JOURNAL OF AMASYA UNIVERSITY THE INSTITUTE OF SCIENCES AND TECNOLOGY (JAUIST)

VOLUME: 4 ISSUE: 2

YEAR: 2023

ISSN: 2717-8900

Journal of Amasya University the Institute of Sciences and Technology (abbreviated as J. Amasya Univ. Inst. Sci. Technol. or JAUIST) is a science journal focusing on new science theories or their applications.

J. Amasya Univ. Inst. Sci. Technol. is an international, peer-reviewed, and open-access journal.

JAUIST was founded in 2020, and its first issue was published on 02 July 2020.

Language: As of 2024, JAUIST accepts contributions in **American English** only.

Editor-in-Chief: Ümit YILDIRIM

Email: amasyascienceandtechnology@gmail.com

APC: JAUIST incurs no article processing charges.

Review Process: Blind Peer Review

The Policy of Screening for Plagiarism: JAUIST accepts submissions for pre-review only if their similarity rate on iThenticate, excluding Sources less than 1% and Bibliography, is **30% or less**.

Publication Ethics: The governance structure of the Journal of Amasya University the Institute of Sciences and Technology and its acceptance procedures are transparent and designed to ensure the highest quality of published material. JAUIST adheres to the international standards developed by the Committee on Publication Ethics (COPE).

Contents

Journal of Amasya University the Institute of Sciences and Technology 4(2) (2023) 72-81

JOURNAL OF AMASYA UNIVERSITY THE INSTITUTE OF SCIENCES AND TECHNOLOGY (JAUIST)

ISSN: 2717-8900 https://doi.org/10.54559/jauist.1312261

Pseudoparallel invariant submanifolds of Kenmotsu manifolds

Nurnisa Karaman[1], Mehmet Atçeken[2,*]

[1,2]Department of Mathematics, Faculty of Arts and Sciences, Aksaray University, Aksaray, Türkiye

Abstract — In this paper, we consider pseudoparallel invariant submanifolds, a particular class of invariant submanifolds of Kenmotsu manifolds, on W_8 curvature tensor and investigate some of their basic characterizations, such as W_8 pseudoparallel, W_8-2 pseudoparallel, W_8-Ricci generalized pseudoparallel, and W_8-2 Ricci generalized pseudoparallel. Moreover, we present some relations between these pseudoparallel invariant submanifolds and semi-parallel invariant submanifolds. We finally discuss the need for further research.

Keywords: *Kenmotsu manifold, pseudoparallel invariant submanifold,* 2-*pseudoparallel invariant submanifolds*

Subject Classification (2020): 53C15, 53D10

1. Introduction

In 1972, Kenmotsu [1] studied a class of contact Riemannian manifolds and called them Kenmotsu manifolds. He proved that if a Kenmotsu manifold satisfies the condition $R(a_1, a_2)\cdot R = 0$, then the manifold has negative curvature -1, where R is the Riemannian curvature tensor of $(1,3)$-type and $R(a_1, a_2)$ denotes the derivation of the tensor algebra at each point of the tangent space. Then, the properties of Kenmotsu manifolds have been studied by several authors, such as Haseeb [2], Wang and Liu [3], Wang and Wang [4], Özgür and De [5], Tripathi and Gupta [6], Singh et al. [7], Parakasha and Hadimani [8], De and De [9], and De and Pathak [10]. Afterward, the geometry of submanifolds has been examined on different manifolds, and many essential properties have been obtained [11–13].

The curvature tensor is one of the most important concepts used to learn the characterization of a manifold. The properties of manifolds, important for mathematics, physics, and engineering, are handled with the help of curvature tensors. One of the essential geometrical manifolds is the Kenmotsu manifolds. Kenmotsu manifolds are one-dimensional versions of complex manifolds. If an almost contact metric manifold satisfies the condition

$$(\nabla_{a_1}\varphi)a_2 = g(\varphi a_1, a_2)\xi - \eta(a_2)\varphi a_1$$

Then, the manifold is called a Kenmotsu manifold.

In this article, pseudoparallel invariant submanifolds for Kenmotsu manifolds are investigated. The Kenmotsu manifold is considered on the W_8-curvature tensor. Submanifolds of these manifolds with properties, such as W_8-pseudoparallel, W_8-2 pseudoparallel, W_8-Ricci generalized pseudoparallel, and W_8-2 Ricci generalized pseudoparallel has been investigated.

*Corresponding Author

[1]nurnisakaraman@gmail.com; [2]mehmetatceken@aksaray.edu.tr

Article History: Received: 09 Jun 2023 — Accepted: 30 Nov 2023 — Published: 31 Dec 2023

2. Preliminaries

This section provides some basic properties to be required in the following sections. Let M be a $(2n+1)$-dimensional differentiable manifold. If it admits a tensor field φ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying the following conditions:

$$\varphi^2 a_1 = -a_1 + \eta(a_1)\xi \text{ and } \eta(\xi) = 1 \tag{2.1}$$

for all $a_1, a_2, \xi \in \chi(M)$, then (φ, ξ, η) is called almost contact structure and (M, φ, ξ, η) is called almost contact manifold. If there is a g metric on the $(2n+1)$-dimensional, almost contact manifold satisfying

$$g(a_1, \xi) = \eta(a_1) \tag{2.2}$$

and

$$g(\varphi a_1, \varphi a_2) = g(a_1, a_2) - \eta(a_1)\eta(a_2) \tag{2.3}$$

for all $a_1, a_2 \in \chi(M)$, then (φ, ξ, η, g) is called an almost contact metric structure and $(M, \varphi, \xi, \eta, g)$ is called an almost contact metric manifold.

On a $(2n+1)$-dimensional manifold M, we have

$$g(\varphi a_1, a_2) = -g(a_1, \varphi a_2)$$

for all $a_1, a_2 \in \chi(M)$, that is, φ is an anti-symmetric tensor field according to the g metric. The transformation Φ defined as

$$\Phi(a_1, a_2) = g(a_1, \phi a_2)$$

for all $a_1, a_2 \in \chi(M)$, is called the fundamental 2-form of the (φ, ξ, η, g) almost contact metric structure where

$$\eta \wedge \Phi^n \neq 0$$

Let M be a $(2n+1)$-dimensional almost contact metric manifold given the structure (φ, ξ, η, g). If $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ on an almost contact metric manifold M, then M is called an almost Kenmotsu manifold.

Let M be $(2n+1)$-dimensional almost contact metric manifold providing the structure (φ, ξ, η, g). If it satisfies the following conditions:

$$\begin{cases} \varphi^2 a_1 = -a_1 + \eta(a_1)\xi, \varphi\xi = 0 \\ \eta(\varphi a_1) = 0, \eta(a_1) = g(a_1, \xi)\eta(\xi) = 1 \\ \left(\nabla_{a_1}\varphi\right)a_2 = g(\varphi a_1, a_2)\xi - \eta(a_2)\varphi a_1 \end{cases} \tag{2.4}$$

for all $a_1, a_2, \xi \in \chi(M)$, then M is called the Kenmotsu manifold.

Lemma 2.1. [1] Let M be a $(2n+1)$-dimensional Kenmotsu manifold. In this case, the following equations are obtained.

$$\nabla_{a_1}\xi = -a_1 + \eta(a_1)\xi \tag{2.5}$$

$$\left(\nabla_{a_1}\eta\right)a_2 = g(a_1, a_2) - \eta(a_1)\eta(a_2) \tag{2.6}$$

$$R(a_1, a_2)\xi = \eta(a_1)a_2 - \eta(a_2)a_1 \tag{2.7}$$

$$R(\xi, a_1)a_2 = -g(a_1, a_2)\xi + \eta(a_2)a_1 \tag{2.8}$$

$$R(a_1, \xi)a_2 = g(a_1, a_2)\xi - \eta(a_2)a_1 \tag{2.9}$$

$$S(a_1, \xi) = -2n\eta(a_1) \tag{2.10}$$

and

$$Q\xi = -2n\xi \tag{2.11}$$

where ∇, R, S, and Q are Levi-Civita connections on M, Riemann tensor, Ricci tensor, and Ricci operator, respectively.

Definition 2.2. [6] Let M be a $(2n+1)$-dimensional semi-Riemannian manifold. Then, the W_8-curvature tensor is defined as

$$W_8(a_1, a_2)a_3 = R(a_1, a_2)a_3 - \frac{1}{2n}[S(a_2, a_3)a_1 - S(a_1, a_3)a_2] \tag{2.12}$$

for all $a_1, a_2, a_3 \in \chi(M)$.

If we choose $a_1 = \xi$, $a_2 = \xi$, and $a_3 = \xi$ in (2.12) for $(2n+1)$-dimensional Kenmotsu manifold respectively, we get

$$W_8(\xi, a_2)a_3 = -g(a_2, a_3)\xi + \eta(a_3)a_2 - \eta(a_2)a_3 - \frac{1}{2n}S(a_2, a_3)\xi \tag{2.13}$$

$$W_8(a_1, \xi)a_3 = g(a_1, a_3)\xi - \eta(a_1)a_3 \tag{2.14}$$

and

$$W_8(a_1, a_2)\xi = \eta(a_1)a_2 + \frac{1}{2n}S(a_1, a_2)\xi \tag{2.15}$$

Let $\widetilde{M}$ be immersed submanifold of a Kenmotsu manifold $M(\phi, \xi, \eta, g)$. Moreover, let the tangent and normal subspaces of $\widetilde{M}$ in $M(\phi, \xi, \eta, g)$ be $\Gamma(T\widetilde{M})$ and $\Gamma(T^{\perp}\widetilde{M})$, respectively. Then, Gauss and Weingarten formulas for $\Gamma(T\widetilde{M})$ and $\Gamma(T^{\perp}\widetilde{M})$ are

$$\nabla_{a_1} a_2 = \widetilde{\nabla}_{a_1} a_2 + \sigma(a_1, a_2) \tag{2.16}$$

and

$$\nabla_{a_1} a_5 = -A_{a_5}a_1 + \widetilde{\nabla}^{\perp}_{a_1} a_5 \tag{2.17}$$

respectively, for all $a_1, a_2 \in \Gamma(T\widetilde{M})$ and $a_5 \in \Gamma(T^{\perp}\widetilde{M})$, where $\widetilde{\nabla}$ and $\widetilde{\nabla}^{\perp}$ are the connections on $\widetilde{M}$ and $\Gamma(T^{\perp}\widetilde{M})$, respectively, σ and A are the second fundamental form and the shape operator of $\widetilde{M}$. There is a relation

$$g(A_{a_5}a_1, a_2) = g(\sigma(a_1, a_2), a_5)$$

between the second basic form and shape operator defined as above. The covariant derivative of the second fundamental form σ is defined as

$$(\nabla_{a_1}\sigma)(a_2, a_3) = \widetilde{\nabla}^{\perp}_{a_1} \sigma(a_2, a_3) - \sigma(\widetilde{\nabla}_{a_1} a_2, a_3) - \sigma(a_2, \widetilde{\nabla}_{a_1} a_3) \tag{2.18}$$

Specifically, if $\nabla\sigma = 0$, $\widetilde{M}$ is said to be a parallel second fundamental form.

Let $\tilde{R}$ be Riemann curvature tensor of $\widetilde{M}$. In this case, the Gauss equation can be expressed as

$$R(a_1,a_2)a_3 = \tilde{R}(a_1,a_2)a_3 + A_{\sigma(a_1,a_3)}a_2 - A_{\sigma(a_2,a_3)}a_1 + \left(\nabla_{a_1}\sigma\right)(a_2,a_3) - \left(\nabla_{a_2}\sigma\right)(a_1,a_3)$$

for all $a_1,a_2,a_3 \in \Gamma\left(T\widetilde{M}\right)$, where

$$\left(\widetilde{\nabla}_{a_1}\sigma\right)(a_2,a_3) - \left(\widetilde{\nabla}_{a_2}\sigma\right)(a_1,a_3) = 0$$

then it is called a curvature-invariant submanifold. Let $\widetilde{M}$ be a Riemannian manifold, T be $(0,k)$-type tensor field and A be $(0,2)$-type tensor field. In this case, Tachibana tensor field $Q(A,T)$ is defined as

$$Q(A,T)(X_1,\cdots,X_k;a_1,a_2) = -T\left((a_1 \wedge_A a_2)X_1,\cdots,X_k\right) - \cdots - T(X_1,\cdots,X_{k-1},(a_1 \wedge_A a_2)X_k) \qquad (2.19)$$

where

$$(a_1 \wedge_A a_2)a_3 = A(a_2,a_3)a_1 - A(a_1,a_3)a_2,\ k \geq 1,\ X_1,X_2,\cdots,X_k,a_1,a_2 \in \Gamma\left(T\widetilde{M}\right) \qquad (2.20)$$

Definition 2.3. [5] A submanifold of a Riemannian manifold (M,g) is said to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel, and 2-Ricci generalized pseudoparallel if

$$R\cdot\sigma \text{ and } Q(g,\sigma)$$

$$R\cdot\nabla\sigma \text{ and } Q(g,\nabla\sigma)$$

$$R\cdot\sigma \text{ and } Q(S,\sigma)$$

$$R\cdot\nabla\sigma \text{ and } Q(S,\nabla\sigma)$$

are linearly dependent, respectively.

3. Pseudoparallel Invariant Submanifolds of Kenmotsu Manifold

Let $\widetilde{M}$ be the immersed submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $\phi\left(T_{a_1}M\right) \subset T_{a_1}M$ in every a_1 point, the $\widetilde{M}$ manifold is called an invariant submanifold. We note that all of the properties of an invariant submanifold inherit the ambient manifold. Throughout this paper, let $\widetilde{M}$ be an invariant submanifold of the Kenmotsu manifold $M(\phi,\xi,\eta,g)$. Therefore, for all $a_1,a_2 \in \Gamma\left(T\widetilde{M}\right)$,

$$\sigma(\phi a_1,a_2) = \sigma(a_1,\phi a_2) = \phi\sigma(a_1,a_2) \qquad (3.1)$$

and

$$\sigma(a_1,\xi) = 0 \qquad (3.2)$$

Lemma 3.1. Let $\widetilde{M}$ be the invariant submanifold of the $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. The second fundamental form σ of $\widetilde{M}$ is parallel if and only if $\widetilde{M}$ is the total geodesic submanifold.

Consider the invariant submanifolds of the Kenmotsu manifold on the W_8-curvature tensor.

Definition 3.2. Let $\widetilde{M}$ be the invariant submanifold of the $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $W_8\cdot\sigma$ and $Q(g,\sigma)$ are linearly dependent, $\widetilde{M}$ is called W_8-pseudoparallel submanifold.

In this mean, it can be said that there is a function k_1 on the set $M_1 = \left\{x \in \widetilde{M} \mid \sigma(x) \neq g(x)\right\}$ such that

$$W_8\cdot\sigma = k_1 Q(g,\sigma)$$

If $k_1 = 0$ specifically, $\widetilde{M}$ is called a W_8-semi-parallel submanifold.

Theorem 3.3. Let $\widetilde{M}$ be an invariant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $\widetilde{M}$ is W_8-pseudoparallel submanifold, then $\widetilde{M}$ is either a total geodesic or $k_1 = -1$.

Proof.

Assume that $\widetilde{M}$ is a W_8-pseudoparallel submanifold. Thus, for all $a_1, a_2, a_4, a_5 \in \Gamma(T\widetilde{M})$,

$$(W_8(a_1,a_2)\cdot\sigma)(a_4,a_5) = k_1 Q(g,\sigma)(a_4,a_5;a_1,a_2) \tag{3.3}$$

From (3.3), it is clear that

$$R^{\perp}(a_1,a_2)\sigma(a_4,a_5) - \sigma(W_8(a_1,a_2)a_4,a_5) - \sigma(a_4,W_8(a_1,a_2)a_5) = -k_1\left\{\sigma\left((a_1\wedge_g a_2)a_4,a_5\right) + \sigma\left(a_4,(a_1\wedge_g a_2)a_5\right)\right\}$$

That is, we can write

$$\begin{gathered} R^{\perp}(a_1,a_2)\sigma(a_4,a_5) - \sigma(W_8(a_1,a_2)a_4,a_5) - \sigma(a_4,W_8(a_1,a_2)a_5) = \\ -k_1\{g(a_2,a_4)\sigma(a_1,a_5) - g(a_1,a_4)\sigma(a_2,a_5) + g(a_2,a_5)\sigma(a_4,a_1) - g(a_1,a_5)\sigma(a_4,a_2)\} \end{gathered} \tag{3.4}$$

If we choose $a_5 = \xi$ in (3.4) and make use of (2.15) and (3.2), then

$$\eta(a_1)\sigma(a_4,a_2) = k_1[\eta(a_2)\sigma(a_4,a_1) - \eta(a_1)\sigma(a_4,a_2)] \tag{3.5}$$

If we choose $a_1 = \xi$ in (3.5), we obtain

$$(k_1+1)\sigma(a_4,a_2) = 0$$

□

Corollary 3.4. Let $\widetilde{M}$ be an pseudoparallel invariant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. Then, $\widetilde{M}$ *is* W_8-semi-parallel if and only if $\widetilde{M}$ is totally geodesic.

Definition 3.5. Let $\widetilde{M}$ be the invariant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $W_8\cdot\nabla\sigma$ and $Q(g,\nabla\sigma)$ are linearly dependent, then $\widetilde{M}$ is called W_8-2 pseudoparallel submanifold.

In this case, there is a function k_2 on the set $M_2 = \{x \in \widetilde{M} \mid \nabla\sigma(x) \neq g(x)\}$ such that

$$W_8\cdot\nabla\sigma = k_2 Q(g,\nabla\sigma)$$

If $k_2 = 0$ specifically, $\widetilde{M}$ is called a W_8-2 semiparallel submanifold.

Theorem 3.6. Let $\widetilde{M}$ be an invariant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $\widetilde{M}$ is W_8-2 pseudoparallel submanifold, then $\widetilde{M}$ is either a total geodesic submanifold or $k_2 = -1$.

Proof.

Assume that $\widetilde{M}$ is a W_8-2 pseudoparallel submanifold. Therefore,

$$(W_8(a_1,a_2)\cdot\nabla\sigma)(a_4,a_5,a_3) = k_2 Q(g,\nabla\sigma)(a_4,a_5,a_3;a_1,a_2) \tag{3.6}$$

for all $a_1, a_2, a_4, a_5, a_3 \in \Gamma(T\widetilde{M})$. If we choose $a_1 = a_3 = \xi$ in (3.6), then

$$R^{\perp}(\xi,a_2)(\nabla_{a_4}\sigma)(a_5,\xi)-(\nabla_{W_8(\xi,a_2)a_4}\sigma)(a_5,\xi)-(\nabla_{a_4}\sigma)(W_8(\xi,a_2)a_5,\xi)-(\nabla_{a_4}\sigma)(a_5,W_8(\xi,a_2)\xi)= \\ -k_2\left\{\left(\nabla_{(\xi\wedge_g a_2)a_4}\sigma\right)(a_5,\xi)+(\nabla_{a_4}\sigma)\left((\xi\wedge_g a_2)a_5,\xi\right)+(\nabla_{a_4}\sigma)(a_5,(\xi\wedge_g a_2)\xi)\right\} \tag{3.7}$$

Calculate all the expressions in (3.7). Thus, by using (2.18) and taking into account Lemma 3.1, we can write

$$\begin{aligned} R^{\perp}(\xi,a_2)(\nabla_{a_4}\sigma)(a_5,\xi) &= R^{\perp}(\xi,a_2)\{\widetilde{\nabla}^{\perp}_{a_4}\sigma(a_5,\xi)-\sigma(\widetilde{\nabla}_{a_4}a_5,\xi)-\sigma(a_5,\widetilde{\nabla}_{a_4}\xi)\} \\ &= -R^{\perp}(\xi,a_2)\sigma(a_5,-a_4+\eta(a_4)\xi) \\ &= R^{\perp}(\xi,a_2)\sigma(a_5,a_4) \end{aligned} \tag{3.8}$$

secondly,

$$\begin{aligned} (\nabla_{W_8(\xi,a_2)a_4}\sigma)(a_5,\xi) &= \widetilde{\nabla}^{\perp}_{W_8(\xi,a_2)a_4}\sigma(a_5,\xi)-\sigma(\widetilde{\nabla}_{W_8(\xi,a_2)a_4}a_5,\xi)-\sigma(a_5,\widetilde{\nabla}_{W_8(\xi,a_2)a_4}\xi) \\ &= -\sigma(a_5,-W_8(\xi,a_2)a_4+\eta(W_8(\xi,a_2)a_4)\xi) \\ &= \eta(a_4)\sigma(a_5,a_2)-\eta(a_2)\sigma(a_5,a_4) \end{aligned} \tag{3.9}$$

Moreover,

$$\begin{aligned} (\nabla_{a_4}\sigma)(W_8(\xi,a_2)a_5,\xi) &= \widetilde{\nabla}^{\perp}_{a_4}\sigma(W_8(\xi,a_2)a_5,\xi)-\sigma(\widetilde{\nabla}_{a_4}W_8(\xi,a_2)a_5,\xi)-\sigma(W_8(\xi,a_2)a_5,\widetilde{\nabla}_{a_4}\xi) \\ &= -\sigma\left(-g(a_2,a_5)\xi+\eta(a_5)a_2-\eta(a_2)a_5-\frac{1}{2n}S(a_2,a_5)\xi,-a_4+\eta(a_4)\xi\right) \\ &= \eta(a_5)\sigma(a_2,a_4)-\eta(a_2)\sigma(a_5,a_4) \end{aligned} \tag{3.10}$$

$$\begin{aligned} (\nabla_{a_4}\sigma)(a_5,W_8(\xi,a_2)\xi) &= (\nabla_{a_4}\sigma)(a_5,a_2-\eta(a_2)\xi) \\ &= (\nabla_{a_4}\sigma)(a_5,a_2)-(\nabla_{a_4}\sigma)(a_5,\eta(a_2)\xi) \\ &= (\nabla_{a_4}\sigma)(a_5,a_2)-\widetilde{\nabla}^{\perp}_{a_4}\sigma(a_5,\eta(a_2)\xi)+\sigma(\widetilde{\nabla}_{a_4}a_5,\eta(a_2)\xi)+\sigma(a_5,\widetilde{\nabla}_{a_4}\eta(a_2)\xi) \\ &= (\nabla_{a_4}\sigma)(a_5,a_2)-\eta(a_2)\sigma(a_5,a_4) \end{aligned} \tag{3.11}$$

$$\begin{aligned} \left(\nabla_{(\xi\wedge_g a_2)a_4}\sigma\right)(a_5,\xi) &= \widetilde{\nabla}^{\perp}_{(\xi\wedge_g a_2)a_4}\sigma(a_5,\xi)-\sigma\left(\widetilde{\nabla}_{(\xi\wedge_g a_2)a_4}a_5,\xi\right)-\sigma\left(a_5,\widetilde{\nabla}_{(\xi\wedge_g a_2)a_4}\xi\right) \\ &= -\eta(a_4)\sigma(a_5,a_2) \end{aligned} \tag{3.12}$$

$$\begin{aligned} (\nabla_{a_4}\sigma)\left((\xi\wedge_g a_2)a_5,\xi\right) &= \widetilde{\nabla}^{\perp}_{a_4}\sigma\left((\xi\wedge_g a_2)a_5,\xi\right)-\sigma(\widetilde{\nabla}_{a_4}(\xi\wedge_g a_2)a_5,\xi)-\sigma\left((\xi\wedge_g a_2)a_5,\widetilde{\nabla}_{a_4}\xi\right) \\ &= -\sigma(g(a_2,a_5)\xi-g(\xi,a_5)a_2,-a_4+\eta(a_4)\xi) \\ &= -\eta(a_5)\sigma(a_2,a_4) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} (\nabla_{a_4}\sigma)(a_5,(\xi\wedge_g a_2)\xi) &= (\nabla_{a_4}\sigma)(a_5,\eta(a_2)\xi-a_2) \\ &= (\nabla_{a_4}\sigma)(a_5,\eta(a_2)\xi)-(\nabla_{a_4}\sigma)(a_5,a_2) \\ &= \eta(a_2)\sigma(a_5,a_4)-(\nabla_{a_4}\sigma)(a_5,a_2) \end{aligned} \tag{3.14}$$

If we substitute (3.8)-(3.14) in (3.7), we have

$$\begin{gathered} R^{\perp}(\xi,a_2)\sigma(a_5,a_4)-\eta(a_4)\sigma(a_5,a_2)-\eta(a_5)\sigma(a_2,a_4)+3\eta(a_2)\sigma(a_5,a_4)-\left(\nabla_{a_4}\sigma\right)(a_5,a_2)= \\ -k_2\{-\eta(a_5)\sigma(a_4,a_2)-\eta(a_4)\sigma(a_2,a_5)+\eta(a_2)\sigma(a_5,a_4)-\left(\nabla_{a_4}\sigma\right)(a_5,a_2)\} \end{gathered} \tag{3.15}$$

If we choose $a_5=\xi$ in (3.5) and use,

$$\left(\nabla_{a_4}\sigma\right)(\xi,a_2)=\sigma(a_4,a_2)$$

we get

$$[k_2+1]\sigma(a_2,a_4)=0$$

which proves the assertions.

□

Corollary 3.7. Let $\widetilde{M}$ be an invariant pseudoparallel submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. Then, $\widetilde{M}$ *is* W_8-2 semi-parallel if and only if $\widetilde{M}$ is totally geodesic.

Definition 3.8. Let $\widetilde{M}$ be the invariant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $W_8\cdot\sigma$ and $Q(S,\sigma)$ are linearly dependent, $\widetilde{M}$ is called W_8-Ricci generalized pseudoparallel submanifold.

In this case, there is a function k_3 on the set $M_3=\left\{x\in\widetilde{M}\mid\sigma(x)\neq S(x)\right\}$ such that

$$W_8\cdot\sigma=k_3Q(S,\sigma)$$

If $k_3=0$ specifically, $\widetilde{M}$ is called a W_8-Ricci generalized semiparallel submanifold.

Theorem 3.9. Let $\widetilde{M}$ be the invariant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $\widetilde{M}$ is W_8-Ricci generalized pseudoparallel submanifold, then $\widetilde{M}$ is either a total geodesic or $k_3=\frac{1}{2n}$.

Proof.

Assume that $\widetilde{M}$ is a W_8-Ricci generalized pseudoparallel submanifold. Therefore, we have

$$(W_8(a_1,a_2)\cdot\sigma)(a_4,a_5)=k_3Q(S,\sigma)(a_4,a_5;a_1,a_2)$$

that is

$$\begin{gathered} R^{\perp}(a_1,a_2)\sigma(a_4,a_5)-\sigma(W_8(a_1,a_2)a_4,a_5)-\sigma(a_4,W_8(a_1,a_2)a_5)= \\ -k_3\{\sigma\left((a_1\wedge_S a_2)a_4,a_5\right)+\sigma(a_4,(a_1\wedge_S a_2)a_5)\} \end{gathered} \tag{3.16}$$

for all $a_1,a_2,a_4,a_5\in\Gamma\left(T\widetilde{M}\right)$. If we choose $a_1=a_5=\xi$ in (3.6), we get

$$-\sigma(a_4,W_8(\xi,a_2)\xi)=k_3S(\xi,\xi)\sigma(a_4,a_2) \tag{3.17}$$

If we use (2.10) and (2.15) in (3.7), we have

$$(1-2nk_3)\sigma(a_4,a_2)=0$$

□

Corollary 3.10. Let $\widetilde{M}$ be an invariant pseudoparallel submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. Then, $\widetilde{M}$ is W_8-Ricci generalized semi-parallel if and only if $\widetilde{M}$ is totally geodesic.

Definition 3.11. *Let* $\widetilde{M}$ be an invariant pseudoparallel submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $W_8 \cdot \widetilde{\nabla}\sigma$ and $Q(S,\widetilde{\nabla}\sigma)$ are linearly dependent, $\widetilde{M}$ is called W_8-2 Ricci generalized pseudoparallel submanifold.

Then, there is a function k_4 on the set $M_4 = \{x \in \widetilde{M} \mid \nabla\sigma(x) \neq S(x)\}$ such that

$$W_8 \cdot \nabla\sigma = k_4 Q(S,\nabla\sigma)$$

If specifically, $k_4 = 0$, M is called a W_8-2 Ricci generalized semiparallel submanifold.

Theorem 3.12. Let $\widetilde{M}$ be an invariant pseudoparallel submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $\widetilde{M}$ is W_8-2 Ricci generalized pseudoparallel submanifold, then $\widetilde{M}$ is either a total geodesic or $k_4 = \frac{1}{2n}$.

Proof.

Assume that $\widetilde{M}$ is a W_8-2 Ricci generalized pseudoparallel submanifold. Thus, we can write

$$(W_8(a_1,a_2)\cdot\nabla\sigma)(a_4,a_5,a_3) = k_4 Q(S,\nabla\sigma)(a_4,a_5,a_3;a_1,a_2) \tag{3.18}$$

for all $a_1,a_2,a_4,a_5,a_3 \in \Gamma(T\widetilde{M})$. If we choose $a_1 = a_5 = \xi$ in (3.18), we can write

$$\begin{aligned} &R^{\perp}(\xi,a_2)(\nabla_{a_4}\sigma)(\xi,a_3) - (\nabla_{W_8(\xi,a_2)a_4}\sigma)(\xi,a_3) - (\nabla_{a_4}\sigma)(W_8(\xi,a_2)\xi,a_3) - (\nabla_{a_4}h)(\xi,W_8(\xi,a_2)a_3) \\ &= -k_4\{(\nabla_{(\xi\wedge_S a_2)a_4}\sigma)(\xi,a_3) + (\nabla_{a_4}\sigma)((\xi\wedge_S a_2)\xi,a_3) + (\widetilde{\nabla}_{a_4}\sigma)(\xi,(\xi\wedge_S a_2)a_3)\} \end{aligned} \tag{3.19}$$

Calculate all the expressions in (3.9). Firstly, making use of (2.18), (3.1), and Lemma 3.1, we can write

$$\begin{aligned} R^{\perp}(\xi,a_2)(\nabla_{a_4}\sigma)(\xi,a_3) &= R^{\perp}(\xi,a_2)\{\widetilde{\nabla}^{\perp}_{a_4}\sigma(\xi,a_3) - \sigma(\widetilde{\nabla}_{a_4}a_3,\xi) - \sigma(a_3,\widetilde{\nabla}_{a_4}\xi)\} \\ &= R^{\perp}(\xi,a_2)\sigma(a_3,a_4) \end{aligned} \tag{3.20}$$

For the same reason, we can write

$$\begin{aligned} (\nabla_{W_8(\xi,a_2)a_4}\sigma)(\xi,a_3) &= \widetilde{\nabla}^{\perp}_{W_8(\xi,a_2)a_4}\sigma(\xi,a_3) - \sigma(\widetilde{\nabla}_{W_8(\xi,a_2)a_4}\xi,a_3) - \sigma(\xi,\widetilde{\nabla}_{W_8(\xi,a_2)a_4}a_3) \\ &= \eta(a_4)\sigma(a_2,a_3) - \eta(a_2)\sigma(a_3,a_4) \end{aligned} \tag{3.21}$$

$$\begin{aligned} (\nabla_{a_4}\sigma)(W_8(\xi,a_2)\xi,a_3) &= (\nabla_{a_4}\sigma)(a_2+\eta(a_2)\xi,a_3) \\ &= (\nabla_{a_4}\sigma)(a_2,a_3) - \eta(a_2)\sigma(a_4,a_3) \end{aligned} \tag{3.22}$$

$$\begin{aligned} (\nabla_{a_4}\sigma)(\xi,W_8(\xi,a_2)a_3) &= \widetilde{\nabla}^{\perp}_{a_4}\sigma(\xi,W_8(\xi,a_2)a_3) - \sigma(\widetilde{\nabla}_{a_4}\xi,W_8(\xi,a_2)a_3) - \sigma(\xi,\widetilde{\nabla}_{a_4}W_8(\xi,a_2)a_3) \\ &= \eta(a_3)\sigma(a_4,a_2) - \eta(a_2)\sigma(a_3,a_4) \end{aligned} \tag{3.23}$$

$$\begin{aligned} (\nabla_{(\xi\wedge_S a_2)a_4}\sigma)(\xi,a_3) &= \widetilde{\nabla}^{\perp}_{(\xi\wedge_S a_2)a_4}\sigma(\xi,a_3) - \sigma(\widetilde{\nabla}_{(\xi\wedge_S a_2)a_4}\xi,a_3) - \sigma(\xi,\widetilde{\nabla}_{(\xi\wedge_S a_2)a_4}a_3) \\ &= 2n\eta(a_4)\sigma(a_2,a_3) \end{aligned} \tag{3.24}$$

$$\begin{aligned} (\nabla_{a_4}\sigma)((\xi\wedge_S a_2)\xi,a_3) &= (\nabla_{a_4}h)(S(a_2,\xi)\xi - S(\xi,\xi)a_2,a_3) \\ &= (\nabla_{a_4}h)(-2n\eta(a_2)\xi + 2na_2,a_3) \\ &= -2n\{\widetilde{\nabla}^{\perp}_{a_4}\sigma(\eta(a_2)\xi,a_3) - \sigma(\widetilde{\nabla}_{a_4}\eta(a_2)\xi,a_3) - \sigma(\eta(a_2)\xi,\widetilde{\nabla}_{a_4}a_3) + 2n(\nabla_{a_4}\sigma)(a_2,a_3)\} \\ &= 2n(\nabla_{a_4}\sigma)(a_2,a_3) - 2n\eta(a_2)\sigma(a_4,a_3) \end{aligned} \tag{3.25}$$

In the same way, we have

$$\begin{aligned}(\nabla_{a_4}\sigma)(\xi,(\xi\wedge_S a_2)a_3) &= (\nabla_{a_4}\sigma)(\xi,S(a_2,a_3)\xi - S(\xi,a_3)a_2)\\ &= (\nabla_{a_4}\sigma)(\xi,S(a_2,a_3)\xi) + (\widetilde{\nabla}_{a_4}\,\sigma)(\xi,2n\eta(a_3)a_2)\\ &= 2n\eta(a_3)\sigma(a_4,a_2)\end{aligned} \tag{3.26}$$

Consequently, we substitute (3.20)-(3.26) in (3.19), we obtain

$$\begin{gathered}R^{\perp}(\xi,a_2)\sigma(a_3,a_4) - \eta(a_4)\sigma(a_2,a_3) + 3\eta(a_2)\sigma(a_4,a_3) - \eta(a_3)\sigma(a_4,a_2) - (\nabla_{a_4}\sigma)(a_2,a_3) =\\ -k_4\{2n\eta(a_4)\sigma(a_2,a_3) - 2n\eta(a_2)\sigma(a_4,a_3) + 2n\eta(a_3)\sigma(a_4,a_2) + 2n(\nabla_{a_4}\sigma)(a_2,a_3)\}\end{gathered} \tag{3.27}$$

If we choose $a_3 = \xi$ in (3.27) and use

$$(\nabla_{a_4}\sigma)(a_2,\xi) = \sigma(a_4,a_2)$$

we get

$$[2nk_4 - 1]\sigma(a_4,a_2) = 0$$

□

Corollary 3.13. Let $\widetilde{M}$ be a pseudoparallel invariant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. Then, $\widetilde{M}$ is W_8-2 Ricci generalized semi-parallel if and only if $\widetilde{M}$ is totally geodesic.

4. Conclusion

This paper considered pseudoparallel invariant submanifolds of Kenmotsu manifolds on W_8 curvature tensor and researched some basic characterizations, such as W_8 pseudoparallel, W_8-2 pseudoparallel, W_8-Ricci generalized pseudoparallel, and W_8-2 Ricci generalized pseudoparallel. In addition, the paper provided some relations between these pseudoparallel invariant submanifolds and semi-parallel invariant submanifolds. In future studies, pseudoparallel invariant submanifolds of Kenmotsu manifolds can also be characterized using other curvature tensors. Besides, this topic is worth studying on the other manifolds.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's master's thesis supervised by the second author. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

References

[1] K. Kenmotsu, *A class of contact Riemannian manifolds*, Tohoku Mathematical Journal 24 (1) (1972) 93–103.

[2] A. Haseeb, *Some results on projective curvature tensor in an ε-Kenmotsu manifold*, Palestine Journal of Mathematics 6 (Special Issue: II) (2017) 196–203.

[3] Y. Wang, X. Liu, *Second order parallel tensors on almost Kenmotsu manifolds satisfying the nullity distributions*, Filomat 28 (4) (2014) 839–847.

[4] Y. Wang, W. Wang, *An Einstein-like metric on almost Kenmotsu manifold*, Filomat 31 (15) (2017) 4695–4702.

[5] C. Özgür, U. C. De, *On the quasi-conformal curvature tensor of a Kenmotsu manifold*, Mathematica Pannonica 17 (2) (2006) 221–228.

[6] M. M. Tripathi, P. Gupta, *τ-curvature tensor on a semi-Riemannian manifold*, Journal of Advanced Mathematical Studies 4 (1) (2011) 117–129.

[7] R. N. Singh, S. K. Pandey, G. Pandey, *On W_2-curvature tensor in a Kenmotsu manifold*, Tamsui Oxford Journal of Information and Mathematical Sciences 29 (2) (2013) 129–141.

[8] D. G. Prakasha, B. S. Hadimani, *On the conharmonic curvature tensor of Kenmotsu manifolds with generalized Tanaka-Webster connection*, Miskole Mathematical Notes 19 (1) (2018) 491–503.

[9] U. C. De, G. Pathak, *On 3-dimensional Kenmotsu manifolds*, Indian Journal of Pure and Applied Mathematics 35 (2) (2004) 159–165.

[10] K. De, U. C. De, *Conharmonic curvature tensor on Kenmotsu manifolds*, Bulletin of the Transilvania University of Braşov Series III: Mathematics and Computer Science 6 (55) (1) (2013) 9–22.

[11] M. Atçeken, T. Mert, *Characterizations for totally geodesic submanifolds of a K-paracontact manifold*, AIMS Mathematics 6 (7) (2021) 7320–7332.

[12] P. Uygun, S. Dirik, M. Atçeken, T. Mert, *Some characterizations invariant submanifolds of a (κ,μ)-para contact space*, Journal of Engineering Research and Applied Science 1 (11) (2022) 1967–1972.

[13] T. Mert, M. Atçeken, *A note on pseudoparallel submanifolds of Lorentzian para-Kenmotsu manifolds*, Filomat 37 (15) (2023) 5095–5107.

Journal of Amasya University the Institute of Sciences and Technology 4(2) (2023) 82-89

JOURNAL OF AMASYA UNIVERSITY THE INSTITUTE OF SCIENCES AND TECHNOLOGY (JAUIST)

ISSN: 2717-8900 https://doi.org/10.54559/jauist.1379597

Effect of fractional time derivatives to pressure-driven flow through the horizontal microchannel

[1]Muhammad Lawan Kaurangini[1], Huzaifa Muhammad Tahir[2], Umar Muhammad Abubakar[3,*]

[1,2,3]Department of Mathematics, Faculty of Computing and Mathematical Science, Aliko Dangote University of Science and Technology, Wudil, Nigeria

Abstract — This research applies fractional time derivatives to fluid flow through a horizontal microchannel. It uses fractional time derivatives with the Laplace transform technique and method of undetermined coefficient to analyze and obtain solutions of the governing equations in the Laplace domain. To this end, the solutions are reversed in the time domain using Riemann-sum approximation methods. In order to obtain the solutions for the pressure-driven flow, the time factional derivative in the Caputo sense is employed. Here, the influence of each governing parameter is explained with a line graph. Results show that with the decreases in fractional order (α), the velocity decreases within the interval $0 < \alpha < 1$. The fluid velocity increases and decreases as the Knudsen number (kn) changes. Besides, transient wall-skin frictions for different times (t) and Knudsen number (kn) with a fixed value of fractional order (α) are observed.

Keywords: *Fractional time derivative, pressure-driven flow, Knudsen number, Laplace transform, Couette flow*

Subject Classification (2020): 76D55, 34A08

1. Introduction

Fractional calculus is a generalization of ordinary differential and integral of non-integer order α. It was first introduced by L'Hospital and Leibnitz in 1695 after they proposed what would happen if the ordinary derivative of integer order was changed to fractional order by L'Hospital. Then, Leibnitz first used the notation $d^{(1/2)}y$ in 1697. Many mathematicians have suggested their interest in its application, as Lacroix 1819 mentioned fractional derivatives in his text on differential and integral calculus. Euler and Fourier mentioned fractional derivatives but did not give any application or example. The first and popular definition, the Riemann-Liouville definition, was proposed by Riemann and Liouville, after which Caputo proposed a second popular definition called the Caputo Fractional Derivative. There are many definitions of Fractional calculus, such as Jumarie, Wely, Eudelyikober, Hadamard, and the Riesz fractional derivative (see, for example, Srivastava and Saxena [1] and Kaur [2]).

In recent decades, many researchers have been devoted to its applications in science and engineering. Fractional calculus has been recognized as a practical modeling approach in fields such as electrical networks, electrochemistry, and viscoelastic deformation, solving linear, nonlinear partial fractional differential equations (see Farid et al. [3]). According to Ali et al. [4], Newtonian and non-Newtonian fluids depend on their deformation. Newtonian fluids are fluids that obey Newton's law of viscosities. In contrast,

*Corresponding Author

[1]kaurangini@yahoo.com; [2]huzaifamtahir1234@gmail.com; [3]uabubakar@kustwudil.edu.ng
Article History: Received: 22 Oct 2023 — Accepted: 27 Dec 2023 — Published: 31 Dec 2023

non-Newtonian fluids do not follow Newton's law of viscosities, having many applications in fields such as industries, medical treatment, and engineering worldwide.

A micro-channel is defined in Djordjevic [5] as a flow channel with a hydraulic diameter of less than 1 mm and characterized by the rarefaction effect, which includes the Knudsen number $kn = \lambda/l$, where λ is the mean free path of the molecules and l is the characteristic length scale (height h of the channel or the radius an of the tube), then the Knudsen number is the quantity that helps to know which fluid dynamic formula to use to model a situation in both statistical mechanics or continuum mechanics for the continuum hypothesis to be valid, the Knudsen number must be less than 0.1 for pressure driven. Saqib et al. [6] explain that the temperature and velocity fields can be reduced for any value of α between the interval $0 < \alpha < 1$ with memory and heredity profile, which are more generally flexible and reliable, in the presence of variation in the thermal boundary layer when increasing volume fraction of carbon nanotubes (CNTs) the temperature profile increases and decreases with increases of fractional order α in both cases.

Ellahia et al. [7] change more than one parameter in the channel and find that the fluid velocity decreases as β increases the fixed value of the channel length L. In the same paper, they also found that for a fixed value of β and growing L, the velocity also increases. Farooq et al. [8] studied the generalized Couette flow by varying the various parameters of temperature distribution and velocity, where it was observed that as the fluid moved from a fixed plate to a movable plate, the velocity and temperature increased. Arif et al. [9] open channel for Couple Stress Fluid (CSF) using integral transform (Laplace and Fourier) for the comparative analysis of Caputo (C), Caputo-Fabrizio (CF), Atangana Baleanu (AB), and classical CSF where it is observed the velocity of the CSF of C, CF has less influence of fluid dynamics than the velocity of the CSF concerning AB, which clearly shows that AB fractional derivatives has a better memory effect than C, CF and increasing the constant pressure gradient of CSF improves the velocity. Maitia et al. [10] discovered the impact of the Caputo-Fabrizio derivative of the fractional order model on blood flow, where they found that increasing the value of the fractional parameter value decreased velocity and temperature when memory counters average speed of blood flow medium effect to drive faster. The velocity profile improves as the stress jump coefficient increases. When the stress jump coefficient β is close to zero, non-Newtonian characteristics become more effective, and β approaches infinity, the model becomes a Newtonian when viscosity increases. Blood flow decreases with a decrease in β. In the same paper, they also find that the shear stress increases in the wall due to chemical reactions.

The governing equation for this present study is in fractional order, focusing on pressure-driven flow through a horizontal channel with many applications in applied science and engineering, such as biological research, geophysical engineering, DNA sequencing for drug delivery, and micro-electro-mechanical systems (MECSs). In the presence of pressure gradient and upper plate motion of the boundary layer (generalized Couette flow), the steady and unsteady flow has been studied related to skin friction on the pressure gradient and dependence on the interface velocity (see Kaurangini and Jha [11]).

2. Analytical Solution

The governing equations [12] are derived from the classical equations and modified by replacing the ordinary time derivatives with the fractional calculus operators. This generalization allows us to define non-integer order integrals or derivatives precisely. Figure 1 manifests the fluid flow in the microchannel is induced by pressure-driven flow.

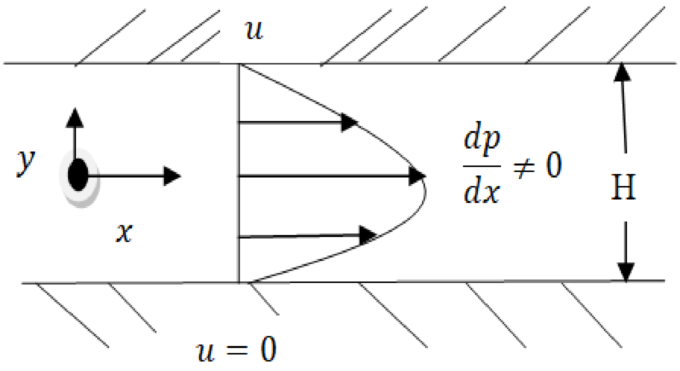

Figure 1. Schematic diagram of pressure-driven flow

The governing equation [12] for the flow is as follows:

$$D_t^{\alpha} u(y,t) = \gamma \frac{\partial^2 u(y,t)}{\partial y^2} + p \tag{2.1}$$

with initial and boundary conditions and $D_t^{\alpha} u(y,t)$ is the Caputo fractional derivative,

$$t \leq 0: u = 0, \text{for all } y \tag{2.2}$$

and

$$t > 0: u(y,t) = \begin{cases} +\beta_v k_n \dfrac{du}{dy}, & y = 0 \\ -\beta_v k_n \dfrac{du}{dy}, & y = H \end{cases} \tag{2.3}$$

with the analysis technique mentioned above, we have the following solution approach:

Taking the Laplace transform of both sides of (2.1) together with (2.2), we have

$$\frac{d^2 \overline{U}(y,s)}{dy^2} - \frac{s^{\alpha} \overline{U}(y,s)}{\gamma} = -\frac{p}{\gamma s} \tag{2.4}$$

Solving (2.4) by the method of undetermined coefficient to obtain the general solution,

$$\overline{U}(y,s) = c_1 \cosh k_1 y + c_2 \sinh k_1 y + k_2 \tag{2.5}$$

Similarly, we apply the Laplace transform to (2.3), the boundary conditions become

$$\frac{1}{s^2} > 0: \overline{U}(y,s) = \begin{cases} +\beta_v k_n \dfrac{d}{dy} \overline{U}(y,s), & y = 0 \\ -\beta_v k_n \dfrac{d}{dy} \overline{U}(y,s), & y = H \end{cases} \tag{2.6}$$

Using (2.5) on (2.6), we obtained the following solutions:

$$\overline{U}(y,s) = k_7 \cosh k_1 y + k_6 \sinh k_1 y + k_2 \tag{2.7}$$

where

$$\begin{cases} k_1 = \left(\dfrac{s^{\alpha}}{\gamma}\right)^{\frac{1}{2}} \\ k_2 = \dfrac{P}{s^{\alpha+1}} \\ k_3 = \cosh k_1 H + \beta_v k_n k_1 \sinh k_1 H \\ k_4 = \sinh k_1 H + \beta_v k_n k_1 \cosh k_1 H \\ k_5 = \beta_v k_n k_1 k_3 + k_4 \\ k_6 = \dfrac{k_2 (k_3 - 1)}{k_5} \\ k_7 = \beta_v k_n k_1 k_6 - k_2 \end{cases} \tag{2.8}$$

3. Skin Friction for Pressure Driven Flow

From (2.7), the skin frictions at the wall of the channel are obtained

$$\hat{\tau}_0 = \left.\frac{d\bar{U}(y,s)}{dy}\right|_{y=0} = (k_1 k_7 \sinh k_1 y + k_1 k_6 \cosh k_1 y)|_{y=0} = K_1 K_6 \tag{3.1}$$

$$\hat{\tau}_1 = \left.\frac{d\bar{U}(y,s)}{dy}\right|_{y=1} = (k_1 k_7 \sinh k_1 y + k_1 k_6 \cosh k_1 y)|_{y=1} = k_1 k_7 \sinh k_1 + k_1 k_6 \cosh k_1 \tag{3.2}$$

4. Results and Discussions

4.1. Numerical Results

This was done to simulate numerical solutions for transient skin friction at different walls using the computational software MATLAB R2014a. We obtained the following results.

Table 1. Transient skin frictions at the walls for different time t and Knudsen number kn of Pressure driven flow

	$\beta v = 0.5, a = 0.5$ $p = 2, \gamma = 1, kn = 0.0$		$\beta v = 0.5, a = 0.5$ $Y = 1, kn = 0.04$		$\beta v = 0.5, a = 0.5, p = 2$ $Y = 1, kn = 0.08$	
t	τ_0	τ_1	τ_0	τ_1	τ_0	τ_1
0.1	0.0140	−0.0140	0.0134	−0.0134	0.0129	−0.0129
0.2	0.0310	−0.0310	0.0299	−0.0299	0.0289	−0.0289
0.3	0.0487	−0.0487	0.0473	−0.0473	0.0460	−0.0460
0.4	0.0670	−0.0670	0.0652	−0.0652	0.0636	−0.0636
0.5	0.0856	−0.0856	0.0835	−0.0835	0.0815	−0.0815
0.6	0.1044	−0.1044	0.1020	−0.1020	0.0998	−0.0998
0.7	0.1234	−0.1234	0.1208	−0.1208	0.1183	−0.1183
0.8	0.1425	−0.1425	0.1397	−0.1397	0.1370	−0.1370
0.9	0.1618	−0.1618	0.1587	−0.1587	0.1558	−0.1558
1.0	0.1812	−0.1812	0.1779	−0.1779	0.1747	−0.1747

4.2. 2D-Plots Presentation

Figures 2-5 describe the velocity profiles for various parameters, such as fractional α, pressure p, Knudsen number kn, and time t, caused by the pressure gradient. Figure 2 shows that the reduction of fractional order reduces the velocity of the fluid, which implies that the velocity can be slowed down by decreasing the fractional order. In addition, it shows the advantages of fractional derivatives over integer derivatives. Figure 3 shows that the transient velocity increases with time, which implies that with a constant driving force, the velocity can increase as time goes on. Figure 4 shows that as the Knudsen number increased, the fluid velocity increased, indicating that enlarging the length scale allowed the mean free path to enlarge and the velocity to increase. Similarly, Figure 5 shows the velocity profiles for different Knudsen numbers kn. Figure 6 depicts the variation of fractional order α with velocity U, which that the velocity of the fluid decreases when the fractional order decreases with the interval of 0.3, and between 0.6 and 0.3, the velocity reduces unlike from 0.9 to 0.6, which clearly shows the effect of the fractional order. Figure 7 shows the variation of velocity U with time t, which shows that increasing time makes velocity also increase and converge at both walls, which leads the velocity to decrease between 0.5 to 0.8, and as time goes on, the velocity will be steady. It should be noted that the fluid velocity slows down as the Knudsen number decreases. Table 1 shows that skin friction increases evenly on both walls but in opposite directions with increasing time and Knudsen number.

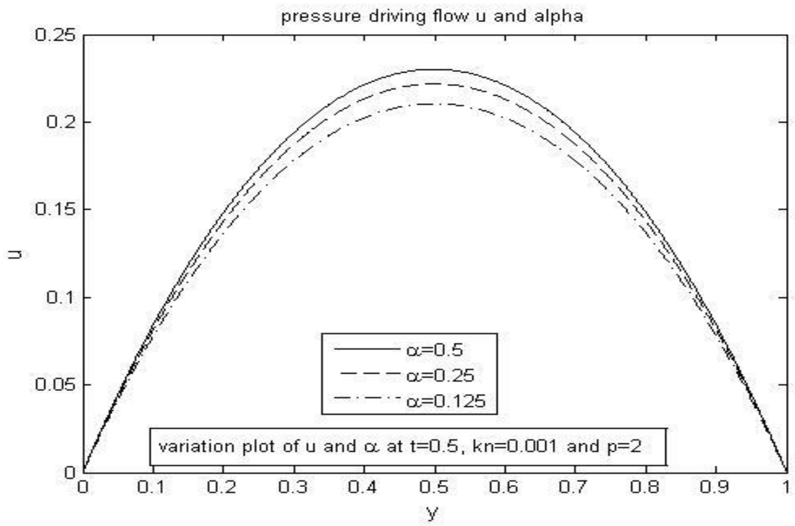

Figure 2. Variation of fractional order α against velocity $U$when the fluid flow as a result of pressure gradient

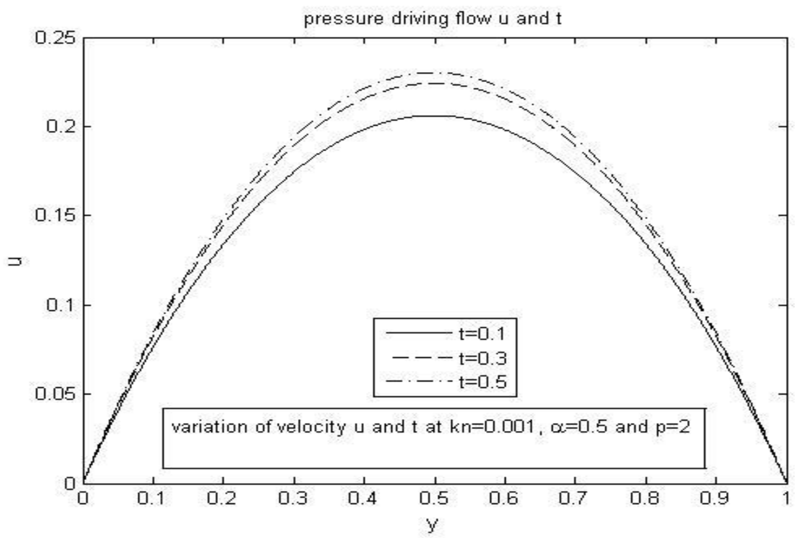

Figure 3. Variation of time t against velocity U when the fluid flows as a result of pressure gradient

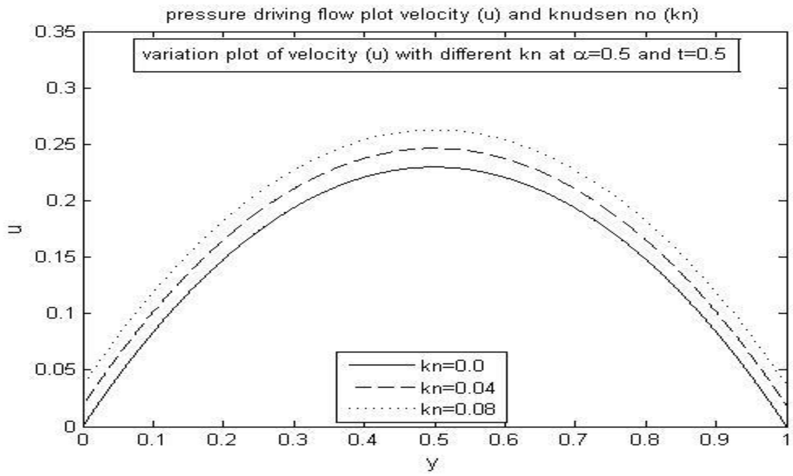

Figure 4. Variation of Knudsen number kn against velocity U as a result of pressure gradient

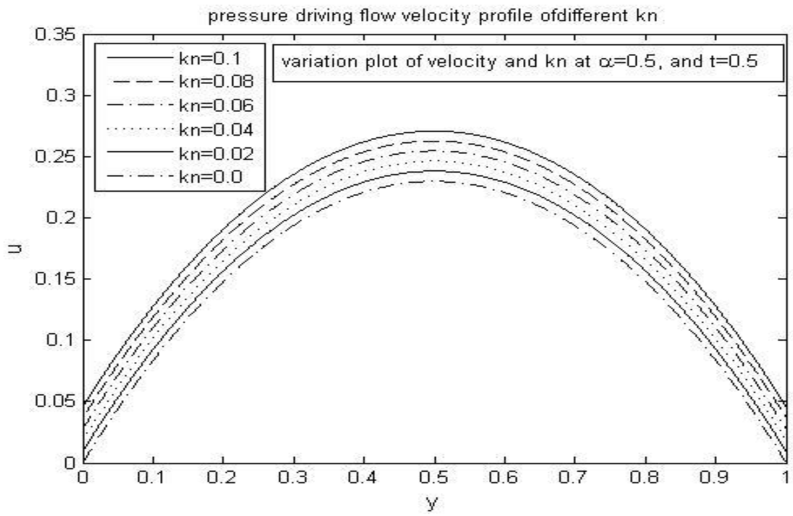

Figure 5. Variation of Knudsen number kn against velocity U as a result of pressure gradient

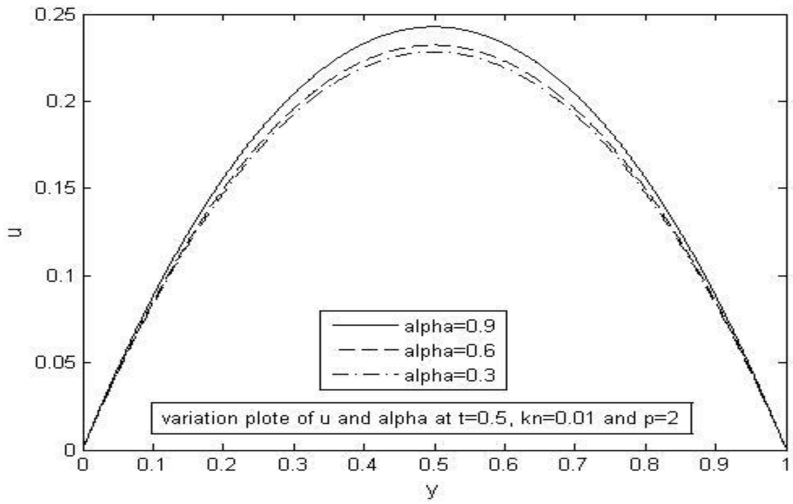

Figure 6. Variation of fractional order α against velocity U when the fluid flows as a result of pressure gradient

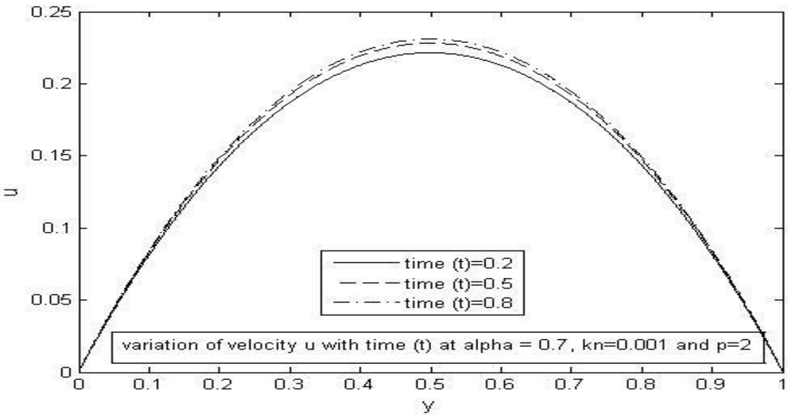

Figure 7. Variation of time t against velocity U when the fluid flows as a result of pressure gradient

5. Conclusion

In this paper, the effect of varying the governing parameters was considered to study the velocity profile induced by pressure-driven flow in the microchannel. The transient skin friction uniformly increases at both walls but in opposite directions. It has been discovered that the velocity of a fluid flow can be controlled by adjusting the fractional order (α). The advantages of fractional derivatives over integer derivatives have been studied. It has been observed that the velocity can increase with a constant driven force as time goes on. Enlarging the length scale enlarged the mean free path and increased velocity. It can be seen that with a continuous decrease of the Knudsen number $kn,$ at $\beta v = -0.5$, the velocity increases but remains constant at $\beta v = 0.0$ while at $\beta v\ =\ 0.5$, the velocity decreases. The results obtained from this study are significant for industries, as they contribute to a better comprehension of various applications such as oil reservoirs, nuclear reactors, groundwater flow, filtration, and geothermal systems. Furthermore, these findings present opportunities for further investigation through the inclusion of porous mediums, suction, and injection velocities.

Abbreviations

D	fractional derivative
α	fractional order
t	time
u	velocity of fluid flow
y	dimensionless y coordinate
p	dimensionless pressure gradient
kn	Knudsen number
β	stress jump coefficient
ρ	fluid density
λ	molecular mean free path
v_f	kinematics viscosity of the fluid
y'	dimensional y-coordinate
u_f	dimensionless velocity in the clear fluid region
u_f'	dimensional velocity in the clear fluid region
u_t	transient velocity
$\frac{\delta p'}{\delta z'}$	dimensional pressure gradient
β_v	dimensionless variable

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

References

[1] H. M. Srivastava, R. K. Saxena, *Operations of fractional integration and their applications*, Applied Mathematics and Computation 118 (1) (2001) 1–52.

[2] D. Kaur, *A detail study on fractional calculus*, in: S. Negi, A. Kumar, T. Singh (Eds.), International Multidisciplinary Conference, Chandigarh, 2022, pp. 54–59.

[3] G. Farid, N. Latif, M. Anwar, A. Imran, M. Ozair, M. Nawaz, *On applications of Caputo k-fractional derivatives*, Advances in Difference Equations 2019 (2019) 439 16 pages.

[4] F. Ali, Z. Ahmad, M. Arif, I. Khan, K. S. Nisar, *A time fractional model of generalized Couette flow of couple stress nanofluid with heat and mass transfer: Applications in engine oil*, IEEE Access 8 (2020) 146944–146966.

[5] V. Djordjevic, *Modeling of the slip boundary condition in micro-channel/pipe flow via fractional derivative*, Monograph of Academy of Nonlinear Sciences. Advances in Nonlinear Sciences II-Sciences and Applications, Belgrade 2 (2008) 136–158.

[6] M. Saqib, A. R. M. Kasim, N. F. Mohammad, D. L. C. Ching, S. Shafie, *Application of fractional derivative without singular and local kernel to enhanced heat transfer in CNTs nanofluid over an inclined plate*, Symmetry 12 (5) (2020) 768 22 pages.

[7] R. Ellahi, T. Hayat, F. M. Mahomed, *Generalized Couette flow of a third-grade fluid with slip: the exact solutions*, Zeitschrift für Naturforschung A 65 (12) (2010) 1071–1076.

[8] M. Farooq, A. Khan, R. Nawaz, S. Islam, M. Ayaz, Y.-M. Chu, *Comparative study of generalized couette flow of couple stress fluid using optimal homotopy asymptotic method and new iterative method*, Scientific Reports 11 (2021) 3478 20 pages.

[9] M. Arif, P. Kumam, W. Kumam, M. B. Riaz, D. Khan, *A comparative analysis of multiple fractional solutions of generalized Couette flow of couple stress fluid in a channel*, Heat Transfer 51 (8) (2022) 7348–7368.

[10] S. Maiti, S. Shaw, G. C. Shit, *Fractional order model for thermochemical flow of blood with Dufour and Soret effects under magnetic and vibration environment*, Colloids and Surfaces B: Biointerfaces, 197 (2021) 1–18.

[11] M. L. Kaurangini, B. K. Jha, *Unsteady generalized Couette flow in composite microchannel*, Applied Mathematics and Mechanics 32 (2011) 23–32.

[12] S. Chen, L. Zheng, C. Li, J. Sui, *Time-space dependent fractional viscoelastic MHD fluid flow and heat transfer over accelerating plate with slip boundary*, Thermal Science 21 (6 Part A) (2017) 2337–2345.

Journal of Amasya University the Institute of Sciences and Technology 4(2) (2023) 90-99

JOURNAL OF AMASYA UNIVERSITY THE INSTITUTE OF SCIENCES AND TECHNOLOGY (JAUIST)

ISSN: 2717-8900 https://doi.org/10.54559/jauist.1395447

Some identities of bivariate Pell and bivariate Pell-Lucas polynomials

Yashwant Kumar Panwar[1,*]

[1]Department of Mathematics, Government Model College, Jhabua, India

Abstract — In this paper, we obtain some identities for the bivariate Pell polynomials and bivariate Pell-Lucas polynomials. We establish some sums and connection formulas involving them. Moreover, we present its two cross two matrices representation and find some of its properties, such as the b^{th} power of the matrix. We finally derive the identities by using Binet's formula, generating function, and induction method.

Keywords: *Bivariate Pell polynomials, bivariate Pell-Lucas polynomials, Binet's formula, generating function*

Subject Classification (2020): 11B37, 11B39

1. Introduction

Sequences and polynomials have a wide range of applications in applied mathematics and physics. Bivariate polynomials are widely used in theoretical physics for modeling physical processes. Catalani [1-3] defined generalized bivariate polynomials from which, by specifying initial conditions, the bivariate Fibonacci and Lucas polynomials are obtained, and many interesting identities are derived. Belbachir and Bencherif [4] generalized to bivariate Fibonacci and Lucas polynomials properties obtained for Chebyshev polynomials. They proved that the coordinates of the bivariate polynomials over an appropriate basis are families of integers satisfying remarkable recurrence relations. Tuğlu et al. [5] presented generalized bivariate Fibonacci and Lucas p-polynomials, which are general forms of the Fibonacci, Lucas, Pell, Jacobsthal, Pell-Lucas, Jacobsthal-Lucas sequences, as well as Fibonacci, Lucas, Pell, Jacobsthal, Pell-Lucas, Jacobsthal-Lucas, bivariate Fibonacci and Lucas, first and second type of Chebyshev polynomials, and many others. Halıcı and Akyüz [6] derived some identities and some sum formulas for the bivariate Pell polynomials using different matrices. Saba and Boussayoud [7] introduced a symmetric function to derive a new generating function of bivariate Pell Lucas polynomials, also derived new symmetric functions, and gave some interesting properties. This study defines the identities of bivariate Pell and bivariate Pell-Lucas polynomials.

2. Preliminaries

For $n \geq 2$, the bivariate Pell polynomials sequence [6] is defined by

$$P_n(x,y) = 2xyP_{n-1}(x,y) + yP_{n-2}(x,y) \tag{2.1}$$

*Corresponding Author

[1]yashwantpanwar@gmail.com
Article History: Received: 24 Nov 2023 — Accepted: 25 Dec 2023 — Published: 31 Dec 2023

Therefore, the first few bivariate Pell polynomials are

$$\{P_n(x,y)\} = \{0,1,2xy,4x^2y^2+y,8x^3y^3+4xy^2,16x^4y^4+12x^2y^3+y^2,\cdots\}$$

Binet's formula,

$$P_n(x,y) = \frac{\ell_1^n - \ell_2^n}{\ell_1 - \ell_2}$$

Generating function,

$$P_n(x,y) = \frac{t}{(1-2xyt-yt^2)}$$

For $n \geq 2$, the bivariate Pell-Lucas polynomials sequence [6] is defined by

$$Q_n(x,y) = 2xyQ_{n-1}(x,y) + yQ_{n-2}(x,y) \tag{2.2}$$

Therefore, the first few bivariate Pell-Lucas polynomials are

$$\{Q_n(x,y)\} = \{2,2xy,4x^2y^2+2y,8x^3y^3+6xy^2,16x^4y^4+16x^2y^3+2y^2,\cdots\}$$

Binet's formula,

$$Q_n(x,y) = \ell_1^n + \ell_2^n$$

Generating function,

$$Q_n(x,y) = \frac{2+2xyt(x-1)}{(1-2xyt-yt^2)}$$

The characteristic equation of recurrence relations (2.1) and (2.2) is

$$t^2 - 2xyt - y = 0$$

where $x \neq 0$, $y \neq 0$, and $x^2y^2 + y \neq 0$. This equation has two real roots: $\ell_1 = xy + \sqrt{x^2y^2+y}$ and $\ell_2 = xy - \sqrt{x^2y^2+y}$. Note that $\ell_1 + \ell_2 = 2xy$, $\ell_1\ell_2 = -y$, and $\ell_1 - \ell_2 = \sqrt{x^2y^2+y}$. Moreover, $P_{-n}(x,y) = \frac{-1}{(-y)^n}P_n(x,y)$ and $Q_{-n}(x,y) = \frac{1}{(-y)^n}Q_n(x,y)$. The main objective of this study is to describe sums and connection formulas. Moreover, we introduce the special sums and prove them using Binet's formula.

3. Results and Discussions

We first establish sums and relations for bivariate Pell and bivariate Pell-Lucas polynomials. The motivation of this work comes from the study of [8-11].

Proposition 3.1. For $\vartheta, \omega \in \mathbb{Z}$, we get

$$\sum_{b=0}^{\vartheta+\omega} \binom{\vartheta+\omega}{b} (2xy)^b y^{\vartheta+\omega-b}\, P_b(x,y) = P_{2\vartheta+2\omega}(x,y)$$

Proof.

By Binet's formula,

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\,P_b(x,y) = \sum_{b=0}^{c+d}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\left(\frac{\ell_1^b-\ell_2^b}{\ell_1-\ell_2}\right)$$

$$= \frac{1}{\ell_1-\ell_2}\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\left(\ell_1^b-\ell_2^b\right)$$

$$= \frac{1}{\ell_1-\ell_2}\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}\{(2xy\ell_1)^b-(2xy\ell_2)^b\}y^{\vartheta+\omega-b}$$

$$= \frac{1}{\ell_1-\ell_2}\left\{(2xy\ell_1+y)^{\vartheta+\omega}-(2xy\ell_2+y)^{\vartheta+\omega}\right\}$$

Since ℓ_1 and ℓ_2 are the roots of $t^2-2xyt-y=0$,

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\,P_b(x,y) = \frac{(\ell_1^2)^{\vartheta+\omega}-(\ell_2^2)^{\vartheta+\omega}}{\ell_1-\ell_2} = P_{2\vartheta+2\omega}(x,y)$$

□

Proposition 3.2. For $\vartheta,\omega\in\mathbb{Z}$, we get

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b(-y)^{\vartheta+\omega-b}\,P_b(x,y) = \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(-2y)^b\,P_{2\vartheta+2\omega-2b}(x,y)$$

Proof.

By Binet's formula,

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b(-y)^{\vartheta+\omega-b}\,P_b(x,y) = \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b(-y)^{\vartheta+\omega-b}\left(\frac{\ell_1^b-\ell_2^b}{\ell_1-\ell_2}\right)$$

$$= \frac{1}{\ell_1-\ell_2}\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b(-y)^{\vartheta+\omega-b}\left(\ell_1^b-\ell_2^b\right)$$

$$= \frac{1}{\ell_1-\ell_2}\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}\{(2xy\ell_1)^b-(2xy\ell_2)^b\}(-y)^{\vartheta+\omega-b}$$

$$= \frac{1}{\ell_1-\ell_2}\left\{(2xy\ell_1-y)^{\vartheta+\omega}-(2xy\ell_2-y)^{\vartheta+\omega}\right\}$$

Since ℓ_1 and ℓ_2 are the roots of $t^2-2xyt-y=0$,

$$2xy\ell_1-y=\ell_1^2-2y$$

and

$$2xy\ell_2-y=\ell_2^2-2y$$

Thus,

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b(-y)^{\vartheta+\omega-b}\,P_b(x,y) = \frac{(\ell_1^2-2y)^{\vartheta+\omega}-(\ell_2^2-2y)^{\vartheta+\omega}}{\ell_1-\ell_2}$$

$$= \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(-2y)^b\,P_{2\vartheta+2\omega-2b}(x,y)$$

□

Proposition 3.3. For $\vartheta,\omega\in\mathbb{Z}$, we get

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^{\vartheta+\omega-b}(-1)^b\,Q_b(x,y) = Q_{\vartheta+\omega}(x,y)$$

Proof.

By Binet's formula,

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^{\vartheta+\omega-b}(-1)^b\,Q_b(x,y) = \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^{\vartheta+\omega-b}(-1)^b\left(\ell_1^b+\ell_2^b\right)$$

$$= \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^{\vartheta+\omega-b}(-1)^b\left(-\ell_1^b\right)+\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^{\vartheta+\omega-b}(-1)^b\left(-\ell_2^b\right)$$

$$= (2xy-\ell_1)^{\vartheta+\omega}+(2xy-\ell_2)^{\vartheta+\omega}$$

$$= \left(\frac{-y}{\ell_1}\right)^{\vartheta+\omega}+\left(\frac{-y}{\ell_2}\right)^{\vartheta+\omega}$$

$$= (-y)^{\vartheta+\omega}\frac{\ell_1^{\vartheta+\omega}+\ell_2^{\vartheta+\omega}}{(\ell_1\ell_2)^{\vartheta+\omega}}$$

$$= Q_{\vartheta+\omega}(x,y)$$

□

Proposition 3.4. For $\vartheta,\omega\in\mathbb{Z}$, we get

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\,Q_b(x,y) = Q_{2\vartheta+2\omega}(x,y)$$

Proof.

By Binet's formula,

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\,Q_b(x,y) = \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\,Q_b(x,y)\left(\ell_1^b+\ell_2^b\right)$$

$$= \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy\ell_1)^b y^{\vartheta+\omega-b}+\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy\ell_2)^b y^{\vartheta+\omega-b}$$

$$= (2xy\ell_1+y)^{\vartheta+\omega}+(2xy\ell_2+y)^{\vartheta+\omega}$$

Since ℓ_1 and ℓ_2 are the roots of $t^2 - 2xyt - y = 0$,

$$\sum_{b=0}^{\vartheta+\omega} \binom{\vartheta+\omega}{b} (2xy)^b y^{\vartheta+\omega-b}\, Q_b(x,y) = (\ell_1^2)^{\vartheta+\omega} + (\ell_2^2)^{\vartheta+\omega}$$

$$= Q_{2\vartheta+2\omega}(x,y)$$

□

Proposition 3.5. If $P_b(x,y)$ and $Q_b(x,y)$ are Bivariate Pell and Bivariate Pell-Lucas polynomials, then for $b \geq \vartheta + \omega$,

$$P_{b+\vartheta+\omega}(x,y) - (-y)^{\vartheta+\omega} P_{b-\vartheta-\omega}(x,y) = P_{\vartheta+\omega}(x,y) Q_b(x,y)$$

Proof.

By Binet's formula,

$$\begin{aligned}
P_{b+\vartheta+\omega}(x,y) - (-y)^{\vartheta+\omega} P_{b-\vartheta-\omega}(x,y) &= \frac{\ell_1^{b+\vartheta+\omega} - \ell_2^{b+\vartheta+\omega}}{\ell_1 - \ell_2} - (-y)^{\vartheta+\omega} \left(\frac{\ell_1^{b-\vartheta-\omega} - \ell_2^{b-\vartheta-\omega}}{\ell_1 - \ell_2} \right) \\
&= \frac{\left(\ell_1^{b+\vartheta+\omega} - \ell_2^{b+\vartheta+\omega}\right) - (-y)^{\vartheta+\omega}\left(\ell_1^{b-\vartheta-\omega} - \ell_2^{b-\vartheta-\omega}\right)}{\ell_1 - \ell_2} \\
&= \frac{\left(\ell_1^{b+\vartheta+\omega} - \ell_2^{b+\vartheta+\omega}\right) - (\ell_1\ell_2)^{\vartheta+\omega}\left(\ell_1^{b-\vartheta-\omega} - \ell_2^{b-\vartheta-\omega}\right)}{\ell_1 - \ell_2} \\
&= \frac{\left(\ell_1^{b+\vartheta+\omega} - \ell_2^{b+\vartheta+\omega}\right) - \left(\ell_1^{b}\ell_2^{\vartheta+\omega} - \ell_1^{\vartheta+\omega}\ell_2^{b}\right)}{\ell_1 - \ell_2} \\
&= \left(\frac{\ell_1^{\vartheta+\omega} - \ell_2^{\vartheta+\omega}}{\ell_1 - \ell_2} \right) \left(\ell_1^b + \ell_2^b\right) \\
&= P_{\vartheta+\omega}(x,y) Q_b(x,y)
\end{aligned}$$

□

Secondly, we investigate sums for Bivariate Pell and Pell-Lucas polynomials with negative indices.

Theorem 3.6. For $\vartheta \geq 1$ and ω any integer, we get

$$\sum_{i=0}^{b} (-y)^i P_{-i\vartheta-\omega}(x,y) = \begin{cases} \dfrac{P_{\vartheta b+\vartheta+\omega}(x,y) - (-y)^{\vartheta} P_{\vartheta b+\omega}(x,y) - (-y)^{\omega} P_{\vartheta-\omega}(x,y) + P_{\omega}(x,y)}{(-1)^{\vartheta} - Q_{\vartheta}(x,y) + 1}, & \omega < \vartheta \\ \dfrac{P_{\vartheta b+\vartheta+\omega}(x,y) - (-y)^{\vartheta} P_{\vartheta b+\omega}(x,y) + (-y)^{\omega} P_{\vartheta-\omega}(x,y) + P_{\omega}(x,y)}{(-1)^{\vartheta} - Q_{\vartheta}(x,y) + 1}, & \text{otherwise} \end{cases}$$

Proof.

Since

$$\sum_{i=0}^{b} (-y)^i P_{-i\vartheta-\omega}(x,y) = -\sum_{i=0}^{b} P_{i\vartheta+\omega}(x,y)$$

By Binet's formula,

$$\sum_{i=0}^{b}(-y)^i P_{-i\vartheta-\omega}(x,y) = -\sum_{i=0}^{b}\frac{\ell_1^{i\vartheta+\omega}-\ell_2^{i\vartheta+\omega}}{\ell_1-\ell_2}$$

$$=\frac{-1}{\ell_1-\ell_2}\left(\ell_1^{\omega}\sum_{i=0}^{b}\ell_1^{\vartheta i}-\ell_2^{\omega}\sum_{i=0}^{b}\ell_2^{\vartheta i}\right)$$

$$=\frac{-1}{\ell_1-\ell_2}\left[\frac{\ell_1^{\vartheta b+\vartheta+\omega}-\ell_1^{\omega}}{\ell_1^{\vartheta}-1}-\frac{\ell_2^{\vartheta b+\vartheta+\omega}-\ell_2^{\omega}}{\ell_2^{\vartheta}-1}\right]$$

$$=\frac{\left(\ell_1^{\vartheta b+\vartheta+\omega}-\ell_2^{\vartheta b+\vartheta+\omega}\right)-(\ell_1\ell_2)^{\vartheta}\left(\ell_1^{\vartheta b+\omega}-\ell_2^{\vartheta b+\omega}\right)-\left(\ell_1^{\vartheta}\ell_2^{\omega}-\ell_1^{\omega}\ell_2^{\vartheta}\right)+\left(\ell_1^{\omega}-\ell_2^{\omega}\right)}{(\ell_1-\ell_2)\left\{(\ell_1\ell_2)^{\vartheta}-\left(\ell_1^{\vartheta}+\ell_2^{\vartheta}\right)+1\right\}}$$

$$=\begin{cases}\dfrac{P_{\vartheta b+\vartheta+\omega}(x,y)-(-y)^{\vartheta}P_{\vartheta b+\omega}(x,y)-(-y)^{\omega}P_{\vartheta-\omega}(x,y)+P_{\omega}(x,y)}{(-1)^{\vartheta}-Q_{\vartheta}(x,y)+1}, & \omega<\vartheta\\[2ex] \dfrac{P_{\vartheta b+\vartheta+\omega}(x,y)-(-y)^{\vartheta}P_{\vartheta b+\omega}(x,y)+(-y)^{\omega}P_{\vartheta-\omega}(x,y)+P_{\omega}(x,y)}{(-1)^{\vartheta}-Q_{\vartheta}(x,y)+1}, & \text{otherwise}\end{cases}$$

□

Theorem 3.7. For $\vartheta \geq 1$ and ω any integer, we get

$$\sum_{i=0}^{b}(-y)^i Q_{-i\vartheta-\omega}(x,y)=\begin{cases}\dfrac{(-y)^{\vartheta}Q_{\vartheta b+\omega}(x,y)-Q_{\vartheta b+\vartheta+\omega}(x,y)-(-y)^{\omega}Q_{\vartheta-\omega}(x,y)+Q_{\omega}(x,y)}{(-1)^{\vartheta}-Q_{\vartheta}(x,y)+1}, & \omega<\vartheta\\[2ex] \dfrac{(-y)^{\vartheta}Q_{\vartheta b+\omega}(x,y)-Q_{\vartheta b+\vartheta+\omega}(x,y)+(-y)^{\omega}Q_{\vartheta-\omega}(x,y)+Q_{\omega}(x,y)}{(-1)^{\vartheta}-Q_{\vartheta}(x,y)+1}, & \text{otherwise}\end{cases}$$

Proof.

Since

$$\sum_{i=0}^{b}(-y)^i Q_{-i\vartheta-\omega}(x,y)=\sum_{i=0}^{b}Q_{i\vartheta+\omega}(x,y)$$

By Binet's formula,

$$\sum_{i=0}^{b}(-y)^i Q_{-i\vartheta-\omega}(x,y)=\sum_{i=0}^{b}\left(\ell_1^{i\vartheta+\omega}+\ell_2^{i\vartheta+\omega}\right)$$

$$=\ell_1^{\omega}\sum_{i=0}^{b}\ell_1^{\vartheta i}+\ell_2^{\omega}\sum_{i=0}^{b}\ell_2^{\vartheta i}$$

$$=\frac{\ell_1^{\vartheta b+\vartheta+\omega}-\ell_1^{\omega}}{\ell_1^{\vartheta}-1}+\frac{\ell_2^{\vartheta b+\vartheta+\omega}-\ell_2^{\omega}}{\ell_2^{\vartheta}-1}$$

$$=\frac{(\ell_1\ell_2)^{\vartheta}\left(\ell_1^{\vartheta b+\omega}+\ell_2^{\vartheta b+\omega}\right)-\left(\ell_1^{\vartheta b+\vartheta+\omega}+\ell_2^{\vartheta b+\vartheta+\omega}\right)-\left(\ell_1^{\vartheta}\ell_2^{\omega}+\ell_1^{\omega}\ell_2^{\vartheta}\right)+\left(\ell_1^{\omega}+\ell_2^{\omega}\right)}{(\ell_1-\ell_2)\left\{(\ell_1\ell_2)^{\vartheta}-\left(\ell_1^{\vartheta}+\ell_2^{\vartheta}\right)+1\right\}}$$

$$= \begin{cases} \dfrac{(-y)^{\vartheta} Q_{\vartheta b+\omega}(x,y) - Q_{\vartheta b+\vartheta+\omega}(x,y) - (-y)^{\omega} Q_{\vartheta-\omega}(x,y) + Q_{\omega}(x,y)}{(-1)^{\vartheta} - Q_{\vartheta}(x,y) + 1}, & \omega < \vartheta \\ \dfrac{(-y)^{\vartheta} Q_{\vartheta b+\omega}(x,y) - Q_{\vartheta b+\vartheta+\omega}(x,y) + (-y)^{\omega} Q_{\vartheta-\omega}(x,y) + Q_{\omega}(x,y)}{(-1)^{\vartheta} - Q_{\vartheta}(x,y) + 1}, & \text{otherwise} \end{cases}$$

□

Thirdly, we establish some identities for bivariate Pell and bivariate Pell-Lucas polynomials.

Theorem 3.8. For $\vartheta, \omega \in \mathbb{Z}$, we get

$$\sum_{b=0}^{\vartheta+\omega} P_b(x,y) t^{-b} = \frac{1}{t^{\vartheta+\omega}(t^2 - 2xyt - y)} \{ t^{\vartheta+\omega+1} - tP_{\vartheta+\omega+1}(x,y) - yP_{\vartheta+\omega}(x,y) \}$$

Proof.

By Binet's formula,

$$\sum_{b=0}^{\vartheta+\omega} P_b(x,y) t^{-b} = \sum_{b=0}^{\vartheta+\omega} \left(\frac{\ell_1^b - \ell_2^b}{\ell_1 - \ell_2} \right) t^{-b}$$

$$= \frac{1}{\ell_1 - \ell_2} \sum_{b=0}^{\vartheta+\omega} \left\{ \left(\frac{\ell_1}{t} \right)^b - \left(\frac{\ell_2}{t} \right)^b \right\}$$

$$= \frac{1}{\ell_1 - \ell_2} \left\{ \frac{1 - \left(\frac{\ell_1}{t} \right)^{\vartheta+\omega+1}}{1 - \frac{\ell_1}{t}} - \frac{1 - \left(\frac{\ell_2}{t} \right)^{\vartheta+\omega+1}}{1 - \frac{\ell_2}{t}} \right\}$$

$$= \frac{1}{(\ell_1 - \ell_2) t^{\vartheta+\omega}} \left(\frac{t^{\vartheta+\omega+1} - \ell_1^{\vartheta+\omega+1}}{t - \ell_1} - \frac{t^{\vartheta+\omega+1} - \ell_2^{\vartheta+\omega+1}}{t - \ell_2} \right)$$

$$= \frac{1}{(\ell_1 - \ell_2) t^{\vartheta+\omega}} \left\{ \frac{t^{\vartheta+\omega+1}(\ell_1 - \ell_2) - t\left(\ell_1^{\vartheta+\omega+1} - \ell_2^{c+d+1}\right) - y\left(\ell_1^{\vartheta+\omega} - \ell_2^{\vartheta+\omega}\right)}{(t - \ell_1)(t - \ell_2)} \right\}$$

$$= \frac{1}{t^{\vartheta+\omega}(t^2 - 2xyt - y)} \{ t^{\vartheta+\omega+1} - tP_{\vartheta+\omega+1}(x,y) - yP_{\vartheta+\omega}(x,y) \}$$

□

Theorem 3.9. For $\vartheta, \omega \in \mathbb{Z}$, we get

$$\sum_{b=0}^{\vartheta+\omega} Q_b(x,y) t^{-b} = \frac{2t^2 - xt}{(t^2 - 2xyt - y)} - \frac{1}{t^{\vartheta+\omega}(t^2 - 2xyt - y)} \{ tQ_{\vartheta+\omega+1}(x,y) + yQ_{\vartheta+\omega}(x,y) \}$$

Proof.

By Binet's formula,

$$\sum_{b=0}^{\vartheta+\omega} Q_b(x,y) t^{-b} = \sum_{b=0}^{\vartheta+\omega} \left(\ell_1^b + \ell_2^b \right) t^{-b}$$

$$= \sum_{b=0}^{\vartheta+\omega}\left\{\left(\frac{\ell_1}{t}\right)^b + \left(\frac{\ell_2}{t}\right)^b\right\}$$

$$= \frac{1-\left(\frac{\ell_1}{t}\right)^{\vartheta+\omega+1}}{1-\frac{\ell_1}{t}} + \frac{1-\left(\frac{\ell_2}{t}\right)^{\vartheta+\omega+1}}{1-\frac{\ell_2}{t}}$$

$$= \frac{1}{t^{\vartheta+\omega}}\left(\frac{t^{\vartheta+\omega+1}-\ell_1^{\vartheta+\omega+1}}{t-\ell_1} + \frac{t^{\vartheta+\omega+1}-\ell_2^{\vartheta+\omega+1}}{t-\ell_2}\right)$$

$$= \frac{2t^{\vartheta+\omega+2} - t\left(\ell_1^{\vartheta+\omega+1}+\ell_2^{\vartheta+\omega+1}\right) - t^{\vartheta+\omega+1}(\ell_1+\ell_2) + \ell_1\ell_2\left(\ell_1^{\vartheta+\omega}+\ell_2^{\vartheta+\omega}\right)}{t^{\vartheta+\omega}(t-\ell_1)(t-\ell_2)}$$

$$= \frac{2t^2 - xt}{(t^2-2xyt-y)} - \frac{1}{t^{\vartheta+\omega}(t^2-2xyt-y)}\{tQ_{\vartheta+\omega+1}(x,y) + yQ_{\vartheta+\omega}(x,y)\}$$

□

Fourthly, we define identities involving common factors of bivariate Pell and Pell-Lucas polynomials.

Theorem 3.10. If $P_b(x,y)$ and $Q_b(x,y)$ are Bivariate Pell and Pell-Lucas polynomials, then holds for every b and s,

i. $P_{2b+s}(x,y)Q_{2b+1}(x,y) = P_{4b+s+1}(x,y) + (-y)^{2b+1}P_{s-1}(x,y)$

ii. $P_{2b+s}(x,y)Q_{2b+2}(x,y) = P_{4b+s+2}(x,y) + y^{2b+2}P_{s-2}(x,y)$

iii. $P_{2b+s}(x,y)Q_{2b}(x,y) = P_{4b+s}(x,y) + y^{2b}P_s(x,y)$

iv. $P_{2b-s}(x,y)Q_{2b+1}(x,y) = P_{4b-s+1}(x,y) + (-y)^{2b+1}P_{-s-1}(x,y)$

v. $P_{2b-s}(x,y)Q_{2b-1}(x,y) = P_{4b-s-1}(x,y) + (-y)^{2b-1}P_{1-s}(x,y)$

vi. $P_{2b-s}(x,y)Q_{2b}(x,y) = P_{4b-s}(x,y) + (-y)^{2b}P_{-s}(x,y)$

vii. $P_{2b}(x,y)Q_{2b+s}(x,y) = P_{4b-s}(x,y) - (-y)^{2b}P_s(x,y)$

viii. $(x^2y^2+y)P_{2b}(x,y)Q_{2b+s}(x,y) = Q_{4b+s}(x,y) - (-y)^{2b}Q_s(x,y)$

ix. $Q_{2b}(x,y)Q_{2b+s}(x,y) = Q_{4b+s}(x,y) + (-y)^{2b}Q_s(x,y)$

Proof.

Using Binet's formula of Bivariate Pell and Pell-Lucas polynomials and Principle of Mathematical Induction (PMI) on b and s, the proof is clear. □

Finally, we present two cross two matrix for bivariate Pell and Pell-Lucas polynomials by $B = \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix}$. Then, we can write, $B^n = \begin{bmatrix} P_{n+1}(x,y) & P_n(x,y) \\ yP_n(x,y) & yP_{n-1}(x,y) \end{bmatrix}$ and we get $det\ (B^n) = (-1)^n(y^n)$ (Cassini's identity).

Many authors introduce and present matrices properties and identities of bivariate polynomials [1,2,4].

Theorem 3.11. Let $b \in \mathbb{N}$. Then,

$$\begin{bmatrix} P_{b+1}(x,y) \\ yP_b(x,y) \end{bmatrix} = B\begin{bmatrix} P_b(x,y) \\ yP_{b-1}(x,y) \end{bmatrix}$$

Proof.

Let $b \in \mathbb{N}$. For $b = 1$,

$$\begin{bmatrix} P_2(x,y) \\ yP_1(x,y) \end{bmatrix} = B \begin{bmatrix} P_1(x,y) \\ yP_0(x,y) \end{bmatrix} = B \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The identity is valid for $b = 1$.

For the mathematical induction on b, suppose that the identity is true for b. Thus,

$$\begin{aligned} \begin{bmatrix} P_{b+2}(x,y) \\ yP_{b+1}(x,y) \end{bmatrix} &= \begin{bmatrix} 2xyP_{b+1}(x,y) + yP_b(x,y) \\ yP_{b+1}(x,y) \end{bmatrix} \\ &= \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} P_{b+1}(x,y) \\ yP_b(x,y) \end{bmatrix} \\ &= \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} P_b(x,y) \\ yP_{b-1}(x,y) \end{bmatrix} \\ &= \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} 2xyP_b(x,y) + yP_{b-1}(x,y) \\ yP_b(x,y) \end{bmatrix} \\ &= \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} P_{n+1}(x,y) \\ yP_n(x,y) \end{bmatrix} \\ &= B \begin{bmatrix} P_{b+1}(x,y) \\ yP_b(x,y) \end{bmatrix} \end{aligned}$$

□

Theorem 3.12. Let $b \in \mathbb{N}$. Then,

$$\begin{bmatrix} Q_{b+1}(x,y) \\ yQ_b(x,y) \end{bmatrix} = B \begin{bmatrix} Q_b(x,y) \\ yQ_{b-1}(x,y) \end{bmatrix}$$

Theorem 3.13. Let $b \in \mathbb{N}$. Then,

$$\begin{bmatrix} P_{b+1}(x,y) \\ yP_b(x,y) \end{bmatrix} = B^b \begin{bmatrix} P_1(x,y) \\ yP_0(x,y) \end{bmatrix}$$

Theorem 3.14. Let $b \in \mathbb{N}$. Then,

$$\begin{bmatrix} Q_{b+1}(x,y) \\ yQ_b(x,y) \end{bmatrix} = B^b \begin{bmatrix} Q_1(x,y) \\ yQ_0(x,y) \end{bmatrix}$$

4. Conclusion

In this paper, we present sums of bivariate Pell and Pell-Lucas polynomials. Moreover, we describe sums with negative indices, some connection formulas, and two cross two matrix representation and give several interesting identities involving them.

Author Contributions

The author read and approved the final version of the paper.

Conflict of Interest

The author declares no conflict of interest.

References

[1] M. Catalani, *Some formulae for bivariate Fibonacci and Lucas polynomials* (2004) 9 pages, https://arxiv.org/abs/math/0406323.

[2] M. Catalani, *Identities for Fibonacci and Lucas polynomials derived from a book of Gould* (2004) 7 pages, https://arxiv.org/abs/math/0407105.

[3] M. Catalani, *Generalized bivariate Fibonacci polynomials* (2004) 11 pages, https://arxiv.org/abs/math/0211366.

[4] H. Belbachir, F. Bencherif, *On some properties of bivariate Fibonacci and Lucas polynomials*, Journal of Integer Sequences 11 (2) (2008) 08.2.6 10 pages.

[5] N. Tuğlu, E. G. Koçer, A. Stakhov, *Bivariate Fibonacci like p-polynomials*, Applied Mathematics and Computation 217 (24) (2011) 10239–10246.

[6] S. Halıcı, Z. Akyüz, *On sum formulae for bivariate Pell polynomials*, Far East Journal of Applied Mathematics 41 (2) (2010) 101–110.

[7] N. Saba, A. Boussayoud, *Complete homogeneous symmetric functions of Gauss Fibonacci polynomials and bivariate Pell polynomials*, Open Journal of Mathematical Sciences 4 (1) (2020) 179–185.

[8] T. Machenry, *A subgroup of the group of units in the ring of arithmetic functions*, Rocky Mountain Journal of Mathematics 29 (3) (1999) 1055–1065.

[9] T. Machenry, *Generalized Fibonacci and Lucas polynomials and multiplicative arithmetic functions*, Fibonacci Quarterly 38 (2) (2000) 167–173.

[10] T. Machenry, G. Tudose, *Reflections on symmetric polynomials and arithmetic functions*, Rocky Mountain Journal of Mathematics 35 (3) (2006) 901–928.

[11] T. Machenry, K. Wong, *Degree k linear recursions mod(p) and number fields*, Rocky Mountain Journal of Mathematics 41 (4) (2011) 1303–1327.

JOURNAL OF AMASYA UNIVERSITY THE INSTITUTE OF SCIENCES AND TECHNOLOGY (JAUIST)

ISSN: 2717-8900 https://doi.org/10.54559/jauist.1400386

Statistical Riesz and Nörlund convergence for sequences of fuzzy numbers

Burak Arslan[1,*], **Samira Jalali**[2], **Serdar Enginoğlu**[3]

[1,2,3]Department of Mathematics, Faculty of Science, Çanakkale Onsekiz Mart University, Çanakkale, Türkiye

Abstract — Nuray and Savaş proposed statistical convergence of fuzzy number sequences. Afterward, Tripathy and Baruah presented Riesz and Nörlund convergence for sequences of fuzzy numbers. This paper defines statistical Riesz and Nörlund convergence of fuzzy number sequences. It then shows that if a sequence of fuzzy numbers is convergent, then it is statistical Riesz/Nörlund convergent, but the converse is not always true. Finally, this paper discusses the need for further research.

Keywords: *Statistical convergence, statistical Riesz convergence, statistical Nörlund convergence, sequences of fuzzy numbers*

Subject Classification (2020): 40A35, 46S40

1. Introduction

Thus far, many studies, such as de la Vallée-Poussin, Cesàro, Riesz, and Nörlund convergence [1-3], have been conducted on sequences of fuzzy numbers [4,5]. New convergence types for nonconvergent sequences of fuzzy numbers via Cesàro, Riesz, and Nörlund means have been proposed in these studies. Another useful type of convergence introduced for fuzzy number sequences is statistical convergence [6]. Afterward, statistical convergence and statistical Cesàro and p-Cesàro convergence [7-9] have been investigated. This study defines statistical Riesz and Nörlund convergence for fuzzy number sequences.

Section 2 of the present study provides some basic definitions to be required in the next section. Section 3 defines statistical Riesz and Nörlund convergence of fuzzy number sequences. Moreover, it shows that convergent sequences are statistical Riesz/Nörlund convergent, but the converse is not always correct. Finally, we discuss the need for further research.

2. Preliminaries

This section presents some basic notions to be needed for the following section.

Definition 2.1. A fuzzy set μ over $\mathbb{R}$ is called a fuzzy number if

i. there exists an $x \in \mathbb{R}$ such that $\mu(x) = 1$

ii. $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$

iii. for all $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $|x - a| < \delta \Rightarrow \mu(x) - \mu(a) < \varepsilon$

iv. the closure of $\{x \in \mathbb{R}: \mu(x) > 0\}$, denoted by $\overline{\text{supp}(\mu)}$, in the usual topology of $\mathbb{R}$ is compact

*Corresponding Author

[1]tburakarslan@gmail.com; [2]samira.jalali83@gmail.com; [3]serdarenginoglu@gmail.com
Article History: Received: 5 Dec 20223 — Accepted: 25 Dec 2023 — Published: 31 Dec 2023

Throughout this paper, the set of all the fuzzy numbers over $\mathbb{R}$ is denoted by $FN(\mathbb{R})$.

Definition 2.2. The α-level set $[\mu]_\alpha$ of $\mu \in FN(\mathbb{R})$ is defined by

$$[\mu]_\alpha := \begin{cases} \{x \in \mathbb{R}: \mu(x) \geq \alpha\}, & 0 < \alpha \leq 1 \\ \overline{\text{supp}(\mu)}, & \alpha = 0 \end{cases}$$

Proposition 2.3. Let $\mu \in FN(\mathbb{R})$. Then, the set $[\mu]_\alpha$, denoted by $[\mu^-(\alpha), \mu^+(\alpha)]$, is a closed, bounded, and non-empty interval for all $\alpha \in [0,1]$.

Proposition 2.4. The function D defined by, for all $\mu, \nu \in FN(\mathbb{R})$,

$$D(\mu, \nu) \coloneqq \sup_{\alpha \in [0,1]} \max\{|\mu^-(\alpha) - \nu^-(\alpha)|, |\mu^+(\alpha) - \nu^+(\alpha)|\}$$

is a metric on $FN(\mathbb{R})$, and $(FN(\mathbb{R}), D)$ is a complete metric space.

Proposition 2.5. Let $\mu, \nu, \eta, \omega \in FN(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then,

i. $D(\lambda\mu, \lambda\nu) = |\lambda| D(\mu, \nu)$

ii. $D(\mu + \nu, \eta + \nu) = D(\mu, \eta)$

iii. $D(\mu + \nu, \eta + \omega) \leq D(\mu, \eta) + D(\nu, \omega)$

Definition 2.6. A sequence (u_k) of fuzzy numbers is a function u from $\mathbb{N}$ to $FN(\mathbb{R})$. The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called the k^{th} term of the sequence.

Across this study, the set of all the sequences of fuzzy numbers is denoted by $w(F)$.

Definition 2.7. A sequence $(u_k) \in w(F)$ is called convergent to $u \in FN(\mathbb{R})$ if, for all $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $D(u_k, u) < \varepsilon$, for all $k \geq n_0$.

Hereinafter, the set of all the convergent sequences of fuzzy numbers is denoted by $c(F)$.

Definition 2.8. [3] Let $(u_k) \in w(F)$, (q_k) be a sequence of non-negative real numbers, not all zero and $q_1 > 0$, and $Q_n \coloneqq q_1 + q_2 + \cdots + q_n$, for all $n \in \mathbb{N}$. If $\lim_n R_n(q, u) = u_0 \in FN(\mathbb{R})$, then (u_k) is called Riesz-convergent to fuzzy number u_0 and denoted by R-$\lim_k u_k = u_0$ or $u_k \xrightarrow{R} u_0$ where

$$R_n(q, u) = \frac{1}{Q_n} \sum_{k=1}^{n} q_k u_k, \quad n \in \mathbb{N}$$

Definition 2.9. [3] Let $(u_k) \in w(F)$, (q_k) be a sequence of non-negative real numbers, not all zero and $q_1 > 0$, and $Q_n \coloneqq q_1 + q_2 + \cdots + q_n$, for all $n \in \mathbb{N}$. If $\lim_n N_n(q, u) = u_0 \in FN(\mathbb{R})$, then (u_k) is called Nörlund-convergent to fuzzy number u_0 and denoted by N-$\lim_k u_k = u_0$ or $u_k \xrightarrow{N} u_0$ where

$$N_n(q, u) = \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k+1} u_k, \quad n \in \mathbb{N}$$

From now on, the set of all the Riesz and Nörlund convergent sequences of fuzzy numbers are denoted by $Rc(F)$ and $Nc(F)$, respectively.

Definition 2.10. The natural density of a set $K \subseteq \mathbb{N}$ is defined by $\delta(K) \coloneqq \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|$ where $|\,.\,|$ denotes the cardinality of a set.

Definition 2.11. [6] A sequence $(u_k) \in w(F)$ is called statistical convergent (or briefly *st*-convergent) to $u_0 \in FN(\mathbb{R})$ and denoted by *st*-$\lim_k u_k = u_0$ or $u_k \xrightarrow{st} u_0$ if

$$\text{for all } \varepsilon > 0 \text{ and for all } k \text{ except for a set of natural density zero, } D(u_k, u_0) < \varepsilon$$

or

$$\text{for all } \varepsilon > 0,\ \delta(\{k \leq n : D(u_k, u_0) \geq \varepsilon\}) = 0$$

or

$$\text{there exists a subsequence } (u_{n_k}) \text{ such that } \lim_k \frac{k}{n_k} = 1 \text{ and } u_{n_k} \to u_0$$

Proposition 2.12. Let $(u_k) \in c(F)$. Then, $(u_k) \in sc(F)$, $(u_k) \in Rc(F)$, and $(u_k) \in Nc(F)$.

3. Statistical Riesz and Nörlund Convergence for Fuzzy Number Sequences

This section proposes statistical Riesz and Nörlund convergence of sequences of fuzzy numbers and investigates their properties.

Definition 3.1. Let $(u_k) \in w(F)$ and (u_{n_k}) be a Riesz convergent subsequence of (u_k) to $u_0 \in FN(\mathbb{R})$ such that $\lim_k \frac{k}{n_k} = 1$. Then, (u_k) is called statistical Riesz convergent (or briefly *stR*-convergent) to u_0 and denoted by *stR*-$\lim_k u_k = u_0$ or $u_k \xrightarrow{stR} u_0$. In other words,

$$stR\text{-}\lim_k u_k = u_0 \Leftrightarrow \exists (u_{n_k}) \ni R\text{-}\lim_{n_k} u_{n_k} = u_0 \wedge \lim_k \frac{k}{n_k} = 1$$

Throughout this study, the set of all the *stR*-convergent sequences of fuzzy numbers is denoted by $stRc(F)$.

Definition 3.2. Let $(u_k) \in w(F)$ and (u_{n_k}) be a Nörlund convergent subsequence of (u_k) to $u_0 \in FN(\mathbb{R})$ such that $\lim_k \frac{k}{n_k} = 1$. Then, (u_k) is called statistical Nörlund convergent (or briefly *stN*-convergent) to u_0 and denoted by *stN*-$\lim_k u_k = u_0$ or $u_k \xrightarrow{stN} u_0$. In other words,

$$stN\text{-}\lim_k u_k = u_0 \Leftrightarrow \exists (u_{n_k}) \ni N\text{-}\lim_{n_k} u_{n_k} = u_0 \wedge \lim_k \frac{k}{n_k} = 1$$

Across this study, the set of all the *stN*-convergent sequences of fuzzy numbers is denoted by $stNc(F)$.

Theorem 3.3. Let $(u_k) \in stc(F)$. Then, $(u_k) \in stRc(F)$.

Proof.

Let $u_k \xrightarrow{st} u_0$. Then, there exists a (u_{n_k}) such that $\lim_k \frac{k}{n_k} = 1$ and $u_{n_k} \to u_0$. From Proposition 2.12, $u_{n_k} \xrightarrow{R} u_0$.

□

The converse of Theorem 3.3 is not always correct.

Example 3.4. Let $(w_k) \in w(\mathbb{R})$ defined by

$$w_k(x) = \begin{cases} v_k, & \exists n \in \mathbb{N} \ni k = n^2 \\ u_k, & \forall n \in \mathbb{N}, k \neq n^2 \end{cases}$$

such that

$$u_k(x) = \begin{cases} \frac{k-2+x}{k}, & x \in [2-k, 2] \\ \frac{k+2-x}{k}, & x \in (2, 2+k] \\ 0, & \text{otherwise} \end{cases}$$

and

$$v_k(x)=\begin{cases} x-k, & x\in[k,k+1] \\ k+2-x, & x\in(k+1,k+2] \\ 0, & \text{otherwise}\end{cases}$$

Then, the α-level sets of u_k and v_k, for all $k\in\mathbb{N}$ and for all $\alpha\in[0,1]$, are as follows:

$$[u_k]_\alpha=[k\alpha-(k-2),(k+2)-k\alpha]$$

and

$$[v_k]_\alpha=[\alpha+k,k+2-\alpha]$$

Therefore, (w_k) is not convergent and also not statistical convergent because

$$\lim_k[u_k]_\alpha=\lim_k[2-k(1-\alpha),2+k(1-\alpha)]=\infty$$

Consider a sequence of real numbers $(q_k)=\left(\frac{k}{2^k}\right)$. Thus,

$$stR\text{-}\lim_k w_k=R\text{-}\lim_k u_k=\lim_k R_k(q,u)$$

where

$$R_k(q,u)=\begin{cases} \dfrac{2\left(2^k-1\right)(x+1)-k(k+x+2)}{6(2^k-1)-k(k+4)}, & x\in\left[\dfrac{k(k+1)}{2^{k+1}-(k+2)}-1,2\right] \\ \dfrac{k(-k+x-6)-2\left(2^k-1\right)(x-5)}{6(2^k-1)-k(k+4)}, & x\in\left(2,\dfrac{k(k+1)}{(k+2)-2^{k+1}}+5\right] \\ 0 & \text{otherwise}\end{cases}$$

and its α-level sets, for all $\alpha\in[0,1]$,

$$[R_k(q,u)]_\alpha=\left[3\alpha-1-\frac{(\alpha-1)k(k+1)}{2^{k+1}-(k+2)},5-3\alpha+\frac{(\alpha-1)k(k+1)}{2^{k+1}-(k+2)}\right]$$

because

$$[u_1]_\alpha=[\alpha+1,3-\alpha]$$

$$[u_2]_\alpha=[2\alpha,4-2\alpha]$$

$$[u_3]_\alpha=[3\alpha-1,5-3\alpha]$$

$$[u_4]_\alpha=[4\alpha-2,6-4\alpha]$$

$$[u_5]_\alpha=[5\alpha-3,7-5\alpha]$$

$$[u_6]_\alpha=[6\alpha-4,8-6\alpha]$$

$$\vdots$$

$$[u_k]_\alpha=[k\alpha-(k-2),(k+2)-k\alpha]$$

and

$$[R_1(q,u)]_\alpha=\frac{q_1u_1}{q_1}=u_1=[\alpha+1,3-\alpha]$$

$$[R_2(q,u)]_\alpha=\frac{q_1u_1+q_2u_2}{q_1+q_2}=\left[\frac{6\alpha+2}{4},\frac{14-6\alpha}{4}\right]$$

$$[R_3(q,u)]_\alpha=\frac{q_1u_1+q_2u_2+q_3u_3}{q_1+q_2+q_3}=\left[\frac{21\alpha+1}{11},\frac{43-21\alpha}{11}\right]$$

$$[R_4(q,u)]_\alpha = \frac{q_1u_1+q_2u_2+q_3u_3+q_4u_4}{q_1+q_2+q_3+q_4} = \left[\frac{58\alpha-6}{26}, \frac{110-58\alpha}{26}\right]$$

$$[R_5(q,u)]_\alpha = \frac{q_1u_1+q_2u_2+q_3u_3+q_4u_4+q_5u_5}{q_1+q_2+q_3+q_4+q_5} = \left[\frac{141\alpha-27}{57}, \frac{255-141\alpha}{57}\right]$$

$$[R_6(q,u)]_\alpha = \frac{q_1u_1+q_2u_2+q_3u_3+q_4u_4+q_5u_5+q_6u_6}{q_1+q_2+q_3+q_4+q_5+q_6} = \left[\frac{318\alpha-78}{120}, \frac{558-318\alpha}{120}\right]$$

$$\vdots$$

$$[R_k(q,u)]_\alpha = \left[\frac{(3\cdot 2^{k+1}-k^2-4k-6)\alpha-(2^{k+1}-k^2-2k-2)}{2^{k+1}-(k+2)}, \frac{5\cdot 2^{k+1}-k^2-6k-10-(3\cdot 2^{k+1}-k^2-4k-6)\alpha}{2^{k+1}-(k+2)}\right]$$

$$= \left[3\alpha-1-\frac{(\alpha-1)k(k+1)}{2^{k+1}-(k+2)}, 5-3\alpha+\frac{(\alpha-1)k(k+1)}{2^{k+1}-(k+2)}\right]$$

Thus,

$$\lim_k [R_k(q,u)]_\alpha = \lim_k \left[3\alpha-1-\frac{(\alpha-1)k(k+1)}{2^{k+1}-(k+2)}, 5-3\alpha+\frac{(\alpha-1)k(k+1)}{2^{k+1}-(k+2)}\right]$$

$$= [3\alpha-1, 5-3\alpha]$$

Hence,

$$\lim_k R_k(q,u) = \begin{cases} \frac{x+1}{3}, & x\in[-1,2] \\ \frac{5-x}{3}, & x\in(2,5] \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$Rst\text{-}\lim_k w_k = \begin{cases} \frac{x+1}{3}, & x\in[-1,2] \\ \frac{5-x}{3}, & x\in(2,5] \\ 0, & \text{otherwise} \end{cases}$$

Consequently, although (w_k) is not convergent and not statistical convergent, (w_k) is statistical Riesz convergent.

Corollary 3.5. Let $(u_k) \in c(F)$. Then, $(u_k) \in stRc(F)$.

Theorem 3.6. Let $(u_k) \in sc(F)$. Then, $(u_k) \in stNc(F)$.

The proof is similar to the proof of Theorem 3.3. The converse of Theorem 3.6 is not always correct.

Example 3.7. Consider (w_k) provided in Example 3.4 and $(q_k) = (2^k)$. Thus,

$$stN\text{-}\lim_k w_k = N\text{-}\lim_k u_k = \lim_k N_k(q,u)$$

where

$$N_k(q,u) = \begin{cases} \dfrac{(2^k-1)x-k}{2^{k+1}-(k+2)}, & x \in \left[\dfrac{k}{2^k-1}, 2\right] \\ \dfrac{-(2^k-1)x+k}{2^{k+1}-2-k}+2, & x \in \left(2, 4-\dfrac{k}{2^k-1}\right] \\ 0 & \text{otherwise} \end{cases}$$

and its α-level sets, for all $\alpha \in [0,1]$,

$$[N_k(q,u)]_\alpha = \left[2\alpha + \frac{k(1-\alpha)}{2^k-1}, 4-2\alpha+\frac{k(\alpha-1)}{2^k-1}\right]$$

because

$$[u_1]_\alpha = [\alpha+1, 3-\alpha]$$

$$[u_2]_\alpha = [2\alpha, 4-2\alpha]$$

$$[u_3]_\alpha = [3\alpha-1, 5-3\alpha]$$

$$[u_4]_\alpha = [4\alpha-2, 6-4\alpha]$$

$$[u_5]_\alpha = [5\alpha-3, 7-5\alpha]$$

$$[u_6]_\alpha = [6\alpha-4, 8-6\alpha]$$

$$\vdots$$

$$[u_k]_\alpha = [k\alpha-(k-2), (k+2)-k\alpha]$$

and

$$[N_1(q,u)]_\alpha = \frac{q_1u_1}{q_1} = u_1 = [\alpha+1, 3-\alpha]$$

$$[N_2(q,u)]_\alpha = \frac{q_2u_1+q_1u_2}{q_1+q_2} = \left[\frac{4\alpha+2}{3}, \frac{10-4\alpha}{3}\right]$$

$$[N_3(q,u)]_\alpha = \frac{q_3u_1+q_2u_2+q_1u_3}{q_1+q_2+q_3} = \left[\frac{11\alpha+3}{7}, \frac{25-11\alpha}{7}\right]$$

$$[N_4(q,u)]_\alpha = \frac{q_4u_1+q_3u_2+q_2u_3+q_1u_4}{q_1+q_2+q_3+q_4} = \left[\frac{26\alpha+4}{15}, \frac{56-26\alpha}{15}\right]$$

$$[N_5(q,u)]_\alpha = \frac{q_5u_1+q_4u_2+q_3u_3+q_2u_4+q_1u_5}{q_1+q_2+q_3+q_4+q_5} = \left[\frac{57\alpha+5}{31}, \frac{119-57\alpha}{31}\right]$$

$$[N_6(q,u)]_\alpha = \frac{q_6u_1+q_5u_2+q_4u_3+q_3u_4+q_2u_5+q_1u_6}{q_1+q_2+q_3+q_4+q_5+q_6} = \left[\frac{120\alpha+6}{63}, \frac{246-120\alpha}{63}\right]$$

$$\vdots$$

$$[N_k(q,u)]_\alpha = \left[\frac{(2^{k+1}-2-k)\alpha+k}{2^k-1}, \frac{2^{k+2}-4-k-(2^{k+1}-2-k)\alpha}{2^k-1}\right]$$

$$= \left[2\alpha + \frac{k(1-\alpha)}{2^k-1}, 4-2\alpha+\frac{k(\alpha-1)}{2^k-1}\right]$$

Thus,

$$\lim_k [N_k(q,u)]_\alpha = [2\alpha, 4-2\alpha]$$

Hence,

$$\lim_k N_k(q,u) = \begin{cases} \dfrac{x}{2}, & x \in [0,2] \\ \dfrac{4-x}{2}, & x \in (2,4] \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$stN\text{-}\lim_k w_k = \begin{cases} \frac{x}{2}, & x \in [0,2] \\ \frac{4-x}{2}, & x \in (2,4] \\ 0, & \text{otherwise} \end{cases}$$

Consequently, although (w_k) is not convergent and not statistical convergent, (w_k) is statistical Nörlund convergent.

Corollary 3.8. Let $(u_k) \in c(F)$. Then, $(u_k) \in stNc(F)$.

4. Conclusion

This paper proposed statistical Riesz and Nörlund convergence of sequences of fuzzy numbers. It then showed that if a sequence of fuzzy numbers is convergent, then it is statistical Riesz/Nörlund convergent, and the converse is not always correct by two examples. In the future, the Tauberian conditions for a statistical Riesz/Nörlund convergent sequence to be convergent/statistical convergent and Korovkin-type theorems can be studied.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

References

[1] V. Loku, N. L. Braha, M. Et, A. Tato, *Tauberian theorems for the generalized de la Vallée-Poussin mean-convergent sequences of fuzzy numbers*, Bulletin of Mathematical Analysis and Applications 9 (2) (2017) 45–56.

[2] S. Enginoğlu, S. Demiriz, *A comparison with the convergent, Cesàro convergent and Riesz convergent sequences of fuzzy numbers*, in: İ. Yüksek, F. Yiğit, H. Baraçlı (Eds.), FUZZYSS'15 The 4th International Fuzzy Systems Symposium, İstanbul, 2015, pp. 416–419.

[3] B. C. Tripathy, A. Baruah, *Nörlund and Riesz mean of sequences of fuzzy real numbers*, Applied Mathematics Letters 23 (5) (2010) 651–655.

[4] M. Matloka, *Sequences of fuzzy numbers*, BUSEFAL 28 (1) (1986) 28–37.

[5] S. Nanda, *On sequences of fuzzy numbers*, Fuzzy Sets and Systems 33 (1) (1989) 123–126.

[6] F. Nuray, E. Savaş, *Statistical convergence of sequences of fuzzy numbers*, Mathematica Slovaca 45 (3) (1995) 269–273.

[7] J-S. Kwon, *On statistical and p-Cesàro convergence of fuzzy numbers*, Korean Journal of Computational and Applied Mathematics 7 (1) (2000) 195–203.

[8] Y. Altın, M. Mursaleen, H. Altınok, *Statistical summability (C,1) for sequences of fuzzy real numbers and a Tauberian theorem*, Journal of Intelligent & Fuzzy Systems 21 (6) (2010) 379–858.

[9] S. Aytar, S. Pehlivan, *Statistical convergence of sequences of fuzzy numbers and sequences of α-cuts*, International Journal of General Systems 37 (2) (2008) 231–237.

Journal of Amasya University the Institute of Sciences and Technology 4(2) (2023) 108-115

JOURNAL OF AMASYA UNIVERSITY THE INSTITUTE OF SCIENCES AND TECHNOLOGY (JAUIST)

ISSN: 2717-8900 https://doi.org/10.54559/jauist.1404602

Investigation of the potential use of halloysite nanotube doped chitosan films for food packaging

Filiz Uğur Nigiz[1,*], Buket Onat[2]

[1,2]Department of Chemical Engineering, Faculty of Engineering, Çanakkale Onsekiz Mart University, Çanakkale, Türkiye

Abstract − Polymer-based food packaging is widely used and causes serious environmental problems due to the chemical ingredients. Therefore, these packages should be replaced by biodegradable alternatives in order to prevent environmental pollution. Many biodegradable polymers are used in food packaging. Among them, chitosan is gaining attention since it is bio-sourced and biodegradable. In this study, the usability of chitosan films as physical and chemical tests investigated food packaging. In order to improve the packaging properties of the films, halloysite nanotube was used as filler with a concentration range of 1-4 wt.%. It was observed that the halloysite significantly increased the opacity, mechanical strength, water resistance, and antioxidant properties of the films.

Keywords: *Chitosan films, composite films, food packaging, halloysite nanotube*

Subject Classification (2020):

1. Introduction

The plastic-based environmental problems are increasing day by day in accordance with plastic consumption depending on the human population. Petrochemical-derived packaging materials, especially those that cannot be recycled after use, are known to remain in the soil and seas for a long time. It is also known that these are broken down into phthalates with sunlight and are harmful to the entire ecosystem. In addition, it has also been proven that chemicals in contact with food undergo plastic transfer to food under light and heat. Therefore, the use of bioplastics for food packaging will have a positive impact on both health and the environment [1-3]. However, it is known that the mechanical strength and water resistance of biopolymers are relatively low compared to the petrochemical-based packaging. For this reason, these deficiencies are tried to be overcome with strong fillers.

Chitin is one of the most abundant biopolymers in the world. It is found in the exoskeletons of insects, arthropods such as crabs, shrimp, and the cell walls of fungi [4]. Chitosan is also biocompatible, completely degradable, soluble in water, and forms a colloidal solution. It can be used in hydrogel or film form [5]. It is seen as a suitable polymer for many fields of study due to its antimicrobial properties, metal binding ability, high mechanical strength, non-toxicity, and biodegradability.

Halloysite nanotube (HNT) is a tubular natural clay with a large surface area. It is used as a nanofiller material in film formation and improves the mechanical properties of the film. Depending on the negatively charged surface of HNT, it enables the slow release of antimicrobial substances in its structure, allowing their effects

*Corresponding Author

[1]filiz.ugur@comu.edu.tr; [2]buketonatt@gmail.com

Article History: Received: 14 Dec 2023 — Accepted: 29 Dec 2023 — Published: 31 Dec 2023

to last for a long time [6] It has been revealed that it has a potential to replace traditional films in recent years due to its long-term durability and food spoilage retarding properties. In the literature, limited studies have performed on use of HNT in chitosan [7]. In studies where it was used other than chitosan, it was observed that it increased many properties of films simultaneously.

Salmos et al. [8] aimed to extend the shelf life of kiwi fruit by obtaining a biodegradable film by dispersing thymol-enriched halloysite nanotube structure in chitosan/polyvinyl alcohol gel. Mechanical properties, transparency, antimicrobial and antioxidant properties were investigated. It was reported that the HNT simultaneously increased both parameters. Risyon et al. [9] aimed to strengthen the low thermal resistance of biopolymers with nanostructure additives by producing polylactic acid/halloysite nanotube (HNT) films. When the mechanical, thermal, and barrier properties were examined, it was concluded that the optimum HNT doping was obtained as 3% by weight. The critical point in composite polymeric films is the homogeneous distribution of the filler through the films. Homogeneous distribution of filler improves the properties throughout the film. For this, it is necessary to determine the appropriate ratio.

In this study, HNT doped chitosan films were prepared, and their usability in food packaging was investigated. The HNT ratio was kept between 1-5% by weight. The effect of HNT doping on film opacity, moisture content, swelling, mechanical strength, and antioxidant properties was investigated.

2. Materials and Methods

Medium molecular weight Chitosan powder was purchased from Aldrich Chemicals. The HNT nanoparticles were kindly supplied from Esan Eczacıbası, Türkiye. Acetic acid (analytic grade) was purchased from Merck Chemical.

2.1. Film Preparation

Films were prepared by solution casting method. The aqueous solution containing 1 wt.% chitosan was stirred at room temperature for 24 hours. The solution contains 2% acetic acid. After the homogeneous mixture was obtained. HNT particles were added with the weight concentration of 1-5% and stirred for three hours. The solution was degassed under vacuum and casted on a polymethyl methacrylate plate. After casting, the films were allowed to dry at room temperature for 2 days and peeled off gently. The films were named according to the concentration of HNT (CS for the pristine chitosan, CS-HNT1, CS-HNT2, CS-HNT3, CS-HNT4 for the filled chitosan films)

2.2. Characterization

FTIR analysis of the membranes was performed with The Agilent Cary 630 FTIR spectrometer. This test was performed to determine the structural moisture retention properties of membranes and to examine their chemical bond structures. The test was performed in the wavelength ranges of 650-4000 cm^{-1}.

The light transmittance of the prepared films is determined by opacity tests. For this test, the opacity of the films cut in certain sizes was examined by measuring the absorbance at 600 nm in UV/Vis spectrophotometer (Shimadzu-1280). The opacity was calculated as shown in (2.1)

$$Opacity = \frac{Absorbance}{Thickness\ of\ films} \tag{2.1}$$

The percentage of moisture trapping of the films in standard media was determined by moisture content tests. The films were dried at 105°C for 24 hours until to constant weight and measured gravimetrically (M_i). The films were kept on a water bath without contact at room temperature and the percentage of weight gain (W_g) was calculated as moisture content as shown in (2.2).

$$Moisture\ content = \frac{Wg}{Wi} * 100 \tag{2.2}$$

The swelling test is used to determine the water resistance of films. To test the swelling properties of the films, each sample was dried in an oven at 65 °C for 12 hours before the test. The films were soaked in 25 mL deionized water for 12 hours and the values before (Wi) and after (Wf) water retention were recorded as shown in (2.3).

$$Swelling\ Degree = \frac{Wf - Wi}{Wi} * 100 \tag{2.3}$$

The antioxidant properties of the added additives were determined by antioxidant activity test. According to this test, films weighing 0.1 gram were placed in 70% methanol and 2,2-diphenyl-1-picrylhydrazyl (DPPH) stable radical (10 ml 70% methanol, 0.2 mg DPPH) and kept in the dark for 90 minutes. When this radical reacts with the antioxidant, a change from violet to yellow is observed and the antioxidant activity is calculated by measuring the absorbance at 520 nm with UV-Vis Spectrophotometer [10].

The mechanical analysis of the films was analyzed in Ankarin brand mechanical analyzer according to ASTM D882 standard. The mechanical strengths were performed by measuring the mechanical strength (stress) and elongation (strain) from the force and elongation at break of the strips cut in 4*1 cm size. Experiments were performed at a tensile speed of 3 mm/min.

3. Results and Discussion

The FTIR spectrum of the films are shown in Figure 1. A prominent band within the range of 3000–3620 cm^{-1} is indicative of N–H and O–H stretching. It is clear that the intensity of the peak decreasing by increasing HNT depending on the reducing moisture content.

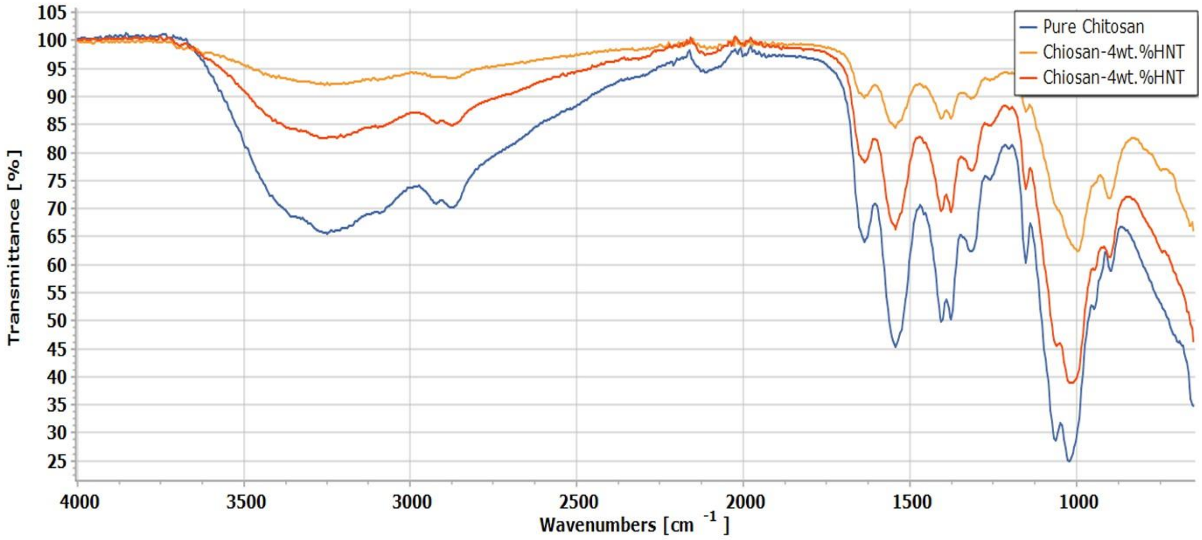

Figure 1. FTIR spectra of the pure and HNT loaded films

C-H symmetric and asymmetric stretching are responsible for the absorption bands at approximately 2920 and 2876 cm^{-1}, respectively. These bands are typical polysaccharide properties and are present in the spectra of various polysaccharides. The bands at about 1638 cm^{-1} (C=O stretching) and 1318 cm^{-1} (C-N stretching) respectively confirmed the presence of residual N-acetyl groups [11].

The light transmittance of the packaging films produced is a very important parameter [12]. While light can be degrading for some foods, an optimal degree of opacity is desired as a high degree of opacity reduces the visibility of the food [13,14]. Especially considering the freshness of the food, it is desired that it refracts and does not transmit too much light. Figure 2 shows the opacity of the films. Accordingly, all additive ratios increased the opacity values compared to pure chitosan film.

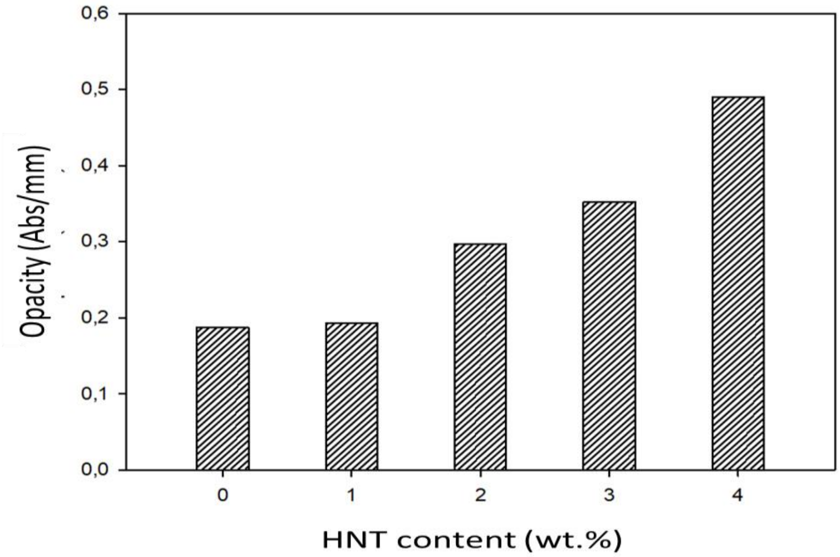

Figure 2. Opacity results of films with and without HNT loading

Figure 3a shows the results of the swelling tests of the additives in pure water for 30 minutes. Swelling tests are normally performed for very long hours, but in this study chitosan was not crosslinked. This is because the cross-linking process prevents understanding the antimicrobial activity or determining the actual character of the films. Therefore, the swelling tests were short because pure chitosan dissolves in water and disintegrates after 20-30 minutes. As can be seen in the figure, when the HNT content in the membrane increased from zero to 2 percentage, its resistance to water was significantly increased. However, there is a small increase of 3 percent and 4 percent. This shows that when the films are loaded with excess HNT, the HNT structure also interacts with water and fills its pores.

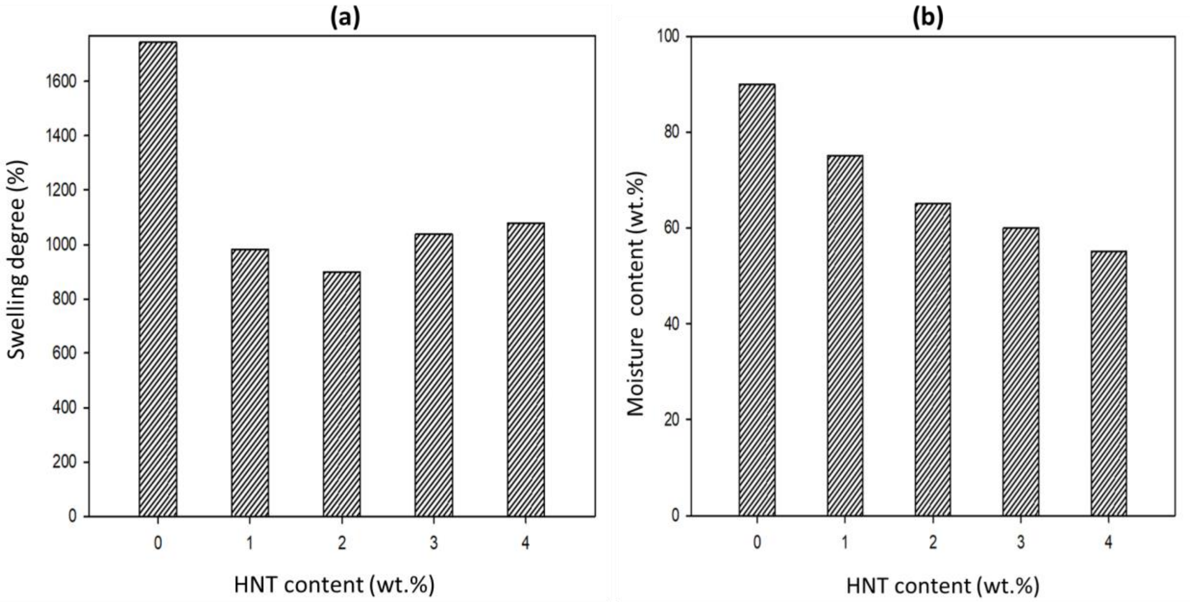

Figure 3. Swelling degree (a) and Moisture content (b) results of films with and without HNT loading

Similar to the swelling test, the moisture retention (content) test is a measure of the resistance of films to water vapor. Unlike swelling, in this test the films interact only with the vapor at room temperature. Moisture retention value is actually expected similar results to swelling. However, since there is no direct contact with water, the structure of the HNT particles is not expected to be filled with water. Therefore, as seen in Figure 3b, the vapor contents gradually decreased with the addition of HNT. While the pure film retains a lot of vapor, this value decreased significantly in HNT doped films. This shows that the resistance of the films to moisture retention increases under room conditions, and therefore the degree of degradation decreases.

Oxidative reactions change the basic properties of foods and cause them to deteriorate. Antioxidants are microcomponents that can scavenge free radicals by terminating oxidative chain reactions. Therefore, antioxidant activity is an important feature for food packaging [15-17]. In this study, DPPH free radical scavenging method is used to evaluate the antioxidant capacity of the chitosan films. Figure 4 shows the antioxidant values of the films depending on the DPPH reagent. This value is also an indicator of the resistance of the films against oxidation and is expected to be as high as possible. As seen in the figure, while the antioxidant value of the film was 20% in the pure film, this value increased to over 24% in the 4% added film. that is, more than 20% antioxidant properties were improved with 4% additive.

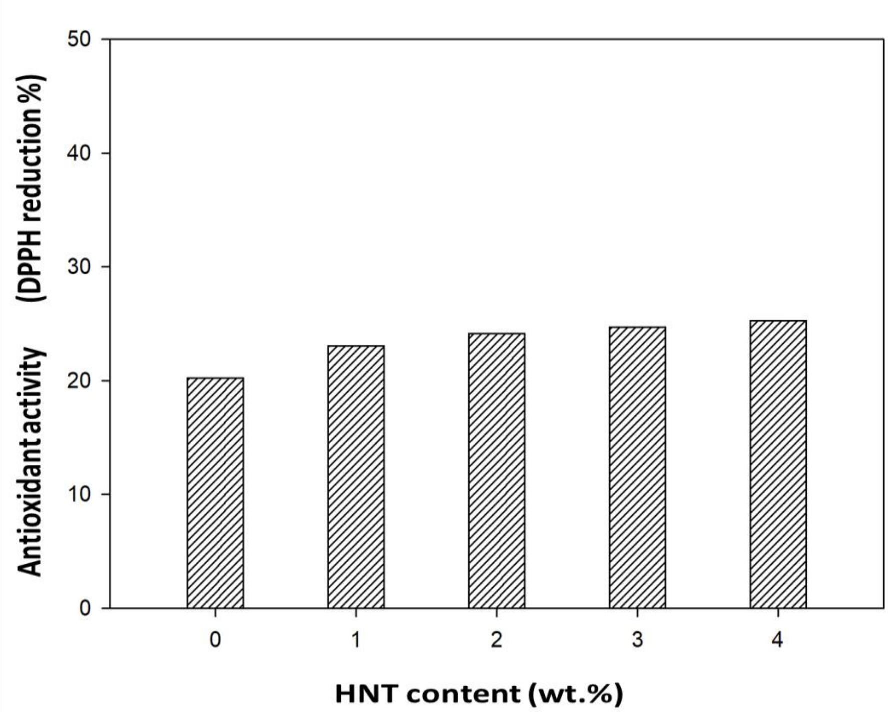

Figure 4. Antioxidant results of films with and without HNT loading

Figure 5 shows the mechanical analysis results of the additive ratios. Mechanical analysis is evaluated in two ways. The first is the stress, which is an indicator of the force applied at break, and the other is the strain value calculated based on the amount of elongation at break. The strength of the films is described by the stress value. Strain gives more information about elasticity. As seen in the figure, the stress value of pure chitosan is around 5.7 MPa. As seen in Figure 5, this value increased to 10.8 MPa with the addition of HNT. However, it decreased after this value. The reason for this is that the additive material added to the polymeric matrix increases the mechanical strength, contributes to load transfer in homogeneous distribution, but due to agglomeration caused by overloading, the load cannot be distributed properly and weak points are formed and cause rupture [18,19].

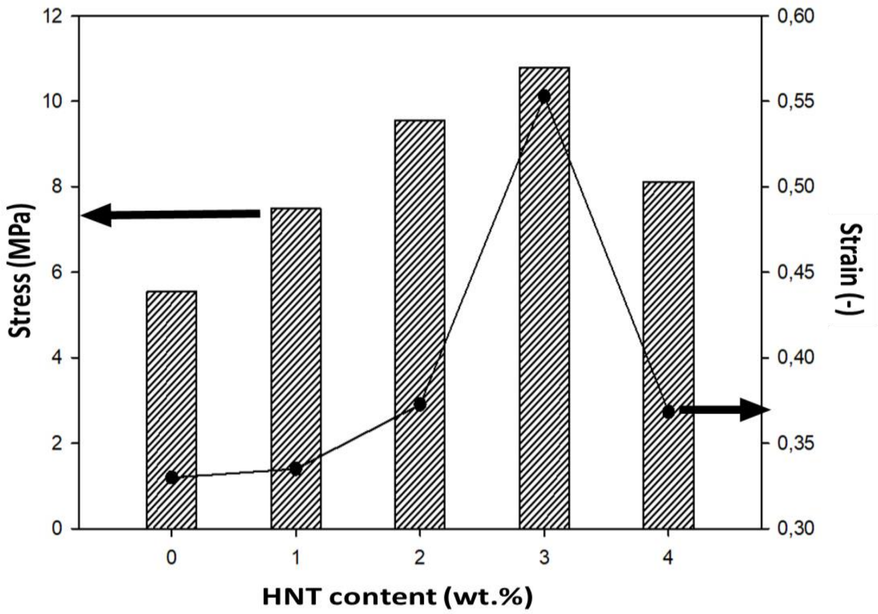

Figure 5. Stress-strain results of films with and without HNT loading

4. Conclusion

The environmental impact of reducing the use of plastic and replacing it with biodegradable packaging is increasing day by day. In this way, carcinogenic compounds resulting from plastic degradation are not formed, and naturally occurring films reduce the carbon footprint in nature. In this study, chitosan-based films were prepared for use in food packaging. In order to improve the mechanical properties, swelling and moisture retention properties, opacity, antioxidant and antimicrobial properties of the films, HNT additives were added between 1-4%. As a result of the study,

i. The HNT additives increased the opacity values depending on the increasing ratios and since this is an indicator that reduces the light transmittance of the films, it has an inhibitory effect on food degradation.

ii. The increasing ratio of HNT decreased the swelling and solubility of the films in water.

iii. Increasing ratios of HNT in films significantly decreased moisture retention. This is a factor preventing the degradation of the films and thus the food when exposed to moisture and steam.

iv. Increasing HNT content in films increased the mechanical strength and generally above 3% the mechanical strength decreased due to agglomeration.

v. Increasing HNT content improved the antioxidant properties.

In the next stage of the study, the degradation processes of the produced films in soil and atmospheric environments will be examined, and accordingly, their sustainability will be examined according to biodegradability criteria.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

Acknowledgment

This study was supported by the Scientific Research Projects Coordination at Çanakkale Onsekiz Mart University, Grant number: FHD-2023-4418.

References

[1] S. Mangaraj, A. Yadav, L. M. Bal, S. K. Dash, N. K. Mahanti, *Application of biodegradable polymers in food packaging industry: A comprehensive review*, Journal of Packaging Technology and Research 3 (1) (2019) 77–96.

[2] A. P. A. de Carvalho, C. A. C. Junior, *Green strategies for active food packagings: A systematic review on active properties of graphene-based nanomaterials and biodegradable polymers*, Trends in Food Science Technology 103 (2020) 130–143.

[3] S. V. N. Vijayendra, T. R. Shamala, *Film forming microbial biopolymers for commercial applications-A review*, Critical Reviews in Biotechnology 34 (4) (2014) 338–357.

[4] S. K. Shukla, A. K. Mishra, O. A. Arotiba, B. B. Mamba, *Chitosan-based nanomaterials: A state-of-the-art review*, International journal of biological macromolecules 59 (2013) 46–58.

[5] N. Wrońska, A. Anouar, M. E. Achaby, K. Zawadzka, M. Kędzierska, K. Miłowska, N. Katir, K. Draoui, S. Różalska, I. Piwoński, M. Bryszewska, A. E. Kadib, K. Lisowska, *Chitosan-functionalized graphene nanocomposite films: interfacial interplay and biological activity*, Materials 13 (4) (2020) 998 16 pages.

[6] Q. Li, T. Ren, P. Perkins, X. Hu, X. Wang, *Applications of halloysite nanotubes in food packaging for improving film performance and food preservation*, Food Control 124 (2021) 107876 8 pages.

[7] D. Huang, Z. Zhang, Y. Zheng, Q. Quan, W. Wang, A. Wang, *Synergistic effect of chitosan and halloysite nanotubes on improving agar film properties*, Food Hydrocolloids 101 (2020) 105471 7 pages.

[8] C. E. Salmas, A. E. Giannakas, D. Moschovas, E. Kollia, S. Georgopoulos, C. Gioti, A. Leontiou, A. Avgeropoulos, A. Kopsacheili, L. Avdylaj, C. Proestos, *Kiwi Fruits Preservation Using Novel Edible Active Coatings Based on Rich Thymol Halloysite Nanostructures and Chitosan/Polyvinyl Alcohol Gels*, Gels 8 (12) (2022) 823 23 pages.

[9] N. P. Risyon, S. H. Othman, R. K. Basha, R. A. Talib, *Characterization of polylactic acid/halloysite nanotubes bionanocomposite films for food packaging*, Food Packaging and Shelf Life 23 (2020) 100450 10 pages.

[10] A. Riaz, C. Lagnika, H. Luo, Z. Dai, M. Nie, M. M. Hashim, C. Liu, J. Song, D. Li, *Chitosan-based biodegradable active food packaging film containing Chinese chive (Allium tuberosum) root extract for food application*, International Journal of Biological Macromolecules 150 (2020) 595–604.

[11] S. Moalla, I. Ammar, M. L. Fauconnier, S. Danthine, C. Blecker, S. Besbes, H. Attia, *Development and characterization of chitosan films carrying Artemisia campestris antioxidants for potential use as active food packaging materials*, International Journal of Biological Macromolecules183 (2021) 254–266.

[12] J. Sutharsan, C. A. Boyer, J. Zhao, *Physicochemical properties of chitosan edible films incorporated with different classes of flavonoids*, Carbohydrate Polymer Technologies and Applications 4 (2022) 100232 11 pages.

[13] G. Yuan, H. Lv, B. Yang, X. Chen, H. Sun, *Physical properties, antioxidant and antimicrobial activity of chitosan films containing carvacrol and pomegranate peel extract*, Molecules 20 (6) (2015) 11034–11045.

[14] S. Ma, Y. Zheng, R. Zhou, M. Ma, *Characterization of chitosan films incorporated with different substances of konjac glucomannan, cassava starch, maltodextrin and gelatin, and application in Mongolian cheese packaging*, Coatings 11 (1) (2021) 84 16 pages.

[15] J. Liu, S. Liu, Y. Chen, L. Zhang, J. Kan, C. Jin, *Physical, mechanical and antioxidant properties of chitosan films grafted with different hydroxybenzoic acids*, Food Hydrocolloids 71 (2017) 176–186.

[16] J. Liu, S. Liu, Q. Wu, Y. Gu, J. Kan, C. Jin, *Effect of protocatechuic acid incorporation on the physical, mechanical, structural and antioxidant properties of chitosan film*, Food Hydrocolloids 73 (2017) 90–100.

[17] M. A. López-Mata, S. Ruiz-Cruz, N. P. Silva-Beltrán, J. D. J. Ornelas-Paz, P. B. Zamudio-Flores, S. E. Burruel-Ibarra, *Physicochemical, antimicrobial and antioxidant properties of chitosan films incorporated with carvacrol*, Molecules 18 (11) (2013) 13735–13753.

[18] C. M. Yoshida, E. N. O. Junior, T. T. Franco, *Chitosan tailor-made films: the effects of additives on barrier and mechanical properties*, Packaging Technology and Science: An International Journal 22 (3) (2009) 161–170.

[19] C. Hu, F. Shen, D. Zhu, H. Zhang, J. Xue, X. Han, *Characteristics of Ti3C2X–chitosan films with enhanced mechanical properties*, Frontiers in Energy Research 4 (2017) 41 5 pages.