{3.19}$$

Handle the left-hand side of the equality: $\left[(K, P)\overset{*}{\tilde{\gamma}}\,(L, R)\right] \tilde{\cap}\, [(G, C) \,\tilde{\cap}\, (L, R)]$. Let $(K, P)\overset{*}{\tilde{\gamma}}(L, R) = (V, P)$ where for all $I \in P$

$$V(I) = \begin{cases} K'(I), & I \in P - R \\ K'(I) \cap L(I), & I \in P \cap R \end{cases}$$

Suppose that $(G, C) \,\tilde{\cap}\, (L, R) = (W, C)$ where for all $I \in C$

$$W(I)=\begin{cases} G(I), & I\in C-R \\ G(I)\cap L(I), & I\in C\cap R\end{cases}$$

Let $(V,P)\,\tilde{\cap}\,(W,R)=(T,P)$ where for all $I\in P$

$$T(I)=\begin{cases} V(I), & I\in P-R \\ V(I)\cap W(I), & I\in P\cap R\end{cases}$$

Thus,

$$T(I)=\begin{cases} K'(I), & I\in(P-R)-C=P\cap C'\cap R' \\ K'(I)\cap L(I), & I\in(P\cap R)-C=P\cap C'\cap R \\ K'(I)\cap G(I), & I\in(P-R)\cap(C-R)=P\cap C\cap R' \\ K'(I)\cap[G(I)\cap L(I)], & I\in(P-R)\cap(C\cap R)=\emptyset \\ [K'(I)\cap L(I)]\cap G(I), & I\in(P\cap R)\cap(C-R)=\emptyset \\ [K'(I)\cap L(I)]\cap[G(I)\cap L(I)], & I\in(P\cap R)\cap(C\cap R)=P\cap C\cap R\end{cases} \tag{3.20}$$

It can be observed that (3.19)=(3.20). □

iii. $(K,P)\,\tilde{\cup}\left[(G,C)\overset{*}{\tilde{\gamma}}(L,R)\right]=\left[(K,P)\tilde{\lambda}(G,C)\right]\tilde{\cap}\,[(L,R)\,\tilde{\cup}\,(K,P)]$

Proof.

Handle the left-hand side of the equality and let $(G,C)\overset{*}{\tilde{\gamma}}(L,R)=(M,C)$ where for all $I\in C$

$$M(I)=\begin{cases} G'(I), & I\in C-R \\ G'(I)\cap L(I), & I\in C\cap R\end{cases}$$

Let $(K,P)\,\tilde{\cup}\,(M,C)=(N,P)$ where for all $I\in P$

$$N(I)=\begin{cases} K(I), & I\in P-C \\ K(I)\cup M(I), & I\in P\cap C\end{cases}$$

Thus,

$$N(I)=\begin{cases} K(I), & I\in P-C \\ K(I)\cup G'(I), & I\in P\cap(C-R)=P\cap C\cap R' \\ K(I)\cup[G'(I)\cap L(I)], & I\in P\cap(C\cap R)=P\cap C\cap R\end{cases} \tag{3.21}$$

Handle the left-hand side of the equality: $\left[(K,P)\tilde{\lambda}(G,C)\right]\tilde{\cap}\,[(L,R)\,\tilde{\cup}\,(K,P)]$. Let $(K,P)\tilde{\lambda}(G,C)=(V,P)$ where for all $I\in P$

$$V(I)=\begin{cases} K(I), & I\in P-C \\ K(I)\cup G'(I), & I\in P\cap C\end{cases}$$

Suppose that $(L,R)\,\tilde{\cup}\,(K,P)=(W,R)$ where for all $I\in R$

$$W(I)=\begin{cases} L(I), & I\in R-P \\ L(I)\cup K(I), & I\in R\cap P\end{cases}$$

Let $(V,P)\,\tilde{\cap}\,(W,R)=(T,P)$. Then, for all $I\in P$

$$T(I)=\begin{cases} V(I), & I\in P-R \\ V(I)\cap W(I), & I\in P\cap R\end{cases}$$

Thus,

$$T(I)=\begin{cases} K(I), & I\in(P-C)-R=P\cap C'\cap R' \\ K(I)\cup G'(I), & I\in(P\cap C)-R=P\cap C\cap R' \\ K(I)\cap L(I), & I\in(P-C)\cap(R-P)=\emptyset \\ K(I)\cap[L(I)\cup K(I)], & I\in(P-C)\cap(R\cap P)=P\cap C'\cap R \\ [K(I)\cup G'(I)]\cap L(I), & I\in(P\cap C)\cap(R-P)=\emptyset \\ [K(I)\cup G'(I)]\cap[L(I)\cup K(I)], & I\in(P\cap C)\cap(R\cap P)=P\cap C\cap R \end{cases}$$

Therefore,

$$T(I)=\begin{cases} K(I), & I\in(P-C)-R=P\cap C'\cap R' \\ K(I)\cup G'(I), & I\in(P\cap C)-R=P\cap C\cap R' \\ K(I)\cap L(I), & I\in(P-C)\cap(R-P)=\emptyset \\ K(I), & I\in(P-C)\cap(R\cap P)=P\cap C'\cap R \\ [K(I)\cup G'(I)]\cap L(I), & I\in(P\cap C)\cap(R-P)=\emptyset \\ [K(I)\cup G'(I)]\cap[L(I)\cup K(I)], & I\in(P\cap C)\cap(R\cap P)=P\cap C\cap R \end{cases} \tag{3.22}$$

It can be observed that (3.21)=(3.22). □

iv. $\left[(K,P)\overset{*}{\tilde{\gamma}}(G,C)\right]\,\tilde{\cup}\,(L,R)=\left[(K,P)\overset{*}{\tilde{\mp}}(L,R)\right]\,\tilde{\cap}\,[(G,C)\,\tilde{\cup}\,(L,R)]$

Proof.

Handle the left-hand side of the equality and let $(K,P)\overset{*}{\tilde{\gamma}}(G,C)=(M,P)$ where for all $I\in P$

$$M(I)=\begin{cases} K'(I), & I\in P-C \\ K'(I)\cap G(I), & I\in P\cap C \end{cases}$$

Let $(M,P)\,\tilde{\cup}\,(L,R)=(N,P)$ where for all $I\in P$

$$N(I)=\begin{cases} M(I), & I\in P-R \\ M(I)\cup L(I), & I\in P\cap R \end{cases}$$

Thus,

$$N(I)=\begin{cases} K'(I), & I\in(P-C)-R=P\cap C'\cap R' \\ K'(I)\cap G(I), & I\in(P\cap C)-R=P\cap C\cap R' \\ K'(I)\cup L(I), & I\in(P-C)\cap R=P\cap C'\cap R \\ [K'(I)\cap G(I)]\cup L(I), & I\in P\cap(C\cap R)=P\cap C\cap R \end{cases} \tag{3.23}$$

Handle the left-hand side of the equality: $\left[(K,P)\overset{*}{\tilde{\mp}}(L,R)\right]\,\tilde{\cap}\,[(G,C)\,\tilde{\cup}\,(L,R)]$. Let $(K,P)\overset{*}{\tilde{\mp}}(L,R)=(V,P)$ where for all $I\in P$

$$V(I)=\begin{cases} K'(I), & I\in P-R \\ K'(I)\cup L(I), & I\in P\cap R \end{cases}$$

Suppose that $(G,C)\,\tilde{\cup}\,(L,R)=(W,C)$ where for all $I\in C$

$$W(I)=\begin{cases} G(I), & I\in C-R \\ G(I)\cup L(I), & I\in C\cap R \end{cases}$$

Let $(V,P)\,\tilde{\cap}\,(W,R)=(T,P)$ where for all $I\in P$

$$T(I)=\begin{cases} V(I), & I\in P-R \\ V(I)\cap W(I), & I\in P\cap R \end{cases}$$

Thus,

$$T(I)=\begin{cases} K'(I), & I\in(P-R)-C=P\cap C'\cap R' \\ K'(I)\cup L(I), & I\in(P\cap R)-C=P\cap C'\cap R \\ K'(I)\cap G(I), & I\in(P-R)\cap(C-R)=P\cap C\cap R' \\ K'(I)\cap[G(I)\cup L(I)], & I\in(P-R)\cap(C\cap R)=\emptyset \\ [K'(I)\cup L(I)]\cap G(I), & I\in(P\cap R)\cap(C-R)=\emptyset \\ [K'(I)\cup L(I)]\cap[G(I)\cup L(I)], & I\in(P\cap R)\cap(C\cap R)=P\cap C\cap R \end{cases} \tag{3.24}$$

It can be observed that (3.23)=(3.24). □

v. $(K,P)\tilde{\setminus}\left[(G,C)\overset{*}{\tilde{\gamma}}(L,R)\right]=[(K,P)\,\tilde{\cap}\,(G,C)]\,\tilde{\cup}\,[(L,R)\tilde{\gamma}(K,P)]$ where $P\cap C\cap R=\emptyset$

Proof.

Handle the left-hand side of the equality and let $(G,C)\overset{*}{\tilde{\gamma}}(L,R)=(M,C)$ where for all $I\in C$

$$M(I)=\begin{cases} G'(I), & I\in C-R \\ G'(I)\cap L(I), & I\in C\cap R \end{cases}$$

Let $(K,P)\tilde{\setminus}(M,C)=(N,P)$ where for all $I\in P$

$$N(I)=\begin{cases} K(I), & I\in P-C \\ K(I)\cap M'(I), & I\in P\cap C \end{cases}$$

Thus,

$$N(I)=\begin{cases} K(I), & I\in P-C \\ K(I)\cap G(I), & I\in P\cap(C-R)=P\cap C\cap R' \\ K(I)\cap[G(I)\cup L'(I)], & I\in P\cap(C\cap R)=P\cap C\cap R \end{cases} \tag{3.25}$$

Handle the left-hand side of the equality: $[(K,P)\,\tilde{\cap}\,(G,C)]\,\tilde{\cup}\,[(L,R)\tilde{\gamma}(K,P)]$. Let $(K,P)\,\tilde{\cap}\,(G,C)=(V,P)$ where for all $I\in P$

$$V(I)=\begin{cases} K(I), & I\in P-C \\ K(I)\cap G(I), & I\in P\cap C \end{cases}$$

Suppose that $(L,R)\tilde{\gamma}(K,P)=(W,R)$ where for all $I\in R$

$$W(I)=\begin{cases} L(I), & I\in R-P \\ L'(I)\cap K(I), & I\in R\cap P \end{cases}$$

Let $(V,P)\,\tilde{\cup}\,(W,R)=(T,P)$. Then, for all $I\in P$

$$T(I)=\begin{cases} V(I), & I\in P-R \\ V(I)\cup W(I), & I\in P\cap R \end{cases}$$

Thus,

$$T(I)=\begin{cases} K(I), & I\in(P-C)-R=P\cap C'\cap R' \\ K(I)\cap G(I), & I\in(P\cap C)-R=P\cap C\cap R' \\ K(I)\cup L(I), & I\in(P-C)\cap(R-P)=\emptyset \\ K(I)\cup[L'(I)\cap K(I)], & I\in(P-C)\cap(R\cap P)=P\cap C'\cap R \\ [K(I)\cap G(I)]\cup L(I), & I\in(P\cap C)\cap(R-P)=\emptyset \\ [K(I)\cap G(I)]\cup[L'(I)\cap K(I)], & I\in(P\cap C)\cap(R\cap P)=P\cap C\cap R \end{cases}$$

Therefore,

$$T(I)=\begin{cases} K(I), & I\in(P-C)-R=P\cap C'\cap R' \\ K(I)\cap G(I), & I\in(P\cap C)-R=P\cap C\cap R' \\ K(I)\cup L(I), & I\in(P-C)\cap(R-P)=\emptyset \\ K(I), & I\in(P-C)\cap(R\cap P)=P\cap C'\cap R \\ [K(I)\cap G(I)]\cup L(I), & I\in(P\cap C)\cap(R-P)=\emptyset \\ [K(I)\cap G(I)]\cup[L'(I)\cap K(I)], & I\in(P\cap C)\cap(R\cap P)=P\cap C\cap R \end{cases} \quad (3.26)$$

It can be observed that (3.25)=(3.26). ◻

vi. $\left[(K,P)\overset{*}{\tilde{\gamma}}(G,C)\right]\tilde{\setminus}(L,R)=\left[(K,P)\overset{*}{\tilde{\theta}}(L,R)\right]\tilde{\cap}\left[(G,C)\tilde{\setminus}(L,R)\right]$

Proof.

Let first handle the left-hand side of the equality and let $(K,P)\overset{*}{\tilde{\gamma}}(G,C)=(M,P)$ where for all $I\in P$

$$M(I)=\begin{cases} K'(I), & I\in P-C \\ K'(I)\cap G(I), & I\in P\cap C \end{cases}$$

Let $(M,P)\tilde{\setminus}(L,R)=(N,P)$ where for all $I\in P$

$$N(I)=\begin{cases} M(I), & I\in P-R \\ M(I)\cap L'(I), & I\in P\cap R \end{cases}$$

Thus,

$$N(I)=\begin{cases} K'(I), & I\in(P-C)-R=P\cap C'\cap R' \\ K'(I)\cap G(I), & I\in(P\cap C)-R=P\cap C\cap R' \\ K'(I)\cap L'(I), & I\in(P-C)\cap R=P\cap C'\cap R \\ [K'(I)\cap G(I)]\cap L'(I), & I\in(P\cap C)\cap R=P\cap C\cap R \end{cases} \quad (3.27)$$

Handle the left-hand side of the equality: $\left[(K,P)\overset{*}{\tilde{\theta}}(L,R)\right]\tilde{\cap}\left[(G,C)\tilde{\setminus}(L,R)\right]$. Let $(K,P)\overset{*}{\tilde{\theta}}(L,R)=(V,P)$ where for all $I\in P$

$$V(I)=\begin{cases} K'(I), & I\in P-R \\ K'(I)\cap L'(I), & I\in P\cap R \end{cases}$$

Suppose that $(G,C)\tilde{\setminus}(L,R)=(W,C)$ where for all $I\in C$

$$W(I)=\begin{cases} G(I), & I\in C-R \\ G(I)\cap L'(I), & I\in C\cap R \end{cases}$$

Let $(V,P)\tilde{\cap}(W,R)=(T,P)$ where for all $I\in P$

$$T(I)=\begin{cases} V(I), & I\in P-R \\ V(I)\cap W(I), & I\in P\cap R \end{cases}$$

Thus,

$$T(I)=\begin{cases} K'(I), & I\in(P-R)-C=P\cap C'\cap R' \\ K'(I)\cap L'(I), & I\in(P\cap R)-C=P\cap C'\cap R \\ K'(I)\cap G(I), & I\in(P-R)\cap(C-R)=P\cap C\cap R' \\ K'(I)\cap[G(I)\cap L'(I)], & I\in(P-R)\cap(C\cap R)=\emptyset \\ [K'(I)\cap L'(I)]\cap G(I), & I\in(P\cap R)\cap(C-R)=\emptyset \\ [K'(I)\cap L'(I)]\cap[G(I)\cap L'(I)], & I\in(P\cap R)\cap(C\cap R)=P\cap C\cap R \end{cases} \quad (3.28)$$

It can be observed that (3.27)=(3.28). ◻

vii. $(K,P)\tilde{\lambda}\left[(G,C)\overset{*}{\tilde{\gamma}}(L,R)\right] = [(K,P)\ \tilde{\cup}\ (G,C)]\ \tilde{\cap}\ [(L,R)\tilde{\mp}(K,P)]$ where $P \cap C \cap R = \emptyset$

Proof.

Handle the left-hand side of the equality and let $(G,C)\overset{*}{\tilde{\gamma}}(L,R) = (M,C)$ where for all $I \in C$

$$M(I) = \begin{cases} G'(I), & I \in C - R \\ G'(I) \cap L(I), & I \in C \cap R \end{cases}$$

Let $(K,P)\tilde{\lambda}(M,C) = (N,P)$ where for all $I \in P$

$$N(I) = \begin{cases} K(I), & I \in P - C \\ K(I) \cup M'(I), & I \in P \cap C \end{cases}$$

Thus,

$$N(I) = \begin{cases} K(I), & I \in P - C \\ K(I) \cup G(I), & I \in P \cap (C - R) = P \cap C \cap R' \\ K(I) \cup [G(I) \cup L'(I)], & I \in P \cap (C \cap R) = P \cap C \cap R \end{cases} \tag{3.29}$$

Handle the left-hand side of the equality: $[(K,P)\ \tilde{\cup}\ (G,C)]\ \tilde{\cap}\ [(L,R)\tilde{\mp}(K,P)]$. Let $(K,P)\ \tilde{\cup}\ (G,C) = (V,P)$ where for all $I \in P$

$$V(I) = \begin{cases} K(I), & I \in P - C \\ K(I) \cup G(I), & I \in P \cap C \end{cases}$$

Suppose that $(L,R)\tilde{\mp}(K,P) = (W,R)$ where for all $I \in R$

$$W(I) = \begin{cases} L(I), & I \in R - P \\ L'(I) \cup K(I), & I \in R \cap P \end{cases}$$

Let $(V,P)\ \tilde{\cap}\ (W,R) = (T,P)$. Then, for all $I \in P$

$$T(I) = \begin{cases} V(I), & I \in P - R \\ V(I) \cap W(I), & I \in P \cap R \end{cases}$$

Thus,

$$T(I) = \begin{cases} K(I), & I \in (P - C) - R = P \cap C' \cap R' \\ K(I) \cup G(I), & I \in (P \cap C) - R = P \cap C \cap R' \\ K(I) \cap L(I), & I \in (P - C) \cap (R - P) = \emptyset \\ K(I) \cap [L'(I) \cup K(I)], & I \in (P - C) \cap (R \cap P) = P \cap C' \cap R \\ [K(I) \cup G(I)] \cap L(I), & I \in (P \cap C) \cap (R - P) = \emptyset \\ [K(I) \cup G(I)] \cap [L'(I) \cup K(I)], & I \in (P \cap C) \cap (R \cap P) = P \cap C \cap R \end{cases}$$

Therefore,

$$T(I) = \begin{cases} K(I), & I \in (P - C) - R = P \cap C' \cap R' \\ K(I) \cup G(I), & I \in (P \cap C) - R = P \cap C \cap R' \\ K(I) \cap L(I), & I \in (P - C) \cap (R - P) = \emptyset \\ K(I), & I \in (P - C) \cap (R \cap P) = P \cap C' \cap R \\ [K(I) \cup G(I)] \cap L(I), & I \in (P \cap C) \cap (R - P) = \emptyset \\ [K(I) \cup G(I)] \cap [L'(I) \cup K(I)], & I \in (P \cap C) \cap (R \cap P) = P \cap C \cap R \end{cases} \tag{3.30}$$

It can be observed that (3.29)=(3.30). ◻

viii. $\left[(K,P)\overset{*}{\tilde{\gamma}}(G,C)\right]\tilde{\lambda}(L,R) = \left[(K,P)\ \overset{*}{\tilde{*}}\ (L,R)\right]\ \tilde{\cap}\ [(G,C)\tilde{\lambda}(L,R)]$

Proof.

Handle the left-hand side of the equality and let$(K,P)\overset{*}{\tilde{\gamma}}(G,C)=(M,P)$ where for all $I\in P$

$$M(I)=\begin{cases}K'(I), & I\in P-C\\ K'(I)\cap G(I), & I\in P\cap C\end{cases}$$

Let $(M,P)\tilde{\lambda}(L,R)=(N,P)$ where for all $I\in P$

$$N(I)=\begin{cases}M(I), & I\in P-R\\ M(I)\cup L'(I), & I\in P\cap R\end{cases}$$

Thus,

$$N(I)=\begin{cases}K'(I), & I\in(P-C)-R=P\cap C'\cap R'\\ K'(I)\cap G(I), & I\in(P\cap C)-R=P\cap C\cap R'\\ K'(I)\cup L'(I), & I\in(P-C)\cap R=P\cap C'\cap R\\ [K'(I)\cap G(I)]\cup L'(I), & I\in(P\cap C)\cap R=P\cap C\cap R\end{cases} \tag{3.31}$$

Handle the left-hand side of the equality: $\left[(K,P)\overset{*}{\tilde{*}}(L,R)\right]\tilde{\cap}\left[(G,C)\tilde{\lambda}(L,R)\right]$. Let $(K,P)\overset{*}{\tilde{*}}(L,R)=(V,P)$ where for all $I\in P$

$$V(I)=\begin{cases}K'(I), & I\in P-R\\ K'(I)\cup L'(I), & I\in P\cap R\end{cases}$$

Suppose that $(G,C)\tilde{\lambda}(L,R)=(W,C)$ where for all $I\in C$

$$W(I)=\begin{cases}G(I), & I\in C-R\\ G(I)\cup L'(I), & I\in C\cap R\end{cases}$$

Let $(V,P)\tilde{\cap}(W,R)=(T,P)$ where for all $I\in P$

$$T(I)=\begin{cases}V(I), & I\in P-R\\ V(I)\cap W(I), & I\in P\cap R\end{cases}$$

Thus,

$$T(I)=\begin{cases}K'(I), & I\in(P-R)-C=P\cap C'\cap R'\\ K'(I)\cup L'(I), & I\in(P\cap R)-C=P\cap C'\cap R\\ K'(I)\cap G(I), & I\in(P-R)\cap(C-R)=P\cap C\cap R'\\ K'(I)\cap[G(I)\cup L'(I)], & I\in(P-R)\cap(C\cap R)=\emptyset\\ [K'(I)\cup L'(I)]\cap G(I), & I\in(P\cap R)\cap(C-R)=\emptyset\\ [K'(I)\cup L'(I)]\cap[G(I)\cup L'(I)], & I\in(P\cap R)\cap(C\cap R)=P\cap C\cap R\end{cases} \tag{3.32}$$

It can be observed that (3.31)=(3.32). ◻

4. Conclusion

In this paper, we explore more about complementary soft binary piecewise plus and gamma operation by investigating the relationships between these soft set operations and soft binary piecewise operations. In this paper, it is aimed to contribute to the soft set literature by obtaining the distributions of soft binary piecewise operations over complementary soft binary piecewise plus and gamma operations. This is a theoretical study for soft sets and some future studies may continue by investigating the distributions of soft binary piecewise operations over other complementary soft piecewise operations.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

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Development of pulp and paper using stem and fruit stem of *Musa Species*

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Abstract — Paper is a thin material mainly used for writing, printing, and packaging. The stem and fruit stem of two *Musa species*, *Musa Acuminata Balbisiana* (Banana stem) and *Musa Paradisiaca* (Plantain stem), were expanded. This research aimed to find other alternatives to reduce the use of wood fibers that are being converted into paper. The stem chips were charged into a digester with different percentages (5, 10, and 15%) of sodium hydroxide (NaOH) and were subjected to the impregnating temperature (60^{o}C) for 1 hour before heating to 100^{o}C for 3 hours. The stems were pounded finely with mortar and pestle and then bleached using 15% hydrogen peroxide. Calcium carbonate (10%) was added to both pulps obtained as filler, while 5% glue was added as a sizing agent. The mixture was agitated, pressurized, and air-dried, and mechanical tests were carried out. It was shown that the tearing resistance, tensile strength, and elongation tests obtained using 5% NaOH for banana, plantain and banana fruit stems were 0.84 mN, 17.82 N/m^2, and 4.90 mm; 1.35 mN, 14.95 N/m, and 2.17 mm; and 1.90 mN, 24.77 N/m^2, and 5.49 mm while the pulp yields were 36.7%, 35.5%, and 38.5%, respectively. The results obtained using 10% NaOH for banana, plantain, and banana fruit stems were 0.80 mN, 17.30 N/m^2, and 4.85 mm; 1.25 mN, 14.73 N/m^2, and 2.0 mm; 1.85 mN, 23.60 N/m^2, and 5.35 mm while the pulp yields were 35.80%, 34.12%, and 32.25% in that order. Moreover, using 15% of NaOH for banana, plantain, and banana fruit stems gave 0.70 mN, 6.89 N/m^2, and 1.86 mm; 0.79 mN, 8.70 N/m^2, and 2.90 mm; and 1.5 mN, 12.62 N/m^2, and 3.03 mm while the pulp yields were 33.8, 33.11, and 31.03%, respectively. This showed that banana fruit stems pulped at 5% NaOH gave better results than banana and plantain stems. In conclusion, the pulp is suitable for producing fiberboards and cartons.

Keywords: *Banana stem, plantain stem, pulp, pressurized, filler, tensile strength*

Subject Classification (2020):

1. Introduction

Man derives his livelihood directly from the natural endowment in which trees fall into this category with more economical and biotechnological values. Naturally, wood has diverse applications for humanity, such as paper production, which has increased tremendously over the past four decades. Other agricultural residues can also be used for pulp and paper production. The high demand for pulp and paper has dramatically reduced pulp wood with a decline in forest-based material. This has prompted many researchers to look for alternatives to pulp wood, such as waste of bagasse and rice straw for producing paper. It is observed that among the agro waste, much research has not been carried out using banana stems [1]. Paper is a slender material produced by pressing together moist fibers derived from cellulose pulp capable of writing or packaging. In developing

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countries like Nigeria, the availability of a steady supply of paper is an essential tool for promoting education and increasing literacy. The development triggers an increased demand for the local production of paper and paper products in food processing, export packaging, hygiene, and household items [2]. The research focused on producing paper from the Pulp obtained from banana stem waste, plantain stem waste, and banana fruit stem waste.

2. Preliminaries

2.1. Chemistry of Paper

The basic structure of pulp and paper sheets is a felted mat of cellulose fibers held together by hydrogen bonds. Cellulose is a polysaccharide with 600 to 1,500 repeated sugar units. It is the structural component of the primary cell wall of green plants and many forms of algae. Cellulose is the most common organic compound on Earth. About 33% of all plant matter is cellulose. Cellulose fibers have high tensile strength, will absorb the additives used to modify pulp into paper and board products, and are supple and chemically stable. The purpose of pulping is to separate cellulose fibers from the other components of the fiber source. In the case of wood, these include hemicelluloses (with 15 to 90 repeated sugar units), lignin (highly polymerized and complex, mainly phenyl propane units; they act as the “glue” that cements the fibers together), extractives (fats, waxes, alcohols, phenols, aromatic acids, essential oils, oleoresins, sterols, alkaloids and pigments), and minerals and other inorganic. The relative proportions of these components vary according to the fiber source, as shown in Table 1 [3].

Table 1. Chemical constituents of pulp and paper fiber sources

Woods / Carbohydrates	Softwood	Hardwood	Straw	Bamboo	Cotton
α-cellulose	38–46	38–49	28–42	26–43	80–85
Hemicelluloses	23–31	20–40	23–38	15–26	nd
Lignin	22–34	16–30	12–21	20–32	nd
Extractives	1-5	2-8	1-2	0.2-5	nd
Minerals	0.1-7	0.1-11	3-20	1-10	0.8-2

Sourced from Anya and Teschke [4]

2.2. Wood Pulp

Wood pulp is a dry fibrous material prepared by chemically or mechanically separating the fibers which make up wood. [5] revealed that pulping is the process by which the bonds within the wood structure are mechanically or chemically ruptured. Chemical pulps can be produced by either alkaline sulfate, Kraft, or acidic (sulfite) processes. The highest proportion of pulp is produced by the sulfate method, followed by mechanical (including semi-chemical, thermomechanical, and mechanical) and sulfite methods. Pulping processes differ in the yield and quality of the product, and for chemical methods, in the chemicals used and the proportion that can be recovered for reuse [6].

2.2.1. Mechanical Pulping

[7] revealed that mechanical pulps are produced by grinding wood against a stone or between metal plates, thereby separating the wood into individual fibers. Shearing breaks cellulose fibers and the resulting pulp is weaker than chemically separated pulps. The lignin connecting cellulose to hemicelluloses is not dissolved; it merely softens, allowing the fibers to be ground out of the wood matrix. The main difference between chemical and mechanical pulp is the yield ratio. Mechanical pulp uses 80 to 95% of the wood fiber, while chemical pulp

uses approximately 45 to 55%. The critical characteristic of mechanical pulp is that it is primarily used for producing paper products where quality is not a significant concern (e.g., newsprint). Mechanical pulping is additionally the only option for processing recovered paper for pulp production. Several mechanical pulping techniques exist [8].

Refiner mechanical pulping (RMP): Its key characteristics are the high yield and the fact that fibers are not too short. RMP can use chips as raw material, processed by two grooved discs. The fibers produced with this technique are lighter than usual; thus, the ratio of paper produced per wood use is increased compared to other pulping methods [9].

Chemi-thermomechanical pulping (CTMP): This technique is characterized by using chemicals in the refining process and the increased flexibility and brightness of the fiber produced. The main advantage of this pulp type, while also a significant drawback, is the relatively high energy demand. The main disadvantage of mechanical pulping is that it is an energy-intensive process and is the most effective energy consumer per product quantity compared to other pulping options. Some additional drawbacks are the relatively short fibers of pulp produced, the low ratio of impurities removal, and the paper products' low strength and brightness characteristics [9].

2.2.2. Chemical Pulping

Chemical pulps are produced by chemically dissolving the lignin between the wood fibers, enabling the fibers to separate relatively undamaged. Because most non-fibrous wood components are removed in these processes, yields are usually 40 to 55%. In chemical pulping, chips and chemicals in an aqueous solution are cooked together in a pressure vessel (digester), which can be operated on a batch or continuous basis. In batch cooking, the digester is filled with chips through a top opening, the digestion chemicals are added, and the contents are cooked at elevated temperature and pressure. Once the cooking is complete, the pressure is released, "blowing" the de-lignified pulp out of the digester and into a holding tank. The sequence is then repeated. In continuous digesting, pre-steamed chips are fed into the digester continuously. Chips and chemicals are mixed together in the impregnation zone at the top of the digester and then proceed through the upper, lower, and washing zones before being blown into the blow tank [10].

2.3. Sulphate Pulping

The sulfate or Kraft pulping process produces a stronger, darker pulp than other methods and requires chemical recovery to compete economically. The method evolved from soda pulping (which uses only sodium hydroxide for digestion). It began to gain prominence in the industry from the 1930s to 1950s with the development of chlorine dioxide bleaching and chemical recovery processes. Developing corrosion-proof metals, such as stainless steel, to handle the acidic and alkaline pulp mill environments also played a role [11]. The cooking liquor for the sulfate process is a solution of sodium hydroxide (NaOH) and sodium sulfide (Na_2S). The NaOH dissolves some of the non-fibrous materials. Others decomposed upon being heated, forming acids as displayed in (2.1). These acids react with the base NaOH to form compounds soluble in water. The NaOH also reacts with the resins in the wood, forming water-soluble soaps. Thus, in one way or another, the non-fibrous materials are dissolved and separated from the cellulose fibers. As sodium hydroxide is consumed, the sodium sulfide reacts with water to produce more sodium hydroxide [12].

$$Na_2S + H_2O \rightarrow NaHS + NaOH \quad (2.1)$$

2.4. Bleaching the Pulp

[13] stated that bleaching is a multi-stage process that refines and brightens raw pulp. The objective is to dissolve (chemical pulps) or modify (mechanical pulps) the brown-colored lignin that was not removed during pulping while maintaining the integrity of the pulp fibers. A mill produces customized pulp by varying the bleaching agents' order, concentration and reaction time. Each bleaching stage is defined by its bleaching agent, pH (acidity), temperature, and duration. After each bleaching stage, the pulp may be washed with caustic to remove spent bleaching chemicals and dissolved lignin before progressing to the next stage. After the last stage, the pulp is pumped through screens and cleaners to remove contaminants such as dirt or plastic. It is then concentrated and conveyed to storage [14].

2.5. Stock Preparation and Paper Making

There are two types of stock preparation systems used. In one, the stock is treated first in a beater and then in a Jordan conical refiner. Continuous stock preparation, consisting of disk-type refining and conical refining, is employed in high-production mills. What happens to the cellulose fibers during stock preparation has an essential effect on the characteristics of paper produced. The beaters and the refiners roughen the individual fibers and fray their ends; this condition is desirable. When such fibers are used to create paper, the fibers interlock to make a strong paper. Secondly, the beating also breaks down the water-resistant outer walls of the fibers, thereby exposing the inner fibrils. This effect is called fibrillation. Once it takes place, the fibers take on water and swell. This effect is called hydration. The longer the refining process continues, the more the fibers are hydrated and the stronger the resulting paper becomes [10].

2.6. Fillers

Materials called fillers are added to the pulp during stock preparation. Printing papers may contain 15 -25% of fillers by weight. Other papers designed for strength and rugged use, such as bond and ledger, may have 2-6% fillers. The three materials most commonly used for fillers are clay, a naturally occurring Alumino-Silicate; Titanium Dioxide, TiO_2; and Calcium Carbonate, $CaCO_3$. The principal reason for adding fillers is to increase opacity, brightness, and smoothness and to reduce ink-strike through [10].

2.7. Drying of Pulp and Paper

Drying involves using air or heat to remove water from the paper sheet. In the earliest days of paper-making, this was done by hanging the paper sheets like laundry. In more modern times, various forms of heated drying mechanisms are used. On the paper machine, the most common is the steam-heated can dryer. These dryers can heat to temperatures above 93°C and are used in long sequences of more than 40 cans. The heat produced by these can quickly dry the paper to less than 6% moisture [15].

2.8. Finishing

The paper may undergo sizing to alter its physical properties for various applications. The paper, at this point, is uncoated. Coated paper has a thin layer of material such as calcium carbonate or China clay applied to one or both sides to create a surface more suitable for high-resolution halftone screens. Coated or uncoated papers may have their surfaces polished by calendaring [16]. This research aimed to produce paper from the pulp obtained from banana, plantain, and fruit stem wastes and to evaluate the mechanical properties of the paper produced.

3. Materials and Methods

Papermaking begins with the collection of raw materials. It continues through preparing the raw material, making pulp, screening out the sheets of paper, drying the paper, and finally finishing it.

3.1. Chemicals Used

Sodium Hydroxide (NaOH), Hydrogen peroxide (H_2O_2)

Calcium Carbonate ($CaCO_3$), Binder (Top Bond with compositions of polyvinyl formal, calcium carbonate and water)

3.2. Raw Materials Used

Banana stem waste

Plantain stem waste and

Banana fruit stem waste

3.3. Pulping Process

The raw materials, which are *Musa Species* (banana stem waste, plantain stem waste, banana fruit stem waste, banana peel, and plantain peel), were obtained from Yaba College of Technology staff quarters, where they were locally grown. The freshly-cut sample parts were cut into an average of 2.0 cm chips, pounded with mortar and pestle for about 10 minutes, and then squeezed to remove some juice.

3.3.1. Chemical Method

The freshly cut and pounded (685.76g) samples chips of average length of 2.0 cm were charged into the digester (1000 ml Beaker) with the required amount of chemical solution of liquor to goods ratio (LR) of 5:1. Different percentages (5%, 10% and 15%) of Sodium Hydroxide concentration were used as cooking liquor. The pulping consisted of two stages. In the first stage, the crushed sample stems were heated to the impregnating temperature 60^0C and maintained at this temperature for 1 hour (60 minutes) so that the cooking liquor could penetrate the sample before it was heated to a boiling point 100^0C and maintained at this temperature for 3 hours (180 minutes) for the digestion to be completed. At the end of the cooking (digestion), the pulp, which at this stage was dark brown and called black liquor, was washed several times with water. The resulting pulp was filtered, pressed, and passed (to neutralize any residual alkaline) once with hot water and several times with cold water. The pulp collected was defibrated and kept for further processing (bleaching or paper making).

3.3.2. Chemical/ Mechanical Method

A small sample quantity was collected and weighed in the chemical-mechanical method. The weighed sample was cut and chopped into about 2 mm and then pounded with mortar and pestle until it became a fine slurry (pulp). The sample was weighed in a beaker containing 5%, 10%, and 15% sodium hydroxide solution. The content was then boiled for an hour (60 minutes) and stirred occasionally. After 1 hour of boiling, the sample was removed from the alkaline solution and rinsed with water to remove the black liquor of the sodium lignite and the unused alkali. The washed sample was then pounded with mortar and pestle until it became slurry (pulp). The pounded sample (pulp) was then washed, filtered, defibrated, and kept for further processing (bleaching or paper making). The established standard used for this work was the TAPPI standard.

3.3.3. Bleaching Stage

The slurry (pulp) was bleached with hydrogen peroxide by boiling for 30 minutes to increase the brightness of the pulp. After this, the bleached pulp was washed in running water, and the pulp water slurry was adjusted to contain 5% fiber and 95% water. It was then left as stock for paper making.

3.4. Production of Handmade Papers

Procedure

Banana stem waste was cut and weighed (685.76g). The sample was cut into 10-20 mm chips, pounded with mortar and pestle for about 10 minutes, then squeezed to remove some juice from it and reweighed. The banana chips were charged into the digester (1 Litre Beaker) with a 5% concentration of Sodium Hydroxide at liquor to goods ratio (LR) of 5:1. The pulping consisted of two stages. In the first stage, the crushed sample stems were heated to the impregnating temperature 60^0C and maintained at this temperature for 1 hour (60 minutes) before it was heated to a boiling point 100^0C and kept at this temperature for 3 hours (180 minutes). At the end of the cooking (digestion), the pulp, which at this stage was dark brown, is called black liquor. The resulting pulp was filtered, pressed, washed several times in running water, and filtered. The sample was then bleached by boiling with hydrogen peroxide (15% w/v mass of pulp) for 30 minutes, filtered, and then the pH was taken.

5% of binder " Top Bond" and 10% $CaCO_3$ were added to the pulp. The mixture was appropriately attired with a stirrer to defibrate (separate) the fibers with other chemicals of different proportions, as shown in Table 2. The mixture (pulp) was then transferred to the paper-making mold screen. The paper produced was dried in the open air for 2-3 hours, hot pressed, and then calendered to smoothen the surface.

The pulp yield in stem and pulp yield in residue indicates the mass (weight) amount of material recovered after a specific process compared to the starting amount of material before the process. The recovery from pulping wood is commonly expressed as the percentage, by oven-dry weight, of pulp obtained from the original wood weight.

Table 2. The mixing proportion of Stems and Chemicals

Chemical/Parameters	NaOH (%)	H_2O_2 (%w/v)	$CaCO_3$ (%w/v)
Banana Stem	5 10 15	15	10
Plantain Stem	5 10 15	15	10
Banana Fruit Stem	5	15	10

3.5. Calculations Involved

Digestion Stage:

Weight of the Banana stem after chopping = 685.76g

Weight of Banana stem after pounding & squeezing = 208.38g

Liquor to goods ratio (LR) = 5:1

$\text{Total volume of bath} = \text{weight of goods} \times \text{ liquor ratio}$

= 208.38 x 5 = 1042 ml

Mass of sodium hydroxide (NaOH) = 5% × Weight of Sample

$= \frac{5}{100}\, x\ 208.38 = 10.42\text{g}$

The volume of water for digestion = Total volume of bath =1042 ml

Impregnating Temperature = 60 ^{0}C

Impregnating Duration = 60 minutes

Cooking Temperature = 100 ^{0}C

Cooking Duration = 180 minutes

Mass of pulp obtained after digestion = 76.47g

pH of pulp = 11.05

Bleaching stage:

$\text{Volume of Hydrogen Peroxide } (H_2O_2) = 15\% \frac{w}{v} \text{ of weight of pulp} \times \text{ liquor ratio}$

=15/100 × 76.47 × 5 = 57.35 ml

$\text{Total Bath} = \text{Mass of pulp} \times \text{ L. R (goods to liquor ratio)}$

= 76.47 × 5 = 382 ml

$\text{Volume of Water for bleaching} = \text{Total bath} - \text{Volume of Hydrogen Peroxide}$

= 382ml – 57.35 ml = 324.65 ml

Duration of bleaching = 30 minutes

Mass of pulp obtained after bleaching = 68.80g

Paper Making Stage:

Mass of pulp for paper making = 9.16g

Mass of Calcium Carbonate (10%w/w $CaCO_3$) added = 10% of mass of pulp

= 10/100 × 9.16 = 0.916g

Mass of binder added = 5 % mass of pulp

= 5/100 × 9.16 = 0.458g

$\text{Volume of water added to bleach sample to form paper slurry} = 5 \times \text{Mass of pulp}$

= 5 × 9.16 = 46 ml

Condition of Drying = Normal Atmospheric Condition

Duration of Drying = 2 hours

3.6. Mechanical Properties

Mechanical tests such as tearing resistance, tensile strength, and elongation were carried out on the produced paper using an Instron Universal Testing Machine with Model Number 3369 and its tenacity. The test procedure started in load control until the sample was loaded with a force of 0.25kN. Then, the procedure was carried out entirely under displacement control through the vertical LVDT at a rate of 3μm/s.

4. Results and Discussions

4.1. The Production of Pulp and Paper

Figures 1-3 show the pulp and paper produced by different mixing proportions of raw materials. These were made from chemical/mechanical pulp of freshly harvested banana, plantain, and banana fruit stems pulped with 5, 10, and 15% of sodium hydroxide (NaOH), 10% of calcium carbonate (CaC03), 5% of binder, 15% of Hydrogen peroxide (H_2O_2), the required amount of chemical solution of liquor to goods ratio (LR) of 5:1 while the mass of pulp was 9.16g. Generally, the samples pulped with 5% sodium hydroxide (NaOH) showed lighter and smoother surfaces, while samples pulped with 15% NaOH were darker with rough surfaces. The brightness of the papers produced from plantain, banana, and banana fruit stems decreased steadily in that order [6].

(a) (b) (c)

Figure 1. Banana stems; (a) 5% of NaOH, (b) 10% of NaOH, and (c) 15% of NaOH

(a) (b) (c)

Figure 2. Plantain stems; (a) 5% of NaOH, (b) 10% of NaOH, and (c) 15% of NaOH

(a) (b) (c)

Figure 3. Banana fruit stems; (a) 5% of NaOH, (b) 10% of NaOH, and (c) 15% of NaOH

4.2. *Musa Species* Pulped with three different percentages of Caustic Soda (NaOH)

The delignification of the chopped raw materials (banana, plantain, and banana fruit stem wastes) was varied according to independent parameters. The independent variables obtained after the operation by mixing 5% of Sodium Hydroxide (NaOH) are displayed in Table 3, while 10% and 15% (NaOH) mixture proportions are shown in Tables 4 and 5, respectively. Each factor was studied with pulp yield, and different types of handmade paper were produced from the pulp. In Table 3, the quantity of pulp (yield) in residue for banana, plantain, and banana fruit stems was 36.70, 35.50, and 38.5%, respectively, while the quantity of pulp (yield) in the stem for banana, plantain, and banana fruit stems were 11.15, 11.22, and 14.19% in that order. Moreover, in Table 4, the quantity of pulp (yield) in residue for banana, plantain, and banana fruit stems was 35.80, 34.12, and 32.25, respectively, while the quantity of pulp (yield) in the stem were 10.86, 10.51, and 11.90% in that order. Table 5 displayed that the quantity of pulp (yield) in residue for banana, plantain, and banana fruit stems was 33.80, 33.11, and 31.20%, while the quantity of pulp (yield) in the stem was 10.32, 33.11, and 31.20 % in the same order [6].

Table 3. *Musa Species* Pulped with 5% charge Caustic Soda (NaOH)

Particulars	**Banana Stem**	**Plantain Stem**	**Banana Fruit stem**
Mass of Stem before Pounding (g)	685.76	708.96	427.09
Mass of Stem after pounding (g)	208.38	224.06	157.40
Quantity of residue in Stem for pulping (%)	30.39	31.60	36.8
Volume of Fluid obtained from Stem (ml)	472	481	264
Mass of Fluid obtained from Stem (g)	474.38	482.36	266.81
Quantity of Fluid in Stem (%)	69.20	68.04	62.47
Mass of Pulp obtained (g)	76.47	79.54	60.60
Mass Rejected (Undigested residue)(g)	4.79	5.26	3.81
Quantity of pulp (yield) in residue (%)	36.70	35.50	38.5
Quantity of pulp (yield) in stem (%)	11.15	11.22	14.19
Quantity rejected (Undigested) in residue (%)	2.30	2.35	2.42
Quantity rejected (Undigested) in Stem (%)	0.70	0.70	0.89
pH of Black Liquor	11.05	11.10	11.13

Table 4. *Musa Species* Pulped with 10% charge Caustic Soda (NaOH)

Particulars	**Banana Stem**	**Plantain Stem**	**Banana Fruit stem**
Mass of Stem before Pounding (g)	650.50	650.50	650.50
Mass of Stem after pounding (g)	197.25	200.35	240.00
Quantity of residue in Stem for pulping (%)	30.32	30.80	46.89
Volume of Fluid obtained from Stem (ml)	445	448	407
Mass of Fluid obtained from Stem (g)	449.15	449.40	409.20
Quantity of Fluid in Stem (%)	69.05	69.09	62.91
Mass of Pulp obtained (g)	70.62	68.36	77.40
Mass Rejected (Undigested residue)(g)	4.14	4.31	3.57
Quantity of pulp (yield) in residue (%)	35.80	34.12	32.25
Quantity of pulp (yield) in stem (%)	10.86	10.51	11.90
Quantity rejected (Undigested) in residue (%)	2.10	2.25	2.38
Quantity rejected (Undigested) in Stem (%)	0.64	0.66	0.88
pH of Black Liquor	12.90	12.92	12.95

Table 5. *Musa Species* Pulped with15% charge Caustic Soda (NaOH)

Particulars	**Banana Stem**	**Plantain Stem**	**Banana Fruit Stem**
Mass of Stem before Pounding (g)	650.50	650.50	650.50
Mass of Stem after pounding (g)	198.50	200.35	240.00
Quantity of residue in Stem for pulping (%)	30.51	30.80	36.92
Volume of Fluid obtained from Stem (ml)	447	448	406
Mass of Fluid obtained from Stem (g)	450.47	448.32	409.00
Quantity of Fluid in Stem (%)	69.25	68.92	62.82
Mass of Pulp obtained (g)	67.10	66.33	76.87
Mass Rejected (Undigested residue)(g)	3.57	4.12	5.48
Quantity of pulp (yield) in residue (%)	33.80	33.11	31.2
Quantity of pulp (yield) in stem (%)	10.32	33.11	31.2
Quantity rejected (Undigested) in residue (%)	1.80	2.06	2.28
Quantity rejected (Undigested) in Stem (%)	0.55	0.63	0.84
pH of Black Liquor	13.12	13.17	13.48

4.3. The Mechanical Properties of Produced Pulp and Paper

The mechanical test results from handmade paper produced from pulp from raw samples (banana, plantain, and banana fruit stem waste) pulped with 5, 10, and 15% NaOH are presented in Table 6. Three mechanical properties were examined using the procedure stipulated by the Technical Association of Pulp and Paper Industry (TAPPI), including tear resistance, tensile strength, and elongation tests. The results were then compared with test results from already-made commercial papers. It was discovered that banana fruit stem waste pulped with 5% NaOH gave the highest tearing resistance, tensile strength, and elongation test (1.90 mN, 24.77 N/m^2, and 5.49 mm) while banana stem waste pulped with 15% NaOH gave the lowest tearing resistance (0.5mN). It was noticed that the mechanical properties of the handmade paper decreased as the concentration of the NaOH increased [14].

The results of the tearing resistance, tensile strength and elongation test obtained using 5% NaOH for banana stem, plantain stem and banana fruit stems were 0.84 mN, 17.82 N/m^2, and 4.90 mm; 1.35 mN, 14.95 N/m, and 2.17 mm; and 1.90 mN, 24.77 N/m^2, and 5.49 mm; while the pulp yields were 36.7, 35.5, and 38.5%, respectively. The results obtained using 10 % NaOH for banana stem, plantain stem, and banana fruit stems were 0.80 mN, 17.30 N/m^2, and 4.85 mm; 1.25 mN, 14.73 N/m^2, and 2.0 mm; and 1.85 mN, 23.60 N/m^2, and 5.35 mm; while the pulp yields were 35.80%, 34.12%, and 32.25 % in that order. The results obtained using 15% NaOH for the banana stem, plantain stems, and banana fruit stems were 0.70 mN, 6.89 N/m^2, and 1.86 mm; 0.79 mN, 8.70 N/m^2, 2.90 mm; and 1.5 mN, 12.62 N/m2, and 3.03 mm while the pulp yields were 33.8, 33.11, and 31.03%, respectively. This showed that banana fruit stems pulped at 5% sodium hydroxide gave better results than banana or plantain stems. The study showed that the pulp is suitable for producing corrugated boards, fiberboards, and cartons [6].

The evaluation of the mechanical test on the handmade paper shows that the banana fruit stem pulped with 5% NaOH gave the highest tearing resistance, tensile strength, and elongation test. The banana stem pulped with 15% NaOH gave the lowest tearing resistance, while the Dried Plantain stem pulped with 10% and 15% NaOH gave the most insufficient tensile strength and elongation test. Comparing the pulp yield (which is a relative return of the pulp compared to the raw chips) obtained from all the raw materials (banana stem, plantain stem, and banana fruit stem) pulped with 5%, 10%, and 15% NaOH, it could be seen that banana fruit stem waste pulped with 5% NaOH gave the highest pulp yield (38.5%). The same banana fruit stem waste gave the lowest pulp yield (30.20%) when pulped with 15% NaOH. It was noticed that the pulp yield decreased as the percentages of the NaOH used increased [5].

The result of the mechanical test carried out on the paper produced from *Musa Species*, when compared with that obtained from commercial papers (Universal Extra white 80 grams bond, Newsprint, and File cover), shows that paper produced from banana fruit stem pulped with 5% NaOH gave a higher tearing resistance and elongation test. As a result, paper made from this *Musa Species* could be suitable for wrapping, packaging, and offset printing purposes [6].

Table 6. Mechanical properties of produced paper

Samples	Pulped up with Chemical	Elongation (mm)	Tearing Resistance (mN)	Tensile strength (N/m^2)
Banana stem	5% (NaOH), 10% ($CaCO_3$), 5% Binder,15% (H_2O_2)	0.84	17.82	4.90
Plantain stem		1.35	14.95	2.17
Banana fruit stem		1.90	24.77	5.49
Banana stem	10% (NaOH), 10%($CaCO_3$), 5%, Binder,15% (H_2O_2)	0.80	17.30	4.85
Plantain stem		1.25	14.73	2.00
Banana fruit stem		1.85	23.60	5.35
Banana stem	15%(NaOH), 15% ($CaCO_3$), 5% Binder,15% (H_2O_2)	0.50	8.90	2.21
Plantain stem		0.59	4.14	1.90
Banana fruit stem		0.67	13.84	3.89

The graphical representation of the pulp yield of samples is shown in Figure 2. Comparing the pulp yield (which is a relative return of the pulp compared to the raw chips) obtained from all the raw materials (banana stem, plantain stem, and banana fruit stem) pulped with 5, 10 and 15% of NaOH, it could be seen that banana fruit stem waste pulped with 5% NaOH gave the highest pulp yield (38.5%). The same banana fruit stem waste gave the lowest pulp yield (30.20%) when pulped with 15% NaOH. It could also be observed that the pulp yield decreases as the percentages of the NaOH used increase [14].

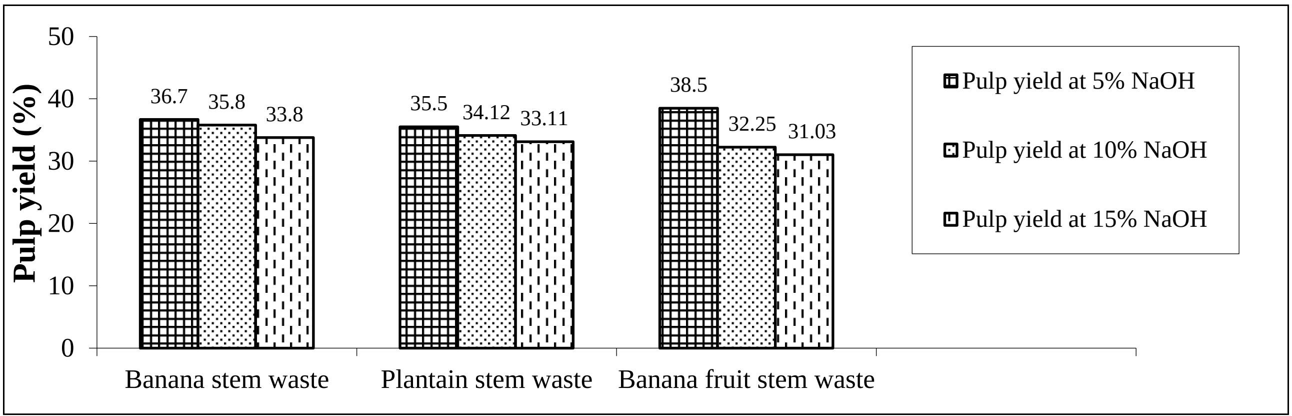

Figure 2. Graphical representation of pulp yield of banana, plantain, and banana fruit stem waste pulped with 5%, 10%, and 15% NaOH

5. Conclusion

The percentage pulp yield from banana stem waste, plantain stem waste, and banana fruit stem waste for pulp yield at 5 %, 10 %, and 15 % NaOH were 36.70, 35.80, and 33.80%; 35.50, 34.12, and 33.11%; and 38.50, 32.25, and 31.03%, respectively which are high enough for industrial pulp and paper making. This study has shown that pulp could be produced from *Musa species* and that the pulp is suitable for the production of corrugated boards, fiberboards, and cartons. It could also be ideal for printing and writing papers when mixed with long fiber pulp.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

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Normal paracontact metric space form on W_0-curvature tensor

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Abstract − In this article, normal paracontact metric space forms are investigated on W_0-curvature tensor. Characterizations of normal paracontact space forms are obtained on W_0-curvature tensor. Special curvature conditions established with the help of Riemann, Ricci, and concircular curvature tensors are discussed on W_0-curvature tensor. Through these curvature conditions, some important characterizations of normal paracontact metric space forms are obtained. Finally, the need for further research is discussed.

Keywords: *W_0-curvature tensors, semisymmetric manifold, normal paracontact space form*

Subject Classification (2020): 53C15, 53C25

1. Introduction

The study of paracontact geometry was initiated by Kenayuki and Williams [1]. Zamkovoy [2] studied paracontact metric manifolds and their subclasses. Recently, Welyczko [3-4] studied curvature and torsion of Frenet Legendre curves in 3-dimensional normal paracontact metric manifolds. In the recent years, contact metric manifolds and their curvature properties have been studied by many authors in [5-7].

In this article, normal paracontact metric space forms are investigated on W_0-curvature tensor. Characterizations of normal paracontact space forms are obtained on W_0-curvature tensor. Special curvature conditions established with the help of Riemann, Ricci, concircular curvature tensors are discussed on W_0-curvature tensor. Through these curvature conditions, some important characterizations of normal paracontact metric space forms are obtained.

2. Preliminaries

Take an n-dimensional differentiable M manifold. If it admits a tensor field ϕ of type $(1,1)$, a contravariant vector field ξ and a 1-form η satisfying the following conditions:

$$\phi^2X = X - \eta(X)\xi, \phi\xi = 0, \eta(\phi X) = 0, \eta(\xi) = 1 \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), g(X, \xi) = \eta(X) \tag{2.2}$$

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for all $X, Y, \xi \in \chi(M)$, (ϕ, ξ, η) is called almost paracontact structure and (M, ϕ, ξ, η) is called almost paracontact metric manifold. If the covariant derivative of ϕ satisfies

$$(\nabla_X \phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{2.3}$$

then, M is called a normal paracontact metric manifold, where ∇ is Levi-Civita connection. From (2.3), we can easily to see that

$$\phi X = \nabla_X \xi \tag{2.4}$$

for any $X \in \chi(M)$ [1].

Moreover, if such a manifold has constant sectional curvature equal to c, then it is the Riemannian curvature tensor is R given by

$$\begin{aligned} R(X,Y)Z = \frac{c+3}{4}[g(Y,Z)X - g(X,Z)Y] + \frac{c-1}{4}[\eta(X)\eta(Z)Y \\ -\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + g(\phi Y, Z)\phi X \\ -g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z] \end{aligned} \tag{2.5}$$

for any vector fields $X, Y, Z \in \chi(M)$ [5].

In a normal paracontact metric space form by direct calculations, we can easily to see that

$$S(X,Y) = \frac{c(n-5) + 3n + 1}{4} g(X,Y) + \frac{(c-1)(5-n)}{4}\eta(X)\eta(Y) \tag{2.6}$$

which implies that

$$QX = \frac{c(n-5) + 4n + 1}{4} X + \frac{(c-1)(5-n)}{4}\eta(X)\xi \tag{2.7}$$

for any $X, Y \in \chi(M)$, where Q is the Ricci operator and S is the Ricci tensor of M.

Lemma 2.1. Let M be an n-dimensional normal paracontact metric manifold. In this case, the following equations hold.

$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X \tag{2.8}$$

$$R(X, \xi)Y = -g(X,Y)\xi + \eta(Y)X \tag{2.9}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.10}$$

$$\eta(R(X,Y)Z) = g(\eta(X)Y - \eta(Y)X, Z) \tag{2.11}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{2.12}$$

$$Q\xi = (n-1)\xi \tag{2.13}$$

where R, S, and Q are Riemann curvature tensor, Ricci curvature tensor, and Ricci operator, respectively.

Tripathi and Gunam [8] described a τ-curvature tensors of the $(1,3)$ type in an n-dimensional (M,g) semi-Riemann manifold. One of these tensors is defined as follows:

Definition 2.1. Let M be an n-dimensional semi-Riemannian manifold. The curvature tensor defined as

$$W_0(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - g(X,Z)QY] \tag{2.14}$$

is called the W_0-curvature tensor.

For the n-dimensional normal paracontact metric space form, if we choose $X=\xi$, $Y=\xi$, and $Z=\xi$, respectively in (2.14), then we get

$$W_0(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - g(X,Z)QY] \tag{2.15}$$

$$W_0(X,\xi)Z = 0 \tag{2.16}$$

$$W_0(X,Y)\xi = \frac{(n-5)(c-1)}{4(n-1)}[\eta(X)Y - \eta(X)\eta(Y)\xi] \tag{2.17}$$

Definition 2.2. Let M be a paracontact manifold. If its Ricci tensor S of type $(0,2)$ is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{2.18}$$

then M is called η-Einstein manifold, where a, b are smooth functions on M. Moreover, if $b=0$, then the manifold is called Einstein.

Definition 2.3. Let (M,g) be a semi-Riemannian manifold and the two-dimensional subspace Π of the tangent space $T_p(M)$. If $K(X_p,Y_p)$ is constant for each $p \in M$ and $X_p, Y_p \in T_p(M)$, then M is called a real space form, where $K(X_p,Y_p)$ is the section curvature of the Π plane.

3. Normal Paracontact Metric Space Forms on W_0-Curvature Tensor

In this section, the characterization of normal paracontact metric space form under special curvature conditions created by W_0-curvature tensor with Riemann, Ricci, concircular curvature tensors will be given. State and prove the following theorems.

Theorem 3.1. Let M be a n-dimensional normal paracontact metric space form. If M is W_0-flat, then M is an Einstein manifold.

Proof.

Assume that manifold M is W_0-flat. From (2.14), we can write

$$W_0(X,Y)Z = 0$$

for each $X,Y,Z \in \chi(M)$. Then, from (2.14), we obtain

$$R(X,Y)Z = \frac{1}{n-1}[S(Y,Z)X - g(X,Z)QY] \tag{3.1}$$

for each $X,Y,Z \in \chi(M)$. If we choose $Z=\xi$ in (3.1) and using (2.10) and (2.12), we obtain

$$\eta(X)QY = (n-1)\eta(X)Y \tag{3.2}$$

If we choose $X = \xi$ in (3.2) and take inner product both sides of the last equation by $Z \in \chi(M)$, then we get

$$S(Y,Z) = (n-1)g(Y,Z)$$

It is clear from the last equation that M is Einstein manifold. □

Theorem 3.2. Let M be the n-dimensional normal paracontact metric space form. If M is W_0-semisymmetric, then M is an Einstein manifold.

Proof.

Assume that M is W_0-semisymmetric. This means

$$(R(X,Y) \cdot W_0)(U,V,Z) = 0$$

for every $X, Y, Z, U, V \in \chi(M)$. Therefore, we can write

$$R(X,Y)W_0(U,V)Z - W_0(R(X,Y)U,V)Z - W_0(U,R(X,Y)V)Z - W_0(U,V)R(X,Y)Z = 0 \tag{3.3}$$

If we choose $X = \xi$ in (3.3) and make use of (2.8), we get

$$\begin{aligned} &g(Y,W_0(U,V)Z)\xi - \eta(W_0(U,V)Z)Y - g(Y,U)W_0(\xi,V)Z \\ &+\eta(U)W_0(Y,V)Z - g(Y,V)W_0(U,\xi)Z + \eta(V)W_0(U,Y)Z \\ &-g(Y,Z)W_0(U,V)\xi + \eta(Z)W_0(U,V)Y = 0 \end{aligned} \tag{3.4}$$

If we use (2.15)-(2.17) in (3.4), we obtain

$$\begin{aligned} &g(Y,W_0(U,V)Z)\xi - \eta(W_0(U,V)Z)Y + Ag(Y,U)g(V,Z)\xi \\ &-Ag(Y,U)\eta(Z)V + \eta(U)W_0(Y,V)Z + \eta(V)W_0(U,Y)Z \\ &-Ag(Y,Z)\eta(U)V + Ag(Y,Z)\eta(U)\eta(V)\xi + \eta(Z)W_0(U,V)Y = 0, \end{aligned} \tag{3.5}$$

where $A = \frac{(n-5)(c-1)}{4(n-1)}$. If we choose $U = \xi$ in (3.5) and use (2.15), we get

$$W_0(Y,V)Z + Ag(V,Z)Y - Ag(Y,Z)V = 0 \tag{3.6}$$

Putting (2.14) in (3.6), we have

$$R(Y,V)Z - \frac{1}{n-1}S(V,Z)Y + \frac{1}{n-1}g(Y,Z)QV + Ag(V,Z)Y - Ag(Y,Z)V = 0 \tag{3.7}$$

If we choose $Z = \xi$ in (3.5) and use (2.10) and (2.12), we get

$$\frac{1}{n-1}\eta(Y)QV + A\eta(V)Y - A\eta(Y)V = 0 \tag{3.8}$$

In (3.8), if we choose $Y = \xi$, and take inner product both sides of the equation by $Z \in \chi(M)$, we then have

$$S(V,Z) = \frac{(n-5)(c-1) + 4(n-1)}{4}g(V,Z) - \frac{(n-5)(c-1)}{4}\eta(V)\eta(Z)$$

□

Theorem 3.3. Let M be the n-dimensional normal paracontact metric space form. If M satisfies the curvature condition $W_0 \cdot R = 0$, then M is a real space form with constant scalar curvature.

Proof.

Assume that

$$(W_0(X,Y) \cdot R)(U,V,Z) = 0$$

for every $X,Y,Z,U,V \in \chi(M)$. Therefore, we can write

$$\begin{aligned} &W_0(X,Y)R(U,V)Z - R(W_0(X,Y)U,V)Z \\ &-R(U,W_0(X,Y)V)Z - R(U,V)W_0(X,Y)Z = 0 \end{aligned} \tag{3.9}$$

If we choose $X = \xi$ in (3.9) and make use of (2.15), we get

$$\begin{aligned} &-Ag(Y,R(U,V)Z)\xi + A\eta(R(U,V)Z)Y + Ag(Y,U)R(\xi,V)Z \\ &-A\eta(U)R(Y,V)Z + Ag(Y,V)R(U,\xi)Z - A\eta(V)R(U,Y)Z \\ &+Ag(Y,Z)R(U,V)\xi - A\eta(Z)R(U,V)Y = 0 \end{aligned} \tag{3.10}$$

If we use (2.8)-(2.10) in (3.10), we obtain

$$\begin{aligned} &-Ag(Y,R(U,V)Z)\xi + A\eta(R(U,V)Z)Y + Ag(Y,U)g(V,Z)\xi \\ &-Ag(Y,U)\eta(Z)V - A\eta(U)R(Y,V)Z - Ag(Y,V)g(U,Z)\xi \\ &+Ag(Y,V)\eta(Z)U - A\eta(V)R(U,Y)Z - A\eta(Z)R(U,V)Y \\ &+Ag(Y,Z)\eta(V)U - Ag(Y,Z)\eta(U)V = 0 \end{aligned} \tag{3.11}$$

If we choose $U = \xi$ in (3.11) and use (2.8), we get

$$-A[R(Y,V)Z - g(V,Z)Y + g(Y,Z)V] = 0 \tag{3.12}$$

□

Theorem 3.4. Let M be the n-dimensional normal paracontact metric space form. If M satisfies the curvature condition $W_0 \cdot W_0 = 0$, then M is an η-Einstein manifold.

Proof.

Assume that

$$(W_0(X,Y) \cdot W_0)(U,V,Z) = 0$$

for every $X,Y,Z,U,V \in \chi(M)$. Therefore, we can write

$$W_0(X,Y)W_0(U,V)Z - W_0(W_0(X,Y)U,V)Z - W_0(U,W_0(X,Y)V)Z - W_0(U,V)W_0(X,Y)Z = 0 \tag{3.13}$$

If we choose $X = \xi$ in (3.13) and make use of (2.15), we get

$$\begin{aligned}&-Ag(Y,W_0(U,V)Z)\xi + A\eta(W_0(U,V)Z)Y + Ag(Y,U)W_0(\xi,V)Z\\&-A\eta(U)W_0(Y,V)Z + Ag(Y,V)W_0(U,\xi)Z - A\eta(V)W_0(U,Y)Z\\&+Ag(Y,Z)W_0(U,V)\xi - A\eta(Z)W_0(U,V)Y = 0\end{aligned} \tag{3.14}$$

If we use (2.15)-(2.17) in (3.14), we obtain

$$\begin{aligned}&-Ag(Y,W_0(U,V)Z)\xi + A\eta(W_0(U,V)Z)Y - A^2g(Y,U)g(V,Z)\xi\\&+A^2g(Y,U)\eta(Z)V - A\eta(U)W_0(Y,V)Z - A\eta(V)W_0(U,Y)Z\\&+A^2g(Y,Z)\eta(U)V - A^2g(Y,Z)\eta(U)\eta(V)\xi - A\eta(Z)W_0(U,V)Y = 0\end{aligned} \tag{3.15}$$

If we choose $U = \xi$ in (3.15) and make the necessary adjustments using (2.15), we get

$$-A\{W_0(Y,V)Z + A[g(V,Z)Y - g(Y,Z)V]\} = 0 \tag{3.16}$$

Putting (2.14) in (3.16) and if we choose $Z = \xi$, we obtain

$$-A\left[A\eta(V)Y - (A+1)\eta(Y)V + \frac{1}{n-1}\eta(Y)QV\right] = 0 \tag{3.17}$$

If we choose $Y = \xi$ in (3.17), then we take inner product both sides of the equation by $Z \in \chi(M)$, we have

$$S(V,Z) = \frac{(n-5)(c-1) + 4(n-1)}{4}g(V,Z) - \frac{(n-5)(c-1)}{4}\eta(V)\eta(Z)$$

□

Corollary 3.1. Let M be the n-dimensional normal paracontact metric space form. If M satisfies the curvature condition $W_0 \cdot W_0 = 0$, then M is an Einstein manifold if and only if M is a real space form with constant scalar curvature $c = 1$.

Definition 3.1. Let M be an n-dimensional Riemannian manifold. The curvature tensor defined as

$$\tilde{Z}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{3.18}$$

is called the concircular curvature tensor.

For the n-dimensional normal paracontact metric space form, if we choose $X = \xi, Y = \xi,$ and $Z = \xi$ in (3.18), respectively, then we get

$$\tilde{Z}(\xi,Y)Z = \left[1 - \frac{r}{n(n-1)}\right][g(Y,Z)\xi - \eta(Z)Y] \tag{3.19}$$

$$\tilde{Z}(X,\xi)Z = \left[1 - \frac{r}{n(n-1)}\right][-g(X,Z)\xi + \eta(Z)Y] \tag{3.20}$$

$$\tilde{Z}(X,Y)\xi = \left[1 - \frac{r}{n(n-1)}\right][\eta(Y)X - \eta(X)Y] \tag{3.21}$$

Theorem 3.5. Let M be the n-dimensional normal paracontact metric space form. If M satisfies the curvature condition $W_0 \cdot \tilde{Z} = 0$, then M is a real space form with constant scalar curvature.

Proof.

Assume that

$$\left(W_0(X,Y) \cdot \tilde{Z}\right)(U,V,Z) = 0$$

for every $X,Y,Z,U,V \in \chi(M)$. Therefore, we can write

$$W_0(X,Y)\tilde{Z}(U,V)Z - \tilde{Z}(W_0(X,Y)U,V)Z - \tilde{Z}(U,W_0(X,Y)V)Z - \tilde{Z}(U,V)W_0(X,Y)Z = 0 \tag{3.22}$$

If we choose $X = \xi$ in (3.22) and make use of (2.15), we get

$$\begin{aligned} &-Ag\left(Y,\tilde{Z}(U,V)Z\right)\xi + A\eta\left(\tilde{Z}(U,V)Z\right)Y + Ag(Y,U)\tilde{Z}(\xi,V)Z \\ &-A\eta(U)\tilde{Z}(Y,V)Z + Ag(Y,V)\tilde{Z}(U,\xi)Z - A\eta(V)\tilde{Z}(U,Y)Z \\ &+Ag(Y,Z)\tilde{Z}(U,V)\xi - A\eta(Z)\tilde{Z}(U,V)Y = 0 \end{aligned} \tag{3.23}$$

If we use (3.19)-(3.21) in (3.23), we obtain

$$\begin{aligned} &-Ag\left(Y,\tilde{Z}(U,V)Z\right)\xi + A\eta\left(\tilde{Z}(U,V)Z\right)Y + ABg(Y,U)\eta g(V,Z)\xi \\ &-ABg(Y,U)\eta(Z)V - A\eta(U)\tilde{Z}(Y,V)Z - ABg(Y,V)g(U,Z)\xi \\ &+ABg(Y,V)\eta(Z)U - A\eta(V)\tilde{Z}(U,Y)Z + ABg(Y,Z)\eta(V)U \\ &-ABg(Y,Z)\eta(U)V - A\eta(Z)\tilde{Z}(U,V)Y = 0 \end{aligned} \tag{3.24}$$

where $B = \left[1 - \frac{r}{n(n-1)}\right]$. If we choose $U = \xi$ in (3.24) and make the necessary adjustments using (3.19), we get

$$-A\{\tilde{Z}(Y,V)Z + B[g(Y,Z)V - g(V,Z)Y]\} = 0 \tag{3.25}$$

If we substitute the (3.18) in (3.25) and we make the necessary arrangements, we obtain

$$-A[R(Y,V)Z - g(V,Z)Y + g(Y,Z)V] = 0$$

□

Theorem 3.6. Let M be the n-dimensional normal paracontact metric space form. If M satisfies the curvature condition $W_0 \cdot S = 0$, then M is an Einstein manifold.

Proof.

Assume that

$$(W_0(X,Y) \cdot S)(U,V) = 0$$

for every $X,Y,U,V \in \chi(M)$. Therefore, we can write

$$S(W_0(X,Y)U,V) + S(U,W_0(X,Y)V) = 0 \tag{3.26}$$

If we choose $X = \xi$ in (3.26) and make use of (2.15), we get

$$\begin{aligned} &-A(n-1)g(Y,U)\eta(V) + A\eta(U)S(Y,V) \\ &-A(n-1)g(Y,V)\eta(U) + A\eta(V)S(U,Y) = 0 \end{aligned} \tag{3.27}$$

If we choose $U = \xi$ in (3.27), we have

$$\frac{(n-5)(c-1)}{4(n-1)}[S(Y,V) - (n-1)g(Y,V)] = 0$$

□

4. Conclusion

In this article, normal paracontact metric space forms are investigated on W_0-curvature tensor. Characterizations of normal paracontact space forms are obtained on W_0-curvature tensor. Special curvature conditions established with the help of Riemann, Ricci, concircular curvature tensors are discussed on W_0-curvature tensor. Through these curvature conditions, important characterizations of normal paracontact metric space forms are obtained.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

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Contact pseudo-slant submanifolds of a para-Sasakian manifold according to type 1, type 2, type 3 cases

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Abstract — This paper aims to present work on contact pseudo-slant submanifolds of para-Sasakian manifolds. The study includes the definitions and some results on type 1, type 2, and type 3 contact pseudo-slant submanifolds. The results were interpreted by taking into account the parallelism and geodesicity of the tensors reduced to the submanifold. Additionally, minimal anti-invariant and invariant submanifolds were evaluated for the type 1, type 2, and type 3 cases of the tensors reduced to the submanifold.

Keywords: *Contact pseudo-slant submanifold, para-Sasakian manifold, geodesic submanifold*

Subject Classification (2020): 53C15, 53C25

1. Introduction

The differential geometry of slant submanifolds has shown an increasing development since Chen [1] defined slant submanifolds in complex manifolds as a natural generalization of both invariant and anti-invariant submanifolds. Since then, many research articles have appeared on the existence of these submanifolds in different known spaces. The slant submanifolds of an almost contact metric manifolds were defined and studied by Lotta [2]. After, these submanifolds were studied by Cabrerizo et al. [3] in the setting of Sasakian manifolds.

The notion of semi-slant submanifolds of an almost Hermitian manifold was introduced by Papagiuc [4]. Hemi-slant submanifolds were first introduced by Carrizo [5], and he called them pseudo-slant submanifolds. Recently, there have been many studies conducted on this subject [6-9]. Finally, Chanyal [10] has studied slant submanifolds on an almost paracontact metric manifold.

In this paper, we study pseudo-slant submanifolds of a para-Sasakian (p-Sasakian) manifold. In Section 2, we review basic formulas and definitions for a p-Sasakian manifold and its submanifolds, which will be used later. In Section 3, we recall the definition and some basic results of a contact pseudo-slant submanifold of almost paracontact metric manifold. We obtain some results for these submanifolds in the setting of a p-Sasakian manifolds. We also research the geodetic states of the distributions.

2. Preliminaries

Let $\widetilde{M}$ be an n-dimensional contact manifold with contact form η, i.e., $\eta \wedge d\eta \neq 0$. It is well known that a contact manifold admits a vector field ξ called the characteristic vector field, shuch that $\eta(\xi) = 1$ and

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$d\eta(X,\xi) = 0$, for every $X \in \Gamma(T\widetilde{M})$. Furthermore, $\widetilde{M}$ admits a Rieman metric g and a vector field ϕ of type (1,1) shuch that

$$\phi^2 X = X - \eta(X)\xi, \eta(X) = g(X,\xi), g(X,\phi Y) = d\eta(X,Y) \tag{2.1}$$

We then say that (ϕ,ξ,η,g) is a contact metric structure. A contact metric is said to be a Sasakian if

$$(\widetilde{\nabla}_X\phi)Y = g(X,Y)\xi - \eta(Y)X \tag{2.2}$$

in which case

$$\widetilde{\nabla}_X\xi = \phi X, \hat{\mathrm{R}}(\mathrm{X},\mathrm{Y})\xi = \eta(Y)X - \eta(X)Y \tag{2.3}$$

We provide a structure similar to Sasakian but not having contact.

An n-dimensional diferentiable manifold is said to admit an almost paracontact Rieman structure (ϕ,ξ,η,g), where ϕ of type (1,1) tensor field ξ is a vector field, η ia a 1-form and g is a Rieman metric on $\widetilde{M}$ such that

$$\phi\xi = 0, \eta(\phi X) = 0, \eta(\xi) = 1, \eta(X) = g(X,\xi) \tag{2.4}$$

$$\phi^2 X = X - \eta(X)\xi, g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.5}$$

for any vector fields X, Y on $\widetilde{M}$. The equation $\eta(\xi) = 1$ is equivalent to $|\eta| \equiv 1$, and then ξ is just the metric dual of η. If (ϕ,ξ,η,g) satisfy the equations

$$d\eta = 0, \widetilde{\nabla}_X\xi = \phi X \tag{2.6}$$

$$\left(\widetilde{\nabla}_X\phi\right)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(Y)\eta(Y) \tag{2.7}$$

then $\widetilde{M}$ is called a p-Sasakian manifold or briefly, a p-Sasakian, especially, a p-Sasakian manifold $\widetilde{M}$ is called a special p-Sasakian manifold or briefly a sp-Sasakian manifold if $\widetilde{M}$ admits a 1-form η satisfiying

$$\left(\widetilde{\nabla}_X\eta\right)Y = -g(X,Y) + \eta(Y)\eta(X)$$

where $\widetilde{\nabla}$ is the Levi-Civita connections of g.

Let M denotes an immersed submanifold of a p-Sasakian manifold $\widetilde{M}$. Considering the non degenerate metric induced on M by the same symbol g as on $\widetilde{M}$. Further, the Gauss and Weingarten formulas are respectively given as,

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \tag{2.8}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla_X^{\perp} V \tag{2.9}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$ where,

i. $X, Y \in \Gamma(TM)$ (tangent bundle) and $V \in \Gamma(T^{\perp}M)$ (normal bundle),

ii. Induced Levi-Civita connection ∇ on M,

iii. Normal connection $\nabla^{\perp}$ on $\Gamma(T^{\perp}M)$

iv. Second fundamental form σ on M,

v. Shape operator A_V associated with the normal section V.

Moreover, the second fundamental form σ and shape operator A_V are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V) \tag{2.10}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

The mean curvature vector H of M is given by

$$H = \frac{1}{m}\sum_{i=1}^{n} \sigma(e_i, e_i) \tag{2.11}$$

where m is the dimension of M and $\{e_1, e_2, \ldots, e_m\}$ is a local orthonormal frame of M.

A submanifold M of an paracontact metric manifold $\widetilde{M}$ is said to be totally umbilical if

$$\sigma(X, Y) = g(X, Y)H \tag{2.12}$$

where H is the mean curvature vector. A submanifold M is said to be totally geodesic if $\sigma(X, Y) = 0$, for each $X, Y \in \Gamma(TM)$ and M is said to be minimal if $H = 0$.

Let M be a submanifold of an almost paracontact metric manifold $\widetilde{M}$. Then, for any $X \in \Gamma(TM)$, we can write

$$\phi X = TX + NX \tag{2.13}$$

where TX is the tangential component and NX is the normal component of ϕX.

Similary, for $V \in \Gamma(T^{\perp}M)$, we can write

$$\phi V = tV + nV \tag{2.14}$$

where tV is the tangential component and nV is the normal component of ϕV.

Furthermore, for any $X, Y \in \Gamma(TM)$, we have $g(TX, Y) = -g(X, TY)$, $g(NX, Y) = -g(X, NY)$, and $V, U \in \Gamma(T^{\perp}M)$, we get $g(U, nV) = -g(nU, V)$. These relations show that N and n are also skew-symmetric tensor fields. Moreover, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we have $g(NX, V) = -g(X, tV)$, which gives the relation between N and t.

Thus, by using (2.1), (2.13), and (2.14), we obtain

$$T^2 = I - \eta \otimes \xi - tN, NT + nN = 0 \tag{2.15}$$

and

$$Tt + tn = 0, NT + n^2 = I \tag{2.16}$$

where the covariant derivatives of the tensor field T, N, t, and n are, respectively, defined by

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y \tag{2.17}$$

$$(\nabla_X N)Y = \nabla_X^{\perp} NY - N\nabla_X Y \tag{2.18}$$

$$(\nabla_X t)V = \nabla_X tV - t\nabla_X^{\perp} V \tag{2.19}$$

and

$$(\nabla_X n)V = \nabla_X^{\perp} nV - n\nabla_X^{\perp} V \tag{2.20}$$

for any $X, Y \in \Gamma(TM)$ and for any $V \in \Gamma(T^{\perp}M)$.

By direct calculations, we obtain the following formulas

$$(\nabla_X T)Y = A_{NY}X + t\sigma(X,Y) + g(X,Y)\xi - \eta(Y)X \tag{2.21}$$

and

$$(\nabla_X N)Y = n\sigma(X,Y) - \sigma(X,TY) \tag{2.22}$$

for any $X, Y \in \Gamma(TM)$,

Similary, we obtain

$$(\nabla_X t)V = A_{nV}X - TA_V X \tag{2.23}$$

and

$$(\nabla_X n)V = -NA_V X - \sigma(tV, X) \tag{2.24}$$

for any $X \in \Gamma(TM)$ and for any $V \in \Gamma(T^{\perp}M)$.

Lemma 2.1. If M is an immersed submanifold of a p-Sasakian manifold $\widetilde{M}$ with $\xi \in \Gamma(TM)$, then

$$\nabla_X \xi = TX \tag{2.25}$$

and

$$\sigma(X,\xi) = NX \tag{2.26}$$

$$A_V \xi = -tV \tag{2.27}$$

$X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Proof.

In (2.8), if $Y = \xi$ is written, we have

$$\widetilde{\nabla}_X \xi = \nabla_X \xi + \sigma(X,\xi)$$

Using (2.13), the tangential and normal parts of the last equation give, respectively, us

$$\nabla_X \xi = TX$$

and

$$\sigma(X,\xi) = NX$$

Besides, in (2.10), if $Y = \xi$ is written and from (2.26), we have

$$g(A_V X,\xi) = g(NX,V) = -g(X,\mathrm{t}V)$$ □

Definition 2.2. [2] Let M be a submanifold of a p-Sasakian manifold $\widetilde{M}$. For each non-zero vector X tangent to M at x, the angle $\theta(x)$, $\theta(x) \in \left[0,\frac{\pi}{2}\right]$, between ϕX and $T_x M$ is called the slant angle or the Wirtinger angle of M. If the slant angle is constant, then the submanifold is also called the slant submanifold. If $\theta = 0$ the submanifold is invariant submanifold. If $\theta = \frac{\pi}{2}$, then it is called anti-invariant submanifold. If $\theta(x) \in \left(0, \frac{\pi}{2}\right)$, then it is called proper-slant submanifold.

We prove the following characterization theorem for slant submanifold.

Theorem 2.3. [10] Let M be a slant submanifold of an almost paracontact metric manifold $(\widetilde{M}, \phi, \xi, \eta, g)$ such that $\xi \in TM$. Then,

i. M is slant of type 1 if and only if for any time like (space-like) vector field $X \in \chi(M) - \langle\xi\rangle$, TX is time like (space-like), and there exists a constant $\lambda \in (1, +\infty)$ such that

$$T^2 = \lambda(I - \eta \otimes \xi)$$

We write $\lambda = \cosh^2\theta$, with $\theta\rangle 0$.

ii. M is slant of type 2 if and only if for any time like (space-like) vector field $X \in \chi(M) - \langle\xi\rangle$, TX is time like (space-like), and there exists a constant $\lambda \in (0,1)$ such that

$$T^2 = \lambda(I - \eta \otimes \xi)$$

We write $\lambda = \cos^2\theta$, with $\theta \in \left(0, \frac{\Pi}{2}\right)$.

iii. M is slant of type 3 if and only if for any time like (space-like) vector field $X \in \chi(M) - \langle\xi\rangle$, TX is time like (space-like), and there exists a constant $\lambda \in (-\infty, 0)$ such that

$$T^2 = \lambda(I - \eta \otimes \xi)$$

We write $\lambda = -\sinh^2\theta$, with $\theta\rangle 0$. In each case θ is called the slant angle.

Corollary 2.4. [10] Let M be a slant submanifold of an almost paracontact metric manifold $(\widetilde{M}, \phi, \xi, \eta, g)$ with slant angle θ. Then, for any $X, Y \in \Gamma(TM)$, we have

If M is of type 1, then

$$g(TX, TY) = -\cosh^2\theta\{g(X, Y) - \eta(X)\eta(Y)\} \tag{2.28}$$

and

$$g(NX, NY) = \sinh^2\theta\{g(X, Y) - \eta(X)\eta(Y)\} \tag{2.29}$$

If M is of type 2, then

$$g(TX, TY) = -\cos^2\theta\{g(X, Y) - \eta(X)\eta(Y)\} \tag{2.30}$$

and

$$g(NX, NY) = -\sin^2\theta\{g(X, Y) - \eta(X)\eta(Y)\} \tag{2.31}$$

If M is of type 3, then

$$g(TX, TY) = \sinh^2\theta\{g(X, Y) - \eta(X)\eta(Y)\} \tag{2.32}$$

and

$$g(NX, NY) = -\cosh^2\theta\{g(X, Y) - \eta(X)\eta(Y)\} \tag{2.33}$$

Proof.

From the anti-symetry of T and Theorem 2.3, we have

$$g(TX,TY) = -g(T^2X,Y) = -\lambda\{g(X,Y) - \eta(X)\eta(Y)\} = \lambda\{g(\phi X, \phi Y)\}$$

(2.13) yields

$$g(\phi X, \phi Y) = g(TX,TY) + g(NX,NY)$$

from last two equations, we obtain

$$g(NX,NY) = (1-\lambda)g(\phi X, \phi Y)$$

Hence, the corallary follows from the values of λ in the Theorem 2.3. □

3. Contact Pseudo-Slant Submanifolds of a Para-Sasakian Manifold

Definition 3.1. [11] We say that M is a contact pseudo-slant submanifold of an almost paracontact metric manifold $\widetilde{M}$ if there exist two orthogonal distributions D_θ and $D^\perp$ on M such that

i. TM admits the orthogonal direct decomposition $TM = D^\perp \oplus D_\theta$, $\xi \in \Gamma(D_\theta)$,

ii. The distribution $D^\perp$ is anti-invariant (totally-real), i.e., $\phi D^\perp \subset (T^\perp M)$,

iii. The distribution D_θ is a slant with slant angle $\theta \neq \frac{\pi}{2}$, that is, the angle between D_θ and $\phi(D_\theta)$ is a constant.

From the definition, it is clear that if $\theta = 0$, then the contac pseudo-slant submanifold is a semi-invariant submanifold, $\theta = \frac{\pi}{2}$ submaifold becomes an anti-invariant.

We suppose that M is a contact pseudo-slant submanifold of an almost paracontact metric manifold $\widetilde{M}$.

Furthermore, let $d_1 = \dim(D^\perp)$ and $d_2 = \dim(D_\theta)$. We distinguish the following six cases.

i. If $d_2 = 0$, then M is an anti-invariant submanifold.

ii. If $d_1 = 0$ and $\theta = 0$, then M is invariant submanifold.

iii. If $d_1 = 0$ and $\theta \in \left(0, \frac{\pi}{2}\right)$, then M is a proper slant submanifold.

iv. If $\theta = \frac{\pi}{2}$ then, M is an anti-invariant submanifold.

v. If $d_2 d_1 \neq 0$ and $\theta = 0$, then M is a semi-invariant submanifold.

vi. If $d_2 d_1 \neq 0$ and $\theta \in \left(0, \frac{\pi}{2}\right)$, then M is a contact pseudo-slant submanifold.

If we denote the orthogonal complementary of φTM in $T^\perp M$ by μ, then the normal bundle $T^\perp M$ can be decomposed as follows:

$$T^\perp M = N(D^\perp) \oplus N(D_\theta) \oplus \mu \tag{3.1}$$

Theorem 3.2. The necessary ond sufficient condition for submanifold M of a p-Sasakian manifold $\widetilde{M}$ to be a contact pseudo-slant submanifold is that $\exists$ a distribution D on M and a constant $\lambda \in (-\infty, +\infty)$ satisfiying

i. $D = \{X \in \Gamma(TM): T^2X = -\lambda X\}$

ii. $TX = 0$, for tangent vectorfield X orthogonal to D

Further, λ can be $\cosh^2\theta$, $\cos^2\theta$, or $-\sinh^2\theta$ [10].

Proof.

From Theorem 2.3 (*i*-*iii*), the proof of the theorem is obvious. □

Definition 3.3. A contact pseudo-slant submanifold M of p-Sasakian manifold $\widetilde{M}$ is said to be D_θ-geodesic (resp. $D^\perp$-geodesic) if $\sigma(X,Y)=0$, for $X,Y\in\Gamma(D_\theta)$ (resp. $\sigma(Z,W)=0$, for $Z,W\in\Gamma(D^\perp)$. If for any $X\in\Gamma(D_\theta)$ and $Z\in\Gamma(D^\perp)$, $\sigma(X,Z)=0$, the M is called mixed geodesic submanifold.

Theorem 3.4. Let M be a proper contact pseudo-slant submanifold of a p-Sasakian manifold $\widetilde{M}$. If t is parallel, then

i. For type 2, M is anti-invariant submanifold.

ii. For type 3, M is invariant submanifold.

iii. M is a mixed-geodesic submanifold.

Proof.

Consider (2.22) and (2.23) which gives the relation between t and N. If t is parallel, then N is parallel, we obtain

$$n\sigma(X,Y)=0$$

for any $X\in\Gamma(D_\theta)$ and $Y\in\Gamma(D^\perp)$. Replacing X by Y in (2.22) and taking into account to N being parallel, we have

$$n\sigma(Y,TX)-\sigma(Y,T^2X)=\cos^2\theta\,\sigma(X,Y)=\sinh^2\theta\,\sigma(X,Y)=0$$

From type 2, we write

$$\cos^2\theta\,\sigma(X,Y)=0\ (\theta=\tfrac{\pi}{2}\ M \text{ is anti-invariant})$$

From type 3, we write

$$\sinh^2\theta\,\sigma(X,Y)=0$$

Thus,

$$2\sinh\theta=e^{\theta}-e^{-\theta}=0\ (\theta=0\ M \text{ is invariant})$$

Besides, for any $X\in\Gamma(D_\theta)$ and $Y\in\Gamma(D^\perp)$, $\sigma(X,Y)=0$, M is a mixed geodesic submanifold. This proves our assertion. □

Theorem 3.5. Let M be a proper contact pseudo-slant submanifold of a p-Sasakian manifold $\widetilde{M}$. If N is parallel, then either M is a $D^\perp$-geodesic or an anti-invariant submanifold of $\widetilde{M}$.

Proof.

Consider (2.22) and (2.23) which gives the relation between t and N. If t is parallel, then N is parallel, we obtain

$$TA_{NY}Z=0$$

for any $Y,Z\in\Gamma(D^\perp)$. This implies that M is either anti-invariant or $A_{NY}Z=0$. Therefore, we obtain

$$g(\sigma(Z,W),NY)=0$$

for any $Y,Z,W\in\Gamma(D^\perp)$. Moreover, by using (2.23), we conclude that

$$g(A_{nV}Z,Y)-g(TA_VZ,Y)=g(\sigma(Y,Z),nV)=0$$

for any $V \in \Gamma(T^{\perp}M)$. This tells us that M is either $D^{\perp}$-geodesic or it is an anti-invariant submanifold. □

Theorem 3.6. Let M be a contact pseudo-slant submanifold of a p-Sasakian manifold $\widetilde{M}$. If N is parallel on D_θ, then either M is a D_θ-geodesic submanifold or $\sigma(X,Y)$ is an eigenvector of n^2 with eigenvalues are $\cosh^2\theta$, $\cos^2\theta$, or $-\sinh^2\theta$, for type 1, type 2, and type 3, respectively.

Proof.

For all $Y, Z \in \Gamma(D_\theta)$. From (2.22), we have

$$n\sigma(Z,Y) - \sigma(Z,TY) = 0 \tag{3.2}$$

Besides, since D_θ is slant distribution, we get

$$n\sigma(Z, Y - \eta(Y)\xi) - \sigma(Z, T(Y - \eta(Y)\xi)) = 0$$

From (2.4) and (2.13), we get

$$n\sigma(Z, Y - \eta(Y)\xi) - \sigma(Z,TY) = 0 \tag{3.3}$$

Applying n to (3.3), we have

$$n^2\sigma(Z, Y - \eta(Y)\xi) - n\sigma(Z,TY) = 0$$

Moreover, by interchanging of Y and TY in (3.2), we have

$$n\sigma(Z,TY) - \sigma(Z,T^2Y) = 0$$

Hence, using Theorem 2.3, we obtain

$$\begin{aligned} n^2\sigma(Z, Y - \eta(Y)\xi) &= n\sigma(Z,TY) \\ &= \sigma(Z,T^2Y) \\ &= \cosh^2\theta\, \sigma(Z, Y - \eta(Y)\xi) \\ &= \cos^2\theta\, \sigma(Z, Y - \eta(Y)\xi) \\ &= -\sinh^2\theta\, \sigma(Z, Y - \eta(Y)\xi) \end{aligned}$$

This implies that either $\sigma = 0$ on D_θ or σ in an eigenvector of n^2 with eigenvalues $\cosh^2\theta$, $\cos^2\theta$, or $-\sinh^2\theta$.□

Theorem 3.7. Let M be a totally umbilical proper contact pseudo-slant submanifold of a p-Sasakian manifold $\widetilde{M}$. If t is parallel, then either M is a minimal or an anti-invariant and invariant submanifold of $\widetilde{M}$.

Proof.

For all $Y \in \Gamma(D^{\perp})$ and $X \in \Gamma(D_\theta)$. Consider (2.22) and (2.23) which gives the relation between t and N. If t is parallel then N is parallel, we obtain

$$n\sigma(X,Y) - \sigma(X,TY) = 0$$

Replacing X by TX in above equation, we get

$$n\sigma(TX,Y) - \sigma(TX,TY) = 0$$

For $Y \in \Gamma(D^{\perp})$, $TY = 0$. Thus,

$$n\sigma(TX,Y) = 0$$

Since M is totally umbilical, from (2.12), we have

$$ng(TX,Y)H = 0$$

Replacing X by TX in above equation and from Theorem 2.3, we obtain

$$\begin{aligned} ng(T^2X,Y)H &= -ng(TX,TY)H \\ &= -n\cosh^2\theta\, g(X,Y)H \\ &= -n\cos^2\theta\, g(X,Y)H \\ &= n\sinh^2\theta\, g(X,Y)H \\ &= 0 \end{aligned}$$

Hence, from type 2 and type 3, we have either $\theta = \frac{\pi}{2}$ (M is anti invariant), $\theta = 0$ (M is invariant), or $H = 0$ (M is minimal). □

4. Conclusion

In this article, interesting results have been obtained regarding the contact pseudo-slant submanifolds of para-Sasakian manifolds, taking into account the geodesic and parallelism situations of the tensors. These situations can be investigated on other contact metric manifolds.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

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Coherent hybrid block method for approximating fourth-order ordinary differential equations

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Abstract — Conventionally, the most used method of solving fourth-order initial value problems of ordinary differential is to first reduce to a system of first-order differential equations. This approach affects the effectiveness and convergence of the numerical method due to the transformation. This paper comprises the derivation, analysis, and implementation of a new hybrid block method for direct solution of fourth-order equations. The method is derived by collocation and interpolation of an assumed basis function. The basic properties of the block method, including zero stability, error constants, consistency, order, and convergence, were analyzed. From the analysis, the block method derived was found to be zero-stable, consistent, and convergent. Errors were computed for the proposed method, and they were proven to produce approximations that agree with exact solutions and as such this shows improvement with those of existing works.

Keywords: *Hybrid methods, block method, linear stability, errors*

Subject Classification (2020): 65L05, 65L70

1. Introduction

Differential Equations are among the essential tools used in producing models in engineering, mathematics, physics, aeronautics, elasticity, astronomy, dynamics, biology, chemistry, medicine, environmental sciences, social sciences, and banking. We study several differential equations in calculus to get closed-form solutions, but not all differential equations possess finite solutions. It is not easy to get even if they possess closed-form solutions. In such situations, depending on the need, numerical solutions of the differential equations are also sought [1,2]. In general, equations arising from modeling physical phenomena do not have analytical or exact solutions. Only a few can be solved analytically; hence, developing numerical methods becomes necessary. Numerical methods play a key role in providing approximate solutions to differential equations due to the difficulty in obtaining the exact solutions. Many numerical techniques, as found in [3-5], have been developed and implemented. Implementing the numerical method in the predictor-corrector approach has some setbacks, including lengthy computational time due to more function evaluations needed per step and computational burden which may affect the method's accuracy in terms of error [6]. Numerical methods are necessary tools that provide solutions despite the complexities of problems. This study seeks to derive Hybrid Block Linear

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Multi-Step Method for the direct solution of general fourth-order initial value problem of ordinary differential equations of the form:

$$\begin{gathered} y^{iv}(x) = f(x, y', y'', y''') \\ y(x_0) = \phi_0, \quad y'(x_0) = \phi_1, \quad y''(x_0) = \phi_2, \text{ and } y'''(x_0) = \phi_3 \end{gathered} \tag{1.1}$$

where $f: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ is continuous, x_0 is the initial point, $y \in \mathbb{R}$ is an n-dimensional vector, x is a scalar variable and a set of equally spaced points on the integration interval defined by; $x_0 < x_1 < x_2 < \cdots < x_n < \cdots < x_{n+k} < x_N$ with a specified positive integer step number $k > 0$. According to Awoyemi [1], Continuous Linear Multi-step Methods (CLMMs) have greater advantages over the discrete methods since they give better error estimates and provide simplified form, allowing easy solution approximation at all interior points of the integration interval. Block methods for approximating the numerical solution have been proposed by scholars for the solution of initial value problems using different polynomial trial functions ranging from power series, Lagrange polynomial, and Chebychev polynomial. Among these methods are in [4,6]. In particular, fourth-order differential equations arise in many physical problems, such as in ship dynamics, deflection of beams, control theory, and mechanics. Therefore, fourth-order equations have attracted significant interest from researchers. Thereby, theoretical and numerical studies dealing with (1.1) have recently appeared in the literature.

2. Specification and Derivation of the Numerical Scheme

We consider a power series of a single variable x as an approximate solution to (1.1) as

$$y(x) = \sum_{j=0}^{t+c-1} \alpha_j x^j \tag{2.1}$$

where $\alpha_j \in \Re$, $j \in \{0,1,\dots,t+c-1\}$, $y \in C^m$, t is the interpolation points, and c is the collocation points. The derivatives of (2.1) are given as:

$$y'(x) = \sum_{j=0}^{t+c-1} j\alpha_j x^{j-1} \tag{2.2}$$

$$y''(x) = \sum_{j=0}^{t+c-1} j(j-1)\alpha_j x^{j-2} \tag{2.3}$$

$$y'''(x) = \sum_{j=0}^{t+c-1} j(j-1)(j-2)\alpha_j x^{j-3} \tag{2.4}$$

$$y^{(iv)}(x) = \sum_{j=0}^{t+c-1} j(j-1)(j-2)(j-3)\alpha_j x^{j-4} \tag{2.5}$$

From (1.1) and (2.2), we have

$$f(x, y, y', y'', y''') = \sum_{j=0}^{t+c-1} j(j-1)(j-2)(j-3)\alpha_j x^{j-4} \tag{2.6}$$

such that $\alpha_j \in \mathbb{R}$ are parameters to be determined, $c \in \left\{0, \frac{1}{10}, 1, \frac{11}{10}, \frac{19}{10}, 2\right\}$ are points of collocation, and $t \in \left\{0, \frac{1}{10}, \frac{11}{10}, \frac{19}{10}\right\}$ are points of interpolation. Collocating (2.2) at $x = x_{n+i}$ such that $i \in \left\{0, \frac{1}{10}, 1, \frac{11}{10}, \frac{19}{10}, 2\right\}$ interpolating (2.2) as well at $x = x_{n+i}$ such that $i \in \left\{0, \frac{1}{10}, \frac{11}{10}, \frac{19}{10}\right\}$ and evaluating at the end point $x = x_{n+i}$ such that $i = 2$ gives a system of nonlinear equations that were solved and then substituted into (2.1), which yields a continuous linear multi-step method in the form:

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^4 \left(\sum_{j=0}^{k} \beta_j(x) f_{n+j} + \beta_w(x) f_{n+w} + \beta_v(x) f_{n+v} + \beta_u(x) f_{n+u} \right) \tag{2.7}$$

where the numerical solution of the initial value problem is approximated to be equivalent to the true solution $y(x)$, $w = \frac{1}{10}$, $v = \frac{11}{10}$, $u = \frac{19}{10}$, α_j, and β_j are constants, $y_{n+j} = y(x_n + jh)$, and

$$f_{n+j} = f(x_n + jh, y_n' + jh, y_n'' + jh, y_n''' + jh)$$

Substituting the obtained coefficient and evaluating the resulting method (2.7) produces the continuous linear multi-step method of the form:

$$y(x) = a_0 y_0 + a_{\frac{1}{10}} y_{\frac{1}{10}} + a_{\frac{11}{10}} y_{\frac{11}{10}} + a_{\frac{19}{10}} y_{\frac{19}{10}} + \beta_0 f_0 + \beta_{\frac{1}{10}} y_{\frac{1}{10}} + \beta_1 f_1 + \beta_{\frac{11}{10}} y_{\frac{11}{10}} + \beta_{\frac{19}{10}} y_{\frac{19}{10}} + \beta_2 f_2 \tag{2.8}$$

Evaluating (2.8) at the non-interpolation point (evaluating point) $x = x_{n+1}$ and $x = x_{n+2}$ gives, respectively, the discrete schemes below:

$$\begin{aligned} y_{n+1} = &-\frac{81}{209} y_n + \frac{1}{2} y_{n+\frac{1}{10}} + \frac{81}{88} y_{n+\frac{11}{10}} - \frac{5}{152} y_{n+\frac{19}{10}} - \frac{58941}{76000000} h^4 f_n + \frac{368459}{273600000} h^4 f_{n+\frac{1}{10}} \\ &+ \frac{213557}{36000000} h^4 f_{n+1} - \frac{21501}{6400000} h^4 f_{n+\frac{11}{10}} + \frac{145063}{218880000} h^4 f_{n+\frac{19}{10}} - \frac{8199}{19000000} h^4 f_{n+2} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} y_{n+2} = &-\frac{9}{11} y_n + y_{n+\frac{1}{10}} - \frac{19}{44} y_{n+\frac{11}{10}} + \frac{5}{4} y_{n+\frac{19}{10}} - \frac{8963}{4000000} h^4 f_n + \frac{81283}{21600000} h^4 f_{n+\frac{1}{10}} \\ &+ \frac{255683}{27000000} h^4 f_{n+1} + \frac{45391}{28800000} h^4 f_{n+\frac{11}{10}} + \frac{70571}{17280000} h^4 f_{n+\frac{19}{10}} - \frac{86477}{36000000} h^4 f_{n+2} \end{aligned} \tag{2.10}$$

Finding the first derivative of (2.8) and evaluating all the collocating points $x = x_n$, $x = x_{n+\frac{1}{10}}$, $x = x_{n+1}$, $x = x_{n+\frac{11}{10}}$, $x = x_{n+\frac{19}{10}}$, and $x = x_{n+2}$ and let $r_{n+j} = y_{n+j}'$ such that $j \in \left\{0, \frac{1}{10}, 1, \frac{11}{10}, \frac{19}{10}, 2\right\}$ gives the following discrete schemes:

$$\begin{aligned} r_n = \frac{1}{34128864000000h} \Big(&-390277440000000 y_n + 396274032000000 y_{n+\frac{1}{10}} - 7368732000000 y_{n+\frac{11}{10}} \\ &+ 1372140000000 y_{n+\frac{19}{10}} + 110860639956 h^4 f_n - 218693892340 h^4 f_{n+\frac{1}{10}} - 593503880992 h^4 f_{n+1} \\ &+ 426421129365 h^4 f_{n+\frac{11}{10}} - 66216430225 h^4 f_{n+\frac{19}{10}} + 43926910236 h^4 f_{n+2} \Big) \end{aligned} \tag{2.11}$$

$$r_{n+\frac{1}{10}} = -\frac{1}{94802400000h}\Big(816480000000y_n - 800553600000y_{n+\frac{1}{10}} - 19391400000y_{n+\frac{11}{10}} +3465000000y_{n+\frac{19}{10}} + 265118103h^4f_n - 506277640h^4f_{n+\frac{1}{10}} - 1452235246h^4f_{n+1} +1036749915h^4f_{n+\frac{11}{10}} -161328475h^4f_{n+\frac{19}{10}} + 106955343h^4f_{n+2}\Big) \quad (2.12)$$

$$r_{n+1} = \frac{1}{1264032000000h}\Big(4898880000000y_n - 6390384000000y_{n+\frac{1}{10}} + 1163484000000y_{n+\frac{11}{10}} +328020000000y_{n+\frac{19}{10}} + 9440623548h^4f_n - 16443847420h^4f_{n+\frac{1}{10}} -71846823136h^4f_{n+1} +43138143495h^4f_{n+\frac{11}{10}} - 7747315675h^4f_{n+\frac{19}{10}} + 5064247188h^4f_{n+2}\Big) \quad (2.13)$$

$$r_{n+\frac{11}{10}} = \frac{1}{71101800000h}\Big(272160000000y_n - 347608800000y_{n+\frac{1}{10}} + 46862550000y_{n+\frac{11}{10}} +28586250000y_{n+\frac{19}{10}} + 563697981h^4f_n - 976068280h^4f_{n+\frac{1}{10}} - 4303171642h^4f_{n+1} +2284734705h^4f_{n+\frac{11}{10}} - 501523825h^4f_{n+\frac{19}{10}} + 325265061h^4f_{n+2}\Big) \quad (2.14)$$

$$r_{n+\frac{19}{10}} = -\frac{1}{118503000000h}\Big(816480000000y_n - 1000692000000y_{n+\frac{1}{10}} + 460545750000y_{n+\frac{11}{10}} -276333750000y_{n+\frac{19}{10}} + 2172127671h^4f_n -473460525h^4f_{n+\frac{11}{10}} - 3589256275h^4f_{n+\frac{19}{10}} + 2155609071h^4f_{n+2}\Big) \quad (2.15)$$

$$r_{n+2} = -\frac{1}{11376288000000h}\Big(108319680000000y_n - 132091344000000y_{n+\frac{1}{10}} +54166644000000y_{n+\frac{11}{10}} - 30394980000000y_{n+\frac{19}{10}} + 303775046388h^4f_n -508773529220h^4f_{n+\frac{1}{10}} - 1179138420416h^4f_{n+1} - 330876154455h^4f_{n+\frac{11}{10}} -598023427925h^4f_{n+\frac{19}{10}} + 345412673628h^4f_{n+2}\Big) \quad (2.16)$$

Finding the second derivative of (2.8) and evaluating all the collocating points $x = x_n$, $x = x_{n+\frac{1}{10}}$, $x = x_{n+1}$, $x = x_{n+\frac{11}{10}}$, $x = x_{n+\frac{11}{10}}$, and $x = x_{n+2}$ and let $s_{n+j} = y''_{n+j}$ such that $j \in \left\{0, \frac{1}{10}, 1, \frac{11}{10}, \frac{19}{10}, 2\right\}$ gives the following discrete schemes:

$$s_n = -\frac{1}{284407200000h^2}\Big(-8436960000000y_n + 9480240000000y_{n+\frac{1}{10}} - 1292760000000y_{n+\frac{11}{10}} +249480000000y_{n+\frac{19}{10}} + 20917549323h^4f_n - 42632273200h^4f_{n+\frac{1}{10}} - 110678017846h^4f_{n+1} +79914551475h^4f_{n+\frac{11}{10}} - 12388620475h^4f_{n+\frac{19}{10}} + 8222376723h^4f_{n+2}\Big) \quad (2.17)$$

$$s_{n+\frac{1}{10}} = -\frac{1}{213305400000h^2}\Big(-5715360000000y_n + 6399162000000y_{n+\frac{1}{10}} - 824134500000y_{n+\frac{11}{10}} +140332500000y_{n+\frac{19}{10}} + 9941764959h^4f_n - 18346830920h^4f_{n+\frac{1}{10}} - 56496792638h^4f_{n+1} +39989691495h^4f_{n+\frac{11}{10}} - 6239749175h^4f_{n+\frac{19}{10}} + 4133232279h^4f_{n+2}\Big) \quad (2.18)$$

$$s_{n+1} = \frac{1}{853221600000h^2}\Big(816480000000y_n - 1939140000000y_{n+\frac{11}{10}} + 1122660000000y_{n+\frac{19}{10}} +6842497761h^4f_n - 11187319000h^4f_{n+\frac{1}{10}} - 63806750722h^4f_{n+1} +7077591675h^4f_{n+\frac{11}{10}} - 9746416075h^4f_{n+\frac{19}{10}} + 6117758361h^4f_{n+2}\Big) \quad (2.19)$$

$$s_{n+\frac{11}{10}} = \frac{1}{426610800000h^2}\Big(-816480000000y_n + 1422036000000y_{n+\frac{1}{10}} - 1260441000000y_{n+\frac{11}{10}} +654885000000y_{n+\frac{19}{10}} + 500908617h^4f_n - 542479960h^4f_{n+\frac{1}{10}} +842757806h^4f_{n+1} - 20192751315h^4f_{n+\frac{11}{10}} - 3020740525h^4f_{n+\frac{19}{10}} + 1792783377h^4f_{n+2}\Big) \quad (2.20)$$

$$s_{n+\frac{19}{10}} = -\frac{1}{426610800000h^2}\Big(10614240000000y_n - 12798324000000y_{n+\frac{1}{10}} + 3587409000000y_{n+\frac{11}{10}} -1403325000000y_{n+\frac{19}{10}} + 32936727627h^4f_n - 54572970760h^4f_{n+\frac{1}{10}} - 83172019414h^4f_{n+1} -87056974665h^4f_{n+\frac{11}{10}} - 80228037175h^4f_{n+\frac{19}{10}} + 45278532387h^4f_{n+2}\Big) \quad (2.21)$$

$$s_{n+2} = -\frac{1}{853221600000h^2}\Big(23677920000000y_n - 28440720000000y_{n+\frac{1}{10}} + 7756560000000y_{n+\frac{11}{10}} -2993760000000y_{n+\frac{19}{10}} + 76992556509h^4f_n - 126928542400h^4f_{n+\frac{1}{10}} - 146034445018h^4f_{n+1} -253898837775h^4f_{n+\frac{11}{10}} - 220541306425h^4f_{n+\frac{19}{10}} + 116527553109h^4f_{n+2}\Big) \quad (2.22)$$

Finding the third derivative of (2.8) and evaluating all the collocating points $x = x_n$, $x = x_{n+\frac{1}{10}}$, $x = x_{n+1}$, $x = x_{n+\frac{11}{10}}$, $x = x_{n+\frac{10}{10}}$, and $x = x_{n+2}$ and let $t_{n+j} = y''_{n+j}$ such that $j \in \left\{0, \frac{1}{10}, 1, \frac{11}{10}, \frac{19}{10}, 2\right\}$ gives the following discrete schemes:

$$t_n = \frac{1}{85322160000h^3}\Big(-2449440000000y_n + 2844072000000y_{n+\frac{1}{10}} - 581742000000y_{n+11}^{10} +187110000000y_{n+\frac{19}{10}} + 20341524573h^4f_n - 56174813860h^4f_{n+\frac{1}{10}} - 105707500426h^4f_{n+1} +79470524610h^4f_{n+\frac{11}{10}} - 12149227750h^4f_{n+\frac{19}{10}} + 8094818853h^4f_{n+2}\Big) \quad (2.23)$$

$$t_{n+\frac{1}{10}} = \frac{1}{85322160000h^3}\Big(-2449440000000y_n + 2844072000000y_{n+\frac{1}{10}} - 581742000000y_{n+\frac{11}{10}} + 187110000000y_{n+\frac{19}{10}} + 24211867707h^4f_n - 51427648360h^4f_{n+\frac{1}{10}} - 106372925494h^4f_{n+1} + 80086291335h^4f_{n+\frac{11}{10}} - 12261672775h^4f_{n+\frac{19}{10}} + 8171629587h^4f_{n+2}\Big) \quad (2.24)$$

$$t_{n+1} = -\frac{1}{85322160000h^3}\Big(2449440000000y_n - 2844072000000y_{n+\frac{1}{10}} + 581742000000y_{n+\frac{11}{10}} - 187110000000y_{n+\frac{10}{10}} + 5853875427h^4f_n - 10122186140h^4f_{n+\frac{1}{10}} \quad (2.25)$$

$$t_{n+\frac{11}{10}} = -\frac{1}{85322160000h^3}\Big(2449440000000y_n - 2844072000000y_{n+\frac{1}{10}} + 581742000000y_{n+\frac{11}{10}} - 187110000000y_{n+\frac{19}{10}} + 5827792293h^4f_n - 10083031640h^4f_{n+\frac{1}{10}} - 66904794506h^4f_{n+1} + 44593228665h^4f_{n+\frac{11}{10}} - 3677327225h^4f_{n+\frac{19}{10}} + 2514430413h^4f_{n+2}\Big) \quad (2.26)$$

$$t_{n+\frac{19}{10}} = -\frac{1}{85322160000h^3}\Big(2449440000000y_n - 2844072000000y_{n+\frac{1}{10}} + 581742000000y_{n+1\frac{11}{10}} - 187110000000y_{n+\frac{19}{10}} + 11143839141h^4f_n - 17818020440h^4f_{n+\frac{1}{10}} + 20572961398h^4f_{n+1} - 80086291335h^4f_{n+\frac{11}{10}} - 56983996025h^4f_{n+\frac{19}{10}} + 27184077261h^4f_{n+2}\Big) \quad (2.27)$$

$$t_{n+2} = -\frac{1}{85322160000h^3}\Big(2449440000000y_n - 2844072000000y_{n+\frac{1}{10}} + 581742000000y_{n+\frac{11}{10}} + 187110000000y_{n+\frac{10}{10}} + 11092955427h^4f_n - 17745186140h^4f_{n+\frac{1}{10}} + 20034220426h^4f_{n+1} - 79470524610h^4f_{n+\frac{11}{10}} - 61770772250h^4f_{n+\frac{10}{10}} + 23339661147h^4f_{n+2}\Big) \quad (2.28)$$

Solving the resulting system for the unknown variables y_{n+j}, r_{n+j}, s_{n+j}, and t_{n+j} such that $j \in \left\{\frac{1}{10}, 1, \frac{11}{10}, \frac{19}{10}, 2\right\}$ gives the discrete schemes which are combined to form the required method below:

$$y_{n+\frac{1}{10}} = \frac{1637}{51710400000}h^4f_{n+2} - \frac{4927}{106375680000}h^4f_{n+\frac{19}{10}} + \frac{301157}{1197504000000}h^4f_{n+\frac{11}{10}} - \frac{949}{3499200000}h^4f_{n+1} + \frac{4774487}{4653936000000}h^4f_{n+\frac{1}{10}} + \frac{100337}{31600800000}h^4f_n + \frac{1}{6000}h^3t_n + \frac{1}{200}h^2s_n + \frac{1}{10}hr_n + y_n \quad (2.29)$$

$$y_{n+1} = -\frac{71}{23085}h^4f_{n+2} + \frac{1325}{290871}h^4f_{n+\frac{19}{10}} - \frac{1145}{37422}h^4f_{n+\frac{11}{10}} + \frac{4315}{122472}h^4f_{n+1} + \frac{3535}{83106}h^4f_{n+\frac{1}{10}} - \frac{2759}{395010}h^4f_n + \frac{1}{6}h^3t_n + \frac{1}{2}h^2s_n + hr_n + y_n \quad (2.30)$$

$$y_{n+\frac{11}{10}} = -\frac{2570535011}{517104000000}h^4f_{n+2} + \frac{288764443}{39191040000}h^4f_{n+\frac{19}{10}} - \frac{5483098423}{108864000000}h^4f_{n+\frac{11}{10}} + \frac{511981129}{8748000000}h^4f_{n+1} + \frac{290459826877}{4653936000000}h^4f_{n+\frac{1}{10}} - \frac{5168273}{432000000}h^4f_n + \frac{1331}{6000}h^3t_n + \frac{121}{200}h^2s_n + \frac{11}{10}hr_n + y_n \quad (2.31)$$

$$y_{n+\frac{19}{10}} = -\frac{1141481639}{19440000000}h^4f_{n+2} + \frac{17212667359}{195955200000}h^4f_{n+\frac{19}{10}} - \frac{696137901157}{1197504000000}h^4f_{n+\frac{11}{10}} + \frac{228690027541}{306180000000}h^4f_{n+1} + \frac{121063908407}{244944000000}h^4f_{n+\frac{1}{10}} - \frac{3467972131}{23760000000}h^4f_n + \frac{6859}{6000}h^3t_n + \frac{361}{200}h^2s_n + \frac{19}{10}hr_n + y_n \quad (2.32)$$

$$y_{n+2} = -\frac{11902}{161595}h^4f_{n+2} + \frac{4600}{41553}h^4f_{n+\frac{19}{10}} - \frac{1880}{2673}h^4f_{n+\frac{11}{10}} + \frac{70256}{76545}h^4f_{n+1} + \frac{172960}{290871}h^4f_{n+\frac{1}{10}} - \frac{35456}{197505}h^4f_n + \frac{4}{3}h^3t_n + 2h^2s_n + 2hr_n + y_n \quad (2.33)$$

$$r_{n+\frac{1}{10}} = \frac{1182947}{10032000000}h^3f_n - \frac{164531}{13608000000}h^3f_{n+1} + \frac{80891}{57456000000}h^3f_{n+2} + \frac{2600291}{51710400000}h^3f_{n+\frac{1}{10}} + \frac{149111}{13305600000}h^3f_{n+\frac{11}{10}} - \frac{85223}{41368320000}h^3f_{n+\frac{19}{10}} + \frac{1}{200}t_nh^2 + \frac{1}{10}s_nh + r_n \quad (2.34)$$

$$r_{n+1} = -\frac{14549}{351120}h^3f_n + \frac{6493}{34020}h^3f_{n+1} - \frac{239}{15120}h^3f_{n+2} + \frac{1175}{6804}h^3f_{n+\frac{1}{10}} - \frac{775}{4752}h^3f_{n+\frac{11}{10}} + \frac{12125}{517104}h^3f_{n+\frac{19}{10}} + \frac{1}{2}t_nh^2 + s_nh + r_n \quad (2.35)$$

$$r_{n+\frac{11}{10}} = -\frac{124817187}{2128000000}h^3f_n + \frac{3787758469}{13608000000}h^3f_{n+1} - \frac{61038329}{2736000000}h^3f_{n+2} + \frac{11692463651}{51710400000}h^3f_{n+\frac{1}{10}} - \frac{94664713}{403200000}h^3f_{n+\frac{11}{10}} + \frac{1370060857}{41368320000}h^3f_{n+\frac{19}{10}} + \frac{121}{200}t_nh^2 + \frac{11}{10}s_nh + r_n \quad (2.36)$$

$$r_{n+\frac{19}{10}} = -\frac{1153759249}{3696000000}h^3f_n + \frac{21767386309}{13608000000}h^3f_{n+1} - \frac{410641471}{3024000000}h^3f_{n+2} + \frac{365811047}{388800000}h^3f_{n+\frac{1}{10}} - \frac{15379050889}{13305600000}h^3f_{n+\frac{11}{10}} + \frac{89914631}{435456000}h^3f_{n+\frac{19}{10}} + \frac{361}{200}t_nh^2 + \frac{19}{10}s_nh + r_n \quad (2.37)$$

$$r_{n+2} = -\frac{2631}{7315}h^3f_n + \frac{2212}{1215}h^3f_{n+1} - \frac{979}{5985}h^3f_{n+2} + \frac{34520}{32319}h^3f_{n+\frac{1}{10}} - \frac{890}{693}h^3f_{n+\frac{11}{10}} + \frac{8150}{32319}h^3f_{n+\frac{10}{10}} + 2t_nh^2 + 2s_nh + r_n \quad (2.38)$$

$$s_{n+\frac{1}{10}} = \frac{14317}{4620000}h^2f_n - \frac{3383}{8505000}h^2f_{n+1} + \frac{221}{4788000}h^2f_{n+2} + \frac{2523209}{1292760000}h^2f_{n+\frac{1}{10}} + \frac{40853}{110880000}h^2f_{n+\frac{11}{10}} - \frac{3677}{54432000}h^2f_{n+\frac{19}{10}} + \frac{1}{10}t_nh + s_n \quad (2.39)$$

$$s_{n+1} = -\frac{5507}{35112}h^2f_n + \frac{5273}{6804}h^2f_{n+1} - \frac{563}{9576}h^2f_{n+2} + \frac{16010}{32319}h^2f_{n+\frac{1}{10}} - \frac{445}{693}h^2f_{n+\frac{11}{10}} + \frac{2825}{32319}h^2f_{n+\frac{19}{10}} + t_nh + s_n \quad (2.40)$$

$$s_{n+\frac{11}{10}} = -\frac{199529}{1064000}h^2 f_n + \frac{33191147}{34020000}h^2 f_{n+1} - \frac{3411353}{47880000}h^2 f_{n+2} + \frac{740819959}{1292760000}h^2 f_{n+\frac{1}{10}} - \frac{7972327}{10080000}h^2 f_{n+\frac{11}{10}} + \frac{109617167}{1034208000}h^2 f_{n+\frac{19}{10}} + \frac{11}{10}t_n h + s_n \quad (2.41)$$

$$s_{n+\frac{19}{10}} = -\frac{21096479}{46200000}h^2 f_n + \frac{367374899}{170100000}h^2 f_{n+1} - \frac{3244307}{12600000}h^2 f_{n+2} + \frac{83618069}{68040000}h^2 f_{n+\frac{1}{10}} - \frac{142440853}{110880000}h^2 f_{n+\frac{11}{10}} + \frac{22591741}{54432000}h^2 f_{n+\frac{10}{10}} + \frac{19}{10}t_n h + s_n \quad (2.42)$$

$$s_{n+2} = -\frac{38}{77}h^2 f_n + \frac{3844}{1701}h^2 f_{n+1} - \frac{356}{1197}h^2 f_{n+2} + \frac{42520}{32319}h^2 f_{n+\frac{1}{10}} + \frac{850}{1701}h^2 f_{n+\frac{19}{10}} + s_n - \frac{890}{693}h^2 f_{n+\frac{11}{10}} + 2t_n h \quad (2.43)$$

$$t_{n+\frac{1}{10}} = \frac{1137667}{25080000}hf_n - \frac{37903}{4860000}hf_{n+1} + \frac{18473}{20520000}hf_{n+2} + \frac{41101}{738720}hf_{n+\frac{1}{10}} + \frac{6859}{950400}hf_{n+\frac{11}{10}} - \frac{19471}{14774400}hf_{n+\frac{19}{10}} + t_n \quad (2.44)$$

$$t_{n+1} = -\frac{35}{114}hf_n + \frac{959}{486}hf_{n+1} - \frac{64}{513}hf_{n+2} + \frac{7175}{9234}hf_{n+\frac{1}{10}} - \frac{325}{216}hf_{n+\frac{11}{10}} + \frac{6875}{36936}hf_{n+\frac{19}{10}} + t_n \quad (2.45)$$

$$t_{n+\frac{11}{10}} = -\frac{233101}{760000}hf_n + \frac{9832097}{4860000}hf_{n+1} - \frac{94501}{760000}hf_{n+2} + \frac{573661}{738720}hf_{n+\frac{1}{10}} - \frac{4653}{3200}hf_{n+\frac{11}{10}} + \frac{2740529}{14774400}hf_{n+\frac{19}{10}} + t_n \quad (2.46)$$

$$t_{n+\frac{19}{10}} = -\frac{487103}{1320000}hf_n + \frac{4849313}{4860000}hf_{n+1} - \frac{446557}{1080000}hf_{n+2} + \frac{168587}{194400}hf_{n+\frac{1}{10}} + \frac{6859}{950400}hf_{n+\frac{11}{10}} + \frac{630059}{777600}hf_{n+\frac{19}{10}} + t_n \quad (2.47)$$

$$t_{n+2} = -\frac{7}{19}hf_n + \frac{244}{243}hf_{n+1} - \frac{7}{19}hf_{n+2} + \frac{4000}{4617}hf_{n+\frac{1}{10}} + \frac{4000}{4617}hf_{n+\frac{19}{10}} + t_n \quad (2.48)$$

2.1. Order and Error Constant of the Block Methods

The linear operator associated with the methods is defined as:

$$L[y(x), h] = \sum_{j=0}^{k} \left[\alpha_j y(x + jh) - h^4 \beta_j y^{iv}(x + jh)\right] \quad (2.49)$$

where the function $y(x)$ is assumed to have continuous derivatives of sufficiently high order. Therefore, expanding (2.49) in the Taylor series about the point x to obtain the expression

$$L[y(x), h] = C_0 y(x) + hC_1 y'(x) + h^2 C_2 y''(x) + \cdots + h^{p+4} C_{p+4} y^{p+4}(x) \quad (2.50)$$

and

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=0}^{k} j\alpha_j$$

$$C_2 = \frac{1}{2!}\sum_{j=0}^{k} j^2\alpha_j$$

$$\vdots$$

$$C_q = \frac{1}{q!}\left(\sum_{j=0}^{k} j^q q(q-1)(q-2)(q-3)\alpha_j \sum_{j=1}^{k} \beta_j j^{q-4}\right), \quad q \in \{0,1,2,3,\cdots,p+4\} \qquad (2.51)$$

In the sense of (2.49), we say that the methods are of order p and error constant C_{p+4} if

$$C_0 = C_1 = C_2 = C_3 = \cdots = C_p = C_{p+1} = C_{p+2} = C_{p+3} = 0, C_{p+4} \neq 0$$

Considering (2.30)

$$y_{n+1} = -\frac{71}{23085}h^4 f_{n+2} + \frac{1325}{290871}h^4 f_{n+\frac{19}{10}} - \frac{1145}{37422}h^4 f_{n+\frac{11}{10}} + \frac{4315}{122472}h^4 f_{n+1} + \frac{3535}{83106}h^4 f_{n+\frac{1}{10}} - \frac{2759}{395010}h^4 f_n + \frac{1}{6}h^3 t_n + \frac{1}{2}h^2 s_n + hr_n + y_n$$

and expanding in Taylor's series about x_n, and collecting the coefficient of the like powers of h gives where $D = \frac{d}{dx}$

$$-y_n(x+h) = -\frac{487103}{1320000}hf_n + \frac{4849313}{4860000}hf_{n+1} - \frac{446557}{1080000}hf_{n+2} + \frac{168587}{194400}hf_{n+\frac{1}{10}} + \frac{6859}{950400}hf_{n+\frac{11}{10}} + \frac{630059}{777600}hf_{n+\frac{19}{10}} - \frac{1}{720}D^6(y_n)(x)h^6 - \frac{1}{5040}D^7(y_n)(x)h^7 - \frac{1}{40320}D^8(y_n)(x)h^8 - \frac{1}{362880}D^9(y_n)(x)h^9 - \frac{1}{3628800}D^{10}(y_n)(x)h^{10} + O(h^{11})$$

$$\frac{1}{6}D^3(y_n)(x)h^3 = \frac{1}{6}D^3(y_n)(x)h^3$$

$$\frac{1}{2}D^2(y_n)(x)h^2 = \frac{1}{2}D^2(y_n)(x)h^2$$

$$D^1(y_n)(x)h^1 = D^1(y_n)(x)h^1$$

$$\frac{-2759}{395010}D^4(y_{n+0h})(x)h^4 = \frac{-2759}{395010}D^4(y_n)(x)h^4$$

$$\frac{3535}{83106}D^4(y_n)\left(x+\frac{h}{10}\right)h^4 = \frac{3535}{83106}D^4(y_n)(x)h^4 + \frac{707}{166212}D^5(y_n)(x)h^5 + \frac{707}{3324240}D^6(y_n)(x)h^6 + \frac{707}{99727200}D^7(y_n)(x)h^7 + \frac{707}{3989088000}D^8(y_n)(x)h^8 + \frac{707}{199454400000}D^9(y_n)(x)h^9 + \frac{707}{11967264000000}D^{10}(y_n)(x)h^{10} + O(h^{11})$$

$$\frac{4315}{122472}D^4(y_n)(x+h)h^4 = \frac{4315}{122472}D^4(y_n)(x)h^4 + \frac{4315}{122472}D^5(y_n)(x)h^5 + \frac{4315}{244944}D^6(y_n)(x)h^6 + \frac{4315}{734832}D^7(y_n)(x)h^7 + \frac{4315}{2939328}D^8(y_n)(x)h^8 + \frac{863}{2939328}D^9(y_n)(x)h^9 + \frac{863}{17635968}D^{10}(y_n)(x)h^{10} + O(h^{11})$$

$$\frac{-1145}{37422}D^4(y_n)\left(x+\frac{11h}{10}\right)h^4 = \frac{-1145}{37422}D^4(y_n)(x)h^4 + \frac{-229}{6804}D^5(y_n)(x)h^5 + \frac{-2519}{136080}D^6(y_n)(x)h^6 + \frac{-27709}{4082400}D^7(y_n)(x)h^7 + \frac{-304799}{163296000}D^8(y_n)(x)h^8 + \frac{-3352789}{8164800000}D^9(y_n)(x)h^9 + \frac{-36880679}{489888000000}D^{10}(y_n)(x)h^{10} + O(h^{11})$$

$$\frac{1325}{290871}D^4(y_n)\left(x+\frac{11h}{10}\right)h^4 = \frac{1325}{290871}D^4(y_n)(x)h^4 + \frac{265}{30618}D^5(y_n)(x)h^5 + \frac{1007}{122472}D^6(y_n)(x)h^6 + \frac{19133}{3674160}D^7(y_n)(x)h^7 + \frac{363527}{146966400}D^8(y_n)(x)h^8 + \frac{6907013}{7348320000}D^9(y_n)(x)h^9 + \frac{131233247}{440899200000}D^{10}(y_n)(x)h^{10} + O(h^{11})$$

$$\frac{-71}{23085}D^4(y_n)\left(x+\frac{11h}{10}\right)h^4 = \frac{-71}{23085}D^4(y_n)(x)h^4 + \frac{-142}{23085}D^5(y_n)(x)h^5 + \frac{-142}{23085}D^6(y_n)(x)h^6 + \frac{-284}{69255}D^7(y_n)(x)h^7 + \frac{-142}{69255}D^8(y_n)(x)h^8 + \frac{-284}{346275}D^9(y_n)(x)h^9 + \frac{-284}{1038825}D^{10}(y_n)(x)h^{10} + O(h^{11})$$

and

$$y_n(x+oh) = y_n$$

Applying (2.30) and collecting the coefficients of like terms of h, we have

$$C_0 = -1+1 = 0$$

$$C_1 = -1+1 = 0$$

$$C_2 = -\frac{1}{2}+\frac{1}{2} = 0$$

$$C_3 = -\frac{1}{6}+\frac{1}{6} = 0$$

$$C_4 = \frac{-1}{24} - \frac{2759}{395010} + \frac{3535}{83106} + \frac{4315}{122472} - \frac{1145}{37422} + \frac{1325}{290871} - \frac{71}{23085} = 0$$

$$C_5 = \frac{-1}{120} + \frac{707}{166212} + \frac{4315}{122472} - \frac{229}{6804} + \frac{265}{30618} - \frac{142}{23085} = 0$$

$$C_6 = -\frac{1}{720} + \frac{707}{3324240} + \frac{4315}{244944} - \frac{2519}{136080} + \frac{1007}{122472} - \frac{142}{23085} = 0$$

$$C_7 = -\frac{1}{5040} + \frac{707}{99727200} + \frac{4315}{734832} - \frac{27709}{4082400} + \frac{19133}{3674160} - \frac{284}{69255} = 0$$

$$C_8 = -\frac{1}{40320} + \frac{707}{3989088000} + \frac{4315}{2939328} - \frac{304799}{163296000} + \frac{363527}{146966400} - \frac{142}{69255} = 0$$

$$C_9 = -\frac{1}{362880} + \frac{707}{199454400000} + \frac{863}{2939328} - \frac{3352789}{8164800000} + \frac{6907013}{7348320000} - \frac{284}{346275} = 0$$

and

$$C_{10} = \frac{-12857}{5443200000} = C_{p+4}$$

which implies

$$p + 4 = 10$$

such that

$$p = 6$$

The method is of order $p = 6$ with an error constant $C_{10} = C_{p+4} = \frac{-12857}{5443200000}$. This procedure will be adopted for other schemes derived. The orders and error constants of the schemes are presented in Table 1.

Table 1. Order and error constant of the derived schemes

Schemes	Order	Error Constant
$y_{n+\frac{1}{10}} = \frac{1637}{51710400000}h^4f_{n+2} - \frac{4927}{106375680000}h^4f_{n+\frac{19}{10}} + \frac{30157}{1197504000000}h^4f_{n+\frac{11}{10}} - \frac{949}{3499200000}h^4f_{n+1} + \frac{4774487}{4653936000000}h^4f_{n+\frac{1}{10}} + \frac{100337}{31600800000}h^4f_n + \frac{1}{6000}t_nh^3 + \frac{1}{200}s_nh^2 + \frac{1}{10}r_nh + y_n$	6	$\frac{3228263}{1088640} * 10^{-11}$
$y_{n+\frac{11}{10}} = -\frac{2570535011}{517104000000}h^4f_{n+2} + \frac{288764443}{39191040000}h^4f_{n+\frac{19}{10}} - \frac{5483098423}{108864000000}h^4f_{n+\frac{11}{10}} + \frac{511981129}{8748000000}h^4f_{n+1} + \frac{290495826877}{4653936000000}h^4f_{n+\frac{1}{10}} - \frac{5168273}{432000000}h^4f_n + \frac{1331}{6000}t_nh^3 + \frac{121}{200}s_nh^2 + \frac{11}{10}r_nh + y_n$	6	$\frac{-411546021227}{1088640} * 10^{-11}$
$y_{n+\frac{19}{10}} = -\frac{1141481639}{19440000000}h^4f_{n+2} + \frac{17212667359}{195955200000}h^4f_{n+\frac{19}{10}} - \frac{696137901157}{1197504000000}h^4f_{n+\frac{11}{10}} + \frac{228690027541}{306180000000}h^4f_{n+1} + \frac{121063908407}{244944000000}h^4f_{n+\frac{1}{10}} - \frac{3467972131}{23760000000}h^4f_n + \frac{6859}{6000}t_nh^3 + \frac{361}{200}s_nh^2 + \frac{19}{10}r_nh + y_n$	6	$\frac{-4543267774051}{1088640} * 10^{-11}$
$y_{n+2} = -\frac{11902}{161595}h^4f_{n+2} + \frac{4600}{41553}h^4f_{n+\frac{19}{10}} - \frac{1880}{2673}h^4f_{n+\frac{11}{10}} + \frac{70256}{76545}h^4f_{n+1} + \frac{172960}{290871}h^4f_{n+\frac{1}{10}} - \frac{35456}{197505}h^4f_n + 4/3h^3t_n + 2h^2s_n + 2hr_n + y_n$	6	$\frac{-2189}{42525000}$

2.2. Consistency

A linear multi-step method is said to be consistent if the order $p \geq 1$ and obeys the following axioms:

i. $a_0 + a_1 + a_2 + \cdots + a_k = 0$
ii. $\rho(1) = \rho'(1) = \rho''(1) = 0$
iii. $\rho^{iv}(r) = 4!\,\sigma(r)$

where $\rho(r) = \sum_{j=0}^{k} a_j r^j$ and $\sigma(r) = \sum_{j=0}^{k} \beta_j r^j$. Considering (2.33),

$$y_{n+2} = -\frac{9}{11}y_n + y_{n+\frac{1}{10}} - \frac{19}{44}y_{n+\frac{11}{10}} + \frac{5}{4y_{n+\frac{19}{10}}} - \frac{8963}{4000000}h^4f_n + \frac{81283}{21600000}h^4f_{n+\frac{1}{10}} + \frac{255683}{27000000}h^4f_{n+1} + \frac{45391}{28800000}h^4f_{n+\frac{11}{10}} + \frac{70571}{17280000}h^4f_{n+\frac{19}{10}} - \frac{86477}{36000000}h^4f_{n+2}$$

we have

i.

$$\sum_{j=0}^{4} a_j = a_0 + a_{\frac{1}{10}} + a_1 + a_{\frac{11}{10}} + a_{\frac{19}{10}} + a_2 = \frac{9}{11} + 1 + 0 + \frac{19}{44} - \frac{5}{4} - 1 = 0$$

ii.

$$\rho(r) = \frac{9}{11} + 1r^{\frac{1}{10}} + 0r^1 + \frac{19}{44}r^{\frac{11}{10}} - \frac{5}{4}r^{\frac{19}{10}} - 1r^2$$

$$\rho(1) = \frac{9}{11} + 1(1)^{\frac{1}{10}} + 0(1)^1 + \frac{19}{44}(1)^{\frac{11}{10}} - \frac{5}{4}(1)^{\frac{19}{10}} - 1(1)^2 = 0$$

$$\rho'(r) = 2r - \frac{1}{10}r^{-\frac{9}{10}} + \frac{19}{40}r^{\frac{1}{10}} - \frac{19}{8}r^{\frac{9}{10}}$$

$$\rho'(1) = 2(1) - \frac{1}{10}(1)^{-\frac{9}{10}} + \frac{19}{40}(1)^{\frac{1}{10}} - \frac{19}{8}(1)^{\frac{9}{10}} = 0$$

iii. For the LHS

$$\rho^{iv}(r) = \frac{4959}{10000}r^{-\frac{39}{10}} + \frac{3249}{40000}r^{-\frac{29}{10}} - \frac{1881}{8000}r^{-\frac{21}{10}}$$

and

$$\rho^{iv}(1) = \frac{4959}{10000}(1)^{-\frac{39}{10}} + \frac{3249}{40000}(1)^{-\frac{29}{10}} - \frac{1881}{8000}(1)^{-\frac{21}{10}} = \frac{171}{500}$$

and for the RHS

$$\sigma(r) = \sum_{j=0}^{4} \beta_j r^j = \beta_0 r^0 + \beta_{1/10} r^{\frac{1}{10}} + \beta_1 r^1 + \beta_{\frac{11}{10}} r^{\frac{11}{10}} + \beta_{\frac{19}{10}} r^{\frac{19}{10}} + \beta_2 r^2$$

$$\sigma(r) = \frac{-8963}{4000000}r^0 + \frac{81283}{21600000}r^{\frac{1}{10}} + \frac{255683}{27000000}r^1 + \frac{45391}{28800000}r^{\frac{11}{10}} + \frac{70571}{17280000}r^{\frac{19}{10}} + \frac{-86477}{36000000}r^2$$

$$\sigma(1) = \frac{-8963}{4000000}(1)^0 + \frac{81283}{21600000}(1)^{\frac{1}{10}} + \frac{255683}{27000000}(1)^1 + \frac{45391}{28800000}(1)^{\frac{11}{10}} + \frac{70571}{17280000}(1)^{\frac{19}{10}} + \frac{-86477}{36000000}(1)^2 = \frac{57}{4000}$$

then

$$4!\,\sigma(1) = 24 * \frac{57}{4000} = \frac{171}{500}$$

and

$$LHS = RHS$$

This implies that $\rho^4(r) = 4!\,\sigma(r)$ for the principal root, $r = 1$, and also since the order $p = 6$, hence it satisfies $p \geq 1$. Therefore, the derived schemes are consistent. Hence, the method is consistent.

2.3. Zero Stability of the Blocks

A numerical method is said to be zero-stable if the roots $z \in \{1,2,3,\cdots,N\}$ of the characteristics polynomial $p(z) = \det(zA^0 - A')$, satisfies $|z| \leq 1$, and the roots $|z| = 1$ have multiplicity not exceeding the order of the differential equation, which is 4. Moreover, as $h^\gamma \to 0p(z) = z^{r-\gamma}(\lambda - 1)$, where γ is the order of the differential equation. From the resulting schemes, we have that

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1/10} \\ y_{n+1} \\ y_{n+11/10} \\ y_{n+19/10} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-1/10} \\ y_{n-1} \\ y_{n-11/10} \\ y_{n-19/10} \\ y_n \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & 0 & 1/10 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 11/10 \\ 0 & 0 & 0 & 0 & 19/10 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} r_{n-1/10} \\ r_{n-1} \\ r_{n-11/10} \\ r_{n-19/10} \\ r_n \end{pmatrix}$$

$$+h^2\begin{pmatrix}0&0&0&0&1/200\\0&0&0&0&1/2\\0&0&0&0&121/200\\0&0&0&0&361/200\\0&0&0&0&2\end{pmatrix}\begin{pmatrix}s_{n-1/10}\\s_{n-1}\\s_{n-11/10}\\s_{n-19/10}\\s_n\end{pmatrix}+h^3\begin{pmatrix}0&0&0&0&1/6000\\0&0&0&0&1/6\\0&0&0&0&1331/6000\\0&0&0&0&6859/6000\\0&0&0&0&4/3\end{pmatrix}\begin{pmatrix}t_{n-1/10}\\t_{n-1}\\t_{n-11/10}\\t_{n-19/10}\\t_n\end{pmatrix}$$

$$+h^4\begin{pmatrix}0&0&0&0&100337/31600800000\\0&0&0&0&-2759/395010\\0&0&0&0&-5168273/432000000\\0&0&0&0&-3467972131/23760000000\\0&0&0&0&-35456/197505\end{pmatrix}\begin{pmatrix}f_{n-1/10}\\f_{n-1}\\f_{n-11/10}\\f_{n-19/10}\\f_n\end{pmatrix}$$

$$+h^4\begin{pmatrix}\frac{4774487}{465393600000}&-\frac{949}{3499200000}&\frac{301157}{1197504000000}&-\frac{4927}{106375680000}&\frac{1637}{51710400000}\\\frac{3535}{83106}&\frac{4315}{122472}&-\frac{1145}{3742}&\frac{1325}{29071}&-\frac{71}{23085}\\\frac{290459826877}{4653936000000}&\frac{511981129}{8748000000}&-\frac{5483098423}{108864000000}&\frac{288764443}{39191040000}&-\frac{2570535011}{517104000000}\\\frac{121063908407}{244944000000}&\frac{228690027541}{306180000000}&-\frac{696137901157}{1197504000000}&\frac{17212667359}{195955200000}&-\frac{1141481639}{19440000000}\\\frac{172960}{290871}&\frac{70256}{76545}&-\frac{1880}{2673}&\frac{4600}{41553}&-\frac{11902}{161595}\end{pmatrix}\begin{pmatrix}f_{n+1/10}\\f_{n+1}\\f_{n+11/10}\\f_{n+19/10}\\f_{n+2}\end{pmatrix}$$

which is expressed in the form

$$A^0Y_N = A'Y_{N-1} + hA''Y'_{N-1} + h^2B'Y''_{N-1} + h^3B''Y'''_{N-1} + h^4(E^0F_N + E'F_{N-1}) \tag{2.52}$$

where

$$Y_N = \begin{pmatrix}y_{n+\frac{1}{10}}\\y_{n+1}\\y_{n+\frac{11}{10}}\\y_{n+\frac{19}{10}}\\y_{n+2}\end{pmatrix}, Y_{N-1} = \begin{pmatrix}y_{n-\frac{1}{10}}\\y_{n-1}\\y_{n-\frac{11}{10}}\\y_{n-\frac{19}{10}}\\y_n\end{pmatrix}, Y'_{N-1} = \begin{pmatrix}r_{n-\frac{1}{10}}\\r_{n-1}\\r_{n-\frac{11}{10}}\\r_{n-\frac{19}{10}}\\r_n\end{pmatrix}, Y''_{N-1} = \begin{pmatrix}s_{n-\frac{1}{10}}\\s_{n-1}\\s_{n-\frac{11}{10}}\\s_{n-\frac{19}{10}}\\s_n\end{pmatrix}, Y'''_{N-1} = \begin{pmatrix}t_{n-\frac{1}{10}}\\t_{n-1}\\t_{n-\frac{11}{10}}\\t_{n-\frac{19}{10}}\\t_n\end{pmatrix},$$

$$F_N = \begin{pmatrix}f_{n-\frac{1}{10}}\\f_{n-1}\\f_{n-\frac{11}{10}}\\f_{n-\frac{19}{10}}\\f_n\end{pmatrix}, F_{N-1} = \begin{pmatrix}f_{n+\frac{1}{10}}\\f_{n+1}\\f_{n+\frac{11}{10}}\\f_{n+\frac{19}{10}}\\f_{n+2}\end{pmatrix},$$

$$A^0 = \begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, A' = \begin{pmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{pmatrix}, A'' = \begin{pmatrix}0&0&0&0&\frac{1}{10}\\0&0&0&0&1\\0&0&0&0&\frac{11}{10}\\0&0&0&0&\frac{19}{10}\\0&0&0&0&2\end{pmatrix},$$

$$B' = \begin{pmatrix}0&0&0&0&\frac{1}{200}\\0&0&0&0&\frac{1}{2}\\0&0&0&0&\frac{121}{200}\\0&0&0&0&\frac{361}{200}\\0&0&0&0&2\end{pmatrix}, B'' = \begin{pmatrix}0&0&0&0&\frac{1}{6000}\\0&0&0&0&\frac{1}{6}\\0&0&0&0&\frac{1331}{6000}\\0&0&0&0&\frac{6859}{6000}\\0&0&0&0&\frac{4}{3}\end{pmatrix},$$

$$E^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 100337/31600800000 \\ 0 & 0 & 0 & 0 & -2759/395010 \\ 0 & 0 & 0 & 0 & -5168273/432000000 \\ 0 & 0 & 0 & 0 & -3467972131/23760000000 \\ 0 & 0 & 0 & 0 & -35456/197505 \end{pmatrix}$$

and

$$E' \begin{pmatrix} \frac{4774487}{4653936000000} & -\frac{949}{3499200000} & \frac{301157}{1197504000000} & -\frac{4927}{106375680000} & \frac{1637}{51710400000} \\ \frac{3535}{83106} & \frac{4315}{122472} & -\frac{1145}{37422} & \frac{1325}{290871} & -\frac{71}{23085} \\ \frac{290459826877}{4653936000000} & \frac{511981129}{8748000000} & -\frac{5483098423}{108864000000} & \frac{288764443}{39191040000} & -\frac{2570535011}{517104000000} \\ \frac{121063908407}{244944000000} & \frac{228690027541}{306180000000} & -\frac{696137901157}{1197504000000} & \frac{17212667359}{195955200000} & -\frac{1141481639}{19440000000} \\ \frac{172960}{290871} & \frac{70256}{76545} & -\frac{1880}{2673} & \frac{4600}{41553} & -\frac{11902}{161595} \end{pmatrix}$$

From the condition that,

$$\begin{aligned} p(z) &= \det(zA^0 - A') \\ &= \det\left[z \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] \\ &= \det\left[\begin{pmatrix} z & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & z \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] \\ &= \det \begin{pmatrix} z & 0 & 0 & 0 & -1 \\ 0 & z & 0 & 0 & -1 \\ 0 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & z & -1 \\ 0 & 0 & 0 & 0 & z-1 \end{pmatrix} \\ &= \begin{vmatrix} z & 0 & 0 & 0 & -1 \\ 0 & z & 0 & 0 & -1 \\ 0 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & z & -1 \\ 0 & 0 & 0 & 0 & z-1 \end{vmatrix} \\ &= z\left(z \begin{vmatrix} z & 0 & -1 \\ 0 & z & -1 \\ 0 & 0 & z-1 \end{vmatrix} - 1 \begin{vmatrix} 0 & z & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{vmatrix} \right) - 1\left(-z \begin{vmatrix} 0 & z & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{vmatrix} \right) \\ &= z\left(z\left(z^2(z-1)\right)\right) \\ &= z^4(z-1) \end{aligned}$$

Hence, $z_{1,2,3,4,5} = 0{,}0{,}0{,}0{,}1$. Therefore, the new methods are zero-stable since $|z| = 1$, i.e., simple, and the magnitude of other roots are zeros $|z| = 0$.

2.4. Convergence

A linear multi-step method is said to be convergent if it is consistent and zero-stable (or if it satisfies the root conditions). Since our methods are zero-stable and consistent, then the methods are convergent.

3. Numerical Scheme Implementation and Results

3.1. Numerical Experiments

This subsection tests the accuracy of the proposed methods with some numerical problems and compares the results with the existing methods. Error is defined as

$$\text{Error} = \max_{a\le x\le b} |y(x) - y_N(x)|,\ N \ge 1,2,3,\ldots \quad (2.53)$$

where $y(x)$ is the exact solution and $y_N(x)(N \ge 1)$ are the approximate solutions. The following examples are considered:

Example 3.1. Consider $y^{iv} - 4y'' = 0$ with initial conditions $y'''(0) = 16, y''(0) = 0, y'(0) = 3, y(0) = 1$, and $h = 0.003125$. Then, the exact solution is

$$y(x) = e^{2x} - e^{-2x} - x + 1$$

Example 3.2. Consider $y^{iv} = x, t \in [0,1]$ with initial conditions, $y'''(0) = 0, y''(0) = 0, y'(0) = 1, y(0) = 0$, and $h = \frac{1}{10}$. Then, the exact solution is

$$y(x) = \frac{x^5}{120} + x$$

Example 3.3. Consider $y^{iv} = \cos x - \sin x$ with initial conditions $y'''(0) = 7$, $y''(0) = -1$, $y'(0) = -1$, $y(0) = 0$, and $h = \frac{1}{320}$. Then, the exact solution is

$$y(x) = \cos x - \sin x + x^3 - 1$$

Table 2 shows the exact result, the computed result from the proposed schemes using power series (PPSS), and the result in [3]. Table 3 shows errors in the proposed scheme and [3] for Example 3.1.

Table 2. Numerical results for Example 3.1

X	**Exact**	**PPSS**	[3]
0	1	1.00000	1.000000
0.003125	1.0093750813803672792	1.0093750813803672792	1.009375081380367264
0.006250	1.0187506510467529486	1.0187506510467529486	1.018750651046752800
0.009375	1.0281271973042491331	1.0281271973042491331	1.028127197304248000
0.012500	1.0375052084960961721	1.0375052084960961721	1.037505208496096000
0.015625	1.0468851730227585890	1.0468851730227585891	1.046885173022758387
0.018750	1.0562675793610032975	1.0562675793610032975	1.056267579361001125
0.021875	1.0656529160829807860	1.0656529160829807861	1.065652916082977682
0.025000	1.0750416718753100306	1.0750416718753100306	1.075041671875305141
0.028125	1.0844343355581678774	1.0844343355581678774	1.084434335558156293
0.031250	1.0938313961043836435	1.0938313961043836434	1.093831396104360521
0.034375	1.1032333426585396797	1.1032333426585396797	1.1032333426585396797
0.037500	1.1126406645560786435	1.1126406645560786435	1.1126406645560786435
⋮	⋮	⋮	⋮
0.31250	2.0204845289132321645	2.0204845289132321645	2.0204845289132321645
0.62500	3.5788381606016512758	3.5788381606016512756	3.5788381606016512756
1	7.2537208156940375353	7.2537208156940375326	7.2537208156940375326

Table 3. Comparison of errors with numerical results of Example 3.1

n	x	**PPSS Error**	**Error** [3]
1	0.003125	$0.0000\text{E}+00$	$1.5000\text{E}-17$
2	0.006250	$0.0000\text{E}+00$	$1.4900\text{E}-16$
3	0.009375	$0.0000\text{E}+00$	$1.1330\text{E}-15$
4	0.012500	$0.0000\text{E}+00$	$1.7200\text{E}-16$
5	0.015625	$1.0000\text{E}-20$	$2.0200\text{E}-16$
6	0.018750	$0.0000\text{E}+00$	$2.1720\text{E}-15$
7	0.021875	$1.0000\text{E}-20$	$3.1040\text{E}-15$
8	0.025000	$0.0000\text{E}+00$	$4.8900\text{E}-15$
9	0.028125	$0.0000\text{E}+00$	$1.1584\text{E}-14$
10	0.031250	$1.0000\text{E}-20$	$2.3123\text{E}-14$
11	0.034375	$0.0000\text{E}+00$	-
12	0.037500	$0.0000\text{E}+00$	-
13	0.31250	$0.0000\text{E}+00$	-
14	0.62500	$2.1000\text{E}-98$	-
15	1	$2.7000\text{E}-18$	-

Table 4 shows the exact and computed results from the proposed schemes using power series (PPSS) for Example 3.2. Table 5 shows errors in the proposed scheme and [7] for Example 3.2.

Table 4. Numerical results for Example 3.2

x	**Exact Solutions**	**PPSS**
0.1	0.100000083333333333	0.1000000833333333333333
0.2	0.200002666666666667	0.2000026666666666666667
0.3	0.300020250000000000	0.3000202500000000000000
0.4	0.400085333333333333	0.4000853333333333333333
0.5	0.500260416666666667	0.5002604166666666666667
0.6	0.600648000000000000	0.6006480000000000000000
0.7	0.701400583333333333	0.7014005833333333333333
0.8	0.802730666666666667	0.8027306666666666666667
0.9	0.904920750000000000	0.9049207500000000000000
1.0	1.008333333333333333	1.0083333333333333333333

Table 5. Comparison of errors with numerical results of Example 3.2

n	[7]	[8]	[9]	**PPSS Error**
1	$1.658\text{E}-13$	$7.000\text{E}-10$	$0.000\text{E}-00$	$0.000\text{E}-00$
2	$3.316\text{E}-12$	$8.999\text{E}-10$	$0.000\text{E}-00$	$0.000\text{E}-00$
3	$7.183\text{E}-12$	$2.999\text{E}-09$	$0.000\text{E}-00$	$0.000\text{E}-00$
4	$6.649\text{E}-11$	$5.100\text{E}-09$	$0.000\text{E}-00$	$0.000\text{E}-00$
5	$9.906\text{E}-11$	$7.799\text{E}-09$	$0.000\text{E}-00$	$0.000\text{E}-00$
6	$3.217\text{E}-11$	$1.180\text{E}-08$	$0.000\text{E}-00$	$0.000\text{E}-00$
7	$2.432\text{E}-10$	$1.240\text{E}-08$	$0.000\text{E}-00$	$0.000\text{E}-00$
8	$3.202\text{E}-10$	$1.410\text{E}-08$	$2.000\text{E}-18$	$0.000\text{E}-00$
9	$2.540\text{E}-10$	$1.880\text{E}-08$	$2.000\text{E}-18$	$0.000\text{E}-00$
10	$2.020\text{E}-10$	$2.600\text{E}-08$	$1.000\text{E}-17$	$0.000\text{E}-00$

Table 6 shows the exact result, the computed result from the proposed schemes using power series (PPSS), and the result in [3]. Table 7 shows errors in the proposed scheme and [3] for Example 3.3.

Table 6. Numerical results for Example 3.3

x	**Exact**	**PPSS**	[3]
0	0	0.00000	-
$\frac{1}{320}$	-0.00312984720468769600	-0.00312984720468769600	-0.00312984720468770183
$\frac{2}{320}$	-0.00626924635577210114	-0.00626924635577210114	-0.00626924635577214781
$\frac{3}{320}$	-0.00941798368752841945	-0.00941798368752841944	-0.00941798368752885697
$\frac{4}{320}$	-0.01257584533946248273	-0.01257584533946248273	-0.0125758453394627160
$\frac{5}{320}$	-0.01574261735661109244	-0.01574261735661109244	-0.0157426173566113317
$\frac{6}{320}$	-0.01891808568984328399	-0.01891808568984328399	-0.0189180856898435642
$\frac{7}{320}$	-0.02210203619616251069	-0.02210203619616251069	-0.0221020361961631824
$\frac{8}{320}$	-0.02529425463900974441	-0.02529425463900974441	-0.025294254639010215
$\frac{9}{320}$	-0.02849452668856748983	-0.02849452668856748983	-0.0284945266885679628
$\frac{10}{320}$	-0.03170263792206470950	-0.03170263792206470950	-0.00312984720468770183

Table 7. Comparison of errors with numerical results of Example 3.3

x	**PPSS Error**	**Error** [3]
0	0	-
$\frac{1}{320}$	$0.0000\mathrm{E}+00$	$5.8350\mathrm{E}-18$
$\frac{2}{320}$	$0.0000\mathrm{E}+00$	$4.6712\mathrm{E}-17$
$\frac{3}{320}$	$1.0000\mathrm{E}-20$	$4.3748\mathrm{E}-16$
$\frac{4}{320}$	$0.0000\mathrm{E}+00$	$2.3340\mathrm{E}-16$
$\frac{5}{320}$	$0.0000\mathrm{E}+00$	$2.3920\mathrm{E}-16$
$\frac{6}{320}$	$0.0000\mathrm{E}+00$	$2.8020\mathrm{E}-16$
$\frac{7}{320}$	$0.0000\mathrm{E}+00$	$6.7177\mathrm{E}-16$
$\frac{8}{320}$	$0.0000\mathrm{E}+00$	$4.6706\mathrm{E}-16$
$\frac{9}{320}$	$0.0000\mathrm{E}+00$	$5.1408\mathrm{E}-16$
$\frac{10}{320}$	$1.0000\mathrm{E}-20$	$5.8350\mathrm{E}-18$

Figures 1-3 present the behavior of the exact solutions compared with the PPSS for Example 3.1-3.3, respectively.

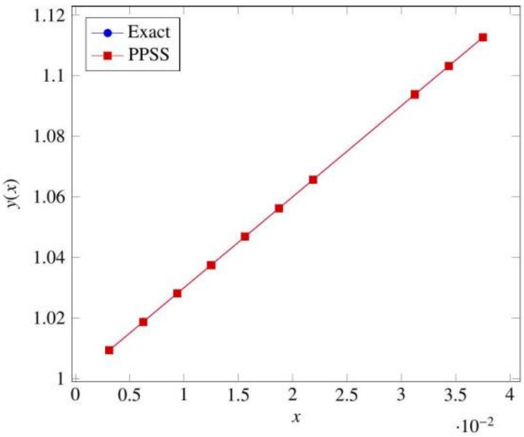

Figure 1. The behavior of the exact solution compared with the PPSS for Example 3.1

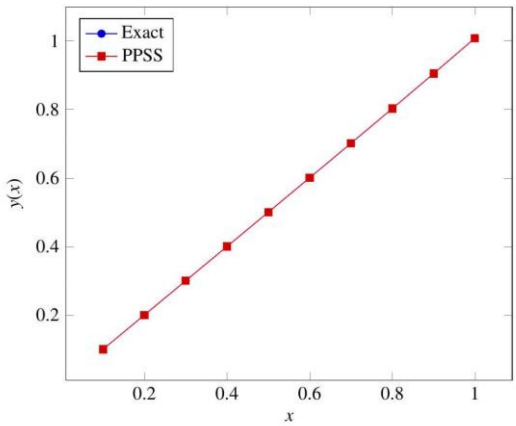

Figure 2. The behavior of the exact solution compared with the PPSS for Example 3.2

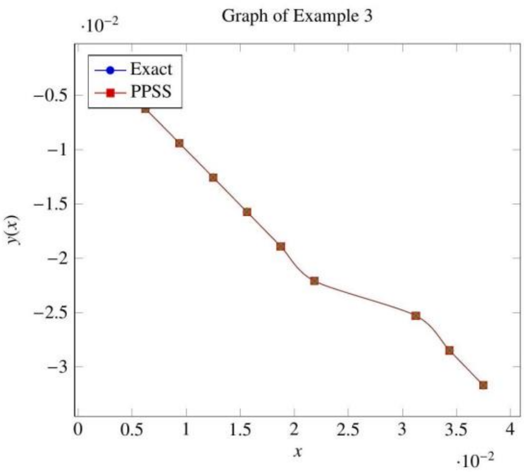

Figure 3. The behavior of the exact solution compared with the PPSS for Example 3.3

4. Conclusion

We have incorporated off-step points for collocation and interpolation to develop a more accurate Two-step Block Hybrid Linear Multi-step method for the numerical solution of initial value problems of fourth-order differential equations. In developing the numerical method with two steps ($k = 2$), we used three off-step points at interpolation and collocation points. The order and error constants were obtained using the method employed by [6]. The Two-step Block Hybrid Linear Multi-step method has six orders of accuracy. Moreover, the zero stability of the Block Hybrid Method was analyzed using the concept of [6], and the method was zero stable. Hence, block hybrid methods are convergent. The method was used to solve some problems adapted from [3], and the computed result and the exact solution were compared with the solution from [3]. Errors were computed, and the proposed method produced approximations closer to the exact solution than the reviewed work. The introduction of three off-step points at both collocation and interpolation has proven to produce more accurate results than the literature results by those who used one off-step point at both collocation and interpolation. Therefore, introducing more off-step points at both collocation and interpolation in developing block methods results in a better approximation.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

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